Mortality Linked Derivatives and their Pricing

Raj Kumari Bahl

Doctor of Philosophy
The University of Edinburgh
July 2017

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. The work has not been submitted for any other degree or professional qualication.

(*Raj Kumari Bahl*)

To my parents and my most beloved husband...

Abstract

This thesis addresses the absence of explicit pricing formulae and the complexity of proposed models (incomplete markets framework) in the area of mortality risk management requiring the application of advanced techniques from the realm of Financial Mathematics and Actuarial Science. In fact, this is a multi-essay dissertation contributing in the direction of designing and pricing mortality-linked derivatives and offering the state of art solutions to manage longevity risk. The first essay investigates the valuation of Catastrophic Mortality Bonds and, in particular, the case of the Swiss Re Mortality Bond 2003 as a primary example of this class of assets. This bond was the first Catastrophic Mortality Bond to be launched in the market and encapsulates the behaviour of a well-defined mortality index to generate payoffs for bondholders. Pricing this type of bond is a challenging task and no closed form solution exists in the literature. In my approach, we adapt the payoff of such a bond in terms of the payoff of an Asian put option and present a new methodology to derive model-independent bounds for catastrophic mortality bonds by exploiting the theory of comonotonicity.

While managing catastrophic mortality risk is an upheaval task for insurers and re-insurers, the insurance industry is facing an even bigger challenge - the challenge of coping up with increased life expectancy. The recent years have witnessed unprecedented changes in mortality rate. As a result academicians and practitioners have started treating mortality in a stochastic manner. Moreover, the assumption of independence between mortality and interest rate has now been replaced by the observation that there is indeed a correlation between the two rates. Therefore, my second essay studies valuation of Guaranteed Annuity Options (GAOs) under the most generalized modeling framework where both interest rate and mortality risk are stochastic and correlated. Pricing these types of options in the correlated environment is an arduous task and a closed form solution is non-existent. In my approach, I employ the use of doubly stochastic stopping times to incorporate the randomness about the time of death and employ a suitable change of measure to facilitate the valuation of survival benefit, there by adapting the payoff of the GAO in terms of the payoff of a basket call option. I then derive general price bounds for GAOs by employing the theory of comonotonicity and the Rogers-Shi (Rogers and Shi, 1995) approach. Moreover, I suggest some 'model-robust' tight bounds based on the moment generating function (m.g.f.) and characteristic function (c.f.) under the affine set up. The strength of these bounds is their computational speed which makes them indispensable for annuity providers who rely heavily on Monte Carlo simulations to calculate the fair market value of Guaranteed Annuity Options. In fact, *sans* Monte Carlo, the academic literature does not offer any solution for the pricing of the GAOs. I illustrate the performance of the bounds for a variety of affine processes governing the evolution of mortality and the interest rate by comparing them with the benchmark Monte Carlo estimates.

Through my work, I have been able to express the payoffs of two well known modern mortality products in terms of payoffs of financial derivatives, there by filling the gaps in the literature and offering state of art techniques for pricing of these sophisticated instruments.

Keywords: Catastrophic Mortality Bonds, model-independent bounds, Asian options, comonotonicity, Guaranteed Annuity Option (GAO), model-robust bounds, Affine Processes, interest rate risk, mortality risk, change of measure, Basket option.

AMS subject classifications (MSC2010): Primary 91G20; secondary 60G44, 60J25, 68U20.

Lay Summary

Since the beginning of mankind, man has been striving hard to become immortal. Though, immortality is still a dream, modern times have been a witness to significant improvements in life expectancy. While this is appealing to hear, increasing life expectancy generates risk for economic agents such as pension funds, insurance companies and governments, because of increased liabilities. Not only this, mortality advancements, especially at older ages make individuals with inadequate pension arrangements vulnerable to poverty threats. This is called 'Longevity Risk':- the risk that a population will live longer than expected, so that adequate savings are unavailable to fulfill the needs of the population.

Knowing the right trend improvements in life expectancy, is the key to handle longevity risk. But, it has been hard to realize this trend in the past; even official agencies have systematically underestimated previous mortality improvements. Pension plan and annuity providers are now raising the question whether longevity is a risk they should be assuming on an unhedged basis. The capital markets are trying their best to offer remedies for handling and unloading longevity risk. These issues came to the limelight after the debacle of the oldest U.K. life insurer 'Equitable Life Assurance Society' (ELAS) due to mis-pricing of longevity linked securities called 'Guaranteed Annuity Options' (GAOs) which offer the flexibility to the buyer to choose between a lump sum or a life-long guaranteed annuity scheme at the time of retirement. When many of these guarantees were written in the UK in the 1970s and 1980s: long-term interest rates were high and mortality tables did not include an explicit allowance for future mortality improvements (longevity). Ever since, however, long-term interest rates declined and mortality improved significantly for lives on which these policies were sold, leading to a lot of these policies to be in money and therefore getting exercised, putting undue burden on the insurer. As a result, pricing of these instruments is a key to solving the 'Longevity Puzzle'. Keeping this in mind, we offer state of art solutions for pricing GAOs under the most general framework, allowing longevity to be stochastic and letting it interact with the financial markets. The strength of our methodologies lies in their versatility, speed and accuracy. These solutions will prove to be an indispensable tool for pension and annuity providers in the future.

While managing longevity improvements is proving to be an upheaval task for pension planners and annuity providers, there is another shade to the uncertainty connected to human lifespan that is threatening the life insurance industry:- the fear of a huge number of deaths due to a catastrophic event such as a pandemic, a natural disaster, a terrorist attack or any other man-made error. This is called 'Mortality Risk'. The occurrence of such events has increased exponentially over the last few decades. These catastrophes have the power to cripple the life insurance industry and therefore large reinsurance giants such as Swiss Re have designed sophisticated instruments called 'Mortality Bonds' to sell the mortality risk to markets. Though these bonds have been successfully subscribed, their complex structure makes it hard to put a fair price on these. We have addressed this challenge by offering 'model-independent' crisp price bounds for the first such mortality bond called 'VITA I'. Comparison of these bounds with the benchmark Monte Carlo estimates illustrates their sharpness around the true value. Once again, we solve a complex pricing problem and provide a powerful methodology in the hands of re-insurers such as Swiss Re, which can be used to test under a variety of models to protect against the 'worst case scenario' and shield against, otherwise difficult to avoid: 'model risk'.

Thus, mortality and longevity: the two sides of the same coin; are causing a lot of distress to financial institutions and addressing the problems related to their management is the need of the hour. The results derived in this thesis offer efficient solutions for pricing complex mortality and longevity linked securities and therefore should appeal alike to 'Financial Mathematicians', 'Actuarial Experts' and practitioners.

Acknowledgements

First of all, I would like to express my deepest gratitude to my supervisor, Dr. Sotirios Sabanis, whose expertise, enthusiasm for research, understanding and encouragement added considerably to the completion of my thesis. I appreciate not only his mathematical comprehension and vast knowledge in many areas, but also his kindness and amicable way of mentoring. He aroused my joy of mathematical research, inspired me with his invaluable ideas, motivated me with fascinating discussions, encouraged me to attend international conferences and workshops and enabled me to deepen my understanding of stochastic analysis and mathematical finance. His guidance has helped me to achieve unbelievable results and would continue to be a source of motivation in all my future endeavours.

My special thanks also go to my second supervisor Professor István, Gyöngy, who provided me with helpful advice and gave invaluable inputs to solve complex questions that I encountered while building my foundations in mathematical finance. The collaboration with him considerably enriched my research experience and led to a very fruitful scientific work.

I would also like to acknowledge Institute and Faculty of Actuaries (IFoA) and the University of Edinburgh who not only provided me with financial support for my Ph.D. studies but also helped me to disseminate my research around the globe with travel grants assistance. I am particularly thankful to Mr. Kevin McIver, former 'Research Relationship Manager', IFoA who was instrumental in encouraging me to participate and present in conferences and workshops connected to my research topic by providing me with a summary of all upcoming events and helping with the travel grants applications. He was the key in organizing my research presentation in the 'Knowledge Sharing Scotland -2015' (KSS) event in Edinburgh that gave me an invaluable experience by interaction with actuarial practitioners.

I would also like to thank Professor Jan Dhaene for his invaluable inputs and suggestions on my first research paper that I presented at the 20th 'Insurance: Mathematics and Economics - 2016' Conference (IME - 2016). His promptness and kindness are unparalleled. Not only this, I also thank him for agreeing to become the external examiner for my thesis.

I also express my extreme gratitude to Professor Finn Lindgren, Chair of Statistics at the School of Mathematics, University of Edinburgh for assuming the responsibility of becoming the internal examiner for my thesis and arranging my Viva. He has been extremely kind to take time out of his busy schedule to arrange the Viva date.

I also thank Professor Hansjoerg Albrecher at Department of Actuarial Science, Université de Lausanne for discussion on some interesting properties of Lévy processes. Further, I am also thankful to the organizers of 'Probability Working Seminars' which provided excellent opportunity to breathe in new ideas of mathematical research. My colleague Mr. Umesh Kumar from University of Delhi who is pursuing Ph.D. from King's College, London also deserves a great deal of thanks for some very useful discussions on various topics of stochastic analysis.

I am also grateful to the administrative staff at JCMB who has always extended their help with a smile. In particular, it is my pleasure to thank Mrs. Gill Law for her excellent support and promptness in dealing with administrative work. I would also like to thank Mr. Chris Jowett for his help and support on administrative matters. Mr. Martin Delaney at the

MTO (Mathematics Teaching Office) also deserves applaud for his generous help in organizing workshops and handling technical issues. During the tenure of my Ph.D., I participated in a number of international conferences, winter schools, workshops and seminars. I am thankful to the 'Laura Wisewell Travel Fund' and the School of Mathematics student travel fund for providing me the necessary financial support.

I also acknowledge the support I got from the 'Mathworks' team online to learn new features of latest versions of MATLAB. In particular, I am extremely grateful to Walter Roberson and Torsten.

I am also thankful to all my colleagues at the Department of Mathematics and Statistics, Ramjas College, University of Delhi who have always encouraged me to move on. In particular, I express my heart felt gratitude to Dr. Babita Goyal, who has always showered me with love and blessings.

Moreover, I am especially indebted to my husband Amit ji for being a strong pillar of strength and support throughout my journey. His love and constant motivation has helped me to achieve the most challenging tasks. I also express my sincere and enormous thanks to my parents and my maternal uncle for their unconditional moral support and encouragement to follow my dreams and accomplish my goals. Last but not the least, I would like to thank the Almighty who has always bestowed his blessings on me and has helped me to successfully complete this thesis.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Since the beginning of evolution, mankind has always cherished a dream - the dream of being immortal. Though immortality is still an illusion, modern times have been a witness to significant improvements in life expectancy. As an example, data from the HMD (2014) depicts that life expectancy at birth in England and Wales increased from a meager 41.59 years in 1841 to a mammoth 81.04 years in 2011.

This increase in life expectancy, though appealing to hear and clearly a sign of social progress, generates risk for governments, economic agents such as pension funds and insurance companies and even individuals because of increased liabilities such as pension for the former group and health costs for the latter. Since everything in this world comes for a cost, longevity is associated with disabilities and therefore individuals need money to ensure a good quality of living standard at old ages and companies need a greater pool of funds to pay the regular stream of payments to individuals for a much longer time. However, this increased financial risk is not just an outcome of the increase in life expectancy. If there was a possibility of fully anticipating mortality improvements, then appropriate methods could be devised for their management. Rather, the real challenge emerges as a result of the uncertainty surrounding the increases in life expectancy. The magnitude of this uncertainty is being widely debated by demographers who are keen to gauge the extent to which past increases in life expectancy will continue into the future. While one school of thought (Oeppen and Vaupel, 2002) argues based on recent mortality trends, that no natural limit can be fixed for life expectancy, the second school of thought (Olshansky et al., 2005) suggests that life expectancy might level off or even decline.

As a result, the uncertainty of future mortality trends, i.e. "longevity risk", can result in important economic implications for individuals, pension plans, annuity providers and social insurance programs such as Social Security. On the one hand, mortality advancements especially at older ages makes individuals with inadequate pension arrangements vulnerable to poverty threats as they face the risk of outliving their savings. On the other hand, pension plans and annuity providers may incur significant financial losses if they consistently underestimate the survival probabilities of their pensioners and policyholders. Moreover, in addition to having substantial pension obligations such as social security programs, governments act as residual risk bearers of last resort and are becoming increasingly concerned about the grave financial consequences of citizens outliving their resources. These potential consequences of longevity risk are particularly acute in current times when the baby boom generation is starting to retire and real interest rates are at low levels.

Besides the uncertainty in life expectancy, there is another major cause that causes grave financial implications for life insurers and that is "mortality risk". While longevity risk described above denotes adverse financial consequences that arise when an individual or group lives longer than expected (i.e. their mortality rate is lower than what was expected at the time that the financial balancing of assets, set aside for future consumption or future payments, was made), its counterpart mortality risk describes serious financial complications arising due to a shorter life time than anticipated of an individual or group (i.e. their mortality rate is higher than what was in the premium/benefit balancing equation). More explicitly, life insurance companies provide protection to their policyholders in the form of a payout made in the event of a policyholder's death, in exchange for a premium. Extreme mortality events, such as a severe pandemic or a natural catastrophe or a large terrorist attack, could result in a life insurance company needing to make sudden payouts to many policyholders. This large payout would be exacerbated in that the investment portfolio would not yet have delivered sufficient returns so that the payouts to policyholders are made sooner than expected. Therefore it is crucial for life insurers, and life re-insurers, to manage their exposure to extreme mortality risks where insurance portfolio diversification by itself is insufficient.

The International Actuarial Association defines four components of longevity/mortality risk viz. *level, trend, volatility and catastrophe.* The four components are classified into two groups which are *systematic risk* and *specific risk* or *idiosyncratic risk* (c.f. Crawford et al., 2008). Systematic risk is defined as the underestimation or overestimation of the base assumption of mortality rates, including the level component and the trend component. Specific risk is taken to be the volatility around the base assumption, including the volatility component and the catastrophe component. According to the famous law of large numbers in Statistics, specific risks can be minimized by diversifying with a large pool of lives; however it is not possible to reduce systematic risk by diversification.

Although mortality risk and longevity risk are two sides of the same coin and even if a certain mutualization exists between these risks, it is hard to obtain a significant risk reduction between the two because of the diversity in their nature. While mortality risk is a short-term risk having a 1 to 5 year maturity and possessing a catastrophic component, longevity risk is a long term risk with maturities ranging from 20 to 80 years and is primarily connected to changes in trend. The successful addressing of the challenges derived from longevity and mortality risk has lead to the development of the area of 'Mortality Risk Management' (Rejda, 2005) which has manifested itself in terms of 'Mortality-linked Derivatives'[1]. The pricing of these derivatives hinges upon the success of portraying the dynamics of future stochastic mortality. The difficulty in pricing these derivatives also stems due to the incompleteness of the market associated with these securities which rules out the existence of a unique 'Equivalent Martingale Measure' or EMM[2]. for pricing them. This PhD thesis is devoted to the investigation of the techniques for managing mortality and longevity risk and in particular to the designing and pricing of mortality-linked derivatives.

In the next few sections, we explore the historical background of morality linked securities, look at the causes and remedies of the mortality and longevity risk, throw light on the motivation for the present work and explain the organization of the thesis.

1.1 Historical Background

We break the historical background into two sections: talking about the history of mortality linked securities and about the models used for describing mortality. For a good review of the historical background, interested readers can refer to Bauer (2008) and Chuang (2013).

1.1.1 Mortality Contingent Securities

Mortality contingent securities have been into existence for a long time. In ancient Rome, contracts called *annua* offered the option to receive annual, life-long payments against a one-time, upfront premium. These "annual stipends" were the precursor of single premium life annuities. Moreover, so-called *collegia funeratica* allowed the poor class of the Roman society to finance

[1]This is a general term to encompass both mortality and longevity linked instruments. We will mention the specific demarcation where required.

[2]A martingale measure is desired here in order to serve as a risk-neutral measure for arbitrage free asset pricing in an incomplete market such as that governing mortality-linked derivatives which in general do not have an underlying trading security upon which they are based.

costly funeral ceremonies by paying small periodic premiums; as such, these confraternities were the first funeral expense funds and may be regarded as the first mutual life insurers. However, the first recorded life insurance policy in modern times was issued in the year 1583; though, it was not until 1759 and 1762 that the first pure life insurance companies started business in the United States and the United Kingdom, respectively, namely the *Presbyterian Ministers' Fund* (U.S.) and the famous *Equitable Society for the Assurance of Life and Survivorship* (U.K.) (c.f. Depew, 1895).

Since then, there has been an exponential growth in the number of life insurance organizations. The interesting fact is that apart from the number of companies, the last few decades have witnessed a huge variety and complexity of the products offered by these organizations. These days, life insurance and life annuity products not only offer protection against the risk of an untimely death, but they also serve as sophisticated investment vehicles containing various options and guarantees. However, this increasing complexity of products is accompanied by unacquainted, non-obvious risks which tested the capacity of insurers and which they either failed to pass or are struggling to come to terms with. As an example, the "Equitable Life" closed to new business in 2000 because they underestimated the amount of risks arising due to interest rate and mortality sensitive guarantees within their products. As indicated above, 'The Equitable Life Assurance Society' (ELAS), a mutual insurer founded in 1762, primarily sold with-profits policies (equivalent to participating policies in the U.S.), in which the Society's surplus was shared with its policyholders. The policies that led to the troubles were issued from 1957-1988 and contained a guaranteed annuity rate. These policies typically guaranteed that £100 cash at retirement could be converted into a £10 per annum annuity, regardless of external financial conditions at the time. The Society's Guaranteed Annuity Rates (GARs) were more generous and flexible (c.f. Baranoff and O'Brien, 2016) and lead to its downfall as life expectancy improved and these options were encashed. This led to the highlighting of the problem of 'Longevity Risk'. We consider this in detail in Section 1.3.3.

The main cause behind the problems of life insurers and annuity providers has been the usage of outdated valuation and risk management methods. As a result in recent years life insurance industry has started to revise their valuation approaches, which is also due to regulatory changes demanding apt valuation procedures, such as the upcoming Solvency II requirements of the European Union and Phase II of the International Financial Reporting Standard 4. In particular, methods from modern mathematical finance are being applied. The drift from 'Deterministic' to 'Stochastic' mortality was the first big step towards a modern approach to actuarial valuations connected to mortality.

Soininen (1995) was among the first to consider the impact of stochastic mortality rates on life insurance premiums and reserves in a functional analytic framework, but his work did not attract much attention. In contrast, the articles by Milevsky and Promislow (2001) and Olivieri (2001) were highlighted and initiated a flow of contributions on stochastic mortality modeling with a focus on applications in actuarial science. Again, many of those employ mathematical methods from time-series modeling or the theory of stochastic differential equations, particularly modeling approaches from mathematical finance. This also brings us to the history of mortality modeling discussed in the next sub-section.

1.1.2 Mortality Modeling

Graunt (1662) was the pioneer of modeling mortality who examined the London Bills of Mortality. He showed that the life span of individuals was predictable in the aggregate and he developed a life table to describe this structured process of death. Halley (1693) had better data and was the first one to illustrate the construction of a rigorous (essentially modern) mortality table from empirical data and its application to price life annuities. Later on in the 18th century, De Moivre (1725) first postulated a functional form mortality table based on uniform distribution of deaths model and demonstrated annuity table calculations using this mathematical model. Then, in the 19th century, Laplace (1820) applied theoretical basics from probability theory, to actuarial modeling. He was followed by Gompertz (1825) who adopted a biological approach and the mathematical formulation came as a consequence to give way to the famous Gompertz law.

The above mortality models are static in nature, i.e., they involve the age x of the person at a fixed point (year) in time but ignore the evolution of mortality over time and do not pay attention to random variation in mortality. Recent mortality modeling attempts to incorporate both the age at death as well as temporal dimension and is mostly based on the two dimensional (age and time) structure supplied by Lee and Carter (1992). In the Lee-Carter (LC) model, while the temporal effect reflects changes in mortality rates across all ages as the time dimension varies, the age component gauges the mortality changes with increasing age when the time dimension is fixed. More specifically, the LC model portrays the evolution of mortality over time by an auto-regressive integrated moving average process, with the special case random walk with drift being the most commonly used. What this modeling method captures is "diffusion risk", which arises from the variations around a fixed drift that determines the expected trend, but not the uncertainty connected with the trend itself.

Although the LC model furnishes the basic framework for mortality modeling, it fails to completely account for rare events such as war, epidemics or natural disasters generally called catastrophes. Such catastrophes create a sudden abnormal increase in the mortality rate referred to in the literature by the name of 'mortality jump'. To accommodate these jumps, many authors have suggested extensions to the LC model[3]. Brouhns et al. (2002) made some modifications to the LC model by using a Poisson random variation for the number of deaths instead of the additive error term in the original model. This is considered to be more practical since the mortality rate is much more variable at older ages than at younger ages. Another extension added to the LC model was to incorporate another term which accounts for the cohort effect found by Renshaw and Haberman (2006). This cohort effect extension attempts to capture the observed phenomenon that during certain times and in some places people born during a certain time period or a particular year (say 1972) will experience a similar mortality rate pattern which differs from those of their immediate predecessors (born in 1971) or immediate successors (born in 1973).

While the Lee-Carter model and each extension is successful in improving mortality forecasting, an unintended consequence that it brings along inherent complexity which renders these extended models unsuitable for pricing of sophisticated mortality linked derivatives. To adequately price a mortality/longevity linked financial derivative, one needs a stochastic mortality model along with an EMM.

Affine[4] processes and in particular Lévy processes have been used by a number of authors to model mortality rate and enable martingale pricing. For example, Biffis (2005) introduces mortality intensity as a sum of deterministic function and an affine jump diffusion process. Luciano and Vigna (2005) utilize affine processes to depict the evolution of mortality rate and furnish detailed calibration using UK data. In their paper, they advocate that a non mean-reverting process is more appropriate to model mortality rate than a mean-reverting one. Moreover, the addition of negative jumps into the diffusion process appears better to fit the real mortality data and forecast the mortality trend. Ballotta and Haberman (2006) propose a model for the force of mortality having a perturbation part driven by a Brownian motion. Mortality intensity, as proposed by Hainaut and Devolder (2008), is a sum of a deterministic function as in Biffis (2005) and a mean reverting stochastic process, driven by tempered alpha-stable subordinators (see Cont and Tankov, 2003, for an introduction). Ahmadi and Gaillardetz (2015) use a Poisson Generalized Liner Model consisting of Gamma and Variance-Gamma processes to model perturbations. Poisson Generalized Linear Models found their application in actuarial science in mortality modeling context in work of Renshaw and Haberman (2003). Wang et al. (2011) report that mortality indices in the LC model possess thicker tails than a normal distribution and appear to be skewed. They propose to use Generalized Hyperbolic

[3]See for example Renshaw et al. (1996), Sithole et al. (2000), Milevsky and Promislow (2001), Ballotta and Haberman (2006) and Deng et al. (2012).

[4]We discuss Affine processes in Chapter 4.

and Classical Tempered Stable distributions to model the mortality indices in the Lee-Carter model.

Another well-known stochastic model portraying mortality evolution was proposed by Cairns, Blake and Dowd in 2006, (c.f. Cairns et al., 2006b) and is known as the CBD model in literature. This model exploits the relative simplicity of mortality curve at higher ages but does not cater to younger ages. This model is extremely popular for pension plan valuations in the UK due to its good performance in capturing mortality dynamics of older age groups. In contrast with the LC model, it contains two factors in describing mortality evolution. The first factor influences mortality rate dynamics at all ages in the same manner whilst the other one affects mortality rate dynamics at higher ages more significantly in comparison to lower ages. Empirical evidence favours the theory that both factors are necessary to achieve a satisfactory fit over the entire mortality term structure. A considerable advantage of their methodology is that it incorporates varied improvements at different ages and at different times - something which was missing in the LC model. Analysis and comparison of some well known mortality models appears in Cairns et al. (2009, 2011) and Dowd et al. (2010b,a).

Research advances in modern finance have inspired research developments in the field of insurance. Milidonis et al. (2011) initialized the concept of 'Regime-Switching' (RS) approach into mortality rate modeling. They introduce a stochastic drift by modeling the time-varying factor in the LC model using a regime-switching log-normal process with two regimes. The drift term in each regime is allowed to be different, so that the drift of the process may vary as the system switches between the two regimes under an assumed Markov chain. Through the examination of the US population mortality index, they illustrated that there were structural changes in the underlying death probability for all age cohorts from all death causes. Moreover, they utilized the concept of regime-switching to model the error term of mortality index in the LC model. This captures the disturbances caused by extreme observations over time and makes the error term non-normal. However, as indicated in their estimation results, it is the volatility rather than the drift that separates the two regimes. Therefore, the regime-switching process may be appropriate for portraying short term catastrophic mortality events which are generally accompanied with high mortality volatility, but may not be sufficient for capturing the risk arising from drift changes. Another attempt to introduce a stochastic drift was made by Sweeting (2011), who considered a piece-wise linear regression. Although the slope of the regression line is allowed to vary in the future, the probability and extent of future slope variations are computed in an ad hoc manner. In fact, the probability is taken as the ratio of the number of observed break points to the total number of data points, while the extent is estimated using the root mean square of the annual changes in the underlying dynamic factor. Alternative mortality models that deal with the risk associated with trend changes in a comprehensive manner are yet to be formulated. In a recent paper, Yanxin and Li (2017) propose a new model called the 'Locally Linear CBD' (LLCBD) model based on the CBD model, in which the drifts that govern the expected mortality trend are allowed to follow a stochastic process and gives the user a chance to quantify not only "diffusion risk" but also "drift risk". It is also a good paper to review the recent upto date developments in stochastic modeling of mortality.

Other cause specific models have also been proposed and we would mention the important ones under the suitable heads. In the next couple of sections we throw light on the causes and effects of the two sides of the Mortality Risk Management.

1.2 Mortality Risk: Cause and Remedies

As pointed out earlier, managing mortality risk is a key objective of the life insurance and reinsurance companies in the present day world. In this section, we investigate the causes behind mortality risk and look at some well known remedies.

1.2.1 The Cause: Catastrophes

Mortality risk is the risk that actual mortality will turn out to be greater than projected. The factors that affect mortality risk of a life insurer or re-insurer (c.f. Garvey, 2011) are:

- *Random Statistical Fluctuations*
- *Mis-estimation of General Mortality trends*
- *Data Issues and Miscalculation of Claim Levels*
- *Catastrophic Events.*

Out of these, the last one is the most important. Catastrophic mortality events are characterized by a sudden and concentrated increase in mortality and as such present a major threat to the life insurance industry due to the potential for a substantial rise in claims over a short period of time. As a result, severe adverse financial consequences can potentially arise, such as breaches in regulatory solvency and capital requirements (see Cox and Hu, 2004, for details). Before throwing light on the main sources of catastrophic risks, we first try to understand the exact definition of a catastrophe.

A catastrophic event can be defined as *"any natural or man-made incident, including terrorism, which results in extraordinary levels of mass casualties, damage, or disruption severely affecting the population, infrastructure, environment, economy, national morale, and/or government functions."*[5] In the context of life insurance and particularly for the purposes of reinsurance, a catastrophic event has a specific meaning or definition, similar to *"one event or occurrence claiming more than an agreed number (a common figure is 5) of lives insured within a given period, usually 24-72 hours"* (IAAust, 2009). However, for the purpose of this thesis, we are referring to an event which has the potential of causing widespread loss of life, possibly leading to substantial risks of insolvency for a life insurer.

The number of catastrophic events has risen sharply in the last four decades. In the 1970s there were roughly 100 catastrophic events per year. This number has consistently more than tripled in the last decade. Between 1994 and 2013, EM-DAT[6] recorded 6,873 natural disasters which claimed 1.35 million lives or almost 68,000 lives on average each year. In addition, 218 million people were affected by natural disasters on average per annum during this 20-year period. We list below the possible mortality catastrophes

- *Infectious diseases/ Pandemics*
- *Natural Disasters*
- *Terrorist Attacks*
- *Wars*
- *Meteorite Crashes*
- *Accidents: Industrial, Transport, Aviation or others.*

Of these, pandemics arising from influenza are considered to be the most serious threat to the life insurance industry due to their capacity to cause a major increase in claims. For a good review of the features and mortality implications of an influenza pandemic for life insurers and details about other catastrophes, the interested readers can refer to Huynh et al. (2013).

Catastrophes lead to spikes in the mortality curve. Figure 1.1 clearly shows the spikes caused in the US insured Age Standardized Mortality Curve, particularly the sharpest spike, arising due to 1918 influenza pandemic. In fact there have 13 or more influenza pandemics since 1500 out of four occurred in the 20th century viz.

[5] http://www.definitions.net/definition/catastrophic event, accessed April 18, 2017.

[6] The Emergency Events Database (**EM-DAT**) is a free and fully search-able database containing worldwide data on the occurrence and impact of over 20,000 natural and technological disasters from 1900 to date.

- *Spanish Flu (1918)*
- *Asian Flu (1957)*
- *Hong Kong Flu (1968)*
- *Russian Flu (1977)*.

It is possible to classify influenza pandemics according to their case-fatality ratio[7] (CFR), when the data are available. The pandemic severity index introduced by the United States Department of Health and Human Services in 2007 categorizes pandemics into 5 different classes according to their CFR, ranging from less that 0.1% in Category 1 to 2% or higher in Category 5. The Spanish flu of 1918 is by far the most severe influenza pandemic to date and is placed in Category 5 while the Asian and Hong Kong flu are set into Category 2. It is difficult to obtain the pandemic severity index for all the past influenza pandemics since the data are not readily available. A good reference here is Taubenberger and Morens (2009) and they identify the pandemics in 1557, 1580, 1729 and 1889 as severe pandemics with high fatality rates.

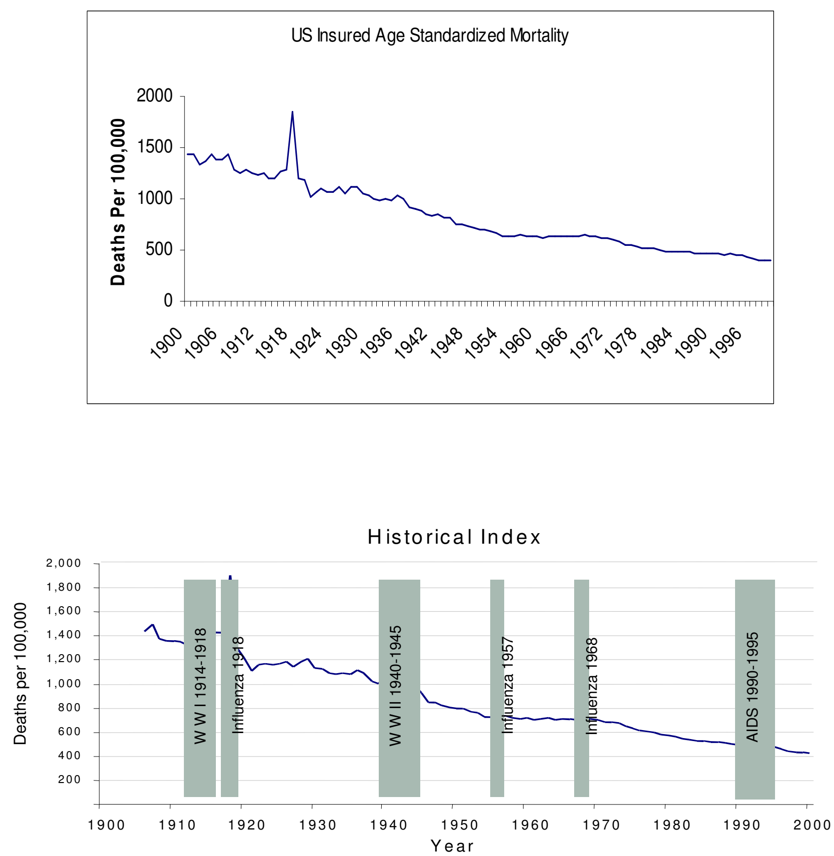


Figure 1.1: Spikes in the US insured Age Standardized Mortality Curve (Source: Klein, 2005)

The important question that comes to mind is that can a flu happen again? According to modern virologists and epidemiologists the answer is affirmative (c.f. Webster and Walker, 2009). In the 21st century, we saw the outbreak of Swine Flu in 2009. According to RMS

[7]The percentage of deaths out of the total reported cases of the disease in the U.S.

pandemic model[8] the risk of a pandemic outbreak in any given year is 1-in-30, while the chance of a 1957-magnitude outbreak is 1-in-40 and that for a 1918-caliber pandemic is 1-in-475. As a result it is reasonable to assume that similar influenza pandemics will occur in future. Also for reasons such as inter-species transmission, intra-species variation and altered virulence, the timings and severity of future pandemics and hence mortality jumps is unpredictable (c.f. Cox et al., 2003).

Mortality jumps are not common but when they occur they trigger a large number of deaths and hence a large number of unexpected death claims, thereby crippling the financial strength of the life insurance industry. In fact, the positive jumps in the mortality curve can be transitory, as caused by natural or man-made catastrophes or pandemics or can be persistent and more lasting, as caused by diseases such as AIDS, SARS, antibiotic resistant strain of tuberculosis etc. The following table from Dacorogna and Cadena (2015) highlights the importance of this discussion.

Table 1.1: List of the main Pandemic, the year it started and Deaths attributed to them (in million) in the last 100 years

Year	1918	1957	1968	1981	2002	2006	2014
Type	Spanish Flu	Asian Flu	Hong Kong Flu	AIDS	SARS	Avian Flu	Ebola
Deaths	30	4	2	25	.008	.002	0.006

Stracke and Heinen (2006) estimated that the worst pandemic would result in approximately €45 billion of additional claim expenses in Germany. This amount is indeed equivalent to five times the total annual gross profit or 100% of the policyholder bonus reserves in the German life insurance market. Toole (2007) highlighted that in a severe pandemic scenario, additional claim expenses would consume 25% of the Risk Based Capital (RBC) of the entire US life insurance industry so that companies with less than 100 % of RBC are at the risk of being insolvent. To understand more clearly, the financial implications of pandemics, we furnish parts of an interesting table from Huynh et al. (2013) in Table 1.2.

Here we have just discussed the potential impact of an influenza pandemic on the life insurance industry. The other possible natural and man-made catastrophes (particularly terrorism[9]) listed above are equally deadly and possess the strength to send tremors across the life insurance business. As an example, in a very recent research report, Swiss Re (Lucia et al., 2017) have published that "Total economic losses and global insured losses from natural catastrophes and man-made disasters in 2016 were the highest since 2012, reversing the downtrend of the previous four years. Globally there were 327 disaster events in 2016, of which 191 were natural catastrophes and 136 were man-made. In total, the disasters resulted in economic losses of USD 175 billion, almost double the level in 2015. In terms of devastation wreaked, there were large-scale disaster events across all regions, including earthquakes in Japan, Ecuador, Tanzania, Italy and New Zealand. In Canada, a wildfire across the wide expanses of Alberta and Saskatchewan turned out to be the country's biggest insurance loss event ever and the second costliest wildfire on sigma records globally. Worldwide, around 11,000 people lost their lives or went missing in disasters in 2016. There were a number of severe flood events in 2016, in the US, Europe and Asia. Hurricane Matthew, the first Category 5 storm to form over the North Atlantic since 2007, was also a major humanitarian disaster causing more than 700 deaths, mostly in Haiti - compared to all single events this year." In context of man-made disasters, 2017 has

[8] http://gcportal.guycarp.com/portal/extranet/getDoc

[9] Very close to writing this thesis the London Bridge, Manchester and Westminister attacks though small in scale took place in London in 2017. The biggest terrorist attack was the World Trade Centre attack on 11th September 2011 claiming 3000 lives. Clearly, more deadly attacks as those involving the use of biological weapons have the potential to inflict a large number of casualities.

already showed the venom that an incident like Grenfell Tower fire in London could spit within hours, killing seventy nine people and testing the strength of insurance industry. Looking at the seriousness of the problem, it is extremely important for life insurers and re-insurers to manage their exposure to catastrophic mortality risk. However it is not a straight road because the probability of such an event occurring in any year is low while the potential for devastating losses is high. Moreover, since catastrophes such as influenza pandemics are rare events, there is scarce data for forming assumptions within the range of internal risk models adopted by companies, and as such calibration of required parameters leaves many loopholes. This would typically be investigated with a range of stress tests, scenario testing, and sensitivity testing (c.f. Baumgart et al., 2007). In the next section we investigate several possibilities relating to the mitigation of risk arising from exposure to catastrophes.

Table 1.2: Summary of Studies Examining the Potential Impact of an Influenza Pandemic on the Life Insurance Industry

Author(s)	Country	Severity	EMRR[10]%	Duration (Years)	AGC[11] or ANC[12]
APRA (2007)	Australia	Severe	100	1	AGC: AUD 1.2 bn
Dreyer et al. (2007)	S.Africa	Mild Moderate	Group life: 70	1 1	AGC: ZAR 0.8 bn AGC: ZAR 2.7 bn
S & H[14] (2006)	Germany	Severe Severe	Ind. life:40[13] 100	1 1	AGC: ZAR 37.6 bn ANC: EUR 5.1 bn
Toole (2007)	U.S	Moderate Severe	57.1 76.9	1 1	ANC: US $2.8 bn ANC: US $64.3 bn
Weisbart (2006)	U.S	Moderate Severe	100 100	1 1	AGC: US $31 bn AGC: US $133 bn

1.2.2 Taming the CAT

When a fire breaks out in a city, there needs to be a prompt firefighting response to contain the fire and prevent it from spreading. The outbreak of a major fire is the wrong time to hold discussions on the pay of firefighters, to raise money for improving the fire service or to consider fire insurance. It is too late in the day to do all that.

As discussed above, just like fire, infectious diseases also spread at an exponential rate. On March 21, 2014, an outbreak of Ebola was confirmed in Guinea. In April, the World Health Organization (WHO) declared that it would cost a modest sum of $5 million to control the disease. In July this cost of control touched $100 million and by October it had ballooned to $1 billion. Ebola acted both as a serial killer and loan shark. If money in not made available readily enough to deal with the outbreak of an epidemic, its magnitude and intensity may go

[10]Excess Mortality Rate Ratio of Insured to General Population is the difference between observed mortality rate and expected baseline mortality rate in the absence of an influenza pandemic (c.f. Simonsen et al., 1997).
[11]Additional Gross Claims
[12]Additional Net Claims
[13]EMRR for Dreyer et al. (2007) is the same across all three scenarios but is differentiated into group life and individual life products.
[14]Stracke and Heinen (2006)

out of hand. However in general this scenario has been repeating itself with many pandemics and the largest casualty is the insurance business.

Several possibilities exist relating to the risk reduction arising from catastrophes. Again a good reference in this direction is Huynh et al. (2013). We list below the possible remedies employed by insurers and re-insurers to safeguard themselves from the calamity of increased claims caused due to a catastrophe and then discuss a few of them in detail with the first one being the most important in the context of this thesis.

- *Catastrophic or Extreme Mortality Bonds*
- *Risk Transfer Mechanisms such as reinsurance or retrocession*[15]
- *Self insuring or retaining the risk through holding greater levels of capital*[16]
- *Natural Hedging by balancing mortality risk with longevity risk*
- *Diversification along other lines of businesses*
- *Appropriate pricing for the risk*

While the first method is a recent innovation, the others are traditional methods of risk mitigation. We will discuss the first one in detail in the next sub-section. In particular, reinsurance or retrocession has been the most popular method of offloading the risk. This consists in transferring the risk from a smaller and less diversified insurer to a larger re-insurer with a more diversified portfolio. However, the ceding party ultimately lands up with the same risk it seeks to transfer, via the credit risk of the counter party re-insurer. This is due to the inherent possibility of reinsurer and retrocessionaire defaulting when faced with widespread catastrophic losses such as in a pandemic. Reinsurance is essentially pure mortality risk business, and the usual advantage conferred by reinsurers' geographical diversification is significantly lost in the event of a pandemic (APRA, 2007; Dreyer et al., 2007; Cummins and Trainar, 2009) since an influenza pandemic is likely to affect multiple geographical regions around the world, compared, for example, to a single earthquake. In other words, while trying to eliminate mortality risk, credit risk comes into the picture. Thus the capacity of reinsurance is rather limited. An alternative to reinsurance are catastrophic mortality bonds, which essentially eradicate credit risk. These catastrophic-mortality securitization instruments offer several advantages and disadvantages compared to reinsurance and are described in the next section.

1.2.3 Catastrophic Mortality Bonds (CMBs)

As mentioned above catastrophic mortality bonds offer mortality securitization. Securitization consists in the isolation of a pool of assets or rights to a set of cash flows and the repackaging of the assets or cash flows into securities that are traded in the capital markets (c.f. Cowley and Cummins, 2005). Insurance-linked securities (ILS) are instruments designed to transfer insurance risk to the capital markets (c.f. Cummins and Trainar, 2009). Life securitizations have been predominantly used as a financing tool although some have facilitated risk management. On the other hand, non-life securitizations such as earthquake bonds and windstorm bonds have typically been used to transfer extreme risk arising due to a catastrophic event into the capital markets for a number of years (c.f. Ernst and Young, 2011).

The market for ILS has grown significantly in recent years, expanding at 40-50 % per year since 1997 (Hartwig et al., 2008). Since extreme mortality can be modeled in the same way as other catastrophic risks (see Johnson, 2013), it has become another of many offerings in a menu of perils from which investors choose and mix. Tables 1.3, 1.4 and 1.5 summarize all the catastrophic mortality bonds issued to date. To the end of 2016, there have been thirteen public catastrophic mortality bonds transactions with a total bond issuance value of approximately U.S. $3.5 billion, with the last issue being that of VITA VI in December 2015 by Swiss Re.

[15] Reinsurance refers to the insurance purchased by an insurer from a re-insurer to transfer risk. Retrocession refers to the purchase of insurance by re-insurers from other reinsurance companies to transfer risk (Bellis et al., 2010).

[16] c.f. Baumgart et al. (2007)

Table 1.3: Summary of Earliest Catastrophic Mortality Bond Transactions

Year	Special Purpose Vehicle	Sponsor	Maturity (Years)	Principal Amount (Millions)	S & P Rating at Issuance	Initial Spread to 3-Month LIBOR/ EURIBOR (bps)	Attachment/ Exhaustion Point(%)	Covered Area
2003	Vita Capital I	Swiss Re	4	U.S. $400	A+	135	130/150	U.S. 70%, U.K. 15% France 7.5%, Italy 5% and Switzerland 2.5%
2006	Vita Capital II	Swiss Re	5 5 5	U.S. $62 U.S. $200 U.S. $100	A- BBB+ BBB-	90 140 140	120/125 115/120 110/115	U.S. 62.5%, U.K. 17.5% Germany 7.5%, Japan 7.5% and Canada 5%
2006	Tartan Capital	Scottish Re	3 3	U.S. $75* U.S. $80	AAA BBB+	19 300	115/120 110/115	U.S. 100%
2006	Osiris Capital	AXA	4 4 4 4	EUR 100* EUR 50 U.S. $150 U.S. $100	AAA A- BBB BB+	20 120 285 500	114/119 114/119 110/114 106/110	France 60%, Japan 25% and U.S. 15%
2006	Vita Capital III	Swiss Re	4 4 4 4 4	U.S. $100* U.S. $100* U.S. $90 EUR 30 EUR 55*	AAA AAA A A AAA	21 21 110 110 22	125/145 125/145 120/125 120/125 120/125	U.S. 62.5%, U.K. 17.5% Germany 7.5%, Japan 7.5% and Canada 5%

(Continued)

Table 1.4: Summary of Middle Stage Catastrophic Mortality Bond Transactions Continued

Year	Special Purpose Vehicle	Sponsor	Maturity (Years)	Principal Amount (Millions)	S & P Rating at Issuance	Initial Spread to 3-Month LIBOR/ EURIBOR (bps)	Attachment/ Exhaustion Point(%)	Covered Area
			5 5 5 5	U.S. \$100*[a] EUR 55 U.S. \$50* U.S. \$50	AAA AA- AAA A	20 80 21 112	125/145 125/145 120/125 120/125	
2008	Nathan	Munich Re	5	U.S. \$100	A-	135	120/130	U.S. 45%, U.K. 25% Canada 25% and Germany 5%
2009 to 2011	Vita Capital IV	Swiss Re	5	U.S. \$75	BB+	650	U.K.: 112.5/120 & U.S.: 105/110	U.K. and U.S.
			4	U.S. \$50	BB+	525	U.K.: 112.5/120 & U.S.: 105/110	U.K. and U.S.
			5	U.S. \$100	BB+	375	Japan: 107.5/115 & U.S.: 105/110	Japan and U.S.
			5	U.S. \$75	BB+	370	Canada: 111.5/120 & Germany: 110/115	Canada and Germany
			5	U.S. \$100	BBB-	N/A	Canada: 120/130 & Germany: 125/135	Canada and Germany
			5	U.S. \$80	BB+	N/A	Canada/Germany: 110/115, U.K.:115/120 ans U.S.: 105/110	Canada, Germany, U.K. and U.S.

[a]*These tranches have been credit enhanced by "monoline" insurers who guarantee the interest and principal payment.

Table 1.5: The Latest Catastrophic Mortality Bond Transactions

Year	Special Purpose Vehicle	Sponsor	Maturity (Years)	Principal Amount (Millions)	S & P Rating at Issuance	Initial Spread to 3-Month LIBOR/ EURIBOR (bps)/Coupon(%)	Attachment/ Exhaustion Point(%)	Covered Area
2012	Vita Capital V D-1	Swiss Re	5	U.S. $125	BB-	97	Australia: 135 & Canada: 120/110	Australia and Canada
	E-1		5	U.S. $150	BB+	97	Australia: 120 Canada: 110 & U.S.: 105	Australia Canada and U.S.
2012	Mythen Re U.S. Hurricane U.K. Mortality	Swiss Re	4	U.S. $120	B+	8.50%	US PCS[a]: 620.2/845.5 & UK: 125/135	U.S. and U.K.
			5	U.S. $80	B-	11.75%	US PCS: 372.7/511.4	U.S.
2013	Atlas IX	SCOR Re	5	U.S. $180	BB	3.25%	102/104	U.S.
2015	Benu Capital Excess Mortality	AXA	5 5	EUR 135 EUR 150	BB+sf BBsf	2.55% 3.35%	France: 116/152.5 Japan: 116/140.8 & U.S.: 108/120.4	Drop down Level France 110%, Japan 110% & U.S. 106%
2015	Vita Capital VI	Swiss Re	5	U.S. $100	BB(sf)	2.90%	Australia: 120 Canada: 115 & U.K.: 120	Australia Canada and U.K.

[a]Property Claim Services

Infact Swiss Re was the pioneer to launch the first catastrophic mortality bond VITA I in December 2003 which was extremely successful. The reinsurance giant has undoubtedly dominated the market in this sector and has experimented even with a 'Longevity Trend Bond' called 'Kortis' in 2010 and a 'Multiple Peril Bond' called 'Mythen Re' in 2012 which was a hybrid of a hurricane and a mortality bond. Moreover, the medium-term outlook for catastrophic mortality bonds remains positive and the market is estimated to reach U.S. $5-20 billion by 2019 (Frey et al., 2009; Weistroffer et al., 2010).

In all these tables the fourth column represents the maturity of the various *tranches*, where tranches are parts of a security that can be broken apart and sold in pieces. Catastrophic mortality bonds have primarily appealed to huge, globally diversified insurers and re-insurers, and have predominantly been used in developed countries. Undoubtedly, these bonds enhance the capacity of the life insurance industry to write mortality risk business by transferring catastrophic losses from the insurance industry to the capital markets (Lin and Cox, 2008; Bouriaux and MacMinn, 2009).

Key Features

The basic transaction structure of catastrophic mortality bonds has remained reasonably generic over most of the thirteen public transactions that have occurred to the end of 2015. The following figure shows the detailed structure.

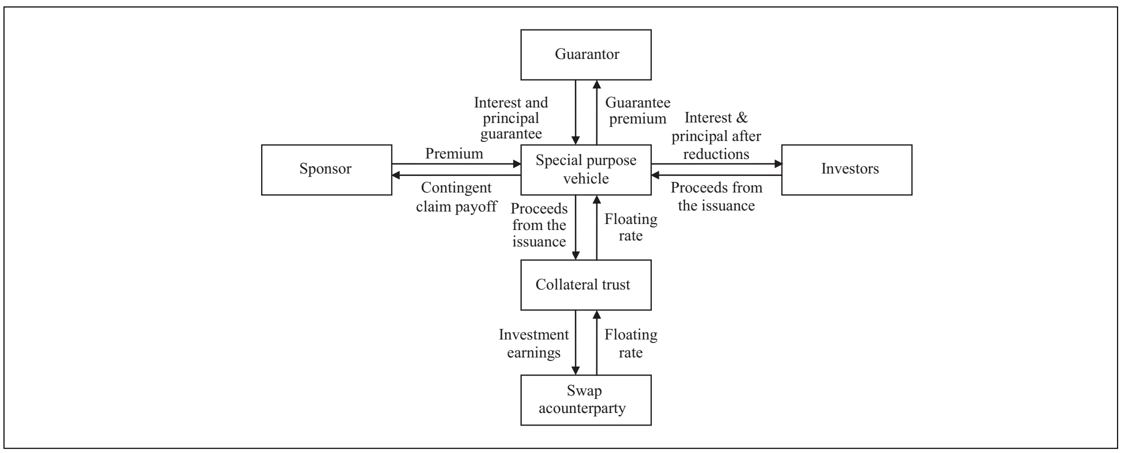


Figure 1.2: Basic Catastrophic Mortality Bond Structure (Source: Linfoot (2007))

The transaction involves three parties

- *The Ceding company or Sponsor*
- *Special Purpose Vehicle (SPV) or issuer*
- *Investors (generally large institutional buyers*[17]*)*

The transaction begins with the formation of a SPV which issues bonds to investors and invests the received capital in high quality securities such as government bonds or AAA corporate bonds which are held in a trust account. The coupon paid to the buyer comprises of investment returns from this account plus the risk premium paid by the ceding company.

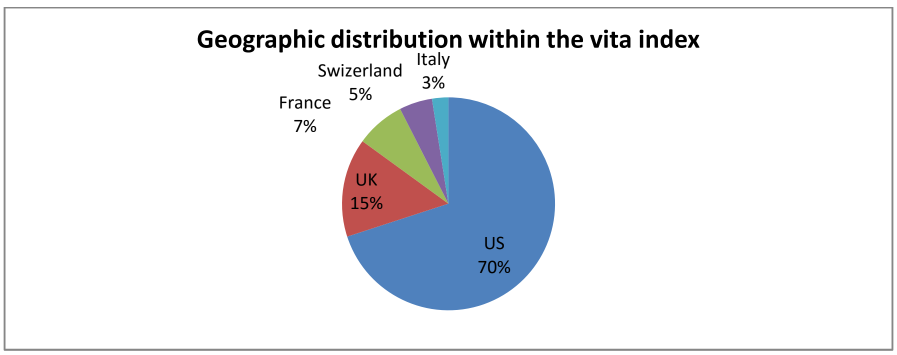
Embedded in these bonds is a call option that is triggered by a defined catastrophic event. Like the transaction structure, the contingent claim payoff mechanism has remained more or less the same for all transactions. The key components of the contingent claim payoff mechanism are

- *principal amount*

[17]Commonly pension funds with a view to hedge their position in terms of longevity and mortality

- *coupon*
- *mortality index*
- *attachment/trigger point*
- *exhaustion point.*

The principal amount represents the maximum payoff that the sponsor can receive if the bond is triggered and this has typically ranged from U.S. \$50 to U.S. \$100 million per tranche. There are well defined attachment and exhaustion points and generally a very attractive periodic coupon to attract investors. The mortality index, attachment point, and exhaustion point determine whether the bond is triggered and if so, what percentage of the principal is paid. The mortality index is generally defined over a 2-calendar-year period in order to mitigate the chance that an influenza pandemic will be cut off by the end of the measurement period. This index is computed using general population mortality rates published by official public reporting sources weighted by age, gender and sometimes country (Rooney, 2008). The weights are specified by the sponsor to broadly reflect their exposure (in terms of data availability and chance of a catastrophic occurrence) to an insured population and are fixed throughout the duration of the period over which the catastrophic bond provides coverage called the *risk period* (Standard and Poors, 2011). As an example, the following figure 1.3 represents the weight distribution of the very first CMB 'Vita I' issued by the Swiss Re in 2003.



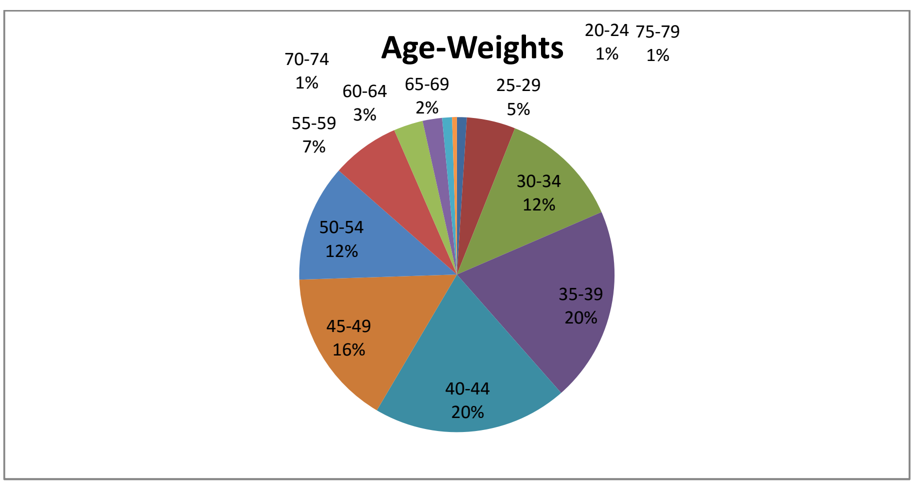


Figure 1.3: Weight Distribution of the VITA I Index

The attachment and exhaustion points are expressed as a percentage of the mortality index at issuance. To date, the lowest attachment point has been 105 percent while the highest exhaustion point has been 150 percent. The reduction mechanism for the principal amount is triggered if the mortality index value exceeds the attachment point otherwise the full principal amount is returned to the investor at maturity. Once the attachment point is exceeded, the reduction in the principal amount increases linearly between the attachment and exhaustion points until the index exceeds the exhaustion point and the full principal is lost by the investor (Bridet, 2009). For the VITA I bond discussed above the attachment and exhaustion points were 130 percent and 150 percent and the capital repayment or erosion phenomenon is shown in figure 1.4. Clearly these bonds are principal-at-risk instruments. The higher the trigger point is, the lower the chance that the event would actually happen and this would mean the lower the returns for the investor and vice-versa based on the principle of 'high risk-high reward'.

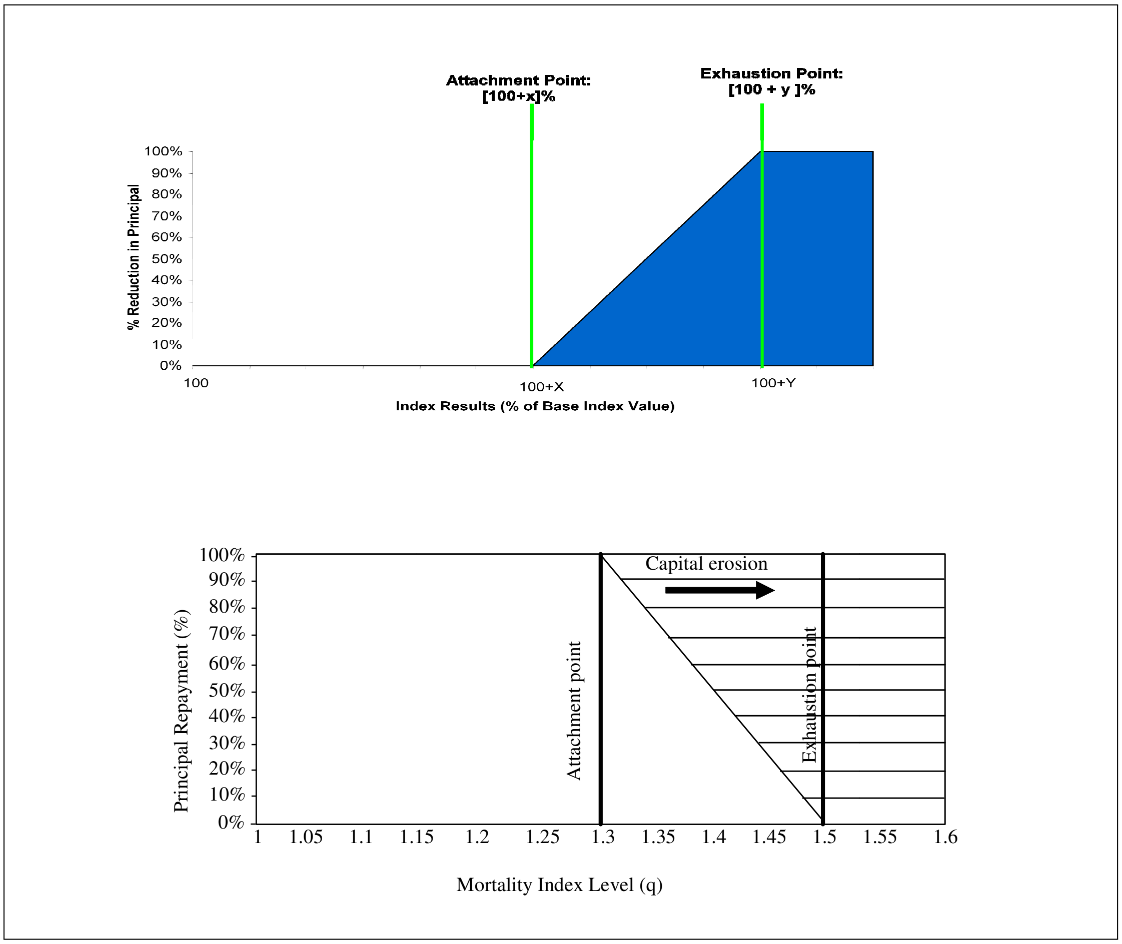


Figure 1.4: Capital Repayment/ Erosion for the Swiss Re 2003 CATM Bond (Source: Klein (2005) and Blake et al. (2006))

The choice of an index-based payoff trigger is driven by investors' appetite for transparent, easy-to-understand, and hard-to-manipulate triggers (Weistroffer et al., 2010) such as the ones based on indemnity[18]. Index-based payoff triggers can be standardized more easily than their indemnity-based counterparts[19] and they reduce moral hazard because the sponsor still has an incentive to limit losses as the payoffs are based on an independent metric rather than the sponsor's actual losses. Moreover, there is a reduction in adverse selection as payoffs are based on publicly available data and there are few informational asymmetries to be exploited (Helfenstein and Holzheu, 2006; Bouriaux and MacMinn, 2009).

[18]Indemnity: triggered by the issuer's actual losses, so the sponsor is indemnified, as if they had purchased traditional catastrophe reinsurance. If the layer specified in the cat bond is $100 million excess of $500 million, and the total claims add up to more than $500 million, then the bond is triggered (c.f. Wikipedia, 2011).

[19]Queensgate and ALPS II are examples of life indemnity bonds issued by Swiss Re.

Valuation

Given how mortality-linked bonds are structured, their pricing requires expertise in actuarial science (for assessing the impact of changes in mortality), econometrics (for modeling the random evolution of mortality rates) and finance (for turning simulation results to prices). A lot of valuation approaches to price CMBs have germinated and we discuss these in Chapter 3. To model extreme mortality risk and value CMBs, researchers have devised a number of stochastic models that incorporate jump effects. These models include some superb contributions by Bauer and Kramer (2009), Biffis (2005), Chen (2013), Chen and Cummins (2010), Chen and Cox (2009), Chen et al. (2010, 2014), Cox et al. (2006, 2010), Deng et al. (2012), Hainaut and Devolder (2008), Lin and Cox (2008), Lin et al. (2013) and Zhou et al. (2013b). Various features of mortality jumps in terms of its occurrence and severity have been investigated in great depth. For example, Chen and Cox (2009) and Chen et al. (2010) use independent Bernoulli distributions, Cox et al. (2010) employed Poisson jump counts while Lin and Cox (2008) considered a discrete-time Markov chain.

Moreover, the first generation of econometric models for valuing catastrophic mortality bonds is univariate, modeling one population at a time. Models belonging to this category encompass those that were proposed by Chen and Cox (2009), Cox et al. (2010), Deng et al. (2012) and Liu and Li (2015b). Although these models capture characteristics such as skewness and leptokurtosis, they are unable to accommodate the potential static cross-correlations among the mortality dynamics of different populations. Therefore, these models may not be efficient enough to price most of catastrophic mortality bonds, including Swiss Re's Vita I, which are linked to the mortality experiences of multiple populations. The second generation of models is multivariate, modeling all populations in question simultaneously. Models that fall into this category have been proposed by Zhou et al. (2013b) who applied a combination of univariate time-series models with correlated innovations, Lin et al. (2013) who employed a model with a common jump effect and correlated idiosyncratic risks, Chen et al. (2015) who considered a combination of uni-variate GARCH models and a factor copula, Wang et al. (2015) who linked uni-variate ARMA models with a dynamic copula and Wang and Li (2016) who propose a DCC-GARCH model, in which the correlations are captured within the model structure (rather than externally through a copula) and are permitted to vary over time.

Advantages and Disadvantages

Catastrophic mortality bonds offer several advantages over traditional reinsurance for circumventing the exposure to catastrophic mortality losses.

- Advantages to the Re-insurer
  - *Additional Tool in the Risk Management Kitty:* These bonds help the issuer to achieve 'Mortality Risk Transfer' so that they secure protection for insurance liabilities when claims are horrendous in the advent of a catastrophe. This is successfully achieved by getting counter parties to offload mortality risk and hence eliminate any dependence on retrocessionaire. These are called *Alternative Risk Transfer* instruments.
  - *Edge over Stop-Loss Reinsurance*[20]*:* CMBs act as a form of collateralized stop-loss reinsurance, which essentially eliminates the credit risk exposure for sponsors (c.f. Bagus, 2007). In comparison to the 1-year coverage usually provided by stop-loss reinsurance, they allow the sponsor to secure fixed cost multi-year coverage, typically ranging from 3 to 5 years, which in turn allows sponsors to spread the fixed cost of issuance over several years (c.f. Cummins, 2008).

[20]Stop-loss reinsurance is a form of excess of loss reinsurance under which the re-insurer's liability commences when the aggregate claims experience on the re-insured portfolio during a specified time period exceeds a predefined level (IAAust, 2009).

- *Flexibility to Access Capital Markets:* These bonds come with the equipment to tap the capital markets when required by using shelf programs.[21] This means that the favourable conditions in the market can be exploited. This has the potential to avoid market disruptions caused by reinsurance prices and availability cycles (c.f. Cummins and Trainar, 2009).

- Advantages to the Buyer
  - *High Yields Offered:* CMBs offer high risk and high reward. Since such bonds are available in many variations, investors can purchase them according to their risk appetites. Depending on the probability of extreme event, higher yields are available from these bonds. In fact, these bonds offer a considerably higher return than similarly rated floating rate securities (c.f. Blake et al., 2006).
  - *Diversification to the Portfolio:* These bonds are extremely lucrative to the investors because of their potential of providing diversification to the portfolio because mortality risk does not bear any correlation with the return on other investments, such as fixed income or equities or with typical market risk factors such as foreign exchange, commodities and credit.
  - *Hedging for Pension Funds:* Pension funds could view this bond as a powerful hedging instrument. The underlying mortality risk associated with the bond is correlated with the mortality risk of the active members of a pension plan. If a catastrophe occurs, the reduction in the principal would be offset by reduction in pension liability of these pension funds.
  - *Charity for the Rich:* On the lighter side, many investors purchase these bonds for a charitable cause. If the unfortunate calamity occurs, the invested money goes to the benefit of the suffering individual/family making them better off in testing times. If the event does not occur, investors get paid off with higher returns, which they take as a reward for their risky venture with an underlying charitable cause.

- Few Disadvantages
  As with every innovation, there are a few disadvantages of CMBs. We discuss these below.
  - *Significant up-front transaction costs:* CMBs have significant up-front transaction costs such as legal, risk modeling, broker, rating agency, and bank fees that require minimal transaction sizes for the issuance to be economical (Helfenstein and Holzheu, 2006), whereas traditional reinsurance is generally free of up-front costs aside from brokerage fees (PartnerRe, 2008).
  - *Basis Risk:* Basis risk exists for catastrophic mortality bonds since the payoff trigger is index based and the actual loss suffered is unlikely to be perfectly matched by the bond payoff. This contrasts with traditional reinsurance, which has no basis risk since it is indemnity based and offers complete coverage for re-insured losses (Hartwig et al., 2008).
  - *Reduced Capital Credit:* The capital credit given by regulators and rating agencies may be reduced for catastrophic mortality bonds in comparison to traditional reinsurance (Standard and Poors, 2008).
  - *Fixed Terms:* For CMBs, terms are fixed throughout the duration of coverage but can be adjusted every year for traditional reinsurance allowing for short term commitment and flexibility.

1.2.4 Experiments in the CAT Bond Niche

We have provided an exhaustive discussion on catastrophic mortality bonds. Interestingly on the two ends of the first CATM bond (VITA I) and the last one (VITA VI), lie instruments which are innovative in a number of ways. We provide a sneak peek here.

[21] Shelf programs are formulated such that all the legal, modeling, rating, and other structuring costs are done for a very large bond issue. However, the entire bond capacity is not issued initially and some is left to be issued at later time when needed by the sponsor. This lowers the issuance cost for subsequent issues and reduces the time to access capital markets (Helfenstein and Holzheu, 2006).

The first one pertains to terrorism, which did not qualify to be called a catastrophe insurance risk prior to September 11th, 2001. Since terrorism cover is comparatively scarce and expensive, post the World Trade Centre attack, Kunreuther (2002), proposed the idea of a terrorism catastrophe bond. Although, conceptually such a financial instrument would be successful in providing a viable alternative to insurance, the associated risk ambiguity is too much for a terrorism bond to be palatable for the rating agencies and investors. A further uncertainty was the investment appetite for terrorism risk. It is questionable that would any investor be interested or authorized to buy a terrorism bond. Even if such investors did exist, would they expect a double-digit coupon spread, e.g. in excess of 10%, in exchange for accepting the terrorism risk ambiguity?

A suitable terrorism securitization opportunity came up in regards to the cancellation risk of the football World Cup, organized by FIFA, (the Fèdèration Internationale de Football Association). Following 9/11, AXA withdrew its insurance coverage for the 2002 FIFA World Cup in Korea/Japan. As a result, obtaining appropriate replacement coverage was an upheaval task for FIFA. This withdrawal was a clear reflection of widespread insurance market anxiety over the terrorist threat after 9/11. The 2002 World Cup was eventually covered by the Berkshire Hathaway Group subsidiary National Indemnity Company (c.f. Woo, 2004) but the huge cost of this coverage, persuaded FIFA to seek the alternative solution of securitization for the next football World Cup, to be hosted by Germany.

CSFB (Credit Suisse First Boston), the investment bank arm of the Credit Suisse Group was instrumental in designing the solution through $262 million transaction with the set up of SPV-Golden Goal Finance Ltd. This deal helped FIFA to securitize about $262 million of future sponsorship revenue, which required that the event cancellation risk be diminished as far as possible, either through insurance or a catastrophe bond. Both options were considered, but the latter turned out to be cost-effective. If the tournament is cancelled because of terrorism risk, the investors will lose 75% of their money invested in the bonds[22].

This $262 million deal remains to this day the only stand-alone securitization of terrorism risk. However, according to Julian Enoizi, the Chief Executive Officer (CEO) of Pool Re, the UK's government-backed mutual terrorism re-insurer: "The terrorism protection gap is now larger than ever and expanding all the time, underlining the need for innovative and effective solutions, with the insurance-linked securities (ILS) space a logical evolution". Julian, addressing an audience in London in early December 2016 at an ILS industry event, remarked that the terrorism marketplace should embrace the ILS sector's features and capacity. 'Terrorism Bonds' could be the new buzz word to counter fight terrorist attacks. Some third-party reinsurance and ILS capital is already being allocated to terrorism risks at a number of ILS funds which deploy investment capital into private collateralized reinsurance contracts. These ILS managers have become comfortable with a certain level of exposure to terrorism risks and that confidence is likely to expand, particularly if rates improve[23]. One example is Swiss-based ILS manager Twelve Capital which generated attractive returns from transactions exposed to global terrorism. Another example is Lancashire Holdings new third-party reinsurance capital management unit Kinesis Capital Management which includes some terror risk within its third-party capital backed underwriting. Kinesis is already offering its clients collateralized, third-party capital backed, solutions for short to medium tail lines of business, including some terrorism risks. As shown by the success of Golden Goal Finance Ltd., new capital markets investors may be attracted towards the purchase of catastrophe bonds exposed to other perils. Terrorism risk, as embedded within event cancellation risk, workers compensation risk, or mortality risk, is securitizable (also see Bruggeman, 2007; Bouriaux and Scott, 2004).

The testing of investor appetite for novel forms of alternative risk transfer is allowing the boundaries of catastrophe bond issuance to be extended gradually. A very recent experimenta-

[22] Accessed on 5th Sep.2017, http://www.artemis.bm/deal_directory/golden-goal-finance-ltd/. ARTEMIS is an online website since 1999. Artemis provides news, analysis and data on catastrophe bonds, insurance-linked securities and alternative reinsurance capital.

[23] http://www.artemis.bm/blog/2017/01/18/ils-terrorism-bonds-a-logical-next-step-for-pool-re-enoizi/. Accessed 5th Sep. 2017

tion in July 2017 has been done by The World Bank which launched specialized bonds aimed at providing financial support to the 'Pandemic Emergency Financing Facility' (PEF), a facility created by the World Bank to channel surge funding to developing countries facing the risk of a pandemic. The pandemic bonds work like this: Investors buy the bonds and receive regular coupons payments in return. If there is an outbreak of disease, the investors don't get their initial money back. There are two varieties of debt, both scheduled to mature in July 2020. The first bond raised $225 million and features an interest rate of around 7%. Payout on the bond is suspended if there is an outbreak of new influenza viruses or coronaviridae (SARS, MERS). The second, riskier bond raised $95 million at an interest rate of more than 11%. This bond keeps investors' money if there is an outbreak of Filovirus, Coronavirus, Lassa Fever, Rift Valley Fever, and/or Crimean Congo Hemorrhagic Fever. The World Bank also issued $105 million in swap derivatives that work in a similar way. These bonds are similar to catastrophe bonds, a $90 billion market used by insurance companies to shift risks of hurricanes, earthquakes, and other natural disasters onto the financial markets. The World Bank's bond sale was 200% oversubscribed, with investors eager to get their hands on the high-yield returns on offer. The majority of buyers was from Europe and included dedicated catastrophe-bond investors, pension funds, and asset managers[24].

With this discussion, we conclude this section on mortality risk and look at the root causes of longevity risk and search for pertinent solutions.

1.3 Longevity Risk: Cause and Remedies

In the twentieth century, being a centenarian was considered to be a matter of great pride and almost an impossible feat to achieve. So much so, that about a century ago, the British monarch started sending anniversary messages to "current citizens of [the monarch's] realms or UK Overseas Territories" who reached the age of 100. In 1917, King George V sent a total of 24 celebratory messages to centenarians. By 1952 this had increased more than 10-fold to 255, and in 2016, it has exploded to nearly 60-fold to 14500 (c.f. National Statistics, 2016). The million dollar question is: Where will it end?

In a recent study based on data from Office of National Statistics, UK (c.f. National Statistics, 2012), Appleby (2013) concluded that, the straight line increase in the numbers of UK citizens reaching an age of 100 years seems set to continue. According to the latest posting on the Official Statistics website[25], one in three babies born in the year 2016 will live to see their 100th birthday. Interestingly, female life expectancy is higher than their male counterpart(s). Around 13% of girls born in 1951 are expected to be alive in 2051. For girls born in 2016 the figure is estimated to be 35% and around 60% of girls born in 2060, might expect to live long enough to receive a message from the reigning monarch. At this rate, the number of centenarians is also projected to continue rising reaching a mammoth 83,300 in 2039 which is more than enough to keep any future monarch busy!

This interesting excerpt highlights the gravity of the problem that is looming large over financial institutions today viz. longevity risk - the risk that people outlive their expected lifetimes. Longevity risk is a considerable risk that affects adversely both the willingness and ability of financial institutions to supply retired households with financial products to deal with wealth decumulation in retirement. Depending on the scenario and need, longevity risk can be defined in a variety of ways. A statistical perspective of the definition is furnished by Coughlan et al. (2013) who provide the following concise yet complete definition: "*It is a combination of*

- *uncertainty surrounding the trend increases in life expectancy*

  and

[24] http://treasury.worldbank.org/cmd/htm/World-Bank-Launches-First-Ever-Pandemic-Bonds-to-Support-500-Million-Pandemic-Emergenc.html. Accessed 5th Sep. 2017

[25] Accessed on 24th June 2017

- *variations around this uncertain trend that is the real problem.*

This is what is meant by longevity risk and it arises as a result of unanticipated changes in mortality rates". Longevity risk is borne by every institution making payments that depend on the life span of individuals. These include Defined Benefit (DB) pension plan sponsors, insurance companies selling life annuities, and governments through the social security pension system and the salary-related pension plans of public-sector employees. The present scenario is particularly acute for insurance companies operating in the European Union (EU) where a new regulatory regime, Solvency II, was introduced in 2016. This requires insurers to possess a pool of significant additional capital to back their annuity liabilities if longevity risk cannot be hedged effectively or marked to market. In the next couple of sections we throw light on the causes of longevity risk and see what possible solutions can be proposed.

1.3.1 The Cause: Increased Life Expectancy

The main reason that is responsible for longevity risk is increased life expectancy. The main propeller for this advancing expectancy of survival is the postponement of senescence (the increase of mortality with age). In fact, there have been three well known theories on the frontier of survival.

- *The Fixed Frontier of Survival: Limited Lifespans - Aristotle 350 BC (cited in Fries (1980))*
- *Breaking through the Frontier of Survival: Secrets of Longevity (Cornaro (1558))*
- *The Advancing Frontier of Survival: Unrecognized Progress (Vaupel et al. (1979))*

We are presently following the last theory and a remarkable finding suggested by the same is the explosion in the number of centenarians (c.f. Vaupel, 2010). We present below certain very alarming self-explanatory figures that support the last theory. The relevant sources have been indicated and a good reference to read further about theories of longevity is Brown (2016).

Just like mortality, longevity can also experience jumps mostly caused by pharmaceutical or medical innovations and generally having long-term gentle effect in contrast to mortality jumps which are short term and intensified. However the survival jump effect could also be transitory, if a new drug or treatment loses its effectiveness. A huge longevity jump may occur in future if an effective treatment of coronary heart disease emerges or a successful cure of cancer can be found. In fact these two amount to 50% of deaths for humans over the age of 40 (c.f. Johnson et al., 2005). Cox et al. (2010) and Deng et al. (2012) have discussed the modeling of longevity jumps. Interestingly, the introduction of life-saving anti-HIV (HAART: highly active antiretroviral therapy) medications led to a large positive longevity shock for those living with HIV. As life expectancies rose, 'Viatical Settlements'[26] disappeared quickly (c.f. Vinals, 2012). Investors in viatical settlements experienced a significant realization of longevity risk, with associated losses, as they were required to continue to pay premiums for much longer than expected and were faced with delayed payouts.

The implications of these developments are indeed grave. In fact, Blake et al. (2014) have pointed out that Global private-sector pension liabilities are of the order of $ 25 trillion. According to the authors in the United Kingdom alone, private-sector 'Defined Benefit' (DB) pension liabilities equal £1,340 billion, while 'Defined Contribution' (DC) (c.f. Section 1.3.2 for DB and DC) pension assets are equal to £737 billion (including £150 billion in annuities with insurance companies). It has been estimated that every additional year of life expectancy at age 65 adds around 3% or £33 billion to the present value of DB pension liabilities in the United Kingdom, with a similar impact on lifetime annuities.

[26] An arrangement whereby a person with a terminal illness sells their life insurance policy to a third party for less than its mature value, in order to benefit from the proceeds while alive.

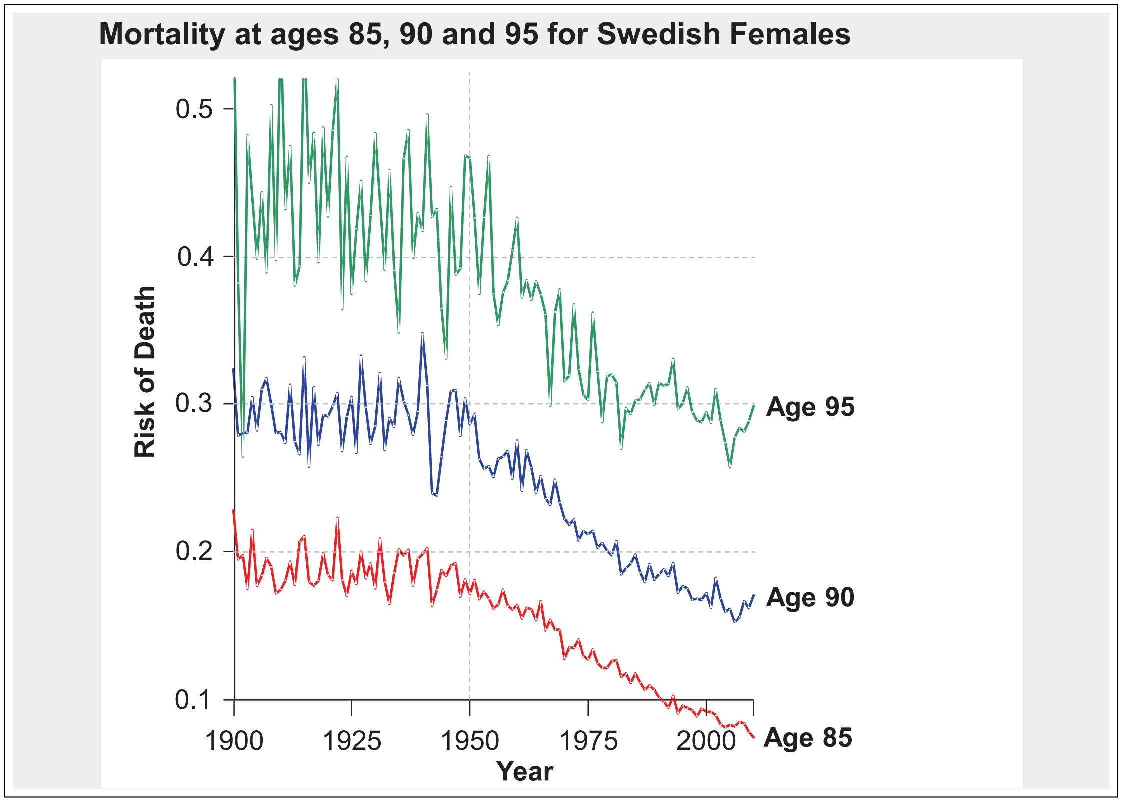


Figure 1.5: Discovery of the Advancing Frontier of Survival (Source: Vaupel and Lundström (1994))

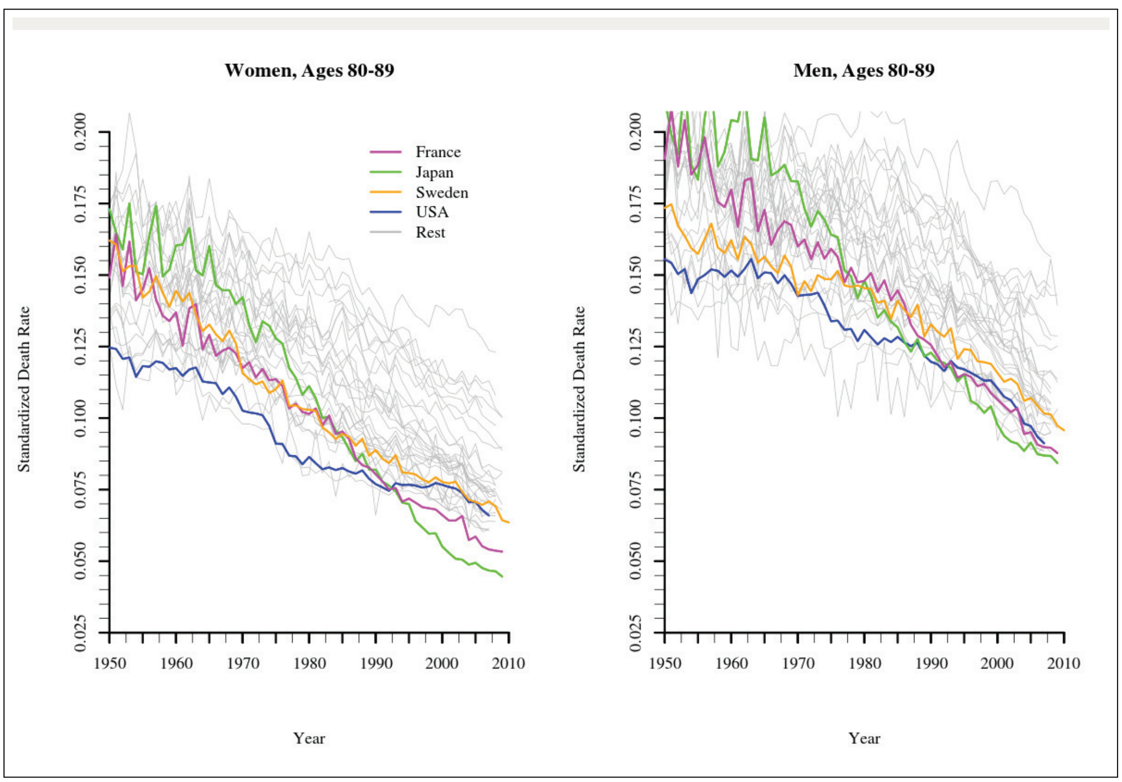


Figure 1.6: The Decline in Octogenarian Mortality (Source: Roland Rau (unpublished) based on HMD: can be found in Brown (2016))

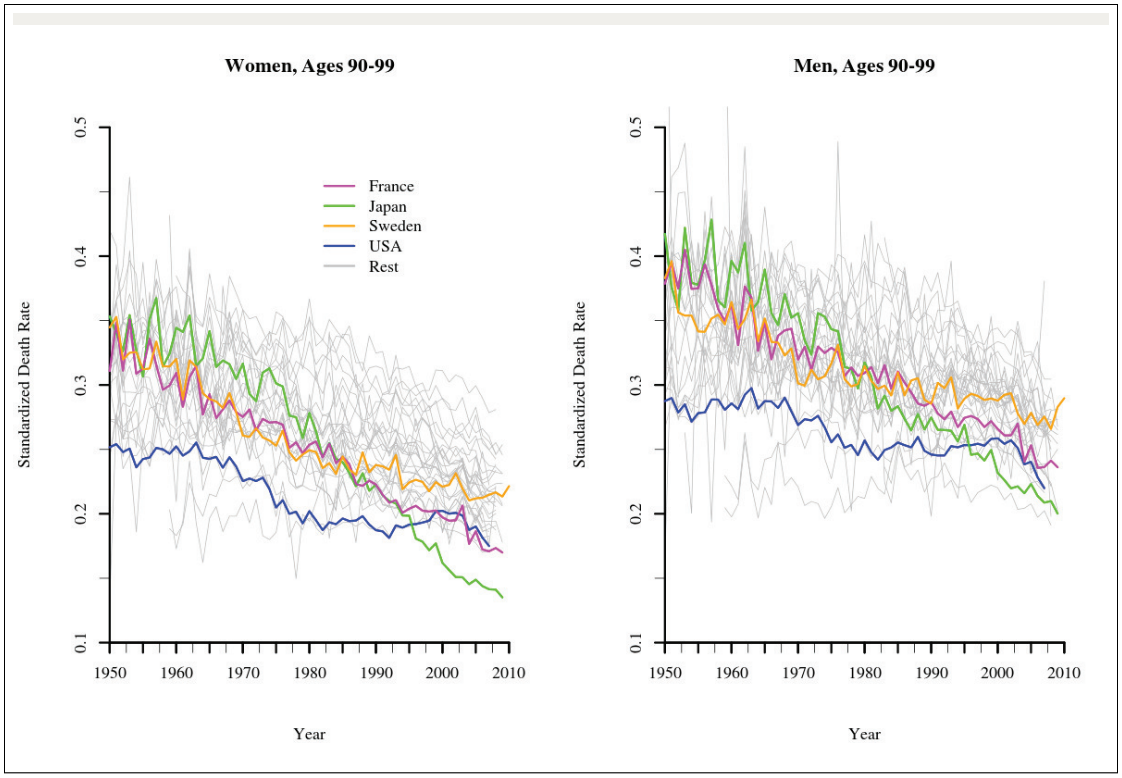


Figure 1.7: The Decline in Nonagenarian Mortality (Source: Roland Rau (unpublished) based on HMD: can be found in Brown (2016))

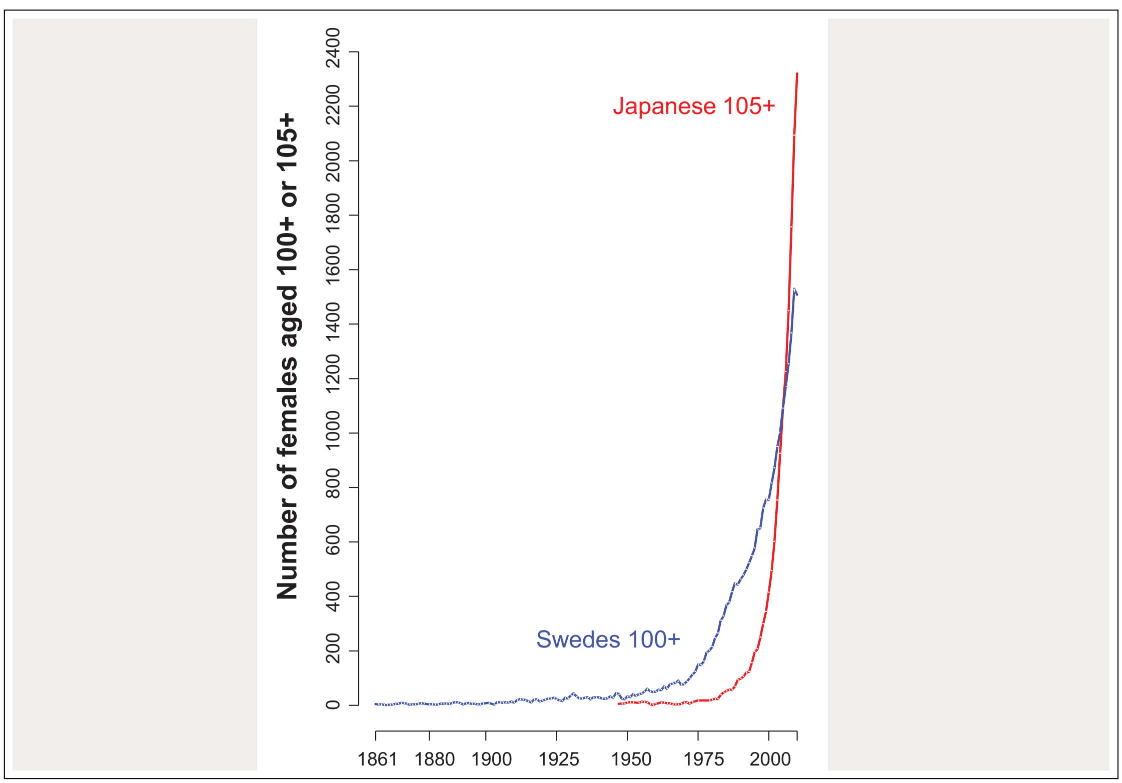


Figure 1.8: The Explosion of Centenarians (Source: Vaupel (2010))

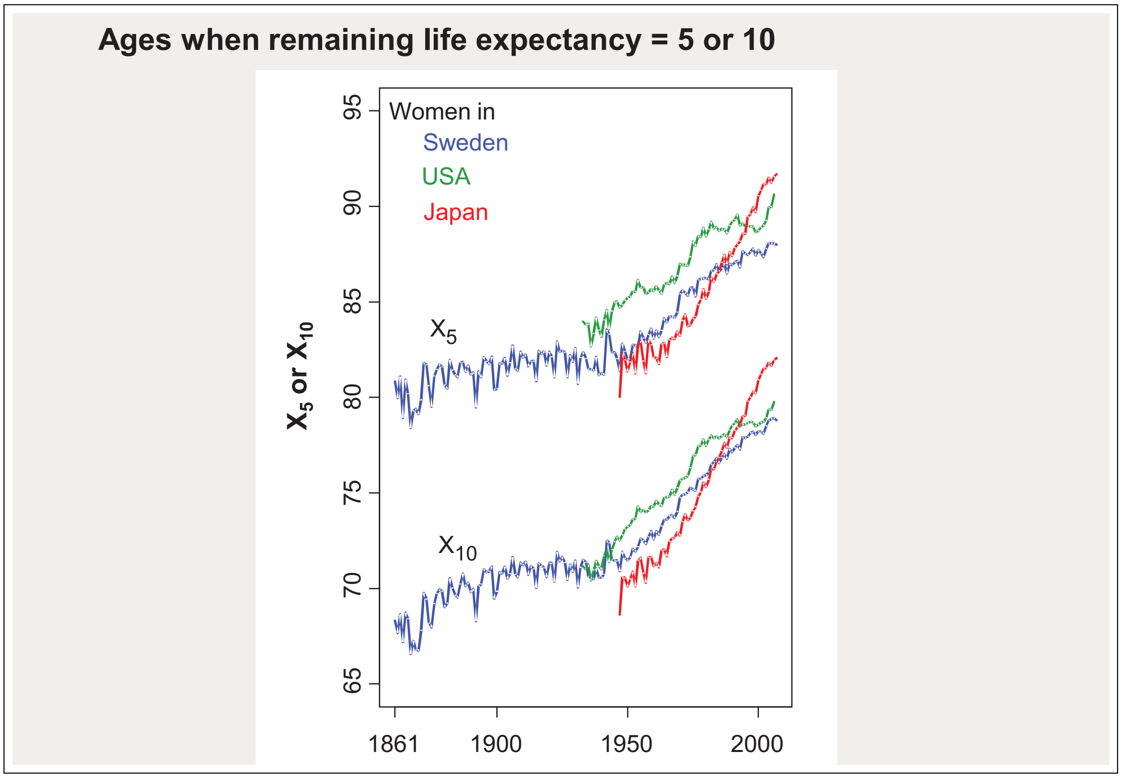


Figure 1.9: Discovery of the Postponement of Senescence (Source: Vaupel (2010))

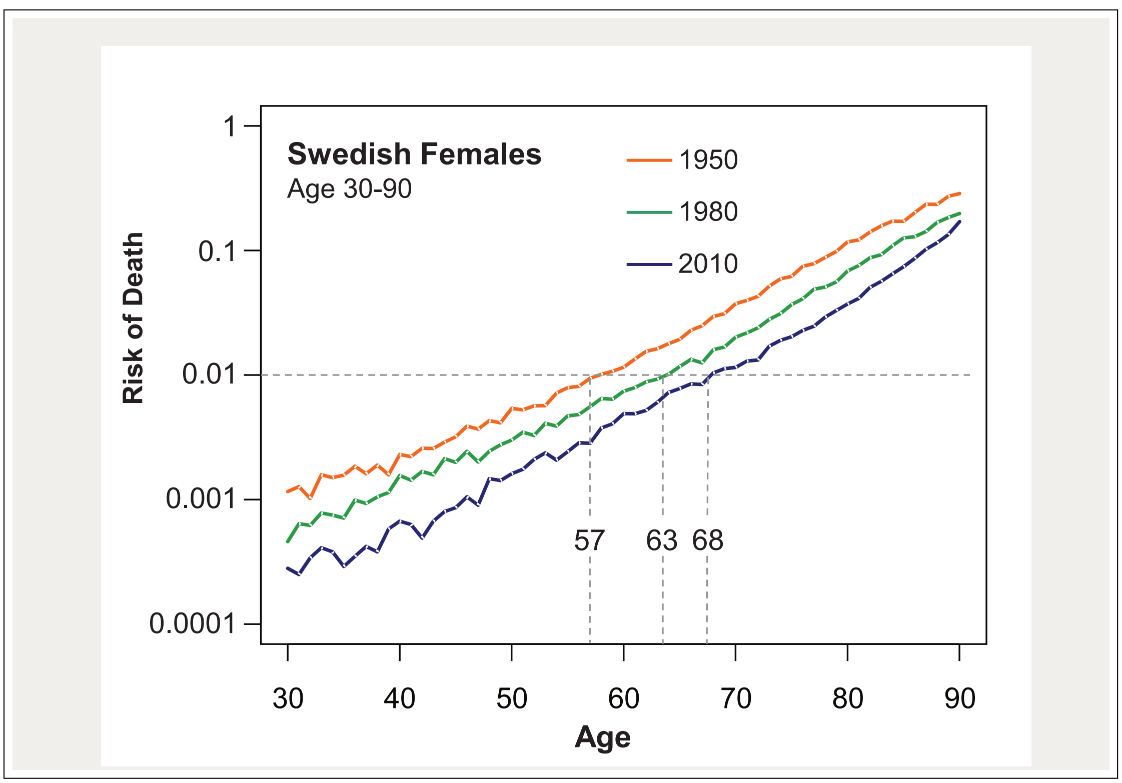


Figure 1.10: The Postponement of Senescence: Evidence from Sweden (Source: Calculations based on HMD by Elisabetta Barbi and Giancarlo Camarda (unpublished): can be found in Brown (2016))

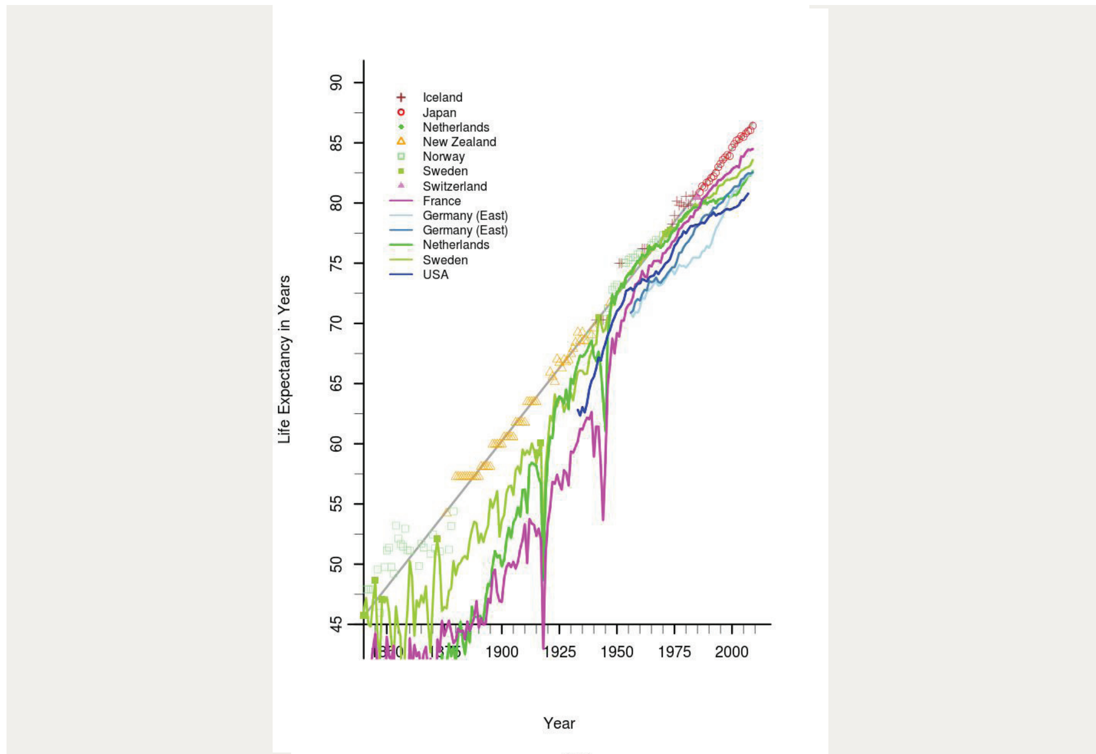


Figure 1.11: The Linear Rise of Record Life Expectancy (Source: Vaupel (2010))

Blake et al. (2014) higlight that the most recent estimates for U.K. state pension liabilities were £3,843 billion in respect of social security pensions, £852 billion in respect of the unfunded pension plans of public-sector employees, and £313 billion in respect of the funded plans of public-sector employees (mainly local government employees). This means that U.K. government-backed longevity-linked liabilities exceed £5 trillion.

Also Blake et al. (2014) comment that longevity risk in the private sector is beginning to become concentrated, especially in the United Kingdom. Private-sector companies in the United Kingdom are distancing themselves from DB pension provision and replacing them with occupational DC plans the equivalent of 401(k) plans in the United States and, as a result, companies are passing the longevity risk back to their employees. Moreover, these companies are trying to offload the legacy longevity risk that they still hold either by buying in annuities from life companies to protect their pensions-in-payment or by undertaking bulk buyouts of their liabilities, again with life companies. In providing these indemnification solutions for DB pension plans, insurance companies are playing an important role in aggregating longevity risk in the economy. So individuals should be anxious because there is a real risk that they will outlive their wealth. In countries such as the United Kingdom and Chile, where annuitization of DC pension pots is either compulsory or greatly incentivized, it will again be life companies that provide the annuities. So, all the trends in pension provision viz.

- *exploding demand from DB plans to use annuities to back their pensions in payment*
- *the increasing demand from DB plans for bulk buyouts*
- *the overall growth in both the number and size of DC pension funds*
- *the related growth in the number of pensioners with DC funds reaching retirement*

highlight a big increase in demand for annuities provided by insurance companies. There are two serious issues connected with this increased demand.

- *Possibility of unhealthy concentration of risk among a small number of insurance companies*

- *Unavailability of capital in the insurance/reinsurance industry to deal with total global private-sector longevity risk.*

Under Solvency II directives, it is advised that insurance liabilities are increased by the addition of a market value margin (MVM) reflecting the cost of capital to cover "non-hedgeable" risks. For annuity companies this is nothing else but the longevity risk. It is currently proposed that in the absence of a hedging instrument for longevity risk, EU insurers will have to charge a 6% cost of capital above the risk-free rate when computing the MVM. Since annuities are long-dated, this calculation could lead to reserving approximately double the amount of capital for longevity risk in comparison to current levels. The resultant extra capital for longevity risk and other Solvency II impacts would have to be passed on to customers, and the money's worth of annuities could roll down by up to 10%.

To cut a long story short, more and more individuals have to manage their own personal longevity risk. In the U.S., the baby boomers are entering their retirement age and need to take care of living a long life taking into account the scarcity of retirement plan payments. The retirees need to strike a balance between investment and consumption of their accumulated wealth. An important point that needs to be taken care of is that extended lifespan may erode their accumulated wealth. *Underestimation* as well as *overestimation* of the personal life expectancy can negatively impact a retiree's life style. This is because

- *If the retiree overestimates his longevity, he will spend less than he could if he has purchased an annuity.*
- *If the retiree underestimates his longevity, he will spend aggressively and might outlive the accumulated wealth.*

In a survey conducted by Society of Actuaries, US (c.f. Deng, 2011), it was discovered that more than 40% of both pre-retirees and retirees underestimate average life expectancy by five or more years. Only 33% of retirees and 39% of pre-retirees have purchased or plan to buy a product or choose a plan option that will generate guaranteed income for life. Further evidence in support of these observations is provided by by Scotti and Effenberger (2007) who find out that the UK retirement market suggests that individuals underestimate their own mortality by as much as five years on average. A defined benefit plan provides individuals with a guaranteed stream of payments, which reduces the chance of outliving the individuals' assets. The Scotti and Effenberger's study shows that individuals that retire without a pension plan have over an 80% likelihood of outliving their assets, and on the contrary those with a defined benefit pension plan have only an 18% chance of outliving their assets.

Individuals without DB pension plans either

- *choose to self-insure against longevity risk by making their own investment and consumption decisions*

  or

- *choose to do nothing because their income does not allow it.*

In fact the old age planning comprises of two important decisions

- *Investment decision: to select an appropriate asset allocation strategy among different investment instruments so as to diversify financial risk and minimize the chance of a portfolio shortfall*
- *Consumption decision: to choose the level of withdrawal from the asset pool to reach a satisfactory life style.*

If individuals who opt for self-insurance fail to do so efficiently and run out of money, they will be forced to lead an impoverished life without income. This would force them to rely on children, relatives, or even federal programs to live out their remaining life, which is certainly an undesirable and painful situation for these self-insured individuals. So, a more reliable and scientific approach to manage the longevity risk for individuals is to transfer and diversify longevity risk with annuities and other financial instruments.

Since a major trend in the retirement market is the declining number of defined benefit plans, as discussed above, the only realistic way of handling the problem of leading longer lives is to pass some of the longevity risk onto governments and the capital markets. Truly, there is an emerging opportunity for the expansion of the private market solution using financial instruments for individual longevity risk management. In the next section, we investigate several possibilities relating to the mitigation of the longevity risk for both individuals and institutions.

1.3.2 Living with Longevity

Longevity is not a curse. Significant medical progress, improved living standards, healthier lifestyles that include organic food, the absence of global wars and control on pandemics are some of the prime reasons for the increase in life expectancy. This is indeed a great achievement for the human race. However, the need of the hour is not to let the elixir spill. The key for preserving and progressing further is to realize that life expectancy is a function of education and income but however as Brown (2016) feels that a causal factor is indeed the robustness of that income and studies are going on to prove that the hypothesis a person with a Defined Benefit pension will live longer than a person with a bank account. In other words financial income security drives longevity[27]. According to Brown (2016) testing is going on to prove this hypothesis. This is enticing as it means the actuarial profession has a causal role to play in enhancement of life expectancy. Interestingly, Dr. Solomon Stephen Huebner (1882-1964), the "Father of Insurance Education" who initiated the concept of 'human life value' had similar views. He believed that annuitants are long livers and freedom from financial worry and fear go a long way in enhancing longevity. According to him, annuities serve in old age, much the same economic purpose that periodic medical examinations do during the working years of life (also see Huebner, 1921). With these thoughts, we now present 'Longevity Risk Solutions' for individuals and companies in the two sub-sections that follow.

Solutions for Individuals

For individuals interested in insuring at least a portion of their longevity risk, there are several products that offer lifetime guarantees. These products include:

- *Immediate Annuities*
  - *Single Premium Immediate Annuity (SPIA)*
  - *Impaired Life Annuity*
- *Variable or Deferred Annuities*
  - *Guaranteed Minimum Withdrawal Benefits (GMWB)*
  - *Guaranteed Minimum Income Benefits (GMIB)*
    - *Guaranteed Annuity Option (GAO)*
- *Advanced Life Delayed Annuities*
- *Corporate Pensions*
- *Reverse Mortgages*

[27] A good example in this context is of the world longest lived person, Jeanne Calment, who died in 1997 aged 122. At an age of 90, she sold her apartment to a lawyer named Andreè-Francois Raffray aged 47 agreeing to a contract where Raffray will pay Calment 2500 francs/month until she died. As a result of this agreement Calment received more than twice the value of the apartment (Richards and Jones, 2004) with payments in the last couple of years being made by Raffray's widow. Clearly the regular stream of payments was an important factor in Calament's longevity.

- *Structured Settlements*
- *Life Settlements*

A good review of the characteristics of these products appears in Deng (2011). Out of these, variable annuities are the most innovative and popular products. A comprehensive summary of the pricing approaches on variable annuities appears in Fenga et al. (2017). Bauer et al. (2008) and Bacinello et al. (2011) present an extensive treatment of pricing of major product designs of guaranteed benefits by simulations. Moreover, the guaranteed benefit that has attracted a lot of attention in Europe is the Guaranteed Annuity Option (GAO). We discuss these in detail later in this chapter.

In regards to corporate pensions there are two categories: defined benefit (DB) and defined contribution (DC) plans and a good discussion appears in Deng (2011). The retirement plans have certain tax advantages, and employers furnish a portion of the employee's contribution. Funds usually cannot be withdrawn without penalty prior to retirement. With a DB plan, the employer sets up a trust and pays in money annually in amounts enough to pay a defined retirement benefit to each employee. The employee receives a fixed income stream after retirement contingent on his/her salary, years of employment, retirement age and other factors. The fund is set up and managed by the employer and the individual employee accounts are not demarcated. With a DC plan, contributions are paid into individual accounts by each employee and the employer may contribute an extra amount. At retirement, a lump sum amount equal to the current account value is available. Defined contribution plans are versatile. The retiree has the options of creating a wide variety of ways to draw down the fund value including simple ad hoc withdrawals or more well-defined withdrawals of a certain percentage of the value per annum. Since DB plans guarantee a fixed stream of cash flows until the death of the policy holder, DB plans have a significant exposure to longevity risk and this risk is born by the employer who has set up the DB plan and administers it. Additionally, the pool of people in the same DB plans generally has similar features, including age, industry, occupation and location. These same risk characteristics can enhance the longevity risk presented by DB plans. Additionally, life expectancy is different for different socioeconomic groups (Villegas, 2015).

Products for Companies

Institutions have experimented with a number of products to transfer longevity risk to markets or governments or back to individuals and this has lead to a large number of innovations mentioned below

- *Longevity Bonds*
- *Longevity Swaps*
- *Longevity or survivor: s-forwards or mortality: q-forwards*
- *Longevity Options*
- *Buy-Ins, Buy-Outs*

For a more comprehensive list of existing and hypothetical longevity products, interested readers can refer to Blake et al. (2006) and Boyer and Stentoft (2012). For pricing 'Buy-Ins' and 'Buy-Outs' interested readers can refer to Lin et al. (2017) and Arik et al. (2017). The maiden attempt of launching a *longevity bond* was in November 2004 when BNP Paribas announced the issue of a 25-year bond linked to a cohort survivor index based on the realized mortality rates of English and Welsh males aged 65 in 2003. The bond was issued by European Investment Bank (EIB) and is commonly known as the EIB bond. However, it was only partially subscribed and later withdrawn for redesign. This performance of EIB bond was in sharp contrast to the first catastrophic mortality bond - the Swiss Re Mortality Bond 2003 discussed earlier in this chapter. A good review of the bond and the potential reasons for its failure can be found in Blake et al. (2006).

Learning lessons from the failure of the EIB bond and the success of mortality catastrophe bonds, Swiss Re launched a longevity trend bond termed "Kortis" in 2010 with a structure having close affinity to catastrophic mortality bonds. For detailed analysis of the Kortis bond, interested readers can refer to Hunt and Blake (2015) and Chen et al. (2017). Apart from longevity bonds, longevity swaps are also becoming popular instruments for managing longevity risks. In fact, a lot of innovations are being tried in the market. For a comprehensive review of the longevity solutions, an excellent reference is Tan et al. (2015). A lot of activity is also going on in the academic sphere with academicians proposing newer products to tackle the bulky longevity risk. In a recent article, Bensusan et al. (2016) propose partial splitting of longevity and financial risk and introduce a longevity novelty called 'Longevity Nominal Choosing Swaption' (LNCS) while Biffis and Blake (2014) argue that a natural way for offloading longevity risk is through suitably designed principal-at-risk bonds. In the passing, we mention that in a series of new instruments being tried in the longevity space, an interesting experiment was the *Longevity Experience Option (LEO)* launched by Deutsche Bank in November 2013. It is formulated as an out-of-the-money call option spread on 10-year forward survival rates and has a 10-year maturity. The survival rates are based on males and females in five-year age cohorts (between 50 and 79) captured from the England and Wales and Netherlands Life and Longevity Markets Association (LLMA) longevity indices. LEOs are being traded over-the-counter (OTC) under a standard ISDA (International Swaps and Derivatives Association) contract. They allow a transfer of longevity risk between pension funds, insurance companies and investors. The intention was to offer a cheaper and more liquid alternative to bespoke longevity swaps which are generally costly and time consuming to implement. Purchasers of the option spread, such as a pension fund, will gain if realized survival rates are greater than the forward rates, but the gains will be limited, thereby providing some solace to the investors providing the longevity hedge. Deutsche Bank believes that the 10-year maturity is the maximum that investors will intake in the current stage in the development of a market in longevity risk transfers (c.f. Tan et al., 2015, for details).

As can be seen in this last paragraph, LEO hinges upon survival index, published by the LLMA. Looking at the need to develop a market to trade longevity and mortality, the Longevity and Life Market Association was set up in 2010. The LLMA promotes the development of a liquid trading market in longevity and mortality-related risk, in a manner similar to those that exist for Insurance Linked Securities (ILS) and other large trend risks like interest rates. There have been a few longevity indices created by various parties associated with LLMA but we still lack a benchmark. Mènioux (2008) and Chan et al. (2014) throw light on some of them. The latter also advocate new mortality indices based on the parameters of Cairns-Blake-Dowd (CBD) (c.f. Cairns et al., 2006b) model laying down some important characteristics that mortality indices should satisfy.

The Emergence of a New Life Market

As discussed above, the evolution of the products being structured to hedge longevity risk indicate a strong potential for index-based instruments to be used more widely to achieve an effective reduction in longevity risk for pension funds and annuity providers while attracting a wide investor base in the capital markets (OECD, 2014). On the other hand, researchers such as Ragnar Norberg (Emeritus Professor of Statistics at the University of Lyon) feel that the aggregate longevity risk in the pension market is so huge that the financial market can only absorb (hedge) a minor part of this risk. This statement is well supported by a "consultative document" on 'Longevity Risk Transfer (LRT) Markets' published by the Basel Committee on Banking Supervision in December 2013 (Basel, 2013) which extensively discusses the impediments to the growth of these markets. According to this report, asymmetric "lemon"[28] risk; regulatory risk and the dearth of critical information, are the key factors hindering the growth of LRT volumes. Perhaps, the need of the hour is to give the additional push towards the more rapid development of the *Life Market* by taking smaller steps to facilitate the standardization and transparency of these instruments (OECD, 2014; Biffis and Blake, 2014).

[28] A "lemon" is something, such as a second hand car, that is revealed to be faulty only after it has been purchased (Akerlof, 1970).

Blake et al. (2008) throw light on the birth of the life market while Loeys et al. (2007) explain that for a new capital market to be established and to succeed, "it must provide effective exposure, or hedging, to a state of the world that is economically important and that cannot be hedged through existing market instruments, and it must use a homogeneous and transparent contract to permit exchange between agents." They feel that "longevity meets the basic conditions for a successful market innovation." Blake et al. (2013) provide further useful insights that could lead to a flourishing life market and in the appendix they furnish a comprehensive list of key organizations that have called on governments to support the development of the life market. To address the problem of population basis risk that is inherent in hedges constructed to offload longevity risk exposures by trading securities that are linked to broad-based mortality indexes, Chan et al. (2016) furnish a graphical population basis risk metric. According to them population basis risk is a major obstacle the development of a liquid longevity market.

In a recent speech at the 11th annual Longevity Risk and Capital Market Solutions Conference senior vice president and head of Longevity Reinsurance at US insurer Prudential Financial Amy Kessler (c.f. Kessler, 2015) talked about the rapid growth in the global pension de-risking market, and her prediction is that the market could double in the coming years. She commented that \$260bn (£170bn) of longevity risk has been offloaded to insurers and re-insurers globally and around two thirds of that from UK schemes. According to her eye-catching deals like BT's £16bn longevity swap (which passed the risk onto Kessler's firm through a captive insurer owned by the scheme) and growing interest in other countries suggest that the forecast of the doubling in size is not far-fetched.

For up-to-date developments on emerging capital market solutions to offset longevity risk, see the annual updates published by the 'Pensions Institute' and the references therein; for example, the latest annual update by Blake and Morales (2017).

From the point of view of the present work, our interest lies in the pricing of Guaranteed Annuity Options (GAOs). We look at these interesting options in the next sub-section.

1.3.3 The Intelligent Options

Interestingly, the product that was responsible for bringing longevity risk into limelight was Guaranteed Annuity Option (GAO) through the closure of the world's oldest life office, the Equitable Life Assurance Society (ELAS) in December 2000. Between 1957 and 1988, ELAS had sold a type of pension annuities with the so-called "Guaranteed Annuity Options (GAOs)" as an embedded feature of the contracts. A guaranteed annuity option (GAO) gives the policy holder a right to convert his accumulated fund at retirement at a guaranteed rate rather than at market annuity rate. At the time of issuance, the value of these GAOs was considered worthless, but they became very valuable at the time of maturity, due to two factors:

- *Reductions in market interest rates*

  and

- *Unanticipated falls in mortality rates at the oldest ages.*

The resultant liability obligations from the guarantees, resulted in serious solvency concerns for ELAS, requiring the setting up of extra reserves, and finally lead to unforeseen financial crisis for the firm (c.f. Baranoff and O'Brien, 2016). Although it appears that the reason behind the problem was poor risk management of the company, and that the problems could be avoided if ELAS had hedged its exposure to both interest rate risk and longevity risk. However, Blake et al. (2006) have clearly pointed out that, even if ELAS had anticipated the problem, it still lacked good instruments to hedge its exposure to both risks, particularly longevity risk, back to that time. Therefore, this is in fact not only the problem of ELAS. During the late 1970s and 1980s, guaranteed annuity rate between cash and pension was a common feature of individual pension policies in the UK and was sold by more than 40 companies in the market. Although these pension policies are no longer being sold in the UK now, there are similar guarantees existing in the corresponding policies in other countries. For example, in the United States' *variable annuity* market, there are

- *'Guaranteed Annuity Rate' (GAR)*
- *'Guaranteed Minimum Income Benefits' (GMIB)*
  and
- *'Guaranteed Minimum Accumulation Benefit' (GMAB)*

contracts. A GAR contract is identical to a GAO. In fact, a GAR contract can also be seen as a particular case of *'Guaranteed Minimum Income Benefits (GMIB)* where the policyholder can choose to obtain the account value (without guarantee) or annuitize the account value at current market conditions (again without a guarantee). However, the policyholder may annuitize some guaranteed amount at annuitization rates, that have been decided upfront. This leads to a GAR. A GMAB contract includes the additional feature that the cash benefit available at retirement is guaranteed to be at least a pre-specified amount. For a detailed overview of these contracts and their pricing by simulation, interested readers can refer to Bauer et al. (2008) and Bacinello et al. (2011). In fact, GAOs with limit have also been introduced by putting a ceiling on the guaranteed rate (c.f. Kling et al., 2014).

However, the problem that led to the downfall of ELAS is still haunting the market. In Jan 1997 a working party sponsor: Life Board of IFoA was set up to take care of these issues (c.f. Bolton et al., 1997). Fortunately, this incident has stimulated a lot of research into the investigation of the issues related to mortality risk, and opened the doors to the development of a longevity derivative market.

Valuation of Guaranteed Annuity Options

The flourishing market of sophisticated insurance products with benefits linked to financial variables along with various guarantees has given impetus to the active use of stochastic modeling of both interest and mortality rates in the valuation of annuity-related products. In this subsection, we present a brief recap of the research carried out in the last two decades in context of GAOs. A good reference in this regard is Gao (2014).

The pioneering contribution on GAOs, by Bolton et al. (1997), throws light on reserving for annuity guarantees. Next, O'Brien (2002) discuss five issues of resolution with particular focus on the possible investment strategies, the solvency of the insurance companies and the GAO liabilities. They conclude that GAOs are important both for policyholder - for whom they provide guarantees and for life offices and therefore appropriate pricing is a must. Boyle and Hardy (2003) examine three major risks involved in GAOs. In their paper, interest rate risk and equity risk are portrayed using correlated affine processes whilst mortality rate remained deterministic and independent with the other two risks. They compute GAO price via the change of measure technique assuming a swaption hedge[29]. Ballotta and Haberman (2006) investigate the valuation of annuity-contingent options and extend the line of research adopted in Ballotta and Haberman (2003), which assumed unsystematic mortality risk. In the later publication, they consider an integrated framework to value GAO exploiting option pricing methodology of modern finance. They assume a stochastic model for the evolution of mortality rate while the term structure of interest rate evolves according to a single-factor Heath-Jarrow-Morton (HJM) model (c.f. Heath et al., 1990). A fair value for GAO was derived through the change of measure technique and Monte-Carlo simulation. Moreover, they examine the sensitivity of GAO prices with respect to key parameters. However, while the two types of risks viz. mortality and financial are stochastic in their valuation, they are still assumed to be independent. A similar assumption is considered in Biffis and Millossovich (2006), who consider doubly stochastic stopping times and affine processes to value the GAO. They emphasize that the exercise decision made by the policyholder in regards to the GAO may not be rational from the insurer's point of view.

[29] A swaption is an option giving the right but not the obligation to engage in a swap. Boyle and Hardy (2003) feel that the interest rate exposure in a GAO is similar to that under a long dated swaption and so, it is instructive to examine some results on hedging swaptions.

A different approach, in regards to the pricing and the hedging for policies with guaranteed annuity options, is offered by Wilkie et al. (2003) and Pelsser (2003). These approaches lay emphasis on modeling the annuity price. In particular, Wilkie et al. (2003) examine the feasibility of employing option pricing methodology to dynamically hedge a GAO. Pelsser (2003) derives a market value for with-profit GAO, using martingale modeling techniques and explains the construction of a static replicating portfolio of vanilla swaptions that replicate the with-profit GAO. In fact, the issue of hedging mortality derivatives is an important but challenging problem (c.f. Liu et al., 2014). A major hindrance in constructing an effective hedging strategy for GAOs is the unavailability of a trading market for mortality risk. In addition, the options written by insurance companies often have very long maturities usually from 10 to 30 years, which complicates the modeling of underlying risks. As a result, static option replication considered by Pelsser (2003) offers a partial solution for insurance companies to hedge their exposure to embedded options in their portfolios. More recently, Luciano et al. (2012) discuss the delta-gamma hedging of mortality and interest rate risks under the independence assumption of these risks.

Chu and Kwok (2007), propose three analytic approximation methods for pricing a GAO, viz. the stochastic duration approach, Edgeworth expansion and multi-factor affine interest rate model setting. The stochastic duration approach relies on the minimization of the price error whilst the Edgeworth expansion method approximates the probability distribution of the annuity value at maturity of the contract. For the affine approximation, a hyperplane is used to approximate the concave exercise boundary to achieve the exercise probability of the annuity option. The authors furnish results on comparison of the methods in terms of numerical accuracy and computational efficiency. A sensitivity analysis of GAO prices is also performed. The paper of Liu et al. (2011) experiments with the evaluation of the annuity rate which is defined as the conditional expected present value random variable of the annuity's future payments. The two risk factors namely mortality and interest rate are modeled as stochastic processes with the former following the LC model and the latter evolving according to a Vasicek model. Comonotonicity is employed to furnish convex-order lower and upper bounds of the annuity rate. The accuracy of the bounds is tested via numerical analysis. The approach has the power of mathematical tractability in computing the distribution function for the sum of comonotonic random variables, and has been adopted in the calculation of other annuity-linked products (c.f. Liu et al., 2013, 2014). Piscopo and Haberman (2012) throw light on valuation of a related product, called 'Guaranteed Lifelong Withdrawal Benefit' (GLWB) option with variable annuity. The authors let the equity risk follow a geometric Brownian motion and let the mortality rate be based on the standard mortality tables with allowance for the possible perturbations having a regime-switching feature. They compute the fair value through Monte Carlo simulations under different scenarios and carry out extensive sensitivity analysis to show the relation between the variation of parameters and the value of the product. Gao et al. (2015b) discuss three ways of postulating a regime-switching approach in modeling the evolution of mortality rates for pricing a GAO sticking to the independence assumption of mortality and interest rate.

In the present era, considering mortality to be independent of financial markets appears to be a far- fetched assumption and a more realistic belief is that the two underlying risks are correlated. This belief is supported by researchers and practitioners. For example, Ang and Maddaloni (2003) investigate the effect of demographic changes on risk premiums. Favero et al. (2011) examine the likelihood that the slowly evolving mean in the log dividend-price ratio is related to demographic trends. Maurer (2014) investigates how demographic changes affect the value of financial assets. He experiments with a continuous time overlapping generations model having stochastic birth and mortality rates. His model suggests that demographic transitions have an important role to play in explaining parts of the time variation in the real interest rate, equity premium and conditional stock price volatility. Moreover, he provides adequate conditions for the interest rate to be decreasing in the birth rate and increasing in the death rate. In Dacorogna and Cadena (2015), the authors furnish some empirical evidence of a changing behavior of the economy and the financial markets during periods of extreme mortality. The authors advocate the use of lead-lag correlation analysis where they lead and lag the economic variables in connection with the mortality indices during those extreme years to see if there are retarded effects. Correlation is computed over 5 years lag and 5 years lead. This analysis has twin benefits: one it shows, if any of the economic or financial indicators have an effect on the mortality indices (lag-analysis) and secondly it clarifies if the mortality indices have lagged effects on the economic or financial indices (lead-analysis). Further, Dacorogna and Apicella (2016) explore existence of this dependence within the Feller process framework[30]. Jevtie et al. (2017) consider a partial equilibrium model for pricing a longevity linked bond in a model with stochastic mortality intensity that affects the income of economic agents. Arik et al. (2017) concentrate on the pricing of pension buy-outs under dependence between interest and mortality rates risks with an explicit correlation structure in a continuous time framework. To take care of this scenario, EU's Solvency II Directive has laid out new insurance risk management practices for capital adequacy requirements based on the assumption of dependence between financial markets and life/health insurance markets including the correlation between the two underpinning risks viz. interest rate and mortality (c.f. Quantitative Impact Study 5:Technical Specifications QIS5, 2010). Jalen and Mamon (2009) introduced a pricing framework in which the dependence between the mortality and the interest rates is explicitly modeled. In their methodology, the mortality rate was modeled as an affine-type diffusion process just like the short rate process. They derived analytic expressions for mortality-linked insurance products employing the change of measure technique. Their approach paved the path for new perspectives and methodology in the valuation of other insurance products under a more reasonable assumption that risk factors are dependent.

This line of research has triggered research on evaluation of GAOs under the assumption of correlation between mortality and financial risks. Liu et al. (2013, 2014), were the pioneers to consider the correlated framework for valuing a GAO and they developed a pricing formula where the interest rate and mortality processes follow bivariate Gaussian dynamics. In their setting, the dependence between mortality and interest rates is described by one constant, namely the pairwise linear correlation coefficient. In fact in Liu et al. (2013) they use the theory of comonotonic bounds in approximating the sums of lognormal random variables to obtain convex price bounds for GAOs in the Gaussian setting. Gao et al. (2015a) propose a modeling framework, where the interest and mortality rates are correlated and the dynamics of each risk factor possess regime-switching affine structures, to facilitate the GAO valuation. The correlation introduced through the diffusion components of the risk factors and the underlying Markov chain driving the switching of regimes provides an explanation of the rates' relation and dynamics. A different measure called endowment-risk-adjusted measure, which first appeared in Liu et al. (2013) and was subsequently used in Gao et al. (2015b) under several competing models, is employed to price the GAO.

More recently, Deelstra et al. (2016) scrutinize the consequences of the dependence assumption on the pricing of a GAO. They assume that mortality and interest rates are driven by systematic and idiosyncratic factors, modeled by affine models which remain positive such as the multi-CIR and the Wishart models. They employ the above mentioned change of measure to value the GAO using Monte Carlo methodology. Their investigation reveals that for an advanced affine model (such as the Wishart one) that permits a non-trivial dependence between the mortality and the interest rates, the value of a GAO cannot be explained only in terms of the initial pairwise linear correlation and this fact plays an important role in risk management in the presence of an unknown dependence. Finally, Gao et al. (2017) address the problem of setting capital reserves for a guaranteed annuity option (GAO). They formulate the modeling framework for the loss function of a GAO. A one-decrement actuarial model having death as the only decrement is employed. Once again, the interest and mortality risk factors follow correlated affine structures. Risk measures are calculated using a moment-based density method and compared with the Monte-Carlo simulation. Bootstrap technique is used to assess the variability of risk measure estimates. The authors also establish the relation between a desired level of risk measure accuracy and required sample size under the constraints of computing time and memory. A sensitivity analysis of parameters is also conducted. Their numerical investigations

[30] Also see Chapter 5

furnish practical considerations for insurers to abide by certain regulatory requirements. Thus, dealing with Guaranteed Annuity Options under the correlation assumption of mortality and financial risks offers a fertile ground for future research.

1.3.4 The Future of Longevity?

In the beginning of this chapter, we had remarked that the actual problem in longevity risk management stems from the uncertainty surrounding the improvements in mortality. Oeppen and Vaupel (2002) had depicted record female life expectancy from 1840 to the present but with horizontal black lines showing asserted ceilings on life expectancy, with a short vertical line indicating the year of publication. This is shown in Figure 1.12.

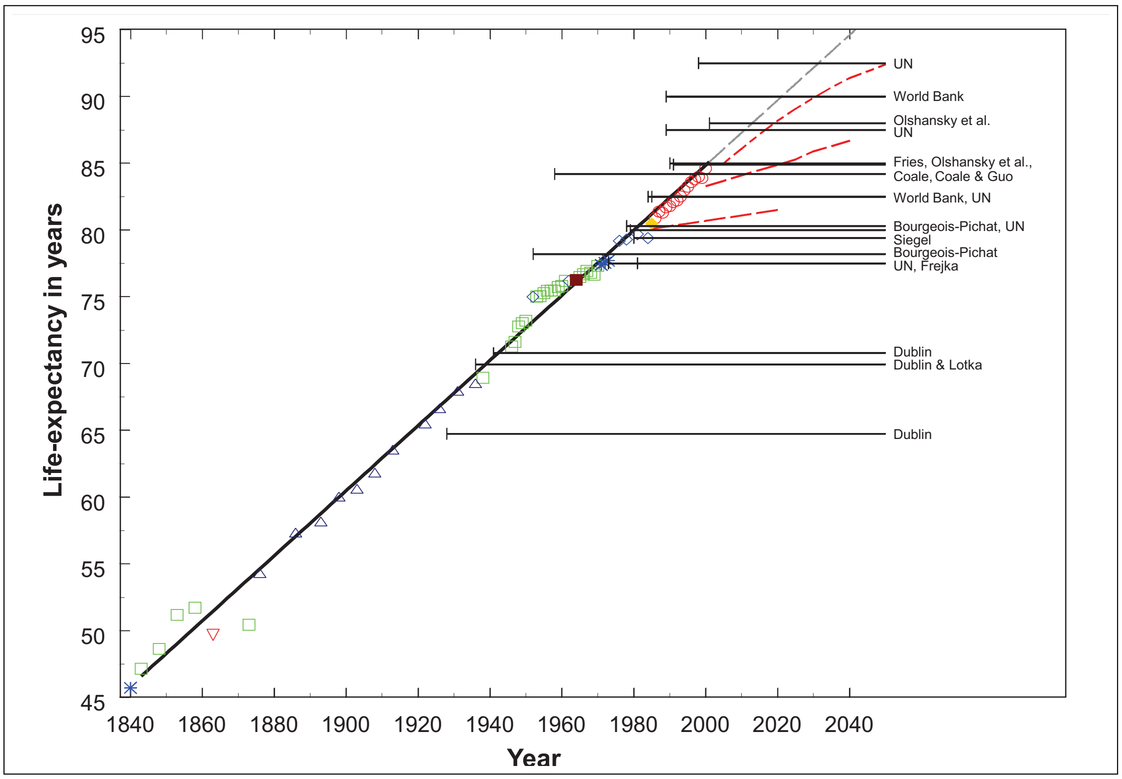


Figure 1.12: The Sorry Saga of Looming Limits to Life Expectancy (Source: Oeppen and Vaupel (2002))

Recently on 4th May 2017, Financial Times (see Cumbo (2017) for details) has reported on basis of estimates of Price Water Coopers (PwC) that a sharp slowdown in the improvement to life expectancy could wipe £310bn from the pension deficits of thousands of UK companies with final salary schemes. These forecasts are based on recent updated calculations published by 'Continuous Mortality Investigation' (CMI, 2016) of the Institute and Faculty of Actuaries' (IFoA) which reduced the projected life expectancy for a 65-year-old man by almost four months from the estimates made in 2015, and that of a 65-year-old woman by almost six months. Professor Andrew Cairns, Director of the Actuarial Research Centre (ARC), the IFoA along with few other leading researchers wrote an interesting letter entitled 'Estimated reductions in schemes' liabilities must be placed in context' in response to this Financial Times article. The British Medical Journal (Fransham and Dorling, 2017) has also published an interesting article detailing the reasons for this downfall in life expectancy and the differences between the results published by National Statistics (2017) and CMI (2016). We present below the interesting Figure 1.13 depicting the stalled improvement in mortality rate.

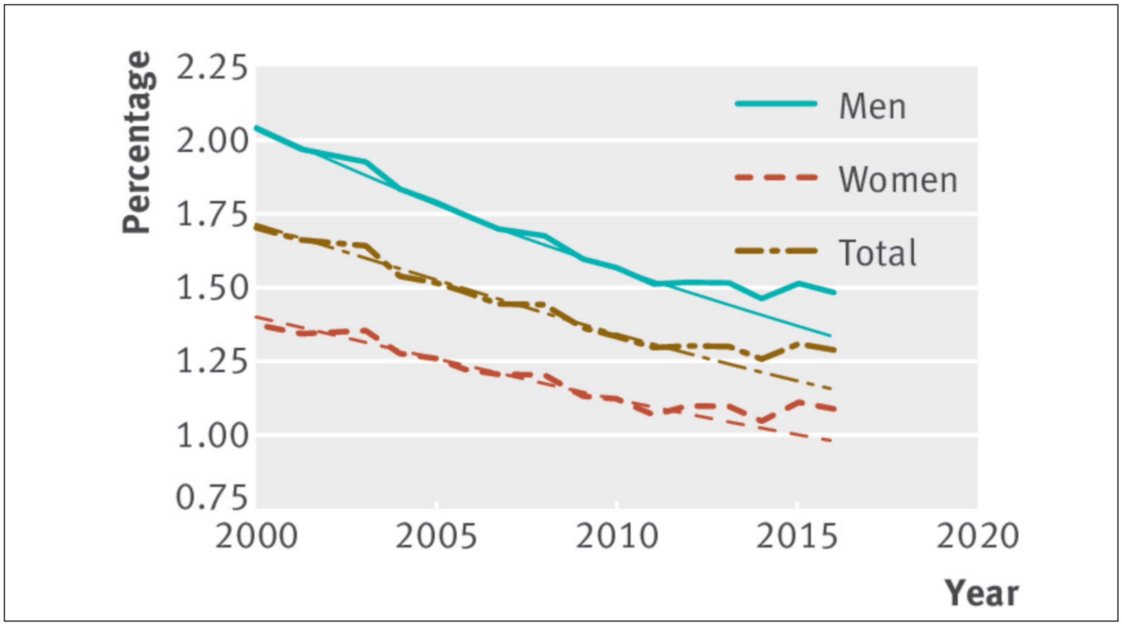


Figure 1.13: Standardised mortality ratios for people aged 20 to 100 years (solid lines) and trends (dashed) 2000-11. (Source: CMI (2016))

Thus, the future of longevity is uncertain and this uncertainty in mortality trends is the driving force for the current work.

1.4 Motivation for the Present Work

As can be seen from the discussion undertaken in this chapter, in the present times, mankind is facing unforeseen challenges in regards to mortality and longevity. On one hand, lives have become so cheap that people are burning like pieces of paper (Grenfell Tower fire in London, 14th June 2017), mankind is killing mankind (London Bridge, 3rd May 2017; Manchester, 22nd May 2017 and Westminister, 22nd March 2017 terrorist attacks) and on the other hand the British Monarchy is having a tough time sending celebration messages to centenarians (Monarchy, 2013). Thus, while on one side, catastrophic events are testing the endurance of the insurance industry, on the other, the improvement in mortality rates is burning a hole in the pockets of annuity and pension providers. Mortality and longevity risk management is the new buzz word in insurance industry. Clearly as pointed out earlier, actuaries and financial mathematicians have the potential to play a key role in enhancing longevity by offering state of art solutions to manage longevity risk and enable retired individuals to maintain a good standard of living by helping them to procure a regular stream of payments.

The present work draws encouragement from the prevailing scenario. It looks at both the sides of the coin. In regards to catastrophic mortality, it looks at the pricing of the first ever catastrophic mortality bond - the Swiss Re VITA I mortality bond launched in 2003. The complexity of the payoff of the bond has been a hindrance in its pricing and to the best of our knowledge only model-dependent solutions in regards to its pricing have been proposed earlier. This became the starting point of our research and in Bahl and Sabanis (2016), we express the payoff of this bond in terms of an Asian put option and present model-independent bounds for pricing of the Swiss Re bond. We exploit the theory of comonotonicity for devising tight lower and upper bounds and test these bounds in the backdrop of a number of models against the benchmark Monte Carlo value. Interestingly, it is the same bond that triggered the popular conference series 'International Longevity Risk and Capital Markets Solutions' with the first edition abbreviated Longevity 1 (L1) being held at Cass Business School in London in February 2005 (c.f. Tan et al., 2015).

Further motivation for research came from the emerging longevity risk. While catastrophic bonds were used by re-insurers to hedge themselves from extreme mortality, we wanted to look at the other side and see which products appealed to the consumer for protecting themselves from extreme longevity. Just like Swiss Re Mortality bond sowed the seeds for the growth in mortality related research, the products that highlighted the seriousness of longevity risk to academicians and practitioners were Guaranteed Annuity Options. To the best of our knowledge, only Monte Carlo estimation of GAO price is available in literature for complex models. However, Monte Carlo simulation-based techniques are sampling methodologies that furnish statistical estimations bearing a sampling error that reduces by the reciprocal of the square root of the sample size. To improve the efficiency by even one significant digit, the sample size is to be magnified a hundredfold (c.f. Fenga et al., 2017). There is a strong bargain between time and accuracy. Many industrial surveys, such as Farr et al. (2008), have discussed the emerging difficulties arising due to inefficient simulation procedures which make it extremely challenging to obtain meaningful information and reach worthy decisions on pricing and risk management in a timely manner. Many a times, accuracy takes a heavy toll due to the lack of time. Recognizing the need of the hour, in Bahl and Sabanis (2017), we offer extremely efficient tight bounds for GAOs. These bounds are model free and model robust and extremely simple to apply particularly in case of affine models. Moreover they have been created in the most generalized pricing framework that allows for correlation between mortality and interest rate risk. Also this latter scenario has escaped the eyes of academicians and practitioners probably both due to its complexity and lack of available information. The available literature in the correlated framework is thin and this gave the impetus to strengthen this area.

1.5 Organization of the Thesis

This thesis is composed of 7 chapters including this introductory chapter. The succeeding chapters are the compilation of related research papers (published on ArXiv) and the necessary foundations required for building the theory.

In chapter 2, we present details of the mathematical framework required in this thesis with particular attention to the concepts of comonotonicity and stochastic ordering.

In chapter 3, we present our findings for the valuation of catastrophic mortality bonds with particular attention to Swiss Re Mortality bond 2003 obtained in Bahl and Sabanis (2016). This bond was the first Catastrophic Mortality Bond to be launched in the market and encapsulates the behaviour of a well-defined mortality index to generate payoffs for bondholders. We propose model-independent price bounds for the Swiss Re Mortality bond 2003 by adapting its payoff in terms of the payoff of an Asian put option. Comonotonicity theory is then applied to derive lower and upper bounds for this bond.

In chapter 4, we consider at the basics of Affine processes with particular attention to Wishart processes.

Chapter 5 deals with pricing of Guaranteed Annuity Options in a generalised framework assuming mortality and interest rates are correlated. It is a reflection of the results obtained in Bahl and Sabanis (2017). Pricing GAOs in the correlated environment is a challenging task and no closed form solution exists in the literature. We employ the use of doubly stochastic stopping times to incorporate the randomness about the time of death and employ a suitable change of measure to facilitate the valuation of survival benefit, there by adapting the payoff of the GAO in terms of the payoff of a basket call option. We derive general price bounds for GAOs by employing the theory of comonotonicity, the Rogers-Shi (Rogers and Shi, 1995) approach and a conditioning approach for the lower bound and arithmetic-geometric mean inequality and comonotonicity for the upper bound. The theory is then applied to affine models to present some very interesting formulae for the bounds under the affine set up.

Chapter 6 presents numerical results to illustrate and verify the theoretical findings of chapters 3 and 5. Experiments are conducted to obtain price bounds for the Swiss Re catastrophic mortality bond for a variety of models and these are benchmarked against Monte Carlo estimates of the bond price. Similarly, numerical examples are furnished and tested against Monte Carlo simulations to estimate the price of a GAO for a variety of affine processes governing the evolution of mortality and the interest rate.

In chapter 7, we summarize the main findings and contributions of the thesis and cast light on future research perspectives.

Chapter 2

Building Blocks

This thesis thrives upon many mathematical concepts, statistical distributions and stochastic processes. We present the most important ones here and the other related results are furnished in appendix A. The concepts discussed here serve as essential tools for efficient working in the chapters that follow.

We begin by furnishing a self-contained set of results which will provide a framework for valuing stop-loss transforms based upon arithmetic sums. Dhaene et al. (2002a) and Shaked and Shanthikumar (2007) serve as vital resources for these results.

2.1 Ordering Random Variables

The distribution function of the sum of risks which are not independent, is hard to determine. This leads us to the quest for a random variable which has a simpler structure and whose distribution function is easier to determine. Such a random variable is generally referred to as 'less attractive' in financial and actuarial literature. Useful citations in this area are Goovaerts et al. (1990), Kaas et al. (1994) and Denuit et al. (1999). The term 'less attractive' requires an explanation. In an attempt to do so, we first define the concept of "stop-loss premium".

Definition 1. ***Stop Loss Premium:*** *The stop-loss premium with retention d of a random variable X is defined as $\boldsymbol{E}\left[(X-d)^+\right]$.*

In the following proposition we present an interesting result regarding the stop-loss premium defined above.

Proposition 2. ***(Stop-Loss Premium with Retention** d**)*** *The stop loss premium with retention d is given as*

$$\boldsymbol{E}\left[(X-d)^+\right] = \int_d^\infty (1 - F_X(x))\, dx \tag{2.1.1}$$

where $F_X(.)$ denotes the distribution function of X.

Equation (2.1.1) clearly shows that stop-loss premium with retention d can be considered as the weight of an upper tail of (the distribution function of) X. Also, it is evident that $\mathbf{E}\left[(X-d)^+\right]$ is a decreasing continuous function of d, with derivative $(F_X(d) - 1)$ at d which vanishes at $+\infty$.

This naturally leads to the definition of stop loss order between random variables.

Definition 3. ***Stop-Loss Order:*** *Consider two random variables X and Y. Then X is said to precede Y in the stop-loss order sense, written as $X \leq_{sl} Y$, if and only if X has lower stop-loss premiums than Y:*

$$\boldsymbol{E}\left[(X-d)^+\right] \leq \boldsymbol{E}\left[(Y-d)^+\right], \quad \forall -\infty < d < \infty \tag{2.1.2}$$

Thus, $X \leq_{sl} Y$ means that X has uniformly smaller upper tails than Y, which in turn implies that payment Y is less attractive than a payment X.

Let us now connect stop loss premium to our situation. In most of the work undertaken in this thesis, we are interested in evaluating expectation of the type

$$\mathbf{E}\left[\left(\sum_{i=1}^{n} X_i - d\right)^+\right] , \quad d \in \mathbb{R} \tag{2.1.3}$$

This is essentially the stop-loss premium of $\sum_{i=1}^{n} X_i$ with retention d. In this equation $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is a random variable in $\mathbb{R}^n$ whose individual components are not mutually independent and d is typically positive. We assume that the marginal distribution functions are known, but the problem we encounter is that the joint distribution of $\mathbf{X}$ is either unknown or too cumbersome to work with. Another significant assumption that we make is that that the marginal distribution functions denoted by F_{X_i}, $i = 1, 2, \ldots, n$ are injective and strictly increasing. In case if we wish to drop this assumption, then we would mention it explicitly.

A popular methodology that is adopted to estimate equation (2.1.3) is to replace $\mathbf{X}$ by another random vector $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$, for which the distribution function $F_{\sum_{i=1}^{n} Y_i}$ can be worked out with less trouble. The selection of $\mathbf{Y}$ is done keeping in mind that the sum of its components can form a stochastic bound on the sum of the components of $\mathbf{X}$ denoted by S, i.e., $S = \sum_{i=1}^{n} X_i$. We have already discussed one such stochastic bound in the stop-loss order sense and now we introduce a stronger relation than stop-loss order.

Definition 4. ***Convex Order:*** *X is said to precede Y in terms of convex order, written as $X \leq_{cx} Y$, if and only if $X \leq_{sl} Y$ and $\boldsymbol{E}[X] = \boldsymbol{E}[Y]$.*

In fact "stop-loss order" is also referred to as "increasing convex order" by Dhaene et al. (2002a) and Chen et al. (2008) and is denoted by $\leq_{icx}$.

In the next section, we introduce the concept of inverse distribution function.

2.2 Inverse Distribution Function

We know that for a given random variable $X\colon \Omega \to \mathbb{R}$, the cumulative distribution function abbreviated as c.d.f., defined as $F_X(x) = P[X \leq x]$ is a right-continuous, non-decreasing function, which is bounded by 0 and 1 such that

$$F_X(-\infty) = \lim_{x \to -\infty} F_X(x) = 0, \quad F_X(+\infty) = \lim_{x \to +\infty} F_X(x) = 1$$

We now define the inverse of a distribution function.

Definition 5. ***Inverse Distribution Function:*** *The inverse of a distribution function is the non-decreasing and left-continuous function $F_X^{-1}(p)$ defined as*

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} | F_X(x) \geq p\}, \quad p \in [0,1] \tag{2.2.1}$$

with $\inf \phi = +\infty$ *by convention.*

Dhaene et al. (2002a) use a more generalized definition of inverse distribution function which we define below.

Definition 6. ***Generalized Inverse Distribution Function:*** *For any real $p \in [0,1]$, a possible choice for the inverse of F_X in p is any point in the closed interval*

$$[\inf\{x \in \mathbb{R} | F_X(x) \geq p\}, \sup\{x \in \mathbb{R} | F_X(x) \leq p\}].$$

Like the previous definition $\inf \phi = +\infty$ and further, $\sup \phi = -\infty$. Considering the left-hand border of this interval to be the value of the inverse cdf at p, we get $F_X^{-1}(p)$. On the same lines Dhaene et al. (2002a) define the right-hand border of this interval as $F_X^{+1}(p)$, i.e.,

$$F_X^{-1+}(p) = \sup\{x \in \mathbb{R} | F_X(x) \leq p\}, \quad p \in [0,1] \tag{2.2.2}$$

which is a non-decreasing and right continuous function. Clearly, $F_X^{-1+}(0) = -\infty$ and $F_X^{-1}(1) = \infty$ and therefore the entire mass of X is contained in the interval $\left[F_X^{-1+}(0), F_X^{-1}(1)\right]$. Moreover, $F_X^{-1}(p)$ and $F_X^{-1+}(p)$ are finite for all $p \in (0,1)$ and thus Dhaene et al. (2002a) use $p \in (0,1)$. We finally define the α-mixed inverse of the cumulative distribution function considered by the aforesaid authors.

Definition 7. ***α-mixed Inverse of C.D.F.:*** *For any $\alpha \in [0,1]$, the α-mixed inverse of the c.d.f. is defines as*

$$F_X^{-1(\alpha)}(p) = \alpha F_X^{-1}(p) + (1-\alpha) F_X^{-1+}(p). \tag{2.2.3}$$

This is a non-decreasing function. It is evident that, $F_X^{-1(0)}(p) = F_X^{-1+}(p)$ and $F_X^{-1(1)}(p) = F_X^{-1}(p)$ and $\forall \alpha \in [0,1]$

$$F_X^{-1}(p) \leq F_X^{-1(\alpha)}(p) \leq F_X^{-1+}(p)$$

In the passing, we note that if the distribution function of a random variable is injective, then its inverse can be obtained directly. An important theorem on inverse distribution functions is documented as theorem 1 in Dhaene et al. (2002a). We outline a simplified version of this theorem.

Theorem 8. *(**Inverse Distribution Function of** $g(X)$) Let X and $g(X)$ be real valued random variables and let $0 < p < 1$. Assume that g and F_X are strictly increasing, then*

$$F_{g(X)}^{-1}(p) = g\left(F_X^{-1}(p)\right). \tag{2.2.4}$$

Proof. Let

$$p = F_{g(X)}(y) = \mathbf{P}\left[g(X) \leq y\right] = \mathbf{P}\left[X \leq g^{-1}(y)\right] = F_X\left(g^{-1}(y)\right)$$

$$\Rightarrow g^{-1}(y) = F_X^{-1}(p)$$

$$\Rightarrow y = g\left(F_X^{-1}(p)\right) = F_{g(X)}^{-1}(p)$$

where the last equality follows from the definition of p. □

We now move to the definition of Comonotonicity.

2.3 Comonotonicity

Over the last decade, it has been shown that the concept of comonotonicity is a handy tool for solving several research and practical problems in the domain of finance and insurance. Deelstra et al. (2011) present an excellent review of applications of comonotonicity in financial literature. The theory of comonotonicity has been exploited extensively to carry out research in this thesis. We begin by defining comonotonicity of a set of n-vectors in $\mathbb{R}^n$.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be any two vectors in $\mathbb{R}^n$. Then by $\mathbf{x} \leq \mathbf{y}$, we mean component wise order which is defined by $x_i \leq y_i$, for all $i = 1, 2, \ldots, n$.

Definition 9. ***Comonotonic Set:*** *A set $A \subseteq \mathbb{R}^n$ is said to be comonotonic if for every $\boldsymbol{x}$ and $\boldsymbol{y}$ in A, either $\boldsymbol{x} \leq \boldsymbol{y}$ or $\boldsymbol{y} \leq \boldsymbol{x}$.*

Another way to interpret this definition is that a set $A \subseteq \mathbb{R}^n$ is said to be comonotonic if for any $\mathbf{x}$ and $\mathbf{y}$ in A, if $x_i < y_i$ for some i, then $\mathbf{x} \leq \mathbf{y}$ must hold. Hence, a comonotonic set is in a way simultaneously monotone in each component. As a result, a comonotonic set is a "thin" set in the sense that it cannot contain any subset of dimension greater than 1. In fact, any subset of a comonotonic set is also comonotonic. Dhaene et al. (2002a) offer a good characterization of comonotonic sets.

Next we introduce the concept of a comonotonic random vector. For this we first define

Definition 10. ***Support of an n-dimensional Random Vector:*** *Let $\boldsymbol{X}$ be an n-dimensional random vector. Any set $A \subseteq \mathbb{R}^n$ is called the support of $\boldsymbol{X}$ if $\boldsymbol{X} \in A$ with probability 1. In fact the smallest support can be regarded as the image of $\boldsymbol{X} : \Omega \to \mathbb{R}^n$. In other words, the support of a random vector is the set of all possible outcomes of $\boldsymbol{X}$.*

Definition 11. ***Comonotonic Random Vector:*** *A random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ is said to be comonotonic if it has a comonotonic support.*

For examples and applications of comonotonicity the reader can refer to Dhaene et al. (2002a) and Dhaene et al. (2002b) respectively. We present below theorem 2 of Dhaene et al. (2002a) which provides equivalent characterizations for comonotonicity of a random vector $\mathbf{X}$.

Theorem 12. ***(Characterizations for Comonotonicity)*** *A random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ is comonotonic if and only if one of the following equivalent conditions holds:*

1. *$\boldsymbol{X}$ has a comonotonic support.*
2. *For all $\boldsymbol{x} = (x_1, \ldots, x_n)$, we have*

$$F_X(x) = \min_{j=1,2,..,n} F_{X_j}(x_j) \tag{2.3.1}$$

3. *For $U \sim Uniform(0,1)$, we have*

$$\boldsymbol{X} \stackrel{d}{=} \left(F_{X_1}^{-1}(U),\, F_{X_2}^{-1}(U), \ldots,\, F_{X_n}^{-1}(U)\right) \tag{2.3.2}$$

4. *There exist a random variable Z and non-decreasing functions f_i $(i = 1, 2, ..., n)$ such that*

$$\boldsymbol{X} \stackrel{d}{=} (f_1(Z),\, f_2(Z), \ldots,\, f_n(Z)) \tag{2.3.3}$$

where $\stackrel{d}{=}$ means equality in distribution.

Proof. We provide a proof of (1) $\Rightarrow$ (2). For the proofs of other stated relationships, one can refer to Dhaene et al. (2002a) and Vynke (2003). Let us suppose that $\mathbf{X}$ has a comonotonic support B. Let $\mathbf{x} \in \mathbb{R}^n$ and define a new set $A_j \subset B$ as follows:

$$A_j = \{\mathbf{y} \in B |\, y_j \leq x_j\}$$

Since B is comonotonic and $A_j \subset B$, it automatically implies that A_j are comonotonic and since a comonotonic set can't have dimension greater than 1, they form 1-dimensional sets. For each j, let $\mathbf{y}_j = \max\{\mathbf{y} \in A_j\}$. Let

$$\mathbf{y}_i = \min_{j=1,\ldots,n} \mathbf{y}_j = \min_{j=1,\ldots,n} \max\{\mathbf{y} \in A_j\}.$$

Then for every $\mathbf{y} \in A_i$, we have: $\mathbf{y} \leq \mathbf{y}_i$ and so $\mathbf{y} \leq \mathbf{y}_j$. Hence, $\mathbf{y} \in A_j$ for every j and so $A_i = \cap_{j=1}^n A_j$. As a result, we have

$$F_X(x) = \mathbf{P}\left[\cap_{j=1}^n \{X_j \leq x_j\}\right] = \mathbf{P}\left[X \in \cap_{j=1}^n A_j\right] = \mathbf{P}\left[X \in A_i\right) = F_{X_i}(x_i]$$

The final step follows from the fact that the probability function $\mathbf{P}$ is a monotone function and $A_i \subseteq A_j$, so that

$$F_{X_i}(x_i) = \mathbf{P}\left[X_i \leq x_i\right] = \mathbf{P}\left[X \in A_i\right) \leq \mathbf{P}\left[X \in A_j\right] = F_{X_j}(x_j)\ \forall j.$$

$\square$

An alternative proof for this result appears in Karniychuk (2006) which is as follows: Let us suppose that $\mathbf{X}$ has a comonotonic support B. Let $\mathbf{x} \in \mathbb{R}^n$ and define a new set $A_j \subset B$ as follows:

$$A_j = \{\mathbf{y} \in B |\, y_j \leq x_j\}$$

Since B is comonotonic, there exists an i such that $A_i = \cap_{j=1}^{n} A_j$. This claim can be proved as follows. Clearly,

$$\cap_{j=1}^{n} A_j = \{\mathbf{y} \in B |\, y_1 \leq x_1,\ y_2 \leq x_2, ..., y_n \leq x_n, \}\,.$$

Now, quite obviously $\cap_{j=1}^{n} A_j \subseteq A_i$ for all i. The only thing which must be shown is that there exist an i in $[1, 2, ..., n]$ such that $A_i \subseteq \cap_{j=1}^{n} A_j$. This can be proved using contradiction. Assume that A_1 does not belong to the intersection or A_2 does not belong to the intersection and so on. For the first case, there exists $z^1 \in A_1$ with $z^1 \leq x_1$, $z^2 > x_2$or... $z^n > x_n$. As a result $z_k^1 > x_1 \geq z_k^k$. Due to comonotonicity, it follows that $z_m^1 \geq z_m^k$, $m = 1, 2, ..., n$ and putting $m = 1$, our assumption falls apart. The same thing can be shown for other cases.

For a characterization property on comonotonic random vectors, the reader is referred to Dhaene et al. (2002a). We conclude this section with the definition of a *comonotonic counterpart.*

Definition 13. ***Comonotonic Counterpart:*** *Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a random vector in $\mathbb{R}^n$. A random vector with the same marginal distributions as $\boldsymbol{X}$ and possessing the comonotonic dependency structure is called a comonotonic counterpart. The notation used for this random vector is $\boldsymbol{X}^c = (X_1^c, \ldots, X_n^c)$. Moreover, we have*

$$\boldsymbol{X}^c \stackrel{d}{=} \left(F_{X_1}^{-1}(U)\,,\, F_{X_2}^{-1}(U)\,, \ldots,\, F_{X_n}^{-1}(U)\right)$$

where $U \sim Uniform\,(0, 1)$.

In the next section we summarize certain useful results on sums of comonotonic random variables taken from Dhaene et al. (2002a).

2.4 Useful Results on Comonotonicity

Let S^c denote the sum of the components of the comonotonic counterpart $\mathbf{X}^c$ of X, i.e.,

$$S^c = X_1^c + X_2^c + \ldots + X_n^c$$

We give below the simplified version of Theorem 5 of Dhaene et al. (2002a) which proves that the inverse distribution function of a sum of comonotonic random variables is simply the sum of the inverse distribution functions of the marginal distributions.

Theorem 14. ***(Inverse c.d.f. of Comonotonic Sum)*** *The inverse distribution function $F_{S^c}^{-1}(p)$ of the sum S^c of the components of the comonotonic random vector $\boldsymbol{X}^c$ is given by*

$$F_{S^c}^{-1}(p) = \sum_{i=1}^{n} F_{X_i}^{-1}(p)\,, \quad 0 < p < 1 \tag{2.4.1}$$

Proof. Clearly as $\mathbf{X}^c$ is a comonotonic counterpart of $\mathbf{X}$, we have by definition 13 that $F_{X_i^c}(x) = F_{X_i}(x) = p\,(\text{say})$ and as a result, we also have $F_{X_i^c}^{-1}(p) = F_{X_i}^{-1}(p)$ for every i. Invoking theorem 12(3), we get the following

$$S^c \stackrel{d}{=} \sum_{i=1}^{n} F_{X_i}^{-1}(p) =: \ g(U)\,, \tag{2.4.2}$$

with U being uniformly distributed on $(0, 1)$. It is evident from the definition of g that $g(U)$ is an increasing function and so using theorem 8 we arrive at:

$$F_{X^c}^{-1}(p) = F_{g(U)}^{-1}(p) = g\left(F_U^{-1}(p)\right) = g(p)\,, \quad \forall p \in (0, 1)$$

where in the last equality, we exploit the fact that if $U \sim Uniform\,(0,1)$, then $F_U\,(p) = p,\ \forall p \in (0,1)$. Thus we have achieved the required result. $\square$

This result then gives way to another powerful result provided by Dhaene et al. (2002a) which partitions the stop-loss premiums of a sum of comonotonic random variables into stop-loss premiums of the terms. We present the theorem in a manner similar to the authors. This theorem and its corollary are extremely useful to derive the results in this thesis. We prove this theorem without considering any restrictive assumptions for the c.d.f.

Theorem 15. ***(Stop-loss Premium of Comonotonic Sum)*** *The stop-loss premiums of the sum S^c of the components of the comonotonic random vector $\boldsymbol{X}^c = (X_1^c, \ldots, X_n^c)$ are given by*

$$\boldsymbol{E}\left[(S^c - d)^+\right] = \sum_{i=1}^{n} \boldsymbol{E}\left[(X_i - d_i)^+\right], \quad \left(F_{S^c}^{-1+}(0) < d < F_{S^c}^{-1}(1)\right) \tag{2.4.3}$$

where the d_i, $i = 1, 2, \ldots, n$ are defined by the following equation:

$$d_i = F_{X_i}^{-1(\alpha_d)}\left(F_{S^c}(d)\right) \tag{2.4.4}$$

and $\alpha_d \in [0,1]$ determined by

$$F_{S^c}^{-1(\alpha_d)}\left(F_{S^c}(d)\right) = d. \tag{2.4.5}$$

Proof. First of all, by the definition of a comonotonic random vector, it is evident that the support of $\mathbf{X}^c$ say B is comonotonic. Further, a comonotonic set can have at most one point of intersection with the hyperplane $H = \{\mathbf{x} : x_1 + \ldots + x_d\}$. The reason for this is that diminishing any one of the x_i of $\mathbf{x} \in H$ would need a simultaneous increment in x_j, where $i \neq j$, since the sum of the $x_i's$ is fixed. However, this violates the comonotonicity assumption. Let us denote this point of intersection as $\mathbf{d} = (d_1, \ldots, d_n)$, so that $\mathbf{d} = B \cap H$. Also, as $0 < F_{S^c}(d) < 1$, has to be true, we know from the definition 7, that there exists an α_d that fulfills the condition (2.4.5). Further, since $\mathbf{d} \in H$, by definition of H, $d = d_1 + \ldots + d_n$. Let us further choose an arbitrary $\mathbf{x} \in H$. Then the following equality is true

$$(x_1 + x_2 + \ldots + x_n - d)^+ = (x_1 - d_1)^+ + (x_2 - d_2)^+ + \ldots + (x_n - d_n)^+. \tag{2.4.6}$$

This can be reasoned out as follows; $\mathbf{x}$ and $\mathbf{d}$ are both elements of B which is a comonotonic set. If there exists any j such that $x_j > d_j$ holds, then $x_k \geq d_k$, $\forall k$, and the L.H.S. equals the R.H.S. because $d = d_1 + \ldots + d_n$. The other possibility is that $x_j \leq d_j$, $\forall j$ which renders both sides to 0. As a final step, let $p = F_{S^c}(d)$. Invoking theorem 5 of Dhaene et al. (2002a), we obtain

$$d = F_{S^c}^{-1(\alpha_d)}(p) = \sum_{i=1}^{n} F_{X_i}^{-1(\alpha_d)}(p) = \sum_{i=1}^{n} F_{X_i}^{-1(\alpha_d)}\left(F_{S^c}(d)\right) = \sum_{i=1}^{n} d_i$$

We substitute X_i^c in place of x_i and take expectations and finally replace X_i^c by X_i exploiting the fact that marginal distributions of the two are identical for each i. This yields the desired result. $\square$

We now prove a very important corollary of this theorem, which we bank upon to undertake the research in this thesis.

Corollary 16. ***(Simplified Expression for Stop-Loss Premium of Comonotonic Sum)*** *The stop-loss premium of a comonotonic sum S^c of the components of the comonotonic random vector $\boldsymbol{X}^c = (X_1^c, \ldots, X_n^c)$ can be written in a simplified manner in terms of the usual inverse distribution functions. For any retention $d \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$,*

$$\boldsymbol{E}\left[(S^c - d)^+\right] = \sum_{i=1}^{n} \boldsymbol{E}\left[\left(X_i - F_{X_i}^{-1}\left(F_{S^c}(d)\right)\right)^+\right] - \left(d - F_{S^c}^{-1}\left(F_{S^c}(d)\right)\right)\left(1 - F_{S^c}(d)\right). \tag{2.4.7}$$

If the marginal c.d.f.s F_{X_i} are strictly increasing, the expression (2.4.7) *reduces to*

$$\boldsymbol{E}\left[(S^c - d)^+\right] = \sum_{i=1}^{n} \boldsymbol{E}\left[\left(X_i - F_{X_i}^{-1}\left(F_{S^c}\left(d\right)\right)\right)^+\right] \tag{2.4.8}$$

Proof. Clearly,

$$\begin{aligned}
\mathbf{E}\left[(X_i - d_i)^+\right] &= \int_{d_i}^{\infty} (x_i - d_i)\, dF_{X_i}(x_i) \\
&= \int_{d_i}^{\infty} \left(x_i - F_{X_i}^{-1}\left(F_{S^c}(d)\right) + F_{X_i}^{-1}\left(F_{S^c}(d)\right) - d_i\right) dF_{X_i}(x_i) \\
&= \int_{d_i}^{\infty} \left(x_i - F_{X_i}^{-1}\left(F_{S^c}(d)\right)\right) dF_{X_i}(x_i) \\
&\quad + \left(d_i - F_{X_i}^{-1}\left(F_{S^c}(d)\right)\right) \int_{d_i}^{\infty} dF_{X_i}(x_i)
\end{aligned} \tag{2.4.9}$$

Consider the second integral in equation (2.4.9). We have

$$\int_{d_i}^{\infty} dF_{X_i}(x_i) = P\left[X_i > d_i\right] \tag{2.4.10}$$

Using the proof of the theorem which follows from the definition of comonotonicity, it is evident that if $X_i > d_i$ holds, then we also have $X_k \geq d_k\ \forall k$, and so we have from equation (2.4.10)

$$\begin{aligned}
\int_{d_i}^{\infty} dF_{X_i}(x_i) &= P\left[\sum_{i=1}^{n} X_i > \sum_{i=1}^{n} d_i\right] \\
&= P\left[S^c > d\right] \\
&= 1 - F_{S^c}(d)
\end{aligned} \tag{2.4.11}$$

In regards to the first integral in equation (2.4.9), we note that the event

$$\left[X_i > d_i\right] \Longleftrightarrow \left[X_i > F_{X_i}^{-1}\left(F_{S^c}(d)\right)\right]. \tag{2.4.12}$$

This can be reasoned as follows. Let $F_{S^c}(d) = p$ and $\sum_{i=1}^{n} X_i = X$. Now we employ definition of comonotonicity. Assume that $X_i > d_i \Rightarrow X > d \Rightarrow F_{S^c}(X) > p$. Let us now assume that $X_i < F_{X_i}^{-1}\left(F_{S^c}(d)\right)$, which in turn on summing both sides and using theorem 14 implies that $X < F_{S^c}^{-1}\left(F_{S^c}(d)\right) \Rightarrow F_{S^c}(X) < p$ which is a contradiction. On the same lines, one can begin with $X_i > F_{X_i}^{-1}\left(F_{S^c}(d)\right)$ to conclude that $X_i > d_i$, which leads to the statement (2.4.12). Substituting the equation (2.4.11) in equation (2.4.9) and considering the statement (2.4.12), we get

$$\begin{aligned}
\mathbf{E}\left[(X_i - d_i)^+\right] &= \int_{F_{X_i}^{-1}(F_{S^c}(d))}^{\infty} \left(x_i - F_{X_i}^{-1}\left(F_{S^c}(d)\right)\right) dF_{X_i}(x_i) \\
&\quad + \left(d_i - F_{X_i}^{-1}\left(F_{S^c}(d)\right)\right)\left(1 - F_{S^c}(d)\right) \\
&= \mathbf{E}\left[\left(X_i - F_{X_i}^{-1}\left(F_{S^c}(d)\right)\right)^+\right] + \left(d_i - F_{X_i}^{-1}\left(F_{S^c}(d)\right)\right)\left(1 - F_{S^c}(d)\right).
\end{aligned} \tag{2.4.13}$$

Summing over both sides and using Theorems 15 and 14 on L.H.S. and R.H.S. respectively and using the definition of d as in Theorem 15, we see that for any retention $d \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$, equation (2.4.7) holds. In case, if the marginal c.d.f.s F_{X_i} are strictly increasing, so that F_{S^c} is strictly increasing and hence $F_{S^c}^{-1}\left(F_{S^c}(d)\right) = d$ (c.f. Dhaene et al., 2002a), we get equation (2.4.8). This completes the proof of the corollary. □

In the next section of this chapter, we discuss about convex bounds for sums of random variables without imposing any restrictions on the distribution functions.

2.5 Convex Bounds for Sums of Random Variables

The previous section serves as a foundation for deriving bounds for the sums $S = X_1 + X_2 + ... + X_n$ of random variables $X_1, X_2, \dots, X_n$ for which the marginal distributions are readily available. The bounds are actually random variables that are greater or smaller than S in the sense of convex order. The main motivation behind this exercise, as explicitly stated in the first section of this chapter is is that the joint distribution of the random vector $(X_1, X_2, \dots, X_n)$ is either unknown or too hard to be worked out.

Dhaene et al. (2002a) have worked out an upper bound that is attainable in the class of all random vectors with given marginals and have proved that the comonotonic upper bound acts as a supremum in the sense of convex order. We begin this section with theorem 7 of the aforesaid paper.

Theorem 17. *(**Upper Bound for a Sum of Random Variables**) For any random vector $(X_1, X_2, \dots, X_n)$, with its comonotonic counterpart being defined by*

$$\boldsymbol{X}^c = \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \dots, F_{X_n}^{-1}(U)\right)$$

we have

$$X_1 + X_2 + ... + X_n \leq_{cx} X_1^c + X_2^c + ... + X_n^c \tag{2.5.1}$$

or in other words $S \leq_{cx} S^c$

Proof. We begin the proof by observing that a random vector and its comonotonic counterpart have the same marginals and thus the expected value of their sums are equal. Thus, we have to just prove the result in the sense of stop loss order as the convex order would follow automatically. For this we have to prove that

$$\mathbf{E}\left[\left(\sum_{i=1}^{n} X_i - d\right)^+\right] \leq \mathbf{E}\left[\left(\sum_{i=1}^{n} X_i^c - d\right)^+\right]$$

or

$$\mathbf{E}\left[(S-d)^+\right] \leq \mathbf{E}\left[(S^c-d)^+\right]$$

holds for any retention d. The following result is true for all $(x_1, x_2, \dots, x_n)$ when $d = d_1 + \dots + d_n$:

$$\left(\sum_{i=1}^{n} x_i - d\right)^+ = \left(\sum_{i=1}^{n} (x_i - d_i)\right)^+ \leq \left(\sum_{i=1}^{n} (x_i - d_i)^+\right)^+ = \left(\sum_{i=1}^{n} (x_i - d_i)^+\right),$$

where the last step follows from the fact that $a^+ = a$ if $a \geq 0$. Now, replacing constants by the corresponding random variables in the above inequality and taking expectations we have

$$\mathbf{E}\left[\left(\sum_{i=1}^{n} X_i - d\right)^+\right] \leq \sum_{i=1}^{n} \mathbf{E}\left[(X_i - d_i)^+\right]$$

for all d and d_i such that $d = \sum_{i=1}^{n} d_i$. In the concluding step, we choose d_i as in theorem 15, i.e., $d_i = F_{X_i}^{-1(\alpha_d)}(F_{S^c}(d))$ and as a result of this theorem we obtain

$$\mathbf{E}\left[\left(\sum_{i=1}^{n} X_i - d\right)^+\right] \leq \sum_{i=1}^{n} \mathbf{E}\left[\left(X_i - F_{X_i}^{-1(\alpha_d)}(F_{S^c}(d))\right)^+\right] = \mathbf{E}\left[(S^c - d)^+\right] \tag{2.5.2}$$

which is the desired result. □

In light of the corollary 16, we can write the upper bound in terms of usual inverse distribution functions.

An interesting outcome of theorems 15 and 17 is the observation that the comonotonic upper bound appearing in equation (2.5.2) serves as the least upper bound having the form $\sum_{i=1}^{n} \mathbf{E}\left[(X_i - d_i)^+\right]$ with $\sum_{i=1}^{n} d_i = d$.

Dhaene et al. (2002a) suggest that the upper bound obtained in theorem 17 can be improved if some extra information regarding the stochastic nature of $(X_1, X_2, \ldots, X_n)$ is available. More specifically if we can lay hands on some random variable Λ, with a given distribution function such that the individual conditional cdfs of $X_i|\left(\Lambda = \lambda\right)$ are known for all i and all possible values of λ, we can derive sharper upper bounds in terms of convex order for S which are smaller in convex order in comparison to the upper bound S^c.

Finally, Dhaene et al. (2002a) use the same infrastructure as for the improved upper bound to furnish a lower bound for S. Banking on theorem 10 of this paper we reproduce this lower bound. Let $S = \sum_{i=1}^{n} X_i$ as before and define $S^l = \sum_{i=1}^{n} \mathbf{E}\left(X_i|\Lambda\right)$. Now, employing Jensen's inequality, we have for any convex function g,

$$\mathbf{E}\left[g\left(\mathbf{X}\right)\right] = \mathbf{E}\left[\mathbf{E}\left(g\left(\mathbf{X}\right)|\Lambda\right)\right] \geq \mathbf{E}\left[g\left(\mathbf{E}\left(\mathbf{X}|\Lambda\right)\right)\right]$$

Now, choosing the convex function as $g\left(\mathbf{x}\right) = \left(\sum_{i=1}^{n} x_i - d\right)^+$, we have

$$\mathbf{E}\left[\left(\sum_{i=1}^{n} X_i - d\right)^+\right] \geq \mathbf{E}\left[\left(\sum_{i=1}^{n} \mathbf{E}\left(X_i|\Lambda\right) - d\right)^+\right] \tag{2.5.3}$$

or

$$\mathbf{E}\left[(S-d)^+\right] \geq \mathbf{E}\left[\left(S^l - d\right)^+\right] \tag{2.5.4}$$

Employing the tower property we see that $\mathbf{E}\left[S^l\right] = \mathbf{E}\left[S\right]$ and this together with equation (2.5.4) is sufficient to show $S^l \leq_{cx} S$. As a final step, we can show that the lower bound for S can be formulated as the sum of stop-loss premiums. This task becomes trivial if we can choose the conditioning variable Λ in such a way that $\mathbf{E}\left[X_i|\Lambda\right]$ is either increasing or decreasing for every i, so that the vector: $\mathbf{X^l} = \left(\mathbf{E}\left[X_1|\Lambda\right], \ldots, \mathbf{E}\left[X_n|\Lambda\right]\right)$ is comonotonic. For this suitable choice of Λ, using Theorem 5 of Dhaene et al. (2002a), we see that d abides by

$$d = F_{S^l}^{-1(\alpha_d)}\left(F_{S^l}\left(d\right)\right) = \sum_{i=1}^{n} F_{\mathbf{E}(X_i|\Lambda)}^{-1(\alpha_d)}\left(F_{S^l}\left(d\right)\right).$$

Now, utilizing theorem 15, we achieve the desired decomposition

$$\begin{aligned} \mathbf{E}\left[(S-d)^+\right] &\geq \mathbf{E}\left[\left(S^l - d\right)^+\right] \\ &= \mathbf{E}\left[\left(\sum_{i=1}^{n} \mathbf{E}\left(X_i|\Lambda\right) - d\right)^+\right] \\ &= \sum_{i=1}^{n} \mathbf{E}\left[\left(\mathbf{E}\left(X_i|\Lambda\right) - F_{\mathbf{E}(X_i|\Lambda)}^{-1(\alpha_d)}\left(F_{S^l}\left(d\right)\right)\right)^+\right]. \end{aligned} \tag{2.5.5}$$

In light of the corollary 16, we can write this lower bound in terms of usual inverse distribution functions. Thus, we have been able to obtain both lower and upper convex order stochastic bounds for the sum of dependent risks $S = \sum_{i=1}^{n} X_i$. With the background theory presented in this chapter, we now dwell into its application to find appropriate lower and upper bounds for longevity and mortality instruments in the next couple of chapters.

As a next step, we present some important results particularly Bayes' formula along with the proof, which plays a vital role for the construction of doubly stochastic stopping times in Chapter 5.

2.6 Some Important Results

This thesis thrives upon many important mathematical formulae. We present a list below:

2.6.1 Jensen's Inequality

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X be an integrable real-valued random variable, $\mathcal{G}$ be a sub-sigma algebra of $\mathcal{F}$ and $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function. Then:

$$E\left[\phi\left(X\right)|\mathcal{G}\right] \geq \phi\left(E\left[X|\mathcal{G}\right]\right) \tag{2.6.1}$$

2.6.2 Bayes' Formula

The following result from Billingsley (1995) plays a very important role in constructing the unified market framework combining financial variables and mortality risk in Chapter 5.

Proposition 18. *(Bayes' Theorem) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{G} \subset \mathcal{F}$, $\mathcal{H}$ be a σ-algebra generated by a partition $B_1, B_2, ...,$ and let $\mathcal{G} \vee \mathcal{H} = \sigma\left(\mathcal{G} \cup \mathcal{H}\right)$, then with probability 1 we have*

$$P[A|\mathcal{G} \vee \mathcal{H}] = \sum_i \mathbb{1}_{B_i} \frac{P[A \cap B_i|\mathcal{G}]}{P[B_i|\mathcal{G}]} \quad \forall A \in \mathcal{F} \tag{2.6.2}$$

Proof. First of all define

$$P_i\left(A\right) = P\left(A|B_i\right). \tag{2.6.3}$$

and E_i the corresponding expectation. Then, we have the following *claim*

$$P_i\left(A|\mathcal{G}\right) = \frac{P\left(A \cap B_i|\mathcal{G}\right)}{P\left(B_i|\mathcal{G}\right)}. \tag{2.6.4}$$

Proof of the claim:
We first have for any function f

$$\int_\Omega f d\mathbb{P}_i = \frac{\int_{B_i} f d\mathbb{P}}{P\left(B_i\right)} \tag{2.6.5}$$

Then for any $G \in \mathcal{G}$, we have that

$$\begin{aligned}
\int_G P_i\left(A|\mathcal{G}\right) P\left(B_i|\mathcal{G}\right) d\mathbb{P} &= \int_G P_i\left(A|\mathcal{G}\right) E[\mathbb{1}_{B_i}|\mathcal{G}] d\mathbb{P} \\
&= \int_G P_i\left(A|\mathcal{G}\right) \mathbb{1}_{B_i}\, d\mathbb{P} \\
&= \int_\Omega \mathbb{1}_G \mathbb{1}_{B_i} P_i\left(A|\mathcal{G}\right) d\mathbb{P} \\
&= \int_{B_i} \mathbb{1}_G P_i\left(A|\mathcal{G}\right) d\mathbb{P} \\
&= P\left(B_i\right) \int_\Omega \mathbb{1}_G P_i\left(A|\mathcal{G}\right) d\mathbb{P}_i \\
&= P\left(B_i\right) \int_G E_i[\mathbb{1}_A|\mathcal{G}] d\mathbb{P}_i \\
&= P\left(B_i\right) \int_G \mathbb{1}_A\, d\mathbb{P}_i \\
&= P\left(B_i\right) P_i\left(A \cap G\right) \\
&= P\left(A \cap G \cap B_i\right) \\
&= E[\mathbb{1}_{A \cap G \cap B_i}] \\
&= \int_G \mathbb{1}_{A \cap B_i}\, d\mathbb{P} \\
&= \int_G E[\mathbb{1}_{A \cap B_i}|\mathcal{G}]\, d\mathbb{P}
\end{aligned}$$

$$= \int_G P(A \cap B_i|\mathcal{G})\ d\mathbb{P} \tag{2.6.6}$$

So, it follows that

$$P_i(A|\mathcal{G})\,P(B_i|\mathcal{G}) = P(A \cap B_i|\mathcal{G}) \tag{2.6.7}$$

and that proves the claim.
Further, for a general set G in $\mathcal{G}$

$$\begin{aligned}
\int_{G \cap B_i} P_i(A|\mathcal{G})\, d\mathbb{P} &= \int_{B_i} \mathbb{1}_G P_i(A|\mathcal{G})\, d\mathbb{P} \\
&= P(B_i) \int_\Omega \mathbb{1}_G\ P_i(A|\mathcal{G})\, d\mathbb{P}_i \\
&= P(B_i) \int_G P_i(A|\mathcal{G})\, d\mathbb{P}_i \\
&= P(B_i) \int_G E_i[\mathbb{1}_A|\mathcal{G}] d\mathbb{P}_i \\
&= P(B_i) \int_G \mathbb{1}_A\ d\mathbb{P}_i \\
&= P(B_i)\, P_i(A \cap G) \\
&= P(A \cap G \cap B_i) \\
&= \int_{G \cap B_i} \mathbb{1}_A\ d\mathbb{P} \\
&= \int_{G \cap B_i} E[\mathbb{1}_A|\mathcal{G} \vee \mathcal{H}]\ d\mathbb{P} \\
&= \int_{G \cap B_i} P(A|\mathcal{G} \vee \mathcal{H})\ d\mathbb{P} \qquad (2.6.8)
\end{aligned}$$

As a result we have shown that

$$\int_C \mathbb{1}_{B_i}\ P_i(A|\mathcal{G})\, d\mathbb{P} = \int_C \mathbb{1}_{B_i}\ P[A|\mathcal{G} \vee \mathcal{H}]\ d\mathbb{P}\ if\ C = G \cap B_i \tag{2.6.9}$$

and this also holds if $C = G \cap B_j; j \neq i$.

But $C's$ of this form constitute a π-system generating $\mathcal{G} \vee \mathcal{H}$ and hence

$$\mathbb{1}_{B_i} P_i(A|\mathcal{G}) = \mathbb{1}_{B_i} P[A|\mathcal{G} \vee \mathcal{H}] \tag{2.6.10}$$

on a set of P-measure 1.

Now, summing both sides of (2.6.10) over i, using the definition of $P_i(A|\mathcal{G})$ from (2.6.4) and remembering that $B_i's$ form a partition of $\mathcal{H}$, swapping sides, we get the desired result, i.e.,

$$P[A|\mathcal{G} \vee \mathcal{H}] = \sum_i \mathbb{1}_{B_i} \frac{P[A \cap B_i|\mathcal{G}]}{P[B_i|\mathcal{G}]}. \tag{2.6.11}$$

$\square$

2.7 Some Important Distributions

We present below the details about some well-known distributions which are indispensable for our research in the order in which they have been utilized for the purpose of derivation.

1. **Normal Distribution**
   A continuous random variable X is said to follow Normal distribution with parameters μ

(mean) and σ^2 (variance) written as $X \sim N\left(\mu, \sigma^2\right)$ if its probability density function (p.d.f.) takes the form:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \;\; x \in \mathbb{R},\; \mu \in \mathbb{R},\; \sigma > 0 \tag{2.7.1}$$

2. **Multivariate Normal Distribution**
A random vector $\mathbf{X} \in \mathbb{R}^k$ is said to follow a multivariate Normal distribution with mean vector $\boldsymbol{\mu}$ and variance-covariance matrix Σ denoted by $\mathbf{X} \sim N_k(\boldsymbol{\mu}, \Sigma)$ if its p.d.f. is given as:

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{k}{2}} \sum^{\frac{1}{2}}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})' \, \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})}, \;\; \mathbf{x} \in \mathbb{R}^k,\; \boldsymbol{\mu} \in \mathbb{R}^k,\; \Sigma \in S_k^{++}, \tag{2.7.2}$$

where S_k^{++} is the set of positive definite symmetric matrices of order k.

If we take $k = 2$, we get the **Bivariate Normal Distribution** (BVN), which can be defined as follows:
A two-dimensional random vector $(X, Y) \sim BVN\left(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho\right)$ where BVN stands for bivariate Normal distribution if its p.d.f. is given as:

$$f(x) = \frac{1}{(2\pi)\,\sigma_1\sigma_2\sqrt{(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right\}}, \;\; (x, y) \in \mathbb{R}^2 \tag{2.7.3}$$

where $\rho\,(\neq 0)$ is the correlation coefficient between X and Y.

3. **Matrix Variate Normal Distribution**
A $p \times n$-random matrix X is said to have a matrix variate Normal distribution with mean $M \in \mathcal{M}_{p,n}$ and covariance $\Sigma \otimes \Psi$, $\Sigma \in S_p^{++}$, $\Psi \in S_n^{++}$, with $\mathcal{M}_{p,n}$ being the set of all $m \times n$ real matrices and $\otimes$ denoting the Kronecker product if $\mathrm{vec}\,(X') \sim N_{pn}\,(\mathrm{vec}\,(M'), \Sigma \otimes \Psi)$, where $'$ denotes the transpose of a matrix, N_{pn} denotes the multivariate Normal distribution on $\mathbb{R}_{pn}$ with mean vector $\mathrm{vec}\,(M')$ and covariance $\Sigma \otimes \Psi$ and $\mathrm{vec}\,(.)$ for a matrix $A \in \mathcal{M}_{p,n}$ with columns $a_i \in \mathbb{R}_m$; $i = 1, 2, ..., n$ defined as

$$vec\,(A) = \begin{pmatrix} a_1 \\ . \\ . \\ . \\ a_n \end{pmatrix} \tag{2.7.4}$$

We use the notation $X \sim N_{p,n}\,(M, \Sigma \otimes \Psi)$.

4. **Lognormal Distribution**
A continuous random variable X is said to follow lognormal distribution with parameters μ and σ^2 written as $X \sim \log N\left(\mu, \sigma^2\right)$ if its p.d.f. is given as:

$$f(x) = \frac{1}{\sqrt{2\pi}x\sigma} e^{-\frac{(\log_e x-\mu)^2}{2\sigma^2}}, \;\; x \in \mathbb{R}^+,\; \mu \in \mathbb{R}^+,\; \sigma > 0 \tag{2.7.5}$$

5. S_u **Distribution**
A continuous random variable X is said to obey the four-parameter transformed Normal (S_u) Distribution (for details see Johnson, 1949; Johnson et al., 1994) if

$$\sinh^{-1}\left(\frac{X-\alpha}{\beta}\right) = Y \sim N\left(\mu, \sigma^2\right), \tag{2.7.6}$$

where α, β, μ and σ are parameters ($\beta, \sigma > 0,\ \mu \in \mathbb{R}$) and $\sinh^{-1}$ is the inverse hyperbolic sine function defined as

$$\sinh^{-1}(z) = \log_e\left(z + \sqrt{1+z^2}\right) \tag{2.7.7}$$

The probability density function of X is given as

$$f(x) = \frac{1}{\sqrt{2\pi}\beta\sigma\sqrt{1+\left(\frac{x-\alpha}{\beta}\right)^2}} e^{-\frac{\left(sinh^{-1}\left(\frac{x-\alpha}{\beta}\right)-\mu\right)^2}{2\sigma^2}}, \quad x \in \mathbb{R} \tag{2.7.8}$$

6. **Gamma Distribution**
A continuous random variable X is said to follow a Gamma Distribution with parameters p and a written as $x \sim Gamma(p, a)$ if its p.d.f. is given as

$$f(x) = \frac{a^p}{\Gamma(p)} x^{p-1} e^{-ax}, \quad x \in \mathbb{R}^+, p \in \mathbb{R}^+, a \in \mathbb{R}^+ \tag{2.7.9}$$

and $\Gamma(.)$ is the Gamma function defined as

$$\Gamma(p) = \int_0^\infty z^{p-1} e^{-z} dz \tag{2.7.10}$$

7. **Log Gamma Distribution**
The log Gamma distribution is a particular type of transformed Gamma distribution. A random variable X is said to follow log Gamma distribution if

$$\frac{\log_e X - \mu}{\sigma} = Y \sim Gamma(p, a), \tag{2.7.11}$$

where μ, σ, p and a are parameters (> 0) and $\log_e$ is the natural logarithm. Useful references for reading about transformed gamma distribution are Johnson et al. (1994), Vitiello and Poon (2010) and Cheng et al. (2014). The p.d.f. of X is given as

$$f(x) = \left(\frac{a}{\sigma}\right)^p \frac{1}{x\Gamma(p)} (\log_e x - \mu)^{p-1} e^{-\frac{a}{\sigma}(\log_e x - \mu)}, \quad x \in \mathbb{R}^+ \tag{2.7.12}$$

8. **Chi Square Distribution**
A continuous random variable X is said to follow a Chi Square Distribution with n degrees of freedom (d.f.) written as $x \sim \chi^2_{(n)}$ if its p.d.f. is given as

$$f(x) = \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}; \quad x \in \mathbb{R}^+, n \in \mathbb{Z}^+, \tag{2.7.13}$$

where $\mathbb{Z}^+$ denotes the set of positive integers. In fact, comparing with the p.d.f. of gamma distribution given in equation (2.7.9), we see that we have a $Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$ distribution. The chi square distribution is also directly linked to the normal distribution in the sense that

"If $Y_i,\ i = 1, 2, ..., n$ are n independent normal random variables with respective means and

variance as μ_i and σ_i^2 then

$$X = \sum_{i=1}^{n} \left(\frac{Y_i - \mu_i}{\sigma_i} \right)^2 \sim \chi^2_{(n)}.\text{''} \tag{2.7.14}$$

In fact, chi square distribution is also referred to as *central* chi square distribution in literature.

9. **Non-Central Chi Square Distribution**
A continuous random variable X is said to follow a Non-Central Chi Square Distribution with n degrees of freedom and non-centrality parameter λ written as $x \sim \chi'^2(n, \lambda)$ (c.f. Johnson et al., 1995) if its p.d.f. is given as

$$f(x) = \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda}{2}} \left(\frac{\lambda}{2}\right)^j}{j!} h_{Y_{n+2j}}(x), \quad x \in \mathbb{R}^+, n \in \mathbb{Z}^+, \tag{2.7.15}$$

where $h_{Y_{n+2j}}(.)$ is the p.d.f. of a random variable Y_{n+2j} distributed according to central χ^2 distribution with $(n+2j)$ d.f.; $j = 0, 1, 2, ...$ From this representation, the non-central chi-squared distribution is seen to be a Poisson-weighted mixture of central chi-squared distributions since the coefficient of the density $h_{Y_{n+2j}}(.)$ is in fact, the probability of a *Poisson($\frac{\lambda}{2}$)* random variable taking the value j. Like its central counterpart, the non-central chi square distribution is also directly linked to the normal distribution in the sense that

"If Z_i, $i = 1, 2, ..., n$ are n independent normal random variables with respective means μ_i and and unit variances then

$$X = \sum_{i=1}^{n} Z_i^2 \sim \chi'^2(n, \lambda), \tag{2.7.16}$$

where the non-centrality parameter λ is given as

$$\lambda = \sum_{i=1}^{n} \mu_i^2.\text{''} \tag{2.7.17}$$

10. **Non-Central Wishart Distribution**
A $p \times p$ random matrix X is said to have a non-central Wishart distribution with parameters p, n, Σ and Θ and written as $X \sim \mathcal{W}_p(n, \Sigma, \Theta)$, (c.f. Pfaffel, 2012; Gupta and Nagar, 2000; Johnson and Kotz, 1972) if its p.d.f. is given by

$$f_X(S) = \left(2^{\frac{1}{2}np} \Gamma_p\left(\frac{n}{2}\right) \det(\Sigma)^{\frac{n}{2}}\right)^{-1} e^{Tr\left[-\frac{1}{2}\left(\Theta + \Sigma^{-1}S\right)\right]} \det(S)^{\frac{1}{2}(n-p-1)} {}_0F_1\left(\frac{n}{2}; \frac{1}{4}\Theta\Sigma^{-1}S\right) \tag{2.7.18}$$

where $S \in S_p^{++}$, the set of positive definite symmetric matrices and $p \in \mathbb{Z}^+$, $n \geq p$, $\Sigma \in S_p^{++}$ and $\Theta \in M_p$, where M_p denote the set of real $p \times p$ matrices. Further, $\Gamma_p(.)$ is the matrix variate Gamma function defined as

$$\Gamma_p(a) = \int_0^{\infty} e^{Tr[-A]} \det(A)^{a - \frac{1}{2}(p+1)} dA, \ \ \forall \ a > \frac{p-1}{2} \tag{2.7.19}$$

and ${}_0F_1$ is the hypergeometric function, where in general, we define, hypergeometric function of matrix argument as

$${}_mF_n(a_1, ..., a_m; b_1, ..., b_n; S) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa ... (a_m)_\kappa C_\kappa(B)}{(b_1)_\kappa ... (b_n)_\kappa k!} \tag{2.7.20}$$

where $a_i,\ b_j \in \mathbb{R}$, B is a symmetric $p \times p$ matrix and $\sum_\kappa$ denotes the summation over all *partitions* κ of k, where by a *partition* of k, we mean a p-tuple $\kappa = (k_1, k_2, ..., k_p)$ such that $k_1 \geq ... \geq k_p \geq 0$ and $k_1 + ... + k_p = k$. Further, $(a)_\kappa = \prod_{j=1}^{p} \left(a - \frac{1}{2}(j-1)\right)_{k_j}$ denotes the generalized hypergeometric coefficient, with $(x)_{k_j} = x(x+1)...(x+k_j-1)$ (c.f. Gupta and Nagar, 2000). Finally, $C_\kappa(B)$ denotes zonal polynomial which is the component of $Tr\left[(B)^k\right]$ in the subspace V_κ of the space V_k, where V_k is the space of all *symmetric homogeneous polynomials* of degree k in the $\frac{p(p+1)}{2}$ distinct elements of $B \in S_p^{++}$, where by a *symmetric homogeneous polynomial* of degree k in $y_1, y_2, ..., y_m$, we mean a polynomial which is unaltered by a permutation of the subscripts and such that every term in the polynomial has degree k (c.f. Muirhead, 2005). Then

$$Tr\left[(B)^k\right] = (B_{11} + ... + B_{pp})^k$$

is an element of V_k. According to Gupta and Nagar (2000), the space V_k can be decomposed into a direct sum of irreducible invariant sub-spaces V_κ. This along with the definition of zonal polynomials implies that

$$Tr\left[(B)^k\right] = \sum_\kappa C_\kappa(B). \tag{2.7.21}$$

As per Gupta and Nagar (2000), a sufficient condition for the hypergeometric function of matrix argument given in equation (2.7.20) to be well defined is $m < n + 1$.

In regards to the p.d.f. of Non-Central Wishart distribution given in equation (2.7.18), it does not exist if $\Sigma \in S_p^{+} \backslash S_p^{++}$, which denotes the boundary of the cone of symmetric positive semi-definite matrices with S_p^{+} denoting the set of symmetric positive semi-definite matrices. However, in this case, one can still define the non-central Wishart distribution using its characteristic function defined in equation (5.7.110) in Chapter 5. A good reference to read about the existence of non-central Wishart distribution is Mayerhofer (2013).

Further, in equation (2.7.18), if $\Theta = 0$, X is said to have a Central Wishart distribution with parameters p, n and Σ written as $X \sim \mathcal{W}_p(n, \Sigma)$.

Finally, the next Lemma from Gupta and Nagar (2000) shows that Wishart distribution is the square of a matrix variate normally distributed random matrix. Hence, it is the matrix variate extension of non-central chi-square distribution.

Lemma 19. *Let* $X \sim N_{p,n}(M, \Sigma \otimes \Psi)$, $n \in \{p, p{+}1, ...\}$. *Then* $XX' \sim \mathcal{W}_p\left(n, \Sigma, \Sigma^{-1}MM'\right)$.

2.8 Some Basic Stochastic Processes

We present below the details about some well-known stochastic processes which are the building blocks for pricing of financial products.

To begin, let $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{P}$ as the real world probability measure, be a complete probability space coupled with the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$. Then, we consider the following stochastic processes.

1. **Lévy Process:** A stochastic process $X = \{X_t\}_{t \geq 0}$ is said to be a Lévy Process if:
   - $X_0 = 0$
   - For any sequence of 'time points' $0 \leq t_0 < t_1 < \ldots \leq t_n$, the 'increments' $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent for integer $n \geq 2$.
   - For $0 \leq s < t$, the random variable $X_t - X_s$ has the same distribution as X_{t-s} (stationary increments).

- For $0 \leq s < t$, X_t is almost surely right continuous with left limits (admits càdlàg).

Excellent references on Lévy processes are Applebaum (2004) and Sato (1999). Wiener process or Brownian Motion is a well known example of Lévy processes and is defined below.

2. Exponential Lévy Model: To ensure positivity as well as the independence and stationarity of log-returns, prices of financial products are usually modeled as exponential of Lévy processes, i.e.,

$$S_t = S_0 e^{X_t}, \tag{2.8.1}$$

where X_t is a Lévy process, S_t is the product price at time t and S_0 is the initial product price.

3. Wiener Process / Standard Brownian Motion: A family of random variables $W = (W_t)_{t \geq 0}$ is called a Wiener process if the following conditions are satisfied:

- The 'increments' $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent random variables for any sequence of 'time points' $0 \leq t_0 < t_1 < \ldots \leq t_n$ for integer $n \geq 2$.
- For $0 \leq s < t$, the random variable $W_t - W_s \sim N(0, t-s)$.
- $W_t(\omega)$ is continuous in t for every $\omega \in \Omega$.
- $W_0 = 0$

We also consider *Wishart Processes* which we discuss in detail in Chapters 4 and 5.

Chapter 3

Model-Independent Price Bounds for Catastrophic Mortality Bonds

This chapter has been published on Arxiv. The full reference follows:

R. K. Bahl and S. Sabanis. Model-Independent Price Bounds for Catastrophic Mortality Bonds. *Working Paper*, *Arxiv*, 2016. URL **arXiv:1607.07108[q-fin.PR]**.

Previous versions of this chapter were presented at the following conferences, workshops and seminars:

- December 2014. Actuarial Teachers and Researcher Conference (ATRC), School of Mathematics, University of Edinburgh. Poster Presentation "Price Bounds for the Swiss Re Bond 2003: A Model-Independent Approach".

- January 2015. Perspectives on Actuarial Risks in Talks of Young Researchers (PARTY), Liverpool, UK. "Price Bounds for the Swiss Re Bond 2003".

- October 2015. Knowledge Sharing Scotland (KSS) Event, Institute and Faculty of Actuaries (IFoA), Edinburgh, UK. "Model-independent Price Bounds for the Swiss Re Mortality Bond - 2003".

- March 2016. Mathematical and Statistical Methods for Actuarial Sciences and Finance (MAF), University Dauphine of Paris, France. "Model-independent Price Bounds for the Swiss Re Mortality Bond 2003".

- June 2016. Pensions, Risk and Investment Conference 2016 with AFIR /ERM (organized by IFoA), Edinburgh, UK. "Mortality from Modeling to Pricing: Challenges and Solutions".

- July 2016. International Congress on Insurance: Mathematics and Economics (IME), Georgia State University, Atlanta, USA. "Model-Independent Price Bounds for the Swiss Re Mortality Bond 2003".

- September 2016. International Mortality and Longevity Symposium (organized by IFoA), Royal Holloway, University of London, UK. Paper and poster presentation entitled "Model-independent Price Bounds for Catastrophic Mortality Bonds".

- September 2016. General Insurance Research Organization (GIRO) Conference (organized by IFoA), Convention Centre, Dublin, Ireland. "'Mortality from Modeling to Pricing: Challenges and Solutions".

- December 2016. International Conference on Actuarial and Financial Mathematics (ICAFM), Sydney, Australia. "Model-Independent Price Bounds for the Swiss Re Mortality Bond 2003". Conferred with the **Best Presentation Award**.

- February 2017. Statistics Seminar Series, University College Dublin, Ireland. Seminar Session entitled "Valuation of Catastrophic Mortality Bonds".

- May 2017. Maxwell Institute Probability Day at ICMS, University of Edinburgh, UK. "Valuation of Catastrophic Mortality Bonds".

In this chapter, we are concerned with the valuation of Catastrophic Mortality Bonds and, in particular, we examine the case of the Swiss Re Mortality Bond 2003 as a primary example of this class of assets. This bond was the first Catastrophic Mortality Bond to be launched in the market and encapsulates the behaviour of a well-defined mortality index to generate payoffs for bondholders. Pricing these type of bonds is a challenging task and no closed form solution exists in the literature. In our approach, we adapt the payoff of such a bond in terms of the payoff of an Asian put option and present a new approach to derive model-independent bounds exploiting comonotonic theory as illustrated in Albrecher et al. (2008), Dhaene et al. (2002b) and Simon et al. (2000) for the pricing of Asian options. We carry out Monte Carlo simulations to estimate the bond price and illustrate the strength of the bounds.

3.1 Introduction

In the present day world, many financial institutions face the risk of unexpected fluctuations in human mortality and clearly, this risk has two aspects. On one side, life insurers paying death benefits will suffer an economic loss if actual rates of mortality are in excess of those expected, due to catastrophic events such as a severe outbreak of an epidemic or a major man-made or natural disaster. This side of the risk is known in the literature by the name of *mortality risk*. On the other hand, pension plan sponsors, as well as insurance companies providing retirement annuities, are subject to *longevity risk*, that is, the risk that people outlive their expected lifetimes. For these institutions, the longer the life-span of people, the greater the period of time over which retirement income must be paid and, hence, the larger the financial liability.

An unanticipated change in mortality rates will affect all policies in force. Therefore, as opposed to the random variations between lifetimes of individuals, it cannot be diversified away by increasing the size of the portfolio. Reinsurance is one possible solution to the problem, but its capacity is usually limited. Alternatively, the risk may be naturally hedged or reduced through balancing products. For example, an insurance company may sell life insurance to the same customers who are buying life annuities. The resulting combination would then reduce the company's exposure to future changes in mortality, consequently permitting a reduction of capital reserves held in respect of mortality or longevity risk. This idea of compensating longevity risk by mortality risk is often referred to as *natural hedging*. However, this strategy, as Cox and Lin (2007) pointed out, may be cost prohibitive and may not be practical in some circumstances particularly due to the underlying difference in the nature of the two risks: mortality risk being a short term risk having a 1 to 5 year maturity and possessing a catastrophic component while longevity risk being a long term risk with maturities ranging from 20 to 80 years and is primarily connected to changes in trend. .

As a result, a natural remedy to tackle these risks has emerged in the form of what is known as mortality securitization which manifests itself in the form of *mortality-linked securities* abbreviated in the literature as *MLSs*. These securities provide a tool in the hands of insurers to transfer their mortality-sensitive exposures to a vested number of investors in the capital market, offering them a reasonable risk premium in return. Mortality-linked securities differ from their longevity counterparts in the sense that while the former have their cash flows linked to a mortality index, the latter are based upon survivor index. For a more detailed review of the two type of bonds, one can refer to Melnick and Everitt (2008). In fact mortality-linked securities are also known as *Extreme Mortality Bonds* or *EMBs* or *Catastrophe (CAT) Mortality Bonds* or *CATM bonds* since they are triggered by a catastrophic evolution of death rates of one or more populations. These bonds are extremely lucrative to the investors because of their potential of providing diversification to the portfolio. The generous return on these bonds generally does not bear any correlation with the return on other investments, such as fixed income or equities. From the point of view of the reinsurer these instruments act as 'Alternative Risk Transfer' (ART) mechanisms.

The pioneering MLS was the Swiss Re mortality bond (Vita I) issued in 2003 which is the prime focus of this chapter. This was followed up by the EIB/BNP longevity bond issued in

2004 (Blake et al., 2006; Lane, 2011). For the former, the principal of the bond would have been reduced if there had been a catastrophic mortality event during the life of the bond, therefore allowing Swiss Re to reduce some of its exposure to extreme mortality risk. On the contrary, the latter was a 25-year longevity bond, which was intended for UK pension funds with exposures to longevity risk. This bond took the form of an annuity bond with annual coupon payments tied to the realized survival rates for some English and Welsh males. However, it did not get the same reception as the Swiss Re bond. Swiss Re followed up the success of VITA I by launching five more series of VITA bonds with the latest one being VITA VI which will cover extreme mortality events in Australia, Canada and the UK over a 5 year term from January 2016. Apart from this Swiss Re also experimented with a multi-peril bond called "Mythen Re" which synthesized catastrophe and mortality risks, obtaining 200 million US dollars in protection for North Atlantic hurricane and UK extreme mortality risk. Many other reinsurance giants such as Scottish Re and Munich Re have also issued a score of other mortality bonds. We refer readers to Blake et al. (2008), Coughlan (2009), Zhou and Li (2013) and Chen et al. (2014) for further details. In fact it is interesting to note that Swiss Re has also launched an innovative 'Longevity Trend Bond' called the Swiss Re Kortis bond in December 2010. Interested readers can refer to Chen et al. (2014) and Hunt and Blake (2015). A more up to date list of developments connected to mortality and longevity securities and markets can be found in Tan et al. (2015) and Liu and Li (2015a). As an aftereffect of these innovative securities, a number of valuation approaches on MLSs have germinated. Huang et al. (2014) classify the approaches into the following four heads:

- *Risk-adjusted process or no-arbitrage pricing:* Under this approach, the first step is to estimate the distribution of future mortality rates in the real-world probability measure. Then the real-world distribution is transformed to its risk-neutral counterpart, on the basis of the actual prices of mortality-linked securities observed in the market. Finally, the price of a mortality-linked security can be calculated by discounting, at the risk-free interest rate, its expected payoff under the identified risk-neutral probability measure. An important point underlying this approach is that it takes into account the actual prices as given. The need of market prices makes the implementation of this approach difficult. One way to effectively use the no arbitrage approach is to use a stochastic mortality model, which is, at the very beginning, defined in the real-world measure and fitted to past data. The model is then calibrated to market prices, yielding a risk-neutral mortality process from which security prices are calculated. For instance, Cairns et al. (2006b) calibrate a two-factor mortality model to the price of the BNP/EIB longevity bond.

- *The Wang transform:* It is the approach given by Wang (2000), Wang (2002) which consists of employing a distortion operator that transforms the underlying distribution into a risk-adjusted distribution and the MLS price is the expected value under the risk-adjusted probability discounted by risk-free rate. The Wang transform was first employed for mortality-linked securities by Lin and Cox (2005), and subsequently by other researchers including Dowd et al. (2006) and Denuit et al. (2007). Based on the positive dependence characteristic of the mortality in catastrophe areas, Shang et al. (2009) develop a pricing model for catastrophe mortality bonds with comonotonicity and a jump-difusion process. Pointing out there is no unique risk-neutral probability in this incomplete market settings, they use the Wang transform method to price the bond. Unless a very simple mortality model is assumed, parameters in the distortion operator are not unique if we are not given sufficient market price data. For example, when Chen and Cox (2009) used their extended Lee-Carter model with transitory jump effects to price a mortality bond, they were required to estimate three parameters in the Wang transform. To solve for these three parameters, Chen and Cox assumed that they were equal, but such an assumption is not easy to justify. In fact Pelsser (2008) has questioned the Wang transform by stating that it is not a universal financial measure for financial and insurance pricing. For more details one can refer to Goovaerts and Laeven (2008) and Lauschagne and Offwood (2010).

- *Instantaneous Sharpe Ratio:* Milevsky et al. (2005) propose that the expected return on the MLS equals the risk free rate plus the Sharp ratio times its standard deviation.

- *The utility-based valuation:* The utility based method defines an investor's utility function and maximizes an agent's expected utility subject to wealth constraints to obtain the MLS equilibrium. For an elaborate discussion one can review Tsai and Tzeng (2013), Cox et al. (2010), Hainaut and Devolder (2008) and Dahl and Moller (2006).

Apart from the aforesaid methods Beelders and Colorassi (2004) and Chen and Cummins (2010) use the extreme value theory to measure mortality risk of the 2003 Swiss Re Bond. For an interesting summary of other methods to price MLSs one can refer to Shang et al. (2011), Zhou et al. (2013a), Tan et al. (2015) and Liu and Li (2015a).

The methods available in literature for the pricing of MLSs offer only a limited application due to restrictions such as availability of price information or specific utility functions. The difficulty in pricing MLSs stems from the fact that the MLS market is incomplete as the underlying mortality rates are usually untradeable in financial markets. As a result, the usual no-arbitrage pricing method can only provide a price range or a price bound, instead of a single value.

Surprisingly, mortality linked securities, apart from their present day form seem to have a long history. In the 17th and 18th centuries, so-called *'tontines'*, which were named after the Neapolitan banker Lorenzo Tonti, had been offered by several governments (Weir, 1989; McKeever, 2009). Within these schemes, investors made a one-time payment, and annual dividends were distributed among the survivors. Hence, while still relying on the investor's survival, his payoffs were connected to the mortality experience among the pool of subscribers. These issues were particularly successful in France, but due to high interest payments, they soon became precarious for the crown's financial situation (see Jennings and Trout, 1982). However, this was not only the case with tontines; life annuities, which presented another large share of the royal debt, were also offered at highly favourable conditions from the investors' perspective. This carelessness was exploited by the Genevan entrepreneur Jacob Bouthillier Beaumont in the scheme attributed to him (c.f. Jennings and Trout, 1982). Here, annuities were subscribed on the lives of a group of Genevan girls for the account of Genevan investors. Thus, their payoffs were directly linked to the survival of the Genevan "madmoiselles", and due to the "generous" assumptions of the French authorities, the schemes were initially highly profitable for the Genevans, the real victim being the French taxpayer. These speculations came to an abrupt end with the French Revolution in 1789, for which the budgetary crises caused by the careless borrowing was, undoubtedly, one major reason. Until the beginning of this century, there has not been another public issue of a mortality linked security, however, there are indications of recent private transactions resembling the tontine scheme (see Dowd et al., 2006). For a more detailed overview of the history of mortality contingent securities the reader is referred to Bauer (2008) and Luis (2016).

Today, all around the world, investment banks and other financial service providers are working on the idea of trading longevity risk, and the first mortality trading desks have been installed solidifying that "betting on the time of death is set".[1]

This chapter is concerned with finding price bounds for the Swiss Re mortality catastrophe bond by expressing its payoff in the form of an Asian put option and using the theory of comonotonicity. Such a methodology has been adovocated by Simon et al. (2000), Dhaene et al. (2002b) and Albrecher et al. (2008) to find a price range for Asian options. For more details on comonotonicity and its applications, one can refer to Kaas et al. (2000) and Dhaene et al. (2002a).

The rest of this chapter is organized as follows: the next section describes the structure of the Swiss Re Bond and expresses its payoff in the form of an Asian put option. Section 3 shows derivations of the lower bound for the aforesaid bond using comonotonicity. In Section 4, we use the same to derive upper bounds for the Swiss Re Bond. In section 5, we illustrate the computation of bounds by choosing specific models for mortality index. The numerical

[1] The Business, 08/15/2007, "Betting on the time of death is set", by P. Thornton.

results for the derived theory and comparisons with Monte Carlo estimate of the bond price are furnished in Chapter 6. Appropriate figures that highlight comparisons among the bounds have also been furnished in Chapter 6.

3.2 Design of the Swiss Re Bond

As pointed out in the introduction, the financial capacity of the life insurance industry to pay catastrophic death losses from natural or man-made disasters is limited. To expand its capacity to pay catastrophic mortality losses, Swiss Re procured about 400 million in coverage from institutional investors in lieu of its first pure mortality security. The reinsurance giant issued a three year bond in December 2003 with maturity on January 1, 2007. To carry out the transaction, Swiss Re set up a special purpose vehicle (SPV) called Vita Capital Ltd. This enabled the corresponding cash flows to be kept off Swiss Re's balance sheet. The principal is subject to mortality risk which is defined in terms of an index q_{t_i} in year t_i. This mortality index was constructed as a weighted average of mortality rates (deaths per 100,000) over age, sex (male 65% and female 35%) and nationality (US 70%, UK 15%, France 7.5%, Italy 5% and Switzerland 2.5%) and is given below.

$$q_{t_i} = \sum_j C_j \sum_k A_k \left(G^m q^m_{k,j,t_i} + G^f q^f_{k,j,t_i} \right) \tag{3.2.1}$$

where q^m_{k,j,t_i} and q^f_{k,j,t_i} are the respective mortality rates (deaths per 100,000) for males and females in the age group k for country j, C_j is the weight attached to country j, A_k is the weight attributed to age group k (same for males and females) and G^m and G^f are the gender weights applied to males and females respectively.

The Swiss Re bond was a principal-at-risk bond. If the index q_{t_i} (t_i = 2004, 2005 or 2006 for $i = 1, 2, 3$ respectively) exceeds K_1 of the actual 2002 level, q_0, then the investors will have a reduced principal payment. The following equation describes the principal loss percentage, in year t_i:

$$L_i = \begin{cases} 0 & \text{if } q_{t_i} \le K_1 q_0 \\ \frac{(q_{t_i} - K_1 q_0)}{(K_2 - K_1) q_0} & \text{if } K_1 q_0 < q_{t_i} \le K_2 q_0 \\ 1 & \text{if } q_{t_i} > K_2 q_0 \end{cases} \tag{3.2.2}$$

In particular, for the case of Swiss Re Bond, $K_1 = 1.3$ and $K_2 = 1.5$. In lieu of having their principal at risk, investors received quarterly coupons equal to the three-month U.S. LIBOR plus 135 basis points. There were 12 coupons in all with a coupon value of

$$CO_j = \begin{cases} \left(\frac{SP + LI_j}{4} \right) .C & \text{if } j = \frac{1}{4}, \frac{2}{4}, ..., \frac{11}{4}, \\ \left(\frac{SP + LI_j}{4} .C + X_T \right) & \text{if } j = 3, \end{cases} \tag{3.2.3}$$

where SP is the spread value which is 1.35%, LI_j are the LIBOR rates, C = \$400 million, $T = t_3$ and X_T is a random variable representing the proportion of the principal returned to the bondholders on the maturity date such that

$$X_T = C \left(1 - \sum_{i=1}^{3} L_i \right)^+, \tag{3.2.4}$$

where $\sum_{i=1}^{3} L_i$ is the aggregate loss ratio at t_3. However, there was no catastrophe during the term of the bond. The discounted cash flow (DC) of payments is given by

$$DC(r) = \sum_{i=1}^{12} \frac{CO_{\frac{i}{4}}}{\left(1 + \frac{r}{4}\right)^i} \tag{3.2.5}$$

where r is the nominal annual interest rate.

Further define

$$Y_T = -\int_0^T \rho(t)\,dt$$

where $\rho(t)$ is the US LIBOR at time t. As a result, the risk-neutral value at time 0 of the random principal returned at the termination of the bond is

$$P = \mathrm{E}_Q\left[e^{-Y_T} X_T\right]$$

where Q is the risk-neutral measure. However, under the assumption of independence of Y_T and X_T, this reduces to

$$P = \mathrm{E}_Q\left[e^{-Y_T}\right]\mathrm{E}_Q[X_T]$$

The fact that the independence assumption under the physical world measure $\mathbb{P}$ and the risk-neutral measure Q are in general unrelated discussed extensively in Dhaene et al. (2013). Since the market is incomplete, one chooses to price under a risk neutral measure that preserves independence between market and mortality risks. In order to proceed, we represent $\mathrm{E}_Q\left[e^{-Y_T}\right]$ as e^{-rT}, which implies

$$P = e^{-rT}\mathrm{E}_Q[X_T] \tag{3.2.6}$$

where r is the risk-free rate of interest. In subsequent writing, we drop Q from the above expression.

3.2.1 The Principal Payoff of Swiss Re Bond as that of an Asian-type Put Option

In fact, we can write X_T given in (3.2.4) in a more compact form similar to the payoff of the Asian put option (see Appendix A.7.1) as shown below:

$$X_T = D\left(q_0 - \sum_{i=1}^{3} 5\left(q_{t_i} - 1.3q_0\right)^+\right)^+ \tag{3.2.7}$$

with

$$D = \frac{C}{q_0} \tag{3.2.8}$$

and the strike price equal to q_0. For the sake of simplicity, we use q_i in place of q_{t_i} and define

$$S_i = 5\left(q_i - 1.3q_0\right)^+ \tag{3.2.9}$$

and

$$S = \sum_{i=1}^{3} S_i \tag{3.2.10}$$

Using (3.2.9)-(3.2.10) in (3.2.7) and plugging the result into (3.2.6), we have:

$$P = De^{-rT}\mathrm{E}\left[\left(q_0 - S\right)^+\right] \tag{3.2.11}$$

It is naturally assumed that the inequalities $S \geq q_0$ almost surely (a.s.) and $S \leq q_0$ a.s. do not hold, otherwise the problem has a trivial solution. This means that $q_0 \in \left(F_S^{-1+}(0), F_S^{-1}(1)\right)$, where as in Dhaene et al. (2002a), F_X^{-1} is the generalized inverse of the cumulative distribution function (cdf), i.e.,

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} | F_X(x) \geq p\}, \;\; p \in [0,1] \tag{3.2.12}$$

and F_X^{-1+} is a more sophisticated inverse defined as

$$F_X^{-1+}(p) = \sup\{x \in \mathbb{R} | F_X(x) \leq p\}, \;\; p \in [0,1]\,. \tag{3.2.13}$$

Our interest lies in the calculation of reasonable bounds for P. We invoke Jensen's inequality for computing the lower bounds and present our findings in the subsequent sections. We exploit this inequality twice and note that in order to maintain uniformity of having a convex function at each step, it is beneficial to consider the call counterpart of the payoff of Swiss Re Bond rather than (3.2.11). We nomenclate this payoff as P_1, i.e., we have

$$P_1 = De^{-rT}\mathrm{E}\Big[(S - q_0)^+\Big] \tag{3.2.14}$$

We then exploit the put-call parity for Asian options to achieve the bounds for the payoff in question.

3.2.2 Put-Call Parity for the Swiss Re Bond

We now derive the put-call parity relationship for the Swiss Re Bond. For any real number a, we have:

$$(a)^+ - (-a)^+ = a \tag{3.2.15}$$

So we obtain

$$e^{-rT}\left(\sum_{i=1}^{3} S_i - q_0\right)^+ - e^{-rT}\left(q_0 - \sum_{i=1}^{3} S_i\right)^+ = e^{-rT}\left(\sum_{i=1}^{3} S_i - q_0\right).$$

On taking expectations on both sides, we obtain

$$e^{-rT}\mathbf{E}\left[\left(\sum_{i=1}^{3} S_i - q_0\right)^+\right] - e^{-rT}\mathbf{E}\left[\left(q_0 - \sum_{i=1}^{3} S_i\right)^+\right] = e^{-rT}\mathbf{E}\left[\sum_{i=1}^{3} S_i - q_0\right].$$

Finally, on multiplying by D and expanding the definition of S_i, we have

$$\begin{aligned} P_1 - P &= De^{-rT}\mathbf{E}\left[\sum_{i=1}^{3} 5\left(q_i - 1.3q_0\right)^+ - q_0\right] \\ \Rightarrow P_1 - P &= De^{-rT}\left[5\sum_{i=1}^{3} e^{rt_i} C\left(1.3q_0, t_i\right) - q_0\right], \end{aligned} \tag{3.2.16}$$

where $C(K, t_i)$ denotes the price of a European call on the mortality index with strike K, maturity t_i and current mortality value q_0. This option would be in-the-money if the mortality index is more than $1.3q_0$ which is the trigger level of Swiss Re bond. Clearly, such instruments are not available for trading in the market at present. However, concrete steps towards a more liquid and transparent life market are being undertaken and perhaps such securities will soon be introduced (c.f. Blake et al., 2013, 2008). The pay-off structures, i.e. the design of the issued securities and the mortality contingent payments should be developed to appear attractive to investors and the re-insurer. Although, the Swiss Re bond was fully subscribed and press reports highlight that investors were quite satisfied with it (e.g. *Euroweek*, 19 December 2003), the market for mortality linked securities still needs innovations such as vanilla options on mortality index to provide flexible hedging solutions. Investors of the Swiss Re bond included a large number of pension funds as they could view this bond as a powerful hedging instrument. The underlying mortality risk associated with the bond is correlated with the mortality risk of the active members of a pension plan. If a catastrophe occurs, the reduction in the principal would be offset by reduction in pension liability of these pension funds. Moreover, the bond offers a considerably higher return than similarly rated floating rate securities (c.f. Blake et al., 2006). In a manner similar to Bauer (2008), we feel the success of the life market hinges upon flexibility. As a result, such option-type structures enable re-insurer to keep most of the capital while at the same time being hedged against catastrophic mortality situation. Cox et al. (2006) present an interesting note on the trigger level of $1.3q_0$ in context of 2004 tsunami in Asia and Africa. A mortality option of the above type would become extremely useful in such a case. Tsai and Tzeng (2013) and Cheng et al. (2014) decompose the terminal payoff of the Swiss Re bond into two call options.

Equation (3.2.16) gives the required put-call parity relation between the Swiss Re mortality bound and its call counterpart. Define

$$G = De^{-rT}\left[5\sum_{i=1}^{3} e^{rt_i} C\left(1.3q_0, t_i\right) - q_0\right]. \tag{3.2.17}$$

Clearly, if we bound P_1 by bounds l_1 and u_1, then the corresponding bounds for the Swiss Re mortality bond are as follows

$$\left(l_1 - G\right)^+ \leq P \leq \left(u_1 - G\right)^+. \tag{3.2.18}$$

3.3 Lower Bounds for the Swiss Re Bond

We now proceed to work out appropriate lower bounds for the terminal value of the principal paid in the Swiss Re Bond. For this, we first calculate bounds for the following Asian-type call option

$$P_1 = De^{-rT}\mathbf{E}\left[\left(\sum_{i=1}^{n} S_i - q_0\right)^+\right] \tag{3.3.1}$$

with $T = t_n$ and $n = 3$. The interval $[0, T]$ consists of the monitoring times $t_1, t_2, ..., t_{n-1}$. The undercurrent of the theory presented in this section is the paper by Albrecher et al. (2008). In an attempt to estimate the value of the Asian call option, the authors derive four lower bounds namely trivial, LB_1, $LB_t^{(1)}$ and $LB_t^{(2)}$, which are sharper in increasing order in sense of their proximity to the actual value of the Asian call. The underlying assumption they make in deriving these bounds is that European call prices with arbitrary strikes and maturities are available in the market. Although, as our previous discussion indicates, such securities with the underlying as the mortality index have not appeared on the horizon as yet, but would be indispensable for the development of a complete life market. The first step towards designing of such securities is the need for a benchmark longevity index. The formation of Life and Longevity Markets Association (LLMA) in 2010 was an important milestone in this direction. The LLMA promotes the development of a liquid trading market in longevity and mortality-related risk, of the type that exists for Insurance Linked Securities (ILS) and other large trend risks like interest rates and inflation. There have been a few mortality indices created by various parties but we still lack a benchmark. Mènioux (2008) throws light on various longevity indices. Invoking Jensen's inequality, in (3.3.1), we have

$$\mathbf{E}\left[\left(\sum_{i=1}^{n} S_i - q_0\right)^+\right] \quad \geq \quad \mathbf{E}\left[\left(5\sum_{i=1}^{n}\left(\mathbf{E}\left(q_i|\Lambda\right) - 1.3q_0\right)^+ - q_0\right)^+\right]. \tag{3.3.2}$$

The general derivation concerning lower bounds for stop loss premium of a sum of random variables based on Jensen's inequality can be found in Simon et al. (2000) and for its application to arithmetic Asian options, one can refer to Dhaene et al. (2002b). We now define

$$Z_i = 5\left(\mathbf{E}\left(q_i|\Lambda\right) - 1.3q_0\right)^+; i = 1, 2, ..., n. \tag{3.3.3}$$

As a result in (3.3.2), we have obtained

$$\mathbf{E}\left[\left(\sum_{i=1}^{n} S_i - q_0\right)^+\right] \geq \mathbf{E}\left[\left(\sum_{i=1}^{n} Z_i - q_0\right)^+\right]. \tag{3.3.4}$$

On investigating the relationship between $\mathbf{E}\left[\sum_{i=1}^{n} S_i\right]$ and $\mathbf{E}\left[\sum_{i=1}^{n} Z_i\right]$, we find that

$$\mathbf{E}\left[\sum_{i=1}^{n} S_i\right] \quad \geq \quad \mathbf{E}\left[\sum_{i=1}^{n} Z_i\right]. \tag{3.3.5}$$

On lines of (3.2.10), define

$$Z = \sum_{i=1}^{n} Z_i \tag{3.3.6}$$

so that we can rewrite (3.3.4) as

$$\mathrm{E}\Big[(S - q_0)^+\Big] \geq \mathrm{E}\Big[(Z - q_0)^+\Big]. \tag{3.3.7}$$

In fact, the two sides of the inequality in (3.3.7) are essentially the stop-loss premiums of S and Z. Thus, we have obtained

$$S \geq_{sl} Z \tag{3.3.8}$$

or

$$S \geq_{\mathrm{sl}} \sum_{i=1}^{n} \left(\mathbf{E}\left(q_i|\Lambda\right) - 1.3q_0\right)^+ .$$

Now, suitably tailoring the inequality (3.3.7) to suit our need of the Asian-type call option by multiplying by the discount factor at time T, we obtain

$$P_1 \geq De^{-rT}\mathbf{E}\left[\left(\sum_{i=1}^{n} 5\left(\mathbf{E}\left(q_i|\Lambda\right) - 1.3q_0\right)^+ - q_0\right)^+\right]. \tag{3.3.9}$$

To exploit the theory of comonotonicity see for example in Dhaene et al. (2002a), we now have to show that the lower bound for S, can be formulated as a sum of stop-loss premiums. This task becomes trivial if we can choose the conditioning variable Λ in such a way that $\mathbf{E}\left(q_i|\Lambda\right)$ is either increasing or decreasing for every i, so that the vector: $\mathbf{q}^{\mathbf{l}} = \left(\mathbf{E}\left(q_1|\Lambda\right), \ldots, \mathbf{E}\left(q_n|\Lambda\right)\right)$ is comonotonic. This automatically implies that the vector: $\mathbf{Z}^{\mathbf{l}} = \left(Z_1, \ldots, Z_n\right)$ is comonotonic. From this point onwards, we assume that $q_0 \in \left(F_Z^{-1+}(0), F_Z^{-1}(1)\right)$ which is not at all a restriction for all practical purposes as pointed out in section 3.2.1. As a result on using comonotonicity (c.f. Corollary 16, Chapter 2), we have

$$\mathbf{E}\left[(S - q_0)^+\right] \quad \geq \quad \sum_{i=1}^{n}\mathbf{E}\left[\left(Z_i - F_{Z_i}^{-1}\left(F_Z\left(q_0\right)\right)\right)^+\right] - \left(q_0 - F_Z^{-1}\left(F_Z\left(q_0\right)\right)\right)\left(1 - F_Z\left(q_0\right)\right). \tag{3.3.10}$$

In case if the marginal cdfs F_{Z_i} are strictly increasing, we have the following compact expression

$$\mathbf{E}\left[(S - q_0)^+\right] \quad \geq \quad \sum_{i=1}^{n}\mathbf{E}\left[\left(Z_i - F_{Z_i}^{-1}\left(F_Z\left(q_0\right)\right)\right)^+\right], \quad q_0 \in \left(F_Z^{-1+}(0), F_Z^{-1}(1)\right). \tag{3.3.11}$$

Note from (3.3.3) and (3.3.6) that the $Z_i's$ and subsequently Z are non-negative and this automatically implies $q_0 \geq 0$. Further, by the definition of cdf, we have

$$F_Z\left(q_0\right) = \mathbf{P}\left[Z \leq q_0\right] = \mathbf{P}\left[\sum_{j=1}^{n} Z_j \leq q_0\right] = \mathbf{P}\left[\sum_{j=1}^{n} 5\left(\mathbf{E}\left(q_j|\Lambda\right) - 1.3q_0\right)^+ \leq q_0\right]. \tag{3.3.12}$$

Thus we have been able to obtain a stop-loss lower bound for $S = \sum_{i=1}^{n} S_i$ by conditioning on an arbitrary random variable Λ, i.e.,

$$P_1 \geq De^{-rT}\sum_{i=1}^{n}\mathbf{E}\left[\left(5\left(\mathbf{E}\left(q_i|\Lambda\right)-1.3q_0\right)^{+}-F_{Z_i}^{-1}\left(F_Z\left(q_0\right)\right)\right)^{+}\right]-K_1, \tag{3.3.13}$$

where

$$K_1 = De^{-rT}\left(q_0 - F_Z^{-1}\left(F_Z\left(q_0\right)\right)\right)\left(1-F_Z\left(q_0\right)\right). \tag{3.3.14}$$

3.3.1 The Trivial Lower Bound

In case, if the random variable Λ is independent of the mortality evolution $\{q_t\}_{t\geq 0}$, the bound in (3.3.9) simply reduces to:

$$P_1 \geq De^{-rT}\mathbf{E}\left[\left(\sum_{i=1}^{n}5\left(\mathbf{E}\left(q_i\right)-1.3q_0\right)^{+}-q_0\right)^{+}\right] \tag{3.3.15}$$

or even more precisely as the outer expectation is redundant

$$P_1 \geq De^{-rT}\left(\sum_{i=1}^{n}5\left(\mathbf{E}\left(q_i\right)-1.3q_0\right)^{+}-q_0\right)^{+}. \tag{3.3.16}$$

Under the assumption of the existence of an Equivalent Martingale Measure (EMM), Q, the discounted mortality process is a martingale, so that

$$\mathbf{E}\left[q_t\right] = q_0 e^{rt}. \tag{3.3.17}$$

If we substitute this in equation (3.3.16), we obtain a very rough lower bound for the Asian-type call option

$$P_1 \geq Ce^{-rT}\left(\sum_{i=1}^{n}5\left(e^{rt_i}-1.3\right)^{+}-1\right)^{+} =: \text{ lb}_0. \tag{3.3.18}$$

In the light of put-call parity derived in section 3.2.2, the trivial lower bound for the Swiss Re mortality bond is given as

$$P \geq \left(\text{ lb}_0 - G\right)^{+} =: \text{ SWLB}_0. \tag{3.3.19}$$

where G is defined in (3.2.17).

3.3.2 The Lower Bound SWLB$_1$

To improve upon the trivial lower bound, we choose $\Lambda = q_1$ in (3.3.13). Using the martingale argument for the discounted mortality process

$$\mathbf{E}\left[q_i|q_1\right] = \mathbf{E}\left[e^{rt_i}e^{-rt_i}q_i|q_1\right] = e^{r(t_i-t_1)}q_1$$

and so from (3.3.3)

$$Z_i = 5\left(e^{r(t_i-t_1)}q_1 - 1.3q_0\right)^{+}; i=1,2,...,n. \tag{3.3.20}$$

Then the random vector $\mathbf{q^l} = \left(q_1, e^{r(t_2-t_1)}q_1, \ldots, e^{r(t_n-t_1)}q_1\right)$ is comonotone. Equation (3.3.13) then reduces to

$$P_1 \geq De^{-rT}\sum_{i=1}^{n}\mathbf{E}\left[\left(5\left(e^{r(t_i-t_1)}q_1-1.3q_0\right)^{+}-F_{Z_i}^{-1}\left(F_Z\left(q_0\right)\right)\right)^{+}\right]-K_1, \tag{3.3.21}$$

where K_1 is given in (3.3.14) and by the definition of cdf, we have

$$F_Z\left(q_0\right) = \mathbf{P}\left[Z \leq q_0\right] = \mathbf{P}\left[\sum_{j=1}^{n}5\left(e^{r(t_j-t_1)}q_1-1.3q_0\right)^{+} \leq q_0\right]$$

$$\Rightarrow F_Z(q_0) = \mathbf{P}\left[\sum_{j=1}^{n} 5\left(e^{r(t_j-t_1)}\frac{q_1}{q_0} - 1.3\right)^+ \leq 1\right].$$

Now, as the left hand side of the inequality within the probability is an increasing function in q_1/q_0, we have that $Z \leq q_0$ if and only if $q_1 \leq xq_0$, where we substitute x for q_1/q_0 in the above probability and obtain its value by solving

$$\sum_{i=1}^{n}\left(e^{r(t_i-t_1)}x - 1.3\right)^+ = 0.2. \tag{3.3.22}$$

As a result, we have

$$F_Z(q_0) = F_{q_1}(xq_0) = F_{Z_i}\left(5q_0\left(e^{r(t_i-t_1)}x - 1.3\right)^+\right) = F_{Z_i}(k_i) \quad \forall i \tag{3.3.23}$$

where

$$k_i = 5q_0\left(e^{r(t_i-t_1)}x - 1.3\right)^+. \tag{3.3.24}$$

Plugging (3.3.23) into (3.3.21), and noting that $Z_i's$ are non-negative, we have

$$\begin{aligned}
P_1 &\geq 5De^{-rT}\sum_{i=1}^{n}\mathbf{E}\left[\left(\left(e^{r(t_i-t_1)}q_1 - 1.3q_0\right)^+ - \frac{1}{5}F_{Z_i}^{-1}\left(F_{Z_i}(k_i)\right)\right)^+\right] - K_1 \\
&= 5De^{-rT}\sum_{i=1}^{n}e^{r(t_i-t_1)}\mathbf{E}\left[\left(\left(q_1 - \frac{1.3q_0}{e^{r(t_i-t_1)}}\right)^+ - \frac{1}{5e^{r(t_i-t_1)}}F_{Z_i}^{-1}\left(F_{Z_i}(k_i)\right)\right)^+\right] - K_1 \\
&= 5De^{-rT}\sum_{i=1}^{n}e^{r(t_i-t_1)}\mathbf{E}\left[\left(q_1 - \frac{q_0}{e^{r(t_i-t_1)}}\left(1.3 + \frac{1}{5q_0}F_{Z_i}^{-1}\left(F_{Z_i}(k_i)\right)\right)\right)^+\right] - K_1 \\
&= 5D\sum_{i=1}^{n}e^{-r(T-t_i)}C\left(\frac{q_0}{e^{r(t_i-t_1)}}\left(1.3 + \frac{1}{5q_0}F_{Z_i}^{-1}\left(F_{Z_i}\left(5q_0\left(e^{r(t_i-t_1)}x - 1.3\right)^+\right)\right)\right),\ t_1\right) \\
&\quad -K_1 \\
&=: \mathrm{lb}_1.
\end{aligned} \tag{3.3.25}$$

where $q_0 \geq 0$ and $C(K, t_1)$ denotes the price of a European call on the mortality index with strike K, maturity t_1 and current mortality index q_0. The function lb_1 provides a lower bound for the Asian-type call option in terms of European calls at each of the times such that these contracts have maturity t_1 and strike $\frac{q_0}{e^{r(t_i-t_1)}}\left(1.3 + \frac{1}{5q_0}F_{Z_i}^{-1}\left(F_{Z_i}\left(5q_0\left(e^{r(t_i-t_1)}x - 1.3\right)^+\right)\right)\right)$ at the ith time point. This bound holds for any arbitrage-free market model and is a significant improvement over the trivial bound given in (3.3.18). Invoking the put-call parity derived in section 3.2.2, the corresponding lower bound for the Swiss Re mortality bond is given as

$$P \geq (\mathrm{lb}_1 - G)^+ =: \mathrm{SWLB}_1, \tag{3.3.26}$$

where G is defined in (3.2.17). In case if the marginal cdfs F_{Z_i} are strictly increasing, we have

$$\mathrm{lb}_1 = 5D\sum_{i=1}^{n}e^{-r(T-t_i)}C\left(q_0\max\left(x,\ \frac{1.3}{e^{r(t_i-t_1)}}\right),\ t_1\right). \tag{3.3.27}$$

3.3.3 A Model-independent Lower Bound

As the next step, we suggest that the bound SWLB_1[2] can be improved by imposing the following additional assumption[3]

[2]An additional lower bound under certain assumptions was also computed and is furnished in Appendix B.1

[3]Albrecher et al. (2008) have proved in proposition 1 of their publication that this assumption holds for exponential Lévy models. We present a detailed proof in Appendix A.1 in regards to our set up.

$$\sum_{i=1}^{n} q_i \geq_{sl} \left(\sum_{i=1}^{j-1} q_0^{(1-t_i/t)} q_t^{t_i/t} + \sum_{i=j}^{n} e^{r(t_i-t)} q_t \right) \tag{3.3.28}$$

for $0 \leq t \leq T$ and $j = \min\{i : t_i \geq t\}$. Clearly,

$$\begin{aligned}
\sum_{i=1}^{n} 5\left(\mathbf{E}\left(q_i|q_t\right) - 1.3q_0\right)^+ &= \sum_{i=1}^{j-1} 5\left(\mathbf{E}\left(q_i|q_t\right) - 1.3q_0\right)^+ + \sum_{i=j}^{n} 5\left(\mathbf{E}\left(q_i|q_t\right) - 1.3q_0\right)^+ \\
&\geq \sum_{i=1}^{j-1} 5q_0 \left(\left(\frac{q_t}{q_0}\right)^{t_i/t} - 1.3 \right)^+ + \sum_{i=j}^{n} 5q_0 \left(\frac{q_t}{q_0} e^{r(t_i-t)} - 1.3 \right)^+ \\
&=: S^{l_2}.
\end{aligned} \tag{3.3.29}$$

Evidently, S^{l_2} is the same as Z in (3.3.6) with Λ being replaced by q_t and thus from (3.3.8), we have

$$S \geq_{sl} S^{l_2} \tag{3.3.30}$$

As before, let $j = \min\{i : t_i \geq t\}$. Consider the components of S^{l_2} in equation (3.3.29) and define $\mathbf{Y} = (Y_1, \ldots, Y_n)$, where

$$Y_i = \begin{cases} 5q_0 \left(\left(\frac{q_t}{q_0}\right)^{t_i/t} - 1.3 \right)^+ & i < j \\ 5q_0 \left(\left(\frac{q_t}{q_0}\right) e^{r(t_i-t)} - 1.3 \right)^+ & i \geq j \end{cases}$$

$i = 1, 2, ..., n$. Clearly, $\mathbf{Y}$ is comonotonic since its components are strictly increasing functions of a single variable q_t. So, the stop-loss transform of S^{l_2} can be written as the sum of stop-loss transform of its components (see for example in Dhaene et al., 2002a), i.e.,

$$\mathbf{E}\left[\left(S^{l_2} - q_0\right)^+\right] = \sum_{i=1}^{n} \mathbf{E}\left[\left(Y_i - F_{Y_i}^{-1}\left(F_{S^{l_2}}\left(q_0\right)\right)\right)^+\right] - \left(q_0 - F_{S^{l_2}}^{-1}\left(F_{S^{l_2}}\left(q_0\right)\right)\right)\left(1 - F_{S^{l_2}}\left(q_0\right)\right) \tag{3.3.31}$$

where as before it is natural that $q_0 \in \left(F_{S^{l_2}}^{-1+}(0), F_{S^{l_2}}^{-1}(1)\right)$ and $F_{S^{l_2}}(q_0)$ is the distribution function of S^{l_2} evaluated at q_0 such that for an arbitrary t, we have:

$$\begin{aligned}
F_{S^{l_2}}(q_0) &= \mathbf{P}\left[S^{l_2} \leq q_0\right] \\
&= \mathbf{P}\left[\sum_{i=1}^{j-1} 5q_0 \left(\left(\frac{q_t}{q_0}\right)^{t_i/t} - 1.3 \right)^+ + \sum_{i=j}^{n} 5q_0 \left(\left(\frac{q_t}{q_0}\right) e^{r(t_i-t)} - 1.3 \right)^+ \leq q_0 \right] \\
&= \mathbf{P}\left[\sum_{i=1}^{j-1} \left(\left(\frac{q_t}{q_0}\right)^{t_i/t} - 1.3 \right)^+ + \sum_{i=j}^{n} \left(\left(\frac{q_t}{q_0}\right) e^{r(t_i-t)} - 1.3 \right)^+ \leq 0.2 \right].
\end{aligned}$$

Clearly, $S^{l_2} \leq q_0$ if and only if $q_t \leq xq_0$, where we substitute x for q_t/q_0 in the above expression and obtain its value by solving:

$$\sum_{i=1}^{j-1} \left(x^{t_i/t} - 1.3\right)^+ + \sum_{i=j}^{n} \left(xe^{r(t_i-t)} - 1.3\right)^+ = 0.2. \tag{3.3.32}$$

As a result, we have:

$$F_{S^{l_2}}(q_0) = F_{q_t}(xq_0) = \begin{cases} F_{Y_i}\left(5q_0\left(x^{t_i/t} - 1.3\right)^+\right) = F_{Y_i}(l_i) & i < j \\ F_{Y_i}\left(5q_0\left(xe^{r(t_i-t)} - 1.3\right)^+\right) = F_{Y_i}(m_i) & i \geq j \end{cases} \tag{3.3.33}$$

where

$$l_i = 5q_0\left(x^{t_i/t} - 1.3\right)^+ ; i < j \tag{3.3.34}$$

and

$$m_i = 5q_0\left(xe^{r(t_i - t)} - 1.3\right)^+ ; i > j. \tag{3.3.35}$$

Using this result in equation (3.3.31) and recalling the definition of the Asian-type call option given in (3.3.1) along with the stop-loss order relationship between S and S^{l_2} as given by equation (3.3.30) and noting that $Y_i's$ are non-negative, we obtain,

$$\begin{aligned}
P_1 &\geq De^{-rT}\left(\sum_{i=1}^{n}\mathbf{E}\left[\left(Y_i - F_{Y_i}^{-1}\left(F_{S^{l_2}}\left(q_0\right)\right)\right)^+\right]\right) - K_2 \\
&= Ce^{-rT}\left(\sum_{i=1}^{j-1}\mathbf{E}\left[\left(5\left(\left(\frac{q_t}{q_0}\right)^{t_i/t} - 1.3\right)^+ - \frac{1}{q_0}F_{Y_i}^{-1}\left(F_{Y_i}\left(l_i\right)\right)\right)^+\right]\right. \\
&\quad \left. + \sum_{i=j}^{n}\mathbf{E}\left[\left(5\left(\left(\frac{q_t}{q_0}\right)e^{r(t_i - t)} - 1.3\right)^+ - \frac{1}{q_0}F_{Y_i}^{-1}\left(F_{Y_i}\left(m_i\right)\right)\right)^+\right]\right) - K_2 \\
&= 5Ce^{-rT}\left(\sum_{i=1}^{j-1}\frac{1}{q_0^{t_i/t}}\mathbf{E}\left[\left(q_t^{t_i/t} - q_0^{t_i/t}\left(1.3 + \frac{1}{5q_0}F_{Y_i}^{-1}\left(F_{Y_i}\left(l_i\right)\right)\right)\right)^+\right]\right. \\
&\quad \left. + \sum_{i=j}^{n}\frac{e^{r(t_i - t)}}{q_0}\mathbf{E}\left[\left(q_t - \frac{q_0}{e^{r(t_i - t)}}\left(1.3 + \frac{1}{5q_0}F_{Y_i}^{-1}\left(F_{Y_i}\left(m_i\right)\right)\right)\right)^+\right]\right) - K_2 \\
&= 5De^{-rT}\left(\sum_{i=1}^{j-1}q_0^{1 - t_i/t}\mathbf{E}\left[\left(q_t^{t_i/t} - q_0^{t_i/t}.\left(1.3 + \frac{1}{5q_0}F_{Y_i}^{-1}\left(F_{Y_i}\left(l_i\right)\right)\right)\right)^+\right]\right. \\
&\quad \left. + \sum_{i=j}^{n}e^{rt_i}C\left(\frac{q_0}{e^{r(t_i - t)}}\left(1.3 + \frac{1}{5q_0}F_{Y_i}^{-1}\left(F_{Y_i}\left(5q_0\left(xe^{r(t_i - t)} - 1.3\right)^+\right)\right)\right), t\right)\right) \\
&\quad -K_2 \\
&=: \mathrm{lb}_t^{(2)}
\end{aligned} \tag{3.3.36}$$

where l_i is defined in (3.3.34) and

$$K_2 = De^{-rT}\left(q_0 - F_{S^{l_2}}^{-1}\left(F_{S^{l_2}}\left(q_0\right)\right)\right)\left(1 - F_{S^{l_2}}\left(q_0\right)\right) \tag{3.3.37}$$

In fact, $\mathrm{lb}_t^{(2)}$ is a lower bound for all t and so it can be maximized with respect to t to yield the optimal lower bound as given below:

$$P_1 \geq \max_{0 \leq t \leq T} \mathrm{lb}_t^{(2)}. \tag{3.3.38}$$

On choosing $t = t_1$ implies $j = 1$ and so equation (3.3.32) reduces to (3.3.22) and we obtain

$$\mathrm{lb}_1^{(2)} = \mathrm{lb}_1. \tag{3.3.39}$$

As a result we have

$$\max_{0 \leq t \leq T} \mathrm{lb}_t^{(2)} \geq \mathrm{lb}_1.$$

Clearly, once again, as in the previous sections, we have

$$P \geq \left(\mathrm{lb}_t^{(2)} - G\right)^+ =: \mathrm{SWLB}_t^{(2)}, \tag{3.3.40}$$

where G is defined in (3.2.17). In case if the marginal cdfs F_{Y_i} are strictly increasing, we have

$$
\begin{aligned}
\mathrm{lb}_t^{(2)} &= 5De^{-rT}\left(\sum_{i=1}^{j-1} q_0^{1-t_i/t}\mathbf{E}\left[\left(q_t^{t_i/t} - q_0^{t_i/t}.\max\left(x^{t_i/t},\, 1.3\right)\right)^+\right]\right. \\
&\quad \left. + \sum_{i=j}^{n} e^{rt_i} C\left(q_0.\max\left(x,\, \frac{1.3}{e^{r(t_i-t)}}\right),\, t\right)\right). \qquad (3.3.41)
\end{aligned}
$$

We now move on to the derivation of an upper bound for the price of Swiss Re bond in the next section.

3.4 Upper Bounds for the Swiss Re Bond

We derive a couple of upper bounds for the Swiss Re bond.

3.4.1 A First Upper Bound

This section will focus on finding an upper bound for the bond in question by using comonotonicity theory in a manner similar to Kaas et al. (2000), Dhaene et al. (2000) and Chen et al. (2008)[4]. Define the comonotonic counterpart of $\mathbf{q} = (q_1, ..., q_n)$ as $\mathbf{q^u} = \left(F_{S_1}^{-1}(U), ..., F_{S_n}^{-1}(U)\right)$ where $U \sim U(0,1)$. Further define

$$
S^c = \sum_{i=1}^{n} F_{S_i}^{-1}(U) = \sum_{i=1}^{n} S_i^c. \qquad (3.4.1)
$$

Clearly,

$$
S \leq_{cx} S^c \qquad (3.4.2)
$$

where cx denotes convex ordering (see for example in Dhaene et al. (2002a)). In other words,

$$
\mathbf{E}\left[\left(\sum_{i=1}^{n} S_i - q_0\right)^+\right] \leq \mathbf{E}\left[\left(\sum_{i=1}^{n} S_i^c - q_0\right)^+\right] \qquad (3.4.3)
$$

and we have

$$
\mathbf{E}\left[\left(\sum_{i=1}^{n} S_i^c - q_0\right)^+\right] = \sum_{i=1}^{n}\mathbf{E}\left[\left(S_i - F_{S_i}^{-1}\left(F_{S^c}(q_0)\right)\right)^+\right] - \left(q_0 - F_{S^c}^{-1}\left(F_{S^c}(q_0)\right)\right)\left(1 - F_{S^c}(q_0)\right), \qquad (3.4.4)
$$

where it is understood that $q_0 \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$. As a result, an upper bound for the call counterpart of the Swiss Re bond is given as

$$
\begin{aligned}
P_1 &\leq De^{-rT}\sum_{i=1}^{n}\mathbf{E}\left[\left(S_i - F_{S_i}^{-1}\left(F_{S^c}(q_0)\right)\right)^+\right] - K_3 \\
&= 5De^{-rT}\sum_{i=1}^{n}\mathbf{E}\left[\left(q_i - \left(1.3q_0 + \frac{F_{S_i}^{-1}\left(F_{S^c}(q_0)\right)}{5}\right)\right)^+\right] - K_3 \\
&= 5De^{-rT}\sum_{i=1}^{n} e^{rt_i} C\left(1.3q_0 + \frac{F_{S_i}^{-1}\left(F_{S^c}(q_0)\right)}{5}, t_i\right) - K_3, \qquad (3.4.5)
\end{aligned}
$$

where

$$
K_3 = De^{-rT}\left(q_0 - F_{S^c}^{-1}\left(F_{S^c}(q_0)\right)\right)\left(1 - F_{S^c}(q_0)\right). \qquad (3.4.6)
$$

[4]Hobson et al. (2005) use the method of Lagrange multipliers to find an upper bound for basket options. We furnish an upper bound based on this approach in the Appendix B.3 and we essentially obtain the same result.

As a result we can write the upper bound given above as

$$P_1 \leq 5De^{-rT}\sum_{i=1}^{n} e^{rt_i} C\left(1.3q_0 + \frac{F_{S_i}^{-1}(x)}{5}, t_i\right) - K_3, \tag{3.4.7}$$

where $x \in (0,1)$ is the solution of the equation

$$\sum_{i=1}^{n} F_{S_i}^{-1}(x) = q_0. \tag{3.4.8}$$

We now seek to express the inverse distribution function of S_i in terms of that of q_i. Let

$$y_i = F_{S_i}^{-1}(x);\ y_i \geq 0 \tag{3.4.9}$$

$$\begin{aligned}
\Rightarrow x &= F_{S_i}(y_i) \\
&= P\left[5\left(q_i - 1.3q_0\right)^+ \leq y_i\right] \\
&= 1 - P\left[5\left(q_i - 1.3q_0\right)^+ > y_i\right] \\
&= 1 - P\left[q_i > 1.3q_0 + \frac{y_i}{5}\right] \\
&= F_{q_i}\left(1.3q_0 + \frac{y_i}{5}\right).
\end{aligned} \tag{3.4.10}$$

$$\Rightarrow y_i = 5\left(F_{q_i}^{-1}(x) - 1.3q_0\right). \tag{3.4.11}$$

From equations (3.4.7), (3.4.9) and (3.4.11), we conclude that the upper bound is given as

$$P_1 \leq 5De^{-rT}\sum_{i=1}^{n} e^{rt_i} C\left(F_{q_i}^{-1}(x), t_i\right) - K_3 =: \mathrm{ub}_1. \tag{3.4.12}$$

where using equations (3.4.8) and (3.4.11), we see that x solves the following equation

$$\sum_{i=1}^{n} F_{q_i}^{-1}(x) = \frac{q_0}{5}(1 + 6.5n). \tag{3.4.13}$$

As in the case of lower bounds, invoking the put-call parity of section 3.2.2, we have for the Swiss Re bond

$$P \leq (\mathrm{ub}_1 - G)^+ =: \mathrm{SWUB}_1. \tag{3.4.14}$$

where G is defined in (3.2.17). In case if the marginal cdfs F_{S_i} are strictly increasing, we have

$$\mathrm{ub}_1 = 5De^{-rT}\sum_{i=1}^{n} e^{rt_i} C\left(F_{q_i}^{-1}(x), t_i\right) \tag{3.4.15}$$

where x solves the equation (3.4.13).

3.4.2 An Improved Upper Bound by conditioning

We now seek to obtain a sharper upper bound for the Swiss Re bond. This is possible if we assume that some additional information is available concerning the stochastic nature of $(q_1, q_2, ..., q_n)$. That is, if we can find a random variable Λ, with a known distribution, such that the individual conditional distributions of q_i given the event $\Lambda = \lambda$ are known for all i and all possible values of λ. Such an approach can be found in Kaas et al. (2000), Dhaene et al. (2002a) and Dhaene et al. (2002b).

Define

$$S^u = \sum_{i=1}^{n} F_{S_i|\Lambda}^{-1}(U) = \sum_{i=1}^{n} S_i^u, \tag{3.4.16}$$

where $U \sim U(0,1)$. Then we have

$$S \leq_{cx} S^u \leq_{cx} S^c, \tag{3.4.17}$$

Now let $\mathbf{q^u} = (S_1^u, ..., S_n^u)$. Since $\left(F_{S_1|\Lambda=\lambda}^{-1}, ..., F_{S_n|\Lambda=\lambda}^{-1}\right)$ is comonotonic, we have,

$$F_{S^u|\Lambda=\lambda}^{-1}(p) = \sum_{i=1}^{n} F_{S_i|\Lambda=\lambda}^{-1}(p), \ p \in (0,1). \tag{3.4.18}$$

It follows that, in this case

$$\sum_{i=1}^{n} F_{S_i|\Lambda=\lambda}^{-1}\left(F_{S^u|\Lambda=\lambda}(q_0)\right) = q_0. \tag{3.4.19}$$

and so we have

$$\begin{aligned}
f(\lambda) &= \mathbf{E}\left[\left(\sum_{i=1}^{n} S_i^u - q_0\right)^+ \middle| \Lambda = \lambda\right] \\
&= \sum_{i=1}^{n} \mathbf{E}\left[\left(S_i - F_{S_i|\Lambda=\lambda}^{-1}\left(F_{S^u|\Lambda=\lambda}(q_0)\right)\right)^+ \middle| \Lambda = \lambda\right] \\
&\quad - \left(q_0 - F_{S^u|\Lambda=\lambda}^{-1}\left(F_{S^u|\Lambda=\lambda}(q_0)\right)\right)\left(1 - F_{S^u|\Lambda=\lambda}(q_0)\right),
\end{aligned} \tag{3.4.20}$$

where it is clear that $q_0 \in \left(F_{S^u|\Lambda=\lambda}^{-1+}(0), F_{S^u|\Lambda=\lambda}^{-1}(1)\right)$. By applying the tower property and using the convex order relationship given by (3.4.17), we obtain an upper bound for the call counterpart of the Swiss Re bond, i.e.,

$$\begin{aligned}
P_1 &\leq De^{-rT}\mathbf{E}\left[(S^u - q_0)^+\right] \\
&= De^{-rT}\mathbf{E}\left[f(\lambda)\right] \\
&= De^{-rT}\sum_{i=1}^{n}\int_{-\infty}^{\infty} \mathbf{E}\left[\left(S_i - F_{S_i|\Lambda=\lambda}^{-1}\left(F_{S^u|\Lambda=\lambda}(q_0)\right)\right)^+ \middle| \Lambda = \lambda\right] dF_\Lambda(\lambda) - K_4 \\
&= 5De^{-rT}\sum_{i=1}^{n}\int_{-\infty}^{\infty} \mathbf{E}\left[\left(q_i - \left(1.3q_0 + \frac{F_{S_i|\Lambda=\lambda}^{-1}\left(F_{S^u|\Lambda=\lambda}(q_0)\right)}{5}\right)\right)^+ \middle| \Lambda = \lambda\right] dF_\Lambda(\lambda) - K_4,
\end{aligned}$$

where

$$K_4 = \left(q_0 - F_{S^u|\Lambda=\lambda}^{-1}\left(F_{S^u|\Lambda=\lambda}(q_0)\right)\right)\left(1 - F_{S^u|\Lambda=\lambda}(q_0)\right). \tag{3.4.21}$$

Given the event $\Lambda = \lambda$, let x be the solution to the following equation

$$\sum_{i=1}^{n} F_{S_i|\Lambda=\lambda}^{-1}(x) = q_0. \tag{3.4.22}$$

Further, we see from equation (3.4.19), that $x = F_{S^u|\Lambda=\lambda}(q_0)$. It therefore follows, as a result of equation 93 of Dhaene et al. (2002a) that an upper bound for the call counterpart of the Swiss Re bond is given as

$$P_1 \leq 5De^{-rT}\sum_{i=1}^{n}\int_{-\infty}^{\infty} \mathbf{E}\left[\left(q_i - \left(1.3q_0 + \frac{F_{S_i|\Lambda=\lambda}^{-1}(x)}{5}\right)\right)^+ \middle| \Lambda = \lambda\right] dF_\Lambda(\lambda) - K_4, \tag{3.4.23}$$

where x is obtained by solving (3.4.22). Moreover, it is straightforward to write

$$F^{-1}_{S_i|\Lambda=\lambda}(x) = 5\left(F^{-1}_{q_i|\Lambda=\lambda}(x) - 1.3q_0\right). \tag{3.4.24}$$

As a result, the upper bound can be rewritten as

$$P_1 \leq 5De^{-rT}\sum_{i=1}^{n}\int_{-\infty}^{\infty}\mathbf{E}\left[\left(q_i - F^{-1}_{q_i|\Lambda=\lambda}(x)\right)^+\middle|\Lambda=\lambda\right]dF_\Lambda(\lambda) - K_4 =: \mathrm{ub}_t^{(1)}, \tag{3.4.25}$$

where $x \in (0,1)$ can be obtained by solving the equation

$$\sum_{i=1}^{n}F^{-1}_{q_i|\Lambda=\lambda}(x) = \frac{q_0}{5}(1+6.5n). \tag{3.4.26}$$

Since this is is an upper bound for all t, it follows that we can find the optimal upper bound by minimising equation (3.4.25) over $t \in [0,T]$. As before, invoking the put-call parity of section 3.2.2, we have for the Swiss Re bond

$$P \leq \left(\mathrm{ub}_t^{(1)} - G\right)^+ =: \mathrm{SWUB}_t^{(1)}, \tag{3.4.27}$$

where G is defined in (3.2.17). As remarked earlier, this bound improves upon the unconditional bound given by (3.4.14). In case if the marginal cdfs $F_{S_i|\Lambda}$ are strictly increasing, one can put $K_4 = 0$ in (3.4.25) to obtain the upper bound.

3.5 Examples

We now derive lower and upper bounds by choosing specific models for the mortality index.

3.5.1 Black-Scholes Model

Let us consider the case where the mortality evolution process $\{q_t\}_{t\geq 0}$ follows the Black-Scholes model (c.f. Black and Scholes, 1973, and Appendix A.4) which we write as $q_t = e^{U_t}$, where $\{U_t\}_{t\geq 0}$ is defined as:

$$U_t = \log_e(q_0) + \left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t^*, \tag{3.5.1}$$

where $\{W_t^*\}_{t\geq 0}$ denotes a standard Brownian motion so that $W_t^* \sim N(0,t)$. As a result

$$U_t \sim N\left(\log_e q_0 + \left(r - \frac{\sigma^2}{2}\right)t,\, \sigma^2 t\right). \tag{3.5.2}$$

We now derive lower and upper bounds for this model on the lines of $\mathrm{SWLB}_t^{(2)}$ and $\mathrm{SWUB}_t^{(1)}$ respectively.

The Lower Bound $\mathrm{SWLB}_t^{(BS)}$

We know that if $(X, Y) \sim \mathrm{BVN}\left(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho\right)$ where BVN stands for bivariate normal distribution, the conditional distribution of the lognormal random variable e^X, given the event $e^Y = y$ is given as

$$F_{e^X|e^Y=y}(x) = \Phi\left(\frac{\log_e x - \left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(\log_e y - \mu_Y)\right)}{\sigma_X\sqrt{1-\rho^2}}\right). \tag{3.5.3}$$

where Φ denotes the c.d.f. of standard normal distribution. Given the time points t_i, t for each i, let ρ be the correlation between U_{t_i} and U_t. Then, from (3.5.2), it is evident that:

$(U_{t_i}, U_t) \sim \mathrm{BVN}\left(\mu_{U_{t_i}}, \mu_{U_t}, \sigma^2_{U_{t_i}}, \sigma^2_{U_t}, \rho\right)$, where the same equation specifies $\mu_{U_{t_i}}, \mu_{U_t}, \sigma^2_{U_{t_i}}$ and $\sigma^2_{U_t}$. Also as $q_t = e^{U_t}$, we have from equation (3.5.3) that the distribution function of q_i conditional on the event $q_t = s_t$ is given as

$$F_{q_i|q_t=s_t}(x) = \Phi(a(x))$$

where $a(x)$ is given by

$$a(x) = \frac{\log_e x - \left(\log\left(q_0\left(\frac{s_t}{q_0}\right)^{\rho\sqrt{\frac{t_i}{t}}}\right) + \left(r - \frac{\sigma^2}{2}\right)\left(t_i - \rho\sqrt{t_i t}\right)\right)}{\sigma\sqrt{t_i(1-\rho^2)}}. \tag{3.5.4}$$

As the differentiation of c.d.f. yields the p.d.f., therefore the conditional density function of q_i given $q_t = s_t$ satisfies the following equation:

$$f_{q_i|q_t=s_t}(x) = \frac{1}{x\sigma\sqrt{t_i(1-\rho^2)}}\phi(a(x)), \tag{3.5.5}$$

where ϕ denotes the p.d.f. of standard normal distribution.

Under the assumption that the mortality evolution process $\{q_t\}_{t\geq 0}$ is defined as $q_t = e^{U_t}$ where U_t is given in equation (3.5.1), the conditional expectation of q_i given q_t is given by the expression

$$\mathbf{E}(q_i|q_t) = \begin{cases} q_0\left(\frac{q_t}{q_0}\right)^{\frac{t_i}{t}} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} & t_i < t, \\ q_t e^{r(t_i-t)} & t_i \geq t. \end{cases} \tag{3.5.6}$$

We utilize this expression to obtain a lower bound for Asian call option under the Black-Scholes setting. Define: $S^{l_3} = \sum_{i=1}^n Y_i$, where exploiting (3.5.6), under the Black-Scholes case, Y_i, $i = 1, 2, ..., n$ are given by

$$Y_i = \begin{cases} 5q_0\left(\left(\frac{q_t}{q_0}\right)^{t_i/t} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} - 1.3\right)^+ & i < j \\ 5q_0\left(\left(\frac{q_t}{q_0}\right) e^{r(t_i-t)} - 1.3\right)^+ & i \geq j \end{cases}$$

Evidently, $\mathbf{Y} = (Y_1, \ldots, Y_n)$ is comonotonic and so we have

$$\mathbf{E}\left[\left(S^{l_3} - q_0\right)^+\right] = \sum_{i=1}^n \mathbf{E}\left[\left(Y_i - F^{-1}_{Y_i}\left(F_{S^{l_3}}(q_0)\right)\right)^+\right], \tag{3.5.7}$$

where $F_{S^{l_3}}(q_0)$ is the distribution function of S^{l_3} evaluated at q_0. For an arbitrary t, we have

$$\begin{aligned} F_{S^{l_3}}(q_0) &= \mathbf{P}\left[S^{l_3} \leq q_0\right] \\ &= \mathbf{P}\left[\sum_{i=1}^{j-1} 5q_0\left(\left(\frac{q_t}{q_0}\right)^{t_i/t} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} - 1.3\right)^+ + \sum_{i=j}^n 5q_0\left(\left(\frac{q_t}{q_0}\right) e^{r(t_i-t)} - 1.3\right)^+ \leq q_0\right] \\ &= \mathbf{P}\left[\sum_{i=1}^{j-1}\left(\left(\frac{q_t}{q_0}\right)^{t_i/t} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} - 1.3\right)^+ + \sum_{i=j}^n\left(\left(\frac{q_t}{q_0}\right) e^{r(t_i-t)} - 1.3\right)^+ \leq 0.2\right]. \end{aligned} \tag{3.5.8}$$

As in the previous section, we substitute x for q_t/q_0 and solve for x, using the equation:

$$\sum_{i=1}^{j-1}\left(x^{t_i/t} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} - 1.3\right)^+ + \sum_{i=j}^n\left(x e^{r(t_i-t)} - 1.3\right)^+ = 0.2. \tag{3.5.9}$$

This is indeed straight forward, noting that the left hand side of this equation is strictly increasing in x. This yields:

$$F_{S^{l_3}}(q_0) = F_{q_t}(xq_0) = \begin{cases} F_{Y_i}\left(5q_0\left(x^{t_i/t}e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} - 1.3\right)^+\right) & i < j, \\ F_{Y_i}\left(5q_0\left(xe^{r(t_i-t)} - 1.3\right)^+\right) & i \geq j. \end{cases}$$

Substituting this in equation (3.5.7), recalling the stop-loss order relationship between S and S^{l_2} as given by equation (3.3.30), applying it for S^{l_3}, splitting the terms and multiplying by the averaged discount factor as done in the last section and noting that the marginal cdfs F_{Y_i} are strictly increasing, we obtain

$$\begin{aligned}
P_1 \quad &\geq \quad De^{-rT}\left(\sum_{i=1}^{n}\mathbf{E}\left[\left(Y_i - F_{Y_i}^{-1}\left(F_{S^{l_3}}(q_0)\right)\right)^+\right]\right) \\
&= \quad Ce^{-rT}\left(\sum_{i=1}^{j-1}\mathbf{E}\left[\left(5\left(\left(\frac{q_t}{q_0}\right)^{t_i/t}e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} - 1.3\right)^+ - 5\left(x^{t_i/t}e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} - 1.3\right)^+\right)^+\right]\right. \\
&\qquad \left. + \sum_{i=j}^{n}\mathbf{E}\left[\left(5\left(\left(\frac{q_t}{q_0}\right)e^{r(t_i-t)} - 1.3\right)^+ - 5\left(xe^{r(t_i-t)} - 1.3\right)^+\right)^+\right]\right) \\
&= \quad 5Ce^{-rT}\left(\sum_{i=1}^{j-1}\frac{1}{q_0^{t_i/t}}\mathbf{E}\left[\left(q_t^{t_i/t}e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} - q_0^{t_i/t}\left(1.3 + \left(x^{t_i/t}e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} - 1.3\right)^+\right)\right)^+\right]\right. \\
&\qquad \left. + \sum_{i=j}^{n}\frac{e^{r(t_i-t)}}{q_0}\mathbf{E}\left[\left(q_t - q_0\left(\frac{1.3}{e^{r(t_i-t)}} + \left(x - \frac{1.3}{e^{r(t_i-t)}}\right)^+\right)\right)^+\right]\right) \\
&= \quad 5De^{-rT}\left(\sum_{i=1}^{j-1}q_0^{1-t_i/t}\mathbf{E}\left[\left(q_t^{t_i/t}e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} - q_0^{t_i/t}\max\left(x^{t_i/t}e^{\frac{\sigma^2 t_i}{2t}(t-t_i)}, 1.3\right)\right)^+\right]\right. \\
&\qquad \left. + \sum_{i=j}^{n}e^{rt_i}C\left(q_0\max\left(x, \frac{1.3}{e^{r(t_i-t)}}\right), t\right)\right).
\end{aligned} \tag{3.5.10}$$

We denote the term within the first summation as E_1 and its value is given below.

$$\mathbf{E}_1 = 5q_0\left(e^{rt_i}\Phi(d_{1ai}) - \max\left(x^{t_i/t}e^{\frac{\sigma^2 t_i}{2t}(t-t_i)}, 1.3\right)\Phi(d_{2ai})\right), \tag{3.5.11}$$

where d_{2ai} and d_{1ai} are given respectively as

$$d_{2ai} = \frac{-\log_e\left(\frac{da_i}{q_0}\right) + \left(r - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} \tag{3.5.12}$$

$$d_{1ai} = d_{2ai} + \sigma\frac{t_i}{\sqrt{t}} \tag{3.5.13}$$

and da_i is given as

$$da_i = q_0\left(\max\left(x^{t_i/t}, \frac{1.3}{e^{\frac{\sigma^2 t_i}{2t}(t-t_i)}}\right)\right)^{t/t_i}. \tag{3.5.14}$$

Inserting (3.5.11) in (3.5.10), we achieve the lower bound $\mathrm{lb}_t^{(BS)}$ as follows

$$P_1 \quad \geq \quad 5De^{-rT}\left(\sum_{i=1}^{j-1}q_0\left(e^{rt_i}\Phi(d_{1ai}) - \max\left(x^{t_i/t}e^{\frac{\sigma^2 t_i}{2t}(t-t_i)}, 1.3\right)\Phi(d_{2ai})\right)\right.$$

$$
+\sum_{i=j}^{n} e^{rt_i} C\left(q_0 \max\left(x,\ \frac{1.3}{e^{r(t_i-t)}}\right),\ t\right)\Bigg)
$$

$$
=: \quad \mathrm{lb}_t^{(BS)}. \tag{3.5.15}
$$

The bound $\mathrm{lb}_t^{(BS)}$ can undergo treatment similar to $\mathrm{lb}_t^{(2)}$ in sense of maximization with respect to t yielding

$$
P_1 \geq \max_{0\leq t\leq T} \mathrm{lb}_t^{(BS)}. \tag{3.5.16}
$$

An interesting comment in the passing is that as we calculate $\mathbf{E}\left[q_i|q_t\right]$ explicitly, rather than finding a lower bound for it, clearly $\mathrm{lb}_t^{(BS)}$ improves on $\mathrm{lb}_t^{(2)}$ in the case where $\{q_t\}$ follows the Black-Scholes model. Again, as before, exploiting the put-call parity,

$$
P \geq \left(\mathrm{lb}_t^{(BS)} - G\right)^+ =: \mathrm{SWLB}_t^{(BS)}, \tag{3.5.17}
$$

where G is defined in (3.2.17).

The Upper Bound $\mathbf{SWUB}_t^{(BS)}$

In section 3.4.2, we have shown that the upper bound SWUB_1 can be improved by assuming that there exists a random variable Λ such that $\mathrm{Cov}\left(X_i, \Lambda\right) \neq 0\ \forall i$. Suppose this assumption is true here and the mortality index $\{q_t\}_{t\geq 0}$ depends on an underlying standard Brownian motion $\{W_t\}_{t\in[0,T]}$. Then, from equation (3.4.25) noting that the marginal cdfs $F_{q_i|W_t=w}$ are strictly increasing so that $K_4 = 0$, we see that an upper bound for the call counterpart of the Swiss Re bond is given as

$$
P_1 \leq 5De^{-rT}\sum_{i=1}^{n}\int_{-\infty}^{\infty} \mathbf{E}\left[\left(q_i - F^{-1}_{q_i|W_t=w}(x)\right)^+ \middle| W_t = w\right] d\Phi\left(\frac{w}{\sqrt{t}}\right), \tag{3.5.18}
$$

where using (3.4.26), we see that x is obtained by solving the following equation

$$
\sum_{i=1}^{n} F^{-1}_{q_i|W_t=w}(x) = \frac{q_0}{5}\left(1 + 6.5n\right). \tag{3.5.19}
$$

An explicit formula for the conditional inverse distribution function of q_i given the event $W_t = w$, is provided by the following result.

Proposition 20. *Under the assumptions of the Black-Scholes model, conditional on the event $W_t = w$, the conditional distribution function of q_i is given by*

$$
F^{-1}_{q_i|W_t=w} = \begin{cases} q_0 e^{\left(r-\frac{\sigma^2}{2}\right)t_i + \sigma\frac{t_i}{t}w + \sigma\sqrt{\frac{t_i}{t}(t-t_i)}\,\Phi^{-1}(x)} & i < j, \\ q_0 e^{\left(r-\frac{\sigma^2}{2}\right)t_i + \sigma w + \sigma\sqrt{(t_i-t)}\,\Phi^{-1}(x)} & i \geq j. \end{cases} \tag{3.5.20}
$$

where $j = min\{i : t_i \geq t\}$.

Proof. Let us set $X = \sigma W_{t_i}$, $Y = W_t$ and $y = e^w$ in (3.5.3). Then we obtain the following expression for the conditional distribution function of $e^{\sigma W_{t_i}}$ given the event $W_t = w$.

$$
F_{e^{\sigma W_{t_i}}|W_t=w}(s) = \Phi\left(\frac{\log_e s - \rho\sigma\sqrt{\frac{t_i}{t}}w}{\sigma\sqrt{t_i\left(1-\rho^2\right)}}\right). \tag{3.5.21}
$$

It then follows that $F_{e^{\sigma W_{t_i}}|W_t=w}(s) = x$ if and only if

$$
s = F^{-1}_{e^{\sigma W_{t_i}}|W_t=w}(x) = e^{\rho\sigma\sqrt{\frac{t_i}{t}}w + \sigma\sqrt{t_i(1-\rho^2)}\,\Phi^{-1}(x)}
$$

We can then obtain equation (3.5.20) by noting that $\rho = \sqrt{(t_i \wedge t)(t_i \vee t)}$ and the following expression for the inverse conditional distribution function of q_i given $W_t = w$.

$$F^{-1}_{q_i|W_t=w} = q_0 e^{\left(r-\frac{\sigma^2}{2}\right)t_i} F^{-1}_{e^{\sigma W_{t_i}}|W_t=w}$$

This completes the proof. $\square$

It is of note that $F^{-1}_{q_i|W_t=w}$ is continuous when $t = t_i$ (that is if, for some i, we have $i = j$). From equation (3.5.19), we then wish to solve the following for x.

$$\sum_{i=1}^{j-1} e^{\left(r-\frac{\sigma^2}{2}\right)t_i+\sigma\frac{t_i}{t}w+\sigma\sqrt{\frac{t_i}{t}(t-t_i)}\Phi^{-1}(x)} + \sum_{i=j}^{n} e^{\left(r-\frac{\sigma^2}{2}\right)t_i+\sigma w+\sigma\sqrt{(t_i-t)}\Phi^{-1}(x)} = 0.2 + 1.3n. \quad (3.5.22)$$

As a result, using equation(3.5.18), the improved upper bound for the call counterpart of the Swiss Re bond in the Black-Scholes case is given by the following set of equations

$$\begin{aligned} P_1 &\leq 5Ce^{-rT}\int_{-\infty}^{\infty}\left(\sum_{i=1}^{n} e^{\left(r-\frac{\sigma^2(t_i\wedge t)^2}{2t_i t}\right)t_i+\sigma\frac{t_i\wedge t}{t}w}\Phi\left(c_1^{(i)}\right) - (0.2+1.3n)(1-x)\right)d\Phi\left(\frac{w}{\sqrt{t}}\right) \\ &=: \mathrm{ub}_t^{(BS)}, \end{aligned} \quad (3.5.23)$$

$$c_1^{(i)} = \begin{cases} \sigma\sqrt{\frac{t_i}{t}(t-t_i)} - \Phi^{-1}(x) & i < j, \\ \sigma\sqrt{(t_i-t)} - \Phi^{-1}(x) & i \geq j. \end{cases} \quad (3.5.24)$$

where $x \in (0,1)$ solves equation (3.5.22). The optimal upper bound in this case is then given by minimising equation (3.5.23) over $t \in [0,T]$. As before, invoking the put-call parity of section 3.2.2, we have for the Swiss Re bond

$$P \leq \left(\mathrm{ub}_t^1 - G\right)^+ =: \mathrm{SWUB}_t^{(BS)}, \quad (3.5.25)$$

where G is defined in (3.2.17).

3.5.2 Log Gamma Distribution

The log Gamma distribution is a particular type of transformed Gamma distribution. The mortality index 'q_t', $t = 1, 2, 3$ follows log Gamma distribution if

$$\frac{\log_e q_t - \mu_t}{\sigma_t} = x_t \sim Gamma\,(p_t,\ a_t)\,, \quad (3.5.26)$$

where μ_t, σ_t, p_t and a_t are parameters (> 0) and *log* is the natural logarithm. Useful references for reading about transformed gamma distribution are Johnson et al. (1994), Vitiello and Poon (2010) and Cheng et al. (2014).

The Lower Bound $\mathrm{SWLB}_t^{(LG)}$

In this case the marginal cdfs F_{Y_i} are strictly increasing. So, for the log-gamma distribution we obtain the following compact expression for $lb_t^{(2)}$ and then subtract G from it to obtain $\mathrm{SWLB}_t^{(LG)}$ (c.f. Appendix A.6 for option pricing under log Gamma distribution).

$$\begin{aligned} \mathrm{lb}_t^{(2)} &= 5Ce^{-rT}\left(\sum_{i=1}^{j-1} q_0^{-t_i/t}\left(\frac{e^{\frac{t_i}{t}\mu_i}}{(\sigma_i^{"})^{p_i}}\left[1-G\left(d'_{2i},\ p_i,\sigma_i^{"}\right)\right] - K_{1i}\left[1-G\left(d'_{2i},\ p_i\right)\right]\right)\right. \\ &\left.+\sum_{i=j}^{n}\frac{e^{r(t_i-t)}}{q_0}\left(q_0e^{rt}\left[1-G\left(d_{1i},\ p_i\right)\right] - K_{2i}\left[1-G\left(d_{2i},\ p_i\right)\right]\right)\right) \end{aligned} \quad (3.5.27)$$

where we have for $i = 1, 2, ..., n$

$$\sigma_i^{"} = 1 - \sigma_i^{'}\frac{t_i}{t},\ \sigma_i^{'} = 1 - \left(q_0 e^{rt_i - \mu_i}\right)^{1/p_i},\ d_{2i}^{'} = \frac{\log_e d_{1i}^{'} - \mu_i}{\sigma_i}$$

,

$$d_{1i}^{'} = q_0 \max\left(x^{t_i/t}, 1.3\right)^{t/t_i},\ K_{1i} = \left(d_{1i}^{'}\right)^{t_i/t},$$

$$K_{2i} = q_0 \max\left(x, \frac{1.3}{e^{r(t_i - t)}}\right),\ d_{1i} = \frac{\log_e K_{2i} - \mu_i}{q_0 e^{rt - \mu_i} - 1},\ d_{2i} = d_{1i} + \log_e K_{2i} - \mu_i,$$

$$G(x, p_i) = \int_0^x \frac{1}{\Gamma(p_i)} x^{p_i - 1} e^{-x} dx,$$

$$G\left(x, p_i, \sigma_i^{"}\right) = \int_0^x \frac{\left(\sigma_i^{"}\right)^{p_i}}{\Gamma(p_i)} x^{p_i - 1} e^{-\left(\sigma_i^{"} x\right)} dx$$

The numerical calculations based on the theory are presented in Chapter 6 while a summary of the key findings and further research avenues is furnished in Chapter 7. We end this chapter with some interesting reflections.

3.6 Final Remarks

The theory developed in this chapter presents a key breakthrough in the pricing of catastrophic mortality bonds. For the first time in the history of pricing of these interesting financial instruments, we have been successful in devising *model-free* or *model-independent* price bounds. The term model-free itself deserves an explanation. In a *model-based* approach, exotic option prices are determined via simulation techniques or via an appropriate approximation technique (also see Linders et al., 2012). A second approach consists of computing lower and upper bounds for European-type exotic options, which are only based on available market information, without postulating any specific model for the underlying mortality index (or assets in general). Such an approach is called *model-free*. All the lower bounds and the first upper bound are model-free subject to availability of appropriate European call option prices written on the mortality index of the catastrophic mortality bond. Although these securities are hypothetical today, they are likely to be introduced soon in the future with the development of a more voluminous and liquid life market. The only word of caution here is that we are assuming an *infinite market case*, where the prices of the options on the mortality index of the catastrophic mortality bond are available for all strikes. An interesting exercise for the future could be to look at the *finite market case.*

Another important observation here is that the improved upper bound for Swiss Re mortality bond developed in this chapter, though not model-free in the sense mentioned above, since it depends on the additional information of the conditional distribution of the mortality index given an additional random variable, has been derived in the most general set up without keeping in mind any particular model and therefore qualifies to be 'model-independent' or 'model-robust' in this sense. Even in a model-based approach, determining the price of catastrophic mortality bonds analytically is a cumbersome exercise, mainly because of the dependence that exists between the mortality indices of various years comprising the term of the bond. Therefore, the use of easily computable lower and upper bounds in terms of the option prices involved may also in a model-based approach be very helpful as an approximation for the real price of the CATM bond. The power of all the derived bounds also stems from the ease of their calculation since they are much faster to compute in comparison to the competing Monte Carlo estimates particularly in the case of complex models. In a nutshell, through this research, we have been successful in adding powerful tools to the armoury of re-insurers like Swiss Re who could implement this methodology for many different scenarios/models (due to its fast implementation) so as to calculate the 'worst case bounds' for a large class of models and in this way reduce model risk.

Chapter 4

Affine Processes

The aim of this chapter is to draw a comprehensive picture of the theory of affine processes with a view to utilize them in the pricing of actuarial products such as the guaranteed annuity options (GAOs). The affine class is a particular class of continuous time Markov processes such that the Fourier-Laplace transform of the marginal distributions depends on the initial state in an exponential affine way. These processes have a high degree of analytical tractability in the sense that the expected value of a large class of payoff functions is explicitly known. This is the basic reason why many of the well-known models used in mathematical finance fall in this class. Also as a particular class of jump diffusions, they allow for simultaneous modeling of diffusive, jump and stochastic volatility phenomena in a multivariate set up.

From the mathematical perspective, the affine class contains Lévy processes, Ornstein-Uhlenbeck processes, continuous branching processes and their multidimensional generalizations such as the matrix-valued Wishart processes. More specifically, the classical bond pricing models of Vasicek (1977) and Cox-Ingersoll-Ross (CIR) (c.f. Cox et al., 1985), as well as the stochastic volatility model of Heston (1993), all exhibit the affine property. The origin of discussion of affine processes dates back to a paper by Kawazu and Watanabe (1971), which studies processes arising as continuous-time limit of Galton-Watson branching processes with immigration. The resulting class of processes is called CBI (continuously branching with immigration), and exhibits the mentioned 'affine property' in one dimension. The requirement of more complex models encouraged the introduction of higher dimensional affine jump diffusions on the so called canonical state space $\mathbb{R}_+^m \times \mathbb{R}^{n-m}$ and one can refer to Duffie and Kan (1996), Dai and Singleton (2000), Duffie et al. (2000) and Duffie et al. (2003) for details.

The development of multivariate stochastic volatility models has lead to the germination of applications of affine processes on non-canonical state spaces, in particular on the cone of positive semi-definite matrices. The study of such matrix-valued affine processes was pioneered by Bru (1989) and Bru (1991) who introduced the so-called Wishart processes, which are multidimensional analogs of squared Bessel processes. A natural generalization of positive semi-definite matrices are so-called symmetric cones, which are cones of squares in Jordan algebras (see the work of Faraut and Koranyi, 1994). Grasselli and Tebaldi (2008) were the first ones to introduce affine diffusion processes on these symmetric cones with the objective to highlight the methodology of solving the corresponding Riccati differential equations. A brilliant discussion of affine processes on the cone of positive semi-definite matrices appears in Cuchiero et al. (2011). Interested readers can also refer to Keller-Ressel and Mayerhofer (2015) for details regarding a unified approach on affine processes.

Affine processes have been used for modeling mortality and pricing actuarial products by a number of authors like Dahl (2004), Biffis (2005), Denuit and Devolder (2006), Luciano and Vigna (2005), Schrager (2006), Cairns et al. (2006a) and Luciano and Vigna (2008). To the best of our knowledge Deelstra et al. (2016) are the only ones to utiliize Wishart processes for modeling the force of mortality and interest rate to examine the role of the dependence between mortality and interest rates when pricing GAOs.

In the next chapter we present an innovative application of the theory of affine processes to devise tight bounds for Guaranteed Annuity Options. In the next few sections we present the background notation and set up to build up the required mathematical formulation.

4.1 Notations

For the stochastic foundations and notations we refer to standard text books such as Jacod and Shiryaev (2003) and Revuz and Yor (1999) and the for the notations in context of affine processes, we refer to Cuchiero (2011) and Cuchiero et al. (2011).

We write $\mathbb{R}_+$ for $[0,\infty)$, $\mathbb{R}_{++}$ for $(0,\infty)$ and $\mathbb{Q}_+$ for non-negative rational numbers. Moreover, let V denote an n-dimensional real vector space with scalar product $\langle \cdot, \cdot \rangle$. Further let $S(V)$ and $S_+(V)$ denote respectively symmetric matrices and the positive semi-definite matrices over V and $\mathcal{L}(V)$ correspond to the space of linear maps on V. For the case $V = \mathbb{R}_d$, we write respectively S_d and S_d^+ for $S(V)$ and $S_+(V)$, and I_d denotes the $d \times d$ identity matrix. Let S_d^{++} denote the interior of the cone of positive semi-definite symmetric $d \times d$ matrices. Further let M_d denote the set of real $d \times d$ matrices while GL_d denote the set of invertible real $d \times d$ matrices. The symbol $\| \cdot \|$ denotes the norm induced by the scalar product $\langle \cdot, \cdot \rangle$. Also, in line with the literature, let $\mathcal{C}^2$ denote the class of twice continuously differentiable functions.

On the lines of Cuchiero (2011), we define affine processes as a particular class of time-homogeneous Markov processes with state space $D \subseteq V$, which is some closed, non-empty subset of the real vector space V. $\mathcal{D}$ denotes the Borel σ-algebra of D. The next section reviews the basic ingredients of the theory of time-homogeneous Markov processes and contains all the necessary definitions.

4.2 The Basic Set Up

We shall not assume the underlying process to be conservative and so we adjoin to the state space D a point Δ, called cemetery state, and set $D_\Delta = D \cup \{\Delta\}$ as well as $\mathcal{D}_\Delta = \sigma(\mathcal{D}, \{\Delta\})$. We make the convention $\| \Delta \| := \infty$ and set $f(\Delta) = 0$ for any other function f on D.

Consider the following objects on a space Ω:

(i) a stochastic process $X = (X_t)_{t \geq 0}$ taking values in $\mathcal{D}_\Delta$ such that

if $X_s(\omega) = \Delta$, then $X_t(\omega) = \Delta$ for all $t \geq s$ and all $\omega \in \Omega$;

(ii) the filtration generated by X, that is, $\mathcal{F}_t^0 = \sigma(X_s, s \leq t)$, where we set $\mathcal{F}^0 = \bigvee_{t \in \mathbb{R}_+} \mathcal{F}_t^0$

(iii) a family of probability measures $(\mathbb{P}_x)_{x \in D_\Delta}$ on $(\Omega, \mathcal{F}^0)$.

Definition 21. Markov Process *A time-homogeneous Markov process*

$$X = \left(\Omega, (\mathcal{F}_t^0)_{t \geq 0}, (X_t)_{t \geq 0}, (p_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in D_\Delta}\right)$$

with state space $(D, \mathcal{D})$ (augmented by Δ) is a D_Δ - valued stochastic process such that, for all $s, t \geq 0$, $x \in D_\Delta$ and all bounded D_Δ-measurable functions $f : D_\Delta \to \mathbb{R}$,

$$\mathbb{E}_x\left[f(X_{t+s}) | \mathcal{F}_s^0\right] = \mathbb{E}_{X_s}\left[f(X_t)\right], \quad \mathbb{P}_x\text{-}a.s. \tag{4.2.1}$$

where $\mathbb{E}_x$ denotes the expectation with respect to $\mathbb{P}_x$ and $(p_t)_{t \geq 0}$ is a transition function on $(D_\Delta, \mathcal{D}_\Delta)$. A transition function is a family of kernels $p_t : D_\Delta \times \mathcal{D}_\Delta \to [0,1]$ such that *(i) for all $t \geq 0$ and $x \in D_\Delta$, $p_t(x,\cdot)$ is a measure on $\mathcal{D}_\Delta$ with $p_t(x,D) \leq 1$, $p_t(x,\{\Delta\}) = 1 - p_t(x,D)$ and $p_t(\Delta,\{\Delta\}) = 1$;*

(ii) for all $x \in D_\Delta$, $p_0(x,\{\Delta\}) = \delta_x(\cdot)$, where $\delta_x(\cdot)$ denotes the Dirac measure;

(iii) for all $t \geq 0$ and $\Gamma \in D_\Delta$, $x \mapsto p_t(x,\Gamma)$ is $\mathcal{D}_\Delta$-measurable;

(iv) for all $s,t \geq 0$, $x \in D_\Delta$, and $\Gamma \in D_\Delta$, the Chapman-Kolmogrov equation holds, that is,

$$p_{t+s}(x,\Gamma) = \int_{\mathcal{D}_\Delta} p_s(x,d\xi)\, p_t(\xi,\Gamma).$$

If $(\mathcal{F}_t)_{t\geq 0}$ is a filtration with $\mathcal{F}_t^0 \subset \mathcal{F}_t$, $t \geq 0$ then X is a time-homogeneous Markov process relative to $(\mathcal{F}_t)$ if (4.2.1) holds with $\mathcal{F}_s^0$ replaced by $\mathcal{F}_s$.

Another way to look at the transition function is to induce a measurable contraction semi-group $(P_t)_{t\geq 0}$ defined by

$$P_t f(x) := \mathbb{E}_x[f(X_t)] = \int_D f(\xi)\, p_t(x,d\xi), \quad x \in D_\Delta,$$

for all bounded $\mathcal{D}_\Delta$-measurable functions $f : D_\Delta \to \mathbb{R}$.

We now define an *affine process* assuming that $V = \mathbb{R}_d$.

Definition 22. Affine Process *A time-homogeneous Markov process X relative to some filtration $(\mathcal{F}_s)$ and with state space $(D,\mathcal{D})$ (augmented by Δ) is called affine if*

(i) it is stochastically continuous, that is, $\lim_{s\to t} p_s(x,\cdot) = p_t(x,\cdot)$ for all $t \geq 0$ and $x \in D$, and

(ii) its Fourier-Laplace transform has exponential affine dependence on the initial state. This means that there exist functions $\phi : \mathbb{R}_+ \times S_d^+ \to \mathbb{R}_+$ and $\psi : \mathbb{R}_+ \times S_d^+ \to S_d^+$ such that

$$\mathbb{E}_x\left[e^{\langle u,X_t\rangle}\right] = P_t e^{\langle u,x\rangle} = \int_D e^{\langle u,\xi\rangle} p_t(x,d\xi) = e^{-\phi(t,u)-\langle\psi(t,u),x\rangle}, \tag{4.2.2}$$

for all $x \in D$ and $(t,u) \in \mathbb{R}_+ \times \mathbb{R}_d$

Remark 1. *The stochastic continuity of X implies that*
(i) $\phi(t,u)$ and $\psi(t,u)$ are jointly continuous in (t,u) (c.f. Lemma 3.2(iii) in Cuchiero et al., 2011).

(ii) the process X is regular in the following sense (c.f. Proposition 3.4 in Cuchiero et al., 2011).

Definition 23. Regularity *An affine process X is called regular if the derivatives*

$$\Im(u) = \left.\frac{\partial\phi(t,u)}{\partial t}\right|_{t=0+}, \quad \Re(u) = \left.\frac{\partial\psi(t,u)}{\partial t}\right|_{t=0+} \tag{4.2.3}$$

exist and are continuous at $u = 0$.

An alternative characterization of the affine property in the sense of Cuchiero et al. (2011) can be provided in terms of the *generator* of the process X, a formal definition of which is given below. One can refer to Alfonsi (2015) for a detailed discussion on generators.

Definition 24. Generator *For an affine process X taking values in $S_d^+ \subset S_d$ the infinitesimal generator is defined as*

$$\mathcal{A}f(x) = \lim_{t \to 0^+} \frac{\mathbb{E}\left[f\left(X_t^x\right)\right] - f(x)}{t} \text{ for } x \in S_d^+,\ f \in \mathcal{C}^2\left(S_d, \mathbb{R}_d\right) \text{ with bounded derivatives.} \tag{4.2.4}$$

The diffusion, drift, jump and killing characteristics of an affine process X depend in an affine way on the underlying state as shown in Theorem 26 below. To proceed to this result we need the following important definition.

Definition 25. Truncation Function *Let $\chi : S_d \to S_d$ be some bounded continuous truncation function with $\chi(\xi) = \xi$ in the neighborhood of 0. An admissible parameter set given by $\left(\alpha, b, \beta^{ij}, c, \gamma, m, \mu\right)$ associated with χ consists of:*

- *a linear diffusion coefficient*
$$\alpha \in S_d^+, \tag{4.2.5}$$
- *a constant drift term*
$$b \succeq (d-1)\,\alpha, \tag{4.2.6}$$
- *a constant killing rate term*
$$c \in \mathbb{R}^+, \tag{4.2.7}$$
- *a linear killing rate coefficient*
$$\gamma \in S_d^+, \tag{4.2.8}$$
- *a constant jump term: a Borel measure m on $S_d^+ \setminus \{0\}$ satisfying*
$$\int_{S_d^+ \setminus \{0\}} \left(\parallel \xi \parallel \wedge 1\right) m\left(d\xi\right) < \infty, \tag{4.2.9}$$
- *a linear jump coefficient: a $d \times d$ matrix $\mu = \left(\mu_{ij}\right)$ of finite signed measures on $S_d^+ \setminus \{0\}$ such that $\mu(E) \in S_d^+$ for all $E \in \mathcal{B}\left(S_d^+ \setminus \{0\}\right)$ and the kernel*
$$M\left(x, d\xi\right) := \frac{\langle x, \mu\left(d\xi\right)\rangle}{\parallel \xi \parallel^2 \wedge 1} \tag{4.2.10}$$
satisfies
$$\int_{S_d^+ \setminus \{0\}} \langle \chi(\xi), u\rangle M\left(x, d\xi\right) < \infty \text{ for all } x, u \in S_d^+ \text{ with } \langle x, u\rangle = 0, \tag{4.2.11}$$
- *a linear drift coefficient: a family $\beta^{ij} = \beta^{ji} \in S_d$ such that the linear map $B : S_d \to S_d$ of the form*
$$B(x) = \sum_{i,j} \beta^{ij} x_{ij} \tag{4.2.12}$$
satisfies
$$\langle B(x), u\rangle - \int_{S_d^+ \setminus \{0\}} \langle \chi(\xi), u\rangle M\left(x, d\xi\right) \geq 0 \text{ for all } x, u \in S_d^+ \text{ with } \langle x, u\rangle = 0. \tag{4.2.13}$$

We now state without proof Theorem 2.4 of Cuchiero et al. (2011) which we require to make further advancement with the application of the theory of affine process in pricing actuarial products like GAOs. Let $\mathcal{D}(\mathcal{A})$ denote the domain of the generator $\mathcal{A}$ and this is taken to be the space $\mathcal{S}_+$ of rapidly decreasing C^∞-functions on S_d^+.

Theorem 26. *Suppose X is an affine process on S_d^+. Then X is regular and has the Feller property. Let $\mathcal{A}$ be the infinitesimal generator on $C_0\left(S_d^+\right)$. Then $\mathcal{S}_+ \subset \mathcal{D}(\mathcal{A})$ and there exists an admissible parameter set $\left(\alpha, b, \beta^{ij}, c, \gamma, m, \mu\right)$ such that for $f \in \mathcal{S}_+$, we have*

$$\begin{aligned}
\mathcal{A} f(x) = & \frac{1}{2} \sum_{i,j,k,l} A_{ijkl}(x) \frac{\partial^2 f(x)}{\partial x_{ij} \partial x_{kl}} \\
& + \sum_{i,j} \left(b_{ij} + B_{ij}(x)\right) \frac{\partial f(x)}{\partial x_{ij}} - \left(c + \langle \gamma, x \rangle\right) f(x) \\
& + \int_{S_d^+ \backslash \{0\}} \left(f(x+\xi) - f(x)\right) m(d\xi) \\
& + \int_{S_d^+ \backslash \{0\}} \left(f(x+\xi) - f(x) - \langle \chi(\xi), \nabla f(x) \rangle\right) M(x, d\xi), \qquad (4.2.14)
\end{aligned}$$

where $B(x)$ is defined in (4.2.12), $M(x, d\xi)$ by (4.2.10) and

$$A_{ijkl}(x) = x_{ik}\alpha_{jl} + x_{il}\alpha_{jk} + x_{jk}\alpha_{il} + x_{jl}\alpha_{ik}. \qquad (4.2.15)$$

Moreover $\phi(t,u)$ and $\psi(t,u)$ in Definition 22 solve the generalized Riccatti differential equations for $u \in S_d^+$,

$$\frac{\partial \phi(t,u)}{\partial t} = \Im\left(\psi(t,u)\right), \quad \phi(0,u) = 0, \qquad (4.2.16)$$

$$\frac{\partial \psi(t,u)}{\partial t} = \Re\left(\psi(t,u)\right), \quad \psi(0,u) = u, \qquad (4.2.17)$$

with

$$\Im(u) = \langle b, u \rangle + c - \int_{S_d^+ \backslash \{0\}} \left(e^{-\langle u, \xi \rangle} - 1\right) m(d\xi) \qquad (4.2.18)$$

and

$$\begin{aligned}
\Re(u) = & -2u\alpha u + B^T(u) + \gamma \\
& - \int_{S_d^+ \backslash \{0\}} \left(\frac{e^{-\langle u, \xi \rangle} - 1 + \langle \chi(\xi), u \rangle}{\| \xi \|^2 \wedge 1}\right) \mu(d\xi), \qquad (4.2.19)
\end{aligned}$$

where $B_{ij}^T(u) = \langle \beta^{ij}, u \rangle$.

Conversely, let $\left(\alpha, b, \beta^{ij}, c, \gamma, m, \mu\right)$ be an admissible parameter set. Then there exists a unique affine process on S_d^+ with infinitesimal generator given in (4.2.14) and (4.2.2) holds for all $(t,u) \in \mathbb{R}_+ \times S_d^+$, where $\phi(t,u)$ and $\psi(t,u)$ are given by (4.2.16) and (4.2.17).

Remark 2. *(c.f. Remark 2.5 in Cuchiero et al., 2011) A sufficient condition for X to be conservative is $c = 0$ and $\gamma = 0$ and*

$$\int_{S_d^+ \cap \{\|\xi\| \geq 1\}} \| \xi \| \left(\mu_{ij}^+(d\xi) + \mu_{ij}^-(d\xi)\right) < \infty \text{ for all } 1 \leq i < j \leq d,$$

where $\mu_{ij} = \mu_{ij}^+ - \mu_{ij}^-$ denotes the Jordan decomposition of μ_{ij}.

In the next sections, we move to the application of the theory of affine processes to price Zero Coupon Bonds by considering the short rate to be driven by a stochastic process on the cone of positive semi-definite matrices and assuming them to be correlated. In the next chapter we utilize this theory to price 'Survival Zero Coupon Bonds' or in short SZCBs and then use the technique to determine a fair price for GAOs. A good reference to build the foundations is Gnoatto (2012).

4.3 The General Affine Pricing Model

4.3.1 The Set Up

To apply the theory of affine process to the pricing of actuarial products, we focus on models where the short rate is given as

$$r_t = \bar{r} + \langle R, X_t \rangle \tag{4.3.1}$$

where $\bar{r} \in \mathbb{R}_+$, $R \in \mathbb{R}_d$ and X is a time homogeneous affine Markov process taking values in a non-empty convex subset E of $\mathbb{R}_d$, $(d \geq 1)$ endowed with the scalar product $\langle, \cdot, \cdot \rangle$. This means that the interest rate is a linear projection of X along constant directions given by the parameter R. We will be interested in the cases where the X is a classical affine process on the state space $\mathbb{R}_+^m \times \mathbb{R}^n$ or an affine Wishart process on the state space S_d^+. In the latter case (4.3.1) is called the Wishart short rate model and $R \in S_d^+$. In the next chapter we use a similar set up to model force of mortality and then price actuarial products under the assumption of correlation between interest rate and mortality rate.

In the next section we will provide a fairly general formula for zero coupon bonds for this family of models. Then we will derive the same under Wishart short rate model. We will follow the approach used in Gnoatto (2012). A similar type of model appears in Gourieroux and Sufana (2003) and Grasselli and Tebaldi (2008). Buraschi et al. (2008) have examined the properties of this model with respect to many issues concerning the yield curve and interest rate derivatives. Chiarella et al. (2010) have analysed the impact of specification of the specification of the risk premium in regards to this model. Deelstra et al. (2016) use this model to price actuarial products assuming that interest rate and mortality rate are correlated.

4.3.2 Zero Coupon Bond Pricing

Before we begin the mathematical formulation of bond pricing, it is very important to discuss the choice of the risk neutral measure that will be used for pricing purpose. Clearly,the short rate is not a traded asset, hence the bond market is arbitrage free but not complete. This implies the existence of many risk neutral pricing measures. Following the standard practice, it is appropriate to assume that the right risk neutral measure $\mathbb{Q}$ can be chosen on the basis of available market data and so will result from a calibration procedure.

We now state without proof a slight modification of Proposition 2.1 of Gnoatto (2012) which provides the necessary methodology to compute the price of a zero coupon bond under the model (4.3.1).

Proposition 27. *Let X be a conservative affine process on S_d^+ under the risk neutral measure $\mathbb{Q}$. Let the short rate be given in accordance with* (4.3.1)*. Let $\tau = T - t$, then the price of a zero-coupon bond is given by*

$$\begin{aligned} P(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T(\bar{r}+\langle R, X_u\rangle)du} | \mathcal{F}_t\right] \\ &= e^{-\bar{r}\tau} e^{-\tilde{\phi}(\tau,R) - \langle \tilde{\psi}(\tau,R), X_t\rangle}, \end{aligned} \tag{4.3.2}$$

where $\tilde{\phi}$ and $\tilde{\psi}$ satisfy the following Ordinary Differential Equations (ODEs):

$$\frac{\partial \tilde{\phi}}{\partial \tau} = \tilde{\Im}\left(\tilde{\psi}(\tau,R)\right), \quad \tilde{\phi}(0,R) = 0, \tag{4.3.3}$$

$$\frac{\partial \tilde{\psi}}{\partial \tau} = \tilde{\Re}\left(\tilde{\psi}(\tau,R)\right), \quad \tilde{\psi}(0,R) = 0, \tag{4.3.4}$$

with

$$\tilde{\Im}\left(\tilde{\psi}(\tau,R)\right) = \langle b, \tilde{\psi}(\tau,R)\rangle - \int_{S_d^+\setminus\{0\}} \left(e^{-\langle\tilde{\psi}(\tau,R),\xi\rangle} - 1\right) m(d\xi), \tag{4.3.5}$$

and

$$\tilde{\Re}\left(\tilde{\psi}\left(\tau,R\right)\right) = -2\tilde{\psi}\left(\tau,R\right)\alpha\tilde{\psi}\left(\tau,R\right)+B^{T}\left(\tilde{\psi}\left(\tau,R\right)\right) - \int_{S_d^+\setminus\{0\}}\left(\frac{e^{-\langle\tilde{\psi}(\tau,R),\xi\rangle}-1+\langle\chi\left(\xi\right),\tilde{\psi}\left(\tau,R\right)\rangle}{\parallel\xi\parallel^{2}\wedge 1}\right)\mu\left(d\xi\right)+R \quad (4.3.6)$$

We now consider the case where the process X in (4.3.1) is a Wishart Process and then apply the above proposition to obtain the zero coupon bond price under the Wishart set up.

4.4 The Wishart Short Rate Model

4.4.1 The Set Up

In this section, we assume that the affine process (X_t) is a d-dimensional Wishart process. Given a $d\times d$ matrix Brownian motion W (i.e a matrix whose entries are independent Brownian motions) the Wishart process X_t (without jumps) is defined as the solution of the $d\times d$-dimensional stochastic differential equation

$$dX_t=\left(\beta Q^TQ+HX_t+X_tH^T\right)dt+\sqrt{X_t}dW_tQ+Q^TdW_t^T\sqrt{X_t},\ t\geq 0, \quad (4.4.1)$$

where $X_0=x\in S_d^+$, $\beta\geq d-1$, $H\in M_d$, $Q\in GL_d$ and Q^T denotes its transpose. In short, we assume that the law of X_t is $WIS_d\left(x_0,\beta,H,Q\right)$.

4.4.2 Existence and Uniqueness of Solution

This process was pioneered by Bru (1991) and she showed the existence and uniqueness of a weak solution for equation (4.4.1). She also established the existence of a unique strong solution taking values in S_d^{++}, i.e. the interior of the cone of positive semi-definite symmetric $d\times d$ matrices that we have denoted by S_d^+.

4.4.3 Generator

Bru (1991) has calculated the infinitesimal generator of the Wishart process as:

$$\mathcal{A}=Tr\left(\left(\beta Q^TQ+Hx+xH^T\right)D^S+2xD^SQ^TQD^S\right), \quad (4.4.2)$$

where Tr stands for trace and $D^S=\left(\partial/\partial x_{ij}\right)_{1\leq i,j\leq d}$. A good reference for understanding the detailed derivation of this generator is Alfonsi (2015).

4.4.4 Zero Coupon Bond Pricing

Continuing with the same ideas as in Section 4.3.2 we prove the main result of this chapter, which gives us an explicit formula for calculating the zero coupon bond price under the Wishart short rate model.

Theorem 28. *Let the short rate be given in accordance with equation* (4.3.1) *as*

$$r_t=\bar{r}+Tr\left(R,X_t\right) \quad (4.4.3)$$

for a process X_t with law $WIS_d\left(x_0,\beta,H,Q\right)$. Let $R\in S_d^{++}$ and $\tau=T-t$, then the price of a zero-coupon bond under the Wishart short rate model (4.4.3) *is given by*

$$\begin{aligned} P_t\left(\tau\right) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T\left(\bar{r}+Tr\left(RX_u\right)\right)du}|\mathcal{F}_t\right] \\ &= e^{-\tilde{\phi}(\tau,R)-Tr\left(\tilde{\psi}(\tau,R)X_t\right)}, \end{aligned} \quad (4.4.4)$$

where $\tilde{\phi}$ and $\tilde{\psi}$ satisfy the following system of ODEs:

$$\begin{cases} \frac{\partial \tilde{\phi}}{\partial \tau} = Tr\left[\beta Q^T Q \tilde{\psi}(\tau, R)\right] + \bar{r}, \\ \tilde{\phi}(0, R) = 0, \\ \frac{\partial \tilde{\psi}}{\partial \tau} = \tilde{\psi}(\tau, R) H + H^T \tilde{\psi}(\tau, R), \\ \qquad -2\tilde{\psi}(\tau, R) Q^T Q \tilde{\psi}(\tau, R) + R, \\ \tilde{\psi}(0, R) = 0. \end{cases} \tag{4.4.5}$$

Proof. A detailed proof appears in Chapter 5 for the integrated market framework. □

Remark 3. *The methodology of solving the system of Riccati equations given in* (4.4.5) *appears in Da Fonseca et al. (2008) where the authors propose that matrix Riccati equations can be linearized by doubling the dimension of the problem, Interested readers can also refer to Grasselli and Tebaldi (2008) and Deelstra et al. (2016). We state without proof the solution in the following proposition.*

Proposition 29. *The functions $\tilde{\phi}$ and $\tilde{\psi}$ in Theorem 28 are given by*

$$\begin{cases} \tilde{\psi}(\tau, R) = A_{22}^{-1}(\tau) A_{21}(\tau), \\ \tilde{\phi}(\tau) = \frac{\beta}{2}\left(\log\left(\det\left(A_{22}(\tau)\right)\right) + \tau Tr\left[H^T\right]\right). \end{cases} \tag{4.4.6}$$

where

$$\begin{bmatrix} A_{11}(\tau) & A_{12}(\tau) \\ A_{21}(\tau) & A_{22}(\tau) \end{bmatrix} = \exp\left[\tau \begin{bmatrix} H & 2Q^T Q \\ R & -H^T \end{bmatrix}\right] \tag{4.4.7}$$

Alternative approaches for the pricing of zero coupon bond under the Wishart short rate model can be found in Grasselli and Tebaldi (2008) and Gnoatto and Grasselli (2014).

In the next chapter we seek to apply the above theory to devise a methodology to compute sharp bounds for Guaranteed Annuity Options.

Chapter 5

General Price Bounds for Guaranteed Annuity Options

This chapter has been published on Arxiv. The full reference follows:

R. K. Bahl and S. Sabanis. General Price Bounds for Guaranteed Annuity Options. *Working Paper*, *Arxiv*, 2017. URL **arXiv:1707.00807v1[q-fin.PR]**.

Previous versions of this chapter were presented at the following winter schools, workshops and conferences:

- January 2017. Perspectives on Actuarial Risks in Talks of Young Researchers (PARTY), Winter School (organized by Institute of Actuarial Sciences, Universit de Lausanne) at Monte Veritá, Ascona, Switzerland. "General Price Bounds for Guaranteed Annuity Options".

- May 2017. Workshop on Recent Developments in Dependence Modeling with Applications in Finance and Insurance: Fourth Edition (organized by Vrije Universiteit Brussel (VUB)) at Hotel Danae, Aegina, Greece. "General Price Bounds for Guaranteed Annuity Options when Mortality and Interest Rate Risks are Correlated".

- July 2017. International Congress on Insurance: Mathematics and Economics (IME), TU Wien (Technische Universität Wien), Vienna, Austria. "General Price Bounds for Guaranteed Annuity Options".

In this chapter, we are concerned with the valuation of Guaranteed Annuity Options (GAOs) under the most generalised modelling framework where both interest rate and mortality risk are stochastic and correlated. Pricing these type of options in the correlated environment is a challenging task and no closed form solution exists in the literature. We employ the use of doubly stochastic stopping times to incorporate the randomness about the time of death and employ a suitable change of measure to facilitate the valuation of survival benefit, there by adapting the payoff of the GAO in terms of the payoff of a basket call option. We derive general price bounds for GAOs by employing the theory of comonotonicity, the Rogers-Shi (Rogers and Shi, 1995) approach and the general closed form basket option pricing bounds as discussed in Caldana et al. (2016). The theory is then applied to affine models to present some very interesting formulae for the bounds under the affine set up. Numerical examples are provided in Chapter 6 and they are benchmarked against Monte Carlo simulations to estimate the price of a GAO for a variety of affine processes governing the evolution of mortality and the interest rate.

5.1 Introduction

In the present era when financial institutions are facing serious challenges in the advent of improving life expectancy, pricing of key products such as 'Guaranteed Annuity Options' which involve survival benefit has gained a lot of momentum. It is the need of the hour to equip the longevity product designers with an insight to efficient pricing of these instruments. This involves designing an apparatus that provides state of art solutions to measure the random impulse of mortality, which indeed calls for looking at mortality in a stochastic sense. Till, very lately the conventional approach of actuaries consisted in treating mortality in a deterministic way in contrast to interest rates which were assumed to possess a stochastic nature. Post this came the era of the assumption that mortality evolves in a stochastic manner but is independent of interest rates (see for example Biffis, 2005). However, the latter assumption is also far from being realistic. This is because both extreme mortality events such as catastrophes and pandemics as well as improving life expectancy go a long way in influencing the value of interest rate. While the former shows a stronger effect in a short term, the latter affects the financial market in a gradual manner. Interested readers can refer to Deelstra et al. (2016), Liu et al. (2014), Liu et al. (2013) and Jalen and Mamon (2009) and the references therein. To the best of our knowledge, Miltersen and Persson (2005) were the first ones to introduce dependence between mortality and interest rates in the actuarial world. In the context of the real world, a study by Nicolini (2004) to understand the relation between these two underlying risks demonstrates that the decline of interest rate in pre-industrial England was perhaps triggered by the decline of adult mortality at the end of the 17th century. More recently Dacorogna and Cadena (2015) examine correlation between mortality and market risks in periods of extremes such as a severe pandemic outbreak while Dacorogna and Apicella (2016) explore existence of this dependence within the Feller process framework.

As remarked in the beginning of this section 'The Life Expectancy Revolution' has pressurised social security programs of various nations thereby triggering fiscal crisis for governments who find it hard to fulfill the needs of an ever growing aging population. The price for this imbalance affects the financial markets adversely leading to a downtrend in returns on investments. To take care of these issues, EU's Solvency II Directive has laid out new insurance risk management practices for capital adequacy requirements based on the assumption of dependence between financial markets and life/health insurance markets including the correlation between the two underpinning risks viz. interest rate and mortality (c.f. Quantitative Impact Study 5:Technical Specifications QIS5, 2010).

In this chapter, we consider the most generalised modelling framework where both interest and mortality risks are stochastic and correlated. In a set up similar to Biffis (2005), we advocate the use of doubly stochastic stopping times to incorporate the randomness about the time of death.

We then utilize this set up and the theory of comonotonicity to devise model-independent price bounds for Guaranteed Annuity Options (GAOs). These are options embedded in certain pension policies that provide the policyholders the right to choose between a lump sum at time of retirement/maturity or to convert the proceeds into an annuity at a guaranteed rate. The reports of the Insurance Institute of London (1972) (c.f. IIL, 1972) show that the origin of GAOs dates back to 1839. However these instruments came into the limelight in UK in the era of 1970-1990. In the advent of increased life expectancy, the research on pricing of GAOs has gained a lot of momentum as the under pricing of such guarantees has already caused serious solvency problems to insurers, for example in the UK, as an after effect of encashment of too many GAOs, the world's oldest life insurer - Equitable Life had to close to new business in 2000.

The existing literature in the direction of pricing of GAOs under the correlation assumption is very thin and only Monte Carlo estimation of the GAO price is available for sophisticated models (c.f. Deelstra et al., 2016). But Monte Carlo method is generally extremely time consuming for complex models (c.f. Fenga et al., 2017). This chapter is a concrete step in the direction of pricing of GAOs under the correlation direction. It investigates the designing of price bounds for GAOs under the assumption of dependence between mortality and interest rate risks and provides a much needed confidence interval for the pricing of these options. Moreover the proposed bounds are model-free or general in the sense they are applicable for all kinds of models and in particular suitable for the affine set up. Keeping pace with the relevant literature (c.f. Liu et al., 2013; Deelstra et al., 2016), we applied a change of probability measure with the 'Survival Zero Coupon Bond' as numéraire for the valuation of the GAO. This change of measure facilitates computation and enhances efficiency (c.f. Liu et al., 2014). The organization of the chapter follows. In section 2 we introduce the market framework with the necessary notations. In section 3 we define GAOs and show that their payoff is similar to that of a basket option. This is followed by Section 4 which highlights the technicalities of affine processes. Sections 5 and 6 are the core sections which present details on finding lower and upper bounds for GAOs. In section 7 we present examples while numerical investigations in support of the developed theory appear in Chapter 6.

5.2 The Market Framework

In this section, we introduce the necessary set up required to construct the mathematical interplay between financial market and the mortality model. We denote by $\mathbb{P}$, the physical world measure and we utilize the fact that in the absence of arbitrage, at least one equivalent martingale measure (EMM) $\mathbb{Q}$ exists. We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ such that the filtration is large enough to support a process X in $\mathbb{R}^k$, representing the evolution of financial variables and a process Y in $\mathbb{R}^d$, representing the evolution of mortality. We take as given an adapted short rate process $r = \{r_t\}_{t\geq 0}$ such that it satisfies the technical condition $\int_0^t r_s ds < \infty$ a.s. for all $t \geq 0$. The short rate process r represents the continuously compounded rate of interest of a risk-less security. Moreover, we concentrate on an insured life aged x at time 0, with random residual lifetime denoted by τ_x which is an $\mathcal{F}_t$-stopping time.

The filtration $\mathbb{F}$ includes knowledge of the evolution of all state variables up to each time t and of whether the policyholder has died by that time. More explicitly, we have:

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$$

where

$$\mathcal{G}_t \vee \mathcal{H}_t = \sigma\left(\mathcal{G}_t \cup \mathcal{H}_t\right)$$

with

$$\mathcal{G}_t = \sigma\left(Z_s :\ 0 \leq s \leq t\right), \quad \mathcal{H}_t = \sigma\left(\mathbb{1}_{\{\tau \leq s\}} :\ 0 \leq s \leq t\right)$$

and where $Z = (X, Y)$ is the joint state variables process in $\mathbb{R}^{k+d}$. Thus we have

$$\mathcal{G}_t = \mathcal{G}_t^X \vee \mathcal{G}_t^Y.$$

In fact $\mathbb{H} = \{\mathcal{H}_t\}_{t\geq 0}$ is the smallest filtration with respect to which τ is a stopping time. In other words $\mathbb{H}$ makes $\mathbb{F}$ the smallest enlargement of $\mathbb{G} = \{\mathcal{G}_t\}_{t\geq 0}$ with respect to which τ is a stopping time, i.e.,

$$\mathcal{F}_t = \cap_{s>t}\mathcal{G}_s \vee \sigma(\tau \wedge s), \ \forall t.$$

We may think of $\mathcal{G}_t$ as carrying information captured from medical/demographical data collected at population/ industry level and of $\mathcal{H}_t$ as recording the actual occurrence of death in an insurance portfolio.

To make the set up more robust, we assume that τ_x is the first jump-time of a nonexplosive $\mathcal{F}_t$-counting process N recording at each time $t \geq 0$ whether the individual has died ($N_t \neq 0$) or not ($N_t = 0$). The stopping time τ_x is said to admit an intensity μ_x if N does, i.e. if μ_x is a non-negative predictable process such that $\int_0^t \mu_x(s)\,ds < \infty$ a.s. for all $t \geq 0$ and such that the compensated process $M = \{N_t - \int_0^t \mu_x(s)\,ds : t \geq 0\}$ is a local $\mathcal{F}_t$-martingale. Our next assumption is that N is a doubly stochastic process or Cox Process driven by a subfiltration $\mathcal{G}_t$ of $\mathcal{F}_t$, with $\mathcal{G}_t$-predictable intensity μ. This implies that on any particular trajectory $t \mapsto \mu_t(\omega)$ of μ, the counting process N is a Poisson-inhomogeneous process with parameter $\int_0^{\cdot} \mu_s(\omega)\,ds$, i.e., we have that for all $t \in [0,T]$ and non-negative integer k,

$$\mathbb{P}(N_T - N_t = k|\mathcal{F}_t \vee \mathcal{G}_T) = \frac{\left(\int_t^T \mu_s ds\right)^k}{k!} e^{-\int_t^T \mu_s ds}. \tag{5.2.1}$$

The main reason for the consideration of a strict sub-filtration $\mathcal{G}_T$ of $\mathcal{F}_t$ is that it provides enough information about the evolution of the intensity of mortality, i.e., about the likelihood of death happening, but not enough information about the actual occurrence of death. Such information is carried by the larger filtration $\mathcal{F}_t$, with respect to which τ is a stopping time. From (5.2.1) by putting $k = 0$, we now proceed to compute the 'probability of survival' up to time $T \geq t$, on the set $\{\tau > t\}$. Let A be the event of no death in the interval $t \in [0,T]$, i.e., $A \equiv \{N_T - N_t = 0\}$, then the tower property of conditional expectation tells us that

$$\begin{aligned}
\mathbb{P}(\tau > T|\mathcal{F}_t) &= E[\mathbb{1}_A|\mathcal{F}_t] \\
&= E\left[E(\mathbb{1}_A|\mathcal{F}_t \vee \mathcal{G}_T)|\mathcal{F}_t\right] \\
&= E\left[\mathbb{P}(N_T - N_t = 0|\mathcal{F}_t \vee \mathcal{G}_T)|\mathcal{F}_t\right] \\
&= E\left[e^{-\int_t^T \mu_s ds}|\mathcal{F}_t\right].
\end{aligned} \tag{5.2.2}$$

In fact, we characterize the conditional law of τ in several steps. Given the non-negative $\mathcal{G}_t$-predictable process μ is satisfying $\int_0^t \mu_x(s)\,ds < \infty$ a.s. for all $t > 0$, we consider an exponential random variable Φ with parameter 1, independent of $\mathcal{G}_\infty$ and define the random time of death τ as the first time when the process $\int_0^t \mu_s ds$ is above the random threshold Φ, i.e.,

$$\tau \doteq \{t \in \mathbb{R}^+ : \int_0^t \mu_s(s)\,ds \geq \Phi\}. \tag{5.2.3}$$

It is evident from (5.2.3) that $\{\tau > T\} = \{\int_0^t \mu_s ds < \Phi\}$, for $T \geq 0$. Next, we work out $\mathbb{P}(\tau > T|\mathcal{G}_t)$ for $T \geq t \geq 0$ by using tower property of conditional expectation, independence of Φ and $\mathcal{G}_\infty$ and facts that μ is a $\mathcal{G}_t$-predictable process and $\Phi \sim \mathit{Exponential}\,(1)$, i.e.,

$$\mathbb{P}(\tau > T|\mathcal{G}_t) = E\left[e^{-\int_0^T \mu_s ds}|\mathcal{G}_t\right]. \tag{5.2.4}$$

In fact, the same result holds for $0 \leq T < t$. Further, we observe that $\{\tau > t\}$ is an atom of $\mathcal{H}_t$. As a result, in a manner similar to Biffis (2005), we have constructed a doubly stochastic $\mathcal{F}_t$-stopping time driven by $\mathcal{G}_t \subset \mathcal{F}_t$ in the following way (c.f. Billingsley, 1995, ex 34.4, p.455):

$$\begin{aligned}
\mathbb{P}(\tau > T|\mathcal{G}_T \vee \mathcal{F}_t) &= \mathbb{1}_{\{\tau > t\}} E\left[\mathbb{1}_{\{\tau > T\}}|\mathcal{G}_T \vee \mathcal{H}_t\right] \\
&= \mathbb{1}_{\{\tau > t\}} e^{-\int_t^T \mu_s ds}.
\end{aligned} \tag{5.2.5}$$

Next, the conditioning on $\mathcal{F}_t$ can be replaced by conditioning on $\mathcal{G}_t$ as shown in the Appendix C of Biffis (2005) and Appendix A.2 of this thesis. We remark that, we do not take $\mathcal{G}_t \vee \sigma(\Phi)$ as our filtration $\mathcal{G}_t$ because, in that case, the stopping time τ would be predictable and would not admit an intensity. The construction potrayed here guarantees that τ is a totally inaccessible stopping time, a concept intuitively meaning that the insureds death arrives as a total surprise to the insurer (see Protter, 1990, Chapter III.2, for details). With this, we move to the focal point of this chapter viz. GAOs.

5.3 Guaranteed Annuity Options

5.3.1 Introduction

A Guaranteed Annuity Option(GAO) is a contract that gives the policyholder the flexibility to convert his/her survival benefit into an annuity at a pre-specified conversion rate. The guaranteed conversion rate denoted by g, can be quoted as an annuity/cash value ratio. According to Bolton et al. (1997), the most popular choice for for the guaranteed conversion rate g for males aged 65 in the UK in the 1980s was $g = \frac{1}{9}$, which means that per £1000 cash value can be converted into an annuity of £111 per annum. The GAO would have a positive value if the guaranteed conversion rate is higher than the available conversion rate; otherwise the GAO is worthless since the policyholder could use the cash to obtain higher value of annuity from the primary market. As a result, the moneyness of the GAO at maturity depends on the price of annuity available in the market at that time and this in turn is calculated using the prevailing interest and mortality rates.

5.3.2 Mathematical Formulation

Consider an x year old policyholder at time 0 who has an access to a unity amount at his retirement age R_x. Then, a GAO gives the policyholder a choice to choose at time $T = R_x - x$ between an annual payment of g or a cash payment of 1. Let $\ddot{a}_x(T)$ denote a whole life annuity due for a person aged x at time 0, which gives an annual payment of one unit amount at the start of each year, this payment beginning from time T and conditional on survival. If w is the largest possible survival age then we have

$$\begin{aligned} \ddot{a}_x(T) &= \sum_{j=0}^{w-(T+x)-1} \mathbb{E}\left[e^{-\int_T^{T+j}(r_s+\mu_s)ds} | \mathcal{G}_T\right] \\ &= \sum_{j=0}^{w-(T+x)-1} \tilde{P}(T, T+j), \end{aligned} \tag{5.3.1}$$

where

$$\tilde{P}(t,T) = \mathbb{E}\left[e^{-\int_t^T(r_s+\mu_s)ds} | \mathcal{G}_t\right] \tag{5.3.2}$$

denotes the price at time t of a pure endowment insurance with maturity T for an insured of age x at time 0 who is still alive at time t. This insurance instrument is nomenclated as a *survival zero-coupon bond* abbreviated as SZCB by Deelstra et al. (2016) and the authors remark that it can be used as a numèraire because it can be replicated by a strategy that involves longevity bonds (c.f. Lin and Cox, 2005) in analogy with the usual bootstrapping methodology used to find the zero rate curve starting by coupon bonds. This insurance instrument pays one unit of money at time T upon the survival of the insured at that time. In fact $r + \mu$ can be viewed as a fictitious short rate or yield to compare these instruments with their financial counterparts.

At time T, the value of the contract having the above embedded GAO can be described by the following decomposition

$$V(T) = \max(g\ddot{a}_x(T), 1)$$

$$= \quad 1 + g \max \left(\ddot{a}_x (T) - \frac{1}{g} \right). \tag{5.3.3}$$

In order to apply risk neutral evaluation, we state a result from Biffis (2005) to compute the fair values of a basic payoff involved by standard insurance contracts. These are benefits, of amount possibly linked to other security prices, contingent on survival over a given time period. We require the short rate process r and the intensity of mortality μ to satisfy the technical conditions stated in section 5.2.

Proposition 30. *(Survival benefit). Let C be a bounded $\mathcal{G}_t$-adapted process. Then, the time-t fair value $SB_t(C_T;T)$ of the time-T survival benefit of amount C_T, with $0 \leq t \leq T$, is given by:*

$$SB_t(C_T;T) = \mathbb{E}\left[e^{-\int_t^T r_s ds}\mathbb{1}_{\{\tau>T\}}C_T|\mathcal{F}_t\right] = \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[e^{-\int_t^T (r_s+\mu_s)ds}C_T|\mathcal{G}_t\right] \tag{5.3.4}$$

In particular, if C is $\mathcal{G}_t^X$-adapted and X and Y are independent, then, the following holds

$$SB_t(C_T;T) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[e^{-\int_t^T r_s ds}C_T|\mathcal{G}_t^X]\mathbb{E}[e^{-\int_t^T \mu_s ds}|\mathcal{G}_t^Y\right] \tag{5.3.5}$$

Proof. A comprehensive proof can be found in Biffis (2005) and is summarized in Appendix A.3. $\square$

Thus, we have the value at time $t = 0$ of the second term in (5.3.3), which is called the GAO option price entered by an x-year policyholder at time $t = 0$ as

$$C(0,x,T) = \mathbb{E}\left[e^{-\int_0^T (r_s+\mu_s)ds} g\left(\ddot{a}_x(T) - \frac{1}{g}\right)^+\right]. \tag{5.3.6}$$

In order to facilitate calculation, we adopt the following change of measure.

5.3.3 Change of Measure

We advocate a change of measure similar to the one adopted in Deelstra et al. (2016). We define a new probability measure $\tilde{Q}$ with the Radon-Nikodym derivative of $\tilde{Q}$ w.r.t $\mathbb{Q}$ as:

$$\frac{d\tilde{Q}}{d\mathbb{Q}} := \eta_T = \frac{e^{-\int_0^T (r_s+\mu_s)ds}}{\mathbb{E}\left[e^{-\int_0^T (r_s+\mu_s)ds}\right]} \tag{5.3.7}$$

where $\mathbb{E}$ denotes the usual expectation w.r.t the EMM $\mathbb{Q}$ and we will use $\tilde{E}$ to denote the expectation w.r.t the new probability measure $\tilde{Q}$. Further on using Bayes' Rule for conditional expectation, the survival benefit in (5.3.4) can be rewritten as

$$SB_t(C_T;T) = \mathbb{1}_{\{\tau>t\}}\tilde{P}(t,T)\,\tilde{E}[C_T|\mathcal{G}_t] \tag{5.3.8}$$

The advantage of the change of measure approach is that the complex expectation appearing in the survival benefit given in (5.3.4) has been decomposed into two simpler expectations: the first one corresponds to the price of the SZCB given in (5.3.2) and the second one is connected to the expected value of the survival benefit C_T under the new probability measure $\tilde{Q}$ which needs to be determined. In the passing, one notes that in (5.3.8) if $C_T = 1$, we get a very interesting relationship

$$SB_t(1;T) = \mathbb{1}_{\{\tau>t\}}\tilde{P}(t,T). \tag{5.3.9}$$

In particular

$$SB_0(1;T) = \mathbb{1}_{\{\tau>t\}}\tilde{P}(0,T). \tag{5.3.10}$$

A similar change of measure has been employed by Liu et al. (2014) and Liu et al. (2013) with the only difference that they use the unitary survival benefit given in (5.3.9) as the numèraire.

On the contrary, Jalen and Mamon (2009) have used a twin change of measure to compute value of a GAO.

5.3.4 Payoff

Under the new probability measure $\tilde{Q}$ defined in (5.3.7), the GAO option price decomposes into the following product

$$C(0,x,T) = g\tilde{P}\left(0,T\right)\tilde{E}\left[\left(\ddot{a}_x\left(T\right) - \frac{1}{g}\right)^+\right] \tag{5.3.11}$$

where $\tilde{P}\left(0,T\right)$ is defined in (5.3.2). To develop ideas further, we express the payoff in a more appealing form as follows:

$$C(0,x,T) = g\tilde{P}\left(0,T\right)\tilde{E}\left[\left(\sum_{i=1}^{n-1} S_T^{(i)} - (K-1)\right)^+\right] \tag{5.3.12}$$

where we utilize the fact that $\tilde{P}\left(T,T\right) = 1$ and define $n = w - (T+x)$ and

$$S_T^{(i)} = \tilde{P}\left(T, T+i\right);\ i = 1, 2, ..., n-1. \tag{5.3.13}$$

The last term on the R.H.S in the payoff of the GAO resembles the payoff of a basket option (see Appendix A.7.2) having unit weights and the SZCBs, maturing at times $T+1, T+2, ..., w-x-1$ acting as the underlying assets. We seek to evaluate tight model-independent bounds for the GAOs in the ensuing sections. To the best of our knowledge, the equations (5.3.6) and (5.3.11) have only been valued by Monte Carlo simulations for specific choice of models. In Liu et al. (2014), numerical experiments in the Gaussian setting have shown that (5.3.11) is a little bit more precise and in particular it is less time consuming than the implementation of (5.3.6). Deelstra et al. (2016) have investigated these calculations for different affine models such as the multi-CIR and the Wishart cases. Liu et al. (2013) have computed very specific comonotonic bounds for GAOs in the Gaussian framework.

5.4 Affine Processes

Affine processes are essentially Markov processes with conditional characteristic function of the affine form. A thorough discussion of these processes on canonical state space appears in Duffie et al. (2003) and Filipovic (2005). More recently the development of multivariate stochastic volatility models has lead to the evolution of applications of affine processes on non-canonical state spaces, in particular on the cone of positive semi-definite matrices. A plethora of research papers are available to explore and interested readers can refer to Cuchiero et al. (2011) for details. A unified approach on affine processes is presented in Keller-Ressel and Mayerhofer (2015). We have already presented the details of the affine processes in Chapter 4. We outline some important results below. In regards to the evolution of interest rates and the force of mortality we consider a set up similar to Deelstra et al. (2016).

Suppose we have a time-homogeneous affine Markov process X taking values in a non-empty convex subset E of $\mathbb{R}^d$, $(d \geq 1)$ equipped with the inner product $\langle \cdot, \cdot \rangle$. We then assume that the dynamics of the interest rate and force of mortality are given respectively as follows.

$$r_t = \bar{r} + \langle R, X_t \rangle \tag{5.4.1}$$

and

$$\mu_t = \bar{\mu} + \langle M, X_t \rangle \tag{5.4.2}$$

where $\bar{r}, \bar{\mu} \in \mathbb{R}$, $M, R \in \mathbb{R}_d$ or M_d where M_d is the set of real square matrices of order d.

This means that the interest rate and mortality are linear projections of the common stochastic factor X along constant directions given by the parameter R and M respectively. We will be interested in the cases where the X is a classical affine process on the state space $\mathbb{R}_+^m \times \mathbb{R}^n$ or an affine Wishart process on the state space S_d^+, which is the set of $d \times d$ symmetric positive semi-definite matrices. The inner product possesses the flexibility to condense into scalar product or trace depending on the nature of R and M being respectively vectors or matrices. In the former set up we consider multi-dimensional CIR case (c.f. Cox et al., 1985). In the case of Vasicek model (c.f. Vasicek, 1977), the affine set up is uni-dimensional. A very good reference to show that the stochastic processes underlying the Vasicek and CIR models fall under the affine set up is Keller-Ressel (2011).

In the passing it is important to note that the affiness of the underlying model is preserved as we move from the physical world to the risk neutral environment, although new affine dynamics emerge (c.f. Biffis and Millossovich, 2006; Duffie et al., 2000). In fact, more recently Dhaene et al. (2013) examine the conditions under which it is possible or not to translate the independence assumption from the physical world to the pricing world.

We now state without proof the following proposition which presents the methodology to value SZCBs and in turn GAOs. A detailed proof appears in Gnoatto (2012) and the necessary notations are defined in Chapter 4.

Proposition 31. *Let X be a conservative affine process on S_d^+ under the risk neutral measure $\mathbb{Q}$. Let the short rate be given in accordance with* (5.4.1). *Let $\tau = T - t$, then the price of a zero-coupon bond is given by*

$$\begin{aligned}\tilde{P}(t,T) &= \mathbb{E}\left[e^{-\int_t^T(\bar{r}+\bar{\mu}+\langle R+M,X_u\rangle)du}|\mathcal{F}_t\right] \\ &= e^{-(\bar{r}+\bar{\mu})\tau}e^{-\tilde{\phi}(\tau,R+M)-\langle\tilde{\psi}(\tau,R+M),X_t\rangle}, \end{aligned} \tag{5.4.3}$$

where $\tilde{\phi}$ and $\tilde{\psi}$ satisfy the following Ordinary Differential Equations (ODEs) which are known also as Riccati ODEs.

$$\frac{\partial\tilde{\phi}}{\partial\tau} = \tilde{\Im}\left(\tilde{\psi}(\tau,R+M)\right), \ \ \tilde{\phi}(0,R+M) = 0, \tag{5.4.4}$$

$$\frac{\partial\tilde{\psi}}{\partial\tau} = \tilde{\Re}\left(\tilde{\psi}(\tau,R+M)\right), \ \ \tilde{\psi}(0,R+M) = 0, \tag{5.4.5}$$

with

$$\tilde{\Im}\left(\tilde{\psi}(\tau,R+M)\right) = \langle b,\tilde{\psi}(\tau,R+M)\rangle - \int_{S_d^+\setminus\{0\}}\left(e^{-\langle\tilde{\psi}(\tau,R+M),\xi\rangle}-1\right)m(d\xi), \tag{5.4.6}$$

and

$$\begin{aligned}\tilde{\Re}\left(\tilde{\psi}(\tau,R+M)\right) &= -2\tilde{\psi}(\tau,R+M)\alpha\tilde{\psi}(\tau,R+M) + B^T\left(\tilde{\psi}(\tau,R+M)\right) \\ &\quad - \int_{S_d^+\setminus\{0\}}\left(\frac{e^{-\langle\tilde{\psi}(\tau,R+M),\xi\rangle}-1+\langle\chi(\xi),\tilde{\psi}(\tau,R+M)\rangle}{\parallel\xi\parallel^2\wedge 1}\right)\mu(d\xi) \\ &\quad + R + M. \end{aligned} \tag{5.4.7}$$

In fact it is interesting to note that assuming this kind of affine structure means that our fictitious yield model is "affine" in the sense that there is, for each maturity T, an affine mapping $Z_T : \mathbb{R}^n \to \mathbb{R}$ such that, at any time t, the yield of any SZCB of maturity T is $Z_T(X_t)$ echoing the results obtained in the seminal paper of Duffie and Kan (1996).

As a result we have for $i = 1, 2, ..., n-1$,

$$S_T^{(i)} = e^{-(\bar{r}+\bar{\mu})i}e^{-\tilde{\phi}(i,R+M)-\langle\tilde{\psi}(i,R+M),X_T\rangle}, \tag{5.4.8}$$

where $\tilde{\phi}(i, R+M)$ and $\tilde{\psi}(i, R+M)$ satisfy the equations (5.4.4) and (5.4.5) with $\tau = i$. Alternatively, one may write

$$S_T^{(i)} = S_0^{(i)} e^{X_T^{(i)}};\ i = 1, 2, ..., n-1, \tag{5.4.9}$$

with

$$S_0^{(i)} = e^{-\left((\bar{r}+\bar{\mu})i + \tilde{\phi}(i,R+M)\right)} \tag{5.4.10}$$

and

$$X_T^{(i)} = -\langle \tilde{\psi}(i, R+M), X_T \rangle. \tag{5.4.11}$$

As a result in the affine case, by using equation (5.4.8) in (5.3.12) the formula for GAO payoff can be written in a very compact form as shown below.

$$C(0, x, T) = g\tilde{P}(0, T)\, \tilde{E}\left[\left(\sum_{i=1}^{n-1} e^{-(\bar{r}+\bar{\mu})i} e^{-\tilde{\phi}(i,R+M) - \langle\tilde{\psi}(i,R+M), X_T\rangle} - (K-1)\right)^+\right], \tag{5.4.12}$$

where $\tilde{P}(0, T)$ given by equation (5.4.3) with $\tau = T$. As a result in the affine case, our quest of bounds for the GAO becomes simplified as we are dealing only with X_T.

The analytical tractability of affine processes is essentially linked to generalized Riccati equations as given above which can be in general solved by numerical methods although explicit solutions are available in the Vasicek (c.f. Vasicek, 1977) and CIR (c.f. Cox et al., 1985) models without jumps.

5.5 Lower Bounds for Guaranteed Annuity Options

We now proceed to work out appropriate lower bounds for the payoff of the GAO as given in (5.3.12). Invoking Jensen's inequality , we have

$$\tilde{E}\left[\left(\sum_{i=1}^{n-1} S_T^{(i)} - (K-1)\right)^+\right] \quad \geq \quad \tilde{E}\left[\left(\sum_{i=1}^{n-1} \tilde{E}\left(S_T^{(i)}|\Lambda\right) - (K-1)\right)^+\right]. \tag{5.5.1}$$

The general derivation concerning lower bounds for stop loss premium of a sum of random variables based on Jensen's inequality can be found in Simon et al. (2000) and for its application to Asian basket options, one can refer to Deelstra et al. (2008). Define

$$S = \sum_{i=1}^{n-1} S_T^{(i)} \tag{5.5.2}$$

and

$$S^l = \sum_{i=1}^{n-1} \tilde{E}\left(S_T^{(i)}|\Lambda\right) \tag{5.5.3}$$

Thus, we have obtained

$$S \geq_{cx} S^l. \tag{5.5.4}$$

Now, suitably tailoring the inequality (5.5.1), we obtain

$$C(0, x, T) \geq g\tilde{P}(0, T)\, \tilde{E}\left[\left(\sum_{i=1}^{n-1} \tilde{E}\left(S_T^{(i)}|\Lambda\right) - (K-1)\right)^+\right]. \tag{5.5.5}$$

To exploit the theory of comonotonicity see for example in Dhaene et al. (2002a), we now have to show that the lower bound for S, i.e. S^l can be expressed as a sum of stop-loss premiums. This task becomes trivial if we can choose the conditioning variable Λ in such a way that $\tilde{E}\left(S_T^{(i)}|\Lambda\right)$ is either increasing or decreasing for every i, so that the vector: $\mathbf{S^l} =$

$\left(\tilde{E}\left(S_T^{(1)}|\Lambda\right),\ldots,\tilde{E}\left(S_T^{(n-1)}|\Lambda\right)\right)$ is comonotonic. Before proceeding, we define as in Dhaene et al. (2002a), F_X^{-1} as the generalized inverse of the cumulative distribution function (cdf), i.e.,

$$F_X^{-1}(p)=\inf\{x\in\mathbb{R}|F_X(x)\geq p\},\quad p\in[0,1] \tag{5.5.6}$$

and F_X^{-1+} is a more sophisticated inverse defined as

$$F_X^{-1+}(p)=\sup\{x\in\mathbb{R}|F_X(x)\leq p\},\quad p\in[0,1]\,. \tag{5.5.7}$$

From this point onwards, we assume that $K-1\in\left(F_{S^l}^{-1+}(0),F_{S^l}^{-1}(1)\right)$ which is not at all a restriction for all practical purposes. Using comonotonicity (c.f. Corollary 16, Chapter 2),

$$\tilde{E}\left[(S-(K-1))^+\right] \quad\geq\quad \sum_{i=1}^{n-1}\tilde{E}\left[\left(\tilde{E}\left(S_T^{(i)}|\Lambda\right)-F^{-1}_{\tilde{E}\left(S_T^{(i)}|\Lambda\right)}\left(F_{S^l}(K-1)\right)\right)^+\right]-K_1 \tag{5.5.8}$$

where

$$K_1=\left((K-1)-F_{S^l}^{-1}\left(F_{S^l}(K-1)\right)\right)\left(1-F_{S^l}(K-1)\right) \tag{5.5.9}$$

In case if the marginal cdfs $F_{\tilde{E}\left(S_T^{(i)}|\Lambda\right)}$ are strictly increasing, we have the following compact expression for $K-1\in\left(F_{S^l}^{-1+}(0),F_{S^l}^{-1}(1)\right)$

$$\tilde{E}\left[(S-(K-1))^+\right] \quad\geq\quad \sum_{i=1}^{n-1}\tilde{E}\left[\left(\tilde{E}\left(S_T^{(i)}|\Lambda\right)-F^{-1}_{\tilde{E}\left(S_T^{(i)}|\Lambda\right)}\left(F_{S^l}(K-1)\right)\right)^+\right]. \tag{5.5.10}$$

Thus we have been able to obtain a convex lower bound for GAO by conditioning on an arbitrary random variable Λ, i.e.,

$$C(0,x,T)\geq g\tilde{P}(0,T)\left(\sum_{i=1}^{n-1}\tilde{E}\left[\left(\tilde{E}\left(S_T^{(i)}|\Lambda\right)-F^{-1}_{\tilde{E}\left(S_T^{(i)}|\Lambda\right)}\left(F_{S^l}(K-1)\right)\right)^+\right]-K_1\right) \tag{5.5.11}$$

where K_1 is defined in (5.5.9).

However, if the comonotonic approach can not be applied, one may consider a non-comonotonic lower bound as discussed in Section 5.5.3.

5.5.1 A First Lower Bound

In case, if the random variable Λ is independent of the prices of pure endowments having term periods $1,2,...,n-1$ at the time T, i.e., of $S_T^{(i)};\ i=1,2,...,n-1$, respectively, the bound in (5.5.5) simply reduces to:

$$C(0,x,T)\geq g\tilde{P}(0,T)\,\tilde{E}\left[\left(\sum_{i=1}^{n-1}\tilde{E}\left(S_T^{(i)}\right)-(K-1)\right)^+\right]. \tag{5.5.12}$$

or even more precisely as the outer expectation is redundant, we obtain a lower bound for GAO expressed in terms of expectation of $S_T^{(i)}$, i.e.,

$$C(0,x,T)\geq g\tilde{P}(0,T)\left(\sum_{i=1}^{n-1}\tilde{E}\left(S_T^{(i)}\right)-(K-1)\right)^+ =:\text{GAOLB}. \tag{5.5.13}$$

The First Lower Bound under the Affine Set Up

Under the affine set up of section 5.4 (c.f. equation (5.4.8)), the lower bound given in equation (5.5.13) reduces to

$$\mathrm{GAOLB}^{aff} = g\tilde{P}(0,T)\left(\sum_{i=1}^{n-1}\left(e^{-\left((\bar{r}+\bar{\mu})i+\tilde{\phi}(i,R+M)\right)}\mathcal{L}\left(\tilde{\psi}(i,R+M)\right)\right)-(K-1)\right)^{+} \quad (5.5.14)$$

where $\mathcal{L}$ denotes the Laplace transform of X_T with parameter $\tilde{\psi}(i,R+M)$ under the transformed measure $\tilde{Q}$. This means that if one can lay hands on the distribution of X_T, this bound has a very compact form.

5.5.2 The Comonotonic Lower Bound

As the next step, we obtain a tighter lower bound by assuming that the endowment products S_i have an asset price process given in terms of exponential Lévy model as follows:

$$S_T^{(i)} = S_0^{(i)}\exp\left(X_T^{(i)}\right);\ i=1,2,...,n-1, \quad (5.5.15)$$

where $X_T^{(i)}$ is a Lévy process observed at time T and $S_0^{(i)}$ is the price of pure endowment of term i years at time 0. For each i, let μ_i and σ_i^2 represent the expectation and variance of X_i respectively. Further, let ρ_{ij} denote the correlation of $X_T^{(i)}$ and $X_T^{(j)}$ and assume that, for all $i,\ j$, this is non-negative. Again using Jensen's inequality, one may write

$$\tilde{E}\left[S_T^{(i)}|S_T^{(j)}=s\right] \geq S_0^{(i)}\exp\left(\tilde{E}\left[X_T^{(i)}|X_T^{(j)}=\log_e\left(\frac{s}{S_0^{(j)}}\right)\right]\right) \quad (5.5.16)$$

Further, we assume that the Lévy process has no jumps so that

$$\tilde{E}\left[X_T^{(i)}|X_T^{(j)}=x_j\right] = \mu^{(i)}+\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}\left(x_j-\mu^{(j)}\right) \quad (5.5.17)$$

where

$$\mu^{(i)} = \tilde{E}\left[X_T^{(i)}\right] = \tilde{E}\left[\log_e\left(\frac{S_T^{(i)}}{S_0^{(i)}}\right)\right];\ i=1,2,...,n-1 \quad (5.5.18)$$

$$\left(\sigma^{(i)}\right)^2 = \mathrm{Var}\left[X_T^{(i)}\right] = \mathrm{Var}\left[\log_e\left(\frac{S_T^{(i)}}{S_0^{(i)}}\right)\right];\ i=1,2,...,n-1 \quad (5.5.19)$$

$$\rho^{(ij)} = \mathrm{Corr}\left[X_T^{(i)},X_T^{(j)}\right] = \mathrm{Corr}\left[\log_e\left(\frac{S_T^{(i)}}{S_0^{(i)}}\right),\log_e\left(\frac{S_T^{(j)}}{S_0^{(j)}}\right)\right];\ i\neq j=1,2,...,n-1 \quad (5.5.20)$$

Also from Dhaene et al. (2002a), we know that

$$S \geq_{cx} \sum_{i=1}^{n-1}\tilde{E}\left(S_T^{(i)}|S_T^{(j)}\right). \quad (5.5.21)$$

Combining (5.5.16), (5.5.17) and (5.5.21), we get that

$$S \geq_{sl} \sum_{i=1}^{n-1}S_0^{(i)}\left(\frac{S_T^{(j)}}{S_0^{(j)}}\right)^{\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}}\exp\left(\mu^{(i)}-\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}\mu^{(j)}\right). \quad (5.5.22)$$

On comparing (5.5.21) with (5.5.3) and (5.5.4), we see that $S_T^{(j)}$ is in fact playing the role of Λ. Further, let $Y_T^{(ij)}$ denote the individual components of the sum on the right hand side of equation (5.5.22). Since we have assumed that $\rho^{(ij)} \geq 0\ \forall i,j$, it follows that the vector

$\left(Y_T^{(1j)}, Y_T^{(2j)}, ..., Y_T^{((n-1)j)}\right)$ is comonotonic since its components are strictly increasing functions of a single variable $S_T^{(j)}$ and so we define

$$S_j^{l_2} = \sum_{i=1}^{n-1} Y_T^{(ij)} \tag{5.5.23}$$

and from (5.5.22) and (5.5.23), it is evident that

$$S \geq_{sl} S_j^{l_2}. \tag{5.5.24}$$

Further, the stop-loss transform of $S_j^{l_2}$ can be written as the sum of stop-loss transform of its components (see for example in Dhaene et al. (2002a)), i.e.,

$$\tilde{E}\left[\left(S_j^{l_2} - (K-1)\right)^+\right] = \sum_{i=1}^{n-1} \tilde{E}\left[\left(Y_T^{(ij)} - F_{Y_T^{(ij)}}^{-1}\left(F_{S_j^{l_2}}(K-1)\right)\right)^+\right] - K_2 \tag{5.5.25}$$

where

$$K_2 = \left((K-1) - F_{S_j^{l_2}}^{-1}\left(F_{S_j^{l_2}}(K-1)\right)\right)\left(1 - F_{S_j^{l_2}}(K-1)\right) \tag{5.5.26}$$

and $(K-1) \in \left(F_{S_j^{l_2}}^{-1+}(0), F_{S j^{l_2}}^{-1}(1)\right)$. Further, $F_{S_j^{l_2}}(K-1)$ is the distribution function of S^{l_2} evaluated at $K-1$ so that we have:

$$\begin{aligned} F_{S_j^{l_2}}(K-1) &= \mathbf{P}\left[S_j^{l_2} \leq (K-1)\right] \\ &= \mathbf{P}\left[\sum_{i=1}^{n-1} S_0^{(i)} \left(\frac{S_T^{(j)}}{S_0^{(j)}}\right)^{\rho^{(ij)}\frac{\sigma_i}{\sigma^{(j)}}} \exp\left(\mu^{(i)} - \rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}\mu^{(j)}\right) \leq (K-1)\right] \end{aligned} \tag{5.5.27}$$

In fact $S_j^{l_2} \leq (K-1)$ if and only if $S_T^{(j)} \leq x S_0^{(j)}$ provided that $\rho^{(ij)} \geq 0 \ \forall \ i, j$, where we substitute x for $S_j/S_0^{(j)}$ in the above expression and obtain its value by solving the following equation

$$\sum_{i=1}^{n-1} S_0^{(i)} (x)^{\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}} \exp\left(\mu^{(i)} - \rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}\mu^{(j)}\right) - (K-1) = 0. \tag{5.5.28}$$

As a result, we have:

$$\begin{aligned} F_{S_j^{l_2}}(K-1) &= F_{S_T^{(j)}}\left(x S_0^{(j)}\right) \\ &= F_{Y_T^{(ij)}}\left(S_0^{(i)} (x)^{\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}} \exp\left(\mu^{(i)} - \rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}\mu^{(j)}\right)\right). \end{aligned} \tag{5.5.29}$$

Using this result in (5.5.25) along with the stop-loss order relationship between S and $S_j^{l_2}$ as given by equation (5.5.24), we obtain

$$\begin{aligned} C(0,x,T) &\geq g\tilde{P}(0,T)\left(\sum_{i=1}^{n-1} S_0^{(i)} \left(S_0^{(j)}\right)^{-\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}} \exp\left(\mu^{(i)} - \rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}\mu^{(j)}\right) \times \right. \\ &\qquad \left. P\left(x S_0^{(j)}, T, \rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}, j\right) - K_2\right) \\ &=: \mathrm{GAOLB}_j^{(2)}, \end{aligned} \tag{5.5.30}$$

where K_2 is defined in (5.5.26). Further, $\mu^{(i)}$, $\left(\sigma^{(i)}\right)^2$ and $\rho^{(ij)}$ are given respectively in (5.5.18)-(5.5.20) and P is defined as the *asymmetric power expectation* function given by

$$P\left(x,t,z,j\right)=\tilde{E}\left[\left(\left(S_t^{(j)}\right)^z-x^z\right)^+\right], \tag{5.5.31}$$

where in our case $t=T$ and we are using $S_T^{(j)}$ in place of $S_t^{(j)}$. Since the above lower bound is a lower bound for every j, we can maximise this for $j\in\{1,2,...,n-1\}$ to obtain an optimal lower bound for GAO.

We have derived this lower bound under the assumption of positive correlation between the objects viz. pure endowments in the basket. Although from the point of view of stochastic processes, this assumption may be restrictive, in reality, it is quite a reasonable assumption. This is because we talking about SZCBs or pure endowments of different duration issued to the same set of lives at the same time T.

5.5.3 The General Lower Bound

To obtain a more general bound, we now relax the assumption of positive correlation between the pure endowments. We adapt the approach undertaken by Deelstra et al. (2008) for Asian basket options for GAOs. This approach considers a non-comonotonic sum based on the methodology of Rogers and Shi (1995) for Asian options.

Let us define $X_T^{(i)}$ in the same way as we have done in the comonotonic case. Next, we choose a single random variable Λ such that $\left(X_T^{(i)},\Lambda\right)$ for every $i\in\{1,2,...,n-1\}$ is Bivariate Normally Distributed (BVN) with correlation coefficient given by $\rho^{(i\Lambda)}$. Clearly a simple application of Jensen's inequality yields the following convex order lower bound for S_i given any random variable Λ.

$$S_T^{(i)}\geq_{cx}\tilde{E}\left(S_T^{(i)}|\Lambda\right). \tag{5.5.32}$$

As a result

$$S\geq_{cx}S^l:=\sum_{i=1}^{n-1}\tilde{E}\left(S_T^{(i)}|\Lambda\right). \tag{5.5.33}$$

We know that if $(X,Y)\sim\text{BVN}\left(\mu_X,\mu_Y,\sigma_X^2,\sigma_Y^2,\rho\right)$, the conditional distribution of the lognormal random variable e^X, given the event $Y=y$ is given as

$$F_{e^X|Y=y}\left(x\right)=\Phi\left(\frac{\log_e x-\left(\mu_X+\rho\frac{\sigma_X}{\sigma_Y}\left(y-\mu_Y\right)\right)}{\sigma_X\sqrt{1-\rho^2}}\right). \tag{5.5.34}$$

where Φ denotes the c.d.f. of standard normal distribution. In our case by assumption, we have $(X_i,\Lambda)\sim\text{BVN}\left(\mu^{(i)},\mu_\Lambda,\left(\sigma^{(i)}\right)^2,\sigma_\Lambda^2,\rho^{(i\Lambda)}\right)$. As a result, the distribution function of $S_T^{(i)}$ conditional on the event $\Lambda=\lambda$ is given as

$$F_{S_T^{(i)}|\Lambda=\lambda}\left(x\right)=\Phi\left(a\left(x\right)\right) \tag{5.5.35}$$

where $a\left(x\right)$ is given by

$$a\left(x\right)=\frac{\log_e x-\left(\log\left(S_0^{(i)}\right)+\mu^{(i)}+\rho^{(i\Lambda)}\frac{\sigma^{(i)}}{\sigma_\Lambda}\left(\lambda-\mu_\Lambda\right)\right)}{\sigma^{(i)}\sqrt{\left(1-\left(\rho^{(i\Lambda)}\right)^2\right)}}. \tag{5.5.36}$$

As the differentiation of c.d.f. yields the p.d.f., therefore the conditional density function of S_i

given $\Lambda = \lambda$ satisfies the following equation:

$$f_{S_T^{(i)}|\Lambda=\lambda}(x) = \frac{1}{x\sigma^{(i)}\sqrt{\left(1-\left(\rho^{(i\Lambda)^2}\right)\right)}}\phi\left(a\left(x\right)\right), \tag{5.5.37}$$

where ϕ denotes the p.d.f. of standard normal distribution. As a result, the conditional expectation of $S_T^{(i)}$ given $\Lambda = \lambda$ is given by the expression

$$\tilde{E}\left(S_T^{(i)}|\Lambda = \lambda\right) = S_0^{(i)} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2\left(1-\left(\rho^{(i\Lambda)}\right)^2\right)}{2} + \rho^{(i\Lambda)}\sigma^{(i)}\frac{(\lambda-\mu_\Lambda)}{\sigma_\Lambda}}. \tag{5.5.38}$$

We utilize this expression to obtain a lower bound for Guaranteed Annuity Option under the above setting. Clearly, using (5.5.5) and (5.5.33), we have

$$C(0,x,T) \geq g\tilde{P}(0,T)\,\tilde{E}\left[\left(\sum_{i=1}^{n-1} S_0^{(i)} e^{\mu_i + \frac{\sigma_i^2\left(1-\rho_{i\Lambda}^2\right)}{2} + \rho_{i\Lambda}\sigma_i\frac{(\lambda-\mu_\Lambda)}{\sigma_\Lambda}} - (K-1)\right)^+\right]. \tag{5.5.39}$$

To obtain the lower bound in a more compact form, we define

$$f(v) = \sum_{i=1}^{n-1} S_0^{(i)} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2\left(1-\rho_{i\Lambda}^2\right)}{2} + \rho^{(i\Lambda)}\sigma^{(i)}\Phi^{-1}(v)} - (K-1) \tag{5.5.40}$$

where

$$v = \Phi\left(\frac{\Lambda-\mu_\Lambda}{\sigma_\Lambda}\right) \tag{5.5.41}$$

Then

$$\tilde{E}\left[\left(S^l-(K-1)\right)^+\right] = \tilde{E}\left[\left(f\left(V\right)\right)^+\right] \tag{5.5.42}$$

with V being uniformly distributed on $(0,1)$. An important consideration in the valuation of $\tilde{E}\left[\left(f\left(V\right)\right)^+\right]$ will be the interval upon which f is positive. This can be obtained by using the following result. Clearly, $f(v)$ is no longer a monotone function of v as in the comonotonic case when not all $\rho^{(i\Lambda)}$ have the same sign.

Proposition 32. *If $\rho^{(i\Lambda)} \geq 0$ for every i, then f has a unique root in $(0,1)$. Otherwise, $f(v)$ has two solutions if and only if $\inf_{v\in(0,1)} f(v) < 0$.*

Proof. Let us first assume that $\rho_{i\Lambda} \geq 0$ for every i. Then, f is a continuous, strictly increasing function of v. Furthermore, we see that f tends to $-(K-1) < 0$ as $v \downarrow 0$ and ∞ as $v \uparrow 1$. Therefore, by applying the Intermediate Value Theorem, we see that f has a single root in $(0,1)$.

On the other hand, if $\rho^{(i\Lambda)}$ and $\rho^{(j\Lambda)}$ are of opposite sign for some $i \neq j$, then observe that the derivative of f with respect to v satisfies

$$f'(v) = \frac{1}{\phi\left(\Phi^{-1}\left(v\right)\right)}\sum_{i=1}^{n-1} S_0^{(i)}\rho^{(i\Lambda)}\sigma^{(i)} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2\left(1-\left(\rho^{(i\Lambda)}\right)^2\right)}{2} + \rho^{(i\Lambda)}\sigma^{(i)}\Phi^{-1}(v)} \tag{5.5.43}$$

where as before ϕ denotes the standard normal density function. We see, that the denominator of $f'(v)$ is strictly positive for $v \in (0,1)$. Let as denote its numerator by $N(v)$. We see that

$$N'(v) = \frac{1}{\phi\left(\Phi^{-1}\left(v\right)\right)}\sum_{i=1}^{n-1} S_0^{(i)}\left(\rho^{(i\Lambda)}\right)^2\left(\sigma^{(i)}\right)^2 e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2\left(1-\left(\rho^{(i\Lambda)}\right)^2\right)}{2} + \rho^{(i\Lambda)}\sigma^{(i)}\Phi^{-1}(v)} \tag{5.5.44}$$

is positive for $v \in (0,1)$. This means that $N(v)$ is an increasing function of v. Moreover, if there exist $\rho^{(i\Lambda)}$, $\rho^{(j\Lambda)}$ of opposite sign, for some $i \neq j$, then

$$\lim_{v \downarrow 0} N(v) = -\infty \text{ and } \lim_{v \uparrow 1} N(v) = +\infty.$$

Therefore, there exists a unique $v^* \in (0,1)$ such that $N(v^*) = 0$ and hence $f'(v^*) = 0$. Also

$$\lim_{v \downarrow 0} f(v) = +\infty \text{ and } \lim_{v \uparrow 1} f(v) = +\infty.$$

So that, $f(v)$ is either positive upon the whole interval $[0,1]$ or has a strictly negative minimum $f(v^*)$. We therefore obtain the following result concerning the infimum of f.

$$\inf_{v \in (0,1)} f(v) = f(v^*). \tag{5.5.45}$$

If $f(v^*) < 0$, then f is a continuous, strictly decreasing function over the interval $(0, v^*)$, which tends to ∞ as $v \downarrow 0$. Hence, there exists a unique $v_1 \in (0, v^*)$ such that $f(v_1) = 0$. Moreover, f is a continuous, strictly increasing function on $(v^*, 1)$, which tends to ∞ as $v \uparrow 1$. Therefore, from the Intermediate Value Theorem, we obtain an additional $v_2 \in (v^*, 1)$ such that $f(v_2) = 0$. If $\inf\limits_{v \in (0,1)} f(v) \geq 0$, then it is immediate that f can only have at most one root. This completes the proof. □

We see from Proposition 32 that either $f(v) \geq 0$ for all $v \in (0,1)$ or there exist $v_1 < v_2$ such that $f(v) \leq 0$ for all $v \in [v_1, v_2]$, with $f(v)$ positive otherwise. This then leads to the following lower bound for guaranteed annuity options.

Theorem 33. *Let $S_T^{(i)}$ be a process given in terms of exponential Lévy model, i.e., $S_T^{(i)} = S_0^{(i)} e^{X_T^{(i)}}$ where $i = 1, 2, ..., n-1$ and let Λ be a normally distributed random variable such that $\left(X_T^{(i)}, \Lambda\right) \sim BVN\left(\mu^{(i)}, \mu_\Lambda, \left(\sigma^{(i)}\right)^2, \sigma_\Lambda^2, \rho^{(i\Lambda)}\right)$. Let $f(v)$ be defined according to equation* (5.5.40). *Then a lower bound for the value of a GAO bought by a life of present age x with guaranteed rate g is given by*

$$C(0, x, T) \leq \ GAOLB_3,$$

where

$$GAOLB_3 = g\tilde{P}(0,T)\left(\sum_{i=1}^{n-1} S_0^{(i)} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2}{2}} - (K-1)\right) \tag{5.5.46}$$

if $f(v) \geq 0$ for all $v \in (0,1)$. Otherwise,

$$GAOLB_3 = g\tilde{P}(0,T)\left(\sum_{i=1}^{n-1} S_0^{(i)} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2}{2}} \Phi\left(\rho^{(i\Lambda)}\sigma^{(i)} - z_2\right) - (K-1)\Phi(-z_2)\right) \tag{5.5.47}$$

if $\rho^{(i\Lambda)}$ are all of positive sign and

$$\begin{aligned} GAOLB_3 \quad = \quad & g\tilde{P}(0,T)\left(\sum_{i=1}^{n-1} S_0^{(i)} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2}{2}} \left(\Phi\left(z_1 - \rho^{(i\Lambda)}\sigma_i\right) + \Phi\left(\rho^{(i\Lambda)}\sigma^{(i)} - z_2\right)\right)\right. \\ & \left. -(K-1)\left(\Phi(z_1) + \Phi(-z_2)\right)\right) \end{aligned} \tag{5.5.48}$$

otherwise, where $z_1 \leq z_2$ solve the following equation in z

$$\sum_{i=1}^{n-1} S_0^{(i)} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2\left(1 - \left(\rho^{(i\Lambda)}\right)^2\right)}{2} + \rho^{(i\Lambda)}\sigma^{(i)} z} - (K-1) = 0. \tag{5.5.49}$$

Proof. The case where $f(v) \geq 0$ is trivial. In the case where $f(v) < 0$ for some v, we see from Proposition 3 that $f(v) = 0$ has one solution in $(0,1)$ if the $\rho^{(i\Lambda)}$ are of the same sign and two otherwise. By setting $z_1 = \Phi^{-1}(v)$ for each i, we obtain the solutions to equation (5.5.49) (where the case with $\rho_{i\Lambda} > 0$ for every i is analogous to setting $z_1 = -\infty$). Let z_1 and z_2 solve equation (5.5.49) and set $v = \Phi(z)$. Then, defining $I = (-\infty, z_1) \cup (z_2, \infty)$, we can write the stop-loss transform of S^l defined in equation (5.5.33) in the following way:

$$\begin{aligned}
\Psi\left(S^l, (K-1)\right) &= \tilde{E}\left[\left(\sum_{i=1}^{n-1} S_0^{(i)} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2\left(1-\left(\rho^{(i\Lambda)}\right)^2\right)}{2} + \rho^{(i\Lambda)}\sigma^{(i)} Z} - (K-1)\right) \mathbb{1}_{\{Z\in I\}}\right] \\
&= \sum_{i=1}^{n-1} S_0^{(i)} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2\left(1-\left(\rho^{(i\Lambda)}\right)^2\right)}{2}} \left(\int_{-\infty}^{z_1} e^{\rho^{(i\Lambda)} z}\phi(z)\,dz + \int_{z_2}^{\infty} e^{\rho^{(i\Lambda)} z}\phi(z)\,dz\right) \\
&\quad - (K-1)\left(\Phi(z_1) + \Phi(-z_2)\right). \qquad (5.5.50)
\end{aligned}$$

Therefore, we obtain equations (5.5.48) and (5.5.47). □

5.6 Upper Bounds for Guaranteed Annuity Options

We derive a couple of upper bounds for the Guaranteed Annuity Options.

5.6.1 A First Upper Bound

This section will focus on finding an upper bound for Guaranteed Annuity Options by using comonotonicity theory in a manner similar to Kaas et al. (2000), Dhaene et al. (2000), Chen et al. (2008) and Linders et al. (2012). Hobson et al. (2005) uses the method of Lagrange multipliers to find an upper bound for basket options.

Define the comonotonic counterpart of $\mathbf{S} = \left(S_T^{(1)}, ..., S_T^{(n-1)}\right)$ with $U \sim U(0,1)$ as $\mathbf{S^u} = \left(F^{-1}_{S_T^{(1)}}(U), ..., F^{-1}_{S_T^{(n-1)}}(U)\right)$. Further define

$$S^c = \sum_{i=1}^{n-1} F^{-1}_{S_T^{(i)}}(U) = \sum_{i=1}^{n-1} S_i^c. \qquad (5.6.1)$$

Clearly (see for example in Dhaene et al., 2002a),

$$S \leq_{cx} S^c \qquad (5.6.2)$$

In other words,

$$\tilde{E}\left[\left(\sum_{i=1}^{n-1} S_T^{(i)} - (K-1)\right)^+\right] \leq \tilde{E}\left[\left(\sum_{i=1}^{n-1} S_i^c - (K-1)\right)^+\right] \qquad (5.6.3)$$

and we have

$$\tilde{E}\left[\left(\sum_{i=1}^{n-1} S_i^c - (K-1)\right)^+\right] = \sum_{i=1}^{n-1} \tilde{E}\left[\left(S_T^{(i)} - F^{-1}_{S_T^{(i)}}\left(F_{S^c}((K-1))\right)\right)^+\right] - K_3 \qquad (5.6.4)$$

where

$$K_3 = \left((K-1) - F^{-1}_{S^c}\left(F_{S^c}(K-1)\right)\right)\left(1 - F_{S^c}(K-1)\right) \qquad (5.6.5)$$

and it is understood that $(K-1) \in \left(F^{-1+}_{S^c}(0), F^{-1}_{S^c}(1)\right)$. As a result, an upper bound for GAO is given as

$$C(0,x,T) \leq g\tilde{P}(0,T)\left(\sum_{i=1}^{n-1} \tilde{E}\left[\left(S_i - F^{-1}_{S_T^{(i)}}\left(F_{S^c}(K-1)\right)\right)^+\right] - K_3\right) \qquad (5.6.6)$$

where K_3 is defined in (5.6.5). Further we write the upper bound given above as

$$C(0,x,T) \leq g\tilde{P}(0,T)\left(\sum_{i=1}^{n-1}\tilde{E}\left[\left(S_T^{(i)} - F_{S_T^{(i)}}^{-1}(x)\right)^+\right] - K_3\right) := \text{ GAOUB}_1 \tag{5.6.7}$$

where $x \in (0,1)$ (see for example Dhaene et al. (2002a)) is the solution of the equation

$$\sum_{i=1}^{n-1} F_{S_T^{(i)}}^{-1}(x) = K - 1. \tag{5.6.8}$$

5.6.2 An Improved Upper Bound by conditioning

We can improve on the upper bound obtained above by finding a conditioning variable Λ under which the S_i are dependent. This is discussed in detail for basket options in Deelstra et al. (2004) using a choice of Λ such that $\Lambda \geq d_\Lambda$ implies $\sum_{i=1}^{n-1} S_T^{(i)} \geq (K-1)$. We use a different more simplified approach for GAOs. We assume that some additional information is available concerning the stochastic nature of $\left(S_T^{(1)}, S_T^{(2)}, ..., S_T^{(n-1)}\right)$. That is, we can find a random variable Λ, with a known distribution, such that the individual conditional distributions of $S_T^{(i)}$ given the event $\Lambda = \lambda$ are known for all i and all possible values of λ. Such an approach can be found in Kaas et al. (2000), Dhaene et al. (2002a) and Dhaene et al. (2002b).

Define

$$S^u = \sum_{i=1}^{n-1} F_{S_T^{(i)}|\Lambda}^{-1}(U) = \sum_{i=1}^{n-1} S_i^u, \tag{5.6.9}$$

where $U \sim U(0,1)$. Then we have

$$S \leq_{cx} S^u \leq_{cx} S^c \tag{5.6.10}$$

Now let $\mathbf{S^u} = \left(S_1^u, ..., S_{n-1}^u\right)$. Since $\left(F_{S_T^{(1)}|\Lambda=\lambda}^{-1}, ..., F_{S_T^{(n-1)}|\Lambda=\lambda}^{-1}\right)$ is comonotonic, we have,

$$F_{S^u|\Lambda=\lambda}^{-1}(p) = \sum_{i=1}^{n-1} F_{S_T^{(i)}|\Lambda=\lambda}^{-1}(p),\ p \in (0,1). \tag{5.6.11}$$

It follows that, in this case

$$\sum_{i=1}^{n-1} F_{S_T^{(i)}|\Lambda=\lambda}^{-1}\left(F_{S^u|\Lambda=\lambda}(K-1)\right) = K - 1. \tag{5.6.12}$$

and so we have

$$\begin{aligned} f(\lambda) &= \tilde{E}\left[\left(\sum_{i=1}^{n-1} S_i^u - (K-1)\right)^+ \middle| \Lambda = \lambda\right] \\ &= \sum_{i=1}^{n-1}\tilde{E}\left[\left(S_T^{(i)} - F_{S_T^{(i)}|\Lambda=\lambda}^{-1}\left(F_{S^u|\Lambda=\lambda}(K-1)\right)\right)^+ \middle| \Lambda = \lambda\right] - K_4, \end{aligned} \tag{5.6.13}$$

where

$$K_4 = \left((K-1) - F_{S^u|\Lambda=\lambda}^{-1}\left(F_{S^u|\Lambda=\lambda}(K-1)\right)\right)\left(1 - F_{S^u|\Lambda=\lambda}(K-1)\right) \tag{5.6.14}$$

and it is clear that $(K-1) \in \left(F_{S^u|\Lambda=\lambda}^{-1+}(0), F_{S^u|\Lambda=\lambda}^{-1}(1)\right)$. By applying the tower property and using the convex order relationship given by (5.6.10), we obtain an upper bound for GAO, i.e.,

$$\begin{aligned}C(0,x,T) &\leq g\tilde{P}(0,T)\,\tilde{E}\left[(S^u-(K-1))^+\right]\\&= g\tilde{P}(0,T)\,\tilde{E}\left[f(\lambda)\right]\\&= g\tilde{P}(0,T)\times\\&\quad\left(\sum_{i=1}^{n-1}\int_{-\infty}^{\infty}\tilde{E}\left[\left(S_T^{(i)}-F^{-1}_{S_T^{(i)}|\Lambda=\lambda}\left(F_{S^u|\Lambda=\lambda}(K-1)\right)\right)^+\middle|\Lambda=\lambda\right]dF_\Lambda(\lambda)-K_4\right)\end{aligned}\tag{5.6.15}$$

where K_4 is defined in (5.6.14). Given the event $\Lambda=\lambda$, let x be the solution to the following equation

$$\sum_{i=1}^{n-1}F^{-1}_{S_T^{(i)}|\Lambda=\lambda}(x)=K-1.\tag{5.6.16}$$

Further, we see from equation (5.6.12), that $x=F_{S^u|\Lambda=\lambda}(K-1)$. It therefore follows, as a result of equation 93 of Dhaene et al. (2002a) that an upper bound for GAO is given as

$$\begin{aligned}C(0,x,T) &\leq g\tilde{P}(0,T)\left(\sum_{i=1}^{n-1}\int_{-\infty}^{\infty}\tilde{E}\left[\left(S_T^{(i)}-F^{-1}_{S_T^{(i)}|\Lambda=\lambda}(x)\right)^+\middle|\Lambda=\lambda\right]dF_\Lambda(\lambda)-K_4\right)\\&=: \mathrm{GAOUB}_j^{(2)},\end{aligned}\tag{5.6.17}$$

where x is obtained by solving (5.6.16).

Since the above upper bound is an upper bound for all j, it follows that we can find the optimal upper bound by maximizing equation (5.6.17) over $j\in\{1,2,...,n-1\}$. As remarked earlier, this bound improves upon the unconditional bound given by (5.6.7). In case if the marginal cdfs $F_{S_T^{(i)}|\Lambda}$ are strictly increasing, one can put $K_4=0$ in (5.6.17) to obtain the upper bound.

5.6.3 An Upper Bound based on the Arithmetic-Geometric Mean Inequality

In order to obtain an upper bound for GAOs which is directly applicable to the affine set up, we make use of arithmetic-geometric mean inequality in a manner similar to Caldana et al. (2016) who used this methodology to arrive at an upper bound for basket options.

Let us first define the arithmetic and geometric mean of the $(n-1)$ pure endowments appearing in the payoff of GAO (c.f. (5.3.12)) respectively as

$$A_T^{(n-1)}=\frac{1}{n-1}\sum_{i=1}^{n-1}S_T^{(i)}\tag{5.6.18}$$

and

$$G_T^{(n-1)}=\left(\prod_{i=1}^{n-1}S_T^{(i)}\right)^{\frac{1}{n-1}},\tag{5.6.19}$$

where $S_T^{(i)};\ i=1,2,...,n-1$ are defined in equation (5.3.13). It is well known that

$$A_T^{(n-1)}\geq G_T^{(n-1)}\quad a.s.\tag{5.6.20}$$

Further, let us define the log-geometric average as

$$Y_T^{(n-1)}=\frac{1}{n-1}\sum_{i=1}^{n-1}\ln S_T^{(i)}.\tag{5.6.21}$$

Next we define as in equation (5.4.9),

$$X_T^{(i)} = \ln\left(\frac{S_T^{(i)}}{S_0^{(i)}}\right);\ i = 1, 2, ..., n-1. \tag{5.6.22}$$

Further, we assume that the joint characteristic function of $\left(X_T^{(1)}, ..., X_T^{(n-1)}\right)$ can be obtained under the transformed measure $\tilde{Q}$, where we define

$$\phi_T(\boldsymbol{\gamma}) = \tilde{E}\left[e^{i\sum_{k=1}^{n-1}\gamma_k X_T^{(k)}}\right] \tag{5.6.23}$$

with $\boldsymbol{\gamma} = [\gamma_1, \gamma_2, ..., \gamma_{n-1}]$. As the next step, we obtain the relationship between log-geometric average and $X_T^{(i)}$'s as follows

$$\begin{aligned} Y_T^{(n-1)} &= \frac{1}{n-1}\sum_{i=1}^{n-1}\ln\left(\frac{S_T^{(i)}}{S_0^{(i)}}S_0^{(i)}\right) \\ &= \frac{1}{n-1}\sum_{i=1}^{n-1}X_T^{(i)} + Y_0^{(n-1)}. \end{aligned} \tag{5.6.24}$$

Next, we try to express the characteristic function of log-geometric average under the transformed measure $\tilde{Q}$ in terms of the joint characteristic function of $X_T^{(i)}$'s viz. $\phi_T(\boldsymbol{\gamma})$ defined in equation (5.6.23). Let $\phi_{Y_T}(\gamma_0)$ denote the characteristic function of log-geometric average $Y_T^{(n-1)}$ with parameter γ_0. Then we have

$$\begin{aligned} \phi_{Y_T}(\gamma_0) &= \tilde{E}\left[e^{i\gamma_0 Y_T^{(n-1)}}\right] \\ &= \tilde{E}\left[e^{i\gamma_0 Y_0^{(n-1)} + i\sum_{k=1}^{n-1}\left(\frac{\gamma_0}{n-1}\right)X_T^{(k)}}\right] \\ &= e^{i\gamma_0 Y_0^{(n-1)}}\phi_T\left(\frac{\gamma_0}{n-1}\mathbf{1}\right) \end{aligned} \tag{5.6.25}$$

where $\mathbf{1} = (1, 1, ..., 1)$ is a $1 \times (n-1)$ vector of 1's, so that $\frac{\gamma_0}{n-1}\mathbf{1}$ is $1 \times (n-1)$ vector with components $\frac{\gamma_0}{n-1}$ and $\phi_T(\boldsymbol{\gamma})$ is defined in (5.6.23). In light of equation (5.6.18), we can express the GAO payoff formula given in equation (5.3.12) as

$$C(0, x, T) = g(n-1)\tilde{P}(0, T)\tilde{E}\left[\left(A_T^{(n-1)} - K'\right)^+\right], \tag{5.6.26}$$

where

$$K' = \frac{K-1}{n-1}. \tag{5.6.27}$$

Adding and subtracting $G_T^{(n-1)}$ within the *max* function on R.H.S. of equation (5.6.26), and exploiting equation (5.6.20), we obtain an upper bound of GAO as

$$\begin{aligned} C(0, x, T) &\leq g(n-1)\tilde{P}(0, T)\left(\tilde{E}\left[\left(G_T^{(n-1)} - K'\right)^+\right] + \tilde{E}\left[A_T^{(n-1)}\right] - \tilde{E}\left[G_T^{(n-1)}\right]\right) \\ &=: \text{GAOUB} \end{aligned} \tag{5.6.28}$$

We make use of Fourier inversion to compute the call type expectation involved in the upper bound and we state the result in the following proposition.

Proposition 34. *Given the geometric mean of $n-1$ pure endowments defined in equation* (5.6.19) *and $K' > 0$,*

$$\tilde{E}\left[\left(G_T^{(n-1)} - K'\right)^+\right] = \frac{e^{-\delta\ln K'}}{\pi}\int_0^\infty e^{-i\eta\ln K'}\Psi_T^G(\eta;\delta)\,d\eta \tag{5.6.29}$$

where $\Psi_T^G(\eta;\delta)$ *denotes the Fourier transform of* $\tilde{E}\left[\left(G_T^{(n-1)}-K^{'}\right)^{+}\right]$ *with respect to* $\ln K^{'}$ *along with the damping factor* $e^{\delta \ln K^{'}}$ *such that*

$$\Psi_T^G(\eta;\delta)=e^{i(\eta-i(\delta+1))Y_0^{(n-1)}}\frac{\phi_T\left(\frac{\eta-i(\delta+1)}{n-1}\mathbf{1}\right)}{\delta^2+\delta-\eta^2+i\eta(2\delta+1)}, \tag{5.6.30}$$

where the parameter δ *tunes the damping factor (c.f. Carr and Madan, 1998; Caldana et al., 2016) and* $\phi_T(.)$ *is defined in equation* (5.6.23).

Proof. Let $f_{Y_T}(y)$ denote the probability density function (p.d.f.) of the log-geometric average $Y_T^{(n-1)}$. We introduce the damping factor in accordance with Carr and Madan (1998). Then, by definition, the Fourier transform of $\tilde{E}\left[\left(G_T^{(n-1)}-K^{'}\right)^{+}\right]$ with respect to $\ln K^{'}$ along with the damping factor $e^{\delta\ln K^{'}}$ is given as

$$\begin{aligned}
\Psi_T^G(\eta;\delta) &= \int_{\mathbb{R}} e^{i\eta\ln K^{'}+\delta\ln K^{'}}\tilde{E}\left[\left(e^{Y_T^{(n-1)}}-K^{'}\right)^{+}\right]d\ln K^{'}\\
&= \int_{\mathbb{R}} e^{i\eta\ln K^{'}+\delta\ln K^{'}}\int_{\ln K^{'}}^{\infty}\left(e^{y}-K^{'}\right)f_{Y_T}(y)\ dy\ d\ln K^{'}\\
&= \int_{\mathbb{R}} e^{i\eta\ln K^{'}+\delta\ln K^{'}}\int_{\ln K^{'}}^{\infty}e^{y}f_{Y_T}(y)\ dy\ d\ln K^{'}\\
&\quad -\int_{\mathbb{R}} e^{i\eta\ln K^{'}+\delta\ln K^{'}}\int_{\ln K^{'}}^{\infty}K^{'}f_{Y_T}(y)\ dy\ d\ln K^{'}\\
&= \Psi_T^{G_1}(\eta;\delta)-\Psi_T^{G_2}(\eta;\delta).
\end{aligned} \tag{5.6.31}$$

We evaluate both integrals by adopting a change of order of integration, as detailed below

$$\begin{aligned}
\Psi_T^{G_1}(\eta;\delta) &= \int_{\mathbb{R}} e^{y}\left(\int_{-\infty}^{y}e^{i\eta\ln K^{'}+\delta\ln K^{'}}d\ln K^{'}\right)f_{Y_T}(y)\,dy\\
&= \frac{1}{i\eta+\delta}\int_{\mathbb{R}}e^{i(\eta-i(\delta+1))y}f_{Y_T}(y)\,dy\\
&= \frac{\phi_{Y_T}(\eta-i(\delta+1))}{i\eta+\delta}\\
&= e^{i(\eta-i(\delta+1))Y_0^{(n-1)}}\frac{\phi_T\left(\frac{\eta-i(\delta+1)}{n-1}\mathbf{1}\right)}{i\eta+\delta}.
\end{aligned} \tag{5.6.32}$$

where the last couple of statements follow from the definition of the characteristic function of $Y_0^{(n-1)}$ given in (5.6.25) and its link to the joint characteristic function of joint characteristic function of $\left(X_T^{(1)},...,X_T^{(n-1)}\right)$ defined in (5.6.23). On the same lines we have

$$\Psi_T^{G_2}(\eta;\delta)=e^{i(\eta-i(\delta+1))Y_0^{(n-1)}}\frac{\phi_T\left(\frac{\eta-i(\delta+1)}{n-1}\mathbf{1}\right)}{i\eta+(\delta+1)}. \tag{5.6.33}$$

Substituting $\Psi_T^{G_1}(\eta;\delta)$ and $\Psi_T^{G_2}(\eta;\delta)$ in equation (5.6.31), remembering the damping factor we get the requisite result given in equation (5.6.29). □

In a similar manner we obtain

$$\tilde{E}\left[G_T^{(n-1)}\right]=e^{Y_0^{(n-1)}}\phi_T\left(\frac{-i}{n-1}\mathbf{1}\right). \tag{5.6.34}$$

We then plug the formulae (5.6.29) and (5.6.34) into equation (5.6.28) to obtain the upper bound GAOUB.

The Upper Bound under the Affine Set Up

Consider the affine set up of section 5.4 (c.f. equations (5.4.8)-(5.4.11)). Let ϕ_{X_T} denote the characteristic function of X_T with parameter Λ under the transformed measure $\tilde{Q}$ so that

$$\phi_{X_T}(\Lambda) = \tilde{E}\left[e^{i\langle\Lambda,\, X_T\rangle}\right]. \tag{5.6.35}$$

Now using equation (5.4.11), we see that the joint characteristic function of $\left(X_T^{(1)}, ..., X_T^{(n-1)}\right)$ under the transformed measure $\tilde{Q}$, given in equation (5.6.23) becomes ,

$$\phi_T^{aff}(\boldsymbol{\gamma}) = \phi_{X_T}\left(-\sum_{k=1}^{n-1}\gamma_k\tilde{\psi}(k, R+M)\right), \tag{5.6.36}$$

where $\left(-\sum_{k=1}^{n-1}\gamma_k\tilde{\psi}(k, R+M)\right)$ is the parameter of the characteristic function, with $\tilde{\psi}(k, R+M)$ satisfying the equations (5.4.5) with $\tau = k$. As a result, $\Psi_T^G(\eta;\delta)$ given in equation (5.6.30) can be written in a more compact way as

$$\Psi_T^{G^{aff}}(\eta;\delta) = e^{i(\eta - i(\delta+1))Y_0^{(n-1)}}\frac{\phi_{X_T}\left(-\frac{(\eta-i(\delta+1))}{n-1}\sum_{k=1}^{n-1}\tilde{\psi}(k, R+M)\right)}{\delta^2+\delta-\eta^2+i\eta(2\delta+1)}. \tag{5.6.37}$$

Similarly, we have from equation (5.6.34),

$$\tilde{E}^{aff}\left[G_T^{(n-1)}\right] = e^{Y_0^{(n-1)}}\phi_{X_T}\left(\frac{i}{n-1}\sum_{k=1}^{n-1}\tilde{\psi}(k, R+M)\right). \tag{5.6.38}$$

Moreover, using the definition of arithmetic average given in equation (5.6.18) and utilizing (5.4.8), we see that

$$\tilde{E}^{aff}\left[A_T^{(n-1)}\right] = \frac{1}{n-1}\sum_{k=1}^{n-1}\left(e^{-\left((\bar{r}+\bar{\mu})k+\tilde{\phi}(k,R+M)\right)}\mathcal{L}\left(\tilde{\psi}(k, R+M)\right)\right), \tag{5.6.39}$$

where as defined in Section 5.5.1, $\mathcal{L}$ denotes the Laplace transform of X_T with parameter $\tilde{\psi}(k, R+M)$ under the transformed measure $\tilde{Q}$. Finally we substitute equation (5.6.37) in the expression (5.6.29) and then the result and equations (5.6.38)-(5.6.39) into (5.6.28) to get

$$\begin{aligned}
\text{GAOUB}^{aff} \;=\; & g(n-1)\tilde{P}(0,T)\left(\frac{1}{n-1}\sum_{k=1}^{n-1}\left(e^{-\left((\bar{r}+\bar{\mu})k+\tilde{\phi}(k,R+M)\right)}\mathcal{L}\left(\tilde{\psi}(k, R+M)\right)\right)\right.\\
& -e^{Y_0^{(n-1)}}\phi_{X_T}\left(\frac{i}{n-1}\sum_{k=1}^{n-1}\tilde{\psi}(k, R+M)\right)\\
& +\frac{e^{-\delta\ln K'}}{\pi}\int_0^\infty\frac{e^{-i\left(\eta\ln K' - (\eta - i(\delta+1))Y_0^{(n-1)}\right)}}{\delta^2+\delta-\eta^2+i\eta(2\delta+1)}\times\\
& \left.\phi_{X_T}\left(-\frac{(\eta-i(\delta+1))}{n-1}\sum_{k=1}^{n-1}\tilde{\psi}(k, R+M)\right)d\eta\right),
\end{aligned} \tag{5.6.40}$$

where $\phi_{X_T}(.)$ is defined in equation (5.6.35) and $\mathcal{L}$ denotes the Laplace transform of X_T under the transformed measure $\tilde{Q}$.

5.7 Examples

We now derive lower and upper bounds by choosing specific models for the interest rate and force of mortality.

5.7.1 Vasicek Model

Let us consider the case where the interest rate (r_t) and the force of mortality (μ_t) for an insured aged x at time 0 obey the Vasicek model (c.f. Vasicek, 1977), with dynamics given by

$$dr_t = a\left(b - r_t\right)dt + \sigma dW_t^1, \tag{5.7.1}$$

where a, b and σ are positive constants and W_t^1 is a standard Brownian motion and

$$d\mu_t = c\mu_t dt + \xi dZ_t, \tag{5.7.2}$$

where c and ξ are positive constants and Z_t is a standard Brownian motion correlated with W_t^1 so that

$$dW_t^1 dZ_t = \rho dt. \tag{5.7.3}$$

This means that $Z_t = \rho W_t^1 + \sqrt{1-\rho^2}W_t^2$, where W_t^2 is a standard Brownian motion independent of W_t^1. It is important to fine tune the model in case of mortality by choosing c and ξ properly to avoid the possibility of negative mortality rates. In fact, under this model, we have (c.f. Liu et al., 2013, for details)

$$S_T^{(i)} = S_0^{(i)} e^{X_T^{(i)}} \tag{5.7.4}$$

where

$$S_0^{(i)} = \alpha^{(i)} \tag{5.7.5}$$

with

$$\alpha^{(i)} = e^{D^{(i)} + \tilde{H}^{(i)}} \tag{5.7.6}$$

such that for $i = 1, 2, ..., n-1$

$$D^{(i)} = \left(b - \frac{\sigma^2}{2a^2}\right)\left(A^{(i)} - i\right) - \frac{\sigma^2}{4a}\left(A^{(i)}\right)^2 \tag{5.7.7}$$

with

$$A^{(i)} = \frac{1 - e^{-ai}}{a} \tag{5.7.8}$$

and

$$\tilde{H}^{(i)} = \left(\frac{\rho\sigma\xi}{ac} - \frac{\xi^2}{2c^2}\right)\left(\tilde{G}^{(i)} - i\right) + \frac{\rho\sigma\xi}{ac}\left(A^{(i)} - \phi^{(i)}\right) + \frac{\xi^2}{4c}\left(\tilde{G}^{(i)}\right)^2 \tag{5.7.9}$$

with

$$\tilde{G}^{(i)} = \frac{e^{ci} - 1}{c} \tag{5.7.10}$$

and

$$\phi^{(i)} = \frac{1 - e^{(a-c)i}}{a - c}. \tag{5.7.11}$$

Further $\left\{X_T^{(i)}\right\}_{i=1,2,\ldots,n-1}$ is defined as:

$$X_T^{(i)} = -\left(A^{(i)} r_T + \tilde{G}^{(i)} \mu_T\right), \tag{5.7.12}$$

where $A^{(i)}$ and $\tilde{G}^{(i)}$ are defined respectively in equations (5.7.8) and (5.7.10). Here we have (c.f. Liu et al., 2013)

$$\left(r_T,\, \mu_T\right) \sim \text{BVN}\left(\mu_{r_T}, \mu_{\mu_T}, \sigma^2_{r_T}, \sigma^2_{\mu_T}, \rho\left(r_T, \mu_T\right)\right) \tag{5.7.13}$$

where BVN stands for bivariate normal distribution and

$$\mu_{r_T} = \tilde{E}\left[r_T\right] = e^{-aT} r_0 + b\left(1 - e^{-aT}\right) - \frac{\sigma^2}{2a^2}\left(1 - e^{-aT}\right)^2 - \frac{\rho\sigma\xi}{c}\left[\frac{e^{cT}\left(e^{-cT} - e^{-aT}\right)}{a - c} - \frac{1 - e^{-aT}}{a}\right], \tag{5.7.14}$$

$$\sigma_{r_T}^2 = \frac{\sigma^2}{2a}\left(1 - e^{-2aT}\right), \tag{5.7.15}$$

$$\mu_{\mu_T} = \tilde{E}\left[\mu_T\right] = e^{cT}\mu_0 - \frac{\xi^2}{2c^2}\left(1 - e^{cT}\right)^2 - \frac{\rho\sigma\xi}{a}\left[\frac{e^{-aT}\left(e^{aT} - e^{cT}\right)}{a-c} - \frac{e^{cT}-1}{c}\right], \tag{5.7.16}$$

$$\sigma_{\mu_T}^2 = \frac{\xi^2}{2c}\left(e^{2cT} - 1\right), \tag{5.7.17}$$

and

$$Cov\left[r_T, \mu_T\right] = \frac{\rho\sigma\xi}{a-c}\left(1 - e^{-(a-c)t}\right) \tag{5.7.18}$$

with Cov standing for covariance. In light of equation (5.7.12) and (5.7.13), it is clear that

$$X_T^{(i)} \sim N\left(\mu^{(i)}, \left(\sigma^{(i)}\right)^2\right) \tag{5.7.19}$$

where $\mu^{(i)}$ and $\left(\sigma^{(i)}\right)^2$ are defined respectively in equations (5.5.18) and (5.5.19) are given as follows in the context of the Vasicek model.

$$\mu^{(i)} = -\left(A^{(i)}\mu_{r_T} + \tilde{G}^{(i)}\mu_{\mu_T}\right) \tag{5.7.20}$$

$$\left(\sigma^{(i)}\right)^2 = \left(A^{(i)}\right)^2\sigma_{r_T}^2 + \left(\tilde{G}^{(i)}\right)^2\sigma_{\mu_T}^2 + 2A^{(i)}\tilde{G}^{(i)}Cov\left[r_T, \mu_T\right]. \tag{5.7.21}$$

In fact, one may write

$$X_T^{(i)} = -W_T^{(i)}, \tag{5.7.22}$$

where $W_T^{(i)} \sim N\left(-\mu^{(i)}, \left(\sigma^{(i)}\right)^2\right)$. Finally, for $i \neq j$ we note that

$$\rho^{(ij)} = Corr\left(X_T^{(i)}, X_T^{(j)}\right) = Corr\left(W_T^{(i)}, W_T^{(j)}\right) \tag{5.7.23}$$

where $Corr$ stands for correlation and for $i \neq j = 1, 2, ..., n-1$

$$\rho^{(ij)} = \frac{A^{(i)}A^{(j)}\sigma_{r_T}^2 + \left(A_i\tilde{G}^{(j)} + A^{(j)}\tilde{G}^{(i)}\right)Cov\left[r_T, \mu_T\right] + \tilde{G}^{(i)}\tilde{G}^{(j)}\sigma_{\mu_T}^2}{\sigma^{(i)}\sigma^{(j)}}. \tag{5.7.24}$$

The computation of the price bounds for GAO hinges upon the availability of the price of SZCBs $\tilde{P}(0,T)$. We refer to Liu et al. (2013) for the price of these instruments under the Vasicek model and note that

$$\tilde{P}(0,T) = \alpha^{(0)}e^{V^{(0)}} \tag{5.7.25}$$

with

$$\alpha^{(0)} = e^{D^{(0)} + \tilde{H}^{(0)}} \tag{5.7.26}$$

where

$$D^{(0)} = \left(b - \frac{\sigma^2}{2a^2}\right)\left(A^{(0)} - T\right) - \frac{\sigma^2}{4a}\left(A^{(0)}\right)^2 \tag{5.7.27}$$

with

$$A^{(0)} = \frac{1 - e^{-aT}}{a} \tag{5.7.28}$$

and

$$\tilde{H}^{(0)} = \left(\frac{\rho\sigma\xi}{ac} - \frac{\xi^2}{2c^2}\right)\left(\tilde{G}^{(0)} - T\right) + \frac{\rho\sigma\xi}{ac}\left(A^{(0)} - \phi^{(0)}\right) + \frac{\xi^2}{4c}\left(\tilde{G}^{(0)}\right)^2 \tag{5.7.29}$$

with

$$\tilde{G}^{(0)} = \frac{e^{cT} - 1}{c} \tag{5.7.30}$$

and

$$\phi^{(0)} = \frac{1 - e^{(a-c)T}}{a - c}. \tag{5.7.31}$$

and finally

$$V^{(0)} = -\left(A^{(0)} r_0 + \tilde{G}^{(0)} \mu_0\right), \tag{5.7.32}$$

where $A^{(0)}$ and $\tilde{G}^{(0)}$ are defined respectively in equations (5.7.28) and (5.7.30) and r_0 and μ_0 are the initial (time 0) values of the interest rate and mortality rate. We now derive lower and upper bounds for the Vasicek model on the lines of $\mathrm{GAOLB}_j^{(2)}$ and $\mathrm{GAOUB}_j^{(2)}$ respectively.

The Lower Bound $\mathrm{GAOLB}_j^{(VS)}$

We know that if $(X, Y) \sim \mathrm{BVN}\left(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho\right)$, the conditional distribution of the lognormal random variable e^X, given the event $e^Y = y$ is given as

$$F_{e^X|e^Y=y}(x) = \Phi\left(\frac{\log_e x - \left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}\left(\log_e y - \mu_Y\right)\right)}{\sigma_X\sqrt{1-\rho^2}}\right). \tag{5.7.33}$$

where Φ denotes the c.d.f. of standard normal distribution. Clearly for two assets, say the ith and jth asset in the basket considered above, it is evident from (5.7.19) and (5.7.23) that $(X_i, X_j) \sim \mathrm{BVN}\left(\mu^{(i)}, \mu^{(j)}, \left(\sigma^{(i)}\right)^2, \left(\sigma^{(j)}\right)^2, \rho^{(ij)}\right)$. Further from equation (5.7.4) as $S_T^{(i)} = S_0^{(i)} e^{X_T^{(i)}}$, we have from equation (5.7.33) that the distribution function of $S_T^{(i)}$ conditional on the event $S_T^{(j)} = s$ is given as

$$F_{S_T^{(i)}|S_T^{(j)}=s}(x) = \Phi(a(x))$$

where $a(x)$ is given by

$$a(x) = \frac{\log_e x - \left(\log\left(S_0^{(i)}\right) + \mu^{(i)} + \rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}\left(\log\left(\frac{s}{S_0^{(j)}}\right) - \mu^{(j)}\right)\right)}{\sigma^{(i)}\sqrt{\left(1-\left(\rho^{(ij)}\right)^2\right)}}. \tag{5.7.34}$$

As the differentiation of c.d.f. yields the p.d.f., therefore the conditional density function of $S_T^{(i)}$ given $S_T^{(j)} = s$ satisfies the following equation:

$$f_{S_T^{(i)}|S_T^{(j)}=s}(x) = \frac{1}{x\sigma^{(i)}\sqrt{\left(1-\left(\rho^{(ij)}\right)^2\right)}}\phi(a(x)), \tag{5.7.35}$$

where ϕ denotes the p.d.f. of standard normal distribution. As a result, the conditional expectation of $S_T^{(i)}$ given $S_T^{(j)}$ is given by the expression

$$\tilde{E}\left(S_T^{(i)}|S_T^{(j)}\right) = S_0^{(i)}\left(\frac{S_T^{(j)}}{S_0^{(j)}}\right)^{\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2\left(1-\left(\rho^{(ij)}\right)^2\right)}{2} - \rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}\mu^{(j)}} \tag{5.7.36}$$

Invoking equation (5.5.21) and denoting the individual components of the sum on the r.h.s. of equation (3.5.6) as Y_{ij}, we see that under the assumption $\rho^{(ij)} \geq 0\ \forall i, j$, the vector $\left(Y_{1j}, Y_{2j}, ..., Y_{(n-1)j}\right)$ is comonotonic, and so define

$$S_j^{l_3} = \sum_{i=1}^{n-1} Y_{ij} \tag{5.7.37}$$

and from (5.7.36), (5.5.21) and (5.7.37), it is evident that

$$S \geq_{cx} S_j^{l_3}. \tag{5.7.38}$$

Further, the stop-loss transform of $S_j^{l_3}$ can be written as the sum of stop-loss transform of its components (see for example in Dhaene et al., 2002a), i.e.,

$$\tilde{E}\left[\left(S_j^{l_3} - (K-1)\right)^+\right] = \sum_{i=1}^{n-1} \tilde{E}\left[\left(Y_{ij} - F_{Y_{ij}}^{-1}\left(F_{S_j^{l_3}}(K-1)\right)\right)^+\right] \tag{5.7.39}$$

where $F_{S_j^{l_3}}(K-1)$ is the distribution function of S^{l_3} evaluated at $K-1$ so that we have:

$$\begin{aligned} F_{S_j^{l_3}}(K-1) &= \mathbf{P}\left[S_j^{l_3} \leq (K-1)\right] \\ &= \mathbf{P}\left[\sum_{i=1}^{n-1} S_0^{(i)} \left(\frac{S_T^{(j)}}{S_0^{(j)}}\right)^{\rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}}} e^{\mu^{(i)} + \frac{\left(1-\left(\rho^{(ij)}\right)^2\right)}{2} - \rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}} \mu^{(j)}} \leq (K-1)\right] \end{aligned} \tag{5.7.40}$$

In fact $S_j^{l_3} \leq (K-1)$ if and only if $S_T^{(j)} \leq x S_0^{(j)}$ provided that $\rho^{(ij)} \geq 0 \ \forall \ i,j$, where we substitute x for $S_j/S_0^{(j)}$ in the above expression and obtain its value by solving the following equation

$$\sum_{i=1}^{n-1} S_0^{(i)} (x)^{\rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}}} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2 \left(1-\left(\rho^{(ij)}\right)^2\right)}{2} - \rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}} \mu^{(j)}} - (K-1) = 0. \tag{5.7.41}$$

As a result, we have:

$$\begin{aligned} F_{S_j^{l_3}}(K-1) &= F_{S_j}\left(x S_0^{(j)}\right) \\ &= F_{Y_T^{(ij)}}\left(S_0^{(i)} (x)^{\rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}}} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2 \left(1-\left(\rho^{(ij)}\right)^2\right)}{2} - \rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}} \mu^{(j)}}\right). \end{aligned} \tag{5.7.42}$$

Using this result in (5.7.39) along with the convex order relationship between S and $S_j^{l_3}$ as given by equation (5.7.38), we obtain

$$\begin{aligned} C(0,x,T) \geq\ & g\tilde{P}(0,T)\left(\sum_{i=1}^{n-1} S_0^{(i)} \left(S_0^{(j)}\right)^{-\rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}}} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2 \left(1-\rho_{ij}^2\right)}{2} - \rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}} \mu^{(j)}} \times \right. \\ & \left. P\left(x S_0^{(j)}, T, \rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}}, j\right)\right) \end{aligned} \tag{5.7.43}$$

where $\mu^{(i)}$, $\left(\sigma^{(i)}\right)^2$ and $\rho^{(ij)}$ for the Vasicek model are given respectively in (5.7.20), (5.7.21) and (5.7.23) and P is defined in (5.5.31) so that we have

$$\begin{aligned} P\left(x S_0^{(j)}, T, \rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}}, j\right) &= \left(S_0^{(j)}\right)^{\rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}}} \left(e^{\rho^{(ij)} \frac{\sigma^{(i)}}{2\sigma^{(i)}} \left(\rho^{(ij)} \sigma^{(i)} \sigma^{(j)} - 2\mu^{(j)}\right)} \Phi\left(d^{(1j)}\right) \right. \\ & \left. - x^{\rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}}} \Phi\left(d^{(2j)}\right)\right), \end{aligned} \tag{5.7.44}$$

where d_{2j} and d_{1j} are given respectively as

$$d^{(2j)} = \frac{\log_e\left(\frac{1}{x}\right) - \mu^{(j)}}{\sigma^{(j)}} \tag{5.7.45}$$

$$d^{(1j)} = d^{(2j)} + \rho^{(ij)}\sigma^{(i)} \tag{5.7.46}$$

Inserting (5.7.44) in (5.7.43), we achieve the lower bound $\text{GAOLB}_j^{(VS)}$ as follows

$$\begin{aligned} C(0,x,T) \quad &\geq \quad g\tilde{P}(0,T)\left(\sum_{i=1}^{n-1} S_0^{(i)} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2\left(1-\left(\rho^{(ij)}\right)^2\right)}{2} - \rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}\mu^{(j)}} \times \right. \\ &\left. \left(e^{\rho^{(ij)}\frac{\sigma^{(i)}}{2\sigma^{(j)}}\left(\rho^{(ij)}\sigma^{(i)}\sigma^{(j)} - 2\mu^{(j)}\right)} \Phi\left(d^{(1j)}\right) - x^{\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}} \Phi\left(d^{(2j)}\right)\right)\right) \\ &=: \quad \text{GAOLB}_j^{(VS)}. \end{aligned} \tag{5.7.47}$$

Since the above lower bound is a lower bound for every j, we can maximise this for $j \in \{1, 2, ..., n-1\}$ to obtain an optimal lower bound for GAO in the Vasicek Case.

The Improved Upper Bound $\textbf{GAOUB}_j^{(VS)}$

In section 5.6.2, we have shown that the upper bound SWUB_1 can be improved by assuming that there exists a random variable Λ such that $\text{Cov}\left(S_T^{(i)}, \Lambda\right) \neq 0 \; \forall i$. Suppose this assumption is true here and we choose

$$\Lambda = \sum_{k=1}^{n-1} Y_T^{(k)} \tag{5.7.48}$$

where

$$Y_T^{(k)} = \frac{X_T^{(k)} - \mu^{(k)}}{\sigma^{(k)}} \tag{5.7.49}$$

where in the context of the Vasicek Model, $X_T^{(k)}$, $\mu^{(k)}$ and $\sigma^{(k)}$ are defined respectively in equations (5.7.12), (5.7.20) and (5.7.21) and it is evident from (5.7.19) that

$$Y_T^{(k)} \sim N(0,1); \quad k = 1, 2, ..., n-1 \tag{5.7.50}$$

and as a result by the definition of Λ in equation (5.7.48)

$$\Lambda \sim N\left(0, \sigma_\Lambda^2\right) \tag{5.7.51}$$

where

$$\sigma_\Lambda^2 = (n-1) + \sum_{k=1}^{n-1}\sum_{\substack{l=1 \\ k\neq l}}^{n-1} \rho^{(kl)} \tag{5.7.52}$$

where $\rho^{(kl)}$ is defined in equation (5.7.23). Also simple calculations show that the correlation coefficient between $X_T^{(k)}$ and Λ is given by

$$\rho_{k\Lambda} = \frac{\sum_{l=1}^{n-1} \rho^{(kl)}}{\sqrt{(n-1) + \sum_{k=1}^{n-1}\sum_{\substack{l=1 \\ k\neq l}}^{n-1} \rho^{(kl)}}}; \quad k = 1, 2, ..., n-1. \tag{5.7.53}$$

As a result

$$\left(X_T^{(k)}, \Lambda\right) \sim \text{BVN}\left(\mu^{(k)}, \; 0, \; \left(\sigma^{(k)}\right)^2, \; \sigma_\Lambda^2, \; \rho_{k\Lambda}\right). \tag{5.7.54}$$

Now, from equation (5.6.17) noting that the marginal cdfs $F_{S_T^{(i)}|\Lambda=\lambda}$ are strictly increasing so that $K_4 = 0$, we see that an upper bound for GAO is given as

$$C(0,x,T) \leq g\tilde{P}(0,T) \sum_{i=1}^{n-1} \int_{-\infty}^{\infty} \mathbf{E}\left[\left(S_T^{(i)} - F^{-1}_{S_T^{(i)}|\Lambda=\lambda}(x)\right)^+ \middle| \Lambda = \lambda\right] d\Phi\left(\frac{\lambda}{\sigma_\Lambda}\right), \tag{5.7.55}$$

where using equation (5.6.16), we see that x is obtained by solving the following equation

$$\sum_{i=1}^{n-1} F^{-1}_{S_T^{(i)}|\Lambda=\lambda}(x) = K-1. \tag{5.7.56}$$

An explicit formula for the conditional inverse distribution function of $S_T^{(i)}$ given the event $\Lambda=\lambda$, is provided by the following result.

Proposition 35. *Under the assumptions of the Vasicek model, conditional on the event $\Lambda=\lambda$, the conditional inverse distribution function of $S_T^{(i)}$ for $i=1,2,...,n-1$ is given by*

$$F^{-1}_{S_T^{(i)}|\Lambda=\lambda} = S_0^{(i)} e^{\mu^{(i)}+\rho_{i\Lambda}\frac{\sigma^{(i)}}{\sigma_\Lambda}\lambda+\sigma^{(i)}\sqrt{1-\rho_{i\Lambda}^2}\Phi^{-1}(x)}. \tag{5.7.57}$$

Proof. The proof follows directly from equations (5.5.35) and (5.5.36) of Section 5.5.3. □

From equation (5.7.56), we then wish to solve the following for x

$$\sum_{i=1}^{n-1} S_0^{(i)} e^{\mu^{(i)}+\rho_{i\Lambda}\frac{\sigma^{(i)}}{\sigma_\Lambda}\lambda+\sigma^{(i)}\sqrt{1-\rho_{i\Lambda}^2}\Phi^{-1}(x)} = K-1. \tag{5.7.58}$$

As a result, using equation (5.7.55), the improved upper bound for Guaranteed Annuity Option is given by the following set of equations

$$\begin{aligned} C(0,x,T) \quad &\leq \quad g\tilde{P}(0,T)\int_{-\infty}^{\infty}\left(\sum_{i=1}^{n-1} S_0^{(i)} e^{\mu^{(i)}+\rho_{i\Lambda}\frac{\sigma^{(i)}}{\sigma_\Lambda}\lambda+\frac{1}{2}\left(\sigma^{(i)}\right)^2\left(1-\rho_{i\Lambda}^2\right)}\Phi\left(c_1^{(i)}\right)\right. \\ &\qquad \left. -\left(K-1\right)\left(1-x\right)\right) d\Phi\left(\frac{\lambda}{\sigma_\Lambda}\right) \\ &=: \quad \mathrm{GAOUB}_j^{(VS)} \end{aligned} \tag{5.7.59}$$

and

$$c_1^{(i)} = \sigma^{(i)}\sqrt{\left(1-\rho_{i\Lambda}^2\right)} - \Phi^{-1}(x) \quad i=1,2,...,n-1 \tag{5.7.60}$$

where $x \in (0,1)$ solves equation (5.7.58).

5.7.2 The Multi-CIR Model

We now consider a p-dimensional affine process $X := (X_t)_{t\geq 0}$ having independent components $(X_{it})_{t\geq 0}$ that function according to the following CIR risk-neutral dynamics:

$$dX_{it} = k_i\left(\theta_i - X_{it}\right)dt + \sigma_i\sqrt{X_{it}}dW_{it}^{\mathbb{Q}},\ i=1,...,p. \tag{5.7.61}$$

One can refer to Deelstra et al. (2016) to show that this model fits into the general affine framework.

Survival Zero Coupon Bond Pricing

Adhering to the notations of the affine set-up defined in section 5.4, in context of mortality and interest rate, let $M,R \in \mathbb{R}_n$ with respective components $M_i, R_i;\ i=1,2,...,p$. The price of a zero-coupon bond under the multi CIR model (5.7.61) is given by

$$\begin{aligned} \tilde{P}(t,T) \quad &= \quad \mathbb{E}\left[e^{-\int_t^T(\bar{r}+\bar{\mu})+\langle(R+M),X_s\rangle ds}|\mathcal{F}_t\right] \\ &= \quad e^{-(\bar{r}+\bar{\mu})(T-t)}\prod_{i=1}^{p}\mathbb{E}\left[e^{-\int_t^T\langle(R_i+M_i),X_{is}\rangle ds}|\mathcal{F}_t\right] \end{aligned}$$

$$= \quad e^{-(\bar{r}+\bar{\mu})(T-t)}\prod_{i=1}^{p}e^{-\tilde{\phi}_i(T-t,R_i+M_i)-\tilde{\psi}_i(T-t,R_i+M_i)X_{it}} \tag{5.7.62}$$

where $\tilde{\phi}_i$ and $\tilde{\psi}_i$ satisfy the following Riccatti equations for every $i = 1, 2, ..., p$ (c.f. Duffie et al., 2000):

$$\begin{cases} \frac{\partial\tilde{\psi}(\tau,u_i)}{\partial\tau} = 1 - k_i\tilde{\psi}_i(\tau,u_i) + \frac{u_i\sigma_i^2}{2}\tilde{\psi}_i(\tau,u_i)^2, \\ \frac{\partial\tilde{\phi}(\tau,u_i)}{\partial\tau} = k_i\theta_i u_i\tilde{\psi}_i(\tau,u_i), \end{cases} \tag{5.7.63}$$

with $\tau = T - t$, $u_i = R_i + M_i$ and initial conditions $\tilde{\psi}_i(0,u_i) = 0$ and $\tilde{\phi}_i(0,u_i) = 0$. The solutions of this system with $i = 1, 2, ..., p$ are

$$\begin{aligned} \tilde{\psi}_i(\tau,u_i) &= \frac{2u_i}{\eta(u_i)+k_i} - \frac{4u_i+\eta(u_i)}{\eta(u_i)+k_i} \\ &\quad\times\frac{1}{(\eta(u_i)+k_i)\exp[\eta(u_i)\tau]+\eta(u_i)-k_i} \end{aligned} \tag{5.7.64}$$

$$\begin{aligned} \tilde{\phi}_i(\tau,u_i) &= \frac{k_i\theta_i}{\sigma_i^2}[\eta(u_i)k_i]\tau \\ &\quad+\frac{2k_i\theta_i}{\sigma_i^2}\log[(\eta(u_i)+k_i)\exp[\eta(u_i)\tau]+\eta(u_i)-k_i] \\ &\quad-\frac{2k_i\theta_i}{\sigma_i^2}\log(2\eta(u_i)) \end{aligned} \tag{5.7.65}$$

where $\eta(u_i) = \sqrt{k_i^2 + 2u_i\sigma_i^2}$.

Price of the GAO

We use equations (5.4.12) and (5.7.62) to obtain the price of the GAO under the transformed measure $\tilde{Q}$ as

$$C(0,x,T) = g\tilde{P}(0,T)\tilde{E}\left[\left(\sum_{i=1}^{n-1}e^{-(\bar{r}+\bar{\mu})i}\prod_{j=1}^{p}e^{-\tilde{\phi}_j(i,R_j+M_j)-\tilde{\psi}_j(i,R_j+M_j)X_{jT}} - (K-1)\right)^+\right] \tag{5.7.66}$$

where $\tilde{P}(0,T)$ given by equation (5.7.62) with $\tau = T$ while $\tilde{\psi}_j(i,R_j+M_j)$ and $\tilde{\phi}_j(i,R_j+M_j)$ are given by equations (5.7.64) and (5.7.65).

Distribution of X_T

In order to obtain explicit bounds for the GAO in the multidimensional CIR case, we need to obtain the distribution of X_{jT} under the transformed measure $\tilde{Q}$. We state this in the following proposition (c.f. Cox et al., 1985; Deelstra et al., 2016, for details).

Proposition 36. *The dynamics of the CIR process X_{jt} defined in equation* (5.7.61) *under the transformed measure $\tilde{Q}$ are given by*

$$dX_{jt} = k_j'\left(\theta_j' - X_{jt}\right)dt + \sigma_j\sqrt{X_{jt}}dW_{jt}', \ j = 1, ..., p. \tag{5.7.67}$$

where

$$k_j' = k_j + \sigma_j^2\tilde{\psi}_j(0,R_j+M_j), \tag{5.7.68}$$

$$\theta_j' = \frac{k_j\theta_j}{k_j+\sigma_j^2\tilde{\psi}_j(0,R_j+M_j)} \tag{5.7.69}$$

and X_{j0}; $j = 1, 2, ..., p$ is the initial value of the process. Then the density function of X_{jT} is *given by*

$$f_{X_{jT}}(x) = f_{\frac{\chi^2\left(\nu_{jT},\lambda_{jT}\right)}{c_{jT}}}(x) = c_{jT} f_{\chi^2(\nu_{jT},\,\lambda_{jT})}(c_{jT}x) \tag{5.7.70}$$

where $f_{\chi^2(\nu_{jT},\,\lambda_{jT})}$; $j = 1, 2, ..., p$ *is the p.d.f. of non-central* χ^2 *with degrees of freedom* ν_j *and non-centrality parameter* λ_{jT} *such that*

$$c_{jT} = \frac{4k_j^{'}}{\sigma_j^2\left(1 - e^{-k_j^{'}T}\right)}, \tag{5.7.71}$$

$$\nu_{jT} = \frac{4k_j^{'}\theta_j^{'}}{\sigma_j^2} \tag{5.7.72}$$

and

$$\lambda_{jT} = c_{jT} X_{j0} e^{-k_j^{'}T}. \tag{5.7.73}$$

The moment generating function (m.g.f.) of X_{jT} has a very interesting exposition as detailed below (c.f. Dufresne, 2001, for details).

$$\mathcal{M}_{X_{jT}}(s_j) = (\beta(s_j))^{\bar{\nu_j}} e^{\lambda_{jT}^{'}(\beta(s_j)-1)} \tag{5.7.74}$$

where

$$\beta(s_j) = (1 - s_j\mu_{jT})^{-1}, \tag{5.7.75}$$

with

$$\mu_{jT} = \frac{2}{c_{jT}}, \tag{5.7.76}$$

$$\bar{\nu_j} = \frac{\nu_{jT}}{2} \tag{5.7.77}$$

and

$$\lambda_{jT}^{'} = 2\lambda_{jT}. \tag{5.7.78}$$

The Lower Bound GAOLB$^{(MCIR)}$

The lower bound GAOLB obtained in equation (5.5.13) condenses into a very compact formula for the Multi-CIR case in a manner similar to the formula (5.5.14) under the affine set up. Before unravelling the same, we see that in light of the notations defined in section 5.4, one can write

$$S_T^{(i)} = S_0^{(i)} e^{X_T^{(i)}};\ i = 1, 2, ..., n-1, \tag{5.7.79}$$

where

$$S_0^{(i)} = e^{-\left((\bar{r}+\bar{\mu})i + \sum_{j=1}^p \tilde{\phi}_j(i, R_j+M_j)\right)} \tag{5.7.80}$$

and

$$X_T^{(i)} = -\sum_{j=1}^p \tilde{\psi}_j(i, R_j + M_j) X_{jT}, \tag{5.7.81}$$

where $\tilde{\phi}_j(i, R_j + M_j)$ and $\tilde{\psi}_j(i, R_j + M_j)$ for $i = 1, 2, ..., n-1$ and $j = 1, 2, ..., p$ are given in equations (5.7.64)-(5.7.65) with τ replaced by i. Further, X_{jT}; $j = 1, 2, ..., p$ are independent random variables and their m.g.f. is given in equation (5.7.74). This leads us to the formulation of the lower bound for the Multi-CIR case presented in the form of the following proposition:

Proposition 37. *The lower bound under the multi-CIR case is*

$$\begin{aligned} GAOLB^{MCIR} \quad = \quad & g\tilde{P}(0,T)\left(\sum_{i=1}^{n-1}\left(e^{-\left((\bar{r}+\bar{\mu})i+\sum_{j=1}^p \tilde{\phi}_j(i,R_j+M_j)\right)+\sum_{j=1}^p \lambda_{jT}^{'}\left(\beta\left(-\tilde{\psi}_j(i,R_j+M_j)\right)-1\right)}\right.\right. \\ & \left.\left.\left(\prod_{j=1}^p\left(\beta\left(-\tilde{\psi}_j(i, R_j+M_j)\right)\right)^{\bar{\nu_j}}\right)\right) - (K-1)\right)^+ \end{aligned} \tag{5.7.82}$$

where $\beta(.)$ *is defined in* (5.7.75) *and* $\bar{\nu_j}$ *and* λ'_{jT} *are given in equations* (5.7.77)-(5.7.78) *and* $\tilde{\psi}_j(i, R_j + M_j)$ *for* $i = 1, 2, ..., n-1; j = 1, 2, ..., p$ *are given in* (5.7.65).

Proof. Using the formula for lower bound given in equation (5.5.13)

$$\text{GAOLB} = g\tilde{P}(0,T)\left(\sum_{i=1}^{n-1}\tilde{E}\left(S_T^{(i)}\right) - (K-1)\right)^+ \tag{5.7.83}$$

Using the formula of $S_T^{(i)}$ given in equations (5.7.79)-(5.7.81) we have

$$\begin{aligned}\text{GAOLB}^{MCIR} &= g\tilde{P}(0,T)\left(\sum_{i=1}^{n-1}\left(e^{-\left((\bar{r}+\bar{\mu})i+\sum_{j=1}^{p}\tilde{\phi}_j(i,R_j+M_j)\right)}\right.\right.\\ &\quad\left.\left.\times\prod_{j=1}^{p}\mathcal{M}_{X_{jT}}\left(-\tilde{\psi}_j(i,R_j+M_j)\right)\right) - (K-1)\right)^+ .\end{aligned} \tag{5.7.84}$$

Using the definition of m.g.f. of $X_{jT};\ j = 1, 2, ..., p$ given in equation (5.7.74) we obtain the requisite result. □

The Upper Bound GAOUB$^{(MCIR)}$

Under the formulation of the pure endowments constituting the GAO basket under the MCIR case ((5.7.79)-(5.7.81)), we write

$$Y_0^{(n-1)} = -\frac{(\bar{r}+\bar{\mu})n}{2} - \frac{1}{n-1}\sum_{k=1}^{n-1}\sum_{j=1}^{p}\tilde{\phi}_j(k, R_j + M_j) \tag{5.7.85}$$

using the definition of log-geometric average $Y_T^{(n-1)}$ given in equation (5.6.21) in Section 5.6.3. We now exploit the set up of the upper bound under the affine case given in section 5.6.3 and note that here instead of $\tilde{\phi}(k, R+M)$ and $\tilde{\psi}(k, R+M)$, we have respectively $\sum_{j=1}^{p}\tilde{\phi}_j(k, R_j + M_j)$ and $\sum_{j=1}^{p}\tilde{\psi}_j(k, R_j + M_j)$ since we are dealing with a p-dimensional CIR process. Thus the joint characteristic function of $\left(X_T^{(1)}, ..., X_T^{(n-1)}\right)$ under the transformed measure $\tilde{Q}$, given in equation (5.6.23) becomes ,

$$\phi_T^{MCIR}(\boldsymbol{\gamma}) = \prod_{j=1}^{p}\phi_{X_{jT}}\left(-\sum_{k=1}^{n-1}\gamma_k\tilde{\psi}_j(k, R_j + M_j)\right), \tag{5.7.86}$$

where $\boldsymbol{\gamma} = [\gamma_1, \gamma_2, ..., \gamma_{n-1}]$, $\phi_{X_{jT}};\ j = 1, 2, ..., p$ denotes the characteristic function of the X_{jT} with parameter $\left(-\sum_{k=1}^{n-1}\gamma_k\tilde{\psi}_j(k, R_j + M_j)\right)$ for $j = 1, 2, ..., p$, with $\tilde{\psi}_j(k, R_j + M_j)$ for $k = 1, 2, ..., n-1;\ j = 1, 2, ..., p$ are given in equation (5.7.65) with τ replaced by k. $\phi_{X_{jT}}(s)$ can be obtained from the formula of its m.g.f. given in equation (5.7.74) by replacing s by is. Further, we see that $\Psi_T^G(\eta; \delta)$ given in equation (5.6.30) reduces to

$$\Psi_T^{G^{MCIR}}(\eta;\delta) = e^{i(\eta - i(\delta+1))Y_0^{(n-1)}}\frac{\prod_{j=1}^{p}\phi_{X_{jT}}\left(-\frac{(\eta-i(\delta+1))}{n-1}\sum_{k=1}^{n-1}\tilde{\psi}_j(k, R_j + M_j)\right)}{\delta^2 + \delta - \eta^2 + i\eta(2\delta+1)}. \tag{5.7.87}$$

Next, we obtain $\tilde{E}^{MCIR}\left[G_T^{(n-1)}\right]$ from equation (5.6.37) by utilizing (5.7.86). Further, we compute

$$\tilde{E}^{MCIR}\left[A_T^{(n-1)}\right] = \frac{1}{n-1}\sum_{k=1}^{n-1}\left(e^{-\left((\bar{r}+\bar{\mu})k+\sum_{j=1}^{p}\tilde{\phi}_j(k,R_j+M_j)\right)}\prod_{j=1}^{p}\mathcal{M}_{X_{jT}}\left(-\tilde{\psi}_j(k, R_j + M_j)\right)\right). \tag{5.7.88}$$

Finally, we plug in the components one by one into equation (5.6.28) to obtain the upper bound GAOUB$^{(MCIR)}$.

5.7.3 The Wishart Short Rate Model

The Set Up

In this section, we assume that the affine process $X := (X_t)_{t\geq 0}$ is a d-dimensional Wishart process. Given a $d\times d$ matrix Brownian motion W (i.e a matrix whose entries are independent Brownian motions) the Wishart process X (without jumps) is defined as the solution of the $d\times d$-dimensional stochastic differential equation

$$dX_t = \left(\beta Q^T Q + HX_t + X_t H^T\right) dt + \sqrt{X_t} dW_t Q + Q^T dW_t^T \sqrt{X_t},\ t \geq 0, \tag{5.7.89}$$

where $X_0 = x \in S_d^+$, $\beta \geq d-1$, $H \in M_d$, $Q \in GL_d$ and Q^T denotes its transpose. M_d has been defined in section 5.4 while GL_d denote the set of invertible real $d\times d$ matrices. In short, we assume that the law of X is $WIS_d(x_0, \beta, H, Q)$.

Existence and Uniqueness of Solution

This process was pioneered by Bru (1991) and she showed the existence and uniqueness of a weak solution for equation (5.7.89). She also established the existence of a unique strong solution taking values in S_d^{++}, i.e. the interior of the cone of positive semi-definite symmetric $d\times d$ matrices that we have denoted by S_d^+.

Generator

Bru (1991) has calculated the infinitesimal generator of the Wishart process as:

$$\mathcal{A} = Tr\left(\left(\beta Q^T Q + Hx + xH^T\right) D^S + 2xD^S Q^T Q D^S\right), \tag{5.7.90}$$

where Tr stands for trace and $D^S = (\partial/\partial x_{ij})_{1\leq i,j\leq d}$. A good reference for understanding the detailed derivation of this generator is Alfonsi (2015) and following this reference we have defined generator in Chapter 4.

Survival Zero Coupon Bond Pricing

We now give an explicit formula for calculating the survival zero coupon bond price under the Wishart short rate model.

Theorem 38. *Let the dynamics for short rate and mortality rate be given in accordance with equation* (5.4.1) *respecively as*

$$r_t = \bar{r} + Tr\left[RX_t\right] \tag{5.7.91}$$

and

$$\mu_t = \bar{\mu} + Tr\left[MX_t\right] \tag{5.7.92}$$

for a process X with law $WIS_d(x_0, \beta, H, Q)$. Let $R, M \in S_d^{++}$ and $\tau = T - t$, then the price of a zero-coupon bond under the Wishart short rate model ((5.7.91)-(5.7.92)*) is given by*

$$\begin{aligned} \tilde{P}(t,T) &= \mathbb{E}\left[e^{-\int_t^T (\bar{r}+\bar{\mu}+Tr[(R+M)X_u])du} | \mathcal{F}_t\right] \\ &= e^{-\tilde{\phi}(\tau,R+M)-Tr\left[\tilde{\psi}(\tau,R+M)X_t\right]}, \end{aligned} \tag{5.7.93}$$

where $\tilde{\phi}$ and $\tilde{\psi}$ satisfy the following system of ODEs:

$$\begin{cases} \frac{\partial \tilde{\phi}}{\partial \tau} = Tr\left[\beta Q^T Q \tilde{\psi}(\tau, R+M)\right] + \bar{r} + \bar{\mu}, \\ \tilde{\phi}(0, R+M) = 0, \\ \frac{\partial \tilde{\psi}}{\partial \tau} = \tilde{\psi}(\tau, R+M) H + H^T \tilde{\psi}(\tau, R+M), \\ \quad -2\tilde{\psi}(\tau, R+M) Q^T Q \tilde{\psi}(\tau, R+M) + R + M, \\ \tilde{\psi}(0, R+M) = 0. \end{cases} \tag{5.7.94}$$

Proof. Consider the expectation in equation (5.7.93). As remarked in section 5.2, the conditioning on $\mathcal{F}_t$ can be reduced to that on $\mathcal{G}_t$ and so we define $t \leq T$, define

$$F(t, X_t) = f(\tau, X_t) = \mathbb{E}\left[e^{-\int_t^T (\bar{r}+\bar{\mu}+Tr[(R+M)X_u])du} | X_t\right]. \tag{5.7.95}$$

This conditional expectation is the Feynman-Kac representation which satisfies the following Partial Differential Equation (PDE):

$$\begin{cases} \frac{\partial f(\tau,x)}{\partial \tau} = \mathcal{A} f(\tau, x) - (\bar{r} + \bar{\mu} + Tr[(R+M)x]) f(\tau, x), \\ f(0, x) = 1, \end{cases} \tag{5.7.96}$$

for all $x \in S_d^+$, where $\mathcal{A}$ is the infinitesimal generator of the Wishart process given in equation (5.7.90). We introduce a candidate solution given below

$$f(\tau, x) = e^{-\tilde{\phi}(\tau,R+M) - Tr\left[\tilde{\psi}(\tau,R+M)x\right]} \tag{5.7.97}$$

so that

$$\frac{\partial f(\tau, x)}{\partial \tau} = \left(-\frac{\partial \tilde{\phi}}{\partial \tau} - Tr\left[\frac{\partial \tilde{\psi}}{\partial \tau} x\right]\right) f(\tau, x) \tag{5.7.98}$$

Also it is clear that

$$\mathcal{A} e^{-\tilde{\phi}(\tau,R+M) - Tr\left[\tilde{\psi}(\tau,R+M)x\right]} = e^{-\tilde{\phi}(\tau,R+M)} \mathcal{A} e^{-Tr\left[\tilde{\psi}(\tau,R+M)x\right]}, \tag{5.7.99}$$

where on using the generator of the Wishart process given in equation (5.7.90), we have

$$\begin{aligned} \mathcal{A} e^{-Tr\left(\tilde{\psi}(\tau,R+M)x\right)} &= \Bigg(-Tr\left[\beta Q^T Q \tilde{\psi}(\tau, R+M)\right] + Tr\Bigg[\Bigg(2\tilde{\psi}(\tau, R+M) Q^T Q \tilde{\psi}(\tau, R+M) \\ &\quad -\tilde{\psi}(\tau, R+M) H - H^T \tilde{\psi}(\tau, R+M)\Bigg) x\Bigg]\Bigg) e^{-Tr\left(\tilde{\psi}(\tau,R+M)x\right)} \end{aligned} \tag{5.7.100}$$

Using equations (5.7.98)-(5.7.100) in equation (5.7.95) and canceling $f(\tau, x)$ throughout, we get

$$\begin{aligned} -\frac{\partial \tilde{\phi}}{\partial \tau} - Tr\left[\frac{\partial \tilde{\psi}}{\partial \tau} x\right] &= -Tr\left[\beta Q^T Q \tilde{\psi}(\tau, R+M)\right] - (\bar{r} + Tr[(R+M)x]) + Tr\Bigg[\Bigg(2\tilde{\psi}(\tau, R+M) \\ &\quad \times Q^T Q \tilde{\psi}(\tau, R+M) - \tilde{\psi}(\tau, R+M) H - H^T \tilde{\psi}(\tau, R+M)\Bigg) x\Bigg]\Bigg) \end{aligned} \tag{5.7.101}$$

Comparing the terms independent of x and the coefficients of x on both sides of equation (5.7.101), we get the required system of ODEs given in equation (5.7.94). This completes the proof. □

The methodology of solving the system of Riccati equations given in (5.7.94) appears in Da Fonseca et al. (2008) where the authors propose that matrix Riccati equations can be linearized by doubling the dimension of the problem (see Grasselli and Tebaldi, 2008; Deelstra et al., 2016). We state without proof the solution in the following proposition.

Proposition 39. *The functions $\tilde{\phi}$ and $\tilde{\psi}$ in Theorem 38 are given by*

$$\begin{cases} \tilde{\psi}(\tau, R+M) = A_{22}^{-1}(\tau) A_{21}(\tau), \\ \tilde{\phi}(\tau, R+M) = \frac{\beta}{2}\left(\log\left(\det\left(A_{22}(\tau)\right)\right) + \tau Tr\left[H^T\right]\right) + \bar{r} + \bar{\mu}. \end{cases} \tag{5.7.102}$$

where

$$\begin{pmatrix} A_{11}(\tau) & A_{12}(\tau) \\ A_{21}(\tau) & A_{22}(\tau) \end{pmatrix} = \exp\left(\tau \begin{pmatrix} H & 2Q^TQ \\ R+M & -H^T \end{pmatrix}\right) \tag{5.7.103}$$

Alternative approaches for the pricing of zero coupon bond under the Wishart short rate model can be found in Grasselli and Tebaldi (2008) and Gnoatto and Grasselli (2014).

Price of the GAO

We use Theorem 38 and equation (5.4.12) to obtain the price of the GAO under the transformed measure $\tilde{Q}$ as

$$C(0,x,T) = g\tilde{P}(0,T)\,\tilde{E}\left[\left(\sum_{i=1}^{n-1} e^{-(\bar{r}+\bar{\mu})i} e^{-\tilde{\phi}(i,R+M) - Tr\left[\tilde{\psi}(i,R+M)X_T\right]} - (K-1)\right)^+\right], \tag{5.7.104}$$

where $\tilde{P}(0,T)$ is given by equation (5.7.94) with $\tau = T$ while $\tilde{\psi}(i, R+M)$ and $\tilde{\phi}(i, R+M)$ for $i = 1, 2, ..., n-1$ are given by the system of equations (5.7.102) with $\tau = i$.

Distribution of X_T

In order to obtain explicit bounds for the GAO in the Wishart case, we need to obtain the distribution of X_T under the transformed measure $\tilde{Q}$. We state this in the following proposition (c.f. Deelstra et al., 2016; Kang and Kang, 2013, for details).

Proposition 40. *The dynamics of the Wishart process X defined in equation* (5.7.89) *under the transformed measure $\tilde{Q}$ are given by*

$$dX_t = \left(\beta Q^TQ + H(t)X_t + X_tH(t)^T\right)dt + \sqrt{X_t}dW_tQ + Q^TdW_t^T\sqrt{X_t},\ t \geq 0, \tag{5.7.105}$$

where

$$H(t) = H - Q^TQ\tilde{\psi}(\tau, R+M), \tag{5.7.106}$$

$X_0 = x \in S_d^+$, $\beta \geq d-1$, $H \in M_d$ *and* $Q \in GL_d$. *Then*

$$X_T \sim \mathcal{W}_d\left(\beta, V(0), V(0)^{-1}\psi(0)^T x\psi(0)\right), \tag{5.7.107}$$

where $\mathcal{W}_d$ stands for non-central Wishart Distribution with parameters as d, β, $V(0)$ and $\psi(0)^T x\psi(0)$ with the last parameter known as non-centrality parameter and is denoted by Θ. Moreover $V(t)$ and $\psi(t)$ solve the following system of ODEs

$$\begin{cases} \frac{d}{dt}\psi(t) = -H(t)^T\psi(t), \\ \frac{d}{dt}V(t) = -\psi(t)^T Q^TQ\psi(t), \end{cases} \tag{5.7.108}$$

with terminal conditions $\psi(T) = I_d$ and $V(T) = 0$.

We now state two propositions in context of non-central Wishart Distribution which are very important for the derivation of bounds for the GAO in the Wishart case (c.f. Pfaffel, 2012; Gupta and Nagar, 2000).

Proposition 41. *(Laplace Transform of Non-Central Wishart Distribution) Consider $X_T \sim \mathcal{W}_d(\beta, V(0), \Theta)$ with $\Theta = V(0)^{-1}\psi(0)^T x\psi(0)$. Then the Laplace transform of X_T is given* *by*

$$\mathcal{L}(U) = \tilde{E}\left[e^{Tr[-UX_T]}\right] = \det\left(I_d + 2V(0)U\right)^{-\frac{\beta}{2}} e^{Tr\left[-\Theta(I_d+2V(0)U)^{-1}V(0)U\right]} \tag{5.7.109}$$

where $U \in S_d^+$.

Proposition 42. *(Characteristic Function of Non-Central Wishart Distribution) Consider* $X_T \sim \mathcal{W}_d(\beta, V(0), \Theta)$ *with* $\Theta = V(0)^{-1}\psi(0)^T x\psi(0)$. *Then the Characteristic Function of* X_T *is given by*

$$\phi_{X_T}(\Lambda) = \tilde{E}\left[e^{Tr[i\Lambda X_T]}\right] = \det\left(I_d - 2iV(0)\Lambda\right)^{-\frac{\beta}{2}} e^{Tr\left[i\Theta(I_d-2V(0)\Lambda)^{-1}V(0)\Lambda\right]} \tag{5.7.110}$$

where $\Lambda \in M_d$.

The Lower Bound GAOLB$^{(WIS)}$

Under the Wishart set up, lower bound GAOLB obtained in equation (5.5.13) reduces to a very neat form. Before arriving at the formula, we define the following notations in the spirit of section 5.4.

$$S_T^{(i)} = S_0^{(i)} e^{X_T^{(i)}};\ i = 1, 2, ..., n-1, \tag{5.7.111}$$

where

$$S_0^{(i)} = e^{-\left((\bar{r}+\bar{\mu})i+\tilde{\phi}(i,R+M)\right)} \tag{5.7.112}$$

and

$$X_T^{(i)} = -Tr\left[\tilde{\psi}(i, R+M)\, X_T\right], \tag{5.7.113}$$

where $\tilde{\psi}(i, R+M)$ and $\tilde{\phi}(i, R+M)$ for $i = 1, 2, ..., n-1$ are given by the system of equations (5.7.102) with $\tau = i$. Further, X_T has a non-central Wishart distribution with Laplace transform given in equation (5.7.109). This result along with the formula (5.5.14), the lower bound for the Wishart case manifests itself into the following form.

$$\begin{aligned} \text{GAOLB}^{(WIS)} &= g\tilde{P}(0,T)\left(\sum_{i=1}^{n-1}\left(e^{-\left((\bar{r}+\bar{\mu})i+\tilde{\phi}(i,R+M)\right)} \det\left(I_d + 2V(0)\tilde{\psi}(i,R+M)\right)^{-\frac{\beta}{2}}\right.\right. \\ &\quad \left.\left. \times\, e^{Tr\left[-\Theta\left(I_d+2V(0)\tilde{\psi}(i,R+M)\right)^{-1}V(0)\tilde{\psi}(i,R+M)\right]}\right) - (K-1)\right)^+ \end{aligned} \tag{5.7.114}$$

The Upper Bound GAOUB$^{(WIS)}$

Under the formulation of the assets in the basket under the Wishart case ((5.7.111)-(5.7.113)), we have

$$Y_0^{(n-1)} = -\frac{(\bar{r}+\bar{\mu})\,n}{2} - \frac{1}{n-1}\sum_{k=1}^{n-1}\tilde{\phi}(k, R+M) \tag{5.7.115}$$

using the definition of log-geometric average $Y_T^{(n-1)}$ given in equation (5.6.21) in section 5.6.3. Further, obtaining the upper bound for the GAO in the Wishart set up is a straightforward exercise as one can exploit the upper bound GAO formula under the affine case given in equation (5.6.40) by calculating the Laplace transform given in equation (5.7.109) such that for $k = 1, 2, ..., n-1$,

$$\mathcal{L}\left(\tilde{\psi}(k,R+M)\right) = \det\left(I_d + 2V(0)\tilde{\psi}(k,R+M)\right)^{-\frac{\beta}{2}} e^{Tr\left[-\Theta\left(I_d+2V(0)\tilde{\psi}(k,R+M)\right)^{-1}V(0)\tilde{\psi}(k,R+M)\right]} \tag{5.7.116}$$

and using the formula (5.7.110) to calculate the functions $\phi_{X_T}\left(-\frac{(\eta-i(\delta+1))}{n-1}\sum_{k=1}^{n-1}\tilde{\psi}(k,R+M)\right)$ and $\phi_{X_T}\left(\frac{i}{n-1}\sum_{k=1}^{n-1}\tilde{\psi}(k,R+M)\right)$, by replacing Λ by $-\frac{(\eta-i(\delta+1))}{n-1}\sum_{k=1}^{n-1}\tilde{\psi}(k,R+M)$ and $\frac{i}{n-1}\sum_{k=1}^{n-1}\tilde{\psi}(k,R+M)$ respectively.

Numerical computations to support the theory are presented in Chapter 6 while a review of the key findings and further research avenues appears in Chapter 7. We end this chapter with some important concluding remarks and observations.

5.8 Final Remarks

The theory developed in this chapter is a very important milestone in the direction of pricing of Guaranteed Annuity Options in a scenario where mortality rate and interest rate are correlated. We have been successful in developing bounds for GAOs under the most general framework and for some mild additional model assumptions, improvements of these bounds have been achieved. The underlying theory does not pin point to any particular model and in this sense offers the freedom for choosing the model. We show that the affine class of models is particularly conducive in obtaining the bounds as it reduces the dimensionality of computation to uni-fold. Moreover the bounds in the affine case have an edge over the Monte Carlo counterpart as the former can be computed in the blink of an eye while the latter can be extremely time consuming for complex models such as Wishart. These observations are discussed in greater detail in Chapters 6 and 7. For complex affine models, one just needs to lay hands on the 'Laplace Transform' and the 'Characteristic Function' to get the bounds and these are readily available in most cases. One could implement this methodology for many different scenarios/models due to its quick implementation so as to calculate the 'worst case bounds' for a large class of models and in this way reduce model risk. Thus, the important point to note here is that the strength of these bounds is their wider applicability beyond model specifications and, clearly, for a fully specified model one can improve the bounds considerably. In fact even if the model is fully specified, determining the price of the GAO analytically in most cases is still not a straightforward exercise because of the dependence that exists between the endowment prices and the correlation of mortality rate and financial markets.

In fact, in order to get some illustration of the numerical performance of the bounds, we first experiment with a Vasicek model with specified parameters chosen carefully so as to circumvent the scenario of producing negative mortality rates. We then follow this up with Multi CIR and Wishart Models. In fact we perform sensitivity analysis to see how a changing correlation structure between mortality and interest rate influences the GAO price bounds obtaining some interesting results for Wishart case where all kinds of behaviours are visible, viz. an increase in correlation may lead to an increase or decrease or a non-monotonic change in the GAO price. These observations pave the way for further research with a variety of models for GAOs and other similar instruments. Clearly the results obtained in this research should ring bells for pension planners and annuity designers who should do multi-scenario testing to get the most appropriate price range for a correlated mortality and financial set up.

Chapter 6

Numerical Results

This thesis details novel ways of valuing mortality and longevity linked derivatives under a wide variety of underlying models. In this chapter, we attempt to describe how our bounds derived in the previous chapters behave numerically by comparing each one of them to the benchmark Monte Carlo value. We begin by considering the results obtained in Chapter 3 to obtain tight bounds for the Swiss Re Mortality Bond 2003. This is followed up by the computation of bounds for Guaranteed Annuity Options derived in Chapter 5.

MATLAB codes for the execution of the theory in respect of Swiss Re Bond and Guaranteed Annuity Options have been furnished respectively in Appendices C and D.

6.1 Bounds for the Swiss Re Bond

The stage is now set to investigate the applications of the theory derived in Chapter 3. We have successfully obtained a number of lower bounds and an upper bound for the Swiss Re bond in sections 3 and 4 of this chapter. In section 3.5 we have furnished a couple of examples. We now test these vis-a-vis the well-known Monte Carlo estimate for the Swiss Re bond. We assume that $C = 1$ in all the examples. We first carry out this working under the well known Black and Scholes (1973) model in finance and then for a couple of transformed distributions. The nomenclature for the bounds has already been specified in sections 3.3 and 3.4. In all the examples, the marginal cdfs are strictly increasing.

6.1.1 Black-Scholes Model

In tables 6.1 and 6.2, we assume that the mortality evolution process $\{q_t\}_{t\geq 0}$ obeys the Black-Scholes model, specified by the following stochastic differential equation (SDE)

$$dq_t = rq_t dt + \sigma q_t dW_t.$$

In order to simulate a path, we will consider the value of the mortality index in the three years that form the term of the bond, i.e., $n = 3$. In fact we consider the time points as $t_1 = 1, ..., t_n = T = 3$. We invoke the following equation to generate the mortality evolution:

$$q_{t_j} = q_{t_{j-1}} \exp\left[\left(r - \frac{1}{2}\sigma^2\right)\delta t + \sigma\sqrt{\delta t} Z_j\right] \quad Z_j \sim N(0,1), \quad j = 1, 2, \ldots, n. \tag{6.1.1}$$

We highlight below the parameter choices in accordance with Lin and Cox (2008). The value of the interest rate is varied in table 6.1 while table 6.2 experiments with the variation in the base value of the mortality index while assuming a zero interest rate. Parameter choices for tables 6.1 and 6.2 with t specified in terms of years are:

$$q_0 = 0.008453,\ T = 3,\ t_0 = 0,\ n = 3,\ \sigma = 0.0388.$$

Table 6.2 is followed by figures 6.1-6.3. While figures 6.1 and 6.2 depict comparisons between the bounds, figure 6.3 portrays the price bounds for the Swiss Re bond generated by the Black-Scholes model. We will let MC denote the Monte Carlo estimate for the Swiss Re bond.

Table 6.1 reflects that the relative difference ($=\frac{|bound-MC|}{MC}$) between any bound and the benchmark Monte Carlo estimate increases with an increase in the interest rate for a fixed value of the base mortality index q_0. This observation is echoed by figure 6.1. On the other hand, figure 6.2 depicts the difference between the Monte Carlo estimate of the Swiss Re bond and the derived bounds. The bound $\mathrm{SWLB}_t^{(BS)}$ fares much better than SWLB_1. The absolute difference between the estimated price and the bounds increase as the value of the base mortality index is increased and then there is a switch and this gap begins to diminish. This observation is supported by the fact that an increase in the starting value of mortality increases the possibility of a catastrophe which leads to the washing out of the principal or in other words the option goes out of money.

6.1.2 Transformed Normal (S_u) Distribution

We now consider an additional example. Assume that the mortality rate $\{q_t\}_{t\geq 0}$, $t = 1, 2, 3$ obeys the four-parameter transformed Normal (S_u) Distribution (for details see Johnson, 1949; Johnson et al., 1994, Chapter 2 and Appendix A.5) which is defined as follows

$$\sinh^{-1}\left(\frac{q_t - \alpha_t}{\beta_t}\right) = x_t \sim N\left(\mu_t, \sigma_t^2\right), \tag{6.1.2}$$

where α_t, β_t, μ_t and σ_t are parameters ($\beta_t, \sigma_t > 0$) and $\sinh^{-1}$ is the inverse hyperbolic sine function.

For table 6.3, we vary the interest rate as in table 6.1 and use the parameter set employed by Tsai and Tzeng (2013). The aforesaid authors use the mortality catastrophe model of Lin and Cox (2008) to generate the data and then utilize the quantile-based estimation of Slifker and Shapiro (1980) to estimate the parameters of the S_u-fit. The initial mortality rate and time points are same as for tables 6.1 and 6.2. The following arrays present the values of the parameters for the three years 2004, 2005 and 2006 that were covered by the Swiss Re bond.

$$\alpha = [0.008399, 0.008169, 0.007905],\ \beta = [0.000298, 0.000613, 0.000904],$$

$$\mu = [0.70780, 0.58728, 0.58743] \text{ and } \sigma = [0.67281, 0.50654, 0.42218].$$

The value of $\mathrm{SWLB}_t^{(2)}$ in table 6.3 has been calculated by using 'Numerical Integration' in MATLAB since the first term in (3.3.36) can not be calculated mathematically. Table 6.3 adds weight to the claim that the bounds are extremely tight for a large class of models assuming a variety of distributions for the mortality index.

6.1.3 Log Gamma Distribution

Finally in tables 6.4 and 6.5, we experiment with log gamma distribution by varying the interest rate in table 6.4 and the base mortality rate in the latter. The parameters are chosen as in Cheng et al. (2014) who employ an approach similar to Tsai and Tzeng (2013) outlined above with $q_0 = .0088$ but use maximum likelihood estimation to obtain the parameters of the fitted log gamma distribution. As before, the following arrays present the year wise parameters

$$p = [61.6326, 64.2902, 71.8574],\ a = [0.0103, 0.0098, 0.0080],$$

$$\mu = [-5.2452, -5.4600, -5.7238] \text{ and } \sigma = [7.4 \times 10^{-5}, 9.5 \times 10^{-5}, 9.4 \times 10^{-5}].$$

Tables 6.4 and 6.5 clearly show that even for non-normal universe, the bounds are extremely precise. Figures 6.4-6.6 are drawn on the lines of figures 6.1-6.3 and strongly support our observation.

Table 6.1: Lower Bounds and Upper Bounds for the Swiss Re Mortality Bond under the Black-Scholes Model with $q_0 = 0.008453$ and $\sigma = 0.0388$ in accordance with Lin and Cox (2008). MC Simulations:5000000 iterations (Antithetic Method)

r	SWLB_0	SWLB_1	$\mathrm{SWLB}_t^{(BS)}$	MC	$\mathrm{SWUB}_t^{(BS)}$	SWUB_1
0.035	0.899130889131	0.899130889153	0.899131577419	0.899130939229	0.899131588500	0.899131637780
0.030	0.913324024542	0.913324024546	0.913324256506	0.913324120543	0.913324317265	0.913324320930
0.025	0.927447505802	0.927447505803	0.927447580428	0.927447582074	0.927447605312	0.927447619324
0.020	0.941626342686	0.941626342687	0.941626365600	0.941626356704	0.941626369727	0.941626384749
0.015	0.955935721003	0.955935721003	0.955935727716	0.955935715489	0.955935732230	0.955935736078
0.010	0.970419124546	0.970419124546	0.970419126422	0.970419112046	0.970419126802	0.970419129772
0.005	0.985101139986	0.985101139986	0.985101140486	0.985101142704	0.985101140840	0.985101141738
0.000	0.999995778016	0.999995778016	0.999995778143	0.999995730679	0.999995778175	0.999995778584

Table 6.2: Lower Bounds and Upper Bounds for the Swiss Re Mortality Bond under the Black-Scholes Model with $r = 0.0$ and $\sigma = 0.0388$ in accordance with Lin and Cox (2008). MC Simulations:5000000 iterations (Antithetic Method)

q_0	SWLB_0	SWLB_1	$\mathrm{SWLB}_t^{(BS)}$	MC	$\mathrm{SWUB}_t^{(BS)}$	SWUB_1
0.007	1.000000000000	1.000000000000	1.000000000000	1.000000000000	1.000000000000	1.000000000000
0.008	0.999999915252	0.999999915252	0.999999915252	0.999999935586	0.999999915253	0.999999915253
0.008453	0.999995778016	0.999995778016	0.999995778143	0.999995730679	0.999995778175	0.999995778584
0.009	0.999821987943	0.999821987950	0.999822025863	0.999816103329	0.999822374801	0.999822875816
0.010	0.978292691035	0.978310383929	0.978503560221	0.978738658828	0.978292691184	0.986262918347
0.011	0.572750782004	0.610962124258	0.610962123857	0.652440509315	0.572755594265	0.877336305502
0.012	0.000000000000	0.040209774144	0.040209770810	0.094615386164	0.000000000000	0.395672911251
0.013	0.000000000000	0.000000000000	0.000000000000	0.001662471990	0.000000000000	0.083466184427
0.014	0.000000000000	0.000000000000	0.000000000000	0.000003376858	0.000000000000	0.008942985848

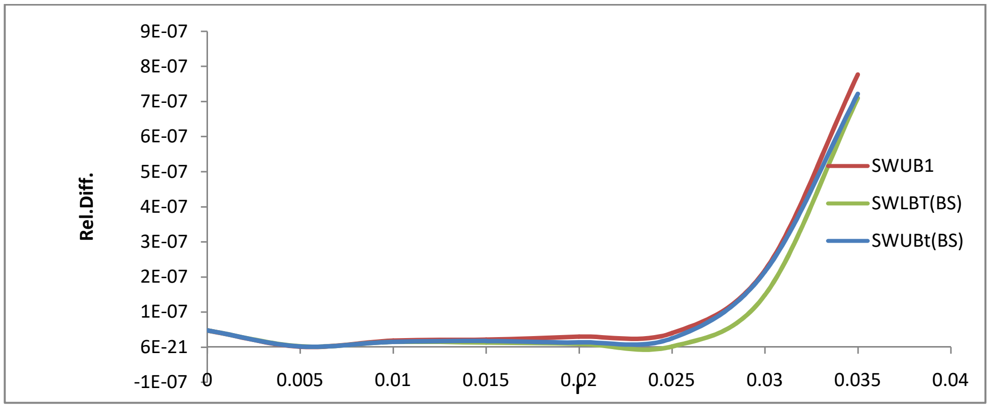


Figure 6.1: Relative Difference of $\mathrm{SWLB}_t^{(BS)}$, $\mathrm{SWUB}_t^{(BS)}$ and SWUB_1 w.r.t. MC estimate under Black-Scholes model

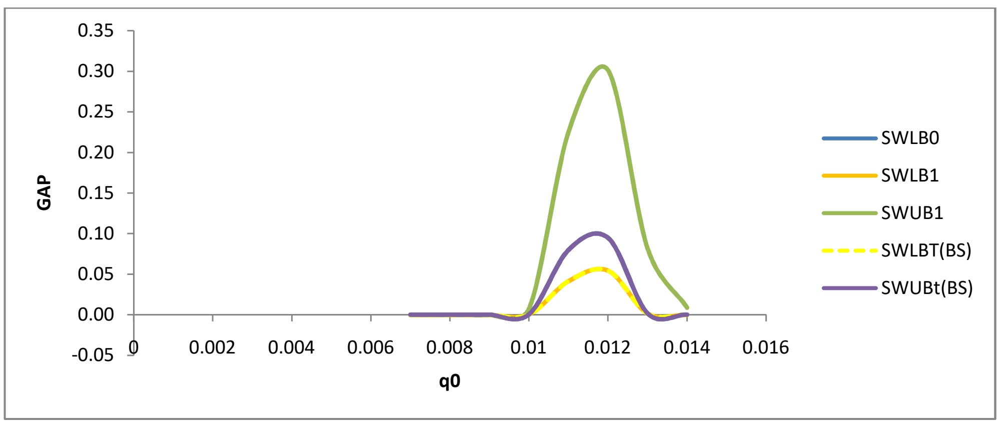


Figure 6.2: Comparison of different bounds under B-S model in terms of difference from MC estimate for r=0

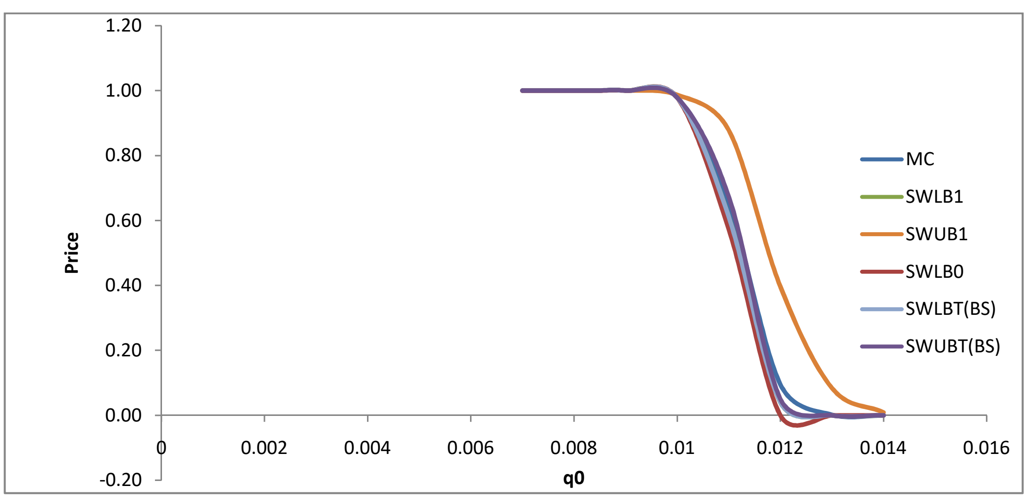


Figure 6.3: Price Bounds under Black-Scholes model for the parameter choice of Lin and Cox(2008) Model

Table 6.3: Lower Bounds and Upper Bound SWUB_1 for the Swiss Re Mortality Bond under the S_u distribution with $q_0 = 0.008453$ and parameter choice in accordance with Tsai and Tzeng (2013). MC Simulations:2000000 iterations (Antithetic Method)

r	SWLB_0	SWLB_1	$\mathrm{SWLB}_t^{(2)}$	MC	SWUB_1
0.035	0.883255461690	0.884321427702	0.885548150429	0.884689900254	0.886806565750
0.030	0.903403981323	0.904010021303	0.904693957669	0.904223406591	0.905481788285
0.025	0.921607066867	0.921935518851	0.922291170235	0.922030679118	0.922759498340
0.020	0.938407830149	0.938576980454	0.938747560828	0.938598989786	0.939010425492
0.015	0.954287129641	0.954369722665	0.954444088119	0.954415686473	0.954582647473
0.010	0.969639544072	0.969677756802	0.969706604343	0.969683647402	0.969774875755
0.005	0.984762743262	0.984779521693	0.984789115795	0.984784143646	0.984820459036
0.000	0.999861354235	0.999868375732	0.999870879263	0.999871208429	0.999884274666

Table 6.4: Lower Bounds and Upper Bound SWUB_1 for the Swiss Re Mortality Bond under the transformed gamma distribution with $q_0 = 0.0088$ and parameter choice in accordance with Cheng et al. (2014). MC Simulations:100000 iterations

r	SWLB_0	SWLB_1	$\mathrm{SWLB}_t^{(LG)}$	MC	SWUB_1
0.035	0.848032774815	0.848424044790	0.855969730838	0.854167495147	0.866104360048
0.030	0.873577023530	0.873813448730	0.879110918003	0.878026709161	0.887240130128
0.025	0.897102805167	0.897242672829	0.900881660116	0.900486935408	0.907283088297
0.020	0.918896959517	0.918977921696	0.921421185493	0.921030195924	0.926366403383
0.015	0.939240965474	0.939286791779	0.940888331577	0.941092453291	0.944633306794
0.010	0.958403723326	0.958429070674	0.959452704643	0.959485386732	0.962230654370
0.005	0.976635430514	0.976649121750	0.977286229664	0.977322136745	0.979302971605
0.000	0.994162849651	0.994170066411	0.994555652671	0.994698510161	0.995987334250

Table 6.5: Lower Bounds and Upper Bound $SWUB_1$ for the Swiss Re Mortality Bond under the transformed gamma distribution with $r = 0.0$ and parameter choice in accordance with Cheng et al. (2014). MC Simulations:100000 iterations

q_0	$SWLB_0$	$SWLB_1$	$SWLB_t^{(LG)}$	MC	$SWUB_1$
0.008	0.999766066714	0.999766066846	0.999772840362	0.999793281502	0.999779562417
0.0088	0.994162849651	0.994170066411	0.994555652671	0.994686720835	0.995987334250
0.009	0.989104987071	0.989146149900	0.989952105693	0.990012775483	0.993383346708
0.01	0.876692543049	0.888049181230	0.896376305638	0.891609413788	0.958189590379
0.011	0.410971060715	0.596089667857	0.596089667857	0.568675584083	0.837207974723
0.012	0.000000000000	0.271045973760	0.271045973760	0.207081909248	0.613838720959
0.013	0.000000000000	0.082740708460	0.082740708460	0.045779872978	0.381822437531
0.014	0.000000000000	0.012702023135	0.012702023135	0.006694089214	0.212229375395
0.015	0.000000000000	0.000000000000	0.000000000000	0.000883157236	0.110420349200
0.016	0.000000000000	0.000000000000	0.000000000000	0.000084710726	0.055539272591
0.017	0.000000000000	0.000000000000	0.000000000000	0.000004497045	0.027576845294
0.018	0.000000000000	0.000000000000	0.000000000000	0.000000019842	0.013697961783

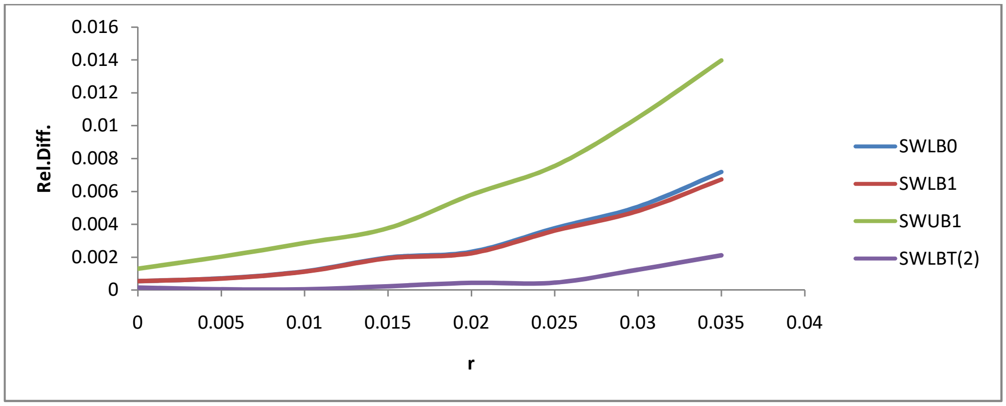


Figure 6.4: Relative Difference of Lower Bounds and SWUB1 w.r.t. MC estimate under Transformed Gamma Distribution

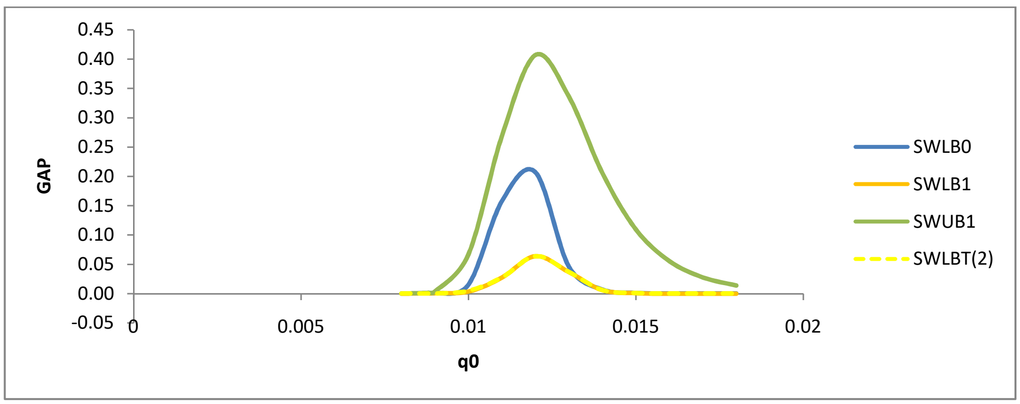


Figure 6.5: Comparison of different bounds under Transformed Gamma Distribution in terms of difference from MC estimate for r=0

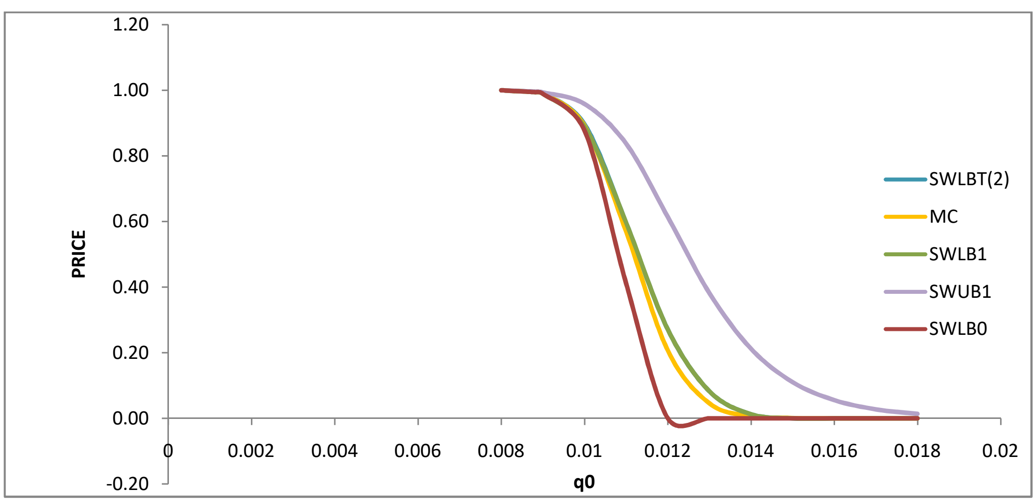


Figure 6.6: Price Bounds under Transformed Gamma Distribution for the parameter choice of Lin and Cox(2008) Model

6.2 Bounds for Guaranteed Annuity Options

Now we investigate the applications of the theory derived in Chapter 5. We have successfully obtained a number of lower bounds and an upper bound for GAOs in sections 5.5 and 5.6. We now test these vis-a-vis the well-known Monte Carlo estimate for the GAO. We first carry out this working under the well known Vasicek (1977) model in finance and then for a couple of more general affine models. The nomenclature for the bounds has already been specified in sections 5.3 - 5.7. In all the examples, the marginal cdfs are strictly increasing and we have the following 'Contract Specification':

$$g = 11.1\%,\ T = 15,\ n = 35;$$

6.2.1 Vasicek Model

In table 6.6, we assume that the interest rate (r_t) and the force of mortality (μ_t) for an insured aged x at time 0 obey the Vasicek model, with dynamics given by the specifications in equations (5.7.1)-(5.7.3). We highlight below the parameter choices in accordance with Liu et al. (2013). The value of the correlation coefficient between the interest rate and the force of mortality is varied in table 6.6. Parameter choices for table 6.6 are

Interest Rate Model:

$$a = 0.15\%,\ b = 0.045,\ \sigma = 0.03,\ r_0 = b;$$

Mortality Model:

$$c = 0.1\%,\ b = 0.045,\ \xi = 0.0003,\ \mu_0 = 0.006.$$

Using equations (5.7.15) and (5.7.17)-(5.7.18), we see that

$$Corr\left[r_T, \mu_T\right] = \frac{2\rho\sqrt{ac}}{(a-c)} \frac{\left(1 - e^{-(a-c)t}\right)}{\sqrt{\left(1 - e^{-2aT}\right)\left(e^{2cT} - 1\right)}} \tag{6.2.1}$$

As a result, we infer that the correlation between mortality and interest rate is directly proportional to the ρ which depicts the correlation between the underlying Brownian motions governing these two risks. In table 6.6, we vary ρ from -0.9 to 0.9 and investigate the effect of changing correlation between the two aforementioned risks on the lower and upper bounds and Monte Carlo estimate for the GAO price under the Vasicek model. To obtain the general lower bound given in section 5.5.3 we adhere to the same choice of Λ as that for the improved upper bound for the Vasicek case given in section 5.7.1. It is evident that when the correlation between these underlying rates grows, the prices of the GAO begin to swell. This finding is in line with the results of Liu et al. (2013). However, while the bounds obtained by these authors are vague, we succeed in deriving tight lower and upper bounds for the GAO price. The numerical findings of Table 6.6 are portrayed in figures 6.7-6.9. While figures 6.7 and 6.8 depict comparisons between the bounds, figure 6.9 portrays the price bounds for the GAO price under the Vasicek Model. We do not work out the third upper bound GAOUB for the Vasicek case since inversion of the distribution function is possible here yielding extremely tight upper bounds especially using the conditioning approach. Table 6.6 reflects that the relative difference ($= \frac{|bound - MC|}{MC}$) between any bound and the benchmark Monte Carlo estimate decreases with an increase in the correlation between mortality and interest rate. This observation is echoed by figure 6.7. On the other hand, figure 6.8 depicts the difference between the Monte Carlo estimate of the GAO price and the derived bounds. The bound $\text{GAOLB}_j^{(VS)}$ fares much better than GAOLB_3, although the former is restricted to the assumption of positive correlation between the two competing risks viz. mortality and interest rate which is by all means a very sensible assumption. The absolute difference between the estimated price and the bounds diminishes as the value of the correlation is increased. The competing worms in figures 6.7 and 6.8 show the efficacy of additional information as the ones exploiting extra knowledge completely outperform the thread of trivial lower bound by a huge margin. Finally figure 6.9 is a testimony to the fact that the bounds are extremely tight. There is indeed a clustering of the bounds around the line depicting Monte Carlo estimator.

Table 6.6: Lower Bounds and Upper Bounds for Guaranteed Annuity Option under the Vasicek Model with parameter choice in accordance with Liu et al. (2013). MC Simulations: 5000000 iterations (Antithetic Method)

ρ	GAOLB	GAOLB_3	$\mathrm{GAOLB}_j^{(VS)}$	MC	$\mathrm{GAOUB}_j^{(VS)}$	GAOUB_1
-0.9	0.082045744190341	0.088077585794385	—	0.089791711422800	0.089917068847054	0.090175032433513
-0.8	0.084191157665060	0.090393766053360	—	0.092053136968117	0.092370520060066	0.092373455587703
-0.7	0.086360371346849	0.092724338785360	—	0.094296354929367	0.094564734798698	0.094589506436884
-0.6	0.088553773983543	0.095070053289969	—	0.096550434401450	0.096649828192622	0.096824263010638
-0.5	0.090771763488456	0.097431774839569	—	0.098697144757000	0.099074543167257	0.099078692704665
-0.4	0.093014747224636	0.099810411041299	—	0.100832911943510	0.101256527508185	0.101353681427553
-0.3	0.095283142299906	0.102206875882750	—	0.103465845623342	0.103534711046787	0.103650054057831
-0.2	0.097577375873131	0.104622072640947	—	0.105710611150318	0.105744051774293	0.105968589290319
-0.1	0.099897885472241	0.107056886502845	—	0.108095470765208	0.108077361040687	0.108310030722257
0.0	0.102245119324506	0.109512182255336	0.110394181853836	0.110556954719044	0.110664647140070	0.110675095339724
0.1	0.104619536699618	0.111988804613683	0.112798007321230	0.112603738148081	0.113014934700507	0.113064480158935
0.2	0.107021608266156	0.114487579886000	0.115227439828916	0.115251590026012	0.115246545493725	0.115478867528274
0.3	0.109451816462018	0.117009318266658	0.117683066385172	0.117779468492292	0.117907094390104	0.117918929439538
0.4	0.111910655879470	0.119554816373780	0.120165448997058	0.120078033512172	0.120145436621166	0.120385331093984
0.5	0.114398633665466	0.122124859823794	0.122675178900746	0.122423792552651	0.122678769521784	0.122878733900060
0.6	0.116916269937927	0.124720225735538	0.125212855487838	0.125015700950805	0.125160614293534	0.125399798032477
0.7	0.119464098218718	0.127341685112358	0.127779087093470	0.128095109318917	0.127815863881931	0.127949184649358
0.8	0.122042665884089	0.129990005082130	0.130374491769752	0.130239118247019	0.130479176675164	0.130527557840740
0.9	0.124652534633377	0.132665950992500	0.132999698048989	0.132970337092151	0.133119674608691	0.133135586364709

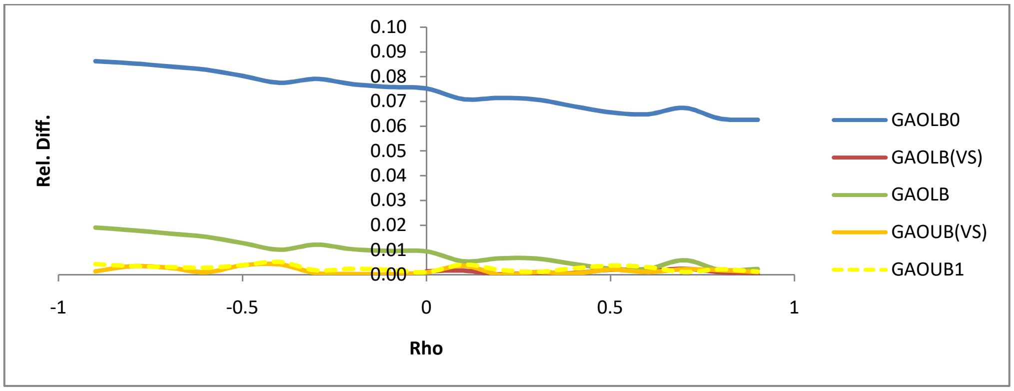


Figure 6.7: Relative Difference of Lower and Upper Bounds w.r.t. MC estimate under Vasicek model with GAOLB0 denoting GAOLB and GAOLB denoting $GAOLB_3$

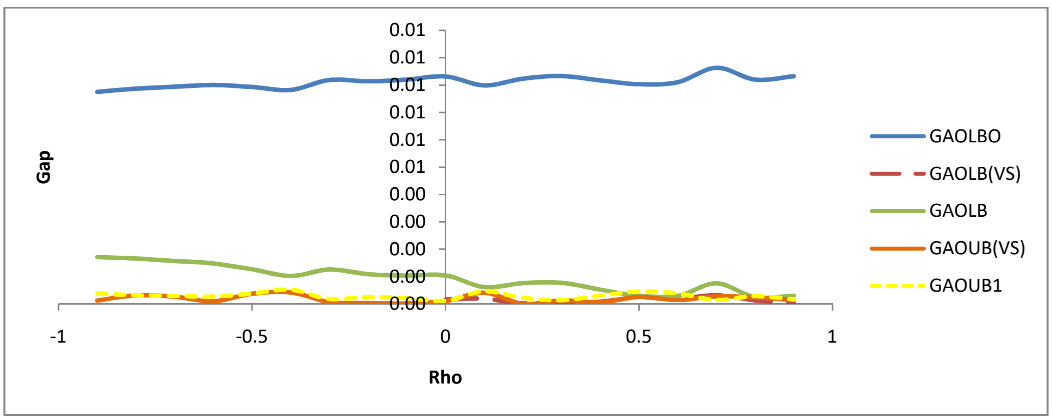


Figure 6.8: Comparison of different bounds under Vasicek Model in terms of difference from MC estimate with GAOLBO denoting GAOLB and GAOLB denoting $GAOLB_3$

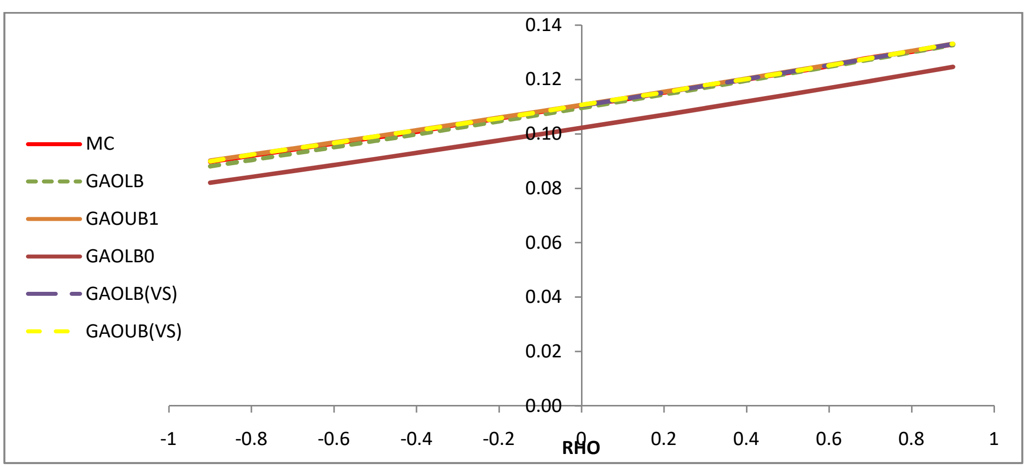


Figure 6.9: GAO Price Bounds under Vasicek model for the parameter choice of Liu(2013) with GAOLB0 denoting GAOLB and GAOLB denoting $GAOLB_3$

6.2.2 Multi CIR Model

First we consider a 3-dimensional CIR process $X := (X_t)_{t\geq 0}$ having independent components $(X_{it})_{t\geq 0}$, $i = 1, 2, 3$ (c.f. Deelstra et al., 2016, for details). We assume the following dynamics for the interest rate process and the mortality process.

$$r_t = \bar{r} + X_{1t} + X_{2t} \tag{6.2.2}$$

and

$$\mu_t = \bar{\mu} + m_2 X_{2t} + m_3 X_{3t}, \tag{6.2.3}$$

where $\bar{r}$, $\bar{\mu}$, m_2 and m_3 are constants. We use model specifications similar to Deelstra et al. (2016) and make a minute alteration in the parameter set. We fix the value of m_2 and obtain the value of m_3 such that the expectation of the mortality is fixed to a specified level denoted by $C_x(T)$ which is predicted by e.g. a Gompertz-Makeham model (c.f. Dickson et al., 2013) at age $x + T$ for an individual aged x at time 0, i.e.,

$$\mathbb{E}[\mu_t] = C_x(T), \tag{6.2.4}$$

Applying expectation on both sides of equation (6.2.3) and substituting in (6.2.4) we get

$$\bar{\mu} + m_2\mathbb{E}[X_{2t}] + m_3\mathbb{E}[X_{3t}] = C_x(T), \tag{6.2.5}$$

where as X_{it}; $i = 1, 2, 3$ is obtained using the Stochastic Differential Equation (SDE) given by (5.7.61), we have

$$\mathbb{E}[X_{it}] = X_{i,0}e^{-k_iT} + \theta_i\left(1 - e^{-k_iT}\right). \tag{6.2.6}$$

Using our contract specifications outlined in the beginning of this section we fix the expected value in (6.2.4) to the level $C_{50}(15) = 0.0125$. A very good discussion in regards to the validity of the model to be used for mortality appears in Deelstra et al. (2016). In fact this model was completely calibrated in Chiarella et al. (2016).

Using the set up defined by equations (6.2.2)-(6.2.3), the linear pairwise correlation between $(r_t)_{t\geq 0}$ and $(\mu_t)_{t\geq 0}$, denoted by ρ_t forms a stochastic process given by

$$\rho_t = \frac{m_2\sigma_2^2 X_{2t}}{\sqrt{\sigma_1^2 X_{1t} + \sigma_2^2 X_{2t}}\sqrt{m_2^2\sigma_2^2 X_{2t} + m_2^2\sigma_2^2 X_{2t}}}. \tag{6.2.7}$$

We vary the value of m_2 and therefore obtain the value of m_3 using equation (6.2.4) and this finally yields the value of ρ. Further in line with Deelstra et al. (2016), we make the following parameter specifications

$$\bar{r} = -0.12332, \quad \bar{\mu} = 0$$

Table 6.7: Parameter Values for the 3-dimensional CIR process

CIR process	Parameters			
X_1	$k_1 = 0.3731$	$\theta_1 = 0.074484$	$\sigma_1 = 0.0452$	$X_{1,0} = 0.0510234$
X_2	$k_2 = 0.011$	$\theta_1 = 0.245455$	$\sigma_2 = 0.0368$	$X_{2,0} = 0.0890707$
X_3	$k_3 = 0.01$	$\theta_1 = 0.0013$	$\sigma_3 = 0.0015$	$X_{3,0} = 0.0004$

Table 6.8 depicts the lower bound, the upper bound based on arithmetic geometric inequality and the Monte Carlo estimate of the GAO price for different values of m_2 and therefore for different values of the initial pairwise linear correlation coefficient ρ_0. The nomenclature of the bounds appear in section 5.7.2. Due to the lack of availability of the inverse distribution

function of weighted sum of non-central chi-square variates, we have to restrict ourselves to calculate only the third upper bound. We find that an increase in the value of ρ_0 enhances the value of the GAO in a manner similar to the Vasicek model shown in Table 6.6. The lower bound is extremely sharp. On the other hand, upper bound is slightly wider. The results of Table 6.8 are portrayed in figures 6.10-6.12.

Figure 6.10 reflects that the relative difference ($= \frac{|bound-MC|}{MC}$) between the upper bound and the benchmark Monte Carlo estimate decreases with an increase in the correlation between mortality and interest rate while the relative difference for the lower bound almost remains constant with varying ρ_0. On the other hand, figure 6.11 depicts the absolute difference between the Monte Carlo estimate of the GAO price and the derived bounds which remain more or less constant. The lower bound fares much better than $GAOUB$. Finally figure 6.12 shows the price bounds and in fact the lower bound stick completely camouflages with that of the MC estimate which is a testimony to the tightness of the lower bound.

Table 6.8: Lower Bound and Upper Bound for Guaranteed Annuity Option under the MCIR Model with partial parameter choice in accordance with Deelstra et al. (2016). MC Simulations: 50000

m_2	ρ	$\text{GAOLB}^{(MCIR)}$	MC	$\text{GAOUB}^{(MCIR)}$
-0.300	-0.570960646515027	0.153351236437789	0.153431631010533	0.216286630652776
-0.100	-0.460513730466363	0.181641413947461	0.181871723226662	0.243710313225013
-0.070	-0.403426257094426	0.187186872445969	0.187285214852833	0.249173899703122
-0.060	-0.376271648827787	0.189122188373390	0.189373949402726	0.251083730739797
-0.050	-0.343007585286942	0.191102351580502	0.191474297920361	0.253039217047040
-0.040	-0.301756813619030	0.193128263182051	0.195421232722993	0.255041205164633
-0.030	-0.250041147986350	0.195200853300304	0.195132243321684	0.257090572307459
-0.020	-0.184739400604580	0.197321081986930	0.197531098187496	0.259188227324353
-0.010	-0.102346730178820	0.199489940182500	0.199619257104038	0.261335111674878
-0.001	-0.011167160239806	0.201484335480591	0.201710195921424	0.263310203859562
0.000	0.000000000000000	0.201708450715130	0.201879045816498	0.263532200435103
0.001	0.011370596893292	0.201933073002533	0.202090425152612	0.263754709122691
0.010	0.122142590872118	0.203977669339908	0.204292134604299	0.265780503352825
0.020	0.257493768936871	0.206298685820891	0.206369996912367	0.268081065972310
0.030	0.391761086281179	0.208672625057373	0.208709896009824	0.270434970824946
0.040	0.508145173072700	0.211100648256358	0.211180180724315	0.272843338639536
0.050	0.596334605305204	0.213583954153270	0.213584231985838	0.275307329501738
0.060	0.656025897318996	0.216123780282872	0.216228415988778	0.277828143936307
0.070	0.693071640464574	0.218721404302618	0.218840241843838	0.280407024005732
0.100	0.730953349866014	0.226874471461256	0.226934772658478	0.288505131181583

6.2.3 Wishart Model

As a final step we test our bounds in the backdrop of the celebrated Wishart model for mortality and interest rate. The functional form of the model for the two aforesaid risks has been detailed in equations (5.7.91)-(5.7.92). To present the application of our methodology we stick to a 2-dimensional Wishart process, i.e., $d = 2$ due to the fact that higher dimensional Wishart processes are difficult to implement. The law for the underlying process X governing the mortality and interest rate processes has been outlined in equation (5.7.89). We consider partial choice of the parameter set in accordance with Deelstra et al. (2016). For all examples considered below, let

$$\beta = 3, \;\; \bar{r} = 0.04, \;\; \bar{\mu} = 0,$$

$$H = \begin{pmatrix} -0.5 & 0.4 \\ 0.007 & -0.008 \end{pmatrix}, \;\; M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \;\; R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{6.2.8}$$

in equations, (4.4.1) and (5.7.91)-(5.7.92).

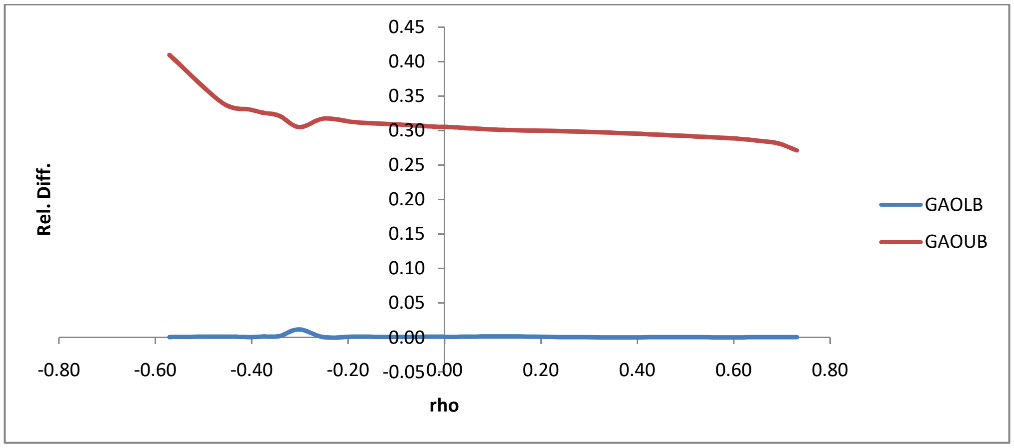


Figure 6.10: Relative Difference of Lower and Upper Bounds w.r.t. MC estimate under MCIR model

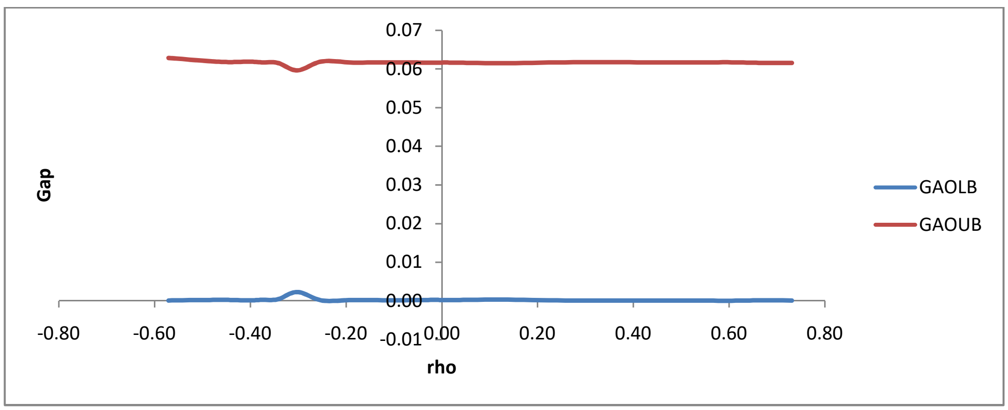


Figure 6.11: Comparison of different bounds under MCIR model in terms of difference from MC estimate

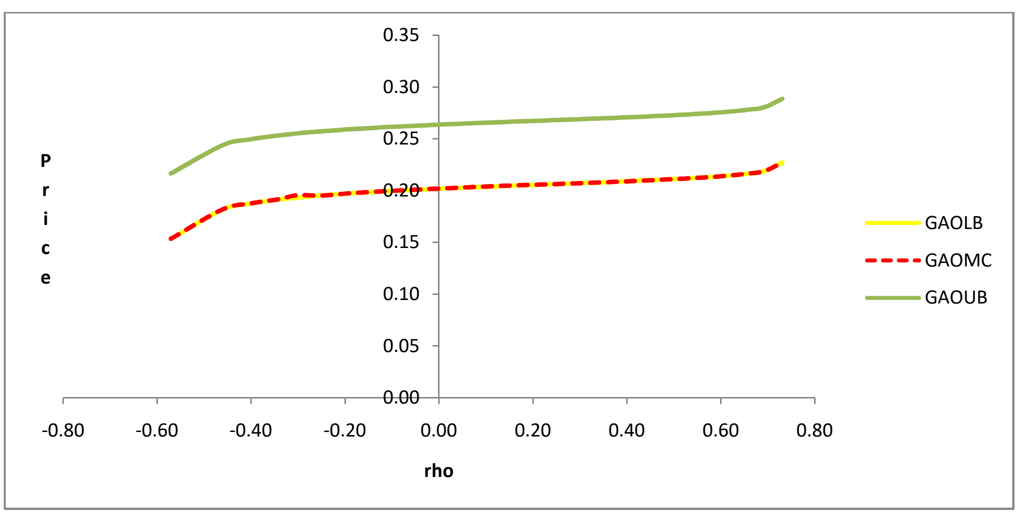


Figure 6.12: GAO Price Bounds under MCIR model

In light of this data considered for the Wishart model, the stochastic correlation between $(r_t)_{t\geq 0}$ and $(\mu_t)_{t\geq 0}$, denoted by ρ_t forms a stochastic process given by

$$\rho_t = \frac{(Q_{11}Q_{12} + Q_{22}Q_{21})\, X_t^{12}}{\sqrt{(Q_{11}^2 + Q_{21}^2)\, X_t^{11}\, (Q_{22}^2 + Q_{12}^2)\, X_t^{22}}}. \tag{6.2.9}$$

As is evident from (6.2.9), using a Wishart formulation for underlying process X produces a more richer dependence structure for the underlying risks than was available under the multi-dimensional CIR case. This calls for carrying out a more sophisticated sensitivity analysis in regards to the involved parameters. In the same spirit as Deelstra et al. (2016), we carry out a two-fold testing

- the first one by varying the off-diagonal elements of the initial Wishart process X_0 and investigating the impact on the prices of the GAO,
- the second one by experimenting with the off-diagonal elements of the matrix Q.

In each case, we compute the first lower bound, the upper bound based on arithmetic geometric inequality and the Monte Carlo estimate of the GAO price which is computed using 20000 simulations. The nomenclature of the bounds appear in section 5.7.3. Due to the lack of availability of the inverse distribution function of weighted sum of non-central chi-square random variables, we have to restrict ourselves to calculate only the third upper bound. For stability checks in relation to the expected values of the interest rate and mortality intensity w.r.t. varying correlation, interested readers can refer to Deelstra et al. (2016).

Effect of a Change in Initial Value X_0

In order to see the behaviour of the price bounds for the GAO price vis-a-vis change in the initial value of the Wishart process, we experiment with two cases:

- Negative off-diagonal elements in the volatility matrix Q
- Positive off-diagonal elements in the volatility matrix Q.

Example 1. In this case we consider the following Wishart process:

$$Q = \begin{pmatrix} 0.06 & -0.0006 \\ -0.06 & 0.006 \end{pmatrix}, \quad X_0 = \begin{pmatrix} 0.01 & X_0^{12} \\ X_0^{12} & 0.001 \end{pmatrix}. \tag{6.2.10}$$

Table 6.9 portrays the lower bound, the upper bound and the Monte Carlo estimate of the GAO price for different values of X_0^{12} and therefore for different values of the initial pairwise linear correlation coefficient ρ_0. We find that an increase in the value of ρ_0 enhances the value of the GAO in a fashion similar to the one shown for the Multi-CIR set up in Table 6.8 and the Gaussian set-up in Table 6.6. In this case both the lower and the upper bounds show close proximity to the GAO value.

Example 2. In the second investigation, we consider the following Wishart process:

$$Q = \begin{pmatrix} 0.06 & 0.00001 \\ 0.0002 & 0.006 \end{pmatrix}, \quad X_0 = \begin{pmatrix} 0.01 & X_0^{12} \\ X_0^{12} & 0.001 \end{pmatrix}. \tag{6.2.11}$$

As can be seen, in this example, we consider positive off-diagonal elements for the matrix Q. Table 6.10 portrays the lower bound, the upper bound and the Monte Carlo estimate of the GAO price for different values of X_0^{12} and therefore for different values of the initial pairwise linear correlation coefficient ρ_0. The results obtained present a sharp contrast to those obtained in Table 6.9 and the value of the GAO price and the corresponding bounds begin to drop as the value of ρ_0 is increases. Both the bounds continue to perform well even on this occasion.

A good justification of the behaviour of the GAO price in the first two examples (also see Deelstra et al., 2016) vis-a-vis the values of X_0^{12} can be provided by noting that under dynamics

Table 6.9: Lower Bound and Upper Bound for Guaranteed Annuity Option under the Wishart Model Example 1 with parameter choice in accordance with Deelstra et al. (2016). MC Simulations: 20000

X_0^{12}	ρ	$\text{GAOLB}^{(WIS)}$	MC	$\text{GAOUB}^{(WIS)}$
-0.003	0.734240363158475	0.241898614923743	0.241247798732840	0.241898616247735
-0.002	0.489493575438983	0.241133565561902	0.240529742517039	0.241133567256078
-0.0015	0.367120181579237	0.240751892681841	0.239890712272120	0.240751894570155
-0.0005	0.122373393859746	0.239990246464251	0.239141473598451	0.239990248759807
0	0.000000000000000	0.239610271506445	0.238621509824004	0.239610274015476
0.0005	-0.122373393859746	0.239230860904335	0.238198077279364	0.239230863633664
0.0015	-0.367120181579237	0.238473729539084	0.237679950746197	0.238473732730283
0.002	-0.489493575438983	0.238096007164703	0.237331879397777	0.238096010597887
0.003	-0.734240363158475	0.237342245012764	0.236699850447918	0.237342248953105

of the Wishart process (5.7.89) the positive factors (i.e. X_t^{11} & X_t^{22}) swell on an average when the initial value X_0^{12} increases. Moreover, for the aforementioned parameter choice, the models for mortality and interest rate for $t \geq 0$ are given as

$$r_t = 0.04 + X_t^{11} \tag{6.2.12}$$

and

$$\mu_t = X_t^{22}. \tag{6.2.13}$$

Now, it is clear from the formula for GAO price given in equation (5.7.104), that the exponential term containing r_t and μ_t decays when X_0^{12} increases and this causes the GAO price and corresponding bounds to diminish when X_0^{12} soars.

Table 6.10: Lower Bound and Upper Bound for Guaranteed Annuity Option under the Wishart Model Example 2. MC Simulations: 20000

X_0^{12}	ρ	$\text{GAOLB}^{(WIS)}$	MC	$\text{GAOUB}^{(WIS)}$
-0.003	-0.004743383550130	0.332948404889575	0.341196353690094	0.332948737923275
-0.002	-0.003162255700087	0.331667762094902	0.340868095614857	0.331668129236460
-0.0015	-0.002371691775065	0.331029148831226	0.339651861654315	0.331029534152954
-0.0005	-0.000790563925022	0.329755328754714	0.339133498851769	0.329755752815905
0	0.000000000000000	0.329120118025352	0.338246665341653	0.329120562698337
0.0005	0.000790563925022	0.328486037563152	0.337667153845205	0.328486503712013
0.0015	0.002371691775065	0.327221259643971	0.336913730200477	0.327221771448801
0.002	0.003162255700087	0.326590558297188	0.336554330647556	0.326591094339721
0.003	0.004743383550130	0.325332521129667	0.335045167150194	0.325333108616048

Effect of a Change in Volatility Matrix Q

We now carry out an experiment to vary the off-diagonal elements of the volatility matrix Q which we assume to be symmetric while specifying the initial value X_0 of the Wishart process.

Example 3. Here the Wishart process is as follows:

$$Q = \begin{pmatrix} 0.06 & Q_{12} \\ Q_{12} & 0.006 \end{pmatrix}, \quad X_0 = \begin{pmatrix} 0.01 & 0.001 \\ 0.001 & 0.001 \end{pmatrix}. \tag{6.2.14}$$

Table 6.11 depicts the lower bound, the upper bound and the Monte Carlo estimate of the GAO price for different values of Q_{12} and therefore for different values of the initial pairwise linear correlation coefficient ρ_0. The results obtained show that the value of the GAO price and the corresponding bounds do not show a monotone behaviour in respect of the linear correlation between mortality and interest rate risks. The tightness of the bounds around the Monte Carlo estimate still remains intact. These observations are echoed in Figure 6.15. In addition Figure 6.13 reflects that the relative difference $\left(= \frac{|bound-MC|}{MC}\right)$ between the lower bound and the benchmark Monte Carlo estimate increases with an increase in the correlation ρ_0 between mortality and interest rate. For example looking at table 6.11, we see that the relative difference for GAOLB increases from a meagre 0.2% for $\rho_0 = -0.3$ to about 7.7% for a $\rho_0 = 0.3$. However, under the same set, the relative difference between the estimated GAO price and the upper bound increases and then there is a switch at $\rho_0 = 0.3$ and this gap begins to diminish. The last observation is also seen in Figure 6.14 for the absolute difference between the bounds and the MC estimate of GAO price.

The reason for this behaviour of the GAO price lies in the structure of the matrix $Q^T Q$ (also see Deelstra et al. (2016)). It is clear that the diagonal elements of $Q^T Q$ increase with a rise in the absolute value of Q_{12}. A glance at the law of the Wishart process given in equation (4.4.1) and equations (6.2.12)-(6.2.13) brings out the fact that the drift and in particular the long term value of the positive factors of the Wishart process and in turn the drift of mortality and interest rate process is an increasing function of the absolute value of Q_{12}. Thus an upward rise in the value of Q_{12} will enhance the positive factors. As a result, it is evident from equation (5.7.104) describing the GAO price in the Wishart case, that the exponential term containing r_t and μ_t decreases when Q_{12} moves away from zero and this causes the GAO price and corresponding bounds to diminish.

Overall our numerical experiments provide strong evidence in support of the extremely adequate performance of our proposed bounds.

Table 6.11: Lower Bound and Upper Bound for Guaranteed Annuity Option under the Wishart Model Example 3 with parameter choice in accordance with Deelstra et al. (2016). MC Simulations: 20000

Q_{12}	ρ	GAOLB$^{(WIS)}$	MC	GAOUB$^{(WIS)}$
-0.01	-0.294220967543866	0.290016256883993	0.290601398401997	0.290593286187411
-0.006	-0.244746787719492	0.331837945818948	0.332421093218907	0.331843140669134
-0.002	-0.109938939767707	0.339526376457815	0.344143066326585	0.339526466816062
0.002	0.109938939767707	0.308919593324378	0.322579113504993	0.308928343737340
0.006	0.244746787719492	0.257040019380241	0.274376988651895	0.257891897298705
0.01	0.294220967543866	0.196440417823759	0.212744888444368	0.204994244625801

6.2.4 Computational Speed of the Bounds

We summarize the time consumed in computation of the bounds and the Monte Carlo estimate in Table 6.12. Further, these observations are portrayed in the figure 6.16 which shows an phenomenal increase in the time taken by Monte Carlo estimation as the number of simulations increase.

Table 6.12: Time taken in seconds for Bounds and Simulations

			Number of	Simulations	for	Monte Carlo
Example	GAOLB	GAOUB	1000	10000	20000	50000
MCIR	0	1	44	352	696	1800
Wishart 1	0	1	47	369	749	2100
Wishart 2	0	1	49	379	757	2200
Wishart 3	0	1	43	359	724	2000

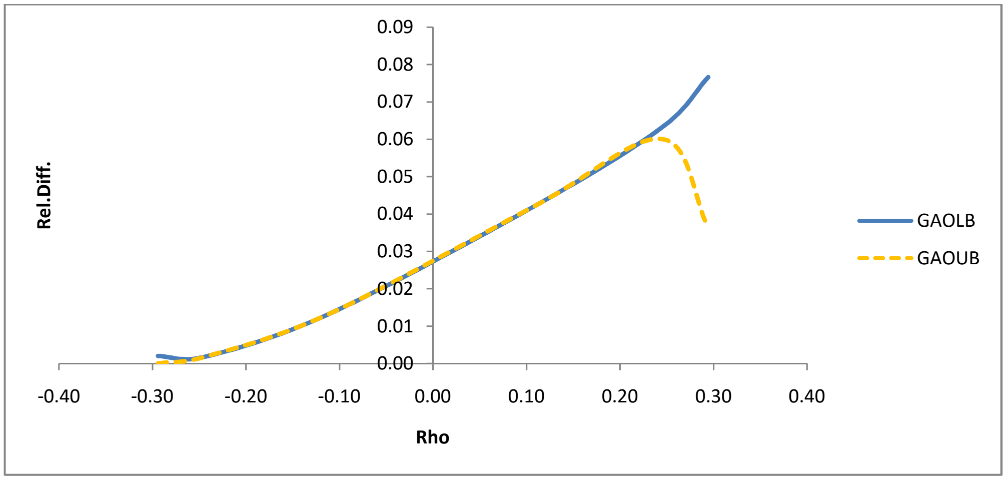


Figure 6.13: Relative Difference of Lower and Upper Bounds w.r.t. MC estimate under Wishart Example 3

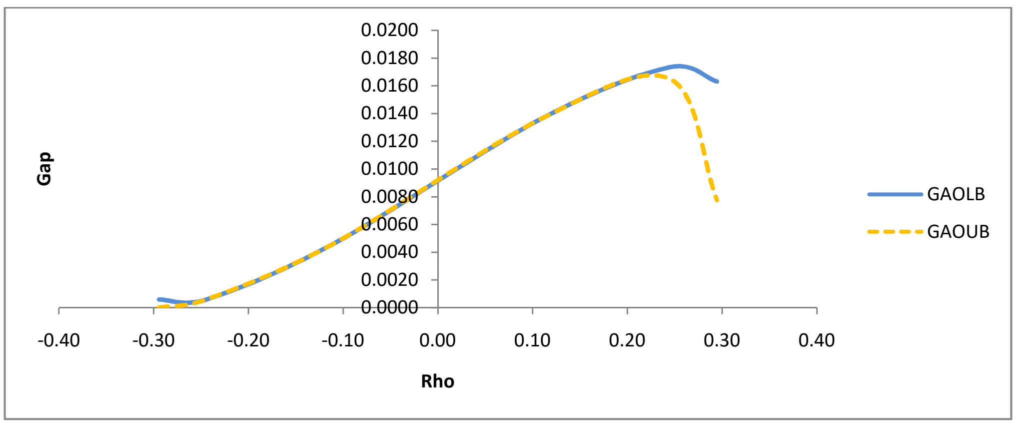


Figure 6.14: Comparison of different bounds under Wishart Example 3 in terms of difference from MC estimate

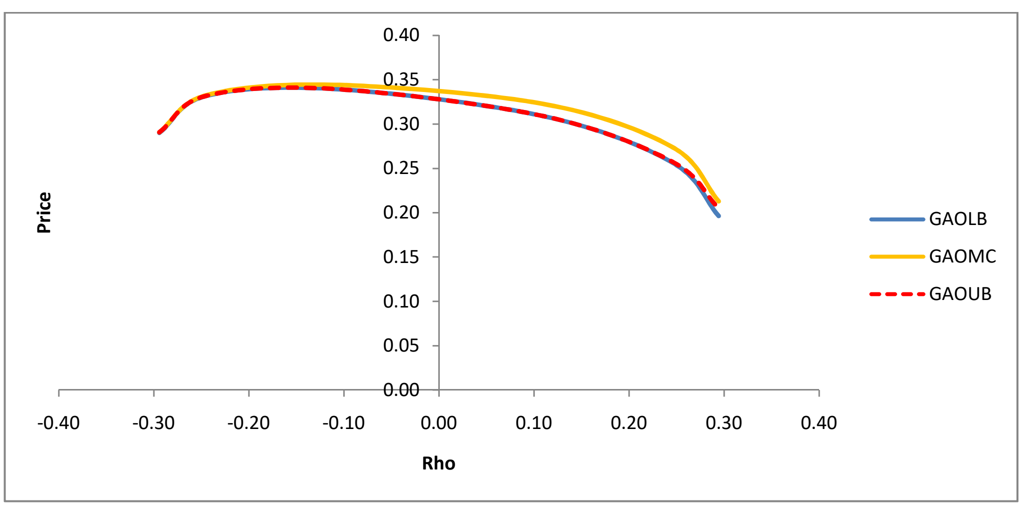


Figure 6.15: Price Bounds under Wishart Example 3

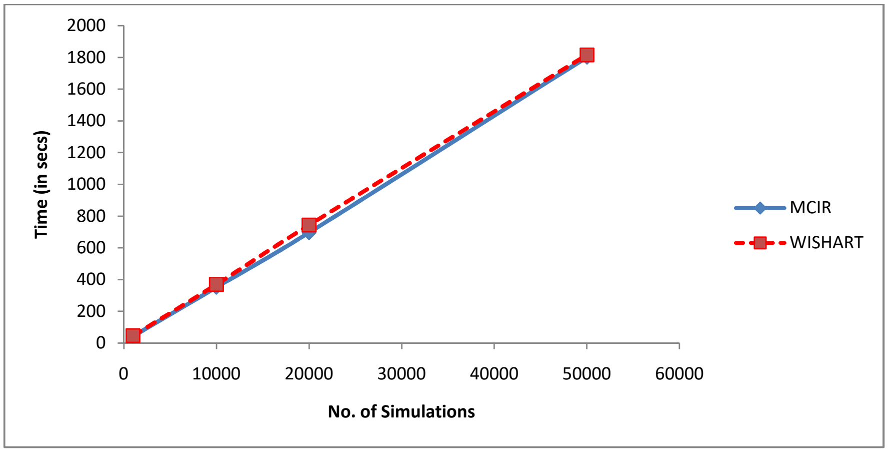


Figure 6.16: The CPU time (seconds) for MCIR and Wishart (average for 3 cases)

All computations in section 6.2.4 are carried out on a personal laptop with Intel(R) Core(TM) i5 CPU-M450 at 2.40 GHz and a RAM of 4.00 GB.

Chapter 7

Conclusions

The gains from this research are manifold. We have been successful in obtaining model-free price bounds for two very useful mortality-linked products viz. 'Catastrophic Mortality Bonds' and 'Guaranteed Annuity Options'. We have utilized up to date methodology available in the literature for the pricing of these products. We summarize the conclusions of these innovative experiments and throw light on future research perspectives in the ensuing sections.

7.1 Mortality Pricing

Mortality forecasts are extremely significant in the management of life insurers and private pension plans. Securitization and construction of mortality bonds has become an important part of capital market solutions. Prior to the launch of the Swiss Re bond in 2003, life insurance securitization was not designed to handle mortality risk.

Our research investigates the designing of price bounds for the Swiss Re mortality bond 2003. As stated in Deng et al. (2012), an incomplete mortality market that has no arbitrage guarantees the existence of at least one risk-neutral measure termed the equivalent martingale measure Q that can be used for calculating fair prices of mortality securities. We rely on this fact and devise model-independent bounds for the mortality security in question. To the best of our knowledge, there is only one earlier publication by Huang et al. (2014) in direction of price bounds for the Swiss Re bond. However, these authors propose gain-loss bonds that suffer from model risk. Our results assume the trading of vanilla options written on the mortality index, as in that case one can use the market price of these options to create bounds which are truly model independent. A worthy observation is that the stimulant for the present work is the theory of comonotonicity. One can therefore easily extend this approach for computing tight bounds for other mortality and longevity linked securities.

7.2 Longevity Pricing

We have derived some very general bounds for the valuation of GAOs under the assumption of a prevailing correlation between mortality and interest rate risk. These bounds serve as a useful tool for financial institutions which are striving hard to find methodologies that offer efficient pricing of longevity linked securities. The techniques used in this research are successful in circumventing the issue of dealing with sums of a large number of correlated variables. Moreover they are extremely useful in reducing the burden of dealing with cumbersome stochastic processes. In fact, the bounds are extremely tight particularly when the underpinning risks are governed by Vasicek models.

The most successful finding of this research is that in the affine case, both the lower and the upper bound depend on the properties of the distribution of the random variables connected to the transformed stochastic processes underlying mortality and interest rate. Moreover the lower bound manifests itself in form of Laplace transform of the underlying random variable while the upper bound reveals itself in the form of the associated characteristic function. Both of these tools are the most conveniently obtained vital statistics for any distribution. The most satisfying aspect is that we need to work in one dimension, in contrast to what would have been atleast a 34-dimensional set up, assuming that a person lives atleast 100 years making $n = 35$. As a result in cases where inversion of the distribution function is unavailable, an upper bound can still be find provided the characteristic function of the log prices of the underlying assets viz. pure endowments in our case is known. This is generally available as an explicit expression in case of affine processes subject to the solution of a system of a generalized Riccati equations.

Another feather in the cap of the bounds is their computational speed. As indicated in section 6.2.4, the Monte Carlo method is extremely slow for large number of simulations in case of sophisticated models. As a result given the same time budget, Monte Carlo estimates are deemed to be extremely inaccurate. Moreover for highly sophisticated multivariate distributions like non-central Wishart, generating random samples generally involves complex algorithms, which are not inbuilt in libraries of packages such as MATLAB. It is indeed not less than magical that our lower bound takes just 0.133 or 0.192 seconds on an average to execute in the MCIR and Wishart case while an average of about 0.286 or 0.659 seconds are required by the upper bound respectively in the two cases. In other words pricing of complex GAOs can be done in no time.

Last but not the least the fact that the upper bound performs much better in the case of Wishart is a noteworthy observation since the Wishart model is much more intricate in terms of gauging the mesh of correlation between mortality and interest rate.

The sensitivity analysis done in this chapter reiterates the fact that it is not possible to explain the value of a GAO completely in terms on the initial pairwise linear correlation between mortality and interest rate risks as highlighted by the Wishart model (c.f. Deelstra et al., 2016, for earlier work in this direction) . This finding sends alarm signals for the risk management in the presence of an unknown dependence as various scenarios are possible.

If the prices of a GAO increase with the (initial) linear correlation coefficient as in the multi-CIR model or the Wishart specifications in Example 1, then the most risk-averse methodology when pricing a GAO would be to take the linear correlation coefficient equal to unity. This will protect the seller from an awkward scenario of underestimation of the GAO price in the event of a high correlation. However, Example 2 of the Wishart case, presents an opposite scenario where prices decrease with increasing initial linear correlation and therefore, risk-adverse seller would adopt the opposite rule in that situation. Example 3 in the Wishart case portrays prices which are not monotone with respect to the correlation, but which seem to lead to the highest prices for zero correlation. Therefore, in this situation, choosing zero correlation might be the appropriate risk-averse choice. In fact the Wishart model comes across the most versatile model presenting all possible dependence scenarios.

The methodology proposed in our research is extremely flexible and can be easily extended to value other insurance products such as indexed annuities or to instruments with option embedded features such as equity-linked annuities, equity-indexed annuities and variable annuities.

7.3 Future Perspectives

The essence of this thesis is the computation of model-free robust bounds for complex mortality and longevity linked products. The thesis utilizes a number of mathematical and statistical tools to obtain the desired results with the theory of comonotonicity being the front runner in simplifying mathematical complexity. This research has a lot of potential to be exploited for valuation of other mortality and longevity linked derivatives.

Applying the techniques used for the pricing of Swiss Re Mortality Bond to the valuation of the ‘Longevity Trend Bond’-‘Kortis’ is a project that is already in the pipeline. The latter has structural similarities to Catastrophic Mortality Bonds with a payoff contingent on the performance of an index called the 'Longevity Divergence Index'. A more easier application of the derived theory is to obtain the price of any other catastrophic mortality bond by suitably adapting the settings.

Experiments to incorporate jumps in the models for mortality to price GAOs will also be very interesting. Moreover, the methodology employed for furnishing bounds for GAOs can be extended to obtain bounds for Guaranteed Minimum Income Benefits (GMIBs).

Hedging risks for insurance companies is far more complicated than hedging financial risks due to the longevity market being in its formative age and lacking liquidity in terms of trading mortality-linked derivatives. A future aim of this research is to devise appropriate hedging techniques for these newly emerging mortality and longevity-linked products.

Appendix A

Some More Building Blocks

We furnish here some more fundamental results that provide clarity regarding the calculations undertaken in this research work.

A.1 The Assumption for SWLB$_t^{(2)}$

We discuss the relation of the equation (3.3.28) to exponential Lévy models.

Proposition 43. *The following assumption holds for exponential Lévy models with mortality evolution $q_t = q_0 e^{X_t}$, where $(X_t)_{t\geq 0}$ is a Lévy process*

$$S^{'} = \sum_{i=1}^{n} q_i \geq_{sl} \sum_{i=1}^{n} \boldsymbol{E}[q_i|q_t] \geq \sum_{i=1}^{j-1} q_0^{(1-t_i/t)} q_t^{t_i/t} + \sum_{i=j}^{n} e^{r(t_i-t)} q_t, \tag{A.1.1}$$

Proof. Assume $t > t_i$. Clearly we have,

$$\begin{aligned}
\mathbf{E}\left[q_i|q_t = s\right] &= q_0 \mathbf{E}\left[e^{X_{t_i}}|q_0 e^{X_t} = s\right] \\
&= q_0 \mathbf{E}\left[e^{X_{t_i}}|X_t = \log_e\left(\frac{s}{q_0}\right)\right] \\
&\geq q_0 e^{\left(\mathbf{E}\left[X_{t_i}|X_t = \log_e\left(\frac{s}{q_0}\right)\right]\right)},
\end{aligned} \tag{A.1.2}$$

where the last step follows from Jensen's inequality. Clearly

$$\mathbf{E}\left[X_t|X_t = \log_e\left(\frac{s}{q_0}\right)\right] = \log_e\left(\frac{s}{q_0}\right) \tag{A.1.3}$$

Clearly, using that for a Lévy process $X_0 = 0$,

$$X_t = (X_t - X_{t-1}) + (X_{t-1} - X_{t-2}) + ... + (X_1 - X_0). \tag{A.1.4}$$

Now, on using the independence and stationarity of increments of a Lévy process (c.f. Chapter 2) on the R.H.S. of (A.1.4), we have that $X_i - X_{i-1} \overset{d}{=} X_1$ for $i = 1, 2, ..., t$ are independently and identically distributed (i.i.d.) random variables. Thus for any $i = 1, 2, ..., t$

$$\mathbf{E}\left[X_i - X_{i-1}|\left(X_t - X_{t-1}\right) + ... + (X_1 - X_0) = \log_e\left(\frac{s}{q_0}\right)\right] = \mathbf{E}\left[X_1|X_t = \log_e\left(\frac{s}{q_0}\right)\right] \tag{A.1.5}$$

Summing both sides over $i = 1, 2, ..., t$ and taking summation inside the expectation on the L.H.S. and noting that we have a telescopic sum, we have

$$\mathbf{E}\left[X_t|X_t = \log_e\left(\frac{s}{q_0}\right)\right] = t\mathbf{E}\left[X_1|X_t = \log_e\left(\frac{s}{q_0}\right)\right] \tag{A.1.6}$$

From equations (A.1.3) and equation (A.1.6), we get

$$\mathbf{E}\left[X_1|X_t = \log_e\left(\frac{s}{q_0}\right)\right] = \frac{1}{t}\log_e\left(\frac{s}{q_0}\right). \tag{A.1.7}$$

On the same lines we have for $t_i < t$

$$\begin{aligned}\mathbf{E}\left[X_{t_i}|X_t = \log_e\left(\frac{s}{q_0}\right)\right] &= \mathbf{E}\left[(X_{t_i} - X_{t_i-1}) + ... + (X_1 - X_0)\,|X_t = \log_e\left(\frac{s}{q_0}\right)\right] \\ &= t_i\mathbf{E}\left[X_1|X_t = \log_e\left(\frac{s}{q_0}\right)\right] \qquad \text{(A.1.8)}\end{aligned}$$

Substituting equation (A.1.7) in equation (A.1.8), we get

$$\mathbf{E}\left[X_{t_i}|X_t = \log_e\left(\frac{s}{q_0}\right)\right] = \frac{t_i}{t}\log_e\left(\frac{s}{q_0}\right). \tag{A.1.9}$$

Substituting equation (A.1.9) in equation (A.1.2), we get

$$\begin{aligned}\mathbf{E}\left[q_i|q_t = s\right] &\geq q_0 e^{\left(\frac{t_i}{t}\log_e\left(\frac{s}{q_0}\right)\right)} \\ &= q_0^{1-\frac{t_i}{t}} s^{\frac{t_i}{t}}. \qquad \text{(A.1.10)}\end{aligned}$$

This implies that for all $t > t_i$, we have

$$\mathbf{E}\left(q_i|q_t\right) \geq q_0^{1-\frac{t_i}{t}} q_t^{\frac{t_i}{t}}. \tag{A.1.11}$$

For $t_i \geq t$, using the fact that discounted asset prices are martingales, we can write $\mathbf{E}\left(q_i|q_t\right) = e^{r(t_i-t)}q_t$. Combining these results we get the following stop-loss order relationship:

$$S^{'} = \sum_{i=1}^{n} q_i \geq_{\text{sl}} \sum_{i=1}^{n}\mathbf{E}\left[q_i|q_t\right] \geq \sum_{i=1}^{j-1} q_0^{(1-t_i/t)} q_t^{t_i/t} + \sum_{i=j}^{n} e^{r(t_i-t)}q_t, \tag{A.1.12}$$

where $j = \min\{i\,:\, t_i \geq t\}$, which is the requisite assumption (A.1.1). $\square$

A.2 Interchange of Sigma Algebras in Combined Market Framework

We consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ that is large enough to support a process X in $\mathbb{R}^k$, representing the evolution of financial variables and a process Y in $\mathbb{R}^d$, representing the evolution of mortality. Moreover, we concentrate on an insured life aged x at time 0, with random residual lifetime denoted by τ_x which is an $\mathcal{F}_t$-stopping time.

The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ includes knowledge of the evolution of all state variables upto each time t and of whether the policyholder has died by that time. More explicitly, we have:

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$$

where

$$\mathcal{G}_t \vee \mathcal{H}_t = \sigma\left(\mathcal{G}_t \cup \mathcal{H}_t\right)$$

with

$$\mathcal{G}_t = \sigma\left(Z_s :\ 0 \leq s \leq t\right), \quad \mathcal{H}_t = \sigma\left(\mathbb{1}_{\{\tau \leq s\}} :\ 0 \leq s \leq t\right)$$

and where $Z = (X, Y)$ is the joint state variables process in $\mathbb{R}^{k+d}$. Thus we have

$$\mathcal{G}_t = \mathcal{G}_t^X \vee \mathcal{G}_t^Y.$$

In fact $\{\mathcal{H}_t\}_{t\geq 0}$ is the smallest filtration with respect to which τ is a stopping time. In other words $\{\mathcal{H}_t\}_{t\geq 0}$ makes $\{\mathcal{F}_t\}_{t\geq 0}$ the smallest enlargement of $\{\mathcal{G}_t\}_{t\geq 0}$ with respect to which τ is a stopping time, i.e.,

$$\mathcal{F}_t = \cap_{s>t}\mathcal{G}_s \vee \sigma\left(\tau \wedge s\right),\ \forall t.$$

We may think of $\mathcal{G}_t$ as carrying information captured from medical/demographical data collected at population/ industry level and of $\mathcal{H}_t$ as recording the actual occurrence of death in an insurance portfolio.

We now show that we can carry out conditioning interchange between $\mathcal{F}_t$ and $\mathcal{G}_t$.

We characterize the conditional law of τ in several steps. We assume that a non-negative $\mathcal{G}_t$-predictable process μ is given satisfying $\int_0^t \mu_x\left(s\right)ds < \infty$ a.s. for all $t > 0$. We then consider an exponential random variable Φ with parameter 1, independent of $\mathcal{G}_\infty$ and define the random time of death τ as the first time when the process $\int_0^t \mu_s ds$ is above the random threshold Φ, i.e.,

$$\tau \doteq \{t \in \mathbb{R}^+ : \int_0^t \mu_s\left(s\right)ds \geq \Phi\}. \tag{A.2.1}$$

It is evident from (A.2.1) that $\{\tau > T\} = \{\int_0^t \mu_s ds < \Phi\}$, for $T \geq 0$. Next, we work out $\mathbb{P}\left(\tau > T|\mathcal{G}_t\right)$ for $T \geq t \geq 0$ by using tower property of conditional expectation, independence of Φ and $\mathcal{G}_\infty$ and facts that μ is a $\mathcal{G}_t$-predictable process and $\Phi \sim \textit{Exponential}\left(1\right)$, i.e.,

$$\begin{aligned}
\mathbb{P}\left(\tau > T|\mathcal{G}_t\right) &= E\left[\mathbb{1}_{\{\Phi>\int_0^T \mu_s ds\}}|\mathcal{G}_t\right] \\
&= E\left[E\left(\mathbb{1}_{\{\Phi>\int_0^T \mu_s ds\}}|\mathcal{G}_T\right)|\mathcal{G}_t\right] \\
&= E\left[\mathbb{P}\left(\Phi > \int_0^T \mu_s ds\right)|\mathcal{G}_t\right] \\
&= E\left[e^{-\int_0^T \mu_s ds}|\mathcal{G}_t\right]
\end{aligned} \tag{A.2.2}$$

In fact, the same result holds for $0 \leq T < t$. Further, we observe that $\{\tau > t\}$ is an atom of $\mathcal{H}_t$. As a result, we have constructed a doubly stochastic $\mathcal{F}_t$-stopping time driven by $\mathcal{G}_t \subset \mathcal{F}_t$ in the following way (c.f. Billingsley, 1995, ex 34.4, p.455):

$$\begin{aligned}
\mathbb{P}\left(\tau > T|\mathcal{G}_T \vee \mathcal{F}_t\right) &= \mathbb{1}_{\{\tau>t\}}E\left[\mathbb{1}_{\{\tau>T\}}|\mathcal{G}_T \vee \mathcal{H}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{P}\left(\{\tau > T\}\cap\{\tau > t\}|\mathcal{G}_T\right)}{\mathbb{P}\left(\{\tau > t\}|\mathcal{G}_T\right)} \\
&= \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{P}\left(\{\tau > T\}|\mathcal{G}_T\right)}{\mathbb{P}\left(\{\tau > t\}|\mathcal{G}_T\right)} \\
&= \mathbb{1}_{\{\tau>t\}}e^{-\int_t^T \mu_s ds}
\end{aligned} \tag{A.2.3}$$

Next, we show that the conditioning on $\mathcal{F}_t$ can be replaced by conditioning on $\mathcal{G}_t$. We first note that $e^{-\int_t^T \mu_s ds}$ is $\mathcal{G}_T$-measurable and so $\sigma\left(e^{-\int_t^T \mu_s ds}\right) \subset \mathcal{G}_T$ and observe that the independence of Φ and $\mathcal{G}_T$ implies independence of Φ and $\sigma\left(e^{-\int_t^T \mu_s ds}\right)$ and in turn of Φ and $\mathcal{G}_t \vee \sigma\left(e^{-\int_t^T \mu_s ds}\right)$. This also implies that $e^{-\int_t^T \mu_s ds}$ is independent of $\sigma\left(\Phi\right)$. As a result, we can write:

$$E\left[e^{-\int_0^T \mu_s ds}|\mathcal{G}_t \vee \sigma\left(\Phi\right)\right] = E\left[e^{-\int_0^T \mu_s ds}|\mathcal{G}_t\right], \tag{A.2.4}$$

for all $T \geq t \geq 0$. Finally we note that $\mathcal{H}_t \subset \sigma\left(\Phi\right)$ and so $\mathcal{F}_t \subset \mathcal{G}_t \vee \sigma\left(\Phi\right)$. Thus, on using tower property and (A.2.4), we have:

$$\begin{aligned} E\left[e^{-\int_0^T \mu_s ds}|\mathcal{F}_t\right] &= E\left[E\left(e^{-\int_0^T \mu_s ds}|\mathcal{G}_t \vee \sigma(\Phi)\right)|\mathcal{F}_t\right] \\ &= E\left[E\left(e^{-\int_0^T \mu_s ds}|\mathcal{G}_t\right)|\mathcal{F}_t\right] \\ &= E\left[e^{-\int_0^T \mu_s ds}|\mathcal{G}_t\right] \end{aligned} \tag{A.2.5}$$

which proves the conditioning interchange between $\mathcal{F}_t$ and $\mathcal{G}_t$.

We remark that, we do not take $\mathcal{G}_t \vee \sigma(\Phi)$ as our filtration $\mathcal{G}_t$ because, in that case, the stopping time τ would be predictable and would not admit an intensity. The construction portrayed here guarantees that τ is a totally inaccessible stopping time, a concept intuitively meaning that the insured's death arrives as a total surprise to the insurer (see Protter, 1990, Chapter III.2, for details).

A.3 Survival Benefit

We now compute the fair value of a basic payoff - 'Survival Benefit' involved by standard insurance contracts. These are benefits, of amount possibly linked to other security prices, contingent on survival or death over a given time period. The modelling set up is as described in Chapter 5. We require the short rate process r and the intensity of mortality μ to satisfy the technical conditions stated in Section 5.2

Proposition 44. *(Survival benefit). Let C be a bounded $\mathcal{G}_t$-adapted process. Then, the time-t fair value $SB_t(C_T;T)$ of the time-T survival benefit of amount C_T, with $0 \leq t \leq T$, is given by:*

$$SB_t(C_T;T) = E\left[e^{-\int_t^T r_s ds}\mathbb{1}_{\{\tau>T\}}C_T|\mathcal{F}_t\right] = \mathbb{1}_{\{\tau>t\}}E\left[e^{-\int_t^T (r_s+\mu_s)ds}C_T|\mathcal{G}_t\right] \tag{A.3.1}$$

In particular, if C is $\mathcal{G}_t^X$-adapted and X and Y are independent, then, the following holds

$$SB_t(C_T;T) = \mathbb{1}_{\{\tau>t\}}E\left[e^{-\int_t^T r_s ds}C_T|\mathcal{G}_t^X]E[e^{-\int_t^T \mu_s ds}|\mathcal{G}_t^Y\right] \tag{A.3.2}$$

Proof. First of all, we note that if we are looking at the time T survival benefit, then the life must have survived upto time T and so

$$\begin{aligned} SB_t(C_T;T) &= E\left[e^{-\int_t^T r_s ds}\mathbb{1}_{\{\tau>T\}}C_T|\mathcal{F}_t\right] \\ &= E[e^{-\int_t^T r_s ds}C_T E\left[\mathbb{1}_{\{\tau>T\}}|\mathcal{F}_t \vee \mathcal{G}_T\right]|\mathcal{F}_t] \end{aligned} \tag{A.3.3}$$

where, we have used the law of iterated expectations and the fact that r and C are $\mathcal{G}_t$-adapted. Equation (A.3.3) implies that $SB_t(C_T;T)$ is zero on the set $\{\tau \leq t\}$. Concentrating on the set $\{\tau > t\}$, we exploit the doubly stochastic property given in (A.2.3), to obtain:

$$E\left[\mathbb{1}_{\{\tau>T\}}|\mathcal{F}_t \vee \mathcal{G}_T\right] = \mathbb{1}_{\{\tau>t\}}e^{-\int_t^T \mu_s ds} \tag{A.3.4}$$

Substituting equation (A.3.4) in equation (A.3.3), and noting that $\mathbb{1}_{\{\tau>t\}}$ is $\mathcal{F}_t$-measurable, we obtain

$$SB_t(C_T;T) = \mathbb{1}_{\{\tau>t\}}E\left[e^{-\int_t^T (r_s+\mu_s)ds}C_T|\mathcal{F}_t\right] \tag{A.3.5}$$

and then we replace $\mathcal{F}_t$ by $\mathcal{G}_t$.

Expression (A.3.2) is finally obtained by exploiting the independence of the filtrations $\mathcal{G}_t^X$ and $\mathcal{G}_t^Y$. We use the following fact: given two independent filtrations $\mathcal{F}_t^1$ and $\mathcal{F}_t^2$, for any bounded random variables V^1 and V^2 that are respectively $\mathcal{F}^1_\infty$ and $\mathcal{F}^2_\infty$-measurable,we have

$$E\left[V^1V^2|\mathcal{F}^1_t \vee \mathcal{F}^2_t\right] = E\left[V^1|\mathcal{F}^1_t\right]E[V^2|\mathcal{F}^2_t] \tag{A.3.6}$$

for all $t \geq 0$. □

A.4 The Black Scholes Model

To begin, let $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{P}$ as the real world probability measure, be a complete probability space coupled with the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$. Let $\mathbb{Q}$ be the risk-neutral measure which is equivalent to the real world probability measure $\mathbb{P}$ and is known as the 'Equivalent Martingale Measure' (EMM). Assume that there are two primary traded securities: a stock S_t and a bond (risk-free asset) B_t. Then we specify in detail the Black-Scholes Model.

Assumptions

The standard Black-Scholes economy is based on the assumptions of constant volatility, no arbitrage, no dividends, no friction, constant interest rate, efficient markets and stock price process following a Geometric Brownian Motion.

The Model

In the Black-Scholes model, the risk-free asset and the stock price have dynamics given by

$$dB_t = rB_tdt$$

$$dS_t = \mu S_tdt + \sigma S_tdW_t,$$

where $\mu \in \mathbb{R}$ is a constant appreciation rate of the stock price called the 'drift' and $\sigma > 0$ is a constant volatility coefficient and $W_t, t \in [0, T]$ is the standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

Equivalent Martingale Measure

The unique EMM $\mathbb{Q}$ which is equivalent to the real world probability measure $\mathbb{P}$ is calculated as shown below by employing Radon-Nikodym derivative[1]

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left[\left(\frac{r-\mu}{\sigma}\right)W^*_T - \frac{1}{2}\frac{(r-\mu)^2}{\sigma^2}T\right]$$

and $W^*_t = W_t - \frac{(r-\mu)}{\sigma}t$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$. The dynamics of the price under $\mathbb{Q}$ are:

$$dS_t = rS_tdt + \sigma S_tdW^*_t, \quad S_0 > 0.$$

Thus, in the Black-Scholes model $B_t = B_0e^{rt}$ and $S_t = S_0 \exp\left[\sigma W^*_t + \left(r - \frac{1}{2}\sigma^2\right)t\right]$.

Black-Scholes Option Pricing Formula

According to Black and Scholes (1973), the current price $C(K, T)$ of a European call option written on a product S_t with exercise price K and maturity T is given as:

$$C(K, T) = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2), \tag{A.4.1}$$

where

$$d_1 = \frac{\log_e\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \tag{A.4.2}$$

[1]By Girsanov's Theorem; Refer to Baxter and Rennie (1996)

and

$$d_2 = d_1 - \sigma\sqrt{T}, \tag{A.4.3}$$

where Φ is the standard normal distribution function S_0 is the product price at time 0.

A.5 Call Option Pricing Formula under S_u Distribution

Suppose that the product price 'S_t' obeys the four-parameter transformed Normal (S_u) Distribution specified in Chapter 2, with parameters α, β, μ and σ. Assuming the complete filtration as in the Black-scholes case with the EMM as $\mathbb{Q}$, we refer to Tsai and Tzeng (2013) to furnish the current price $C(K,T)$ of a European call option written on a product S_t with exercise price K and maturity T, which is given as:

$$C(K,T) = e^{-rT}\left(\frac{\beta}{2}e^{\mu_{\mathbb{Q}}+\frac{1}{2}\sigma^2}\Phi(d_1) - \frac{\beta}{2}e^{-\mu_{\mathbb{Q}}+\frac{1}{2}\sigma^2}\Phi(d_2) + (\alpha - K)\Phi(d_3)\right), \tag{A.5.1}$$

where

$$d_1 = \frac{-\sinh^{-1}\left(\frac{K-\alpha}{\beta}\right) + \mu_{\mathbb{Q}}}{\sigma} + \sigma, \tag{A.5.2}$$

$$d_2 = \frac{-\sinh^{-1}\left(\frac{K-\alpha}{\beta}\right) + \mu_{\mathbb{Q}}}{\sigma} - \sigma, \tag{A.5.3}$$

$$d_3 = \frac{-\sinh^{-1}\left(\frac{K-\alpha}{\beta}\right) + \mu_{\mathbb{Q}}}{\sigma} \tag{A.5.4}$$

and where

$$\mu_{\mathbb{Q}} = \sinh^{-1}\left(\frac{1}{\beta}e^{-\frac{1}{2}\sigma^2}\left(S_0 e^{rT} - \alpha\right)\right) \tag{A.5.5}$$

is the risk-adjusted mean. S_0 is the price of the product at time 0.

A.6 Call Option Pricing Formula under Log Gamma Distribution

Suppose that the product price 'S_t' obeys the four-parameter log Gamma Distribution specified in Chapter 2 with parameters μ, σ, p and a. Assuming the complete filtration as in the Black-scholes case with the EMM as $\mathbb{Q}$, we refer to Vitiello and Poon (2010) to furnish the current price $C(K,T)$ of a European call option written on a product S_t with exercise price K and maturity T, which is given as:

$$C(K,T) = S_0\left[1 - G(d_1, p)\right] - Ke^{-rT}\left[1 - G(d_2, p)\right], \tag{A.6.1}$$

where $G(.,.)$ is the Gamma c.d.f. given as

$$G(X,p) = \frac{\Gamma(X,p)}{\Gamma(p)} \tag{A.6.2}$$

where $\Gamma(.,.)$ is the incomplete gamma function defined as

$$\Gamma(X,p) = \int_0^X z^{p-1}e^{-z}dz \tag{A.6.3}$$

and $\Gamma(.)$ is the Gamma function defined in equation (2.7.10) in Chapter 2. Further

$$d_1 = \frac{\log_e K - \mu}{\left(S_0 e^{rT-\mu}\right)^{\frac{1}{p}} - 1}, \tag{A.6.4}$$

$$d_2 = d_1 + (\log_e K - \mu) \tag{A.6.5}$$

with S_0 is the price of the product at time 0. Vitiello and Poon (2010) also show that the risk-neutral density under the measure $\mathbb{Q}$ in this case is

$$f^{\mathbb{Q}}(x) = \left(\frac{1}{\sigma'}\right)^p \frac{1}{x\Gamma(p)} (\log_e x - \mu)^{p-1} e^{-\frac{1}{\sigma'}(\log_e x - \mu)}, \quad x \in \mathbb{R}^+ \tag{A.6.6}$$

where

$$\sigma' = 1 - \left(S_0 e^{rT-\mu}\right)^{-1/p}. \tag{A.6.7}$$

A.7 The Key Exotic Options

This thesis dwells upon expressing the payoff of mortality contingent securities in terms of exotic options (c.f. Rubinstein, 1992, for the term 'exotic') such as Asian and Basket. We present the basic definition and payoff structure of these options below assuming that $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{P}$ as the real world probability measure, is a complete probability space coupled with the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ and $\mathbb{Q}$ is the EMM.

A.7.1 Asian Options

An Asian option also known as an "average option" in literature is a financial instrument whose payoff depends on the average price of the underlying asset during the span of the option.

We are interested in *discrete sampled fixed strike* European-type Asian call option, having the payoff:

$$\left(\frac{1}{n}\sum_{i=1}^{n} S_{t_i} - K\right)^+,$$

where $\{S_t\}$ is price of a product observed at the *monitoring times* $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n = T$.

The value of the Asian option given above is:

$$AC(K,T) = e^{-rT}\mathbf{E}_Q\left[\left(\frac{1}{n}\sum_{i=1}^{n} S_{t_i} - K\right)^+\right]. \tag{A.7.1}$$

The corresponding put option payoff can be obtained by multiplying the term inside the *maximum* function by (-1).

A.7.2 Basket Options

Given a vector of weights $\mathbf{w} = (w_1, w_2, ..., w_n) \in \mathbb{R}^n$, a basket is defined as the weighted sum of the n product prices $S_T^{(i)}$; $i = 1, 2, ..., n$ at time T, i.e,

$$A_T^{(n)} = \sum_{i=1}^{n} w_i S_T^{(i)}.$$

A basket call option gives the holder the right, but not the obligation, to purchase a portfolio of assets at a fixed price K, known as the option's strike price. We consider European-style options, where the buyer has the option to exercise the option only at maturity T. As a result, the basket option payoff at time T is

$$\left(A_T^{(n)} - K\right)^+$$

The value of the basket option given above is:

$$BC\left(K,T\right)=e^{-rT}\mathbf{E}_{Q}\left[\left(A_{T}^{(n)}-K\right)^{+}\right]. \tag{A.7.2}$$

Appendix B

Some Interesting Results

Apart from the results that have been presented in the thesis, there were some other connected results that we have derived an can be exploited to a greater detail in future research work. We furnish these results here.

B.1 The Lower Bound $\mathbf{SWLB}_t^{(1)}$

To further improve the bounds post the Lower Bound SWLB_1, for the Swiss Re Bond, additional assumptions are required. The following inequality holds for every random variable Y and every constant c

$$\mathbf{E}\left[a^+\right] \geq \mathbf{E}\left[a\mathbb{1}_{\{Y \geq c\}}\right]. \tag{B.1.1}$$

Motivated by Albrecher et al. (2008), we choose $a = \left(\sum_{i=1}^{n} S_i - q_0\right)$ and $Y = q_t$, where they make an appropriate choice for t later on. This leads to

$$P_1 \geq De^{-rT}\mathbf{E}\left[\left(\sum_{i=1}^{n} S_i - q_0\right)\mathbb{1}_{\{q_t \geq c\}}\right]. \tag{B.1.2}$$

This reduces to:

$$P_1 \geq De^{-rT}\mathbf{E}\left[\sum_{i=1}^{n} S_i \mathbb{1}_{\{q_t \geq c\}} - q_0 \mathbb{1}_{\{q_t \geq c\}}\right]. \tag{B.1.3}$$

Now, again utilizing (B.1.1) we choose: $a = 5\left(q_i - 1.3q_0\right)$ and $Y = q_t$ so that (B.1.3) along with the definition of S_i in (3.2.9) yields:

$$P_1 \geq De^{-rT}\mathbf{E}\left[\sum_{i=1}^{n} 5\left(q_i - 1.3q_0\right)\mathbb{1}_{\{q_t \geq c\}} - q_0 \mathbb{1}_{\{q_t \geq c\}}\right]. \tag{B.1.4}$$

We then split the first sum into two parts at $j = \min\{i : t_i \geq t\}$ and condition the second part on the information available up to time t depicted by $\mathcal{F}_t$ so as to yield

$$\begin{aligned} P_1 \quad \geq \quad & De^{-rT}\Bigg(5\sum_{i=1}^{j-1}\mathbf{E}\left[q_i \mathbb{1}_{\{q_t \geq c\}}\right] + 5\sum_{i=j}^{n}\mathbf{E}\left[\mathbb{1}_{\{q_t \geq c\}} e^{r(t_i - t)} q_t\right] \\ & - 6.5q_0\sum_{i=1}^{n}\mathbf{P}\left[q_t \geq c\right] - q_0\mathbf{P}\left[q_t \geq c\right]\Bigg), \end{aligned} \tag{B.1.5}$$

where in the last equation, in the second term, we utilize the fact that discounted asset prices are martingales. We further modify the second term as follows.

$$\begin{aligned}
\mathbf{E}\left[\mathbb{1}_{\{q_t\geq c\}}e^{r(t_i-t)}q_t\right] &= \mathbf{E}\left[\mathbb{1}_{[q_t\geq c]}e^{rt_i}e^{-rt}\left(q_t-c+c\right)\right]\\
&= e^{rt_i}e^{-rt}\mathbf{E}\left[\mathbb{1}_{\{q_t\geq c\}}\left(q_t-c\right)\right]+ce^{r(t_i-t)}\mathbf{P}\left[q_t\geq c\right]\\
&= e^{rt_i}e^{-rt}\mathbf{E}\left[\left(q_t-c\right)^+\right]+ce^{r(t_i-t)}\mathbf{P}\left[q_t\geq c\right]\\
&= e^{rt_i}C\left(c,t\right)+ce^{r(t_i-t)}\mathbf{P}\left[q_t\geq c\right]. \qquad \text{(B.1.6)}
\end{aligned}$$

where $C(c,t)$ denotes the price of a European call on the mortality index with strike c, maturity t and current mortality value q_0. Putting equation (B.1.6) in equation (B.1.5), we obtain

$$\begin{aligned}
P_1 \geq De^{-rT}\Bigg(&5\sum_{i=1}^{j-1}\mathbf{E}\left[q_i\mathbb{1}_{\{q_t\geq c\}}\right]+5\sum_{i=j}^{n}e^{rt_i}C\left(c,t\right)\\
&-\mathbf{P}\left[q_t\geq c\right]\left(q_0\left(1+6.5n\right)-5c\sum_{i=j}^{n}e^{r(t_i-t)}\right)\Bigg). \qquad \text{(B.1.7)}
\end{aligned}$$

Further, we assume that q_i and $\mathbb{1}_{\{q_t\geq c\}}$ are non-negatively correlated for $t>t_i$

$$\Rightarrow \operatorname{Cov}\left(q_i,\ \mathbb{1}_{\{q_t\geq c\}}\right)\geq 0 \Rightarrow \mathbf{E}\left[q_i\mathbb{1}_{\{q_t\geq c\}}\right]\geq \mathbf{E}\left[q_i\right]\mathbf{E}\left[\mathbb{1}_{\{q_t\geq c\}}\right].$$

Using equation (3.3.17) we can bound the first term in equation (B.1.7) from below as follows.

$$\mathbf{E}\left[q_i\mathbb{1}_{\{q_t\geq c\}}\right]\geq q_0e^{rt_i}\mathbf{P}\left[q_t\geq c\right]$$

and this finally yields:

$$P_1\geq De^{-rT}\left(5\sum_{i=j}^{n}e^{rt_i}C\left(c,t\right)-\mathbf{P}\left[q_t\geq c\right]\left(q_0\left(1+6.5n\right)-5q_0\sum_{i=1}^{j-1}e^{rt_i}-5c\sum_{i=j}^{n}e^{r(t_i-t)}\right)\right). \qquad \text{(B.1.8)}$$

Clearly, we have

$$C\left(c,t\right)=e^{-rt}\mathbf{E}\left[\left(q_t-c\right)^+\right]=e^{-rt}\mathbf{E}\left[\mathbb{1}_{\{q_t\geq c\}}\left(q_t-c\right)\right],$$

which finally leads to:

$$C\left(c,t\right)=e^{-rt}\mathbf{E}\left[\mathbb{1}_{\{q_t\geq c\}}q_t\right]-ce^{-rt}\mathbf{P}\left[q_t\geq c\right].$$

Let $g(.)$ denote the probability density function (p.d.f.) of the mortality index q_t. Then, we can write the above equation as

$$C\left(c,t\right)=e^{-rt}\left[\int_c^\infty x\,g\left(x\right)dx-c\int_c^\infty g\left(x\right)dx\right]. \qquad \text{(B.1.9)}$$

Differentiating $C(c,t)$ w.r.t c, using Leibnitz rule for differentiation under the integral sign, since the limit involves c, we obtain

$$\frac{\partial}{\partial c}C\left(c,t\right)=e^{-rt}\left[-c\,g\left(c\right)-\mathbf{P}\left[q_t\geq c\right]+c\,g\left(c\right)\right]$$

$$\Rightarrow \mathbf{P}\left[q_t\geq c\right]=-e^{rt}\frac{\partial}{\partial K}C\left(K,t\right)\Big|_{K=c}=:-e^{rt}C_K\left(c,t\right)$$

Substituting $\mathbf{P}[q_t \geq c]$ in equation (B.1.8) and rearranging the terms, we achieve

$$P_1 \geq 5De^{-rT}\sum_{i=j}^{n} e^{rt_i}\left(C(c,t) + C_K(c,t)\left(\frac{(0.2+1.3n)\,q_0 - \sum_{i=1}^{j-1} e^{rt_i} q_0}{\sum_{i=j}^{n} e^{r(t_i - t)}} - c\right)\right). \quad \text{(B.1.10)}$$

Now define:

$$\tilde{c}_t = \frac{(0.2+1.3n)\,q_0 - \sum_{i=1}^{j-1} e^{rt_i} q_0}{\sum_{i=j}^{n} e^{r(t_i - t)}}. \quad \text{(B.1.11)}$$

Clearly, the right-hand side would be maximal if $c = \tilde{c}_t$ is given by (B.1.11). Hence, the optimal lower bound for the Asian-type call option is given by:

$$P_1 \geq 5De^{-rT}\max_{0\leq t\leq T} C\left(\tilde{c}_t, t\right)\sum_{i=j}^{n} e^{rt_i} =: \mathrm{lb}_t^{(1)} \quad \text{(B.1.12)}$$

where $c = \tilde{c}_t$ is given by (B.1.11) and $j = \min\{i : t_i \geq t\}$.

The existence of $\mathrm{lb}_t^{(1)}$ hinges upon the assumption of non-negative correlation between q_{t_i} and $\mathbb{1}_{\{q_t \geq c\}}$ for $t > t_i$. Finally, in the light of put-call parity derived in section 3.2.2, the trivial lower bound for the Swiss Re mortality bond is given as

$$P \geq \left(\mathrm{lb}_t^{(1)} - G\right)^+ =: \mathrm{SWLB}_t^{(1)}. \quad \text{(B.1.13)}$$

where G is defined in (3.2.17).

B.2 Performance of $\mathrm{SWLB}_t^{(1)}$

We present below in tables B.1 and B.2, the values of lower bound $\mathrm{SWLB}_t^{(1)}$ for various models considered in section 6.1 of Chapter 6.

Table B.1: The Lower Bound $\mathrm{SWLB}_t^{(1)}$ for the Swiss Re Mortality Bond under the Black-Scholes (B-S) Model, Transformed Normal (S_u) Distribution and Transformed Gamma Distribution with parameter specifications in Tables 6.1, 6.3 and 6.4 respectively

r	B-S Model	S_u Distn.	T. Gamma Distn.
0.035	0.899130889163	0.884321427702	0.848490721687
0.030	0.913324024548	0.904010021303	0.873845296963
0.025	0.927447505803	0.921935518851	0.897255685548
0.020	0.941626342687	0.938576980454	0.918981602796
0.015	0.955935721003	0.954369722665	0.939286791779
0.010	0.970419124546	0.969677756802	0.958429070674
0.005	0.985101139986	0.984779521693	0.976649121750
0.000	0.999995778016	0.999868375732	0.994170066411

The bound is extremely tight around the Monte Carlo values as can be compared from the respective tables in Chapter 6.

B.3 An Alternative Method to obtain the First Upper Bound SWUB_1

This section will focus on finding the upper bound SWUB_1 for the Swiss Re Bond 2003 by employing a Lagrange optimization technique. Before formally beginning the derivation of the upper bound, we throw light on an interesting proposition regarding the convexity of the value of a call option which would help us to arrive at the requisite upper bound.

Table B.2: The Lower Bound $\mathrm{SWLB}_t^{(1)}$ for the Swiss Re Mortality Bond under the Black-Scholes Model and Transformed Gamma Distribution with parameter specifications in Tables 6.2 and 6.5 respectively

r	B-S Model	T. Gamma Distn.
0.008	0.999999915252	0.999766071152
0.009	0.999821987950	0.989146149900
0.01	0.978310383929	0.888049181230
0.011	0.610962123857	0.596089667857
0.012	0.040209770810	0.271045973760
0.013	0.000000000000	0.082740708460
0.014	0.000000000000	0.012702023135

Proposition 45. *The payoff of the call option is a convex function*[1] *of the strike price, i.e.,* $\boldsymbol{E}\left[(X-x)^{+}\right]$ *is convex in* x.

Proof.

$$\begin{aligned}\mathbf{E}\left[(X-(ax+(1-a)\,y))^{+}\right] &= \mathbf{E}\left[((a+1-a)\,X-(ax+(1-a)\,y))^{+}\right]\\ &= \mathbf{E}\left[(a\,(X-x)+(1-a)\,(X-y))^{+}\right]\\ &\leq \mathbf{E}\left[a\,(X-x)^{+}+(1-a)\,(X-y)^{+}\right]\\ &= a\mathbf{E}\left[(X-x)^{+}\right]+(1-a)\,\mathbf{E}\left[(X-y)^{+}\right].\end{aligned}$$

□

Now, to begin, consider a vector $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)$ such that $\lambda_i\in\mathbb{R}$ and $\sum_{i=1}^{n}\lambda_i=1$. Now, with the help of $\boldsymbol{\lambda}$ we can write the payoff of the Asian-type call option as shown below.

$$P_1=Ce^{-rT}\mathrm{E}\left[\left(\sum_{i=1}^{n}\left(5\left(\tfrac{q_i}{q_0}-1.3\right)^{+}-\lambda_i\right)\right)^{+}\right]. \tag{B.3.1}$$

Using the above proposition, equation (B.3.1) implies that the upper bound for the above Asian-type call can be expressed as follows:

$$P_1\leq Ce^{-rT}\sum_{i=1}^{n}\mathrm{E}\left[\left(5\left(\tfrac{q_i}{q_0}-1.3\right)^{+}-\lambda_i\right)^{+}\right] \tag{B.3.2}$$

$$\begin{aligned}\Rightarrow P_1 &\leq 5De^{-rT}\sum_{i=1}^{n}\mathrm{E}\left[\left(q_i-q_0\left(1.3+\tfrac{\lambda_i}{5}\right)\right)^{+}\right]\\ &= 5De^{-rT}\sum_{i=1}^{n}e^{rt_i}C\left(q_0\left(1.3+\frac{\lambda_i}{5}\right),t_i\right).\end{aligned} \tag{B.3.3}$$

As the λ_i are arbitrary, the goal is then to minimise this bound over all possible $\boldsymbol{\lambda}$ ensuring that $q_0\left(1.3+\frac{\lambda_i}{5}\right)>0$. This in turn is equivalent to minimising the sum $\sum_{i=1}^{n}e^{rt_i}C\left(q_0\left(1.3+\frac{\lambda_i}{5}\right),t_i\right)$. In order to achieve this, we assume that the European call option payoff viz. $C(K,T)>0$ for every positive K, T and that $C(K,T)\downarrow 0$ as $K\to\infty$. Then C is a convex, strictly decreasing

[1] A function $f:I\to\mathbb{R}$, where I is an interval in $\mathbb{R}$, is convex if and only if $f(ax+(1-a)\,y)\leq af(x)+(1-a)\,f(y)\quad\forall a\in[0,1]$ and any pair of elements $x,y\in I$.

function of K with a continuous, strictly increasing derivative $\partial C/\partial K < 0$. We define

$$d_i = q_0 \left(1.3 + \frac{\lambda_i}{5}\right);\ i = 1, 2, ..., n. \tag{B.3.4}$$

Next, we define the Lagrangian as

$$L(\boldsymbol{\lambda}, \phi) = \frac{5}{q_0} \sum_{i=1}^{n} e^{rt_i} C(d_i, t_i) + \phi \left(\sum_{i=1}^{n} \lambda_i - 1\right)$$

where ϕ is the Lagrange's multiplier. We wish to find λ_i for each i, that minimises L. Differentiating L w.r.t λ_i, we obtain

$$\frac{\partial L}{\partial \lambda_i} = -\mathbf{P}[q_i \geq d_i] + \phi.$$

Thus, it is evident that the function L has a point of maxima or minima when λ_i solves the following equation for every i, i.e.,

$$\lambda_i = \frac{5}{q_0} \left(F_{q_i}^{-1}(1 - \phi) - 1.3 q_0\right) \tag{B.3.5}$$

where $F_{q_i}^{-1}$ is the inverse distribution function of the mortality index q_i. Further more, since $q_i \geq 0$, we have that the strike prices of the call viz. $d_i = q_0 \left(1.3 + \frac{\lambda_i}{5}\right) > 0,\ \forall i$. The next aim is to check that the constraint $\sum_{i=1}^{n} \lambda_i = 1$ is satisfied. For this, we define H as

$$H(\phi) = \sum_{i=1}^{n} \lambda_i - 1 = \frac{5}{q_0} \sum_{i=1}^{n} \left(F_{q_i}^{-1}(1 - \phi) - 1.3 q_0\right) - 1. \tag{B.3.6}$$

Under the aforesaid assumptions, H is a continuous function of ϕ. Moreover, since by assumption F_{q_i} is injective for all $t_i,\ i = 1, 2, ..., n$, it follows that H is strictly decreasing in ϕ. Hence, a solution to $H(\phi) = 0$ exists if $\inf\ H(\phi) < 0 < \sup H(\phi)$. For $\phi = 1$, $H(\phi) = -6.5n - 1$ and for the Swiss Re bond as $n = 3$, we have $H(1) = -20.5$. As far as searching for such a value of ϕ is concerned, for which $H(\phi) > 0$, we can immediately see that $F_{q_i}(K) = 1$, only when $K \to \infty$. Thus, $\lim_{\phi \downarrow 0} H(\phi) = \infty$ and so the application of intermediate value theorem ensures that we can find ϕ^* that satisfies $H(\phi^*) = 0$. Also ϕ^* is unique since H is strictly decreasing.

The final task is to check that the stationary point of L, which is obtained when $\boldsymbol{\lambda} = \lambda(\phi*)$, is a point of minima. This is indeed straightforward because $\partial C/\partial K$ is strictly increasing. This implies that on using equation (B.3.5) in conjunction with equation (B.3.3), a minimal upper bound for the call counterpart of the Swiss Re bond is given by

$$P_1 \leq 5De^{-rT} \sum_{i=1}^{n} e^{rt_i} C\left(F_{q_i}^{-1}(1 - \phi^*), t_i\right). \tag{B.3.7}$$

Evidently, the argument of $F_{q_i}^{-1}$ in this result is identical to the one in equation (B.3.6) and this allows us to rewrite the upper bound as

$$P_1 \leq 5De^{-rT} \sum_{i=1}^{n} e^{rt_i} C\left(F_{q_i}^{-1}(x), t_i\right) =: \mathrm{ub}_1. \tag{B.3.8}$$

where $x \in (0, 1)$ is the solution of the equation

$$\sum_{i=1}^{n} F_{q_i}^{-1}(x) = \frac{q_0}{5}(1 + 6.5n) \tag{B.3.9}$$

which is a direct consequence of (B.3.6).

As in the case of lower bounds, invoking the put-call parity of section 3.2.2, we have for the

Swiss Re bond

$$P \leq (\mathrm{ub}_1 - G)^+ =: \mathrm{SWUB}_1. \tag{B.3.10}$$

where G is defined in (3.2.17). This provides an alternative methodology in comparison to comonotonicity approach employed in section 3.4.1 to obtain SWUB_1.

Appendix C

MATLAB Codes

We now present the relevant algorithms that have been used to produce the numerical results in Chapter 6.

C.1 Bounds for the Swiss Re Bond

We now present programs written on the basis of theory derived in Chapter 3 to furnish bounds for the Swiss Re Mortality Bond 2003 presented in Chapter 6.

C.1.1 The Black-Scholes Model

The parameter choices for the tables 6.1 - 6.2 have been indicated in Chapter 6. We vary the time points according to the structure of the Swiss Re Bond in the codes presented below. We present below the parameter choice for these tables and this snippet is a part of every code in this sub-section. To obtain Table 6.1, we vary the values of interest rate while fixing the base value of the mortality index, i.e., $q_0 = 0.008453$. In Table 6.2, we vary the base values of the mortality index while specifying the interest rate as $r = 0.0$. We present the MATLAB codes below. These will take care of both scenarios. The varying base mortality index is denoted by w_0.

Snippet C.1.1.1: Parameter Choice for Table 6.1

```
% Parameters
n=3; %Duration of the Swiss Re Bond
q0=0.008453; %Base Value of the Mortality Index
w0=.008453; %Value of the Base Mortality Index which is varied for Table 6.2
r=0.035;  %Risk Free Rate
s=0.0388; %Volatility of the B—S Model
```

The snippet given below follows Snippet C.1.1.1 to initialize arrays and values used in every program. For the program to compute, Monte Carlo estimate of bond price, we do not use this snippet.

Snippet C.1.1.2: Initializing

```
t=zeros(n,1); %Vector of Time Points
C1=zeros(n,1); %Vector of Call Options in equation (3.2.16)
G1=0; %Initial Value of Sum used in G
```

The snippet that we now present is an integral part of each of the codes in this section and specifies the time points at which the principal loss percentage viz. L_i; $i = 1, 2, 3$, is computed (c.f. equation (3.2.2)) and as a result S_is are calculated at these time points (c.f. equations (3.2.9) - (3.2.11)). It also computes the value of G given in equation (3.2.17) which is required to compute price bounds for Swiss Re Bond 2003 exploiting put-call parity. For the program to compute Monte Carlo estimate of bond price, we do not use this snippet.

Snippet C.1.1.3: Specification of Time Points and Computation of G

```
%Specifying Time Points and Computing G given in equation (3.2.17)
for i=1:n;
    t(i)=i;
end
for i=1:n;
    C1(i)=blsprice(w0,1.3*q0,r,t(i),s,0); %The Computation of Call option in (3.2.17)
    G1=G1+C1(i)*exp(r*t(i));
end
G=exp(-r*3)*(5*G1/q0-1)
```

1. The Trivial Lower Bound $\mathbf{SWLB}_0$

```
%Snippet C.1.1.1
%Snippet C.1.1.2
s1=zeros(n,1); %Initializing the vector of terms in the Sum in (3.3.18)
SUM1=0; %Initializing the Sum in SWLB0
%Snippet C.1.1.3
%Computation of SWLB0
for i=1:n;
    s1(i)=5*max((w0*exp(r*t(i))-1.3*q0),0);
    SUM1=SUM1+s1(i);
end
lb0=exp(-3*r)*max((SUM1/q0-1),0); %Equation (3.3.18)
format 'long'
LB0=max(lb0-G,0) %To ensure positive value
format 'short'
```

2. The Lower Bound $\mathbf{SWLB}_1$

```
%Snippet C.1.1.1
%Snippet C.1.1.2
x_lower=0; %First initial guess for equation (3.3.22)
x_upper=1000; %Second initial guess for equation (3.3.22)
stn=zeros(n,1); %Initializing the vector of coefficients of x in (3.3.22)
AVl=0; %Initializing the sum on L.H.S. of (3.3.22) for lower guess
AVu=0; %Initializing the sum on L.H.S. of (3.3.22) for upper guess
AVm=0; %Initializing the sum on L.H.S. of (3.3.22) for midpoint of guesses
terml=zeros(n,1); %Initializing the summand on L.H.S. of (3.3.22) for lower guess
termu=zeros(n,1); %Initializing the summand on L.H.S. of (3.3.22) for upper guess
termm=zeros(n,1); %Initializing summand on L.H.S.:(3.3.22) for mid pt. of guesses
st=zeros(n,1); %Initializing the vector of coefficients of call in (3.3.27)
POW=zeros(n,1); %Defining the vector of terms in the sum in equation (3.3.27)
SPOW=0; %Initializing the terms in the sum in equation (3.3.27)
K=zeros(n,1); %Defining the vector of strike prices of the calls in (3.3.27)
%Snippet C.1.1.3
%Code to find the Solution of the equation (3.3.22) using Bisection Method
for i=1:n;
    stn(i)=exp(r*(t(i)-t(1)));
    terml(i)=max((stn(i)*x_lower-1.3),0);
    termu(i)=max((stn(i)*x_upper-1.3),0);
    AVl=AVl+terml(i);
    AVu=AVu+termu(i);
end
fl=AVl-0.2; %L.H.S.-R.H.S. value for equation (3.3.22) for lower initial guess
fu=AVu-0.2; %L.H.S.-R.H.S. value for equation (3.3.22) for upper initial guess
while x_upper-x_lower>0.000000001;
      x_mid=(x_lower+x_upper)/2; %Average of Guesses
      AVm=0;
```

```
        for i=1:n;
            termm(i)=max((stn(i)*x_mid-1.3),0);
            AVm=AVm+termm(i);
        end
        fm=AVm-0.2; %L.H.S.-R.H.S. for (3.3.22) for mid pt. of guesses
        if (fm*fu)<0;
            x_lower=x_mid;
            fl=fm;
        else
            x_upper=x_mid;
            fu=fm;
        end
end
soln=x_mid; %The solution of equation (3.3.22)
% Code to compute lb1 using equation (3.3.27)
for i=1:n;
    K(i)=q0*(1.3/stn(i)+max((soln-1.3/stn(i)),0)); %Strike of the call in (3.3.27)
    POW(i)=blsprice(w0,K(i),r,t(1),s,0); %The terms in the sum in (3.3.27)
    SPOW=SPOW+st(i)*POW(i); %Computation of the sum in (3.3.27)
end
lb1=5*exp(-r*3)*SPOW/q0;
format 'long'
SWLB1=max(lb1-G,0) %To ensure positive value
format 'short'
```

3. The Lower Bound $\mathbf{SWLB}_t^{(BS)}$

```
%Snippet C.1.1.1
%Snippet C.1.1.2
x_lower=zeros(n,1); %Initialising the vector of Initial guess for equation (3.5.9)
x_upper=zeros(n,1); %Initialising vector of second Initial guess for (3.5.9)
x_mid=zeros(n,1); %Initialising vector of Average of Two Initial Guesses for (3.5.9)
soln=zeros(n,1); %Vector of Solutions for n time points
fl=zeros(n,1); %Vector of the L.H.S.-R.H.S. value of (3.5.9) for lower initial guess
fu=zeros(n,1); %Vector of the L.H.S.-R.H.S. value of (3.5.9) for upper initial guess
fm=zeros(n,1); %Vector of the L.H.S.-R.H.S. value of (3.5.9) for midpt. of guesses
stl=zeros(n,1); %Vector of first term for lower guess in the sum on L.H.S of (3.5.9)
stu=zeros(n,1); %Vector of first term for upper guess in the sum on L.H.S of (3.5.9)
stm=zeros(n,1); %Vector of first term for mid pt. of guesses in sum on L.H.S:(3.5.9)
AVl=zeros(n,1); %Initializing vector of 1st sum:L.H.S.(3.5.9) lower guess
AVu=zeros(n,1); %Initializing vector of 1st sum:L.H.S.(3.5.9) for upper guess
AVm=zeros(n,1); %Initializing vector of 1st sum:L.H.S.(3.5.9) for midpt. of guesses
std=zeros(n,1); %The Vector of Coeff. of x inside the Second sum on L.H.S. of (3.5.9)
AVdl=zeros(n,1); %Initializing vector of 2nd sum:L.H.S.(3.5.9) for lower guess
AVdu=zeros(n,1); %Initializing vector of 2nd sum:L.H.S.(3.5.9) for upper guess
AVdm=zeros(n,1); %Initializing vector of 2nd sum:L.H.S.(3.5.9) for midpt. of guesses
stdl=zeros(n,1); %Vector of 2nd term for lower guess in the sum on L.H.S of (3.5.9)
stdu=zeros(n,1); %Vector of 2nd term for upper guess in the sum on L.H.S of (3.5.9)
stdm=zeros(n,1); %Vector of 2nd term for mid pt. guesses in sum on L.H.S of (3.5.9)
stau=zeros(n,1); %Vector of square root of time points
POW1=zeros(n,1); %Initializing the vector of terms in 1st sum in lbt(BS) in (3.5.15)
SPOW1=zeros(n,1); %1st sum in lbt(BS) in (3.5.15)
POW2=zeros(n,1); %Initializing vector of terms in 2nd sum in lbt(BS) in (3.5.15)
SPOW2=zeros(n,1); %2nd sum in lbt(BS) in (3.5.15)
d=zeros(n,1); %Initializing the sum in lbt(BS) in (3.5.15)
C2=zeros(n,1); %Initializing vector of calls:2nd term in lbt(BS) in (3.5.15)
%Snippet C.1.1.3
%Code to find the Solution of the equation (3.5.9) using Bisection Method
for j=1:n;
    x_lower(j)=0;
    x_upper(j)=2;
    for i=j:n;
        std(i)=exp(r*(t(i)-t(j)));
        stdl(i)=max((x_lower(j)*std(i)-1.3),0);
        stdu(i)=max((x_upper(j)*std(i)-1.3),0);
        AVdl(j)=AVdl(j)+stdl(i);
        AVdu(j)=AVdu(j)+stdu(i);
    end
    for k=1:j-1;
```

```
        stl(k)=max((x_lower(j)^(t(k)/t(j))*exp(s^2*t(k)*(t(j)-t(k))/...
        (2*t(j)))-1.3),0);
        AVl(j)=AVl(j)+stl(k);
        stu(k)=max((x_upper(j)^(t(k)/t(j))*exp(s^2*t(k)*(t(j)-t(k))/...
        (2*t(j)))-1.3),0);
        AVu(j)=AVu(j)+stu(k);
    end
    fl(j)=AVl(j)+AVdl(j)-0.2; %L.H.S.-R.H.S. for (3.5.9) for lower guess
    fu(j)=AVu(j)+AVdu(j)-0.2; %L.H.S.-R.H.S. for (3.5.9) for upper guess
    while x_upper(j)-x_lower(j)>0.000000000000001;
          x_mid(j)=(x_lower(j)+x_upper(j))/2;
          AVm(j)=0;
          AVdm(j)=0;
          for m=1:j-1;
              stm(m)=max((x_mid(j)^(t(m)/t(j))*exp(s^2*t(m)*(t(j)-t(m))/...
              (2*t(j)))-1.3),0);
              AVm(j)=AVm(j)+stm(m);
          end
          for m1=j:n;
              std(m1)=exp(r*(t(m1)-t(j)));
              stdm(m1)=max((x_mid(j)*std(m1)-1.3),0);
              AVdm(j)=AVdm(j)+stdm(m1);
          end
          fm(j)=AVm(j)+AVdm(j)-0.2; %L.H.S.-R.H.S. in (3.5.9) for mid pt. of guesses
          if (fm(j)*fu(j))<0;
             x_lower(j)=x_mid(j);
             fl(j)=fm(j);
          else
             x_upper(j)=x_mid(j);
             fu(j)=fm(j);
          end
    end
    format 'long'
    soln(j)=x_mid(j)
end
%Computation of the second Term in lbt(BS) in (3.5.15)
for j=1:n;
    for k=j:n;
        modisoln=max((soln(j)-1.3/exp(r*(t(k)-t(j)))),0);
        dnk=q0*(1.3/exp(r*(t(k)-t(j)))+modisoln); %Strike for call:2nd term (3.5.15)
        C2(k)=blsprice(w0,dnk,r,t(j),s,0); %Call price:2nd term (3.5.15)
        POW2(k)=5*exp(r*t(k))*C2(k)/q0; %Summand of the 2nd term in lbt(BS)
        SPOW2(j)=SPOW2(j)+POW2(k); %Computation of the second sum in lbt(BS)
    end
end
%Computation of the first term in lbt(BS) in (3.5.15)
for j=1:n
    stau(j)=sqrt(t(j));
    for i=1:j-1;
        modi2soln=max(((soln(j))^(t(i)/t(j))-1.3/exp(s^2*t(i)*(t(j)-t(i))...
        /(2*t(j)))),0);
        di=q0*(1.3/exp(s^2*t(i)*(t(j)-t(i))/(2*t(j)))+modi2soln)^(t(j)/t(i));
        d2i=(-log(di/w0)+(r-s^2/2)*t(j))/(s*stau(j));
        d1i=d2i+(s*t(i))/stau(j);
        p1=normcdf(d1i);
        p2=normcdf(d2i);
        POW1(i)=5*((w0/q0)^(t(i)/t(j))*exp(r*t(i))*p1-(1.3...
        +max(((soln(j))^(t(i)/t(j))*exp(s^2*t(i)*(t(j)-t(i))/(2*t(j)))-1.3), 0))*p2);
        SPOW1(j)=SPOW1(j)+POW1(i); %Computation of the first sum in lbt(BS)
    end
    d(j)=SPOW1(j)+SPOW2(j); %Computation of the consolidated lbt(BS) in (3.5.15)
end
m=max(d);
format 'long'
lbt2=exp(-3*r)*m; %The Optimal Lower Bound lbt(BS) given by equation (3.5.16)
format 'long'
SWLBtBS=max(lbt2-G,0)
format 'short'
```

4. The Upper Bound SWUB_1

```
%Snippet C.1.2.1
%Snippet C.1.2.2
x_lower=-6.5; %First initial guess for equation (3.4.15)
x_upper=10000; %Second initial guess for equation (3.4.15)
stn=zeros(n,1); %Initializing the vector of coefficients of call in (3.4.15)
AVl=0; %Initializing the sum on L.H.S. of (3.4.13) for lower guess
AVu=0; %Initializing the sum on L.H.S. of (3.4.13) for upper guess
AVm=0; %Initializing the sum on L.H.S. of (3.4.13) for midpoint of guesses
POW=zeros(n,1); %Defining the vector of calls in ub1 in (3.4.15)
SPOW=0; %Initializing the terms in the sum in equation (3.4.15)
K=zeros(n,1); %Defining the vector of strike prices of the calls in (3.4.15)
%Snippet C.1.1.3
%Code to find the Solution of the equation (3.4.13) using Bisection Method
for i=1:n;
    mu1(i)=(r-s^2/2)*t(i); %Parameter in inverse distribution function
    sigma1(i)=sqrt(s^2*t(i)); %Parameter in inverse distribution function
    AVl=AVl+exp(mu1(i)+sigma1(i)*x_lower);
    AVu=AVu+exp(mu1(i)+sigma1(i)*x_upper);
end
fl=AVl-(0.2+1.3*n); %L.H.S.-R.H.S. for equation (3.4.13) for lower initial guess
fu=AVu-(0.2+1.3*n); %L.H.S.-R.H.S. for equation (3.4.13) for upper initial guess
while x_upper-x_lower>0.000000000000001;
      x_mid=(x_lower+x_upper)/2;
      AVm=0;
      for i=1:n;
          AVm=AVm+exp(mu1(i)+sigma1(i)*x_mid);
      end
      fm=AVm-(0.2+1.3*3); %L.H.S.-R.H.S. for (3.4.13) for mid pt. of guesses
      if (fm*fu)<0;
          x_lower=x_mid;
          fl=fm;
      else
          x_upper=x_mid;
          fu=fm;
      end
end
soln=x_mid %The solution of equation(3.4.13)
% Code to compute ub1 using equation (3.4.15)
for i=1:n;
    stn(i)=exp(r*t(i)); %Coefficient of call in (3.4.15)
    K(i)=q0*exp(mu1(i)+sigma1(i)*soln) %Strike of the call in (3.4.15)
    POW(i)=blsprice(w0,K(i),r,t(i),s,0); %The call price in (3.4.15)
    SPOW=SPOW=SPOW+stn(i)*POW(i); %Computation of the sum in (3.4.15)
end
ub1=5*exp(-3*r)*SPOW/q0;
format 'long'
SWUB1=max(ub1-G,0) %To ensure positive value
format 'short'
```

5. The Upper Bound $\mathbf{SWUB}_t^{(BS)}$

```
%Snippet C.1.1.1
%Snippet C.1.1.2
x_lower=zeros(n,1); %Initialising the vector of Initial guess for equation (3.5.22)
x_upper=zeros(n,1); %Initialising vector of second Initial guess for (3.5.22)
x_mid=zeros(n,1); %Initialising vector of Average of Two Initial Guesses for (3.5.22)
soln=zeros(n,1); %Vector of Solutions for n time points
fl=zeros(n,1); %Vector of the L.H.S.-R.H.S. value of (3.5.22) for lower initial guess
fu=zeros(n,1); %Vector of the L.H.S.-R.H.S. value of (3.5.22) for upper initial guess
fm=zeros(n,1); %Vector of the L.H.S.-R.H.S. value of (3.5.22) for midpt. of guesses
stl=zeros(n,1); %Vector of first term for lower guess in the sum on L.H.S of (3.5.22)
stu=zeros(n,1); %Vector of first term for upper guess in the sum on L.H.S of (3.5.22)
stm=zeros(n,1); %Vector of first term for mid pt. of guesses in sum on L.H.S:(3.5.22)
AVl=zeros(n,1); %Initializing vector of 1st sum:L.H.S.(3.5.22) lower guess
AVu=zeros(n,1); %Initializing vector of 1st sum:L.H.S.(3.5.22) for upper guess
AVm=zeros(n,1); %Initializing vector of 1st sum:L.H.S.(3.5.22) for midpt. of guesses
std=zeros(n,1); %Vector of Coeff. of x inside Second sum on L.H.S. of (3.5.22)
AVdl=zeros(n,1); %Initializing vector of 2nd sum:L.H.S.(3.5.22) for lower guess
AVdu=zeros(n,1); %Initializing vector of 2nd sum:L.H.S.(3.5.22) for upper guess
```

```
AVdm=zeros(n,1); %Initializing vector of 2nd sum:L.H.S.(3.5.22) for midpt. of guesses
stdl=zeros(n,1); %Vector of 2nd term for lower guess in the sum on L.H.S of (3.5.22)
stdu=zeros(n,1); %Vector of 2nd term for upper guess in the sum on L.H.S of (3.5.22)
stdm=zeros(n,1); %Vector of 2nd term for mid pt. guesses in sum on L.H.S of (3.5.22)
x=zeros(n,1); %Inializing Vector of NORMAL cdf of solution
y=zeros(n,1); %Inializing second term in (3.5.23)
%The same arrays as above for ANTITHETIC CASE
x_lower2=zeros(n,1); %Initialising the vector of Initial guess for equation (3.5.22)
x_upper2=zeros(n,1); %Initialising vector of second Initial guess for (3.5.22)
x_mid2=zeros(n,1); %Initial. vector of Average of Two Initial Guesses for (3.5.22)
soln2=zeros(n,1); %Vector of Solutions for n time points
fl2=zeros(n,1); %Vector of L.H.S.-R.H.S. value of (3.5.22) for lower initial guess
fu2=zeros(n,1); %Vector of L.H.S.-R.H.S. value of (3.5.22) for upper initial guess
fm2=zeros(n,1); %Vector of L.H.S.-R.H.S. value of (3.5.22) for midpt. of guesses
stl2=zeros(n,1); %Vector of first term for lower guess in sum on L.H.S of (3.5.22)
stu2=zeros(n,1); %Vector of first term for upper guess in sum:L.H.S of (3.5.22)
stm2=zeros(n,1); %Vector of first term for mid pt. guess in sum:L.H.S:(3.5.22)
AVl2=zeros(n,1); %Initializing vector of 1st sum:L.H.S.(3.5.22) lower guess
AVu2=zeros(n,1); %Initializing vector of 1st sum:L.H.S.(3.5.22) for upper guess
AVm2=zeros(n,1); %Initializing vector of 1st sum:L.H.S.(3.5.22) for midpt. of guesses
std2=zeros(n,1); %Vector of Coeff. of x inside Second sum on L.H.S. of (3.5.22)
AVdl2=zeros(n,1); %Initializing vector of 2nd sum:L.H.S.(3.5.22) for lower guess
AVdu2=zeros(n,1); %Initializing vector of 2nd sum:L.H.S.(3.5.22) for upper guess
AVdm2=zeros(n,1); %Initializing vector of 2nd sum:L.H.S.(3.5.22) for midpt. guess
stdl2=zeros(n,1); %Vector of 2nd term for lower guess in the sum on L.H.S of (3.5.22)
stdu2=zeros(n,1); %Vector of 2nd term for upper guess in the sum on L.H.S of (3.5.22)
stdm2=zeros(n,1); %Vector of 2nd term for mid pt. guesses in sum on L.H.S of (3.5.22)
%The other initializations
M=1000000; %Twice the number of Simulations
M1=M/2; %The number of Simulations
%Calculation of the option for the first case
POW1=zeros(n,1);
SPOW1=zeros(n,1);
POW2=zeros(n,1);
SPOW2=zeros(n,1);
d=zeros(n,1);
C2=zeros(n,1); %(3.5.23): i=1:j-1
C2a=zeros(n,1); %(3.5.23): i=j:n
w=zeros(n,M1); %First set of random normal numbers
w2=zeros(n,M1); %Second set of random normal numbers
%Calculation of the option for the Antithetic Case
POW12=zeros(n,1);
SPOW12=zeros(n,1);
POW22=zeros(n,1);
SPOW22=zeros(n,1);
d2=zeros(n,1);
C22=zeros(n,1); %(3.5.23): i=1:j-1
C22a=zeros(n,1); %(3.5.23): i=j:n
%The  other arrays required for (3.5.23)
UBC2=zeros(n,1); %Initializing (3.5.24)
UBC1=zeros(n,1); %Initializing Normal c.d.f. of (3.5.24)
UBC22=zeros(n,1); %Initializing (3.5.24): Antithetic Case
UBC12=zeros(n,1); %Initializing Normal c.d.f. of (3.5.24): Antithetic Case
UBt=zeros(n,1); %Initializing the array of upper bounds
ubt=zeros(n,1);
SUBt=zeros(n,1);
UBt2=zeros(n,1); %The array of upper bounds: Antithetic case
ubt2=zeros(n,1);
SUBt2=zeros(n,1);
CUBt=zeros(n,1); %Summing the original and antithetic case upper bounds
%Snippet C.1.1.3
%Code to find Solution of (3.5.22) using Bisection Method:Both cases
for j=1:n;
    for is=1:M1;
        w(j,is)=sqrt(t(j))*randn;
        w2(j,is)=-(w(j,is));
        x_lower(j)=-1000;
        x_upper(j)=10000;
        x_lower2(j)=-1000;
        x_upper2(j)=100000;
        for i=j:n;
```

```
        stdl(i)=std(i)*exp(s*w(j,is)+s*sqrt(t(i)-t(j))*x_lower(j));
        stdu(i)=std(i)*exp(s*w(j,is)+s*sqrt(t(i)-t(j))*x_upper(j));
        AVdl(j)=AVdl(j)+stdl(i);
        AVdu(j)=AVdu(j)+stdu(i);
        stdl2(i)=std(i)*exp(s*w2(j,is)+s*sqrt(t(i)-t(j))*x_lower2(j));
        stdu2(i)=std(i)*exp(s*w2(j,is)+s*sqrt(t(i)-t(j))*x_upper2(j));
        AVdl2(j)=AVdl2(j)+stdl2(i);
        AVdu2(j)=AVdu2(j)+stdu2(i);
    end
    for k=1:j-1;
        stl(k)=std(k)*exp(s*t(k)*w(j,is)/t(j)+...
        s*sqrt(t(k)*(t(j)-t(k))/t(j))*x_lower(j));
        AVl(j)=AVl(j)+stl(k);
        stu(k)=std(k)*exp(s*t(k)*w(j,is)/t(j)+...
        s*sqrt(t(k)*(t(j)-t(k))/t(j))*x_upper(j));
        AVu(j)=AVu(j)+stu(k);
        stl2(k)=std(k)*exp(s*t(k)*w2(j,is)/t(j)+...
        s*sqrt(t(k)*(t(j)-t(k))/t(j))*x_lower2(j));
        AVl2(j)=AVl2(j)+stl2(k);
        stu2(k)=std(k)*exp(s*t(k)*w2(j,is)/t(j)+...
        s*sqrt(t(k)*(t(j)-t(k))/t(j))*x_upper2(j));
        AVu2(j)=AVu2(j)+stu2(k);
    end
    fl(j)=AVl(j)+AVdl(j)-(0.2+1.3*3); %L.H.S.-R.H.S.(3.5.22):lower guess
    fu(j)=AVu(j)+AVdu(j)-(0.2+1.3*3); %L.H.S.-R.H.S.(3.5.22):upper guess
    fl2(j)=AVl2(j)+AVdl2(j)-(0.2+1.3*3); %L.H.S.-R.H.S.(3.5.22):guess1 (Anti)
    fu2(j)=AVu2(j)+AVdu2(j)-(0.2+1.3*3); %L.H.S.-R.H.S.(3.5.22):guess2 (Anti)
    while x_upper(j)-x_lower(j)>0.00000000000001;
          x_mid(j)=(x_lower(j)+x_upper(j))/2;
          AVm(j)=0;
          AVdm(j)=0;
          for m=1:j-1;
              stm(m)=std(m)*exp(s*t(m)*w(j,is)/t(j)+...
              s*sqrt(t(m)*(t(j)-t(m))/t(j))*x_mid(j));
              AVm(j)=AVm(j)+stm(m);
          end
          for m1=j:n;
              stdm(m1)=std(m1)*exp(s*w(j,is)+s*sqrt(t(m1)-t(j))*x_mid(j));
              AVdm(j)=AVdm(j)+stdm(m1);
          end
          fm(j)=AVm(j)+AVdm(j)-(0.2+1.3*3);
          if (fm(j)*fu(j))<0;
              x_lower(j)=x_mid(j);
              fl(j)=fm(j);
          else
              x_upper(j)=x_mid(j);
              fu(j)=fm(j);
          end
     end
     while x_upper2(j)-x_lower2(j)>0.00000000000001;
      x_mid2(j)=(x_lower2(j)+x_upper2(j))/2;
      AVm2(j)=0;
      AVdm2(j)=0;
  for m=1:j-1;
      stm2(m)=std(m)*exp(s*t(m)*w2(j,is)/t(j)+...
      s*sqrt(t(m)*(t(j)-t(m))/t(j))*x_mid2(j));
      AVm2(j)=AVm2(j)+stm2(m);
  end
  for m1=j:n;
      stdm2(m1)=std(m1)*exp(s*w2(j,is)+s*sqrt(t(m1)-t(j))*x_mid2(j));
      AVdm2(j)=AVdm2(j)+stdm2(m1);
  end
  fm2(j)=AVm2(j)+AVdm2(j)-(0.2+1.3*3);
  if (fm2(j)*fu2(j))<0;
      x_lower2(j)=x_mid2(j);
      fl2(j)=fm2(j);
  else
      x_upper2(j)=x_mid2(j);
      fu2(j)=fm2(j);
  end
end
```

```
    format 'long'
    soln(j)=x_mid(j);
    soln2(j)=x_mid2(j);
    x(j)=normcdf(soln(j));
    y(j)=1-x(j);
    cons(j)=(0.2+1.3*3)*y(j);
    x2(j)=normcdf(soln2(j));
    y2(j)=1-x2(j);
    cons2(j)=(0.2+1.3*3)*y2(j);
    for k=j:n;
        UBC2(k)=s*sqrt(t(k)-t(j))-soln(j);
        UBC1(k)=normcdf(UBC2(k));
        UBC22(k)=s*sqrt(t(k)-t(j))-soln2(j);
        UBC12(k)=normcdf(UBC22(k));
        C2(k)=w0*exp((r-(s^2*t(j))/(2*t(k)))*t(k)+s*w(j,is));
        POW2(k)=C2(k)*UBC1(k);
        SPOW2(j)=SPOW2(j)+POW2(k);
        C22(k)=w0*exp((r-(s^2*t(j))/(2*t(k)))*t(k)+s*w2(j,is));
        POW22(k)=C22(k)*UBC12(k);
        SPOW22(j)=SPOW22(j)+POW22(k);
    end
    for i=1:j-1;
        UBC2(i)=s*sqrt(t(i)*(t(j)-t(i))/t(j))-soln(j);
        UBC1(i)=normcdf(UBC2(i));
        UBC22(i)=s*sqrt(t(i)*(t(j)-t(i))/t(j))-soln2(j);
        UBC12(i)=normcdf(UBC22(i));
        C2a(i)=w0*exp((r-(s^2*t(i))/(2*t(j)))*t(i)+s*t(i)*w(j,is)/t(j));
        POW1(i)=C2a(i)*UBC1(i);
        SPOW1(j)=SPOW1(j)+POW1(i);
        C22a(i)=w0*exp((r-(s^2*t(i))/(2*t(j)))*t(i)+s*t(i)*w2(j,is)/t(j));
        POW12(i)=C22a(i)*UBC12(i);
        SPOW12(j)=SPOW12(j)+POW12(i);
    end
    d(j)=SPOW1(j)+SPOW2(j)-q0*cons(j);
    d2(j)=SPOW12(j)+SPOW22(j)-q0*cons2(j);
    ubt(j)=exp(-3*r)*5*d(j)/q0;
    ubt2(j)=exp(-3*r)*5*d2(j)/q0;
    format 'long'
    UBt(j)=max(ubt(j)-G,0); %To ensure positive value
    SUBt(j)=SUBt(j)+UBt(j);
    UBt2(j)=max(ubt2(j)-G,0); %To ensure positive value
    SUBt2(j)=SUBt2(j)+UBt2(j);
end
format 'long'
CUBt(j)=(SUBt(j)+SUBt2(j))/M
end
m=min(CUBt)
format 'short'
```

6. The Monte Carlo Estimate (Antithetic Method)

```
%Snippet C.1.1.1
M=5000000; %Twice the Number of Simulations/Paths
M1=M/2; %Number of Simulations/Paths
t=zeros(n,1); %Vector of Time Points
q=zeros(M1,n+1); %Matrix of Mortality Indices at the n Time points for...
                 %M Paths beginning at time zero
Z=zeros(M1,n); %Same as S but Value begins at time point t(1)
X=zeros(M1,n); %Initializing the call counterpart of Swiss Re Bond
q2=zeros(M1,n+1); %Same as q but paths generated using Antithetic Variate
Z2=zeros(M1,n); %Same as q2 but Value begins at time point t(1)
AV=zeros(M1,1); %Initializing the sum of calls along each path
AV2=zeros(M1,1); %Initializing the sum of calls along...
                 %each path obtained using Antithetic Variate
swiss=zeros(M1,1); %Payoff of the Swiss Re Bond along each path
swiss2=zeros(M1,1); %Same as above but paths generated
                    %using Antithetic Variate
V=0; %Initializing the sum of bond payoffs to 0
%Specifying Time Points
```

```
for i=1:n;
    t(i)=i;
end
%Simulating the Black-Scholes Model using the Antithetic Method
for i=1:M1;
    q(i,1)=w0;
    q2(i,1)=w0;
    for j=1:n
        q(i,j+1)=q(i,j)*exp((r-s^2/2)*h+s*sqrt(h)*r1);
        Z(i,j)=q(i,j+1);
        X(i,j)=5*max((Z(i,j)/q0-1.3),0);
        q2(i,j+1)=q2(i,j)*exp((r-s^2/2)*h+s*sqrt(h)*(-r1)); %(-r1) acts...
                                                          %as antithetic variate
        Z2(i,j)=q2(i,j+1);
        X2(i,j)=5*max((Z2(i,j)/q0-1.3),0);
    end
end
%Monte Carlo Estimation of the Swiss Re Mortality Bond 2003
for i=1:M1;
    for j=1:n;
        AV(i)=AV(i)+X(i,j); %Computing the Total Price along each path
        AV2(i)=AV2(i)+X2(i,j); %Same as above but along path generated...
        %using Antithetic Variate
    end
    swiss(i)=max((1-AV(i)),0); %Computing Payoff of Swiss Re along
                               %each path
    swiss2(i)=max((1-AV2(i)),0); %Same as above but path generated using...
                                 %Antithetic Variate
    V=V+swiss(i)+swiss2(i); %Summing the Payoffs of different paths
end
format 'long'
disp(' The Monte Carlo Estimate is')
MC=exp(-(r*T))*V/(2*M1) %Monte Carlo Estimate of the Swiss Re Bond 2003
format 'short'
```

7. The Lower Bound $\mathbf{SWLB}_t^{(1)}$

```
%Snippet C.1.1.1
%Snippet C.1.1.2
std=zeros(n,1); %The vector of terms in the denominator of Strike Price...
                %in Call Option Payoff in lbt1
AVd=zeros(n,1); %Initializing the above referred sum for each j
stn=zeros(n,1); %The Vector of Terms inside the Sum in lbt1 in (B.1.12)
AVn=zeros(n,1); %Initializing the above referred sum for each j
st=zeros(n,1); %The vector of terms in the numerator of Strike Price in...
               %Call Option Payoff in lbt1
AV=zeros(n,1); %Initializing the above referred sum for each j
ct=zeros(n,1); %The Vector of Strike Prices in the Call Function for...
               %different j
d=zeros(n,1); %R.H.S. of (B.1.12) for each j before discounting and...
              %dividing by n
%Snippet C.1.1.3
%Computation of Various Sums involved in lbt1 given in (B.1.12)
for j=1:n;
    for i=j:n;
        std(i)=exp(r*(t(i)-t(j)));
        stn(i)=exp(r*t(i));
        AVd(j)=AVd(j)+std(i);
        AVn(j)=AVn(j)+stn(i); %computes the multiplication factor in lbt1
    end
    for k=1:j-1;
        st(k)=exp(r*t(k));
        AV(j)=AV(j)+st(k);
    end
end
%Computation of Strike Prices for each j; j=1,2,...,n
for l=1:n;
    ct(l)=((0.2+1.3*3)*q0-q0*AV(l))/AVd(l); %computes the strikes
    if (ct(l)>0);
```

```
        C2=blsprice(w0,ct(l),r,t(l),s,0); %Computation of Call...
                                                %Price in lbt1
        d(l)=C2*AVn(l);
    else
        d(l)=0;
    end
end
m=max(d);
format 'long'
lbt1=5*exp(-3*r)*m/q0
format 'long'

lbt1=5*exp(-3*r)*m/q0; %The Optimal Lower Bound lbt1 given by (B.1.12)
SWLBt1=max(lbt1-G,0) %Ensures positive value
format 'short'
```

C.1.2 Transformed Normal (S_u) Distribution

The parameter choices for the table 6.3 have been indicated in Chapter 6. We vary the time points according to the structure of the Swiss Re Bond in the codes furnished below. We present below the parameter choice for this table and this snippet is a part of every code in this sub-section. The annual mortality index is assumed to follow S_u) distribution specified in section 6.1.2. To obtain Table 6.1, we vary the values of interest rate while fixing the base value of the mortality index, i.e., $q_0 = 0.008453$. We present the MATLAB codes for this table below.

Snippet C.1.2.1: Parameter Choice for Table 6.1

```
% Parameters
n=3; %Duration of the Swiss Re Bond
q0=0.008453; %Base Value of the Mortality Index
r=0.035;  %Risk Free Rate
%Parameters of Su Distn. in (6.1.2) for q1, q2 and q3
alpha=[.008399, .008169, .007905];
beta=[.000298, .000613, .000904];
mu=[.70780, .58728, .58743];
sigma=[.67281, .50654, .42218];
```

After this, we have the snippet C.1.2.2 to initialize arrays and values used in every program except for the program to compute Monte Carlo estimate of bond price.

Snippet C.1.2.2: Initializing

```
t=zeros(n,1); %Vector of Time Points
C1=zeros(n,1); %Vector of Call Options in equation (3.2.16)
G1=0; %Initial Value of Sum used in G
%Arrays required for computation of (3.2.17) using (A.5.1)
muQ=zeros(n,1); %Parameter of the risk neutral density
d1=zeros(n,1);
d2=zeros(n,1);
d3=zeros(n,1);
num=zeros(n,1);
```

The snippet that follows is an integral part of each of the codes in this section except for the program to compute Monte Carlo estimate of bond price. This snippet is constructed on the same lines as snippet C.1.1.3.

Snippet C.1.2.3: Specification of Time Points and Computation of G

```
%Specifying Time Points and Computing G given in equation (3.2.17)
for i=1:n;
    t(i)=i;
```

```
end
K1=1.3*q0;
for i=1:n;
    muQ(i)=asinh(exp(-sigma(i)^2/2)*(q0*exp(r*t(i))-alpha(i))/beta(i));
    num(i)=-asinh((K1-alpha(i))/beta(i))+muQ(i);
    d1(i)=num(i)/sigma(i)+sigma(i);
    d2(i)=num(i)/sigma(i)-sigma(i);
    d3(i)=num(i)/sigma(i);
    C1(i)=beta(i)/2*exp(muQ(i)+sigma(i)^2/2)*normcdf(d1(i))-beta(i)...
    /2*exp(-muQ(i)+sigma(i)^2/2)*normcdf(d2(i))+(alpha(i)-K1)*normcdf(d3(i));
    G1=G1+C1(i);
end
G=exp(-r*3)*(5*G1/q0-1);
```

1. The Trivial Lower Bound SWLB_0

The MATLAB code for SWLB_0 is similar to the one for Black-Scholes Model. So, we do not rewrite it here.

2. The Lower Bound SWLB_1

```
%Snippet C.1.2.1
%Snippet C.1.2.2
x_lower=-1000; %First initial guess for equation (3.3.22)
x_upper=10; %Second initial guess for equation (3.3.22)
stn=zeros(n,1); %Initializing the vector of coefficients of x in (3.3.22)
AVl=0; %Initializing the sum on L.H.S. of (3.3.22) for lower guess
AVu=0; %Initializing the sum on L.H.S. of (3.3.22) for upper guess
AVm=0; %Initializing the sum on L.H.S. of (3.3.22) for midpoint of guesses
terml=zeros(n,1); %Initializing the summand on L.H.S. of (3.3.22) for lower guess
termu=zeros(n,1); %Initializing the summand on L.H.S. of (3.3.22) for upper guess
termm=zeros(n,1); %Initializing summand on L.H.S.:(3.3.22) for mid pt. of guesses
st=zeros(n,1); %Initializing the vector of coefficients of call in (3.3.27)
POW=zeros(n,1); %Defining the vector of terms in the sum in equation (3.3.27)
SPOW=0; %Initializing the terms in the sum in equation (3.3.27)
K=zeros(n,1); %Defining the vector of strike prices of the calls in (3.3.27)
%Arrays to compute lb1 under Su Distribution
muQlb1=zeros(n,1); %Parameter of the risk neutral density
d1lb1=zeros(n,1);
d2lb1=zeros(n,1);
d3lb1=zeros(n,1);
numlb1=zeros(n,1);
%Snippet C.1.2.3
%Code to find the Solution of the equation (3.3.22) using Bisection Method
for i=1:n;
    stn(i)=exp(r*(t(i)-t(1)));
    terml(i)=max((stn(i)*x_lower-1.3),0);
    termu(i)=max((stn(i)*x_upper-1.3),0);
    AVl=AVl+terml(i);
    AVu=AVu+termu(i);
end
fl=AVl-0.2; %L.H.S.-R.H.S. value for equation (3.3.22) for lower initial guess
fu=AVu-0.2; %L.H.S.-R.H.S. value for equation (3.3.22) for upper initial guess
while x_upper-x_lower>0.000000001;
      x_mid=(x_lower+x_upper)/2; %Average of Guesses
      AVm=0;
      for i=1:n;
          termm(i)=max((stn(i)*x_mid-1.3),0);
          AVm=AVm+termm(i);
      end
      fm=AVm-0.2; %L.H.S.-R.H.S. for (3.3.22) for mid pt. of guesses
      if (fm*fu)<0;
          x_lower=x_mid;
          fl=fm;
      else
```

```
            x_upper=x_mid;
            fu=fm;
        end
end
soln=x_mid; %The solution of equation (3.3.22)
% Code to compute lb1 using equation (3.3.27)
for i=1:n;
    K(i)=q0*(1.3/stn(i)+max((soln-1.3/stn(i)),0)); %Strike of call in (3.3.27)
    muQlb1(i)=asinh(exp(-sigma(1)^2/2)*(q0*exp(r*t(1))-alpha(1))/beta(1));
    numlb1(i)=-asinh((K(i)-alpha(1))/beta(1))+muQlb1(i);
    d1lb1(i)=numlb1(i)/sigma(1)+sigma(1);
    d2lb1(i)=numlb1(i)/sigma(1)-sigma(1);
    d3lb1(i)=numlb1(i)/sigma(1);
    POW(i)=beta(1)/2*exp(muQlb1(i)+sigma(1)^2/2)*normcdf(d1lb1(i))-beta(1)...
    /2*exp(-muQlb1(i)+sigma(1)^2/2)*normcdf(d2lb1(i))+...
    (alpha(1)-K(i))*normcdf(d3lb1(i)) %Computation of the sum in (3.3.27)
    SPOW=SPOW+stn(i)*POW(i); %Computation of the sum in (3.3.27)
end
lb1=5*exp(-r*3)*SPOW/q0;
format 'long'
SWLB1=max(lb1-G,0) %To ensure positive value
format 'short'
```

3. The Lower Bound $\mathrm{SWLB}_t^{(2)}$

```
%Snippet C.1.2.1
%Snippet C.1.2.2
x_lower=zeros(n,1); %Initializing the vector of Initial guess for equation (3.3.32)
x_upper=zeros(n,1); %Initializing vector of second Initial guess for (3.3.32)
x_mid=zeros(n,1); %Initializing vector of Average of Two Initial Guesses for (3.3.32)
soln=zeros(n,1); %Vector of Solutions for n time points
fl=zeros(n,1); %Vector of the L.H.S.-R.H.S. value of (3.3.32) for lower initial guess
fu=zeros(n,1); %Vector of the L.H.S.-R.H.S. value of (3.3.32) for upper initial guess
fm=zeros(n,1); %Vector of the L.H.S.-R.H.S. value of (3.3.32) for midpt. of guesses
stl=zeros(n,1); %Vector of first term for lower guess in the sum on L.H.S of (3.3.32)
stu=zeros(n,1); %Vector of first term for upper guess in the sum on L.H.S of (3.3.32)
stm=zeros(n,1); %Vector of first term for mid pt. of guesses in sum on L.H.S:(3.3.32)
AVl=zeros(n,1); %Initializing vector of 1st sum:L.H.S.(3.3.32) lower guess
AVu=zeros(n,1); %Initializing vector of 1st sum:L.H.S.(3.3.32) for upper guess
AVm=zeros(n,1); %Initializing vector of 1st sum:L.H.S.(3.3.32) for midpt. of guesses
std=zeros(n,1); %Vector of Coeff. of x inside 2nd sum on L.H.S. of (3.3.32)
AVdl=zeros(n,1); %Initializing vector of 2nd sum:L.H.S.(3.3.32) for lower guess
AVdu=zeros(n,1); %Initializing vector of 2nd sum:L.H.S.(3.3.32) for upper guess
AVdm=zeros(n,1); %Initializing vector of 2nd sum:L.H.S.(3.3.32) for midpt. of guesses
stdl=zeros(n,1); %Vector of 2nd term for lower guess in the sum on L.H.S of (3.3.32)
stdu=zeros(n,1); %Vector of 2nd term for upper guess in the sum on L.H.S of (3.3.32)
stdm=zeros(n,1); %Vector of 2nd term for mid pt. guesses in sum on L.H.S of (3.3.32)
stau=zeros(n,1); %Vector of square root of time points
POW1=zeros(n,1); %Initializing the vector of terms in 1st sum in lbt(2) in (3.3.41)
SPOW1=zeros(n,1); %1st sum in lbt(2) in (3.3.41)
POW2=zeros(n,1); %Initializing vector of terms in 2nd sum in lbt(2) in (3.3.41)
SPOW2=zeros(n,1); %2nd sum in lbt(2) in (3.3.41)
d=zeros(n,1); %Initializing the sum in lbt(2) in (3.3.41)
C2=zeros(n,1); %Initializing vector of calls:2nd term in lbt(2) in (3.3.41)
%Arrays to compute lbt(2) in (3.3.41) under Su Distribution
muQlb2=zeros(n,1); %Parameter of the risk neutral density
d1lb2=zeros(n,1);
d2lb2=zeros(n,1);
d3lb2=zeros(n,1);
numlb2=zeros(n,1);
T2=zeros(n,1);
T3=zeros(n,1);
%Snippet C.1.2.3
%Code to find the Solution of the equation (3.3.32) using Bisection Method
for j=1:n;
    x_lower(j)=0;
    x_upper(j)=10;
    for i=j:n;
        std(i)=exp(r*(t(i)-t(j)));
```

```
        stdl(i)=max((x_lower(j)*std(i)-1.3),0);
        stdu(i)=max((x_upper(j)*std(i)-1.3),0);
        AVdl(j)=AVdl(j)+stdl(i);
        AVdu(j)=AVdu(j)+stdu(i);
    end
    for k=1:j-1;
        stl(k)=max((x_lower(j)^(t(k)/t(j))-1.3),0);
        AVl(j)=AVl(j)+stl(k);
        stu(k)=max((x_upper(j)^(t(k)/t(j))-1.3),0);
        AVu(j)=AVu(j)+stu(k);
    end
    fl(j)=AVl(j)+AVdl(j)-0.2; %L.H.S.-R.H.S. for (3.3.32) for lower guess
    fu(j)=AVu(j)+AVdu(j)-0.2; %L.H.S.-R.H.S. for (3.3.32) for upper guess
    while x_upper(j)-x_lower(j)>0.000000000000001;
          x_mid(j)=(x_lower(j)+x_upper(j))/2;
          AVm(j)=0;
          AVdm(j)=0;
          for m=1:j-1;
              stm(m)=max((x_mid(j)^(t(m)/t(j))-1.3),0);
              AVm(j)=AVm(j)+stm(m);
          end
          for m1=j:n;
              std(m1)=exp(r*(t(m1)-t(j)));
              stdm(m1)=max((x_mid(j)*std(m1)-1.3),0);
              AVdm(j)=AVdm(j)+stdm(m1);
          end
          fm(j)=AVm(j)+AVdm(j)-0.2; %L.H.S.-R.H.S. in (3.3.32) for mid pt. of guesses
          if (fm(j)*fu(j))<0;
             x_lower(j)=x_mid(j);
             fl(j)=fm(j);
          else
             x_upper(j)=x_mid(j);
             fu(j)=fm(j);
          end
    end
    format 'long'
    soln(j)=x_mid(j)
end
%Computation of the second Term in lbt(2) in (3.3.41)
for j=1:n;
    for k=j:n;
        modisoln=max((soln(j)-1.3/exp(r*(t(k)-t(j)))),0);
        dnk=q0*(1.3/exp(r*(t(k)-t(j)))+modisoln); %Strike for call:2nd term (3.3.41)
        muQlb2(j)=asinh(exp(-sigma(j)^2/2)*(q0*exp(r*t(j))-alpha(j))/beta(j));
        numlb2(k)=-asinh((dnk-alpha(j))/beta(j))+muQlb2(j);
        d1lb2(k)=numlb2(k)/sigma(j)+sigma(j);
        d2lb2(k)=numlb2(k)/sigma(j)-sigma(j);
        d3lb2(k)=numlb2(k)/sigma(j);
        C2(k)=beta(j)/2*exp(muQlb2(j)+sigma(j)^2/2)*normcdf(d1lb2(k))-...
        beta(j)/2*exp(-muQlb2(j)+sigma(j)^2/2)*normcdf(d2lb2(k))+...
        (alpha(j)-dnk)*normcdf(d3lb2(k)) %Call Price :2nd term (3.3.41)
        POW2(k)=5*exp(r*(t(k)-t(j)))*C2(k)/q0 %Summand of the 2nd term in lbt(2)
        SPOW2(j)=SPOW2(j)+POW2(k); %Computation of the second sum in lbt(2)
    end
end

%Computation of the first term in lbt(2) in (3.3.41)
for j=1:n
    for i=1:j-1;
        modi2soln=max(((soln(j))^(t(i)/t(j))-1.3),0);
        di=q0*(1.3+modi2soln)^(t(j)/t(i));
        Ki=di^(t(i)/t(j));
        muQlb2(j)=asinh(exp(-sigma(j)^2/2)*(q0*exp(r*t(j))-alpha(j))/beta(j));
        num3i=-asinh((di-alpha(j))/beta(j))+muQlb2(j);
        d3i=num3i/sigma(j);
        d2i(i)=-num3i+muQlb2(j);
        T2(i)=Ki*normcdf(d3i);
        fun = @(x) (alpha(j)+beta(j).*(sinh(x))).^(t(i)./t(j)).*...
        exp(-(x-muQlb2(j)).^2./(2.*sigma(j).^2))./(2.*pi).^0.5;
        qu(i)= integral(fun,d2i(i),300); %Numerical Integration
        T3(i)=qu(i);
```

```
        POW1(i)=5*(T3(i)-T2(i))/(q0^(t(i)/t(j)));
        SPOW1(j)=SPOW1(j)+POW1(i); %Computation of the first sum in lbt(2)
    end
    d(j)=SPOW1(j)+SPOW2(j); %Computation of the consolidated lbt(2) in (3.3.41)
end
m=max(d);
format 'long'
lbt2=exp(-3*r)*m; %The Optimal Lower Bound lbt(2) given by equation (3.3.38)
format 'long'
SWLBt2=max(lbt2-G,0)
format 'short'
```

4. The Upper Bound $SWUB_1$

```
%Snippet C.1.2.1
%Snippet C.1.2.2
x_lower=-1000; %First initial guess for equation (3.4.15)
x_upper=100000; %Second initial guess for equation (3.4.15)
AVl=0; %Initializing the sum on L.H.S. of (3.4.13) for lower guess
AVu=0; %Initializing the sum on L.H.S. of (3.4.13) for upper guess
AVm=0; %Initializing the sum on L.H.S. of (3.4.13) for midpoint of guesses
POW=zeros(n,1); %Defining the vector of calls in ub1 in (3.4.15)
SPOW=0; %Initializing the terms in the sum in equation (3.4.15)
K=zeros(n,1); %Defining the vector of strike prices of the calls in (3.4.15)
%Arrays to compute ub1 under Su Distribution
muQub1=zeros(n,1); %Parameter of the risk neutral density
d1ub1=zeros(n,1);
d2ub1=zeros(n,1);
d3ub1=zeros(n,1);
numub1=zeros(n,1);
%Snippet C.1.2.3
%Code to find the Solution of the equation (3.4.13) using Bisection Method
for i=1:n;
    muQub1(i)=asinh(exp(-sigma(i)^2/2)*(q0*exp(r*t(i))-alpha(i))/beta(i)); %Pmt
                                              %in inverse distribution function
    AVl=AVl+(alpha(i)+beta(i)*sinh(muQub1(i)+sigma(i)*x_lower));
    AVu=AVu+(alpha(i)+beta(i)*sinh(muQub1(i)+sigma(i)*x_upper));
end
fl=AVl-(0.2+1.3*n); %L.H.S.-R.H.S. for equation (3.4.13) for lower initial guess
fu=AVu-(0.2+1.3*n); %L.H.S.-R.H.S. for equation (3.4.13) for upper initial guess
while x_upper-x_lower>0.000000000000001;
      x_mid=(x_lower+x_upper)/2;
      AVm=0;
      for i=1:n;
          AVm=AVm+(alpha(i)+beta(i)*sinh(muQub1(i)+sigma(i)*x_mid));
      end
      fm=AVm-(0.2+1.3*3); %L.H.S.-R.H.S. for (3.4.13) for mid pt. of guesses
      if (fm*fu)<0;
          x_lower=x_mid;
          fl=fm;
      else
          x_upper=x_mid;
          fu=fm;
      end
end
soln=x_mid %The solution of equation(3.4.13)
% Code to compute ub1 using equation (3.4.15)
for i=1:n;
    K(i)=alpha(i)+beta(i)*sinh(muQub1(i)+sigma(i)*soln) %Call strike:(3.4.15)
    numub1(i)=-asinh((K(i)-alpha(i))/beta(i))+muQ(i);
    d1ub1(i)=numub1(i)/sigma(i)+sigma(i);
    d2ub1(i)=numub1(i)/sigma(i)-sigma(i);
    d3ub1(i)=numub1(i)/sigma(i);
    POW(i)=beta(i)/2*exp(muQ(i)+sigma(i)^2/2)*normcdf(d1ub1(i))-...
    beta(i)/2*exp(-muQ(i)+sigma(i)^2/2)*normcdf(d2ub1(i))+...
    (alpha(i)-K(i))*normcdf(d3ub1(i)) %The call price in (3.4.15)
    SPOW=SPOW+POW(i); %Computation of the sum in (3.4.15)
end
ub1=5*exp(-3*r)*SPOW/q0;
```

```
format 'long'
SWUB1=max(ub1-G,0) %To ensure positive value
format 'short'
```

5. The Monte Carlo Estimate (Antithetic Method)

```
%Snippet C.1.2.1
M=2000000; %Twice the Number of Simulations/Paths
M1=M/2; %Number of Simulations/Paths
t=zeros(n,1); %Vector of Time Points
q=zeros(M1,n+1); %Matrix of Mortality Indices at the n Time points for...
                 %M Paths beginning at time zero
X=zeros(M1,n); %Initializing the call counterpart of Swiss Re Bond
X2=zeros(M1,n); %Initializing call counterpart of Swiss Re: Anti. Case
q2=zeros(M1,n+1); %Same as q but paths generated using Antithetic Variate
AV=zeros(M1,1); %Initializing the sum of calls along each path
AV2=zeros(M1,1); %Initializing the sum of calls along...
                 %each path obtained using Antithetic Variate
swiss=zeros(M1,1); %Payoff of the Swiss Re Bond along each path
swiss2=zeros(M1,1); %Same as above but paths generated
                    %using Antithetic Variate
V=0; %Initializing the sum of bond payoffs to 0
muQ=zeros(n,1); %Parameter of the risk neutral density
%Specifying Time Points
for i=1:n;
    t(i)=i;
end
for i=1:n;
    muQ(i)=asinh(exp(-sigma(i)^2/2)*(q0*exp(r*t(i))-alpha(i))/beta(i));
end
%Simulating the Su Distribution using the Antithetic Method
for i=1:M1;
    for j=1:n
        r1=randn;
        q(i,j)=alpha(j)+beta(j)*sinh(muQ(j)+sigma(j)*r1);
        X(i,j)=5*max((q(i,j)/q0-1.3),0);
        q2(i,j)=alpha(j)+beta(j)*sinh(muQ(j)+sigma(j)*(-r1)); %(-r1) acts...
                                                              %as antithetic variate
        X2(i,j)=5*max((q2(i,j)/q0-1.3),0);
    end
end
%Monte Carlo Estimation of the Swiss Re Mortality Bond 2003
for i=1:M1;
    for j=1:n;
        AV(i)=AV(i)+X(i,j); %Computing the Total Price along each path
        AV2(i)=AV2(i)+X2(i,j); %Same as above but along path generated...
        %using Antithetic Variate
    end
    swiss(i)=max((1-AV(i)),0); %Computing Payoff of Swiss Re along
                               %each path
    swiss2(i)=max((1-AV2(i)),0); %Same as above but path generated using...
                                 %Antithetic Variate
    V=V+swiss(i)+swiss2(i); %Summing the Payoffs of different paths
end
format 'long'
disp(' The Monte Carlo Estimate is')
MC=exp(-(r*T))*V/(2*M1) %Monte Carlo Estimate of the Swiss Re Bond 2003
format 'short'
```

6. The Lower Bound $\mathrm{SWLB}_t^{(1)}$

```
%Snippet C.1.2.1
%Snippet C.1.2.2
std=zeros(n,1); %The vector of terms in the denominator of Strike Price...
                %in Call Option Payoff in lbt1
AVd=zeros(n,1); %Initializing the above referred sum for each j
```

```
stn=zeros(n,1); %The Vector of Terms inside the Sum in lbt1 in (B.1.12)
AVn=zeros(n,1); %Initializing the above referred sum for each j
st=zeros(n,1); %The vector of terms in the numerator of Strike Price in...
               %Call Option Payoff in lbt1
AV=zeros(n,1); %Initializing the above referred sum for each j
ct=zeros(n,1); %The Vector of Strike Prices in the Call Function for...
               %different j
d=zeros(n,1); %R.H.S. of (B.1.12) for each j before discounting and...
              %dividing by n
%Arrays required for calculating call price in (B.1.12)
C2a=zeros(n,1);
C2=zeros(n,1);
%Arrays to compute lbt1 in (B.1.12) under Su Distribution
muQlbt1=zeros(n,1); %Parameter of the risk neutral density
d1lbt1=zeros(n,1);
d2lbt1=zeros(n,1);
d3lbt1=zeros(n,1);
numlbt1=zeros(n,1);
%Snippet C.1.2.3
%Computation of Various Sums involved in lbt1 given in (B.1.12)
for j=1:n;
    for i=j:n;
        std(i)=exp(r*(t(i)-t(j)));
        stn(i)=exp(r*t(i));
        AVd(j)=AVd(j)+std(i);
        AVn(j)=AVn(j)+stn(i); %computes the multiplication factor in lbt1
    end
    for k=1:j-1;
        st(k)=exp(r*t(k));
        AV(j)=AV(j)+st(k);
    end
end
%Computation of Strike Prices for each j; j=1,2,...,n
for l=1:n;
    ct(l)=((0.2+1.3*3)*q0-q0*AV(l))/AVd(l); %computes the strikes
    if (ct(l)>0);
        muQlbt1(l)=asinh(exp(-sigma(l)^2/2)*(w0*exp(r*t(l))-alpha(l))/beta(l));
        numlbt1(l)=-asinh((ct(l)-alpha(l))/beta(l))+muQlbt1(l);
        d1lbt1(l)=numlbt1(l)/sigma(l)+sigma(l);
        d2lbt1(l)=numlbt1(l)/sigma(l)-sigma(l);
        d3lbt1(l)=numlbt1(l)/sigma(l);
        C2a(l)=beta(l)/2*exp(muQlbt1(l)+sigma(l)^2/2)*normcdf(d1lbt1(l))-...
        beta(l)/2*exp(-muQlbt1(l)+sigma(l)^2/2)*normcdf(d2lbt1(l))+...
        (alpha(l)-ct(l))*normcdf(d3lbt1(l)) %Expectation in (B.1.12)
        C2(l)=exp(-r*t(l))*C2a(l); %Computation of Call Price in lbt1
        d(l)=C2*AVn(l);
    else
        d(l)=0;
    end
end
m=max(d);
format 'long'
lbt1=5*exp(-3*r)*m/q0; %The Optimal Lower Bound lbt1 given by (B.1.12)
SWLBt1=max(lbt1-G,0) %Ensures positive value
format 'short'
```

C.1.3 Log Gamma Distribution

The parameter choices for the tables 6.4 - 6.5 have been indicated in Chapter 6. We vary the time points according to the structure of the Swiss Re Bond in the codes presented below. We present below the parameter choice for these tables and this snippet is a part of every code in this sub-section. To obtain Table 6.4, we vary the values of interest rate while fixing the base value of the mortality index, i.e., $q_0 = 0.0088$. In Table 6.5, we vary the base values of the mortality index while specifying the interest rate as $r = 0.0$. The MATLAB codes that we present below, take care of both scenarios. The varying base mortality index for table 6.5 is denoted by w_0.

Snippet C.1.3.1: Parameter Choice for Table 6.1

```
% Parameters
n=3; %Duration of the Swiss Re Bond
q0=0.0088; %Base Value of the Mortality Index
w0=.008453; %Value of the Base Mortality Index which is varied for Table 6.5
r=0.035;  %Risk Free Rate
%Parameters of Log Gamma Distn. in (3.5.26) for q1, q2 and q3
p=[61.6326, 64.2902, 71.8574];
a=[.0103, .0098, .0080];
mu=[-5.2452, -5.4600, -5.7238];
sigma=[0.000074, .000095, .000094];
```

The snippet given below follows Snippet C.1.3.1 to initialize arrays and values used in every program. For the program to compute, Monte Carlo estimate of bond price, we do not use this snippet.

Snippet C.1.3.2: Initializing

```
t=zeros(n,1); %Vector of Time Points
C1=zeros(n,1); %Vector of Call Options in equation (3.2.16)
G1=0; %Initial Value of Sum used in G
%Arrays required for computation of (3.2.17) using (A.6.1)
d1=zeros(n,1);
d2=zeros(n,1);
```

The snippet that we now present is an integral part of each of the codes in this section and specifies the time points at which the principal loss percentage viz. L_i; $i = 1, 2, 3$, is computed (c.f. equation (3.2.2)) and as a result S_is are calculated at these time points (c.f. equations (3.2.9) - (3.2.11)). It also computes the value of G given in equation (3.2.17) which is required to compute price bounds for Swiss Re Bond 2003 exploiting put-call parity. For the program to compute Monte Carlo estimate of bond price, we do not use this snippet.

Snippet C.1.3.3: Specification of Time Points and Computation of G

```
%Specifying Time Points and Computing G given in equation (3.2.17)
for i=1:n;
    t(i)=i;
end
K=1.3*q0;
for i=1:n;
    d1(i)=(log(K)-mu(i))/((w0*exp(r*t(i)-mu(i)))^(1/p(i))-1)
    d2(i)=d1(i)+log(K)-mu(i)
    C1(i)=w0*exp(r*t(i))*(1-gamcdf(d1(i),p(i),1))-...
    K*(1-gamcdf(d2(i),p(i),1)) %The Computation of Call option in (3.2.17)
    G1=G1+C1(i)*exp(r*t(i));
end
G=exp(-r*3)*(5*G1/q0-1)
```

1. The Trivial Lower Bound SWLB_0

The MATLAB code for SWLB_0 is similar to the one for Black-Scholes Model. So, we do not rewrite it here.

2. The Lower Bound $SWLB_1$

```
%Snippet C.1.3.1
%Snippet C.1.3.2
x_lower=-1000; %First initial guess for equation (3.3.22)
x_upper=10; %Second initial guess for equation (3.3.22)
stn=zeros(n,1); %Initializing the vector of coefficients of x in (3.3.22)
AVl=0; %Initializing the sum on L.H.S. of (3.3.22) for lower guess
AVu=0; %Initializing the sum on L.H.S. of (3.3.22) for upper guess
AVm=0; %Initializing the sum on L.H.S. of (3.3.22) for midpoint of guesses
terml=zeros(n,1); %Initializing the summand on L.H.S. of (3.3.22) for lower guess
termu=zeros(n,1); %Initializing the summand on L.H.S. of (3.3.22) for upper guess
termm=zeros(n,1); %Initializing summand on L.H.S.:(3.3.22) for mid pt. of guesses
st=zeros(n,1); %Initializing the vector of coefficients of call in (3.3.27)
POW=zeros(n,1); %Defining the vector of terms in the sum in equation (3.3.27)
SPOW=0; %Initializing the terms in the sum in equation (3.3.27)
K1=zeros(n,1); %Defining the vector of strike prices of the calls in (3.3.27)
%Arrays to compute lb1 under log-Gamma Distribution
muQlb1=zeros(n,1); %Parameter of the risk neutral density
d1lb1=zeros(n,1);
d2lb1=zeros(n,1);
d3lb1=zeros(n,1);
numlb1=zeros(n,1);
%Snippet C.1.3.3
%Code to find the Solution of the equation (3.3.22) using Bisection Method
for i=1:n;
    stn(i)=exp(r*(t(i)-t(1)));
    terml(i)=max((stn(i)*x_lower-1.3),0);
    termu(i)=max((stn(i)*x_upper-1.3),0);
    AVl=AVl+terml(i);
    AVu=AVu+termu(i);
end
fl=AVl-0.2; %L.H.S.-R.H.S. value for equation (3.3.22) for lower initial guess
fu=AVu-0.2; %L.H.S.-R.H.S. value for equation (3.3.22) for upper initial guess
while x_upper-x_lower>0.000000001;
      x_mid=(x_lower+x_upper)/2; %Average of Guesses
      AVm=0;
      for i=1:n;
          termm(i)=max((stn(i)*x_mid-1.3),0);
          AVm=AVm+termm(i);
      end
      fm=AVm-0.2; %L.H.S.-R.H.S. for (3.3.22) for mid pt. of guesses
      if (fm*fu)<0;
          x_lower=x_mid;
          fl=fm;
      else
          x_upper=x_mid;
          fu=fm;
      end
end
soln=x_mid; %The solution of equation (3.3.22)
% Code to compute lb1 using equation (3.3.27)
for i=1:n;
    K(i)=q0*(1.3/stn(i)+max((soln-1.3/stn(i)),0)); %Strike of call in (3.3.27)
    d1lb1(i)=(log(K1(i))-mu(1))/((w0*exp(r*t(1)-mu(1)))^(1/p(1))-1);
    d2lb1(i)=d1lb1(i)+log(K1(i))-mu(1);
    POW(i)=w0*exp(r*t(1))*(1-gamcdf(d1lb1(i),p(1),1))-...
    K1(i)*(1-gamcdf(d2lb1(i),p(1),1)); %Computation of the sum in (3.3.27)
    SPOW=SPOW+stn(i)*POW(i); %Computation of the sum in (3.3.27)
end
lb1=5*exp(-r*3)*SPOW/q0;
format 'long'
SWLB1=max(lb1-G,0) %To ensure positive value
format 'short'
```

3. The Lower Bound $\mathrm{SWLB}_t^{(2)}$

```
%Snippet C.1.3.1
%Snippet C.1.3.2
x_lower=zeros(n,1); %Initializing the vector of Initial guess for equation (3.3.32)
x_upper=zeros(n,1); %Initializing vector of second Initial guess for (3.3.32)
x_mid=zeros(n,1); %Initializing vector of Average of Two Initial Guesses for (3.3.32)
soln=zeros(n,1); %Vector of Solutions for n time points
fl=zeros(n,1); %Vector of the L.H.S.-R.H.S. value of (3.3.32) for lower initial guess
fu=zeros(n,1); %Vector of the L.H.S.-R.H.S. value of (3.3.32) for upper initial guess
fm=zeros(n,1); %Vector of the L.H.S.-R.H.S. value of (3.3.32) for midpt. of guesses
stl=zeros(n,1); %Vector of first term for lower guess in the sum on L.H.S of (3.3.32)
stu=zeros(n,1); %Vector of first term for upper guess in the sum on L.H.S of (3.3.32)
stm=zeros(n,1); %Vector of first term for mid pt. of guesses in sum on L.H.S:(3.3.32)
AVl=zeros(n,1); %Initializing vector of 1st sum:L.H.S.(3.3.32) lower guess
AVu=zeros(n,1); %Initializing vector of 1st sum:L.H.S.(3.3.32) for upper guess
AVm=zeros(n,1); %Initializing vector of 1st sum:L.H.S.(3.3.32) for midpt. of guesses
std=zeros(n,1); %Vector of Coeff. of x inside 2nd sum on L.H.S. of (3.3.32)
AVdl=zeros(n,1); %Initializing vector of 2nd sum:L.H.S.(3.3.32) for lower guess
AVdu=zeros(n,1); %Initializing vector of 2nd sum:L.H.S.(3.3.32) for upper guess
AVdm=zeros(n,1); %Initializing vector of 2nd sum:L.H.S.(3.3.32) for midpt. of guesses
stdl=zeros(n,1); %Vector of 2nd term for lower guess in the sum on L.H.S of (3.3.32)
stdu=zeros(n,1); %Vector of 2nd term for upper guess in the sum on L.H.S of (3.3.32)
stdm=zeros(n,1); %Vector of 2nd term for mid pt. guesses in sum on L.H.S of (3.3.32)
stau=zeros(n,1); %Vector of square root of time points
POW1=zeros(n,1); %Initializing the vector of terms in 1st sum in lbt(2) in (3.3.41)
SPOW1=zeros(n,1); %1st sum in lbt(2) in (3.3.41)
POW2=zeros(n,1); %Initializing vector of terms in 2nd sum in lbt(2) in (3.3.41)
SPOW2=zeros(n,1); %2nd sum in lbt(2) in (3.3.41)
d=zeros(n,1); %Initializing the sum in lbt(2) in (3.3.41)
C2=zeros(n,1); %Initializing vector of calls:2nd term in lbt(2) in (3.3.41)
%Arrays to compute lbt(2) in (3.3.41) under log Gamma Distribution
sigman=zeros(n,1); %sigma' in (3.5.27)
e1=zeros(n,1); %sigma'' in (3.5.27)
d1lbt2=zeros(n,1);
d2lbt2=zeros(n,1);
T2=zeros(n,1);
T3=zeros(n,1);
%Snippet C.1.3.3
%Code to find the Solution of the equation (3.3.32) using Bisection Method
for j=1:n;
    x_lower(j)=0;
    x_upper(j)=10;
    for i=j:n;
        std(i)=exp(r*(t(i)-t(j)));
        stdl(i)=max((x_lower(j)*std(i)-1.3),0);
        stdu(i)=max((x_upper(j)*std(i)-1.3),0);
        AVdl(j)=AVdl(j)+stdl(i);
        AVdu(j)=AVdu(j)+stdu(i);
    end
    for k=1:j-1;
        stl(k)=max((x_lower(j)^(t(k)/t(j))-1.3),0);
        AVl(j)=AVl(j)+stl(k);
        stu(k)=max((x_upper(j)^(t(k)/t(j))-1.3),0);
        AVu(j)=AVu(j)+stu(k);
    end
    fl(j)=AVl(j)+AVdl(j)-0.2; %L.H.S.-R.H.S. for (3.3.32) for lower guess
    fu(j)=AVu(j)+AVdu(j)-0.2; %L.H.S.-R.H.S. for (3.3.32) for upper guess
    while x_upper(j)-x_lower(j)>0.000000000000001;
          x_mid(j)=(x_lower(j)+x_upper(j))/2;
          AVm(j)=0;
          AVdm(j)=0;
          for m=1:j-1;
              stm(m)=max((x_mid(j)^(t(m)/t(j))-1.3),0);
              AVm(j)=AVm(j)+stm(m);
          end
          for m1=j:n;
              std(m1)=exp(r*(t(m1)-t(j)));
              stdm(m1)=max((x_mid(j)*std(m1)-1.3),0);
              AVdm(j)=AVdm(j)+stdm(m1);
```

```
            end
            fm(j)=AVm(j)+AVdm(j)-0.2; %L.H.S.-R.H.S. in (3.3.32) for mid pt. of guesses
            if (fm(j)*fu(j))<0;
               x_lower(j)=x_mid(j);
               fl(j)=fm(j);
            else
               x_upper(j)=x_mid(j);
               fu(j)=fm(j);
            end
      end
      format 'long'
      soln(j)=x_mid(j)
end
for i=1:n;
      sigman(i)=(1-1/((w0*exp(r*t(i)-mu(i)))^(1/p(i))));
end
%Computation of the second Term in lbt(2) in (3.3.41)
for j=1:n;
      for k=j:n;
          modisoln=max((soln(j)-1.3/exp(r*(t(k)-t(j)))),0);
          dnk=q0*(1.3/exp(r*(t(k)-t(j)))+modisoln); %Strike for call:2nd term (3.3.41)
          d1lbt2(k)=(log(dnk)-mu(j))/((w0*exp(r*t(j)-mu(j)))^(1/p(j))-1);
          d2lbt2(k)=d1lbt2(k)+log(dnk)-mu(j);
          C2(k)=w0*exp(r*t(j))*(1-gamcdf(d1lbt2(k),p(j),1))-...
          dnk*(1-gamcdf(d2lbt2(k),p(j),1)); %Call Price :2nd term (3.3.41)
          POW2(k)=5*exp(r*(t(k)-t(j)))*C2(k)/q0 %Summand of the 2nd term in lbt(2)
          SPOW2(j)=SPOW2(j)+POW2(k); %Computation of the second sum in lbt(2)
      end
end
%Computation of the first term in lbt(2) in (3.3.41)
for j=1:n
      for i=1:j-1;
          e1(i)=1-(t(i)/t(j))*sigman(j);
          if (e1(i)>0)
              modi2soln=max(((soln(j))^(t(i)/t(j))-1.3),0);
              di=q0*(1.3+modi2soln)^(t(j)/t(i));
              Ki=di^(t(i)/t(j));
              d2i=(log(di)-mu(j))/sigman(j);
              d3i=d2i*e1(i);
              T2(i)=Ki*(1-gamcdf(d2i,p(j),1));
              T3(i)=(exp((t(i)/t(j))*mu(j))*(1-gammainc(d3i,p(j))))/((e1(i))^p(j));
              POW1(i)=5*(T3(i)-T2(i))/(q0^(t(i)/t(j)));
              SPOW1(j)=SPOW1(j)+POW1(i); %Computation of the first sum in lbt(2)
          else
              SPOW1(j)=0;
          end
      end
      d(j)=SPOW1(j)+SPOW2(j); %Computation of consolidated lbt(2):(3.3.41)
end
m=max(d);
format 'long'
lbt2=exp(-3*r)*m; %The Optimal Lower Bound lbt(2) given by equation (3.3.38)
format 'long'
SWLBt2=max(lbt2-G,0)
format 'short'
```

4. The Upper Bound SWUB$_1$

```
%Snippet C.1.3.1
%Snippet C.1.3.2
x_lower=-1000; %First initial guess for equation (3.4.15)
x_upper=100000; %Second initial guess for equation (3.4.15)
AVl=0; %Initializing the sum on L.H.S. of (3.4.13) for lower guess
AVu=0; %Initializing the sum on L.H.S. of (3.4.13) for upper guess
AVm=0; %Initializing the sum on L.H.S. of (3.4.13) for midpoint of guesses
POW=zeros(n,1); %Defining the vector of calls in ub1 in (3.4.15)
SPOW=0; %Initializing the terms in the sum in equation (3.4.15)
K2=zeros(n,1); %Defining the vector of strike prices of the calls in (3.4.15)
%Arrays to compute ub1 under log-Gamma Distribution
```

```
sigman=zeros(n,1); %sigma' in (3.5.27): used in ub1
d1ub1=zeros(n,1);
d2ub1=zeros(n,1);
%Snippet C.1.3.3
for i=1:n;
    sigman(i)=(1-1/((w0*exp(r*t(i)-mu(i)))^(1/p(i))))
end
%Code to find the Solution of the equation (3.4.13) using Bisection Method
for i=1:n;
    AVl=AVl+exp(mu(i)+sigman(i)*x_lower);
    AVu=AVu+exp(mu(i)+sigman(i)*x_upper);
end
fl=AVl-(0.2+1.3*n); %L.H.S.-R.H.S. for equation (3.4.13) for lower initial guess
fu=AVu-(0.2+1.3*n); %L.H.S.-R.H.S. for equation (3.4.13) for upper initial guess
while x_upper-x_lower>0.000000000000001;
      x_mid=(x_lower+x_upper)/2;
      AVm=0;
      for i=1:n;
          AVm=AVm+exp(mu(i)+sigman(i)*x_mid);
      end
      fm=AVm-(0.2+1.3*3); %L.H.S.-R.H.S. for (3.4.13) for mid pt. of guesses
      if (fm*fu)<0;
          x_lower=x_mid;
          fl=fm;
      else
          x_upper=x_mid;
          fu=fm;
      end
end
soln=x_mid %The solution of equation(3.4.13)
% Code to compute ub1 using equation (3.4.15)
for i=1:n;
    K2(i)=exp(mu(i)+sigman(i)*soln); %Call strike:(3.4.15)
    d1ub1(i)=(log(K2(i))-mu(i))/((w0*exp(r*t(i)-mu(i)))^(1/p(i))-1);
    d2ub1(i)=d1ub1(i)+log(K2(i))-mu(i);
    POW(i)=w0*exp(r*t(i))*(1-gamcdf(d1ub1(i),p(i),1))-...
    K2(i)*(1-gamcdf(d2ub1(i),p(i),1)); %The call price in (3.4.15)
    SPOW=SPOW+POW(i); %Computation of the sum in (3.4.15)
end
ub1=5*exp(-3*r)*SPOW/q0;
format 'long'
SWUB1=max(ub1-G,0) %To ensure positive value
format 'short'
```

5. The Monte Carlo Estimate

```
%Snippet C.1.3.1
M=100000; %Number of Simulations/Paths
t=zeros(n,1); %Vector of Time Points
q=zeros(M1,n+1); %Matrix of Mortality Indices at the n Time points for...
                 %M Paths beginning at time zero
X=zeros(M1,n); %Initializing the call counterpart of Swiss Re Bond
AV=zeros(M1,1); %Initializing the sum of calls along each path
swiss=zeros(M1,1); %Payoff of the Swiss Re Bond along each path
V=0; %Initializing the sum of bond payoffs to 0
sigman=zeros(n,1); %sigma' in (3.5.27): used in MC estimate
%Specifying Time Points
for i=1:n;
    t(i)=i;
end
for i=1:n;
    sigman(i)=(1-1/((w0*exp(r*t(i)-mu(i)))^(1/p(i))));
end
%Simulating the log-Gamma Distribution
for i=1:M;
    for j=1:n
        pn=p(j);
        an=a(j);
        r1=gamrnd(pn,1);
```

```
        r1=randn;
        q(i,j)=exp(mu(j)+sigman(j)*r1);
        X(i,j)=5*max((q(i,j)/q0-1.3),0);
    end
end
%Monte Carlo Estimation of the Swiss Re Mortality Bond 2003
for i=1:M;
    for j=1:n;
        AV(i)=AV(i)+X(i,j); %Computing the Total Price along each path
    end
    swiss(i)=max((1-AV(i)),0); %Computing Payoff of Swiss Re along
                               %each path
    V=V+swiss(i); %Summing the Payoffs of different paths
end
format 'long'
disp(' The Monte Carlo Estimate is')
MC=exp(-(r*T))*V/M %Monte Carlo Estimate of the Swiss Re Bond 2003
format 'short'
```

6. The Lower Bound $\mathbf{SWLB}_t^{(1)}$

```
%Snippet C.1.3.1
%Snippet C.1.3.2
std=zeros(n,1); %The vector of terms in the denominator of Strike Price...
                %in Call Option Payoff in lbt1
AVd=zeros(n,1); %Initializing the above referred sum for each j
stn=zeros(n,1); %The Vector of Terms inside the Sum in lbt1 in (B.1.12)
AVn=zeros(n,1); %Initializing the above referred sum for each j
st=zeros(n,1); %The vector of terms in the numerator of Strike Price in...
               %Call Option Payoff in lbt1
AV=zeros(n,1); %Initializing the above referred sum for each j
ct=zeros(n,1); %The Vector of Strike Prices in the Call Function for...
               %different j
d=zeros(n,1); %R.H.S. of (B.1.12) for each j before discounting and...
              %dividing by n
%Arrays required for calculating call price in (B.1.12)
C2a=zeros(n,1);
C2=zeros(n,1);
%Arrays to compute lbt1 in (B.1.12) under log-Gamma Distribution
d1lbt1=zeros(n,1);
d2lbt1=zeros(n,1);
%Snippet C.1.3.3
%Computation of Various Sums involved in lbt1 given in (B.1.12)
for j=1:n;
    for i=j:n;
        std(i)=exp(r*(t(i)-t(j)));
        stn(i)=exp(r*t(i));
        AVd(j)=AVd(j)+std(i);
        AVn(j)=AVn(j)+stn(i); %computes the multiplication factor in lbt1
    end
    for k=1:j-1;
        st(k)=exp(r*t(k));
        AV(j)=AV(j)+st(k);
    end
end
%Computation of Strike Prices for each j; j=1,2,...,n
for l=1:n;
    ct(l)=((0.2+1.3*3)*q0-q0*AV(l))/AVd(l); %computes the strikes
    if (ct(l)>0);
        d1lbt1(l)=(log(ct(l))-mu(l))/((w0*exp(r*t(l)-mu(l)))^(1/p(l))-1);
        d2lbt1(l)=d1lbt1(l)+log(ct(l))-mu(l);
        C2a(l)=w0*exp(r*t(l))*(1-gamcdf(d1lbt1(l),p(l),1))-...
        ct(l)*(1-gamcdf(d2lbt1(l),p(l),1)); %Expectation in (B.1.12)
        C2(l)=exp(-r*t(l))*C2a(l); %Computation of Call Price in lbt1
        d(l)=C2*AVn(l);
    else
        d(l)=0;
    end
end
```

```
m=max(d);
format 'long'
lbt1=5*exp(-3*r)*m/q0; %The Optimal Lower Bound lbt1 given by (B.1.12)
SWLBt1=max(lbt1-G,0) %Ensures positive value
format 'short'
```

C.2 Bounds for Guaranteed Annuity Options

We now furnish programs written on the basis of theory derived in Chapter 5 to furnish bounds for the GAOs for a variety of models which were presented in Chapter 6.

C.2.1 Vasicek Model

In Chapter 6, table 6.6 presents the price bounds for GAO under the Vasicek Model. The parameter choices for this table have already been detailed in the aforesaid chapter and we present them in the form of snippet C.2.1.1 given below. This snippet is an integral part of every code in this sub-section. In fact, table 6.6 depicts the changes in the price bounds for GAO with respect to the variation in the value of the correlation coefficient between the interest rate and the force of mortality. The MATLAB codes for this table are furnished below.

Snippet C.2.1.1: Parameter Choice for Table 6.6

```
% Parameters of GAO
n=35; % Number of years of life post the pension age
T=15; %Number of years to retirement
g=.111; %Guaranteed Annuity Rate (GAR)
K=1/g;
rho=0.9; %Corr. b\w interest and mortality rates varied in the codes
% Parameters of the Interest rate Model in (5.7.1)
a=0.15;
b=0.045;
sigma=0.03;
r0=b;
% Parameters of the Mortality Model in (5.7.2)
c=0.1;
xi=0.0003;
mu0=0.006;
```

The snippet given below, follows Snippet C.2.1.1 to initialize arrays used in every program.

Snippet C.2.1.2: Initializing

```
% The arrays required to compute ST(i) in (5.7.4)
A=zeros(n,1);
Gcurl=zeros(n,1);
D=zeros(n,1);
PHI=zeros(n,1);
Hcurl=zeros(n,1);
beta=zeros(n,1);
alpha=zeros(n,1);
```

The snippet that we now present is an integral part of each of the codes in this section and specifies the computation of basic parameters given in equations (5.7.6) - (5.7.11).

Snippet C.2.1.3: Calculation of Basic Parameters

```
% Computation of the basic parameters
for i=1:n;
    A(i)=(1-exp(-a*(i-1)))/a;
```

```
    Gcurl(i)=(exp(c*(i-1))-1)/c;
    D(i)=(b-sigma^2/(2*a^2))*(A(i)-(i-1))-sigma^2*A(i)^2/(4*a);
    PHI(i)=(1-exp((a-c)*(i-1)))/(a-c);
    Hcurl(i)=(-1)*(xi^2/(2*c^2))*(Gcurl(i)-(i-1))+...
    (rho*sigma*xi)*(A(i)-PHI(i))/(a*c)+xi^2*Gcurl(i)^2/(4*c);
    beta(i)=exp(D(i)+Hcurl(i));
    alpha(i)=beta(i);
end
```

The next snippet is used to compute mean and variance for mortality and interest rate as well as the covariance between them at time T specified by equations (5.7.14) - (5.7.18).

Snippet C.2.1.4: Calculation of Parameters of Distributions

```
% Computation of mean, variances and covariance
merT=exp(-a*T)*r0+b*(1-exp(-a*T))-sigma^2*(1-exp(-a*T))^2/(2*a^2)-...
(rho*sigma*xi)*((exp(c*T)*(exp(-c*T)-exp(-a*T)))/(a-c)-(1-exp(-a*T))/a)/c;
memuT=exp(c*T)*mu0-xi^2*(1-exp(c*T))^2/(2*c^2)+...
(rho*sigma*xi)*((exp(-a*T)*(exp(a*T)-exp(c*T)))/(a-c)-(exp(c*T)-1)/c)/a;
VarrT=sigma^2*(1-exp(-2*a*T))/(2*a);
VarmuT=xi^2*(exp(2*c*T)-1)/(2*c);
CovrTmuT=rho*sigma*xi*(1-exp(-(a-c)*T))/(a-c)
```

The computation of the SZCB $\tilde{P}(0,T)$ given in equation (5.3.2) is an integral part of every program. The following snippet based on equations (5.7.25)-(5.7.32) details its calculation.

Snippet C.2.1.5: Calculation of $\tilde{P}(0,T)$

```
% Computation of PCURL(0,T)
A0=(1-exp(-a*T))/a;
Gcurl0=(exp(c*T)-1)/c;
D0=(b-sigma^2/(2*a^2))*(A0-T)-sigma^2*A0^2/(4*a);
PHI0=(1-exp(-(a-c)*T))/(a-c);
Hcurl0=(((rho*sigma*xi)/(a*c))-(xi^2/(2*c^2)))*(Gcurl0-T)+...
(rho*sigma*xi)*(A0-PHI0)/(a*c)+xi^2*Gcurl0^2/(4*c);
beta0=exp(D0+Hcurl0);
V0=-(A0*r0+Gcurl0*mu0);
Pcurl0=beta0*exp(V0);
```

1. The Lower Bound GAOLB$_1$

```
%Snippet C.2.1.1
%Snippet C.2.1.2
% Defining arrays for lower bound
MrTmuT=zeros(n,1); %Initializing the m.g.f.
POW=zeros(n,1); %Initializing vector of terms in the Sum in GAOLB in(5.5.13)
SPOW=0; %Initializing the Sum in GAOLB
%Snippet C.2.1.3
%Snippet C.2.1.4
%Snippet C.2.1.5
% Computation of JOINT MGF of rT and muT
for i=1:n;
    MrTmuT(i)=exp(-(A(i)*merT+Gcurl(i)*memuT)+0.5*(A(i)^2*VarrT+...
    Gcurl(i)^2*VarmuT+2*A(i)*Gcurl(i)*CovrTmuT));
end
for i=2:n; %To ensure 1 to (n-1)
    POW(i)=alpha(i)*MrTmuT(i); %Expectation in (5.5.13)
    SPOW=SPOW+POW(i);
end
SPOW1=max((SPOW-(K-1)),0);
format 'long'
GAOLB1=g*Pcurl0*SPOW1
format 'short'
```

The next two snippets are used to compute mean and variance for $X_T^{(i)}$ as well as the covariance between $X_T^{(i)}$ and $X_T^{(j)}$ such that $i \neq j$ specified by equations (5.7.20),(5.7.21) and (5.7.24) and are used in some of the programs below.

Snippet C.2.1.6: Calculation of Mean and Variance of $X_T^{(i)}$

```
% Defining the arrays for Mean and Variance of XT(i)
E1=zeros(n,1); %Mean
V1=zeros(n,1); %Variance
% Computation of mean and variance of XT(i)
%Define Wi=A(i)*rT+Gcurl(i)*muT =-XT(i):(5.7.22)
%Define E1 and V1 as the mean and variance of W respectively
for i=1:n;
E1(i)=A(i)*merT+Gcurl(i)*memuT; %(5.7.20)
V1(i)=A(i)^2*VarrT+Gcurl(i)^2*VarmuT+2*A(i)*Gcurl(i)*CovrTmuT; %(5.7.21)
end
```

Snippet C.2.1.7: Calculation of Covariance b/w $X_T^{(i)}$ and $X_T^{(j)}$

```
%Initializing
COV=zeros(n,n);  %Covariance of Xk and Xj:Num. of (5.7.24)
CORR=zeros(n,n); %Correlation of Xk and Xj
SCORR1=zeros(n,1); %Sum of correlation between Xk and Xl
SCORR2=0; %Denominator of rho(k,lambda) if lambda is used
% Computation of Correlation of Xk and Xj (5.7.24)
%which is also the correlation of Wk and Wj
for k=2:n;
    for j=2:n;
        COV(k,j)=A(k)*A(j)*VarrT+(A(k)*Gcurl(j)+...
        A(j)*Gcurl(k))*CovrTmuT+Gcurl(k)*Gcurl(j)*VarmuT;
        CORR(k,j)=COV(k,j)/sqrt(V1(k)*V1(j));
        SCORR1(k)=SCORR1(k)+CORR(k,j);
        if (COV(k,j)<0)
            A2=k;
            A3=j;
            C2=0;
        else
            C2=COV(k,j);
        end
    end
    SCORR2=SCORR2+SCORR1(k);
end
```

2. The Lower Bound GAOLB

```
%Snippet C.2.1.1
%Snippet C.2.1.2
CORRkLAMBDA=zeros(n,1); %Correlation between Xk and LAMBDA
flag=0; %Indicator to test sign of covariance
% Defining arrays for lower bound
y=zeros(n,1);
d2=zeros(n,1); Array for c.d.f argument in sum in (5.5.47)
POW=zeros(n,1);
SPOW=0;
%Initializing for solving the equation
x_lower=-6.5; %First initial guess for equation (5.5.49)
x_upper=10000; %Second initial guess for equation (5.5.49)
AVl=0; %Initializing the sum on L.H.S. of (5.5.49) for 1st guess
AVu=0; %Initializing the sum on L.H.S. of (5.5.49) for 2nd guess
AVm=0; %Initializing the sum on L.H.S. of (5.5.49) for midpt. guess
%Snippet C.2.1.3
%Snippet C.2.1.4
%Snippet C.2.1.5
%Snippet C.2.1.6
%Snippet C.2.1.7
```

```
% Calculation of Correlation between Xk and LAMBDA
for k=2:n;
    CORRkLAMBDA(k)=SCORR1(k)/sqrt(SCORR2);
    if (CORRkLAMBDA(k)<0)
        flag=1
    else
        f=flag
    end
end
% Solution of the equation (5.5.49)
for i=2:n;
    AVl=AVl+alpha(i)*exp(-(E1(i))+0.5*V1(i)*(1-CORRkLAMBDA(i)^2)+...
    sqrt(V1(i))*CORRkLAMBDA(i)*x_lower);
    AVu=AVu+alpha(i)*exp(-(E1(i))+0.5*V1(i)*(1-CORRkLAMBDA(i)^2)+...
    sqrt(V1(i))*CORRkLAMBDA(i)*x_upper);
end
fl=AVl-K+1; %L.H.S.-R.H.S. value for (5.5.49) for 1st guess
fu=AVu-K+1; %L.H.S.-R.H.S. value for (5.5.49) for 2nd guess
while x_upper-x_lower>0.000000000000001;
      x_mid=(x_lower+x_upper)/2;
      AVm=0;
      for i=2:n;
          AVm=AVm+alpha(i)*exp(-(E1(i))+0.5*V1(i)*(1-CORRkLAMBDA(i)^2)+...
          sqrt(V1(i))*CORRkLAMBDA(i)*x_mid);
      end
      fm=AVm-K+1;
      if (fm*fu)<0;
          x_lower=x_mid;
          fl=fm;
      else
          x_upper=x_mid;
          fu=fm;
      end
end
soln=x_mid;
% Computation of lower bound GAOLB
%Since the value of f=0, we present the following section
for i=2:n;
    y(i)=alpha(i)*exp(-(E1(i))+0.5*V1(i)); %Coeff. of c.d.f. in Term 1 in (5.5.47)
    d2(i)=-soln+CORRkLAMBDA(i)*sqrt(V1(i));
end
for i=2:n;
    POW(i)=y(i)*normcdf(d2(i)); %Terms of sum in (5.5.47)
    SPOW=SPOW+POW(i);
end
format 'long'
GAOLB=g*Pcurl0*(SPOW-(K-1)*normcdf(-soln)) %(5.5.47)
format 'short'
```

3. The Lower Bound $\mathrm{GAOLB}_j^{(VS)}$

```
%Snippet C.2.1.1
%Snippet C.2.1.2
% Defining arrays for lower bound
y=zeros(n,1);
y2=zeros(n,1);
d1=zeros(n,1); %d2 in (5.7.45)
d2=zeros(n,1); %d1 in (5.7.46)
P=zeros(n,1); %Initializing Power expectation of (5.7.44)
POW=zeros(n,1); %Initializing elements of sum in (5.7.43)
SPOW=zeros(n,1);
%Defining the arrays for solving the equation
x_lower=zeros(n,1); %Array for 1st guess for equation (5.7.41)
x_upper=zeros(n,1); %Array for 2nd guess for equation (5.7.41)
x_mid=zeros(n,1); %Array for midpt. guess for equation (5.7.41)
soln=zeros(n,1); %Initializing array of solutions for each j
fl=zeros(n,1); %Vector of the L.H.S.-R.H.S. value of (5.7.41) for lower initial guess
fu=zeros(n,1); %Vector of the L.H.S.-R.H.S. value of (5.7.41) for upper initial guess
```

```
fm=zeros(n,1); %Vector of the L.H.S.—R.H.S. value of (5.7.41) for midpt. of guesses
stl=zeros(n,1); %Vector of first term for lower guess in the sum on L.H.S.:(5.7.41)
stu=zeros(n,1); %Vector of first term for upper guess in the sum on L.H.S.:(5.7.41)
stm=zeros(n,1); %Vector of first term for mid pt. of guesses in sum on L.H.S:(5.7.41)
AVl=zeros(n,1); %Initializing vector of 1st sum:L.H.S.(5.7.41) lower guess
AVu=zeros(n,1); %Initializing vector of 1st sum:L.H.S.(5.7.41) for upper guess
AVm=zeros(n,1); %Initializing vector of 1st sum:L.H.S.(5.7.41) for midpt. of guesses
%Snippet C.2.1.3
%Snippet C.2.1.4
%Snippet C.2.1.5
%Snippet C.2.1.6
%Snippet C.2.1.7
% Solution of the equation (5.7.41)
for j=2:n;
    x_lower(j)=—100;
    x_upper(j)=100;
    for i=2:n;
        stl(i)=alpha(i)*(x_lower(j))^(COV(i,j)/V1(j))*exp(—E1(i)+...
        (COV(i,j)/V1(j))*E1(j)+0.5*V1(i)*(1—CORR(i,j)^2));
        stu(i)=alpha(i)*(x_upper(j))^(COV(i,j)/V1(j))*exp(—E1(i)+...
        (COV(i,j)/V1(j))*E1(j)+0.5*V1(i)*(1—CORR(i,j)^2));
        AVl(j)=AVl(j)+stl(i);
        AVu(j)=AVu(j)+stu(i);
    end
    fl(j)=AVl(j)—K+1;
    fu(j)=AVu(j)—K+1;
    while x_upper(j)—x_lower(j)>0.0000000000001;
          x_mid(j)=(x_lower(j)+x_upper(j))/2;
          AVm(j)=0;
          for k=2:n;
              stm(k)=alpha(k)*(x_mid(j))^(COV(k,j)/V1(j))*exp(—E1(k)+...
              (COV(k,j)/V1(j))*E1(j)+0.5*V1(k)*(1—CORR(k,j)^2));
              AVm(j)=AVm(j)+stm(k);
          end
          fm(j)=AVm(j)—K+1;
          if (fm(j)*fu(j))<0;
              x_lower(j)=x_mid(j);
              fl(j)=fm(j);
          else
              x_upper(j)=x_mid(j);
              fu(j)=fm(j);
          end
    end
    format 'long'
    soln(j)=x_mid(j)
end
% Computation of lower bound
for j=2:n;
    d1(j)=(log(1/soln(j))—E1(j))/sqrt(V1(j)); %(5.7.45)
    y(j)=normcdf(d1(j));
    for i=2:n;
        d2(i)=d1(j)+COV(i,j)/sqrt(V1(j)); %(5.7.46)
        y2(i)=exp(COV(i,j)*(COV(i,j)—2*E1(j))/(2*V1(j)))*normcdf(d2(i));
        P(i)=(y2(i)—(soln(j))^(COV(i,j)/V1(j))*y(j)); %(5.7.44)
        POW(i)=alpha(i)*exp(—E1(i)+(COV(i,j)/V1(j))*E1(j)+...
        0.5*V1(i)*(1—CORR(i,j)^2))*P(i); %(5.7.43)
        SPOW(j)=SPOW(j)+POW(i);
    end
end
m=max(SPOW);
format 'long'
GAOLBVS=g*Pcurl0*m
format 'short'
```

4. The First Upper Bound GAOUB$_1$

```
%Snippet C.2.1.1
%Snippet C.2.1.2
% Defining arrays for upper bound
```

```
y=zeros(n,1);
d1=zeros(n,1);
d2=zeros(n,1);
POW=zeros(n,1); %Elements of sum in (5.6.7)
SPOW=0;
%Initializing for solving the equation (5.6.8)
x_lower=-6.5; %First initial guess for equation (5.6.8)
x_upper=10000; %Second initial guess for equation (5.6.8)
AVl=0; %Initializing the sum on L.H.S. of (5.6.8) for 1st guess
AVu=0; %Initializing the sum on L.H.S. of (5.6.8) for 2nd guess
AVm=0; %Initializing the sum on L.H.S. of (5.6.8) for midpt. guess
%Snippet C.2.1.3
%Snippet C.2.1.4
%Snippet C.2.1.5
%Snippet C.2.1.6
% Solution of the equation (5.6.8) (REMARK: x=norminv(1-x))
for i=2:n;
    AVl=AVl+alpha(i)*exp(-(E1(i)+sqrt(V1(i))*x_lower));
    AVu=AVu+alpha(i)*exp(-(E1(i)+sqrt(V1(i))*x_upper));
end
fl=AVl-K+1;
fu=AVu-K+1;
while x_upper-x_lower>0.000000000000001;
      x_mid=(x_lower+x_upper)/2;
      AVm=0;
      for i=2:n;
          AVm=AVm+alpha(i)*exp(-(E1(i)+sqrt(V1(i))*x_mid));
      end
      fm=AVm-K+1;
      if (fm*fu)<0;
          x_lower=x_mid;
          fl=fm;
      else
          x_upper=x_mid;
          fu=fm;
      end
end
soln=x_mid
for i=2:n;
    y(i)=alpha(i)*exp(-(E1(i)+sqrt(V1(i))*soln));
    d1(i)=(log(alpha(i)/y(i))-E1(i))/sqrt(V1(i));
    d2(i)=d1(i)+sqrt(V1(i));
end
% Computation of upper bound
for i=2:n;
    POW(i)=alpha(i)*exp((V1(i)-2*E1(i))/2)*normcdf(d2(i))-...
    y(i)*normcdf(d1(i)); %Elements of sum in (5.6.7)
    SPOW=SPOW+POW(i);
end
format 'long'
GAOUB1=g*Pcurl0*SPOW
format 'short'
```

5. The Improved Upper Bound $\mathbf{GAOUB}_j^{(VS)}$

```
%Snippet C.2.1.1
%Snippet C.2.1.2
%The number of SIMULATIONS
M=10000;
%Defining the array for Monte Carlo Simulation
LAMBDA=zeros(M,1);
CORRkLAMBDA=zeros(n,1); %Correlation between Xk and LAMBDA
flag=0; %Indicator to test sign of covariance
% Defining arrays etc. for improved upper bound
y=zeros(n,1);
y2=zeros(n,1);
d1=zeros(n,1);
d2=zeros(n,1);
```

```
POW=zeros(n,1); %Elements of sum in (5.7.59)
SPOW=zeros(M,1);
UB1=zeros(M,1);
SUB1=0;
%Defining the arrays for solving the equation (5.7.58)
AVl=zeros(M,1); %Initializing sum on L.H.S. of (5.7.58) for 1st guess
AVu=zeros(M,1); %Initializing sum on L.H.S. of (5.7.58) for 2nd guess
AVm=zeros(M,1); %Initializing sum on L.H.S. of (5.7.58) for midpt. guess
stl=zeros(n,1); %Vector of elements in sum:L.H.S of (5.7.58) for 1st guess
stu=zeros(n,1); %Vector of elements in sum:L.H.S of (5.7.58) for 2nd guess
stm=zeros(n,1); %Vector of elements in sum:L.H.S of (5.7.58) for midpt. guess
%Snippet C.2.1.3
%Snippet C.2.1.4
%Snippet C.2.1.5
%Snippet C.2.1.6
%Snippet C.2.1.7
MEANLAMB=0;
SDLAMB=sqrt(SCORR2);
% Calculation of Correlation between Xk and LAMBDA
for k=2:n;
    CORRkLAMBDA(k)=SCORR1(k)/sqrt(SCORR2);
    if (CORRkLAMBDA(k)<0)
        flag=1
    else
        f=flag
    end
end
% Solution of the equation (5.7.58)
for ik=1:M;
    LAMBDA(ik)=normrnd(0,SDLAMB);
    x_lower=-100000;
    x_upper=100000;
    for i=2:n;
        stl(i)=alpha(i)*exp(-(E1(i))+CORRkLAMBDA(i)*...
        sqrt(V1(i)/SCORR2)*(LAMBDA(ik)-MEANLAMB)+...
        sqrt(V1(i)*abs(1-CORRkLAMBDA(i)^2))*x_lower);
        stu(i)=alpha(i)*exp(-(E1(i))+CORRkLAMBDA(i)*...
        sqrt(V1(i)/SCORR2)*(LAMBDA(ik)-MEANLAMB)+...
        sqrt(V1(i)*abs(1-CORRkLAMBDA(i)^2))*x_upper);
        AVl(ik)=AVl(ik)+stl(i);
        AVu(ik)=AVu(ik)+stu(i);
     end
     fl=AVl(ik)-K+1;
     fu=AVu(ik)-K+1;
     while x_upper-x_lower>0.00000000000001;
           x_mid=(x_lower+x_upper)/2;
           AVm(ik)=0;
           for i=2:n;
               stm(i)=alpha(i)*exp(-(E1(i))+CORRkLAMBDA(i)*...
               sqrt(V1(i)/SCORR2)*(LAMBDA(ik)-MEANLAMB)+...
               sqrt(V1(i)*abs(1-CORRkLAMBDA(i)^2))*x_mid);
               AVm(ik)=AVm(ik)+stm(i);
           end
           fm=AVm(ik)-K+1;
           if (fm*fu)<0;
               x_lower=x_mid;
               fl=fm;
           else
               x_upper=x_mid;
               fu=fm;
           end
      end
      soln=x_mid;
      % Computation of improved upper bound in (5.7.59) for (ik)th simulation
      for i=2:n;
          y(i)=alpha(i)*exp(-(E1(i))+CORRkLAMBDA(i)*sqrt(V1(i)/SCORR2)*...
          (LAMBDA(ik)-MEANLAMB)+sqrt(V1(i)*abs(1-CORRkLAMBDA(i)^2))*soln);
          d1(i)=soln;
          d2(i)=d1(i)-sqrt(V1(i)*abs(1-CORRkLAMBDA(i)^2)); %(5.7.52) modified
          POW(i)=alpha(i)*exp((V1(i)*abs(1-CORRkLAMBDA(i)^2))/2-...
          E1(i)+CORRkLAMBDA(i)*sqrt(V1(i)/SCORR2)*(LAMBDA(ik)-MEANLAMB))*...
```

```
        (1-normcdf(d2(i)))-y(i)*(1-normcdf(d1(i)));
         SPOW(ik)=SPOW(ik)+POW(i);
      end
      UB1(ik)=g*Pcurl0*(SPOW(ik)); %For each simulation
      SUB1=SUB1+UB1(ik);
end
format 'long'
GAOUBVS=SUB1/M
format 'short'
```

6. The Monte Carlo Estimate (Antithetic Method)

```
%Snippet C.2.1.1
%Snippet C.2.1.2
N2=5000000; %Half the Number of Simulations/Paths
S=0; %The GRAND SUM of all simulations
% Defining arrays for MC Estimate
POW=zeros(n,1); %Initializing the GAO value using (5.7.4)
SPOW=zeros(N2,1);
POW2=zeros(n,1); %Initializing the GAO value using (5.7.4) for Antithetic case
SPOW2=zeros(N2,1);
SPOWK=zeros(N2,1);
SPOWK2=zeros(N2,1);
R=zeros(N2,1);
MU=zeros(N2,1);
%Snippet C.2.1.3
%Snippet C.2.1.4
%Snippet C.2.1.5
% Computation of MC Estimate
for i=1:N2;
    R(i)=randn;
    MU(i)=randn;
    R1=merT+sqrt(VarrT)*R(i);
    R2=merT+sqrt(VarrT)*(-R(i));
    MU1=memuT+sqrt(VarmuT)*(Corr*R(i)+sqrt(1-Corr^2)*MU(i));
    MU2=memuT+sqrt(VarmuT)*(Corr*(-R(i))+sqrt(1-Corr^2)*(-MU(i)));
    for k=1:n;
        POW(k)=alpha(k)*exp(-(A(k)*R1+Gcurl(k)*MU1));
        POW2(k)=alpha(k)*exp(-(A(k)*R2+Gcurl(k)*MU2));
        SPOW(i)=SPOW(i)+POW(k);
        SPOW2(i)=SPOW2(i)+POW2(k);
    end
    SPOWK(i)=max((SPOW(i)-K),0);
    SPOWK2(i)=max((SPOW2(i)-K),0);
    S=S+SPOWK(i)+SPOWK2(i);
end
format 'long'
GAOMC=g*Pcurl0*S/(2*N2)
format 'short'
```

C.2.2 Multi CIR Model

In Chapter 6, table 6.8 presents the price bounds for GAO for the Multi-CIR Model. The parameter choices for this table have already been detailed in the aforesaid chapter and we present them in the form of snippet C.2.2.1 given below. This snippet is an integral part of every code in this sub-section. In fact, table 6.8 depicts the changes in the price bounds for GAO with respect to the variation in the value of the parameter m_2 and hence the initial correlation coefficient between the interest rate and the force of mortality. The MATLAB codes for this table are furnished below.

Snippet C.2.2.1: Parameter Choice for Table 6.8

```
% Parameters of GAO
n=35; % Number of years of life post the pension age
T=15; %Number of years to retirement
```

```
g=.111; %Guaranteed Annuity Rate (GAR)
K=1/g;
rho=0.9; %Corr. b\w interest and mortality rates varied in the codes
% Parameters of the Multi CIR Model in (5.7.61)
m=3; %Dimension of the CIR process
k=[0.3731, 0.011, 0.01];
theta=[0.074484, 0.245455, 0.0013];
sigma=[0.0452, 0.0368, 0.0015];
rbar=-0.12332;
mubar=0;
Xi0=[0.0510234, 0.0890707, 0.0004];
R=[1, 1, 0];
m2=0.1;
```

The snippet given below, follows Snippet C.2.2.1 to initialize arrays used in every program.

Snippet C.2.2.2: Initializing

```
EQXiT=zeros(m,1); %(6.2.6)
C50=0.0125; %Used to compute m3
% The arrays required to compute ST(i) in (5.7.4)
u=zeros(m,1);
curl=zeros(m,1);
psiT=zeros(m,1);
phiT=zeros(m,1);
psi=zeros(n,m);
phi=zeros(n,m);
S1=0;
S2=0;
S1phi=zeros(n,1);
psi0=zeros(m,1); %Computing at T-T for transformed distribution
kcurl=zeros(m,1); %(5.7.68)
thetacurl=zeros(m,1);%(5.7.69)
```

The snippet given below, is used to compute the value of parameter m_3 and hence specify M (c.f. section 6.2.2) and is used in every program. It also details the calculation of the initial correlation ρ_0 between mortality and interest rate (c.f. equation (6.2.7)).

Snippet C.2.2.3: Computation of m_3 and ρ_0

```
for i=1:m;
    EQXiT(i)=Xi0(i)*exp(-(k(i)*T))+theta(i)*(1-exp(-(k(i)*T)));
end
m3=(0.0125-mubar-m2*EQXiT(2))/EQXiT(3)
M=[0, m2, m3];
%Computation of Rho0
Rho0=(m2*sigma(2)^2*Xi0(2))/(sqrt((sigma(1)^2*Xi0(1)+...
sigma(2)^2*Xi0(2))*(m2^2*sigma(2)^2*Xi0(2)+m3^2*sigma(3)^2*Xi0(3))))
```

The snippet that we now present is an integral part of each of the codes in this section and specifies the computation of basic parameters given in equations (5.7.64) - (5.7.65) obtained by solving the Riccatti equations.

Snippet C.2.2.4: Calculation of Basic Parameters

```
%Computation of the general psi and phis: tau=T+i-1-T = i-1
for i=1:m;
    u(i)=R(i)+M(i); %used in (5.7.64)-(5.7.65)
    curl(i)=sqrt(k(i)^2+2*u(i)*sigma(i)^2); %eta in (5.7.64)-(5.7.65)
end
for i=1:n;
    for j=1:m; %j stands for dimension of CIR model
```

```
        psi(i,j)=(2*u(j))/(curl(j)+k(j))-...
        ((4*u(j)*curl(j))/(curl(j)+k(j)))*1/((curl(j)+k(j))*...
        exp(curl(j)*(i-1))+curl(j)-k(j));
        phi(i,j)=-((k(j)*theta(j))*(curl(j)+k(j))*(i-1))/...
        (sigma(j)^2)+(2*k(j)*theta(j)*log((curl(j)+...
        k(j))*exp(curl(j)*(i-1))+curl(j)-k(j)))/(sigma(j)^2)-...
        ((2*k(j)*theta(j))*log(2*curl(j)))/(sigma(j)^2);
        S1phi(i)=S1phi(i)+phi(i,j);
    end
end
```

The computation of the SZCB $\tilde{P}(0,T)$ given in equation (5.3.2) is an integral part of every program. The following snippet details its calculation.

Snippet C.2.2.5: Calculation of $\tilde{P}(0,T)$

```
%Calculation of Pcurl(0,T) under the original measure Q:
%tau=T-0=T for calculating psi and phi
for i=1:m;
     psiT(i)=(2*u(i))/(curl(i)+k(i))-((4*u(i)*curl(i))/...
     (curl(i)+k(i)))*1/((curl(i)+k(i))*exp(curl(i)*T)+curl(i)-k(i));
     phiT(i)=-((k(i)*theta(i))*(curl(i)+k(i))*T)/(sigma(i)^2)+(2*k(i)*...
     theta(i)*log((curl(i)+k(i))*exp(curl(i)*T)+curl(i)-k(i)))/(sigma(i)^2)-...
     ((2*k(i)*theta(i))*log(2*curl(i)))/(sigma(i)^2);
     S1=S1+phiT(i);
     S2=S2+psiT(i)*Xi0(i);
end
Pcurl0=exp(0.12332*T)*exp(-(S1+S2));
```

To compute any bound for the GAO price under the Multi CIR case, we need to obtain the distribution of X_T under the transformed measure $\tilde{Q}$. The following snippet executes equations (5.7.68) - (5.7.69) and appears in every program given below.

Snippet C.2.2.6: Calculation of Parameters of Transformed Distribution of X_T

```
%Calculation of the transformed parameters under the change of measure:
%tau= T-T = 0 for calculating psi and phi
for i=1:m;
     psi0(i)=(2*u(i))/(curl(i)+k(i))-((4*u(i)*curl(i))/...
     (curl(i)+k(i)))*1/((curl(i)+k(i))*1+curl(i)-k(i));
     kcurl(i)=k(i)+psi0(i)*sigma(i)^2;
     thetacurl(i)=(k(i)*theta(i))/kcurl(i);
 end
```

1. The Lower Bound GAOLB$^{(MCIR)}$

```
%Snippet C.2.2.1
%Snippet C.2.2.2
%Arrays for mgf
muiT=zeros(m,1);
lambdaT=zeros(m,1);
vbar=zeros(m,1);
phipsil=zeros(n,m);
MGF=zeros(n,m);
% Defining arrays for lower bound
PROD1=ones(n,1); %Product of m.g.f.s
EcurlSi=zeros(n,1); %(5.7.83)-(5.7.84)
POW=zeros(n,1); %Initializing vector of terms in the Sum in GAOLB in (5.7.84)
SPOW=0; %Initializing the Sum in GAOLB
%Snippet C.2.2.3
%Snippet C.2.2.4
```

```
%Snippet C.2.2.5
%Snippet C.2.2.6
%Computation of mgf with parameter -phipsil for the three dim of the process
for i=1:m;
    vbar(i)=(2*kcurl(i)*thetacurl(i))/(sigma(i)^2);
    lambdaT(i)=(2*kcurl(i)*Xi0(i))/(sigma(i)^2*(exp(kcurl(i)*T)-1));
    muiT(i)=(sigma(i)^2*(1-exp(-kcurl(i)*T)))/(2*kcurl(i));
end
%Computation of EcurlSi
for i=1:n;
    for j=1:m;
        %the next equation computes the parameter of mgf: s=-phipsil(i,j)
        phipsil(i,j)=(1+psi(i,j)*muiT(j))^(-1); %(5.7.75)
        %Equation (5.7.74)
        MGF(i,j)=(phipsil(i,j))^(vbar(j))*exp(lambdaT(j)*(phipsil(i,j)-1));
        PROD1(i)=PROD1(i)*MGF(i,j);
    end
    EcurlSi(i)=exp(-(rbar+mubar)*(i-1))*exp(-S1phi(i))*PROD1(i); %(5.7.84)
end
for i=2:n;
    POW(i)=EcurlSi(i);
    SPOW=SPOW+POW(i);
end
SPOW1=max((SPOW-(K-1)),0);
format 'long'
GAOLBMCIR=g*Pcurl0*SPOW1
format 'short'
```

2. The Upper Bound GAOUB$^{(MCIR)}$

```
%Snippet C.2.2.1
%Snippet C.2.2.2
%Arrays for mgf
muiT=zeros(m,1);
lambdaT=zeros(m,1);
vbar=zeros(m,1);
phipsil=zeros(n,m);
MGF=zeros(n,m);
% Defining arrays for upper bound
PROD1=ones(n,1); %Product of m.g.f.s
EcurlSi=zeros(n,1); %(5.7.83)-(5.7.84)
POW=zeros(n,1); %Initializing vector of terms in the Sum in GAOLB in(5.7.84)
SPOW=0; %Initializing the Sum in GAOLB
CFPMT=zeros(m,1);
phiCFPMT=zeros(m,1);
MGFGM=zeros(m,1);
S02=zeros(n,1); %The exponent part of S0(i)
YT0old=0;
Δ=0.75; %The damping factor
%Snippet C.2.2.3
%Snippet C.2.2.4
%Snippet C.2.2.5
%Snippet C.2.2.6
%Computation of mgf with parameter -phipsil for the three dim of the process
for i=1:m;
    vbar(i)=(2*kcurl(i)*thetacurl(i))/(sigma(i)^2);
    lambdaT(i)=(2*kcurl(i)*Xi0(i))/(sigma(i)^2*(exp(kcurl(i)*T)-1));
    muiT(i)=(sigma(i)^2*(1-exp(-kcurl(i)*T)))/(2*kcurl(i));
end
%Computation of EcurlSi
for ii=2:n;
    for j=1:m;
        %the next equation computes the parameter of mgf: s=-phipsil(i,j)
        phipsil(ii,j)=(1+psi(ii,j)*muiT(j))^(-1);
        MGF(ii,j)=(phipsil(ii,j))^(vbar(j))*exp(lambdaT(j)*(phipsil(ii,j)-1));
        PROD1(ii)=PROD1(ii)*MGF(ii,j);  %Product of mgf of 3 components
    end
    EcurlSi(ii)=exp(-(rbar+mubar)*(ii-1))*exp(-S1phi(ii))*PROD1(ii);
    S02(ii)=-(rbar+mubar)*(ii-1)-S1phi(ii);
```

```
    YT0old=YT0old+S02(ii);
end
%Computation of CFPMT: Parameter for c.f. and m.g.f.: Sum of psi's
for j=1:m; %note that j stands for the dimension index of the CIR model
    for ii=2:n;
        CFPMT(j)=CFPMT(j)+psi(ii,j);
    end
end
%Computation of MGF for EGNT
% No looping on i is required since i IS SUMMED WITHIN THE PMT
for j=1:m;
%the next equation computes the parameter of mgf: s=-phiCFPMT(j)
%note -i^2=1 so -CFPMT(j) is retained making it + within beta
    phiCFPMT(j)=(1+((CFPMT(j)*muiT(j))/(n-1)))^(-1);
    MGFGM(j)=(phiCFPMT(j))^(vbar(j))*exp(lambdaT(j)*(phiCFPMT(j)-1));
    PROD2=PROD2*MGFGM(j);
end
for ii=2:n;
    POW(ii)=EcurlSi(ii);
    SPOW=SPOW+POW(ii);
end
SPOW5=SPOW/(n-1); %This gives E[AT]
YT0=YT0old/(n-1);
EGNT=exp(YT0)*PROD2; %This gives E[GT]
%Product of three cf's is being inverted just once
fun7=@(gamma1) real(exp(-(1i.*gamma1).*log(Knew)).*exp((1i.*gamma1+(Δ+1))....
     *YT0).*(((1+1i.*(((gamma1-1i.*(Δ+1))).*CFPMT(1).*muiT(1)./(n-1))).^(-...
     (vbar(1)))).*exp(lambdaT(1)*((1+1i.*(((gamma1-1i.*(Δ+1))).*CFPMT(1).*...
     muiT(1)./(n-1))).^(-1)-1)).*((1+1i.*(((gamma1-1i.*(Δ+1))).*CFPMT(2).*...
     muiT(2)./(n-1))).^(-(vbar(2)))).*exp(lambdaT(2).*((1+1i.*(((gamma1-1i.*...
     (Δ+1))).*CFPMT(2).*muiT(2)./(n-1))).^(-1)-1)).*((1+1i.*(((gamma1-1i.*...
     (Δ+1))).*CFPMT(3).*muiT(3)./(n-1))).^(-(vbar(3)))).*exp(lambdaT(3).*...
     ((1+1i.*(((gamma1-1i.*(Δ+1))).*CFPMT(3).*muiT(3)./(n-1))).^(-1)-1)))...
     ./(Δ.^2+Δ-gamma1.^2+(1i.*gamma1.*(2.*Δ+1))));
qty9 = quadgk(fun7,0,inf);
qty7 = exp(-(Δ*log(Knew)))*qty9/pi;
format 'long'
GAOUBMCIR=g*Pcurl0*(n-1)*(qty7+SPOW5-EGNT)
format 'short'
```

3. The Monte Carlo Estimate

```
%Snippet C.2.2.1
%Snippet C.2.2.2
%Initializing
M1=50000 %Number of Simulations
%Arrays for mgf
muiT=zeros(m,1);
lambdaT=zeros(m,1);
vbar=zeros(m,1);
phipsil=zeros(n,m);
MGF=zeros(n,m);
% Defining arrays for lower bound
PROD1=ones(n,1); %Product of m.g.f.s
EcurlSi=zeros(n,1); %(5.7.83)-(5.7.84)
POW=zeros(n,1); %Initializing vector of terms in the Sum in GAOLB in(5.7.84)
SPOW=0; %Initializing the Sum in GAOLB
%Snippet C.2.2.3
%Snippet C.2.2.4
%Snippet C.2.2.5
%Snippet C.2.2.6
%Defining pmts. of non-central chi-square variables defined in sum
for i=1:m;
    vbar(i)=(2*kcurl(i)*thetacurl(i))/(sigma(i)^2);
    lambdaT(i)=(2*kcurl(i)*Xi0(i))/(sigma(i)^2*(exp(kcurl(i)*T)-1));
    muiT(i)=(sigma(i)^2*(1-exp(-kcurl(i)*T)))/(2*kcurl(i));
    DELTA(i)=2*lambdaT(i);
    sqDELTA(i)=sqrt(DELTA(i));
    H(i)=2*vbar(i);
```

```
end
for i=1:n;
    for j=1:m;
        %The parameter used for converting the generated
        %non-central chi square variates into weighted sum of XiT's
        LAMBDA(i,j)=(psi(i,j)*muiT(j))/2;
    end
end
% Computing the MC estimate
for l=1:M1;
    SPOW(l)=0;
    for i=2:n;
        S1MC(i)=0;
        for j=1:m;
            R1 = ncx2rnd(H(j),DELTA(j));
            SPEXP1(j)=LAMBDA(i,j)*R1;
            S1MC(i)=S1MC(i)+SPEXP1(j);
        end
        POW(i)=S0i(i)*exp(-S1MC(i));
        SPOW(l)=SPOW(l)+POW(i);
    end
    SPOWK(l)=max((SPOW(l)-(K-1)),0);
    SMC=SMC+SPOWK(l);
end
format 'long'
GAOMC=g*Pcurl0*SMC/M1
format 'short'
```

C.2.3 Wishart Model

The parameter choices for the tables 6.9 - 6.11 have been indicated in Chapter 6. While the first two tables experiment the effect on the price bounds of a change in the initial value X_0, the final table depicts the influence of a change in the volatility matrix Q on the bounds. In the latter case, we assumed a symmetric volatility matrix Q and varied its off-diagonal elements. As a result, we considered a total of 3 examples. The MATLAB codes for these examples are similar with the corresponding change being made in the parameter snippet. We present below the codes in respect of Example 1, i.e., table 6.9. The parameter choices for this example have been indicated in section 6.2.3. We present the same through the following snippet.

Snippet C.2.3.1: Parameter Choice for Table 6.9

```
% Parameters of GAO
n=35; % Number of years of life post the pension age
T=15; %Number of years to retirement
g=.111; %Guaranteed Annuity Rate (GAR)
K=1/g;
rho=0.9; %Corr. b\w interest and mortality rates varied in the codes
% Parameters of the Wishart Model in (5.7.90)
rbar=0.04;
mubar=0;
R=[1, 0; 0, 0];
M=[0, 0; 0, 1];
X012=0.002;
X0=[0.01, X012; X012, 0.001]; %X012 is varied for Table 6.9
H=[-0.5, 0.4; 0.007, -0.008];
Q=[0.06 -0.0006; -0.06, 0.006];
beta=3;
d=2 %Dimension of Non-Central Wishart
```

The snippet given below, follows Snippet C.2.3.1 to initialize arrays used in every program.

Snippet C.2.3.2: Initializing

```
%Specifications of the arrays for the Levy Assumption
```

```
S0i=zeros(n,1);
PSIi = cell(1, n);
PHIi=zeros(n,1);
SPSI=0;
```

The snippet given below, is used to compute the initial correlation ρ_0 between mortality and interest rate (c.f. equation (6.2.9)).

Snippet C.2.3.3: Computation of ρ_0

```
% Computation of Rho0
Num=(Q(1,1)*Q(1,2)+Q(2,2)*Q(2,1))*X0(1,2);
Denom=sqrt((Q(1,1)^2+Q(2,1)^2)*X0(1,1)*(Q(2,2)^2+Q(1,2)^2)*X0(2,2));
Rho0=Num/Denom
```

The snippet that we now present is an integral part of each of the codes in this section and specifies the computation of basic parameters given in the system of equations (5.7.94).

Snippet C.2.3.4: Calculation of Basic Parameters

```
%Computation of the general psi and phis: tau=T+i-1-T = i-1
for ii=2:n;
         %Exponential of the matrix as in (5.7.103)
         APHNEW=expm([(ii-1).*H,(ii-1).*(2*(Q'*Q));(ii-1).*(R+M),(ii-1).*(-(H'))]);
         APHNEW11=APHNEW(1:2,1:2);
         APHNEW21=APHNEW(3:4,1:2);
         APHNEW12=APHNEW(1:2,3:4);
         APHNEW22=APHNEW(3:4,3:4);
         % Computation of PSIi and PHIi as in (5.7.102)
         PSIi{ii}=APHNEW22\APHNEW21;
         SPSI=SPSI+PSIi{ii};
         PHIi(ii)=beta*(log(det(APHNEW22))+(ii-1)*trace(H'))/2;
end
```

The computation of the SZCB $\tilde{P}(0,T)$ given in equation (5.3.2) is an integral part of every program. The following snippet details its calculation.

Snippet C.2.3.5: Calculation of $\tilde{P}(0,T)$

```
% Specification of new MATRICES Aij's USING (5.7.103)
%tau=T-0=T for calculating psi and phi
ANEW=expm([T.*H,T.*(2*(Q'*Q));T.*(R+M),T.*(-(H'))]);
A11=ANEW(1:2,1:2);
A21=ANEW(3:4,1:2);
A12=ANEW(1:2,3:4);%intersection
A22=ANEW(3:4,3:4);
% Computation of psi(T) and phi(T)
PSIT=A22\A21;
PHIT=beta*(log(det(A22))+T*trace(H'))/2;
% Computation of SZCBs: Pcurl(0,T):(5.7.102)
C=trace(PSIT*X0);
SZCB=exp(-(rbar+mubar)*T)*exp(-PHIT-C);
```

The next couple of snippets are executed independently to solve simultaneously the systems of equations (5.7.94) and (5.7.108) and therefore provide the parameters of the non-central Wishart distribution under the transformed measure $\tilde{Q}$ given in (5.7.107). These use the inbuilt *ode45* tool of MATLAB to solve the equations.

Snippet C.2.3.6: Defining the System of Equations (5.7.94) and (5.7.108)

```
function dy = mnewwish7(¬, XYV, A, B, R)
    [r,c] = size(A);
    XYV = reshape(XYV, [r, c, 3]);
    X = XYV(:,:,1);
    Y = XYV(:,:,2);
    V = XYV(:,:,3);

    dy = [-A.'*X - X*A + 2*X*(B.'*B)*X - R, -(A-(B.'*B)*X).'*Y, -Y.'*(B.'*B)*Y];

    dy = dy(:);

end
```

Snippet C.2.3.7: Solution of Equations Using *ode45*

```
%Solving systems (5.7.92) and (5.7.108)
A=[-0.5, 0.4; 0.007, -0.008];
B=[0.06 -0.0006; -0.06, 0.006];
R = [1 0; 0 1]; %R is actually R+M
X0 = [0, 0; 0, 0];
Y0 = [1 0; 0 1];
V0 = [0, 0; 0, 0];
[r, c] = size(A);

initial_value = [X0, Y0, V0];
[T, XYV] = ode45(@(t,XYV)mnewwish7(t, XYV, A, B, R), [15 0], initial_value(:));
nXYV = reshape(XYV, [], r, c, 3);
format 'long'
X = permute( nXYV(:,:,:,1), [2 3 1])
Y = permute( nXYV(:,:,:,2), [2 3 1])
V = permute( nXYV(:,:,:,3), [2 3 1])

pX = reshape( nXYV(:,:,:,1), [], r*c);
pY = reshape( nXYV(:,:,:,2), [], r*c);
pV = reshape( nXYV(:,:,:,3), [], r*c);
```

On the basis of the last snippet, we obtain the following parameter set which forms an integral part of every program in this section.

Snippet C.2.3.8: Parameters of Distribution of X_T under the Measure $\tilde{Q}$

```
%(5.7.107) and we call PSI as PSICURL
SIGMA=[0.006811791307233, -0.000407806990090; -0.000407806990090, 0.000392912436263];
PSICURL=[0.011526035149236, 0.013935191202735; 0.758273970303934, 0.955423605940771];
```

1. The Lower Bound GAOLB$^{(WIS)}$

```
%Snippet C.2.3.1
%Snippet C.2.3.2
%Other Initializations
%Specifications of array for the Levy Assumption
S0i=zeros(n,1);
% Defining arrays for lower bound
POW=zeros(n,1); %Initializing vector of terms in the Sum in GAOLB in (5.7.114)
SPOW=0; %Initializing the Sum in GAOLB
%Snippet C.2.3.3
%Snippet C.2.3.4
%Snippet C.2.3.5
%Snippet C.2.3.8
%Computation of the non-centrality parameter THETA1
```

```
THETA1=SIGMA\(PSICURL'*X0*PSICURL);
%Computation of Sum in Lower Bound in (5.7.114)
for ii=2:n;
    S0i(ii)=exp(-((rbar+mubar)*(ii-1)+PHIi(ii))); %(5.7.112)
    %Elements in (5.7.114)
    POW(ii)=S0i(ii)*exp(trace(-((THETA1/(eye(2)+2.*SIGMA*PSIi{ii}))*...
    SIGMA*PSIi{ii})))/((det(eye(2)+2.*SIGMA*PSIi{ii}))^(beta/2));
    SPOW=SPOW+POW(ii);
end
format 'long'
SPOW1=max((SPOW-(K-1)),0);
GAOLBWIS=g*SZCB*SPOW1
format 'short'
```

2. The Upper Bound GAOUB$^{(WIS)}$

```
function UB7
qty = quadgk(@fun4,0,inf)
[y, Δ, n, Knew, g, SZCB, SPOW, EGNT] = fun4(1);
qty1 = exp(-(Δ*log(Knew)))*qty/pi;
format 'long'
GAOUBWIS=g*SZCB*(n-1)*(qty1+SPOW-EGNT)
format 'short'

function [y, Δ, n, Knew, g, SZCB, SPOW, EGNT] = fun4(gamma1vec)
%Snippet C.2.3.1
%Snippet C.2.3.2
%Other Initializations
%Specifications of array for the Levy Assumption
S0i=zeros(n,1);
% Defining arrays for upper bound
POW=zeros(n,1); %Initializing vector of terms in the Sum in GAOUB in (5.6.40)
SPOW1=0; %Initializing the Sum in GAOUB
S0i2=zeros(n,1);
Yn0old=0;
Δ=0.75; %The damping factor
%Snippet C.2.3.3
%Snippet C.2.3.4
%Snippet C.2.3.5
%Snippet C.2.3.8
%Knew is our K' and it is equal to (K-1)/(n-1)
Knew=(K-1)/(n-1);
%Computation of the non-centrality parameter THETA1
THETA1=SIGMA\(PSICURL'*X0*PSICURL);
%Computation of sums for E[AT] and E[GT]
for ii=2:n;
    S0i(ii)=exp(-((rbar+mubar)*(ii-1)+PHIi(ii))); %(5.7.112)
    %Elements in (5.6.40)
    POW(ii)=S0i(ii)*exp(trace(-((THETA1/(eye(2)+2.*SIGMA*PSIi{ii}))*...
    SIGMA*PSIi{ii})))/((det(eye(2)+2.*SIGMA*PSIi{ii}))^(beta/2));
    SPOW1=SPOW1+POW(ii);
    S0i2(ii)=(-((rbar+mubar)*(ii-1)+PHIi(ii)));
end
Yn0=Yn0old/(n-1);
SPOW=SPOW1/(n-1); %This gives E[AT]
%Characteristic Function of Non-Central Wishart given by (5.7.110)
fun0=@(Gamma) exp(trace(1i.*((THETA1/(eye(2)-2i.*SIGMA*Gamma))*SIGMA*Gamma)))./...
     ((det(eye(2)-2i.*SIGMA*Gamma))^(beta/2));
EGNT=exp(Yn0)*fun0((1./(n-1)).*1i.*SPSI);  %This gives E[GT] using (5.6.34)
for j=1:numel(gamma1vec)
    gamma1=gamma1vec(j);
    y(j)=real(exp(-(1i.*gamma1).*log(Knew)).*exp((1i.*gamma1+(Δ+1)).*Yn0).*...
    (exp(trace(1i.*((THETA1/(eye(2)-2i.*SIGMA*((-(1./(n-1)).*(gamma1-1i.*...
    (Δ+1))).*SPSI)))*SIGMA*((-(1./(n-1)).*(gamma1-1i.*(Δ+1))).*SPSI))))./...
    ((det(eye(2)-2i.*SIGMA*((-(1./(n-1)).*(gamma1-1i.*(Δ+1))).*...
    SPSI))).^(beta./2))))./(Δ.^2+Δ-gamma1.^2+(1i.*gamma1.*(2.*Δ+1))));
end
```

3. The Monte Carlo Estimate

```
%Snippet C.2.3.1
%Snippet C.2.3.2
%Other Initializations
M1=20000; %Number of Simulations
%Specifications of array for the Levy Assumption
S0i=zeros(n,1);
Yi=zeros(n,1);
% Defining arrays etc. for MC estimate
S1MC=zeros(n,1);
POW=zeros(n,1);
SPOW=zeros(M1,1);
SPOWK=zeros(M1,1);
SMC=0;
%Snippet C.2.3.3
%Snippet C.2.3.4
%Snippet C.2.3.5
%Snippet C.2.3.8
%Computation of a new parameter OMEGA: for random sampling from NCW
OMEGA=PSICURL'*X0*PSICURL;
%For random sample generation from non-Central Wishart (NCW)
%Computation of the Cholesky decomposition of SIGMA and OMEGA
SIG=chol(SIGMA);
CHECK=SIG'*SIG;
D=[chol(SIGMA)]';
OMG=chol(OMEGA);
Mhat=[chol(OMEGA)]';
Mtilde=D\Mhat;
% Snippet for computing the MC estimate
for l=1:M1;
    Y=[Mtilde zeros(2,1)]+randn(2,3);
    XT=D*(Y*Y')*D'; %Simulating NCW random matrix
    for ii=2:n;
        S0i(ii)=exp(-((rbar+mubar)*(ii-1)+PHIi(ii)));
        Yi(ii)=trace(PSIi{ii}*XT); %XT(i) of Levy without negative sign
        POW(ii)=S0i(ii)*exp(-Yi(ii));
        SPOW(l)=SPOW(l)+POW(ii);
    end
    SPOWK(l)=max((SPOW(l)-(K-1)),0);
    SMC=SMC+SPOWK(l);
end
format 'long'
GAOMC=g*SZCB*SMC/M1
format 'short'
```

C.2.4 Computational Speed of the Bounds

To gauge the efficiency of our working for the bounds in the Multi CIR case and the Wishart case and generate table 6.12, we invoke the ‘Run and Time’ command on the MATLAB taskbar and this generates the ‘Profile Summary’ depicting the execution time.

Bibliography

S. S. Ahmadi and P. Gaillardetz. Modeling Mortality and Pricing Life Annuities with Lèvy Processes. *Insurance: Mathematics and Economics*, 64:337–350, 2015.

G. A. Akerlof. The Market for “Lemons”: Quality Uncertainty and the Market Mechanism. *Quarterly Journal of Economics*, 84:488–500, 1970.

H. Albrecher, P. A. Mayer, and W. Schoutens. General Lower Bounds for Arithmetic Asian Option Prices. *Applied Mathematical Finance*, 15(2):123–149, 2008.

A. Alfonsi. *Affine Diffusions and Related Processes: Simulation, Theory and Applications, Vol. 6*. Bocconi and Springer Series, New York, 2015.

A. Ang and A. Maddaloni. Do Demographic Changes affect Risk Premiums? Evidence from International Data. Technical report, National Bureau of Economic Research, 2003.

D. Applebaum. *Lévy Processes and Stochastic Calculus*. Cambridge University Press, Cambridge, 2004.

J. Appleby. How Long Can We Expect to Live? *BMJ*, 346, 2013.

APRA. Australian Prudential Regulation Authority: APRAs Pandemic Stress Test of the Insurance Industry. *APRA Insight*, 3:2–7, 2007.

A. Arik, Y. Yolcu-Okur, S. Sahin, and O. Ugur. Pricing Pension Buy-outs under Stochastic Interest and Mortality Rates. *Scandinavian Actuarial Journal*, 2017. URL `http://dx.doi.org/10.1080/03461238.2017.1328370`.

A. R. Bacinello, P. Millossovich, A. Olivieri, and E. Pitacco. Variable Annuities: A Unifying Valuation Approach. *Insurance: Mathematics and Economics*, 49(3):285–297, 2011.

G. Bagus. Taming the Cat: Securitizing Pandemic Losses Offers Life Insurers a Guaranteed Way to Fund Potential Claims. Bests Review, September:2007, 2007. URL `http://publications.milliman.com/publications/lifepublished/pdfs/taming-the-cat-PA09-01-07.pdf`.

R. K. Bahl and S. Sabanis. Model-Independent Price Bounds for Catastrophic Mortality Bonds. *Working Paper, Arxiv*, 2016. URL `arXiv:1607.07108[q-fin.PR]`.

R. K. Bahl and S. Sabanis. General Price Bounds for Guaranteed Annuity Options. *Working Paper, Arxiv*, 2017. URL `arXiv:1707.00807v1[q-fin.PR]`.

L. Ballotta and S. Haberman. Valuation of Guaranteed Annuity Conversion Options. *Insurance: Mathematics and Economics*, 33:87–108, 2003.

L. Ballotta and S. Haberman. The Fair Valuation Problem of Guaranteed Annuity Option: The Stochastic Mortality Environment Case. *Insurance: Mathematics and Economics*, 38: 195–214, 2006.

D. Baranoff and C. O’Brien. Equitable Life U.K.: A Decade of Regulations and Restructuring. The Geneva Association, 2016. URL `www.genevaassociation.org/media/952811/20161006_equitablelife_full.pdf`.

C. Basel. Longevity Risk Transfer Markets: Market Structure, Growth Drivers and Impediments, and Potential Rsks. Bank for International Settlements, 2013. URL `http://www.bis.org/publ/joint34.htm`.

D. Bauer. *Stochastic Mortality Modelling and Securitizaton of Mortality Risk*. PhD thesis, ifa-Verlag, Ulm, Germany, 2008.

D. Bauer and F. W. Kramer. Risk and Valuation of Mortality Contigent Bonds. *Working Paper, Georgia State University*, 2009.

D. Bauer, A. Kling, and J. Rub. A Universal Pricing Framework for Guaranteed Minimum Benefits in Variable Annuities. *Astin Bulletin*, 38(2):621–651, 2008.

C. Baumgart, R. Lempertseder, A. Riswadkar, K. Woolnough, and M. Zweim¨ uller. *Haug, A. ed. CRO Briefing Emerging Risks InitiativePosition Paper*, chapter Influenza Pandemic, pages 1–13. Amstelveen: CRO Forum, 2007.

M. Baxter and A. Rennie. *Financial Calculus: An Introduction to Derivative Pricing*. Cambridge University Press, 1996.

O. Beelders and D. Colorassi. Modelling Mortality Risk with Extreme Value Theory: The Case of Swiss Re's Mortality-Indexed Bond. *Global Association of Risk Professionals*, 4:26–30, 2004.

C. Bellis, R. Lyon, S. Klugman, and J. Shepherd. *Understanding Actuarial Management: The Actuarial Control Cycle*. Sydney: The Institute of Actuaries of Australia, 2nd edition, 2010.

H. Bensusan, N. E. Karoui, S. Loisel, and Y. Salhi. Partial Splitting of Longevity and Financial Risks: The Longevity Nominal Choosing Swaption. *Insurance: Mathematics and Economics*, 68:61–72, 2016.

E. Biffis. Affine Processes for Dynamic Mortality and Actuarial Valuations. *Insurance: Mathematics and Economics*, 37(3):443–468, 2005.

E. Biffis and D. Blake. Keeping Some Skin in the Game: How to Start a Capital Market in Longevity Risk Transfers. *North American Actuarial Journal*, 18(1):14–21, 2014.

E. Biffis and P. Millossovich. The Fair Value of Guaranteed Annuity Option. *Scandinavian Actuarial Journal*, 1:23–41, 2006.

B. Billingsley. *Probability and Measure, 3rd ed.* Wiley, New York, 1995.

F. Black and M. Scholes. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81(3):637–659, 1973.

D. Blake and M. Morales. Longevity Risk and Capital Markets: The 2014-15 Update. Pensions Institute (February 2017), 2017. URL `https://www.pensions-institute.org/workingpapers/wp1702.pdf`.

D. Blake, A. J. G. Cairns, and K. Dowd. Living with Mortality: Longevity Bonds and Other Mortality-linked Securities. *British Actuarial Journal*, 12:153–197, 2006.

D. Blake, A. Cairns, and K. Dowd. The Birth of the Life Market. *Asia-Pacific Journal of Risk and Insurance*, 3:6–36, 2008.

D. Blake, A. Cairns, G. Coughlan, K. Dowd, and R. MacMinn. The New Life Market. *The Journal of Risk and Insurance*, 80(3):501–558, 2013.

D. Blake, T. Boardman, and A. J. G. Cairns. Sharing Longevity Risk: Why Governments Should Issue Longevity Bonds. *North American Actuarial Journal*, 18(1):258–277, 2014.

M. Bolton, D. Carr, P. Collis, C. George, V. Knowles, and A. Whitehouse. Reserving for Annuity Guarantees. In: The Report of the Annuity Guarantees. *Working Party*, pages 1–36, 1997.

S. Bouriaux and R. MacMinn. Securitization of Catastrophe Risk: New Developments. *Journal of Insurance Issues*, 32(1):1–34, 2009.

S. Bouriaux and W. L. Scott. Capital Market Solutions to Terrorism Risk Coverage: A Feasibility Study. *The Journal of Risk Finance incorporating Balance Sheet*, 5(4):33–44, 2004.

M. M. Boyer and L. Stentoft. "If We can Simulate It We Can Insure It: An Application to Longevity Risk Management". *CIRANO Scientific Publications*, s(08), 2012.

P. Boyle and M. Hardy. Guaranteed Annuity Options. *Astin Bulletin*, 33(2):125–152, 2003.

R. Bridet. Extreme Mortality Bonds. Paper presented at the International Actuarial Association AFIR/LIFE Colloquium, Munich, 2009.

N. Brouhns, M. Denuit, and J. Vermunt. A Poisson Log-Bilinear Approach to the Construction of Projected Life Tables. *Insurance: Mathematics and Economics*, 31:373–393, 2002.

R. L. Brown. Theories of Longevity. Online, 2016. [online] Available HTTP://https://www.actuaries.org.uk/documents/plenary-2-international-perspective.

M.-F. Bru. Diffusions of Perturbed Principal Component Analysis. *Journal of Multivariate Analysis*, 29(1):127–136, 1989.

M.-F. Bru. Wishart Processes. *Journal of Theoretical Probability*, 4(4):725–751, 1991. [online] Available dx.doi.org/10.1007/BF01259552.

V. Bruggeman. Capital Market Instruments for Catastrophe Risk Financing. Annual Meeting of American Risk and Insurance Association in Quebec City, Canada, 2007.

A. Buraschi, A. Cieslak, and Trojani. Correlation Risk and the Term Structure of Interest Rates. *Working Paper, Imperial College and University of St. Gallen.*, 2008.

A. J. G. Cairns, D. Blake, and K. Dowd. Pricing Death: Framework for the Valuation and Securitization of Mortality Risk. *ASTIN Bulletin*, 36:79–120, 2006a.

A. J. G. Cairns, D. Blake, and K. Dowd. A Two-factor Model for Stochastic Mortality with Parameter Uncertainty: Theory and Calibration. *Journal of Risk and Insurance*, 73:687–718, 2006b.

A. J. G. Cairns, D. Blake, K. Dowd, G. D. Coughlan, D. Epstein, A. Ong, and I. Balevich. A Quantitative Comparison of Stochastic Mortality Models Using Data from England and Wales and the United States. *North American Actuarial Journal*, 13:1–35, 2009.

A. J. G. Cairns, D. Blake, K. Dowd, G. D. Coughlan, D. Epstein, and M. Khalaf-Allah. Mortality Density Forecasts: An Analysis of Six Stochastic Mortality Models. *Insurance: Mathematics and Economics*, 48:355–367, 2011.

R. Caldana, G. Fusai, A. Gnoatto, and M. Grasselli. General Closed-form Basket Option Pricing Bounds. *Quantitative Finance*, 16(4):535–554, 2016.

P. Carr and D. B. Madan. Option Valuation Using the Fast Fourier Transform. *Journal of Computational Finance*, 2:61–73, 1998.

W.-S. Chan, J. S.-H. Li, and J. Li. The CBD Mortality Indexes: Modeling and Applications. *North American Actuarial Journal*, 18(1):38–58, 2014.

W.-S. Chan, J. S.-H. Li, K. Zhou, and R. Zhou. Towards a Large and Liquid Longevity Market: A Graphical Population Basis Risk Metric. *The Geneva Papers on Risk and Insurance*, 41 (1):118–127, 2016.

H. Chen. A Family of Mortality Jump Models applied to US data. *Asia Pacific Journal of Risk Insurance*, 8:105–121, 2013.

H. Chen and S. H. Cox. Modeling Mortality with Jumps: Application to Mortality Securitization. *Journal of Risk and Insurance*, 76:727–751, 2009.

H. Chen and S. D. Cummins. Longevity Bond Premiums: The Extreme Value Approach and Risk Cubic Pricing. *Insurance: Mathematics and Economics*, 46(1):150–161, 2010.

H. Chen, S. H. Cox, and S. S. Wang. Is The Home Equity Conversion Program in the United States sustainable? Evidence from Pricing Mortgage Insurance Premiums and Non-Course Provisions using the Conditional Esscher Transform. *Insurance: Mathematics and Economics*, 46:371–384, 2010.

H. Chen, R. D. MacMinn, and T. Sun. Mortality Dependence and Longevity Bond Pricing: A Dynamic Factor Copula Mortality Model with the GAS Structure. *Working Paper, cass.city.ac.uk*, 2014.

H. Chen, R. MacMinn, and T. Sun. Multi-Population Mortality Models: A Factor Copula Approach. *Insurance: Mathematics and Economics*, 63:135–146, 2015.

H. Chen, R. D. MacMinn, and T. Sun. Mortality Dependence and Longevity Bond Pricing: A Dynamic Factor Copula Mortality Model with the GAS Structure. *The Journal of Risk and Insurance*, 84:393–415, 2017.

X. Chen, G. Deelstra, J. Dhaene, and M. Vanmaele. Static Super-Replicating Strategies for a Class of Exotic Options. *Insurance: Mathematics and Economics*, 42(3):1067–1085, 2008.

H. W. Cheng, C.-F. Tzeng, M.-H. Hsieh, and J. T. Tsai. Pricing of Mortality-Linked Securities with Transformed Gamma Distribution. *Academia Economic Papers*, 42(2):271–303, 2014.

C. Chiarella, C. Hsiao, and T. To. Risk Premia and Wishart Term Structure Models. *Working Paper*, 2010.

C. Chiarella, C. Hsiao, and T. To. Risk Premia and Wishart Term Structure Models. *Journal of Empirical Finance*, 37:59–78, 2016.

C. Chu and Y. Kwok. Valuation of Guaranteed Annuity Options in Affine Term and Structure Models. *International Journal of Theoretical and Applied Finance*, 10:368–387, 2007.

S.-L. Chuang. *Stochastic Mortality Modelling and Pricing of Mortality/Longevity Linked Derivatives*. PhD thesis, The University of Texas, Austin, 2013.

CMI. Mortality Projections Committee Working Paper 97. CMI mortality projections model, 2016. URL `https://www.actuaries.org.uk/learn-and-develop/continuousmortality-investigation/cmi-working-papers/mortality-projections/cmi-working-papers-97-98-and-99`.

R. Cont and P. Tankov. *Financial Modelling with Jump Processes*. Chapman and Hall/CRC Financial Mathematical Series, 2003.

L. Cornaro. *Art of Living Long*. Springer 2005, 1558.

G. Coughlan. *The Handbook of Insurance-linked Securities* Eds. In Barrieu, P. and Albertini, L., chapter Longevity Risk Transfer: Indices and Capital Market Solutions. West Sussex: John Wiley and Sons, 2009.

G. Coughlan, D. Blake, R. Macminn, A. J. G. Cairns, and K. Dowd. *Handbook of Insurance* Eds. In Dionne, G., chapter Longevity Risk and Hedging Solutions, pages 997–1035. New York: Springer, 2013.

A. Cowley and J. D. Cummins. Securitization of Life Insurance Assets and Liabilities. *Journal of Risk and Insurance*, 72(2):193–226, 2005.

J. C. Cox, J. E. Ingersoll, and S. Ross. A Theory on the Term Structure of Interest Rates. *Econometrica*, 53(2):385–407, 1985.

N. J. Cox, S. E. Tamblyn, and T. Tam. Influenza Pandemic Planning. *Vaccine*, 21:1801–1803, 2003.

S. H. Cox and Y. Hu. Modeling Mortality Risk from Exposure to a Potential Future Extreme Event and Its Impact on Life Insurance. Atlanta: Department of Risk Management and Insurance J. Mack Robinson College of Business, 2004.

S. H. Cox and Y. Lin. Natural Hedging of Life and Annuity Mortalty Risks. *North American Actuarial Journal*, 11(3):1–15, 2007.

S. H. Cox, Y. Lin, and S. S. Wang. Multivariate Exponential Tilting and Pricing Implications for Mortality Securitization. *The Journal of Risk and Insurance*, 73(4):719–736, 2006.

S. H. Cox, Y. Lin, and H. Pedersen. Mortality Risk Modelling: Applications to Insurance Securitization. *Insurance: Mathematics and Economics*, 46(1):242–253, 2010.

T. Crawford, R. d. Haan, and C. Runchey. Longevity Risk Quantication and Management: A Review of Relevant Literature. Society of Actuaries, 2008.

C. Cuchiero. *Affine and Polynomial Processes.* PhD thesis, ETH Zürich, 2011.

C. Cuchiero, D. Filipović, E. Mayerhofer, and J. Teichmann. Affine Processes on Positive Semidefinite Matrices. *The Annals of Applied Probability*, 21(2):397–463, 2011.

J. Cumbo. Life Expectancy Shift 'Could Cut Pension Deficits by £310bn'. Financial Times, 2017. URL `https://www.ft.com/content/77fa62fe-2feb-11e7-9555-23ef563ecf9a?mhq5j=e1`.

J. D. Cummins. Cat bonds and Other Risk-Linked Securities: State of the Market and Recent Developments. *Risk Management and Insurance Review*, 11(1):23–47, 2008.

J. D. Cummins and P. Trainar. Securitization, Insurance and Reinsurance. *Journal of Risk and Insurance*, 76(3):463–492, 2009.

J. Da Fonseca, M. Grasselli, and C. Tebaldi. A Multifactor Volatility Heston Model. *Quantitative Finance*, 8(6):591–604, 2008.

M. Dacorogna and G. Apicella. A general Framework for Modeling Mortality to Better Estimate its Relationship to Interest Rate Risks. *SCOR Papers*, 2016.

M. Dacorogna and M. Cadena. Exploring the Dependence between Mortality and Market Risks. *SCOR Papers*, 2015.

M. Dahl. Stochastic Mortality in Life Insurance: Market Reserves and Mortality-linked Insurance Contracts. *Insurance: Mathematics and Economics*, 35(1):113–136, 2004.

M. Dahl and T. Moller. Valuation and Hedging of Life Insurance Liablities with Systematic Mortality Risk. *Insurance: Mathematics and Economics*, 39(2):193–217, 2006.

Q. Dai and K. J. Singleton. Specification analysis of Affine Term Structure Models. *The Journal of Finance*, 55(5):1943–1977, 2000.

A. De Moivre. *Annuities upon Lives.* Printed by W.P. and sold by Francis Fayram, London, 1725.

G. Deelstra, J. Liinev, and M. Vanmaele. Pricing of Arithmetic Basket Options by Conditioning. *Insurance: Mathematics and Economics*, 34(1):55–77, 2004.

G. Deelstra, I. Diallo, and M. Vanmaele. Bounds for Asian Basket Options. *Journal of Computational and Applied Mathematics*, 218:215–228, 2008.

G. Deelstra, J. Dhaene, and M. Vanmaele. *Advanced Mathematical Methods for Finance* Eds. Nunno, G. Di and Øksendal, B., chapter An Overview of Comonotonicity and its Applications in Finance and Insurance. Springer, 2011.

G. Deelstra, M. Grasselli, and C. V. Weverberg. The Role of the Dependence between Mortality and Interest Rates when Pricing Guaranteed Annuity Options. *Insurance: Mathematics and Economics*, 71:205–219, 2016.

Y. Deng. *Longevity Risk Modeling, Securities Pricing and Other Related Issues.* PhD thesis, University of Texas, 2011.

Y. Deng, P. L. Brockett, and R. D. MacMinn. Longevity/Mortality Risk Modeling and Securities Pricing. *The Journal of Risk and Insurance*, 79(3):697–721, 2012.

M. Denuit and P. Devolder. Continuous Time Stochastic Mortality and Securitization of Longevity Risk. *Working Paper 06-02, Institut des Sciences Actuarielles,* - Universite Catholique de Louvain, Louvain-la-Neuve, 2006.

M. Denuit, F. De Vylder, and C. Lefèvre. Extrema with respect to s-convex Orderings in Moment Spaces: A General Solution. *Insurance: Mathematics and Economics*, 24(3):201–217, 1999.

M. Denuit, P. Devolder, and A.-C. Goderniaux. Securitization of Longevity Risk: Pricing Survivor Bonds with Wang Transform in the Lee-Carter Framework. *Insurance: Mathematics and Economics*, 74:87–113, 2007.

C. M. Depew. *1795-1895 One Hundred Years of American Commerce, Vol. II.* D.O Haynes and Co., New York, 1895.

J. Dhaene, S. Wang, V. R. Young, and M. J. Goovaerts. Comonotonicity and Maximum Stop Loss Premiums. *Bulletin of the Swiss Association of Actuaries*, 2000(2):99–113, 2000.

J. Dhaene, M. Denuit, M. Goovaerts, R. Kaas, and D. Vyncke. The Concept of Comonotonicity in Actuarial Science and Finance: Theory. *Insurance: Mathematics and Economics*, 31(1): 3–33, 2002a.

J. Dhaene, M. Denuit, M. Goovaerts, R. Kaas, and D. Vyncke. The Concept of Comonotonicity in Actuarial Science and Finance: Application. *Insurance: Mathematics and Economics*, 31 (2):133–161, 2002b.

J. Dhaene, A. Kukush, E. Luciano, W. Schoutens, and B. Stassen. On the (In-) dependence between Financial and Actuarial Risks. *Insurance: Mathematics and Economics*, 53(2):522–531, 2013.

D. Dickson, M. Hardy, and R. Waters. *Actuarial Mathematics for Life Contingent Risks.* Cambridge University Press, 2013.

K. Dowd, D. Blake, A. J. G. Cairns, and P. Dawson. Survivor Swaps. *Journal of Risk and Insurance*, 73:1–17, 2006.

K. Dowd, A. Cairns, D. Blake, G. Coughlan, D. Epstein, and M. Khalaf-Allah. Backtesting Stochastic Mortality Models: An Ex-Post Evaluation of Multi-Period-Ahead Density Forecasts. *North American Actuarial Journal*, 14:281–298, 2010a.

K. Dowd, A. J. G. Cairns, D. Blake, G. D. Coughlan, D. Epstein, and M. Khalaf-Allah. Evaluating the Goodness of Fit of Stochastic Mortality Models. *Insurance: Mathematics and Economics*, 47:255–265, 2010b.

A. Dreyer, G. Kritzinger, and J. Decker. Assessing the Impact of a Pandemic on the Life Insurance Industry in South Africa. Paper presented at the 1st IAA Life Colloquium, Stockholm, 2007.

D. Duffie and R. Kan. A Yield Factor Model of Interest Rates. *Mathematical Finance*, 6(4): 379–406, 1996.

D. Duffie, J. Pan, and K. J. Singleton. Transform Analysis and Asset Pricing for Affine Jump-diffusions. *Econometrica*, 68(6):1343–1376, 2000.

D. Duffie, D. Filipovic, and W. Schachermayer. Affine Processes and Applications in Finance. *The Annals of Applied Probability*, 13(3):984–1053, 2003.

D. Dufresne. The Integrated Square-Root Process. Research Paper 90, The University of Melbourne, 2001.

Ernst and Young. The Resurgence of Insurance-Linked Securities. Sydney: Ernst & Young Australia, 2011.

J. Faraut and A. Koranyi. Analysis on Symmetric Cones. The Clarendon Press Oxford University, New York, Oxford Science Publications, 1994.

I. Farr, H. Mueller, M. Scanlon, and S. Stronkhorst. Economic Capital for Life Insurance Companies. SOA Monograph, 2008.

C. A. Favero, A. E. Gozluklu, and A. Tamoni. Demographic Trends, the Dividend-Price Ratio, and the Predictability of Long-run Stock Market Returns. *Journal of Financial and Quantitative Analysis*, 46(5):1493–1520, 2011.

R. Fenga, X. Jing, and J. Dhaene. Comonotonic Approximations of Risk Measures for Variable Annuity Guaranteed Benefits with Dynamic Policyholder Behavior. *Journal of Computational and Applied Mathematics*, 311:272–292, 2017.

D. Filipovic. Time-inhomogeneous Affine Processes. *Stochastic Processes and their Applications*, 115:639–659, 2005.

M. Fransham and D. Dorling. Have Mortality Improvements Stalled in England? *BMJ*, 357, 2017.

A. Frey, M. Kirova, and C. Schmidt. The Role of Indices in Transferring Insurance Risks to the Capital Markets. *in: T. Hess, ed. Sigma (Zurich: Swiss Re)*, 4, 2009. URL `http://media.swissre.com/documents/sigma4_2009_en.pdf`.

J. F. Fries. Aging, Natural Death and the Compression of Morbidity. *New England Journal of Medicine*, 303:130–135, 1980.

H. Gao. *Valuation and Risk Measurement of Guaranteed Annuity Options under Stochastic Environment*. PhD thesis, The University of Western Ontario, 2014.

H. Gao, R. Mamon, and X. Liu. Pricing a Guaranteed Annuity Option under Correlated and Regime-Switching Risk Factors. *European Actuarial Journal*, 5:309–326, 2015a.

H. Gao, R. Mamon, X. Liu, and A. Tenyakov. Mortality Modeling with Regime-Switching for the Valuation of a Guaranteed Annuity Option. *Insurance: Mathematics and Economics*, 63: 108–120, 2015b.

H. Gao, R. Mamon, and X. Liu. Risk Measurement of a Guaranteed Annuity Option under A Stochastic Modeling Framework. *Mathematics and Computers in Simulation*, 132:100–119, 2017.

J. Garvey. Securitisation of Extreme Mortality Risk, 2011.

A. Gnoatto. The Wishart Short Rate Model. *International Journal of Theoretical and Applied Finance*, 15(8), 2012.

A. Gnoatto and M. Grasselli. The Explicit Laplace Transform for the Wishart Process. *Journal of Applied Probability*, 51:640–656, 2014.

B. Gompertz. On the Nature of the Function Expressive of the Law of Human Mortality, and on a New Mode of Determining the Value of Life Contingencies. *Philosophical Transactions of the Royal Society of London*, 115:513–585, 1825.

M. J. Goovaerts and R. Laeven. Actuarial Risk Measures for Financal Dervative Pricing. *Insurance: Mathematics and Economics*, 42:540–547, 2008.

M. J. Goovaerts, R. Kaas, A. Van Heerwaarden, and T. Bauwelinckx. Effective Actuarial Methods. *Insurance Series, North Holland, Amsterdam*, 3, 1990.

C. Gourieroux and R. Sufana. Wishart Quadratic Term Structure Models. *SSRN eLibrary*, 2003.

M. Grasselli and C. Tebaldi. Solvable Affine Term Structure Models. *Mathematical Finance*, 18(1):135–153, 2008. ISSN 0960-1627. doi: 10.1111/j.1467-9965.2007.00325.x. URL http://www.researchgate.net/profile/Claudio_Tebaldi/publication/227611192_SOLVABLE_AFFINE_TERM_STRUCTURE_MODELS/links/00b7d5263a7a59cca8000000.pdfhttp://dse.univr.it/safe/Papers/GrasselliTebaldi04b.pdf.

J. Graunt. Natural and Political Observations made upon the Bills of Mortality. London: Martin, Allestry, and Dicas, 1662.

A. K. Gupta and D. K. Nagar. *Matrix Variate Distributions*. Chapman & Hall/CRC, 2000.

D. Hainaut and P. Devolder. Mortality Modelling with Lèvy Processes. *Insurance: Mathematics and Economics*, 42:409–418, 2008.

E. Halley. An Estimate of the Degrees of the Mortality of Mankind, drawn from Curious Tables of the Births and Funerals at the City of Breslaw; with an Attempt to Ascertain the Price of Annuities upon Lives. *Philosophical Transactions*, 196:596–610, 1693.

K. Hartwig, K. Karl, S. Strauss, and T. Watson. Convergence of Insurance and Capital Markets. New York: World Economic Forum, 2008.

D. Heath, R. Jarrow, and A. Morton. Bond Pricing and Term Structure of Interest Rates: A Discrete Time Approximation. *Journal of Financial and Quantitative Analysis*, 25:419–440, 1990.

R. Helfenstein and T. Holzheu. SecuritizationNew Opportunities for Insurers and Investors. *in: T. Hess, ed. Sigma (Zurich: Swiss Re)*, 4, 2006. URL http://media.swissre.com/documents/sigma7_2006_en.pdf.

S. Heston. A Closed-form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *The Review of Financial Studies*, 6(2):327–343, 1993.

HMD. Human Mortality Database. *University of California, Berkeley (USA)and Max Planck Institute for Demographic Research (Germany)*, 2014. [online] Available HTTP: www.mortality.org.

D. Hobson, P. Laurence, and T. H. Wang. Static-arbitrage Upper Bounds for the Prices of Basket Options. *Quantitative Finance*, 5(4):329–342, 2005.

Y. L. Huang, J. T. Tsai, S. S. Yang, and H. W. Cheng. Price Bounds of Mortality-linked Security in Incomplete Insurance Market. *Insurance: Mathematics and Economics*, 55:30–39, 2014.

S. Huebner. *Life Insurance*. D. Appleton and Company, 1921.

A. Hunt and D. Blake. Modelling Longevity Bonds: Analysing the Swiss Re Kortis Bond. *Insurance: Mathematics and Economics*, 63:12–29, 2015.

A. Huynh, A. Bruhn, and B. Browne. A Review of Catastrophic Risks for Life Insurers. *Risk Management and Insurance Review*, 16(2):233–266, 2013. ISSN 0960-1627. doi: 10.1111/rmir.12011.

IAAust. The Practice of Life Insurance. Sydney: The Institute of Actuaries of Australia, 2009.

IIL. History of Options under Life Policies. *Insurance Institute of London, London*, 1972.

J. Jacod and A. N. Shiryaev. *Limit Theorems for Stochastic Processes, volume 288 of Fundamental Principles of Mathematical Sciences*. Springer-Verlag, Berlin, second edition, 2003.

L. Jalen and R. Mamon. Valuation of Contingent Claims with Mortality and Interest Rate Risks. *Mathematical and Computer Modelling*, 49:1893–1904, 2009.

R. M. Jennings and A. P. Trout. The Tontine: From the Reign of Louis Xiv to the French Revolutionary Era. Monograph 12, S. S. Huebner Foundation, The Wharton School, University of Pennsylvania, Philadelphia, 1982.

P. Jevtie, M. Kwak, and T. A. Pirvu. Longevity Bond Pricing in Equilibrium. *Working Paper,* Montreal Institute of Structured Finance and Derivatives, 2017. URL `http://ifsid.ca/en/wp-content/uploads/sites/2/2017/05/WP-17_04_Pirvu_LongevityDerivativePaper.pdf`.

L. Johnson. Catastrophe Bonds and Financial Risk: Securing Capital and Rule through Contingency. *Geoforum*, 45:30–40, 2013.

M. L. Johnson, V. L. Bengston, P. G. Coleman, and T. Kirkwood. *The Cambridge Handbook of Age and Ageing*. Cambridge University Press, Cambridge, 2005.

N. L. Johnson. Systems of Frequency Curves Generated by Methods of Translation. *Biometrika*, 36:149–176, 1949.

N. L. Johnson and S. Kotz. *Continuous Multivariate Distributions*. Wiley, 1972.

N. L. Johnson, S. Kotz, and N. Balakrishnan. *Continuous Univariate Distributions, Vol. 1.* Wiley, New York, 1994.

N. L. Johnson, S. Kotz, and N. Balakrishnan. *Continuous Univariate Distributions, Vol. 2.* Wiley, New York, 1995.

R. Kaas, A. Van Heerwaarden, and M. J. Goovaerts. Ordering of Actuarial Risks. Institute for Actuarial Science and Econometrics, University of Amsterdam, Amsterdam, 1994.

R. Kaas, J. Dhaene, and M. Goovaerts. Upper and Lower Bounds for Sums of Random Variables. *Insurance: Mathematics and Economics*, 27(2):151–168, 2000.

C. Kang and W. Kang. Exact Simulation of Wishart Multidimensional Stochastic Volatility Model. Working paper, 2013.

M. Karniychuk. Comparing Approximations for Risk Measures Related to Sums of Correlated Lognormal Random Variables. Master's thesis, Technische Universität Chemnitz, 2006.

K. Kawazu and S. Watanabe. Branching Processes with Immigration and Related Limit Theorems. *Theory of Probability and its Applications*, 16(1):36–54, 1971.

M. Keller-Ressel. Affine Processes, 2011. URL `https://www.math.tu-dresden.de/~mkeller/docs/affine_process_minicourse.pdf`.

M. Keller-Ressel and E. Mayerhofer. Exponential Moments of affine Processes. *The Annals of Applied Probability*, 25(2):714–752, 2015.

A. Kessler. Why the Longevity Transfer Market is Set to double? online, 2015. URL `http://pensionrisk.prudential.com/pdfs/pp-amy-kessler-article_2015-10-16.pdf`.

R. Klein. Life Insurance Securitization. Online, 2005. URL `https://www.pensions-institute.org/conferences/longevity/Klein_Ronnie.pdf`.

A. Kling, J. Rub, and K. Schilling. Risk Analysis of Annuity Conversion Options in a Stochastic Mortality Environment. *Astin Bulletin*, 44(2):197–236, 2014.

H. Kunreuther. The Role of Insurance in Managing Extreme Events: Terrorism. RISQUES, 2002.

M. Lane. Longevity Risk from the Perspective of the Ils Markets. *Geneva Papers Risk Insurance-Issues and Practice*, 36:501–515, 2011.

P.-S. Laplace. *Tome VII Théorie Analytiques des Probabilités*. Ouvres, 3rd Edition, 1820.

C. Lauschagne and T. Offwood. A Note on the Connection Between the EsscherGirsanov Transform and the Wang Transform. *Insurance: Mathematics and Economics*, 47:385–390, 2010.

R. D. Lee and L. R. Carter. Modeling and Forecasting U.s. Mortality. *Journalof the American Statistical Association*, 87:659–675, 1992.

Y. Lin and S. H. Cox. Securitization of Mortality Risks in Life Annuities. *Journal of Risk and Insurance*, 72:227–252, 2005.

Y. Lin and S. H. Cox. Securitization of Catastrophic Mortality Risks. *Insurance: Mathematics and Economics*, 42(2):628–637, 2008.

Y. Lin, S. Liu, and J. Yu. Pricing of Mortality Securities with Correlated Mortality Indices. *Insurance: Mathematics and Economics*, 80:921–948, 2013.

Y. Lin, T. Shi, and A. Arik. Pricing buy-ins and buy-outs. *The Journal of Risk and Insurance*, 84:367–392, 2017.

D. Linders, J. Dhaene, H. Hounnon, and M. Vanmaele. Index Options: A Model-free Approach. Research report, afi-1265 feb, Leuven: KU Leuven - Faculty of Business and Economics, 2012.

A. Linfoot. Financing Catastrophic Risk: Mortality Bond Case Study (Scottish Re), 2007. URL `http://www.actuaries.jp/lib/meeting/reikai18-4-siryo-en.pdf`.

X. Liu, J. Jang, and S. Kim. An Application of Comonotonicity Theory in a Stochastic Life Annuity Framework. *Insurance: Mathematics and Economics*, 48:271–279, 2011.

X. Liu, R. Mamon, and H. Gao. A Comonotonicity-based Valuation Method for Guaranteed Annuity Options. *Journal of Computational and Applied Mathematics*, 250:58–69, 2013.

X. Liu, R. Mamon, and H. Gao. A Generalized Pricing Framework Addressing Correlated Mortality and Interest Risks: a change of Probability Measure Approach. *Stochastics*, 86(4): 594–608, 2014.

Y. Liu and J. S.-H. Li. The Age Pattern of Transitory Mortality Jumps and its Impact on the Pricing of Catastrophic Mortality Bonds. *Insurance: Mathematics and Economics*, 64: 135–150, 2015a.

Y. Liu and J. S.-H. Li. The Age Pattern of Transitory Mortality Jumps and its Impact on the Pricing of Catastrophic Mortality Bonds. *Insurance: Mathematics and Economics*, 64: 135–150, 2015b. URL `http://dx.doi.org/10.1016/j.insmatheco.2015.05.005`.

J. Loeys, N. Panigirtzoglou, and R. M. Ribeiro. Longevity: A Market in the Making. London: J.P. Morgan Securities Ltd., 2007. URL `www.lifemetrics.com.`

B. Lucia, R. Sharan, , S. Barrett, and C. Honegger. Natural Catastrophes and Man-Made Disasters in 2016: A year of widespread damages. *in: K. Karl, ed. Sigma (Zurich: Swiss Re)*, 2, 2017. URL `http://media.swissre.com/documents/sigma2_2017_en.pdf`.

E. Luciano and E. Vigna. Non Mean Reverting Affine Processes for Stochastic Mortality. *Carlo Alberto Notebook 30/06 and ICER WP 4/05*, 2005.

E. Luciano and E. Vigna. Mortality Risk via Affine Stochastic Intensities:Calibration and Empirical Relevance. *Belgium Actuarial Bulletin*, 8:5–16, 2008.

E. Luciano, L. Regis, and E. Vigna. Delta-Gamma Hedging of Mortality and Interest Rate Risk. *Insurance: Mathematics and Economics*, 50(3):402–412, 2012.

L.-G. Luis. *Insuring Life: Value, Security and Risk*. Routledge, Oxon, 2016.

T. A. Maurer. Asset Pricing Implications of Demographic Change. *Working Paper*, 2014.

E. Mayerhofer. On the Existence of Non-Central Wishart Distributions. *Journal of Multivariate Analysis*, 114:448–456, 2013.

K. McKeever. A Short History of Tontines. *Fordham Journal of Corporate and Financial Law*, 15(2), 2009.

E. L. Melnick and B. S. Everitt. *Encyclopedia of Quantitative Risk Analysis and Assessment.* Wiley, 2008. Vol. 1-4.

J. Mènioux. Securitization of Life risks: The Cedant's Point of View. *In proceedings of the Workshop on Mortality and Longevity Risks*, 2008. [online] Available at: www.axa.com/lib/axa/uploads/GRM/PS2.Menioux Slides.pdf, 1 February 2008.

M. A. Milevsky and S. D. Promislow. Mortality Derivatives and the Option to Annuitize. *Insurance: Mathematics and Economics*, 29(3):299–318, 2001.

M. A. Milevsky, S. D. Promislow, and V. R. Young. Financial Valuation of Mortality Risk via the Instantaneous Sharpe Ratio: Applications to Pricing Pure Endowments. *Working Paper, Department of Mathematics,* Unversity of Michigan, 2005. [online] Available HTTP: `arxiv.org/abs/0705.1302`, [14/07/2014].

A. Milidonis, Y. Lin, and S. H. Cox. Mortality Regimes and Pricing. *North American Actuarial Journal*, 15:266–289, 2011.

K. Miltersen and S.-A. Persson. Is Mortality Dead? Stochastic Forward Force of Mortality Determined by No Arbitrage. *Working Paper, University of Bergen*, 2005.

B. Monarchy. Queen and Anniversary Messages: Facts and Figures. Online, 2013. [online] Available HTTP: www.royal.gov.uk/HMTheQueen/Queenandanniversarymessages/Factsandfigures.aspx.

R. J. Muirhead. *Aspects of Multivariate Statistical Theory.* Wiley, 2005.

O. f. National Statistics. What are the Chances of Surviving to Age 100? Online, 2012. URL `www.ons.gov.uk/ons/dcp171776_260525.pdf.`

O. f. National Statistics. What are the Chances of Surviving to Age 100? Online, 2016. URL `HTTP://visual.ons.gov.uk/what-are-your-chances-of-living-to-100/`.

O. f. National Statistics. Quarterly Mortality Reports, England: Data up to December 2016. Newport: Office for National Statistics, 2017, 2017. [online] Available HTTP://www.ons.gov.uk/releases/quarterlymortalityreportsdatauptodec2016.

E. Nicolini. Mortality, Interest Rates, Investment, and Agricultural Production in 18th Century England. *Journal of Financial and Quantitative Analysis*, 41(2):130–155, 2004.

C. O'Brien. Guaranteed Annuity Options: Five Issues for Resolution. CRIS Discussion Paper Series 8, 2002.

OECD. *Mortality Assumptions and Longevity Risk: Implications for Pension Funds and Annuity Providers.* OECD Publishing, 2014.

J. Oeppen and J. W. Vaupel. Broken Limits to Life Expectancy. *Science*, 296(5570):1029–1031, 2002.

A. Olivieri. Uncertainty in Mortality Projections: an Actuarial Perspective. *Insurance: Mathematics and Economics*, 29:231–245, 2001.

S. Olshansky, D. Passaro, R. Hershow, J. Layden, B. Carnes, J. Brody, L. Hayflick, R. Butler, D. Allison, and D. Ludwig. A Potential Decline in Life Expectancy in the United States in the 21st Century. *New England Journal of Medicine*, 352(11):11381145, 2005.

PartnerRe. A Balanced Discussion on Insurance Linked Securities. Permbroke: PartnerRe, 2008.

A. Pelsser. Pricing and Hedging Guaranteed Annuity Options via Static Option. *Insurance: Mathematics and Economics*, 33:283–296, 2003.

A. Pelsser. On the Applicability of the Wang Transform for Pricing Financial Risks. *ASTIN Bulletin*, 38:171–181, 2008.

O. Pfaffel. Wishart Processes. *Working Paper, Arxiv*, 2012. URL `arXiv:1201.3256v1[math.PR]`.

G. Piscopo and S. Haberman. The Valuation of Guaranteed Lifelong Withdrawal Benefit Options in Variable Annuity Contracts and the Impact of Mortality Risk. *North American Actuarial Journal*, 15(1):59–76, 2012.

P. Protter. *Stochastic Integration and Differential Equations, 2nd ed.* Springer-Verlag, 1990.

QIS5. Quantitative Impact Study 5:Technical Specifications. European Commission Internal Market and Services:Financial Institutions, Insurance and Pensions, Brussels, 2010. URL `http://www.ceiops.eu/index.php?option=content\&task=view\&id=732`.

G. E. Rejda. *Principles of Risk Management and Insurance,* Ninth Ed. Addison-Wesley, New York, 2005.

A. Renshaw and S. Haberman. A Cohort Based Extension to the Lee-Carter model for Mortality Reduction Factors. *Insurance: Mathematics and Economics*, 38:556–570, 2006.

A. E. Renshaw and S. Haberman. Lee-Carter Mortality Forecasting with Age-Specific Enhancement. *Insurance: Mathematics and Economics*, 33(2):255–272, 2003.

A. E. Renshaw, S. Haberman, and P. Hatzoupoulos. The Modeling of Recent Mortality Trends in United Kingdom Male Assured Lives. *British Actuarial Journal*, 2(11):449–477, 1996.

D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion, volume 293 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

S. Richards and J. Jones. Financial Aspects of Longevity Risk. *Staple Inn Actuarial Society, London*, 2004.

L. C. G. Rogers and Z. Shi. The Value of an Asian Option. *Journal of Applied Probability*, 32 (4):1077–1088, 1995.

S. Rooney. Extreme Mortality Bonds. Paper presented at the Securities Industry and Financial Markets Association Insurance & Risk-Linked Securities Conference, New York, 2008.

M. Rubinstein. Exotic Options (with Eric Reiner). *Research Program in Finance Working Papers* "RPF-220", University of California, Berkeley, 1992.

K.-I. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge, 1999.

D. Schrager. Affine Stochastic Mortality. *Insurance: Mathematics and Economics*, 38:81–97, 2006.

V. Scotti and D. Effenberger. Annuities - A Private Solution to Longevity Risk. *Sigma*, 3, 2007. URL `http://www.swissre.com/resources/38ac420048e52fb78327af983ae020a8sigma3_2007_e_rev.pdf`.

M. Shaked and J. G. Shanthikumar. *Stochastic Orders*. Springer Series in Statistics. Springer, New York, 2007.

Q. Shang, X. Qin, and Y. Wang. Design of Catastrophe Mortality Bonds based on the Comonotonicity Theory and Jump-Difusion Process. *International Journal of Innovative Computing Information and Control*, 5(4):991–1000, 2009.

Z. Shang, M. J. Goovaerts, and J. Dhaene. A Recursive Approach to Mortality-linked Derivative Pricing. *Insurance: Mathematics and Economics*, 49:240–248, 2011.

S. Simon, M. J. Goovaerts, and J. Dhaene. An Easy Computable Upper Bound for the Price of an Arithmetic Asian Option. *Insurance: Mathematics and Economics*, 26:175–184, 2000.

L. Simonsen, M. J. Clarke, G. D. Williamson, D. F. Stroup, N. H. Arden, and L. B. Schonberger. The Impact of Influenza Epidemics on Mortality: Introducing a Severity Index. *American Journal of Public Health*, 87(12):1944–1950, 1997.

T. Z. Sithole, S. Haberman, and R. J. Verrall. An Investigation into Parametric Models for Mortality Projections, with Applications to Immediate Annuitants and Life Office Pensioners Data. *Insurance: Mathematics and Economics*, 27:285–312, 2000.

J. F. Slifker and S. S. Shapiro. The Johnson System: Selection and Parameter Estimation. *Technometrics*, 22:239–246, 1980.

D. Soininen. Stochastic Variation of Interest and Mortality. Proceedings of the 5th AFIR International Colloquium, 1995. 871-903.

Standard and Poors. Insurance Linked Securities: Mortality Catastrophe Bonds. Global Credit Portal: Ratings Direct, 2008. URL `https://www.globalcreditportal.com/ratingsdirect/`.

Standard and Poors. Insurance Linked Securities, 2011. URL `http://www.standardandpoors.com/ratings/ils/en/us`. (Accessed April 25, 2017).

A. Stracke and W. Heinen. Influenza Pandemic: The impact on an Insured Lives Life Insurance Portfolio. *The Actuary June*, 2006.

P. J. Sweeting. A Trend Change Extension of the Cairns Blake Dowd Model. *Annals of Actuarial Science*, 5:143–162, 2011.

K. S. Tan, D. Blake, and R. MacMinn. Longevity Risk and Capital Markets: the 2013 - 14 Update. *Insurance: Mathematics and Economics*, 63:1–11, 2015.

J. Taubenberger and D. Morens. Pandemic Influenza - Including a Risk Assessment of H5N1. *Revue Scientifique et Technique*, 28(1):187–202, 2009.

J. Toole. Potential Impact of Pandemic Influenza on the Us Life Insurance Industry. Research Report, Society of Actuaries, 2007.

J. T. Tsai and L. Y. Tzeng. Pricing of Mortality-linked Contigent Claims: an Equilibrium Approach. *Astin Bulletin*, 43(2):97–121, 2013.

O. Vasicek. An Equilibrium Characterization of the Term Structure. *Journal of Financial Economics*, 5:177–188, 1977.

J. W. Vaupel. Biodemography of Human Ageing. *Nature*, 464:536–542, 2010.

J. W. Vaupel and H. Lundström. *Studies in the Economics of Aging* Ed. Wise, D. A., chapter Longer Life Expectancy? Evidence from Sweden of Reductions in Mortality Rates at Advanced Ages. University of Chicago Press, 1994.

J. W. Vaupel, K. G. Manton, and E. Stallard. The Impact of Heterogeneity in Individual Frailty on the Dynamics of Mortality. *Demography*, 16:439–454, 1979.

A. Villegas. *Mortality: Modelling, Socio-Economic Differences and Basis Risk*. PhD thesis, City University London, 2015.

J. Vinals. The Financial Impact of Longevity Risk. Chapter 4 in Global Financial Stability Report, 2012. URL `https://www.imf.org/en/Publications/GFSR/Issues/2016/12/31/Global-Financial-Stability-Report-April-2012-The-Quest-for-Lasting-Stability-25343`.

L. Vitiello and S.-H. Poon. General Equilibrium and Preference Free Model for Pricing Options under Transformed Gamma Distribution. *Journal of Future Markets*, 30(5):409–431, 2010.

D. Vynke. *Comonotonicity: the Perfect Dependence. The Concept of Comonotonicity in Actuarial Science and Finance.* PhD thesis, Katholieke Universitiet Leuven, 2003.

C. W. Wang, H. C. Huang, and I. C. Liu. A Quantitative Comparison of the Lee-Carter Model under Dierent Types of Non-Gaussian Innovations. *Geneva Papers Risk Insurance-Issues and Practice*, 36(4):675–696, 2011.

C. W. Wang, H. C. Huang, and I. C. Liu. Modeling Multi-Country Mortality Dependence and its Application in Pricing Survivor Index Swaps - A Dynamic Factor Copula Approach. *Insurance: Mathematics and Economics*, 63:30–39, 2015.

S. S. Wang. A Class of Distortion Operators for Pricing Financial and Insurance Risks. *Journal of Risk Insurance*, 67(1):15–36, 2000.

S. S. Wang. A Universal Framework for Pricing Financial and Insurance Risks. *Astin Bulletin*, 32(2):213–234, 2002.

Z. Wang and J. S.-H. Li. A DCC-GARCH Multi-Population Mortality Model and its Applications to Pricing Catastrophic Mortality Bonds. *Finance Research Letters*, 16:103–111, 2016.

R. G. Webster and E. J. Walker. Influenza. *American Scientist*, 91(2):122–129, 2009.

D. R. Weir. Tontines, Public Finance, and Revolution in France and England 1688-1789. *The Journal of Economic History*, 49:95–124, 1989.

S. Weisbart. Can the Life Insurance Industry Survive the Avian Flu? New York: Insurance Information Institute, 2006.

C. Weistroffer, B. Speyer, and S. Kaiser. Insurance Linked Securities Deutsche Bank Research. Frankfurt: Deutsche Bank, 2010.

Wikipedia. Catastrophe Bond, 2011. URL `https://en.wikipedia.org/wiki/Catastrophe_bond`. (Accessed April 25, 2017).

A. Wilkie, H. Waters, and S. Yang. Reserving, Pricing and Hedging for Policies with Guaranteed Annuity Options. *British Actuarial Journal*, 9:263–425, 2003.

G. Woo. A Catastrophe Bond Niche: Multiple Event Risk. Risk Management Solutions Ltd. NBER Insurance Workshop, 2004.

L. Yanxin and J. S.-H. Li. The Locally Linear Cairns - Blake - Dowd Model: A Note on Delta Nuga Hedging of Longevity Risk. *Astin Bulletin*, 47(1):79–151, 2017.

R. Zhou and J. S.-H. Li. A Cautionary Note on Pricing Longevity Index Swaps. *ASTIN Bulletin*, 2013(1):1–23, 2013.

R. Zhou, J. S.-H. Li, and K. S. Tan. Economic Pricing of Mortality-Linked Securities: A Tâtonnement Approach. *Journal of Risk and Insurance*, 2013a.

R. Zhou, J. S.-H. Li, and K. S. Tan. Pricing Mortality Risk Securities: A Two Population Model with Transitory Jump Effects. *Journal of Risk and Insurance*, 80:733–774, 2013b.