A GROUP THEORETICAL STUDY OF THE HARMONIC OSCILLATOR QUARK MODEL AND ITS IMPLICATIONS FOR BARYON SPECTROSCOPY

Thesis

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TO

Maria and Bruno

DECLARATION

Much of the work described in this thesis, notably the bulk of the material in Chapters 2-5, was performed in collaboration with at least one of the following: Drs. K.C. Bowler, A.J.G. Hey, P.D. Jarvis and R.C. King. The contents of Chapter 6, however, are entirely my own, except where otherwise indicated by the appropriate reference. Published work includes the following references not acknowledged in the text:

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ABSTRACT

We undertake a group-theoretical study of the specific form of the harmonic oscillator quark model proposed by Isgur and Karl, restricting our attention to the non-strange sector of the baryon spectrum. In particular, we consider the spectrum-generating group, Sp(12,R), appropriate to the study of the 3-quark problem, and demonstrate how it may be used to label the oscillator eigenstates and to provide a new and direct means of constructing wavefunctions of definite orbital angular momentum and permutation - symmetry type. We indicate how Sp(12,R) provides the most appropriate means of classifying the symmetry-breaking induced by an anharmonic perturbation and we derive an algebraic mass formula, involving the quadratic Casimir invariant operators of Sp(12,R) and its relevant subgroups, plus one non-subgroup invariant operator, which successfully reproduces the splitting pattern of the $N = 2$ supermultiplets originally derived in the literature by straightforward perturbative techniques. Some results for the $N = 3$ level are also given together with an outline of the method for generalisation to any degree of excitation of the system. Much of our understanding of the rôle of the spectrum - generating group in this context derives from a parallel study of the simpler case of a 2-particle bound system, which we also describe. We examine the implications of our results for baryon spectroscopy: in particular, we discuss in some detail the possibility that the $\Delta D35(1940)$ resonance is evidence for an $N = 3$ $[\underline{56},1^-]$ supermultiplet corresponding to excitation of new gluonic degrees of freedom. After inclusion of hyperfine effects, and with reasonable values of the parameters in the model, we recover the pertinent features appropriate to the D35, G37 and G39 sectors of a recent preliminary phase-shift analysis by Cutkosky et al.

C O N T E N T S

C O N T E N T S (Contd.)

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PROLOGUE

In recent years, the theory of quantum chromodynamics (QCD) has become the leading candidate to describe the strongly-interacting particles (hadrons). In this theory, hadrons are assumed to be composite particles, their constituents being quarks (q) and antiquarks ($\bar{q}$). However, QCD goes beyond the early quark models (Lichtenberg 1981 and references therein) in that it purports to be a dynamical theory of the interactions of quarks.

At the most naïve level, quarks and antiquarks provide a mnemonic that accounts for properties such as the spin, parity and electric charge of the observed hadrons as their being composed of qqq for baryons ($\bar{q}\,\bar{q}\,\bar{q}$ for antibaryons) and $q\bar{q}$ for mesons. In the naïve model, quarks have finite mass, fractional electric charge and are spin $\frac{1}{2}$ particles (fermions) with two types of internal degree of freedom: flavour and colour. Flavour concerns quantum numbers such as electric charge, isospin and hypercharge, which are directly observed in elementary particle interactions. At present, there is good (but indirect) experimental evidence for five flavours of quark: up (u), down (d), strange (s), charmed (c) and bottom (b), although there is strong theoretical prejudice favouring the existence of a sixth flavour called top (t). Colour was originally introduced to resolve a conflict with the generalised Pauli exclusion principle for the quarks in the ground-state baryons. A quark of a given flavour can exist in any of three possible colour states, say red, green or blue, although these names have no logical connection to the concept. The quarks couple via their colour charges to eight massless vector gluons in such a fashion that the Lagrangian is locally-invariant under the gauge group $SU(3)_{colour}$. The three-

valued colour degree of freedom carried by quarks transforms as the fundamental representation, $\underline{3}$, of $SU(3)_{colour}$. This group is assumed to be an exact symmetry group for the strong interactions, and all physical states are, by hypothesis, colour singlets. It is this latter assumption which resolves the statistics problem of baryon spectroscopy, for the colour-singlet state of three quarks is completely antisymmetric in the colour variable. It is precisely because $SU(3)_{colour}$ has only three invariant tensors, δ^{α}_{β}, $\varepsilon^{\alpha\beta\gamma}$ and $\varepsilon_{\alpha\beta\gamma}$, that all colour-singlet states may be decomposed into systems of mesons, baryons and antibaryons, respectively.

At this stage, the introduction of colour must be considered as an ad hoc hypothesis invented to explain the symmetry of quark space and spin wavefunctions. Nonetheless, the colour hypothesis finds much (indirect) experimental support including (Lichtenberg 1981, Greenberg 1978 and references therein):

(a) the saturation property of hadrons, viz. that low-lying states all have the quantum numbers of qqq for baryons and $q\bar{q}$ for mesons. The quark model without the colour degree of freedom fails to account for saturation;

(b) measurement of the ratio , R, of hadrons to muon pairs ($\mu^+\mu^-$) produced in electron-positron (e^+e^-) collisions. Away from vector meson resonances, the calculated value of R approximately agrees with the experimental value if quarks have three colours, but is about a factor of 3 too small if quarks are colourless;

(c) the decay modes of the tau-lepton (τ). The calculated branching ratio into hadronic and leptonic modes approximately agrees with experiment if quarks have three colours, but disagrees if quarks are colourless. This evidence is perhaps not really independent of the evidence from e^+e^- annihilation, since in the Glashow-Salam-Weinberg

model, the weak interaction responsible for τ decay into hadrons is related to the electromagnetic interaction responsible for hadron production in e^+e^- annihilation;

(d) the decay rate of the neutral pi-meson, π^o. The existence of three colours requires the calculated decay rate of the π^o to be increased by a factor of 9 (Ross 1981 and references therein) and this factor is just what is needed to bring theory and experiment into close agreement;

(e) the Drell-Yan cross-section, which is decreased by a factor of $\frac{1}{3}$ if colour is included in the theory compared to the value without colour. This prediction is favoured but not confirmed by experiment.

It is anticipated that QCD may provide a mechanism to confine quarks and other colour-carrying particles in hadrons permanently, although this remains to be demonstrated. Because confinement is only conjectured to be a consequence of QCD, it is not known whether the observation of free quarks would be compatible with QCD. Indeed, there is one experiment (Fairbank et al. 1977) in which fractional charge appears to have been seen, but this result awaits confirmation. The success of the symmetric quark model for baryon spectroscopy (so-called because each baryon consists of three quarks in a state totally symmetric in the combined space, spin and flavour degrees of freedom) and the non-observation of the extra mesons corresponding to fully exploiting the colour degree of freedom, suggest that one elevates to the level of a principle the notion that only colour-singlet states exist in nature.

An immediate consequence of the non-observation of free quarks is that their masses must be inferred indirectly from the properties of hadrons. Thus, the values of the quark masses depend on theory as well as observation. Furthermore, the value of the mass of a given

quark seems to depend on the nature of the indirect process one is looking at. In this thesis, quark masses are always to be understood as constituent quark masses, i.e. the masses of the quarks determined from the masses and magnetic moments of hadrons in a non-relativistic approximation. The constituent quark masses determined in different ways may vary by 100 MeV or more. Following Lipkin (1980), we can envision a picture of a hadron in which each constituent quark carries its own share of the coloured gluon field within the hadron and has an effective mass determined by the gluon field. This gives an effective mass for a quark which is roughly one-third of the mass of a baryon. In this picture, the mass of a quark, as measured in an experiment which transfers energy and momentum to the quark, depends on how much of the quark's associated gluon field recoils with it and contributes to its inertia. In deep inelastic electron scattering, the process is assumed to take place so rapidly that the field does not recoil with the quark and the quark effectively has zero mass. An isolated quark would carry all its gluon field with it when its momentum changed and would have a very high mass, infinite in models where quarks are permanently confined. In the processes studied in baryon spectroscopy, the quark seems to carry with it a gluon field which consistently gives it its share of the mass of the baryon, about 350 MeV in the case of the up and down quarks.

Aside from its many attractive features, both aesthetic and phenomenological, QCD has in addition the very important feature of being amenable to rigorous tests and therefore of being disproved: whilst it is a strongly-interacting field theory at large distances, at short distances the theory becomes asymptotically free and perturbative techniques can be applied. As in quantum electrodynamics (QED), the strong-interaction coupling constant, α_s, can be generalised

to a strong-interaction running coupling strength, $\alpha_s(Q^2)$, where Q is the 4-momentum transfer of the process being studied. Because a quark has its colour screened by the creation of quark-antiquark pairs in a manner analogous to the screening of electric charge in QED, we might expect that $\alpha_s(Q^2)$ increases at large Q^2. However, in QCD there is another effect which acts in the opposite direction. The gluon field of a quark itself carries colour, and can transport this colour away from the original quark, leaving it with a smaller net colour. In effect, the gluons cause the original point source of colour to be smeared out in space. Therefore, in QCD, two effects (creation of quark-antiquark pairs and creation of gluons) act in opposing directions: the first to strengthen $\alpha_s(Q^2)$ at small distances and the second to weaken it. Which effect wins out depends on how many quark flavours there are. Provided the number of quark flavours with $(\text{mass})^2 \ll Q^2$ is no greater than sixteen, the effect of the gluons wins out and the closer two quarks come together, the smaller will be the effective coupling strength. It is this effect which is loosely termed asymptotic freedom.

Like QED, QCD depends on only one parameter, the so-called Λ-parameter. Unlike QED, however, in which the fine-structure constant is dimensionless, Λ has the dimensions of energy. Empirically, Λ is not very well determined thus far, but probably lies in the range 200 MeV - 500 MeV (Lichtenberg 1981 and references therein). Indeed, the value of Λ inferred from experimental data depends on which renormalisation scheme is used (Ross 1981). One reason why it is so difficult to determine Λ is that at high energy, where perturbation theory works best, $\alpha_s(Q^2)$ is relatively insensitive to the value of Λ. A value of 400 MeV for Λ gives $[\alpha_s\ (1\ \text{GeV})^2] = 0.76$ and $\alpha_s[(10\ \text{GeV})^2] = 0.25$. Even at 10 GeV, the QCD effective coupling

strength is considerably greater than the electromagnetic coupling strength, $\alpha(Q^2)$, which equals 0.0074 at $Q^2 = (10\ \mathrm{GeV})^2$. Despite asymptotic freedom, QCD perturbation theory converges more slowly than QED perturbation theory at all presently-accessible energies (Lichtenberg 1981).

The behaviour of QCD at small Q^2 remains to this day an unsolved problem. However, it has been conjectured that the increasing effective coupling strength makes the interaction so strong at small Q^2 (corresponding to large separation, r) that quarks and gluons (and any other coloured states) are permanently confined to the interior of hadrons. The difficulties involved in obtaining predictions about the long-range behaviour of QCD have led to the investigation of a number of models which assume quark confinement a priori and which embody ideas inferred from QCD.

It is the study of one such quark model, viz. that due to Isgur and Karl (Isgur 1980 and references therein) which forms the subject of this thesis, and we begin the thesis proper with a brief résumé of non-relativistic quark models and their relevance to the problem of quark confinement.

CHAPTER 1

INTRODUCTION

1.1 Confinement

There is strong support from the study of QCD on a lattice (De Rújula et al. 1975 and references therein) that the long-range confining potential is spin- and flavour-independent and depends only on colour variables. In this approach, the quark fields are defined only at the sites of a hypercubical lattice and the gauge fields are associated with the links between neighbouring sites. The gauge symmetries of the model consist of independent $SU(3)_{colour}$ rotations at each lattice site. The interaction energy between two distant static quarks can be written as an expansion in inverse powers of the quark-gluon coupling constant, α_s, which is conjectured to be large for large lattice spacing. The leading term in this expansion is proportional to N, the number of lattice links connecting the quarks, so that the force between distant quarks is constant. Thus, the picture suggested by lattice gauge theories is that of a colour flux tube, outside of which the gluon fields are not allowed to propagate, connecting the two colour charges. This effect makes the dynamics essentially 1-dimensional. Since, in one dimension, the Green's function of Laplace's equation is proportional to the distance, one obtains a linear potential. This linear form of the long-range confining potential in mesons is well supported by data from charmonium studies.

For baryons, where 3-body forces might be important, Dosch and Müller (1976) have shown, again by employing lattice methods, that the interquark potential is well approximated by a sum of 2-body potentials which grow linearly with distance and whose slope is roughly one-half of that in mesons. Thus, in baryons, the picture suggested by lattice

gauge theories is that of three flux tubes joining the quarks. Gromes and Stamatescu (1979) suggest an alternative picture, however. They propose that the gluon fields are confined to the plane defined by the three quarks in the baryon. This flux surface makes the dynamics essentially 2-dimensional, and since, in two dimensions, the Green's function of Laplace's equation varies logarithmically with the distance, a logarithmic potential results.

We shall have very little to say in this thesis about the precise functional form of the long-range confining potential. Following Isgur (1980), we mention briefly a simple model for confinement based on QCD. We consider a non-relativistic, colour-symmetric, 2-body potential of the form:

$$V_{qq}(r_{12}) = -V_{conf.}(r_{12})\ \underline{F}^{(1)} \cdot \underline{F}^{(2)} \qquad (1.1a)$$

$$V_{q\bar{q}}(r_{12}) = +V_{conf.}(r_{12})\ \underline{F}^{(1)} \cdot \underline{F}^{(2)*} \qquad (1.1b)$$

$$V_{\bar{q}\bar{q}}(r_{12}) = -V_{conf.}(r_{12})\ \underline{F}^{(1)*} \cdot \underline{F}^{(2)*} \qquad (1.1c)$$

where $\underline{r}_{12} = \underline{r}_1 - \underline{r}_2$, and $F_\alpha^{(i)}$, $\alpha = 1,2, \ldots, 8$ are the generators of $SU(3)_{colour}$, to be understood as acting on the colour wavefunction of the i^{th} quark or antiquark. $V_{conf}(r)$ is the effective long-range colour confinement potential. We stress that these potentials depend only on colour variables and not on flavour degrees of freedom. This implies that eigenstates of the confining potential alone (before the introduction of spin-dependent effects) have flavour-symmetry-breaking only via the explicit appearance of the quark masses in the kinetic energy term of the Hamiltonian.

The motivation for the potentials in equations (1.1a) - (1.1c) rests in the fact that a cluster of quarks and antiquarks in a given $SU(3)_{colour}$ representation will be confined to another cluster of

quarks and antiquarks in the conjugate representation with a strength proportional to the eigenvalue of the colour quadratic Casimir operator acting on the given representation (Greenberg 1978 and references therein). In particular, this solves the problem of cancelling out the long-range potential between separated hadrons, since the eigenvalue of the colour quadratic Casimir operator vanishes for colour-singlet states. In terms of our picture of coloured flux lines, we view these as quantised, so that all the coloured flux emanating from a given coloured particle ends on another coloured particle, leaving no residual long-range interaction between separated hadrons (colour-singlet states).

By employing the colour-singlet wavefunctions:

$$|ij\rangle_{\text{meson}} = \frac{1}{\sqrt{3}} \delta^{\alpha\beta} q_\alpha^i \bar{q}_\beta^j \qquad (1.2a)$$

$$|ijk\rangle_{\text{baryon}} = \frac{1}{\sqrt{6}} \varepsilon^{\alpha\beta\gamma} q_\alpha^i q_\beta^j q_\gamma^k \ , \qquad (1.2b)$$

where i, j, k label the quarks/antiquarks and α, β, γ are colour indices, we can easily demonstrate that the effective (i.e. colour-averaged) quark-antiquark and quark-quark potentials appropriate to a meson or baryon, respectively, are:

$$V_{q\bar{q}}^{\text{meson}}(r_{ij}) = \frac{4}{3} V_{\text{conf.}}(r_{ij}) \qquad (1.3a)$$

$$V_{qq}^{\text{baryon}}(r_{ij}) = \frac{2}{3} V_{\text{conf.}}(r_{ij}) \qquad (1.3b)$$

so that baryon and meson bound states are governed by related effective potentials. Because a quark in a baryon transforms as $\underline{3}$ of $SU(3)_{\text{colour}}$, the remaining diquark necessarily transforms as $\overline{\underline{3}}$ of $SU(3)_{\text{colour}}$, in order that the baryon remain a colour-singlet state. However, the potentials in equations (1.1a) - (1.1c) are sensitive only to the net

colour of an object, so the diquark effectively serves as an antiquark. It follows that the effective interaction per quark in the diquark is just half the quark-antiquark interaction, which is just the result in equations (1.3a) - (1.3b).

1.2 Short-range forces

While the potentials in equations (1.1a) - (1.1c) express the long-range confinement characteristics of QCD, they are, of course, neither expected nor observed to be the whole story. With only the long-range confining potential, which is both spin- and flavour-independent, one would observe such degeneracies as $\Delta - N$ and $\rho - \pi$. Splittings of this type have long been attributed in quark models to the existence of spin- and flavour-dependent, short-range interactions and it is the form of these interactions which we now examine.

Following De Rújula et al. (1975), we assume that the effective short-range quark-quark interaction arises from one-gluon-exchange, which is represented by the diagram in Figure 1. Whilst the associated short-range potential, together with its relativistic corrections, can be inferred from a non-relativistic reduction of the Bethe-Saltpeter equation (Gromes 1980 and references therein), a much simpler approach provides us with the leading (Coulombic) contribution to the short-range potential (Lichtenberg 1981). Neglecting spin and the effects of identical particles, the scattering amplitude, $A_{q^i q^j}$, for the process illustrated in Figure 1 is given by:

$$A_{q^i q^j} = \underline{F}^{(i)} . \underline{F}^{(j)} \frac{\alpha_s(Q^2)}{2\pi^2 Q^2} \quad . \qquad (1.4)$$

We obtain the leading contribution to the corresponding short-range

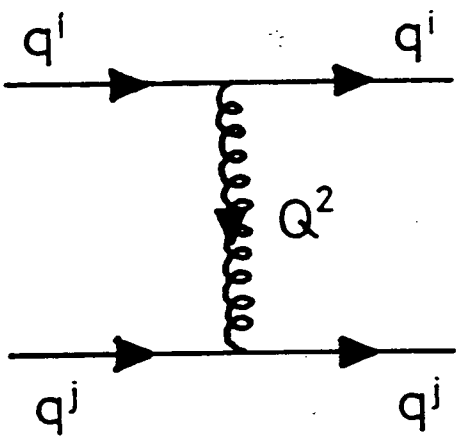


Figure 1 One-gluon-exchange

potential by taking the Fourier transform of $A_{q^i q^j}$ with respect to $\underline{Q}$, having first replaced the 4-momentum transfer, Q, by the 3-momentum transfer $\underline{Q}$. This latter replacement is a non-relativistic approximation giving:

$$V_{\text{short-range}}(r_{ij}) = \frac{\underline{F}^{(i)}.\underline{F}^{(j)}}{2\pi^2} \int d^3\underline{Q}\, e^{i\underline{Q}\cdot\underline{r}_{ij}} \frac{\alpha_s(\underline{Q}^2)}{\underline{Q}^2} . \quad (1.5)$$

The problem with equation (1.5) is that for small $\underline{Q}^2$, $\alpha_s(\underline{Q}^2)$ varies considerably with $\underline{Q}^2$. However, we ignore the difficulty that $\alpha_s(\underline{Q}^2)$ varies rapidly in an important region of integration and replace $\alpha_s(\underline{Q}^2)$ by some appropriate average value, α_s, which can be taken outside the integral. We then obtain:

$$V_{\text{short-range}}(r_{ij}) = (\underline{F}^{(i)}.\underline{F}^{(j)}) \frac{\alpha_s}{r_{ij}} , \quad (1.6)$$

and recalling that in a baryon (c.f. equation (1.3b)):

$$\langle \underline{F}^{(i)}.\underline{F}^{(j)} \rangle_{\text{colour-average}} = -\frac{2}{3} , \quad (1.7)$$

we deduce that the leading, colour-averaged, contribution to the short-range interaction between two quarks in a baryon is:

$$\langle V_{\text{short-range}}(r_{ij}) \rangle_{\text{colour-average}} = -\frac{2\alpha_s}{3r_{ij}} . \quad (1.8)$$

Had we performed a non-relativistic reduction of the Bethe-Saltpeter equation, we would have deduced that :

$$\langle V_{\text{short-range}}(r_{ij}) \rangle_{\text{colour-average}} = -\frac{2\alpha_s}{3} S_{ij} , \quad (1.9)$$

where the Fermi-Breit interaction, S_{ij}, is given by (De Rújula et al. 1975 and references therein):

$$S_{ij} - \frac{1}{r_{ij}} = H_{ij}^{OO} + H_{ij}^{D} + H_{ij}^{C} + H_{ij}^{SO} + H_{ij}^{T} , \quad (1.10)$$

and the relativistic corrections on the RHS of equation (1.10) are:

$$H_{ij}^{OO} = -\frac{1}{2m_i m_j}\left(\frac{\underline{p}_i \cdot \underline{p}_j}{r_{ij}} + \frac{\underline{r}_{ij} \cdot (\underline{r}_{ij} \cdot \underline{p}_i)\underline{p}_j}{r_{ij}^3}\right) \quad \text{(orbit-orbit term)} \qquad (1.11a)$$

$$H_{ij}^{D} = -\frac{\pi}{2}\,\delta^3(\underline{r}_{ij})\left(\frac{1}{m_i^2} + \frac{1}{m_j^2}\right) \quad \text{(Darwin term)} \qquad (1.11b)$$

$$H_{ij}^{C} = -\frac{8\pi}{3m_i m_j}\,\delta^3(\underline{r}_{ij})(\underline{s}_i \cdot \underline{s}_j) \quad \text{(contact term)} \qquad (1.11c)$$

$$H_{ij}^{SO} = -\frac{1}{2r_{ij}^3}\left\{\frac{1}{m_i^2}(\underline{r}_{ij} \times \underline{p}_i)\cdot \underline{s}_i - \frac{1}{m_j^2}(\underline{r}_{ij} \times \underline{p}_j)\cdot \underline{s}_j + \frac{2}{m_i m_j}\left[(\underline{r}_{ij} \times \underline{p}_i)\cdot \underline{s}_j - (\underline{r}_{ij} \times \underline{p}_j)\cdot \underline{s}_i\right]\right\} \quad \text{(spin-orbit term)} \qquad (1.11d)$$

$$H_{ij}^{T} = -\frac{1}{m_i m_j}\cdot\frac{1}{r_{ij}^5}\cdot\left[3(\underline{s}_i \cdot \underline{r}_{ij})(\underline{s}_j \cdot \underline{r}_{ij}) - (\underline{s}_i \cdot \underline{s}_j)\underline{r}_{ij}^2\right] \quad \text{(tensor term)} \qquad (1.11e)$$

In equations (1.11a) - (1.11e), m_i, $\underline{p}_i$ and $\underline{s}_i$ represent the mass, momentum and spin of the i^{th} quark. We stress that we have included in equation (1.10) only relativistic corrections of $O(v^2/c^2)$: higher-order relativistic corrections have been neglected. [Note that, because we work throughout this thesis with natural units, wherein $\hbar = c = 1$, the factors of $\hbar$ and c in equations (1.11a) - (1.11e) have been suppressed.]

Notice that the orbit-orbit and Darwin terms, in common with the leading Coulomb term, are spin-independent. Such terms are difficult to distinguish from the (completely-empirical) spin-independent confinement potential, $V_{conf.}(r_{ij})$, and so, following Isgur (1980), we choose to subsume these terms, together with the long-range behaviour of the confining potential, under the one expression, $V_{conf.}(r_{ij})$. The spin-orbit terms have been discussed extensively by several authors (Reinders 1980 and references therein). However, underlying the model

of Isgur and Karl (Isgur 1980 and references therein) is the notion that spin-orbit effects are relatively unimportant; accordingly, we choose not to include them in our considerations. By way of contrast, we shall have a great deal to say in this thesis about the contact and tensor terms.

1.3 The harmonic oscillator quark model

In the main body of this thesis, we shall be exclusively concerned with studying non-strange baryons within the context of the specific model due to Isgur and Karl (Isgur 1980 and references therein), so that we shall not consider the strange, charmed or bottom quarks in the chapters which follow. We shall, however, include the strange quark in the discussion which forms the remainder of this chapter.

We begin this section by briefly reviewing the harmonic oscillator quark model (Hey 1980 and references therein). The basic assumptions of this model are:

(a) quark dynamics in baryons is non-relativistic. This enables us to extract the centre-of-mass (CM) motion by making the canonical change of variables from the individual quark coordinates, $\underline{r}_i$, to the coordinates:

$$\underline{R} = \frac{1}{3}(\underline{r}_1 + \underline{r}_2 + \underline{r}_3) \tag{1.12a}$$

$$\underline{\rho} = \frac{1}{\sqrt{2}}(\underline{r}_1 - \underline{r}_2) \tag{1.12b}$$

$$\underline{\lambda} = \frac{1}{\sqrt{6}}(\underline{r}_1 + \underline{r}_2 - 2\underline{r}_3) \tag{1.12c}$$

in the case of equal-mass quarks. We remark that the internal coordinates $\underline{\rho}$ and $\underline{\lambda}$ form a basis for the 2-dimensional mixed irreducible representation of the permutation group on three objects, S_3. (For an excellent review of S_3, see Lichtenberg 1970).

(b) the quark-binding forces may be approximated by 2-body harmonic interactions. This approximation is particularly convenient because the $\underline{\rho}$- and $\underline{\lambda}$-modes decouple and the non-relativistic 3-body problem can be solved exactly. In fact, as we shall demonstrate explicitly in this thesis, the calculations can be done group-theoretically.

The resulting zeroth-order Hamiltonian, H_o, may be written:

$$H_o = \sum_i (m_i + \underline{p}_i^2/2m_i) + \sum_{i<j} \tfrac{1}{2}K\underline{r}_{ij}^2 \qquad (1.13a)$$

$$\text{i.e. } H_o = (\Sigma_i \underline{p}_i)^2/2(3m) + 3m + \frac{\underline{p}_\rho^2}{2m} + \frac{\underline{p}_\lambda^2}{2m} + \frac{3K}{2}(\underline{\rho}^2 + \underline{\lambda}^2) \qquad (1.13b)$$

for the case of equal-mass quarks. K is a measure of the oscillator strength and $\underline{p}_\rho = m\dot{\underline{\rho}}$, $\underline{p}_\lambda = m\dot{\underline{\lambda}}$. We may safely drop the first term on the RHS of equation (1.13b) as it simply describes the translational motion of the 3-body system as a whole.

The construction of the allowed states of the harmonic oscillator quark model is a standard problem. The assumption that quarks are spin $\frac{1}{2}$ objects obeying Fermi statistics, together with the hypothesis that all observed states are colour-singlet states, leads to 3-quark state vectors of the form:

$$|3q\rangle = |\text{flavour}\rangle|\text{spin}\rangle|\text{space}\rangle|\text{colour}\rangle . \qquad (1.14)$$

antisymmetric — symmetric — antisymmetric

For the three quark flavours u, d and s, neglecting quark mass differences, we must construct overall symmetric $SU(6)_{\text{flavour}\times\text{spin}} \otimes O(3)$ wavefunctions. This overall symmetry requires that the $SU(6)_{\text{flavour}\times\text{spin}}$ and space states have matching permutation symmetries, so that the allowed $SU(6)_{\text{flavour}\times\text{spin}} \otimes O(3)$ state is:

$$\Psi_{\text{total}} = \Sigma\, \Psi[SU(6)_{\text{flavour}\times\text{spin}}] \times \Psi_{\text{space}} \tag{1.15a}$$

$$= \begin{array}{|c|c|c|} \hline \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \\ \hline \multicolumn{1}{|c}{\multirow{-2}{*}{}} \\ \end{array} \otimes \begin{array}{|c|c|} \hline \\ \hline \multicolumn{1}{|c}{\multirow{-2}{*}{}} \\ \end{array} \oplus \begin{array}{|c|} \hline \\ \hline \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \\ \hline \\ \hline \end{array} \tag{1.15b}$$

$$= \underline{56} \times \text{symmetric} \oplus \underline{70} \times \text{mixed} \oplus \underline{20} \times \text{antisymmetric}, \tag{1.15c}$$

where the Young tableaux in equation (1.15b) give the permutation symmetry and, for the $SU(6)_{\text{flavour}\times\text{spin}}$ states, also label the irreducible representations. (For an extensive review of Young tableaux and their uses in the context of unitary groups, see Lichtenberg 1970).

The use of $SU(6)_{\text{flavour}\times\text{spin}} \otimes O(3)$ for counting states and labelling supermultiplets is convenient on a phenomenological level. However, since $SU(6)_{\text{flavour}\times\text{spin}}$ is broken by quark mass differences, we need not take states with a spatial wavefunction belonging to a single irreducible representation of the permutation group on three objects, S_3, as basis states in an analysis of baryons. Isgur and Karl (Isgur 1980 and references therein) emphasise this point in their study of the strange baryons. Use of broken $SU(6)_{\text{flavour}\times\text{spin}}$ with the generalised Pauli principle is equivalent to treating quarks of different mass as distinguishable; however, for a given calculation, one approach may be simpler than the other.

We display in Figure 2 the first five levels of the spectrum of allowed $SU(6)_{\text{flavour}\times\text{spin}} \otimes O(3)$ supermultiplets in the harmonic oscillator quark model. The masses are labelled by the excitation number, N, which is just the total number of excitations of the $\underline{\rho}$- and $\underline{\lambda}$-oscillators, and the supermultiplets by their $SU(6)_{\text{flavour}\times\text{spin}}$ representation and corresponding orbital angular momentum and parity, L^P. The assignment of the O(3) quantum numbers, L^P, to the various $SU(6)_{\text{flavour}\times\text{spin}}$ multiplets is arrived at from a study of the

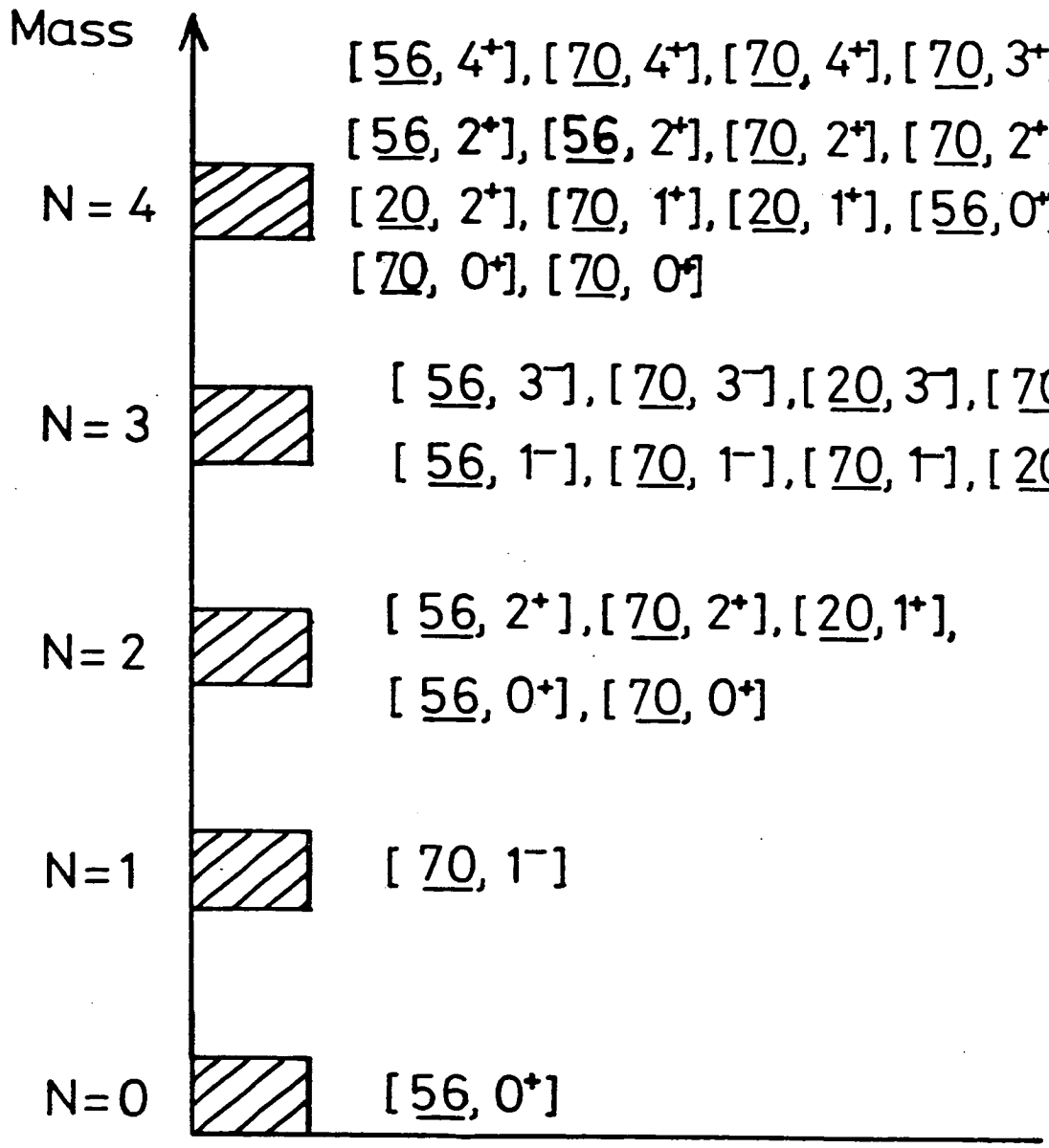


Figure 2 Spectrum of allowed $SU(6)_{\text{flavour} \times \text{spin}} \otimes O(3)$ supermultiplets for N = 0, 1, 2, 3 and 4

permutation symmetry of the spatial wavefunctions: in general, $P = (-1)^N$.

We remark that the degeneracy of the five $SU(6)_{flavour\times spin} \otimes O(3)$ supermultiplets at the $N = 2$ level, the eight $SU(6)_{flavour\times spin} \otimes O(3)$ supermultiplets at the $N = 3$ level and the seventeen $SU(6)_{flavour\times spin} \otimes O(3)$ supermultiplets at the $N = 4$ level is a result specific to the harmonic oscillator potential. It reflects the dynamical symmetry of the harmonic oscillator, about which we shall have much to say in subsequent chapters of this thesis.

We can subdivide the harmonic oscillator quark model into two types of quark model: algebraic and explicit:

(a) algebraic quark models

In such models, detailed dynamical assumptions are avoided as far as possible. Instead, such models rely on symmetries to parameterise the data in terms of relatively few, unknown reduced matrix elements which serve as independent parameters. It is within the context of such a model that Dalitz and collaborators performed their $SU(6)_{flavour\times spin}$ mass-operator analyses (Dalitz and Horgan 1973, Horgan 1974, 1976a, Dalitz et al. 1977a). These analyses are based on non-relativistic quark dynamics, and the assumption that the observed low-lying supermultiplets are consistent with the $SU(6)_{flavour\times spin} \otimes O(3)$ structure of the harmonic oscillator quark model. Mass splittings are accounted for in terms of symmetry-breaking forces which are assumed to be of a 2-body nature. The effect of such terms is calculated in first-order perturbation theory and can be represented by matrix elements of the form:

$$M_{\alpha\beta} = \sum_{i<j} \langle\alpha|M_{ij}|\beta\rangle \qquad (1.16)$$

where $|\alpha\rangle$ and $|\beta\rangle$ are states with a common value of N and i,j label the quarks. Since these states are totally symmetric in the quark variables, $M_{\alpha\beta}$ depends only on the quantum numbers of the subsystem comprising quarks 1 and 2. This subsystem can be classified under $SU(6)_{flavour\times spin}$ as $\underline{21}$ ($\square\square$ ≡ symmetric) or $\underline{15}$ ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$ ≡ antisymmetric) and the full 2-body mass operator, M, can be written:

$$M = \sum_i V_i(\rho^2)T_i, \tag{1.17}$$

where $V_i(\rho^2)$ is a scalar potential and i describes the properties of the mass operator, T_i. The different mass operators are classified as tensors under $SU(6)_{flavour\times spin} \otimes O(3)$, and they must couple to at least one of the following $SU(6)_{flavour\times spin}$ outer products:

$$\overline{\underline{21}} \otimes \underline{21} = \underline{1} \oplus \underline{35} \oplus \underline{405} \tag{1.18a}$$

$$\overline{\underline{15}} \otimes \underline{15} = \underline{1} \oplus \underline{35} \oplus \underline{189}\,. \tag{1.18b}$$

The $SU(3)_{flavour} \otimes SU(2)_{spin}$ subgroup reductions of the $SU(6)_{flavour\times spin}$ irreducible representations appearing on the RHS of equations (1.18a) - (1.18b) define all the possible mass-operator transformation properties. In addition, permissible mass operators must conserve isospin and hypercharge, i.e. they must have $I = Y = 0$, so the possible mass operators are limited to the $\underline{1}$, $\underline{8}$ or $\underline{27}$ irreducible representations of $SU(3)_{flavour}$. Dalitz and collaborators further restrict the set of permissible operators by considering only those which transform under $SU(3)_{flavour}$ as $\underline{1}$ or $\underline{8}$. This assumption of "octet dominance" is motivated by the mass formula for the $N = 0$ $[\underline{56}, 0^+]$ supermultiplet, which is very successful. Beyond this general framework, these authors make an important assumption by retaining spin-orbit forces but neglecting spin-tensor forces. Throughout

this thesis, we shall repeatedly call this assumption into question. To constrain the model still further, Dalitz et al. make the additional dynamical assumption that the confining forces may be approximated by 2-body harmonic oscillator potentials. This has the consequence that:

(a) the wavefunctions separate:

$$\Psi(\underline{\rho}, \underline{\lambda}) = \phi(\underline{\rho})\chi(\underline{\lambda}) \quad ; \tag{1.19}$$

(b) for the same value of N, parameters of different $SU(6)_{flavour \times spin} \otimes O(3)$ supermultiplets are related;

(c) for different values of N, the reduced matrix elements are related.

We shall demonstrate in this thesis that property (b) reflects the dynamical symmetry of the harmonic oscillator and that, as conjectured by Horgan (1976a), property (c) derives from the existence of a spectrum-generating group for the harmonic oscillator.

Despite the very explicit framework underlying the calculations of Dalitz et al. (Horgan 1976a and references therein), the actual fitting procedure for the resonance masses proved very difficult to systematise (Dalitz et al. 1977a). Mixing within, and between, $SU(6)_{flavour \times spin} \otimes O(3)$ supermultiplets allows an enormous number of possible assignments, with the result that whilst the mass formulae derived by Dalitz and collaborators are in good agreement with experiment, mixing angles and decay rates have been predicted which find less happy agreement with the experimental data.

(b) explicit quark models

The phenomenological analysis of Dalitz and collaborators is but a single step towards a definitive theory. The next step, using guesses about the form of the interactions based on QCD, has already allowed

more specific predictions (with fewer attendant parameters) to be made. Such explicit quark models assume:

(a) the long-range confining interaction is spin- and flavour-independent and leads to $SU(6)_{flavour \times spin} \otimes O(3)$ hadron supermultiplets;

(b) $SU(3)_{flavour}$ - breaking occurs only via explicit quark mass differences;

(c) asymptotic freedom to motivate a short-range, spin- and flavour-dependent force arising from the non-relativistic reduction of one-gluon-exchange. This is the essential new feature and gives rise to the standard Fermi-Breit interaction in equations (1.9), (1.10) and (1.11a) - (1.11e). For mesons, the colour factor of $(-\frac{2}{3})$ in equation (1.9) must be replaced by $(-\frac{4}{3})$.

The recent revival of the construction industry for explicit, non-relativistic quark models owes its origin to the pioneering work of De Rújula et al. (1975). These authors assume the Hamiltonian for the three quarks in a baryon is of the form:

$$H = L(\underline{r}_1, \underline{r}_2, \underline{r}_3) + \sum_i (m_i + \frac{\underline{p}_i^2}{2m_i}) - \frac{2\alpha_s}{3} \sum_{i<j} S_{ij} \quad , \qquad (1.20)$$

where $L(\underline{r}_1, \underline{r}_2, \underline{r}_3)$ describes the long-range interaction responsible for the binding of the quarks within the baryon and S_{ij} is as in equation (1.10). The original application of this Hamiltonian held within it no assumptions about the explicit form of the confining forces giving rise to the term $L(\underline{r}_1, \underline{r}_2, \underline{r}_3)$: the calculations were concerned only with mass splittings. More precisely, De Rújula et al. wrote:

$$H = H_o + V \qquad (1.21)$$

where:

$$H_o = L(\underline{r}_1, \underline{r}_2, \underline{r}_3) + \sum_i (m_u + \frac{\underline{p}_i^2}{2m_u}) \quad , \qquad (1.22)$$

V is everything else, and m_u denotes the mass of the "up" quark.

The eigenstates of H_o are the degenerate $SU(6)_{flavour\times spin} \otimes O(3)$ supermultiplets and these serve as basis states for first-order perturbation theory in V, which introduces splittings between the supermultiplets. Lacking detailed information about the zeroth-order eigenstates, these authors could only parametrise the expectation values of V and fit to observed particle masses. Because there were fewer parameters than there were particle masses, a number of mass formulae could be deduced. Indeed, given the simplicity of their model, the phenomenology obtained by these authors for the N = 0 $[\underline{56}, 0^+]$ supermultiplet was a resounding success. Both the Gell-Mann-Okubo mass formula:

$$2M_N + 2M_\Xi = 3M_\Lambda + M_\Sigma \qquad (1.23)$$

and the equal-spacing rule for the decuplet:

$$M_\Delta - M_{\Sigma^*} = M_{\Sigma^*} - M_{\Xi^*} = M_{\Xi^*} - M_\Omega \qquad (1.24)$$

were successfully recovered. These authors also predicted the $SU(6)_{flavour\times spin}$ relation:

$$M_{\Sigma^*} - M_\Sigma = M_{\Xi^*} - M_\Xi \ , \qquad (1.25)$$

together with a unified mechanism for the (Δ-N) and (Σ-Λ) mass differences. By means of the relation:

$$M_\Sigma - M_\Lambda = \frac{2}{3}(1 - \frac{m_u}{m_s})(M_\Delta - M_N) \ , \qquad (1.26)$$

they found the ratio, $\frac{m_u}{m_s}$, of constituent quark masses to be approximately 0.6. This ratio, combined with the value of the proton magnetic moment, μ_p, and the relation:

$$m_u = \frac{M_p}{\mu_p} \ , \qquad (1.27)$$

where M_p denotes the proton mass, gave a value of approximately 330 MeV for the "up"-quark mass, m_u.

The phenomenology of the $N = 1$ $[\underline{70}, 1^-]$ supermultiplet was somewhat confused, however, and the whole question of excited states was taken up by many authors including Reinders (1978), Gromes and Stamatescu (1976, 1979) and Isgur and Karl (1977, 1978a, 1978b, 1979a). Reinders (1978), in common with De Rújula et al. (1975), did not specify the form of the confining potential, but worked in the non-relativistic 3-quark shell model with $SU(6)_{flavour \times spin} \otimes O(3)$ symmetry, as formulated by Horgan and Dalitz (1973), using, in particular, the wavefunctions calculated by these authors without the specific radial dependence determined by their choice of an harmonic oscillator potential as the confining potential. The lack of detailed information about the radial dependence of the wavefunctions forced Reinders to parametrise the expectation values of the perturbation, V, and to fit to the observed particle spectrum. However, unlike De Rújula et al. (1975), who treated the $SU(3)_{flavour}$-breaking only to first-order in the mass difference of the strange and non-strange quarks, Reinders treated the $SU(3)_{flavour}$-breaking in an exact way. By way of contrast, both Gromes and Stamatescu (1976, 1979) and Isgur and Karl (1977, 1978a, 1978b, 1979a) proposed detailed models for $L(\underline{r}_1, \underline{r}_2, \underline{r}_3)$ in terms of 2-body inter-quark potentials. Gromes and Stamatescu (1976) employed a linear quark confining potential using harmonic-oscillator wavefunctions to calculate matrix elements and reproduce the non-strange P-wave baryon spectrum. The most detailed and phenomenologically successful study of both strange and non-strange baryons belongs to Isgur and Karl (Isgur 1980 and references therein), however, and it is to a more detailed discussion of their model that we now turn.

1.4 The Isgur-Karl model

Isgur and Karl (1977, 1978a, 1978b, 1979a, 1979b) assume the baryon Hamiltonian is of the form :

$$H = \sum_i (m_i + \underline{p}_i^2/2m_i) + \sum_{i<j} V(r_{ij}) + \sum_{i<j} H_{ij}^{hyp}, \qquad (1.28)$$

where :

$$V(r_{ij}) = \frac{2}{3} V_{conf.}(r_{ij}) \qquad (1.29)$$

$$H_{ij}^{hyp} = \frac{2\alpha_s}{3m_i m_j} \left[\frac{8\pi}{3}(\underline{s}_i \cdot \underline{s}_j)\delta^3(\underline{r}_{ij}) + \frac{1}{r_{ij}^5}(3(\underline{s}_i \cdot \underline{r}_{ij})(\underline{s}_j \cdot \underline{r}_{ij}) - (\underline{s}_i \cdot \underline{s}_j)\underline{r}_{ij}^2)\right], \qquad (1.30)$$

and $V_{conf.}(r_{ij})$ is the spin- and flavour-independent confining potential appearing in equations (1.1a) - (1.1c) and (1.3a) - (1.3b). Isgur and Karl choose to do perturbation theory around the harmonic oscillator Hamiltonian, H_o, in equation (1.13a). These authors set:

$$V(r_{ij}) = \tfrac{1}{2}K\underline{r}_{ij}^2 + U(r_{ij}) , \qquad (1.31)$$

where $U(r_{ij})$ is an unknown scalar potential which is assumed to include a short-range, attractive (Coulombic) potential and any deviation of the long-range part of the confining potential from the harmonic oscillator form. It is the specific inclusion of the full spin-spin interaction, H_{ij}^{hyp}, (i.e. both contact and tensor terms) and the deliberate neglect of spin-orbit forces in the Fermi-Breit interaction in equation (1.10) which is the key to the phenomenological success enjoyed by the Isgur-Karl model.

We describe in this section some of the successful applications of the Isgur-Karl model relevant to the $N = 0$ and $N = 1$ levels of the harmonic oscillator quark model, reserving for later chapters of this thesis a detailed discussion of the $N = 2$ and $N = 3$ levels. Although in later chapters we shall concern ourselves only with

non-strange baryon resonances, we review both the strange and non-strange sectors of the $N = 0$ and $N = 1$ levels in the remainder of this section. It is convenient to distinguish the cases $S = 0$ and $S = -1$, where S denotes strangeness, since these two cases contain some quite different physical effects. The cases $S = -2$ and $S = -3$ follow trivially from the cases $S = -1$ and $S = 0$, respectively.

(a) S = 0 sector

In this sector, all three quarks may be taken to have a common mass, m, and the relevant zeroth-order Hamiltonian is:

$$H_o = 3m + \frac{\underline{p}_\rho^{\,2}}{2m} + \frac{\underline{p}_\lambda^{\,2}}{2m} + \frac{3K}{2}(\underline{\rho}^2 + \underline{\lambda}^2), \qquad (1.32)$$

where the symbols have the same meaning as in equation (1.13b), and we have neglected a term representing the kinetic energy of translation of the centre-of-mass of the system. It is clear from equation (1.32) that, in the harmonic oscillator quark model, the 3-quark system is equivalent, after elimination of the centre-of-mass motion, to two independent 3-dimensional harmonic oscillators, labelled by $\underline{\rho}$ and $\underline{\lambda}$, respectively. The ground-state baryons correspond to the $\underline{\rho}$- and $\underline{\lambda}$-oscillators simultaneously occupying their respective ground states, so that the total orbital angular momentum of the system is zero. In the first-excited level of the harmonic oscillator quark model, there is a single quantum of excitation localised either in the relative motion of quarks 1 and 2 ($\underline{\rho}$-type excitation) or in the relative motion of quark 3 about the centre-of-mass of quarks 1 and 2 ($\underline{\lambda}$-type excitation). The total orbital angular momentum $L = 1$ for both these possibilities. In the $S = 0$ sector, the $\underline{\rho}$- and $\underline{\lambda}$-oscillators possess the common frequency:

$$\omega_\rho = \omega_\lambda \equiv \omega = \left(\frac{3K}{m}\right)^{\frac{1}{2}}, \qquad (1.33)$$

so that there are two degenerate orbital states in the first-excited level for each value of L_z, the z-component of the total orbital angular momentum (= 1, 0, -1). This degeneracy arises because the Hamiltonian of a system of three equal-mass particles, with identical forces acting between all pairs of particles, is invariant under the permutation group on three objects, S_3.

For $N = 0$ and $N = 1$, where N, the principal quantum number, equals the total number of $\underline{\rho}$-type and $\underline{\lambda}$-type excitations, the eigenstates of H_o may be chosen to be (Isgur 1980):

$$N = 0: \quad \psi^S_{oo} = \frac{\alpha^3}{\pi^{3/2}} \exp[-\tfrac{1}{2}\alpha^2(\underline{\rho}^2+\underline{\lambda}^2)] \tag{1.34}$$

$$N = 1: \quad \psi^{M_\rho}_{11} = -\frac{\alpha^4}{\pi^{3/2}} (\rho_x+i\rho_y)\exp[-\tfrac{1}{2}\alpha^2(\underline{\rho}^2+\underline{\lambda}^2)] \tag{1.35a}$$

$$\psi^{M_\lambda}_{11} = -\frac{\alpha^4}{\pi^{3/2}} (\lambda_x+i\lambda_y)\exp[-\tfrac{1}{2}\alpha^2(\underline{\rho}^2+\underline{\lambda}^2)] \ , \tag{1.35b}$$

where $\alpha^4 = 3Km$. The notation for the spatial wavefunctions is $\psi^P_{LL_z}$, where L is the total orbital angular momentum and L_z its z-component, and P (= S (symmetric), A (antisymmetric), M_ρ (mixed, of type $\underline{\rho}$) or M_λ (mixed, of type $\underline{\lambda}$)) specifies the symmetry of the wavefunction under the permutation group, S_3. Note that, in equations (1.34) and (1.35a) - (1.35b), we display only the highest state of a given orbital angular momentum multiplet. We postpone until the next chapter any discussion of how one classifies and constructs these wavefunctions.

(b) S = -1 sector

In this sector, it is conventional to select the more massive strange quark as the third quark, so that $m_1 = m_2 = m$ and $m_3 = m_s$. The zeroth-order Hamiltonian is:

$$H_o = 2m + m_s + \frac{\underline{p}_\rho^2}{2m} + \frac{\underline{p}_\lambda^2}{2m_\lambda} + \frac{3K}{2}(\underline{\rho}^2+\underline{\lambda}^2) \ , \tag{1.36}$$

where $\underline{p}_\lambda = m_\lambda \dot{\underline{\lambda}}$ and m_λ is defined to be:

$$m_\lambda = \frac{3mm_s}{2m+m_s} \quad . \tag{1.37}$$

It is clear that the zeroth-order Hamiltonian has lost the permutation group, S_3, as its invariance group, since the quarks now have different masses. As a result, the two orbital states which were degenerate in the $S = 0$ sector are now split. It is no longer equivalent, energetically, whether the P-wave excitation is localised in the relative motion of the non-strange pair or in the motion of the strange quark relative to the non-strange pair. While this result holds for a general pair-potential between quarks, it is particularly easy to verify within the harmonic oscillator quark model (Isgur and Karl 1978a). The $\underline{\rho}$- and $\underline{\lambda}$-oscillators now have different frequencies:

$$\omega_\rho \equiv \omega = \left(\frac{3K}{m}\right)^{\frac{1}{2}} \tag{1.38a}$$

$$\omega_\lambda = \left(\frac{3K}{m_\lambda}\right)^{\frac{1}{2}} \tag{1.38b}$$

so that :

$$\omega_\rho - \omega_\lambda = \omega\left[1 - \left(\frac{2x+1}{3}\right)^{\frac{1}{2}}\right] , \tag{1.39}$$

where $x = \frac{m}{m_s} \simeq 0.6$.

The eigenstates of the zeroth-order Hamiltonian are quite distinct from those of the $S = 0$ sector because the degeneracy between the $\underline{\rho}$- and $\underline{\lambda}$-modes has been broken. For $N = 0$ and $N = 1$, the eigenstates may be taken to be (Isgur 1980):

$$N = 0: \quad \psi_{oo} = \frac{\alpha_\rho^{3/2}\alpha_\lambda^{3/2}}{\pi^{3/2}} \exp(-\tfrac{1}{2}\alpha_\rho^2\underline{\rho}^2 - \tfrac{1}{2}\alpha_\lambda^2\underline{\lambda}^2) \tag{1.40}$$

$$N = 1: \quad \psi_{11}^\rho = -\frac{\alpha_\rho^{5/2}\alpha_\lambda^{3/2}}{\pi^{3/2}} (\rho_x + i\rho_y)\exp(-\tfrac{1}{2}\alpha_\rho^2\underline{\rho}^2 - \tfrac{1}{2}\alpha_\lambda^2\underline{\lambda}^2) \tag{1.41a}$$

$$\psi_{11}^\lambda = -\frac{\alpha_\rho^{3/2}\alpha_\lambda^{5/2}}{\pi^{3/2}} (\lambda_x + i\lambda_y)\exp(-\tfrac{1}{2}\alpha_\rho^2\underline{\rho}^2 - \tfrac{1}{2}\alpha_\lambda^2\underline{\lambda}^2), \tag{1.41b}$$

where, once again, we display explicitly only the highest state of an orbital angular momentum multiplet, and where:

$$\alpha_\rho^4 \equiv \alpha^4 = 3Km \tag{1.42a}$$

$$\alpha_\lambda^4 = 3Km_\lambda \, . \tag{1.42b}$$

A simple picture of $S = -1$ baryons emerges once it is realised that, since the strange-quark mass differs from the non-strange-quark mass, it is no longer necessary to construct baryon wavefunctions which are totally antisymmetric in the combined space, spin, flavour and colour degrees of freedom. In this situation, one is free to single out the strange quark **as quark 3 and only** the symmetry of the states under $1 \leftrightarrow 2$ interchange remains relevant. With this in mind, Isgur and Karl (1978b, 1979a) introduce the isospin wavefunctions:

$$\phi_\Sigma = \frac{1}{\sqrt{2}} (ud + du)s \tag{1.43a}$$

$$\phi_\Lambda = \frac{1}{\sqrt{2}} (ud - du)s \tag{1.43b}$$

appropriate to the description of the Σ^o and Λ states (by isospin invariance, these wavefunctions are sufficient to describe the whole $S = -1$ sector). Since the spatial wavefunctions are either symmetric or antisymmetric under the interchange of quarks 1 and 2, one can enlist the aid of the usual spin and colour wavefunctions to construct states (the so-called "uds basis states") which are antisymmetric under $1 \leftrightarrow 2$ interchange.

Most of the machinery introduced by Isgur and Karl to describe the excited baryons does not come into play in their study of the ground-state ($N = 0$) baryons (Isgur and Karl 1979b). The masses of the $N(940)$ and $\Delta(1232)$ states are used as input to fix the two completely-free parameters of the model at the $N = 0$ level: the unperturbed

(i.e. in the absence of the hyperfine interaction) position, E_o, of the non-strange sector and an overall strength parameter, D, for the hyperfine interaction, where D is defined to be:

$$D = \frac{4\alpha_s \alpha^3}{3\sqrt{2\pi}\, m^2} \quad . \tag{1.44}$$

Isgur and Karl take into account second-order effects in the hyperfine interaction by calculating the mixing between the ground-state baryons and the positive-parity excited states associated with the $N = 2$ level of the harmonic oscillator quark model, using the masses and compositions of these excited states as determined from their study of the $N = 2$ level (Isgur and Karl 1979a). As a result, $E_o > \frac{1}{2}(M_N + M_\Delta)$. The agreement with experiment found by Isgur and Karl for the $N = 0$ baryons is such that the discrepancies are practically at the level of electromagnetic corrections. That the agreement is so good is undoubtedly because some of the inadequacies of the model are hidden in the fitting of the parameters of the model; nonetheless, one can rest assured that the model is describing several real effects. In particular, second-order effects play a significant rôle in determining the $N = 0$ baryon spectrum. The naïve (Σ-Λ) mass difference (c.f. equation (1.26)):

$$M_\Sigma - M_\Lambda = \frac{2}{3}\left(1 - \frac{m}{m_s}\right)(M_\Delta - M_N) \tag{1.45}$$

obtained by De Rújula et al. (1975), whilst being numerically correct, is, in fact, modified by two competing effects. Wavefunction distortion due to the attendant heavier mass of the strange quark compared to the non-strange-quark mass serves to bring the strange quark in closer to the other two quarks and tends to compensate for $x = \frac{m}{m_s} < 1$. This effect reduces $M_\Sigma - M_\Lambda$ by approximately 30 MeV. (Isgur and Karl

1979b). On the other hand, the Λ is much more strongly mixed with $N = 2$ states than is the Σ and this tends to open up the $(\Sigma-\Lambda)$ mass gap. These two contributions in practice nearly cancel so that the naïve result is numerically accurate.

In the non-strange sector of the $N = 1$ level, the orbital wavefunctions $\psi^{M_\rho}_{1m}$ and $\psi^{M_\lambda}_{1m}$ $(m = -1, 0, 1)$ are combined with the spin and isospin wavefunctions of the three quarks according to the prescriptions of the symmetric quark model to give the well-known $S = \frac{1}{2}$ N, $S = \frac{3}{2}$ N and $S = \frac{1}{2}\,\Delta$ states; the spin and orbital angular momentum are then coupled to give states of fixed total angular momentum, J.

The contact interaction:

$$H^{\text{contact}} = \frac{16}{9}\pi\alpha_s \sum_{i<j} \frac{1}{m_i m_j} (\underline{s}_i \cdot \underline{s}_j)\delta^3(\underline{r}_{ij}) \qquad (1.46)$$

elevates the $S = \frac{3}{2}$ N states and the $S = \frac{1}{2}\,\Delta$ states relative to the $S = \frac{1}{2}$ N states. This pattern of mass shifts, due to the contact interaction, occurs because the wavefunctions of both the $S = \frac{3}{2}$ N states and the $S = \frac{1}{2}\,\Delta$ states are sums of products of flavour and spin factors which are symmetric and mixed: symmetric spin and mixed flavour for the $S = \frac{3}{2}$ N states and mixed spin and symmetric flavour for the $S = \frac{1}{2}\,\Delta$ states, so that, in both cases, the $\underline{\lambda}$-component of the spatial wavefunction, which is symmetric in the coordinates of quarks 1 and 2, is multiplied by wavefunctions symmetric in both the flavour and spin of quarks 1 and 2. Thus, the contact interaction acts with probability $\frac{1}{2}$ and, when it acts, the pair of quarks 1 and 2 has spin 1. Since, in the non-strange sector, both the form of the contact interaction and the $SU(6)_{\text{flavour}\times\text{spin}} \otimes O(3)$ baryon wavefunctions are symmetric, the matrix elements of the contact interaction in equation (1.46) are proportional to the corresponding matrix elements of $(\underline{s}_1 \cdot \underline{s}_2)$. Therefore, the contact interaction shifts both the $S = \frac{3}{2}$ N states and the $S = \frac{1}{2}\,\Delta$ states by

the same amount. For the $S = \frac{1}{2}$ N states, both the flavour and spin wavefunctions are mixed, and the $\underline{\lambda}$-component of the spatial wavefunction is multiplied by $SU(6)_{flavour \times spin}$ wavefunctions which are both symmetric or both antisymmetric in quarks 1 and 2 with equal probability. Thus, as far as the contact interaction, which still acts with probability $\frac{1}{2}$, is concerned, it is equally likely that the pair of quarks 1 and 2 has spin 0 or spin 1: the contact interaction therefore shifts the $S = \frac{1}{2}$ N states by a different amount from the other states. These considerations, together with the usual calculation of matrix elements of $(\underline{s}_1 \cdot \underline{s}_2)$ for spin 0 and spin 1 states, via the identity:

$$\underline{s}_1 \cdot \underline{s}_2 = \tfrac{1}{2}[(\underline{s}_1 + \underline{s}_2)^2 - \underline{s}_1^2 - \underline{s}_2^2] \qquad (1.47)$$

lead to the result that the $S = \frac{3}{2}$ N and $S = \frac{1}{2}$ Δ states are raised, and the $S = \frac{1}{2}$ N states are lowered, by the same absolute amount (Isgur and Karl 1977). This effect is clearly visible amongst the non-strange members of the $N = 1$ $[\underline{70}, 1^-]$ supermultiplet: the $S = \frac{3}{2}$ states $N(1675)\frac{1}{2}^-$, $N(1700)\frac{3}{2}^-$ and $N(1670)\frac{5}{2}^-$ and the $S = \frac{1}{2}$ states $\Delta(1655)\frac{1}{2}^-$ and $\Delta(1685)\frac{3}{2}^-$ lie about 150 MeV above the $S = \frac{1}{2}$ states $N(1530)\frac{1}{2}^-$ and $N(1520)\frac{3}{2}^-$.

The tensor part of the Fermi-Breit interaction:

$$H^{tensor} = \frac{2\alpha_s}{3} \sum_{i<j} \frac{1}{m_i m_j} \cdot \frac{1}{r_{ij}^5} \cdot [3(\underline{s}_i \cdot \underline{r}_{ij})(\underline{s}_j \cdot \underline{r}_{ij}) - (\underline{s}_i \cdot \underline{s}_j)\underline{r}_{ij}^2] \qquad (1.48)$$

which is absent in the $N = 0$ $[\underline{56}, 0^+]$ supermultiplet, enters here for the quark pairs in a relative $L = 1$ state, and produces significant mixings between $S = \frac{3}{2}$ and $S = \frac{1}{2}$, but only small mass shifts. The value of α_s found from the (Δ-N) mass difference in the $N = 0$ $[\underline{56}, 0^+]$ supermultiplet normalises both of these effects absolutely. The agreement with experiment, particularly for the mixings which were found from decay data, is striking (Isgur and Karl 1977, Cashmore et al.

1975). For $N(1520)\frac{3}{2}^-$, the mixing angle, θ, between $S = \frac{3}{2}$ and $S = \frac{1}{2}$ is 6.3° versus 10° experimentally, and for $N(1530)\frac{1}{2}^-$, the mixing angle is -31.7° versus -32° experimentally, where θ is defined by:

$$|NJ^-> \quad = \quad \sin\theta\,|S=\tfrac{3}{2},J> + \cos\theta\,|S=\tfrac{1}{2},\ J> \quad . \tag{1.49}$$

In the strangeness $S = -1$ sector of the $N = 1$ [$\underline{70}$, 1^-] supermultiplet, Isgur and Karl (1978a) were able to account for the experimental observation that the pair of states $\Lambda\,\frac{5}{2}^-(1830)$ and $\Sigma\,\frac{5}{2}^-(1765)$ is inverted relative to the ground-state pair $\Lambda\frac{1}{2}^+(1115)$ and $\Sigma\frac{1}{2}^+(1190)$. These authors interpret this observation as a direct consequence of the splitting of the $\underline{\rho}$- and $\underline{\lambda}$-normal modes in the strangeness $S = -1$ sector. Both negative-parity states have $J^P = \frac{5}{2}^-$ and are "fully stretched" in the coupling of the orbital angular momentum $L = 1$ to the total quark spin $S = \frac{3}{2}$. The spin wavefunction of both the $\Lambda\frac{5}{2}^-$ and the $\Sigma\frac{5}{2}^-$ is totally symmetric under permutations of the three quarks. The $\Lambda\frac{5}{2}^-$, being an isospin-singlet state, has the isospin wavefunction:

$$\phi_\Lambda \quad = \quad \frac{1}{\sqrt{2}}\ (ud - du)s \tag{1.50}$$

which is antisymmetric under the interchange of the non-strange quarks and therefore must correspond to the $\underline{\rho}$-orbitally excited state which is also antisymmetric under interchange of the non-strange quarks. Similar reasoning leads one to deduce that the $\Sigma\frac{5}{2}^-$, with unit isospin, has the isospin wavefunction:

$$\phi_\Sigma \quad = \quad \frac{1}{\sqrt{2}}\ (ud + du)s \tag{1.51}$$

and therefore must correspond to the $\underline{\lambda}$-orbitally excited state. These two orbital states are non-degenerate in zeroth-order: the $\Sigma\frac{5}{2}^-$ and $\Lambda\frac{5}{2}^-$ are split before the hyperfine interaction is introduced. Using equation (1.39) with $\omega \simeq 520$ MeV and $x \simeq 0.6$, one finds:

$$M_o(\Lambda\tfrac{5}{2}^-) - M_o(\Sigma\tfrac{5}{2}^-) = \omega_\rho - \omega_\lambda \simeq 75 \text{ MeV}, \quad (\hbar = 1) \tag{1.52}$$

where M_o denotes the zeroth-order contribution to the mass of a given state, in reasonable agreement with the observed mass difference (Isgur and Karl 1978a and references therein):

$$M(\Lambda\tfrac{5}{2}^-) - M(\Sigma\tfrac{5}{2}^-) \simeq 50 \pm 15 \text{ MeV} . \tag{1.53}$$

The effect of the hyperfine interaction, while small, accounts for the discrepancy: evaluated with harmonic-oscillator wavefunctions, these interactions reduce the predicted splitting to approximately 50 MeV (Isgur and Karl 1978b). The relative ineffectiveness of the hyperfine interaction in this case is due both to these states having identical spin structure and to their being coupled to the highest total angular momentum possible.

We conclude the introduction to this thesis by briefly assessing the validity of the assumption inherent in the harmonic oscillator quark model that the quarks move with non-relativistic velocities.

1.5 Scales and Relativity

Following Lipkin (1980), we compare constituent quark models for hadrons with analogous constituent models for atoms and nuclei. There are several important differences characterised by a set of different scales. Any bound state has several features with the dimensions of length or mass including:

(1) the mass of the bound state or its Compton wavelength;

(2) the size of the bound state or its Bohr radius;

(3) the excitation energy for orbitally-excited states;

(4) the fine or hyperfine structure arising from spin-dependent interactions.

The four mass scales are listed in Table 1.1 for three different bound state models. (In Table 1.1, A refers to the atomic number of a given nucleus).

Bound states	Mass	Size $(\frac{\hbar c}{r})$	Excitation Energy (ΔE)	Hyperfine Energy
Positronium	1 MeV	$\frac{1}{137}$ MeV	$\frac{1}{(137)^2}$ MeV	$\frac{1}{(137)^3}$ MeV
Nuclei	A GeV	50-100 MeV	5-10 MeV	---
Hadrons	1 GeV	200 MeV	600 MeV	300 MeV

Table 1.1 Scales of bound states

In atomic physics, represented by positronium as the simplest system bound by atomic forces with both constituents having the same mass, the four scales decrease monotonically in steps of $\frac{1}{137}$. In nuclei, the scales decrease monotonically in steps of about an order of magnitude. However, in hadrons, these scales are all approximately equal and the excitation energy is actually larger than the energy defined by the size.

These scales have implications for the validity of the non-relativistic approximation. A particle moving in a non-relativistic orbit has $\frac{v}{c} << 1$. But the velocity, v, is just the product of the radius of the orbit, r, and the angular velocity, ω, and this angular frequency in a quantum system is related to the orbital excitation energy, ΔE. Thus:

$$\frac{v}{c} = \frac{r\omega}{c} = \frac{r\Delta E}{\hbar c} . \qquad (1.54)$$

A more rigorous derivation employing the Heisenberg equations of motion and some matrix algebra (Lipkin 1980) gives:

$$\frac{v}{c} = \frac{r\overline{\Delta E}}{\hbar c} , \qquad (1.55)$$

where $\overline{\Delta E}$ is some mean excitation energy for states of opposite parity from the ground state. Certainly, $\overline{\Delta E}$ must be greater than the excitation energy of the lowest odd-parity excited state. This gives $\frac{v}{c}$ of the order of $\frac{1}{137}$ for positronium, $\frac{1}{10}$ for nuclei, but unity for hadrons. Thus, it would appear that any model for a hadron which fits both the size of the proton as measured by its mean-square-radius, and its excitation spectrum as measured by the excitation energy of its first odd-parity N^*, cannot be non-relativistic (Lipkin 1980). This statement is model-independent.

How, then, is one to justify the use of a non-relativistic quark model? By drawing an analogy with the renormalisation prescription of quantum electrodynamics, Lipkin (1980) suggests that the resolution of this dilemma may lie in some hitherto-unknown principle of relativistic regularisation. In this thesis, we simply assume that a non-relativistic treatment <u>is</u> valid, although we recognise that there are difficulties in trying to justify this assumption. We take heart from observations such as the one made by Isgur (1980) viz. that an harmonic-oscillator wavefunction for the quarks in a proton, if adjusted to reproduce the observed proton charge radius:

$$\langle \Sigma_i q_i \underline{r}_i^2 \rangle^{\frac{1}{2}}_{\text{proton}} = \frac{1}{\alpha} \quad , \tag{1.56}$$

where q_i denotes the electric charge on the i^{th} quark, $\underline{r}_i$ its position and α is the usual harmonic-oscillator constant, gives the quarks (whose mass $\simeq$ 350 MeV) less than 100 MeV of kinetic energy:

$$\langle \underline{p}_3^2 / 2m \rangle_{\text{proton}} = \frac{\alpha^2}{2m} \lesssim 100 \text{ MeV} \quad . \tag{1.57}$$

Further, as Gromes (1980) points out, although one finds in such models that $\langle \underline{p}^2/m^2 \rangle \simeq 1$, so that one might be led to conclude that a non-relativistic approximation is invalid, typical approximations

involved in the derivation of the effective Hamiltonian such as:

$$\sqrt{(m^2 + \underline{p}^2)} \simeq m + \underline{p}^2/2m \tag{1.58}$$

are not so bad for $\underline{p}^2 \simeq m^2$.

CHAPTER 2

CLASSIFICATION AND CONSTRUCTION OF HARMONIC-OSCILLATOR STATE FUNCTIONS

2.1 Classification of State Functions

The group theory of the harmonic oscillator has been discussed extensively in the literature (Wybourne 1974), especially in connection with its application to non-relativistic quark models (Horgan 1976b). In this section, we develop the formalism leading, for the 3-body case, to the unitary dynamical (or degeneracy) group, U(6), and to the symplectic spectrum-generating group, Sp(12,R). These groups provide a novel and direct means of labelling and constructing the harmonic-oscillator state functions of given total orbital angular momentum, parity and permutation symmetry.

We begin by introducing the creation and annihilation operators $a_i^+(\underline{\rho})$, $a_i^+(\underline{\lambda})$, $a_i(\underline{\rho})$ and $a_i(\underline{\lambda})$ $(i = x, y, z)$ associated with the $\underline{\rho}$- and $\underline{\lambda}$-oscillators. In a spherical basis, we define:

$$a_{\pm 1}^+(\underline{\rho}) = \mp \frac{1}{\sqrt{2}} \{a_x^+(\underline{\rho}) \pm i a_y^+(\underline{\rho})\} \tag{2.1a}$$

$$a_o^+(\underline{\rho}) = a_z^+(\underline{\rho}) \tag{2.1b}$$

satisfying the commutation relations:

$$[a_\mu(\underline{\rho}), a_{\mu'}^+(\underline{\rho})] = \delta_{\mu\mu'}, \qquad (\mu, \mu' = +1, 0, -1; \hbar = 1) \tag{2.2}$$

with similar expressions for the $\underline{\lambda}$-mode.

In the non-strange sector, with which we deal exclusively throughout the remainder of this thesis, the general excited state of the zeroth-order Hamiltonian, H_o, in equation (1.32), corresponding to principal quantum number, N, orbital angular momentum, L, and permutation symmetry, P (= S, A, M_ρ or M_λ) may be written:

$$|N\rangle \equiv |\psi^{P}_{LL_z}\rangle \equiv \hat{\psi}^{(P)}_{LL_z}|0\rangle \tag{2.3a}$$

with:

$$|0\rangle \equiv |0\rangle_{\rho}\,|0\rangle_{\lambda} \tag{2.3b}$$

and $\hat{\psi}^{(P)}_{LL_z}$ an Nth-order monomial of creation operators.

The creation and annihilation operators $\underline{a}^{+}(\underline{\rho})$, $\underline{a}^{+}(\underline{\lambda})$, $\underline{a}(\underline{\rho})$ and $\underline{a}(\underline{\lambda})$ may be viewed as components of the 6-vectors:

$$(a_I) = (a_{ia}) = (\underline{a}(\underline{\rho}),\ \underline{a}(\underline{\lambda})) \tag{2.4a}$$

$$(a^{+}_I) = (a^{+}_{ia}) = (\underline{a}^{+}(\underline{\rho}),\ \underline{a}^{+}(\underline{\lambda})) \tag{2.4b}$$

with $i = 1,2,3$; $a = 1,2$; and $I = 1,2, \ldots, 6$. They satisfy the commutation relations:

$$[a_I,\ a_J] = [a^{+}_I,\ a^{+}_J] = 0 \tag{2.5a}$$

$$[a_I,\ a^{+}_J] = [a_{ia},\ a^{+}_{jb}] = \delta_{ij}\cdot\delta_{ab} = \delta_{IJ} \tag{2.5b}$$

for $I, J = 1,2, \ldots, 6$.

With this notation, the Hamiltonian H_o takes the form:

$$H_o - 3m = \omega \sum_{I=1}^{6} \tfrac{1}{2}\{a^{+}_I,\ a_I\}\ , \tag{2.6}$$

where $\omega = (\frac{3K}{m})^{\frac{1}{2}}$. The dynamical (or degeneracy) group is revealed by noting that all 36 bilinear operators:

$$E_{IJ} = \tfrac{1}{2}\{a^{+}_I,\ a_J\} \tag{2.7}$$

commute with H_o and satisfy the commutation relations:

$$[E_{IJ},\ E_{KL}] = \delta_{JK}\,E_{IL} - \delta_{IL}\,E_{KJ} \tag{2.8}$$

of the real Lie algebra of GL(6,C), whose complex form is well-known as the Lie algebra of U(6). It follows that degenerate oscillator

states are associated with unitary, finite - dimensional irreducible representations of U(6). In the canonical labelling scheme based on the structure $U(6) \sim SU(6) \otimes U(1)$, one of the state labels is, of course, the principal quantum number N associated with the U(1) subgroup whose generator $E_{IJ}\cdot\delta^{IJ}$ is proportional to $(H_o - 3m)$.

The spectrum consists of the N = 0 vacuum state $|0>$, transforming as the representation $\{0\} = \underline{1}$ of U(6); the N = 1 states $a_I^+|0>$, transforming as $\{1\} = \underline{6}$ of U(6); the N = 2 states $a_I^+a_J^+|0>$, transforming as $\{2\} = \underline{21}$ of U(6) and so on. [Throughout this thesis, we use the standard notation for irreducible representations of unitary, orthogonal and symplectic groups, namely $\{\lambda_1,\lambda_2, \ldots\}$ $[\lambda_1,\lambda_2, \ldots.]$ and $<\lambda_1,\lambda_2,\ldots.>$, respectively, where $\lambda_1, \lambda_2, \ldots.$ is the partition specifying the Young diagram with row lengths $\lambda_1, \lambda_2, ..$ and so on. We also use the notation $(\lambda_1, \lambda_2, \ldots.)$ for an irreducible representation of the symmetric group, S_n, where $\lambda_1+\lambda_2+ \ldots.$ equals n].

One way to generate this spectrum involves the set of bilinear operators which can be formed from the components a_I and a_I^+ of the 12-vector:

$$(a_A) = (a_{I\alpha}) = (a_{ia\alpha}) = (\underline{a}(\rho), \underline{a}(\lambda), \underline{a}^+(\rho), \underline{a}^+(\lambda)) \tag{2.9}$$

with $I = 1,2, \ldots, 6$; $\alpha = 1$ (annihilation operator), 2 (creation operator); and $A = 1,2, \ldots. 12$. These satisfy the commutation relations:

$$[a_A, a_B] = [a_{I\alpha},a_{J\beta}] = \delta_{IJ}\cdot\varepsilon_{\alpha\beta} = J_{AB} \tag{2.10}$$

for $A,B = 1,2,\ldots., 12$, with:

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } J = \delta \otimes \varepsilon . \tag{2.11}$$

The 78 bilinear operators:

$$S_{AB} = \tfrac{1}{2}\{a_A, a_B\} \tag{2.12}$$

satisfy the commutation relations:

$$[S_{AB},S_{CD}] = J_{BC}\, S_{AD} + J_{AD}\, S_{BC} + J_{AC}\, S_{BD} + J_{BD}\, S_{AC} \tag{2.13}$$

of the real Lie algebra of Sp(12,R). The significance of this enlarged group is that the complete Fock space of the oscillator spectrum decomposes into just two infinite-dimensional, unitary irreducible representations of Sp(12,R): states of even or odd N. This follows from the fact that the Sp(12,R) generators, S_{AB}, are bilinear in the creation and annihilation operators and therefore ladder in steps of two (or zero). The Lie algebra of Sp(12,R) is referred to as a spectrum-generating algebra, analogous for the 3-quark case to the algebra Sp(2N,R) for the single N-dimensional oscillator.

In contrast to this, there exists another way in which the physical oscillator states are manifested as basis states: this time of finite-dimensional, non-unitary irreducible representations of Sp(12,R). This comes about because both the creation and annihilation operators themselves and, more generally, monomials of them form the bases of such representations. This follows from the existence of the map:

$$S_{AB}: \quad a_C \to [S_{AB},a_C] = J_{AC}a_B + J_{BC}a_A \tag{2.14}$$

and its generalisations:

$$S_{AB}: \quad a_C a_D \to [S_{AB},a_C a_D] = J_{AC}a_B a_D + J_{BC}a_A a_D + J_{AD}a_C a_B + J_{BD}a_C a_A \tag{2.15}$$

and so on. It is clear, in particular, that the operators 1, a_A, $(a_A a_B + a_B a_A)$, $(a_A a_B a_C + a_B a_C a_A + a_C a_A a_B + a_B a_A a_C + a_C a_B a_A + a_A a_C a_B)$, ... form bases of the symmetric representations $\langle 0\rangle = 1$, $\langle 1\rangle = \underline{12}$, $\langle 2\rangle = \underline{78}$, $\langle 3\rangle = \underline{364}$, ... of Sp(12,R). Since the variables 1, a_I^+, $a_I^+ a_J^+$, $a_I^+ a_J^+ a_K^+$, ... belong to these bases, the physical oscillator states are indeed associated with finite-dimensional irreducible representations as claimed. The physical states of the representation $\langle N\rangle$ are precisely those for which :

$$S_{12} = S_{I1J2}\cdot\delta^{IJ} = E_{IJ}\cdot\delta^{IJ} \quad , \tag{2.16}$$

acting as in equations (2.14) and (2.15), has eigenvalue N.

This generator, besides being the U(1) generator encountered earlier, belongs to the algebra of the subgroup Sp(2,R) of Sp(12,R), whose generators are:

$$S_{\alpha\beta} = S_{I\alpha J\beta}\cdot\delta^{IJ} \quad . \tag{2.17}$$

This group is locally isomorphic to the pseudo-orthogonal group SO(2,1) as can be seen by noting that $P_1 = \frac{1}{4}(S_{11} + S_{22})$, $P_2 = \frac{i}{4}(S_{11} - S_{22})$ and $P_3 = \frac{1}{2}S_{12}$ satisfy:

$$[P_i, P_j] = i \sum_{k=1}^{3} \varepsilon_{ijk}\, \eta_{kk}\, P_k \quad , \tag{2.18}$$

where :

$$\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad . \tag{2.19}$$

The physical states are those components of the SO(2,1) multiplets of pseudospin $[0] = \underline{1}$, $[\frac{1}{2}] = \underline{2}$, $[1] = \underline{3}$, ..., $[N/2] = \underline{N+1}$, ... with maximum third component $0, \frac{1}{2}, 1, \ldots N/2, \ldots$. The corresponding Sp(2,R) multiplets are denoted by $\langle 0\rangle = \underline{1}$, $\langle 1\rangle = \underline{2}$, $\langle 2\rangle = \underline{3}$, ... $\langle N\rangle = \underline{N+1}$, ...

Just as in the case of a single 3-dimensional oscillator (c.f. Chapter 4), the physical states with fixed pseudospin quantum numbers may be further classified into multiplets of an orthogonal group. In the case of three particles, this is the subgroup $O(6)$ of $Sp(12,R)$, with generators:

$$O_{IJ} = S_{I\alpha J\beta} \cdot \varepsilon^{\alpha\beta} = E_{IJ} - E_{JI} \tag{2.20}$$

satisfying:

$$[O_{IJ}, O_{KL}] = \delta_{JK} O_{IL} + \delta_{IL} O_{JK} - \delta_{IK} O_{JL} - \delta_{JL} O_{IK} . \tag{2.21}$$

The group $O(6)$ contains as a subgroup, $O(3) \otimes O(2)$, whose constituents, $O(3)$ and $O(2)$, respectively, are generated by:

$$O_{ij} = O_{iajb} \cdot \delta^{ab} \tag{2.22}$$

and:

$$O_{ab} = O_{iajb} \cdot \delta^{ij} . \tag{2.23}$$

The former serve to specify the total orbital angular momentum of the physical states through the familiar generators:

$$L_i = i \varepsilon_{ijk} O_{jk} , \tag{2.24}$$

while the latter is the generator of rotations in the 2-dimensional space associated with the $\underline{\rho}$- and $\underline{\lambda}$-modes of oscillation. Typically, a rotation through θ is given in this space by:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} . \tag{2.25}$$

It should be stressed, however, that the full group $O(2)$ also includes the reflection:

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{2.26}$$

This is particularly important because the permutation group, S_3, is

a subgroup of $O(2)$ but not of $SO(2)$. This can be seen by noting that the permutations $P(123)$ and $P(12)$, which generate S_3, are given in the 2-dimensional, faithful irreducible representation $(2,1) = \underline{M}$ by:

$$P(123) = R(^{2\pi}/3) \tag{2.27a}$$

$$P(12) = \sigma \quad . \tag{2.27b}$$

Thus, to summarise, the subgroup chain to be used in labelling the oscillator states is:

$$Sp(12,R) \supset Sp(2,R) \otimes O(6) \supset Sp(2,R) \otimes O(3) \otimes O(2) \supset U(1) \otimes SO(3) \otimes S_3 \ . \tag{2.28}$$

Its key labels N, L and P are associated with the irreducible representations of $U(1)$, $SO(3)$ and S_3, respectively.

In order to enumerate at each level, specified by N, the $O(3) \otimes S_3$ multiplets and hence the $SU(6)_{\text{flavour}\times\text{spin}} \otimes O(3)$ supermultiplets, we require the branching rules for various subgroup embeddings. In the case of the continuous groups, these are given by simple Young diagram techniques (King 1975). These techniques enable one to deduce, for example, how the symmetric tensor representation $\langle N \rangle$ of $Sp(12,R)$ reduces on restriction to $Sp(2,R) \otimes O(6)$. One obtains:

$$N = 0: \quad \underline{1} \supset \underline{1} \otimes \underline{1} \tag{2.29a}$$

$$N = 1: \quad \underline{12} \supset \underline{2} \otimes \underline{6} \tag{2.29b}$$

$$N = 2: \quad \underline{78} \supset \underline{3} \otimes (\underline{20} \oplus \underline{1}) \oplus \underline{1} \otimes \underline{15} \tag{2.29c}$$

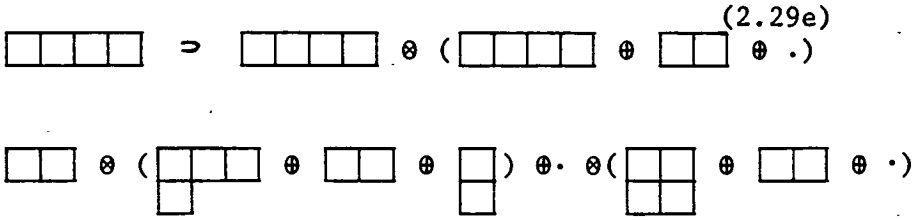
i.e. $\square\square \supset \square\square \otimes (\square\square \oplus \cdot) \oplus \cdot \otimes \begin{smallmatrix}\square\\\square\end{smallmatrix}$

$$N = 3: \quad \underline{364} \supset \underline{4} \otimes (\underline{50} \oplus \underline{6}) \oplus \underline{2} \otimes (\underline{64} \oplus \underline{6}) \tag{2.29d}$$

i.e. $\square\square\square \supset \square\square\square \otimes (\square\square\square \oplus \square) \oplus \square \otimes (\begin{smallmatrix}\square\square\\\square\end{smallmatrix} \oplus \square)$

N=4: $\underline{1365} \supset \underline{5} \otimes (\underline{105} \oplus \underline{20} \oplus \underline{1}) \oplus \underline{3} \otimes (\underline{175} \oplus \underline{20} \oplus \underline{15}) \oplus \underline{1} \otimes (\underline{84} \oplus \underline{20} \oplus \underline{1})$

(2.29e)

i.e. 

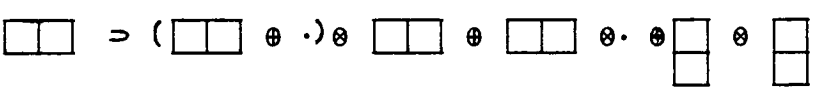
and so on. Since it is required that the corresponding physical oscillator states have pseudospin $\frac{N}{2}$ and be associated with the Sp(2,R) representation <N>, it follows that the relevant O(6) multiplets [M] have M = N, N-2, N-4, ..., where the sequence terminates with either 1 or 0. This is in accordance with the result expected through consideration of the degeneracy group U(6), since the totally symmetric tensor representation {N} of U(6) yields just these representations [M] of O(6) on restriction to this subgroup.

The corresponding branching rule (King 1975) for the reduction $O(6) \supset O(3) \otimes O(2)$ yields in the case of the representation [M] of O(6), the following multiplets of $O(3) \otimes O(2)$:

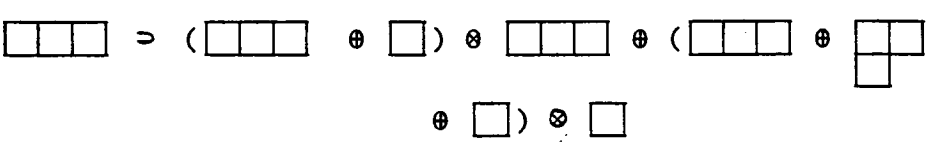
M = 0: $\underline{1} \supset \underline{1} \otimes \underline{1}$ (2.30a)

M = 1: $\underline{6} \supset \underline{3} \otimes \underline{2}_1$ (2.30b)

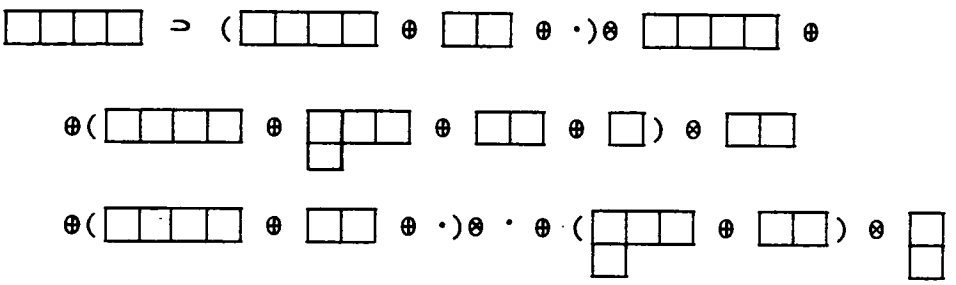
M = 2: $\underline{20} \supset (\underline{5} \oplus \underline{1}) \otimes \underline{2}_2 \oplus \underline{5} \otimes \underline{1} \oplus \underline{3} \otimes \underline{1}^*$ (2.30c)

i.e. 

M = 3: $\underline{50} \supset (\underline{7} \oplus \underline{3}) \otimes \underline{2}_3 \oplus (\underline{7} \oplus \underline{5} \oplus \underline{3}) \otimes \underline{2}_1$ (2.30d)

i.e. 

M=4: $\underline{105} \supset (\underline{9}\oplus\underline{5}\oplus\underline{1})\otimes\underline{2}_4\oplus(\underline{9}\oplus\underline{7}\oplus\underline{5}\oplus\underline{3})\otimes\underline{2}_2\oplus(\underline{9}\oplus\underline{5}\oplus\underline{1})\otimes\underline{1}\oplus(\underline{7}\oplus\underline{5})\otimes\underline{1}^*$ (2.30e)

i.e. 

and so on.

The final reduction from O(2) to S_3 proceeds as follows. The scalar, $[0] = \underline{1}$, and pseudoscalar, $[1^2] = [0]^* = \underline{1}^*$, representations of O(2) are, of course, symmetric and antisymmetric, respectively, under S_3, and thus yield on restriction to this subgroup $(3) = \underline{S}$ and $(1^3) = \underline{A}$. The remaining irreducible representations of O(2) are the doublets $[m] = \underline{2}_m$, labelled by a quantum number m (integer or half-integer, in general) such that $R(\theta)$ is mapped to $R(m\theta)$. In what we consider, only integer values of m occur and it is easy to deduce that under the restriction from O(2) to S_3:

$$[m] \rightarrow \begin{cases} \underline{S} \oplus \underline{A}\,, & \text{if } m \equiv 0 \pmod 3 \\ \underline{M}\,, & \text{if } m \equiv 1,2 \pmod 3 \end{cases} \qquad (2.31)$$

This allows us to complete the reduction procedure and thereby determine the $SU(6)_{\text{flavour}\times\text{spin}} \otimes O(3)$ supermultiplets at each degeneracy level specified by N. The results for N = 0, 1, 2, 3 and 4 are displayed in Table 2.1.

N	O(6)	O(3)⊗O(2)	$[SU(6),L^P]$
0	$\underline{1}$	$\underline{1} \otimes \underline{1}$	$[\underline{56},0^+]$
1	$\underline{6}$	$\underline{3} \otimes \underline{2}_1$	$[\underline{70},1^-]$
2	$\underline{20}$	$(\underline{5}\oplus\underline{1}) \otimes \underline{2}_2$	$[\underline{70},2^+],[\underline{70},0^+]$
		$\underline{3} \otimes \underline{1}^*$	$[\underline{20},1^+]$
		$\underline{5} \otimes \underline{1}$	$[\underline{56},2^+]$
	$\underline{1}$	$\underline{1} \otimes \underline{1}$	$[\underline{56},0^+]$
3	$\underline{50}$	$(\underline{7}\oplus\underline{3}) \otimes \underline{2}_3$	$[\underline{56},3^-],[\underline{20},3^-],[\underline{56},1^-],[\underline{20},1^-]$
		$(\underline{7}\oplus\underline{5}\oplus\underline{3})\otimes\underline{2}_1$	$[\underline{70},3^-],[\underline{70},2^-],[\underline{70},1^-]$
	$\underline{6}$	$\underline{3} \otimes \underline{2}_1$	$[\underline{70},1^-]$
4	$\underline{105}$	$(\underline{9}\oplus\underline{5}\oplus\underline{1})\otimes \underline{2}_4$	$[\underline{70},4^+],[\underline{70},2^+],[\underline{70},0^+]$
		$(\underline{9}\oplus\underline{7}\oplus\underline{5}\oplus\underline{3})\otimes \underline{2}_2$	$[\underline{70},4^+],[\underline{70},3^+],[\underline{70},2^+],[\underline{70},1^+]$
		$(\underline{9}\oplus\underline{5}\oplus\underline{1})\otimes \underline{1}$	$[\underline{56},4^+],[\underline{56},2^+],[\underline{56},0^+]$
		$(\underline{7}\oplus\underline{5})\otimes \underline{1}^*$	$[\underline{20},3^+],[\underline{20},2^+]$
	$\underline{20}$	$(\underline{5}\oplus\underline{1})\otimes \underline{2}_2$	$[\underline{70},2^+],[\underline{70},0^+]$
		$\underline{3} \otimes \underline{1}^*$	$[\underline{20},1^+]$
		$\underline{5} \times \underline{1}$	$[\underline{56},2^+]$
	$\underline{1}$	$\underline{1} \otimes \underline{1}$	$[\underline{56},0^+]$

Table 2.1 $SU(6)_{\text{flavour}\times\text{spin}} \otimes O(3)$ supermultiplet structure of the harmonic oscillator quark model: levels N=0 to N=4

2.2 Construction of state functions

Making use of the variables:

$$\underline{\zeta} = \underline{\rho} + i\underline{\lambda} \tag{2.32a}$$

$$\underline{\eta} = \underline{\rho} - i\underline{\lambda} \ , \tag{2.32b}$$

the physical states are now remarkably easy to construct in terms of auxiliary creation operators which we define by:

$$\underline{a}^{+}(\underline{\zeta}) = \underline{a}^{+}(\underline{\rho}) + i\underline{a}^{+}(\underline{\lambda}) \tag{2.33a}$$

$$\underline{a}^{+}(\underline{\eta}) = \underline{a}^{+}(\underline{\rho}) - i\underline{a}^{+}(\underline{\lambda}) \ . \tag{2.33b}$$

These are the basis states of two 1-dimensional irreducible representations of SO(2). The transformation to this basis then serves to diagonalise all the rotation matrices $R(\theta)$ including $R^{2\pi/3}$). Under the action of the permutations of S_3:

$$P(12)\underline{a}^{+}(\underline{\zeta}) = -\underline{a}^{+}(\underline{\eta}) \tag{2.34a}$$

$$P(12)\underline{a}^{+}(\underline{\eta}) = -\underline{a}^{+}(\underline{\zeta}) \tag{2.34b}$$

and:

$$P(123)\underline{a}^{+}(\underline{\zeta}) = e^{-i\,2\pi/3}\,\underline{a}^{+}(\underline{\zeta}) \tag{2.35a}$$

$$P(123)\underline{a}^{+}(\underline{\eta}) = e^{i\,2\pi/3}\,\underline{a}^{+}(\underline{\eta}) \tag{2.35b}$$

so it is trivial to determine the transformation properties of monomials in these operators. Consider the particular monomial:

$$W(\underline{\zeta},\underline{\eta}) = \{\underline{a}^{+}(\underline{\zeta})\cdot\underline{a}^{+}(\underline{\eta})\}^{a}\{\underline{a}^{+}(\underline{\zeta})\cdot\underline{a}^{+}(\underline{\zeta})\}^{b}\{\underline{a}^{+}(\underline{\eta})\cdot\underline{a}^{+}(\underline{\eta})\}^{c}\{a^{+}_{+1}(\underline{\zeta})\}^{p} \times$$

$$\times \{a^{+}_{+1}(\underline{\eta})\}^{q}[a^{+}(\underline{\zeta})a^{+}(\underline{\eta})]^{\alpha}_{+1} \tag{2.36}$$

where:

$$[a^{+}(\underline{\zeta})a^{+}(\underline{\eta})]_{+1} = a^{+}_{+1}(\underline{\zeta})a^{+}_{o}(\underline{\eta}) - a^{+}_{o}(\underline{\zeta})a^{+}_{+1}(\underline{\eta}) \ . \tag{2.37}$$

The monomial $W(\underline{\zeta},\underline{\eta})$ in equation (2.36) creates a stretched state of orbital angular momentum $L = L_z = p + q + \alpha$, with corresponding principal quantum number $N = 2(a+b+c+\alpha) + (p+q)$. The factor:

$$\underline{a}^+(\underline{\zeta})\cdot\underline{a}^+(\underline{\eta}) \quad = \quad \{\underline{a}^+(\underline{\rho})\}^2 + \{\underline{a}^+(\underline{\lambda})\}^2 \tag{2.38}$$

is an O(6) invariant so that the monomial $W(\underline{\zeta},\underline{\eta})$ belongs to the representation [M] of O(6) with $M \leq N - 2a$. Furthermore, if $m = 2(b-c) + (p-q)$, the corresponding O(2) representation [m] is 2-dimensional with basis states $W(\underline{\zeta},\underline{\eta})$ and $\overline{W}(\underline{\zeta},\underline{\eta})$, where:

$$\overline{W}(\underline{\zeta},\underline{\eta}) \quad = \quad W(\underline{\eta},\underline{\zeta}) \tag{2.39}$$

unless $W(\underline{\zeta},\underline{\eta}) = \pm W(\underline{\eta},\underline{\zeta})$, in which case $m=0$ and the corresponding O(2) representation $[m] = [0]$ or $[0]^*$ is 1-dimensional. From these basis states $W(\underline{\zeta},\underline{\eta})$ and $\overline{W}(\underline{\zeta},\underline{\eta})$ of irreducible representations of SO(2), the basis states of irreducible representations of S_3 are recovered in the form:

$$\mathcal{R}W(\underline{\zeta},\underline{\eta}) \quad = \quad \tfrac{1}{2}\{W(\underline{\zeta},\underline{\eta}) + \overline{W}(\underline{\zeta},\underline{\eta})\} \tag{2.40a}$$

$$\mathcal{I}W(\underline{\zeta},\underline{\eta}) \quad = -\tfrac{i}{2}\{W(\underline{\zeta},\underline{\eta}) - \overline{W}(\underline{\zeta},\underline{\eta})\} . \tag{2.40b}$$

The results depend only upon m (mod 6) and are given in Table 2.2 in which S, A, M_ρ and M_λ signify basis states of the representations $(3) = \underline{S}$, $(1^3) = \underline{A}$ and $(2,1) = \underline{M}$ of S_3, with M_ρ and M_λ transforming under permutations in exactly the same manner as $\underline{\rho}$ and $\underline{\lambda}$.

m(mod 6)	0	1	2	3	4	5
$\mathcal{R}W$	S	M_ρ	M_λ	A	$-M_\lambda$	M_ρ
$\mathcal{I}W$	A	M_λ	M_ρ	S	M_ρ	$-M_\lambda$

Table 2.2 Monomials of definite permutation symmetry: $\mathcal{R}W = \tfrac{1}{2}(W+\overline{W})$ and $\mathcal{I}W = -\tfrac{i}{2}(W-\overline{W})$.

The particular factor $[a^+(\underline{\zeta})a^+(\underline{\eta})]_{+1}$ is antisymmetric in the sense that:

$$P(12)[a^+(\underline{\zeta})a^+(\underline{\eta})]_{+1} = -[a^+(\underline{\zeta})a^+(\underline{\eta})]_{+1} \qquad (2.41a)$$

whilst:

$$P(123)[a^+(\underline{\zeta})a^+(\underline{\eta})]_{+1} = [a^+(\underline{\zeta})a^+(\underline{\eta})]_{+1}. \qquad (2.41b)$$

It is, therefore, a basis state of the type A. This factor, moreover, satisfies a syzygy-like identity:

$$\{[a^+(\underline{\zeta})a^+(\underline{\eta})]_{+1}\}^2 = (\underline{a}^+(\underline{\zeta})\cdot\underline{a}^+(\underline{\zeta}))\{a^+_{+1}(\underline{\eta})\}^2 + (\underline{a}^+(\underline{\eta})\cdot\underline{a}^+(\underline{\eta}))\{a^+_{+1}(\underline{\zeta})\}^2$$
$$- 2(\underline{a}^+(\underline{\zeta})\cdot\underline{a}^+(\underline{\eta}))a^+_{+1}(\underline{\zeta})a^+_{+1}(\underline{\eta}) \quad . \qquad (2.42)$$

The implication of this and the use of W and $\overline{W}$ in equations (2.40a) - (2.40b) is that in constructing all the independent oscillator states for a given value of N, it is only necessary to consider those distinct monomials $W(\underline{\zeta},\underline{\eta})$ of degree N with $m \geq 0$ and $\alpha = 0$ or 1.

The N=2 and N=3 states derived in this way are given explicitly in Tables 2.3 and 2.4:

$[SU(6),L^P]$	$O(3) \otimes O(2)$	S_3	Monomial
$[\underline{56},0^+]$	$\underline{1} \otimes \underline{1}$	S	$\mathfrak{R}(\underline{a}^+(\underline{\zeta})\cdot\underline{a}^+(\underline{\eta}))$
$[\underline{70},0^+]$	$\underline{1} \otimes \underline{2}_2$	M_ρ	$\mathfrak{I}(\underline{a}^+(\underline{\zeta})\cdot\underline{a}^+(\underline{\zeta}))$
		M_λ	$\mathfrak{R}(\underline{a}^+(\underline{\zeta})\cdot\underline{a}^+(\underline{\zeta}))$
$[\underline{20},1^+]$	$\underline{3} \otimes \underline{1}^*$	A	$\mathfrak{I}([a^+(\underline{\zeta})a^+(\underline{\eta})]_{+1})$
$[\underline{56},2^+]$	$\underline{5} \otimes \underline{1}$	S	$\mathfrak{R}(a^+_{+1}(\underline{\zeta})a^+_{+1}(\underline{\eta}))$
$[\underline{70},2^+]$	$\underline{5} \otimes \underline{2}_2$	M_ρ	$\mathfrak{I}(\{a^+_{+1}(\underline{\zeta})\}^2)$
		M_λ	$\mathfrak{R}(\{a^+_{+1}(\underline{\zeta})\}^2)$

Table 2.3 State-function monomials at the N = 2 level

$[SU(6),L^P]$	$O(3) \otimes O(2)$	S_3	Monomial
$[\underline{56},3^-]$	$\underline{7} \otimes \underline{2}_3$	S	$\mathcal{S}(\{a^+_{+1}(\underline{\zeta})\}^3)$
$[\underline{20},3^-]$	$\underline{7} \otimes \underline{2}_3$	A	$\mathcal{A}(\{a^+_{+1}(\underline{\zeta})\}^3)$
$[\underline{70},3^-]$	$\underline{7} \otimes \underline{2}_1$	M_ρ	$\mathcal{A}(\{a^+_{+1}(\underline{\zeta})\}^2 a^+_{+1}(\underline{\eta}))$
		M_λ	$\mathcal{S}(\{a^+_{+1}(\underline{\zeta})\}^2 a^+_{+1}(\underline{\eta}))$
$[\underline{70},2^-]$	$\underline{5} \otimes \underline{2}_1$	M_ρ	$\mathcal{A}(a^+_{+1}(\underline{\zeta})[a^+(\underline{\zeta})a^+(\underline{\eta})]_{+1})$
		M_λ	$\mathcal{S}(a^+_{+1}(\underline{\zeta})[a^+(\underline{\zeta})a^+(\underline{\eta})]_{+1})$
$[\underline{56},1^-]$	$\underline{3} \otimes \underline{2}_3$	S	$\mathcal{S}(\{\underline{a}^+(\underline{\zeta})\cdot\underline{a}^+(\underline{\zeta})\}a^+_{+1}(\underline{\zeta}))$
$[\underline{20},1^-]$	$\underline{3} \otimes \underline{2}_3$	A	$\mathcal{A}(\{\underline{a}^+(\underline{\zeta})\cdot\underline{a}^+(\underline{\zeta})\}a^+_{+1}(\underline{\zeta}))$
$[\underline{70},1^-]$	$\underline{3} \otimes \underline{2}_1$	M_ρ	$\mathcal{A}(\{\underline{a}^+(\underline{\zeta})\cdot\underline{a}^+(\underline{\zeta})\}a^+_{+1}(\underline{\eta}))$
		M_λ	$\mathcal{S}(\{\underline{a}^+(\underline{\zeta})\cdot\underline{a}^+(\underline{\zeta})\}a^+_{+1}(\underline{\eta}))$
$[\underline{70},1^-]$	$\underline{3} \otimes \underline{2}_1$	M_ρ	$\mathcal{A}(\{\underline{a}^+(\underline{\zeta})\cdot\underline{a}^+(\underline{\eta})\}a^+_{+1}(\underline{\zeta}))$
		M_λ	$\mathcal{S}(\{\underline{a}^+(\underline{\zeta})\cdot\underline{a}^+(\underline{\eta})\}a^+_{+1}(\underline{\zeta}))$

Table 2.4 State-function monomials at the N = 3 level

The procedure used in constructing such states of definite orbital angular momentum, parity and permutation symmetry is thus extremely simple and somewhat more direct than previous procedures. The simplification is in large measure due to the use of the $(\underline{\zeta},\underline{\eta})$-basis. This contrasts with the complexities associated with the use of the $(\underline{\rho},\underline{\lambda})$-basis, which are apparent in the construction procedure of Karl and Obryk (1968) based on the reduction:

$$U(6) \supset SO(3) \otimes S_3, \tag{2.43}$$

and even more strikingly in that of Horgan (1976b), based on the subgroup chain:

$$U(6) \supset U(3) \otimes U(2) \supset SO(3) \otimes S_3. \tag{2.44}$$

We shall make use of the subgroup chain:

$$Sp(12,R) \supset U(6) \supset O(6) \supset O(3) \otimes O(2) \supset SO(3) \otimes S_3 \qquad (2.45)$$

which incorporates the group $O(6)$, whose use has been advocated and adopted in this context by Cutkosky and Hendrick (1977a) and which appears naturally in the subgroup chain in equation (2.28) by virtue of equation (2.20). However, our scheme is not ideal in that the states obtained directly from equation (2.36) are not all associated with a unique irreducible representation of $O(6)$. In general, M can take on the values $N-2a$, $N-2a-2$, $N-2a-4$, ... and a more complete labelling scheme involving the specification of M can be obtained merely by orthogonalising states commencing with the state of lowest value of M which corresponds to the largest value of a for a given N in equation (2.36).

At the $N=2$ level, there are no ambiguities and this orthogonalisation is not necessary, but at the $N=3$ level, there are two $P = \underline{M}\ [\underline{70},1^-]$ supermultiplets which may be distinguished by the $O(6)$ labels $[3] = \underline{50}$ and $[1] = \underline{6}$, as can be seen from Table 2.1. The necessary orthogonal combinations of the states given in Table 2.4 are:

$$[3]\quad [\underline{70},1^-]\quad \underline{3} \otimes \underline{2}_1 \begin{cases} M_\rho & \mathcal{R}(\{\underline{a}^+(\underline{\zeta})\cdot\underline{a}^+(\underline{\zeta})\}a^+_{+1}(\underline{\eta}) - \tfrac{1}{2}\{\underline{a}^+(\underline{\zeta})\cdot\underline{a}^+(\underline{\eta})\}a^+_{+1}(\underline{\zeta})) \qquad (2.46a) \\ M_\lambda & \mathcal{I}(\{\underline{a}^+(\underline{\zeta})\cdot\underline{a}^+(\underline{\zeta})\}a^+_{+1}(\underline{\eta}) - \tfrac{1}{2}\{\underline{a}^+(\underline{\zeta})\cdot\underline{a}^+(\underline{\eta})\}a^+_{+1}(\underline{\zeta})) \qquad (2.46b) \end{cases}$$

and:

$$[1]\quad [\underline{70},1^-]\quad \underline{3} \otimes \underline{2}_1 \begin{cases} M_\rho & \mathcal{R}(\{\underline{a}^+(\underline{\zeta})\cdot\underline{a}^+(\underline{\eta})\}a^+_{+1}(\underline{\zeta})) \qquad (2.47a) \\ M_\lambda & \mathcal{I}(\{\underline{a}^+(\underline{\zeta})\cdot\underline{a}^+(\underline{\eta})\}a^+_{+1}(\underline{\zeta})) \ . \qquad (2.47b) \end{cases}$$

A similar orthogonalisation procedure is required at the $N=4$ level to distinguish, for example, the two $P = \underline{S}\ [\underline{56},2^+]$ states labelled by $M = 4$ and $M = 2$. This difficulty is also experienced in making use of

the states of Karl and Obryk (1968) which, in this particular case, coincide with those given by equation (2.36).

In contrast to this, in the case cited by Horgan (1976b) of the $N = 4$, $P = \underline{M}\ [\underline{70},4^+]$ states, the two pairs of states, M_ρ and M_λ, are again not mutually orthogonal in the scheme of Karl and Obryk (1968). Horgan (1976b) constructs orthogonal states by diagonalising a matrix K whose eigenvalues then serve to label the states. However, the method used here leads unambiguously to the four states $|N, M, L, L_z = L, m, P\rangle$:

$$|4,4,4,4,4,M_\rho\rangle = \frac{1}{2\sqrt{3}} (\{a^+_{+1}(\underline{\rho})\}^3 a^+_{+1}(\underline{\lambda}) - a^+_{+1}(\underline{\rho})\{a^+_{+1}(\underline{\lambda})\}^3) \qquad (2.48a)$$

$$|4,4,4,4,4,M_\lambda\rangle = \frac{1}{8\sqrt{3}} (-\{a^+_{+1}(\underline{\rho})\}^4 + 6\{a^+_{+1}(\underline{\rho})\}^2\{a^+_{+1}(\underline{\lambda})\}^2 - \{a^+_{+1}(\underline{\lambda})\}^4) \qquad (2.48b)$$

and:

$$|4,4,4,4,2,M_\rho\rangle = \frac{1}{2\sqrt{3}} (\{a^+_{+1}(\underline{\rho})\}^3 a^+_{+1}(\underline{\lambda}) + a^+_{+1}(\underline{\rho})\{a^+_{+1}(\underline{\lambda})\}^3) \qquad (2.49a)$$

$$|4,4,4,4,2,M_\lambda\rangle = \frac{1}{4\sqrt{3}} (\{a^+_{+1}(\underline{\rho})\}^4 - \{a^+_{+1}(\underline{\lambda})\}^4) \quad , \qquad (2.49b)$$

where equations (2.33a) - (2.33b) have been used to express the states in terms of $\underline{a}^+(\underline{\rho})$ and $\underline{a}^+(\underline{\lambda})$. In this case, it is the label m of the $O(2)$ representation $[m] = \underline{2}_m$ which distinguishes the states and guarantees their orthogonality.

For higher values of N, branching multiplicities in the chain in equation (2.45) lead to other labelling ambiguities and the need to orthogonalise further states (King 1980). For $N \leq 4$, the chain in equation (2.45) does, however, provide a complete labelling scheme.

CHAPTER 3

A PHENOMENOLOGICAL STUDY OF THE N=2 AND N=3 LEVELS OF THE ISGUR-KARL MODEL

3.1 Method of Computation

In this chapter, we take as our starting point the form of the harmonic oscillator quark model suggested by Isgur and Karl (Isgur 1980 and references therein), incorporating anharmonic perturbations, quark mass differences and some effects of the non-relativistic reduction of coloured-gluon exchange, and which is described by the Hamiltonian (c.f. equations (1.28) and (1.31)):

$$H = \sum_i (m_i + \underline{p}_i^2/2m_i) + \sum_{i<j} (\tfrac{1}{2}K\underline{r}_{ij}^2 + U(r_{ij})) + \sum_{i<j} H_{ij}^{hyp}. \quad (3.1)$$

By treating $U(r_{ij})$ and the hyperfine interaction, H_{ij}^{hyp}, perturbatively in the harmonic-oscillator basis, Isgur and Karl (1979a) obtained a good description of both strange and non-strange, positive-parity baryon resonances up to about 2 GeV in mass, which they assigned to the N=2 level of the harmonic oscillator model. In this chapter, we deal only with the non-strange sectors of the N=2 and N=3 levels, wherein all three quarks may be assigned a common mass, m, and we further simplify matters by postponing until Chapter 6 any consideration of the hyperfine interaction. Thus, we base our present analysis on the Hamiltonian:

$$H = H_o + \sum_{i<j} U(r_{ij}) \quad , \quad (3.2)$$

where H_o is as defined in equation (1.32) and $U(r_{ij})$ is an unknown anharmonic perturbation depending only on the magnitude, r_{ij}, of the

separation of quarks i and j. We shall treat $U(r_{ij})$ to first-order in perturbation theory using the eigenstates of H_o as basis states. Thus, we content ourselves for the present with predicting only the mean masses of the various harmonic-oscillator supermultiplets. This approach is already known to lead to an intriguing result for the N=2 supermultiplets (Isgur and Karl 1979a). Figure 3 shows the pattern of the degeneracy-breaking induced by an arbitrary anharmonic perturbation. Apart from an overall sign, the relative splittings are independent of the detailed form of $U(r_{ij})$. [For example, Gromes and Stamatescu (1976, 1979) have shown that, if $U(r_{ij})$ is assumed to have a power-law form:

$$U(r_{ij}) = \lambda r_{ij}^{k} , \qquad \lambda , k > 0 \tag{3.3}$$

then the pattern of splittings is as indicated in Figure 3 if $0 < k < 2$, but is inverted if $k > 2$]. This suggests that the result may be derived from purely group-theoretic considerations and corresponds to the breaking of a symmetry of the unperturbed Hamiltonian, H_o, by the perturbation $\sum_{i<j} U(r_{ij})$. We shall demonstrate that this is indeed the case in Chapter 5. The pattern of splittings at the N=2 level - with the lowering of the radially-excited supermultiplet $[\underline{56}, 0^+]$ below the $[\underline{70}, 0^+]$, $[\underline{56}, 2^+]$ and $[\underline{70}, 2^+]$ supermultiplets - seems to correspond well with the physical situation and lends confidence to the belief that the pattern of splittings at the N=3 level will be of interest.

In the absence of hyperfine interactions, the calculation of the first-order energy shifts induced by the anharmonic perturbation, $\sum_{i<j} U(r_{ij})$, is straightforward. We exploit the permutational symmetry of the $SU(6)_{flavour \times spin} \otimes O(3)$ 3-quark states, $|\phi>$, to reduce the problem to a calculation involving the $\underline{\rho}$-oscillator alone:

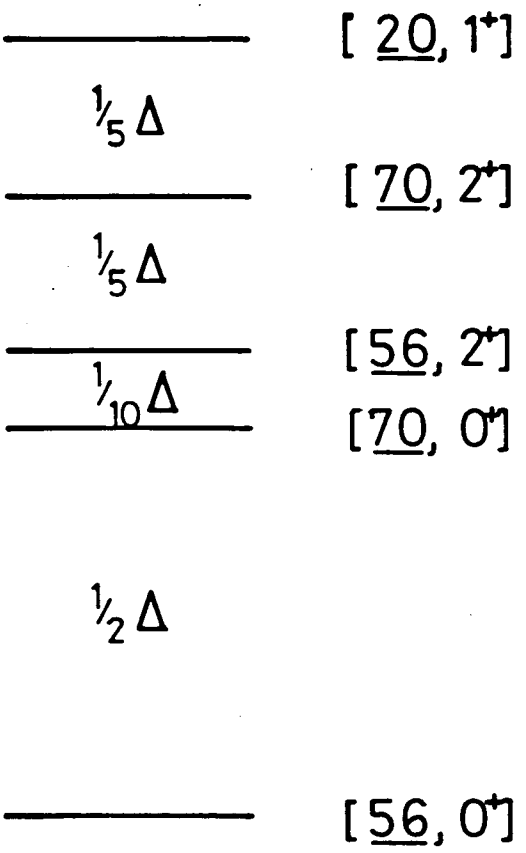


Figure 3 The pattern of splittings of the N = 2 supermultiplets induced by the anharmonic perturbation $\sum_{i<j} U(r_{ij})$

$$\langle\phi|\sum_{i<j} U(r_{ij})|\phi\rangle = 3\langle\phi|U(\sqrt{2}\rho)|\phi\rangle \quad . \tag{3.4}$$

The $\underline{\rho}$-oscillator matrix elements may, of course, be evaluated using explicit oscillator wavefunctions (Isgur and Karl 1979a, Dalitz and Horgan 1973, Gromes and Stamatescu 1976) or, more elegantly, by an algebraic procedure which exploits the commutation relations of the creation and annihilation operators. As the latter method does not appear to be widely used, we give an example from the N=2 level. Table 3.1 gives the correctly-normalised monomials, constructed by the procedure given in Chapter 2, for the five N=2 supermultiplets of the harmonic oscillator model:

$[SU(6),L^P]$	Monomial
$[\underline{56},\ 0^+]$	$\hat{\psi}^{(s)}_{oo} = \frac{1}{2\sqrt{3}}\{[\underline{a}^+(\underline{\rho})]^2 + [\underline{a}^+(\underline{\lambda})]^2\}$
$[\underline{70},\ 0^+]$	$\hat{\psi}^{(M_\rho)}_{oo} = \frac{1}{\sqrt{3}}\ \underline{a}^+(\underline{\rho})\cdot\underline{a}^+(\underline{\lambda})$
	$\hat{\psi}^{(M_\lambda)}_{oo} = \frac{1}{2\sqrt{3}}\{[\underline{a}^+(\underline{\rho})]^2 - [\underline{a}^+(\underline{\lambda})]^2\}$
$[\underline{20},\ 1^+]$	$\hat{\psi}^{(A)}_{11} = -\frac{1}{\sqrt{2}}\{a^+_{+1}(\underline{\rho})a^+_o(\underline{\lambda}) - a^+_{+1}(\underline{\lambda})a^+_o(\underline{\rho})\}$
$[\underline{56},\ 2^+]$	$\hat{\psi}^{(s)}_{22} = \frac{1}{2}\{[a^+_{+1}(\underline{\rho})]^2 + [a^+_{+1}(\underline{\lambda})]^2\}$
$[\underline{70},\ 2^+]$	$\hat{\psi}^{(M_\rho)}_{22} = a^+_{+1}(\underline{\rho})a^+_{+1}(\underline{\lambda})$
	$\hat{\psi}^{(M_\lambda)}_{22} = \frac{1}{2}\{[a^+_{+1}(\underline{\rho})]^2 - [a^+_{+1}(\underline{\lambda})]^2\}$

Table 3.1 The correctly-orthonormalised monomials for the N = 2 states.

Note that only monomials with maximal L_z are given in Table 3.1.

The energy shift for the N = 2 $[\underline{56}, 0^+]$ supermultiplet is given by:

$$\Delta E_{[\underline{56},0^+]} = \frac{1}{4}\{{}_\rho\langle 0|\{[\underline{a}^+(\underline{\rho})]^2\}^+ U(\sqrt{2}\rho)[\underline{a}^+(\underline{\rho})]^2|0\rangle_\rho \; {}_\lambda\langle 0|0\rangle_\lambda$$

$$+ {}_\rho\langle 0|\{[\underline{a}^+(\underline{\rho})]^2\}^+ U(\sqrt{2}\rho)|0\rangle_\rho \; {}_\lambda\langle 0|[\underline{a}^+(\underline{\lambda})]^2|0\rangle_\lambda$$

$$+ {}_\rho\langle 0|U(\sqrt{2}\rho)[\underline{a}^+(\underline{\rho})]^2|0\rangle_\rho \; {}_\lambda\langle 0|\{[\underline{a}^+(\underline{\lambda})]^2\}^+|0\rangle_\lambda$$

$$+ {}_\rho\langle 0|U(\sqrt{2}\rho)|0\rangle_\rho \; {}_\lambda\langle 0|\{[\underline{a}^+(\underline{\lambda})]^2\}^+[\underline{a}^+(\underline{\lambda})]^2|0\rangle_\lambda\} \quad . \quad (3.5)$$

Clearly:

$${}_\lambda\langle 0|[\underline{a}^+(\underline{\lambda})]^2|0\rangle_\lambda = {}_\lambda\langle 0|\{[\underline{a}^+(\underline{\lambda})]^2\}^+|0\rangle_\lambda = 0 \quad (3.6)$$

and:

$${}_\lambda\langle 0|0\rangle_\lambda = 1, \quad (3.7)$$

whilst, by repeated use of the commutation relation:

$$[a_i(\underline{\lambda}), a_j^+(\underline{\lambda})] = \delta_{ij} \quad , \quad (3.8)$$

we readily find that:

$${}_\lambda\langle 0|\{[\underline{a}^+(\underline{\lambda})]^2\}^+[\underline{a}^+(\underline{\lambda})]^2|0\rangle_\lambda = 6 \quad (3.9)$$

so that:

$$\Delta E_{[\underline{56},0^+]} = \frac{3}{2}\,{}_\rho\langle 0|U(\sqrt{2}\rho)|0\rangle_\rho + \frac{1}{4}{}_\rho\langle 0|\{[\underline{a}^+(\underline{\rho})]^2\}^+ U(\sqrt{2}\rho)[\underline{a}^+(\underline{\rho})]^2|0\rangle_\rho \; . \quad (3.10)$$

The RHS of equation (3.10) can be expressed in terms of Gaussian moments of the perturbing potential, as noted by Gromes and Stamatescu (1976), and by Isgur and Karl (1979a). Thus, trivially:

$${}_\rho\langle 0|U(\sqrt{2}\rho)|0\rangle_\rho = \frac{\alpha^3}{\pi^{3/2}} \int d^3\underline{\rho}\, U(\sqrt{2}\rho) e^{-\alpha^2\rho^2} \quad , \quad (3.11)$$

where, as usual, $\alpha^4 = 3Km$, while, with just a little more work:

$$\frac{1}{4\rho}\langle 0|\{[\underline{a}^{+}(\underline{\rho})]^{2}\}^{+}U(\sqrt{2}\rho)[\underline{a}^{+}(\underline{\rho})]^{2}|0\rangle_{\rho} = \frac{\alpha^{7}}{\pi^{3/2}}\int d^{3}\underline{\rho}\ \rho^{4}U(\sqrt{2}\rho)e^{-\alpha^{2}\rho^{2}}$$

$$- \frac{3\alpha^{5}}{\pi^{3/2}}\int d^{3}\underline{\rho}\rho^{2}U(\sqrt{2}\rho)e^{-\alpha^{2}\rho^{2}} + \frac{9\alpha^{3}}{4\pi^{3/2}}\int d^{3}\underline{\rho}\ U(\sqrt{2}\rho)e^{-\alpha^{2}\rho^{2}}. \tag{3.12}$$

Isgur and Karl (1979a) define parameters a, b and c as follows:

$$a = \frac{3\alpha^{3}}{\pi^{3/2}}\int d^{3}\underline{\rho}\ U(\sqrt{2}\rho)e^{-\alpha^{2}\rho^{2}} \tag{3.13a}$$

$$b = \frac{3\alpha^{5}}{\pi^{3/2}}\int d^{3}\underline{\rho}\rho^{2}\ U(\sqrt{2}\rho)e^{-\alpha^{2}\rho^{2}} \tag{3.13b}$$

$$c = \frac{3\alpha^{7}}{\pi^{3/2}}\int d^{3}\underline{\rho}\rho^{4}\ U(\sqrt{2}\rho)e^{-\alpha^{2}\rho^{2}} \tag{3.13c}$$

yielding the result:

$$\Delta E_{[\underline{56},0^{+}]} = \frac{5}{4}a - b + \frac{1}{3}c \quad . \tag{3.14}$$

Thus, we may write:

$$E_{[\underline{56},0^{+}]} = E_{o} + 2\Omega - \Delta \tag{3.15}$$

where:

$$E_{o} = 3m + 3\omega + a \tag{3.16a}$$

$$\Omega = \omega - \frac{1}{2}a + \frac{1}{3}b \tag{3.16b}$$

$$\Delta = -\frac{5}{4}a + \frac{5}{3}b - \frac{1}{3}c\ . \tag{3.16c}$$

The remainder of the N = 2 results are readily obtained:

$$E_{[\underline{70},0^{+}]} = E_{o} + 2\Omega - \frac{1}{2}\Delta \tag{3.17}$$

$$E_{[\underline{56},2^{+}]} = E_{o} + 2\Omega - \frac{2}{5}\Delta \tag{3.18}$$

$$E_{[\underline{70},2^+]} = E_o + 2\Omega - \frac{1}{5}\Delta \qquad (3.19)$$

$$E_{[\underline{20},1^+]} = E_o + 2\Omega \qquad (3.20)$$

giving the pattern of splittings displayed in Figure 3.

We now consider the corresponding calculation for the eight N = 3 supermultiplets. The orthonormalised monomials are given in Table 3.2:

$[SU(6), L^P]$	Monomial
$[\underline{56}, 3^-]$	$\hat{\psi}^{(s)}_{33} = -\frac{1}{2\sqrt{6}}\{[a^+_{+1}(\underline{\lambda})]^3 - 3[a^+_{+1}(\underline{\rho})]^2 a^+_{+1}(\underline{\lambda})\}$
$[\underline{20}, 3^-]$	$\hat{\psi}^{(A)}_{33} = \frac{1}{2\sqrt{6}}\{[a^+_{+1}(\underline{\rho})]^3 - 3[a^+_{+1}(\underline{\lambda})]^2\, a^+_{+1}(\underline{\rho})\}$
$[\underline{70}, 3^-]$	$\hat{\psi}^{(M_\rho)}_{33} = \frac{1}{2\sqrt{2}}\{[a^+_{+1}(\underline{\rho})]^3 + [a^+_{+1}(\underline{\lambda})]^2\, a^+_{+1}(\underline{\rho})\}$
	$\hat{\psi}^{(M_\lambda)}_{33} = \frac{1}{2\sqrt{2}}\{[a^+_{+1}(\underline{\lambda})]^3 + [a^+_{+1}(\underline{\rho})]^2\, a^+_{+1}(\underline{\lambda})\}$
$[\underline{70}, 2^-]$	$\hat{\psi}^{(M_\rho)}_{22} = -\frac{1}{\sqrt{3}}\{a^+_{+1}(\underline{\lambda})a^+_o(\underline{\rho}) - a^+_{+1}(\underline{\rho})a^+_o(\underline{\lambda})\}\, a^+_{+1}(\underline{\lambda})$
	$\hat{\psi}^{(M_\lambda)}_{22} = \frac{1}{\sqrt{3}}\{a^+_{+1}(\underline{\lambda})a^+_o(\underline{\rho}) - a^+_{+1}(\underline{\rho})a^+_o(\underline{\lambda})\}\, a^+_{+1}(\underline{\rho})$
$[\underline{56}, 1^-]$	$\hat{\psi}^{(s)}_{11} = \frac{1}{2\sqrt{10}}\{([\underline{a}^+(\underline{\rho})]^2 - [\underline{a}^+(\underline{\lambda})]^2)a^+_{+1}(\underline{\lambda}) + 2[\underline{a}^+(\underline{\rho}).\underline{a}^+(\underline{\lambda})]\cdot \cdot a^+_{+1}(\underline{\rho})\}$
$[\underline{20}, 1^-]$	$\hat{\psi}^{(A)}_{11} = \frac{1}{2\sqrt{10}}\{([\underline{a}^+(\underline{\rho})]^2 - [\underline{a}^+(\underline{\lambda})]^2)a^+_{+1}(\underline{\rho}) - 2[\underline{a}^+(\underline{\rho})\cdot\underline{a}^+(\underline{\lambda})]a^+_{+1}(\underline{\lambda})\}$
$[\underline{70}, 1^-]$	$\hat{\psi}^{(M_\rho)}_{11} = \frac{1}{4\sqrt{5}}\{([\underline{a}^+(\underline{\rho})]^2 - 3[\underline{a}^+(\underline{\lambda})]^2)a^+_{+1}(\underline{\rho}) + 4[\underline{a}^+(\underline{\rho}).\underline{a}^+(\underline{\lambda})]a^+_{+1}(\underline{\lambda})$
	$\hat{\psi}^{(M_\lambda)}_{11} = \frac{1}{4\sqrt{5}}\{([\underline{a}^+(\underline{\lambda})]^2 - 3[\underline{a}^+(\underline{\rho})]^2)a^+_{+1}(\underline{\lambda}) + 4[\underline{a}^+(\underline{\rho}).\underline{a}^+(\underline{\lambda})]a^+_{+1}(\underline{\rho})$
$[\underline{70}, 1^-]$	$\hat{\psi}^{(M_\rho)}_{11} = \frac{1}{4}\{[\underline{a}^+(\underline{\rho})]^2 + [\underline{a}^+(\underline{\lambda})]^2\}a^+_{+1}(\underline{\rho})$
	$\hat{\psi}^{(M_\lambda)}_{11} = \frac{1}{4}\{[\underline{a}^+(\underline{\rho})]^2 + [\underline{a}^+(\underline{\lambda})]^2\}a^+_{+1}(\underline{\lambda})$

Table 3.2 The correctly-orthonormalised monomials for the N = 3 states

Note that only monomials with maximal L_z are given in Table 3.2, and that, for the two degenerate $[\underline{70}, 1^-]$ supermultiplets, we have taken the particular orthogonal combinations of the monomials that are given by equations (2.46a) - (2.46b) and (2.47a) - (2.47b).

As has been noted in the literature (Horgan 1976a, Dalitz et al. 1977b), three of the perturbed N = 3 supermultiplets depend only on the parameters of the $N \leq 2$ levels. Thus:

$$E_{[\underline{56}, 1^-]} = E_o + 3\Omega - \frac{11}{10}\Delta \tag{3.21}$$

$$E_{[\underline{70}, 2^-]} = E_o + 3\Omega - \frac{2}{5}\Delta \tag{3.22}$$

$$E_{[\underline{56}, 3^-]} = E_o + 3\Omega - \frac{3}{5}\Delta \ . \tag{3.23}$$

The remaining five N = 3 supermultiplets depend on a new parameter, d, where:

$$d = \frac{3\alpha^9}{\pi^{3/2}} \int d^3\underline{\rho}\ \rho^6\ U(\sqrt{2}\rho) e^{-\alpha^2\rho^2} \ . \tag{3.24}$$

If we define the quantity:

$$\delta = \frac{1}{2}b - \frac{2}{5}c + \frac{2}{35}d, \tag{3.25}$$

the remaining perturbed N = 3 energy levels may be written:

$$E_{[\underline{70}, 3^-]} = E_o + 3\Omega - \frac{7}{10}\Delta + \frac{1}{2}\delta \tag{3.26}$$

$$E_{[\underline{20}, 3^-]} = E_o + 3\Omega - \frac{2}{5}\Delta + \frac{1}{3}\delta \tag{3.27}$$

$$E_{[\underline{20}, 1^-]} = E_o + 3\Omega - \frac{2}{5}\Delta + \frac{7}{6}\delta \tag{3.28}$$

$$E_{[\underline{70}, 1^-]} = E_o + 3\Omega - \frac{29}{40}\Delta + \frac{7}{8}\delta \pm \pm \sqrt{\{[\frac{7}{8}(\delta - \frac{1}{15}\Delta)]^2 + \frac{1}{45}\Delta^2\}}, \tag{3.29}$$

with the two (previously-degenerate) $[\underline{70}, 1^-]$ supermultiplets mixed and split by the perturbation.

3.2 Phenomenological considerations

After inclusion of the hyperfine interaction, calculated perturbatively to lowest-order in H_{ij}^{hyp}, Isgur and Karl (Isgur 1980 and references therein) were able to obtain a reasonable phenomenological description of the N = 2 level with $E_o \simeq 1150$ MeV and $\Omega \simeq \Delta \simeq 440$ MeV. Using these values, we find that the mean mass of the non-strange sector of the N = 3 [56, 1^-] supermultiplet, given here by equation (3.21), is around 1985 MeV - close to the mass of the ΔD35 (1940) resonance at 1940 ± 30 MeV (Cutkosky 1980). Given the simplicity of the model and, in particular, our neglect of the hyperfine interaction, this is remarkably good agreement. Of course, the effect of including the hyperfine interaction on the prediction for the ΔD35 (1940) state remains to be seen and we examine this in detail in Chapter 6. Note also that, with $E_o \simeq 1150$ MeV and $\Omega \simeq \Delta \simeq 440$ MeV, the [56, 1^-] supermultiplet necessarily lies lowest of the three levels given by equations (3.21) - (3.23).

Dalitz et al. (1977b) have looked in detail at the question of the assignment of the ΔD35 (1940) state to the N = 3 [56, 1^-] supermultiplet. Instead of just looking at mean masses of supermultiplets in the harmonic oscillator quark model, they attempted to do better than that, and obtained a sum rule relating the mass of the ΔD35 of the N = 3 [56, 1^-] supermultiplet to masses of known Δ states which they assigned to the N = 0 [56, 0^+] and N = 2_Λ [56, 2^+] supermultiplets. Specifically, these authors quote the result:

$$M(\Delta D35) = \frac{3}{5}M(\Delta F37) + \frac{1}{15}M(\Delta P31) + \frac{5}{6}M(\Delta P33^*) - \frac{1}{2}M(\Delta P33) \quad (3.30)$$

relating the masses of the N = 3 [56, 1^-], the N = 2 [56, 0^+] and [56, 2^+], and the N = 0 [56, 0^+] supermultiplets. Identifying the

$\Delta F37$ (1930) and $\Delta P31$ (1940) as belonging to the N = 2 [56, 2^+] supermultiplet, and the $\Delta P33^*$(1690) as belonging to the N = 2 [56, 0^+] supermultiplet, they predict:

$$M(\Delta D35) = 2088 \pm 25 \text{ MeV}, \tag{3.31}$$

some 150 MeV higher in mass than the candidate observed by Cutkosky et al. (Cutkosky 1980). The sum rule in equation (3.30) is derived by performing a spin-average over Δ states within the N = 2 [56, 0^+] and [56, 2^+] supermultiplets, and the RHS of equation (3.30) is actually independent of the magnitude of the spin-orbit effects which these authors consider. In general, however, spin-orbit forces will be expected to mix the $\Delta D35$ states belonging to the N = 3 [56, 1^-] and [70, 2^-] supermultiplets: Dalitz and collaborators (1977b) estimate that such mixing will be small. At first sight, therefore, it seems that the sum rule in equation (3.30) provides a better, and more specific, test of the assignment of the $\Delta D35$ (1940) state to the N = 3 [56, 1^-] supermultiplet than our less ambitious procedure of estimating merely the mean non-strange supermultiplet mass using the parameters of the Isgur-Karl model. However, the whole analysis of Dalitz et al. (1977b) is dependent on the neglect of spin-tensor forces. Such tensor forces can mix Δ states of the same total angular momentum, J, within the N = 2 band and also, of course, they can mix the $\Delta D35$ states of the N = 3 [56, 1^-], [70, 2^-], [56, 3^-] and [70, 3^-] supermultiplets. Since the analysis of Isgur and Karl (Isgur 1980 and references therein) suggests that spin-tensor forces are indeed important in determining the masses and mixing of the individual states of $SU(6)_{flavour \times spin} \otimes O(3)$ supermultiplets, the status of the sum rule in equation (3.30) for the $\Delta D35$ (1940) is somewhat obscure. In fact, the detailed predictions of the Isgur-Karl

model for the N = 2 states (Isgur and Karl 1979a) indicate that the $\Delta P31$ (1940), classified by Dalitz et al. (1977b) as a pure N = 2 $[\underline{56}, 2^+]$state, is actually an almost complete mixture of N = 2 $[\underline{56}, 2^+]$ and $[\underline{70}, 0^+]$ basis states.

In view of all these uncertainties, we content ourselves for the present with examining the zeroth-order, non-strange, mean masses of the N = 3 supermultiplets in an attempt to gain a first indication of whether or not an assignment of the $\Delta D35$ (1940) to the N = 3 $[\underline{56}, 1^-]$ supermultiplet is at all plausible, postponing until Chapter 6 a detailed study of the $\Delta D35$ (1940) state within the framework of the Isgur-Karl model with due inclusion of hyperfine effects. In this respect, our present analysis is more akin to the earlier analysis of Horgan (1976a), who discussed such mean masses in the context of his $SU(6)_{flavour \times spin}$ mass fits. Horgan predicted the central mass value of the N = 3 $[\underline{56}, 1^-]$ supermultiplet to be around 2080 ± 50 MeV, about 100 MeV higher than our value for the mean mass of the non-strange sector. Given the fundamental differences of approach of the Isgur-Karl Hamiltonian, which includes an $SU(6)_{flavour \times spin}$-independent anharmonic perturbation, $U(r_{ij})$, together with spin-tensor interactions, and of Horgan, who introduces $SU(6)_{flavour \times spin}$-dependent anharmonic perturbations and does not include tensor forces, the two estimates are surprisingly close. In fact, the algebraic structure of our results for the N = 3 $[\underline{56}, 1^-]$, $[\underline{70}, 2^-]$ and $[\underline{56}, 3^-]$ supermultiplets may be obtained from the more general results of Horgan (1976a) by identifying the parameter a_4 (in Horgan's notation) with the (in principle, independent) parameter b_4. Phenomenologically, Horgan found the values (Horgan 1976a):

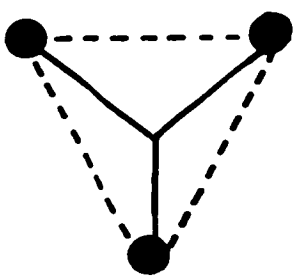
$$a_4 = 2000 \text{ MeV} \qquad (3.32a)$$

$$b_4 = 2100 \text{ MeV}, \qquad (3.32b)$$

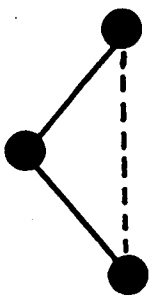
thus lending support to Isgur and Karl's (and our) less general treatment of anharmonic perturbations.

In contrast to these approaches, all based on the non-relativistic harmonic oscillator quark model, Cutkosky and Hendrick (1977a, 1977b) investigated the status of the N = 3 [$\underline{56}$, 1^-] supermultiplet in a quark model based on the relativistic string picture of confinement. Their model depicts a non-strange baryon as a system of three light quarks bound together by "strings" which represent gluon fields. The leading term in the potential energy of the baryon system is proportional to the minimum total string length needed to connect the three quarks in a given configuration. To represent phenomenologically the kinetic energy and momentum carried by the strings, a fourth constituent is added to the baryon: the monad, which is assumed to be a colourless, massless, neutral, scalar particle. Cutkosky and Hendrick refer to the baryon system composed of three quarks and a monad bound by string-length-potential interactions as the 4C model, and to the more conventional picture of a baryon as composed of three quarks interacting via string-length-potentials as the 3C model.

The three quarks need not lie along a single string in either the 3C or 4C models; a 3-string vertex is allowed and at least one string is attached to each quark. In the 3C model, the minimum-length configuration has three strings meeting at 120°, provided all interior angles of the quark triangle are less than 120°, as indicated in Figure 4(a); otherwise, there are two strings meeting at the obtuse-angle vertex, as in Fig. 4(b). In the 4C model, the minimum-length configuration always has two strings attached to the monad. One of these strings joins a single quark to the monad whilst the other string is part of a minimum-string-length system which joins the monad and the other two quarks, as in Figures 5(a)

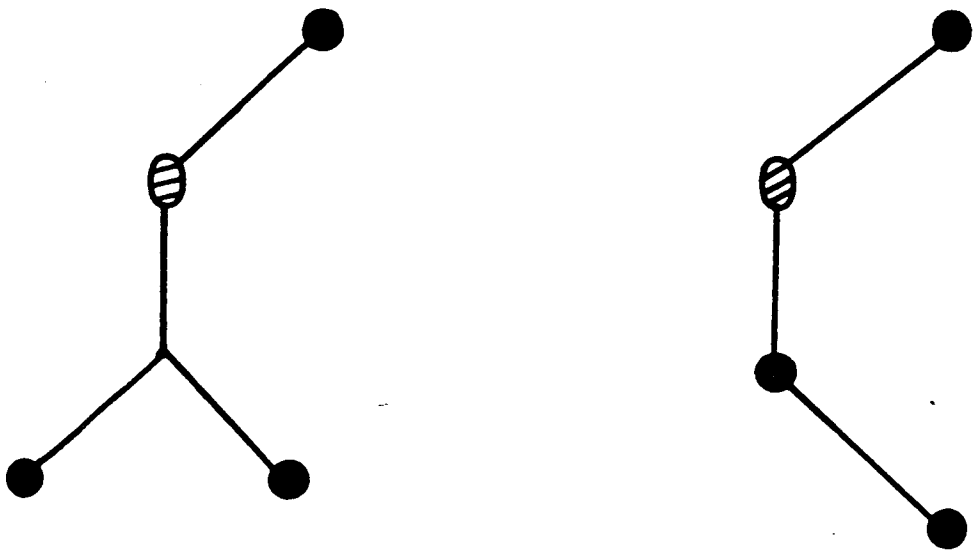


(a)



(b)

Figure 4 Minimum-string-length configurations in the 3C model



(a) (b)

Figure 5 Minimum-string-length configurations in the 4C model

and 5(b). In both Figures 4 and 5, the solid lines represent gluon strings.

Cutkosky and Hendrick (1977b) found that, in the 3C model, the mean mass of the N = 3 $[\underline{56}, 1^-]$ supermultiplet came out more than 200 MeV too high for the ΔD35 (1940) to be accommodated in this supermultiplet, even although the $[\underline{56}, 1^-]$ was found to be the lowest-lying of the N = 3 supermultiplets. These authors obtained qualitative agreement for the mean positions of other $SU(6)_{flavour \times spin} \otimes O(3)$ supermultiplets, but no attempt was made to perform detailed fits including hyperfine splittings. By way of contrast, they found that the 4C model provided an adequate picture of the N = 3 $[56, 1^-]$ supermultiplet in which the monad is in an orbital angular momentum L = 1 state relative to the three quarks in a symmetric ground state, in much the same way that the bag can be in an L = 1 state relative to the symmetric 3-quark state to give a low-lying $[56, 1^-]$ supermultiplet in the MIT bag model (Rebbi 1976 and references therein). In terms of their 4C model, therefore, the ΔD35 (1940) appears to be a good candidate for a new type of baryon, in which gluonic degrees of freedom are excited.

What conclusions can we come to? It is certainly true that the Isgur-Karl Hamiltonian has had more success than any other quark model in fitting the enormous amount of baryon data available for both negative- and positive-parity states to the N = 1 and N = 2 oscillator levels, respectively. In view of this, it seems entirely reasonable to take this model as the most reliable guide to the baryon spectrum, and examine the model's predictions for the N = 3 states. In the approximation of neglecting hyperfine interactions, and taking Isgur and Karl's parameters determined from their fit to the N = 2 level (Isgur 1980 and references therein), we predict a mean mass for

the non-strange sector of the $N = 3$ $[\underline{56}, 1^-]$ supermultiplet only 45 MeV above the quoted mass, 1940 ± 30 MeV, for the $\Delta D35$ (1940) (Cutkosky 1980). While it is clear that hyperfine interactions will mix and shift the masses of the Δ states at the $N = 3$ oscillator level (which effects we shall study in detail in Chapter 6), it seems impressive that such a constrained and simple model as the one described in this chapter can get so close to the mass of the $\Delta D35$ (1940), with no "fine tuning" of the three parameters E_o, Ω and Δ. We therefore conclude that, contrary to the claim of Cutkosky and Hendrick (1977a, 1977b), the $\Delta D35$ (1940) does not constitute unambiguous evidence for some new degree of freedom inside baryons. We shall see in Chapter 6 that a more detailed investigation including hyperfine effects serves to strengthen this conclusion.

CHAPTER 4

THE 2-BODY SYSTEM - A PROTOTYPE CALCULATION

4.1 Introduction

In this chapter, we consider the group-theoretic aspects of the simpler problem of a system of two equal-mass particles. We have in mind, for example, applications to quark-antiquark bound states, but, in view of the present unhappy state of meson spectroscopy, our main purpose must be pedagogic; the calculations presented here serve as prototypes for the 3-quark problem which we discuss in the next chapter.

The 2-body system possesses many interesting symmetries if the particles are bound by harmonic forces, notably the spatial symmetries associated with the dynamical (or degeneracy) group, U(3), and the spectrum-generating group, Sp(6,R), of the 3-dimensional oscillator. These groups can be used to classify the oscillator states in a manner akin to that described in §2.1 for the 3-quark system and which we review briefly in §4.2. In §4.3, we examine the effect on the energy spectrum of allowing a small, anharmonic potential in the Hamiltonian, using first-order perturbation theory and explicit oscillator wave-functions. We then re-interpret this effect as the breaking of the dynamical U(3) symmetry of the unperturbed system by examining explicitly the transformation properties of the perturbing potential under the subgroup, SU(3), of U(3) in §4.4 and by employing the spectrum-generating group, Sp(6,R), in §4.5 to re-derive the perturbed spectrum.

4.2 A brief review of the classification of the 3-dimensional oscillator states

A system of two quarks, each of mass m, bound by an harmonic potential, is described by a Hamiltonian of the form:

$$H = 2m + \sum_{i=1}^{2} \underline{p}_i^2/2m + \tfrac{1}{2}K\underline{r}_{12}^2 \quad , \tag{4.1}$$

where $\underline{r}_{12} = \underline{r}_1 - \underline{r}_2$ and K is a measure of the oscillator strength. We can rewrite H as:

$$H = 2m + \underline{p}^2/2\mu + \tfrac{1}{2}K\,\underline{r}_{12}^2 \quad , \tag{4.2}$$

where the reduced mass of the system $\mu = \frac{1}{2}m$, $\underline{p} = \mu\dot{\underline{r}}_{12}$ and we have neglected a term representing the kinetic energy of translation of the system as a whole.

The general excited state of this Hamiltonian is given by (Dicke and Wittke 1960):

$$\psi_{n\ell m} = N(L_-)^{\ell-m}(a_{+1}^+)^{\ell}(\underline{a}^+\cdot\underline{a}^+)^{(n-\ell)/2}\,\psi_{ooo} \quad , \tag{4.3}$$

where N is a normalisation constant, ψ_{ooo} represents the ground state of the 3-dimensional harmonic oscillator and the creation operators are defined by analogy with equations (2.1a) - (2.1b). Denoting $\underline{r}_{12}$ by $\underline{r}$ throughout the remainder of this chapter, and recalling that:

$$a_i^+ = \frac{\alpha}{\sqrt{2}}\,r_i - \frac{1}{\sqrt{2}\alpha}\,\frac{\partial}{\partial r_i} \quad , \quad i = x,y,z \tag{4.4}$$

where, in the present context, $\alpha^4 = 2K\mu$, we can easily generate the required wavefunctions. The "stretched" states of angular momentum, for which $m = \ell$, are displayed in Table 4.1 for the following values of the principal quantum number, n: 0,1,2,3 and 4:

n	"Stretched" wavefunction
0	$\psi_{ooo} = (\frac{\alpha^3}{\pi^{3/2}})^{\frac{1}{2}} e^{-\frac{1}{2}\alpha^2 \underline{r}^2}$
1	$\psi_{111} = -\alpha(\frac{\alpha^3}{\pi^{3/2}})^{\frac{1}{2}}(x + iy)e^{-\frac{1}{2}\alpha^2 \underline{r}^2}$
2	$\psi_{222} = \frac{\alpha^2}{\sqrt{2}} (\frac{\alpha^3}{\pi^{3/2}})^{\frac{1}{2}}(x + iy)^2 e^{-\frac{1}{2}\alpha^2 \underline{r}^2}$
	$\psi_{200} = \sqrt{\frac{2}{3}}\, \alpha^2(\frac{\alpha^3}{\pi^{3/2}})^{\frac{1}{2}}(\underline{r}^2 - \frac{3}{2}\alpha^{-2})e^{-\frac{1}{2}\alpha^2 \underline{r}^2}$
3	$\psi_{333} = -\frac{\alpha^3}{\sqrt{6}} (\frac{\alpha^3}{\pi^{3/2}})^{\frac{1}{2}} (x + iy)^3 e^{-\frac{1}{2}\alpha^2 \underline{r}^2}$
	$\psi_{311} = -\sqrt{\frac{2}{5}}\, \alpha^3 (\frac{\alpha^3}{\pi^{3/2}})^{\frac{1}{2}}(\underline{r}^2 - \frac{5}{2}\alpha^{-2})(x+iy)e^{-\frac{1}{2}\alpha^2 \underline{r}^2}$
4	$\psi_{444} = \frac{\alpha^4}{2\sqrt{6}} (\frac{\alpha^3}{\pi^{3/2}})^{\frac{1}{2}}(x + iy)^4 e^{-\frac{1}{2}\alpha^2 \underline{r}^2}$
	$\psi_{422} = \frac{\alpha^4}{\sqrt{7}} (\frac{\alpha^3}{\pi^{3/2}})^{\frac{1}{2}}(\underline{r}^2 - \frac{7}{2}\alpha^{-2})(x + iy)^2 e^{-\frac{1}{2}\alpha^2 \underline{r}^2}$
	$\psi_{400} = \frac{\alpha^4}{2\sqrt{30}} (\frac{\alpha^3}{\pi^{3/2}})^{\frac{1}{2}}(4\underline{r}^4 - 20\alpha^{-2}\underline{r}^2 + 15\alpha^{-4})e^{-\frac{1}{2}\alpha^2 \underline{r}^2}$

Table 4.1 "Stretched" wavefunctions of the 3-dimensional harmonic oscillator for n = 0,1,2,3 and 4.

We may immediately infer the following from the considerations of Chapter 2:

A. The nine bilinear quantities:

$$E_{ij} = \tfrac{1}{2}\{a_i^+, a_j\}\ , \qquad i,j = 1,2,3 \tag{4.5}$$

which satisfy the commutation relations:

$$[E_{ij}, E_{k\ell}] = \delta_{jk} E_{i\ell} - \delta_{i\ell} E_{kj} \qquad (4.6)$$

and each of which commutes with the oscillator Hamiltonian in equation (4.2), which in turn is proportional to their diagonal sum:

$$H - 2m = \omega \sum_{i=1}^{3} E_{ii} , \qquad \omega^2 = \frac{2K}{\mu} \qquad (4.7)$$

serve as the generators of the U(3) dynamical symmetry group of the 3-dimensional harmonic oscillator. The physical oscillator states corresponding to principal quantum number n (where n is associated with the U(1) subgroup of U(3) corresponding to the structure $U(3) \sim SU(3) \times U(1)$) transform as the symmetric irreducible representation {n,0,0} of U(3), whose dimension, $\frac{1}{2}(n+1)(n+2)$, is immediately recognisable as the degeneracy of the n^{th} level of the 3-dimensional harmonic oscillator.

B. The twenty-one bilinear quantities:

$$S_{AB} = \tfrac{1}{2}\{a_A, a_B\} , \qquad A,B = 1,2,\ldots, 6 \qquad (4.8)$$

where:

$$a_A = (\underline{a}, \underline{a}^+) \equiv a_{i\alpha} , \qquad i = 1,2,3 \qquad (4.9)$$

$$\alpha = 1 \text{ (annihilation operator)},$$

$$2 \text{ (creation operator)}$$

generate the spectrum-generating group, Sp(6,R), of the 3-dimensional harmonic oscillator. The physical oscillator states corresponding to principal quantum number, n, are associated with the symmetric irreducible representation <n,0,0> of Sp(6,R).

Fradkin (1965) provides us with a direct physical interpretation of the generators of the dynamical symmetry group U(3) in terms of the geometry of the corresponding classical 3-dimensional harmonic

oscillator. By noting that the eigenvalue problem for the quantum-mechanical 3-dimensional harmonic oscillator has a separable solution in terms of both Cartesian and spherical-polar coordinates, Fradkin is able to construct the conserved symmetric tensor operator:

$$\hat{A}_{ij} = \frac{1}{2m}\left[\hat{p}_i\hat{p}_j + (m\omega)^2\,\hat{r}_i\hat{r}_j\right], \qquad i = x,y,z \tag{4.10}$$

in addition to the orbital angular momentum vector operator:

$$\hat{\underline{L}} = \hat{\underline{r}} \times \hat{\underline{p}} \tag{4.11}$$

which is also conserved. Examination of the corresponding classical equation of motion:

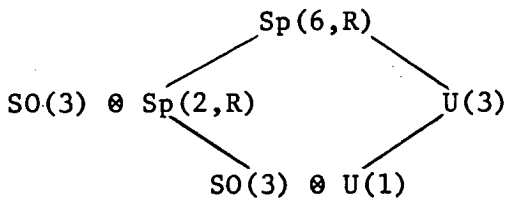
$$\dot{\underline{p}} + m\omega^2\underline{r} = 0, \tag{4.12}$$

whose general solution corresponds to an elliptical orbit, leads Fradkin to deduce that A_{ij}, the classical counterpart of the operator in equation (4.10), completely specifies the orientation of the elliptical orbit (in a manner analogous to the Runge-Lenz vector for the Kepler problem), whilst the orbital angular momentum vector, $\underline{L}$, serves to define the normal to the plane of the orbit. It is precisely because of the periodic nature of the motion, i.e. because the orbit is re-entrant, that one can construct conserved quantities which provide a complete description of the orbit. In the (non-relativistic) Kepler problem, for which the dynamical symmetry group is O(4), the orbit is also elliptical and there exists a conserved vector quantity, $\underline{A}$ (the Runge-Lenz vector) whose direction is from the force centre at one of the foci to the centre of the ellipse and whose magnitude is related to the eccentricity of the ellipse. In contrast, for the 3-dimensional harmonic oscillator problem, the force centre is located at the centre of the elliptical orbit so that the orientation of the major axis cannot

be specified by a vector with a unique sense. In this case, the necessary quantity to define the principal axes of the orbit is a symmetric tensor.

Observing that the trace of the quantum-mechanical symmetric tensor operator, $\hat{A}_{ij}$, is (neglecting the term 2m in equation (4.2)) just the Hamiltonian, Fradkin is able to make the connection with the corresponding dynamical symmetry group U(3) by demonstrating how the remaining five independent components of the traceless symmetric tensor operator derived from $\hat{A}_{ij}$ can be combined to form a new set of five independent operators which, taken together with the three independent components of the orbital angular momentum vector operator, satisfy the commutation relations characteristic of the generators of the subgroup, SU(3), of U(3).

For our part, we shall be concerned in this chapter with the following chain of subgroup embeddings:



Consider, firstly, the subgroup embedding $Sp(6,R) \supset SO(3) \otimes Sp(2,R)$. The irreducible representations of Sp(6,R) of interest to us, and their reductions under this embedding, are as follows:

$$n = 0: \quad \underline{1} \supset \underline{1} \otimes \underline{1} \tag{4.13a}$$

$$n = 1: \quad \underline{6} \supset \underline{3} \otimes \underline{2} \tag{4.13b}$$

$$n = 2: \quad \underline{21} \supset (\underline{5} \oplus \underline{1}) \otimes \underline{3} \oplus \underline{3} \otimes \underline{1} \tag{4.13c}$$

$$n = 3: \quad \underline{56} \supset (\underline{7} \oplus \underline{3}) \otimes \underline{4} \oplus (\underline{5} \oplus \underline{3}) \otimes \underline{2} \tag{4.13d}$$

$$n = 4: \quad \underline{126} \supset (\underline{9} \oplus \underline{5} \oplus \underline{1}) \otimes \underline{5} \oplus (\underline{7} \oplus \underline{5} \oplus \underline{3}) \otimes \underline{3} \oplus (\underline{5} \oplus \underline{1}) \otimes \underline{1} \tag{4.13e}$$

In addition, the physical oscillator states at level n must belong to the symmetric irreducible representation <n> of Sp(2,R), since the creation operators, a_i^+, satisfy Bose commutation relations. The generators of Sp(2,R), $S_{\alpha\beta}$, are given by

$$S_{\alpha\beta} = S_{AB} \cdot \delta^{ij} = S_{i\alpha j\beta} \cdot \delta^{ij} \quad . \tag{4.14}$$

As in Chapter 2, we may think of the $S_{\alpha\beta}$ as carrying a quantum number which we call "pseudospin". In particular, the third component of pseudospin is given by the eigenvalue of the operator:

$$\hat{P}_3 = \tfrac{1}{2}S_{12} = \tfrac{1}{2}S_{i1i2} = \tfrac{1}{4}\{a_{i1}, a_{i2}\} = \tfrac{1}{4}(\underline{a}.\underline{a}^+ + \underline{a}^+.\underline{a}), \tag{4.15}$$

where the eigenvalue of $\hat{P}_3$ acting on the tensor operator, T, is defined to be p_3 if:

$$[\hat{P}_3, T] = p_3 . T \quad . \tag{4.16}$$

We see from equation (4.15) that the operator $\hat{P}_3$ is simply proportional to the unperturbed oscillator Hamiltonian. Since:

$$\left[\hat{P}_3, (a_i^+)^n\right] = \frac{n}{2}(a_i^+)^n \tag{4.17}$$

we assign a pseudospin of $\frac{1}{2}n$ to the physical oscillator states with principal quantum number, n. This assignment of pseudospin is consistent with the dimensionality of the Sp(2,R) representation, <n>, which is equal to $n+1 = 2(\frac{n}{2}) + 1$.

Recalling that the physical states at level n necessarily belong to an (n+1)-dimensional irreducible representation of Sp(2,R), we can immediately deduce from equations (4.13a) - (4.13e) that the angular momentum content of the first five oscillator levels is:

$$n = 0 : \ell = 0 \tag{4.18a}$$

$$n = 1 : \ell = 1 \tag{4.18b}$$

$$n = 2 : \ell = 2 \oplus \ell = 0 \tag{4.18c}$$

$$n = 3 : \ell = 3 \oplus \ell = 1 \tag{4.18d}$$

$$n = 4 : \ell = 4 \oplus \ell = 2 \oplus \ell = 0 \ . \tag{4.18e}$$

Alternatively, we may consider the subgroup embedding $U(3) \supset SO(3) \otimes U(1)$, where the U(1) subgroup is generated by the operator $\hat{P}_3$ or, equivalently, by the number operator, $\hat{n}$. The irreducible representations of U(3) of interest to us, and their reductions under this embedding, are as follows:

$$n = 0 : \ \underline{1} \supset \underline{1} \otimes \underline{1}_0 \tag{4.19a}$$

$$n = 1 : \ \underline{3} \supset \underline{3} \otimes \underline{1}_1 \tag{4.19b}$$

$$n = 2 : \ \underline{6} \supset (\underline{5} \oplus \underline{1}) \otimes \underline{1}_2 \tag{4.19c}$$

$$n = 3 : \ \underline{10} \supset (\underline{7} \oplus \underline{3}) \otimes \underline{1}_3 \tag{4.19d}$$

$$n = 4 : \ \underline{15} \supset (\underline{9} \oplus \underline{5} \oplus \underline{1}) \otimes \underline{1}_4 \ . \tag{4.19e}$$

Thus, for $n = 0,1,2,3$ and 4, we deduce the angular momentum content given by equations (4.18a) - (4.18e). In general, the symmetric irreducible representation $\{n,0,0\}$ of U(3) has angular momentum content:

$$\{n,0,0\} \supset \ell = n \oplus \ell = n-2 \oplus \ell = n-4 \oplus \ldots\ldots \oplus \ell = \{{}^{1}_{0}\} \quad \text{if } n \text{ is } \{{}^{\text{odd}}_{\text{even}}\}. \tag{4.20}$$

4.3 Explicit wavefunction techniques

An harmonic potential is an unlikely candidate for the confining potential between two quarks in the real world. Accordingly, we choose to mimic the model of Isgur and Karl for baryons (Isgur 1980 and references therein) by writing the confining potential between two quarks, V(r), as:

$$V(r) = \tfrac{1}{2}K\underline{r}^2 + U(r) \tag{4.21}$$

so that the Hamiltonian in equation (4.2) becomes:

$$H = 2m + \underline{p}^2/2\mu + \tfrac{1}{2}K\underline{r}^2 + U(r) \quad . \tag{4.22}$$

In the absence of the anharmonic perturbation, $U(r)$, the eigenstates of H are just the oscillator eigenstates, $\psi_{n\ell m}$, given by equation (4.3), with energy:

$$E_n^{(0)} = 2m + (n + \tfrac{3}{2})\omega \quad . \qquad (\hbar = 1) \tag{4.23}$$

We calculate the change in energy, $\Delta E_{n,\ell}$, due to inclusion of the anharmonic term, $U(r)$, in H, using first-order perturbation theory with the oscillator eigenstates as basis states. The resulting energy of the state $\psi_{n\ell m}(r)$ is:

$$E_{n,\ell} = E_n^{(0)} + \Delta E_{n,\ell} = 2m + (n + \tfrac{3}{2})\omega + \langle n,\ell|U(r)|n,\ell\rangle . \tag{4.24}$$

The matrix elements $\langle n,\ell|U(r)|n,\ell\rangle$ which occur for $n = 0,1,2,3$ and 4 are given in Table 4.2.

The notation used in Table 4.2 is as follows:

$$A = \frac{\alpha^3}{\pi^{3/2}} \int d^3\underline{r}\; U(r) e^{-\alpha^2 r^2} \tag{4.25a}$$

$$B = \frac{\alpha^5}{\pi^{3/2}} \int d^3\underline{r}\; r^2\, U(r) e^{-\alpha^2 r^2} \tag{4.25b}$$

$$C = \frac{\alpha^7}{\pi^{3/2}} \int d^3\underline{r}\; r^4\, U(r) e^{-\alpha^2 r^2} \tag{4.25c}$$

$$D = \frac{\alpha^9}{\pi^{3/2}} \int d^3\underline{r}\; r^6\, U(r) e^{-\alpha^2 r^2} \tag{4.25d}$$

$$E = \frac{\alpha^{11}}{\pi^{3/2}} \int d^3\underline{r}\; r^8 U(r) e^{-\alpha^2 r^2} \quad . \tag{4.25e}$$

Note that in proceeding from one level to the next, only one extra parameter appears in the formulae in Table 4.2.

n	$\langle n,\ell\|U(r)\|n,\ell\rangle$
0	$\langle 0,0\|U(r)\|0,0\rangle = A$
1	$\langle 1,1\|U(r)\|1,1\rangle = \frac{2}{3}B$
2	$\langle 2,2\|U(r)\|2,2\rangle = \frac{4}{15}C$
	$\langle 2,0\|U(r)\|2,0\rangle = \frac{3}{2}A - 2B + \frac{2}{3}C$
3	$\langle 3,3\|U(r)\|3,3\rangle = \frac{8}{105}D$
	$\langle 3,1\|U(r)\|3,1\rangle = \frac{5}{3}B - \frac{4}{3}C + \frac{4}{15}D$
4	$\langle 4,4\|U(r)\|4,4\rangle = \frac{16}{945}E$
	$\langle 4,2\|U(r)\|4,2\rangle = \frac{14}{15}C - \frac{8}{15}D + \frac{8}{105}E$
	$\langle 4,0\|U(r)\|4,0\rangle = \frac{15}{8}A - 5B + \frac{13}{3}C - \frac{4}{3}D + \frac{2}{15}E$

Table 4.2 Matrix elements of the perturbing potential, U(r).

4.4 Transformation properties of U(r) under U(3) and SU(3)

It is easy to see, using the Wigner-Eckart theorem, that $\langle n,\ell|U(r)|n,\ell\rangle$ vanishes unless U(r) couples to the U(3) product $\{\overline{n,0,0}\} \otimes \{n,0,0\}$. It is not difficult to show, using Young diagram techniques (Lichtenberg 1970), that in U(3) (or SU(3)):

$$n = 0: \quad \underline{\bar{1}} \otimes \underline{1} = \underline{1} \tag{4.26a}$$

$$n = 1: \quad \underline{\bar{3}} \otimes \underline{3} = \underline{1} \oplus \underline{8} \tag{4.26b}$$

$$n = 2: \quad \underline{\bar{6}} \otimes \underline{6} = \underline{1} \oplus \underline{8} \oplus \underline{27} \tag{4.26c}$$

$$n = 3: \quad \underline{\overline{10}} \otimes \underline{10} = \underline{1} \oplus \underline{8} \oplus \underline{27} \oplus \underline{64} \tag{4.26d}$$

$$n = 4: \quad \underline{\overline{15}} \otimes \underline{15} = \underline{1} \oplus \underline{8} \oplus \underline{27} \oplus \underline{64} \oplus \underline{125}\ . \tag{4.26e}$$

In terms of dimensions, the general case reads:

$$\underline{\overline{\tfrac{1}{2}(n+1)(n+2)}} \otimes \underline{\tfrac{1}{2}(n+1)(n+2)} = \sum_{p=0}^{n} \underline{(p+1)^3} \tag{4.27}$$

We can simplify matters even further by noting that $U(r)$ is a central potential, and therefore must transform as a scalar under the rotation group, $SO(3)$. Thus, $U(r)$ can couple only to those irreducible representations of $SU(3)$ on the RHS of equations (4.26a)-(4.26e) which have an $\ell = 0$ component. These are:

$$\underline{1}, \quad \underline{27} \quad \text{and} \quad \underline{125} \; . \tag{4.28}$$

We turn now to the problem of constructing tensor operators which transform as the $\ell = 0$ component of the $SU(3)$ irreducible representations $\underline{1}$, $\underline{27}$ and $\underline{125}$, out of the creation and annihilation operators, a_i^+ and a_i, respectively. Introduce the notation $a^i \equiv a_i^+$ and recall that an irreducible tensor operator of $SU(3)$ denoted by:

$$T^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} \tag{4.29}$$

is necessarily symmetric in both sets of indices $\{i_1, i_2, \ldots, i_p\}$ and $\{j_1, j_2, \ldots, j_q\}$ and is necessarily traceless (Dalitz 1965).

We illustrate the method with a simple example, viz. the construction of the tensor operator transforming as $\underline{8}$ of $SU(3)$. The required tensor is necessarily of the form:

$$S^i_{\ j} = a^i a_j \; . \tag{4.30}$$

All that remains to do is to construct a traceless tensor, $T^i_{\ j}$, from $S^i_{\ j}$. Clearly:

$$T^i_{\ j} = S^i_{\ j} - \frac{1}{3} \delta^i_{\ j} . S^k_{\ k} \tag{4.31}$$

$$\text{i.e.} \quad T^i_{\ j} = a^i a_j - \frac{1}{3} (\underline{a}^+ . \underline{a}) \, \delta^i_{\ j} \; , \tag{4.32}$$

noting $a^k a_k = \underline{a}^+ . \underline{a}$. In Table 4.3, we have listed the irreducible tensor operators transforming as $\underline{1}$, $\underline{8}$, $\underline{27}$, $\underline{64}$ and $\underline{125}$, respectively, under $SU(3)$:

SU(3) irreducible representation	SU(3) irreducible tensor operator
$[\underline{1}]_{n=0}$	$T = 1$
$[\underline{8}]_{n=1}$	$T^i{}_j = a^i a_j - \frac{1}{3}(\underline{a}^+.\underline{a})\delta^i{}_j$
$[\underline{27}]_{n=2}$	$T^{ij}_{k\ell} = a^i a^j a_k a_\ell - \frac{1}{5}(\underline{a}^+.\underline{a} - 1)[\delta^i{}_k a^j a_\ell + 3 \text{ perms.}]$ $+ \frac{1}{5.4}(\underline{a}^+.\underline{a} - 1)(\underline{a}^+.\underline{a})[\delta^i{}_k \delta^j{}_\ell + 1 \text{ perm.}]$
$[\underline{64}]_{n=3}$	$T^{ijk}_{\ell mn} = a^i a^j a^k a_\ell a_m a_n - \frac{1}{7}(\underline{a}^+.\underline{a} - 2)[\delta^i{}_\ell a^j a^k a_m a_n + 8 \text{ perms.}]$ $+ \frac{1}{7.6}(\underline{a}^+.\underline{a} - 2)(\underline{a}^+.\underline{a} - 1)[\delta^i{}_\ell \delta^j{}_m a^k a_n + 17 \text{ perms.}]$ $- \frac{1}{7.6.5}(\underline{a}^+.\underline{a} - 2)(\underline{a}^+.\underline{a} - 1)(\underline{a}^+.\underline{a})[\delta^i{}_\ell \delta^j{}_m \delta^k{}_n + 5 \text{ perms.}]$
$[\underline{125}]_{n=4}$	$T^{ijk\ell}_{mnpq} = a^i a^j a^k a^\ell a_m a_n a_p a_q - \frac{1}{9}(\underline{a}^+.\underline{a} - 3) \times$ $\times [\delta^i{}_m a^j a^k a^\ell a_n a_p a_q + 15 \text{ perms.}]$ $+ \frac{1}{9.8}(\underline{a}^+.\underline{a} - 3)(\underline{a}^+.\underline{a} - 2)[\delta^i{}_m \delta^j{}_n a^k a^\ell a_p a_q + 17 \text{ perms.}]$ $- \frac{1}{9.8.7}(\underline{a}^+.\underline{a} - 3)(\underline{a}^+.\underline{a} - 2)(\underline{a}^+.\underline{a} - 1) \times$ $\times [\delta^i{}_m \delta^j{}_n \delta^k{}_p a^\ell a_q + 95 \text{ perms.}]$ $+ \frac{1}{9.8.7.6}(\underline{a}^+.\underline{a} - 3)(\underline{a}^+.\underline{a} - 2)(\underline{a}^+.\underline{a} - 1)(\underline{a}^+.\underline{a}) \times$ $\times [\delta^i{}_m \delta^j{}_n \delta^k{}_p \delta^\ell{}_q + 23 \text{ perms.}]$

TABLE 4.3 Irreducible tensor operators of SU(3).

One note of caution, however: the total number of tensor indices before we perform any contractions must equal 2n, where n is the principal quantum number of the relevant level. Thus, strictly speaking, the tensor operator $T^i_{\ j}$ in equation (4.32) represents the $\underline{8}$ in the SU(3) product:

$$\underline{\bar{3}} \otimes \underline{3} = \underline{1} \oplus \underline{8} \quad . \tag{4.33}$$

In order to construct the tensor operator transforming as $\underline{8}$ in e.g. the SU(3) product:

$$\underline{\bar{6}} \otimes \underline{6} = \underline{1} \oplus \underline{8} \oplus \underline{27} \quad , \tag{4.34}$$

we must consider a tensor of the form:

$$S^{i_1 i_2}_{j_1 j_2} = a^{i_1} a^{i_2} a_{j_1} a_{j_2} \quad . \tag{4.35}$$

This is trivially symmetric in $\{i_1,i_2\}$ and $\{j_1,j_2\}$ because of the commutation relations:

$$[a^i,a_j] = -\delta^i_{\ j} \tag{4.36a}$$

$$[a^i,a^j] = 0; \quad [a_i,a_j] = 0 \quad . \tag{4.36b}$$

We must construct from $S^{i_1 i_2}_{j_1 j_2}$ a tensor $T^i_{\ j}$ which is traceless. After some labour, we find the required tensor is:

$$T^i_{\ j} = (\underline{a}^+.\underline{a} - 1)\,[a^i a_j - \tfrac{1}{3}(\underline{a}^+.\underline{a})\delta^i_{\ j}] \quad . \tag{4.37}$$

Note that this is just the $T^i_{\ j}$ of equation (4.32) multiplied by a function of $\underline{a}^+.\underline{a} \equiv \hat{n}$, the number operator.

The general form of expression for the tensor operators $[\underline{1}]_n$, $[\underline{8}]_n$, $[\underline{27}]_n$, $[\underline{64}]_n$ and $[\underline{125}]_n$, where $[\underline{1}]_n$, $[\underline{8}]_n$, ... denotes the tensor operator transforming as $\underline{1}$, $\underline{8}$, ... in the outer product $\overline{\{n,0\}} \otimes \{n,0\}$ of SU(3), is as follows:

$$[\underline{1}]_{n=r} = \hat{n}(\hat{n}-1) \ldots\ldots (\hat{n} - r + 1)\ [\underline{1}]_{n=0} \tag{4.38a}$$

$$[\underline{8}]_{n=r} = (\hat{n}-1)(\hat{n}-2) \ldots\ldots(\hat{n} - r + 1)\ [\underline{8}]_{n=1} \tag{4.38b}$$

$$[\underline{27}]_{n=r} = (\hat{n}-2)(\hat{n}-3)\ldots\ldots (\hat{n} - r + 1)\ [\underline{27}]_{n=2} \tag{4.38c}$$

$$[\underline{64}]_{n=r} = (\hat{n}-3)(\hat{n}-4)\ldots\ldots (\hat{n} - r + 1)\ [\underline{64}]_{n=3} \tag{4.38d}$$

$$[\underline{125}]_{n=r} = (\hat{n}-4)(\hat{n}-5)\ldots\ldots (\hat{n} - r + 1)\ [\underline{125}]_{n=4} \ . \tag{4.38e}$$

Note that in equations (4.38a) - (4.38e), and throughout the remainder of this chapter, we make repeated use of the fact that:

$$\underline{a}^{+}.\underline{a} \equiv \hat{n} \ . \tag{4.39}$$

Of more immediate interest to us are the $\ell = 0$ components of the tensor operators transforming as $\underline{1}$ (trivially, the number 1), $\underline{27}$ and $\underline{125}$. These we obtain by performing further contractions, e.g. in the case of $\underline{27}$ (appearing in the SU(3) product $\overline{\underline{6}} \otimes \underline{6} = \underline{1} \oplus \underline{8} \oplus \underline{27}$), with the corresponding irreducible tensor operator given by (c.f. Table 4.3):

$$T^{i_1 i_2}_{j_1 j_2} \ , \tag{4.40}$$

we contract i_1 with i_2 and j_1 with j_2, to obtain:

$$T^{ii}_{jj} = a^i a^i a_j a_j - \tfrac{1}{2}(\underline{a}^{+}.\underline{a})(\underline{a}^{+}.\underline{a} - 1) \ . \tag{4.41}$$

We can rewrite this in terms of $\hat{n}$ and $\hat{\underline{\ell}}^2$ where:

$$\hat{\underline{\ell}} \equiv \text{angular momentum operator} = i(\underline{a}\times\underline{a}^{+}). \tag{4.42}$$

After some labour, we obtain

$$T^{\ell=0}_{\underline{27}} = T^{ii}_{jj} = \tfrac{1}{2}\hat{n}(\hat{n}+3) - \hat{\underline{\ell}}^2 \ . \tag{4.43}$$

Similarly, we find:

$$T^{\ell=0}_{\underline{125}} = T^{iijj}_{kk\ell\ell} = a^i a^i a^j a^j a_k a_k a_\ell a_\ell + \frac{8}{27}(\underline{a}^+.\underline{a}-2)(\underline{a}^+.\underline{a}-3)a^i a^i a_j a_j$$

$$+ \frac{49}{54}(\underline{a}^+.\underline{a})(\underline{a}^+.\underline{a}-1)(\underline{a}^+.\underline{a}-2)(\underline{a}^+.\underline{a}-3) \qquad (4.44)$$

which reduces to:

$$T^{\ell=0}_{\underline{125}} = \{\hat{n}(\hat{n}+1) - \hat{\underline{\ell}}^2\}^2 + \{\frac{8}{27}(\hat{n}-2)(\hat{n}-3) - 2(2\hat{n}-1)\}\{\hat{n}(\hat{n}+1) - \hat{\underline{\ell}}^2\}$$

$$+ \frac{49}{54}\hat{n}(\hat{n}-1)(\hat{n}-2)(\hat{n}-3) \; . \qquad (4.45)$$

We are now in a position to calculate matrix elements of the perturbing potential:

$$\Delta E_{n,\ell} = \langle n,\ell|U(r)|n,\ell\rangle \; . \qquad (4.46)$$

We consider the cases $n = 0,1,2,3$ and 4 and, in what follows, β, γ, λ, μ, θ, ϕ, ρ, σ and τ denote undetermined reduced matrix elements:

$$n = 0 : \langle 0,0|U(r)|0,0\rangle = \beta\langle 0,0|[T^{\ell=0}_{\underline{1}}]_{n=0}|0,0\rangle = \beta \qquad (4.47)$$

$$n = 1 : \langle 1,1|U(r)|1,1\rangle = \gamma\langle 1,1|\,[T^{\ell=0}_{\underline{1}}]_{n=1}|1,1\rangle \qquad (4.48)$$

$$= \gamma\langle 1,1|1.[T^{\ell=0}_{\underline{1}}]_{n=0}|1,1\rangle = \gamma \qquad (4.49)$$

$$n = 2 : \langle 2,\ell|U(r)|2,\ell\rangle = \lambda\langle 2,\ell|[T^{\ell=0}_{\underline{1}}]_{n=2}|2,\ell\rangle +$$

$$+ \mu\langle 2,\ell|[T^{\ell=0}_{\underline{27}}]_{n=2}|2,\ell\rangle \; , \qquad (4.50)$$

where $\ell = 0,2$

$$= \lambda\langle 2,\ell|2.1[T^{\ell=0}_{\underline{1}}]_{n=0}|2,\ell\rangle +$$

$$+\mu\langle 2,\ell|[T^{\ell=0}_{\underline{27}}]_{n=2}|2,\ell\rangle \qquad (4.51)$$

so that :

$$\langle 2,2|U(r)|2,2\rangle = 2\lambda - \mu \qquad (4.52a)$$

$$\langle 2,0|U(r)|2,0\rangle = 2\lambda + 5\mu \; . \qquad (4.52b)$$

$$n = 3: \quad \langle 3,\ell|U(r)|3,\ell\rangle = \theta\langle 3,\ell|[T_{\underline{1}}^{\ell=0}]_{n=3}|3,\ell\rangle +$$

$$+ \phi\langle 3,\ell|[T_{\underline{27}}^{\ell=0}]_{n=3}|3,\ell\rangle , \qquad (4.53)$$

where $\ell = 1,3$

$$= \theta\langle 3,\ell|3.2.1[T_{\underline{1}}^{\ell=0}]_{n=0}|3,\ell\rangle +$$

$$+ \phi\langle 3,\ell|1.[T_{\underline{27}}^{\ell=0}]_{n=2}|3,\ell\rangle \qquad (4.54)$$

so that:

$$\langle 3,3|U(r)|3,3\rangle = 6\theta - 3\phi \qquad (4.55a)$$

$$\langle 3,1|U(r)|3,1\rangle = 6\theta + 7\phi \ . \qquad (4.55b)$$

$$n = 4: \quad \langle 4,\ell|U(r)|4,\ell\rangle = \rho\langle 4,\ell|[T_{\underline{1}}^{\ell=0}]_{n=4}|4,\ell\rangle + \sigma\langle 4,\ell|[T_{\underline{27}}^{\ell=0}]_{n=4}|4,\ell\rangle$$

$$+ \tau\langle 4,\ell|[T_{\underline{125}}^{\ell=0}]_{n=4}|4,\ell\rangle , \qquad (4.56)$$

where $\ell = 0,2,4$

$$= \rho\langle 4,\ell|4.3.2.1[T_{\underline{1}}^{\ell=0}]_{n=0}|4,\ell\rangle +$$

$$+ \sigma\langle 4,\ell|2.1[T_{\underline{27}}^{\ell=0}]_{n=2}|4,\ell\rangle$$

$$+ \tau\langle 4,\ell|[T_{\underline{125}}^{\ell=0}]_{n=4}|4,\ell\rangle \qquad (4.57)$$

so that $\langle 4,4|U(r)|4,4\rangle = 24\rho - 12\sigma + \frac{196}{9}\tau \qquad (4.58a)$

$$\langle 4,2|U(r)|4,2\rangle = 24\rho + 16\sigma + \frac{812}{27}\tau \qquad (4.58b)$$

$$\langle 4,0|U(r)|4,0\rangle = 24\rho + 28\sigma + \frac{4148}{27}\tau \ . \qquad (4.58c)$$

It is at this point that the limitations of using the dynamical symmetry group, U(3), to predict the spectrum of excited states are fully revealed to us. We can only compare matrix elements of the perturbation U(r) sandwiched between states having the same value of n,

the principal quantum number, i.e. between states belonging to the same irreducible representation of SU(3). We have no way of deciding how the reduced matrix elements which appear in the $n = 2$ case are related to the reduced matrix elements which appear in the $n = 3$ case, for example. Thus, it is not possible to make a direct comparison between the results obtained for $\Delta E_{n,\ell}$ in §4.3 using explicit wavefunction techniques and those obtained from a group theory point of view via the dynamical symmetry group, U(3). However, for $n \geq 4$, we can contrive some sort of check on our results, e.g. for $n = 4$:

$$\text{Define} \quad \Delta^{p}_{q} = \langle 4,p|U(r)|4,p\rangle - \langle 4,q|U(r)|4,q\rangle \quad . \tag{4.59}$$

It serves as a useful check on our results that both the explicit calculations and the group theory considerations predict that the ratio:

$$\frac{10\Delta^{2}_{4} - 7\Delta^{o}_{4}}{3\Delta^{o}_{4} - 10\Delta^{o}_{2}} = 1 \quad . \tag{4.60}$$

4.5 Spectrum-generating group, Sp(6,R) of the 3-dimensional harmonic oscillator

In this section, we re-address the problem of computing group-theoretically the effect on the oscillator energy levels of switching on the anharmonic perturbation, $U(r)$. We are interested in calculating the matrix elements:

$$\Delta E_{n,\ell} = \langle n,\ell|U(r)|n,\ell\rangle \quad . \tag{4.61}$$

Since the states $\{|n,\ell\rangle : \ell = n, n-2, \ldots\}$ are associated with the irreducible representation $\langle n,0,0\rangle$ of $Sp(6,R)$, the Wigner-Eckart

theorem implies that $U(r)$ had better couple to the $Sp(6,R)$ product $\langle n,0,0\rangle \otimes \langle n,0,0\rangle$; otherwise, $\Delta E_{n,\ell} = 0$.

We assume that $U(r)$ admits a power series expansion in powers of $\underline{r}^2$:

$$U(r) = \sum_j \beta^{(2j)} (\underline{r}^2)^j \quad , \tag{4.62}$$

where the strengths of each order, j, are governed by the values of the distinct coupling constants, $\beta^{(2j)}$. The justification for this form is two-fold. Firstly, a very large class of potentials may be expected to have an expansion as in equation (4.62), which is consistent with a perturbation scheme based on the dominant harmonic term having $j = 1$. Secondly, without invoking non-linear realisations, the $Sp(6,R)$ algebra of the operators S_{AB} in equation (4.8) is associated with a Fock space in which only multinomials, bilinear in a_A, have a well-defined action. If we write equation (4.62) in the form:

$$U(r) = \sum_j \beta^{(2j)} U^{(2j)} \quad , \tag{4.63}$$

then the form of equation (4.62) ensures that each term $U^{(2j)}$ transforms as a component of the totally symmetric representation $\langle 2j,0,0\rangle$ of $Sp(6,R)$. Hence $U(r)$ transforms as a sum of symmetric irreducible representations of the group $Sp(6,R)$, so that, at the oscillator level, n, the splitting pattern is controlled by tensor operators transforming under $Sp(6,R)$ as:

$$\langle 2n,0,0\rangle \oplus \langle 2n-2,0,0\rangle \oplus \ldots\ldots\ldots \oplus \langle 4,0,0\rangle \oplus \langle 2,0,0\rangle \oplus \langle 0,0,0\rangle . \tag{4.64}$$

Equation (4.64) explains why only one extra parameter appears in the formulae in Table 4.2 for $\Delta E_{n,\ell}$ as we pass from one level to the

next. As n increases by one unit, one additional tensor operator enters equation (4.64) and contributes to the splitting pattern. For example, at the $n = 1$ level, only tensor operators transforming as $\underline{1}$ $(\langle 0,0,0\rangle)$ or $\underline{21}$ $(\langle 2,0,0\rangle)$ under $Sp(6,R)$ can contribute, whilst, at the $n = 2$ level, the only contributions come from operators transforming as $\underline{1}$, $\underline{21}$ or $\underline{126}$ $(\langle 4,0,0\rangle)$ under $Sp(6,R)$.

The $SO(3)$ transformation properties of $U^{(2j)}$ are easy to deduce. Since $U(r)$ is a central potential, $U^{(2j)}$ must transform as a scalar, $\underline{1}$, under $SO(3)$, for all j.

In order to identify the $Sp(2,R)$ content of $U^{(2j)}$, it is sufficient to note that, since $\underline{r} = \frac{1}{\sqrt{2}\alpha}[\underline{a}(\underline{r}) + \underline{a}^{+}(\underline{r})]$, the expansion of $U^{(2j)}$ yields a monomial of degree $2j$ which is totally symmetric under permutations of the $Sp(2,R)$ indices. It follows that $U^{(2j)}$ transforms under $Sp(2,R)$ as a sum of components of the symmetric representation, $\langle 2j\rangle$. This corresponds to the statement that $U^{(2j)}$ has pseudospin j. It should be noted that not every component of $U^{(2j)}$ contributes to the values of the energy levels. The only effective component of $U^{(2j)}$ is necessarily a $U(1)$ singlet, which ensures that its third component of pseudospin is zero. Any other value merely gives a contribution to the matrix element $\langle n,\ell|U^{(2j)}|n,\ell\rangle$ which automatically vanishes. In terms of the monomial constituting $U^{(2j)}$, this condition corresponds to the fact that, besides being symmetric under the interchange of creation and annihilation operators, it is of the same degree in these operators taken separately.

We only concern ourselves in what follows with the cases $n = 0,1$ and 2. We shall need the following $Sp(6,R)$ outer products:

$$n = 0: \quad \underline{1} \otimes \underline{1} = \underline{1} \tag{4.65a}$$

$$n = 1: \quad \underline{6} \otimes \underline{6} = \underline{1} \oplus \underline{14} \oplus \underline{21} \tag{4.65b}$$

$$n = 2: \quad \underline{21} \otimes \underline{21} = 1 \oplus \underline{14} \oplus \underline{21} \oplus \underline{90} \oplus \underline{126} \oplus \underline{189}\ . \tag{4.65c}$$

In addition, the following $Sp(6,R) \supset SO(3) \otimes Sp(2,R)$ reductions will prove useful:

$$\underline{1} \supset \underline{1} \otimes \underline{1} \tag{4.66a}$$

$$\underline{21} \supset (\underline{5} \oplus \underline{1}) \otimes \underline{3} \oplus \underline{3} \otimes \underline{1} \tag{4.66b}$$

$$\underline{126} \supset (\underline{9} \oplus \underline{5} \oplus \underline{1}) \otimes \underline{5} \oplus (\underline{7} \oplus \underline{5} \oplus \underline{3}) \otimes \underline{3} \oplus (\underline{5} \oplus \underline{1}) \otimes \underline{1}\ . \tag{4.66c}$$

We address each of the levels $n = 0$, 1 and 2, in turn:

$n = 0$: The tensor operator which contributes must transform as $(\underline{1};\underline{1};\underline{1})$ under $(Sp(6,R); SO(3); Sp(2,R))$ and have third component of pseudospin = 0. The Wigner-Eckart theorem then gives:

$$\Delta E_{0,0} = \langle 0,0|U(r)|0,0\rangle = \langle \| 0 \| \rangle .1, \tag{4.67}$$

where $\langle \| 0 \| \rangle$ denotes the appropriate reduced matrix element.

$n = 1$: In addition to the operator which contributes at the $n = 0$ level, the only other operator which can contribute must transform as $(\underline{21};\underline{1};\underline{3})$ under $(Sp(6,R); SO(3); Sp(2,R))$ and have third component of pseudospin = 0. This operator is proportional to the unperturbed oscillator Hamiltonian or, equivalently, the number operator, $\hat{n}$. The Wigner-Eckart theorem then gives:

$$\Delta E_{1,1} = \langle 1,1|U(r)|1,1\rangle = \langle \| 0 \| \rangle .1 + \langle \| 1 \| \rangle .1 \tag{4.68}$$

where $\langle \| 0 \| \rangle$ is the <u>same</u> reduced matrix element as appears in the $n = 0$ case, by virtue of the fact that $Sp(6,R)$ is the spectrum-generating group for the 3-dimensional oscillator - we shall expand

on this point in the next chapter - and $\langle\|1\|\rangle$ is a new reduced matrix element required by the Wigner-Eckart theorem.

n = 2: In addition to the two operators which contribute at the n = 1 level, there is one further operator which contributes. This transforms as $(\underline{126};\underline{1};\underline{5})$ under (Sp(6,R); SO(3); Sp(2,R)) and necessarily has vanishing third component of pseudospin. In order to construct this operator, we must first construct the tensor operator which transforms as $\underline{126}$ under Sp(6,R). This is clearly:

$$X^{\underline{126}}_{ABCD} = a_A a_B a_C a_D + 23 \text{ permutations} \tag{4.69}$$

$$\text{i.e. } X^{\underline{126}}_{ABCD} = 4[\{S_{AB}, S_{CD}\} + \{S_{AC}, S_{BD}\} + \{S_{AD}, S_{BC}\}] . \tag{4.70}$$

The next step is to identify the component of this tensor which transforms as $\underline{1} \otimes \underline{5}$ under $SO(3) \otimes Sp(2,R)$. Using the notation:

$$T^{Sp(6,R)}_{SO(3)\otimes Sp(2,R)}, \tag{4.71}$$

we deduce:

$$T^{\underline{126}}_{\underline{1}\otimes\underline{5}} = X^{\underline{126}}_{i\alpha i\beta j\gamma j\delta} + X^{\underline{126}}_{i\alpha i\gamma j\beta j\delta} + X^{\underline{126}}_{i\alpha i\delta j\beta j\gamma} . \tag{4.72}$$

In particular, the component of $T^{\underline{126}}_{\underline{1}\otimes\underline{5}}$ with vanishing third component of pseudospin is:

$$(T^{\underline{126}}_{\underline{1}\otimes\underline{5}})_{P_3=0} = X^{\underline{126}}_{i1i1j2j2} + 2X^{\underline{126}}_{i1i2j1j2} . \tag{4.73}$$

Omitting terms with $P_3 \neq 0$, we find, on the other hand:

$$4a^4(\underline{r}^2)^2 = \underline{a}^2(\underline{a}^+)^2 + (\underline{a}^+)^2\underline{a}^2 + (\underline{a}.\underline{a}^+ + \underline{a}^+.\underline{a})^2 \tag{4.74}$$

$$\text{i.e } 4a^4(\underline{r}^2)^2 = \frac{1}{12}\left[X^{\underline{126}}_{i1i1j2j2} + 2X^{\underline{126}}_{i1i2j1j2}\right] \tag{4.75}$$

so that we deduce:

$$(T^{\underline{126}})_{\underline{10}\otimes\underline{5}\ P_3=0} = 48\alpha^4(\underline{r}^2)^2 \ . \tag{4.76}$$

It is worthwhile pausing for a moment at this stage to consider the reduction $Sp(6,R) \supset U(3)$. We find, in particular, that:

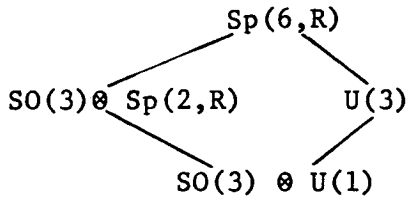
$$\underline{1} \supset \underline{1} \tag{4.77a}$$

$$\underline{21} \supset \underline{1} \oplus \underline{6} \oplus \overline{\underline{6}} \oplus \underline{8} \tag{4.77b}$$

$$\underline{126} \supset \underline{1} \oplus \underline{6} \oplus \overline{\underline{6}} \oplus \underline{8} \oplus \underline{15} \oplus \overline{\underline{15}} \oplus \underline{24} \oplus \overline{\underline{24}} \oplus \underline{27} \ . \tag{4.77c}$$

The only irreducible representations of U(3) occurring on the RHS of equations (4.77a) - (4.77c), which contain an $\ell = 0$, $P_3 = 0$ component, are $\underline{1}$ and $\underline{27}$, in agreement with our conclusions from §4.4 that the only operators contributing to $\Delta E_{2,\ell}$ necessarily transform as $\underline{1}$ or $\underline{27}$ under SU(3) or U(3). Further, we are now in a position to understand the limitations of using the dynamical symmetry group, U(3), to infer the pattern of splittings. We know that the tensor operators responsible for the breaking of the dynamical U(3) symmetry of the $n = 2$ level of the unperturbed system must transform as $\underline{1}$ or $\underline{27}$ under SU(3) or U(3). It is clear from equations (4.77a) - (4.77c) that the symmetry-breaking operator transforming as $\underline{1}$ under U(3) will be a linear combination of the U(3) singlets occurring in the reduction of $\underline{1}$, $\underline{21}$ and $\underline{126}$ of Sp(6,R), whilst the tensor operator transforming as a U(3) singlet and responsible for the symmetry-breaking at the $n = 1$ level will be a different linear combination of the U(3) singlets occurring in the reduction of $\underline{1}$ and $\underline{21}$ of Sp(6,R). Since we have not attempted to determine precisely what these linear combinations are, we cannot deduce any relation between the reduced matrix elements γ, λ and μ occurring in equations (4.48) - (4.52b).

The final stage of the calculation is to construct expectation values of $(T^{\underline{\underline{126}}}_{\underline{\underline{105}}})_{P_3=0}$ by an explicit algebraic formula. In order to achieve this, we must consider the full subgroup labelling chain:



which includes the reduction $U(3) \supset SO(3) \otimes U(1)$, the U(1) being generated by $\hat{P}_3$, or, equivalently, by $\hat{n}$.

We assume that we can expand $(T^{\underline{\underline{126}}}_{\underline{\underline{105}}})_{P_3=0}$ in terms of the quadratic Casimir invariants of Sp(6,R) and the subgroups of Sp(6,R) which we have been considering. We set:

$$\tfrac{1}{4}(T^{\underline{\underline{126}}}_{\underline{\underline{105}}})_{P_3=0} = a\,C_2\{1\} + bC_2\{3\} + cC_2\langle 2\rangle + dC_2\langle 6\rangle + eC_2[3], \tag{4.78}$$

where $\begin{Bmatrix} C_2\{n\} \\ C_2\langle n\rangle \\ C_2[n] \end{Bmatrix}$ represents the quadratic Casimir invariant operator of $\begin{Bmatrix} U(n) \\ Sp(n) \\ SO(n) \end{Bmatrix}$. At this stage a, b, c, d and e are unknown coefficients which we must determine. We do so by considering five matrix elements of $\tfrac{1}{4}(T^{\underline{\underline{126}}}_{\underline{\underline{105}}})_{P_3=0}$ which we know from the Wigner-Eckart theorem must vanish. The five states from which we form matrix elements of $\tfrac{1}{4}(T^{\underline{\underline{126}}}_{\underline{\underline{105}}})_{P_3=0}$ have the following transformation properties under $(Sp(6,R);\ SO(3);\ Sp(2,R);\ U(3);\ U(1)_{P_3})$:

$$(\underline{6}\ ;\ \underline{3}\ ;\ \underline{2}\ ;\ \underline{3}\ ;\ \underline{1}_1) \tag{4.79a}$$

$$(\underline{14};\ \underline{3}\ ;\ \underline{3}\ ;\ \underline{\bar{3}}\ ;\ \underline{1}_2) \tag{4.79b}$$

$$(\underline{14};\ \underline{3}\ ;\ \underline{3}\ ;\ \underline{8}\ ;\ \underline{1}_o) \tag{4.79c}$$

$$(\underline{14};\ \underline{5}\ ;\ \underline{1}\ ;\ \underline{8}\ ;\ \underline{1}_o) \tag{4.79d}$$

$$(\underline{21};\ \underline{3}\ ;\ \underline{1}\ ;\ \underline{8}\ ;\ \underline{1}_o)\ . \tag{4.79e}$$

The eigenvalues of the quadratic Casimir operators are evaluated using the standard formulae for U(n), Sp(n) and SO(n), respectively (Jarvis 1979):

$$C_2\{n\} = \sum_r \lambda_r(\lambda_r + n + 1 - 2r) \tag{4.80a}$$

$$C_2\langle n\rangle = 2\sum_r \lambda_r(\lambda_r + n + 2 - 2r) \tag{4.80b}$$

$$C_2[n] = 2\sum_r \lambda_r(\lambda_r + n - 2r), \tag{4.80c}$$

where $\lambda_1, \lambda_2, \ldots$ is the partition specifying the Young diagram with row lengths $\lambda_1, \lambda_2, \ldots$. The eigenvalues are given in Table 4.4:

State \ Group	Sp(6,R)	SO(3)	Sp(2,R)	U(3)	U(1)
$(\underline{6};\underline{3};\underline{2};\underline{3};\ \underline{1}_1)$	14	4	6	3	1
$(\underline{14}\ ;\underline{3};\underline{3};\underline{\bar{3}};\underline{1}_2)$	24	4	16	4	4
$(\underline{14};\underline{3};\underline{3};\underline{8};\underline{1}_o)$	24	4	16	6	0
$(\underline{14};\underline{5};\underline{1};\underline{8};\underline{1}_o)$	24	12	0	6	0
$(\underline{21};\underline{3};\underline{1};\underline{8};\underline{1}_o)$	32	4	0	6	0

TABLE 4.4 Eigenvalues of quadratic Casimir invariant operators

We obtain the following set of five simultaneous equations in a, b, c, d and e:

$$a + 3b + 6c + 14d + 4e = 0 \qquad (4.81a)$$

$$4a + 4b + 16c + 24d + 4e = 0 \qquad (4.81b)$$

$$6b + 16c + 24d + 4e = 0 \qquad (4.81c)$$

$$6b + 24d + 12e = 0 \qquad (4.81d)$$

$$6b + 32d + 4e = 0 \,. \qquad (4.81e)$$

These have the unique solution:

$$a = 6, \quad b = 12, \quad c = -1, \quad d = -2, \quad e = -2 \qquad (4.82)$$

giving:

$$\tfrac{1}{4}(T^{\underline{126}}_{\underline{10}\otimes\underline{5}})_{P_3=0} = 6C_2\{1\} + 12C_2\{3\} - C_2\langle 2\rangle - 2C_2\langle 6\rangle - 2C_2[3] \,. \qquad (4.83)$$

For physical oscillator states, only symmetric irreducible representations occur and equations (4.80a) - (4.80c) reduce to the simpler set of formulae:

$$C_2\{1\} = \hat{n}^2 \qquad (4.84a)$$

$$C_2\{3\} = \hat{n}(\hat{n}+2) \qquad (4.84b)$$

$$C_2\langle 2\rangle = 2\hat{n}(\hat{n}+2) \qquad (4.84c)$$

$$C_2\langle 6\rangle = 2\hat{n}(\hat{n}+6) \qquad (4.84d)$$

$$C_2[3] = 2\hat{\underline{\ell}}^2 \,, \qquad (4.84e)$$

leading to:

$$\frac{1}{16}(T^{\underline{126}}_{\underline{10}\otimes\underline{5}})_{P_3=0} = \hat{n}(3\hat{n}-1) - \hat{\underline{\ell}}^2 \,. \qquad (4.85)$$

Thus, using the Wigner-Eckart theorem, we deduce for the $n = 2$ level:

$$\Delta E_{2,\ell} = \langle 2,\ell|U(r)|2,\ell\rangle = \langle\|0\|\rangle .1 + \langle\|1\|\rangle .2 + \langle\|2\|\rangle .\{2.5-\ell(\ell+1)\}, \qquad (4.86)$$

where $<\|0\|>$ and $<\|1\|>$ are the same reduced matrix elements as appear in the $n = 1$ case, and $<\|2\|>$ is a new reduced matrix element required by the Wigner-Eckart theorem.

Recalling the results for the $n = 0$ and $n = 1$ levels from equations (4.67) and (4.68), we deduce:

$$<0,0|U(r)|0,0> = <\|0\|> \tag{4.87a}$$

$$<1,1|U(r)|1,1> = <\|0\|> + <\|1\|> \tag{4.87b}$$

$$<2,2|U(r)|2,2> = <\|0\|> + 2<\|1\|> + 4<\|2\|> \tag{4.87c}$$

$$<2,0|U(r)|2,0> = <\|0\|> + 2<\|1\|> +10<\|2\|> . \tag{4.87d}$$

We obtain complete agreement with the results obtained by explicit wavefunction techniques in §4.3 for the $n = 0$, 1 and 2 levels, as displayed in Table 4.2, if we make the correspondence:

$$<\|0\|> = A \tag{4.88a}$$

$$<\|1\|> = -A + \frac{2}{3}B \tag{4.88b}$$

$$<\|2\|> = \frac{1}{4}A - \frac{1}{3}B + \frac{1}{15}C . \tag{4.88c}$$

Again, it is interesting to digress for a moment and consider the subgroup $U(3)$ of $Sp(6,R)$ in a little more detail. We can use the method of expanding an operator in terms of quadratic Casimir invariants to check the validity of equation (4.43). Recalling the embedding $U(3) \supset SO(3) \otimes U(1)$, we set:

$$T^{\ell=0}_{\underline{27}} = C_2\{1\} + xC_2\{3\} + yC_2[3] , \tag{4.89}$$

where x and y are coefficients to be determined, and we have set the coefficient of $C_2\{1\}$ equal to unity, for convenience. Since, in $U(3)$, $\underline{\bar{3}} \otimes \underline{3}$ does not contain $\underline{27}$, we may write:

$$\langle \underline{3} | T^{\ell=0}_{\underline{27}} | \underline{3} \rangle = 0 \tag{4.90a}$$

$$\langle \bar{\underline{3}} | T^{\ell=0}_{\underline{27}} | \bar{\underline{3}} \rangle = 0 , \tag{4.90b}$$

obtaining two simultaneous equations for x and y, viz.:

$$1 + 3x + 4y = 0 \tag{4.91a}$$

$$4 + 4x + 4y = 0 \tag{4.91b}$$

with the unique solution:

$$x = -3, \quad y = 2 . \tag{4.92}$$

Thus:

$$T^{\ell=0}_{\underline{27}} = C_2\{1\} - 3C_2\{3\} + 2C_2[3] . \tag{4.93}$$

It is interesting to note that the precise linear combination of $C_2\{1\}$ and $C_2\{3\}$ which appears in equation (4.93) is simply $(-3)\times$ the quadratic Casimir invariant operator of SU(3). Acting on physical oscillator states, equation (4.93) reduces to:

$$T^{\ell=0}_{\underline{27}} = -2\{\hat{n}(\hat{n}+3) - 2\hat{\ell}^2\} , \tag{4.94}$$

by virtue of equations (4.84a), (4.84b) and (4.84e). Since an overall multiplicative factor in $T^{\ell=0}_{\underline{27}}$ is unimportant, the validity of equation (4.43) is upheld.

Remarkably, it appears not to be possible to construct an expression for $T^{\ell=0}_{\underline{125}}$ solely in terms of Casimir invariant operators. Our considerations for $T^{\ell=0}_{\underline{27}}$ suggest that we should try to expand $T^{\ell=0}_{\underline{125}}$ in terms of the independent Casimir invariant operators of SU(3), SO(3) and U(1), which we may take as $C_2'\{3\}$ and $C_3'\{3\}$, $C_2[3]$ and $C_1\{1\}$, respectively. The notation $C_2'\{3\}$ and $C_3'\{3\}$

denotes, respectively, the quadratic and cubic Casimir invariant operators of SU(3), as distinct from their U(3) counterparts $C_2\{3\}$ and $C_3\{3\}$, and $C_1\{1\}$ represents the linear Casimir invariant operator of U(1) $[C_2\{1\} = (C_1\{1\})^2]$. The most general form we can write down for $T^{\ell=0}_{\underline{125}}$ is then:

$$T^{\ell=0}_{\underline{125}} = X + Y\, C_2[3] + Z(C_2[3])^2 \tag{4.95}$$

where

$$X = A(C_1\{1\})^4 + B(C_1\{1\})^3 + D(C_1\{1\})^2 + FC_1\{1\} + G(C_1\{1\})^2 C_2'\{3\}$$
$$+ HC_1\{1\}C_2'\{3\} + JC_1\{1\}C_3'\{3\} + KC_3'\{3\} + L(C_2'\{3\})^2 + MC_2'\{3\} \tag{4.96a}$$

$$Y = N(C_1\{1\})^2 + PC_1\{1\} + QC_2'\{3\} + R \tag{4.96b}$$

and A, B, D, F, G, H, J, K, L, M, N, P, Q, R and Z denote coefficients to be determined.

We can demonstrate that the expression for $T^{\ell=0}_{\underline{125}}$ in equation (4.95) must be incomplete by noting, in particular, that $\underline{125}$ is not contained in the SU(3) outer product:

$$\overline{\underline{15}}' \otimes \underline{15}' = \underline{1} \oplus \underline{8} \oplus \underline{8} \oplus \underline{10} \oplus \overline{\underline{10}} \oplus \underline{27} \oplus \underline{27} \oplus \underline{35} \oplus \overline{\underline{35}} \oplus \underline{64}\ , \tag{4.97}$$

where $\underline{15}'$ is the SU(3) irreducible representation corresponding to the partition $\lambda_1 = 3$, $\lambda_2 = 1$ [as distinct from $\underline{15}$, which corresponds to $\lambda_1 = 4$, $\lambda_2 = 0$]. This fact, coupled with the reduction:

$$\underline{15}' \supset (\underline{7} \oplus \underline{5} \oplus \underline{3}) \otimes \underline{1}_4 \tag{4.98}$$

under the embedding $U(3) \supset SO(3) \otimes U(1)$, enables one to deduce that the matrix elements of $T^{\ell=0}_{\underline{125}}$ sandwiched between the (SU(3); SO(3); U(1)) states:

$$(\underline{15}' \; ; \; \underline{7} \; ; \; \underline{1}_4) \tag{4.99a}$$

$$(\underline{15}' \; ; \; \underline{5} \; ; \; \underline{1}_4) \tag{4.99b}$$

$$(\underline{15}' \; ; \; \underline{3} \; ; \; \underline{1}_4) \tag{4.99c}$$

necessarily vanish. We thus obtain the following set of simultaneous equations for X, Y and Z:

$$X + 24Y + 576Z = 0 \tag{4.100a}$$

$$X + 12Y + 144Z = 0 \tag{4.100b}$$

$$X + 4Y + 16Z = 0 \tag{4.100c}$$

with the unique (trivial) solution:

$$X = Y = Z = 0 \; . \tag{4.101}$$

This proves that the expression for $T^{\ell=0}_{\underline{125}}$ in equation (4.95) must be incomplete.

Although we do not pursue this particular example any further, we take away from it an important clue which will prove to be of use to us in the next chapter, viz., we must be careful to take proper account of the possibility of including non-Casimir invariant operators in expressions such as the one in equation (4.95). We shall elaborate on this point in the next chapter.

CHAPTER 5

Sp(12,R) AND ANHARMONIC SYMMETRY-BREAKING

The introduction of anharmonic 2-body potentials into the harmonic oscillator quark model for baryons represents a breaking of the symmetry in the U(6) degeneracy group sector. The aim of this chapter is to amplify this assertion. In particular, we shall show how the first-order mass splittings in equations (3.15) and (3.17)-(3.20) for the $N = 2$ level, derived in Chapter 3 by explicit state-function and operator techniques, can be understood both qualitatively and quantitatively as a mass formula of the Gell-Mann - Okubo type. We shall also demonstrate how these techniques can be usefully applied at the $N = 3$ level.

To bring about this understanding, it is necessary to consider not just the degeneracy group U(6) but the spectrum-generating group Sp(12,R) as well. This follows from the fact that the anharmonic perturbation is a function of all twelve components of the vector (a_A) through the dependence of the potential upon $\underline{\rho}$ and $\underline{\lambda}$ and hence upon $\underline{a}(\underline{\rho})$, $\underline{a}^+(\underline{\rho})$, $\underline{a}(\underline{\lambda})$ and $\underline{a}^+(\underline{\lambda})$. We assume that this potential may be cast in the form:

$$V(\underline{\rho},\underline{\lambda}) = U(\sqrt{2}\underline{\rho}) + U(-\frac{1}{\sqrt{2}}\underline{\rho} + \sqrt{\frac{3}{2}}\underline{\lambda}) + U(-\frac{1}{\sqrt{2}}\underline{\rho} - \sqrt{\frac{3}{2}}\underline{\lambda}) \quad (5.1)$$

where (c.f. equation (4.62)):

$$U(\sqrt{2}\,\underline{\rho}) = \sum_j \beta^{(2j)} (\underline{\rho}^2)^j \quad (5.2)$$

and $\beta^{(2j)}$, $j = 0,1,2,\ldots$ are arbitrary coefficients independent of $\underline{\rho}$. The justification for this form, quite apart from the requirement that it be totally symmetric and composed of 2-body contributions, was given in §4.5. It follows that:

$$V(\underline{\rho},\underline{\lambda}) \quad = \quad \sum_j \beta^{(2j)} \, V^{(2j)} \quad , \tag{5.3}$$

where, at each order j, the perturbation $V^{(2j)}$ is realised as an homogeneous polynomial in $\underline{\rho}$ and $\underline{\lambda}$ and hence in a_A, of degree $2j$. The strengths of each order are governed by the values of the distinct coupling constants, $\beta^{(2j)}$. The symmetry of equation (5.1) and the form of equation (5.2) ensure, furthermore, that each term $V^{(2j)}$ transforms as a component of the totally symmetric tensor representation $\langle 2j \rangle$ of $Sp(12,R)$.

It is necessary, in order to achieve a quantitative understanding of the level splitting, to determine the transformation properties of the various terms $V^{(2j)}$ with respect to the subgroups of $Sp(12,R)$ discussed in Chapter 2. By construction, $V^{(2j)}$ is both an $O(3)$ and an S_3 singlet, transforming as $[0] = \underline{1}$ and $(3) = \underline{S}$, respectively. With regard to its $O(2)$ properties, it is convenient to make use once more of the $(\underline{\zeta}, \underline{\eta})$-basis introduced in equations (2.32a) - (2.32b). In terms of these vectors:

$$V^{(2j)} \quad = \quad (\underline{\rho}.\underline{\rho})^j + [(-\tfrac{1}{2}\underline{\rho} + \frac{\sqrt{3}}{2}\underline{\lambda}).(-\tfrac{1}{2}\underline{\rho} + \frac{\sqrt{3}}{2}\underline{\lambda})]^j + \\ + [(-\tfrac{1}{2}\underline{\rho} - \frac{\sqrt{3}}{2}\underline{\lambda}).(-\tfrac{1}{2}\underline{\rho} - \frac{\sqrt{3}}{2}\underline{\lambda})]^j \tag{5.4}$$

i.e.

$$V^{(2j)} \quad = \quad \frac{1}{4^j}\{[(\underline{\zeta}+\underline{\eta}).(\underline{\zeta}+\underline{\eta})]^j + [(\omega^{-1}\underline{\zeta}+\omega\underline{\eta}).(\omega^{-1}\underline{\zeta}+\omega\underline{\eta})]^j \\ + \quad [(\omega\underline{\zeta}+\omega^{-1}\underline{\eta}).(\omega\underline{\zeta}+\omega^{-1}\underline{\eta})]^j\} \tag{5.5}$$

i.e.

$$V^{(2j)} \quad = \quad \frac{1}{4^j}\{(\underline{\zeta}.\underline{\zeta}+2\underline{\zeta}.\underline{\eta}+\underline{\eta}\cdot\underline{\eta})^j + (\omega\underline{\zeta}\cdot\underline{\zeta}+2\underline{\zeta}\cdot\underline{\eta}+\omega^{-1}\underline{\eta}\cdot\underline{\eta})^j + \\ + (\omega^{-1}\underline{\zeta}.\underline{\zeta}+2\underline{\zeta}.\underline{\eta}+\omega\underline{\eta}.\underline{\eta})^j\} \tag{5.6}$$

where $\omega = e^{-i\,2\pi/3}$. Hence:

$$V^{(2)} = \frac{3}{2}(\underline{\zeta}.\underline{\eta}) \tag{5.7}$$

$$V^{(4)} = \frac{3}{8}\{2(\underline{\zeta}.\underline{\eta})^2 + (\underline{\zeta}.\underline{\zeta})(\underline{\eta}.\underline{\eta})\} \tag{5.8}$$

$$V^{(6)} = \frac{3}{64}\{8(\underline{\zeta}.\underline{\eta})^3 + 12(\underline{\zeta}.\underline{\eta})(\underline{\zeta}.\underline{\zeta})(\underline{\eta}.\underline{\eta}) + (\underline{\zeta}.\underline{\zeta})^3 + (\underline{\eta}.\underline{\eta})^3\} \tag{5.9}$$

and so on.

The first of these terms is just the familiar harmonic term:

$$V^{(2)} = \frac{3}{2}(\underline{\rho}^2 + \underline{\lambda}^2) \tag{5.10}$$

which is both an O(2) and an O(6) singlet. However, $V^{(4)}$ is not an O(6) singlet, although it is a linear combination of O(2) singlets. It is, in fact, a linear combination of terms transforming as O(3) ⊗ O(2) singlet members of the O(6) representations [4] = $\underline{105}$ and [0] = $\underline{1}$. The term $V^{(6)}$ is not even an O(2) singlet, involving as it does a term transforming as the S_3 singlet S state of the O(2) representation [6] = $\underline{2}_6$.

In order to identify the Sp(2,R) content of $V^{(2j)}$, it suffices to expand $\underline{\zeta}$ and $\underline{\eta}$ in terms of the annihilation and creation operators $a_A = a_{ia\alpha}$, distinguished by $\alpha = 1$ and 2, respectively. Since $\underline{\rho} = \frac{1}{\sqrt{2}\alpha}[\underline{a}(\underline{\rho}) + \underline{a}^+(\underline{\rho})]$ and $\underline{\lambda} = \frac{1}{\sqrt{2}\alpha}[\underline{a}(\underline{\lambda}) + \underline{a}^+(\underline{\lambda})]$, the expansion of $V^{(2j)}$ yields a monomial of degree $2j$ which is totally symmetric under permutations of the indices, α. It follows that $V^{(2j)}$ transforms under Sp(2,R) as a sum of components of the symmetric representation $\langle 2j \rangle$. In the terminology appropriate to the locally-isomorphic group SO(2,1), this corresponds to the statement that $V^{(2j)}$ has pseudospin, j.

As in §4.5, we stress that not every component of $V^{(2j)}$ contributes to the values of the energy levels. Quite apart from the requirement used in constructing $V^{(2j)}$ that it be an O(3) ⊗ S_3 singlet, the only

effective component is necessarily a U(1) singlet, thus ensuring that its third component of pseudospin is zero. In the case $j = 1$, for example, this implies that the effective part of $V^{(2j)}$, given in equation (5.10), is simply proportional to the Hamiltonian H_o (modulo the term 3m) in equation (2.6).

The fact that the operators $V^{(2j)}$ are not, for each value of j, associated with a single irreducible representation of O(6), makes it convenient to consider other subgroups of Sp(12,R). These include the group Sp(6,R) with generators:

$$S_{PQ} = S_{i\alpha j\beta} = S_{PaQb}\delta^{ab} = S_{ia\alpha\ jb\beta}\ \delta^{ab} \qquad (5.11)$$

with $P, Q = 1,2,\ldots. 6$. This subgroup appears in the labelling chain:

$$Sp(12,R) \supset Sp(6,R) \otimes O(2) \supset Sp(2,R) \otimes O(3) \otimes O(2)$$
$$\supset U(1) \otimes SO(3) \otimes S_3 \qquad (5.12)$$

which is an alternative to the chain in equation (2.28). That the labelling chain in equation (5.12) is useful in dealing with $V^{(2j)}$ is a consequence of this operator transforming as a symmetric, O(3) singlet, pseudospin j, component of the symmetric Sp(12,R) representation $\langle 2j \rangle$. The branching rules for the embeddings $Sp(12,R) \supset Sp(6,R) \otimes O(2)$ and $Sp(6,R) \supset Sp(2,R) \otimes O(3)$ (King 1975) then ensure that this component necessarily belongs to the irreducible representation $\langle 2j \rangle$ of Sp(6,R).

Having established the transformation properties of the terms in the perturbation expansion of the potential in equation (5.3), it is necessary to discuss their rôle in determining the breaking of the degeneracy at each level, N. This involves calculating:

$$\Delta E^{(2j)} = \langle N|V^{(2j)}|N\rangle \qquad j = 0,1,2,\ldots \qquad (5.13)$$

which, by virtue of the Wigner-Eckart theorem, factorises into the product of a reduced matrix element signified by $\langle\!\| 2j \|\!\rangle$ and an appropriate Clebsch-Gordan coefficient. There are many approaches to such calculations: here we describe a method which has been discussed in detail by Jarvis (1979) and which leads to an algebraic formula of the Gell-Mann - Okubo type, together with a very simple method of checking the results.

At each level N, the relevant operators are those coupling to the Sp(12,R) product:

$$\langle N\rangle \otimes \langle N\rangle = \langle 2N\rangle \oplus \langle 2N-2\rangle \oplus \langle 2N-4\rangle \oplus \langle 2N-1,1\rangle \oplus \langle 2N-3,1\rangle \oplus \ldots . \tag{5.14}$$

The operators transforming as $\langle 0\rangle = \underline{1}$ and $\langle 2\rangle = \underline{78}$ of Sp(12,R), corresponding to $j = 0$ and $j = 1$, produce no splitting, since the former gives an overall shift in energy:

$$\Delta E^{(o)} = \langle\!\| 0 \|\!\rangle \tag{5.15a}$$

and the latter is just an harmonic term giving:

$$\Delta E^{(2)} = C_1\{1\} \langle\!\| 2 \|\!\rangle , \tag{5.15b}$$

where $C_1\{1\}$, the linear Casimir invariant operator of U(1), has eigenvalue N. At the $N = 0$ and $N = 1$ levels, there are no further contributions as can be seen from equation (5.14). However, the $N = 2$ level is split by the single operator $V^{(4)}$, transforming as $\langle 4\rangle = \underline{1365}$ of Sp(12,R), whilst the $N = 3$ level is split by the operators $V^{(4)}$ and $V^{(6)}$, transforming as $\langle 4\rangle = \underline{1365}$ and $\langle 6\rangle = \underline{12376}$ of Sp(12,R), respectively.

Concentrating on the lowest-order anharmonic term with $j = 2$, this is labelled with respect to the groups Sp(12,R); Sp(6,R); Sp(2,R); U(1); O(3); O(2); S_3 by:

$$\langle 4\rangle \; ; \; \langle 4\rangle \; ; \; \langle 4\rangle \; ; \; \{0\} \; ; \; [0] \; ; \; [0] \; ; \; (3) \tag{5.16a}$$

$$= \quad \underline{1365} \; ; \; \underline{126} \; ; \; \underline{5} \; ; \; \underline{1} \; ; \; \underline{1} \; ; \; \underline{1} \; ; \; \underline{S} \; . \tag{5.16b}$$

It is a straightforward task to construct the corresponding tensor operator in the enveloping algebra of Sp(12,R). It is a symmetrised second-order product of generators. An arbitrary component of $\langle 4\rangle = \underline{1365}$ is simply:

$$X_{ABCD} \propto \{S_{AB}, S_{CD}\} + \{S_{AC}, S_{BD}\} + \{S_{AD}, S_{BC}\} \; . \tag{5.17}$$

In the Sp(6,R) $\otimes$ O(2) basis of Sp(12,R), representing $A \equiv ia\alpha$ by Pa, the tensor operator transforming as the $\underline{126} \otimes \underline{1}$ component of $\underline{1365}$ is clearly:

$$Y_{PQRS} \propto (X_{PaQbRcSd} + X_{PaRbQcSd} + X_{PaSbQcRd})\delta^{ab} \, \delta^{cd} \tag{5.18}$$

and similarly, in the Sp(2,R) $\otimes$ O(3) basis of Sp(6,R), replacing P by $i\alpha$, the tensor operator transforming as the $\underline{5} \otimes \underline{1}$ component of $\underline{126}$ is

$$Z_{\alpha\beta\gamma\delta} \propto (Y_{i\alpha j\beta k\gamma \ell\delta} + Y_{i\alpha j\gamma k\beta \ell\delta} + Y_{i\alpha j\delta k\beta \ell\gamma})\delta^{ij} \, \delta^{k\ell} \; . \tag{5.19}$$

The $P_3 = 0$ component of $Z_{\alpha\beta\gamma\delta}$, and thus the desired tensor operator, is:

$$V^{(4)} \propto Z_{1122} \; . \tag{5.20}$$

Using these definitions, we can rewrite $V^{(4)}$ in terms of the various combinations $\{S_{AB}, S_{CD}\}$, each of which is a U(1) $\otimes$ O(3) $\otimes$ O(2) invariant. There are fourteen independent ways of using the tensors δ^{ij}, δ^{ab} and $\varepsilon^{\alpha\beta}$, ensuring $\alpha = 1,2$ equally often, to make such invariants, and we find, after some labour, that:

$$V^{(4)} \propto \tfrac{1}{2}\{S_{ialial}, S_{jb2jb2}\} + \{S_{ialibl}, S_{ja2jb2}\} + \{S_{ialjal}, S_{ib2jb2}\}$$
$$+ \{S_{ialjbl}, S_{ia2jb2}\} + \{S_{ialjbl}, S_{ib2ja2}\} + \{S_{ialia2}, S_{jbljb2}\}$$
$$+ \{S_{ialib2}, S_{jaljb2}\} + \{S_{ialib2}, S_{jb1ja2}\} + \{S_{ialja2}, S_{ibljb2}\}$$
$$+ \{S_{ialjb2}, S_{ialjb2}\} + \{S_{ialjb2}, S_{iblja2}\} + \{S_{ialja2}, S_{jblib2}\}$$
$$+ \{S_{ialjb2}, S_{jalib2}\} + \{S_{ialjb2}, S_{jblia2}\} . \qquad (5.21)$$

This operator can be expanded in terms of the set of quadratic Casimir invariant operators of Sp(12,R) and its subgroups. The algebraic formula involves several different labelling chains including those containing U(6), O(6) and Sp(2,R) which we have already mentioned. The required embedding diagram is presented in Figure 6. Thirteen different subgroups are involved and Table 5.1 defines these subgroups by specifying their generators explicitly:

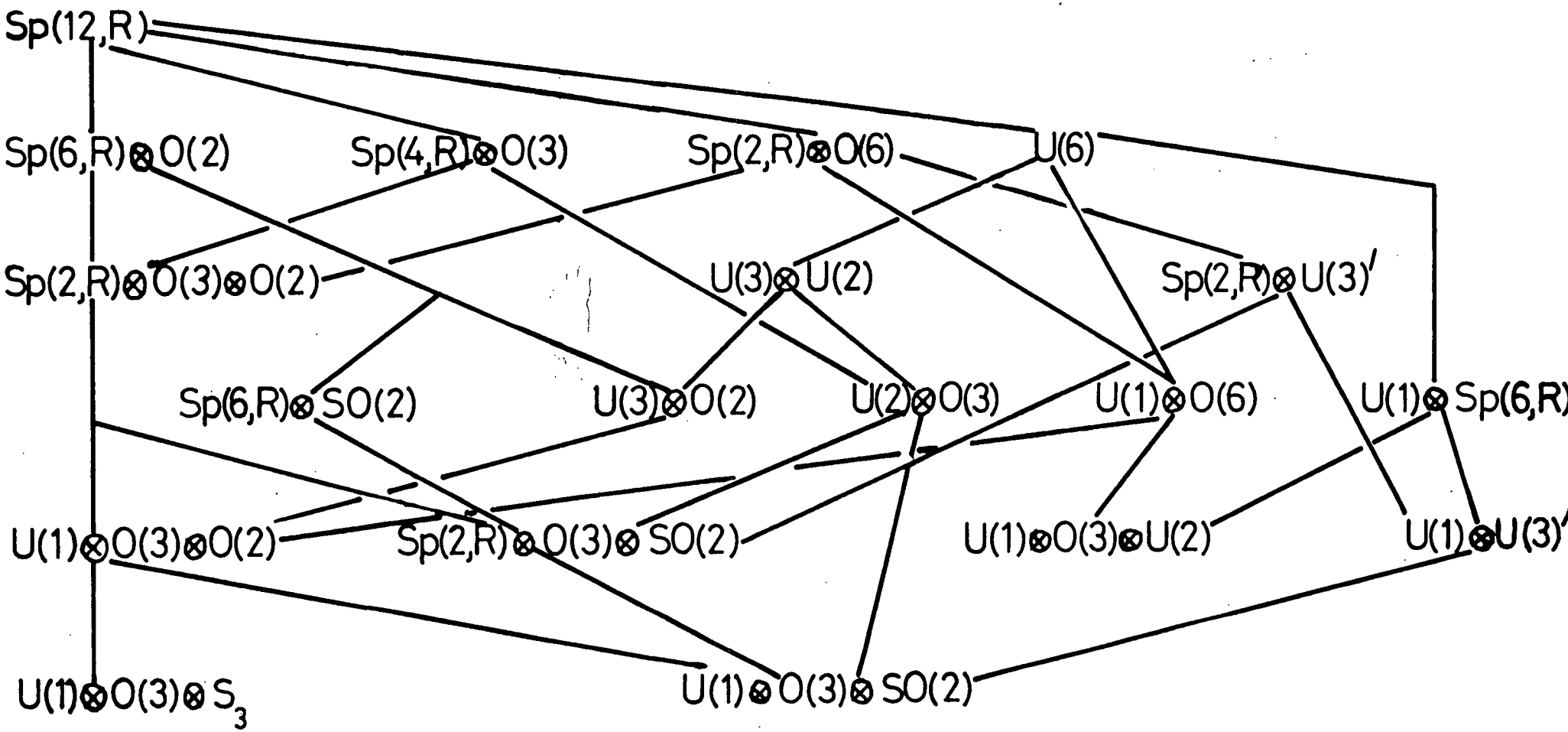


Figure 6 Sp(12,R) labelling chains

Subgroup	Generators
U(1)	$E_{IJ}\ \delta^{IJ}$
U(2)	$E_{iajb}\ \delta^{ij}$
U(3)	$E_{iajb}\ \delta^{ab}$
U(6)	$E_{IJ} = S_{I2J1}$
Sp(2,R)	$S_{I\alpha J\beta}\delta^{IJ}$
Sp(4,R)	$S_{UiVj}\ \delta^{ij}$
Sp(6,R)	$S_{PaQb}\ \delta^{ab}$
O(2)	$O_{iajb}\ \delta^{ij}$
O(3)	$O_{iajb}\ \delta^{ab}$
O(6)	$O_{IJ} = S_{I\alpha J\beta}\varepsilon^{\alpha\beta} = E_{IJ} - E_{JI}$
U(3)'	$\frac{1}{2}(O_{i1j1} + O_{i2j2} + iO_{i1j2} - iO_{i2j1})$
Sp(6,R)'	$J_{IM}E_{MJ} + J_{JM}E_{MI}$

TABLE 5.1 Subgroup generators

In Table 5.1, the Sp(4,R) index $U \equiv a\alpha$.

The corresponding quadratic Casimir invariant operators are defined in Table 5.2.

In addition, the expansion of $V^{(4)}$ as a component of X_{ABCD} involves at least one of the operators:

$$\Sigma = \tfrac{1}{2}\delta^{ij}\delta^{k\ell}\delta^{ab}\delta^{cd}\ \varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}\{S_{ia\alpha kc\gamma},\ S_{jd\delta\ell b\beta}\} \tag{5.22a}$$

and

$$\Sigma' = \tfrac{1}{2}\delta^{ij}\delta^{k\ell}\delta^{ab}\delta^{cd}\ \varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}\{S_{ia\alpha kc\gamma},\ S_{\ell b\delta jd\beta}\}\ . \tag{5.22b}$$

These are invariants of the subgroups $Sp(4,R) \otimes O(3)$ and $Sp(6,R) \otimes O(2)$,

Subgroup	Quadratic Casimir invariant operator
U(1)	$C_2\{1\} = \frac{1}{2}\{S_{ia1\ ia2}, S_{jb1\ jb2}\}$
U(2)	$C_2\{2\} = \frac{1}{2}\{S_{ia1\ ib2}, S_{jb1\ ja2}\}$
U(3)	$C_2\{3\} = \frac{1}{2}\{S_{ia1\ ja2}, S_{jb1\ ib2}\}$
U(6)	$C_2\{6\} = \frac{1}{2}\{S_{ia1\ jb2}, S_{jb1\ ia2}\}$
Sp(2,R)	$C_2\langle 2\rangle = \{S_{ia1\ ia2}, S_{jb1\ jb2}\} - \{S_{ia1\ ia1}, S_{jb2\ jb2}\}$
Sp(4,R)	$C_2\langle 4\rangle = \{S_{ia1\ ib2}, S_{jb1\ ja2}\} - \{S_{ia1\ ib1}, S_{ja2\ jb2}\}$
Sp(6,R)	$C_2\langle 6\rangle = \{S_{ia1\ ja2}, S_{jb1\ ib2}\} - \{S_{ia1\ ja1}, S_{ib2\ jb2}\}$
O(2)	$C_2[2] = \{S_{ia1\ ib2}, S_{jb1\ ja2}\} - \{S_{ia1\ ib2}, S_{ja1\ jb2}\}$
O(3)	$C_2[3] = \{S_{ia1\ ja2}, S_{jb1\ ib2}\} - \{S_{ia1\ ja2}, S_{ib1\ jb2}\}$
O(6)	$C_2[6] = \{S_{ia1\ jb2}, S_{jb1\ ia2}\} - \{S_{ia1\ jb2}, S_{ia1\ jb2}\}$
U(3)'	$C_2\{3'\} = \frac{1}{4}[(\{S_{ia1\ jb2}, S_{ib1\ ja2}\} + \{S_{ia1\ jb2}, S_{jb1\ ia2}\} + \{S_{ia1\ ja2}, S_{jb1\ ib2}\}) - (\{S_{ia1\ jb2}, S_{ja1\ ib2}\} + \{S_{ia1\ jb2}, S_{ia1\ jb2}\} + \{S_{ia1\ ja2}, S_{ib1\ jb2}\})]$
Sp(6,R)'	$C_2\langle 6'\rangle = \{S_{ia1\ jb2}, S_{jb1\ ia2}\} + \{S_{ia1\ jb2}, S_{ib1\ ja2}\} - \{S_{ia1\ ja2}, S_{ib1\ jb2}\}$

TABLE 5.2 Quadratic Casimir invariant operators

respectively, of Sp(12,R), in that they commute with the generators of these groups. However, they do not belong to the enveloping algebra of these groups and are thus not Casimir operators.

The eigenvalues of the quadratic Casimir invariant operators are easily evaluated. For example, the quadratic Casimir operator of

Sp(12,R) is given by :

$$C_2\langle 12\rangle \quad = \quad J^{DB}\, J^{CA}\, S_{AD}\, S_{BC} \quad ; \tag{5.23}$$

the corresponding eigenvalues may be evaluated by making use of the finite-dimensional representations discussed in Chapter 2 and defined by equation (2.14) and its generalisations. This implies that:

$$C_2 : \quad a_P a_Q \ldots\ldots \rightarrow J^{DB}\, J^{CA}\, [S_{AD}, [S_{BC}, a_P a_Q \ldots\ ]] = C_2\langle 12\rangle a_P a_Q \ldots\ . \tag{5.24}$$

Since the results depend only on the commutation relations, they are identical with the finite-dimensional compact case which gives rise to the formulae in equations (4.80a) - (4.80c) (Jarvis 1979); in particular:

$$C_2\langle 12\rangle \quad = \quad 2 \sum_r \lambda_r(\lambda_r + 14 - 2r) \tag{5.25}$$

in the representation $\langle \lambda_1, \lambda_2, \ldots\ \rangle$.

By comparison, the computation of the eigenvalues of the non-Casimir invariant operators, Σ and Σ', is much less straightforward. Interestingly enough, Σ and Σ' are not unrelated: simple manipulation of the definitions reveals that:

$$\Sigma - \Sigma' \quad = \quad C_2[6] + C_2\,[3] - 4C_2\{3'\} \quad . \tag{5.26}$$

Accordingly, we shall treat Σ and Σ' together by assuming that the corresponding symmetry group is $Sp(s) \otimes O(t)$, and denote this generic form by Λ. In common with quadratic Casimir operators, Λ is a bilinear operator of the form $\{S, S'\}$, and must, therefore, be evaluated on tensor operator states, T, by means of double commutators $[S, [S', T]] + [S', [S,T]]$. We define:

$$\Lambda \quad = \quad \tfrac{1}{2} J^{\alpha\beta}\, J^{\gamma\delta}\, G^{IJ}\, G^{KL} \{S_{I\alpha K\gamma}, S_{J\delta L\beta}\} \tag{5.27}$$

for $\alpha, \beta = 1,2, \ldots\ldots s$ and $I,J = 1,2, \ldots\ldots t$, where G and J are the symmetric and antisymmetric metrics of $O(t)$ and $Sp(s)$, respectively. We find for symmetric tensors $X_{I\alpha}$, $X_{I\alpha J\beta}$ and $X_{I\alpha J\beta K\gamma}$, in particular, that:

$$[\Lambda, X_{I\alpha}] = -2(s+t)X_{I\alpha} \tag{5.28a}$$

$$[\Lambda, X_{I\alpha J\beta}] = -4(s+t)X_{I\alpha J\beta} - 4G_{IJ}X_{L\alpha M\beta}\,G^{LM} + 4J_{\alpha\beta}\,X_{I\gamma J\delta}\,J^{\gamma\delta} \tag{5.28b}$$

$$\begin{aligned}[\Lambda, X_{I\alpha J\beta K\gamma}] = &-6(s+t)X_{I\alpha J\beta K\gamma} - 4G_{IJ}X_{L\alpha M\beta K\gamma}\,G^{LM} - 4G_{JK}\,X_{I\alpha L\beta M\gamma}G^{LM} \\ &- 4\,G_{IK}\,X_{L\alpha J\beta M\gamma}\,G^{LM} + 4J_{\alpha\beta}X_{I\sigma J\tau K\gamma}\,J^{\sigma\tau} \\ &+ 4\,J_{\beta\gamma}\,X_{I\alpha J\sigma K\tau}\,J^{\sigma\tau} + 4J_{\alpha\gamma}X_{I\sigma J\beta K\tau}\,J^{\sigma\tau}\,.\end{aligned} \tag{5.28c}$$

If we consider only the "stretched" states of angular momentum, as in equation (2.36), neither the $J^{\sigma\tau}$ traces, nor the G^{LM} traces (other than those corresponding to $[\underline{a}^{+}(\underline{\zeta}).\underline{a}^{+}(\underline{\eta})]^{a}$ in the original tensor) can contribute, and we have explicitly:

$$[\Lambda, X_{I\alpha J\beta}] = -4(s+t)X_{I\alpha J\beta} \tag{5.29a}$$

$$[\Lambda, G^{IJ}X_{I\alpha J\beta}] = -4(s+2t)G^{IJ}X_{I\alpha J\beta} \tag{5.29b}$$

$$[\Lambda, X_{I\alpha J\beta K\gamma}] = -6(s+t)X_{I\alpha J\beta K\gamma} \tag{5.29c}$$

$$[\Lambda, G^{IJ}X_{I\alpha J\beta K\gamma}] = -(6s + 10t + 8)G^{IJ}\,X_{I\alpha J\beta K\gamma}\,. \tag{5.29d}$$

The eigenvalues of both the relevant quadratic Casimir invariant operators and the non-Casimir invariant operator, Σ, are set out for the cases $N = 2$ and $N = 3$ in Tables 5.3 and 5.4, respectively.

$[SU(6), L^P]$	$[\underline{20}, 1^+]$		$[\underline{70}, 2^+]$		$[\underline{56}, 2^+]$		$[\underline{70}, 0^+]$		$[\underline{56}, 0^+]$	
Subgroup H	Representation	$C_2(H)$	Representation	$C_2(H)$	Representation	$C_2(H)$	Representation	$C_2(H)$	Representation	$C_2(H)$
U(1)	$\{2\} = \underline{1}_2$	4	$\{2\} = \underline{1}_2$	4	$\{2\} = \underline{1}_2$	4	$\{2\} = \underline{1}_2$	4	$\{2\} = \underline{1}_2$	4
U(2)	$\{1^2\} = \underline{\bar{1}}$	2	$\{2\} = \underline{3}$	6	$\{2\} = \underline{3}$	6	$\{2\} = \underline{3}$	6	$\{2\} = \underline{3}$	6
U(3)	$\{1^2\} = \underline{\bar{3}}$	4	$\{2\} = \underline{6}$	8	$\{2\} = \underline{6}$	8	$\{2\} = \underline{6}$	8	$\{2\} = \underline{6}$	8
U(3)'	$\{2,1\} = \underline{8}$	6	$\{2\} = \underline{6}$	8	$\{2,1\} = \underline{8}$	6	$\{2\} = \underline{6}$	8	$\{0\} = \underline{1}$	0
U(6)	$\{2\} = \underline{21}$	14	$\{2\} = \underline{21}$	14	$\{2\} = \underline{21}$	14	$\{2\} = \underline{21}$	14	$\{2\} = \underline{21}$	14
Sp(2,R)	$\langle 2\rangle = \underline{3}$	16	$\langle 2\rangle = \underline{3}$	16	$\langle 2\rangle = \underline{3}$	16	$\langle 2\rangle = \underline{3}$	16	$\langle 2\rangle = \underline{3}$	16
Sp(4,R)	$\langle 1^2\rangle = \underline{5}$	16	$\langle 2\rangle = \underline{10}$	24	$\langle 2\rangle = \underline{10}$	24	$\langle 2\rangle = \underline{10}$	24	$\langle 2\rangle = \underline{10}$	24
Sp(6,R)	$\langle 1^2\rangle = \underline{14}$	24	$\langle 2\rangle = \underline{21}$	32	$\langle 2\rangle = \underline{21}$	32	$\langle 2\rangle = \underline{21}$	32	$\langle 2\rangle = \underline{21}$	32
Sp(6,R)'	$\langle 2\rangle = \underline{21}$	32	$\langle 2\rangle = \underline{21}$	32	$\langle 2\rangle = \underline{21}$	32	$\langle 2\rangle = \underline{21}$	32	$\langle 2\rangle = \underline{21}$	32
Sp(12,R)	$\langle 2\rangle = \underline{78}$	56	$\langle 2\rangle = \underline{78}$	56	$\langle 2\rangle = \underline{78}$	56	$\langle 2\rangle = \underline{78}$	56	$\langle 2\rangle = \underline{78}$	56
O(2)	$[0]^* = \underline{1}^*$	0	$[2] = \underline{2}_2$	8	$[0] = \underline{1}$	0	$[2] = \underline{2}_2$	8	$[0] = \underline{1}$	0
O(3)	$[1] = \underline{3}$	4	$[2] = \underline{5}$	12	$[2] = \underline{5}$	12	$[0] = \underline{1}$	0	$[0] = \underline{1}$	0
O(6)	$[2] = \underline{20}$	24	$[2] = \underline{20}$	24	$[2] = \underline{20}$	24	$[2] = \underline{20}$	24	$[0] = \underline{1}$	0
Σ	-28		-28		-28		-40		-40	
$\Delta E^{(4)} / \langle \| 4 \| \rangle$	0		16		32		40		80	

TABLE 5.3 Subgroup representation labels and operator eigenvalues for N = 2

TABLE 5.4: Subgroup representation labels and operator eigenvalues for N = 3.

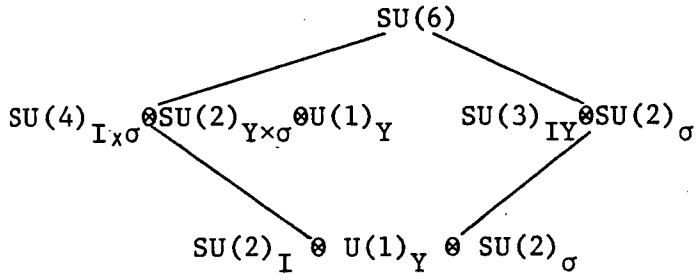
$[SU(6),L^P]$	$[\underline{70},2^-]$		$[\underline{20},3^-],[\underline{56},3^-]$		$[\underline{70},3^-]$		$[\underline{20},1^-],[\underline{56},1^-]$		$[\underline{70},1^-]$		$[\underline{70},1^-]$	
Subgroup H	Representation	$C_2(H)$	Representation	$C_2(H)$	Representation	$C_2(H)$	Representation	$C_2(H)$	Representation	$C_2(H)$	Representation	$C_2(H)$
U(1)	$\{3\} = \underline{1}_3$	9	$\{3\} = \underline{1}_3$	9	$\{3\} = \underline{1}_3$	9	$\{3\} = \underline{1}_3$	9	$\{3\} = \underline{1}_3$	9	$\{3\} = \underline{1}_3$	9
U(2)	$\{2,1\} = \underline{2}'$	6	$\{3\} = \underline{4}$	12	$\{3\} = \underline{4}$	12	$\{3\} = \underline{4}$	12	$\{3\} = \underline{4}$	12	$\{3\} = \underline{4}$	12
									$\{2,1\} = \underline{2}'$	6	$\{2,1\} = \underline{2}'$	6
U(3)	$\{2,1\} = \underline{8}$	9	$\{3\} = \underline{10}$	15	$\{3\} = \underline{10}$	15	$\{3\} = \underline{10}$	15	$\{3\} = \underline{10}$	15	$\{2,1\} = \underline{8}$	9
U(3)'	$\{3,1\} = \underline{15}$	11	$\{3\} = \underline{10}$	15	$\{3,1\} = \underline{15}$	11	$\{3\} = \underline{10}$	15	$\{3,1\} = \underline{15}$	11	$\{3,1\} = \underline{15}$	11
									$\{1\} = \underline{3}$	3	$\{1\} = \underline{3}$	3
U(6)	$\{3\} = \underline{56}$	24	$\{3\} = \underline{56}$	24	$\{3\} = \underline{56}$	24	$\{3\} = \underline{56}$	24	$\{3\} = \underline{56}$	24	$\{3\} = \underline{56}$	24
Sp(2,R)	$\langle 3\rangle = \underline{4}$	30	$\langle 3\rangle = \underline{4}$	30	$\langle 3\rangle = \underline{4}$	30	$\langle 3\rangle = \underline{4}$	30	$\langle 3\rangle = \underline{4}$	30	$\langle 3\rangle = \underline{4}$	30
Sp(4,R)	$\langle 2,1\rangle = \underline{16}$	30	$\langle 3\rangle = \underline{20}$	42	$\langle 3\rangle = \underline{20}$	42	$\langle 3\rangle = \underline{20}$	42	$\langle 3\rangle = \underline{20}$	42	$\langle 3\rangle = \underline{20}$	42
									$\langle 2,1\rangle = \underline{16}$	30	$\langle 2,1\rangle = \underline{16}$	30
Sp(6,R)	$\langle 2,1\rangle = \underline{64}$	42	$\langle 3\rangle = \underline{56}$	54	$\langle 3\rangle = \underline{56}$	54	$\langle 3\rangle = \underline{56}$	54	$\langle 3\rangle = \underline{56}$	54	$\langle 2,1\rangle = \underline{64}$	42
Sp(6,R)'	$\langle 3\rangle = \underline{56}$	54	$\langle 3\rangle = \underline{56}$	54	$\langle 3\rangle = \underline{56}$	54	$\langle 3\rangle = \underline{56}$	54	$\langle 3\rangle = \underline{56}$	54	$\langle 3\rangle = \underline{56}$	54
Sp(12,R)	$\langle 3\rangle = \underline{364}$	90	$\langle 3\rangle = \underline{364}$	90	$\langle 3\rangle = \underline{364}$	90	$\langle 3\rangle = \underline{364}$	90	$\langle 3\rangle = \underline{364}$	90	$\langle 3\rangle = \underline{364}$	90
O(2)	$[1] = \underline{2}_1$	2	$[3] = \underline{2}_3$	18	$[1] = \underline{2}_1$	2	$[3] = \underline{2}_3$	18	$[1] = \underline{2}_1$	2	$[1] = \underline{2}_1$	2
O(3)	$[2] = \underline{5}$	12	$[3] = \underline{7}$	24	$[3] = \underline{7}$	24	$[1] = \underline{3}$	4	$[1] = \underline{3}$	4	$[1] = \underline{3}$	4
O(6)	$[3] = \underline{50}$	42	$[3] = \underline{50}$	42	$[3] = \underline{50}$	42	$[3] = \underline{50}$	42	$[3] = \underline{50}$	42	$[3] = \underline{50}$	42
									$[1] = \underline{6}$	10	$[1] = \underline{6}$	10
Σ	-42		-42		-42		-62		-62		-62	
$\Delta E^{(4)} / \langle \| 4 \| \rangle$	32		48		80		88		-		-	

In terms of these eigenvalues, the resulting mass formula is:

$$\Delta E^{(4)} = \langle \| 4 \| \rangle \{3C_2\{1\} + 6C_2\{2\} + 12C_2\{3\} - 4C_2\{3'\} - \tfrac{1}{2}C_2\langle 2\rangle - C_2\langle 4\rangle$$
$$- C_2\langle 6\rangle + 3C_2\langle 6'\rangle - C_2\langle 12\rangle - C_2[2] - 3C_2[3] + \Sigma\} \qquad (5.30)$$

whence the values of $\Delta E^{(4)}/\langle \| 4 \| \rangle$ displayed in Tables 5.3 and 5.4.

Several aspects of this formula should be noted. Firstly, the use of overcomplete, non-commuting labelling chains is familiar from similar studies of symmetry-breaking in non-relativistic $SU(6)_{\text{flavour}\times\text{spin}}$ models, where the labelling structure is:



where Wigner's $SU(4)_{I\times\sigma}$, and the familiar $SU(2)_{Y\times\sigma}$, are used to place different isospin and hypercharge submultiplets into larger multiplets (Jarvis, 1980). In the present context, the over-completeness means that for $Sp(12,R)$, just as for $SU(6)_{\text{flavour}\times\text{spin}}$, the formula is only useful for states which are associated with a unique irreducible representation of each subgroup. Thus, for example, at the $N = 3$ level, the formula fails for the $[\underline{70}, 1^-]$ states which may be diagonalised with respect to $O(6)$, as in Table 2.1, but not simultaneously with respect to $Sp(6,R)$. This cannot be avoided and is a result of the proliferation of subgroup chains and labels, necessitated by the non-maximal nature of the embedding of the physical symmetry group $U(1) \otimes SO(3) \otimes S_3$ in $Sp(12,R)$.

The validity of equation (5.30) is easy to verify once it is realised that the expansion of $V^{(4)}$ in the form of components of X_{ABCD} in equation (5.17) can involve only quadratic Casimir operators

and Σ (or Σ'). This is accomplished by expanding $V^{(4)}$ in terms of the complete set of fourteen operators with arbitrary coefficients. These coefficients are then fixed by noting that $<N|V^{(4)}|N>$ is necessarily zero for all the states bearing the (Sp(12,R); Sp(6,R); Sp(2,R)) labels:

$$(<1>; <1>; <1>); (<1^2>; <2>; <2>); (<1^2>; <2>; <0>); (<1^2>; <1^2>; <2>);$$
$$(<1^2>; <1^2>; <0>); (<2>; <1^2>; <2>;); (<2>; <1^2>; <0>); (<2>; <2>; <0>), \tag{5.31}$$

in a manner analogous to the derivation of equation (4.83). In fact, the last seven of the sets of labels in equation (5.31) provide enough information, through eighteen conditions, to fix and check the coefficients (King 1980).

Notice, therefore, that the formula for $\Delta E^{(4)}$ in equation (5.30) has been derived in two different ways. The first method involved the explicit construction of $V^{(4)}$ and its re-expression in terms of quadratic Casimir operators and a single non-Casimir invariant operator, Σ. The second approach determined the coefficients of the Casimir operators and Σ by taking matrix elements of $V^{(4)}$ between specific states of the $N = 0$, 1 and 2 levels which necessarily vanish by virtue of the Wigner-Eckart theorem. The equivalence of these two approaches demonstrates explicitly that the reduced matrix element $<\| 4 \|>$ is indeed independent of the state label, N. Recall that we anticipated this result in writing down equations (4.68) and (4.86), stating there that the result was characteristic of a spectrum-generating algebra.

Returning to the application of equation (5.30), the splittings for $N = 2$ are precisely those of Chapter 3 with:

$$<\| 4 \|> \;=\; -\frac{1}{80}\Delta \;=\; \frac{1}{16}(\frac{1}{4}a - \frac{1}{3}b + \frac{1}{15}c), \tag{5.32}$$

where a, b and c are the familiar Isgur-Karl parameters defined in equations (3.13a) - (3.13c). The total effect of the anharmonic perturbation on the N = 2 states is given by:

$$\Delta E = \Delta E^{(0)} + \Delta E^{(2)} + \Delta E^{(4)} \tag{5.33}$$

where:

$$\Delta E^{(0)} = \langle \| 0 \| \rangle \tag{5.34a}$$

$$\Delta E^{(2)} = C_1\{1\} \langle \| 2 \| \rangle \tag{5.34b}$$

and $\Delta E^{(4)}$ is given by equation (5.30). The correspondence between the parameters of Isgur and Karl and the reduced matrix elements is :

$$\langle \| 0 \| \rangle \equiv a \equiv E_o - (3m+3\omega) \tag{5.35a}$$

$$\langle \| 2 \| \rangle \equiv -\frac{1}{2}a + \frac{1}{3}b \equiv \Omega - \omega \tag{5.35b}$$

and $\langle \| 4 \| \rangle$ is given by equation (5.32).

As we established earlier, in the N = 3 case, there are two anharmonic reduced matrix elements, viz. $\langle \| 4 \| \rangle$ and $\langle \| 6 \| \rangle$. In fact, as was demonstrated explicitly in Chapter 3, for the states belonging to the [56, 1^-], [70, 2^-] and [56, 3^-] $SU(6)_{flavour \times spin} \otimes O(3)$ supermultiplets, the Clebsch-Gordan coefficient multiplying $\langle \| 6 \| \rangle$ vanishes, so that the level splittings are again given by equation (5.30) in terms of $\langle \| 4 \| \rangle$ alone. One of the zeroes has the same origin as that appropriate to the N = 2 [20,1^+] supermultiplet for which $\Delta E^{(4)} = 0$ by virtue of this supermultiplet carrying the Sp(6,R) label $\langle 1^2 \rangle$ = 14. The same argument implies that $\Delta E^{(6)} = 0$ for the N = 3 [70,2^-] supermultiplet which carries the Sp(6,R) label $\langle 2,1 \rangle$ = 64 and therefore decouples from $V^{(6)}$ which transforms as a component of the Sp(6,R) representation $\langle 6 \rangle$ = 462.

Other zeroes owe their origin to the $Sp(6,R)_\rho$ subgroup of Sp(12, R)

which we encountered (but did not at the time identify as such) in Chapter 4 in our study of the single-oscillator problem. The relevance of this subgroup stems from the fact that, for physical states, $|\phi\rangle$:

$$\langle\phi|V(\underline{\rho},\underline{\lambda})|\phi\rangle \quad = \quad 3\langle\phi|U(\sqrt{2}\,\rho)|\phi\rangle \quad , \tag{5.36}$$

implying that matrix elements may be calculated merely by looking at the expectation value of $(\underline{\rho}^2)^j$, $j = 0,1,2,\ldots$. For the case $j = 3$, it is clear that the $Sp(6,R)_\rho$ representation associated with $V^{(6)}$ is $\langle 6\rangle = \underline{462}$ and that, at the $N = 3$ level, only those states with maximal $Sp(6,R)_\rho$ assignment $\langle 3\rangle = \underline{56}$ may couple to $V^{(6)}$. For the states constructed in Chapter 2 in terms of $\underline{a}^+(\underline{\zeta})$ and $\underline{a}^+(\underline{\eta})$, it is necessary to examine only the leading power in $\underline{a}^+(\underline{\rho})$. Any factor $[a^+(\underline{\zeta})a^+(\underline{\eta})]_{+1}$ leads to the total $Sp(6,R)$ and $Sp(6,R)_\rho$ being non-maximal: the states belonging to the $N = 3$ $[\underline{70},2^-]$ supermultiplet are of this type. Of the remaining monomials, $\mathcal{R}(\zeta^{\frac{1}{2}(N+m)}\ \eta^{\frac{1}{2}(N-m)})$ has leading power ρ^N and is therefore associated with maximal $Sp(6,R)$, whereas $\mathcal{I}(\zeta^{\frac{1}{2}(N+m)}\ \eta^{\frac{1}{2}(N-m)})$ has leading power ρ^{N-1} and is therefore non-maximal. States of this latter form at the $N = 3$ level belong to the $[\underline{56},1^-]$ and $[\underline{56},3^-]$ supermultiplets. Thus, the states belonging to the $N = 3$ $[\underline{56},1^-]$, $[\underline{70},2^-]$ and $[\underline{56},3^-]$ supermultiplets decouple from $V^{(6)}$. The level splittings produced by $V^{(4)}$ can be calculated from equation (5.30) and the results are indicated in Tables 5.3 and 5.4. When combined with the results obtained from equations (5.34a) - (5.34b) in the manner indicated in equation (5.33), they agree with the explicit state-function and operator calculations of Chapter 3.

However, by virtue of equation (5.36), we need look only to $Sp(6,R)_\rho$ to recover the splitting pattern of the $N = 2$ states displayed in Figure 3. The considerations of Chapter 4 for the

single-oscillator problem imply that the splitting is attributable to a unique symmetry-breaking operator $(T^{126}_{\underline{1}\oplus\underline{5}})_{P_3=0}$ [notation: $(T^{Sp(6,R)_\rho}_{SO(3)_\rho \oplus Sp(2,R)_\rho})$], the form of which we infer directly from equation (4.85) to be:

$$(T^{126}_{\underline{1}\oplus\underline{5}})_{P_3=0} \propto \hat{N}_\rho(3\hat{N}_\rho - 1) - \hat{\underline{L}}^2_\rho \quad , \tag{5.37}$$

where $\hat{N}_\rho$ denotes the number operator for the $\underline{\rho}$-oscillator and $\hat{\underline{L}}_\rho$ is the corresponding orbital angular momentum operator. It only remains to classify the $N = 2$ states under $Sp(6,R)_\rho$ and its relevant subgroups. To this end, we make the following identifications:

$$|0\rangle_\rho \equiv |\underline{1};\ \underline{1};\ \underline{1};\ \underline{1};\ \underline{1}_o\rangle_\rho \equiv |N_\rho = 0,\ L_\rho = 0\rangle \tag{5.38a}$$

$$\underline{a}^+(\underline{\rho})|0\rangle_\rho \equiv |\underline{6};\ \underline{3},\ \underline{3},\ \underline{2};\ \underline{1}_1\rangle_\rho \equiv |N_\rho = 1,\ L_\rho = 1\rangle \tag{5.38b}$$

$$[a^+_{+1}(\underline{\rho})]^2|0\rangle_\rho \equiv |\underline{21};\ \underline{6};\ \underline{5};\ \underline{3};\ \underline{1}_2\rangle_\rho \equiv \sqrt{2}|N_\rho = 2,\ L_\rho = 2\rangle \tag{5.38c}$$

$$[\underline{a}^+(\underline{\rho})]^2|0\rangle_\rho \equiv |\underline{21};\ \underline{6};\ \underline{1};\ \underline{3};\ \underline{1}_2\rangle_\rho \equiv \sqrt{6}|N_\rho = 2,\ L_\rho = 0\rangle. \tag{5.38d}$$

The intermediate step in equations (5.38a) - (5.38d) corresponds to the classification of the various physical states of the $\underline{\rho}$-oscillator under $|\,Sp(6,R)_\rho;\ U(3)_\rho;\ SO(3)_\rho;\ Sp(2,R)_\rho;\ U(1)_\rho\rangle$. Note also the appearance of normalisation factors on the RHS of equations (5.38a) - (5.38d): their inclusion is crucial.

We illustrate the method by reference to a specific example: the $N = 2$ $[\underline{56},\ 0^+]$ supermultiplet. We know from equation (3.10) that:

$$\Delta E_{[\underline{56},0^+]} = \tfrac{3}{2}{}_\rho\langle 0|U(\sqrt{2}\rho)|0\rangle_\rho + \tfrac{1}{4}{}_\rho\langle 0|\{[\underline{a}^+(\underline{\rho})]^2\}^+ U(\sqrt{2}\rho)[\underline{a}^+(\underline{\rho})]^2|0\rangle_\rho \,. \tag{5.39}$$

Since we are interested only in the relative shifts of the $N = 2$ states, for which the operator $(T^{126}_{\underline{1}\oplus\underline{5}})_{P_3=0}$ in equation (5.37) is solely

responsible, we may consider instead:

$$\Delta E'_{[\underline{56},0^+]} \propto \frac{3}{2}\rho\langle \underline{1};\ \underline{1};\ \underline{1};\ \underline{1};\ \underline{1}_o|\ (T^{\underline{126}}_{\underline{10}\underline{5}})_{P_3=0}\ |\underline{1};\ \underline{1};\ \underline{1};\ \underline{1};\ \underline{1}_o\rangle_\rho + \frac{1}{4}\rho\langle \underline{21};\ \underline{6};\ \underline{1};\ \underline{3};\ \underline{1}_2|(T^{\underline{126}}_{\underline{10}\underline{5}})_{P_3=0}\ |\underline{21};\ \underline{6};\ \underline{1};\ \underline{3};\ \underline{1}_2\rangle_\rho\ , \tag{5.40}$$

where we have made use of the intermediate step in equations (5.38a) and (5.38d). The first term on the RHS of equation (5.40) vanishes by virtue of the Wigner-Eckart theorem, and we may use the second step in equation (5.38d), together with equation (5.37), to deduce:

$$\Delta E'_{[\underline{56},0^+]} \propto \frac{3}{2}\langle N_\rho=2,\ L_\rho=0|\{\hat{N}_\rho(3\hat{N}_\rho-1) - \hat{\underline{L}}^2_\rho\}|N_\rho=2,\ L_\rho=0\rangle\ . \tag{5.41}$$

Similarly, one finds:

$$\Delta E'_{[\underline{70},0^+]} \propto \frac{3}{4}\langle N_\rho=2,\ L_\rho=0|\{\hat{N}_\rho(3\hat{N}_\rho-1) - \hat{\underline{L}}^2_\rho\}|N_\rho=2,\ L_\rho=0\rangle \tag{5.42}$$

$$\Delta E'_{[\underline{56},2^+]} \propto \frac{3}{2}\langle N_\rho=2,\ L_\rho=2|\{\hat{N}_\rho(3\hat{N}_\rho-1) - \hat{\underline{L}}^2_\rho\}|N_\rho=2,\ L_\rho=2\rangle \tag{5.43}$$

$$\Delta E'_{[\underline{70},2^+]} \propto \frac{3}{4}\langle N_\rho=2,\ L_\rho=2|\{\hat{N}_\rho(3\hat{N}_\rho-1) - \hat{\underline{L}}^2_\rho\}|N_\rho=2,\ L_\rho=2\rangle \tag{5.44}$$

$$\Delta E'_{[\underline{20},1^+]} = 0 \quad \text{(by virtue of the Wigner-Eckart theorem).} \tag{5.45}$$

The constant of proportionality common to equations (5.41) - (5.45) is just a reduced matrix element, $\langle\|\ \|\rangle$, as required by the Wigner-Eckart theorem. Equations (5.41) - (5.45) then imply:

$$\Delta E'_{[\underline{56},0^+]} = 15\ \langle\|\ \|\rangle \tag{5.46a}$$

$$\Delta E'_{[\underline{70},0^+]} = \frac{15}{2}\ \langle\|\ \|\rangle \tag{5.46b}$$

$$\Delta E'_{[\underline{56},2^+]} = 6\ \langle\|\ \|\rangle \tag{5.46c}$$

$$\Delta E'_{[\underline{70},2^+]} = 3\ \langle\|\ \|\rangle \tag{5.46d}$$

$$\Delta E'_{[\underline{20},1^+]} = 0\ , \tag{5.46e}$$

from which the pattern of splittings of the $N = 2$ states in Figure 3 follows, provided $\langle \| \; \| \rangle < 0$. If $\langle \| \; \| \rangle > 0$, the pattern is inverted.

CHAPTER 6

HYPERFINE INTERACTIONS IN NEGATIVE-PARITY BARYONS

6.1 The ΔD35 (1940) Resonance

We return in this chapter to study in more detail the suggestion by Cutkosky and Hendrick (1977a, 1977b) that the ΔD35(1940) resonance may be evidence for new degrees of freedom in the baryon spectrum. In Chapter 3, we predicted a mean mass of 1985 MeV for the non-strange sector of the $N = 3$ $[\underline{56},1^-]$ supermultiplet, to which the ΔD35(1940) is most plausibly assigned, and concluded that the ΔD35(1940) did not represent unambiguous evidence for gluonic degrees of freedom inside baryons. Whilst we worked strictly within the framework of the Isgur-Karl model in Chapter 3, we failed to take account of hyperfine effects, such as arise from the non-relativistic reduction of one-gluon-exchange. It is the aim of this chapter to investigate the effect of including such hyperfine effects upon the mass and composition predicted for the ΔD35(1940) resonance, and to re-examine the implications of the result of the most recent phase-shift analysis performed by Cutkosky et al. (Cutkosky 1980) for the mass of the ΔD35(1940) resonant state, viz.:

$$M(\Delta D35) = 1940 \pm 30 \text{ MeV}. \tag{6.1}$$

We may rewrite the effective hyperfine interaction between two quarks 1 and 2, arising out of coloured-gluon-exchange, as (c.f. equation (1.30)):

$$H_{12}^{hyp} = \frac{D\sqrt{\pi}}{4\alpha^3}\{\frac{8\pi}{3}(\underline{s}_1\cdot\underline{s}_2)\delta^3(\underline{\rho}) + \rho^{-5}[3(\underline{s}_1\cdot\underline{\rho})(\underline{s}_2\cdot\underline{\rho}) - (\underline{s}_1\cdot\underline{s}_2)\underline{\rho}^2]\}\ , \tag{6.2}$$

where $\underline{s}_1$ and $\underline{s}_2$ are the quark spins, $\sqrt{2}\underline{\rho} = \underline{r}_1 - \underline{r}_2$ is a vector

joining the two quarks, α is the usual harmonic-oscillator constant, and D is a constant to be identified with the $(\Delta-N)$ mass difference in the $N = 0$ $[\underline{56},0^+]$ supermultiplet. As in equation (1.44):

$$D = \frac{4\alpha_s\alpha^3}{3\sqrt{2\pi}\ m^2} \quad . \tag{6.3}$$

Isgur and Karl (1978b, 1979a) have stressed that the two terms in equation (6.2), with the relative strength indicated, are two parts of a single physical interaction, viz. the static interaction of two intrinsic colour-magnetic dipoles. The first term ("contact" term) may be visualised as arising from the $\underline{\mu}_i.\underline{B}_j^{\text{internal}}$ interaction of colour-magnet i with the colour-magnetic field internal to colour-magnet j; the second term ("tensor" term) is just the $\underline{\mu}_i.\underline{B}_j^{\text{external}}$ interaction of colour-magnet i with the external magnetic-dipole field of colour-magnet j and is the colour analogue of the familiar force between two macroscopic magnets (Isgur 1980). The situation is illustrated in Figure 7. The factor of $\frac{8\pi}{3}$ in equation (6.2) is just a geometrical factor, viz. $2 \times (\frac{4\pi}{3})$, where the factor of $\frac{4\pi}{3}$ comes from the volume of the sphere in which the colour-magnetic field, $\underline{B}_j$, of quark j is parallel to the colour-magnetic dipole moment, $\underline{\mu}_i$, of quark i. The contact term, being a scalar operator in both space and spin variables, operates only when the quark pair has zero orbital angular momentum, whilst the tensor term, which is an operator of rank two in both space and spin variables, is operative only when the quark pair has non-zero orbital angular momentum.

The full Hamiltonian can be written (c.f. equations (1.28) and (1.31)):

$$H = H_o + \sum_{i<j} H_{ij}^{hyp} \tag{6.4a}$$

where :

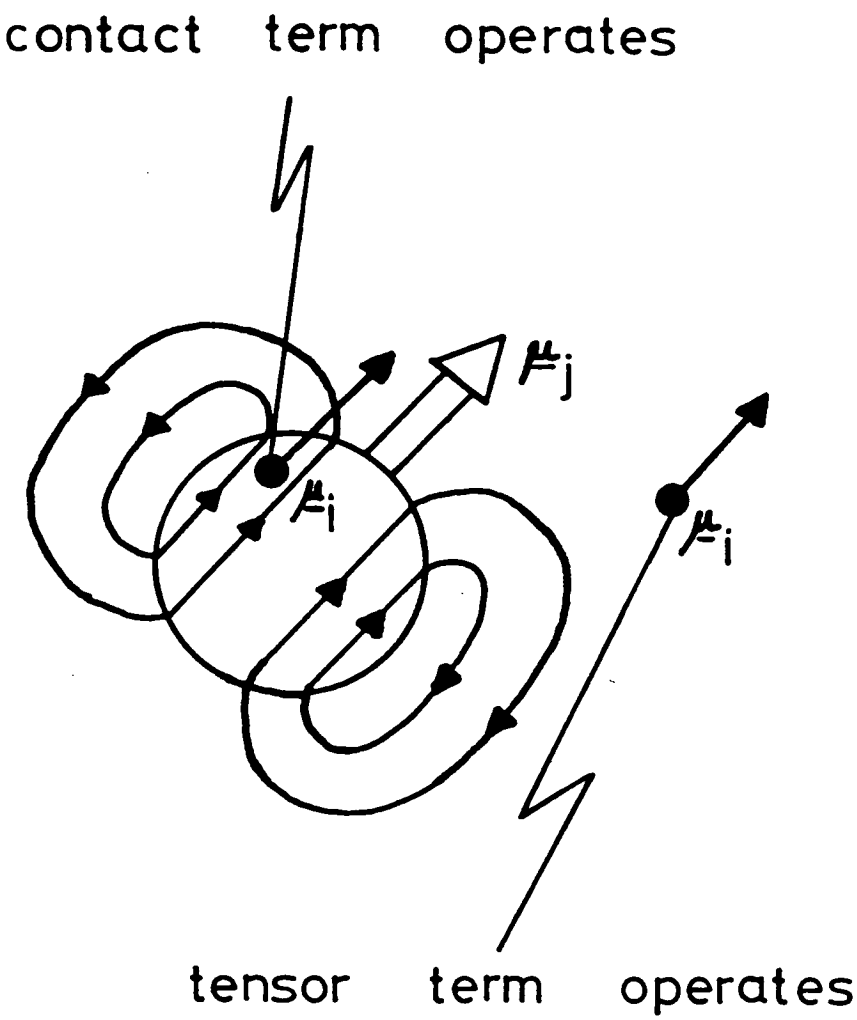


Figure 7 A diagrammatic representation of the origin of the contact and tensor terms

$$H_o = \sum_i (m + \underline{p}_i^2/2m) + \sum_{i<j} (\tfrac{1}{2}K\underline{r}_{ij}^2 + U(r_{ij})). \qquad (6.4b)$$

Note that, since we are concerned only with non-strange resonances, we may take all three quarks to have the same mass, m. Inclusion of the hyperfine term induces splitting and mixing amongst the N = 3 states: in particular, the tensor term will give rise to mixing of the $\Delta D35$ states belonging to the N = 3 $[\underline{56},1^-]$, $[\underline{70},2^-]$, $[\underline{56},3^-]$ and $[\underline{70},3^-]$ supermultiplets. In Appendix A, we explain the nomenclature for non-strange baryon resonances and list, for completeness, all possible mixings amongst N = 3 supermultiplets which exist for these resonances.

The matrix elements of the contact term can be computed by means of the identity:

$$\langle \Delta^{2S+1}L_pJ^- | H^{\text{contact}} | \Delta^{2S'+1}L'_p,J^- \rangle$$
$$= \delta_{LL'}\cdot\delta_{SS'}\cdot \frac{\pi^{3/2}D}{\alpha^3} \, {}^P\langle SS|\{(\underline{s}_1+\underline{s}_2)^2 - \tfrac{3}{2}\}|SS\rangle^{P'} \langle \psi^P_{LL}|\delta^3(\underline{\rho})|\psi^{P'}_{LL}\rangle \,, \qquad (6.5)$$

whereas the matrix elements of the tensor term are found via the identity (Brink and Satchler 1962):

$$\langle \Delta^{2S+1}L_pJ^- | \rho^{-5}[3(\underline{s}_1\cdot\underline{\rho})(\underline{s}_2\cdot\underline{\rho}) - (\underline{s}_1\cdot\underline{s}_2)\underline{\rho}^2] | \Delta^{2S'+1}L'_p,J^- \rangle$$
$$= (-1)^{J-L-S'}(2L+1)^{\frac{1}{2}}(2S+1)^{\frac{1}{2}}W(LL'SS';2J)$$
$$\times {}^P\langle S \| \tfrac{1}{2}(s_{1+}s_{2-} + s_{1-}s_{2+} - 4s_{1z}s_{2z}) \| S'\rangle^{P'}$$
$$\times {}^P\langle L \| \tfrac{1}{2}\rho^{-5}(\underline{\rho}^2 - 3\rho_z{}^2) \| L'\rangle^{P'} \,, \qquad (6.6)$$

where W is a Racah coefficient and the last two factors on the RHS of equation (6.6) are reduced matrix elements of the tensors whose zeroth-components are displayed. $|\Delta^{2S+1}L_pJ^-\rangle$ represents a negative-

parity Δ state of orbital angular momentum, L, total spin, S, and total angular momentum, J. P is the permutation symmetry of the spatial wavefunction. The form of the spatial wavefunction, $|\psi^{P}_{LL}>$, can be inferred from:

$$|\psi^{P}_{LL}> \quad = \quad \hat{\psi}^{(P)}_{LL}|0> \quad ; \tag{6.7}$$

the corresponding monomial of creation operators, $\hat{\psi}^{(P)}_{LL}$, is given in Table 3.2. The required spin wavefunctions are listed in Table 6.1:

Spin state	Spin wavefunction
$\left\|\frac{3}{2}\frac{3}{2}\right>^{S}$	$\|\uparrow\uparrow\uparrow>$
$\left\|\frac{3}{2}\frac{1}{2}\right>^{S}$	$\frac{1}{\sqrt{3}}\{\|\uparrow\uparrow\downarrow> + \|\uparrow\downarrow\uparrow> + \|\downarrow\uparrow\uparrow>\}$
$\left\|\frac{3}{2}-\frac{1}{2}\right>^{S}$	$\frac{1}{\sqrt{3}}\{\|\downarrow\uparrow\downarrow> + \|\downarrow\downarrow\uparrow> + \|\uparrow\downarrow\downarrow>\}$
$\left\|\frac{3}{2}-\frac{3}{2}\right>^{S}$	$\|\downarrow\downarrow\downarrow>$
$\left\|\frac{1}{2}\frac{1}{2}\right>^{M_\rho}$	$\frac{1}{\sqrt{2}}\{\|\uparrow\uparrow\downarrow> - \|\downarrow\uparrow\uparrow>\}$
$\left\|\frac{1}{2}-\frac{1}{2}\right>^{M_\rho}$	$\frac{1}{\sqrt{2}}\{\|\uparrow\downarrow\uparrow> - \|\downarrow\uparrow\downarrow>\}$
$\left\|\frac{1}{2}\frac{1}{2}\right>^{M_\lambda}$	$-\frac{1}{\sqrt{6}}\{\|\uparrow\uparrow\downarrow> + \|\downarrow\uparrow\uparrow> - 2\|\uparrow\uparrow\downarrow>\}$
$\left\|\frac{1}{2}-\frac{1}{2}\right>^{M_\lambda}$	$\frac{1}{\sqrt{6}}\{\|\uparrow\downarrow\downarrow> + \|\downarrow\uparrow\downarrow> - 2\|\downarrow\downarrow\uparrow>\}$

TABLE 6.1 3-quark spin wavefunctions

We illustrate the method with a specific example - the calculation of the matrix element of $\sum_{i<j} H^{hyp}_{ij}$ for the $|\Delta^4 P_s \frac{5}{2}^->$ state or, equivalently, the $\Delta D35$ state belonging to the $N = 3$ $[\underline{56},1^-]$

supermultiplet:

A. Contact term

$$\langle \Delta^4 P_s \tfrac{5}{2}^- | H^{\text{contact}} | \Delta^4 P_s \tfrac{5}{2}^- \rangle \tag{6.8}$$
$$= \frac{\pi^{3/2} D}{\alpha^3} \; {}^s\langle \tfrac{3}{2}\tfrac{3}{2} | \{ (\underline{s}_1+\underline{s}_2)^2 - \tfrac{3}{2} \} | \tfrac{3}{2}\tfrac{3}{2} \rangle^s \langle \psi^s_{11} | \delta^3(\underline{\rho}) | \psi^s_{11} \rangle \, ,$$

using equation (6.5). We have, from Table 6.1:

$$| \tfrac{3}{2}\tfrac{3}{2} \rangle^s = |\uparrow\uparrow\uparrow\rangle \tag{6.9}$$

so that:

$${}^s\langle \tfrac{3}{2}\tfrac{3}{2} | \{ (\underline{s}_1+\underline{s}_2)^2 - \tfrac{3}{2} \} | \tfrac{3}{2}\tfrac{3}{2} \rangle^s = \frac{1}{2} \, . \tag{6.10}$$

The spatial wavefunction, ψ^s_{11}, is given by:

$$|\psi^s_{11}\rangle \equiv \hat{\psi}^{(s)}_{11} |0\rangle = \frac{\alpha^3}{\sqrt{10}} [(\underline{\rho}^2 - \underline{\lambda}^2)(\lambda_x + i\lambda_y)$$
$$+ 2(\underline{\rho}.\underline{\lambda})(\rho_x + i\rho_y)] . \frac{\alpha^3}{\pi^{3/2}} \exp[-\tfrac{1}{2}\alpha^2(\underline{\rho}^2+\underline{\lambda}^2)] \tag{6.11}$$

and a little labour reveals:

$$\langle \psi^s_{11} | \delta^3(\underline{\rho}) | \psi^s_{11} \rangle = \frac{7\alpha^3}{8\pi^{3/2}} \, . \tag{6.12}$$

Thus, we obtain, finally:

$$\langle \Delta^4 P_s \tfrac{5}{2}^- | H^{\text{contact}} | \Delta^4 P_s \tfrac{5}{2}^- \rangle = \frac{\pi^{3/2} D}{\alpha^3} \cdot \frac{1}{2} \cdot \frac{7\alpha^3}{8\pi^{3/2}} = \frac{7}{16} D. \tag{6.13}$$

B. Tensor term

The permutational symmetry of the non-strange sector allows us to write:

$$\langle \Delta^4 P_s \tfrac{5}{2}^- | H^{\text{tensor}} | \Delta^4 P_s \tfrac{5}{2}^- \rangle = 3 \langle \Delta^4 P_s \tfrac{5}{2}^- | H^{\text{tensor}}_{12} | \Delta^4 P_s \tfrac{5}{2}^- \rangle . \tag{6.14}$$

We can use equations (6.2) and (6.6) to rewrite this as:

$$\langle \Delta^4 P_s \, \tfrac{5}{2}^- | H^{\text{tensor}} | \Delta^4 P_s \, \tfrac{5}{2}^- \rangle = 3 . \frac{D\sqrt{\pi}}{4\alpha^3} \cdot (-1)^{(\frac{5}{2}-1-\frac{3}{2})} . 2\sqrt{3} . W(11 \tfrac{3}{2} \tfrac{3}{2}; 2 \tfrac{5}{2})$$

$$\times {}^s\langle \tfrac{3}{2} \| \tfrac{1}{2}(s_{1+}s_{2-} + s_{1-}s_{2+} - 4s_{1z}s_{2z}) \| \tfrac{3}{2} \rangle^s$$

$$\times {}^s\langle 1 \| \tfrac{1}{2} \rho^{-5} (\underline{\rho}^2 - 3\rho_z^2) \| 1 \rangle^s . \quad (6.15)$$

The value of the Racah coefficient is found to be (Biedenharn and Van Dam 1965):

$$W(11 \tfrac{3}{2} \tfrac{3}{2}; 2 \tfrac{5}{2}) = \frac{1}{10\sqrt{6}} , \quad (6.16)$$

and noting that, because of the Wigner-Eckart theorem:

$$ {}^s\langle \tfrac{3}{2} \| \tfrac{1}{2}(s_{1+}s_{2-} + s_{1-}s_{2+} - 4s_{1z}s_{2z}) \| \tfrac{3}{2} \rangle^s$$

$$= \frac{ {}^s\langle \tfrac{3}{2} \tfrac{3}{2} | \tfrac{1}{2}(s_{1+}s_{2-} + s_{1-}s_{2+} - 4s_{1z}s_{2z}) | \tfrac{3}{2} \tfrac{3}{2} \rangle^s}{\langle 2 \tfrac{3}{2} \tfrac{3}{2} \tfrac{3}{2} | 2 \tfrac{3}{2} 0 \tfrac{3}{2} \rangle} \quad (6.17a)$$

and:

$$ {}^s\langle 1 \| \tfrac{1}{2}\rho^{-5} (\underline{\rho}^2 - 3\rho_z^2) \| 1 \rangle^s = \frac{\langle \psi^s_{11} | \frac{1}{2}\rho^{-5}(\underline{\rho}^2 - 3\rho_z^2) | \psi^s_{11} \rangle}{\langle 2111 | 2101 \rangle} , \quad (6.17b)$$

we obtain:

$$\langle \Delta^4 P_s \, \tfrac{5}{2}^- | H^{\text{tensor}} | \Delta^4 P_s \, \tfrac{5}{2}^- \rangle = \frac{3\sqrt{3}}{2} \cdot \frac{D\sqrt{\pi}}{\alpha^3} \cdot \frac{1}{10\sqrt{6}} \cdot \sqrt{5} \cdot \sqrt{10}$$

$$\times {}^s\langle \tfrac{3}{2} \tfrac{3}{2} | \tfrac{1}{2}(s_{1+}s_{2-} + s_{1-}s_{2+} - 4s_{1z}s_{2z}) | \tfrac{3}{2} \tfrac{3}{2} \rangle^s$$

$$\times \langle \psi^s_{11} | \tfrac{1}{2}\rho^{-5}(\underline{\rho}^2 - 3\rho_z^2) | \psi^s_{11} \rangle . \quad (6.18)$$

Note that we have evaluated the Clebsch-Gordan coefficients (Abramowitz and Stegun 1965):

$$\langle 2\ \tfrac{3}{2}\ \tfrac{3}{2}\ \tfrac{3}{2} \mid 2\ \tfrac{3}{2}\ 0\ \tfrac{3}{2}\rangle = \frac{1}{\sqrt{5}} \tag{6.19a}$$

[notation: $\langle j_1 j_2 jm | j_1 j_2 m_1 m_2\rangle$]

$$\langle 2\ 1\ 1\ 1 \mid 2\ 1\ 0\ 1\rangle = \frac{1}{\sqrt{10}} \tag{6.19b}$$

in arriving at equation (6.18). Using equations (6.9) and (6.11), we deduce:

$$^{s}\langle \tfrac{3}{2}\ \tfrac{3}{2} | \tfrac{1}{2}(s_{1+}s_{2-} + s_{1-}s_{2+} - 4s_{1z}s_{2z}) | \tfrac{3}{2}\ \tfrac{3}{2}\rangle^{s} = -\frac{1}{2} \tag{6.20a}$$

whilst:

$$\langle \psi^{s}_{11} | \frac{1}{2}\rho^{-5}(\underline{\rho}^2 - 3\rho_z^2) | \psi^{s}_{11}\rangle = -\frac{2\alpha^3}{75\sqrt{\pi}} . \tag{6.20b}$$

Thus, finally, we have:

$$\langle \Delta^4 P_s\ \tfrac{5}{2}^- | H^{tensor} | \Delta^4 P_s\ \tfrac{5}{2}^-\rangle = \frac{3}{4}\cdot\frac{D\sqrt{\pi}}{\alpha^3}\cdot(-\tfrac{1}{2})\cdot(-\frac{2\alpha^3}{75\sqrt{\pi}}) = \frac{1}{100}D. \tag{6.21}$$

Equations (6.13) and (6.21) together lead to:

$$\langle \Delta^4 P_s\ \tfrac{5}{2}^- | \sum_{i<j} H^{hyp}_{ij} | \Delta^4 P_s\ \tfrac{5}{2}^-\rangle = (\frac{7}{16} + \frac{1}{100})D . \tag{6.22}$$

In exactly the same fashion, we find, in terms of the basis states:

$$(\Delta^4 P_s\ \tfrac{5}{2}^-,\ \Delta^2 D_M\ \tfrac{5}{2}^-,\ \Delta^4 F_s\ \tfrac{5}{2}^-,\ \Delta^2 F_M\ \tfrac{5}{2}^-) \tag{6.23a}$$

corresponding to:

$$(\Delta D35(\underline{56},1^-),\ \Delta D35(\underline{70},2^-),\ \Delta D35(\underline{56},3^-),\ \Delta D35(\underline{70},3^-)): \tag{6.23b}$$

$$H_{D35} = \begin{bmatrix} \{E_o+3\Omega-\frac{11}{10}\Delta+(\frac{7}{16}+\frac{1}{100})D\} & \frac{3\sqrt{5}}{200}D & -\frac{9\sqrt{21}}{700}D & -\frac{9\sqrt{70}}{2800}D \\ \frac{3\sqrt{5}}{200}D & \{E_o+3\Omega-\frac{2}{5}\Delta\} & \frac{\sqrt{105}}{175}D & 0 \\ -\frac{9\sqrt{21}}{700}D & \frac{\sqrt{105}}{175}D & \{E_o+3\Omega-\frac{3}{5}\Delta+ & -\frac{3\sqrt{30}}{350}D \\ & & +(\frac{1}{8}+\frac{9}{350})D\} & \\ -\frac{9\sqrt{70}}{2800}D & 0 & -\frac{3\sqrt{30}}{350}D & \{E_o+3\Omega-\frac{7}{10}\Delta+ \\ & & & +\frac{1}{2}\delta+\frac{3}{16}D\} \end{bmatrix}$$

(6.24)

having first recalled the matrix elements of H_o in equations (3.21) - (3.23) and (3.26). Whilst the lack of data for possible N=3 resonances prevents us fro assigning a best-fit value to δ, we can find approximate values for E_o, Ω, Δ and D.

Isgur and Karl (1978b) obtained a good description of the low-lying, negative-parity baryon states assigned to the $N = 1$ oscillator level using the values $E_o \simeq 1090$ MeV, $\Omega \simeq 520$ MeV and $D \simeq 300$ MeV, whilst the same authors (Isgur and Karl 1979a) found a whole host of positive-parity, excited baryon states could be accommodated neatly into the $N = 2$ oscillator level by choosing $E_o + 2\Omega \simeq 2020$ MeV, $\Delta \simeq 420$ MeV and $D \simeq 300$ MeV. More recently, Isgur and Karl (1979b) have successfully fitted the ground-state baryons to the N=0 oscillator level by setting $E_o \simeq 1135$ MeV and $D \simeq 260$ MeV.

The point we wish to stress is that, as is clear from the fits of Isgur and Karl to the $N = 0$, 1 and 2 oscillator levels, it is not surprising that the most reasonable assignments for the parameters E_o, Ω, Δ and D will vary, albeit slightly, from one oscillator level to the next. We emphasise that the Isgur-Karl model remains, despite its many successes, a simple, naïve model of the interactions between the quarks confined in a baryon. An harmonic oscillator potential is an unlikely

candidate for the confining potential of the real world; if we choose as a first approximation to the confining potential between two quarks an harmonic oscillator potential, we should anticipate that its curvature will change as we pass from one level to the next. Thus, it is entirely reasonable to expect that the effective oscillator-spacing, Ω, will vary, albeit fairly smoothly, from one level to the next (Hey 1981).

We take account of the possibility of such small variations by taking the parameters to lie in the following ranges:

$$E_o \simeq 1150 \pm 50 \text{ MeV} \qquad (6.25a)$$

$$\Omega \simeq 440 \pm 20 \text{ MeV} \qquad (6.25b)$$

$$\Delta \simeq 440 \pm 20 \text{ MeV} \qquad (6.25c)$$

$$D \simeq 280 \pm 20 \text{ MeV} \quad . \qquad (6.25d)$$

We find good agreement with the result of Cutkosky (1980) for the mass of the $\Delta D35(1940)$ resonance, viz.:

$$M(\Delta D35) = 1940 \pm 30 \text{ MeV} \qquad (6.26)$$

by taking $E_o = 1100$ MeV, $\Omega = 420$ MeV, $\Delta = 460$ MeV and $D = 280$ MeV, and diagonalising the matrix H_{D35}, numerically, for a range of values of the parameter δ. Specifically, δ was allowed to vary, in steps of 50 MeV, from -600 MeV to + 600 MeV. The results are shown graphically in Figure 8. We find that the value of the lowest eigenvalue of H_{D35} [to be identified as the prediction of this model for the mass of the $\Delta D35(1940)$ state] and the composition of the corresponding eigenvector are highly insensitive to variations in the value of δ, provided δ remains positive. Let us choose, for the purposes of illustration, a value of $\delta = 50$ MeV: we obtain the spectrum of mean $N = 3$ non-strange supermultiplet masses which is displayed in Figure 9. With $\delta = 50$ MeV,

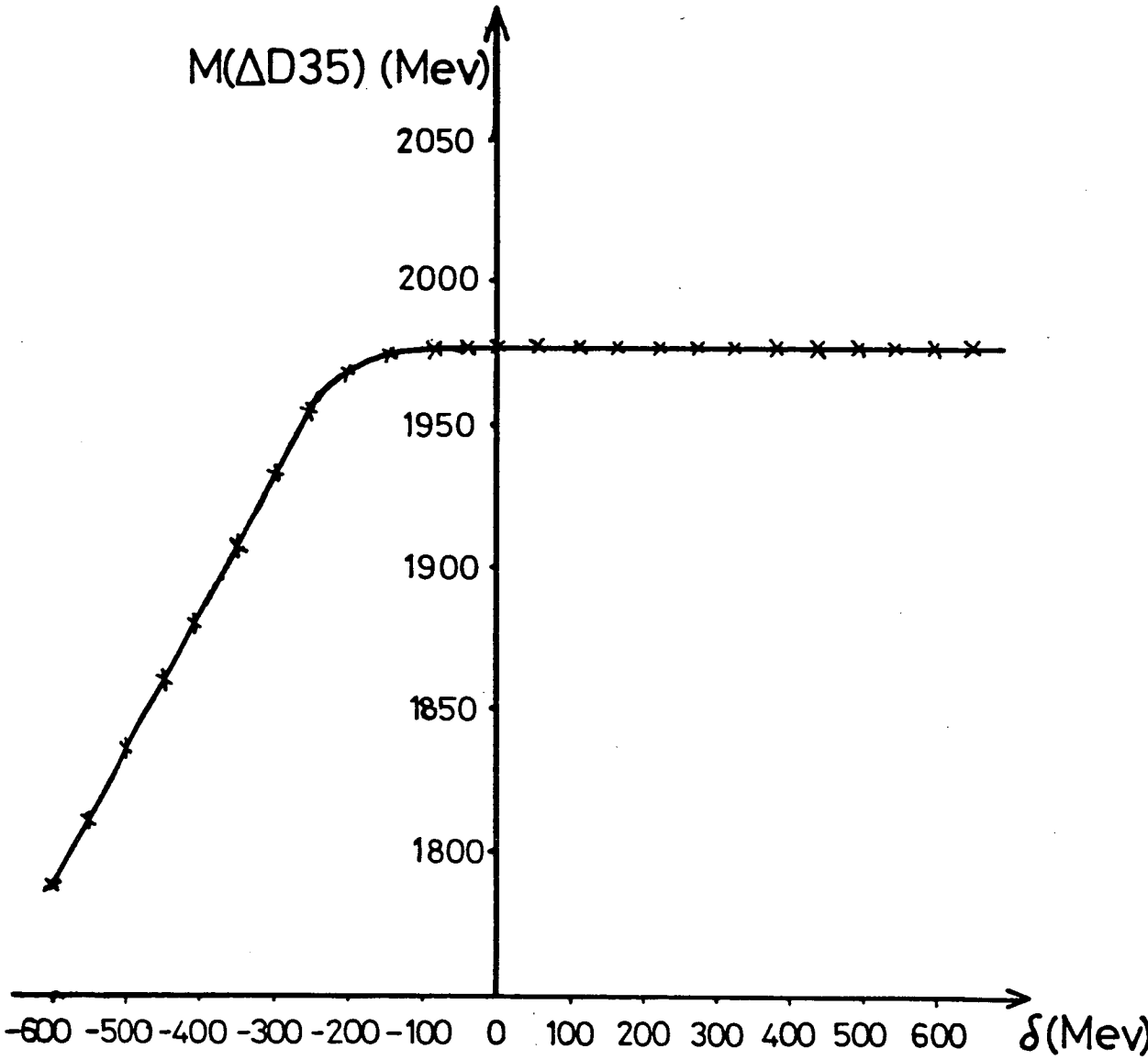


Figure 8 M(ΔD35) as a function of δ (with E_o, Ω, Δ and D fixed)

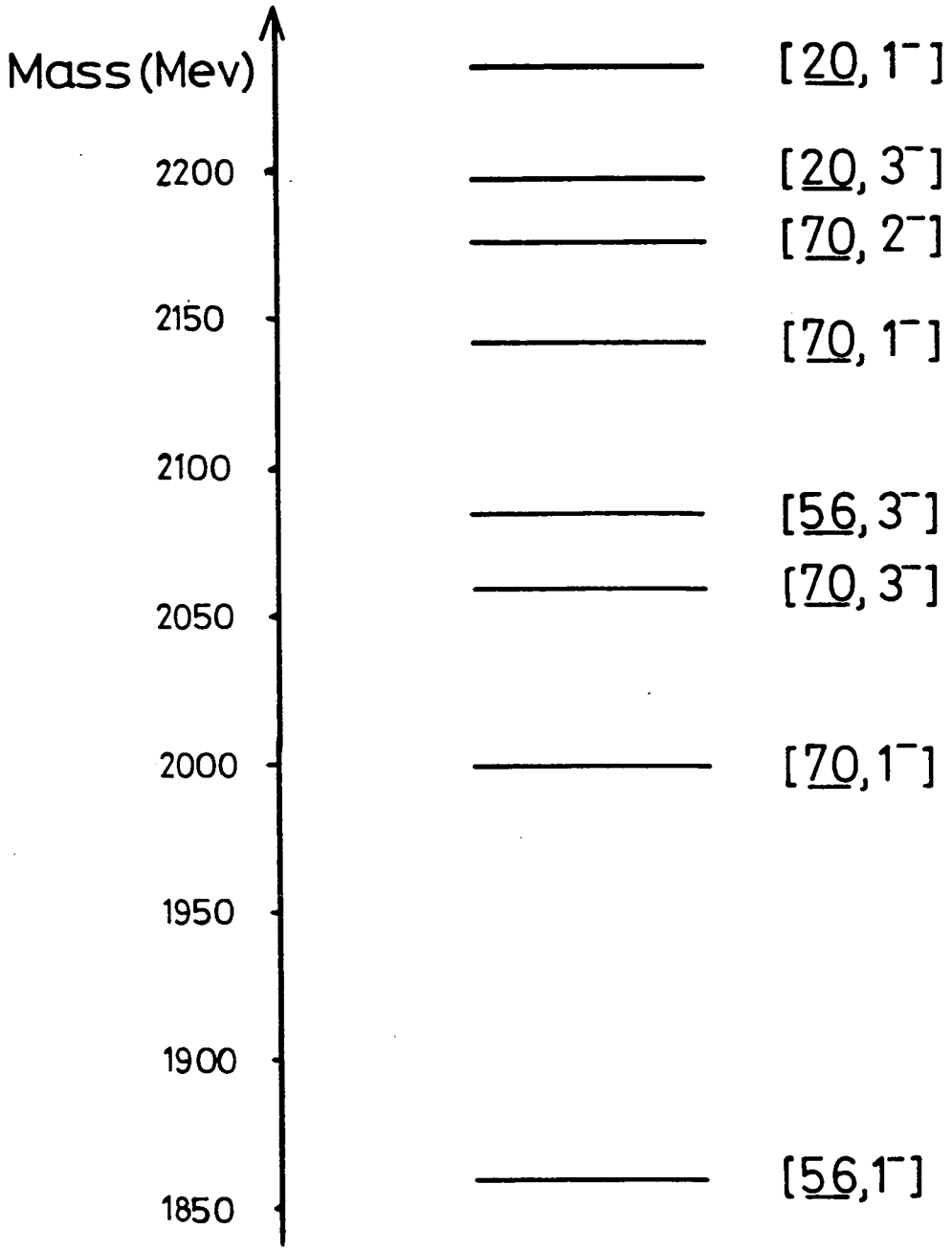


Figure 9 Mean N = 3 non-strange supermultiplet masses when $\delta = 50$ Mev

we find the lowest eigenvalue of H_{D35} to be approximately 1975 MeV and the composition of the corresponding eigenvector to be (0.9881, -0.0585. 0.1144, 0.0839), i.e. the ΔD35(1940) state is predicted to be an almost pure $[\underline{56},1^-]$ state.

We conclude that, after inclusion of hyperfine effects and with reasonable values for the parameters E_o, Ω, Δ, δ and D, the Isgur-Karl model can easily accommodate the result of Cutkosky (1980) for the mass of the ΔD35(1940) state, and that, contrary to previous claims in the literature (Cutkosky and Hendrick 1977a, 1977b), the ΔD35 (1940) resonance does not represent unambiguous evidence for new degrees of freedom inside baryons.

6.2 Negative-parity resonances in the D35, G37 and G39 sectors of the N = 3 spectrum

We demonstrate in this section how further information can be extracted from the calculations performed in §6.1. In particular, we examine within the framework of the Isgur-Karl model the implications of preliminary observations in the G37 and G39 sectors, and the possible observation of a second resonance in the D35 channel, as reported recently by Cutkosky (1980). As demonstrated in §6.1, the tensor term gives rise to mixing of the four ΔD35 states belonging to the N = 3 $[\underline{56},1^-]$, $[\underline{70},2^-]$, $[\underline{56},3^-]$ and $[\underline{70},3^-]$ supermultiplets; it also causes mixing between the two ΔG37 states belonging to the N = 3 $[\underline{56},3^-]$ and $[\underline{70},3^-]$ supermultiplets (c.f. Appendix A). It is a relatively simple matter to infer from the form of H_{D35} in equation (6.24) that, in terms of the basis states:

$$(\Delta^4 F_S \tfrac{7}{2}^-,\ \Delta^2 F_M \tfrac{7}{2}^-) \qquad (6.27a)$$

corresponding to:

$$(\Delta G37(\underline{56},3^-),\ \Delta G37(\underline{70},3^-)) : \tag{6.27b}$$

$$H_{G37} = \begin{bmatrix} \{E_o + 3\Omega - \frac{3}{5}\Delta + (\frac{1}{8} + \frac{3}{35})D & \frac{3\sqrt{2}}{140} D \\ \frac{3\sqrt{2}}{140} D & \{E_o + 3\Omega - \frac{7}{10}\Delta + \frac{1}{2}\delta + \frac{3}{16}D\} \end{bmatrix} . \tag{6.28}$$

Further, since the quantum numbers of the ΔG39 resonance necessarily imply that it be assigned to the $N = 3$ $[\underline{56},3^-]$ supermultiplet, the mass of the ΔG39 state is given by:

$$M(\Delta G39) = E_o + 3\Omega - \frac{3}{5}\Delta + (\frac{1}{8} - \frac{3}{70})D . \tag{6.29}$$

Note that this expression is independent of δ.

The results in equations (6.28) and (6.29) follow from the observations:

A. Matrix elements of the contact term depend only on the orbital angular momenta, L and L', and the total spins, S and S', of the states involved, and not on how L and S (or L' and S') are coupled to form the total angular momentum, J. Thus, for example:

$$\langle \Delta^4 F_s \tfrac{7}{2}^- | H^{contact} | \Delta^4 F_s \tfrac{7}{2}^- \rangle = \langle \Delta^4 F_s \tfrac{5}{2}^- | H^{contact} | \Delta^4 F_s \tfrac{5}{2}^- \rangle = \frac{1}{8}D, \tag{6.30}$$

recalling equation (6.24).

B. Matrix elements of the tensor term depend explicitly on J via a phase factor $(-1)^J$ and a Racah coefficient $W(LL'SS'; 2J)$, as is clear from equation (6.6). Thus, for example:

$$\frac{\langle \Delta^4 F_s \tfrac{7}{2}^- | H^{tensor} | \Delta^4 F_s \tfrac{7}{2}^- \rangle}{\langle \Delta^4 F_s \tfrac{5}{2}^- | H^{tensor} | \Delta^4 F_s \tfrac{5}{2}^- \rangle} = (-1)^{(\frac{7}{2} - \frac{5}{2})} \frac{W(33\ \frac{3}{2}\ \frac{3}{2};\ 2\ \frac{7}{2})}{W(33\ \frac{3}{2}\ \frac{3}{2};\ 2\ \frac{5}{2})} = \frac{10}{3} \tag{6.31}$$

whence:

$$\langle \Delta^4 F_s \, \tfrac{7}{2}^- | H^{\text{tensor}} | \Delta^4 F_s \, \tfrac{7}{2}^- \rangle = \frac{3}{35} D, \qquad (6.32)$$

upon recalling equation (6.24).

Included in the results of the most recent phase-shift analysis by Cutkosky et al. (Cutkosky 1980) are the following observations which are listed in Table 6.2:

State	Mass (MeV)	Rating	$SU(6)_{\text{flavour}\times\text{spin}} \otimes O(3)$
D35	1940 ± 30	****	$[\underline{56},1^-]$
D35	2400 ± 125	**	not quoted
G37	2200 ± 80	**	$[\underline{70},3^-]$
G39	2300 ± 100	**	not quoted

Table 6.2 Experimentally-determined masses and compositions of some N = 3 non-strange baryon resonances in the D35, G37 and G39 channels.

The $SU(6)_{\text{flavour}\times\text{spin}} \otimes O(3)$ assignments in Table 6.2 are taken from an earlier article by Cutkosky et al. (1979).

We present in Table 6.3 the results we obtain by taking E_o = 1100 MeV, Ω = 420 MeV, Δ = 460 MeV, δ = 600 MeV and D = 280 MeV and diagonalising the matrices H_{D35} and H_{G37}, numerically. We stress that a value of 50 MeV was assigned to δ in §6.1 purely for the purposes of illustration and that increasing the value of δ (with the parameters E_o, Ω, Δ and D fixed at the values indicated in §6.1) serves only to improve, albeit slightly, the good agreement found in §6.1 with the result of Cutkosky et al. (Cutkosky 1980) for the mass and composition of the ΔD35 (1940) state.

State	Approx. Mass (MeV)	Approx. composition	$SU(6)_{flavour \times spin} \otimes O(3)$ content
D35	1975	(0.99, -0.06, 0.11, 0.03)	([56,1^-],[70,2^-],[56,3^-],[70,3^-])
D35	2120	(-0.12, -0.25, 0.96, 0.05)	([56,1^-],[70,2^-],[56,3^-],[70,3^-])
D35	2180	(0.03, 0.97, 0.25, 0.02)	([56,1^-],[70,2^-],[56,3^-],[70,3^-])
D35	2390	(-0.02, -0.01, -0.06,0.99)	([56,1^-],[70,2^-],[56,3^-],[70,3^-])
G37	2145	(0.99, -0.04)	([56,3^-], [70,3^-])
G37	2390	(0.04, 0.99)	([56,3^-], [70,3^-])
G39	2105	1.00	[56,3^-]

TABLE 6.3 Predicted masses and compositions of the N = 3 non-strange baryon resonances in the D35, G37 and G39 channels.

The first remark we wish to make is that the value we obtain for the mass of the ΔG39 state is somewhat lower than that observed by Cutkosky et al. (Cutkosky 1980). We believe that this discrepancy simply reflects the fact that the values we have assigned to the parameters E_o, Ω, Δ, δ and D are not best-fit values. A full fit to the N = 3 non-strange spectrum is currently being performed by other authors (Forsyth 1981) who, in a private communication, have already confirmed the validity of the matrix elements in equation (6.24): we are confident that the best-fit values appropriate to the N = 3 level for the parameters E_o, Ω, Δ and D will lie in the ranges quoted in equations (6.25a)-(6.25d). If we substitute these values for E_o, Ω, Δ and D into equation (6.29), we deduce

$$M(\Delta G39) \simeq 2230 \pm 125 \text{ MeV} \qquad (6.33)$$

so that the preliminary result of Cutkosky et al. (Cutkosky 1980), viz.:

$$M(\Delta G39) = 2300 \pm 100 \text{ MeV} \qquad (6.34)$$

can be easily accommodated.

Our second remark concerns the preliminary observation, as reported by Cutkosky (1980), of a resonance in the G37 channel, for which a mass of 2200 ± 80 MeV is quoted. In our scheme, there are two ΔG37 resonances of mass 2145 MeV and 2390 MeV, respectively. The former ΔG37 state is predicted to be an almost pure $[\underline{56},3^-]$ state, whilst the latter ΔG37 state is an almost pure $[\underline{70},3^-]$ state. We find that the mass and composition of the lower-mass ΔG37 state are highly insensitive to variations in the value of δ, provided δ remains above 200 MeV, whilst the mass of the higher-mass ΔG37 resonance increases uniformly with δ. We suggest, therefore, that the ΔG37(2200) state observed by Cutkosky et al. (Cutkosky 1980) may be more naturally assigned to the N = 3 $[\underline{56},3^-]$ supermultiplet rather than to the N = 3 $[\underline{70},3^-]$ supermultiplet as these authors suggest (Cutkosky et al. 1979).

Finally, we remark that we can easily accommodate the preliminary result of Cutkosky et al. (Cutkosky 1980) for the mass of the 2-star ΔD35 resonance, viz.:

$$M(\Delta D35) = 2400 \pm 125 \text{ MeV} \qquad (6.35)$$

provided we identify this resonance with the highest-mass state of our quartet of ΔD35 resonances. We assign an approximate mass of 2390 MeV to this resonance and predict that it be an almost pure $[\underline{70},3^-]$ state. We have no explanation to offer as to why the remaining intermediate-mass ΔD35 resonances in our scheme have not been observed so far. We find that the masses and compositions of the two intermediate-mass ΔD35 resonances are highly insensitive to variations in the value of δ, provided δ remains greater than 200 MeV.

We conclude that we can recover the pertinent features of the most recent preliminary phase-shift analysis performed by Cutkosky et al. (Cutkosky 1980), within the framework of the Isgur-Karl model, with due

inclusion of hyperfine effects arising out of coloured-gluon exchange. The determination of the best-fit values appropriate to the $N = 3$ level for the parameters E_o, Ω, Δ, δ and D remains the outstanding problem.

CHAPTER 7

CONCLUSIONS

The observation by Isgur and Karl (1979a) that the pattern of splittings of the $N = 2$ supermultiplets in Figure 3, induced by an arbitrary anharmonic perturbation, $\sum_{i<j} U(r_{ij})$, is independent of the detailed form of $U(r_{ij})$, suggests that the result may be derived from purely group-theoretical arguments and corresponds to the breaking of a symmetry of the unperturbed system. The principal aim of this thesis has been to justify this assertion.

The results for the $N = 2$ level (Isgur and Karl 1979a) and some results at the $N = 3$ level (Dalitz et al. 1977b) were originally derived using explicit oscillator wavefunction techniques. However, we presented in Chapter 3 an alternative derivation of these results based on an algebraic procedure which relies for its effect on the commutation relations of the creation and annihilation operators of the harmonic oscillator.

As a first step towards recovering these results via group-theoretical techniques, we studied in Chapter 4 the simpler system of two equal-mass particles bound principally by harmonic-oscillator forces. It might be expected that adding an arbitrary anharmonic potential into the Hamiltonian would introduce a large number of parameters. However, direct calculation using first-order perturbation theory revealed that the perturbed energy levels are given in terms of a small number of independent parameters, to be identified as moments of the perturbing potential: on proceeding from one level to the next, only one extra parameter appears. We re-interpreted our results as manifesting the breaking of the dynamical $U(3)$ symmetry

of the unperturbed system and successfully recovered our results for $n \leq 2$ by means of the spectrum-generating group, $Sp(6,R)$.

Encouraged by this success, we extended our considerations to embrace a system of three equal-mass quarks, confident that we could successfully employ the spectrum-generating group, $Sp(12,R)$, to account for the pattern of splittings in Figure 3 in terms of the breaking of the dynamical $U(6)$ symmetry of the unperturbed system. We had previously introduced the necessary formalism in Chapter 2, where we described in detail a novel and particularly simple and direct method of utilising the group $Sp(12,R)$ and its subgroups appearing in the embedding chain in equation (2.28) to classify the unperturbed oscillator eigenstates and construct the corresponding spatial wavefunctions of definite angular momentum and permutation-symmetry type: the $O(2)$ subgroup is particularly convenient for the enumeration and construction of states of definite symmetry under the permutation group, S_3, and the $SO(3)$ subgroup gives the angular momentum content. The relevance of the group $Sp(12,R)$ to the splitting pattern induced by the anharmonic perturbation is that $U(r_{ij})$ is naturally classified under $Sp(12,R)$. For example, at the $N = 2$ level, there is a unique breaker of $Sp(12,R)$ which can contribute to the splitting pattern, and we explicitly constructed in Chapter 5 an algebraic mass formula, involving the quadratic Casimir invariants of the various subgroups which appear in Figure 6, plus one non-Casimir invariant, which successfully reproduces the splitting pattern of Figure 3. We had previously been alerted to the need to include non-Casimir invariants in our considerations in Chapter 4.

Having thus achieved our principal goal, we were naturally led to consider the phenomenological implications of our results for baryon

spectroscopy. Whilst it is true that the non-relativistic harmonic oscillator quark model has been remarkably successful in accounting for many features of the baryon spectrum (Isgur 1980 and references therein), nevertheless the theoretical foundations of this non-relativistic potential model are not well understood - certainly not from a fundamental standpoint like that of QCD. It is therefore important to examine possible deviations from this rather simple picture of baryons. The failure of previous attempts in the literature (Cutkosky and Hendrick 1977a, 1977b; Dalitz et al. 1977b) to account for the $\Delta D35(1940)$ resonant state as a genuine 3-quark excitation - the mass predicted by Dalitz et al. (1977b) for the $\Delta D35(1940)$ state was some 150 MeV higher than the experimentally-determined mass of 1940 ± 30 MeV (Cutkosky 1980) - strongly suggested that a new degree of freedom in the baryon spectrum was being excited. Several possible realisations of such "extra" degrees of freedom exist. For example, the rigid, spherical-cavity approximation to the MIT bag model (De Grand et al. 1975) gives a good description of the ground-state $[\underline{56},0^+]$ supermultiplet. However, in order to generate the negative-parity resonances, the rigidity of the surface must be relaxed to allow small surface oscillations. Rebbi (1976) identified a $[\underline{70},1^-]$ supermultiplet together with extra $[\underline{56},1^-]$ supermultiplets which may be visualised as arising from oscillations of the 3-quark system with respect to the bag walls. One of these $[\underline{56},1^-]$ supermultiplets is identified with the zero mode corresponding to translation of the ground state: the rest are presumably extra non-3-quark physical $[\underline{56},1^-]$ supermultiplets. The detailed calculations of Rebbi (1976) suggested that the first such supermultiplet should lie below 2 GeV.

On the face of it, then, it seemed unlikely that the $\Delta D35(1940)$ state could be a genuine 3-quark state. Nonetheless, the fact that the

sum rule in equation (3.30) derived by Dalitz et al. (1977b) was based on the specific assumption of neglecting spin-tensor forces prompted us to re-examine this result. The success enjoyed by the Isgur-Karl model (Isgur 1980 and references therein) in providing a good semi-quantitative guide to both positive- and negative-parity, strange and non-strange baryon resonances up to about 2 GeV in mass is powerful testimony to the currently-held belief that such spin-tensor forces are important, more important certainly than the spin-orbit forces retained in the detailed $SU(6)_{flavour \times spin}$ mass analyses performed by Horgan and others (Horgan 1976a and references therein).

Our initial approach was considerably less ambitious than that of Dalitz et al. (1977b) in that we contented ourselves with predicting only the mean masses of the non-strange sectors of the various $SU(6)_{flavour \times spin} \otimes O(3)$ supermultiplets, with total neglect of the one-gluon-exchange hyperfine interactions. The interesting new physics occurs at the $N = 3$ level where the following eight supermultiplets are expected: $[\underline{56},3^-]$, $[\underline{56},1^-]$, $[\underline{70},3^-]$, $[\underline{70},2^-]$, $[\underline{70},1^-]$, $[\underline{70},1^-]$, $[\underline{20},3^-]$, and $[\underline{20},1^-]$. Turning on the anharmonic perturbation, $\sum_{i<j} U(r_{ij})$, splits these supermultiplets and introduces a new parameter, δ, which is defined in equation (3.25), specific to the $N = 3$ level. An important feature to note is that the masses of three of the $N = 3$ supermultiplets, viz., the $[\underline{56},1^-]$, $[\underline{70},2^-]$ and $[\underline{56},3^-]$ supermultiplets, are independent of δ: their masses are determined entirely by the $N = 2$ level parameters. Isgur and Karl (Isgur 1980 and references therein) obtained reasonable phenomenology for the $N = 0$, 1 and 2 levels by setting $E_o \simeq 1150$ MeV and $\Omega \simeq \Delta \simeq 440$ MeV. Using these values, we predict that the $[\underline{56},1^-]$ supermultiplet necessarily lies lowest of the $N = 3$ supermultiplets and we assign a mean mass of approximately 1985 MeV to the non-strange sector of the $[\underline{56},1^-]$

supermultiplet, only 45 MeV above the quoted mass, 1940 ± 30 MeV, for the $\Delta D35(1940)$ state (Cutkosky 1980). Given the simplicity of the model and our neglect of hyperfine interactions, this is remarkably good agreement, and lends confidence to our belief that,contrary to previous claims in the literature (Cutkosky and Hendrick 1977a, 1977b), the $\Delta D35(1940)$ does not constitute unambiguous evidence for gluonic degrees of freedom in the baryon spectrum.

Of course, a properly-consistent treatment of the $\Delta D35(1940)$ problem demands consideration of hyperfine effects and we duly considered these in Chapter 6. An important point contained in Chapter 6 is the realisation that the best-fit values for the parameters in the Isgur-Karl model may change, albeit slightly, from one level to the next. With this point firmly in mind, we confidently assert that the best-fit values appropriate to the $N = 3$ level for the parameters E_o, Ω, Δ and D very likely lie in the ranges indicated in equations (6.25a)-(6.25d); the lack of data for possible $N = 3$ baryon resonances, however, precludes the assignment of a best-fit value to the parameter, δ. Instead, we chose to diagonalise the Hamiltonian matrix H_{D35} for a range of values of δ, whilst fixing $E_o = 1100$ MeV, $\Omega = 420$ MeV, $\Delta = 460$ MeV and $D = 280$ MeV. We found that the value of the lowest eigenvalue of H_{D35} and the composition of the corresponding eigenvector are highly insensitive to variations in the value of δ, provided δ remains positive. In particular, setting $\delta = 50$ MeV implies an approximate mass of 1975 MeV for the $\Delta D35(1940)$ state, which, in addition, is predicted to be an almost pure $N = 3$ $[\underline{56},1^-]$ state. We conclude that the $\Delta D35(1940)$ state does not represent unambiguous evidence for the presence of non-3-quark degrees of freedom in baryons.

We turned, finally, in §6.2 to investigate the G37 and G39 sectors

of the $N = 3$ baryon spectrum, reassured in the knowledge that the Hamiltonian matrices for these sectors could be easily inferred from the form of H_{D35}. With $E_o = 1100$ MeV, $\Omega = 420$ MeV, $\Delta = 460$ MeV, $\delta = 600$ MeV and $D = 280$ MeV, we found we could recover the pertinent features appropriate to the D35, G37 and G39 sectors of the most recent phase-shift analysis performed by Cutkosky et al. (Cutkosky 1980). We stress that, in this respect, our aim has been simply to demonstrate that it <u>is</u> possible to assign values, appropriate to the $N = 3$ level, to the parameters of the Isgur-Karl model so as to ensure reasonable agreement with the results reported by Cutkosky (1980). By no means are we claiming that the best-fit value for δ, once it is finally determined, will lie in the near vicinity of 600 MeV. We are not unduly worried by the sizeable discrepancy which exists between our predicted mass of 2105 MeV for the ΔG39 resonant state and that quoted by Cutkosky (1980), viz. 2300 ± 100 MeV. Observing that the predicted mass for this state is independent of the value of δ, and that the range of values in equations (6.25a) - (6.25d) for the parameters E_o, Ω, Δ and D gives $M(\Delta G39) \simeq 2230 \pm 125$ MeV, we are confident that the discrepancy simply reflects the fact that the values we have assigned to these parameters are not the best-fit values appropriate to the $N = 3$ level. The determination of the best-fit values appropriate to the $N = 3$ level for the parameters E_o, Ω, Δ, δ and D remains the outstanding problem.

EPILOGUE

The work described in this thesis, then, has produced no evidence to suggest that the Isgur-Karl model will be any less successful in describing the full N = 3 baryon spectrum than it has been in accounting for the many features of the baryon spectrum up to about 2 GeV in mass (Isgur 1980 and references therein). Indeed, the Isgur-Karl model has been shown to be almost embarrassingly successful, so much so that it is difficult to imagine any model doing very much better. This state of affairs has earned the description of "the great Isgur-Karl disaster" from Hey (1980) who suggested, in somewhat tongue-in-cheek fashion, that there remains little in the context of baryon spectroscopy to be solved. However, to subscribe in earnest to such a view would be both short-sighted of us and an (unintentional) affront to those members of the physics community who have investigated in detail the rôle of spin-orbit forces in determining the baryon spectrum (Reinders 1980 and references therein). Far be it from us to indulge at this stage in any form of iconoclasm; nonetheless, this thesis would be shamefully incomplete without some discussion of spin-orbit forces.

The non-relativistic reduction of an arbitrary potential, V(r), originating from either scalar coupling (g = 0) or from vector coupling (g = 1), gives rise to the spin-orbit interaction (Reinders 1980 and references therein):

$$H_{ij}^{SO} = \frac{1}{2r_{ij}} \frac{dV}{dr_{ij}} \{(2g-1) \left[\frac{(\underline{r}_{ij} \times \underline{p}_i)\cdot\underline{s}_i}{m_i^2} - \frac{(\underline{r}_{ij} \times \underline{p}_j)\cdot\underline{s}_j}{m_j^2} \right] + \\ + \frac{2g}{m_i m_j} \left[(\underline{r}_{ij} \times \underline{p}_i)\cdot\underline{s}_j - (\underline{r}_{ij} \times \underline{p}_j)\cdot\underline{s}_i \right] \} \qquad (E.1)$$

where the g-independent terms are generated by the mechanism of Thomas precession in the potential V. Clearly, if V contains 3-body interactions, there will be corresponding 3-body terms in its spin-orbit component, $\sum_{i<j} H_{ij}^{SO}$, after the non-relativistic reduction; however, such terms have not yet been invoked in the literature (Close and Dalitz 1981 and references therein). In the equal-mass case, where $m_i = m_j = m$, equation (E.1) can be re-arranged to give:

$$H_{ij}^{SO} = \frac{1}{4m^2} \cdot \frac{1}{r_{ij}} \frac{dV}{dr_{ij}} \{(4g-1) \left[\underline{r}_{ij} \times (\underline{p}_i - \underline{p}_j)\right] \cdot (\underline{s}_i + \underline{s}_j) - \left[\underline{r}_{ij} \times (\underline{p}_i + \underline{p}_j)\right] \cdot (\underline{s}_i - \underline{s}_j)\} \quad . \tag{E.2}$$

Notice the presence of a (g-independent) translation-non-invariant term proportional to $(\underline{p}_i + \underline{p}_j)$ in equation (E.2). At first sight, this seems rather worrying. However, Close and Osborn (1970) have stressed that great care is needed concerning the relationship between centre-of-mass-frame (CM) and laboratory-frame variables <u>when the particles have intrinsic spin</u>. Conventionally, the calculation and specification of 2-particle interactions are carried out in the rest frame of the two particles; yet, these self-same interactions are then used to calculate the energy of the 3-particle system, constituting the baryon, <u>in its rest frame</u> which is not the frame in which the 2-particle interaction was originally calculated. Close and Osborn (1970) showed that:

$$\underline{r}_i - \underline{r}_j = \underline{r}_{ij} - \frac{1}{2mM} (\underline{s}_i - \underline{s}_j) \times \underline{P} + \text{higher-order terms} \tag{E.3}$$

where $(\underline{r}_i - \underline{r}_j)$ is the relative position vector of quarks i and j as measured in the rest frame of the baryon, r_{ij} is the genuine internal degree of freedom as measured in the rest frame of quarks i

and j, and M and $\underline{P}$ denote the mass and momentum, respectively, of the whole system. Thus, one must be careful to distinguish between $(\underline{r}_i - \underline{r}_j)$ and $\underline{r}_{ij}$ in the leading (spin-independent) potential term $V(r_i - r_j)$, with the result that:

$$V(r_i - r_j) = V(r_{ij}) + \frac{1}{2mM}\frac{1}{r_{ij}}\frac{dV}{dr_{ij}}(\underline{r}_{ij} \times \underline{P}).(\underline{s}_i - \underline{s}_j) + \ldots \quad (E.4)$$

where ..... denotes higher-order corrections. Notice that the second term on the RHS of equation (E.4) has precisely the form of a spin-orbit interaction and we must ensure, therefore, that it is included in equation (E.2), which then becomes:

$$H_{ij}^{SO} = \frac{1}{4m^2}\cdot\frac{1}{r_{ij}}\frac{dV}{dr_{ij}}\{(4g-1)\left[\underline{r}_{ij} \times (\underline{p}_i - \underline{p}_j)\right].(\underline{s}_i + \underline{s}_j) - \left[\underline{r}_{ij} \times (\underline{p}_i + \underline{p}_j - \frac{2m}{M}\underline{P})\right].(\underline{s}_i - \underline{s}_j)\} \quad . \quad (E.5)$$

In particular, we may write:

$$H_{12}^{SO} = \frac{1}{4m^2}\cdot\frac{1}{\rho}\frac{dV}{d\rho}\{(4g-1)(\underline{\rho} \times \underline{p}_\rho).(\underline{s}_1+\underline{s}_2) - \frac{1}{\sqrt{3}}(\underline{\rho}\times\underline{p}_\lambda).(\underline{s}_1-\underline{s}_2)\} \quad (E.6)$$

where $\underline{\rho}$ and $\underline{\lambda}$ are defined in equations (1.12b) and (1.12c). This expression is clearly translation-invariant. The terms in equation (E.6) proportional to $(\underline{\rho} \times \underline{p}_\rho).(\underline{s}_1+\underline{s}_2)$ and $(\underline{\rho} \times \underline{p}_\lambda).(\underline{s}_1-\underline{s}_2)$ are commonly referred to as 2-body spin-orbit and 3-body spin-orbit interactions, respectively. [However, as Close and Dalitz (1981) point out, the latter term of reference is something of a misnomer, since the physical process of particle exchange which gives rise to V involves only the two particles i and j.]

There are two possible origins for the spin-orbit component of quark-quark interactions in baryons, viz.:

(a) the one-gluon-exchange potential, V_{gluon}

This originates from the exchange of a vector particle (gluon) - see Figure 1 - so that we must set $g = 1$ in equation (E.6). The leading contribution to V_{gluon} is Coulombic:

$$V_{gluon}(ij) = -\frac{2\alpha_s}{3r_{ij}} \tag{E.7}$$

and the resulting spin-orbit contribution is:

$$H^{SO}_{gluon}(12) = \frac{\alpha_s}{2\sqrt{2}\, m^2\rho^3}\left[(\underline{\rho}\times\underline{p}_\rho).(\underline{s}_1+\underline{s}_2) - \frac{1}{3\sqrt{3}}(\underline{\rho}\times\underline{p}_\lambda).(\underline{s}_1-\underline{s}_2)\right]. \tag{E.8}$$

(b) the confining potential, $V_{conf.}$

Although the precise functional form of $V_{conf.}$ is unknown, it is assumed to be of scalar origin (Reinders 1978 and references therein) and to be composed of pairwise interactions between the quarks:

$$V_{conf.} = \sum_{i<j} V_{conf.}(ij) \ . \tag{E.9}$$

In their study of the negative-parity baryons most plausibly assigned to the $N = 1$ oscillator level, Isgur and Karl (1978b) assume an harmonic confining potential:

$$V_{conf.} = \sum_{i<j} \tfrac{1}{2}K\, \underline{r}_{ij}^2 \tag{E.10}$$

which gives rise to the spin-orbit interaction:

$$H^{SO}_{conf.}(12) = -\frac{K}{2m^2}\left[(\underline{\rho}\times\underline{p}_\rho).(\underline{s}_1+\underline{s}_2) + \frac{1}{\sqrt{3}}(\underline{\rho}\times\underline{p}_\lambda).(\underline{s}_1-\underline{s}_2)\right]. \tag{E.11}$$

Following Isgur and Karl (1978b), we may write the full contribution to the spin-orbit interaction as:

$$H^{SO}(12) = H^{SO}_{2B}(12) + H^{SO}_{3B}(12) \qquad (E.12)$$

where :

$$H^{SO}_{2B}(12) = \frac{1}{2m^2}\left(\frac{\alpha_s}{\sqrt{2}\rho^3} - K\right)(\underline{\rho}\times\underline{p}_\rho).(\underline{s}_1+\underline{s}_2) \quad \text{(2-body spin-orbit)} \qquad (E.13a)$$

$$H^{SO}_{3B}(12) = -\frac{1}{6\sqrt{3}m^2}\left(\frac{\alpha_s}{\sqrt{2}\rho^3} + 3K\right)(\underline{\rho}\times\underline{p}_\lambda).(\underline{s}_1-\underline{s}_2). \quad \text{(3-body spin-orbit)} \qquad (E.13b)$$

For the states assigned to the $N = 1$ $[\underline{70},1^-]$ supermultiplet, all relevant matrix elements of H^{SO}_{2B} are proportional to $(D - F)$, whilst all relevant matrix elements of H^{SO}_{3B} are proportional to $(D + 3F)$, where:

$$D = \frac{4\alpha_s\alpha^3}{3\sqrt{2\pi}\,m^2} \quad \text{(c.f. equation (1.44))} \qquad (E.14a)$$

$$F = \frac{K}{m^2} \; . \qquad (E.14b)$$

(Isgur and Karl 1978b). We stress that these results hold true only in the limit of exact $SU(3)_{flavour}$ symmetry, i.e. neglecting the difference in the mass of the strange and non-strange quarks. In their fit to the $N = 1$ baryon spectrum, Isgur and Karl (1978b) assigned the values $(D - F) \simeq 85$ MeV and $(D + 3F) \simeq 945$ Mev.

These assignments, however, raise immediate difficulties for the Isgur-Karl model, as was illustrated by Close and Dalitz (1981) with the following three well-chosen examples:

A. <u>(ΔD33 - ΔS31) mass difference</u>

The contributions of the various spin-dependent interactions to the masses of these two states (Isgur and Karl 1978b) are listed in Table E.1:

State	$H^{contact}$	H^{tensor}	H^{SO}_{2B}	H^{SO}_{3B}
ΔD33	$\frac{1}{4}$ D	0	0	$\frac{1}{12}$(D+3F)
ΔS31	$\frac{1}{4}$ D	0	0	$-\frac{1}{6}$(D+3F)

TABLE E.1 Contributions to the (ΔD33 - ΔS31) mass difference

The results in Table E.1 imply that:

$$M(\Delta D33) - M(\Delta S31) = \frac{1}{4}(D + 3F) \simeq 235 \text{ MeV} \quad . \tag{E.15}$$

The Particle Data Group (1980), however, lists both the ΔD33 and ΔS31 states as 4-star resonances, whose masses lie in the respective ranges 1630 - 1740 MeV and 1600 - 1650 MeV. Thus, the experimental data would appear to be consistent with a mass difference lying anywhere in the region 0 - 140 MeV. Indeed, the Isgur-Karl model (i.e. including the contact and tensor interactions, but neglecting any spin-orbit terms) predicts that these states are degenerate. However, Kelly (1980) has recently emphasised that, although different pion-nucleon-scattering phase-shift analyses do differ quite widely in the absolute mass values which they assign to these resonances, they appear to agree that the mass difference, M(ΔD33)-M(ΔS31), lies in the range 80-100 MeV. The experimental data, then, would appear to suggest that the ΔD33 state is not degenerate with the ΔS31 state, but lies some 90 MeV above it. Thus, the calculated mass splitting in equation (E.15) has the correct sign, but its magnitude is roughly three times too large.

B. (ND15 - ΔS31) mass difference

Table E.2 lists the contributions of the various spin-dependent interactions to the masses of these two states (Isgur and Karl 1978b):

State	$H^{contact}$	H^{tensor}	H^{SO}_{2B}	H^{SO}_{3B}
ND15	$\frac{1}{4}$ D	$-\frac{1}{20}$ D	$\frac{3}{4}$(D-F)	0
ΔS31	$\frac{1}{4}$ D	0	0	$-\frac{1}{6}$(D+3F)

TABLE E.2 Contributions to the (ND15 - ΔS31) mass difference

The results in Table E.2 imply that:

$$M(ND15) - M(\Delta S31) = -\frac{1}{20} D + \frac{3}{4}(D-F) + \frac{1}{6}(D+3F) \simeq 205 \text{ MeV}, \quad (E.16)$$

whereas the experimental data (Particle Data Group 1980) suggest a mass difference of approximately 20 MeV. Whilst the Isgur-Karl model (with no spin-orbit forces), in predicting a mass splitting of $-\frac{1}{20} D \simeq -15$ MeV, finds good agreement with the magnitude of the observed splitting, it fails to account for the sign of the observed mass difference. Including spin-orbit forces, as is clear from equation (E.16), results in a predicted mass splitting which is an order of magnitude too large.

C. (ΛD03 - ΛS01) mass difference

A proper treatment of this mass difference requires that we take into account the mass difference between the strange and non-strange quarks; however, as pointed out by Close and Dalitz (1981), whilst any discrepancies in the masses of these two states which are found to exist will be modified by this effect, they will not be removed. Accordingly, we list in Table E.3 the contributions of the various spin-dependent interactions to the masses of these two states, calculated in the limit of exact $SU(3)_{flavour}$ symmetry (Isgur and Karl 1978b):

State	$H^{contact}$	H^{tensor}	H^{SO}_{2B}	H^{SO}_{3B}
$\Lambda D03$	$-\frac{3}{4}D$	0	$\frac{1}{2}(D-F)$	$-\frac{1}{12}(D+3F)$
$\Lambda S01$	$-\frac{3}{4}D$	0	$-(D-F)$	$\frac{1}{6}(D+3F)$

TABLE E.3 Contributions to the ($\Lambda D03$ - $\Lambda S01$) mass difference

We deduce from the results contained in Table E.3 that:

$$M(\Lambda D03) - M(\Lambda S01) = \frac{3}{2}(D-F) - \frac{1}{4}(D+3F) \simeq -110 \text{ MeV} . \qquad (E.17)$$

In fact, a proper treatment, including quark mass differences, gives $M(\Lambda D03) - M(\Lambda S01) \simeq -190$ MeV (Close and Dalitz 1981). The Particle Data Group (1980), however, reports the existence of a 4-star $\Lambda\,\frac{3}{2}^-$ resonance at 1520 MeV, and a 4-star $\Lambda\,\frac{1}{2}^-$ resonance at 1405 MeV, so that the experimentally-observed mass difference is approximately 115 MeV. Indeed, the $\Lambda D03(1520)$ - $\Lambda S01(1405)$ mass splitting is the only firm evidence at present for the presence of spin-orbit effects in the $N = 1$ $[\underline{70},1^-]$ supermultiplet. As such, it constitutes a very real problem for the Isgur-Karl model, and has prompted Isgur and Karl to suggest that the observed low mass of the $\Lambda S01(1405)$ state may be a direct consequence of its proximity to the kaon-nucleon ($\bar{K}N$) threshold. It is well-known that mass shifts due to mixing with virtual decay channels can be quite strong in the vicinity of a threshold: such an effect could significantly depress the $\Lambda S01(1405)$ state since it is strongly-coupled to $\bar{K}N$ (Isgur and Karl 1978b).

What conclusions are we to form, then, finally? It is beyond contention that, in its modern guise incorporating anharmonic perturbations, quark mass differences and <u>some</u> effects of the non-relativistic reduction of coloured-gluon exchange, the Isgur-Karl model has been

remarkably successful in accounting for many features of the baryon spectrum. And yet this very statement of fact is filled with polemic. The controversy centres on the use of the word "some" and immediately raises the question: why do we need to include only the contact and tensor interactions in our considerations, in order to achieve a successful description of baryon resonances up to about 2 GeV in mass, with no apparent need (nor room) for the spin-orbit interaction? The findings of Isgur and Karl (1978b, 1979a) suggest that spin-orbit effects, if present at all in the baryon spectrum, are at a level much reduced (10% - 20%) from naïve expectations from the non-relativistic reduction of one-gluon-exchange. Yet we know from rather general considerations that the spin-orbit interaction must exist, and indeed the observed $(\Delta D33 - \Delta S31)$ and $(\Lambda D03 - \Lambda S01)$ mass differences require the existence of a spin-orbit interaction in such a model. The situation is further complicated, however, because:

(a) the observed $(\Delta D33 - \Delta S31)$ splitting, for which only a 3-body spin-orbit interaction can be held responsible in such models, suggests that the 3-body spin-orbit force is present only to within approximately 30% of its expected strength;

(b) the observed $(\Lambda D03 - \Lambda S01)$ mass difference, 115 ± 5 MeV, which can be attributed to the existence of a strong spin-orbit interaction, has the opposite sign from the calculated separation.

The experimental evidence to hand, then, provides one with conflicting signals as to the strength and sign of the spin-orbit effects present in shaping the $N = 1$ baryon spectrum. We can prescribe no remedy to resolve this dilemma. However, implicit in the results obtained by Isgur and Karl (1978b) for the spin-orbit matrix elements relevant to the non-strange sector of the $N = 1$ $[\underline{70},1^-]$ supermultiplet is the assumption that it is not inappropriate to employ an harmonic oscillator

form for the confining potential and the corresponding harmonic oscillator orbitals as basis states. Certainly, it is unlikely that the confining potential is an harmonic oscillator potential, although the approximation of using harmonic oscillator orbitals appears to be well supported by the experimental data (Isgur and Karl 1978b). It is instructive to relax both these constraints and the results for the non-strange sector of the $N = 1$ $[\underline{70},1^-]$ supermultiplet which follow are displayed in Table E.4:

State	H^{SO}_{2B}	H^{SO}_{3B}
$^4\underline{8}\ \frac{5}{2}^-$	$\frac{3}{4}(D_1-F_1)$	0
$^2\underline{10}\ \frac{3}{2}^-$	0	$\frac{1}{12}(D_2+3F_2)$
$^4\underline{8}\ \frac{3}{2}^-/^2\underline{8}\ \frac{3}{2}^-$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{10}}{8} \\ -\frac{\sqrt{10}}{8} & \frac{1}{4} \end{pmatrix}(D_1-F_1)$	$\begin{pmatrix} 0 & \frac{\sqrt{10}}{24} \\ \frac{\sqrt{10}}{24} & 0 \end{pmatrix}(D_2+3F_2)$
$^2\underline{1}\ \frac{3}{2}^-$	$\frac{1}{2}(D_1-F_1)$	$-\frac{1}{12}(D_2+3F_2)$
$^2\underline{10}\ \frac{1}{2}^-$	0	$-\frac{1}{6}(D_2+3F_2)$
$^4\underline{8}\ \frac{1}{2}^-/^2\underline{8}\ \frac{1}{2}^-$	$\begin{pmatrix} -\frac{5}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix}(D_1-F_1)$	$\begin{pmatrix} 0 & \frac{1}{12} \\ \frac{1}{12} & 0 \end{pmatrix}(D_2+3F_2)$
$^2\underline{1}\ \frac{1}{2}^-$	$-(D_1-F_1)$	$\frac{1}{6}(D_2+3F_2)$

TABLE E.4 Spin-orbit matrix elements for the $N = 1$ $[\underline{70},1^-]$ supermultiplet calculated in the $SU(3)_{flavour}$ limit for an arbitrary confining potential.

The notation for the states in Table E.4 is $^{2S+1}SU(3)_{\text{flavour}}\ J^-$, where S and J denote the total spin and total angular momentum, respectively. The reduced matrix elements appearing in Table E.4 are defined as follows:

$$D_1 = -\frac{\alpha_s}{2m^2}\ {}^{\rho}\langle 1 \| \rho^{-3}(\underline{\rho} \times \underline{p}_{\rho}) \| 1 \rangle^{\rho} \tag{E.18a}$$

$$D_2 = -\frac{\alpha_s}{m^2}\ {}^{\rho}\langle 1 \| \rho^{-3}(\underline{\rho} \times \underline{p}_{\lambda}) \| 1 \rangle^{\lambda} \tag{E.18b}$$

$$F_1 = -\frac{1}{2\sqrt{2}m^2}\ {}^{\rho}\langle 1 \| \frac{1}{\rho}\frac{dV_{\text{conf.}}}{d\rho}(\underline{\rho} \times \underline{p}_{\rho}) \| 1 \rangle^{\rho} \tag{E.18c}$$

$$F_2 = -\frac{1}{\sqrt{2}m^2}\ {}^{\rho}\langle 1 \| \frac{1}{\rho}\frac{dV_{\text{conf.}}}{d\rho}(\underline{\rho} \times \underline{p}_{\lambda}) \| 1 \rangle^{\lambda} \tag{E.18d}$$

It is easy to show, using either explicit wavefunction techniques or an algebraic method based on the commutation relations satisfied by the creation and annihilation operators (as was employed in Chapter 3), that the assumption of harmonic-oscillator orbitals leads to:

$$D_1 = D_2 = D \tag{E.19a}$$

$$F_1 = F_2\ , \tag{E.19b}$$

where D is defined in equation (E.14a). For example, Gromes and Stamatescu (1979) consider within the context of this assumption the case of a power-law confining potential:

$$V_{\text{conf.}} = \sum_{i<j} \lambda r_{ij}^{k}\ , \qquad \lambda,\ k > 0 \tag{E.20}$$

for which:

$$F_1 = F_2 = \frac{2^{(k/2+1)} \cdot \lambda k}{3\sqrt{\pi}\ \alpha^{(k-2)} m^2} \cdot \left(\frac{k+1}{2}\right)!\ . \tag{E.21}$$

In particular, the case $k = 2$ leads to:

$$F_1 = F_2 = \frac{2\lambda}{m^2} \tag{E.22}$$

which coincides with the expression for F in equation (E.14b) upon setting $\lambda = \frac{1}{2}K$, as it must. [The case of a logarithmic confining potential:

$$V_{conf.} = \sum_{i<j} \lambda \log \left(\frac{r_{ij}}{r_o}\right) , \qquad \lambda > 0 \tag{E.23}$$

where r_o is an arbitrary length-scale, can be dealt with rather neatly by noting:

$$\left[\frac{\partial(r^k)}{\partial k}\right]_{k=0} = \log r, \tag{E.24}$$

with the result that:

$$F_1 = F_2 = \frac{\lambda\alpha^2}{3m^2} \;] \quad . \tag{E.25}$$

Returning to the results in Table E.4, it is clear that we can easily assign values to the parameters D_1, D_2, F_1 and F_2 so as to account for the sign and magnitude, as determined experimentally, of the spin-orbit contributions to mass splittings such as $(\Delta D33 - \Delta S31)$ and $(\Delta D03 - \Delta S01)$. [This procedure has much in common with the analysis of Reinders (1978), who treated the reduced matrix elements as free parameters to be fitted to the experimental data; Reinders, however, did not relax the constraint of using harmonic-oscillator orbitals.] However, we do not find this prescription particularly satisfying. Much of the appeal of the Isgur-Karl model derives from its simplicity: it strikes us that relaxing the restriction to harmonic-oscillator orbitals introduces an unnecessary complication into the model.

The need to resolve this dilemma underlines the importance of determining the precise form of the confining potential, $V_{conf.}$, as is clear from equations (E.18c) and (E.18d): the near-cancellation of 2-body spin-orbit effects in the $N = 1$ baryon spectrum, as reported

by Isgur and Karl (1978b), depends crucially on these authors assuming an harmonic-oscillator form for the confining potential. Yet, this dilemma further serves as a timely reminder that the Isgur-Karl model is no more than a simple and naïve model of the interactions of the quarks in a baryon: one should be wary of falling into the trap of assuming that the Isgur-Karl model is the last word on baryon phenomenology. Indeed, had the Isgur-Karl model provided "perfect" phenomenology, one would have been faced with the even greater dilemma of having to conclude either that the underlying quark dynamics in a baryon is governed by 2-body forces, and not by 3-body forces as QCD would have one believe, or that the underlying forces are 3-body in nature but that baryon spectroscopy can never have anything fundamental to say about quark dynamics, in pretty much the manner of a "black box", the parameters of which have no deep significance and bear no simple relation to the QCD Lagrangian (Hey 1980). What is true is that the confusion currently reigning over the rôle of the spin-orbit interaction in shaping the baryon spectrum poses a very real problem for any attempt to understand how a model such as the Isgur-Karl model might be derived from the fundamental standpoint of QCD. Viewed in this light, the resolution of the spin-orbit problem is seen to be highly desirable, since it may provide important clues leading, ultimately, to a deeper understanding of the strong interaction. For the time being, however, the Isgur-Karl model must remain, despite its myriad successes, "QCD-inspired".

APPENDIX A

We explain briefly the nomenclature used in this thesis to label non-strange baryon resonances. These resonances are observed in πN scattering:

$$\pi N \rightarrow X \rightarrow \text{final state}, \tag{A.1}$$

where X denotes the resonant state. We are not concerned here with the composition of the final state. The notation employed to label the resonance X is:

$$XA(2I)(2J) , \tag{A.2}$$

where $X \equiv N$ (nucleon-type) or Δ (delta-type), $A(= S,P,D,F,.....)$ is the spectroscopic symbol specifying the angular momentum, $L_{\pi N}$ $(= 0,1,2,3,.....)$, of the πN partial-wave amplitude in which the resonance X is observed, and I and J denote the isospin and spin, respectively, of the resonance X. $L_{\pi N}$ and J are not independent, since:

$$J = |L_{\pi N} \pm \tfrac{1}{2}| . \tag{A.3a}$$

The parity, P, of the resonance is given by:

$$P = (-1)^{(L_{\pi N} + 1)} . \tag{A.3b}$$

In Chapter 6, we investigated in detail some negative-parity Δ-resonances which are most plausibly assigned to the $N = 3$ level of the harmonic oscillator quark model. We indicate now how one determines with which $N = 3$ $SU(6)_{flavour \times spin} \otimes O(3)$ supermultiplets a given non-strange resonance can mix. We illustrate the method with reference to a specific example: the $\Delta D35$ state:

ΔD35 state

For this state, $L_{\pi N} = 2$ and $J = \frac{5}{2}$. The $N = 3$ level of the harmonic oscillator quark model comprises eight $SU(6)_{flavour \times spin} \otimes O(3)$ supermultiplets in all (c.f. Figure 2), viz.: the $[\underline{56},3^-]$, $[\underline{70},3^-]$, $[\underline{20},3^-]$, $[\underline{70},2^-]$, $[\underline{56},1^-]$, $[\underline{70},1^-]$, $[\underline{70},1^-]$ and $[\underline{20},1^-]$ supermultiplets. The decompositions:

$$\underline{56} \supset \underline{10} \otimes \underline{4} \oplus \underline{8} \otimes \underline{2} \tag{A.4a}$$

$$\underline{70} \supset \underline{10} \otimes \underline{2} \oplus \underline{8} \otimes \underline{4} \oplus \underline{8} \otimes \underline{2} \oplus \underline{1} \otimes \underline{2} \tag{A.4b}$$

$$\underline{20} \supset \underline{8} \otimes \underline{2} \oplus \underline{1} \otimes \underline{4} \tag{A.4c}$$

under the reduction $SU(6)_{flavour \times spin} \supset SU(3)_{flavour} \otimes SU(2)_{spin}$, reveal that:

(a) a Δ-state belonging to a $\underline{56}_{flavour \times spin}$ $[\underline{70}_{flavour \times spin}]$ multiplet necessarily has total quark spin $S = \frac{3}{2}$ $[S = \frac{1}{2}]$;

(b) a Δ-state cannot belong to a $\underline{20}_{flavour \times spin}$ multiplet.

Recalling that:

$$\underline{J} = \underline{L} + \underline{S} \ , \tag{A.5}$$

where L (not to be confused with $L_{\pi N}$) denotes the orbital angular momentum of a given $[SU(6), L^P]$ supermultiplet, and that $J = \frac{5}{2}$ for the ΔD35 state, we quickly deduce that the ΔD35 state can, in principle, mix with the $N = 3$ $[\underline{56},1^-]$, $[\underline{70},2^-]$, $[\underline{56},3^-]$ and $[\underline{70},3^-]$ supermultiplets.

We list, for completeness, all possible $N = 3$ N- and Δ-type resonances in Table A.1, where we also give the possible mixings with $SU(6)_{flavour \times spin} \otimes O(3)$ supermultiplets:

State	$SU(6)_{flavour\times spin} \otimes O(3)$ content
NS11	$[\underline{70},2^-]^{3/2}$, $[\underline{56},1^-]$, $[\underline{70},1^-]^{1/2}$, $[\underline{70},1^-]^{1/2}$, $[\underline{70},1^-]^{3/2}$, $[\underline{70},1^-]^{3/2}$, $[\underline{20},1^-]$
ND13	$[\underline{70},3^-]^{3/2}$, $[\underline{70},2^-]^{1/2}$, $[\underline{70},2^-]^{3/2}$, $[\underline{56},1^-]$, $[\underline{70},1^-]^{1/2}$ $[\underline{70},1^-]^{1/2}$, $[\underline{70},1^-]^{3/2}$, $[\underline{70},1^-]^{3/2}$, $[\underline{20},1^-]$
ND15	$[\underline{56},3^-]$, $[\underline{70},3^-]^{1/2}$, $[\underline{70},3^-]^{3/2}$, $[\underline{20},3^-]$, $[\underline{70},2^-]^{1/2}$, $[\underline{70},2^-]^{3/2}$ $[\underline{70},1^-]^{3/2}$, $[\underline{70},1^-]^{3/2}$
NG17	$[\underline{56},3^-]$, $[\underline{70},3^-]^{1/2}$, $[\underline{70},3^-]^{3/2}$, $[\underline{20},3^-]$, $[\underline{70},2^-]^{3/2}$
NG19	$[\underline{70},3^-]^{3/2}$
ΔS31	$[\underline{56},1^-]$, $[\underline{70},1^-]^{1/2}$, $[\underline{70},1^-]^{1/2}$
ΔD33	$[\underline{56},3^-]$, $[\underline{70},2^-]^{1/2}$, $[\underline{56},1^-]$, $[\underline{70},1^-]^{1/2}$, $[\underline{70},1^-]^{1/2}$
ΔD35	$[\underline{56},3^-]$, $[\underline{70},3^-]^{1/2}$, $[\underline{70},2^-]^{1/2}$, $[\underline{56},1^-]$
ΔG37	$[\underline{56},3^-]$, $[\underline{70},3^-]^{1/2}$
ΔG39	$[\underline{56},3^-]$

TABLE A.1 N = 3 N- and Δ-type resonances and their possible $SU(6)_{flavour\times spin} \otimes O(3)$ content.

In Table A.1, the notation $[\underline{70},L^-]^S$, where the total quark spin $S = \frac{1}{2}$ or $\frac{3}{2}$, is used.

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