Nonisotropic Operators Arising in the Method of Rotations

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Doctor of Philosophy
University of Edinburgh
2007

For Miho, my greatest discovery

Abstract

This thesis is concerned with the mapping properties of the related objects,

$$\begin{aligned} \mathcal{M}f(x,\omega) &:= \sup_{h>0} h^{-1} \left| \int_{(0,h)} f(x-\delta_t\omega)\,dt \right|, \\ \mathcal{M}_\Gamma f(x) &:= \sup_{h>0} h^{-1} \left| \int_{(0,h)} f(x-\Gamma(t))\,dt \right|, \end{aligned}$$

and their associated singular integral operators, H and H_Γ respectively. Here, $\delta_t := \exp((\log t)P)$ and P is a real d by d matrix whose eigenvalues have positive real part, and $\Gamma : \mathbb{R} \to \mathbb{R}^d$ parameterises a curve.

For p in $(1, \max(2, (d+1)/2)]$, we prove that $\mathcal{M}$ maps L^p to $L^p(L^q)$ for an optimal range of q (modulo an endpoint). For H, the same optimality is achieved for p in $(1, 2]$.

If $\Gamma(t) = (t, P(\gamma(t)))$, where P is a real polynomial and γ is a convex function, then we give sufficient conditions in order for $\mathcal{M}_\Gamma$ and H_Γ to be bounded on L^p, for all p in $(1, \infty)$, with bounds independent of the coefficients of P. We also consider when these operators map $L \log L$ to weak L^1 locally. The same conclusions are shown to hold for the corresponding hypersurface in $\mathbb{R}^{d+1}$ $(d \geq 2)$ under weaker hypotheses on Γ.

We give sufficient conditions on a convex curve Γ in $\mathbb{R}^d$ $(d \geq 2)$ in order for $\mathcal{M}_\Gamma$ and H_Γ to map $L \log L$ to weak L^1 locally. Finally, it is shown that if Γ is a piecewise linear version of a parabola then the best one can expect, in terms of Orlicz spaces locally near L^1, is that $\mathcal{M}_\Gamma$ maps $L(\log L)^{1/2}$ to $L^{1,\infty}$.

Acknowledgements

First and foremost, I express my deepest thanks to my supervisor Jim Wright for his guidance and encouragement throughout the past three years. I couldn't have wished for more. To my room 4607 compatriots past and present: Mike, Daniel, Stefán, Spyros, and Sanjay. To my other friends in Edinburgh; especially to Dom with whom I have shared so much. To my family, who continue to bless me with so much love and support. And finally, the financial support of EPSRC and the Dixie Educational Foundation is gratefully acknowledged.

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Preliminaries

L^p and $L^p(L^q)$ spaces

For a fixed measure space (X,μ) and $p \in [1,\infty]$ we introduce the familiar $L^p(X)$ space as those measurable functions $f : X \to \mathbb{C}$ such that $\|f\|_p$ is finite, where

$$\|f\|_p \;:=\; \left(\int_X |f|^p \, d\mu\right)^{1/p},$$

with the agreement that when p is ∞ we interpret the above expression as the essential supremum of f on X. When there is no danger of confusion, we simply write L^p for $L^p(X)$. Each $p \in [1,\infty]$ has a dual exponent, denoted by p', which satisfies $1/p + 1/p' = 1$.

We also define a specific class of *mixed-norm* spaces for functions defined on measure spaces with a product structure. In particular, for $p, q \in [1,\infty]$ we denote by $L^p(L^q)$ the space of measurable functions $f : \mathbb{R}^d \times S^{d-1} \to \mathbb{C}$ such that $\|f\|_{L^p(L^q)}$ is finite, where

$$\|f\|_{L^p(L^q)} \;:=\; \left(\int_{\mathbb{R}^d} \left(\int_{S^{d-1}} |f(x,\omega)|^q \, d\omega\right)^{p/q} dx\right)^{1/p}.$$

Here, $d\omega$ and dx are the natural Lebesgue measures on S^{d-1} and $\mathbb{R}^d$ respectively, and as above, we can interpret this expression appropriately when either exponent p or q is ∞. Mixed-norm spaces of this type were first introduced by Benedek and Panzone in [2] with greater generality. The above setting is sufficient for our purposes.

Lorentz spaces

Fix a measure space (X,μ). If $f : X \to \mathbb{C}$ is measurable, define the *decreasing rearrangement* of f by

$$f^*(t) = \inf\{\lambda \in (0,\infty) : \mu(\{x \in X : |f(x)| > \lambda\}) \le t\}.$$

Define $L^{p,q}(X)$ to be the space of measurable functions f on X such that $\|f\|_{p,q}$ is finite, where

$$\begin{aligned} \|f\|_{p,q} &:= \left(\frac{q}{p}\int_{(0,\infty)} (t^{1/p}f^*(t))^q\frac{dt}{t}\right)^{1/q} \quad \text{for } p,q\in[1,\infty), \text{ and} \\ \|f\|_{p,\infty} &:= \sup_{t\in(0,\infty)} t^{1/p}f^*(t) \quad \text{for } p\in[1,\infty]. \end{aligned}$$

These spaces were introduced by Lorentz in [39] and [40]. The quantity $\|\cdot\|_{p,q}$ satisfies the triangle inequality only when $1\le q\le p<\infty$, or $p=q=\infty$. Despite this, the spaces arising when $q>p$ will be of most interest to us. Observe that, if $q_1\le q_2$ then $L^{p,q_1}\subseteq L^{p,q_2}$, and also that $L^{p,p}$ coincides with L^p for each $p\in[1,\infty]$, with equality of norms.

Orlicz spaces

Let $Q\subseteq\mathbb{R}^k$ be a fixed unit cube. Suppose $\Phi:[0,\infty)\to[0,\infty)$ is nondecreasing and convex with $\Phi(0)=0$. We define the *Luxemburg norm* of a measurable function f by,

$$\|f\|_{\Phi(L)(Q)} := \inf\left\{\alpha>0 : \int_Q \Phi(|f(x)|/\alpha)\,dx\le 1\right\}. \tag{1}$$

Then we define the corresponding *Orlicz space*, $\Phi(L)(Q)$, as those measurable functions $f:Q\to\mathbb{C}$ such that the norm in (1) is finite. Such spaces were introduced by Orlicz in [53] and the norm in (1) appeared in [41]. One thinks of Orlicz spaces as generalisations of the more widely known $L^p(Q)$ spaces. Indeed, if $\Phi(t)=t^p$ for $p\in[1,\infty)$ then it is easy to check that the norm defined in (1) coincides with the $L^p(Q)$ norm of f. It turns out that Orlicz spaces are complete spaces. For more details on the rich theory of these spaces, we refer the reader to [38].

Notice that we have now introduced two generalisations of the classical L^p spaces; Orlicz spaces and Lorentz spaces. The theory of a further generalisation of these spaces, *Orlicz-Lorentz spaces*, has also emerged (see, for example, [42], [44], [33], and [47]).

Some universal notation

A function $\psi:\mathbb{R}^d\to\mathbb{C}$ belongs to the *Schwartz class*, $\mathcal{S}(\mathbb{R}^d)$, if ψ is infinitely differentiable and, for all $\alpha,\beta\in\mathbb{N}^d$,

$$p_{\alpha,\beta}(\psi) := \sup_{x\in\mathbb{R}^d} |x^\alpha D^\beta\psi(x)| < \infty.$$

The Schwartz class is a Fréchet space, dense in $L^p(\mathbb{R}^d)$ for all $p \in [1,\infty)$ under the following topology given by the seminorms $p_{\alpha,\beta}$: a sequence $(\psi_k)_{k\geq 1}$ converges to the zero function if and only if $p_{\alpha,\beta}(\psi_k)$ tends to zero as k tends to infinity, for all $\alpha, \beta \in \mathbb{N}^d$. The space of *tempered distributions*, $\mathcal{S}'(\mathbb{R}^d)$, is the space of bounded linear functionals on $\mathcal{S}(\mathbb{R}^d)$. The action of a tempered distribution μ on an element of ψ of $\mathcal{S}(\mathbb{R}^d)$ will be denoted by $\langle \mu, \psi \rangle$.

Adopting the notation $x.y$ for the standard inner product of elements x and y in $\mathbb{R}^d$, the Fourier transform of a finite Borel measure μ on $\mathbb{R}^d$ will be defined by

$$\widehat{\mu}(\xi) := \int_{\mathbb{R}^d} e^{ix.\xi}\, d\mu(x). \tag{2}$$

We shall often require the use of Euclidean balls in $\mathbb{R}^d$; that is, the open balls defined by Euclidean distance in $\mathbb{R}^d$. The Euclidean ball of radius r and centre x in $\mathbb{R}^d$ will be denoted by $B_r(x)$. In Chapter 2 and Chapter 4, the reader should also be ready to meet balls in $\mathbb{R}^d$ defined by certain nonisotropic distance functions. The notation will be made clear at the appropriate moment.

If E is a subset of $\mathbb{R}^d$, we shall use $|E|$ to denote either the Lebesgue measure of E or the number of elements in E. There should be no confusion caused by this. Let χ_E denote the characteristic function of E.

For positive numbers A and B, we frequently employ the notation $A \lesssim B$ to dissolve constants, and this notation will be defined in each chapter separately. Automatically, $B \gtrsim A$ means $A \lesssim B$, and $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. Any dependence in a constant that we wish to emphasise will be done so via subscripts or parentheses.

A toolbox

- *(Plancherel's theorem)* Up to an absolute constant, the mapping $f \mapsto \widehat{f}$ is an isometry on $L^2(\mathbb{R}^d)$. To see how to make sense of the Fourier transform defined in (2) for L^p functions, $p \in (1,2]$, see [24].

- *(van der Corput's lemma)* Suppose $\theta : (a,b) \to \mathbb{R}$ and $\psi : (a,b) \to \mathbb{R}$ are smooth, and that $|\theta^{(k)}(t)| \geq 1$ for all $t \in (a,b)$. Then

$$\left| \int_a^b e^{i\lambda\theta(t)} \psi(t)\, dt \right| \leq C(k) \left(|\psi(b)| + \int_a^b |\psi'(t)|\, dt \right) \lambda^{-1/k} \tag{3}$$

  holds when

  1. $k \geq 2$, or
  2. $k = 1$ and θ' is monotonic,

and the constant $C(k)$ is independent of θ and λ. This estimate is due to van der Corput; a proof can be found in [60]. One should bear in mind that, using a simple integration by parts argument, it suffices to show (3) when $\psi = 1$.

- *(Minkowski's inequality)* Let (X, μ) and (Y, ν) be σ-finite measure spaces. Then, for all $p \in [1, \infty)$,

$$\left(\int_X \left| \int_Y f(x,y)\, d\nu(y) \right|^p d\mu(x) \right)^{1/p} \leq \int_Y \left(\int_X |f(x,y)|^p\, d\mu(x) \right)^{1/p} d\nu(y).$$

- *(Hölder's inequality)* Let (X, μ) be a σ-finite measure space. Then, for all $p \in [1, \infty]$,

$$\left| \int_X f(x) g(x)\, d\mu(x) \right| \leq \left(\int_X |f(x)|^p\, d\mu(x) \right)^{1/p} \left(\int_X |g(x)|^{p'}\, d\mu(x) \right)^{1/p'}.$$

- *(Sobolev spaces and Sobolev embedding)* Let ν be a real number. If u is a distribution on S^{d-1}, we shall say that $u \in L^2_\nu(S^{d-1})$ if, for any coordinate patch $U \subseteq S^{d-1}$ and any $\psi \in C_0^\infty(U)$, the distribution ψu belongs to $L^2_\nu(U)$, where U is identified with its image in $\mathbb{R}^{d-1}$. For a definition of the more familiar Sobolev spaces on Euclidean spaces see, for example, [65]. If $\nu = 1$ we can equivalently define $L^2_1(S^{d-1})$ to be the set of all $u \in L^2(S^{d-1})$ such that for any smooth vector field $\mathfrak{X}$ on S^{d-1}, $\mathfrak{X}u \in L^2(S^{d-1})$.

  If ν is a real number such that $2 < (d-1)/\nu$, then the identity mapping from $L^2_\nu(S^{d-1})$ to $L^q(S^{d-1})$ is continuous, if $1/q = 1/2 - \nu/(d-1)$. This result is a particular case of a general theory of Sobolev spaces on manifolds. See [65] for more details.

- *(Interpolation)* Suppose T is a linear operator such that for $i \in \{0, 1\}$, we have

$$\|Tf\|_{L^{p_i}(L^{q_i})} \leq C_i \|f\|_{p_i} \quad \text{for each } f \in L^{p_i}(\mathbb{R}^d),$$

  where $p_i \leq q_i$. It follows that for $\theta \in [0, 1]$,

$$\|Tf\|_{L^p(L^q)} \leq C_0^{1-\theta} C_1^\theta \|f\|_p \quad \text{for each } f \in L^p(\mathbb{R}^d),$$

  where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

  This interpolation theorem was proved in greater generality in [2] using complex interpolation methods. One may also deduce this result using real interpolation and the Peetre K-functional; see [3] for a full treatment on this method. For a fleeting glimpse at the main point in the real interpolation, we have that

$$(L^{p_0}(\mathbb{R}^d), L^{p_1}(\mathbb{R}^d))_{\theta,p} = L^p(\mathbb{R}^d),$$

and

$$(L^{p_0}(L^{q_0}), L^{p_1}(L^{q_1}))_{\theta,p} \subseteq L^p(L^q), \quad \text{since } p_i \leq q_i \text{ for each } i.$$

The notation $(\cdot,\cdot)_{\theta,p}$ we have just used for the intermediate spaces can be found, for example, in [3].

We shall also utilise interpolation theory on Sobolev spaces. Our results follow from complex interpolation methods; in particular from the fact that

$$[L^2(S^{d-1}), L^2_1(S^{d-1})]_\nu = L^2_\nu(S^{d-1}) \quad \text{for each } \nu \in [0,1].$$

We refer the reader to [65] for further details, including the definition of the notation $[\cdot,\cdot]_\nu$ for the intermediate spaces.

Chapter 1

Background and Introduction.

In this thesis we shall be concerned with the boundedness, or mapping properties, of various singular integral and maximal operators. Rather than out of the blue definitions, this preamble is intended to show how our operators arise in a very natural manner from certain classes of partial differential equations via the method of rotations. Those in the know may prefer to move straight to Section 1.1 and Section 1.2.

The isotropic case

Throughout this thesis we shall refer to the dilations $x \mapsto (tx_1, \dots, tx_d)$ on $\mathbb{R}^d$, for $t \in (0, \infty)$, as *isotropic* dilations.

Constant coefficients

Let P be a polynomial on $\mathbb{R}^d$ which is homogeneous of degree n with respect to isotropic dilations; that is to say, $P(\xi) = \sum_{|\alpha|=n} p_\alpha \xi^\alpha$ for some real coefficients p_α and $\xi \in \mathbb{R}^d$. Suppose further that the the polynomial P only vanishes at the origin so that the differential operator $P(D)$ is elliptic. Define the operator Λ by

$$\widehat{\Lambda f}(\xi) := |\xi| \widehat{f}(\xi), \tag{1.1}$$

and the operator T_P by

$$\widehat{T_P f}(\xi) := (-i)^n \frac{P(\xi)}{|\xi|^n} \widehat{f}(\xi).$$

It follows from,

$$\widehat{P(D) f}(\xi) = P(-i\xi) \widehat{f}(\xi),$$

that we can express the differential operator $P(D)$ as

$$P(D) f = T_P(\Lambda^n f). \tag{1.2}$$

The operator Λ is well understood since $-\Lambda^2$ is the Laplacian operator. It is clear that $\xi \mapsto (-i)^n P(\xi)/|\xi|^n$ is homogeneous of degree zero with respect to isotropic dilations, and furthermore, belongs to $C^\infty(\mathbb{R}^d \setminus \{0\})$ from our assumption that $P(D)$ is elliptic. For f belonging to $\mathcal{S}(\mathbb{R}^d)$, it is possible to show that

$$T_P f(x) = af(x) + p.v. \int_{\mathbb{R}^d} K(y) f(x-y)\, dy,$$

where $a \in \mathbb{C}$, and K satisfies the following conditions:

(K1). K is homogeneous of degree $-d$ with respect to isotropic dilations;

(K2). $\int_{S^{d-1}} K(\omega)\, d\omega = 0$;

(K3). K belongs to $C^\infty(\mathbb{R}^d \setminus \{0\})$.

A proof of this fact may be found in [24]. Thus, we have effectively reduced the study of T_P to the study of the operator

$$Tf(x) := p.v. \int_{\mathbb{R}^d} K(y) f(x-y)\, dy, \tag{1.3}$$

where K satisfies (K1), (K2), and (K3). To ensure T is well-defined, we initially restrict f to $\mathcal{S}(\mathbb{R}^d)$. These operators go back to work of Mihlin in [45] and the now classical work of Calderón and Zygmund in [6]. It follows from [6] that T extends to a bounded operator on L^p for each $p \in (1, \infty)$. A point we wish to emphasise here is that one can prove this fact by considering the following associated maximal function of Hardy-Littlewood type,

$$M_{HL} f(x) := \sup_{h>0} |B_h(0)|^{-1} \left| \int_{y \in B_h(0)} f(x-y)\, dy \right|. \tag{1.4}$$

It is known that M_{HL} satisfies the following key distributional estimate: There exist $C < \infty$ such that

$$|\{x \in \mathbb{R}^d : M_{HL} f(x) > \alpha\}| \leq C\alpha^{-1} \|f\|_1 \quad \text{for all } \alpha > 0. \tag{1.5}$$

This fact was proved by Hardy and Littlewood [32] when $d = 1$, and for $d > 1$ by Wiener [67] and Marcinkiewicz and Zygmund [43]. Moreover, one can prove that there exists $C < \infty$ such that

$$|\{x \in \mathbb{R}^d : |Tf(x)| > \alpha\}| \leq C|\{x \in \mathbb{R}^d : M_{HL} f(x) > \alpha\}| + C\alpha^{-2} \int_{|f(x)| \leq \alpha} |f(x)|^2\, dx, \tag{1.6}$$

and therefore we can use (1.5) to deduce the same result for T. The result in (1.6) is the fruit of the much celebrated Calderón-Zygmund theory; a proof is implicitly written in [61]. The boundedness of T on L^p for $p \in (1,\infty)$ now follows by simple arguments involving interpolation and duality.

This is an example of a general expectation that, despite no formal link, the behaviour of a singular integral operator will be determined by the behaviour of the associated maximal operator. If this were not sufficient motivation for the study of maximal operators, one may be further persuaded by their direct connection to pointwise convergence results of the form,

$$\lim_{h\to 0} |\Gamma(x,h)|^{-1} \int_{\Gamma(x,h)} f(y)\,dy = f(x) \quad \text{for almost all } x \in \mathbb{R}^d, \tag{1.7}$$

where $\{\Gamma(x,h) : x \in \mathbb{R}^d, h \in (0,\infty)\}$ are measurable subsets of $\mathbb{R}^d$ (with respect to the appropriate Lebesgue measure) and f belongs to a certain class of functions. The case where $\Gamma(x,h)$ is the Euclidean ball $B_h(x)$ is the classical Lebesgue differentiation theorem and (1.7) holds for all $f \in L^1(\mathbb{R}^d)$. Moreover, the distributional estimate (1.5) is known to be equivalent to (1.7). The problem becomes significantly more difficult when $\Gamma(x,h)$ are lower dimensional subsets of $\mathbb{R}^d$. For example, it is unknown whether (1.7) is true for functions in L^1 if $\Gamma(x,h)$ is a piece of parabola of length h emanating from x, or, if h is restricted to a dyadic subsequence of $(0,\infty)$, the boundary of $B_h(x)$. We return to this matter in Section 1.1.

Let us now demonstrate an alternative proof that the operator in (1.3) is bounded on L^p, for all $p \in (1,\infty)$, if K satisfies (K1),

(K2'). K is an odd function;

(K3'). $\int_{S^{d-1}} |K(\omega)|\,d\omega < \infty$.

Here, (K2') is a stronger cancellation condition than (K2), and (K3') is a weaker smoothness condition than (K3). If $f \in \mathcal{S}(\mathbb{R}^d)$ then, by changing variables to polar coordinates and using the oddness of K,

$$2Tf(x) = \lim_{\varepsilon\to 0} \int_{S^{d-1}} K(\omega) \left(\int_{|r|\in(\varepsilon,\infty)} f(x - r\omega)\frac{dr}{r} \right) d\omega.$$

Because of condition (1) and the smoothness of f, we can use the dominated convergence theorem to deduce that

$$2Tf(x) = \int_{S^{d-1}} K(\omega) Hf(x,\omega)\,d\omega, \tag{1.8}$$

where

$$Hf(x,\omega) := p.v. \int_{\mathbb{R}} f(x - t\omega)\frac{dt}{t}. \tag{1.9}$$

For a fixed $\omega \in S^{d-1}$ this operator is essentially the classical one-dimensional Hilbert transform, H_1, defined *a priori* by

$$H_1 f(s) := p.v. \int_{\mathbb{R}} f(s-t)\frac{dt}{t},$$

for f belonging to $\mathcal{S}(\mathbb{R})$. In particular, if we fix $p \in (1,\infty)$ and $\omega \in S^{d-1}$, and write each $x \in \mathbb{R}^d$ as

$$x = (x.\omega)\omega + (x - (x.\omega)\omega), \tag{1.10}$$

then

$$Hf(x,\omega) = H_1(f(\cdot\,\omega + (x - (x,\omega)\omega)))(x.\omega). \tag{1.11}$$

We can now use the famous theorem of M. Riesz that H_1 is bounded on L^p to see that

$$\begin{aligned}\int_{\mathbb{R}^d} |Hf(x,\omega)|^p \, dx &= \int_{y.\omega=0}\int_{\mathbb{R}} |H_1(f(\cdot\,\omega + y))(\lambda)|^p \, d\lambda dy \\ &\leq C(p)\int_{y.\omega=0}\int_{\mathbb{R}} |f(\cdot\,\omega + y)(\lambda)|^p \, d\lambda dy = C(p)\|f\|_p^p.\end{aligned}$$

Therefore

$$\|Hf(\cdot,\omega)\|_p \leq C(p)\|f\|_p. \tag{1.12}$$

Because the bound in (1.12) is independent of ω, it follows from (1.8), (3'), together with an application of Minkowski's inequality, that T is bounded on L^p. Passing from the expression (1.3) for T to (1.8) is an instance of the *method of rotations*. This approach was introduced by Calderón and Zygmund in [7]. With the aid of Riesz kernels, this method can be used to handle even kernels too.

Nonconstant coefficients

In more a general context, one is led to *variable kernel* singular integral operators of the form

$$Tf(x) := p.v. \int_{\mathbb{R}^d} K(x,y)f(x-y)\, dy, \tag{1.13}$$

where, for each x,

(K1). $K(x,\cdot)$ is homogeneous of degree $-d$ with respect to isotropic dilations;

(K2). $\int_{S^{d-1}} K(x,\omega)\, d\omega = 0$;

and some smoothness condition holds. As an example, one need look no further than a homogeneous polynomial differential operator with *nonconstant* coefficients; an argument akin to the constant coefficient case leads to operators like

(1.13). Observe that if we assume that $K(x,\cdot)$ is odd instead of (K2), then one can apply the method of rotations, and one is reunited with the operator H defined in (1.9). Furthermore, if one assumes for the smoothness condition that $\omega \mapsto \sup_{x\in\mathbb{R}^d} |K(x,\omega)|$ belongs to $L^1(S^{d-1})$, then one may deduce that T is bounded on L^p for all $p \in (1,\infty)$. We will return to the matter of the boundedness of T under weaker smoothness assumptions on K later in the thesis.

Nonisotropic case

Our motivation for this discussion will be the differential operator $P(D)$, where $P(\xi) := \xi_2 - \xi_1^2$; this operator essentially defines the two dimensional heat equation. It is easy to check that

$$S(P(D)f) = D^{(0,1)}f \quad \text{and} \quad T(P(D)f) = D^{(2,0)}f,$$

where

$$\widehat{Sf}(\xi) := \frac{-i\xi_2}{-i\xi_2 + \xi_1^2}\widehat{f}(\xi) \quad \text{and} \quad \widehat{Tf}(\xi) := \frac{-\xi_1^2}{-i\xi_2 + \xi_1^2}\widehat{f}(\xi). \tag{1.14}$$

One can quickly see that both of the multiplier functions which govern S and T in (1.14) are homogeneous of degree zero with respect to the *parabolic* dilations, $x \mapsto (tx_1, t^2x_2)$ on $\mathbb{R}^2$. By considering inverse Fourier transforms, we are thus led to operators of the form (1.3) where K is homogeneous of degree -3 with respect to parabolic dilations; that is, $K(tx_1, t^2x_2) = t^{-3}K(x_1,x_2)$ for each $t \in (0,\infty)$. It is at this point where we have reached a junction at which two directions of pursuit offer themselves. Both are initiated by the method of rotations, and the main body of work in this thesis splits into contributions along both paths.

To be more specific, let us fix a kernel K which is homogeneous of degree -3 with respect to parabolic dilations, and odd. Apply the change of variables $y_1 = t\cos\widetilde{\omega}$ and $y_2 = t^2\sin\widetilde{\omega}$, which are in the spirit of polar coordinates, but better suited to parabolic dilations. Then, the operator T defined by (1.3) may be written as

$$2Tf(x) = \int_{S^1} K(\omega)Hf(x,\omega)(1+\sin^2\widetilde{\omega})\,d\omega,$$

where $\omega := (\cos\widetilde{\omega}, \sin\widetilde{\omega})$,

$$Hf(x,\omega) := p.v. \int_{\mathbb{R}} f(x - \delta_t\omega)\frac{dt}{t}, \tag{1.15}$$

and

$$\delta_t = \begin{pmatrix} t & 0 \\ 0 & \operatorname{sgn}(t)t^2 \end{pmatrix} \quad \text{for } t \in \mathbb{R} \tag{1.16}$$

is our family of dilations. As with the similar looking object in (1.9), one can prove that the analogue of (1.12) holds. Like the isotropic case, if we know that

K belongs to $L^1(S^1)$, then Minkowski's inequality implies that T is bounded on L^p for each $p \in (1, \infty)$.

Remark. For fixed ω, the L^2 boundedness of the operator in (1.15) was originally proved in a thesis of Fabes. In [49], the L^p boundedness was proved for all $p \in (1, \infty)$. The parabolic analogue of (1.12) easily follows by scaling. However, the proof we gave for the isotropic case on page 10 does not work. There is no obvious reduction to a one-dimensional operator since the orthogonal decomposition in (1.10) has no obvious analogue. This issue reappears in Chapter 2 of this thesis.

The first turn at our junction is to consider the dilations in (1.16) as a prototype, then fix ω, and consider the corresponding operators defined in (1.15) as *Hilbert transforms along curves.* Such operators have generated considerable interest in the past thirty years, and we continue this road of discussion in more detail in Section 1.1.

The second route appears if one assumes that the kernel belongs to $L^{q'}(S^1)$ for some q' strictly greater than 1. More generally, suppose we are in the variable kernel case and we assume that there exists a constant $C(q') < \infty$ such that

$$\sup_{x \in \mathbb{R}^d} \int_{S^{d-1}} |K(x, \omega)|^{q'} \, d\omega \leq C(q').$$

An application of Hölder's inequality implies that,

$$\|Tf\|_p \leq C(q') \left(\int_{\mathbb{R}^d} \left(\int_{S^{d-1}} |Hf(x, \omega)|^q \, d\omega \right)^{p/q} dx \right)^{1/p}. \tag{1.17}$$

This begs the question: For what values of p is the mixed-norm quantity on the right hand side of (1.17) controlled by $\|f\|_p$. We discuss this further in Section 1.2.

1.1 Operators on curves

Given an integer $d \geq 2$ and a map $\Gamma : \mathbb{R} \to \mathbb{R}^d$ we define operators H_Γ and $\mathcal{M}_\Gamma$ by

$$H_\Gamma f(x) := p.v. \int_{\mathbb{R}} f(x - \Gamma(t)) \, \frac{dt}{t}, \tag{1.18}$$

$$\mathcal{M}_\Gamma f(x) := \sup_{h > 0} h^{-1} \left| \int_{(0,h)} f(x - \Gamma(t)) \, dt \right|, \tag{1.19}$$

for f belonging to $\mathcal{S}(\mathbb{R}^d)$. We shall refer to H_Γ as the (global) Hilbert transform along Γ and $\mathcal{M}_\Gamma$ as the (global) maximal operator along Γ. We also introduce local versions of these operators, H_Γ^{loc} and $\mathcal{M}_\Gamma^{loc}$, where the integral in (1.18) is restricted to $(-1,1)$ and the supremum in (1.19) is restricted to h in $(0,1)$. It will be convenient for us to work with the following dyadic form of $\mathcal{M}_\Gamma$:

$$M_\Gamma f(x) := \sup_{k\in\mathbb{Z}} \lambda^{-k} \left| \int_{(\lambda^k,\lambda^{k+1})} f(x-\Gamma(t))\,dt \right|, \quad \text{for a fixed } \lambda \in (1,\infty). \tag{1.20}$$

It is clear that there exists $C(\lambda) \in (0,\infty)$ such that $M_\Gamma f \leq C(\lambda)\mathcal{M}_\Gamma|f|$ and $\mathcal{M}_\Gamma f \leq C(\lambda) M_\Gamma |f|$. For our purposes this means $\mathcal{M}_\Gamma$ is equivalent to M_Γ. The local version is defined in the obvious way.

On L^p for $p \in (1,\infty)$

The question of interest here is the following: For which Γ and what range of p can say that either H_Γ or $\mathcal{M}_\Gamma$ (or the local versions) are bounded on L^p? Of course $\mathcal{M}_\Gamma$ is bounded on L^∞, and so we choose to omit this triviality from subsequent theorems on $\mathcal{M}_\Gamma$. We begin with the case that Γ is a polynomial curve in $\mathbb{R}^d$. The following theorem is well known.

Theorem 1.1.1. [60] *Let $\Gamma(t) = (P_1(t),\ldots,P_d(t))$, where $P_1,\ldots,P_d$ are real polynomials on $\mathbb{R}$. Then H_Γ and $\mathcal{M}_\Gamma$ are bounded on L^p for all $p \in (1,\infty)$, with bounds independent of the coefficients of $P_1,\ldots,P_d$.*

A somewhat related problem is the case when Γ is of finite type, that is to say $\{\Gamma^{(k)}(0) : k \geq 1\}$ spans $\mathbb{R}^d$.

Theorem 1.1.2. [62] *If Γ is of finite type then H_Γ^{loc} and $\mathcal{M}_\Gamma^{loc}$ are bounded on L^p for all $p \in (1,\infty)$.*

We may then ask what happens in the case that Γ is not of finite type. This brings us to the simplest case of this kind, where we have $d = 2$, $\Gamma(t) = (t,\gamma(t))$ and all of the derivatives of γ vanish at zero. One such (nonconvex) γ was constructed in [62] for which $\mathcal{M}_\Gamma^{loc}$ is unbounded on L^p for any $p \in (1,\infty)$. Despite this, positive results are possible for such curves when, in particular, we consider convex γ. If we restrict our attention to curves γ satisfying:

$$\gamma \in C^2(0,\infty), \text{ convex on } [0,\infty) \quad \text{and} \quad \gamma(0) = \gamma'(0) = 0, \tag{1.21}$$

and extend γ to a function on $\mathbb{R}$ by stipulating that it must be either even or odd, then the following notions naturally arise.

Definition 1.1.3. 1. A function $f : \mathbb{R} \to \mathbb{R}$ belongs to $\mathcal{C}_1$ if there exists $D \in (1,\infty)$ such that for each $t \in (0,\infty)$ we have $f(Dt) \geq 2f(t)$. Such an f is said to be doubling.

2. A differentiable function $f : \mathbb{R} \to \mathbb{R}$ belongs to $\mathcal{C}_2$ if there exists $\varepsilon_0 > 0$ such that for $t \in (0,\infty)$, $f'(t) \geq \varepsilon_0 f(t)/t$. Such an f is said to be infinitesimally doubling, and if f is nondecreasing on $(0,\infty)$ then $f \in \mathcal{C}_2$ implies $f \in \mathcal{C}_1$.

We shall also need the function h defined for $t \in (0,\infty)$ by $h(t) := t\gamma'(t) - \gamma(t)$. Notice that because γ is convex and $\gamma(0) = 0$ we get the important fact that

$$t\gamma'(t) \geq \gamma(t) \quad \text{for all } t \in (0,\infty) \tag{1.22}$$

(and hence h is nonnegative). We now state a series of known results in this setting.

Theorem 1.1.4. [13] *Suppose γ is even and satisfies (1.21), and $p \in (1,\infty)$. Then H_Γ is L^p bounded if and only if $\gamma' \in \mathcal{C}_1$.*

The L^2 result in Theorem 1.1.4 was proved earlier in [51]. In the context of L^p boundedness for $p \in (1,\infty)$, this is of course the end of the matter for H_Γ when γ is convex and even. In the odd case, the current situation is less satisfactory. We have:

Theorem 1.1.5. [51] *Suppose γ is odd and satisfies (1.21). Then H_Γ is L^2 bounded if and only if $h \in \mathcal{C}_1$.*

This theorem of course means that, for each $p \in (1,\infty)$, $h \in \mathcal{C}_1$ is a necessary condition for H to be L^p bounded. However, it was demonstrated in [9] that this condition is far from sufficient. There they construct a γ such that $h \in \mathcal{C}_1$ yet H_Γ is unbounded on L^p for any $p \in (1,\infty)$ not equal to 2. Some known sufficient conditions in the odd case are given in the following:

Theorem 1.1.6. *Suppose γ is odd and satisfies (1.21), and $p \in (1,\infty)$.*

1. [13] *If $\gamma' \in \mathcal{C}_1$ then H_Γ is L^p bounded.*

2. [9] *If $h \in \mathcal{C}_2$ then H_Γ is L^p bounded.*

For $\mathcal{M}_\Gamma$, a necessary and sufficient condition for L^p boundedness in geometric terms is not known. ,It was demonstrated in [64] (see also [58]) that a convex γ exists for which $\mathcal{M}_\Gamma$ is unbounded on L^p for all $p \in (1,\infty)$. There is however an analogue of Theorem 1.1.6:

Theorem 1.1.7. *Suppose γ satisfies (1.21) and $p \in (1,\infty)$.*

1. [13] *If $\gamma' \in \mathcal{C}_1$ then $\mathcal{M}_\Gamma$ is L^p bounded.*

2. [9] *If $h \in \mathcal{C}_2$ then $\mathcal{M}_\Gamma$ is L^p bounded.*

Remarks. 1. The case where a convex curve on $[0,\infty)$ is extended to be either even or odd is encompassed by the notion of a biconvex balanced curve given in [22]. There it is shown that if the derivative of such a curve satisfies a doubling condition then, for all $p \in (1,\infty)$, we get L^p boundedness of both H_Γ and $\mathcal{M}_\Gamma$ (and also the associated maximal Hilbert transform).

2. Suppose γ satisfies (1.21), and moreover, is infinitely differentiable. We shall say that the curve $(t,\gamma(t))$ is *flat* if all of the derivatives of γ vanish at zero. This may seem a little obvious, but our aim is to avoid any confusion with the following alternative candidate for the term 'flat': If the intervals $\{I_j\}_{j\in\mathcal{J}}$ are disjoint and have $(0,\infty)$ as their union, let $(t,\gamma(t))$ be a curve which is linear on each interval I_j. Such a curve has zero curvature on each piece, and for this reason, stakes a claim to be called flat. However, we shall call such curves *piecewise linear.* Observe that if $I_j = (2^j, 2^{j+1}]$ for each integer j, and $(t,\gamma(t))$ is the parabolic piecewise linear curve defined by $\gamma(2^j) = 2^{2j}$, then the class $\mathcal{C}_1$ admits the function γ' (with $D = 2$). Piecewise linear curves are the focus of attention in Chapter 5 of this thesis.

Motivated by the above theorems, our contribution will be to prove that both H_Γ and $\mathcal{M}_\Gamma$ are bounded on L^p, for all $p \in (1,\infty)$, along a class of *nonconvex* plane curves, Γ. We state and prove our theorem in Chapter 3.

Near L^1

For the curve, $\Gamma(t) = (t,t^2)$, it is clear that H_Γ and $\mathcal{M}_\Gamma$ are not bounded on L^1. A substantial open problem of particular interest to us is the following: Can we enlarge the target space to the Lorentz space $L^{1,\infty}$ and say that these operators are bounded from L^1 to $L^{1,\infty}$? An affirmative answer for the maximal operator would, for instance, imply that for each $f \in L^1(\mathbb{R}^2)$,

$$\lim_{h\to 0} h^{-1}\int_{(0,h)} f(x-(t,t^2))\,dt = f(x) \quad \text{for almost all } x \in \mathbb{R}^2. \tag{1.23}$$

It follows from [48] that (1.23) holds for $f \in L^p(\mathbb{R}^2)$ for each $p \in (1,\infty)$.

Before describing the significant progress for H_Γ and $\mathcal{M}_\Gamma$ along the parabola and near L^1, we set the scene a little. Of interest to us will be Orlicz spaces defined by the family of functions

$$\Phi(t) = \Phi_{i,\sigma}(t) = t(\log^{(i)}(t+100))^\sigma \quad \text{for } i \in \{1,2\},\ \sigma \in [0,\infty), \tag{1.24}$$

where $\log^{(i)}$ denotes the composition of log with itself i times. If $\sigma < \sigma'$ then, for each $\varepsilon > 0$, we have the following chain of inclusions,

$$L^{1+\varepsilon}(Q) \subsetneq \Phi_{1,\sigma'}(L)(Q) \subsetneq \Phi_{1,\sigma}(L)(Q) \subsetneq \Phi_{2,\sigma'}(L)(Q) \subsetneq \Phi_{2,\sigma}(L)(Q) \subsetneq L^1(Q),$$

where Q is some unit cube in $\mathbb{R}^d$. This fact is a consequence of a general result which essentially says that distinct functions give rise to distinct Orlicz spaces. For the precise form of this result, see [38].

Definition 1.1.8. Let Φ be a function belonging to the family in (1.24). Let T be either H_Γ or $\mathcal{M}_\Gamma$ (or their local versions). We shall say that T *is of weak type* $L(\log^{(i)}(L))^\sigma$ if there is a constant C so that the inequality

$$|\{x \in \mathbb{R}^d : |Tf(x)| > \alpha\}| \leq \int_{\mathbb{R}^d} \Phi\left(\frac{C|f(x)|}{\alpha}\right) dx \tag{1.25}$$

holds for all positive α.

Remarks. 1. Suppose Q is a unit cube in $\mathbb{R}^d$ and T is either H_Γ or $\mathcal{M}_\Gamma$. It follows from a remark on page 609 of [57] that if T satisfies (1.25) then the local operator $f \mapsto T^{loc}(f\chi_Q)$ is a bounded map from $\Phi(L)(Q)$ to $L^{1,\infty}$.

2. The distributional estimate in (1.5) is equivalent to saying that M_{HL} is of weak type L (more commonly referred to as weak type $(1,1)$).

In terms of the above setup, the best known result on H_Γ and $\mathcal{M}_\Gamma$ where Γ is a parabola is in [57]. The operators considered in [57] are more general: Let Σ be a smooth compact hypersurface of $\mathbb{R}^d$, and let ν be a smooth and compactly supported density on Σ. The fundamental assumption is that the Gaussian curvature does not vanish to infinite order on Σ. Define the dilations $\{\delta_t : t \in (0,\infty)\}$ by

$$\delta_t := \exp((\log t)P), \tag{1.26}$$

where P is a (fixed) d by d matrix with real entries and eigenvalues with positive real part. Then define the measure ν_k by

$$\langle \nu_k, \psi\rangle := \langle \nu, \psi(\delta_{2^k}\cdot)\rangle.$$

In [57] it is shown that the operator $f \mapsto \sup_{k\in\mathbb{Z}} |\nu_k * f|$ is of weak type $L\log^{(2)} L$. Moreover, if the cancellation condition, $\widehat{\nu}(0) = 0$, holds then it is also shown that the operator $f \mapsto \sum_{k\in\mathbb{Z}} \nu_k * f$ extends to an operator of weak type $L\log^{(2)} L$. Taking $d = 2$ and the matrix P to be $\mathrm{diag}(1,2)$ we essentially recover $\mathcal{M}_\Gamma$ and H_Γ. Therefore, (1.23) holds for functions belonging locally to $L\log^{(2)} L$.

Also known in the parabola case are the following results involving certain *Hardy spaces* and the smoother maximal operator,

$$\widetilde{\mathcal{M}}f(x) := \sup_{h>0} \left| \int_{\mathbb{R}} f(x - (t,t^2))t^{-1}\phi(t^{-1}h)\, dt \right|,$$

where ϕ is, say, a smooth function with compact support. Christ showed in [17] that $\widetilde{\mathcal{M}}$ maps the appropriate Hardy space associated to parabolic dilations to

$L^{1,\infty}$ (and to no $L^{1,q}$ for $q < \infty$). Later, in [56], it was shown that $\widetilde{\mathcal{M}}$ maps the smaller product-type Hardy space $H^1_{prod}(\mathbb{R} \times \mathbb{R})$ to the smaller Lorentz space $L^{1,2}$ (and to no $L^{1,q}$ for $q < 2$). Our focus in this thesis will be on results concerning the above Orlicz spaces, and thus we discuss these Hardy space results no further.

The result in [57] covers the finite type plane curves mentioned in Theorem 1.1.2. However, there are no known extensions to include the classes of flat plane curves which naturally arise in the L^p theory for $p \in (1, \infty)$. Our contribution in Chapter 4 is to show one can go beyond the L^p theory for one such class of flat curves, and furthermore one can extend to include flat curves in higher dimensions.

A further relevant result in this context is the counterexample of Christ in [18] which shows that if we let Γ be the parabolic piecewise linear curve defined earlier, then $\mathcal{M}_\Gamma$ is not of weak type L. Unfortunately, the construction is completely inapplicable to the smooth parabola case. In Chapter 5 of this thesis, we extend Christ's result and prove that $\mathcal{M}_\Gamma$ is at best of weak type $L(\log L)^{1/2}$.

1.2 Mixed-norm estimates

Given an integer $d \geq 2$ and a Schwartz function f on $\mathbb{R}^d$, define operators H and $\mathcal{M}$ by

$$Hf(x,\omega) := p.v. \int_{\mathbb{R}} f(x - \delta_t \omega) \frac{dt}{t}, \tag{1.27}$$

$$\mathcal{M}f(x,\omega) := \sup_{h>0} h^{-1} \left| \int_{(0,h)} f(x - \delta_t \omega)\, dt \right|, \tag{1.28}$$

where $\{\delta_t : t \in (0,\infty)\}$ is defined exactly as in (1.26), and, for $t \in (-\infty, 0)$, we set $\delta_t := -\delta_{-t}$.

Remark. Taking $d = 2$ and $P = \operatorname{diag}(1,2)$ we see that our dilations match those in (1.16) and thus our expressions for H in (1.27) and (1.15) coincide.

Inspired by (1.17), we are interested in the following: For what range of p and q are the operators H and $\mathcal{M}$ bounded from L^p to $L^p(L^q)$? Below, we survey the isotropic situation, giving known results along with a variety of further applications. In Chapter 2 we improve upon all known results in the nonisotropic setting governed by the dilations in (1.16).

Known results for the isotropic case

If we take P to be the identity matrix, then the δ_t generate the isotropic dilations. The isotropic case is thus essentially the same as the case where P is a fixed

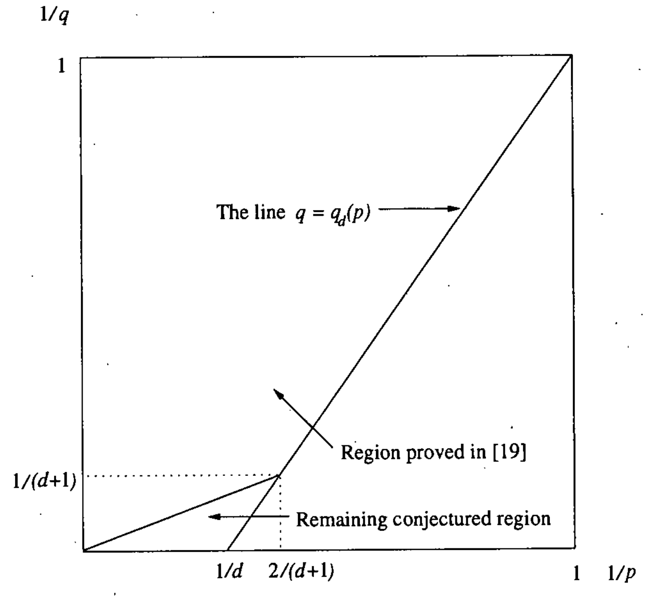

Figure 1.1: The isotropic case

multiple of the identity. As far as we know, the best known result in this case is in [19], and to state the theorem, we use their notation,

$$q_d(p) := \frac{p(d-1)}{d-p}, \quad \text{for } p \in (1, \infty), \tag{1.29}$$

with the agreement that $q_d(p) = \infty$ when $p \geq d$.

Theorem 1.2.1. [19] *Suppose P is a multiple of the identity matrix. Then, for any $d \geq 2$, $p \in (1, \max(2, (d+1)/2)]$, and $q \in [1, q_d(p))$, the operators H and $\mathcal{M}$ are bounded from L^p to $L^p(L^q)$.*

If one tests each operator on the characteristic function of the Euclidean unit ball, then one must have $q \in [1, q_d(p))$. In fact, this gives the conjectured range of p and q for isotropic dilations. Theorem 1.2.1 therefore solves the conjecture for $p \in (1, \max(2, (d+1)/2)]$, and hence when $d = 2$, the conjecture is completely resolved. Figure 1.1 illustrates the isotropic situation (when $d \geq 3$).

Applications in the isotropic case

As one might expect, via the method of rotations, the estimates given by Theorem 1.2.1 give the best known estimates on the variable kernel operator in (1.13) under the following conditions:

(K1). $K(x, \cdot)$ is homogeneous of degree $-d$ with respect to isotropic dilations;

(K2). $K(x, \cdot)$ is an odd function;

(K3). $\sup_{x\in\mathbb{R}^d}\left(\int_{S^{d-1}}|K(x,\omega)|^r\,d\omega\right)^{1/r}<\infty$.

The result states that T is bounded on L^p provided that $p\in(1,\max(2,(d+1)/2)]$ and $r\in((1-1/d)p',\infty)$. In fact, in [19], they show the same conclusion holds if (K2) is replaced by $K(x,\cdot)$ having zero average over S^{d-1} for each $x\in\mathbb{R}^d$. This result improved upon earlier work of Calderón and Zygmund [8] and Cowling and Mauceri [23], and is a sharp result in the stated range of p.

In a different direction, the estimates on $\mathcal{M}$ from Theorem 1.2.1 were used to establish bounds on the *Kakeya maximal operator*. Specifically, if N is a large positive parameter, we let $\mathcal{R}_N$ be the collection of rectangles in $\mathbb{R}^d$ which contain the origin and have one side of length r and $d-1$ sides of length $N^{-1}r$, for all $r\in(0,\infty)$. Then the Kakeya maximal operator, $\mathcal{K}_N$, is defined by

$$\mathcal{K}_Nf(x):=\sup_{R\in\mathcal{R}_N}\frac{1}{|R|}\int_R|f(x-y)|\,dy,$$

and the famous conjecture is that

$$\|\mathcal{K}_Nf\|_p\le C(\log N)^\lambda N^{d/p-1}\|f\|_p,\quad p\in(1,d],\tag{1.30}$$

holds for some $\lambda,C<\infty$ depending on only d and p. In [21], Córdoba established (1.30) for $p\in(1,2]$. It was shown in [19] that the estimate for $\mathcal{M}$ in Theorem 1.2.1 implies (1.30) for $p\in(1,\max(2,(d+1)/2)]$ and thus improved upon Córdoba's result when $d\ge 4$. Spurred on by the work of Bourgain in [5], who further extended the range of p and also found exciting new links with other fundamental open problems in harmonic analysis, (1.30) has since received a large amount of attention. At the time of writing of the fairly recent survey article [35], the best known range of p was $(1,(d+2)/2]$ for $3\le d\le 8$, due to Wolff [70], and $(1,(4d+3)/7)$ for $d\ge 9$, due to Katz and Tao [34]. Recent progress on (1.30) has been achieved through arguments involving geometric combinatorics and arithmetic combinatorics, rather than the Fourier transform based proof of Theorem 1.2.1. We believe that the best known mixed-norm estimates for $\mathcal{M}$ are still those in [19].

Another application was observed by Durán [25] who established a connection between the maximal operator $\mathcal{M}$ and an aspect of numerical approximation, the Bramble-Hilbert lemma. In [27], R. Fefferman proved mixed-norm estimates for $\mathcal{M}$ and extended the result of Durán.

Chapter 2

Mixed-Norm Estimates for a Nonisotropic Maximal Operator Arising in the Method of Rotations

2.1 Introduction

In this chapter, we prove mixed-norm estimates for the operators H and $\mathcal{M}$ defined in (1.27) and (1.28). We suppose throughout this chapter that the matrix P which defines the dilations δ_t is not a multiple of the identity matrix. The notation $q_d(p)$ defined earlier in (1.29) will be adopted without change. For the maximal operator, our main result is as follows.

Theorem 2.1.1. *1. For any $d \geq 2$ and $p \in (1, \infty)$, a necessary condition that $\mathcal{M}$ is a bounded operator from L^p to $L^p(L^q)$ is that $q \in [1, q_d(p)]$.*

2. For any $d \geq 2$, $p \in (1, \max(2, (d+1)/2)]$, and $q \in [1, q_d(p))$, $\mathcal{M}$ is bounded from L^p to $L^p(L^q)$.

It is easy to show Theorem 2.1.1(2) when $1 \leq q \leq p \leq \infty$ (and $p > 1$) . To see this, use Minkowski's inequality and the fact that

$$\|\mathcal{M}f(\cdot, \omega)\|_p \leq C\|f\|_p, \tag{2.1}$$

where the constant C is independent of $\omega \in S^{d-1}$. The estimate in (2.1) for fixed ω was proved by Stein and Wainger in [62]. However, the arguments in this paper can be used to prove the uniform estimate (2.1).

If we can prove Theorem 2.1.1(2) when $p = \max(2, (d+1)/2)$ then the full assertion holds by interpolation with our trivial estimates near $p = 1$ and $q = 1$. We have in mind the mixed-norm interpolation result on page 5 stated for linear operators. The maximal operator $\mathcal{M}$ is not linear, but an easy linearising argument means we can still invoke the result on page 5 as stated. One can also deduce certain results in the range $p \in (\max(2,(d+1)/2),\infty)$. In fact, by interpolation with the trivial estimate when $p = q = \infty$ one gets that $\mathcal{M}$ is bounded from L^p to $L^p(L^q)$ for $p \in (\max(2,(d+1)/2),\infty)$ and $q \in [1,2p)$.

Modulo the endpoint $q = q_d(p)$, Theorem 2.1.1 says that we have the same result for $\mathcal{M}$ whether we have isotropic or nonisotropic dilations. In particular, modulo this endpoint, Theorem 2.1.1 is sharp in all dimensions for $p \in (1,\max(2,(d+1)/2)]$, and when $d = 2$, sharp for $p \in (1,\infty)$.

Our analysis of the singular integral operator H has been less successful. At the moment, the following is known to us.

Theorem 2.1.2. *1. For any $d \geq 2$ and $p \in (1,\infty)$, a necessary condition that H is a bounded operator from L^p to $L^p(L^q)$ is that $q \in [1, q_d(p)]$.*

2. For any $d \geq 2$, $p \in (1,2]$, and $q \in [1,q_d(p))$, H is bounded from L^p to $L^p(L^q)$.

It follows from Theorem 2.1.2 that we have a sharp result for H in all dimensions for $p \in (1,2]$, and, when $d = 2$, for all $p \in (1,\infty)$ (modulo an endpoint). As with $\mathcal{M}$, it suffices to prove Theorem 2.1.2(2) when $p = 2$, and in this case, $q \leq 2$ is trivial.

Best known results in the nonisotropic case

Firstly, we emphasise that the only known results in the nonisotropic setting concern the case that P is a diagonal matrix with distinct real and positive diagonal entries. Let P be such a matrix and write $P = \mathrm{diag}(\alpha_1,\ldots,\alpha_d)$. For $\mathcal{M}$, if we set

$$p_s := \frac{2(d\sum_j \alpha_j - (d-2)\min_j(\alpha_j))}{d\sum_j \alpha_j - (d-4)\min_j(\alpha_j)},$$
$$p_c := \frac{2(d-1+1/d)}{d-1+2/d}, \quad \text{and} \quad q_c := \frac{2(d-1+1/d)}{d-1},$$

then Sato [55] and Chen [16] achieve the range of p and q shown in Figure 2.1. Either result can subsume the other, depending on certain relationships between the numbers d, $\min_j(\alpha_j)$, and $\sum_j \alpha_j$.

For H, we believe that the best known result is the following theorem of Chen, which is restricted to $d = 2$.

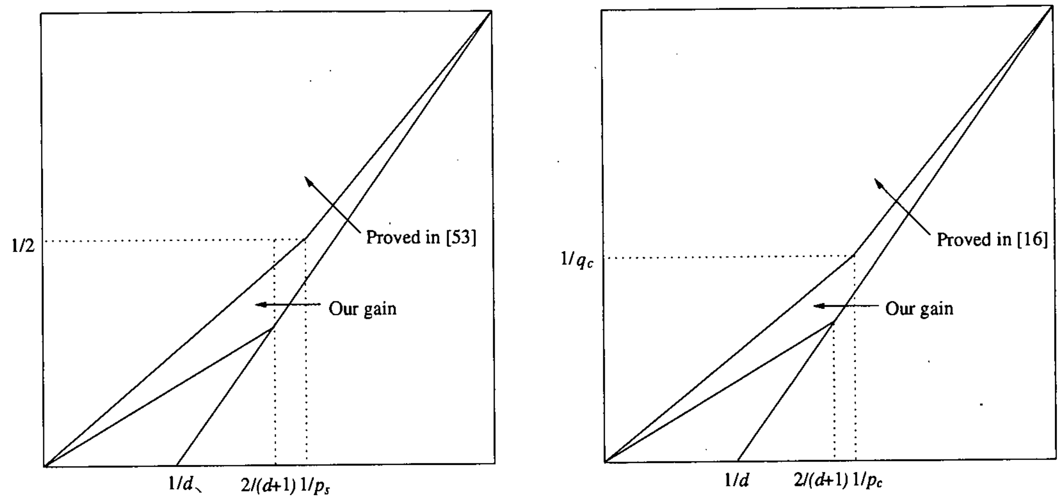

Figure 2.1: Our improvement for the nonisotropic maximal operator

Theorem 2.1.3. [15] *If* $P = \mathrm{diag}(\alpha_1, \alpha_2)$ *and* $1 < \alpha_2/\alpha_1 < 4/3$ *then* H *is bounded from* L^p *to* $L^p(L^q)$ *provided*

1. $p \in (1, 2]$ *and* $q \in (1, 2p/(3-p))$; *or*
2. $p \in (2, \infty)$ *and* $q \in (1, 2p)$.

We shall not highlight the gain from Theorem 2.1.2 by a diagram. It is clear that when $d = 2$ and $p = 2$, Theorem 2.1.2 achieves the optimal range, $q \in [1, \infty)$. Compare this with the range $q \in (1, 4)$ given by Theorem 2.1.3.

Remark. We should emphasise that in [15], Chen actually proved the stronger result that Theorem 2.1.3 is true if one replaces H by the corresponding maximal Hilbert transform.

Preliminaries

We frequently rely on the fact that our dilations δ_t satisfy the following group property:

$$\delta_s \delta_t = \delta_{st} \quad \text{for all } s, t \in (0, \infty). \tag{2.2}$$

Associated to P are smooth P-homogeneous distance functions ϱ; that is, $\varrho \in C^\infty(\mathbb{R}^d \setminus \{0\})$ and $\varrho(\delta_t x) = t\varrho(x)$ for all $t \in (0, \infty)$ and all $x \in \mathbb{R}^d$. For our purposes it is crucial to choose a ϱ with a specific property; namely, we will need the hypersurface,

$$\Sigma_\varrho := \{\omega \in \mathbb{R}^d : \varrho(\omega) = 1\}$$

to have nonvanishing Gaussian curvature. First, take a real symmetric positive definite matrix Q such that, for fixed $x \in \mathbb{R}^d \setminus \{0\}$, the function

$$t \mapsto (Q\delta_t x.\delta_t x)^{1/2}$$

is strictly increasing on $[0,\infty)$. Assuming that such a matrix Q exists for the moment, we may, for each $x \in \mathbb{R}^d \setminus \{0\}$, set $\varrho(x)$ to be the unique $t \in (0,\infty)$ such that

$$Q\delta_{t^{-1}}x.\delta_{t^{-1}}x = 1.$$

When $x = 0$ we set $\varrho(x) = 0$. On the existence of such a matrix Q, one may take

$$Q = \int_{(0,\infty)} \exp(-tP^*)\exp(-tP)\,dt.$$

It is straightforward to check that this has the requisite properties; this rather cute choice can be found in [62]. Note that the choice of Q is certainly not unique.

Notation. Write $A \lesssim B$ for $A \leq CB$, where C depends only on d, the matrices P and Q, and any index p or q that may be present.

We now introduce polar coordinates in our nonisotropic setting: For each nonzero $x \in \mathbb{R}^d$ there exists a unique pair $(r,\omega) \in (0,\infty) \times \Sigma_\varrho$ such that

$$x = \delta_r\omega;$$

where $r = \varrho(x)$ and $\omega = \delta^{-1}_{\varrho(x)}x$. Then the volume element in $\mathbb{R}^d$ is

$$dx = r^{\tau-1}dr\,d\omega, \tag{2.3}$$

where dr is Lebesgue measure on the positive real line, $d\omega$ is a smooth C^∞ measure on Σ_ϱ, and τ is the trace of P. This change of variables will be referred to as passing to nonisotropic polar coordinates. For a proof of (2.3), see [62].

Since Q is a positive definite symmetric matrix, Σ_ϱ is an ellipsoid with nonvanishing Gaussian curvature. Since the measure $d\omega$ is smooth, it follows (see, for example, [60]) that for large $|\xi|$,

$$|\widehat{d\omega}(\xi)| \lesssim |\xi|^{-(d-1)/2}, \tag{2.4}$$

and this is the key estimate we shall need.

Although the triangle inequality will fail in general, there exists a constant $C \geq 1$ such that

$$\varrho(x+y) \leq C(\varrho(x)+\varrho(y)) \quad \text{for all } x,y \in \mathbb{R}^d. \tag{2.5}$$

Define the associated balls

$$B(x,r) := \{y \in \mathbb{R}^d : \varrho(x-y) < r\} \quad \text{for } x \in \mathbb{R}^d, r \in (0,\infty).$$

The following bounds will also be useful (for a proof, see [62]);

$$t^{\alpha_1}|x| \lesssim |\delta_t x| \lesssim t^{\alpha_2}|x| \quad \text{for all } t < 100, \tag{2.6}$$

$$t^{\alpha_3}|x| \lesssim |\delta_t x| \lesssim t^{\alpha_4}|x| \quad \text{for all } t \geq 100, \tag{2.7}$$

where each α_j depends only on P.

Remark. In many circumstances it actually suffices to take Q to be the identity matrix, or equivalently, Σ_ϱ to be S^{d-1}. For example, this is the case if P is a diagonal matrix or if $\|\delta_t\| \leq t$ for each $t \in [0,1]$ (see [54]). To simplify the proofs of Theorem 2.1.1 and Theorem 2.1.2, we shall assume throughout this chapter that $\Sigma_\varrho = S^{d-1}$. It will be clear how to modify the arguments in the more general context.

Overview. In the coming section, we prove the necessity parts of Theorem 2.1.1 and Theorem 2.1.2. Section 2.3 and Section 2.4 are devoted to the sufficiency parts of Theorem 2.1.1 and Theorem 2.1.2 respectively. In Section 2.5 we prove the main oscillatory integral estimate used for these results. Finally, in Section 2.6 we exhibit a few applications of our results.

2.2 Necessity

As far as we know, no necessary conditions have been given in the nonisotropic case. This may be because of difficulties arising from the competing homogeneities of the nonisotropic dilations δ_t and the isotropic nature of the sphere S^{d-1}. To reflect this, we set f_N to be the characteristic function of $\mathfrak{B}_N := \delta_N(B_{CN^{-1}}(0))$, where $C \sim 1$ whose exact value will be revealed later in the proof, and $N \geq 100$. Notice that in the case of isotropic dilations, $\mathfrak{B}_N$ is simply a Euclidean ball which is independent of N, and for parabolic dilations in the plane, $\mathfrak{B}_N$ amounts to the interior of an ellipse of scale 1 in the x direction and scale N in the y direction.

Suppose first that $\mathcal{M}$ is bounded from L^p to $L^p(L^q)$, so that

$$N^{\tau-d} \sim \|f_N\|_p^p \gtrsim \int_{\mathbb{R}^d} \left(\int_{S^{d-1}} \left(\sup_{h>0} \int_0^h f_N(x - \delta_t\omega)\, dt \right)^q d\omega \right)^{p/q} dx. \tag{2.8}$$

Now pass to nonisotropic polar coordinates $x = \delta_r\theta$ for $r \in (0,\infty)$ and $\theta \in S^{d-1}$, so that (2.8) reads

$$\begin{aligned} N^{\tau-d} &\gtrsim \int_0^\infty \int_{S^{d-1}} \left(\int_{S^{d-1}} \left(\sup_{h>0} \frac{1}{h} \int_0^h f_N(\delta_r\theta - \delta_t\omega)\, dt \right)^q d\omega \right)^{p/q} d\theta r^{\tau-1}\, dr \\ &\gtrsim \int_N^{2N} \int_{S^{d-1}} \left(\int_{\mathcal{A}_\theta} \left(\frac{1}{r+1} \int_r^{r+1} f_N(\delta_r\theta - \delta_t\omega)\, dt \right)^q d\omega \right)^{p/q} d\theta r^{\tau-1}\, dr, \end{aligned}$$

where $\mathcal{A}_\theta := S^{d-1} \cap B_{N^{-1}}(\theta)$. For fixed $r \in (N, 2N)$ and $\theta \in S^{d-1}$, we claim that for $\omega \in \mathcal{A}_\theta$ and $t \in (r, r+1)$ we have $\delta_r\theta - \delta_t\omega \in \mathfrak{B}_N$. This claim granted,

$$\begin{aligned} N^{\tau-d} &\gtrsim \frac{1}{N^p} \int_N^{2N} \int_{S^{d-1}} \left(\int_{\mathcal{A}_\theta} d\omega \right)^{p/q} d\theta r^{\tau-1}\, dr \\ &\sim \frac{1}{N^{p+(d-1)p/q}} \int_N^{2N} r^{\tau-1}\, dr \sim N^{\tau-p-(d-1)p/q}, \end{aligned}$$

and this implies that $q \in [1, q_d(p)]$.

To prove the claim, first write $t = r + h$ and $\omega = \theta + \eta$ where $h \in (0,1)$ and $\eta \in B_{N^{-1}}(0)$. Then,

$$\delta_r\theta - \delta_t\omega = \delta_r(I - \delta_{1+h/r})\theta - \delta_{r+h}\eta,$$

which means that $\delta_r\theta - \delta_t\omega \in \mathfrak{B}_N$ if and only if

$$\delta_{r/N}(I - \delta_{1+h/r})\theta - \delta_{(r+h)/N}\eta \in B_{CN^{-1}}(0). \tag{2.9}$$

By (2.6), $|\delta_{(r+h)/N}\eta| \leq C_2(r+h)^{\alpha_2}N^{-\alpha_2-1} \leq C_2 3^{\alpha_2}N^{-1}$, for some $C_2 \sim 1$. It therefore remains to bound $|\delta_{r/N}(I - \delta_{1+h/r})\theta|$, for which we use the following lemma.

Lemma 2.2.1. *There exists $C' \sim 1$ such that for all $t \in (0,1)$ we have $\|\delta_{t+1} - I\| \leq C't$.*

Proof. Write $\delta_{t+1} = I + \log(t+1)P + A(t)$, where, of course, $A(t) := e^{\log(t+1)P} - I - \log(t+1)P$. Clearly,

$$\|A(t)\| = \left\| \sum_{k=2}^{\infty} \frac{(\log(t+1))^k}{k!} P^k \right\| \leq \sum_{k=2}^{\infty} \frac{t^k}{k!}\|P\|^k \leq t\sum_{k=0}^{\infty} \frac{\|P\|^k}{k!} = te^{\|P\|}.$$

Hence, $\|\delta_{t+1} - I\| \leq t(\|P\| + e^{\|P\|})$. □

Using Lemma 2.2.1 and (2.6), $|\delta_{r/N}(I - \delta_{1+h/r})\theta| \leq C_2 2^{\alpha_2}C'N^{-1}$. Theorem 2.1.1(1) is proved by making the choice $C := C_2(3^{\alpha_2} + 2^{\alpha_2}C')$.

To get the necessary condition for H, we also test this operator on the function f_N, for large N. The only difference to the above argument is that one should restrict the θ integral to some smaller subset of S^{d-1} of size ~ 1 to remove the cancellation in the t integral.

2.3 Proof of Theorem 2.1.1(2)

Unlike previous approaches in the nonisotropic setting, we shall use the successful techniques used for the isotropic case in [19]. The proof proceeds in two steps. The first step is to show that Theorem 2.1.1(2) is true when $p = 2$. Secondly, we show that a weaker estimate holds arbitrarily close to the case $(p,q) = ((d+1)/2, d+1)$. Notice that this point lies on the critical line $1/q = 1/q_d(p)$. Some Littlewood-Paley theory will be used to show that our weaker estimate near the critical point together with the L^2 estimate imply that Theorem 2.1.1(2) holds when $p = (d+1)/2$, as required.

Step 1: the case $p = 2$

We setup a square function type argument using a fixed number $\sigma \in (1, \infty)$ which we do not specify but remark that it only depends on the matrix P. Select $\varsigma \in (1, \sigma)$ for which $(\varsigma, \varsigma^2) \subsetneq (1, \sigma)$. Then choose $\psi \in \mathcal{S}(\mathbb{R})$ such that ψ vanishes outside $(1, \sigma)$, ψ is equal to 1 on (ς, ς^2), and $0 \leq \psi \leq 1$. Then let

$$\psi_k(t) := \varsigma^{-k}\psi(\varsigma^{-k}t) \quad \text{for each } k \in \mathbb{Z}.$$

Now choose a positive function $\phi \in \mathcal{S}(\mathbb{R}^d)$ such that $\int \phi = \int \psi$ and such that there exists a decreasing function $\widetilde{\phi}$ defined on $[0, \infty)$ such that $\phi(x) = \widetilde{\phi}(\varrho(x))$ for each $x \in \mathbb{R}^d$. Then let

$$\phi_k(t) := \det \delta_{\varsigma^{-k}} \phi(\delta_{\varsigma^{-k}} x) \quad \text{for each } k \in \mathbb{Z}.$$

Now define, for each $k \in \mathbb{Z}$,

$$A_k f(x, \omega) := \int_{\mathbb{R}} f(x - \delta_t \omega)\psi_k(t)\, dt - \int_{\mathbb{R}^d} f(x - y)\phi_k(y)\, dy.$$

For $f \geq 0$, one certainly has

$$Mf(x, \omega) \lesssim \sup_{k \in \mathbb{Z}} |A_k f(x, \omega)| + \sup_{k \in \mathbb{Z}} \int_{\mathbb{R}^d} f(x - y)\phi_k(y)\, dy. \tag{2.10}$$

The second term on the right hand side of (2.10) is $\lesssim M_{HL} f(x)$, where the Hardy-Littlewood type maximal operator, M_{HL}, is defined by

$$M_{HL} f(x) := \sup_{r > 0} \frac{1}{|\delta_r B_1(0)|} \left| \int_{\delta_r B_1(0)} f(x - z)\, dz \right|.$$

Moreover, M_{HL} is a bounded operator on L^p for all $p \in (1, \infty)$ (see, for example, Chapter 1 of [60]), and thus it suffices to prove

$$\left\| \left(\sum_{k \in \mathbb{Z}} |A_k f|^2 \right)^{1/2} \right\|_{L^2(L^q)} \lesssim \|f\|_2. \tag{2.11}$$

Fix $q \in (2, q_d(2))$ and choose $\nu \in (0, 1/2)$ such that

$$q^{-1} = 1/2 - \nu/(d - 1). \tag{2.12}$$

To prove (2.11), we first invoke Minkowski's inequality and Sobolev embedding to get

$$\left\| \left(\sum_{k \in \mathbb{Z}} |A_k f(x, \cdot)|^2 \right)^{1/2} \right\|_q \leq \left(\sum_{k \in \mathbb{Z}} \|A_k f(x, \cdot)\|_q^2 \right)^{1/2} \lesssim \|\mathfrak{A} f(x)\|_{\mathcal{H}}.$$

Here, the operator $\mathfrak{A}$, defined by $\mathfrak{A}f(x) := \{A_k f(x,\cdot)\}_{k\in\mathbb{Z}}$, is being viewed as a linear operator from L^2 to $L^2(\mathcal{H})$, where $\mathcal{H}$ is the Hilbert space $l^2(L^2_\nu)$. Now,

$$\int_{\mathbb{R}^d} \|\mathfrak{A}f(x)\|^2_{\mathcal{H}}\, dx \sim \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \|m(\delta^*_{\varsigma^k}\xi,\cdot)\|^2_{L^2_\nu}\, d\xi,$$

where

$$m(\xi,\omega) := \int_{\mathbb{R}} \psi(t) e^{i\delta_t\omega.\xi}\, dt - \int_{\mathbb{R}^d} \phi(x) e^{ix.\xi}\, dx.$$

Thus, Theorem 2.1.1(2) for $p = 2$ will be proved once we have shown

$$\sup_{\xi\in\mathbb{R}^d} \sum_{k\in\mathbb{Z}} \|m(\delta^*_{\varsigma^k}\xi,\cdot)\|^2_{L^2_\nu} \lesssim 1. \tag{2.13}$$

If we can show that there exists $\varepsilon > 0$ depending only on ν such that, for almost all $\xi \in \mathbb{R}^d$,

$$\|m(\xi,\cdot)\|_{L^2_\nu} \lesssim \min(|\xi|, |\xi|^{-\varepsilon}), \tag{2.14}$$

then (2.13) follows from (2.6) and (2.7). In fact, we show that (2.14) holds with $\varepsilon = 1/2(1/2 - \nu)$. We shall do so by showing that the following estimates hold almost everywhere:

$$\|m(\xi,\cdot)\|_{L^2_0} \lesssim \min(|\xi|, |\xi|^{\varepsilon-1/2}); \tag{2.15}$$

$$\|m(\xi,\cdot)\|_{L^2_1} \lesssim \min(|\xi|, |\xi|^{\varepsilon+1/2}); \tag{2.16}$$

and then interpolate between the Sobolev spaces L^2_0 and L^2_1. Firstly, for small $|\xi|$, we use the fact that $\int \psi = \int \phi$ to get

$$m(\xi,\omega) = \int_{\mathbb{R}} \psi(t)(e^{i\delta_t\omega.\xi} - 1)dt - \int_{\mathbb{R}^d} \phi(x)(e^{ix.\xi} - 1)\, dx.$$

In modulus, this is $\lesssim |\xi|$ by the mean value theorem. Since the modulus of any first order derivative of $\omega \mapsto \delta_t\omega.\xi$ on S^{d-1} is $\lesssim |\xi|$, the estimates for small $|\xi|$ in (2.15) and (2.16) follow. The estimates in (2.15) and (2.16) for large $|\xi|$ are implied by the following lemma, whose proof is delayed until Section 2.5.

Lemma 2.3.1. *Fix $a \in \{0,1\}$. Suppose that for each fixed $(\xi,\omega) \in \mathbb{R}^d\backslash\{0\}\times S^{d-1}$, the function $\Psi_{(\xi,\omega)}$ is supported in $[1,\sigma]$, smooth on $(1,\sigma)$, and*

$$|\Psi_{(\xi,\omega)}(t)| + |\Psi'_{(\xi,\omega)}(t)| \lesssim |\xi|^a \quad \text{for all } t \in (1,\sigma). \tag{2.17}$$

Then,

$$\int_{S^{d-1}} \left| \int_{\mathbb{R}} \Psi_{(\xi,\omega)}(t) e^{i\delta_t\omega.\xi}\, dt \right|^2 d\omega \lesssim |\xi|^{-1+2a+2\varepsilon}.$$

Step 2: the case (p,q) near to $((d+1)/2, d+1)$

With a very similar methodology to [19], we extend the mixed-norm inequalities for $\mathcal{M}$ to $p \in (2,(d+1)/2]$ by proving the following weaker estimates near the endpoint $(p,q) = ((d+1)/2, d+1)$. Let $\widetilde{\mathcal{C}} \subseteq [0,1]^2$ denote the convex hull of the points $(0,0), (0,1), (1,1)$, and $(2/(d+1), 1/(d+1))$.

Lemma 2.3.2. *There exists a constant $C(p,q) < \infty$ such that for all $k \in \mathbb{Z}$,*

$$\|A_k f\|_{L^p(L^q)} \leq C(p,q)\|f\|_p \tag{2.18}$$

whenever $(1/p, 1/q)$ belongs to the interior, $\widetilde{\mathcal{C}}^o$, of $\widetilde{\mathcal{C}}$.

Proof. Fix $(1/p,1/q) \in \widetilde{\mathcal{C}}^o$. Lemma 2.3.2 is obvious if $q \leq p$ by Minkowski's inequality and therefore we assume throughout this proof that $q > p$.

Since M_{HL} is bounded on L^p, it is immediate from (2.10) that it suffices to prove (2.18) with A_k replaced by T_k, where

$$T_k f(x,\omega) := \int_{\mathbb{R}} f(x - \delta_t\omega)\psi_k(t)\, dt.$$

Observe that

$$\|T_k f\|^p_{L^p(L^q)} = \det \delta_{\varsigma^k} \|T_0(f(\delta_{\varsigma^k}\cdot))\|^p_{L^p(L^q)},$$

and therefore it suffices to prove (2.18) for T_0. It is also clear that it suffices to take $f \geq 0$; indeed $\psi \geq 0$ and thus this is the worst case. Our final reduction is that we may suppose that f is supported on the unit cube centred at the origin. That we may restrict our attention to unit cubes, $\{Q\}$, follows from the fact that T_0 is a local operator; more specifically, there exists $C \sim 1$ such that

$$x - \delta_t\omega \in Q \Rightarrow x \in CQ \quad \text{for all } t \in (1,\sigma) \text{ and } \omega \in S^{d-1},$$

and therefore $T_0 f$ is supported in CQ whenever f is supported in Q. By translation invariance it suffices to consider the unit cube centred at the origin, which we call Q_0. Hölder's inequality now implies that $\|T_0 f\|_{L^p(L^q)} \lesssim \|T_0 f\|_{L^q(L^q)}$, which means it suffices to show

$$\int_{CQ_0}\int_{S^{d-1}} \left(\int_{(1,\sigma)} f(x - \delta_t\omega)\, dt\right)^q d\omega dx \lesssim \|f\|^q_p,$$

or, by duality,

$$\left|\int_{\substack{\omega\in S^{d-1}\\ x\in CQ_0}} \int_{(1,\sigma)} f(x-\delta_t\omega) g(x,\omega)\, dt d\omega dx\right| \lesssim \|f\|_p \left(\int_{\substack{\omega\in S^{d-1}\\ x\in CQ_0}} g(x,\omega)^{q'}\, d\omega dx\right)^{1/q'}. \tag{2.19}$$

To show (2.19) we use a recent theorem of Gressman in [31]. We now describe the general setup and main theorem in [31] and demonstrate that (2.19) follows immediately as a special case.

Let X and Y be smooth manifolds equipped with measures of smooth density and assume the $\dim X < \dim Y$. Let $\mathfrak{M}$ be a smooth $(\dim Y + 1)$-dimensional submanifold of $X \times Y$, also equipped with a measure, and such that the natural projections $\pi_X : \mathfrak{M} \to X$ and $\pi_Y : \mathfrak{M} \to Y$ have everywhere surjective differential maps. Furthermore, let $\mathfrak{X}_1$ and $\mathfrak{Y}_1$ be those vector fields on $\mathfrak{M}$ which are annihilated by $d\pi_X$ and $d\pi_Y$, respectively. Now choose a nonvanishing representative $Y_1 \in \mathfrak{Y}_1$ and define $T(V) := [V, Y_1]$, where $[\cdot,\cdot]$ denotes the Lie bracket. Define $\mathfrak{X}_j$ to be the collection of all vector fields in $\mathfrak{X}_{j-1}$ such that $T(V) \in \mathfrak{X}_{j-1} + \mathfrak{Y}_1$.

Definition 2.3.3. The ensemble $(\mathfrak{M}, X, Y, \pi_X, \pi_Y)$ is said to be nondegenerate through order k at $m \in \mathfrak{M}$ if there are $\dim X - 1$ vector fields $X_j \in \mathfrak{X}_k$ such that $\{\mathfrak{X}_1|_m, \mathfrak{Y}_1|_m, T^k(X_j) : j = 1, \ldots, \dim X - 1\}$ spans the tangent space of $\mathfrak{M}$ at m.

Let $\mathfrak{C}_k \subseteq [0,1]^2$ be the convex hull of the points $(0,1), (1,0), (0,0)$, and

$$\left\{ \left(\frac{2}{j \dim X - j + 2}, 1 - \frac{2}{(j+1)(j \dim X - j + 2)} \right) : j = 1, \ldots, k \right\}. \quad (2.20)$$

Then we have the following.

Theorem 2.3.4. [31] *Let $(\mathfrak{M}, X, Y, \pi_X, \pi_Y)$ be nondegenerate through order k at $m \in \mathfrak{M}$. Then there exists an open set $U \subset M$ containing m and a constant $C(p,q') < \infty$ such that, for any positive functions f_X and f_Y on X and Y, respectively,*

$$\int_U f_X(\pi_X(m)) f_Y(\pi_Y(m))\, dm \leq C(p,q') \|f_X\|_p \|f_Y\|_{q'}$$

whenever $(1/p, 1/q')$ belongs to the interior of $\mathfrak{C}_k$.

To see how (2.19) follows from Theorem 2.3.4, we take

$$X := \mathbb{R}^d, Y := \mathbb{R}^d \times S^{d-1}, \mathfrak{M} := \{(x - \delta_t\omega, x, \omega) : x \in CQ_0, \omega \in S^{d-1}, t \in (1,\sigma)\}, \quad (2.21)$$

each equipped with their natural Lebesgue measure. Since $\mathfrak{M}$ is compact it is clear that that Theorem 2.3.4 implies (2.19) once we demonstrate that, at each point $m \in \mathfrak{M}$, $(\mathfrak{M}, X, Y, \pi_X, \pi_Y)$ is nondegenerate through order 1 at m. To this end we consider m lying in the piece of $\mathfrak{M}$ parameterised by,

$$\begin{aligned} \Phi \; &: \; (1,\sigma) \times CQ_0 \times B_1(0) \subset \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d-1} \longrightarrow \mathfrak{M} \\ &: \; (t, x, y) \longmapsto (x - \delta_t\omega, x, \omega), \end{aligned}$$

where $\omega := (y_1, \ldots, y_{d-1}, (1 - |y|^2)^{1/2})$. We can parameterise the rest of $\mathfrak{M}$ using (a finite number of) maps which are similar to Φ. It will be apparent that the argument which follows can be modified to get the same outcome for the remaining elements of $\mathfrak{M}$. Our computations of the vector fields $\mathfrak{X}_1$ and $\mathfrak{Y}_1$ occur in a Euclidean space and thus appear as $2d$-tuples. Our choice of parameterisation means that it is convenient to write these $2d$-tuples in the form $(t|x|y)$ where $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, and $y \in \mathbb{R}^{d-1}$.

One can easily verify that, if e_j is the jth standard basis vector in $\mathbb{R}^{d-1}$ and $\omega_j := (e_j, -y_j(1 - |y|)^{-1/2}) \in \mathbb{R}^d$, then the vectors

$$\begin{aligned} X_j &:= (0|\delta_t\omega_j|e_j) \quad \text{for } j = 1, \ldots, d-1; \text{ and} \\ X_d &:= (1|t^{-1}P\delta_t\omega|0) \end{aligned}$$

lie in $\mathfrak{X}_1$, and the vector $(1|0|0)$ lies in $\mathfrak{Y}_1$. It is also straightforward to verify that

$$T(X_j) = (0|t^{-1}P\delta_t\omega_j|0) \quad \text{for } j = 1, \ldots, d-1.$$

We claim that for each fixed $(t, x, y) \in (1, \sigma) \times CQ_0 \times B_1(0)$ the set

$$\{Y_1, X_j, X_d, T(X_j) : j = 1 \ldots, d-1\} \tag{2.22}$$

is linearly independent. Upon a dimension count, this implies that $(\mathfrak{M}, X, Y, \pi_X, \pi_Y)$ is nondegenerate through order 1 at m, as claimed.

To see that the set in (2.22) is linearly independent, suppose that

$$\alpha Y_1 + \sum_{j=1}^{d-1} \beta_j X_j + \beta_d X_d + \sum_{j=1}^{d-1} \gamma_j T(X_j) = 0.$$

The last $d-1$ components force $\beta_j = 0$ for $j = 1, \ldots, d-1$. Therefore,

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & t^{-1}P\delta_t\omega & -t^{-1}P\delta_t\omega_1 & \cdots & t^{-1}P\delta_t\omega_{d-1} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta_d \\ \gamma_1 \\ \vdots \\ \gamma_{d-1} \end{pmatrix} = 0, \tag{2.23}$$

and it suffices to show that the determinant of the matrix in (2.23) is nonzero. This determinant is clearly equal to

$$\det(t^{-1}P\delta_t)\det(\omega, -\omega_1, \ldots, -w_{d-1}),$$

and an easy computation shows that this equals,

$$t^{\tau-1}\det(P)(1 - |y|)^{-1/2},$$

which is nonzero for each $(t, x, y) \in (1, \sigma) \times CQ_0 \times B_1(0)$. This completes the proof of Lemma 2.3.2. □

Following [19], we shall combine Lemma 2.3.2 with some Littlewood-Paley theory to complete the proof of Theorem 2.1.1(2). Begin with a smooth compactly supported function η on $\mathbb{R}^d$ such that $0 \leq \eta \leq 1$ and

$$\eta(\xi) = \begin{cases} 1 & \text{for } |\xi| \leq 1, \\ 0 & \text{for } |\xi| \geq 2, \end{cases}$$

and set $\eta_k(\xi) := \eta(\delta^*_{2^k}\xi)$. It can be shown (see, for example, [9]) that there exists a natural number $D \sim 1$ such that if

$$\lambda_k := \eta_{k+D} - \eta_{k-D} \quad \text{and} \quad \widehat{\Lambda_k} := \lambda_k,$$

then the following is true.

Theorem 2.3.5. *1. The Λ_k decompose the identity operator in the following sense:*

$$\sum_{k\in\mathbb{Z}} \lambda_k(\xi) = 2D \quad \textit{for each } \xi \neq 0.$$

2. *There exists a natural number $N \sim 1$ such that for any $\xi \in \mathbb{R}^d$, the number of $k \in \mathbb{Z}$ for which $\lambda_k(\xi) \neq 0$ is at most N.*

3. *If either $|\delta^*_{\varsigma^{k-D}}\xi| \geq 2$ or $|\delta^*_{\varsigma^{k+D}}\xi| \leq 1$ then $\lambda_k(\xi) = 0$.*

4. *For all $p \in (1, \infty)$,*

$$\left\| \left(\sum_{k\in\mathbb{Z}} |\Lambda_k * f|^2 \right)^{1/2} \right\|_p \lesssim \|f\|_p.$$

For any Schwartz function f we have

$$\sup_{k\in\mathbb{Z}} |A_k f(x,\omega)| \sim \sup_{k\in\mathbb{Z}} \left| A_k \left(\sum_{j\in\mathbb{Z}} \Lambda_{j+k} * f \right)(x,\omega) \right| \leq \sum_{j\in\mathbb{Z}} B_j f(x,\omega),$$

where

$$B_j f(x,\omega) := \sup_{k\in\mathbb{Z}} |A_k(\Lambda_{j+k} * f)(x,\omega)|.$$

We claim that it suffices to prove the following inequalities for each Schwartz function f and each $j \in \mathbb{Z}$.

$$\|B_j f\|_{L^p(L^q)} \lesssim \|f\|_p \quad \text{for each } (1/p, 1/q) \in \widetilde{\mathfrak{C}}^o; \tag{2.24}$$

$$\|B_j f\|_{L^2(L^q)} \lesssim \varsigma^{-\alpha(q)|j|}\|f\|_2 \quad \text{for some } \alpha(q) > 0 \text{ and } q < q_d(2). \tag{2.25}$$

In fact, interpolation between (2.24) and (2.25) implies that

$$\|B_j f\|_{L^p(L^q)} \lesssim 2^{-\alpha'(p,q)|j|}\|f\|_p$$

for each $p \in (2, (d+1)/2)$ and $q \in [1, q_d(p))$. Hence, for such p and q,

$$\|\mathcal{M}f\|_{L^p(L^q)} \lesssim \sum_{j\in\mathbb{Z}} \|B_j f\|_{L^p(L^q)} + \|M_{HL}f\|_p \lesssim \|f\|_p + \sum_{j\in\mathbb{Z}} 2^{-\alpha'(p,q)|j|}\|f\|_p \lesssim \|f\|_p.$$

We can now use this estimate and interpolation to achieve the same conclusion when $p_0 = (d+1)/2$ and $q_0 \in [1, q_d(p))$. Indeed, fix such a p_0 and q_0 and interpolate the above estimate for p sufficiently close to p_0 and an appropriate $q \in [1, q_d(p))$, with the trivial estimate $\|\mathcal{M}f\|_{L^\infty(L^\infty)} \lesssim \|f\|_\infty$.

To wrap things up, it only remains to prove (2.24) and (2.25). To see (2.24), first fix $(1/p, 1/q) \in \widetilde{\mathcal{C}}^o$ with $p < q$, and observe the following trivial majorisations:

$$\begin{aligned}
\|B_j f\|^p_{L^p(L^q)} &\leq \int_{\mathbb{R}^d} \left(\sum_{k\in\mathbb{Z}} \int_{S^{d-1}} |A_k(\Lambda_{j+k} * f)(x,\omega)|^q \, d\omega \right)^{p/q} dx \\
&\leq \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^d} \left(\int_{S^{d-1}} |A_k(\Lambda_{j+k} * f)(x,\omega)|^q \, d\omega \right)^{p/q} dx.
\end{aligned}$$

Now (2.24) follows from Lemma 2.3.2, the fact that the $l^p(\mathbb{Z})$ norm is dominated by the $l^2(\mathbb{Z})$ norm, and Theorem 2.3.5(4) in the following way:

$$\begin{aligned}
\|B_j f\|^p_{L^p(L^q)} &\lesssim \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^d} |\Lambda_{j+k} * f(x)|^p \, dx \\
&\lesssim \int_{\mathbb{R}^d} \left(\sum_{k\in\mathbb{Z}} |\Lambda_{j+k} * f(x)|^2 \right)^{p/2} dx \lesssim \|f\|^p_p.
\end{aligned}$$

To show (2.25), we take the same approach that we used to prove (2.11) and also Theorem 2.3.5(3) to get

$$\begin{aligned}
\|B_j f\|^2_{L^2(L^q)} &\lesssim \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^d} |\widehat{\Lambda_{j+k}}(\xi)|^2 |\widehat{f}(\xi)|^2 \min(|\delta^*_{\varsigma^k}\xi|^2, |\delta^*_{\varsigma^k}\xi|^{-2\varepsilon}) \, d\xi \\
&\lesssim \sum_{k\in\mathbb{Z}} \int_{A_k} |\widehat{f}(\xi)|^2 \min(|\delta^*_{\varsigma^k}\xi|^2, |\delta^*_{\varsigma^k}\xi|^{-2\varepsilon}) \, d\xi,
\end{aligned}$$

where

$$A_k := \{\xi \in \mathbb{R}^d : |\delta^*_{\varsigma^{j+k+D}}\xi| > 1 \text{ and } |\delta^*_{\varsigma^{j+k-D}}\xi| < 2\}.$$

It is easy to verify that (2.25) follows from (2.6), (2.7), and Theorem 2.3.5(2). This completes the proof of Theorem 2.1.1.

Remark. The ensemble $(\mathfrak{M}, X, Y, \pi_X, \pi_Y)$ is not nondegenerate through order k for any $k \geq 2$, in the setup of (2.21). Using this, and the fact from [31] that Theorem 2.19 is essentially sharp, means that a different approach is needed to improve upon Theorem 2.1.1.

2.4 Proof of Theorem 2.1.2(2)

In the isotropic case, the schema in [19] is to deduce the same estimates for H from those known for $\mathcal{M}$ (see Lemma 4.1 on pages 197-198). The argument here relies on the fact that H arises from the classical one-dimensional Hilbert transform in the way described in (1.11). As we remarked on page 12, this approach is not available in a nonisotropic setting. However, as an aside, the point at which the argument breaks down throws up an interesting question involving weighted inequalities for operators along curves. Specifically, for fixed $\omega \in S^{d-1}$, what values of $r \in (1,\infty)$ and $s \in (0,\infty)$ is it true that

$$\int_{\mathbb{R}^d} |Hf(x,\omega)|^r Mf(x,\omega)^{-s}\,dx \leq C(r,s,\omega)\int_{\mathbb{R}^d} |f(x)|^r Mf(x,\omega)^{-s}\,dx$$

holds for some finite constant $C(r,s,\omega)$, and if so, how does $C(r,s,\omega)$ depend on ω?

We prove Theorem 2.1.2(2) using a similar technique to Step 1 for the maximal operator. Fix $q \in (2, q_d(2))$ and choose $\nu \in (0,1/2)$ as in (2.12). It suffices to prove

$$\|Hf\|_{L^2(L^q)} \lesssim \|f\|_2 \quad \text{for } q \in (2, q_d(2)),$$

and by Sobolev embedding, it therefore suffices to prove

$$\|Hf\|_{L^2(L^2_\nu)} \lesssim \|f\|_2 \quad \text{for } q \in (2, q_d(2)). \tag{2.26}$$

But, by Plancherel's theorem,

$$\|Hf\|^2_{L^2(L^2_\nu)} \sim \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \|m(\xi,\cdot)\|^2_{L^2_\nu}\,d\xi,$$

where

$$m(\xi,\omega) := p.v.\int_{\mathbb{R}} e^{i\delta_t\omega.\xi}\frac{dt}{t}, \tag{2.27}$$

and therefore (2.26) follows if we can show

$$\sup_{\xi\in\mathbb{R}^d} \|m(\xi,\cdot)\|_{L^2_\nu} \lesssim 1. \tag{2.28}$$

We shall make a dyadic splitting of the integral in (2.27) using the same $\sigma \in (1,\infty)$. So, for each $k \in \mathbb{Z}$, define

$$m_k(\xi,\omega) := \int_{|t|\in[1,\sigma]} e^{i\delta^*_{\sigma^k}\xi.\delta_t\omega}\frac{dt}{t},$$

in such a way that

$$m(\xi,\omega) = \sum_{k\in\mathbb{Z}} \int_{|t|\in[\sigma^k,\sigma^{k+1}]} e^{i\xi.\delta_t\omega}\frac{dt}{t} = \sum_{k\in\mathbb{Z}} \int_{|t|\in[1,\sigma]} e^{i\xi.\delta_{\sigma^k t}\omega}\frac{dt}{t} = \sum_{k\in\mathbb{Z}} m_k(\xi,\omega).$$

We claim that, if $\varepsilon = 1/2(1/2 - \nu)$ then, for almost all ξ,

$$\|m_0(\xi, \cdot)\|_{L^2_\nu} \lesssim \min(|\xi|, |\xi|^{-\varepsilon}). \tag{2.29}$$

It follows from, $m_k(\xi, \cdot) = m_0(\delta^*_{\sigma^k}\xi, \cdot)$, along with (2.6) and (2.7), that (2.29) implies (2.28). We prove (2.29) by showing that (2.15) and (2.16) hold with m_0 replacing the m which appears in these equations (and not m defined in (2.27)), and interpolating. The estimates for small $|\xi|$ are again easy to verify. The estimates for large $|\xi|$ follow from Lemma 2.3.1 and the fact that $\delta_t = -\delta_{-t}$ for negative t.

2.5 Proof of Lemma 2.3.1

Firstly, choose $C_\varepsilon > \sigma$ such that $\log|\xi| \le |\xi|^{2\varepsilon}$ for $|\xi| \ge C_\varepsilon$. Since $C_\varepsilon \sim 1$, it is clear that we only need to consider $|\xi| \ge C_\varepsilon$.

We shall handle the cases $d \ge 3$ and $d = 2$ separately. In the former case we make use of the following well-known estimate on the Fourier transform of surface measure $d\omega$ on S^{d-1}:

$$|\widehat{d\omega}(\eta)| \lesssim \min(1, |\eta|^{-(d-1)/2}). \tag{2.30}$$

The decay exponent in (2.30) is sharp and we shall see that this is the reason for our dimensional dichotomy.

So firstly, suppose $d \ge 3$. We write,

$$\int_{S^{d-1}} \left| \int_1^\sigma \Psi_{(\xi,\omega)}(t) e^{i\xi\cdot\delta_t\omega}\, dt \right|^2 d\omega = \int_1^\sigma \int_1^\sigma \widehat{d\omega}((\delta_t^* - \delta_s^*)\xi) \Psi_{(\xi,\omega)}(t) \overline{\Psi_{(\xi,\omega)}(s)}\, dtds,$$

in order to capitalise on the decay exponent in (2.30). Thus, using (2.30) and also (2.17), we get

$$\begin{aligned}
\int_{S^{d-1}} \left| \int_1^\sigma \Psi_{(\xi,\omega)}(t) e^{i\xi\cdot\delta_t\omega}\, dt \right|^2 d\omega &\lesssim |\xi|^{2a} \int_{\substack{(s,t)\in[1,\sigma]^2 \\ 0<(t-s)|\xi|\le 1}} 1\, dtds \\
&\quad + |\xi|^{2a} \int_{\substack{(s,t)\in[1,\sigma]^2 \\ 1<(t-s)|\xi|}} |(\delta_t^* - \delta_s^*)\xi|^{-(d-1)/2}\, dtds \\
&=: |\xi|^{2a}(I + II).
\end{aligned}$$

Clearly I is comparable to the measure of a rectangle in $\mathbb{R}^2$ with sidelengths $|\xi|^{-1}$ and 1. Hence $I \lesssim |\xi|^{-1}$, and the contribution from this term is suitably under control.

We claim that for all $|\xi| \ge C_\varepsilon$, and all $(s,t) \in [1,\sigma]^2$ with $t > s$ we have,

$$|(\delta_t^* - \delta_s^*)\xi| \gtrsim (t - s)|\xi|. \tag{2.31}$$

Firstly we complete the proof of Lemma 2.3.1 for $d \geq 3$ equipped with (2.31). We may as well suppose $(\sigma - 1)|\xi| > 1$, otherwise there is no II term. If $d \geq 4$ then

$$\begin{aligned} II &\lesssim |\xi|^{-(d-1)/2} \int_1^\sigma \int_{s+|\xi|^{-1}}^\sigma (t-s)^{-(d-1)/2}\, dtds \\ &\leq |\xi|^{-(d-1)/2} \int_1^\sigma |\xi|^{(d-3)/2}\, ds \sim |\xi|^{-1}, \end{aligned}$$

whilst if $d = 3$,

$$II \lesssim |\xi|^{-1} \int_1^\sigma \int_{s+|\xi|^{-1}}^\sigma (t-s)^{-1}\, dtds \lesssim |\xi|^{-1} \int_1^\sigma \log|\xi|\, ds \lesssim |\xi|^{-1+2\varepsilon}.$$

Notice that the estimate for I also holds when $d = 2$. However, a simple computation shows that when $d = 2$ the best one can hope from the term II is the weaker estimate $|\xi|^{-1/2}$. We shall therefore use an alternative argument when $d = 2$ which instead capitalises on the decay from the t-integral for fixed ω. Before moving on to this case, we prove our claim in (2.31). For this, it clearly suffices to prove that for all $(s,t) \in [1,\sigma]^2$ with $t > s$,

$$\|(\delta_t - \delta_s)^{-1}\| \lesssim (t-s)^{-1}. \tag{2.32}$$

So we fix $(s,t) \in [1,\sigma]^2$ with $t > s$ and by writing

$$\delta_t - \delta_s = \delta_s(\delta_{t/s} - I),$$

we seek to get a bound on the norm of the inverse of $\delta_{t/s} - I$. Putting $u = t/s$ for notational convenience, we have $u \in [1,\sigma]$, and

$$\begin{aligned} \delta_u - I &= (\log u)P + \sum_{j=2}^\infty \frac{(\log u)^j}{j!} P^j \\ &= (\log u)P \left(I + \sum_{j=2}^\infty \frac{(\log u)^{j-1}}{j!} P^{j-1} \right). \end{aligned}$$

Setting $B(u) := -\sum_{j=2}^\infty (j!)^{-1} (\log u)^{j-1} P^{j-1}$, then, as long as $\sigma < 2$, we have

$$\begin{aligned} \|B'(v)\| = v^{-1} \left\| \sum_{j=2}^\infty \frac{(j-1)(\log v)^{j-2}}{j!} P^{j-1} \right\| &\lesssim \sum_{j=2}^\infty \frac{(j-1)(\log 2)^{j-2}}{j!} \|P\|^{j-1} \\ &=: C_P < \infty, \end{aligned}$$

for each $v \in (1,\sigma)$. Hence, if we choose $\sigma \in (1, \min(2, 1 + (2C_P)^{-1}))$ then the mean value theorem implies, $\|B(u)\| \leq C_P(u-1) \leq 1/2$. This implies $I - B(u)$ is invertible and moreover $\|(I - B(u))^{-1}\| \leq (1 - \|B(u)\|)^{-1} \leq 2$. Whence,

$$\|(\delta_u - I)^{-1}\| \leq (\log u)^{-1} \|P^{-1}\| \|(I - B(u))^{-1}\| \lesssim (u-1)^{-1}.$$

Therefore,

$$\|(\delta_t-\delta_s)^{-1}\| \le \|(\delta_u - I)^{-1}\| \|\delta_{1/s}\| \lesssim (u-1)^{-1} \sim (t-s)^{-1},$$

which proves (2.32) and consequently completes the proof of Lemma 2.3.1 in the case $d \ge 3$.

For $d=2$, first write $\omega \in S^1$ as $(\cos\theta, \sin\theta)$ for $\theta \in (0, 2\pi)$. We claim that for all $s \in (0, \log\sigma)$, and all $(\xi, \theta) \in \mathbb{R}^2 \setminus \{0\} \times (0, 2\pi)$ with

$$|\xi.(\cos\theta, \sin\theta)| \ge 1,$$

the following is true:

$$\left| \int_0^s e^{i\xi.e^{tP}(\cos\theta,\sin\theta)}\, dt \right|^2 \lesssim \frac{1}{|\xi.(\cos\theta,\sin\theta)|}. \tag{2.33}$$

To see how this would complete the proof of Lemma 2.3.1 in the case $d=2$, first note that

$$\begin{aligned}
&\int_1^\sigma \Psi_{(\xi,\omega)}(t) e^{i\xi.\delta_t(\cos\theta,\sin\theta)}\, dt \\
= &\int_0^{\log\sigma} e^{i\xi.e^{sP}(\cos\theta,\sin\theta)} e^s \Psi_{(\xi,\omega)}(e^s)\, ds \\
= &\int_0^{\log\sigma} \frac{d}{ds}\left(\int_0^s e^{i\xi.e^{tP}(\cos\theta,\sin\theta)}\, dt \right) e^s \Psi_{(\xi,\omega)}(e^s)\, ds \\
= &\ \sigma \Psi_{(\xi,\omega)}(\sigma) \int_0^{\log\sigma} e^{i\xi.e^{tP}(\cos\theta,\sin\theta)}\, dt \\
&- \int_0^{\log\sigma} \left(\int_0^s e^{i\xi.e^{tP}(\cos\theta,\sin\theta)}\, dt \right) e^s(\Psi_{(\xi,\omega)}(e^s) + e^s \Psi'_{(\xi,\omega)}(e^s))\, ds.
\end{aligned}$$

Then (2.17) and (2.33) imply

$$\left| \int_1^\sigma \Psi_{(\xi,\omega)}(t) e^{i\xi.\delta_t(\cos\theta,\sin\theta)}\, dt \right|^2 \lesssim \frac{|\xi|^{2a}}{|\xi.(\cos\theta,\sin\theta)|}. \tag{2.34}$$

A straightforward computation now gives

$$\int_{1 \le |\xi.(\cos\theta,\sin\theta)|} \left| \int_1^\sigma \Psi_{(\xi,\omega)}(t) e^{i\xi.\delta_t(\cos\theta,\sin\theta)}\, dt \right|^2 d\theta \lesssim |\xi|^{-1+2a} \log|\xi| \lesssim |\xi|^{-1+2a+2\varepsilon}.$$

Since we also have the trivial estimate

$$\int_{|\xi.(\cos\theta,\sin\theta)|<1} \left| \int_1^\sigma e^{i\xi.\delta_t(\cos\theta,\sin\theta)}\, dt \right|^2 d\theta \lesssim \int_{|\xi.(\cos\theta,\sin\theta)|<1} 1\, d\theta \lesssim |\xi|^{-1},$$

the proof of Lemma 2.3.1 will be complete once we prove (2.33). We shall accomplish this by fixing ξ and θ, and invoking van der Corput's lemma, with the phase function Θ defined by

$$\Theta(t) := \xi.e^{tP}(\cos\theta, \sin\theta) \quad \text{for } t \in [0, s].$$

Our first observation is an explicit formula for the exponential of a general 2 by 2 matrix. Let

$$P := \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix},$$

and let $\Delta := ((a_1 - a_4)^2 + 4a_2a_3)^{1/2}$. A direct computation gives:

$$e^{tP} = e^{\frac{(a_1+a_4)t}{2}} \begin{pmatrix} \cosh(\frac{\Delta t}{2}) + (a_1 - a_4)\frac{\sinh(\frac{\Delta t}{2})}{\Delta} & 2a_2\frac{\sinh\frac{\Delta t}{2}}{\Delta} \\ 2a_3\frac{\sinh\frac{\Delta t}{2}}{\Delta} & \cosh(\frac{\Delta t}{2}) - (a_1 - a_4)\frac{\sinh(\frac{\Delta t}{2})}{\Delta} \end{pmatrix}$$

We consider the cases where Δ is nonzero and zero separately, and firstly suppose the former. Then we may write

$$\Theta(t) = e^{Ct}(A\sinh(\Delta t/2) + B\cosh(\Delta t/2)), \quad \text{where}$$

$$\begin{aligned} A(\xi,\theta) &:= \Delta^{-1}((a_1 - a_4)\xi_1\cos\theta + 2a_2\xi_1\sin\theta + 2a_3\xi_2\cos\theta - (a_1 - a_4)\xi_2\sin\theta), \\ B(\xi,\theta) &:= \xi.(\cos\theta, \sin\theta), \\ C &:= (a_1 + a_4)/2. \end{aligned}$$

We claim that the following estimates hold on the first and second derivatives of Θ: if $\beta := \det P/(2(C^2 + \Delta^2/4))$ then, for all $t \in [0, s]$,

$$2C|B(\xi,\theta)| \leq (1+\beta)|\Delta||A(\xi,\theta)| \quad \Rightarrow \quad |\Theta''(t)| \gtrsim |B(\xi,\theta)|, \tag{2.35}$$

$$2C|B(\xi,\theta)| \geq (1+\beta)|\Delta||A(\xi,\theta)| \quad \Rightarrow \quad |\Theta'(t)| \gtrsim |B(\xi,\theta)|. \tag{2.36}$$

We shall also show that Θ'' has $\lesssim 1$ zeros on $[0, \log\sigma]$. This allows us to split the integral in (2.33) into $\lesssim 1$ pieces where Θ' is monotone and thus, (4.39) and (4.38) imply (2.33) via van der Corput's lemma.

To begin our proof of the claim, first recall that P has real entries and the eigenvalues of P have positive real part. Therefore the following hold:

1. The eigenvalues of P are $C \pm \Delta/2$ and $C > 0$.

2. $C^2 - \Delta^2/4 = a_1a_4 - a_2a_3 = \det P > 0$.

Thus, β is well defined and is certainly positive. Now, writing $A = A(\xi,\theta)$ and $B = B(\xi,\theta)$, we have

$$\begin{aligned} \Theta'(t) &= e^{Ct}((CA + \Delta B/2)\sinh(\Delta t/2) + (CB + \Delta A/2)\cosh(\Delta t/2)), \quad \text{and} \\ \Theta''(t) &= e^{Ct}((C^2A + \Delta CB + \Delta^2 A/4)\sinh(\Delta t/2) \\ &\qquad + (C^2B + \Delta CA + \Delta^2 B/4)\cosh(\Delta t/2)). \end{aligned}$$

Let us first look at what happens when $t = 0$, and begin with the case $2C|B| \leq (1+\beta)|\Delta||A|$. Then,

$$\begin{aligned} |\Theta''(0)| &= |C^2B + \Delta CA + \Delta^2 B/4| \\ &\geq C|\Delta||A| - (C^2 + \Delta^2/4)|B| \\ &\geq C|\Delta||A| - (C^2 + \Delta^2/4)\frac{(1+\beta)|\Delta||A|}{2C} \\ &= \frac{|\Delta||A|}{2C}(2C^2 - (C^2 + \Delta^2/4)(1+\beta)), \end{aligned}$$

and our choice of β ensures

$$|\Theta''(0)| \geq \frac{|\Delta||A| \det P}{4C} \gtrsim |B|. \tag{2.37}$$

For $2C|B| \geq (1+\beta)|\Delta||A|$ we have

$$|\Theta'(0)| = |CB + \Delta A/2| \geq C|B| - |\Delta||A|/2 \geq \beta|B|/(1+\beta) \gtrsim |B|. \tag{2.38}$$

Next, note that there exists some $t_0 \sim 1$, such that

$$|\cosh(\Delta t/2)| \geq 1/2 \quad \text{and} \quad |\sinh(\Delta t/2)| \leq |\Delta| t \quad \text{for } t \in [0, t_0].$$

If $2C|B| \leq (1+\beta)|\Delta||A|$, then (2.37) implies there is a constant $c \sim 1$, such that

$$|\Theta''(0)| \geq 4c|\Delta||C^2A + \Delta CB + \Delta^2 A/4| \geq 4t|\Delta||C^2A + \Delta CB + \Delta^2 A/4|,$$

as long as σ is chosen such that $\log\sigma \leq c$. Therefore, if we also ensure $\log\sigma \leq t_0$, then (2.37) implies

$$|\Theta''(t)| \geq |\Theta''(0)|/2 - |\Delta||C^2A + \Delta CB + \Delta^2 A/4|t \geq |\Theta''(0)|/4 \gtrsim |B|,$$

which is (4.39). Similarly, if we suppose $2C|B| \geq (1+\beta)|\Delta||A|$, then there exists a constant $c' > 0$ such that

$$|\Theta'(0)| \geq 4c'|\Delta||CA + |\Delta|B/2|,$$

and this, (2.38), and a choice of σ with $\log\sigma \leq c'$, imply

$$|\Theta'(t)| \geq |\Theta'(0)|/2 - |\Delta||CA + \Delta B/2|t \geq |\Theta'(0)|/4 \gtrsim |B|.$$

Thus, we have proved (4.39) and (4.38). It remains to show that the number of zeros of Θ'' on $[0, \log\sigma]$ is $\lesssim 1$. To see this, if we write,

$$\Theta''(t) = e^{Ct}(\tilde{A}\sinh(\Delta t/2) + \tilde{B}\cosh(\Delta t/2)),$$

where

$$\begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix} = \begin{pmatrix} C^2 + \Delta^2/4 & \Delta C \\ \Delta C & C^2 + \Delta^2/4 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix},$$

then we must have that $(\tilde{A}, \tilde{B}) \neq (0,0)$. Otherwise, we would be able to use the fact that $B \neq 0$ to deduce the following nonsense,

$$0 = (C^2 + \Delta^2/4)^2 - \Delta^2 C^2 = (C^2 - \Delta^2/4)^2 = (\det P)^2 > 0.$$

Observe that if $\Theta''(t) = 0$ then $(\tilde{A} + \tilde{B})e^{\Delta t} + \tilde{B} - \tilde{A} = 0$, and consequently, $\tilde{A} + \tilde{B} \neq 0$. Letting $z := (\tilde{A} - \tilde{B})(\tilde{A} + \tilde{B})^{-1}$, and for argument's sake, $\arg(z) \in [0, 2\pi)$, we must have $\Delta t = \log|z| + i(\arg(z) + 2k\pi)$ for some $k \in \mathbb{Z}$. The fact that $|\Delta t| \lesssim 1$ means of course $|k| \lesssim 1$ and therefore the number of possible t such that $\Theta''(t) = 0$ is $\lesssim 1$. This completes the proof of (2.33) when $\Delta \neq 0$.

Suppose finally that $\Delta = 0$, so that the phase function Θ simplifies to $\Theta(t) = e^{Ct}(B + At)$, where

$$A(\xi, \theta) := ((a_1 - a_4)\xi_1 \cos\theta)/2 + a_2\xi_1 \sin\theta + a_3\xi_2 \cos\theta - ((a_1 - a_4)\xi_2 \sin\theta)/2,$$

and $B(\xi, \theta)$ and C are unchanged. One can check that, modulo a suitable choice of σ, $|\Theta''(t)| \gtrsim |B(\xi, \theta)|$ if $3|A(\xi, \theta)| \geq 2C|B(\xi, \theta)|$, and $|\Theta'(t)| \gtrsim |B(\xi, \theta)|$ if $3|A(\xi, \theta)| \leq 2C|B(\xi, \theta)|$. It is straightforward to check that Θ'' has at most one zero in $[0, \log\sigma]$. This concludes our proof of Lemma 2.3.1.

Remark. The proof of Lemma 2.3.1 shows that if $d \geq 4$ and P is a real d by d matrix whose eigenvalues have positive real part, then there exists a number $\sigma \in (1, \infty)$ such that,

$$\int_{S^{d-1}} \left| \int_1^\sigma e^{i\xi.\delta_t\omega}\, dt \right|^2 d\omega \lesssim |\xi|^{-1}. \tag{2.39}$$

The loss of an epsilon power in the statement of Lemma 2.3.1 arose from our arguments for $d = 2$ and $d = 3$. We claim that in the case of parabolic dilations in the plane there is no loss of epsilon. To see this, fix $\theta \in (0, 2\pi)$ and ξ with $|\xi|$ much larger than 1, and let $\Theta(t) := \xi_1 t \cos\theta + \xi_2 t^2 \sin\theta$ for $t \in [1, 2]$ (we are thus choosing $\sigma = 2$, but the claim actually holds for any $\sigma \in (1, \infty)$). We shall apply van der Corput's lemma on the first and second derivatives of Θ; clearly Θ' is monotone.

Split the θ-integral over $[0, 2\pi]$ into disjoint subintervals I_1, I_2, and I_3, where

$$\begin{aligned} I_1 &:= \{\theta : |\xi_1||\cos\theta| \leq |\xi_2||\sin\theta|\}, \\ I_2 &:= \{\theta : |\xi_1||\cos\theta| \geq 8|\xi_2||\sin\theta|\}, \end{aligned}$$

and I_3 is of course the complement of $I_1 \cup I_2$. Notice that for $\theta \in I_1$ we have $|\Theta'(t)| \geq |\xi_2||\sin\theta|$ for all $t \in [1, 2]$ and therefore van der Corput's lemma implies

$$\left| \int_1^2 e^{i\Theta(t)}\, dt \right| \lesssim \frac{1}{|\xi_2||\sin\theta|} \lesssim \frac{1}{|\xi_1||\cos\theta|}. \tag{2.40}$$

When $|\xi_2| \geq |\xi_1|$, the stronger estimate in (2.40) implies

$$\int_{\theta \in I_1} \left| \int_1^2 e^{i\Theta(t)}\, dt \right|^2 d\theta \;\lesssim\; |\xi_2|^{-2} \int_{\substack{\theta \in I_1 \\ |\xi_2||\sin\theta|>1}} |\sin\theta|^{-2}\, d\theta + \int_{\substack{\theta \in I_1 \\ |\xi_2||\sin\theta|\leq 1}} 1\, d\theta$$
$$\lesssim\; |\xi_2|^{-1} \sim |\xi|^{-1}.$$

On the other hand, when $|\xi_1| \geq |\xi_2|$ one can perform a similar argument to the one above using the weaker estimate in (2.40). This takes care of the contribution from I_1. For I_2, an analogous argument works; we spare the reader the details.

For $\theta \in I_3$, observe that $|\Theta''(t)| \sim |\xi_2||\sin\theta| \sim |\xi_1||\cos\theta|$ for all $t \in [1, 2]$. Also notice that if $|\xi_2| \geq |\xi_1|$ then $|\cos\theta| \sim 1$ and $|\xi_2||\sin\theta| \gtrsim 1$. Thus, when $|\xi_2| \geq |\xi_1|$, van der Corput's lemma implies

$$\int_{\theta \in I_3} \left| \int_1^2 e^{i\Theta(t)}\, dt \right|^2 d\theta \lesssim |\xi_2|^{-1} \int_{\theta \in I_3} |\sin\theta|^{-1}\, d\theta \sim |\xi_2|^{-1} \sim |\xi|^{-1}.$$

A similar argument works for the contribution from I_3 if $|\xi_1| \geq |\xi_2|$; this completes the proof of our claim. It may be of interest to establish whether (2.39) holds for all dilations given by (1.26) and all $d \geq 2$.

2.6 Some applications

Variable kernel singular integrals

Recall that τ is the trace of P, and $\mathfrak{J}$ is defined via the change of variables in (2.3). Suppose K, defined on $\mathbb{R}^d \times \mathbb{R}^d$, satisfies the following conditions:

(K1). $K(x, \cdot)$ is homogeneous of degree $-\tau$ with respect to the dilations δ_t for positive t;

(K2). $K(x, \cdot)\mathfrak{J}$, defined on S^{d-1}, is an odd function;

(K3). $\sup_{x \in \mathbb{R}^d} \left(\int_{S^{d-1}} |K(x, \omega)|^r\, d\omega \right)^{1/r} < \infty$.

(Note that the above conditions are not precisely those that appear in Chapter 1, but we reuse the notation in order to maintain a correspondence). Then the following theorem holds via the estimates for H given by Theorem 2.1.2 and the method of rotations.

Theorem 2.6.1. *The operator T defined in (1.13) is bounded on L^p provided*

1. $p \in (1, 2]$ *and* $r \in ((1 - 1/d)p', \infty)$; *or*

2. $p \in (2, \infty)$ *and* $r \in (p(d-1)/(p(d-1) - (d-2)), \infty)$.

Also known in this context is the work of Fabes and Rivière in [26] where a weaker cancellation condition and a substantially stronger smoothness condition are assumed. Specifically, it is shown that T is bounded on L^p for all $p \in (1, \infty)$ under the above homogeneity condition (K1) and the following conditions:

(K2'). $\int_{S^{d-1}} K(x,\omega)\mathfrak{J}(\omega)\, d\omega = 0$;

(K3'). $K(x,\cdot)$ belongs to $C^\infty(\mathbb{R}^d \setminus \{0\})$ and $\sup_{x\in\mathbb{R}^d} \|D^\alpha K(x,\cdot)\|_{L^\infty(S^{d-1})} < \infty$.

Fabes and Rivière proved this result using a spherical harmonic expansion of the kernel, in the spirit of the work of Calderón and Zygmund in [7] (see also [8]). Our approach in the nonisotropic setting is to follow [19]; recall our discussion at the end of Chapter 1, where we highlighted the success of the method of rotations in handling kernels satisfying the weak smoothness condition (K3) above. Notice that for $p \in (1, 2]$, Theorem 2.6.1 shows that the same outcome holds for isotropic and nonisotropic dilations. It would be nice to be able to show that one can prove that Theorem 2.6.1 holds with (K2) replaced with (K2'). The standard approach to handle the even case with isotropic dilations is to make use of the Riesz kernels. At present, we are working on an analogous argument in our nonisotropic setting. We include our next theorem as a potential first step towards this. Indeed, in the isotropic case, the result is crucial to the standard argument for handling even kernels (see [8] and [27]); a nonisotropic version appears in [55] when P is a diagonal matrix.

Theorem 2.6.2. *For $\varepsilon > 0$, define $K_\varepsilon(x,y) := \varepsilon^{-\tau} N(x,y)\Psi(\delta_{\varepsilon^{-1}}(y))$, where*

1. *$N(x,\cdot)$ is homogeneous of degree $-\tau$ with respect to the dilations δ_t for positive t;*
2. *$\sup_{x\in\mathbb{R}^d} \left(\int_{S^{d-1}} |N(x,\omega)|^r\, d\omega\right)^{1/r} < \infty$;*
3. *Ψ is a nonnegative and nonincreasing L^1 function, radial with respect to ϱ; that is, $\Psi = \psi(\varrho(\cdot))$ for some nonnegative and nonincreasing function ψ on $[0, \infty)$.*

Then the operator T^* defined by

$$T^* f(x) := \sup_{\varepsilon>0} \left| \int_{\mathbb{R}^d} K_\varepsilon(x,y) f(x-y)\, dy \right|,$$

is bounded on L^p provided that either (1) or (2) of Theorem 2.6.1 holds.

Proof. Passing to nonisotropic polar coordinates and using the homogeneity of N we get

$$\left|\int_{\mathbb{R}^d} K_\varepsilon(x,y)f(y)\,dy\right| \leq \int_{S^{d-1}} |N(x,\omega)| \left(\varepsilon^{-\tau}\int_0^\infty \psi(\varepsilon^{-1}t)t^{\tau-1}|f(x-\delta_t\omega)|\,dt\right) d\omega, \tag{2.41}$$

for each $\varepsilon > 0$. We claim that, for fixed $\omega \in S^{d-1}$, the term against which $|N(x,\omega)|$ integrates in (2.41) is $\lesssim Mf(x,\omega)$. Given the claim, the proof of Theorem 2.6.2 follows from Hölder's inequality and condition (2) of this theorem.

To prove the claim, first write

$$\varepsilon^{-\tau}\int_0^\infty \psi(\varepsilon^{-1}t)t^{\tau-1}|f(x-\delta_t\omega)|\,dt = \int_0^\infty\int_0^{\varepsilon^{-\tau}\psi(\varepsilon^{-1}t)} t^{\tau-1}|f(x-\delta_t\omega)|\,dsdt. \tag{2.42}$$

For fixed positive s, the set $\{t \in (0,\infty) : s \leq \varepsilon^{-\tau}\psi(\varepsilon^{-1}t)\}$ is some interval $(0, t(s)]$ because ψ is nonincreasing. If we suppose that $\tau \in [1,\infty)$, by changing the order of integration, the quantity in (2.42) is $\leq$

$$\int_0^{\varepsilon^{-\tau}\psi(0)} t(s)^\tau . t(s)^{-1}\int_0^{t(s)} |f(x-\delta_t\omega)|\,dtds \leq \mathcal{M}f(x,\omega)\int_0^{\varepsilon^{-\tau}\psi(0)}\int_0^{t(s)} t^{\tau-1}\,dtds. \tag{2.43}$$

By changing back the order of integration and a change of variables, the right hand side of (2.43) is $\lesssim \|\Psi\|_1\mathcal{M}f(x,\omega)$, which completes the proof of our claim when $\tau \in [1,\infty)$. When $\tau \in (0,1)$, for each positive s we have

$$\begin{aligned}
\int_0^{t(s)} |f(x-\delta_t\omega)|t^{\tau-1}\,dt &= \int_0^{t(s)}\int_0^{t^{\tau-1}} |f(x-\delta_t\omega)|\,dudt \\
&= \int_0^{t(s)^{\tau-1}}\int_0^{t(s)} |f(x-\delta_t\omega)|\,dtdu \\
&\quad + \int_{t(s)^{\tau-1}}^\infty \int_0^{u^{1/(\tau-1)}} |f(x-\delta_t\omega)|\,dtdu \\
&\lesssim t(s)^\tau\mathcal{M}f(x,\omega).
\end{aligned}$$

Therefore, the right hand side of (2.42) is $\lesssim$ the right hand side of (2.43). From this point, we finish the proof of our claim for $\tau \in (0,1)$ as we did for $\tau \in [1,\infty)$. □

A nonisotropic Kakeya maximal function

For a bounded subset F of $\mathbb{R}^d$, define its diameter with respect to P, $\mathrm{diam}_P(F)$, by

$$\mathrm{diam}_P(F) := \sup\{\varrho(x-y) : x, y \in F\},$$

and its eccentricity with respect to P, $\mathcal{E}_P(F)$, by

$$\mathcal{E}_P(F) := \frac{\operatorname{diam}_P(F)^\tau}{|F|}.$$

We also wish to introduce a notion of star-shaped in our nonisotropic context, and in particular with respect to the origin. We shall say that F is star-shaped with respect to the origin and the matrix P if

$$F = \{\delta_r \omega : \omega \in S^{d-1} \text{ and } 0 \le r < R(\omega)\}, \tag{2.44}$$

for some nonnegative measurable function R on S^{d-1}.

Remarks. 1. When P is the identity matrix, the above reduce to the usual definitions of diameter, eccentricity, and star-shapeliness of bounded sets in Euclidean space.

2. Suppose F is star-shaped with respect to the origin and P. Using (2.5),

$$\sup\{\varrho(x) : x \in F\} \le \operatorname{diam}_P(F) \le C_P \sup\{\varrho(x) : x \in F\},$$

and therefore, using the notation of (2.44), homogeneity, and (??),

$$\sup\{R(\omega) : \omega \in S^{d-1}\} \le \operatorname{diam}_P(F) \le C_P \sup\{R(\omega) : \omega \in S^{d-1}\}. \tag{2.45}$$

3. Eccentricity with respect to P is invariant under the action of the dilations δ_r. One can easily verify the following:

$$\mathcal{E}_P(\delta_r F) = \frac{[r\operatorname{diam}_P(F)]^\tau}{\det(\delta_r)|F|} = \mathcal{E}_P(F). \tag{2.46}$$

Example 2.6.3. Suppose $d = 2$ and $P = \operatorname{diag}(1, 2)$, so that we have parabolic dilations. If a point $(x_0, y_0) \in \mathbb{R}^2$ lies in $F \subseteq \mathbb{R}^2$, and $x_0 > 0$, then in order to satisfy (2.44) and be star-shaped with respect to the origin and parabolic dilations, F must contain the section of the parabola $y = (y_0/x_0^2)x^2$ for $x \in [0, x_0]$.

For a positive number N, let $\mathfrak{F}_N$ denote the family of all subsets of $\mathbb{R}^d$ which are star-shaped with respect to the origin and P, and have eccentricity with respect to P no greater than N. Define the following maximal operator,

$$M_{\mathfrak{F}_N} f(x) := \sup_{F \in \mathfrak{F}_N} |F|^{-1} \int_F |f(x-y)|\, dy.$$

With isotropic dilations, L^p estimates on $M_{\mathfrak{F}_N}$ were established in Corollary 3.5 of [19]. These estimates are easily shown to imply that the conjecture in (1.30) concerning the standard Kakeya maximal operator is true for all $p \in$

$(1, \max(2, (d+1)/2)]$. We shall use Theorem 2.1.1 to prove analogous estimates in the nonisotropic setting. Following [19], for fixed p, we shall need to know the dependence on q of the constant $C_{P,d,p,q}$ in the following estimate from Theorem 2.1.1:

$$\|\mathcal{M}f\|_{L^p(L^q)} \leq C_{P,d,p,q}\|f\|_p \quad \text{for } q < q_d(p).$$

To simplify the notation in the remainder of this section, use introduce the following notation.

Notation. For positive numbers A and B, write $A \lessapprox B$ for $A \leq CB$, where C depends only on the matrix P and the ambient dimension d. Also write $A \approx B$ if $A \lessapprox B \lessapprox A$.

We are most interested in the endpoint $p_0 := \max(2, (d+1)/2)$ since all of our estimates on $\mathcal{M}$ in Theorem 2.1.1 follow from our sharp estimates at this point. Recall the theorem of Gressman on page 29 from which we were able to deduce the estimates at p_0 and for $q < q_d(p_0)$ when $d \geq 4$. Gressman proves this theorem in [31] by showing that restricted weak type estimates hold at the endpoints in (2.20). In our application, this set of points reduced to the singleton set containing $(1/p_0, 1/q_d(p_0)')$, and therefore a restricted weak type estimate holds at this point. After unravelling the duality, one can interpolate from the resulting restricted weak type $(p_0, q_d(p_0))$ estimate using Marcinkiewicz interpolation (see, for example, [63]) to get strong type estimates away from the endpoint (these are essentially the estimates in the statement of Lemma 2.3.2). Moreover, the blow up in the constant as we approach the end-point can be computed. As in [19], it follows from our proof of Theorem 2.1.1 that there exists $\lambda \approx 1$ such that

$$\|\mathcal{M}f\|_{L^{p_0}(L^q)} \lessapprox \left(\frac{1}{q} - \frac{1}{q_d(p_0)}\right)^{-\lambda} \|f\|_p, \tag{2.47}$$

for all $q < q_d(p_0)$. Using this fact, we can prove the following theorem.

Theorem 2.6.4. *Fix a large positive number N (say no less than 100). Then, for each $d \geq 2$, one has the following estimate,*

$$\|M_{\mathfrak{F}_N}f\|_{p_0} \lessapprox (\log N)^{\lambda} N^{1/q_d(p_0)}\|f\|_{p_0}, \tag{2.48}$$

where $p_0 := \max(2, (d+1)/2)$.

Proof. For now, let q be any element of $(1, \infty)$. Using the fact that $F = \{\delta_r\omega : \omega \in S^{d-1} \text{ and } 0 \leq r < R(\omega)\}$, we use nonisotropic polar coordinates to get

$$|F|^{-1}\int_F |f(x-y)|\,dy \;\approx\; |F|^{-1}\int_{S^{d-1}}\int_0^{R(\omega)} |f(x-\delta_r\omega)|r^{\tau-1}\,drd\omega.$$

By using arguments from the proof of Theorem 2.6.2 one can show

$$\int_0^{R(\omega)} |f(x-\delta_r\omega)|r^{\tau-1}\,dr \lessapprox \mathcal{M}f(x,\omega)R(\omega)^\tau.$$

Note that $R(\omega)=R(\omega)^{1/q'}R(\omega)^{1/q}\lessapprox R(\omega)^{1/q'}\mathrm{diam}_P(F)^{1/q}$, by (2.45). Therefore, using Hölder's inequality, and the hypothesis that $\mathcal{E}_P(F)\le N$,

$$\begin{aligned}|F|^{-1}\int_F |f(x-y)|\,dy &\lessapprox |F|^{-1}\mathrm{diam}_P(F)^{\tau/q}\left(\int_{S^{d-1}} R(\omega)^\tau\,d\omega\right)^{1/q'}\|\mathcal{M}f(x,\cdot)\|_q\\ &\approx |F|^{-1+1/q'}\mathrm{diam}_P(F)^{\tau/q}\|\mathcal{M}f(x,\cdot)\|_q\\ &\le N^{1/q}\|\mathcal{M}f(x,\cdot)\|_q.\end{aligned}$$

Therefore,

$$\|M_{\mathfrak{F}_N}f\|_{p_0}\lessapprox N^{1/q}\|\mathcal{M}f\|_{L^{p_0}(L^q)}.$$

If we choose $q\in(1,q_d(p_0))$ such that

$$\frac{1}{q}-\frac{1}{q_d(p_0)}=\frac{1}{\log N},$$

then the desired estimate in (2.48) follows from (2.47). □

Maximal operators related to the operator $M_{\mathfrak{F}_N}$ concerning averages over curved sets have been studied by Wisewell in [68] and [69]. Minicozzi and Sogge [46] and Sogge [59] consider the quite different problem of geodesics in curved space. The estimate at $(d+1)/2$ (appearing in Theorem 2.6.4) was achieved in [69] for a very broad class of curves. Rather than the Fourier transform based proof that we used to prove Theorem 2.6.4, Wisewell proves a $(d+1)/2$ bound using more modern geometric techniques; in particular the bush argument of Bourgain. For the curves naturally associated to the dilations δ_t considered in this chapter, it is an interesting question as to whether the $(d+1)/2$ estimate for the maximal operator may be extended. This question was studied in some depth in [68] and [69] for parabolic curves in $\mathbb{R}^d$. It was shown that on the one hand there exist such curves for which the estimate $(d+1)/2$ is (in some sense) best possible. Nevertheless, some necessary conditions on the parabolic curve were given in [69] for which the $(d+1)/2$ bound can be extended. In fact, using recent arguments of Wolff and Katz, Wisewell proves a $(d+2)/2$ maximal operator estimate. Furthermore, using arithmetic methods, progress beyond $(d+1)/2$ was made on the question of the Minkowski dimension of certain related null sets.

For future work, we hope to fully address the question of whether Theorem 2.6.4 can be improved for certain curves naturally associated to the dilations δ_t. Moreover, we hope to investigate whether some of the more recent techniques developed for tackling Kakeya type maximal operators can form a basis for an argument which extends the range of p in the mixed-norm estimates for the *isotropic* directional maximal operator, $\mathcal{M}$, in Theorem 1.2.1. Any progress on this problem would naturally beg the question of whether similar progress could be made in the nonisotropic setting governed by the dilations δ_t considered in this chapter; that is to say, extend the range of p in Theorem 2.1.1.

Chapter 3

L^p-Boundedness of the Hilbert Transform and Maximal Operator Along a Class of Nonconvex Curves

3.1 Introduction

Recall the definitions of H_Γ and $\mathcal{M}_\Gamma$ from (1.18) and (1.19). The following theorem concerning a class of nonconvex curves $\Gamma : \mathbb{R} \to \mathbb{R}^2$ is the main result that we prove in this chapter.

Theorem 3.1.1. *Suppose P is a real polynomial and γ is convex on $[0, \infty)$, twice differentiable, either even or odd, $\gamma(0) = 0$, and $\gamma'(0) \geq 0$. If $\Gamma(t) = (t, P(\gamma(t)))$, $p \in (1, \infty)$, and either (1) $P'(0)$ is zero, or (2) $P'(0)$ is nonzero and $\gamma' \in \mathcal{C}_1$, then*

$$\|H_\Gamma f\|_p \leq C\|f\|_p \quad \text{and} \quad \|\mathcal{M}_\Gamma f\|_p \leq C\|f\|_p.$$

Moreover the constant C depends only on p, γ, and the degree of P.

Remarks. 1. By taking $\gamma(t) = t$ we recover a special case of Theorem 1.1.1 since we can then suppose $P'(0) = 0$. Our proof does not require the 'lifting' technique used in [60] to prove Theorem 1.1.1. Also, taking $P(s) = s$ we recover Theorem 1.1.6(1), Theorem 1.1.7(1), and the sufficiency part of Theorem 1.1.4.

2. Some examples of nonconvex curves were studied in [71], and later these were generalised somewhat through a technical theorem in [66]. Although the class of curves in Theorem 3.1.1 falls within the scope of [66], the bounds obtained from the technical theorem in [66] depend on the coefficients of P. Furthermore, our proof is more direct in this setting.

We shall see that ideas in our proof of Theorem 3.1.1 can be used for certain hypersurfaces instead of curves. Specifically, if $d \geq 2$ and $\Gamma : \mathbb{R}^d \to \mathbb{R}^{d+1}$ parameterises a hypersurface, then, ignoring a slight abuse of notation, we associate to this the corresponding Hilbert transform and maximal operator by

$$H_\Gamma f(x) := p.v. \int_{\mathbb{R}^d} f(x - \Gamma(y))K(y)\, dy,$$

$$\mathcal{M}_\Gamma f(x) := \sup_{h>0} h^{-d} \left| \int_{|y|\in(0,h)} f(x - \Gamma(y))\, dy \right|,$$

where $K : \mathbb{R}^d \to \mathbb{R}$ is a Calderón-Zygmund kernel; that is K is homogeneous of degree $-d$ with respect to isotropic dilations, K is of class C^∞ on $\mathbb{R}^d \setminus \{0\}$, and $\int_{|y|\in(a,b)} K(y)\, dy = 0$ for each $0 < a < b$. Again, it is clear that a dyadic version of the maximal operator, in analogue with (1.20), is equivalent to $\mathcal{M}_\Gamma$. Then we have the following theorem.

Theorem 3.1.2. *Suppose P is a real polynomial and γ is convex on $[0, \infty)$, twice differentiable, either even or odd, $\gamma(0) = 0$, and $\gamma'(0) \geq 0$. If $\Gamma(y) = (y, P(\gamma(|y|)))$ and $p \in (1, \infty)$ then*

$$\|H_\Gamma f\|_p \leq C\|f\|_p \quad \textit{and} \quad \|\mathcal{M}_\Gamma f\|_p \leq C\|f\|_p.$$

Moreover the constant C depends only on p, d, γ, and the degree of P.

Remark. The case $P(s) = s$ was proved in [37]. Notice how in this case the convexity of γ suffices for L^p boundedness, which is in stark contrast to the case $d = 1$ that we alluded to earlier.

Notation. Write $A \lesssim B$ for $A \leq CB$, where C is an absolute constant which may depend on p, γ, d, and the degree of P but is independent of the coefficients of P.

Overview. In the following section we make a suitable decomposition of our operators based on key results concerning polynomials of one variable. The next section contains the fundamental results for the proof of Theorem 3.1.1. In the last section we prove Theorem 3.1.2. Both Theorem 3.1.1 and Theorem 3.1.2 are to appear in [4].

3.2 Preliminaries and reductions

Let $P(s) = \sum_{k=1}^n p_k s^k$ be a real polynomial of degree n, where $n \geq 2$ (it is without loss of generality that we suppose $P(0) = 0$). With the model case that P is a monomial in mind, we let γ^j denote the jth power of γ and note that, using only the convexity of γ, it is simple to verify that $(\gamma^j)' \in \mathfrak{C}_2$ if $j \geq 2$. It will be a continuing theme throughout this chapter that the cases $j \geq 2$ and $j = 1$ will need separate considerations; the latter being the more difficult. If D is the doubling constant for $(\gamma^j)'$ then we consider the dyadic operator M_Γ with $\lambda := \max\{3, D\}$ (recall the role of λ in the dyadic operator M_Γ defined in (1.20)).

We now discuss the decomposition of $(0,\infty)$ crucial to the proof of Theorem 3.1.1. The ideas here originated from work in [10] (see also [29]). First let $z_1, \ldots, z_n$ be the roots of P ordered as $0 = |z_1| \leq |z_2| \leq \ldots \leq |z_n|$. Our decomposition will depend on a constant A which depends only on the degree of P and whose value we fix later. Firstly, we include $G_1 = (0, A^{-1}|z_2|]$. Then, for $j \in \{2, \ldots, n-1\}$, if the interval $(A|z_j|, A^{-1}|z_{j+1}|]$ is nonempty this is also included and called G_j. Finally, we include $G_n = [A|z_n|, \infty)$. Now let $\mathfrak{J} := \{1\} \cup \{n\} \cup \bigcup_{G_j \neq \emptyset} \{j\}$. Observe that $(0,\infty) \setminus \bigcup_{j \in \mathfrak{J}} G_j$ can be written as $\bigcup_{k \in \mathfrak{K}} D_k$ where the D_k are disjoint and, moreover, each $D_k = (\alpha_k, \beta_k)$ enjoys the property that $\alpha_k \sim \beta_k$. The notation is suggestive since the D_k resemble dyadic intervals and, as we are thinking of A as 'large', the G_j are 'long' intervals, or gaps of P. Our decomposition is then:

$$(0,\infty) = \bigcup_{j \in \mathfrak{J}} \gamma|_{(0,\infty)}^{-1}(G_j) \cup \bigcup_{k \in \mathfrak{K}} \gamma|_{(0,\infty)}^{-1}(D_k). \tag{3.1}$$

We of course then get the corresponding decomposition of $\mathbb{R}$ by taking symmetric versions of the intervals in the above decomposition. If I is a subset of $(0,\infty)$ then define H_I and M_I by

$$H_I f(x) := \int_{|t| \in \gamma|_{(0,\infty)}^{-1}(I)} f(x - \Gamma(t)) \frac{dt}{t}, \tag{3.2}$$

$$M_I f(x) := \sup_{k \in \mathbb{Z}} \lambda^{-k} \left| \int_{t \in [\lambda^k, \lambda^{k+1}] \cap \gamma|_{(0,\infty)}^{-1}(I)} f(x - \Gamma(t))\, dt \right|. \tag{3.3}$$

It is easy to see that each H_{D_k} and M_{D_k} are L^p bounded. After an application of Minkowski's inequality, this will follow if $\gamma^{-1}(\beta_k) \lesssim \gamma^{-1}(\alpha_k)$. In fact, (1.22) implies

$$\log \frac{\gamma^{-1}(\beta_k)}{\gamma^{-1}(\alpha_k)} = \int_{\gamma^{-1}(\alpha_k)}^{\gamma^{-1}(\beta_k)} \frac{dt}{t} = \int_{\alpha_k}^{\beta_k} \frac{1}{\gamma^{-1}(s)\gamma'(\gamma^{-1}(s))}\, ds \leq \int_{\alpha_k}^{\beta_k} \frac{ds}{s} = \log \frac{\beta_k}{\alpha_k},$$

and therefore,

$$\frac{\gamma^{-1}(\beta_k)}{\gamma^{-1}(\alpha_k)} \leq \frac{\beta_k}{\alpha_k} \lesssim 1. \tag{3.4}$$

Along with the fact that the cardinalities of $\mathfrak{J}$ and $\mathfrak{K}$ are $\lesssim 1$, Theorem 3.1.1 will follow once we verify that H_{G_j} and M_{G_j} are L^p bounded (with bounds independent of the coefficients of P), for each $j \in \mathfrak{J}$. So for the rest of this chapter, we fix $j \in \mathfrak{J}$, and for $k \in \mathbb{Z}$ we define $I_k := [1, \lambda] \cap \lambda^{-k}\gamma|_{(0,\infty)}^{-1}(G_j)$, and measures H_k and μ_k by:

$$\langle H_k, \psi\rangle := \int_{|t|\in I_k} \psi(\lambda^k t, P(\gamma(\lambda^k t)))\,\frac{dt}{t},\ \langle \mu_k, \psi\rangle := \int_{I_k} \psi(\lambda^k t, P(\gamma(\lambda^k t)))\,dt,$$

for $\psi \in \mathcal{S}(\mathbb{R}^2)$. In order to analyse H_{G_j} and M_{G_j}, we need to understand the behaviour of P on G_j. The following lemma is essentially contained in [10] and [29].

Lemma 3.2.1. *There exists a number $C_n > 1$ such that for any $A \geq C_n$,*

1. $|P(s)| \sim |p_j||s|^j$ *for all* $j \in \mathfrak{J}$ *and* $|s| \in G_j$*;*
2. $P'(s)/P(s) > 0$ *for all* $j \in \mathfrak{J}$ *and* $s \in G_j$*;* $P'(s)/P(s) < 0$ *for all* $j \in \mathfrak{J}$ *and* $-s \in G_j$*;*
3. $|s||P'(s)|/|P(s)| \sim 1$ *for all* $j \in \mathfrak{J}$ *and* $|s| \in G_j$*;*
4. $P''(s)/P(s) > 0$ *and* $s^2P''(s)/P(s) \sim 1$ *for all* $j \in \mathfrak{J} \setminus \{1\}$ *and* $|s| \in G_j$*.*

Proof. For (1)-(3) see Lemma 2.1 in [29] and Lemma 2.5 of [10]. For (4), let $\mathbb{N}_n := \{1, \ldots, n\}$ and define $S_1 := \{(l_1, l_2) \in \mathbb{N}_n \times \mathbb{N}_n : l_1 < l_2 \text{ and } l_2 \leq j\}$ and $S_2 := \{(l_1, l_2) \in \mathbb{N}_n \times \mathbb{N}_n : l_1 < l_2 \text{ and } l_2 \geq j+1\}$. Then write

$$\begin{aligned}\frac{P''(s)}{P(s)} &= 2\sum_{l_1<l_2} \frac{1}{(s-z_{l_1})(s-z_{l_2})}\\ &= 2\sum_{(l_1,l_2)\in S_1} \frac{1}{(s-z_{l_1})(s-z_{l_2})} + 2\sum_{(l_1,l_2)\in S_2} \frac{1}{(s-z_{l_1})(s-z_{l_2})}\\ &=: I + II.\end{aligned}$$

Let $\Re[z]$ denote the real part of z and suppose $A > 10$. Then, for $(l_1, l_2) \in S_1$,

$$\begin{aligned}\Re\left[\frac{1}{(s-z_{l_1})(s-z_{l_2})}\right] &= \frac{\Re[(s-z_{l_1})(s-z_{l_2})]}{|s-z_{l_1}|^2|s-z_{l_2}|^2}\\ &= \frac{s^2 - \Re[(z_{l_1}+z_{l_2})]s + \Re[z_{l_1}z_{l_2}]}{|s-z_{l_1}|^2|s-z_{l_2}|^2}\\ &\geq \frac{(1-2A^{-1}-A^{-2})}{(1+A^{-1})^4}\frac{1}{s^2},\end{aligned}$$

where the last inequality follows because $|z_{l_k}| \leq A^{-1}|s|$ for $k = 1, 2$.

If $l \leq j$ then $|s - z_l| \geq (1 - A^{-1})|s|$ and if $l \geq j+1$ then $|s - z_l| \geq (A-1)|s| \geq (1 - A^{-1})|s|$. Therefore, if $(l_1, l_2) \in S_2$ then

$$\frac{1}{|s-z_{l_1}||s-z_{l_2}|} \leq \frac{1}{A\,(1-A^{-1})^2}\frac{1}{s^2}.$$

If C_n' is twice the cardinality of S_1 and C_n'' is twice the cardinality of S_2 then

$$\frac{P''(s)}{P(s)} = \Re\left[\frac{P''(s)}{P(s)}\right] = \Re[I] + \Re[II]$$
$$\geq \left(\frac{C_n'(1-2A^{-1}-A^{-2})}{(1+A^{-1})^4} - \frac{C_n''}{A(1-A^{-1})^2}\right)\frac{1}{s^2}.$$

It is now clear that there is some $C_n > 1$ for which the first assertion of (4) and the lower bound in the remaining assertion follow for $A \geq C_n$. The upper bound is easier and we leave the details to the reader. □

By (the proof of) Lemma 3.2.1, we can choose A so that for all $|s| \in G_j$,

$$|P(s)| \leq 2|p_j|s^j \quad \text{and} \quad \frac{1}{2}j|p_j|s^{j-1} \leq |P'(s)| \leq 2j|p_j|s^{j-1}. \tag{3.5}$$

In the light of Lemma 3.2.1 it is an appropriate moment to discuss our method of proof of the L^p boundedness of H_{G_j} and M_{G_j}, and hence Theorem 3.1.1. Firstly, $P'(0)$ being zero is equivalent to G_1 being empty. Heuristically Lemma 3.2.1 tells us that on G_j the curve $(t, P(\gamma(t)))$ behaves like $(t, |p_j|\gamma(t)^j)$. Of course, when $j = 1$ some stronger condition than convexity is necessary. When G_1 is nonempty, under the assumption $\gamma' \in \mathfrak{C}_1$, we will be able to follow the proof in [13] or [22] to get L^p bounds for our operators on G_1. We stress here that, under the assumption $h \in \mathfrak{C}_2$ (or the stronger condition $\gamma' \in \mathfrak{C}_2$), the method of proof in [9] fails to work for our operators on G_1. Fundamental to the argument in [9] are dilation matrices and estimates on the Fourier transform of certain measures. However the fact that Lemma 3.2.1(4) does not hold for $j = 1$ means we are unable to achieve such estimates. For $j \geq 2$ either the approach in [13] (and also [22]) or [9] is available to us because $(\gamma^j)' \in \mathfrak{C}_2$. Therefore $(\gamma^j)' \in \mathfrak{C}_1$ and the h-function associated to γ^j belongs to $\mathfrak{C}_2$ (recall the definition of the h-function from page 14).

The following proposition, which can be found on page 384 of [12], lays down the bare essentials of a combination of ideas from [9], [13] and [22]. We use this to prove L^p bounds for H_{G_j} and M_{G_j}, and state it as follows:

Proposition 3.2.2. [12] *Suppose $\{A_k\}_{k\in\mathbb{Z}} \subseteq GL(2,\mathbb{R})$ satisfies*

$$\|A_{k+1}^{-1}A_k\| \leq \alpha < 1. \tag{3.6}$$

Suppose $\{\nu_k\}_{k\in\mathbb{Z}}$ is a family of measures satisfying

$$A_{k+1}^{-1}\operatorname{supp}\nu_k \subseteq B, \tag{3.7}$$

for some fixed ball B,

$$\widehat{\nu_k}(0) = 0, \tag{3.8}$$

and

$$|\widehat{\nu_k}(\xi)| \le C|A_k^*\xi|^{-1} \quad \textit{for } \xi \textit{ outside some cone } \triangle_k. \tag{3.9}$$

If T_k is defined by $\widehat{T_k f}(\xi) = \chi_{\triangle_k}(\xi)\hat{f}(\xi)$ and satisfies

$$\left\| \left(\sum_{k\in\mathbb{Z}} |T_k f|^2 \right)^{1/2} \right\|_p \le C_p \|f\|_p \quad \textit{for } p \in (1,\infty), \tag{3.10}$$

then $f \mapsto \sum_{k\in\mathbb{Z}} \nu_k * f$ is bounded on L^p for $p \in (1,\infty)$ with bound depending only on α, B, C and C_p.

3.3 L^p bounds for M_{G_j} and H_{G_j}

For $t > 0$ let

$$A(t) := \begin{pmatrix} t & 0 \\ 0 & |p_j|\gamma(t)^j \end{pmatrix}.$$

Define the family of dilations $\{A_k\}_{k\in\mathbb{Z}}$ by $A_k := A(\lambda^k)$, where we recall that $\lambda = \max\{3, D\}$ and D is the doubling constant for $(\gamma^j)'$.

We begin with M_{G_j} and create cancellation by introducing measures σ_k defined by:

$$\langle \sigma_k, \psi \rangle := \frac{\widehat{\mu_k}(0)}{|A_{k+1}B|} \int_{A_{k+1}B} \psi(x)\, dx,$$

where $B := \{x \in \mathbb{R}^2 : |x| < 10\}$. To complete the setup of Proposition 3.2.2, we define $\nu_k := \varepsilon_k(\mu_k - \sigma_k)$, where $\{\varepsilon_k\} \subseteq \{-1, 1\}$. Now (1.22) implies that $\gamma(t)^j/\gamma(s)^j \le t/s$ whenever $s \ge t > 0$, and therefore (3.6) holds with $\alpha = 2/\lambda < 1$. By (3.5), if $t \in I_k$ then $|P(\gamma(\lambda^k t))| \le 2|p_j|\gamma(\lambda^k t)^j \le 2|p_j|\gamma(\lambda^{k+1})^j$. Thus,

$$\operatorname{supp} \mu_k = \{(\lambda^k t, P(\gamma(\lambda^k t))) : t \in I_k\} \subseteq A_{k+1}B.$$

Of course σ_k is supported in $A_{k+1}B$, therefore so is ν_k and we have (3.7). It is trivial to verify (3.8). To deal with (3.9) and (3.10) we define $\triangle_k$ to be the set of $\xi = (\xi_1, \xi_2)$ in $\mathbb{R}^2$ satisfying:

$$4|p_j|(\gamma^j)'(\lambda^{k+1}) > \frac{|\xi_1|}{|\xi_2|} > \frac{1}{4}|p_j|(\gamma^j)'(\lambda^k). \tag{3.11}$$

The following lemma is well known.

Lemma 3.3.1. [50] *Let $\{\tau_k\}_{k\in\mathbb{Z}}$ be a sequence of positive real numbers such that for some $R > 1$, $\tau_{k+1} \ge R\tau_k$ for all $k \in \mathbb{Z}$. Let $M > 1$ and define $\triangle_k$ to be the set of all $\xi \in \mathbb{R}^2$ satisfying $M^{-1}\tau_k \le |\xi_1||\xi_2|^{-1} \le M\tau_{k+1}$. If $\widehat{T_k f} = \chi_{\triangle_k}\hat{f}$ then*

$$\left\| \left(\sum_{k\in\mathbb{Z}} |T_k f|^2 \right)^{1/2} \right\|_p \le C_p \|f\|_p,$$

for all $p \in (1,\infty)$.

It is immediate from Lemma 3.3.1 that we now have (3.10) (note there is no issue of the constant C_p depending on p_j because $|p_j|(\gamma^j)'(\lambda^{k+1})(|p_j|(\gamma^j)'(\lambda^k))^{-1} \geq 2$). If we can prove (3.9) then we are done. Indeed, $M_{G_j}f \leq \sup_k |\sigma_k * f| + (\sum_k |(\mu_k - \sigma_k) * f|^2)^{1/2}$. In L^p norm, the latter term is $\lesssim \|f\|_p$ by using a standard Rademacher function argument and the fact that the conclusion of Proposition 3.2.2 holds with bounds independent of ε, and the former term is $\lesssim \|f\|_p$ by Proposition 2.2 of [9] and the fact that $|\widehat{\mu_k}(0)| \lesssim 1$.

Before we prove (3.9) in Lemma 3.3.3 we need the following:

Lemma 3.3.2. *For all $j \in \mathfrak{J} \setminus \{1\}$, the function*

$$t \longmapsto P''(\gamma(\lambda^k t))\gamma'(\lambda^k t)^2 + P'(\gamma(\lambda^k t))\gamma''(\lambda^k t)$$

is singled-signed on I_k.

Proof. By (2) and (4) of Lemma 3.2.1, it must be the case that P' and P'' have the same sign on G_j. The convexity of γ implies $\gamma''(\lambda^k t)$ is nonnegative for $t \in I_k$ and so the result follows. □

Lemma 3.3.3. *If $\xi \notin \triangle_k$ then $|\widehat{\nu_k}(\xi)| \lesssim |A_k\xi|^{-1}$.*

Proof. Since

$$|\widehat{\sigma_k}(\xi)| \lesssim |\widehat{\chi_B}(A_{k+1}\xi)| \lesssim |A_{k+1}\xi|^{-1} \lesssim |A_k\xi|^{-1}, \tag{3.12}$$

we are left to find a decay estimate for $\widehat{\mu_k}$. Let $\theta(t) = \lambda^k t\xi_1 + P(\gamma(\lambda^k t))\xi_2$ for $t \in I_k$. Suppose first that $|\xi_1| > 4|p_j|(\gamma^j)'(\lambda^{k+1})|\xi_2|$. Then, by (3.5),

$$|\theta'(t)| \geq \lambda^k|\xi_1| - |P'(\gamma(\lambda^k t))|\gamma'(\lambda^k t)\lambda^k|\xi_2| \geq \lambda^k|\xi_1| - 2|p_j|(\gamma^j)'(\lambda^k t)\lambda^k|\xi_2| \gtrsim \lambda^k|\xi_1|.$$

Now $\theta''(t) = [P''(\gamma(\lambda^k t))\gamma'(\lambda^k t)^2 + P'(\gamma(\lambda^k t))\gamma''(\lambda^k t)]\lambda^{2k}\xi_2$. For any $j \neq 1$, Lemma 3.3.2 implies that θ'' is singled-signed on I_k and therefore we have that θ' is monotone on I_k. We now invoke van der Corput's lemma for these j to get $|\widehat{\mu_k}(\xi)| \lesssim (\lambda^k|\xi_1|)^{-1} \lesssim |A_k\xi|^{-1}$, where the last inequality follows from (1.22). The situation for $j = 1$ will be dealt with momentarily.

If now $|\xi_1| < \frac{1}{4}|p_j|(\gamma^j)'(\lambda^k)|\xi_2|$ then we use (3.5) to get

$$\begin{aligned}|\theta'(t)| &\geq \frac{1}{2}|p_j|(\gamma^j)'(\lambda^k t)\lambda^k|\xi_2| - \lambda^k|\xi_1| \\ &\geq \frac{1}{4}|p_j|(\gamma^j)'(\lambda^k t)\lambda^k|\xi_2| \geq \frac{1}{4}|p_j|(\gamma^j)'(\lambda^k)\lambda^k|\xi_2|.\end{aligned}$$

Another application of van der Corput's lemma and then (1.22) gives

$$|\widehat{\mu_k}(\xi)| \lesssim (|p_j|\gamma(\lambda^k)^{j-1}\gamma'(\lambda^k)\lambda^k|\xi_2|)^{-1} \lesssim |A_k\xi|^{-1},$$

which completes the proof for $j \neq 1$.

When $j = 1$ we are unable to divide I_k into a suitable number of intervals on which θ' is monotone and therefore we must argue in a slightly different way. Let us again begin with $|\xi_1| > 4|p_1|\gamma'(\lambda^{k+1})|\xi_2|$. Of course we still get $|\theta'(t)| \gtrsim \lambda^k|\xi_1|$ for $t \in I_k$. Using this and integration by parts (which is how the standard proof of van der Corput's lemma proceeds),

$$|\widehat{\mu_k}(\xi)| \lesssim (\lambda^k|\xi|)^{-1} + \int_{I_k} \frac{|\theta''(t)|}{\theta'(t)^2}\, dt.$$

Note $\int_{I_k} |\theta''(t)|/\theta'(t)^2\, dt$ is less than

$$\int_{I_k} \frac{\lambda^{2k}|\xi_2||P'(\gamma(\lambda^k t))|\gamma''(\lambda^k t)}{\theta'(t)^2}\, dt + \int_{I_k} \frac{\lambda^{2k}|\xi_2||P''(\gamma(\lambda^k t))|\gamma'(\lambda^k t)^2}{\theta'(t)^2}\, dt =: \alpha_1 + \alpha_2.$$

For α_1 we introduce $\phi(t) = \lambda^k t|\xi_1| + |p_1|\gamma(\lambda^k t)|\xi_2|$ for $t \in I_k$. Note, $\phi'(t) \sim \lambda^k|\xi_1| \lesssim |\theta'(t)|$ and, again using (3.5), we see that

$$\alpha_1 \lesssim \int_{I_k} \frac{\phi''(t)}{\phi'(t)^2}\, dt \lesssim (\lambda^k|\xi_1|)^{-1}.$$

For α_2, first we write

$$\alpha_2 \leq \int_{I_k} \lambda^k|P''(\gamma(\lambda^k t))|\gamma'(\lambda^k t)\frac{\gamma'(\lambda^{k+1})\lambda^k|\xi_2|}{\theta'(t)^2}\, dt \lesssim (|p_1|\lambda^k|\xi_1|)^{-1}\int_{G_1} |P''(s)|\, ds.$$

Suppose $P'' \geq 0$ on $[s_1, s_2] \subseteq G_1$. Then $\int_{[s_1,s_2]} |P''(s)|\, ds = P'(s_2) - P'(s_1) \lesssim |p_1|$ by Lemma 3.2.1. Similarly if $P'' < 0$ on $[\tilde{s_1}, \tilde{s_2}] \subseteq G_1$. Since G_1 splits into $\lesssim 1$ disjoint such intervals, we get $\alpha_2 \lesssim (\lambda^k|\xi_1|)^{-1}$. Now, (1.22) implies $(\lambda^k|\xi_1|)^{-1} \lesssim |A_k\xi|^{-1}$, so we have $|\widehat{\mu_k}(\xi)| \lesssim |A_k\xi|^{-1}$ in the case $|\xi_1| > 4|p_1|\gamma'(\lambda^{k+1})|\xi_2|$.

Finally, suppose $|\xi_1| < \frac{1}{4}|p_1|\gamma'(\lambda^k)|\xi_2|$. Yet another application of (3.5) gives

$$|\theta'(t)| \geq \frac{1}{4}|p_1|\gamma'(\lambda^k t)\lambda^k|\xi_2| \geq \frac{1}{4}|p_1|\gamma'(\lambda^k)\lambda^k|\xi_2|,$$

for $t \in I_k$. With α_1, α_2, and ϕ as above we have $\phi'(t) \sim |p_1|\gamma'(\lambda^k t)\lambda^k|\xi_2| \lesssim |\theta'(t)|$. The same argument used previously for α_1 gives $\alpha_1 \lesssim (|p_1|\gamma'(\lambda^k)\lambda^k|\xi_2|)^{-1}$. Also

$$\begin{aligned}\alpha_2 &\lesssim \int_{I_k} \lambda^k|P''(\gamma(\lambda^k t))|\gamma'(\lambda^k t)\frac{1}{|p_1|\gamma'(\lambda^k t)\lambda^k|\xi_2|}\, dt \\ &\lesssim (|p_1|\gamma'(\lambda^k)\lambda^k|\xi_2|)^{-1}\int_{G_1} |p_1|^{-1}|P''(s)|\, ds \lesssim (|p_1|\gamma'(\lambda^k)\lambda^k|\xi_2|)^{-1}.\end{aligned}$$

By (1.22) it follows that $|\widehat{\mu_k}(\xi)| \lesssim (|p_1|\gamma'(\lambda^k)\lambda^k|\xi_2|)^{-1} \lesssim |A_k\xi|^{-1}$, and this completes the proof of Lemma 3.3.3. □

Finally, for H_{G_j} we apply Proposition 3.2.2 with A_k and $\triangle_k$ unchanged, and ν_k equal to H_k. Since (3.8) is true, we only need check (3.9). Firstly, if γ is even then this is almost immediate from the work done in the proof of Lemma 3.3.3. Indeed, this and integration by parts gives us the decay for the integral over I_k, while the integral over $-I_k$ is simply a reflection in the vertical axis of the integral over I_k. For odd γ, we claim that Lemma 3.3.2 holds on $-I_k$ as well. To see this, simply observe that P' and P'' have opposing signs on $-G_j$, by (2) and (4) of Lemma 3.2.1, and couple this with the fact that $\gamma'' \leq 0$ on $(-\infty, 0)$. Now, (3.9) will follow if we carry out the argument used in the proof of Lemma 3.3.3 and integration by parts. This completes the proof of Theorem 3.1.1.

3.4 The hypersurface

We again decompose $(0, \infty)$ as in (3.1). If H_{D_k} and M_{D_k} are defined in the analogous way, then

$$\int_{|y|\in\gamma^{-1}(D_k)} |K(y)|\, dy \lesssim \int_{S^{d-1}} |K(\omega)| \int_{r\in\gamma^{-1}(D_k)} \frac{dr}{r}\, d\sigma(\omega) \lesssim 1,$$

and therefore these operators are bounded on L^p. So we fix $j \in \mathfrak{J}$ and turn our attention to showing H_{G_j} and M_{G_j} are L^p bounded operators. Taking $\lambda := d+2$ and I_k as before, define H_k and μ_k by:

$$\begin{aligned} \langle H_k, \psi\rangle &:= \int_{|y|\in I_k} \psi(\lambda^k y, P(\gamma(\lambda^k|y|)))K(y)\, dy, \\ \langle \mu_k, \psi\rangle &:= \int_{|y|\in I_k} \psi(\lambda^k y, P(\gamma(\lambda^k|y|)))\, dy, \end{aligned}$$

for $\psi \in \mathcal{S}(\mathbb{R}^{d+1})$. Also, put $A_k := A(\lambda^k)$ where, for $t > 0$, $A(t)$ is the $d+1$ by $d+1$ diagonal matrix with (r,r)-entry equal to $|p_j|\gamma(t)^j$ when $r = d+1$, and t otherwise.

Lemma 3.4.1. *$|\widehat{H_k}(\xi)| + |\widehat{\mu_k}(\xi)| \lesssim |A_k\xi|^{(1-d)/2}$ for $\xi \neq 0$.*

Proof. We just prove the decay estimate for $\widehat{H_k}$ because the corresponding result for $\widehat{\mu_k}$ can be proved in the same way. If $\xi = (\xi', \xi_{d+1})$ then

$$\begin{aligned} \widehat{H_k}(\xi) &= \int_{|y|\in I_k} e^{i(\lambda^k y.\xi' + P(\gamma(\lambda^k|y|))\xi_{d+1})} K(y)\, dy \\ &= \int_{r\in I_k} e^{iP(\gamma(\lambda^k r))\xi_{d+1}} \int_{S^{d-1}} e^{i\lambda^k r\omega.\xi'} K(\omega)\, d\sigma(\omega) \frac{dr}{r}. \end{aligned}$$

It is well known (see, for example [60]) that because K is smooth away from the origin, for $r \in I_k$,

$$\left| \int_{S^{d-1}} e^{i\lambda^k r\xi'.\omega} K(\omega)\, d\sigma(\omega) \right| \lesssim (\lambda^k r|\xi'|)^{(1-d)/2} \lesssim (\lambda^k|\xi'|)^{(1-d)/2}.$$

Therefore the claim follows for $|p_j|\gamma(\lambda^k)^j|\xi_{d+1}| \leq 4\lambda^k|\xi'|$. Suppose then that $|p_j|\gamma(\lambda^k)^j|\xi_{d+1}| \geq 4\lambda^k|\xi'|$. Fix $\omega \in S^{d-1}$ and let $\theta(r) = \lambda^k r\omega.\xi' + P(\gamma(\lambda^k r))\xi_{d+1}$ for $r \in I_k$. Then (3.5) and (1.22) imply

$$|\theta'(r)| \geq \frac{1}{2}|p_j|(\gamma^j)'(\lambda^k r)\lambda^k|\xi_{d+1}| - \lambda^k|\xi'| \gtrsim |p_j|\gamma(\lambda^k)^j|\xi_{d+1}|.$$

It follows that

$$\left|\int_{r\in I_k} e^{i\theta(r)}\,\frac{dr}{r}\right| \lesssim (|p_j|\gamma(\lambda^k)^j|\xi_{d+1}|)^{-1} \lesssim |A_k\xi|^{-1}$$

(as in the proof of Lemma 3.3.3 this follows by van der Corput's lemma for $j \in \mathfrak{J}\setminus\{1\}$, and the substitute argument for $j = 1$). This completes the proof of Lemma 3.4.1. □

We can now use Proposition 3.2.2 (or a weaker form, given that we in fact have uniform decay estimates) to complete the proof of Theorem 3.1.2.

Chapter 4

Flat Curves in $\mathbb{R}^d$ Near L^1

4.1 Introduction

Suppose that $\Gamma(t) = (t, \gamma(t))$ where γ is odd, belongs to $C^2(0,\infty)$, and is convex on $(0,\infty)$. Recall the definitions of the set $\mathcal{C}_2$ and the function h associated to γ from Chapter 1. By Theorem 1.1.6(2) and Theorem 1.1.7(2) we know that if h belongs to $\mathcal{C}_2$ then H_Γ and $\mathcal{M}_\Gamma$ are bounded operators on L^p for each $p \in (1,\infty)$. In [11] an extension of these results in $\mathbb{R}^d$ for $d \geq 2$ was achieved. Let us begin this chapter with a description of how the notion of convexity was extended to higher dimensions and also how the analogue of the curvature assumption on the function h was formed.

Let $\gamma_2, \ldots, \gamma_d$ belong to $C^d(0,\infty)$. For $k = 1, \ldots, d$ let

$$D_k(t) := \det \begin{pmatrix} 1 & \gamma_2'(t) & \cdots & \gamma_k'(t) \\ 0 & \gamma_2''(t) & \cdots & \gamma_k''(t) \\ \vdots & \vdots & & \vdots \\ 0 & \gamma_2^{(k)}(t) & \cdots & \gamma_k^{(k)}(t) \end{pmatrix},$$

and set $D_0(t) := 1$. For $k = 1, \ldots, d$ define

$$N_k(t) := \det \begin{pmatrix} t & \gamma_2(t) & \cdots & \gamma_k(t) \\ 1 & \gamma_2'(t) & \cdots & \gamma_k'(t) \\ \vdots & \vdots & & \vdots \\ 0 & \gamma_2^{(k-1)}(t) & \cdots & \gamma_k^{(k-1)}(t) \end{pmatrix},$$

$$h_k(t) := \frac{N_k(t)}{D_{k-1}(t)}.$$

Definition 4.1.1. The curve $(t, \gamma_2(t), \ldots, \gamma_d(t))$ is said to be convex if for all $k = 1, \ldots, d$ we have

$$D_k(t) > 0 \quad \text{for all } t \in (0,\infty). \tag{4.1}$$

The curvature assumption is that each h_k belongs to $\mathcal{C}_2$; that is to say, there exists $c(d) > 0$ such that for all $k = 2, \dots, d$ we have

$$th_k'(t) \geq c(d)h_k(t) \quad \text{for all } t \in (0, \infty). \tag{4.2}$$

The higher dimensional analogue of Theorem 1.1.6(2) and Theorem 1.1.7(2) is the following:

Theorem 4.1.2. [11] *Suppose $\Gamma(t) = (t, \gamma_2(t), \dots, \gamma_d(t))$ is odd, $\Gamma(0) = 0$, and (4.1) and (4.2) are satisfied. Then H_Γ and $\mathcal{M}_\Gamma$ are bounded operators on L^p for all $p \in (1, \infty)$.*

In this chapter we consider the mapping properties near L^1 of H_Γ and $\mathcal{M}_\Gamma$, where Γ belongs to the class of curves described in Theorem 4.1.2. Our main result is the following:

Theorem 4.1.3. *Suppose $\Gamma(t) = (t, \gamma_2(t), \dots, \gamma_d(t))$ is odd, $\Gamma(0) = 0$, and (4.1) and (4.2) are satisfied. Then H_Γ and $\mathcal{M}_\Gamma$ are of weak type $L \log L$.*

To see the context in which Theorem 4.1.3 stands, let us consider the prototypical finite type curves in $\mathbb{R}^2$ and $\mathbb{R}^3$; $\Gamma_2(t) := (t, t^2)$ and $\Gamma_3(t) = (t, t^2, t^3)$ respectively. It is known from [20] that H_{Γ_3} and $\mathcal{M}_{\Gamma_3}$ locally map $L \log L$ to $L^{1,\infty}$. It was shown in [57] that H_{Γ_2} and $\mathcal{M}_{\Gamma_2}$ locally map $L \log^{(2)} L$ to $L^{1,\infty}$. The proof of the stronger result for Γ_2 in [57] uses the fact that Γ_2 has codimension 1. It is presently open as to whether the result in [20] for Γ_3 can be extended at all beyond $L \log L$. Until this is achieved, Theorem 4.1.3 has little hope of improvement for $d \geq 3$. The result in [57] offers some hope to extend Theorem 4.1.3 when $d = 2$. However, we have so far been unable to achieve any such improvement; a short discussion on this matter appears at the end of Section 4.2.

Remark. Theorem 4.1.3 implies that if f belongs locally to $L \log L$, then, for almost every $x \in \mathbb{R}^d$,

$$\lim_{h \to 0} h^{-1} \int_{(0,h)} f(x - \Gamma(t))\, dt = f(x).$$

Overview. We prove Theorem 4.1.3 in Section 4.2. In Section 4.3 we consider local mapping properties near L^1 of H_Γ and $\mathcal{M}_\Gamma$ where Γ belongs to the class of nonconvex hypersurfaces studied in Chapter 3.

4.2 Proof of Theorem 4.1.3

The schema to prove Theorem 4.1.3 is the same as that used in Section 3 of [57]. In this setting of flat curves, we shall use the Calderón-Zygmund theory developed in [9]. Before defining the appropriate Calderón-Zygmund cubes, we shall introduce the dilation matrices defined in [11] which are associated to the curve Γ. Our dilations will satisfy the well-known Rivière condition which serves as a substitute for the group property that the dilations δ_t from Chapter 2 enjoy. Thus we are able to define certain 'nice' normalised versions of Γ. These will be nice in the sense that it is possible to prove decay estimates for the Fourier transform of certain measures supported on these normalised curves.

Notation. Write $A \lesssim B$ for $A \leq CB$, where C depends on at most d and Γ.

Dilations and decay estimates

All of the work on the choice of dilations and proving the decay estimates that follow was done in [11]. We shall state their results without proof. We again work with the dyadic maximal operator in (1.20); the choice of λ will be made later in the proof.

The dilation matrices $\{A(t) : t \in (0, \infty)\}$ are defined in terms of the following differential operators:

$$\begin{aligned} R_0 f &:= f, \\ R_k f &:= \left(\frac{f}{h_k}\right)' \frac{h_k^2}{h_k'} \qquad \text{for } k = 1, \ldots, d. \end{aligned}$$

We define

$$A(t) := \begin{pmatrix} t & R_1 t & \cdots & R_{d-1}R_{d-2}\ldots R_1 t \\ \gamma_2(t) & R_1\gamma_2(t) & \cdots & R_{d-1}R_{d-2}\ldots R_1\gamma_2(t) \\ \vdots & \vdots & & \vdots \\ \gamma_d(t) & R_1\gamma_d(t) & \cdots & R_{d-1}R_{d-2}\ldots R_1\gamma_d(t) \end{pmatrix}.$$

Remarks. 1. If $d = 2$, the situation is entirely analogous to that in [9]; we have $\Gamma(t) = (t, \gamma_2(t))$ and $D_2(t) > 0$ implies $\gamma_2''(t) > 0$. Moreover, $h_2(t) = t\gamma_2'(t) - \gamma_2(t)$, so we recover the h-function associated to the plane curve, and the dilation matrices coincide. For a discussion on why these dilations are appropriate see Section 4 in [11].

2. Condition (4.1) implies, via Lemma 1 and Lemma 2 in [52], that $h_k(t) > 0$ and $h_k'(t) > 0$ for $t \in (0, \infty)$ and $k = 1, \ldots, d$. Therefore, $R_1, \ldots, R_d$ are well defined.

3. Each $A(t)$ is lower triangular. In particular, if $A(t) = (A_{i,j}(t))_{1 \leq i,j \leq d}$ then,

$$A_{1,1}(t) = t \quad \text{and, for } j = 2, \ldots, d, \quad A_{j,j}(t) = h_j(t). \tag{4.3}$$

The previous remark and (4.3) imply that each $A(t)$ is invertible. For a proof of (4.3), see the proof of Lemma 5.3 in [11].

Part (1) of the following proposition says that $\{A(t) : t \in (0,\infty)\}$ satisfies the Rivière condition. A proof can be found in Section 5 of [11]; the full strength of the curvature hypothesis (4.2) is not needed to prove Proposition 4.2.1 and the assumption that each h_k belongs to $\mathcal{C}_1$ suffices. The remaining parts of Proposition 4.2.1 are trivial consequences of the first and are only included for emphasis.

Proposition 4.2.1. *There exists $C, \varepsilon \sim 1$ such that for $s \geq t > 0$ and $\xi \in \mathbb{R}^d$,*

1. $\|A(s)^{-1}A(t)\| = \|A(t)^*(A(s)^*)^{-1}\| \leq C\,(t/s)^{\varepsilon}$;
2. $|A(s)^{-1}A(t)\xi| \leq C\,(t/s)^{\varepsilon}\,|\xi|$ *and* $|A(t)^*(A(s)^*)^{-1}\xi| \leq C\,(t/s)^{\varepsilon}\,|\xi|$;
3. $|A(t)^{-1}A(s)\xi| \geq C^{-1}\,(s/t)^{\varepsilon}\,|\xi|$ *and* $|A(s)^*(A(t)^*)^{-1}\xi| \geq C^{-1}\,(s/t)^{\varepsilon}\,|\xi|$.

For each $k \in \mathbb{Z}$, we now define the normalised versions, Γ_k, of Γ by

$$\Gamma_k(t) := A(\lambda^k)^{-1}\Gamma(\lambda^k t) \quad \text{for each } |t| \in [1, \lambda].$$

Also define the following measures:

$$\begin{aligned} \langle \mu^{(k)}, \psi\rangle &:= \int_1^{\lambda} \psi(\Gamma_k(t))\,dt, \quad \langle \mu_k^{(k)}, \psi\rangle := \langle \mu^{(k)}, \psi(A(\lambda^k)\cdot)\rangle, \\ \langle H^{(k)}, \psi\rangle &:= \int_{|t|\in[1,\lambda]} \psi(\Gamma_k(t))\,\frac{dt}{t}, \quad \langle H_k^{(k)}, \psi\rangle := \langle H^{(k)}, \psi(A(\lambda^k)\cdot)\rangle. \end{aligned}$$

Clearly we have $M_\Gamma f = \sup_{k\in\mathbb{Z}} |f * \mu_k^{(k)}|$ and $H_\Gamma f = \sum_{k\in\mathbb{Z}} H_k^{(k)} * f$. The notation $\mu_k^{(k)}$ may seem heavy-handed at first. The intention is to maintain the notation from Section 3 of [57] in the sense that $\mu_k^{(k)}$ is a $A(\lambda^k)$-dilate of the measure $\mu^{(k)}$; a measure that will not in general be fixed as k varies, yet has the property that its Fourier transform satisfies a decay estimate independent of k, and in this sense one can think of $\mu^{(k)}$ as almost fixed. This decay estimate is the content of the subsequent lemma. This was proved in Section 5 of [11] via a variant of van der Corput's lemma (see Proposition 3.1 of [11] for this variant).

Lemma 4.2.2. *There exists $\delta \in (0,1)$ such that for $\xi \neq 0$, $|\widehat{\mu^{(k)}}(\xi)| \lesssim |\xi|^{-\delta}$ and $|\widehat{H^{(k)}}(\xi)| \lesssim |\xi|^{-\delta}$.*

The proof of Lemma 4.2.2 from [11] shows in fact that one can take $\delta = 1/d$.

Calderón-Zygmund theory

In order to utilise the Calderón-Zygmund theory developed in [9], we shall define balls $\{B_k\}_{k\in\mathbb{Z}}$ satisfying,

(B1). $\bigcup_{k\in\mathbb{Z}} B_k = \mathbb{R}^d$;

(B2). $\bigcap_{k\in\mathbb{Z}} B_k = \{0\}$;

(B3). each B_k is open, balanced, convex, and bounded;

(B4). $B_k \subset B_{k+1}$ for each k;

(B5). for each k we have $|B_{k+1}| \sim |B_k|$.

Initially put $\widetilde{B_k} := A(\lambda^k)B_1(0)$; clearly (B3) holds. Now we choose

$$\lambda := 4^{\lceil 1+(\log_2 C)/(2\varepsilon)\rceil},$$

where C and ε are those appearing in Proposition 4.2.1, and $\beta := C/\lambda^\varepsilon$. Notice that our choice of λ ensures $0 < \beta < 1$ and, moreover, by Proposition 4.2.1,

$$\|A(\lambda^{k+1})^{-1}A(\lambda^k)\| \leq \beta. \tag{4.4}$$

For any $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^d$, (4.4) implies

$$|A(\lambda^{k+1})^{-1}\xi| = |A(\lambda^{k+1})^{-1}A(\lambda^k)A(\lambda^k)^{-1}\xi| \leq \beta|A(\lambda^k)^{-1}\xi|, \tag{4.5}$$

which immediately implies (B4). Moreover, it follows inductively that

$$|A(\lambda^{k+l})^{-1}\xi| \leq \beta^l|A(\lambda^k)^{-1}\xi|, \tag{4.6}$$

for all integers $l \geq 0$. We claim that (4.6) implies (B1) and (B2) also hold. To see (B1), take $\xi \in \mathbb{R}^d \setminus \{0\}$ and choose $l_0 \geq 0$ such that $\beta^{l_0} \leq (2|A(1)^{-1}\xi|)^{-1}$. Then (4.6) implies $|A(\lambda^{l_0})^{-1}\xi| \leq \beta^{l_0}|A(1)^{-1}\xi| \leq 1/2$, so that $\xi \in \widetilde{B_{l_0}}$. For (B2), take $\xi \neq 0$ and choose $l_1 \geq 0$ such that $\beta^{l_1} \leq |A(1)^{-1}\xi|$. Then, by (4.6), $|A(\lambda^{-l_1})^{-1}\xi| \geq \beta^{-l_1}|A(1)^{-1}\xi| \geq 1$ and hence $\xi \notin \widetilde{B_{-l_1}}$.

Unfortunately, we cannot guarantee (B5) holds for the $\widetilde{B_k}$. Nonetheless, if we fix $k \in \mathbb{Z}$ then it is possible to choose a finite collection $\{E_k^1, \ldots, E_k^{n(k)}\}$ of open, balanced, convex, and bounded sets so that

$$\widetilde{B_k} = E_k^1 \subset E_k^2 \subset \ldots \subset E_k^{n(k)} = \widetilde{B_{k+1}},$$

and $|E_k^l| \leq 2|E_k^{l-1}|$ for $l = 1, \ldots, n(k) - 1$. Then define the collection of B_k by

$$\{B_k\}_{k\in\mathbb{Z}} := \bigcup_{k\in\mathbb{Z}} \bigcup_{l=1}^{n(k)-1} \{E_k^l\}, \tag{4.7}$$

so that (B1)-(B5) hold for the B_k.

Observe that, for each $k \in \mathbb{Z}$, (B3) allows us to define a norm $\|.\|_k$ such that $B_k = \{x \in \mathbb{R}^d : \|x\|_k < 1\}$. For each $k \in \mathbb{Z}$ define an associated ball with centre $y \in \mathbb{R}^d$ and radius $r > 0$ with respect to $\|.\|_k$ as

$$B(y, k, r) := \{x \in \mathbb{R}^d : \|x - y\|_k < r\}. \tag{4.8}$$

Notation. For each $k \in \mathbb{Z}$, $x \in \mathbb{R}^d$, and nonempty subset S of $\mathbb{R}^d$, define

$$\text{dist}_k(x, S) := \inf\{\|x - s\|_k : s \in S\}.$$

We now state the Whitney type decomposition relative to the balls in (4.8) which appears on page 680 of [9].

Proposition 4.2.3. *There exists $A \sim 1$ such that the following hold.*

1. *If Ω is any nonempty proper open subset of $\mathbb{R}^d$, then $\Omega = \bigcup_{B \in \mathcal{E}} B$, where*

$$\mathcal{E} := \{B(x, k, 1) : x \in \Omega, k \in \mathbb{Z}, 5 < \text{dist}_k(x, \partial\Omega) < A\}.$$

2. *If in addition $|\Omega|$ is finite then we can find a sequence of disjoint balls $Q_i := B(x_i, k_i, 1) \in \mathcal{E}$ such that $\Omega = \bigcup_i B(x_i, k_i, 3)$.*

Taking $A \sim 1$ that appears in Proposition 4.2.3, define the following collection of all translates of the B_k:

$$\mathfrak{B} := \{B(y, k, A) : y \in \mathbb{R}^d, k \in \mathbb{Z}\},$$

and the associated Hardy-Littlewood type maximal function M_{HL} by

$$M_{HL}f(x) := \sup_{x \in B \in \mathfrak{B}} \frac{1}{|B|} \int_B |f(y)|\, dy.$$

By Proposition 2.2 of [9], we know that M_{HL} is of weak type L.

Main estimates

Recall that our goal is to prove the estimate,

$$|\{x \in \mathbb{R}^d : |M_\Gamma f(x)| > \alpha\}| \lesssim \int \frac{|f(x)|}{\alpha} \log\left(\frac{|f(x)|}{\alpha} + 100\right) dx, \qquad (4.9)$$

holds for all $\alpha > 0$.

Fix $\alpha > 0$ and set $\Omega := \{x \in \mathbb{R}^d : M_{HL}f(x) > \alpha\}$ for a fixed f such that the right hand side of (4.9) is finite. From the weak type L of M_{HL} we get $|\Omega| \lesssim \|f\|_1/\alpha$. Next, apply Proposition 4.2.3 to obtain sequences $\{x_i\} \subseteq \Omega$, $\{j_i\} \subseteq \mathbb{Z}$, and disjoint Whitney cubes $\{Q_i\}$ such that the following hold.

(W1). $\Omega = \bigcup_i Q_i$.

(W2). $B(x_i, j_i, 1) \subseteq Q_i \subseteq B(x_i, j_i, 3)$.

(W3). $5 < \inf\{\|x_i - y\|_{j_i} : y \in \partial\Omega\} < A$.

Also define $\Omega^* := \bigcup_i B(x_i, j_i, C+10)$, where C is the constant appearing in the statement of Proposition 4.2.1. Observe that (W1) implies $|\Omega^*| \sim |\Omega|$; in particular

$$|\Omega^*| \lesssim \|f\|_1/\alpha. \tag{4.10}$$

By an analogue of the Lebesgue differentiation theorem we know that $|f(x)| \leq \alpha$ for $x \notin \Omega$. Our first decomposition of f is then,

$$f = g + \sum_i f_{Q_i}, \tag{4.11}$$

where

$$f_{Q_i}(x) := \begin{cases} f(x) & \text{if } x \in Q_i \text{ and } |f(x)| > \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

This decomposition is akin to that of classical Calderón-Zygmund theory; observe that g enjoys good L^∞ properties; in particular

$$|g(x)| \lesssim \alpha,$$

and since $|g(x)| \leq |f(x)|$ for any $x \in \mathbb{R}^d$, we may also deduce at once that

$$\|g\|_2 \lesssim \alpha^{1/2}\|f\|_1^{1/2}. \tag{4.12}$$

Furthermore, for each i, (W3) gives us some $y \in \partial\Omega$ such that $\|x_i - y\|_{j_i} < A$. Thus

$$\frac{1}{|Q_i|}\int_{Q_i} |f_{Q_i}(x)|\,dx \lesssim \frac{1}{|B(x_i,j_i,A)|}\int_{B(x_i,j_i,A)} |f(x)|\,dx \lesssim M_{HL}f(y) \leq \alpha, \tag{4.13}$$

and one has that each f_{Q_i} is, on average, under control.

Next, decompose f_{Q_i} further by letting

$$f_{Q_i}^n(x) := \begin{cases} f_{Q_i}(x) & \text{if } \lambda^{(n-1)\delta}\alpha < |f_{Q_i}(x)| \leq \lambda^{n\delta}\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where δ is the decay exponent from Lemma 4.2.2. Notice that

$$f_{Q_i} = \sum_{n\geq 1} f_{Q_i}^n \tag{4.14}$$

and, by (4.13),

$$\sum_{n\geq 1} \frac{1}{|Q_i|}\int |f_{Q_i}^n(x)|\,dx \lesssim \alpha. \tag{4.15}$$

Now let

$$g_{Q_i}^n(x) := \chi_{Q_i}(x)\frac{1}{|Q_i|}\int_{Q_i} f_{Q_i}^n(y)\,dy, \tag{4.16}$$

$$b_{Q_i}^n(x) := f_{Q_i}^n(x) - g_{Q_i}^n(x), \tag{4.17}$$

and

$$g^n(x) := \sum_i g^n_{Q_i}(x), \quad b^n(x) := \sum_i b^n_{Q_i}(x), \quad f^n(x) := \sum_i f^n_{Q_i}(x). \tag{4.18}$$

Observe that (4.15) implies

$$\sum_{n\geq 1} |g^n_{Q_i}(x)| \leq \chi_{Q_i}(x) \sum_{n\geq 1} \frac{1}{|Q_i|} \int_{Q_i} |f^n_{Q_i}(y)|\, dy \lesssim \alpha \chi_{Q_i}(x). \tag{4.19}$$

Moreover, by (4.15) and (4.19),

$$\sum_{n\geq 1} \|b^n_{Q_i}\|_1 \lesssim \alpha |Q_i|. \tag{4.20}$$

The next step is to decompose the measures $\mu^{(k)}$, first by the following localization: Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ have compact support in $B_{1/2}(0)$ with $\int \phi(x)\, dx = 1$, and $\int x_k \phi(x)\, dx = 0$ for all $k \in \{1, \dots, d\}$. Note that Taylor's theorem implies

$$|\widehat{\phi}(\xi) - 1| \lesssim |\xi|^2, \tag{4.21}$$

under our hypotheses. Also define, for each $n \geq 1$, $\phi_n(x) = \lambda^{nd}\phi(\lambda^n x)$.

To this stage, the proof of Theorem 4.1.3 is the same for M_Γ and H_Γ. We now focus our attention on M_Γ; the proof for H_Γ is very similar, and the necessary changes will be made clear later.

For each $n \geq 1$ let

$$\mu^{(k),0} := \mu^{(k)}, \tag{4.22}$$

$$\mu^{(k),n} := \phi_n * \mu^{(k)}. \tag{4.23}$$

For each $k \in \mathbb{Z}$ and $n \geq 1$ define the following dilates of these localisations:

$$\mu^{(k),n}_k(x) := \det A(\lambda^k)^{-1} \mu^{(k),n}(A(\lambda^k)^{-1}x).$$

Use (4.11), (4.14), and (4.18) to decompose $\mu^{(k)}_k * f$ as

$$\begin{aligned} \mu^{(k)}_k * g + \mu^{(k)}_k * \sum_i \sum_n f^n_{Q_i} &= \mu^{(k)}_k * g + \sum_n \mu^{(k)}_k * f^n \\ &= \mu^{(k)}_k * g + \sum_{n\geq 1} (\mu^{(k)}_k - \mu^{(k),n}_k) * f^n + \sum_{n\geq 1} \mu^{(k),n}_k * f^n, \end{aligned}$$

and then (4.17), (4.18), and (4.22) to continue this decomposition to get

$$\begin{aligned} \mu^{(k)}_k * f &= \mu^{(k)}_k * g + \sum_{n\geq 1} (\mu^{(k)}_k - \mu^{(k),n}_k) * f^n \\ &+ \mu^{(k)}_k * \sum_{n\geq 1} g^n + \sum_{n\geq 1} \sum_{m=0}^{n-1} (\mu^{(k),m+1}_k - \mu^{(k),m}_k) * g^n + \sum_{n\geq 1} \mu^{(k),n}_k * b^n. \end{aligned}$$

Therefore

$$M_\Gamma f(x) = \sup_{k\in\mathbb{Z}} |\mu_k^{(k)} * f| \leq M_{I,1} + M_{I,2} + M_{I,3} + M_{I,4} + M_{II},$$

where

$$\begin{aligned}
M_{I,1} &:= \sup_{k\in\mathbb{Z}} |\mu_k^{(k)} * g|,\\
M_{I,2} &:= \sup_{k\in\mathbb{Z}} \left|\mu_k^{(k)} * \sum_{n\geq 1} g^n\right|,\\
M_{I,3} &:= \sum_{n\geq 1} \sup_{k\in\mathbb{Z}} |(\mu_k^{(k)} - \mu_k^{(k),n}) * f^n|,\\
M_{I,4} &:= \sum_{m\geq 0} \sup_{k\in\mathbb{Z}} \left|(\mu_k^{(k),m+1} - \mu_k^{(k),m}) * \sum_{n>m} g^n\right|,\\
M_{II} &:= \sum_{n\geq 1} \sup_{k\in\mathbb{Z}} |\mu_k^{(k),n} * b^n|.
\end{aligned}$$

In order to handle the terms $M_{I,1}, M_{I,2}, M_{I,3}$, and $M_{I,4}$, we shall show that

$$\sum_{i=1}^{4} \|M_{I,i}\|_2^2 \lesssim \alpha \|f\|_1. \tag{4.24}$$

An application of Chebyshev's inequality gives

$$|\{x \in \mathbb{R}^d : M_{I,i}(x) > \alpha/5\}| \lesssim \alpha^{-2} \|M_{I,i}\|_2^2 \lesssim \alpha^{-1}\|f\|_1,$$

which is clearly dominated by the right hand side of (4.9).

Before proving (4.24) we outline how we control the more difficult term, M_{II}, using L^1 arguments. Recalling the definition of our balls B_j from (4.7), for each i let l_i be the integer satisfying

$$\widetilde{B_{l_i-1}} \subseteq B_{j_i} \subseteq \widetilde{B_{l_i}}. \tag{4.25}$$

For each $n \geq 1$ and i, set

$$S_{n,i} := \{k \in \mathbb{Z} : l_i - 2 \leq k \leq l_i + \varepsilon^{-1} n\}, \tag{4.26}$$

where ε appears in Proposition 4.2.1. Then $M_{II} \leq M_{II,1} + M_{II,2}$, where

$$\begin{aligned}
M_{II,1} &:= \sum_{n\geq 1}\sum_{i}\sum_{k\notin S_{n,i}} |\mu_k^{(k),n} * b_{Q_i}^n|,\\
M_{II,2} &:= \sum_{n\geq 1}\sum_{i}\sum_{k\in S_{n,i}} |\mu_k^{(k),n} * b_{Q_i}^n|.
\end{aligned}$$

We claim that

$$\|M_{II,1}\|_{L^1(\mathbb{R}^d\setminus\Omega^*)} \lesssim \|f\|_1, \tag{4.27}$$

where the set Ω^* was introduced on page 63. By (4.10) and Chebyshev's inequality, this implies

$$\begin{aligned}|\{x \in \mathbb{R}^d : M_{II,1}(x) > \alpha/5\}| &\leq |\{x \in \mathbb{R}^d \setminus \Omega^* : M_{II,1}(x) > \alpha/5\}| + \|f\|_1/\alpha \\ &\lesssim \|f\|_1/\alpha.\end{aligned}$$

To handle the contribution from $M_{II,2}$ we use a very coarse argument. Notice first that since the total variation of $\mu^{(k)}$ is uniformly $\lesssim 1$ we get

$$\|\mu_k^{(k),n}\|_1 = \|\mu^{(k),n}\|_1 \lesssim \|\phi_n\|_1 = \|\phi\|_1 \sim 1,$$

and thus, by Chebyshev's inequality and the fact that, for each i, $|S_{n,i}| \lesssim n$, we get

$$|\{x \in \mathbb{R}^d : M_{II,2}(x) > \alpha/5\}| \lesssim \alpha^{-1} \sum_{n\geq 1} \sum_i n \|b_{Q_i}^n\|_1.$$

Therefore,

$$\begin{aligned}|\{x \in \mathbb{R}^d : M_{II,2}(x) > \alpha/5\}| &\lesssim \alpha^{-1} \sum_{n\geq 1} \sum_i n \int_{x\in Q_i, \lambda^{(n-1)\delta}\alpha < |f(x)| \leq \lambda^{n\delta}\alpha} |f(x)|\, dx \\ &\lesssim \int \frac{|f(x)|}{\alpha} \log\left(\frac{|f(x)|}{\alpha} + 100\right) dx.\end{aligned}$$

The rest of the proof of Theorem 4.1.3 is then dedicated to (4.24) and (4.27).

From Theorem 4.1.2 we know that M_Γ is a bounded operator on L^2. This and (4.12) implies

$$\|M_{I,1}\|_2^2 = \|\sup_{k\in\mathbb{Z}} |\mu_k^{(k)} * g|\|_2^2 \lesssim \|g\|_2^2 \lesssim \alpha\|f\|_1.$$

Now (4.19) clearly implies

$$\left|\sum_{n\geq 1} \sum_i g_{Q_i}^n(x)\right| \lesssim \alpha \sum_i \chi_{Q_i}(x) \leq \alpha,$$

so,

$$\left\|\sum_{n\geq 1} g^n\right\|_2^2 \lesssim \alpha \sum_{n\geq 1} \sum_i \int |g_{Q_i}^n(x)|\, dx = \alpha \sum_{n\geq 1} \sum_i \int |f_{Q_i}^n(x)|\, dx \leq \alpha\|f\|_1. \quad (4.28)$$

Using the L^2 boundedness of M_Γ again, we get

$$\|M_{I,2}\|_2^2 \lesssim \left\|\sup_{k\in\mathbb{Z}} \left|\mu_k^{(k)} * \sum_{n\geq 1} g^n\right|\right\|_2^2 \lesssim \left\|\sum_{n\geq 1} g^n\right\|_2^2 \lesssim \alpha\|f\|_1,$$

as required. To handle $M_{I,3}$ and $M_{I,4}$ we use the following estimates concerning our localised measures.

Lemma 4.2.4. *For each $m \geq 0$,*

$$\left\| \sup_{k\in\mathbb{Z}} |(\mu_k^{(k),m+1} - \mu_k^{(k),m}) * f| \right\|_2 \lesssim \lambda^{-m\delta} \|f\|_2.$$

Proof. Clearly,

$$\begin{aligned}\left\| \sup_{k\in\mathbb{Z}} |(\mu_k^{(k),m+1} - \mu_k^{(k),m}) * f| \right\|_2^2 &\leq \left\| \left(\sum_{k\in\mathbb{Z}} |(\mu_k^{(k),m+1} - \mu_k^{(k),m}) * f|^2 \right)^{1/2} \right\|_2^2 \\ &\sim \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}} |\widehat{\mu_k^{(k),m+1}}(\xi) - \widehat{\mu_k^{(k),m}}(\xi)|^2 |\widehat{f}(\xi)|^2 \, d\xi,\end{aligned}$$

so it suffices to show that, for each $\xi \neq 0$,

$$\sum_{k\in\mathbb{Z}} |\widehat{\mu^{(k),m+1}}(A(\lambda^k)^*\xi) - \widehat{\mu^{(k),m}}(A(\lambda^k)^*\xi)|^2 \lesssim \lambda^{-2m\delta}. \tag{4.29}$$

We claim that, for each $\xi \neq 0$,

$$|\widehat{\mu^{(k),m+1}}(\xi) - \widehat{\mu^{(k),m}}(\xi)| \lesssim \lambda^{-m\delta} \min(\lambda^{-m}|\xi|, (\lambda^{-m}|\xi|)^{-1}). \tag{4.30}$$

That (4.30) implies (4.29) easily follows because Proposition 4.2.1 allows us to estimate the left hand side of (4.29) by a convergent geometric series. To prove (4.30), note that

$$|\widehat{\mu^{(k),m+1}}(\xi) - \widehat{\mu^{(k),m}}(\xi)| = |\widehat{\phi}(\lambda^{-m-1}\xi) - \widehat{\phi}(\lambda^{-m}\xi)||\widehat{\mu^{(k)}}(\xi)|.$$

If $\lambda^{-m}|\xi| \leq 1$ then we can use (4.21) and Lemma 4.2.2 to get

$$|\widehat{\mu^{(k),m+1}}(\xi) - \widehat{\mu^{(k),m}}(\xi)| \lesssim |\lambda^{-m}\xi|^2|\xi|^{-\delta} = \lambda^{-m\delta}(\lambda^{-m}|\xi|)^{2-\delta} \lesssim \lambda^{-m\delta}(\lambda^{-m}|\xi|).$$

On the other hand, if $\lambda^{-m}|\xi| \geq 1$ then we can use the fact that $\phi \in \mathcal{S}(\mathbb{R}^d)$ and Lemma 4.2.2 to get

$$|\widehat{\mu^{(k),m+1}}(\xi) - \widehat{\mu^{(k),m}}(\xi)| \lesssim ((\lambda^{-m-1}|\xi|)^{-1} + (\lambda^{-m}|\xi|)^{-1})\lambda^{-m\delta} \lesssim \lambda^{-m\delta}(\lambda^{-m}|\xi|)^{-1}.$$

This completes the proof of (4.30) and hence Lemma 4.2.4. □

Since we have chosen $\widehat{\phi}(0) = 1$ we may write

$$\mu_k^{(k)} - \mu_k^{(k),n} = \sum_{m\geq n} (\mu_k^{(k),m+1} - \mu_k^{(k),m}).$$

This and Lemma 4.2.4 imply,

$$\|M_{I,3}\|_2 \leq \sum_{n\geq 1} \sum_{m\geq n} \| \sup_{k\in\mathbb{Z}} |(\mu_k^{(k),m+1} - \mu_k^{(k),m}) * f^n| \|_2 \lesssim \sum_{n\geq 1} \lambda^{-n\delta} \|f^n\|_2,$$

and furthermore

$$\|f^n\|_2^2 = \sum_i \int_{x\in Q_i,\, \lambda^{(n-1)\delta}\alpha<|f(x)|\le\lambda^{n\delta}\alpha} |f(x)|^2\, dx \le \lambda^{n\delta}\alpha\|f\|_1.$$

Therefore

$$\|M_{I,3}\|_2 \lesssim \alpha^{1/2}\|f\|_1^{1/2}\sum_{n\ge1}\lambda^{-n\delta/2} \lesssim \alpha^{1/2}\|f\|_1^{1/2},$$

as claimed. For $M_{I,4}$, we use Lemma 4.2.4 to get

$$\begin{aligned}\|M_{I,4}\|_2 &\lesssim \sum_{m\ge0}\left\|\sup_{k\in\mathbb{Z}}\left|(\mu_k^{(k),m+1}-\mu_k^{(k),m}) * \sum_{n>m} g^n\right|\right\|_2\\ &\lesssim \sum_{m\ge0} 2^{-m\delta}\left\|\sum_{n>m} g^n\right\|_2,\end{aligned}$$

and this is $\lesssim \alpha\|f\|_1$ by a similar argument to that used for (4.28). This concludes the proof of (4.24).

We now prove the remaining claim, (4.27). Firstly, we need the following simple, but important, property of the normalised curves.

Lemma 4.2.5. *For any $k\in\mathbb{Z}$ and $t\in[0,1]$ we have $|\Gamma_k(t)|\le C$, where C appears in Proposition 4.2.1.*

Proof. If e_1 is the element of $\mathbb{R}^d$ given by $(1,0,\dots,0)$ then, by Proposition 4.2.1,

$$|\Gamma_k(t)| = |A(\lambda^k)^{-1}A(\lambda^k t)e_1| \le Ct^\varepsilon \le C. \qquad \square$$

Fix i and consider k such that $k\le l_i-2$. We claim that these k do not contribute to $\|M_{II,1}\|_{L^1(\mathbb{R}^d\setminus\Omega^*)}$. To see this, observe that by Lemma 4.2.5,

$$\begin{aligned}A(\lambda^k)\mathrm{supp}\,\mu^{(k)} &= \{\Gamma(\lambda^k t): t\in[1,\lambda]\}\\ &= \{A(\lambda^{k+1})\Gamma_{k+1}(\lambda^{-1}t): t\in[1,\lambda]\}\\ &\subseteq A(\lambda^{k+1})B_C(0).\end{aligned}$$

Therefore,

$$A(\lambda^k)\mathrm{supp}\,\mu^{(k)} \subseteq C\widetilde{B_{k+1}} \subseteq C\widetilde{B_{l_i-1}} \subseteq CB_{j_i}.$$

Also, since ϕ is supported in $B_{1/2}(0)$, we have that,

$$A(\lambda^k)\mathrm{supp}\,\phi_n \subseteq A(\lambda^k)B_1(0) \subseteq B_{j_i} \quad \text{for each } n\ge1.$$

Hence,

$$\begin{aligned}\mathrm{supp}(\mu_k^{(k),n} * b_{Q_i}^n) &\subseteq \mathrm{supp}\, b_{Q_i}^n + A(\lambda^k)\mathrm{supp}\,\mu^{(k)} + A(\lambda^k)\mathrm{supp}\,\phi_n\\ &\subseteq Q_i + B(0,j_i,C+1) \subseteq \Omega^*.\end{aligned}$$

So to prove (4.27) it suffices to prove that

$$\sum_i \sum_{n\geq 1} \sum_{k\geq l_i+\varepsilon^{-1}n} \|\mu_k^{(k),n} * b_{Q_i}^n\|_1 \lesssim \|f\|_1. \tag{4.31}$$

Let $x \in \mathbb{R}^d$ and use the cancellation of $b_{Q_i}^n$ and then Taylor's theorem to get

$$\begin{aligned}
&\mu_k^{(k),n} * b_{Q_i}^n(x)\\
= &\int_{Q_i} b_{Q_i}^n(y)[\mu_k^{(k),n}(x-y) - \mu_k^{(k),n}(x-x_i)]\,dy\\
= &\det A(\lambda^k)^{-1} \int_{Q_i} b_{Q_i}^n(y)[\mu^{(k),n}(A(\lambda^k)^{-1}(x-y)) - \mu^{(k),n}(A(\lambda^k)(x-x_i))]\,dy\\
= &\det A(\lambda^k)^{-1} \int_0^1 \int_{Q_i} b_{Q_i}^n(y) A(\lambda^k)^{-1}(x_i-y).\nabla\mu^{(k),n}(A(\lambda^k)^{-1}(z))\,dydt,
\end{aligned}$$

where $z := x - x_i + t(x_i - y)$. For $y \in Q_i$ we have

$$A(\lambda^k)^{-1}(x_i-y) \in A(\lambda^k)^{-1}A(\lambda^{l_i})B_3(0).$$

Since $k \geq l_i$ it follows by Proposition 4.2.1 that

$$|A(\lambda^k)^{-1}(x_i-y)| \lesssim \lambda^{(l_i-k)\varepsilon}.$$

Also,

$$\mu^{(k),n}(x) = \lambda^{nd}\int \phi(\lambda^n(x-u))\,d\mu^{(k)}(u),$$

so that

$$\int |\nabla\mu^{(k),n}(x)|\,dx \lesssim \lambda^{n(d+1)} \int\int |\nabla\phi(\lambda^n(x-u))|\,dxd\mu^{(k)}(u) \lesssim \lambda^n.$$

Therefore,

$$\int |\mu_k^{(k),n} * b_{Q_i}^n(x)|\,dx \lesssim \lambda^{(l_i-k)\varepsilon+n}\|b_{Q_i}^n\|_1,$$

and one can use this estimate, with the help of (4.20), to deduce (4.31) as follows.

$$\begin{aligned}
\sum_i \sum_{n\geq 1} \sum_{k\geq l_i+\varepsilon^{-1}n} \|\mu_k^{(k),n} * b_{Q_i}^n\|_1 &\lesssim \sum_i \sum_{n\geq 1} \|b_{Q_i}^n\|_1\\
&\lesssim \sum_i \alpha|Q_i| = \alpha|\Omega| \lesssim \|f\|_1.
\end{aligned}$$

This concludes the proof of Theorem 4.1.3 for $\mathcal{M}_\Gamma$.

As noted previously, the proof Theorem 4.1.3 for H_Γ is similar to the one we used for the maximal operator. Firstly, we define

$$H^{(k),0} := H^{(k)} \quad \text{and, for } n \geq 1, \quad H^{(k),n} := \phi_n * H^{(k)};$$

and then

$$H_k^{(k),n}(x) := \det A(\lambda^k)^{-1} H^{(k),n}(A(\lambda^k)^{-1}x).$$

We decompose H_Γ in a similar manner as before. Specifically, if $S_{n,i}$ is defined exactly as in (4.26), we write

$$H_\Gamma f = H_{I,1} + H_{I,2} + H_{I,3} + H_{I,4} + H_{II,1} + H_{II,2},$$

where

$$\begin{aligned}
H_{I,1} &:= \sum_{k\in\mathbb{Z}} H_k^{(k)} * g,\\
H_{I,2} &:= \sum_{k\in\mathbb{Z}} H_k^{(k)} * \sum_{n\geq 1} g^n,\\
H_{I,4} &:= \sum_{m\geq 0}\sum_{k\in\mathbb{Z}} (H_k^{(k),m+1} - H_k^{(k),m}) * \sum_{n>m} g^n,\\
H_{I,3} &:= \sum_{n\geq 1}\sum_{k\in\mathbb{Z}} (H_k^{(k)} - H_k^{(k),n}) * f^n,\\
H_{II,1} &:= \sum_{n\geq 1}\sum_{i}\sum_{k\notin S_{n,i}} H_k^{(k),n} * b_{Q_i}^n,\\
H_{II,2} &:= \sum_{n\geq 1}\sum_{i}\sum_{k\in S_{n,i}} H_k^{(k),n} * b_{Q_i}^n.
\end{aligned}$$

As before, it suffices to prove the following estimates.

$$\sum_{i=1}^{4} \|H_{I,i}\|_2^2 \lesssim \alpha\|f\|_1; \tag{4.32}$$

$$\|H_{II,1}\|_{L^1(\mathbb{R}^2\setminus\Omega^*)} \lesssim \|f\|_1. \tag{4.33}$$

It is easy to see that $\|H_{I,1}\|_2^2+\|H_{I,2}\|_2^2 \lesssim \alpha\|f\|_1$ using the fact that H_Γ is bounded on L^2 by Theorem 4.1.2. Moreover, the following analogue of Lemma 4.2.4,

$$\left\|\sum_{k\in\mathbb{Z}}(H_k^{(k),m+1} - H_k^{(k),m}) * f\right\|_2 \lesssim 2^{-m\delta}\|f\|_2,$$

holds via Plancherel's theorem and Lemma 4.2.2. Thus, we may repeat arguments for $M_{I,3}$ and $M_{I,4}$ to get $\|H_{1,3}\|_2^2 + \|H_{1,4}\|_2^2 \lesssim \alpha\|f\|_1$ and hence (4.32). We may also run the argument that we used to prove (4.24) almost verbatim to deduce (4.33). This completes the proof of Theorem 4.1.3.

Beyond $L\log L$: a stumbling block

We conclude this section with a brief discussion on the possibility of improving Theorem 4.1.3 when $d = 2$. Our motivation is the main theorem in [57]; we encourage the reader to recall the setup of this paper given at the end Chapter 1. If we consider flat plane curves, then we violate the fundamental curvature assumption in [57]. By running through the argument in [57] with $\mu^{(k)}$ taking on the role of the fixed measure ν, and, for example, $\Gamma(t) = (t, 2^{-t^{-2}})$, one sees this violation quite clearly in the sense that the following crucial pointwise estimate

$$|D^\alpha(\mu^{(k)} * \widetilde{\mu^{(k)}})(x)| \lesssim |x|^{-1-|\alpha|} \tag{4.34}$$

fails when $|\alpha| = 1$ (where $\langle\widetilde{\mu^{(k)}}, \psi\rangle := \langle\mu^{(k)}, \psi(-\cdot)\rangle$). Incidentally, the pointwise estimate (4.34) is true when $\alpha = 0$ and $\Gamma(t) = (t, \gamma(t))$ for any convex γ whose derivative belongs to $\mathcal{C}_2$.

It may be more fruitful to move in the direction of Hardy space estimates. Two such results were mentioned at the end of Chapter 1 for the parabola. It is an interesting question whether theses results have analogues for some class of flat curves.

4.3 Nonconvex hypersurfaces

Let $d \geq 1$ and let $\Gamma(y) := (y, P(\gamma(|y|)))$ for $y \in \mathbb{R}^d$, where P is a polynomial with real coefficients of degree no less than 2, and γ satisfies the following conditions.

$$\gamma \in C^2(0,\infty), \text{ convex on } [0,\infty) \quad \text{and} \quad \gamma(0) = 0, \gamma'(0) \geq 0. \tag{4.35}$$

Our main result in this section is the following.

Theorem 4.3.1. *Suppose γ is extended to either an odd or even function on $\mathbb{R}$. Then the operators $\mathcal{M}_\Gamma$ and H_Γ are of weak type $L\log L$ if either*

1. *$d \geq 2$;*
2. *$d = 1$ and $P'(0) = 0$.*

The hypothesis of Theorem 4.3.1 should come with little surprise in the light of the analysis in Chapter 3. Recall that we were unable to suitably handle the second derivative of P on the first gap, in the sense that certain *almost everywhere* Fourier transform estimates were out of reach in the case $d = 1$. However, such estimates are crucial for the argument of Section 4.2. Hence, when $d = 1$ we eliminate this issue with the hypothesis $P'(0) = 0$ since this means that the first gap of P is empty. As in Chapter 3, when $d \geq 2$, this is not necessary because we can make use of the decay of the Fourier transform of surface measure on S^{d-1}.

Near L^1, the case $d = 1$ and $P'(0) \neq 0$ is clearly open; if we allow P to have degree 1, then of course we have a sufficient condition in Theorem 4.1.3, however one should note that, in this case, there is also no first gap. In the light of Theorem 3.1.1, the additional hypothesis $\gamma' \in \mathcal{C}_1$ offers itself as a possibility for a sufficient condition. In the next chapter we shall see some negative results for some examples of such γ, though we will not go so far as to prove that the conclusion of Theorem 4.3.1 cannot hold under these conditions.

Proof of Theorem 4.3.1. We consider the equivalent dyadic operator, M_Γ, which takes averages over $\{y \in \mathbb{R}^d : |y| \in (2^k, 2^{k+1})\}$ for $k \in \mathbb{Z}$.

Without loss of generality, we may take $P(0) = 0$. Write $P(s) = \sum_{k=1}^n p_k s^k$ where each p_k is real. Recall from Section 3.2 the decomposition of $(0, \infty)$ in (3.1) based on the roots of P. Also recall from (3.2) and (3.3) the definitions of the restricted operators H_I and M_I. We claim that the assertions of Theorem 4.3.1 are obvious for H_{D_k} and M_{D_k}, where $D_k = (\alpha_k, \beta_k)$ is a dyadic interval introduced in Section 3.2. This is simply because (3.4) implies that H_{D_k} and M_{D_k} are bounded operators from L^1 to itself.

The preceding observation tells us that to prove Theorem 4.3.1 it suffices prove the same assertions for each H_{G_j} and M_{G_j}; for this, we fix j, and use the same method as in Section 4.2. The appropriate $d+1$ by $d+1$ dilation matrices $\{A(t) = (A(t)_{k,l}) : t \in (0, \infty)\}$ are defined as follows.

$$A(t)_{k,l} := \begin{cases} t & \text{for } k = l \text{ and } \ k = 1, \dots, d, \\ |p_j|\gamma(t)^j & \text{for } k = l = d+1, \\ 0 & \text{for } k \neq l. \end{cases}$$

It follows from (3.4) that Proposition 4.2.1 holds for these dilations with ($C = \varepsilon = 1$). We normalise Γ in the same way:

$$\Gamma_k(y) := A(2^k)^{-1}\Gamma(2^k y) \quad \text{for } y \in \mathbb{R}^d.$$

Therefore,

$$\Gamma_k(y) = (y, \gamma_k(|y|)) \quad \text{where, for } t \in (0, \infty), \quad \gamma_k(t) := \frac{P(\gamma(2^k t))}{|p_j|\gamma(2^k)^j}.$$

Let $I_k := [1, 2] \cap 2^{-k}\gamma|_{(0,\infty)}^{-1}(G_j)$, as in Chapter 3.

Lemma 4.3.2. *Suppose $j \neq 1$ and γ is odd. Then, for all $|t| \in I_k$, we have*

$$|\gamma_k''(t)| \gtrsim |\gamma_k'(t)| \gtrsim |\gamma_k(t)| \gtrsim 1.$$

Proof. It is immediate that Lemma 3.2.1 and (4.35) give $|\gamma_k(t)| \gtrsim 1$ for all $|t| \in I_k$. Also, Lemma 3.2.1 and (1.22) imply that

$$\frac{|\gamma_k'(t)|}{|\gamma_k(t)|} = 2^k \frac{|P'(\gamma(2^k t))|}{|P(\gamma(2^k t))|}\gamma'(2^k t) \gtrsim 2^k \frac{\gamma'(2^k t)}{\gamma(2^k t)} \gtrsim 1,$$

which implies $|\gamma_k'(t)| \gtrsim 1$. For the remaining assertion, observe that

$$\frac{P''(\gamma(2^k t))\gamma'(2^k t)}{P'(\gamma(2^k t))} \quad \text{and} \quad \frac{\gamma''(2^k t)}{\gamma'(2^k t)}$$

are both positive on I_k and both negative on $-I_k$. This follows from Lemma 3.2.1 and the fact that γ is odd. Therefore,

$$\begin{aligned} \frac{|\gamma_k''(t)|}{|\gamma_k'(t)|} &= 2^k \left| \frac{P''(\gamma(2^k t))\gamma'(2^k t)}{P'(\gamma(2^k t))} + \frac{\gamma''(2^k t)}{\gamma'(2^k)} \right| \\ &\geq 2^k \frac{|P''(\gamma(2^k t))|\gamma'(2^k t)}{|P'(\gamma(2^k t))|} \gtrsim 1, \end{aligned}$$

where the last bound follows from another application of Lemma 3.2.1. □

Remark. The estimates in Lemma 4.3.3 up to the first derivative were being used in Chapter 3.

Define the following measures:

$$\begin{aligned} \langle \mu^{(k)}, \psi \rangle &:= \int_{|y| \in I_k} \psi(\Gamma_k(y))\, dy, \quad \langle \mu_k^{(k)}, \psi \rangle := \langle \mu^{(k)}, \psi(A(2^k)\cdot) \rangle, \\ \langle H^{(k)}, \psi \rangle &:= \int_{|y| \in I_k} \psi(\Gamma_k(y)) K(y)\, dy, \quad \langle H_k^{(k)}, \psi \rangle := \langle H^{(k)}, \psi(A(2^k)\cdot) \rangle. \end{aligned}$$

Of course, $M_{G_j} f = \sup_{k \in \mathbb{Z}} |\mu_k^{(k)} * f|$ and $H_{G_j} f = \sum_{k \in \mathbb{Z}} H_k^{(k)} * f$. Then we have the following decay estimates.

Lemma 4.3.3. *For each $\xi \neq 0$ we have $|\widehat{\mu^{(k)}}(\xi)| + |\widehat{H^{(k)}}(\xi)| \lesssim |\xi|^{-1/2}$.*

Proof. Under condition (1) of Theorem 4.3.1, this was essentially proved in Lemma 3.4.1 and we shall not repeat the details. Instead, assume condition (2) of Theorem 4.3.1 holds and, for fixed ξ with $|\xi| \geq 1$, define

$$\theta(t) := t\xi_1 + \gamma_k(t)\xi_2 \quad \text{for } t \in I_k.$$

If $|\xi_2| \geq |\xi_1|$ then, by Lemma 4.3.3, we have $|\theta''(t)| \gtrsim |\xi_2| \sim |\xi|$ for all $t \in I_k$. By van der Corput's lemma,

$$\left| \int_{t \in I_k} e^{i\xi \cdot (t, \gamma_k(t))}\, dt \right| \lesssim |\xi|^{-1/2}. \tag{4.36}$$

We claim that (4.36) also holds when $|\xi_1| \geq |\xi_2|$. To see this, first suppose that $P > 0$ on G_j. Since G_1 is empty, we know from Lemma 3.2.1 that $\gamma_k'' > 0$ and $\gamma_k' > 0$ on I_k. Hence there is at most one solution $t \in I_k$ to the equation

$$\gamma_k'(t) = \frac{|\xi_1|}{2|\xi_2|}. \tag{4.37}$$

If a solution t_0 to (4.37) exists, then for $t \in I_k$ with $t \leq t_0$ we have

$$|\theta'(t)| \geq |\xi_1| - \gamma_k'(t)|\xi_2| \geq |\xi_1|/2 \sim |\xi|; \tag{4.38}$$

and for $t > t_0$, by Lemma 4.3.3,

$$|\theta''(t)| = \gamma_k''(t)|\xi_2| \gtrsim \gamma_k'(t)|\xi_2| \geq |\xi_1|/2 \sim |\xi|. \tag{4.39}$$

Since $\gamma_k'' > 0$ on I_k we know that θ' is monotone on I_k. Thus (4.36), or in fact a better estimate, follows from van der Corput's lemma for $|\xi_1| \geq |\xi_2|$. The case where a solution to (4.37) does not exist is handled as in (4.38) or (4.39).

If γ is even, $|\widehat{\mu^{(k)}}(\xi)| \lesssim |\xi|^{-1/2}$ is immediate from the above. If γ is odd, then one can say that $\gamma_k'' > 0$ and $\gamma_k' < 0$ on I_k. Then the equation

$$|\gamma_k'(t)| = \frac{|\xi_1|}{2|\xi_2|}$$

has at most one solution, and one can argue as above to deduce that $|\widehat{\mu^{(k)}}(\xi)| \lesssim |\xi|^{-1/2}$. A similar argument works when $P < 0$ on G_j, and using integration by parts we get the required decay estimate for $\widehat{H^{(k)}}$. □

The final ingredient in the proof of Theorem 4.3.1 is the appropriate choice of Calderón-Zygmund balls. This is significantly simpler than in the Section 4.2 because the dilation matrices are diagonal. For fixed $k \in \mathbb{Z}$ there exists a finite collection $\{E_k^1, \ldots, E_k^{n(k)}\}$ of open, balanced, convex, and bounded sets such that

$$A(2^k)B_1(0) = E_k^1 \subseteq E_k^2 \subseteq \ldots \subseteq E_k^{n(k)} = A(2^{k+1})B_1(0),$$

and $|E_k^l| \leq 2|E_k^{l+1}|$ for $l = 1, \ldots, n(k)-1$. One can easily verify that (3.4) implies that $A(t)B_1(0) \subset A(s)B_1(0)$ whenever $0 < t < s$, and therefore the E_k^l are well defined. As in Section 4.2, the collection $\{B_k\}_{k\in\mathbb{Z}}$ is chosen to be the collection $\{E_k^l\}_{k\in\mathbb{Z},1\leq l\leq n(k)}$. Then the conditions (B1) - (B5) hold for the B_k and we are free to use the Calderón-Zygmund theory developed in [9]. The main ingredients are now in place, and the argument that we used in Section 4.2 can now be used to complete the proof of Theorem 4.3.1. □

Remarks. 1. Theorem 4.3.1 implies a certain pointwise convergence result for averages over the hypersurfaces considered in this section, for functions belonging locally to $L \log L$.

2. It follows from the proof of Theorem 4.3.1 that there exists a finite constant C, which is independent of the coefficients of P, such that for all unit cubes Q in $\mathbb{R}^d$,

$$\|\mathcal{M}_\Gamma^{loc}(f\chi_Q)\|_{L^{1,\infty}} + \|H_\Gamma^{loc}(f\chi_Q)\|_{L^{1,\infty}} \leq C\|f\|_{L\log L(Q)}.$$

Chapter 5

Piecewise Linear Curves Near L^1

5.1 Introduction

Suppose we are given a plane curve $\{(t, \widetilde{\gamma}(t)) : t \in (0, \infty)\}$ and $\lambda \in (1, \infty)$ is fixed. We can form a continuous piecewise linear version, Γ, of this curve in the following manner: Define a $\Gamma(t) := (t, \gamma(t))$ by stipulating that,

$$\text{for each } k \in \mathbb{Z},\ \gamma(\lambda^k) = \widetilde{\gamma}(\lambda^k) \text{ and } \gamma \text{ is affine on } [\lambda^{k-1}, \lambda^k]; \tag{5.1}$$

see Figure 5.1 for an example. In [18], Christ proves that if the derivative of γ takes infinitely many distinct values then $\mathcal{M}_\Gamma$ is not of weak type L. In fact, this result is a corollary of the more general proposition stated below concerning averages over line segments, in $\mathbb{R}^d$ for $d \geq 2$, which point in distinct directions and may have arbitrary location. To be specific, let $\mathfrak{L}_N := \{l_j : 1 \leq j \leq N\}$ be a collection of N line segments in $\mathbb{R}^d$ of finite length, let ω_j be a unit vector in the same direction as l_j, and let μ_j denote one-dimensional Hausdorff measure on l_j normalised to have total mass 1. Then define the following maximal function,

$$M_{\mathfrak{L}_N} f(x) := \sup_{1 \leq j \leq N} \int_{l_j} |f(x-y)| d\mu_j(y). \tag{5.2}$$

Proposition 5.1.1. [18] *Fix $d \geq 2$, $N \geq 1$, and a collection of line segments $\{l_j : 1 \leq j \leq N\}$ as above with $\omega_j \neq \omega_k$ for each $j \neq k$. Then there exists $\varepsilon > 0$ and a function f_ε in L^1 such that*

$$|\{x \in \mathbb{R}^d : M_{\mathfrak{L}_N} f_\varepsilon(x) > \varepsilon\}| \geq B(d) N \varepsilon^{-1} \|f_\varepsilon\|_1.$$

The main goal of this chapter is to consider the weak type behaviour of $\mathcal{M}_\Gamma$ on $\Phi(L)$, for Φ belonging to the family of functions in (1.24) and certain examples of piecewise linear Γ.

Overview. In Section 5.2 we firstly consider the case $\widetilde{\gamma} = P/Q$, where P and Q are polynomials with rational coefficients. This certainly covers the parabolic case,

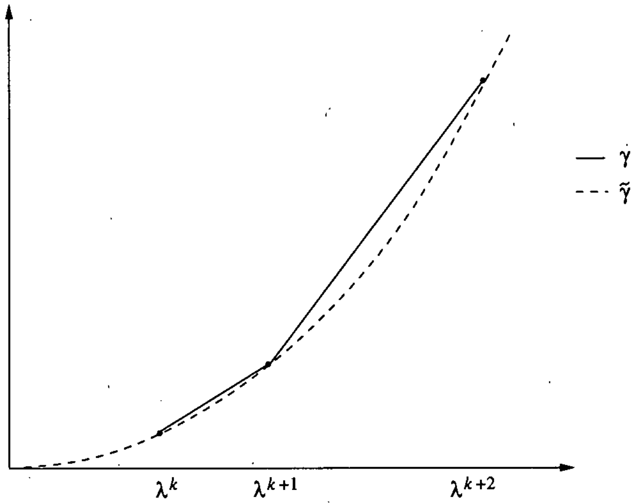

Figure 5.1: A piecewise linear version, γ, of $\widetilde{\gamma}$

$\widetilde{\gamma}(t) = (t, t^2)$, and we note that the derivative of the resulting γ, as defined above in (5.1), belongs to $\mathcal{C}_1$. We include some fragmentary results in the case of real coefficients. Persuaded by the generality of Proposition 5.1.1, we also consider the case where $\widetilde{\gamma}(t)$ is a prototype flat curve, $2^{-t^{-2}}$ for small $t > 0$; again γ' belongs to $\mathcal{C}_1$ (or strictly speaking, some modified local version of $\mathcal{C}_1$). In Section 5.3 we include a very brief discussion on the sharpness of the our results.

Notation. Write $A \lesssim B$ for $A \leq CB$ where the constant C depends on at most Γ. If $v \in \mathbb{R}^2$, define $\mathbb{R}v := \{tv : t \in \mathbb{R}\}$. If E is a finite line segment, denote the length of E by $\mathbb{L}(E)$. Let $\text{dist}(E, F) = \inf\{|x - y| : x \in E \text{ and } y \in F\}$, for nonempty subsets E and F of $\mathbb{R}^2$.

5.2 Main results

Rational coefficients

Theorem 5.2.1. *Suppose $\lambda \in \mathbb{Q} \cap (1, \infty)$, and let $\widetilde{\gamma}(t) = (t, R(t))$ where $R(t) = P(t)/Q(t)$ and P and Q are polynomials with rational coefficients such that R is non-affine. If $\Gamma(t) := (t, \gamma(t))$, where γ satisfies* (5.1)*, then $\mathcal{M}_\Gamma$ is not of weak type $L(\log L)^\sigma$ for each $\sigma \in (0, 1/2)$.*

Remarks. 1. Our proof of Theorem 5.2.1 is completely based on Christ's proof of Proposition 5.1.1 in [18]. We use exactly his construction of the function f_ε. For the specific Γ in Theorem 5.2.1, however, we shall see that it is possible to make a quantitative estimate on how small ε should be; this is in contrast to the indeterminate way ε is chosen in Christ's proof, which is of course demanded by the generality of Proposition 5.1.1.

2. The case where Γ is a curve in $\mathbb{R}^d$ ($d \geq 2$) with (smooth) rational components has been studied in [29] and [30]. In particular, if $R_1, \ldots, R_d$ are real rational functions and $\Gamma(t) = (R_1(t), \ldots, R_d(t))$ then it is shown in [30] that H_Γ and $\mathcal{M}_\Gamma$ are bounded on L^p for all $p \in (1, \infty)$.

3. With reference to the discussion in Section 4.3 on page 71, since $L \log L$ locally sits inside $L(\log L)^{1/2}$, Theorem 5.2.1 does not preclude the condition $\gamma' \in \mathcal{C}_1$ as being sufficient for the maximal operator along the plane convex curve $(t, \gamma(t))$ to be of weak type $L \log L$.

4. When R is affine, $\mathcal{M}_\Gamma$ is essentially the classical one-dimensional Hardy-Littlewood maximal operator, and hence is of weak type L.

Proof of Theorem 5.2.1. Let $\lambda = p(\lambda)/q(\lambda)$, where $p(\lambda), q(\lambda) \in \mathbb{N}$, $p(\lambda) \geq 2$. Write

$$R(t) = \frac{\sum_{r=n_0}^{n_1} p_r t^r}{\sum_{s=m_0}^{m_1} q_s t^s},$$

where each p_r and q_s are rational and $p_{n_0}, p_{n_1}, q_{m_0}, q_{m_1}$ are all nonzero. Without loss of generality, we suppose that $\min(n_0, m_0) \geq 1$ and $p_{n_1} = q_{m_1} = 1$. If we let $n := n_1 - m_1$ then we have the following fact: Given $0 < A_1 < 1 < A_2$ there exists $t_0 \sim 1$ such that

$$A_1 t^n \leq R(t) \leq A_2 t^n \quad \text{for all } t \in (t_0, \infty). \tag{5.3}$$

It should be clear from (5.3) that the cases $n = 0$ and $n = 1$ should cause the most difficulty. Heuristically, these cases are closest to the situation where all the line segments are pointing in the same direction.

Fix a natural number N, which counts the number of line segments. This parameter will later tend to infinity and should be considered large. If k is an integer with $1 \leq k \leq N$, let l_k be the portion of the curve Γ in the interval $[\lambda^{k_0+k-1}, \lambda^{k_0+k}]$. Here, $k_0 \sim 1$ is another fixed natural number whose role is to ensure that we are sufficiently far along the curve so that we have useful information on R, like (5.3). The exact value of k_0 will not be given, but it will be clear from the proof that an appropriate choice can be made.

We parameterise each line segment l_k in the following way:

$$l_k = \{c_k + t(1, \tau_k) : t \in [-t_k, t_k]\}, \tag{5.4}$$

where

$$\tau_k := \frac{R(\lambda^{k_0+k}) - R(\lambda^{k_0+k-1})}{\lambda^{k_0+k} - \lambda^{k_0+k-1}}$$

is the slope of l_k, c_k is the midpoint of l_k, and t_k is some real number. It is straightforward to verify that

$$t_k \sim \lambda^k \quad \text{and} \quad \mathbb{L}(l_k) \sim \begin{cases} \lambda^{nk} & \text{for } n \geq 1, \\ \lambda^k & \text{for } n \leq 0. \end{cases} \tag{5.5}$$

Also define

$$\widetilde{l_k} := \{t(1, \tau_k) : t \in [-t_k/8, t_k/8]\}, \tag{5.6}$$
$$\widehat{l_k} := \{c_k + t(1, \tau_k) : t \in [-t_k/2, t_k/2]\}. \tag{5.7}$$

Clearly $\mathcal{M}_\Gamma$ dominates $M_{\mathcal{L}_N}$; for the majority of this proof, we work with $M_{\mathcal{L}_N}$.

It is crucial that we have some control on the slopes τ_k. The following lemma contains the information we require.

Lemma 5.2.2. *1. For $1 \leq k \leq N$,*

$$|\tau_k| \sim \begin{cases} \lambda^{(n-1)k} & \textit{for } n \geq 2, \\ 1 & \textit{for } n \leq 1, \end{cases}$$

2. For any $j \neq k$, is

$$|\tau_k - \tau_j| \gtrsim \begin{cases} 1 & \textit{for } n \geq 2, \\ \lambda^{-\rho N} & \textit{for } n \leq 1, \end{cases}$$

for some natural number ρ depending only on R, and

$$|\tau_k - \tau_j| \lesssim \begin{cases} \lambda^{(n-1)N} & \textit{for } n \geq 2, \\ 1 & \textit{for } n \leq 1. \end{cases}$$

Proof. The proof of this lemma is easy if $n \geq 2$ or $n \leq -1$. For $n \geq 2$, we have

$$\frac{A_1 - \lambda^{-n} A_2}{1 - \lambda^{-1}} \lambda^{(k_0+k)(n-1)} \leq \tau_k \leq \frac{A_2 - \lambda^{-n} A_1}{1 - \lambda^{-1}} \lambda^{(k_0+k)(n-1)}, \tag{5.8}$$

and so part (1) of the lemma follows by choosing A_1 and A_2 sufficiently close to 1 so that $\lambda^n > A_2/A_1$. We also get from (5.8), and perhaps a refined choice of $A_1, A_2 \sim 1$ sufficiently close to 1, that

$$\frac{\tau_{k+1}}{\tau_k} \geq \frac{A_1 - \lambda^{-n} A_2}{A_2 - \lambda^{-n} A_1} \lambda^{n-1} > 1. \tag{5.9}$$

Therefore, if $k > j$,

$$|\tau_k - \tau_j| = \tau_k - \tau_j \geq \tau_k - \tau_{k-1} = \tau_k(1 - \tau_{k-1}/\tau_k) \gtrsim \tau_k \gtrsim 1,$$

which gives the lower bound in part (2) of the lemma. The upper bound $|\tau_k - \tau_j| \lesssim \lambda^{(n-1)N}$ is trivial by (5.8). A similar argument also works for $n \leq -1$ and so we omit the details.

Suppose now that $n = 1$. Although the above argument still applies to get part (1) we shall need to be a little more careful in order to establish part (2). We have

$$\begin{aligned}\tau_k &= \frac{1}{1-\lambda^{-1}} \frac{\sum_{r=n_0}^{m_1+1}\sum_{s=m_0}^{m_1} p_r q_s(\lambda^{-s}-\lambda^{-r})\lambda^{(r+s-1)(k_0+k)}}{\left(\sum_{s=m_0}^{m_1} q_s\lambda^{s(k_0+k)}\right)\left(\sum_{s'=m_0}^{m_1} q_{s'}\lambda^{s'(k_0+k-1)}\right)} \\ &= 1 + R_k,\end{aligned}$$

where $R_k := P_k/Q_k$, and

$$\begin{aligned}P_k &:= \sum_{r=n_0}^{m_1}\sum_{s=m_0}^{m_1-1} p_r q_s(\lambda^{-s}-\lambda^{-r})\lambda^{(r+s-1)(k_0+k)} \\ &\qquad - \sum_{s=m_0}^{m_1-1}\sum_{s'=m_0}^{m_1-1} q_s q_{s'}(\lambda^{-s'}-\lambda^{-s'-1})\lambda^{(s+s')(k_0+k)}, \\ Q_k &:= \sum_{s=m_0}^{m_1}\sum_{s'=m_0}^{m_1} q_s q_{s'}(\lambda^{-s'}-\lambda^{-s'-1})\lambda^{(s+s')(k_0+k)}.\end{aligned}$$

Therefore,

$$P_k = \sum_{r=\min(n_0+m_0-1,2m_0)}^{r_0} \widetilde{p_r}\lambda^{r(k_0+k)},$$

for some $r_0 \le 2m_1 - 2$, with $\widetilde{p_{r_0}} \ne 0$. Indeed, if all the $\widetilde{p_r}$ were zero then $\tau_k = 1$ for each $k \ge 1$, and this implies that R is affine. It is clear then that, choosing $k_0 \sim 1$ sufficiently large, we can make R_k as close to the quantity

$$\frac{\widetilde{p_{r_0}}}{(\lambda^{-m_1}-\lambda^{-m_1-1})}\lambda^{-(2m_1-r_0)(k_0+k)}$$

as we please; since $\widetilde{p_{r_0}} \sim 1$, this certainly proves part (1) of the lemma when $n = 1$. For part (2), suppose that $\widetilde{p_{r_0}} > 0$. Then $R_k > 0$ for each k and R_{k+1}/R_k is as close to $\lambda^{-(2m_1-r_0)} < 1$ as we please. Hence, if $k > j$,

$$|\tau_k - \tau_j| = |R_k - R_j| = R_j - R_k \ge R_j - R_{j+1} \gtrsim R_j \gtrsim \lambda^{-(2m_1-r_0)N}.$$

A similar argument for the lower bound in part (2) can be used if $\widetilde{p_{r_0}} < 0$. Also it is clear that $|\tau_k| \lesssim 1$ for each k and this implies the upper bound in part (2); this completes the proof of the lemma when $n = 1$. The case $n = 0$ can be handled in a similar way to the case $n = 1$ so we choose to omit the details. □

The next step is to define a set of points in $\mathbb{R}^2$,

$$\{z_j^k : 1 \le k \le N,\ 0 \le j \le \Lambda(k) - 1\}, \tag{5.10}$$

which meet certain conditions. For $1 \le k \le N$, we fix $\Lambda(k)$ to be some integer satisfying

$$\max(\mathbb{L}(l_k), 1) < \Lambda(k) \lesssim \mathbb{L}(l_k). \tag{5.11}$$

Such a choice is certainly possible; for instance, we are free to dilate each of the line segments l_k by a fixed factor (which may depend on N). Thus, we may do so in such a way that $\mathbb{L}(l_k) > 1$ for each k and then the existence of $\Lambda(k)$ is immediate. Assuming the z_j^k have been chosen, we let

$$\mathcal{A} := \left\{ w \in \mathbb{R}^2 : w = \sum_{k=1}^{N} z_{\alpha_k}^k, \ 0 \leq \alpha_k \leq \Lambda(k) - 1 \right\}. \tag{5.12}$$

Define, for each $1 \leq k \leq N$, a kth-equivalence relation on $\mathcal{A}$, $\approx_k$, by

$$\text{if} \quad w, w' \in \mathcal{A} \quad \text{then} \quad w \approx_k w' \quad \text{if and only if} \quad w - w' \in \mathbb{R}(1, \tau_k).$$

Then the following are the required conditions on the z_j^k.

(Z1). $z_0^k = 0$.

(Z2). $z_j^k \in \widetilde{l_k}$.

(Z3). For fixed $1 \leq j \leq N$, if $w, w' \in \mathcal{A}$ and $w = \sum_{k=1}^N z_{\alpha_k}^k$, $w' = \sum_{k=1}^N z_{\beta_k}^k$ then

$$w \approx_j w' \quad \text{if and only if} \quad \alpha_i = \beta_i \quad \text{for all} \quad i \neq j.$$

A simple consequence of condition (Z3) is that

$$|\mathcal{A}| = \prod_{k=1}^{N} \Lambda(k) \sim \begin{cases} \lambda^{nN(N+1)/2} & \text{for } n \geq 1, \\ \lambda^{N(N+1)/2} & \text{for } n \leq 0. \end{cases} \tag{5.13}$$

We shall initially define a set of points

$$\{Z_j^k : 1 \leq k \leq N, \ 0 \leq j \leq \Lambda(k) - 1\}$$

which are manageable in a sense that will become clear later in the proof. These points will satisfy (Z1) and (Z2), but may not satisfy (Z3). Our choice of the z_j^k will then be a small perturbation of the Z_j^k to ensure (Z3), whilst not disturbing the nice properties of the Z_j^k. To define the Z_j^k, we introduce positive numbers $\delta_1, \ldots, \delta_N$ inductively in the following way:

$$\delta_1 := 2^{-C_1 N^2},$$

and for $2 \leq k \leq N$,

$$\delta_k := C_2(N) \sum_{i=1}^{k-1} \delta_i, \tag{5.14}$$

where C_1 and $C_2(N)$ shall be chosen later, with the constraint

$$C_1 \sim 1 \quad \text{and} \quad 1 \lesssim C_2(N)^{N-1} \lesssim \delta_1^{-1}. \tag{5.15}$$

Then we have the following tautological result.

Lemma 5.2.3. *1. For $2 \le k \le N$ we have $\delta_k = C_2(N)\,(1 + C_2(N))^{k-2}\,\delta_1$.*

2. $0 < \delta_1 < \delta_2 < \ldots < \delta_N$.

For $1 \le k \le N$ and $0 \le j \le \Lambda(k) - 1$, define

$$Z_j^k := C_3(N) j (1 + \delta_k)(1, \tau_k), \tag{5.16}$$

where the role of the constant $C_3(N)$ is to ensure that

$$C_3(N)\Lambda(k)(1 + \delta_k) \le t_k/16 \quad \text{for each } 1 \le k \le N. \tag{5.17}$$

It follows from Lemma 5.2.3(1) and (5.15) that there exists a choice

$$C_3(N) \sim \Lambda(N)^{-1} \tag{5.18}$$

which is up to this task.

Now we shall use the Z_j^k to define the z_j^k. We make the following claim: For each $1 \le k \le N$, we can choose real numbers η_r^i for $1 \le i \le k$ and $0 \le r \le \Lambda(r)$ so that

$$0 \le \eta_r^i \le C_4(N) := \min(\{t_l/(16\Lambda(l)) : 1 \le l \le N\} \cup \{\lambda^{-(C_1+2)N^2}\}), \tag{5.19}$$

and if

$$z_r^i := Z_r^i + r\eta_r^i(1, \tau_i), \tag{5.20}$$

then $z_r^i \in \widetilde{l}_i$ and condition $(Z3)_k$ holds; this condition being condition (Z3) with $1 \le j \le N$ but the set $\mathcal{A}$ replaced by all elements of the form

$$\sum_{i=1}^{k} z_{\alpha_i}^i \quad \text{for } 0 \le \alpha_i \le \Lambda(i) - 1.$$

Note that (Z1) is clearly satisfied if we have (5.20). We proceed by induction on k.

If $k = 1$ then we define $\eta_j^1 := 0$ for each $0 \le j \le \Lambda(1)$. One can easily check that each Z_j^1, and therefore z_j^1, lies on $\widetilde{l}_1$. Since each z_j^1 is distinct, condition $(Z3)_1$ is also satisfied.

Suppose the claim has been shown to be true for k. We shall define each η_r^{k+1} in succession, beginning with η_1^{k+1}. One can check that as long as z_1^{k+1} does not belong to one of the following lines

$$\sum_{i=1}^{k}(z_{\alpha_i}^i - z_{\beta_i}^i) + \mathbb{R}(1, \tau_j) \quad \text{for } j \ne k+1 \text{ and } 0 \le \alpha_i, \beta_i \le \Lambda(i) - 1, \tag{5.21}$$

then condition $(Z3)_{k+1}$ will not be violated for $\alpha_{k+1} \in \{0, 1\}$. We do not need to include the case $j = k+1$ in (5.21) because this case is handled by the assumption that the claim is true for k. One can also check that as long as we have (5.19) then we have $Z_1^{k+1}+\eta_1^{k+1}(1,\tau_{k+1}) \in \widetilde{l_{k+1}}$. This means we can choose any η_1^{k+1} satisfying (5.19) except the finitely many possible η_1^{k+1} for which $Z_1^{k+1}+\eta_1^{k+1}(1,\tau_{k+1})$ lies on one of the lines in (5.21). Now fix such a choice of η_1^{k+1}, and hence z_1^{k+1}, and consider η_2^{k+1}. One can again easily verify that as long as z_2^{k+1} does not belong to one of the lines

$$\sum_{i=1}^{k}(z^i_{\alpha_i} - z^i_{\beta_i}) + \mathbb{R}(1,\tau_j) \quad \text{for } j \neq k+1, \tag{5.22}$$

or one of the lines

$$z_1^{k+1} + \sum_{i=1}^{k}(z^i_{\alpha_i} - z^i_{\beta_i}) + \mathbb{R}(1,\tau_j) \quad \text{for } j \neq k+1, \tag{5.23}$$

then condition $(Z3)_{k+1}$ is not violated for $\alpha_{k+1} \in \{0,1,2\}$. Also, if η_2^{k+1} satisfies (5.19) then $Z_2^{k+1}+2\eta_2^{k+1}(1,\tau_{k+1}) \in \widetilde{l_{k+1}}$. Hence we are free to fix any η_2^{k+1} obeying (5.19) except the finitely many for which $Z_2^{k+1}+2\eta_2^{k+1}(1,\tau_{k+1})$ lies on one of the lines in either (5.22) or (5.23). We may continue this procedure to obtain η_r^{k+1} satisfying (5.19) for $0 \leq r \leq \Lambda(k+1)-1$ and which give rise to points z_j^{k+1} via (5.20) which satisfy $(Z3)_{k+1}$. (Note that the bound on the right hand side of (5.19) ensures that if we have (5.20) then we always have $z_j^k \in \widetilde{l_k}$.) This completes our induction.

Henceforth in this proof z_j^k is defined by (5.20), where η_j^k satisfies (5.19), and the z_j^k satisfy (Z1),(Z2), and (Z3).

Remark. We have now introduced four distinguished constants $C_1, C_2(N), C_3(N)$, and $C_4(N)$ involved in the definition of the points z_j^k. These points are absolutely key to the proof of Theorem 5.2.1. To avoid confusion, no other constant which appears in the remainder of this proof will contain a subscript.

Define

$$\varepsilon := \lambda^{-(C_1+100)N^2}, \tag{5.24}$$

and

$$f_N(x) := \sum_{w\in\mathcal{A}} \chi_{B_\varepsilon(w)}(x). \tag{5.25}$$

This is the function f_ε appearing in the statement of Proposition 5.1.1; the subscript has turned into N since this is the crucial parameter in the proof of Theorem 5.2.1. For each $w \in \mathcal{A}$ and $1 \leq k \leq N$ define

$$S(w,k) := \{x \in \mathbb{R}^2 : \operatorname{dist}(x - \widehat{l_k}, w) < \varepsilon/2\}. \tag{5.26}$$

We claim that, for each $w \in \mathcal{A}$ and $1 \leq k \leq N$,

$$S(w,k) \subseteq \{x \in \mathbb{R}^2 : M_{\mathcal{L}_N} f_N(x) \geq \varepsilon/2\}. \tag{5.27}$$

To see this, fix $1 \leq k \leq N$, $w \in \mathcal{A}$, and $x \in S(w,k)$. It suffices to show

$$\mathbb{L}((x - l_k) \cap B_\varepsilon(w')) \geq \varepsilon/2 \quad \text{for all } w' \approx_k w, \tag{5.28}$$

because (5.28), the fact that there are precisely $\Lambda(k)$ elements $w' \in \mathcal{A}$ for which $w' \approx_k w$, and (5.11) give

$$\begin{aligned} M_{\mathcal{L}_N} f_N(x) &\geq \int_{l_k} f_N(x-y)\, d\mu_k(y) \\ &\geq \sum_{w' \approx_k w} \int_{-t_k}^{t_k} \chi_{B_\varepsilon(w')}(x - c_k - t(1,\tau_k)) \frac{dt}{2t_k} \\ &\geq \frac{\varepsilon}{4t_k|(1,\tau_k)|} \sum_{w' \approx_k w} 1 \\ &= \frac{\varepsilon}{2} \cdot \frac{\Lambda(k)}{\mathbb{L}(l_k)} \\ &\geq \varepsilon/2. \end{aligned}$$

To prove (5.28), suppose $w' \approx_k w$ so that, first using condition (Z3) and then condition (Z2), $w' - w = (s' - s)(1, \tau_k)$ where $s, s' \in [-t_k/8, t_k/8]$. Now, $x \in S(w,k)$ and therefore there exists $t \in [-t_k/2, t_k/2]$ for which

$$|x - (c_k + t(1,\tau_k)) - w| = \operatorname{dist}(x - \widehat{l_k}, w) < \varepsilon/2.$$

Hence

$$|x - (c_k + (t - (s' - s))(1,\tau_k)) - w'| = |x - (c_k + t(1,\tau_k)) - w| < \varepsilon/2,$$

and since $|t - (s' - s)| < t_k$ it follows that

$$\operatorname{dist}(x - l_k, w') < \varepsilon/2. \tag{5.29}$$

Obviously $\mathbb{L}(x - l_k) \geq \varepsilon/2$ and therefore (5.28) follows from (5.29).

We have now reduced matters to obtaining a lower estimate on the area of

$$\bigcup_{j=1}^{N} \bigcup_{w \in \mathcal{A}} S(w,j).$$

The bulk of the work for this is contained in the following lemma.

Lemma 5.2.4. *For each $1 \leq j \leq N$,*

$$\left| \bigcup_{w \in \mathcal{A}} S(w,j) \right| \gtrsim \varepsilon |\mathcal{A}|.$$

Proof. Fix $1 \leq j \leq N$ and let

$$\widehat{\mathcal{A}} := \left\{ \widehat{w} \in \mathbb{R}^2 : \widehat{w} = \sum_{k \neq j} z^k_{\alpha_k} \quad \text{and} \quad 0 \leq \alpha_k \leq \Lambda(k) - 1 \right\}.$$

Then we have $|\widehat{\mathcal{A}}| = |\mathcal{A}|/\Lambda(j)$. We claim that if we take distinct elements $\widehat{w_1}$ and $\widehat{w_2}$ from $\widehat{\mathcal{A}}$ then $S(\widehat{w_1}, j)$ and $S(\widehat{w_2}, j)$ are disjoint. This claim granted, the lemma follows easily because

$$\begin{aligned} \left| \bigcup_{w \in \mathcal{A}} S(w,j) \right| &\geq \left| \bigcup_{\widehat{w} \in \widehat{\mathcal{A}}} S(\widehat{w},j) \right| \\ &= \sum_{\widehat{w} \in \widehat{\mathcal{A}}} |S(\widehat{w},j)| \geq \varepsilon \frac{\mathbb{L}(l_j)}{2} . \frac{|\mathcal{A}|}{\Lambda(j)} \gtrsim \varepsilon |\mathcal{A}|, \end{aligned}$$

where the final bound is due to (5.11). To prove the claim, suppose that $S(\widehat{w_1}, j) \cap S(\widehat{w_2}, j)$ is nonempty. Then it follows that

$$\operatorname{dist}(\widehat{w_1} - \widehat{w_2}, \widehat{l_j} - \widehat{l_j}) < \varepsilon. \qquad (5.30)$$

The rest of the proof is therefore dedicated to showing that, using our choice of the z^k_j, (5.30) is a contradiction.

Write $\widehat{w_1} = \sum_{k \neq j} z^k_{\alpha_k}$ and $\widehat{w_2} = \sum_{k \neq j} z^k_{\beta_k}$. Suppose that $\mathcal{K}$ is the set of all $k \in \{1, \ldots, N\} \setminus \{j\}$ for which $\alpha_k \neq \beta_k$. Clearly $\mathcal{K}$ is a nonempty set, and we let k_0 be the largest member of $\mathcal{K}$.

We shall be working with the Z^k_j initially (recall their definition in (5.16)), and we write,

$$\begin{aligned} &\sum_{k \neq j} (Z^k_{\alpha_k} - Z^k_{\beta_k}) \\ = &\Bigg(C_3(N) \sum_{k \neq j} (\alpha_k - \beta_k) + C_3(N) \sum_{k \neq j} \delta_k (\alpha_k - \beta_k), \\ &\quad C_3(N) \sum_{k \neq j} \tau_k (\alpha_k - \beta_k) + C_3(N) \sum_{k \neq j} \tau_k \delta_k (\alpha_k - \beta_k) \Bigg) \\ =: &\ (s + r_1, t + r_2). \end{aligned}$$

A simple computation shows that

$$(1 + \tau_j^2)^{1/2} \operatorname{dist}((s + r_1, t + r_2), \mathbb{R}(1, \tau_j)) = |(\tau_j s - t) + (\tau_j r_1 - r_2)|. \qquad (5.31)$$

When nonzero, $\tau_j s - t$ provides the main contribution to the second term on the right hand side of (5.31). A lower bound is attained in Sublemma 5.2.6 below. First, the following sublemma gives us the required bounds on the 'remainder' term, $|\tau_j r_1 - r_2|$.

Sublemma 5.2.5. *If ρ is from Lemma 5.2.2(2), then*

$$|\tau_j r_1 - r_2| \lesssim \begin{cases} N\lambda^{(n-1)N}\delta_N & \text{for } n \geq 2, \\ N\delta_N & \text{for } n \leq 1, \end{cases}$$

and

$$|\tau_j r_1 - r_2| \gtrsim \begin{cases} C_3(N)\delta_1 & \text{for } n \geq 2, \\ \lambda^{-\rho N}C_3(N)\delta_1 & \text{for } n \leq 1. \end{cases}$$

Proof. For the upper bound, we use Lemma 5.2.2, Lemma 5.2.3, and the fact that the number of points on each $\widetilde{l_k}$ is equal to $\Lambda(k)$. For the lower bound, first consider $n \geq 2$. Note that

$$C_3(N)^{-1}|r_2 - \tau_j r_1| = \left|(\tau_{k_0} - \tau_j)\delta_{k_0}(\alpha_{k_0} - \beta_{k_0}) + \sum_{k\in\mathcal{K}\setminus\{k_0\}} (\tau_k - \tau_j)\delta_k(\alpha_k - \beta_k)\right|,$$

and we deal with the more difficult case where $\mathcal{K} \setminus \{k_0\} \neq \emptyset$ first. If $n \geq 2$, then using the fact that there at most $\Lambda(N)$ points on each $\widetilde{l_k}$ and Lemma 5.2.3 we get

$$C_3(N)^{-1}|r_2 - \tau_j r_1| \gtrsim \delta_{k_0} - C\lambda^{(n-1)N}\Lambda(N)\sum_{k=1}^{k_0-1}\delta_k,$$

for some $C \sim 1$. Now our definition in (5.14) implies that it is possible to choose

$$C_2(N) \sim \lambda^{(n-1)N}\Lambda(N), \tag{5.32}$$

such that $C_3(N)^{-1}|r_2 - \tau_j r_1| \gtrsim \delta_{k_0}$; by Lemma 5.2.3 this implies the lower bound when $n \geq 2$ and $\mathcal{K} \setminus \{k_0\} \neq \emptyset$. Note that (5.32) does not violate (5.15) for a suitably large choice of $C_1 \sim 1$. If $n \geq 2$ and $\mathcal{K} \setminus \{k_0\} = \emptyset$ then, by Lemma 5.2.2 and Lemma 5.2.3, $C_3(N)^{-1}|r_2 - \tau_j r_1| = |\tau_{k_0} - \tau_j|\delta_{k_0}|\alpha_{k_0} - \beta_{k_0}| \gtrsim \delta_1$, as required.

When $n \leq 1$ the same argument applies, and Lemma 5.2.2 moves us to make the choice $C_2(N) \sim \lambda^{\rho N}\Lambda(N)$, which of course does not violate (5.15). □

Sublemma 5.2.6. *If $\tau_j s - t \neq 0$ then for sufficiently large N,*

$$|\tau_j s - t| \gtrsim \lambda^{-4m_1 N(N-1)} q(\lambda)^{-4m_1 N^2}.$$

Proof. It suffices to consider the case where P and Q have integer coefficients since we are free to replace γ with any fixed nonzero multiple. Now,

$$\begin{aligned}(1 - \lambda^{-1})\tau_k &= \frac{\sum_r p_r \lambda^{(r-1)(k_0+k)}}{\sum_s q_s \lambda^{s(k_0+k)}} - \frac{\sum_r p_r \lambda^{(r-1)(k_0+k-1)}}{\sum_s q_s \lambda^{s(k_0+k-1)}} \\ &= \frac{\sum_r \sum_s p_r q_s \lambda^{(k_0+k)(r+s-1)}(\lambda^{-s} - \lambda^{-r})}{\sum_s \sum_{s'} q_s q_{s'} \lambda^{(s+s')(k_0+k)-s}},\end{aligned}$$

and therefore

$$(1-\lambda^{-1})(\tau_j - \tau_k) = \frac{\sum_{r,s,\varsigma,\varsigma'} p_r q_s q_\varsigma q_{\varsigma'} (\lambda^I - \lambda^J)(\lambda^{-s} - \lambda^{-r})}{\sum_{s,s',\varsigma,\varsigma'} q_s q_{s'} q_\varsigma q_{\varsigma'} \lambda^{(s+s')(k_0+j)+(\varsigma+\varsigma')(k_0+k)-s-\varsigma}} =: \frac{\mathcal{N}(j,k)}{\mathcal{D}(j,k)},$$

where,

$$I := I(r,s,\varsigma,\varsigma') := (k_0+j)(r+s-1) + (k_0+k)(\varsigma+\varsigma') - \varsigma \in \mathbb{N},$$

$$J := J(r,s,\varsigma,\varsigma') := (k_0+k)(r+s-1) + (k_0+j)(\varsigma+\varsigma') - \varsigma \in \mathbb{N},$$

for $n_0 \leq r \leq n_1$ and $m_0 \leq s, \varsigma, \varsigma' \leq m_1$. Using the fact that $\max(I,J) \leq (n_1 + 3m_1)(k_0 + N)$ it is easy to see that

$$C(N)\mathcal{N}(j,k) \;:=\; q(\lambda)^{2(n_1+3m_1)(k_0+N)} p(\lambda)^{n_1+m_1} \mathcal{N}(j,k) \in \mathbb{Z}.$$

Similarly, one can check that

$$C'(N)\mathcal{D}(j,k) := q(\lambda)^{4m_1(k_0+N)} \mathcal{D}(j,k) \in \mathbb{Z}.$$

Since

$$\begin{aligned}(1-\lambda^{-1})(\tau_j s - t) &= \sum_{k\neq j} (\alpha_k - \beta_k) \frac{\mathcal{N}(j,k)}{\mathcal{D}(j,k)} \\ &= \frac{1}{\prod_{k''\neq j} \mathcal{D}(j,k'')} \sum_{k\neq j} (\alpha_k - \beta_k) \mathcal{N}(j,k) \left(\prod_{k'\neq j, k'\neq k} \mathcal{D}(j,k') \right),\end{aligned}$$

it follows that

$$(1-\lambda^{-1}) \left(\prod_{k''\neq j} \mathcal{D}(j,k'') \right) C(N) C'(N)^{N-2} (\tau_j s - t) \in \mathbb{Z}.$$

Moreover,

$$\mathcal{D}(j,k) = Q(\lambda^{k_0+j}) Q(\lambda^{k_0+j-1}) Q(\lambda^{k_0+k}) Q(\lambda^{k_0+k-1}),$$

so

$$0 < \mathcal{D}(j,k) \lesssim \lambda^{4m_1 N}.$$

Now we can use the fact that $\tau_j s - t \neq 0$ to deduce

$$|\tau_j s - t| \gtrsim \lambda^{-4m_1 N(N-1)} C(N)^{-1} C'(N)^{-(N-2)} \gtrsim \lambda^{-4m_1 N(N-1)} q(\lambda)^{-4m_1 N^2},$$

for sufficiently large N. This completes the proof of Sublemma 5.2.6. □

We are now in a position to show that (5.30) is a contradiction, and hence complete the proof of Lemma 5.2.4. First, suppose $\tau_j s - t$ is nonzero. It is clear from Sublemma 5.2.5 and Sublemma 5.2.6 that, upon a large enough choice of $C_1 \sim 1$, we have $|\tau_j s - t| \gtrsim |\tau_j r_1 - r_2|$. Therefore, (5.31) and Lemma 5.2.2 imply

$$\operatorname{dist}\left(\sum_{k\neq j}(Z^k_{\alpha_k} - Z^k_{\beta_k}), \mathbb{R}(1,\tau_j)\right) \gtrsim \lambda^{-(n-1)N-4m_1N(N-1)}q(\lambda)^{-4m_1N^2},$$

for sufficiently large N. If on the other hand $\tau_j s - t$ is zero then we use (5.31), Lemma 5.2.2, Sublemma 5.2.5, and the choice of $C_3(N)$ in (5.18) to get

$$\operatorname{dist}\left(\sum_{k\neq j}(Z^k_{\alpha_k} - Z^k_{\beta_k}), \mathbb{R}(1,\tau_j)\right) \gtrsim \lambda^{-(C_1+1)N^2}, \tag{5.33}$$

for sufficiently large N and $C_1 \sim 1$. Thus, in either case, we can conclude that (5.33) holds. Hence there exists a constant $C \sim 1$ such that

$$\begin{aligned}
&\operatorname{dist}(\widehat{w_1} - \widehat{w_2}, \mathbb{R}(1,\tau_j)) \\
&= \operatorname{dist}\left(\sum_{k\neq j}(Z^k_{\alpha_k} - Z^k_{\beta_k}) + \sum_{k\neq j}(\alpha_k\eta^k_{\alpha_k} - \beta_k\eta^k_{\beta_k})(1,\tau_k), \mathbb{R}(1,\tau_j)\right) \\
&\geq \operatorname{dist}\left(\sum_{k\neq j}(Z^k_{\alpha_k} - Z^k_{\beta_k}), \mathbb{R}(1,\tau_j)\right) - \left|\sum_{k\neq j}(\alpha_k\eta^k_{\alpha_k} - \beta_k\eta^k_{\beta_k})(1,\tau_k)\right| \\
&\gtrsim \lambda^{-(C_1+1)N^2} - C\sum_{k\neq j}|\alpha_k\eta^k_{\alpha_k} - \beta_k\eta^k_{\beta_k}||(1,\tau_k)|.
\end{aligned}$$

It follows from (5.19) that $\operatorname{dist}(\widehat{w_1} - \widehat{w_2}, \mathbb{R}(1,\tau_j)) \gtrsim \lambda^{-(C_1+1)N^2}$. But $\widehat{l_j} - \widehat{l_j} \subseteq \mathbb{R}(1,\tau_j)$, so, for sufficiently large N,

$$\operatorname{dist}(\widehat{w_1} - \widehat{w_2}, \widehat{l_j} - \widehat{l_j}) \geq \operatorname{dist}(\widehat{w_1} - \widehat{w_2}, \mathbb{R}(1,\tau_j)) \geq \lambda^{-(C_1+2)N^2} \geq \varepsilon.$$

This contradicts (5.30) and thus completes the proof of Lemma 5.2.4. □

We next intend to use Lemma 5.2.4 and our choice of ε to prove that the sets

$$S_j := \bigcup_{w\in\mathcal{A}} S(w,j) \quad \text{for } 1 \leq j \leq N,$$

are essentially disjoint. We claim that this follows if we can show that there exists some $C \sim 1$ such that whenever $i \neq j$,

$$|S(w,i) \cap S(w',j)| \leq \lambda^{CN}\varepsilon^2, \tag{5.34}$$

where w and w' are allowed to be equal. Since (5.34) is not difficult, we prove this first. Observe that $S(w,j)$ is a tubular neighbourhood around the line segment $w + \widehat{l_j}$. Therefore the maximum overlap of $S(w,i)$ with $S(w',j)$ occurs as shown

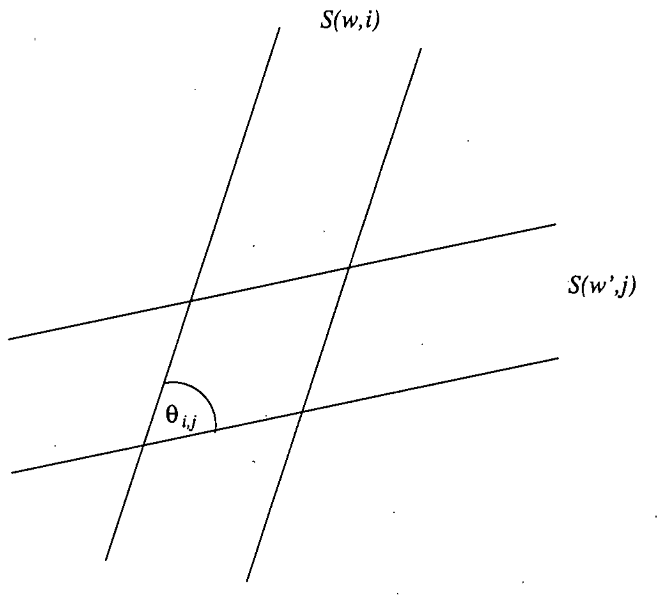

Figure 5.2: The maximum overlap of $S(w,i)$ and $S(w',j)$

in Figure 5.2, where $\theta_{i,j} \in (0,\pi/2)$ is the angle between the vectors $(1,\tau_i)$ and $(1,\tau_j)$. A little elementary geometry gives

$$\sin\theta_{i,j} = \frac{|\tau_i - \tau_j|}{(1+\tau_i^2)^{1/2}(1+\tau_j^2)^{1/2}} \gtrsim \begin{cases} \lambda^{-2(n-1)N} & \text{for } n \geq 2, \\ \lambda^{-\rho N} & \text{for } n \leq 1, \end{cases}$$

from which (5.34) follows.

An elementary consequence of (5.34) is that, for any $r \geq 2$ and $1 \leq i_1 < \ldots < i_r \leq N$,

$$|S_{i_1} \cap \ldots \cap S_{i_r}| \leq \lambda^{CN}\varepsilon^2|\mathcal{A}|^2. \tag{5.35}$$

Using the inclusion-exclusion principle in a very crude way this implies

$$\left|\bigcup_{j=1}^{N} S_j\right| \geq \sum_{j=1}^{N} |S_j| - 2^N\lambda^{CN}\varepsilon^2|\mathcal{A}|^2.$$

By Lemma 5.2.4, there exists $C' \sim 1$ such that

$$\sum_{j=1}^{N} |S_j| \geq 2C'N\varepsilon|\mathcal{A}|. \tag{5.36}$$

Recalling our choice of ε in (5.24), and the the estimate in (5.13) for the cardinality of $\mathcal{A}$, there exists a suitably large choice of $C_1 \sim 1$ such that

$$C' \geq N^{-1}2^N\lambda^{CN}\varepsilon|\mathcal{A}|. \tag{5.37}$$

(The constants may cause some confusion here; recall C_1 is one of our distinguished constants, $C \sim 1$ appears in (5.34), and C' arises in (5.36) above.) Hence,

$$\left|\bigcup_{j=1}^{N}\bigcup_{w\in\mathcal{A}} S(w,j)\right| = \left|\bigcup_{j=1}^{N} S_j\right| \sim \sum_{j=1}^{N} |S_j| \gtrsim N|\mathcal{A}|\varepsilon. \tag{5.38}$$

This bound is sufficient to complete the proof of Theorem 5.2.1: If we suppose that $\mathcal{M}_\Gamma$ is of weak type $L(\log L)^\sigma$ for some $\sigma \in (0, 1/2)$ then, by (5.27) and (5.38),

$$N|\mathcal{A}|\varepsilon \lesssim \int \frac{f_N(x)}{\varepsilon}\left(\log\left(\frac{2f_N(x)}{\varepsilon}+10\right)\right)^\sigma dx. \tag{5.39}$$

However, $0 \leq f_N(x) \leq |\mathcal{A}|$, from which it follows that the right hand side of (5.39) is $\lesssim$

$$\frac{1}{\varepsilon}\left(\log\left(\frac{2|\mathcal{A}|}{\varepsilon}+10\right)\right)^\sigma \sum_{w\in\mathcal{A}} |B_\varepsilon(w)| \lesssim N^{2\sigma}|\mathcal{A}|\varepsilon.$$

Since $\sigma \in (0, 1/2)$, this is clearly nonsense for large enough N, and thus the proof of Theorem 5.2.1 is complete. □

Real coefficients

It is clear from the proof of Theorem 5.2.1 that the proof of Sublemma 5.2.6 was the sole place that we used the condition that the coefficients of the polynomials P and Q were rational. The real coefficient case seems to be tricky if one adopts the same approach. Let us consider the basic case where $P(t) = \sum_{r=1}^n p_r t^r$ where $n \geq 2$ ($p_n \neq 0$), $Q = 1$, and $\lambda = 2$. Then, reusing notation from the proof of Theorem 5.2.1,

$$\begin{aligned}\frac{2^{n-1}(\tau_j s - t)}{C_3(N)} &= \sum_{r=2}^n p_r \sum_{k=1}^N (\alpha_k - \beta_k)(2^n - 2^{n-r})(2^{(k_0+j)(r-1)} - 2^{(k_0+k)(r-1)}) \\ &=: \sum_{r=2}^n p_r I(r, j, N),\end{aligned}$$

where $I(r, j, N)$ is an integer for each r, j, and N. In the case where $\tau_j s - t \neq 0$ we have only been able to control $|\sum_{r=2}^n p_r I(r, j, N)|$ from below in very easy cases using elementary arguments. For example, if we assume that $p_2 = \ldots = p_{n-1} = 0$, or nothing when $n = 2$, then we can easily deduce that

$$|\tau_j s - t| \neq 0 \Rightarrow |\tau_j s - t| \gtrsim C_3(N). \tag{5.40}$$

At the next level of difficulty where precisely one of $p_2, \ldots, p_{n-1}$ is nonzero, say p_{r_0}, then we are looking to control the quantity,

$$|p_{r_0} I(r_0, j, N) + p_n I(n, j, N)|,$$

from below. We may as well assume that $I(r_0, j, N)$ and $I(n, j, N)$ are nonzero; otherwise we immediately get the outcome in (5.40). Thus we are naturally led to the theory of rational approximation and the topic of convergents.

Definition 5.2.7. Let ζ be a real number with continued fraction representation $[a_0; a_1, a_2, \ldots]$ ($a_0 \in \mathbb{Z}$ and $a_j \in \mathbb{N}$ for $j \geq 1$). Then, for $l \geq 1$, the *convergent of order* l of ζ is the (irreducible) rational number A_l/B_l with continued fraction representation $[a_0; a_1, \ldots, a_l]$.

The following theorems contain the crucial results we need concerning convergents. Proofs can be found in [36].

Theorem 5.2.8. *Let ζ be a an irrational number. If A_l/B_l is the convergent of order l of ζ, then*

$$\inf\{|\beta\zeta - \alpha| : \beta \in \{1, \ldots, B_l\}, \alpha \in \mathbb{Z}\} = |B_l\zeta - A_l| \geq \frac{1}{B_{l+1} + B_l},$$

for sufficiently large l.

Theorem 5.2.9. *There exists a null set $\mathcal{N}$ (in the sense of Lebesgue) such that for all ζ belonging to $\mathbb{R} \setminus \mathcal{N}$ there exist real numbers μ and ν in $(1, \infty)$ such that, for sufficiently large l,*

$$\mu^l \leq B_l \leq \nu^l.$$

Suppose that at least one of p_{r_0}/p_n and p_n/p_{r_0} is an irrational number and lies outside the null set $\mathcal{N}$ from Theorem 5.2.9, and for argument's sake suppose the former is such a number. Let A_l/B_l denote the convergent of order l of p_{r_0}/p_n and let μ and ν be the growth constants from Theorem 5.2.9. Without too much work, one has the estimate,

$$|I(r_0, j, N)| \leq 2^{5nN} \leq \mu^{CN},$$

for some $C \sim 1$, and without loss of generality we take C to be a natural number. Hence, by Theorem 5.2.8 and Theorem 5.2.9,

$$\begin{aligned} |I(r_0, j, N)p_{r_0}/p_n + I(n, j, N)| &\geq |B_{CN}p_{r_0}/p_n - A_{CN}| \\ &\geq \frac{1}{B_{CN+1} + B_{CN}} \gtrsim \frac{1}{\nu^{CN}}. \end{aligned}$$

Therefore, we can deduce that

$$|\tau_j s - t| \neq 0 \Rightarrow |\tau_j s - t| \gtrsim C_3(N)\nu^{-CN}.$$

However, we are stuck with the undesirable problem that the null set $\mathcal{N}$ is indeterminate. If instead we had assumed that either p_{r_0}/p_n, or its reciprocal, was an algebraic number then Liouville's classical theorem on rational approximation (see, for example, [1]) will also give a version of Sublemma 5.2.6. (Roth's famous improvement of Liouville's theorem is of no help to us here.) However, this result is much more unsatisfactory since algebraic numbers form a null set in $\mathbb{R}$. We summarise the above observations in the following 'baby theorem'. Further progress in the real coefficient case seems to require a fresh approach, with a view to handling a greater number of nonzero coefficients.

Baby Theorem 5.2.10. *Suppose* $\lambda = 2$ *and* $\tilde{\gamma}(t) = (t, P(t))$ *where* $P(t) = \sum_{r=1}^{n} p_r t^r$ *for some* $n \geq 2$ *and* $p_n \neq 0$. *The following conditions are sufficient to conclude that* $\mathcal{M}_\Gamma$ *is not of weak type* $L(\log L)^\sigma$ *for any* $\sigma \in (0, 1/2)$.

1. $n = 2$.
2. $n \geq 3$ *and* $\{p_2, \ldots, p_{n-1}\} = \{0\}$.
3. $n \geq 3$, $\{p_2, \ldots, p_{n-1}\} = \{p_{r_0}, 0\} \neq \{0\}$ *and* p_{r_0}/p_n, *or its reciprocal, is an irrational number which either belongs to the complement of the null set* $\mathcal{N}$ *arising in Theorem 5.2.9 or is an algebraic number.*

A flat example

In a different direction, we simply state a result concerning our flat curve prototype $\tilde{\gamma}(t) = 2^{-t^{-2}}$ with $\lambda = 2$. We are only interested in the resulting piecewise linear curve Γ near the origin and thus the local operator $\mathcal{M}_\Gamma^{loc}$. Notice that the argument we used to prove Theorem 5.2.1 considered the portion of the curves in question at infinity. However, one can check that we could have also considered the portion of the curves near the origin. This requires blowing everything up by a factor $C(N)$ which does not affect the argument at all, and this approach yields the following theorem.

Theorem 5.2.11. *If* $\sigma \in (0, 1)$ *then* $\mathcal{M}_\Gamma^{loc}$ *is not of weak type* $L(\log^{(2)} L)^\sigma$.

Despite the flatness at the origin of $(t, 2^{-t^{-2}})$, when one forms the piecewise linear version, one still has a good quantitative grip on how the slopes are behaving. Indeed, one can check that as k tends to minus infinity, the slope on $[2^k, 2^{k+1}]$ is essentially $2^{-C2^{-2k}}$ for some $C \sim 1$. This fact determines the conclusion of Theorem 5.2.11.

Remark. Although we have not checked the details, we suspect that all of the main results in this section are also true for the associated singular integral operator H_Γ. In [18], Christ remarks that it is apparent from his construction that H_Γ is not of weak type L when Γ is a piecewise linear curve whose derivative takes infinitely many values.

5.3 Sharpness

For this discussion, let us consider the case $\widetilde{\gamma}(t) = t^2$ which motivated Theorem 5.2.1. We know from this result that, at best, $\mathcal{M}_\Gamma$ locally maps $L(\log L)^{1/2}$ into $L^{1,\infty}$. We suggest that it is far from obvious how one can push Christ's counterexample construction any further. Recall that we needed our choice of ε to satisfy (5.37) in order to prove that the sets S_k are essentially disjoint, and thus avoid the very delicate question of how they overlap. Moreover, we needed (5.37) to be true regardless of what we chose as the definition of the z_j^k. From this point of view, we are forced to take ε to be at most λ^{-CN^2} for some $C \sim 1$. For any improvement, we need ε to be at least $\lambda^{-C'N^s}$ for some $C' \sim 1$ and $s < 2$.

We believe that the best known result in the positive direction is in [14] where it was shown that $\mathcal{M}_\Gamma$ maps L^p to L^p (globally) for all $p > 1$. The same result also follows from a more general result in [13]. The proof in [13] involved a bootstrap argument involving a square function very closely related to the following one:

$$\mathcal{R} : f \mapsto \left(\sum_{k\in\mathbb{Z}} |R_k f|^2 \right)^{1/2},$$

where $\widehat{R_k f}(\xi) = 2\chi_{\Delta_k}(\xi)\widehat{f}(\xi)$ and, for a fixed $\lambda \in (1,\infty)$,

$$\Delta_k := \left\{ \xi \in \mathbb{R}^2 : \frac{1}{\lambda^{k+2}} \leq \frac{\xi_2}{\xi_1} \leq \frac{1}{\lambda^k} \right\}.$$

The Δ_k are angular sections which form a decomposition of the plane and are finitely overlapping. It is certainly not clear to us how a bootstrap argument would apply to the Orlicz spaces near L^1. However, we conclude this chapter with a sketch proof of the potentially useful observation that $\mathcal{R}$ is not of weak type $L(\log L)^\sigma$ for each $\sigma \in [0,1)$. That $\mathcal{R}$ is bounded on L^p for all $p > 1$ is essentially proved in [50]. Also, the smoothed out version of $\mathcal{R}$ is a Marcinkiewicz-type multiplier, and a result of R. Fefferman in [28] implies that $\mathcal{R}$ is of weak type $L \log L$.

Sketch proof of our observation. First notice that

$$R_k f = f + H_k(H_{k+2} f), \tag{5.41}$$

where $H_k f := Hf(\cdot, \omega_k)$ and H is the operator in (1.9), with the dilations δ_t isotropic and $\omega_k := (-1, \lambda^k)/|(-1,\lambda^k)|$. Equality (5.41) follows because $\widehat{H_k f}(\xi) = -i\mathrm{sgn}(\xi.\omega_k)\widehat{f}(\xi)$. Our observation follows simply by evaluation of $\mathcal{R}$ on $\chi_{B_{2\delta}(0)}$, for sufficiently small $\delta > 0$. The point is that if, for $k \geq 1$ the infinite strips Σ_k are Σ_k' are those shown in Figure 5.3, then $\sim \log(1/\delta)$ of these Σ_k are disjoint,

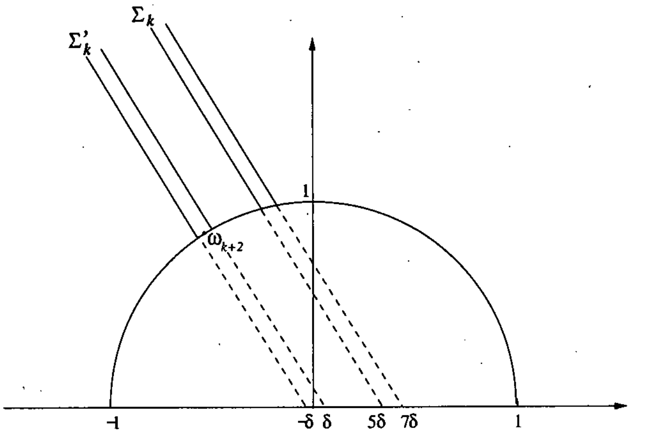

Figure 5.3: The strips Σ_k and Σ'_k

and for x in Σ_k, we have that $|R_k(\chi_{B_{2\delta}(0)})(x)| \gtrsim \delta(\lambda^k x_1 + x_2)^{-1}$. The latter is true since $H_{k+2}(\chi_{B_{2\delta}(0)})(y) \gtrsim \delta|y|^{-1}$ for $y \in \Sigma'_k$. Thus, for small $\delta > 0$ we get,

$$|\{x \in \mathbb{R}^2 : \mathcal{R}f(x) \gtrsim \delta\}| \gtrsim \delta \log(1/\delta),$$

and our observation follows. $\square$

Bibliography

[1] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory, 2nd Ed.*, New York: Springer Verlag (1997).

[2] A. Benedek, R. Panzone, *The spaces L^p, with mixed norm*, Duke Math. J. **28** (1961), 301–324.

[3] C. Bennett, R. Sharpley, *Interpolation of Operators*, Academic Press (1988).

[4] N. Bez, *L^p-boundedness for the Hilbert transform and maximal operator along a class of nonconvex curves*, Proc. Amer. Math. Soc. **135** (2007), 151–161.

[5] J. Bourgain, *Besicovitch-type maximal operators and applications to Fourier analysis*, Geom. Funct. Anal. **1** (1991), 147–187.

[6] A. Calderón, A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952), 85–139.

[7] A. Calderón, A. Zygmund, *On singular integrals*, Amer. J. Math. **78** (1956), 289–309.

[8] A. Calderón, A. Zygmund, *On singular integrals with variable kernels*, Appl. Anal. **7** (1978), 221–238.

[9] A. Carbery, M. Christ, J. Vance, S. Wainger, D. Watson, *Operators associated to flat plane curves: L^p estimates via dilation methods*, Duke Math. J. **59** (1989), 677–700.

[10] A. Carbery, F. Ricci, J. Wright, *Maximal functions and Hilbert transforms associated to polynomials*, Revista Mat. Ibero. **14** (1998), 117–144.

[11] A. Carbery, J. Vance, S. Wainger, D. Watson, *The Hilbert transform and maximal function along flat curves, dilations, and differential equations*, Amer. J. Math. **116** (1994), 1203–1239.

[12] A. Carbery, S. Ziesler, *Hilbert transforms and maximal functions along rough flat curves*, Revista Mat. Ibero. **10** (1994), 379–393.

[13] H. Carlsson, M. Christ, A. Cordoba, J. Duoandikoetxea, J. L. Rubio de Francia, J. Vance, S. Wainger, D. Weinberg, *L^p estimates for maximal functions and Hilbert transforms along flat convex curves in $\mathbb{R}^2$*, Bull. Amer. Math. Soc. **14** (1986), 263–267.

[14] H. Carlsson, S. Wainger, *Maximal functions related to convex polygonal lines*, Indiana Univ. Math. J. **34** (1985), 815–823.

[15] L.-K. Chen, *The singular integrals related to the Calderón-Zygmund method of rotations*, Appl. Anal. **30** (1988), 319–329.

[16] L.-K. Chen, *The maximal operators related to the Calderón-Zygmund method of rotations*, Illinois J. Math. **33** (1989), 268–279.

[17] M. Christ, *Weak type (1,1) bounds for rough operators*, Ann. of Math. **128** (1988), 19–42.

[18] M. Christ, *Examples of singular maximal functions unbounded on L^p*, Conference on Mathematical Analysis (El Escorial, 1989), Publ. Mat. **35** (1991), 269–279.

[19] M. Christ, J. Duoandikoetxea, J. Rubio de Francia, *Maximal operators related to the Radon transform and the Calderón-Zygmund method of rotations*, Duke Math. J. **53** (1986), 189–209.

[20] M. Christ, E. M. Stein, *A remark on singular Calderón-Zygmund theory*, Proc. Amer. Math. Soc. **99** (1987), 71–75.

[21] A. Córdoba, *The Kakeya maximal function and the spherical summation multipliers*, Amer. J. Math. **99** (1977), 1–22.

[22] A. Córdoba, J. L. Rubio de Francia, *Estimates for Wainger's singular integrals along curves*, Revista Mat. Ibero. **2** (1986), 105–117.

[23] M. Cowling, G. Mauceri, *Inequalities for some maximal functions, I*, Trans. Amer. Math. Soc. **287** (1985), 431–455.

[24] J. Duoandikoetxea, *Fourier Analysis*, American Mathematical Society, Providence, Rhode Island (1995).

[25] R. Durán, *On polynomial approximation in Sobolev spaces*, SIAM J. Numer. Anal. **20** (1983), 985–988.

[26] E. B. Fabes, N. M. Rivière, *Singular integrals with mixed homogeneity*, Studia Math. **27** (1966), 19–38.

[27] R. Fefferman, *On an operator arising in the Calderón-Zygmund method of rotations and the Bramble-Hilbert lemma*, Proc. Nat. Acad. Sci. U.S.A. **80** (1983), 3877–3878.

[28] R. Fefferman, *Some topics from harmonic analysis and partial differential equations*, Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991), 175–210, Princeton Math. Ser., 42, Princeton Univ. Press, Princeton, NJ, (1995).

[29] M. Folch-Gabayet, J. Wright, *An oscillatory integral estimate associated to rational phases*, J. Geom. Anal. **13** (2003), 291–299.

[30] M. Folch-Gabayet, J. Wright, *Singular and maximal operators associated to curves with rational components*, to appear in Trans. Amer. Math. Soc.

[31] P. Gressman, *L^p-improving properties of X-ray like transforms*, Math. Res. Lett. **13** (2006), 10001–10017.

[32] G. H. Hardy, J. E. Littlewood, *A maximal theorem with function-theoretic applications*, Acta Math. **54** (1930), 81–116.

[33] A. Kamińska, *Some remarks on Orlicz-Lorentz spaces*, Math. Nachr. **147** (1990), 29–38.

[34] N. Katz, T. Tao, *New bounds for the Kakeya problems*, J. Anal. Math. **87** (2002), 231–263.

[35] N. Katz, T. Tao, *Recent progress on the Kakeya conjecture*, Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations (El Escorial, 2000), Publ. Mat. (2002), 161–179.

[36] A. Ya. Khintchine, *Continued Fractions*, P. Noordhoff (1963).

[37] W. Kim, S. Wainger, J. Wright, S. Ziesler, *Singular integrals and maximal functions associated to surfaces of revolution*, Bull. London Math. Soc. **28** (1996), 291–296.

[38] M. A. Krasnosel'skǐ, Ya. B. Rutickiǐ, *Convex Functions and Orlicz Spaces*, P. Noordhoff (1961).

[39] G. G. Lorentz, *Some new functional spaces*, Ann. of Math. **51** (1950), 37–55.

[40] G. G. Lorentz, *On the theory of spaces* Λ, Pacific J. Math. **1** (1951), 411–429.

[41] W. A. J. Luxemburg, *Banach Function Spaces*, Thesis, Delft Technical Univ. (1955).

[42] L. Maligranda, *Indices and interpolation*, Dissert. Math. **234** (1984), 1–49.

[43] J. Marcinkiewicz, A. Zygmund, *On the summability of double Fourier series*, Fund. Math. **32** (1939), 112–132.

[44] M. Mastyło, *Interpolation of linear operators in Calderón-Lozanovskiĭ spaces*, Comment. Math. Prace Mat. **26** (1986), 247–256.

[45] S. Mihlin, *Singular integral equations*, Amer. Math. Soc. Transl. **10** (1962), 84–198.

[46] W. P. Minicozzi II, C. D. Sogge, *Negative results for Nikodym maximal functions and related oscillatory integrals in curved space*, Math. Res. Lett. **4** (1997), 221–237.

[47] S. J. Montgomery-Smith, *Comparison of Orlicz-Lorentz spaces*, Studia Math. **103** (1992), 161–189.

[48] A. Nagel, N. M. Rivière, S. Wainger, *A maximal function associated to the curve* (t, t^2), Proc. Nat. Acad. Sci. U.S.A. **73** (1976), 1416–1417.

[49] A. Nagel, N. M. Rivière, S. Wainger, *On Hilbert transforms along curves. II.*, Amer. J. Math. **98** (1976), 395–403.

[50] A. Nagel, E. M. Stein, S. Wainger, *Differentiation in lacunary directions*, Proc. Nat. Acad. Sci. U.S.A. **75** (1978), 1060–1062.

[51] A. Nagel, J. Vance, S. Wainger, D. Weinberg, *Hilbert transforms for convex curves*, Duke Math. J. **50** (1983), 735–744.

[52] A. Nagel, J. Vance, S. Wainger, D. Weinberg, *The Hilbert transform for convex curves in* $\mathbb{R}^n$, Amer. J. Math. **108** (1986), 485–504.

[53] W. Orlicz, *Über eine gewisse Klasse von Räumen vom Typus B*, Bull. Intern. Acad. Pol. **8** (1932), 207–220.

[54] N. M. Rivière, *Singular integrals and multiplier operators*, Ark. Mat. **9** (1971), 243-278.

[55] S. Sato, *Maximal functions associated with curves and the Calderón-Zygmund method of rotations*, Trans. Amer. Math. Soc. **293** (1986), 799–806.

[56] A. Seeger, T. Tao, *Sharp Lorentz space estimates for rough operators*, Math. Ann. **320** (2001), 381–415.

[57] A. Seeger, T. Tao, J. Wright, *Singular maximal functions and Radon transforms near L^1*, Amer. J. Math. **126** (2004), 607–647.

[58] A. Seeger, S. Wainger, J. Wright, S. Ziesler, *Singular integral and maximal integral operators associated to hypersurfaces: L^p theory*, Journ. Geom. Anal. **15** (2005), 477–498.

[59] C. D. Sogge, *Concerning Nikodym-type sets in 3-dimensional curved spaces*, J. Amer. Math. Soc. **12** (1999), 1–31.

[60] E. M. Stein, *Harmonic Analysis*, Princeton Univ. Press, Princeton, N. J. (1993).

[61] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princton, N. J. (1970).

[62] E. M. Stein, S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), 1239–1295.

[63] E. M. Stein, G. Weiss, *An Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, N. J. (1971).

[64] J. O. Strömberg, unpublished.

[65] M. E. Taylor, *Partial Differential Equations I*, New York: Springer Verlag (1996).

[66] J. Vance, S. Wainger, J. Wright, *The Hilbert transform and maximal function along nonconvex curves in the plane*, Revista Mat. Ibero. **10** (1994), 93–121.

[67] N. Wiener, *The ergodic theorem*, Duke Math. J. **5** (1939), 1–18.

[68] L. Wisewell, *Oscillatory Integrals and Curved Kakeya Sets*, PhD thesis, University of Edinburgh (2003).

[69] L. Wisewell, *Kakeya sets of curves*, Geom. Funct. Anal. **15** (2005), 1319–1362.

[70] T. Wolff, *An improved bound for Kakeya type maximal functions*, Revista Mat. Ibero. **11** (1995), 651–674.

[71] J. Wright, *L^p estimates for operators associated to oscillating plane curves*, Duke Math. J. **67** (1992), 101–157.