Vector fields on surfaces

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Come Holy Spirit, fill the hearts of Your faithful
and kindle in them the fire of Your love.
Send forth Your spirit and they shall be created,
and shall renew the face of the earth.

To my family. All of them.

Abstract

We consider "minimal" vector fields on a surface Σ with genus g. These are non-degenerate vector fields with the minimal number of vanishing points that satisfy a set of technical conditions to exclude pathological cases. We show that a minimal vector field gives rise to a directed graph with $2g-2$ vertices such that each vertex has two edges entering and leaving it, a "dual" pair of circuit decompositions of equal size and a function that pairs up the circuits of this dual pair. Conversely, we show that given such a graph with a pair of circuit decompositions and such a function we can construct a unique minimal vector field.

This correspondence enables us to classify these vector fields. The proof of the correspondence result requires several invariants, one each from graph theory and the topology of the surface. These invariants are, respectively, the directed graph Γ formed by the non-compact flowlines of the vector field and a neighbourhood of this graph. Invariants arising from the homology of the pair (Σ, V) are also discussed, where V is the set of vertices of the directed graph Γ.

Further, we show how to construct all possible minimal vector fields from a given graph provided the graph satisfies certain natural properties and give an algorithm that identifies which circuit decompositions have a suitable dual.

We obtain some new results on the Martin polynomial, a skein-type polynomial of graphs first identified by P. Martin (1977). Some other combinatorial results concerning polynomials and graphs are proved.

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And to my parents, my sisters and my friends, for putting up with me for the past three years.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Nicholas James O'Neil Creagh*)

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Chapter 0

Introduction

We shall look at some little studied areas of dynamical systems. Here one is concerned with *flows* or *vector fields* on a surface and the properties of such flows up to global diffeomorphism. A common theme running through the study of such systems is that of stability. Such systems are defined using both local and global properties, including a description of the behaviour of the system around each vanishing point, or zero. Stable systems have been central to the study of dynamical systems resulting in many works being written on them, including their classification by Morse and Smale.

Here we shall take a topological look at another type of dynamical system, one that would be considered highly unstable by the above authors. We shall (by making use of the well-known identification of flows with their generators, vector fields) call these unstable systems *minimal vector fields* and it is the intention of this thesis to classify and count the number of minimal vector fields on a surface.

A minimal vector field on a surface is defined in terms of its behaviour near its fixed points, with conditions on the flowlines away from these points. In fact, the vector fields can be described by the following properties:

1. All zero points are saddle points.
2. There are the minimal number of saddle points.
3. There are only finitely many non-compact flowlines (equivalently, all but a finite number of flowlines are periodic).

We shall see that this description gives that minimal vector fields, unlike stable systems, are such that every point is non-wandering and that the set of points that are on periodic orbits is dense in the surface.

We shall show that a minimal vector field can, up to an equivalence to be defined, be described uniquely in terms of an embedded 2-regular digraph. This digraph will be described in terms of the fixed points and non-compact flowlines of the vector field. In order to completely describe the embedded graph, however, we shall make use of the ideas of a *ribboned graph* and a *circuit decomposition* of a 2-regular digraph.

The first chapter of the thesis recalls the concepts in dynamical systems that we shall use. It then proceeds to show that the above conditions are both natural and equivalent to a series of other conditions. This shows that minimal vector fields, whilst rare (indeed, finite up to an equivalence which we define), are still natural animals in the zoo of flows.

In chapter 2 we switch from differential topology to algebraic topology and calculate the automorphism group of the first homology group a surface with n points on it, i.e. $H_1(\Sigma, V)$ where V is the set of points and Σ the surface. As almost all of the flowlines of a minimal vector field are periodic, we may consider the homology classes of such flowlines. Indeed, whilst the set of such classes does not completely define a minimal vector field, the relations developed in this chapter will be essential in proving the final classification theorem.

In chapter 3 we change tack and view the fixed points and the non-compact edges of a minimal vector field as the vertices and edges of an embedded 2-regular digraph. At this point we shall develop the graph theory we need for our classification result. One key idea here is that of a *circuit decomposition* i.e. a partitioning of the edges into circuits. Related to the definition of a circuit decomposition is that of a *dual decomposition* which we shall define. We shall see that the embedded 2-regular digraph of a minimal vector field, together with further combinatorial data concerning circuit decompositions of the graph is enough to classify minimal vector fields up to a surface diffeomorphism. However, not all circuit decompositions will be suitable for our purposes. We will see that only those circuit decompositions of equal cardinality to their dual arise by considering the embedded graph of a minimal vector field. So we shall need a means of distinguishing such circuit decompositions and an algorithm for doing this will be presented. We also introduce a series of polynomials of graphs, beginning with the Martin polynomial, a skein-type polynomial defined using a series of iterative relations. Using this we define a series of related polynomials that identify which decompositions arise from a minimal vector field. We will have then completed one side of the classification result, namely constructing the invariants from the vector field.

In chapter 4 we reverse direction and consider how one might construct a minimal vector field given a 2-regular digraph and a suitable circuit decomposition. We will first construct a ribboned graph, which is a 2-manifold with boundary on which is embedded the graph in such a way that the circuit decompositions of the graph correspond to the boundary components of the manifold. From this, we will construct the minimal vector field. We conclude this chapter with some results concerning when such objects can be constructed.

Chapter 5 consists of the theorems that together make up the classification result. Also included here is a discussion of other invariants of minimal vector fields, including several that arise from the homology groups discussed in chapter 2. By the end of this chapter we shall have completed our classification of minimal vector fields. We shall thus be in a strong position to count them.

Chapter 6 is a detour concerning some new results on the Martin polynomial that arose from the above work, which include extensions of work by Ellis-Monaghan on the undirected Martin polynomial to the directed Martin polynomial. In this chapter we discuss the formation of the Martin polynomial axiomatically and show it is, in some sense, unique (i.e. it is the only polynomial of graphs that satisfies a set of general axioms) and then define it in terms of the incidence matrix of a 2-regular digraph. We conclude this chapter with a discussion of when a polynomial is the Martin polynomial of some graph.

We end with chapter 7, which revolves around a partially successful attempt to enumerate minimal vector fields in terms of the genus g of Σ and contains several results concerning combinatorics. Included here are results on the enumeration of graphs, together with a proof of the result that any finitely generated group is the automorphism group of some 2-regular digraph.

Chapter 1

On flows, vector fields and conditions of minimality

The purpose of this chapter is to establish definitions of differential topology that we need. In this chapter we define

1. a smooth vector field X on a compact surface Σ of genus g.
2. the conditions under which two vector fields are equivalent. We will do this by reference to the equivalent problem for flows.
3. a minimal vector field X, which is the main object of study.

We start with some basic definitions.

1.1 Basic vector field definitions

Definition 1.1.1 (Surface). A (topological) 2-manifold is a paracompact Hausdorff space Σ for which each point p has an open neighbourhood U homeomorphic to an open subset V of Euclidean space $\mathbb{R}^2$, i.e. there exists a homeomorphism $\phi : U \to V$. The triple (ϕ, U, V) is called a *chart* at p. A set of charts whose domains cover Σ is called an *atlas* for Σ.

For each $p, q \in \Sigma$ there exists open subsets U_p, U_q of Σ, open subsets V_p, V_q of $\mathbb{R}^2$ and maps

$$\phi_p : V_p \to U_p$$
$$\phi_q : V_q \to U_q$$

Suppose $U_p \cap U_q \neq \emptyset$. Define $V_{pq} = \phi_p^{-1}(U_p \cap U_q)$ and V_{qp} similarly. Let $\phi_{pq} : V_{pq} \to V_{qp}$ be the function $\phi_{pq} = \phi_q^{-1} \circ \phi_p$. Then we say the manifold is *smooth* if all the maps ϕ_{pq} are diffeomorphisms. A *surface* is a smooth 2-manifold. All our surfaces will be compact.

Definition 1.1.2 (Tangent bundle). The *tangent bundle* of a surface Σ is the manifold

$$T\Sigma = \{(p, v) \in \Sigma \times T_p\Sigma\} = \bigcup_{p\in\Sigma}(p, T_p\Sigma).$$

with a certain topology, which we do not specify here. The tangent bundle is smooth if it is a smooth manifold, i.e. it has a smooth atlas. For the definition of $T_p\Sigma$ and a smooth atlas of a vector bundle, together with the details that $T\Sigma$ is a smooth manifold, see [19] pp2-5, [10] p11 and [13] Appendix A, p208. Notice that a tangent bundle has an associated map $\pi : T\Sigma \to \Sigma$ given by $\pi(p, v) = p$.

Definition 1.1.3 (Vector Field). A smooth *vector field* is a smooth map

$$X : \Sigma \to T\Sigma$$

such that $\pi \circ X$ is the identity on Σ. Intuitively, it is a map which smoothly assigns to each $p \in \Sigma$ a vector $X(p) \in T_p\Sigma$.

Definition 1.1.4 (Zeros of a vector field X). A *zero* of a vector field X is a point $p \in X$ such that $X(p) = 0$. We define the set $\mathcal{Z}(X)$ to be the set of all zeros of X.

Definition 1.1.5 (Integral curves or flowlines of a vector field). An *integral curve* or *flowline* of a vector field is a map $\gamma : I \to \Sigma$ such that

$$X \circ \gamma(t) = \gamma'(t) = \frac{d}{d\tau}\gamma(\tau)\bigg|_{\tau=t}$$

for all $t \in I$, where $I \subset \mathbb{R}$ is an interval containing 0. It is a standard result (for example [8], remark 1.3, Theorem 1.4 p2 and Theorem 1.12, p8) that integral curves both exist and, on a compact manifold without boundary, may be extended uniquely to a map satisfying the above condition for all $t \in \mathbb{R}$.

Notice that a flowline as defined above has both a speed and a starting point. However, we will say that two flowlines are *equivalent* if they are the same up to a time change, i.e. if and only if there exists $\tau \in \mathbb{R}$ such that for all $t \in \mathbb{R}$ $\gamma_1(t) = \gamma_2(t + \tau)$.

If Σ is a surface with non-empty boundary $\partial\Sigma$ then any connected component of $\partial\Sigma$ that is also an integral curve is called a *boundary circuit.*

Definition 1.1.6 (Index of a point w.r.t. a vector field on $\mathbb{R}^2$). The following discussion and definition owes much to [13] pp133-135.

Suppose X is a continuous vector field on an open subset V of the Euclidean plane and suppose γ is a simple closed curve in V that does not pass through any zero of X. Then we can associate with γ an integer, its *index with respect to* X. We can describe this as follows. Consider a point p that moves around γ in the anti-clockwise direction, starting at p_0. The angle $\theta(p)$ that the vector $X(p)$ makes with the x-axis is only defined up to a multiple of 2π. However, if we start with, say $0 \leq \theta(p_0) \leq 2\pi$, we can choose a representative $\theta(p)$ that varies continuously with p. When we return to p_0 after traversing γ once $\theta(p)$ may not return to its original value, $\theta(p_0)$. But it will take a value that differs from $\theta(p_0)$ by $2n\pi$, for some integer n. Thus $2n\pi$ is the *total angular variation* of the vector field around the curve γ. The number n, which obviously does not depend upon the starting point p_0 and the speed with which p moves round γ, is the index of γ.

It is clear that if we deform γ continuously into a curve $\tilde{\gamma}$ through a family of curves, none of which contains a zero of X then the index changes continuously. As it is an integer, it is constant. Thus, for an isolated zero p of X we can unambiguously define the *index of p with respect to X* to be the index of any sufficiently small (i.e. encloses no other zeros) circle with centre p.

Definition 1.1.7 (Index of a point w.r.t. a vector field on a surface). Let X be a continuous vector field on a smooth, oriented surface Σ. Let γ be a simple closed curve in Σ such that $X|_\gamma \neq 0$ and let C be an oriented collar around γ. Let $f : C \to \mathbb{R}^2$ be an orientation-preserving homeomorphism of C into the plane. This sends γ to a simple closed curve in an open subset of $\mathbb{R}^2$. Thus it has an index with respect to the vector field induced from X. We define the index of γ with respect to X, $Ind_X(\gamma)$ to be the index of the embedding of γ with respect to the induced vector field. It can be shown that this is well-defined and independent of the choice of embedding f (see [13] pp133-142, in particular the discussion on p136).

Using this, we can define the index of a point $p \in \Sigma$ with respect to X, $Ind_X(p)$ as the index of a suitable small curve around p. See Figures 1.2 and 1.3 for examples.

In this situation the following, well-known theorem is useful.

Theorem 1.1.8 (Poincaré-Hopf Index theorem). *For a smooth vector field*

X with isolated zeros on a compact surface Σ

$$\sum_{p \in \mathcal{Z}(X)} Ind_X(p) = 2 - 2g$$

where g is the genus of Σ

Proof: This 2-dimensional version of this theorem was proved by Poincaré in 1885. The full n-dimensional theorem was proved by Hopf in [11] in 1926 after earlier partial results by Brouwer and Hadamard.

For a simple and clear (albeit incomplete) proof, see [7, Theorem 8.3]. A complete proof is given in [8] Chapter 5 Theorem 3.1). □

We can extend this result with the following lemma.

Lemma 1.1.9. *If X is a smooth vector field with isolated zeros on a smooth, oriented surface Σ with genus g and k boundary circuits $\gamma_1 \dots \gamma_k$ then*

$$\sum_{p \in \mathcal{Z}(X)} Ind_X(p) = 2 - 2g - \sum_{i=1}^{k} Ind_X(\gamma_i)$$

where $\mathcal{Z}(X) = \{p \in \Sigma : X(p) = 0\}$.

Proof: Span the γ_i's by discs D_i which are disjoint from the interior of Σ and let $v_i \in \text{int}(D_i)$, i.e. the interior of D_i. Then extend $X|_{\gamma_i}$ to a smooth, non-vanishing vector field X' on each $D_i - v_i$. By Definition 1.1.7, $Ind_X(\gamma_i) = Ind_{X'}(\gamma_i) = Ind_{X'}(v_i)$.

By the Poincaré - Hopf Theorem (theorem 1.1.8)

$$\sum_{p \in \mathcal{Z}(X)} Ind_{X'}(p) + \sum_{i=1}^{k} Ind_{X'}(v_i) = 2 - 2g$$

which is

$$\sum_{p \in \mathcal{Z}(X)} Ind_X(p) = 2 - 2g - \sum_{i=1}^{k} Ind_X(\gamma_i)$$

□

Corollary 1.1.10. *If all the boundary circuits of a smooth surface Σ with genus g and boundary circuits $\gamma_1, \dots, \gamma_k$ are flowlines of a smooth vector field X, then*

$$\sum_{p \in \mathcal{Z}(X)} Ind_X(p) = \chi(\Sigma)$$

Proof: As in Lemma 1.1.9 span each boundary circuit γ_i by a disc D_i which is disjoint from the interior of Σ. But in this case γ_i is a flowline of the induced vector field on the disc. So each disc is diffeomorphic to the unit circle. As γ_i is a flowline of X, so the unit circle is a flowline of the induced vector field. Now the total angular variation of the tangent to the unit circle S^1 is 2π. Hence $Ind_X(\gamma_i) = 1$.

So

$$\sum_{i=1}^{k} Ind_X(\gamma_i) = k$$

Substituting this into Lemma 1.1.9 we have

$$\sum_{p \in Z(X)} Ind_X(p) = 2 - 2g - k$$

and the right hand side is equal to $\chi(\Sigma)$, the Euler characteristic of the surface Σ. □

Definition 1.1.11 (Derivative dX_p). As Σ is a smooth manifold, for any point $p \in \Sigma$ there exists an open neighbourhood U of p that is diffeomorphic to some open ball around the origin in the plane $\mathbb{R}^2$. This gives local coordinates (x, y) for U. With respect to these local coordinates the vector field X takes the form $X(p) = (X_1(p), X_2(p))$. As X is smooth, so X_1 and X_2 are smooth. Thus the differential matrix

$$\mathrm{d}X_p = \begin{bmatrix} \frac{\partial X_1}{\partial x}\big|_p & \frac{\partial X_2}{\partial x}\big|_p \\ \frac{\partial X_1}{\partial y}\big|_p & \frac{\partial X_2}{\partial y}\big|_p \end{bmatrix}$$

exists and the sign of the determinant of this matrix is independent of the choice of local coordinates (see, for example [19] p6,7).

Now, using this definition, we can classify zeros of a vector field as follows;

Definition 1.1.12 (Degeneracy). The vector field X is *non-degenerate* at the singular point $p \in \Sigma$ if the linear transformation dX_p is nonsingular and there exists a smooth function $f : \Sigma \longrightarrow \mathbb{R}$ such that for all q in a neighbourhood of p,

$$X(q) = \left(\frac{\partial f}{\partial x}\bigg|_q, \frac{\partial f}{\partial y}\bigg|_q \right).$$

It is *degenerate* otherwise.

Lemma 1.1.13. *The index of X at a non-degenerate zero is either +1 or -1 according as the determinant of dX_p is positive or negative.*

Proof: See [19, p37 Lemma 4] □

Note that it can be shown that there is only one type (up to a local homeomorphism taking orbits to orbits and preserving an orbit's orientation) of non-degenerate zero with negative index, called a saddle point, but there are two with positive index, called maxima, and minima (see Figure 1.1). See Figures 1.2 and 1.3 for the calculation of the index of these zeros.

Notice that Definition 1.1.12 has two conditions on the vector field near p for it to be non-degenerate. Some authors (e.g. [19]) have only the first condition. If we were not to have the second condition (namely the existence of a smooth f) then there would be other non-degenerate zeros with positive index that we would have to consider (for example, the centre discussed later in example 1.3.2). Equally, we would also have to consider that there would be other zeros with negative index. But, by insisting that locally our vector field X is the gradient of some map f, we need only consider the two zeros of index $+1$ and the single zero of index -1 mentioned above.

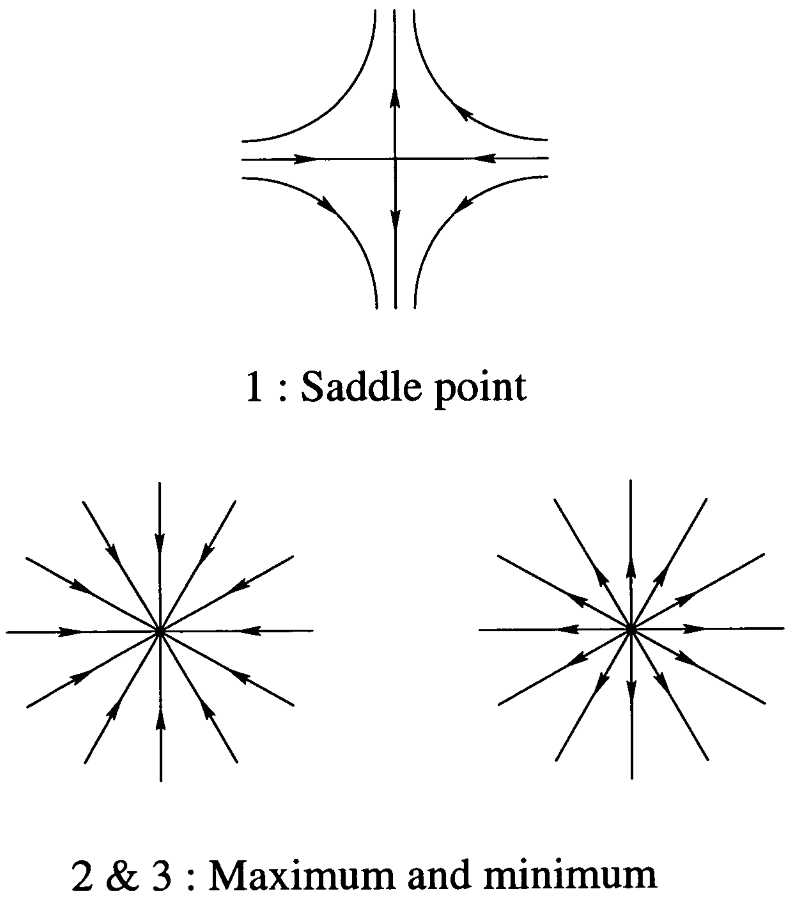


Figure 1.1: Non-degenerate zeros

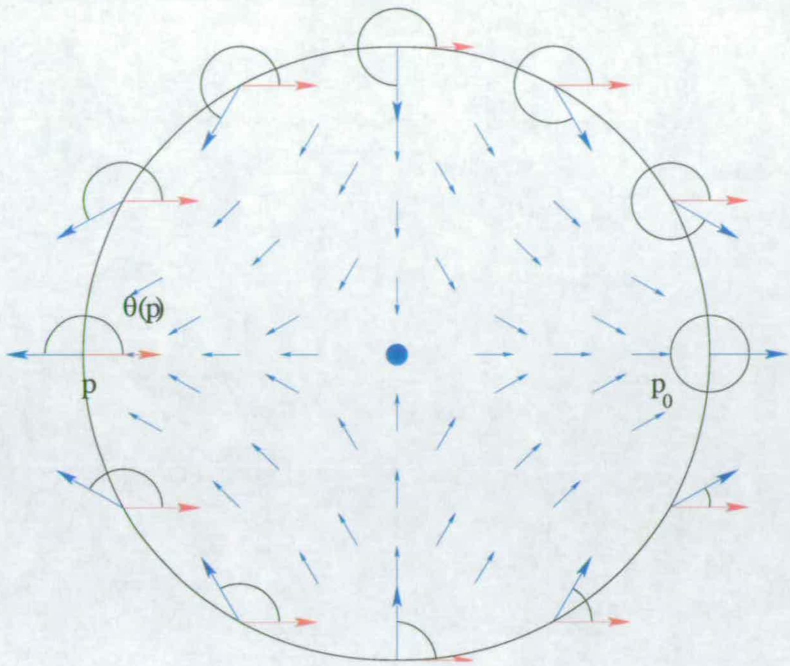


Figure 1.2: How to calculate the index of a saddle point

1.2 Flows and Equivalence of vector fields

By the end of this section we will have defined the conditions under which two (smooth) vector fields are equivalent. We will start with some flow definitions. It is well-known that flows and vector fields are two different ways of describing the same thing. For example, for the 2 dimensional case, see [8], p5 remark 1.3.

As before, let Σ be a smooth, compact surface of genus g. Let X be a vector field on Σ.

Definition 1.2.1 (Flows). A flow is a smooth map $f : \Sigma \times \mathbb{R} \to \Sigma$ with the properties

$$f(p,0) = p \qquad f(p, t_1 + t_2) = f(f(p,t_1), t_2), \qquad p \in \Sigma, t_i \in \mathbb{R}$$

The trajectory of the point p is the set

$$l(p) = \{f(p,t) : t \in \mathbb{R}\}$$

Proposition 1.2.1. *There is a bijective relationship between vector fields and flows on a given surface* Σ

Proof: The bijection is defined as follows. Given a flow f the related vector field is simply the vector field X_f such that, at each point $p \in \Sigma$

$$X_f(p) = \frac{d}{dt} f(p,t)\bigg|_{t=0}$$

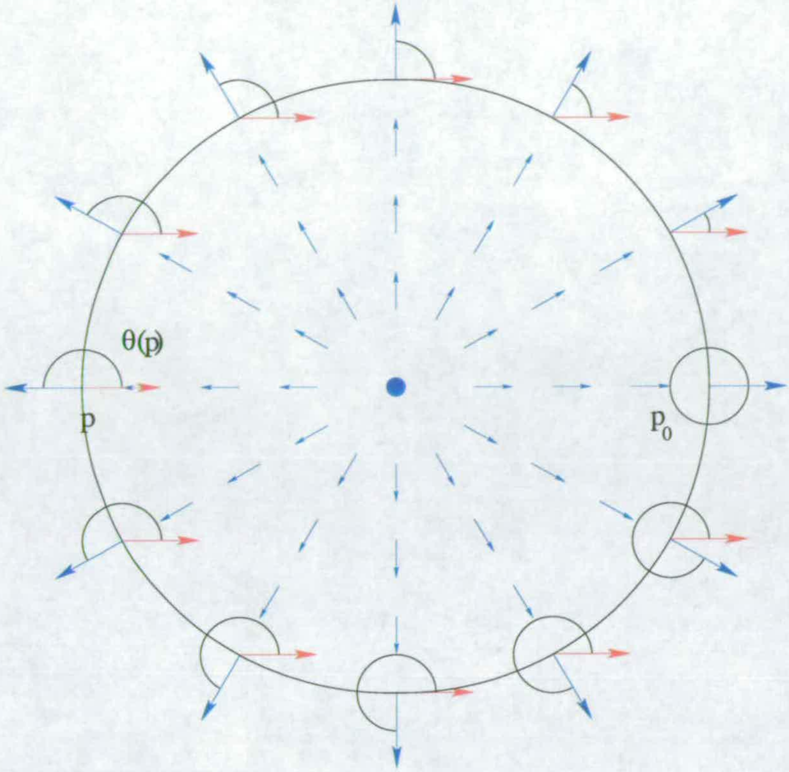


Figure 1.3: How to calculate the index of a maximum (or, by reversing all directions, a minimum)

The inverse is defined corresponding to the vector field X by defining the flow f_X as

$$f_X(p,t) = \gamma(t)$$

where γ is the flowline of X through p. It now remains to show that this is indeed a flow. We need to check the two properties given above. But the condition that $f_X(p,0) = p$ follows immediately from the fact that $f_X(p,t) = \gamma(t)$ where γ is the flowline through p, hence $\gamma(0) = p$. As for the second condition, observe that if $\gamma_1(t)$ is the flowline through p and $\gamma_2(t)$ is the flowline through $f_X(p,t_1)$ then $\gamma_2(t_2) = \gamma_1(t_1 + t_2)$. So, using the uniqueness of flowlines (up to a time change, see Definition 1.1.5)

$$f_X(f_X(p,t_1),t_2) = \gamma_2(t_2) = \gamma_1(t_1 + t_2) = f_X(p, t_1 + t_2)$$

as required.

Hence the above relationship is a bijection. $\square$

Notice, however, that a set of trajectories (where a trajectory is the curve made by a flowline, i.e. the trajectory of the flowline γ is the set $\{\gamma(t) : t \in \mathbb{R}\}$) does not uniquely define a flow. It can only define a flow up to multiplication by a smooth map $F : \Sigma \to \mathbb{R}^+$, as we shall see.

The following is a standard definition (see for example [20], page 3)

Definition 1.2.2 (Flow equivalence). Two flows are *equivalent* if there exists a homeomorphism φ of Σ that takes the trajectories of one onto the trajectories of the other and preserves the direction of travel along trajectories.

This equivalence is commonly called *topological equivalence.* For an example of its definition and extended use, see [13] p32 onwards.

Now, suppose we have two vector fields X, Y such that the two flows f_X, f_Y are equivalent. We can use the equivalence of the flows to define an equivalence relation for the vector fields.

Definition 1.2.3. Two vector fields X, Y are equivalent if and only if the corresponding flows f_X, f_Y are topologically equivalent. In this situation, we say $X \sim Y$.

1.3 Pseudo-Minimal vector fields

In this section we shall begin the process by which we define minimal vector fields. We shall begin by first defining a pseudo-minimal vector field as a "nice" (in a sense to be defined) vector field with the minimum number of "nice" zeros. In a later section we shall then be able to define a minimal vector field as a pseudo-minimal vector field satisfying extra conditions. By the end of this section the reader should understand what a pseudo-minimal vector field is.

All the following definitions are defined using flows. The previous section showed how a definition written in terms of a flow can be re-written in terms of a vector field. For example, a fixed point of a flow becomes a zero of a vector field.

Many of the definitions that follow are adapted from [20], p2.

Definition 1.3.1 (Trajectory definitions). Let f be a flow on a compact surface Σ and let p be a given point in Σ.

The trajectory γ of p is the union of the positive and negative semi-trajectories, i.e. $\gamma = \gamma_+ \cup \gamma_-$, where the positive *semi-trajectory* of p is the set

$$\gamma_+ = \{f(p,t) : t \geq 0\}$$

and the negative *semi-trajectory* of p is the set

$$\gamma_- = \{f(p,t) : t \leq 0\}$$

So the trajectory of a flowline is the set of all points on that flowline. For most purposes these two concepts are interchangeable, hence the tendency to call the trajectory of the flowline γ by the same name, γ. If it could be ambiguous, it will be made clear which concept is being used.

The *limit set* of a trajectory γ is the union of the ω and α limit sets of γ, where the ω*-limit set* of the positive semi-trajectory γ_+ is the set

$$\omega = \{q \in \Sigma : \text{ there is a sequence } (t_r)_{r\geq 0} \text{ tending to } +\infty \\ \text{such that } f(p,t_r) \to q\}$$

and the α*-limit set* of the negative semi-trajectory γ_- is the set

$$\alpha = \{q \in \Sigma : \text{ there is a sequence } (t_r)_{r\geq 0} \text{ tending to } -\infty \\ \text{such that } f(p,t_r) \to q\}$$

A trajectory is *recurrent* if it is contained in its limit set.

A fixed point and a periodic trajectory are considered to be *trivial recurrent* trajectories. Any other recurrent trajectory is called a *non-trivial recurrent trajectory.*

A *quasiminimal set* is the closure of a non-trivial recurrent trajectory.

A flow is *regular* if it has no non-trivial recurrent trajectories.

A vector field is *regular* if the associated flow is regular. So a regular vector field has no quasiminimal sets. All vector fields considered here will be regular and on a compact, oriented surface. Hence they will not contain quasiminimal sets. The definition of non-trivial recurrent trajectories and quasiminimal sets is included for completeness only. For examples of flows with quasiminimal sets, see [20] pp15-16. The examples given there are of Denjoy and Cherry flows.

To illustrate the concepts, consider the following examples.

Example 1.3.2. Consider the two vector fields on $\mathbb{R}^2$ given by

$$X_1(x,y) = (-y,x) \qquad\qquad X_2(x,y) = (x,-y)$$

X_1 is called a centre and X_2 is called a saddle. Both vector fields are zero at the origin, so the origin is a fixed point of both flows, i.e. a trivial recurrent trajectory. In the case of X_1, all flowlines not passing through the origin are periodic, so every flowline of X_1 is a trivial recurrent trajectory. In general, the flowline of X_1 through a point (x_0, y_0) is the circle $x^2 + y^2 = x_0^2 + y_0^2$ and the flowline of X_2 through the same point is the connected component of $xy = x_0y_0$ containing the point (x_0, y_0). Figure 1.4 shows the two vector fields.

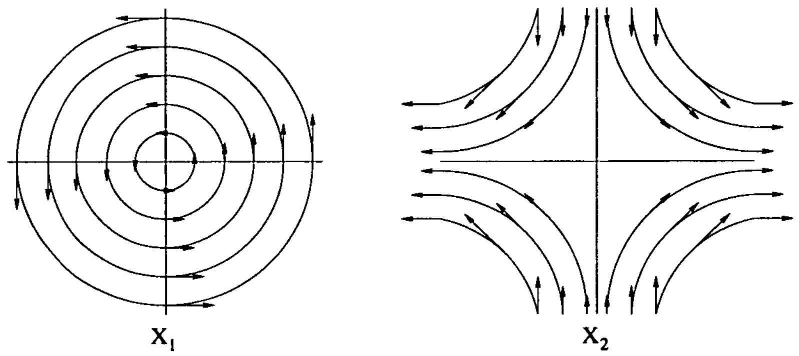


Figure 1.4: The two vector fields X_1 and X_2 from example 1.3.2.

To further illustrate the concepts defined above, consider the flowline γ of X_2 through the point $(0, 1)$. It is given parametrically by $\gamma(t) = (e^t, 0)$ so its positive semi trajectory is homeomorphic to the half-open interval $[1, \infty)$. Its negative semi-trajectory is homeomorphic to the half-open interval $(0, 1]$. Its ω-limit is at infinity whilst its α-limit is the fixed point at the origin.

It is easy to see that in fact X_1 is degenerate at the origin, whilst X_2 is non-degenerate, as the function

$$f(x, y) = \frac{x^2 - y^2}{2}$$

satisfies the conditions of Definition 1.1.12 and the determinant of $\mathrm{d}X_2$ is 1 everywhere. □

Example 1.3.3. Consider the universal covering of the torus, $\tilde{\pi} : \mathbb{R}^2 \to \mathbb{T}^2$ given by $\tilde{\pi}(x, y) = (e^{ix}, e^{iy})$. Let q be irrational. The vector field $X(x, y) = (1, q)$ defines a flow on the torus that has no fixed points. For any point $p \in \mathbb{T}^2$ the flowline through that point is not periodic and the limit set of such a flowline is the whole surface. Thus any flowline is non-trivial recurrent. Thus for this flow the whole surface is quasiminimal.

This example shows that it is relatively easy to construct flows with quasiminimal sets if we allow such sets to be the whole surface. However, the construction of flows with non-trivial quasiminimal sets is known to be hard. Examples of such flows include Denjoy and Cherry flows, and are given in [20] pp15-16. □

Definition 1.3.4 (Non-degenerate vector fields). A non-degenerate vector field is one such that no zero is degenerate.

Hence a non-degenerate, regular vector field is equivalent to a regular flow with non-degenerate fixed points.

Note the following "simplification" theorem, from [20], page 3. This is also referred to by some authors as the "flow box" theorem. The proof that follows is taken from [8], page 13. An alternative proof may be found in [10], page 243.

Theorem 1.3.5 (Rectifiability Theorem). *Let f be a flow on Σ and $p \in \Sigma$ a regular point. Then there is a neighbourhood $U \ni p$ and a diffeomorphism $\phi : U \longrightarrow \mathbb{R}^2$ such that for every trajectory $\gamma, \gamma \cap U \neq \emptyset$, each connected component of $\gamma \cap U$ is mapped by ϕ to the line $y =$constant.*

Proof: As this is a local theorem, it suffices to consider the case of X, a vector field on $\mathbb{R}^2 =< x_1, x_2 >$ such that $X(0,0) = \frac{\partial}{\partial x_1}$. We define ϕ to be the map defined in a small neighbourhood of the origin by

$$\phi(x_1, x_2) = f_X((0, x_2), x_1)$$

Now, the Jacobian of ϕ at the origin is the identity, hence we can apply the Inverse Function Theorem (see [10] Appendix IV for a statement and proof) to get an inverse for ϕ in an open neighbourhood U of the origin that defines the required coordinate system. □

Definition 1.3.6 (Transversal). An arc $\tau \subset \Sigma$ is called a *transversal* to X if for every point $p \in \tau$ there is a rectifying diffeomorphism $\phi : U \longrightarrow \mathbb{R}^2$ that takes $\tau \cap U$ to the line $x =$ constant. In this case notice that $\phi \circ X$ is parallel to $\frac{\partial}{\partial x}$

Definition 1.3.7 (One-sided circuit). A *one-sided circuit* is a simple, closed, one-sided curve, i.e. a circuit for which any neighbourhood is homeomorphic to a Möbius strip. Notice that the existence of a curve with only one side on a surface Σ without boundary implies that the surface is non-orientable.

Note the following classification theorem for flowlines, from [20], page 35.

Theorem 1.3.8. *Let f be a flow with finitely many fixed points on a compact surface Σ and let $\gamma_{+(-)}$ be a positive (negative) semi-trajectory. Then the $\omega(\alpha)-$limit set of $\gamma_{+(-)}$ is one of the following types:*

1. *a fixed point,*

2. *a periodic trajectory,*

3. *a one-sided circuit,*

4. *a set consisting of a finite number saddle points together with separatrices connecting them, or*

5. *a quasiminimal set*

Proof: This result is adapted from a core result in the area of dynamical systems on surfaces. Indeed, the theorem in a form similar to that given above has been constructed in [20] using several cited works. The proof relies on showing that any limit set that is not one of the first four types must be quasiminimal. For the summary of the proof, see [20], page 35. □

However, the following corollary is sufficient for our purposes here.

Corollary 1.3.9. *If f is the flow of X, a non-degenerate, regular vector field with finitely many fixed points on a compact orientable surface Σ then the $\omega(\alpha)-$limit set of a positive (negative) semi-trajectory is one of the following types:*

1. *a fixed point,*

2. *a periodic trajectory*

3. *a set consisting of a finite number saddle points together with separatrices connecting them.*

Proof: By Theorem 1.3.8 we need only show that no semi-trajectory can have, as its limit set, a one-sided circuit or a quasiminimal set. But the first case is excluded by the orientability of Σ and the second by the regularity of X. □

Thus we have the following theorem.

Theorem 1.3.10. *Suppose X is a regular vector field on a compact oriented surface Σ with genus $g > 1$ such that X is non-degenerate. Then X has at least $2g - 2$ zeros.*

Moreover, if there exists an X that has precisely $2g-2$ zeros then all the zeros are saddle points and the $\omega(\alpha)$-limit sets of all half-flowlines of X are either periodic cycles, zeros or sets consisting of a finite number of saddle points together with separatrices connecting them.

Remark 1.3.11. We should note that this theorem only shows that the minimum possible number of zeros of a non-degenerate vector field is $2g - 2$. It does not show that such a vector field exists as it assumes it can be constructed. The main aim of this thesis is to show that such vector fields exist and to classify them.

Proof: We begin by showing that if X has the conditions given, then it has at least $2g - 2$ zeros. But as X is non-degenerate every zero has index ± 1. By Theorem 1.1.8 the sum of the indices of the zeros is precisely $2 - 2g$. So we immediately see that there are at least $2g - 2$ zeros with index -1, hence X has at least $2g - 2$ zeros.

Now suppose X has precisely $2g - 2$ zeros. We need to show that each zero is necessarily a saddle point. But each zero has index ± 1 and the sum of the $2g - 2$ zeros is $2 - 2g$. Hence we immediately see that each zero has index -1. But the only non-degenerate zero with index -1 is a saddle point, so X has precisely $2g - 2$ saddle points, as required.

The condition that the $\omega(\alpha)$-limit sets of all half-flowlines of X are either periodic cycles, zeros or circles of separatrices linking fixed points is due to Corollary 1.3.9.
□

Definition 1.3.12 (Pseudo-minimal vector field). A regular vector field X satisfying the conditions of Theorem 1.3.10 with the added provision that the $\omega(\alpha)$-limit sets of all half-flowlines of X are either periodic cycles or zeros is called *pseudo-minimal.* An example of such a vector field is shown in Figure 1.6. In this case, note that the two vector fields differ only on the left "handle" of the surface. The vector fields can be considered by cutting the handle away. Then it can be seen that the two vector fields are equivalent to the vector fields $X(x, y, z) = (-y, x, 1)$ and $X(x, y, z) = (-y, x, 0)$ on the unit cylinder (i.e. $(x, y, z) \in \mathbb{R}^3$ such that $x^2 + y^2 = 1, z \in [0, 1]$). The two vector fields on the cylinders are shown in Figure 1.5.

1.4 Non-wandering points and the Poincaré Map

In order to move now from a pseudo-minimal vector field to a minimal one, we need to take a slight detour and consider some properties of certain flows. For this we need the following.

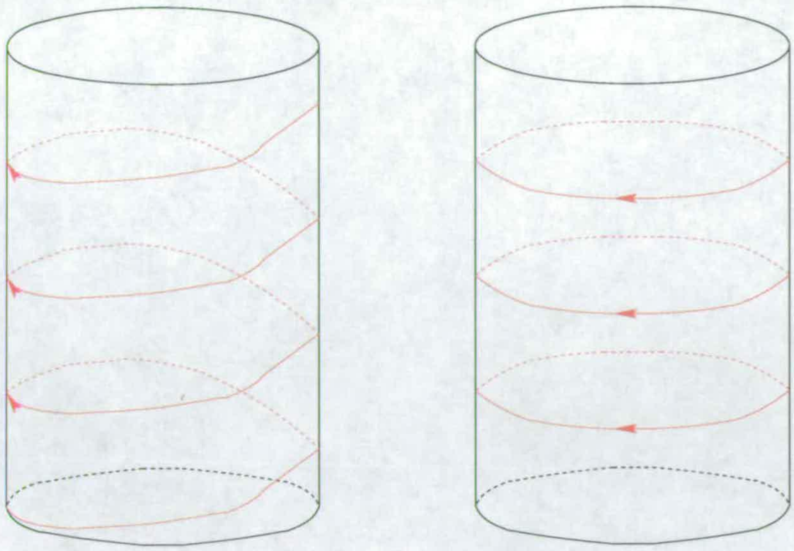


Figure 1.5: Cylinders with the restrictions of two vector fields, both pseudo-minimal, with the second also being minimal.

Definition 1.4.1 (Non-wandering points). A point p is called *non-wandering* if, for any neighbourhood $U(p)$ there is a sequence $(t_r)_{r\geq 0}$ such that $t_r \to +\infty$ or $t_r \to -\infty$ and $U \cap f(U, t_r) \neq \emptyset$ for all r.

We define $\Omega(X)$ to be the set of non-wandering points of X.

We say a vector field (similarly, a flow) is non-wandering if $\Omega(X) = \Sigma$ (similarly $\Omega(X_f) = \Sigma$).

The following theorem is taken from [20], page 142:

Theorem 1.4.2. *Let Σ be a compact surface and suppose f is a non-wandering flow on Σ. Then Σ can be represented as a union $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_k$ of regions with pairwise disjoint interiors and such that the boundary of $\partial\Sigma_i$'s is a union of flowlines γ such that*

$$\lim_{t\to\pm\infty} X \circ \gamma(t) = X(\lim_{t\to\pm\infty} \gamma(t)) = 0$$

and their limit points.

The interior of each Σ_i is one of the following:

1. *a disc region with a centre (i.e. a fixed point with index $+1$ surrounded arbitrarily closely by periodic orbits),*

2. *an annulus filled up with periodic trajectories,*

3. *a Möbius strip filled up with periodic trajectories, or,*

4. *a quasiminimal set.*

Proof: For a proof see [20], page 142. □

Corollary 1.4.3. *Let X be a pseudo-minimal vector field on an orientable surface Σ such that $\Omega(X) = \Sigma$. Then Σ can be represented as a union $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_k$ of regions with pairwise disjoint interiors and such that the boundary of $\partial\Sigma_i$'s is a union of flowlines γ such that $X(\lim_{t\to\pm\infty} \gamma(t)) = 0$ and their limit points.*

Moreover, an interior of Σ_i is an annulus filled up with periodic trajectories.

Proof: As X is pseudo-minimal Σ contains no quasiminimal sets. By Corollary 1.3.10, as X is pseudo-minimal no zero of X is a centre. As Σ is orientable, no subset of Σ can be a Mobius strip. To complete the proof now appeal to Theorem 1.4.2 □

The following lemma of Poincaré and the subsequent definition is quoted from [8] (pp 59-67):

Lemma 1.4.4 (Poincaré). *Let c be a periodic orbit of a period T of X and let p be a point of c. Let τ be a curve on Σ transverse to X through p. The flow generated by X is denoted f_X. Then there exists open $U \subset \tau$, open V with $T \in V \subset \mathbb{R}$ and a function $\alpha : U \to V$, as smooth as X, satisfying the properties:*

1. $\alpha(p) = T$

2. $f_X(x, \alpha(x)) \in \tau$ *for all* $x \in U$.

Proof: See [8] (pp 59-67) □

Definition 1.4.5 (Poincaré Map). This function $\alpha(x)$ may therefore be interpreted as the time of first return to τ of the flowline through x.

The map $g : U \to \tau$ given by $g(x) = f_X(x, \alpha(x))$ is called the *Poincaré map* of τ.

1.5 Minimal vector fields

In this section we move from pseudo-minimal vector fields to minimal ones. The following theorem contains the essence of the definition.

Theorem 1.5.1. *The following conditions are equivalent for a pseudo-minimal vector field X on a compact surface Σ.*

1. *Each flowline of X is one of the following types:*
   (a) *Zeros of X.*
   (b) *Periodic cycles of X.*
   (c) $\gamma : \mathbb{R} \to \Sigma$ *such that* $X(\lim_{t\to\pm\infty}\gamma(t)) = 0$.
2. *X has only finitely many non-compact flowlines.*
3. *The set of points on periodic flowlines is dense in Σ.*
4. *If γ is a flowline of X such that either its $\alpha-$limit set, or its $\omega-$limit set contains a periodic cycle, then γ is a periodic cycle.*
5. $\Omega(X) = \Sigma$
6. *If γ is a flowline of X then $\gamma \cap (\alpha(\gamma) \cup \omega(\gamma)) = \emptyset$ implies that both $\alpha(\gamma)$ and $\omega(\gamma)$ are single points.*

We note in passing that our definition of pseudo-minimal vector fields implies that in a sufficiently small neighbourhood around the separatrices of the saddle points all flowlines satisfy condition 1. However, this proof will show that if any one of the above conditions is true, that the entire surface will be a "sufficiently small neighbourhood" in this context.

Proof: The proof will show that

i. Condition 1 implies condition 2
ii. Condition 2 implies condition 3
iii. Condition 3 implies condition 4
iv. Condition 4 implies condition 1
v. Condition 3 implies condition 5
vi. Condition 5 implies condition 1
vii. Condition 1 implies condition 6
viii. Condition 6 implies condition 4

which demonstrates the equivalence required.

So, part i. condition 1 implies condition 2. Now, the only non-compact flowlines are those given in 1c. As X is pseudo-minimal, each fixed point of X is a saddle point, and there are at most two such flowlines coming in to each saddle. Thus, the number of non-compact flowlines is less than or equal to twice the number of fixed points, which is finite.

In fact, the number of non-compact flowlines is can easily be seen to be twice the number of fixed points by this argument, as each non-compact flowline has precisely one ω-limit, and such a limit is a fixed point of X. But each fixed point of X is, by Definition 1.3.10, a saddle point, which has two such flowlines as its ω-limit.

Part ii. condition 2 implies condition 3. X has finitely many non-compact flowlines implies the compact flowlines are dense. But a flowline is compact if and only if it contains its limit points. Now, by Corollary 1.3.9 and the definition of pseudo-minimality the only limit points are fixed points or periodic cycles and it is easy to see that if γ is a compact flowline that contains a fixed point (periodic cycle) then γ is a fixed point (periodic cycle). We know that as X is non-degenerate the number of fixed points is finite, hence the set of points on periodic cycles is dense in Σ.

Part iii. condition 3 implies condition 4. Suppose c is the (w.l.o.g) $\omega-$limit cycle of some flowline γ. Suppose further, for a contradiction, that $\gamma \neq c$. Let $p \in c$. Consider the transversal τ through p (defined by Definition 1.3.6). Let p_0 be a point in $\gamma \cap \tau$ and define a sequence of points $(p_r)_{r \in \mathbb{N}}$ such that p_r is the r-th point of intersection of γ with τ. It is easy to see that this sequence converges to p. However, as $\gamma \neq c$ we have that for all r, $p_r \notin c$.

As the set of points on the periodic flowlines is dense in Σ for each r there is some point $q_r \in [p_r, p_{r+1}] \subset \tau$ such that c_r, the flowline through q_r, is periodic. It is clear that, by this definition, the sequence $(q_r)_{r \in \mathbb{N}}$ also converges to p.

Now, by Lemma 1.4.4 there exists $U \ni p$, open in Σ and a function $\alpha : U \cap \tau \to \mathbb{R}$, such that α is the time of first return map for τ. U is an open neighbourhood of p, so there exits $R \in \mathbb{N}$ such that for all $r \geq R$, $q_r \in U$. Thus $[p, q_r] \subset U \cap \tau$. It is clear that as the Poincaré map $g : x \to f_X(x, \alpha(x))$ fixes p and q_r and is continuous, so it sends $[p, q_r]$ to itself. Thus the map α has the property that, for all $x \in [p, q_r]$, $f_X(x, \alpha(x)) \in [p, q_r]$, i.e. that the Poincaré map g maps $[p, q_r]$ to itself. It is also clear that it preserves the orientations of $[p, q_r]$.

Thus the sequence $(p_r)_{r \in \mathbb{N}} \in U$ given above is the sequence such that $p_{r+1} =$

$f_X(p_r, \alpha(p_r))$ for all $r \geq R$.

Now, as the Poincaré map g fixes $[p, q_r]$ set-wise and as the periodic cycles are dense in Σ, so the set of points fixed by g is dense in $[p, q_r]$. Thus by continuity g fixes $[p, q_r]$ pointwise. Thus for all r, $p_{r+1} = f_X(p_r, \alpha(p_r)) = p_r$ and so γ is periodic. Thus $\gamma = c$, the required contradiction.

Hence $\gamma = c$ as required.

Part iv. condition 4 implies condition 1. By Corollary 1.3.9 and the definition of pseudo-minimality if γ is a flowline of X then the limit set of γ is either a pair of fixed points, a single fixed point, or a periodic cycle. But by assumption, if the limit set contains a periodic cycle, then γ is that periodic cycle. Thus we need only consider the cases when the limit set contains a fixed point. But it is easy to see that the only two possibilities are then that either γ is a fixed point itself, or γ is a flowline of the type described in condition 1, part 1c

Part v. condition 3 implies condition 5. Let $p \in \Sigma$. Then, as the set of periodic flowlines is dense in Σ for any open neighbourhood U of p there exists some point $x \in U$ that is contained in a periodic flowline of X (γ, say). Thus we may choose our sequence $(t_r)_{r\geq 0}$ to be the r-th multiples of the period of γ so that for all r $f(x, t_r) = x \in f(U, t_r)$. Thus for all r we have $U \cap f(U, t_r) \neq \emptyset$ as required.

Part vi. condition 5 implies condition 1. By assuming that $\Omega(X) = \Sigma$ we have satisfied all the conditions of Corollary 1.4.3. Thus if a flowline γ of X is not a fixed point or such that $X(\lim_{t\to\pm\infty} \gamma(t)) = 0$ then it is in the interior of a region Σ_i as given by Corollary 1.4.3. But this corollary says that any such flowline is a periodic orbit, as required.

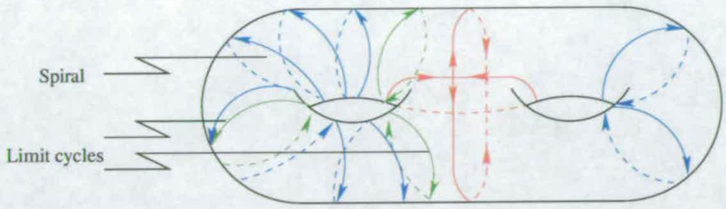
Part vii. condition 1 implies condition 6. Suppose γ is a flowline of X satisfying $\gamma \cap (\alpha(\gamma) \cup \omega(\gamma)) = \emptyset$. Then γ is neither a zero of X not a periodic cycle, as flowlines of both types meet their limit sets. Hence γ must be a flowline of type 1c, which implies that both its α and ω limit sets are single points, as required.

Part viii. condition 6 implies condition 4. Suppose γ is a flowline of X such that (w.l.o.g.) $\alpha(\gamma)$ contains some periodic cycle c. Then condition 6 says that γ must meet its limit sets. Now, as X is pseudo-minimal, $\alpha(\gamma)$ is either a periodic cycle or a zero of X. Hence $\alpha(\gamma)$ is a periodic cycle and γ meets $\alpha(\gamma) \cup \omega(\gamma)$.

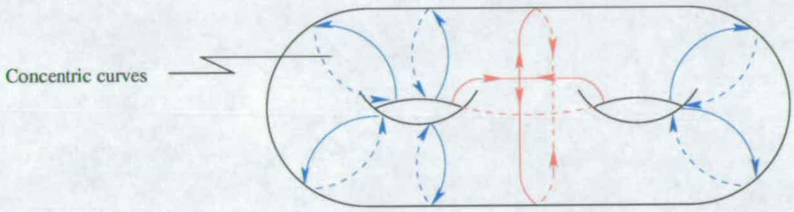
Suppose γ doesn't meet $\alpha(\gamma)$. So $\gamma \cap \omega(\gamma) \neq \emptyset$. Now, as X is pseudo-minimal, by definition 1.3.10 $\omega(\gamma)$ is either fixed or periodic. But if $\omega(\gamma)$ is fixed, then γ contains a fixed point and so both γ and $\alpha(\gamma)$ are also fixed, which contradicts our assumption that $\alpha(\gamma)$ contains a periodic cycle. So $\omega(\gamma)$ is a periodic orbit, γ meets a periodic cycle and hence is a periodic cycle, as required.

□

Definition 1.5.2. A pseudo-minimal vector field that also satisfy the conditions of the above theorem shall be called *minimal.* An example of a minimal vector field is shown in Figure 1.6.



A pseudo–minimal vector field



A minimal vector field

Figure 1.6: Two vector fields, both pseudo-minimal, with the second also being minimal.

For the remained of this thesis we shall discuss the implications of this definition, including a complete classification of minimal vector fields.

Chapter 2

On Dehn twists, embedded graphs and homology

In the next few chapters we shall define and prove concepts and results required for the classification and construction of minimal vector fields. It will be eventually seen that a complete classification of such vector fields requires an invariant constructed from graph theory and one constructed from topology and combinatorics.

In this chapter we shall calculate the automorphism group of the first homology group of a compact oriented surface Σ with n fixed points. We shall show that it is a matrix group consisting of all matrices of the form

$$\begin{bmatrix} \Psi & A \\ 0 & S \end{bmatrix}$$

where $\Psi \in \mathrm{Sp}(2g, \mathbb{Z})$, the symplectic group of $2g \times 2g$ integer-valued matrices, $A \in \mathcal{M}((2g, n-1), \mathbb{Z})$ and $S \in \mathcal{S}(n)$ for $\mathcal{S}(n)$ a group to be specified.

2.1 Preliminaries comments on homology

Let Σ be a compact, oriented surface with genus g and let $V = \{p_1, \ldots, p_n\}$ be a set of n points on Σ. We shall start with some basic comments concerning the homology of the pair (Σ, V).

Consider the homology sequence of the pair (Σ, V) with $\mathbb{Z}$-coefficients.

$$\ldots \longrightarrow H_1(V) \xrightarrow{\alpha} H_1(\Sigma) \xrightarrow{\beta} H_1(\Sigma, V) \xrightarrow{\delta} H_0(V) \longrightarrow H_0(\Sigma) \longrightarrow 0$$

This reduces to

$$0 \xrightarrow{\alpha} \mathbb{Z}^{2g} \xrightarrow{\beta} H_1(\Sigma, V) \xrightarrow{\delta} \mathbb{Z}^n \longrightarrow \mathbb{Z} \longrightarrow 0$$

Thus $H_1(\Sigma, V) \cong \mathbb{Z}^{2g+n-1}$. Note that we can also consider the short exact sequence:

$$0 \longrightarrow H_1(\Sigma) \xrightarrow{\beta} H_1(\Sigma, V) \xrightarrow{\tilde{\delta}} \tilde{H}_0(V) \longrightarrow 0 \tag{2.1}$$

where $\tilde{H}_0(V) = \{\sum \alpha_i v_i : \alpha_i \in \mathbb{Z} \text{ and } \sum \alpha_i = 0\} \subset H_0(V)$.

Now $H_0(V)$ has generators $\{p_1, \ldots, p_n\}$, i.e. we shall not distinguish between the points of V and the corresponding generators of $H_0(V)$. To select a basis of $\tilde{H}_0(V)$ we note that it is generated by elements of the form $p_i - p_j$, which may be thought of as the directed edges of an abstract graph on the vertices $\{p_1, \ldots, p_n\}$. We use this idea to define a basis as follows. Let K_n be the complete directed graph on the vertices $\{p_1, \ldots, p_n\}$. Choose a spanning tree T. This, necessarily, has $n-1$ edges, each of the form $p_i - p_j$. The edges of T form our basis $\mathcal{B}_T = \{f_1, \ldots, f_{n-1}\}$, dependent on the chosen spanning tree T. Note that, as T is a tree, it is contractible. This will be important later. Figure 2.1 shows an example of the selection of a basis using this method. In this figure, the basis $\mathcal{B}$ is show in red. Note that direction arrows are not shown and that for each edge shown here, it would be more accurate to say there are two directed edges, one in each direction.

Notice that the spanning tree T may be embedded on Σ so as to lie within a disc $\mathcal{D}$ on Σ. From now on we will assume that this is in fact this case, i.e. $T \subset \mathcal{D} \subset \Sigma$, where $\mathcal{D}$ is a disc embedded in Σ hence contractible. It will be important later that this is the case. Thus we are choosing our basis in a geometrical way.

We can then write $H_1(\Sigma, V)$ as

$$H_1(\Sigma, V) = < a_1, b_1, a_2, b_2, \ldots, a_g, b_g, f_1, f_2, \ldots, f_{n-1} > \tag{2.2}$$

where $a_1, \ldots, a_g, b_1, \ldots, b_g$ are the "usual" generators for $H_1(\Sigma)$ as shown in Figure 2.2 and each f_k corresponds to an edge of T.

Note 1. Strictly, the a_i, b_i are the images under β of generators of $H_1(\Sigma)$ but we shall ignore this subtlety.

Thus $H_1(\Sigma, V) \cong H_1(\Sigma) \oplus \tilde{H}_0(V)$. We shall refer back to this description of $H_1(\Sigma, V)$ later.

Definition 2.1.1. *$Aut(H_1(\Sigma))$ is defined to be the group of automorphisms of $H_1(\Sigma)$ that can be realised by orientation preserving diffeomorphisms of Σ.*

The group $Aut(H_1(\Sigma, V))$ is defined similarly.

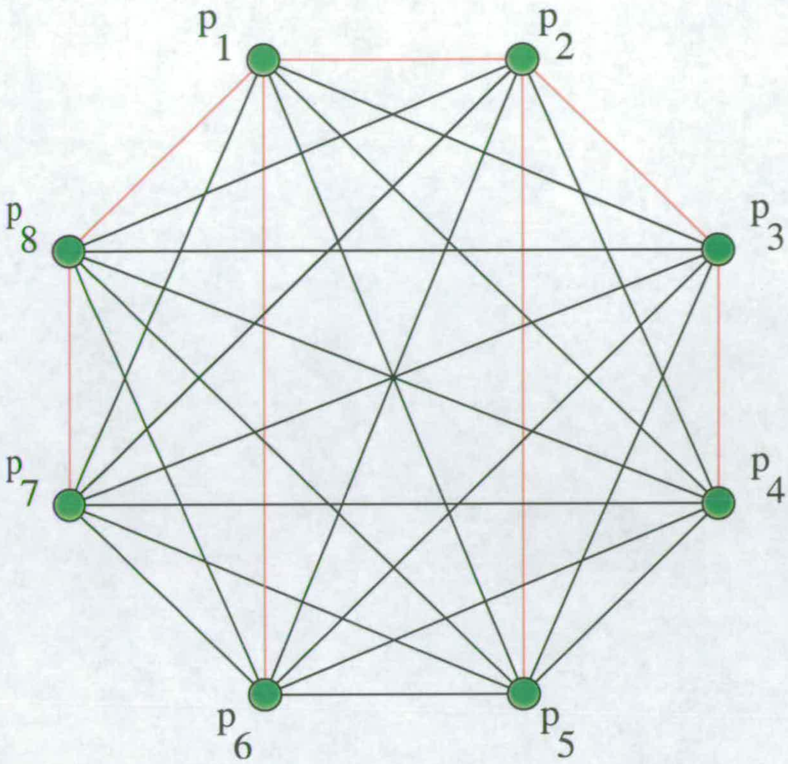


Figure 2.1: An example of how to select a basis of $H_0(V)$ from the complete graph K.

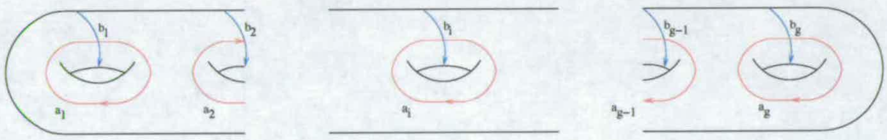


Figure 2.2: The "usual" or canonical basic curves that generate $H_1(\Sigma)$

2.2 Dehn twists

In this section we shall show that a Dehn twist induces a well-defined linear map on $H_1(\Sigma, V)$.

In what follows we shall assume that γ is some fixed, simple, closed curve on Σ and σ and τ are closed curves on Σ though not necessarily fixed. Also f will be an arc connecting two points of V, so that f represents a class in $H_1(\Sigma, V)$ that does not vanish under $\tilde{\delta}$.

Definition 2.2.1 (Dehn twists). A *Dehn twist* D_γ around an oriented, simple, closed curve γ is defined to be a diffeomorphism that is the identity everywhere on Σ save for a small band N_γ (i.e. closed neighbourhood) about γ. The action in this band, as shown in Figure 2.3. is to cut along one edge of the band, twist the entire neighbourhood through a complete turn and reglue along the same edge.

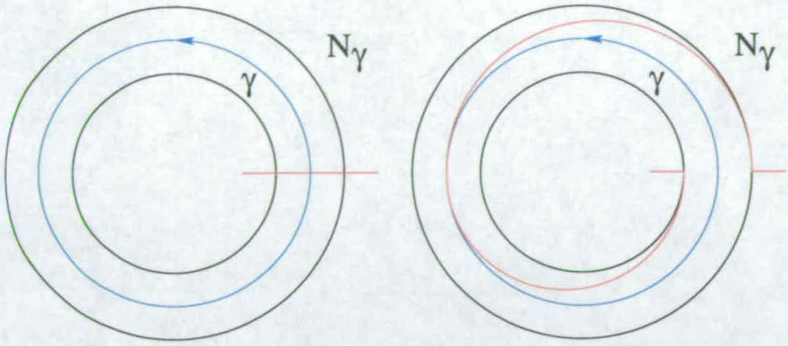


Figure 2.3: Effect of a Dehn twist around a curve γ

Dehn twists were first defined by Max Dehn in the early 1920s but not published by him at the time. They were later used by Goeritz in 1933 and revived by Lickorish in 1962, although we will refrain from referring to them as "Lickorish twists".

Lemma 2.2.2. *The homology class of a simple, closed curve γ is primitive, i.e., if $[\gamma] \in H_1(\Sigma, \mathbb{Z})$ satisfies $[\gamma] = k[\omega]$ for some $k \in \mathbb{Z}, [\omega] \in H_1(\Sigma, \mathbb{Z})$ then $k = \pm 1$. Conversely, if $c \in H_1(\Sigma, \mathbb{Z})$ is primitive then there exists γ, a simple closed curve on Σ such that γ is a representative curve for c (i.e. $[\gamma] = c$).*

Proof: This is a result of Poincaré 1904. See [21] □

Lemma 2.2.3 (Lickorish). *Any piecewise linear, orientation preserving homeomorphism of Σ is isotopic to a product of Dehn twists.*

Proof: See [16], Theorem 1, p536 □

Lemma 2.2.4. *$Aut(H_1(\Sigma, \mathbb{Z}))$ is generated by Dehn twists around simple closed curves.*

Proof: Recall that $Aut(H_1(\Sigma, \mathbb{Z}))$ is the group of automorphisms of $H_1(\Sigma)$ that can be realised by orientation preserving diffeomorphisms of Σ. But it is well known that any orientation preserving diffeomorphism of Σ is isotopic to a piecewise linear, orientation preserving homeomorphism of Σ, which, by Lemma 2.2.3, is isotopic to a product of Dehn twists. So any orientation preserving diffeomorphism of Σ is isotopic to a product of Dehn twists.

But two isotopic homeomorphisms induce the same map on homology. Hence the homology map induced by such a diffeomorphism is the product of the maps induced by a series of Dehn twists. □

Definition 2.2.5 (Intersection Number). Suppose σ and τ are two curves on an oriented Σ such that, when they intersect, do so transversely in a finite number of points. To each intersection we define a weight ± 1 (see Figure 2.4). Then the *intersection number* $\sigma \cdot \tau$ is the weighted sum of the intersections.

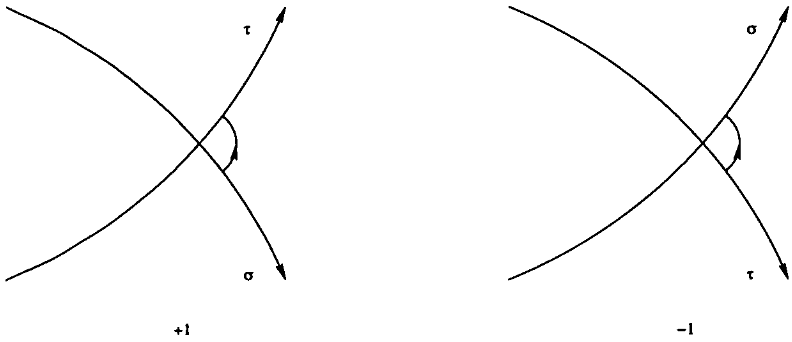


Figure 2.4: Assigning weights at intersections

We make the following observations about the intersection number.

Lemma 2.2.6. *The intersection number is a well-defined bilinear, skew form on the first homology group, i.e. we have a pairing*

$$H_1\Sigma \times H_1\Sigma \to \mathbb{Z}$$

given by

$$([\sigma], [\tau]) \to \sigma \cdot \tau$$

that is well-defined, bilinear and skew. Moreover, using the basis given in (2.2) for $H_1(\Sigma)$, we have that

$$\begin{aligned} a_i \cdot a_j &= 0 \\ b_i \cdot b_j &= 0 \\ a_i \cdot b_j &= \delta_{ij} \end{aligned}$$

where δ_{ij} is the Kronecker delta.

Proof: This is a standard result. For a proof see [7] pp246-256. $\square$

Lemma 2.2.7. *Let γ be an oriented simple closed curve on Σ and*

$$D_\gamma : \Sigma \to \Sigma$$

be a Dehn twist about γ. *Then*

$$(D_\gamma)_* : H_1\Sigma \to H_1\Sigma$$

is linear and is given by the formula

$$(D_\gamma)_*[\sigma] = [\sigma] + (\sigma \cdot \gamma)[\gamma]$$

Moreover, if γ' *is homologous to* γ *then* $(D_{\gamma'})_* = (D_\gamma)_*$

Proof: The linearity of the right-hand side of the formula follows from bilinearity of $(\cdot)$ and the fact that homology classes are linear (i.e. $[\sigma+\gamma] = [\sigma]+[\gamma]$). That the Dehn twist actually has this effect follows by inspection, for every time σ and γ intersect, the Dehn twist about γ will add one copy of γ to σ if the intersection has weight $+1$, and subtract one if the intersection has weight -1. See Figure 2.3.

The proof of the last statement follows from the fact that the intersection number is well-defined on homology. See [7] pp357-359. □

This result can be extended to $H_1(\Sigma, V)$ as follows.

Lemma 2.2.8. *Let* γ *be an oriented simple closed curve on* $\Sigma - V$ *and let*

$$D_\gamma : (\Sigma, V) \to (\Sigma, V)$$

be a Dehn twist about γ. *Then*

$$(D_\gamma)_* : H_1(\Sigma, V) \to H_1(\Sigma, V)$$

is a well-defined linear map given by

$$(D_\gamma)_*[c] = [c] + ([c] \cdot [\gamma])[\gamma]$$

where $[c] \in H_1(\Sigma, V)$
Moreover, if $|V| \leq 1$ *and* γ' *is homologous to* γ *then* $(D_{\gamma'})_* = (D_\gamma)_*$.

Proof: D_γ is a Dehn twist around γ so it is a diffeomorphism of the pair (Σ, V) to itself. As such, there is an induced map on homology

$$(D_\gamma)_* : H_1(\Sigma, V) \to H_1(\Sigma, V)$$

which is linear and additive. So the essence of the claim here is two-fold. Firstly, that the function $p_\gamma : H_1(\Sigma, V) \to H_1(\Sigma, V)$ defined by the formula

$$p_\gamma[c] = [c] + ([c] \cdot [\gamma])[\gamma] \tag{2.3}$$

is well-defined for all $[c] \in H_1(\Sigma, V)$, and secondly that $(D_\gamma)_* \equiv p_\gamma$.

Now, from the previous lemma, we know that p_γ, as defined by the formula given in (2.3) given above, is well-defined on all $[c]$ such that $\tilde{\delta}[c] = 0$, where $\tilde{\delta}$ is defined in (2.1). So suppose $[c]$ is an element of $H_1(\Sigma, V)$ such that $\tilde{\delta}[c] \neq 0$. Then there exists a representative set of curves $c \in (\Sigma, V)$ for $[c]$. We define $c \cdot \gamma$ in the same way as given in Definition 2.2.5, i.e. as the weighted sum of the intersections of c and γ. So all we need show is that this definition is independent of the choice of c.

So suppose c' is another representative of $[c]$. Then $[c] - [c'] = 0$ so, in particular $\tilde{\delta}[c - c'] = 0$. Therefore $[c - c'] \in \text{im } \beta$ and so $((c - c') \cdot \gamma)$ makes sense. But $c - c'$ is homologous to zero, thus $((c - c') \cdot \gamma) = 0$. Thus $(c \cdot \gamma) = (c' \cdot \gamma)$ and the intersection number is well-defined on $H_1(\Sigma, V)$. Thus p_γ is well-defined. It is also clear that p_γ is linear.

So all that remains is to show that $(D_\gamma)_* \equiv p_\gamma$. But this follows by inspection in a similar manner to Lemma 2.2.7, that is, every time c (a representative curve for $[c]$) and γ intersect, a Dehn twist about γ will add one copy of γ to c if the intersection has weight $+1$ and subtract one if the intersection has weight -1. But this shows that a Dehn twist around γ has the effect on homology given by p_γ. But the map $(D_\gamma)_*$ is the induced map of the Dehn twist. Hence result.

Note that if $|V| \leq 1$ then $H_1(\Sigma, V) \cong H_1\Sigma$ and by the previous lemma, γ' is homologous to γ implies that $(D_{\gamma'})_* = (D_\gamma)_*$.

Figure 2.5 shows why $\gamma' \sim \gamma \implies (D_{\gamma'})_* = (D_\gamma)_*$ no longer holds. Here γ' and γ are identical outside the part of Σ shown. Thus the have the same homology class in $H_1(\Sigma)$. But $f \cdot \gamma' \neq f \cdot \gamma$. Hence $(D_{\gamma'})_* \neq (D_\gamma)_*$ on $H_1(\Sigma, V)$. □

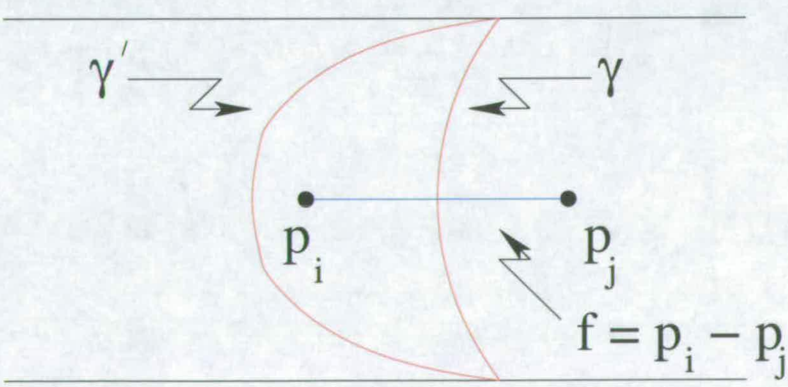


Figure 2.5: Example from Lemma 2.2.8 showing γ, γ' such that $(D_{\gamma'})_* \neq (D_\gamma)_*$

2.3 Transvections and Sympletic groups

We now take a slight detour to consider certain matrix groups to get results that will be required for the main theorem.

We aim to prove that the group of integer-valued symplectic matrices is generated by integer symplectic transvections.

We do not claim that this work is new. Indeed, it has long been known (see [5]) that every symplectic matrix can be written as a product of symplectic transvections. Other work in this field includes [17] on the number of transvection factors of a symplectic matrix and [2] on symplectic transvection in more general cases. Much of this work has also been inspired by [12].

Definition 2.3.1. The matrix group $Sp(2g, \mathbb{R}) \subset GL(2g, \mathbb{R})$, the *real symplectic group*, is the group of all matrices Ψ with that satisfy

$$\Psi^T \mathcal{J}_g \Psi = \mathcal{J}_g$$

where

$$\mathcal{J}_g = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_1 \end{bmatrix}$$

and

$$J_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Such a matrix Ψ is called a *symplectic matrix* The matrix group $Sp(2g, \mathbb{Z}) = GL(2g, \mathbb{Z}) \cap Sp(2g, \mathbb{R})$, the *integer symplectic group* is the group of all symplectic matrices with integer coefficients.

Definition 2.3.2. A matrix Ψ is a *symplectic transvection* if and only if

$$\Psi = I + \lambda \underline{a}\ \underline{a}^T \mathcal{J}_g$$

where $\underline{a} \in \mathbb{R}^{2g}$, I is the $2g \times 2g$ identity matrix and $\lambda \in \mathbb{R}$.

If $\underline{a} \in \mathbb{Z}^{2g}$ and $\lambda \in \mathbb{Z}$ then Ψ is an *integer symplectic transvection.*

If $\underline{a} \in \mathbb{Z}^{2g}$ is not an integer multiple of any other $\underline{b} \in \mathbb{Z}^{2g}$ (i.e. if $\underline{a} = k\underline{b}$ then $k = \pm 1$) then we say $\underline{a}$ is *primitive.*

If $\lambda = \pm 1$ and $\underline{a}$ is primitive (and thus necessarily has integer coefficients) then Ψ is a *unit symplectic transvection* (or *u.s.t.*).

Note 2. The traditional definition of a transvection is of a linear transformation T of a vector space V such that $\exists\ a \in V$, $\phi \in V^*$ such that for all $x \in V$

$$Tx = x + \phi(x)a.$$

So we have a theorem:

Theorem 2.3.3. *Suppose V is a symplectic vector space and T is a transvection which preserves the symplectic structure. Then $\exists\ \lambda \in \mathbb{R}$ such that*

$$Tx = x + \lambda(x \cdot a)a.$$

Proof: If $a = 0$ then set $\lambda = 0$ and the theorem is true. Otherwise, as the symplectic structure is non-degenerate, there exists $x_a \in V$ such that $x_a \cdot a \neq 0$. Let

$$\lambda = \frac{\phi(x_a)}{x_a \cdot a}$$

so $Tx_a = x_a + \phi(x_a)a = x_a + \lambda(x_a \cdot a)a$.

Then, as T preserves the symplectic structure, for all $x \in V$

$$\begin{aligned}
x \cdot x_a &= Tx \cdot Tx_a \\
&= (x + \phi(x)a) \cdot (x_a + \lambda(x_a \cdot a)a) \\
&= (x \cdot x_a) + \phi(x)(a \cdot x_a) + \lambda(x_a \cdot a)(x \cdot a) + \phi(x)\lambda(x_a \cdot a)(a \cdot a)
\end{aligned}$$

But $a \cdot a = 0$ and $a \cdot x_a = -x_a \cdot a$ so $\phi(x)(x_a \cdot a) = \lambda(x_a \cdot a)(x \cdot a)$ for all $x \in V$, i.e. $\phi(x) = \lambda(x \cdot a)$ for all $x \in V$. □

But $x \cdot a = a^T \mathcal{J}_g x$ so $Tx = (I + \lambda a\ a^T \mathcal{J}_g)x$, which motivates our Definition 2.3.2

Claim 2.3.4. *If Ψ is a unit symplectic transvection, then $\Psi \in Sp(2g, \mathbb{Z})$*

Proof: Note that $\mathcal{J}_g^2 = -I$, $\mathcal{J}_g^T = -\mathcal{J}_g$ and $\underline{a}^T \mathcal{J}_g \underline{a} = 0$. Now, as Ψ is a symplectic transvection,

$$\Psi = I + \lambda \underline{a}\ \underline{a}^T \mathcal{J}_g$$

so

$$\begin{aligned}
\Psi^T \mathcal{J}_g \Psi &= (I + \lambda \underline{a}\ \underline{a}^T \mathcal{J}_g)^T \mathcal{J}_g (I + \lambda \underline{a}\ \underline{a}^T \mathcal{J}_g) \\
&= (I + \lambda \mathcal{J}_g^T \underline{a}\ \underline{a}^T)(\mathcal{J}_g + \lambda \mathcal{J}_g \underline{a}\ \underline{a}^T \mathcal{J}_g) \\
&= \mathcal{J}_g + \lambda \mathcal{J}_g^T \underline{a}\ \underline{a}^T \mathcal{J}_g + \lambda \mathcal{J}_g \underline{a}\ \underline{a}^T \mathcal{J}_g + \lambda^2 \mathcal{J}_g^T \underline{a}\ \underline{a}^T \mathcal{J}_g \underline{a}\ \underline{a}^T \mathcal{J}_g \\
&= \mathcal{J}_g
\end{aligned}$$

Thus Ψ is a symplectic matrix. It remains to show that it has integer coefficients. However, $\underline{a}$ has integer coefficients, so $\underline{a}\ \underline{a}^T$ is a $2g \times 2g$ integer matrix. Hence $\lambda\underline{a}\ \underline{a}^T\mathcal{J}_g$ is a $2g \times 2g$ integer matrix. Hence Ψ is. $\square$

From now on, all symplectic matrices will be assumed to have integer coefficients.

Theorem 2.3.5. *$Sp(2g, \mathbb{Z})$ is generated by unit symplectic transvections.*

Proof: Let $a_i = e_{2i-1}$ and $b_i = e_{2i}$ where $1 \leq i \leq g$ and e_j is the unit column vector with $2g$ entries, all zero except for the j-th entry, which is one.

For the sake of clarity, we shall list matrices by considering their actions on the column vectors $a_1, b_1, \ldots, a_g, b_g$.

Note that

$$\begin{aligned} a_j^T\mathcal{J}_g a_i &= 0 \\ b_j^T\mathcal{J}_g b_i &= 0 \\ b_j^T\mathcal{J}_g a_i &= \delta_{ij} \end{aligned}$$

where δ_{ij} is the Kronecker delta, as before.

Consider the following matrices (note that, in the following descriptions of transformations, if a vector is not written then it is mapped to itself):

$\mathcal{K}_i$

$$\begin{aligned} a_i &\longrightarrow b_i \\ b_i &\longrightarrow -a_i \end{aligned}$$

$\mathcal{U}_\sigma$. For any permutation of g points, σ,

$$\begin{aligned} a_i &\longrightarrow a_{\sigma i} \\ b_i &\longrightarrow b_{\sigma i} \end{aligned}$$

$\mathcal{U}_{i,j}$

$$\begin{aligned} a_i &\longrightarrow a_i + a_j \\ b_j &\longrightarrow -b_i + b_j \end{aligned}$$

$\mathcal{T}_i$

$$a_i \longrightarrow a_i + b_i$$

$\mathcal{T}_{i,j}$

$$\begin{aligned} a_i &\longrightarrow a_i + b_j \\ a_j &\longrightarrow b_i + a_j \end{aligned}$$

It is easy to see that each of the above are in fact symplectic matrices, i.e. elements of $Sp(2g, \mathbb{Z})$. Moreover if c is any integer combination of the a_i and b_i such that c is primitive and we define $D_c = I + c\, c^T \mathcal{J}_g$, (a u.s.t.) then by inspection it is easy to see that

$$\begin{aligned} \mathcal{K}_i &= D_{b_i} \circ D_{a_i} \circ D_{b_i} \\ \mathcal{T}_i &= D_{b_i} \\ \mathcal{T}_{i,j} &= D_{b_j} \circ D_{b_i} \circ D^{-1}_{b_i - b_j} \\ \mathcal{U}_{(12)} &= D^{-1}_{a_1 + a_2} \circ S \circ S \qquad\qquad \text{where } S = D_{a_1} \circ D_{a_2} \circ D_{b_1 - b_2} \\ \mathcal{U}_{i,j} &= D^{-1}_{a_j} \circ D^{-1}_{b_i} \circ D_{b_i + a_j} \end{aligned}$$

and as any permutation σ can be decomposed into transpositions of ajacent elements we have that any element of type $\mathcal{U}_\sigma$ is itself generated by elements of the form $\mathcal{U}_{(12)}$ given above.

Thus we have that each of the above matrices is a product of u.s.t's.

In order to prove the theorem we shall need the following claim

Claim 2.3.6. *Suppose Ψ is a symplectic matrix. Then there exists a (necessarily) symplectic matrix $\Omega = \Omega_1 \cdots \Omega_n$ (where, $\forall j\ \Omega_j$ is an u.s.t.) such that*

$\Psi\Omega$

$$\begin{aligned} a_1 &\longrightarrow a_1 \\ b_1 &\longrightarrow b_1 \end{aligned}$$

although it may have some non-trivial effect on the other a_i, b_i.

Proof: Suppose that

Ψ

$$\begin{aligned} a_1 &\longrightarrow \sum_{i=1}^{n} \lambda_i a_i + \mu_i b_i \\ b_1 &\longrightarrow \sum_{j=1}^{n} \nu_j a_j + \eta_j b_j \end{aligned}$$

Using the symplectic maps (which from above we know are themselves generated by u.s.t.'s) $\mathcal{T}_{1,k}^{-1}$ and $\mathcal{K}_k\mathcal{K}_1\mathcal{T}_{1,k}\mathcal{K}_1^{-1}\mathcal{K}_k^{-1}$ we perform the Euclidean Algorithm (EA) to reduce Ψ to the matrix:

$$a_1 \to \sum_{i=1}^{n} \lambda_i' a_i$$
$$b_1 \to \sum_{j=1}^{n} \nu_j' a_j + \eta_j' b_j$$

Then, using the symplectic matrices $\mathcal{U}_{1,k}^{-1}$ and $\mathcal{U}_{k,1}^{-1}$ we perform the EA to further reduce Ψ to the symplectic matrix $\tilde{\Psi}$

$$a_1 \to d_1 a_1$$
$$b_1 \to \sum_{i=1}^{n} p_i a_i + q_i b_i$$

But observe that

$$\begin{aligned}
1 &= b_1^T \mathcal{J}_g a_1 \\
&= b_1^T \tilde{\Psi}^T \mathcal{J}_g \tilde{\Psi} a_1 \qquad\qquad \text{as } \tilde{\Psi} \text{ is a symplectic matrix} \\
&= (\tilde{\Psi} b_1)^T \mathcal{J}_g (\tilde{\Psi} a_1) \\
&= (\sum_{i=1}^{n} p_i a_i + q_i b_i)^T \mathcal{J}_g (d_1 a_1) \\
&= d_1 (\sum_{i=1}^{n} p_i (a_i^T \mathcal{J}_g a_1) + q_i (b_i^T \mathcal{J}_g a_1)) \\
&= d_1 \sum_{i=1}^{n} q_i \delta_{1i} \\
&= d_1 q_1
\end{aligned}$$

But as $d_1, q_1 \in \mathbb{Z}$ we have that $d_1 = q_1 = \pm 1$, and as the matrix $\mathcal{J}_g^2$ sends a_1 to $-a_1$ and b_1 to $-b_1$ we may assume that without loss of generality $d_1 = q_1 = 1$.

Thus we have reduced Ψ to the matrix

$$a_1 \to a_1$$
$$b_1 \to p_1 a_1 + b_1 + \sum_{j=2}^{n} p_j a_j + q_j b_j$$

Now we can use the matrices $\mathcal{U}_{k,1}$ for all $k \geq 2$, followed by the matrices $\mathcal{K}_1\mathcal{K}_k\mathcal{T}_{1,k}\mathcal{K}_k^{-1}\mathcal{K}_1^{-1}$ for all $k \geq 2$ to reduce to the matrix

$$\begin{aligned} a_1 &\longrightarrow a_1 \\ b_1 &\longrightarrow da_1 + b_1 \end{aligned}$$

But this is just the matrix $\mathcal{T}_1^d$ so we're done. $\square$

We can now proceed by induction on g. Our induction hypothesis is that if Ψ is a symplectic $2n \times 2n$ matrix, where $n < g$ then Ψ is composed of u.s.t.'s.

If $2g = 2$ then claim 2.3.6 shows that Ψ is the product of u.s.t.'s.

Suppose now that $g > 1$ and $\Psi \in Sp(2g, \mathbb{Z})$. By claim 2.3.6 there exists a symplectic matrix Ω such that the matrix of $\Psi\Omega$ is

$$\begin{bmatrix} I & 0 \\ B & D \end{bmatrix}$$

where I is the 2×2 identity matrix, D is a $2k - 2 \times 2k - 2$ matrix and B is a $2k - 2 \times 2$ matrix.

Now, $\Psi\Omega$ is symplectic, so $(\Psi\Omega)^T \mathcal{J}_g(\Psi\Omega) = \mathcal{J}_g$. But this implies that $B = 0$ and D is symplectic. Hence, by our induction hypothesis $D = D_1 \cdots D_m$ where $\forall j\ D_j$ is a u.s.t.

Now define Λ_j to be the augmented matrix

$$\begin{bmatrix} I & 0 \\ 0 & D_j \end{bmatrix}$$

which is clearly a u.s.t., and $\Lambda = \Lambda_1 \ldots \Lambda_m$.

Then $\Psi\Omega = \Lambda$ and so $\Psi = \Lambda\Omega^{-1}$, the product of u.s.t's. as required.

$\square$

2.4 Preliminary Theorem

Lemma 2.4.1. *A Dehn twist around a simple closed curve γ induces a unit symplectic transvection on $H_1(\Sigma, \mathbb{Z})$.*

Conversely if Ψ is a unit symplectic transvection on $H_1(\Sigma, \mathbb{Z})$ then there exists a closed simple curve γ in Σ such that $\Psi = D_\gamma$.

Proof: By Lemma 2.2.7 a Dehn twist, D_γ around γ induces an integer symplectic transvection on $H_1(\Sigma, \mathbb{Z})$. But by Lemma 2.2.2 as γ is a simple, closed curve on Σ, $[\gamma]$ is not a multiple of any other homology class. Thus D_γ is a unit symplectic transvection.

Conversely, Ψ is a unit symplectic transvection implies that there exists a primitive vector $\underline{a} \in H_1(\Sigma, \mathbb{Z})$ such that $\Psi = I + \underline{a}\ \underline{a}^T \mathcal{J}_g$. Then, by Lemma 2.2.2 there exists γ, a representative curve for $\underline{a}$ such that γ is a simple, closed, oriented curve and by Lemma 2.2.7 the Dehn twist around γ induces a map $(D_\gamma)_* = I + \underline{a}$ $\underline{a}^T \mathcal{J}_g = \Psi$ on $H_1(\Sigma, \mathbb{Z})$. □

Theorem 2.4.2. *$Aut(H_1(\Sigma, \mathbb{Z})) = Sp(2g, \mathbb{Z})$.*

Proof: By Lemma 2.2.4 $\mathrm{Aut}(H_1(\Sigma, \mathbb{Z}))$ is generated by Dehn twists.

By Theorem 2.3.5 $Sp(2g, \mathbb{Z})$ is generated by u.s.t.'s.

But it was shown in Lemma 2.4.1 that a Dehn twist around a simple, closed curve induces a unit symplectic transvection on $H_1(\Sigma, \mathbb{Z})$. Thus $\mathrm{Aut}(H_1(\Sigma, \mathbb{Z}))$ is generated by unit symplectic transvections.

Hence result. □

2.5 Main Theorem

Theorem 2.5.1. *Let $\mathcal{G}$ be the group*

$$\mathcal{G} = \left\{ \begin{bmatrix} \Psi & A \\ 0 & S \end{bmatrix} \middle| \begin{array}{l} \Psi \in Sp(2g, \mathbb{Z}), A \in \mathcal{M}((2g, n-1), \mathbb{Z}) \\ \text{and } S \in \mathcal{S}(n) \end{array} \right\} \subset GL(2g+n-1, \mathbb{Z})$$

where $\mathcal{M}((2g, n-1), \mathbb{Z})$ is the additive group of $2g \times n-1$ matrices with integer coefficients and $\mathcal{S}(n)$ is the group of $n-1$ square matrices corresponding to permutations of n points.

Then $\mathcal{G} = Aut(H_1(\Sigma, V))$, the group consisting of automorphism of $H_1(\Sigma, V)$ that can be realised by diffeomorphisms of (Σ, V).

Note 3. How can we say that $\mathcal{S}(n)$ is a group of permutations of n points when it is a group of $n-1$ square matrices? It is clear that a diffeomorphism of (Σ, V) restricts to a permutation of V. But what does a permutation of V do to $\tilde{H}_0(V)$? Consider the following short exact sequence.

$$0 \longrightarrow \tilde{H}_0(V) \longrightarrow H_0(V) \xrightarrow{\pi} \mathbb{Z} \longrightarrow 0$$

where, if $H_0(V) =< v_1, \ldots, v_n >$ then π is the map that sends $\sum \lambda_i v_i$ to $\sum \lambda_i$, where $\lambda_i \in \mathbb{Z}$.

Let σ be a permutation of V. Then σ induces an automorphism $\sigma_* : H_0(V) \longrightarrow H_0(V)$. Then $\pi(\sigma_*(x)) = \pi(x)\ \forall x \in H_0(V)$. Hence $\sigma_*(\tilde{H}_0(V)) \subseteq \tilde{H}_0(V)$. So σ_* can be regarded as a map $\tilde{H}_0(V) \longrightarrow \tilde{H}_0(V)$ and with respect to our chosen basis it has some matrix which we refer to as a permutation matrix.

Proof: Recall that $H_1(\Sigma, V) \cong H_1(\Sigma) \oplus \tilde{H}_0(V)$, using the basis chosen for $H_1(\Sigma, V)$ in equation 2.2.

The proof will be split into two parts;

1. Every diffeomorphism of (Σ, V) induces a map of
$$H_1(\Sigma, V) \cong H_1(\Sigma) \oplus \tilde{H}_0(V)$$
with a matrix of the required form to be an element of $\mathcal{G}$.

2. Every matrix
$$\begin{bmatrix} \Psi & A \\ 0 & S \end{bmatrix}$$
is realised by a diffeomorphism (by appropriate construction).

So, part 1. Note that it is clear that any diffeomorphism of (Σ, V) induces a map of $H_1(\Sigma, V) \cong H_1(\Sigma) \oplus \tilde{H}_0(V)$. Now suppose $f : H_1(\Sigma) \oplus \tilde{H}_0(V) \to H_1(\Sigma) \oplus \tilde{H}_0(V)$ is linear and is induced by a diffeomorphism of (Σ, V). Then it is clear that f is made up of 4 maps,

$$\begin{aligned} f_{1,1} &: H_1\Sigma \to H_1\Sigma, \\ f_{1,2} &: \tilde{H}_0(V) \to H_1(\Sigma), \\ f_{2,1} &: H_1(\Sigma) \to \tilde{H}(V)0 \\ f_{2,2} &: \tilde{H}_0(V) \to \tilde{H}_0(V). \end{aligned}$$

Now, by Theorem 2.4.2 $f_{1,1}$ is symplectic (i.e. has a symplectic matrix w.r.t the basis given in (2.2)). By Note 3 $f_{2,2}$ is a permutation matrix. The question remains as to why $f_{2,1}$ is zero. But consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(\Sigma) & \xrightarrow{\beta} & H_1(\Sigma, V) & \xrightarrow{\tilde{\delta}} & \tilde{H}_0(V) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_{1,1}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_{2,2}} & & \\ 0 & \longrightarrow & H_1(\Sigma) & \xrightarrow{\beta} & H_1(\Sigma, V) & \xrightarrow{\tilde{\delta}} & \tilde{H}_0(V) & \longrightarrow & 0 \end{array}$$

The map $f_{2,1}$ is the map $\tilde{\delta} \circ f \circ \beta$ which is, by commutativity, the map $\tilde{\delta} \circ \beta \circ f_{1,1}$. But by exactness, $\tilde{\delta} \circ \beta \equiv 0$ and we're done.

It is also clear, by considering this commutative diagram, that the map $f_{1,2}$ can be anything at all, hence an element of $\mathcal{M}((2g, n-1), \mathbb{Z})$

Alternatively, the reason that the matrix of the diffeomorphism with respect to this basis must have a zero block in the bottom left corner is because, if it didn't, then the diffeomorphism would take closed curves to curves with boundaries, which is a contradiction.

Now, part 2 is proved by constructing the necessary diffeomorphism. We first observe that

$$\begin{bmatrix} \Psi & A \\ 0 & S \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \begin{bmatrix} \Psi & 0 \\ 0 & I \end{bmatrix} \in \mathcal{G}$$

so we need only construct diffeomorphisms for the following matrices:

1. $\begin{bmatrix} \Psi & 0 \\ 0 & I \end{bmatrix}$
2. $\begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix}$
3. $\begin{bmatrix} I & A \\ 0 & I \end{bmatrix}$

So, case 1: Diffeomorphisms that induce matrices of the form

$$\begin{bmatrix} \Psi & 0 \\ 0 & I \end{bmatrix}$$

Notice that these diffeomorphisms fix V pointwise, i.e. are the identity when restricted to V.

The essence of the idea here is that as $\mathcal{D}$ is contractible, any diffeomorphism f may be replaced by another diffeomorphism f' such that f' is identical to f away from $\mathcal{D}$ and f' fixes $\mathcal{D}$ pointwise. So, given a diffeomorphism of Σ that induces the matrix Ψ we can construct a related diffeomorphism of (Σ, V) inducing the required matrix.

For this we require the following claim. Observe, as noted earlier, that we may choose a disc $\mathcal{D}$ such that V lies entirely within its interior.

Claim 2.5.2. *For any Dehn twist around a curve γ there exists a Dehn twist around a curve homotopic in Σ to γ that fixes $\mathcal{D}$ pointwise.*

Proof: The idea is that, because $\mathcal{D}$ is contractible, we may replace any curve γ that passes through $\mathcal{D}$ with another curve homotopic in Σ (i.e. representing the same element in $H_1(\Sigma)$) to it that is contained in $\Sigma - \mathcal{D}$.

We choose γ' in $[\gamma] \in H_1(\Sigma)$ such that $\gamma' \subset \Sigma - \mathcal{D}$ as follows. Suppose there is some part of γ that passes through $\mathcal{D}$. Then we replace this with an arc that follows the boundary of $\mathcal{D}$. These two curves are homotopic because $\mathcal{D}$ is contractible. (For a diagrammatic way of viewing this, consider Figure 2.6. What we do is to "complete" the part of γ that passes through $\mathcal{D}$ and replace that part of γ with the other half of the "completion".) Then $D_{\gamma'}|_{\mathcal{D}} = \mathrm{Id}_{\mathcal{D}}$. □

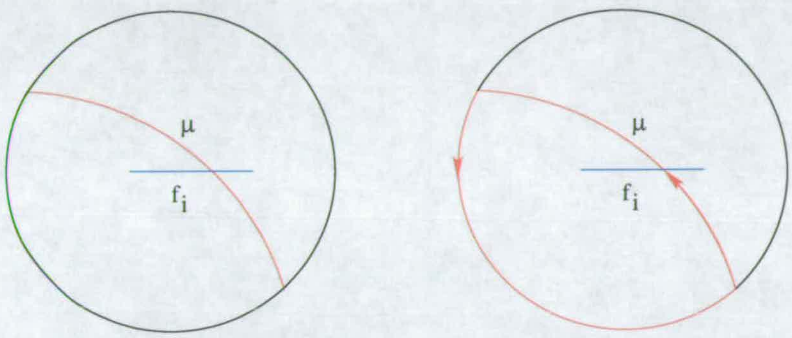


Figure 2.6: Selecting and "completing" μ to form a closed, contractible curve that intersects f_i once and does not intersect any other f_j

We may now continue with the proof of the main theorem.

Let $\begin{bmatrix} \Psi & 0 \\ 0 & I \end{bmatrix}$ be an element of $\mathcal{G}$. We need to show that this is induced by a diffeomorphism of (Σ, V). Now Ψ is symplectic, so by Theorem 2.4.2 it is induced by a diffeomorphism h of Σ. We shall use h to construct a diffeomorphism of (Σ, V) which induces the required map on $H_1(\Sigma, V)$.

Now, by Lemma 2.2.4 we know that h is composed of Dehn twists of Σ around a series of curves $\gamma_1, \ldots, \gamma_m \subset \Sigma$. Now some of these curves γ_i may pass through $\mathcal{D}$. However, Claim 2.5.2 shows that we may replace all such curves γ_i with curves γ_i' homotopic to them in Σ that do not pass through $\mathcal{D}$. So h is homotopic in Σ to a diffeomorphism h' of Σ that fixes $\mathcal{D}$ pointwise. So h and h' both induce the same map of $H_1(\Sigma)$, i.e. Ψ. But h' fixes $\mathcal{D}$ pointwise. So we have a diffeomorphism h' of (Σ, V) which induces the required map of $H_1(\Sigma, V)$ with respect to the basis given.

Thus for any sympletic matrix Ψ we have constructed a diffeomorphism of (Σ, V) which induces an automorphism of $H_1(\Sigma, V)$ with the required matrix. And thus the matrix

$$\begin{bmatrix} \Psi & 0 \\ 0 & I \end{bmatrix} \in \mathcal{G}$$

Case 2: Diffeomorphisms that induce matrices of the form

$$\begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix}$$

Notice that these are diffeomorphisms of Σ with support contained within $\mathcal{D}$ (i.e. diffeomorphism that fix $\Sigma - \mathcal{D}$ pointwise), where $\mathcal{D}$ is the disc chosen such that $V \subset \mathcal{D}$ as before.

Clearly, all we might be able to do to the points of V is to permute them, i.e. act on them with the symmetric group on n objects. We shall show that, in fact, this is precisely what we can do.

Recall that the symmetric group is generated by transpositions. So, all we need do is show that for any two points in V, there exists a diffeomorphism of Σ that switches them round whilst leaving the remaining points of V fixed, fixing the rest of Σ pointwise and thus inducing the identity on $H_1(\Sigma)$.

Let the two points be p and q. Then there exists a curve γ contained within $\mathcal{D}$ that encircles p and q and no other point of V (i.e. if γ is taken to be the boundary of a disc, then the only points of V in the interior of γ are p and q). Take a band around γ with the same condition. Then apply the diffeomorphism shown in Figure 2.7, a "half-Dehn" twist. Notice also the effect this diffeomorphism has on a line passing between the two points.

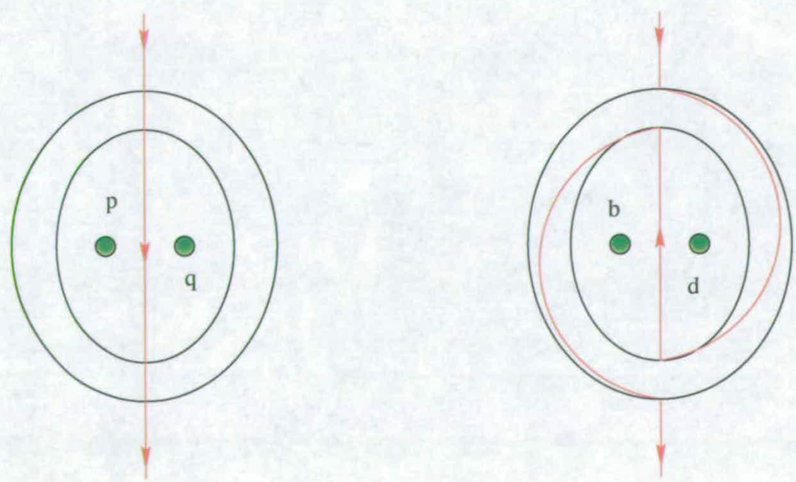


Figure 2.7: The half-Dehn twist that permutes the points p and q.

Thus for any S an element of the $n-1$ square matrix representation of $\mathcal{S}(n)$ we have constructed a diffeomorphism of (Σ, V) that induces the automorphism of $H_1(\Sigma, V)$ with the required matrix. Thus

$$\begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix} \in \mathcal{G}$$

Case 3: Diffeomorphism that induces matrices of the form

$$\begin{bmatrix} I & A \\ 0 & I \end{bmatrix}$$

Notice that these are diffeomorphism that add closed curves to the generators of the pre-image of $\tilde{H}_0(V)$.

To prove this, we will require two results.

Lemma 2.5.3. *Let $E_{i,j} \in \mathcal{M}((2g, n-1), \mathbb{Z})$ be such that $E_{i,j}$ has a 1 in the i,j-th place, but zeros elsewhere. Then there exists c_i, a generator of $H_1(\Sigma)$, f_j an element of the basis $\mathcal{B}_T$ of $\tilde{H}_0(V)$ and a simple closed curve $\tilde{\gamma} \in (\Sigma, V)$ such that a Dehn twist $D_{\tilde{\gamma}}$ around $\tilde{\gamma}$ induces the automorphism of $H_1(\Sigma, V)$ with matrix*

$$\begin{bmatrix} I & E_{i,j} \\ 0 & I \end{bmatrix}$$

with respect to the basis given earlier in (2.2).

Proof: Firstly, we must identify c_i and f_j. But this is easy, as

$$c_i = \begin{cases} a_{(i+1)/2} & \text{if } i \text{ is odd} \\ b_{i/2} & \text{otherwise} \end{cases}$$

and f_j is simply the j-th element of the basis $\mathcal{B}_T$.

Now, the claim here is that this element f_i of the basis $\mathcal{B}_T$ of $\tilde{H}_0(V)$ and for any closed curve γ with homology class $c_{i,j}$ there exists a curve $\tilde{\gamma}$ with the same homology class such that the intersection number of $\tilde{\gamma}$ and a curve in the homology class given by f_j is given by the Kronecker delta δ_{ij}. If this can be shown, then a Dehn twist around a small band around $\tilde{\gamma}$ (chosen to be small enough not to affect any other generator) will send f_j to $f_j + c_{i,j}$, whilst leaving all other generators fixed.

So, we proceed as follows. Firstly, let μ be a curve with initial and final points on the boundary of $\mathcal{D}$, otherwise contained within $\mathcal{D}$, that intersects any representative of f_j only once (from now on we shall not distinguish between f_j and its representative curve, and refer to them both as f_j). Such a curve μ exists because T is a spanning tree, hence contractible. Then "complete" μ, i.e. make μ into a closed curve (which we shall also call μ) by going round the boundary of $\mathcal{D}$ from one the final point of μ to the initial point (see Figure 2.6). Thus we have a contractible curve which intersects f_i once. We then homotope both γ and μ so that $\tilde{\gamma} = \gamma + \mu$ is a simple closed curve. As such homotopies can be done away from the interior of $\mathcal{D}$, so $D_{\tilde{\gamma}}$ has the required property.

Thus we have shown that any matrix of the form:

$$\begin{bmatrix} \Psi & E_{i,j} \\ 0 & I \end{bmatrix} \in \mathcal{G}$$

where Ψ is the sympletic matrix that corresponds to the given Dehn twist applied to the a's and b's, as $\tilde{\gamma}$ may not necessarily have zero intersection number with the generators of $H_1(\Sigma)$. But composition of diffeomorphisms corresponds to matrix multiplication, so matrices of the form:

$$\begin{bmatrix} I & E_{i,j} \\ 0 & I \end{bmatrix} = \begin{bmatrix} \Psi & E_{i,j} \\ 0 & I \end{bmatrix} \begin{bmatrix} \Psi^{-1} & 0 \\ 0 & I \end{bmatrix} \in \mathcal{G}$$

□

Lemma 2.5.4. *Let $A \in \mathcal{M}((2g, n-1), \mathbb{Z})$ be the matrix $A = (a_{i,j})$ where $i = 1 \ldots 2g, j = 1 \ldots n-1$. Then there exists a diffeomorphism f_A of (Σ, V) that induces an automorphism of $H_1(\Sigma, V)$ with matrix*

$$\begin{bmatrix} I & A \\ 0 & I \end{bmatrix}$$

that consists of the composition of diffeomorphisms above.

Proof: The key here is that composition of diffeomorphisms corresponds to matrix multiplication. So, as

$$A = \sum_{i=1}^{2g} \sum_{j=1}^{n-1} E_{i,j}$$

we have that

$$\begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \prod_{i=1}^{2g} \left(\prod_{j=1}^{n-1} \begin{bmatrix} I & E_{i,j} \\ 0 & I \end{bmatrix} \right)$$

so f_A is the composition of Dehn twists around simple closed curves as required and we have constructed a diffeomorphism of (Σ, V) which induces the automorphism of $H_1(\Sigma, V)$ and has the required matrix with respect to the basis (2.2). So any matrix of the form

$$\begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \in \mathcal{G}$$

□

So, in general, if

$$A = E_1 + E_2 + \cdots + E_n$$

then all matrices of the form

$$\begin{bmatrix} \Psi & A \\ 0 & S \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & E_1 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & E_2 \\ 0 & I \end{bmatrix} \cdots \begin{bmatrix} I & E_n \\ 0 & I \end{bmatrix} \begin{bmatrix} \Psi & 0 \\ 0 & I \end{bmatrix} \in \mathcal{G}$$

□

Chapter 3

On the graphical theory of minimal vector fields

It is the intention in this chapter to concentrate on the graph theory required for our classification result.

To this end, we shall restrict ourselves to a particular class of digraphs, which we shall define. In our discussion, we shall consider ways of breaking down a graph into smaller parts, and consider a polynomial which encodes how this may be done. We finish with an algorithm designed to allow us to pass from a graph in its simplest form to a balanced ribboned graph, a concept (which we define) that we shall show in a later chapter to be almost all we need to classify minimal vector fields.

3.1 Definitions

We start with some basic definitions.

Definition 3.1.1 (Graph). A graph Γ consists of a (finite) set V of points, called the set of *vertices*, and a (finite) set E, called the set of *edges*. We also have a map $p : E \to V \times V$ assigning to $e \in E$ the *end points* of Γ.

This is a slightly non-standard definition for a graph, as it is more usual to identify E with its projection under p. However, the reason for the technicality is that we shall need to consider graphs with more than one edge connecting the same pair of vertices. So we do not insist that p is one to one.

In fact, the object defined here is more usually referred to as a *multigraph*. However, as we shall only deal with multigraphs, they can be referred to as graphs without confusion.

Definition 3.1.2 (4-regular graph). A *4-regular graph* is a graph $\Gamma = (V, E)$ such that, for any $v \in V$ the number of edges with end point v is 4.

Definition 3.1.3 (Directed graph). We now put a direction on every edge. A *directed graph* (or *digraph*) is a quadruple $\Gamma = (V, E, \iota, \tau)$ consisting of a set V of vertices, a set E together with a projection map $p : E \to V \times V$ that assigns to each edge $e \in E$ an ordered pair of elements of V and two maps, ι and τ, the initial and terminal maps respectively. The relation between the maps p, ι and τ is that $p(e) = (\iota(e), \tau(e))$, for any $e \in E$.

For most purposes, it will suffice to treat E as merely a set of ordered pairs of elements of V and treat ι and τ as merely the first and second projection maps π_1, π_2 where $\pi_i(a_1, a_2) = a_i$, i.e. suppress the map p.

Definition 3.1.4 (Degree of a vertex). If v is a vertex of Γ then we define $d^+(v)$ (the incoming degree) to be the number of edges e with $\tau(e) = v$ and $d^-(v)$ (the outgoing degree) to be the number of edges e with $\iota(e) = v$. The sum of these two values is the *degree* of the vertex v. So the degree of $v \in V$ can be seen to be equal to $|\iota^{-1}(v)| + |\tau^{-1}(v)|$.

Definition 3.1.5 (2-regular digraph). A *2-regular digraph* is a digraph $\Gamma = (V, E, \iota, \tau)$ such that the incoming degree of any $v \in V$ is equal to the outgoing degree, and both equal 2 (i.e. for any $v \in V$ we have $|\iota^{-1}(v)| = 2 = |\tau^{-1}(v)|$). Thus any such graph has, at each vertex, an edge pattern equivalent to that shown in Figure 3.1

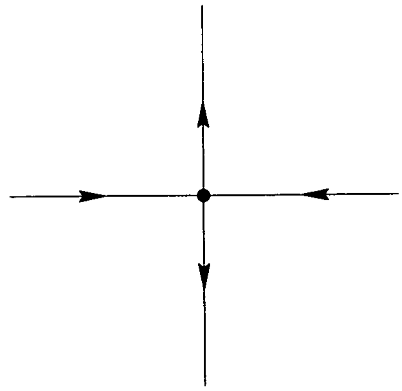


Figure 3.1: Vertex of a 2-regular digraph

The following two definitions are by convention.

Definition 3.1.6. The empty graph E is given as $E = (\emptyset, \emptyset)$ whilst the graph consisting of one single loop with no vertices is denoted L (or 0) and is given as $L = (\emptyset, \{\emptyset\})$, where $\emptyset$ is the empty set.

3.2 Circuit decompositions

In this section, we shall consider a way of partitioning E, using *circuits* to form *circuit decompositions* of a graph.

Definition 3.2.1 (Circuit). A *circuit* of a directed graph is an ordered subset $\mathcal{C} \subset E$, i.e.

$$\mathcal{C} = (e_1, \ldots, e_k) \subset E$$

such that $\tau(e_j) = \iota(e_{j+1})$ and $\tau(e_k) = \iota(e_1)$ for all $j < k$. Note that we do not insist on distinct vertices for a subset to be a circuit, i.e. all the graphs shown in Figure 3.2 are circuits of the graph Γ.

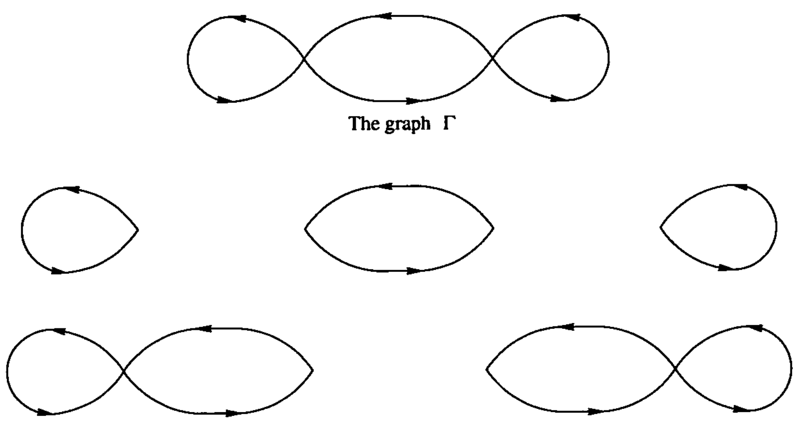


Figure 3.2: Circuits of a graph Γ

Note 4. Notice that in Figure 3.2, Γ is, itself, a circuit. It is always true that for connected 2-regular digraphs, the graph is a circuit of itself, as the following theorem shows.

Theorem 3.2.2. *A connected, directed graph is a circuit of itself if and only if for every vertex v, $d^+(v) = d^-(v)$.*

Proof: An equivalent statement of this theorem states that a directed graph is Eulerian if and only if for every vertex v, $d^+(v) = d^-(v)$. It is in this form that this theorem of Euler is discussed and proved in [4]. □

Definition 3.2.3 (Circuit decompositions). A *circuit decomposition*, $\mathcal{C}$ of a directed graph Γ is a partition of the set of edges E into circuits. The number of circuits in a circuit decomposition $\mathcal{C}$ is denoted by $|\mathcal{C}|$.

Note 5. As decomposing a graph into circuits involves making a choice at each vertex as to how the incoming and outgoing edges are paired up, for a 2-regular digraph the maximum number of circuit decompositions (and indeed the number of labelled circuit decompositions) is $2^{|V|}$ where $|V|$ is the number of vertices (i.e. the cardinality of V). In general, if $C(n)$ is the number of ways of pairing up two sets, each of n objects, then the maximum number of circuit decompositions is

$$\prod_{v\in V} C(d^+(v))\delta_{d^+(v),d^-(v)}$$

where $\delta_{m,n}$ is the Kronecker delta given by

$$\delta_{m,n} = \begin{cases} 1 & m = n \\ 0 & \text{otherwise} \end{cases}$$

Definition 3.2.4 (Dual circuit decomposition). Let $\mathcal{C}$ be a circuit decomposition of a graph Γ. This amounts to a choice at each vertex of how to pair up the incoming and outgoing edges. As at each vertex there are two outgoing edges, there are two such pairings (see Figure 3.3). Define the *dual circuit decomposition* $\mathcal{C}'$ as being the set of circuits in which at each vertex the other choice is made, as in Figure 3.4.

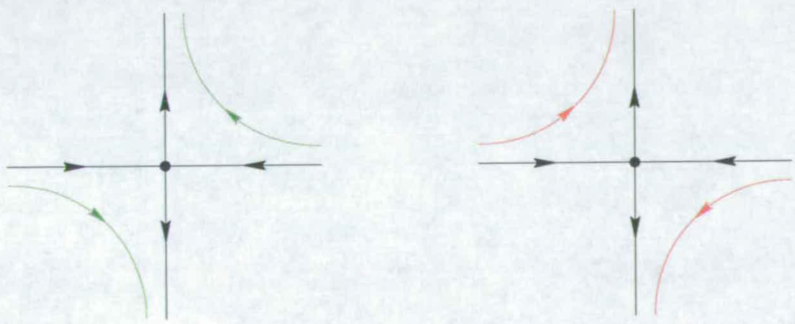


Figure 3.3: The two possible choices for circuits at a vertex.

Note 6. Duality in this case is a local definition. Thus we do not necessarily have that a circuit has the same number of elements as its dual.
The dual circuit decomposition defined above *is* a circuit decomposition, as a decomposition is clearly defined by the pairing of edges at all vertices of a graph.

3.3 Local orientation systems

In this section we shall consider a way of partitioning E to form the circuit decompositions defined above. This method, dependent on *local orientations* is easily seen to be relevant to an embedded graph.

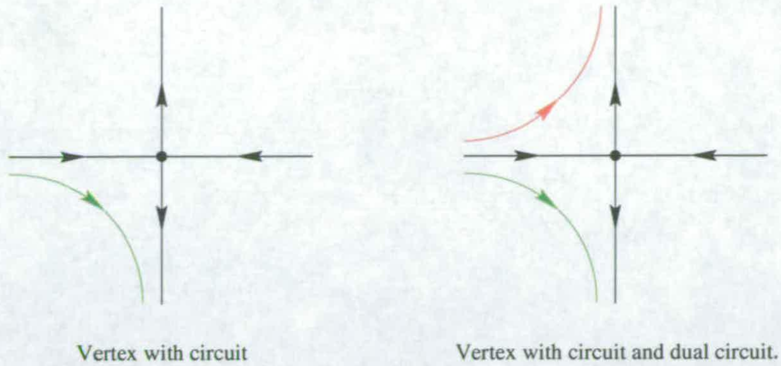


Figure 3.4: A vertex with both a circuit and its dual.

Definition 3.3.1 (Local orientation). For each $v \in V$ there are two incoming edges e_1^i, e_2^i and two outgoing edges e_1^o, e_2^o. An *orientation* or_v at v is an equivalence class of ordered sets, where

$$(e_1^i, e_1^o, e_2^i, e_2^o) \equiv (e_2^i, e_2^o, e_1^i, e_1^o)$$
$$(e_1^i, e_2^o, e_2^i, e_1^o) \equiv (e_2^i, e_1^o, e_1^i, e_2^o)$$

See Figure 3.5 for an illustration of the two possible orientations of a given vertex v. If or_v is the orientation at v then the other possible orientation is called the *dual orientation* $\overline{or}_v$.

Define the set $OR_\Gamma = \{or_v : v \in V\}$. This is a *local system of orientations* for Γ. The set $\overline{OR}_\Gamma = \{\overline{or}_v : v \in V\}$ is the dual orientation system for Γ.

Lemma 3.3.2. *Every local orientation system OR_Γ defines a unique circuit decomposition $\mathcal{C}_{OR_\Gamma}$. Moreover, the dual decomposition $\mathcal{C}'_{OR_\Gamma}$ is the circuit decomposition given by the dual orientation, i.e. $\mathcal{C}'_{OR_\Gamma} = \mathcal{C}_{\overline{OR}_\Gamma}$.*

Conversely, every circuit decomposition defines a unique local orientation system.

Proof: We need to define two operations. One is a function $OR : E \to E$ that sends an edge to the next edge in the orientation i.e. we say an orientation or_v can act on an edge in the following manner:

$$or_v e = \begin{cases} f & \text{if } or_v = (e, f, g, h) \text{ for some other edges } g, h \text{ and } \tau(e) = v \\ \text{undefined} & \text{otherwise} \end{cases}$$

and define the action of OR on e as $ORe = or_{\tau(e)}e$.

The second operation is a relation $\sim_{OR}$. If $e, f \in E$ then $e \sim_{OR} f$ if and only if there exists a sequence of edges $e_0, e_1, \ldots, e_{n-1}, e_n$ such that $e_0 = e, e_n = f$ and for all $i = 0, \ldots, n-1$ $e_{i+1} = or_{\tau(e_i)}e_i$.

We need to prove that $\sim_{OR}$ is an equivalence relation. To do this we first define the concept of a *circuit centered at an edge.* Suppose $e \in E$. Then by continued application of the function OR to e we can construct the circuit $c_e = (e, ORe, OR^2e, \dots, OR^{k-1}e)$ containing e. Notice that $OR^k e = e$ so we may define the *order* $o(e)$ of an edge e as the smallest non-zero integer k such that $OR^k e = e$. So $o(e) =$ the length of the circuit defined by OR_Γ containing e.

So, to prove $e \sim_{OR} e$ construct the circuit centered at e. The required chain is then the chain $(c_e, OR^k e) = (e, ORe, \dots, OR^{k-1}e, OR^k e)$.

To prove that, for two edges e, f if $e \sim_{OR} f$ then $f \sim_{OR} e$ consider the circuit c_e centered at e. As $e \sim_{OR} f$ we know that f is contained within c_e. Thus there exists some integer $k \leq o(e)$ such that $OR^k f = e$. The required chain is then $(OR^k e, OR^{k+1}e, \dots, OR^{o(e)}e) = (f, ORf, \dots, e)$.

To prove that, for three edges e, f, g if $e \sim_{OR} f$ and $f \sim_{OR} g$ then $e \sim_{OR} g$ we simply consider the chains $(e, \dots, f)$ and $(f, \dots, g)$ and concatenate them to form the chain $(e, \dots, f, \dots, g)$ which is the required chain.

Now, as $\sim_{OR}$ is an equivalence relation on E, it partitions E into equivalence classes. But by considering the circuit centered at e we see that the equivalence class containing e is simply the circuit centered at e. Thus the equivalence classes are circuits, and $E/ \sim_{OR}$ is a circuit decomposition, defined uniquely in terms of OR_Γ. We define C_{OR_Γ} to be this circuit decomposition.

That the dual decomposition is the decomposition defined by the dual orientation follows immediately from the definitions of duality.

To prove the converse, notice that at each v, a circuit decomposition is just a pairing of each incoming edge with a corresponding outgoing edge. This pairing is all that is needed to construct a local orientation system. □

Definition 3.3.3 (Automorphism group of a graph). For a 2-regular digraph Γ an automorphism is a pair of maps $\phi = (\phi_V, \phi_E)$ where $\phi_V : V \to V$ and $\phi_E : E \to E$ (i.e. $\phi : \Gamma \to \Gamma$) such that the following diagrams commute:

$$\begin{array}{ccc} E & \xrightarrow{\phi_E} & E \\ \downarrow{\scriptstyle \iota} & & {\scriptstyle \iota}\downarrow \\ V & \xrightarrow[\phi_V]{} & V \end{array} \quad \text{and} \quad \begin{array}{ccc} E & \xrightarrow{\phi_E} & E \\ \downarrow{\scriptstyle \tau} & & {\scriptstyle \tau}\downarrow \\ V & \xrightarrow[\phi_V]{} & V \end{array}$$

and for each $v \in V$ we define the orientation as follows. Suppose the orientation at v is $or_v = (e^i_j, e^o_k, e^i_l, e^o_m)$. Then the orientation at $\phi_V(v)$ is

$$or_{\phi_V(v)} = \phi_E or_v = (\phi_E e^i_j, \phi_E e^o_k, \phi_E e^i_l, \phi_E e^o_m)$$

It is clear that the set of automorphisms form a group under composition. We call this group the *automorphism group of the graph* Γ, or $\text{Aut}(\Gamma)$.

Note that this is a non-standard definition. For the more standard definition we refer the reader to [22], p64.

Definition 3.3.4 (Ribboned graph of Γ). A ribboned graph R is a pair $R = (\Gamma, \{\mathcal{C}, \mathcal{C}'\})$ of a 2-regular digraph Γ together with a circuit decomposition $\mathcal{C}$ and its dual $\mathcal{C}'$.

In this case we say that Γ is the *underlying graph* (or *underlying directed graph*) of R.

R is *balanced* if $|\mathcal{C}| = |\mathcal{C}'|$ and in this case we say that $\mathcal{C}$ is a *balanced circuit decomposition.*

The set of all ribbonings of a graph Γ is denoted $\mathcal{S}_\Gamma$.

Definition 3.3.5 (Pairing of a ribboned graph). Given a ribboned graph $(\Gamma, \{\mathcal{C}, \mathcal{C}'\})$ a *pairing* of the ribboned graph is a bijection $\tau : \mathcal{C} \to \mathcal{C}'$. Note that a necessary and sufficient condition for a ribboned graph to have a pairing is that the cardinality of $\mathcal{C}$ is equal to the cardinality of $\mathcal{C}'$. We shall call a ribboned graph with a pairing a *balanced ribboned graph.*

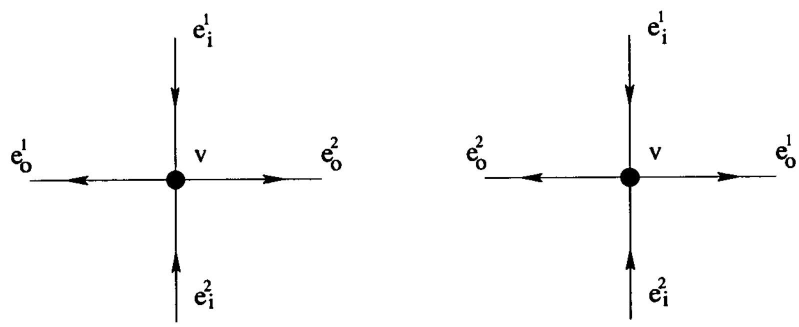


Figure 3.5: The two possible orientations of a vertex v

3.4 Transitions

Now that we understand how to construct a circuit decomposition of directed graph Γ we shall consider how the information contained within such a decomposition can be coded up as a polynomial dependent solely on the graph. To do this we need to consider the concept of a *transition* at a vertex v of Γ.

This can be seen as a reformulation of the previous section. However, it will enable us to define a *transition polynomial* of a graph, as we shall see.

Definition 3.4.1 (Splitting or transition at v). Let Γ be a connected 4 - regular graph and v a vertex of Γ. If we "split" v according to the scheme shown in Figure 3.6 then we obtain another 4-regular graph with one less vertex. We can perform such a splitting in three ways. If one of the three splittings disconnects Γ then v is called a *cut-vertex.*

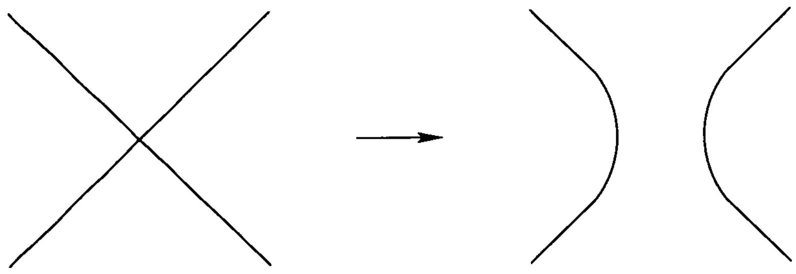


Figure 3.6: Splitting at v

Notice that a circuit decomposition is equivalent to a choice at each vertex v as to how to pair up the incoming edges to the outgoing edges. Such a choice is called a *transition* or *splitting* at v.

Definition 3.4.2. Although there are three potential transitions at each vertex of a 2-regular digraph (i.e. pairings of edges at the vertex) one of these pairs up the two incoming edges and the two outgoing edges. We shall call this transition *incoherent* and the other two pairings *coherent*

See Figure 3.7 for an example of the two coherent and one incoherent transitions.

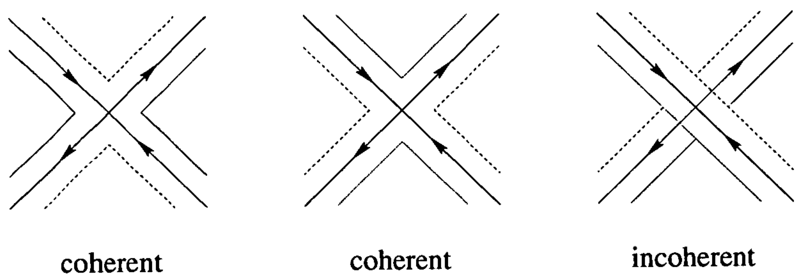


Figure 3.7: Coherent and incoherent transitions

Note 7 (Dual Decomposition). A circuit decomposition is equivalent to a choice of splitting at each vertex v. At each vertex there are two possible coherent splittings and a circuit decomposition C will choose one of them. This leads to a natural definition of a dual decomposition to C, which we will call C', the circuit decomposition we get if we choose the other splitting at each vertex to the one chosen in C. This definition can easily be seen as equivalent to that given in Definition 3.2.4.

3.5 On transition polynomials of 2-regular digraphs

In his thesis [18] of 1977, P. Martin introduced a class of iteratively defined polynomials, the Martin polynomials, that depend solely on a 4-regular graph. We shall use these polynomials to define a slightly different polynomial that also depends on a circuit decomposition of a 2-regular digraph and that gives information on the dual decomposition.

Note that the following discussion owes much to [14].

Definition 3.5.1 (Undirected Martin Polynomial). Let Γ be a 4-regular graph. Define a polynomial $m(\Gamma; \tau)$ on Γ as follows:

1. If v is not a cut-vertex of Γ, then $m(\Gamma; \tau) = m(\Gamma_1; \tau) + m(\Gamma_2; \tau) + m(\Gamma_3; \tau)$, where $\Gamma_1, \Gamma_2, \Gamma_3$ are the results of the three possible splittings at v.

2. If v is a cut-vertex of Γ and Γ_1, Γ_2 are the components of Γ formed by this cut, then $m(\Gamma; \tau) = \tau\, m(\Gamma_1; \tau) m(\Gamma_2; \tau)$

3. If L is a free loop (i.e. the connected graph on no vertices) then $m(L; \tau) = 1$.

Martin introduced a similar polynomial, which we shall denote by m' for 2-regular digraphs, i.e. 4-regular graphs with a specified Eulerian orientation (an orientation that can be split into single loop in such a way that the orientation is preserved). The only way it differs from the previous definition is that there are now only two possible splittings at v (i.e. the two coherent splittings). The formal definition is:

Definition 3.5.2 (Directed Martin Polynomial). The *Directed Martin Polynomial* of a 2-regular digraph Γ is a polynomial $m'(\Gamma; \tau)$ defined on Γ as follows:

1. If v is not a cut-vertex of Γ, then $m'(\Gamma; \tau) = m'(\Gamma_1; \tau) + m'(\Gamma_2; \tau)$, where Γ_1, Γ_2 are the results of the two possible coherent splittings at v.

2. If v is a cut-vertex of Γ and Γ_1, Γ_2 are the components of Γ formed by this cut, then $m'(\Gamma; \tau) = \tau m'(\Gamma_1; \tau) m'(\Gamma_2; \tau)$

3. If L is a free loop (i.e. the connected graph on no vertices) then $m'(L; \tau) = 1$.

Note 8. From now on, Γ will assumed to be directed unless otherwise stated.

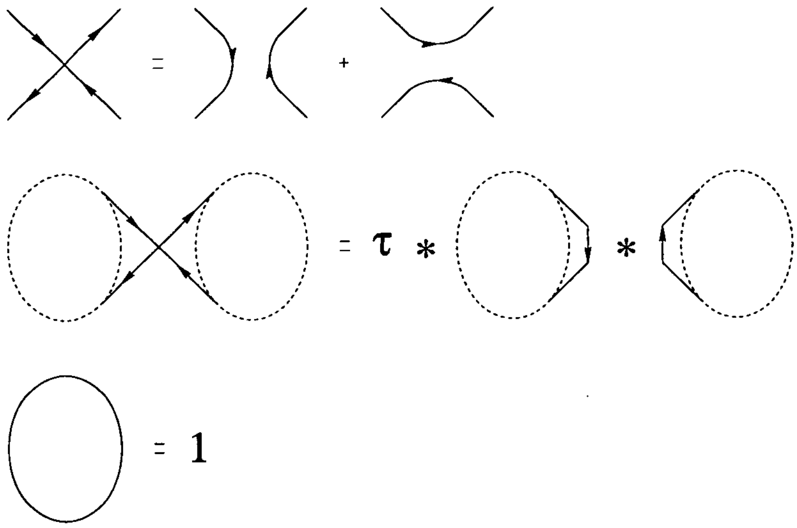


Figure 3.8: Pictorial equations recursively defining m'

It is convenient to visualize these rules in the pictorial form shown in Figure 3.8: Using these definitions we can now define a slightly adapted Martin polynomial.

Definition 3.5.3 (Adapted Martin polynomial). Let Γ be a 2-regular digraph with a given circuit decomposition $\mathcal{C}$. Let α and β be constants. We define the polynomial $p(\Gamma, \mathcal{C}, \alpha, \beta, \tau)$ pictorially in Figure 3.9.

The key idea here is that when we split the graph at v, we have two possible coherent splittings, one of which will agree with the given circuit decomposition, i.e. splits the graph according to the transition given by the circuit decomposition. We label that splitting with α and label the other splitting with β. The equivalent rules would then be:

1. If v is not a cut-vertex of Γ, then

$$p(\Gamma, \mathcal{C}, \alpha, \beta, \tau) = \alpha p(\Gamma_1, \mathcal{C}|_{\Gamma_1}, \alpha, \beta, \tau) + \beta p(\Gamma_2, \mathcal{C}|_{\Gamma_2}, \alpha, \beta, \tau)$$

   where Γ_1, Γ_2 are the results of the two possible coherent splittings at v, with Γ_1 the splitting agreeing with the circuit decomposition. Note that here $\mathcal{C}|_{\Gamma_i}$ is the circuit decomposition left after splitting the graph according to the transition that gives Γ_i.

2. If v is a cut-vertex of Γ and Γ_1, Γ_2 are the components of Γ formed by this cut, then

$$p(\Gamma, \mathcal{C}, \alpha, \beta, \tau) = (\tau\alpha + \beta)(p(\Gamma_1, \mathcal{C}|_{\Gamma_1}, \alpha, \beta, \tau) + p(\Gamma_2, \mathcal{C}|_{\Gamma_2}, \alpha, \beta, \tau)$$

if the transition given by the circuit decomposition at v cuts the graph, and

$$p(\Gamma, C, \alpha, \beta, \tau) = (\tau\beta + \alpha)(p(\Gamma_1, C|_{\Gamma_1}, \alpha, \beta, \tau) + p(\Gamma_2, C|_{\Gamma_2}, \alpha, \beta, \tau)$$

otherwise. Note that here $C|_{\Gamma_i}$ is the circuit decomposition given by restricting C to Γ_i.

3. If L is a free loop (i.e. the connected graph on no vertices) then

$$p(L, C, \alpha, \beta, \tau) = 1$$

where C is the unique circuit decomposition on L.

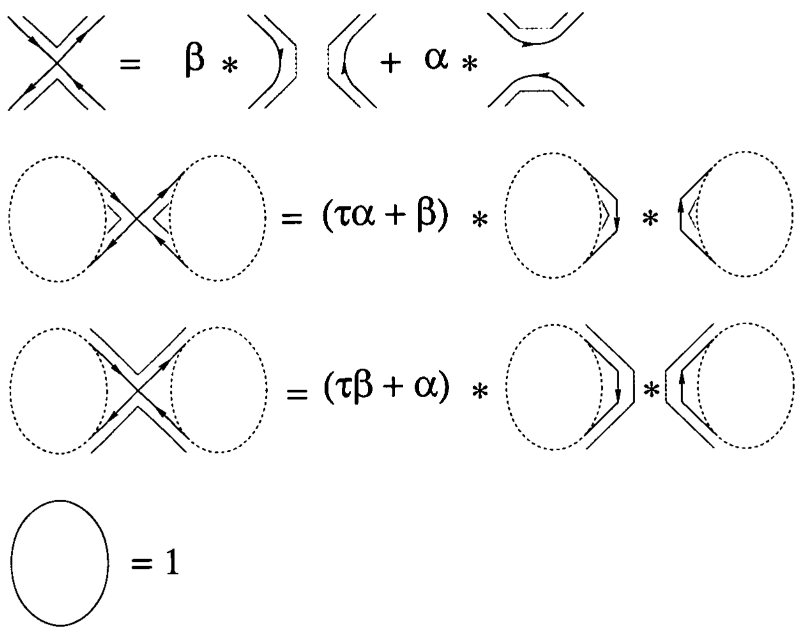


Figure 3.9: Pictorial equations recursively defining the adapted Martin polynomial $p(\Gamma, C, \alpha, \beta, \tau)$

Before we move on we shall note the obvious connection between the Directed Martin polynomial and the Adapted Martin polynomial, namely:

Lemma 3.5.4. *For any circuit decomposition C of Γ*

$$p(\Gamma, C, 1, 1, \tau) = m'(\Gamma, \tau + 1)$$

Proof: Simply compare the equations in Figure 3.8 with those in Figure 3.9 when $\alpha = \beta = 1$. □

We shall now consider the connection between recursive polynomials and transition polynomials. The following definition is taken from [14]:

Definition 3.5.5. Let $T(\Gamma)$ be the set of circuit decompositions of a 4-regular graph Γ and let A be a map (called a *weight function for* Γ) that associates to every transition of Γ a weight chosen in some set of variables or constants. Then the associated *transition polynomial* $Q(\Gamma, A, \tau)$ is defined by:

$$Q(\Gamma, A, \tau) = \sum_{t \in T(\Gamma)} \left(\prod_{v \in V_\Gamma} A(t(v)) \right) \tau^{|\mathcal{C}|-1} \tag{3.1}$$

where $t(v)$ is the splitting, or *transition* at v associated with $\mathcal{C}$.

For reasons of convenience we consider that $P(L)$ contains one trivial circuit decomposition.

The following three results are taken from [14]. The proofs given there are reproduced here for completeness.

The first is straightforward.

Proposition 3.5.1. *Let Γ be a 4-regular graph with weight function A and v a vertex of Γ. Let t_1, t_2, t_3 be the three transitions at v. Then:*

$$Q(\Gamma, A, \tau) = \sum_{i=1,2,3} A(t_i) Q(\Gamma * t_i, A/v, \tau)$$

where $\Gamma * t_i$ is the graph obtained by splitting Γ at v according to the transition t_i and A/v is identified in the obvious way with a weight function on $\Gamma * t_i$.

Proof: This follows immediately from equation (3.1). □

The next two results require us to define two ways in which two weighted graphs may be put together. The first, denoted $\cup$ is the disjoint union. The second, denoted $=$, is a linked union. Both of these are shown in Figure 3.10

Also, if A_1, A_2 are weight functions for Γ_1, Γ_2 respectively, then $A_{1,2}$ denotes the weight function for Γ that restricts to A_i on Γ_i.

Lemma 3.5.6. *Let Γ_1, Γ_2 be two graphs with weight functions A_1, A_2 respectively. Then*

1. $Q(\Gamma_1 \cup \Gamma_2, A_{1,2}, \tau) = \tau Q(\Gamma_1, A_1, \tau) Q(\Gamma_2, A_2, \tau)$
2. $Q(\Gamma_1 = \Gamma_2, A_{1,2}, \tau) = \quad Q(\Gamma_1, A_1, \tau) Q(\Gamma_2, A_2, \tau)$

Proof: Notice first that any circuit decomposition $\mathcal{C}$ of Γ can be identified with a pair of circuit decompositions $(\mathcal{C}_1, \mathcal{C}_2)$ of (Γ_1, Γ_2). Thus

$$\prod_{v \in V_\Gamma} A_{1,2}(\mathcal{C}(v)) = \left(\prod_{v \in V_{\Gamma_1}} A_1(\mathcal{C}_1(v)) \right) \left(\prod_{v \in V_{\Gamma_2}} A_2(\mathcal{C}_2(v)) \right)$$

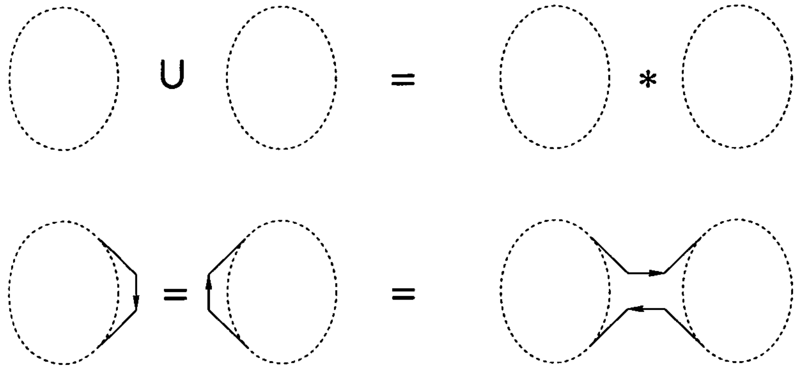


Figure 3.10: Diagram showing the two ways ($\cup$ and $=$) of joining two 4−regular graphs.

Moreover, it is clear that for case 1 $|\mathcal{C}| = |\mathcal{C}_1|+|\mathcal{C}_2|$ and for case 2 $|\mathcal{C}| = |\mathcal{C}_1|+|\mathcal{C}_2|-1$ Then the result follows by plugging these into equation (3.1). We will explicitly do this for the first case. The second is similar.

So, we have that

$$
\begin{aligned}
Q(\Gamma, A_{1,2}, \tau) &= \sum_{\mathcal{C}\in T(\Gamma)} \prod_{v\in V_\Gamma} A_{1,2}(\mathcal{C}(v))\tau^{|\mathcal{C}|-1} \\
&= \sum_{(\mathcal{C}_1,\mathcal{C}_2)\in T(\Gamma_1\cup\Gamma_2)} \left(\prod_{v\in V_{\Gamma_1}} A_{1,2}(\mathcal{C}_1(v))\tau^{|\mathcal{C}_1|-1}\right)\left(\prod_{v\in V_{\Gamma_2}} A_{1,2}(\mathcal{C}_2(v))\tau^{|\mathcal{C}_2|-1}\right)\tau \\
&= \tau \sum_{\mathcal{C}_1\in T(\Gamma_1)} \sum_{\mathcal{C}_2\in T(\Gamma_2)} \left(\prod_{v\in V_{\Gamma_1}} A_{1,2}(\mathcal{C}_1(v))\tau^{|\mathcal{C}_1|-1}\right)\left(\prod_{v\in V_{\Gamma_2}} A_{1,2}(\mathcal{C}_2(v))\tau^{|\mathcal{C}_2|-1}\right) \\
&= \tau \left(\sum_{\mathcal{C}_1\in T(\Gamma_1)} \prod_{v\in V_{\Gamma_1}} A_{1,2}(\mathcal{C}_1(v))\tau^{|\mathcal{C}_1|-1}\right)\left(\sum_{\mathcal{C}_2\in T(\Gamma_2)} \prod_{v\in V_{\Gamma_2}} A_{1,2}(\mathcal{C}_2(v))\tau^{|\mathcal{C}_2|-1}\right) \\
&= \tau\, Q(\Gamma_1, A_1, \tau) Q(\Gamma_2, A_2, \tau)
\end{aligned}
$$

□

The pictorial representation of this lemma is shown in Figure 3.11

Proposition 3.5.2. *Let Γ be a 4-regular graph with weight function A and v a cut-vertex of Γ. Let t_1 be the separating transition at v and let t_2, t_3 be the other transitions at v. Let Γ_1, Γ_2 be the two connected components of $\Gamma * t_1$. Then:*

$$Q(\Gamma, A, \tau) = (A(t_1)\tau + A(t_2) + A(t_3))Q(\Gamma_1, A_1, \tau)Q(\Gamma_2, A_2, \tau)$$

where A_i is the restriction of A to the transitions at vertices of Γ_i.

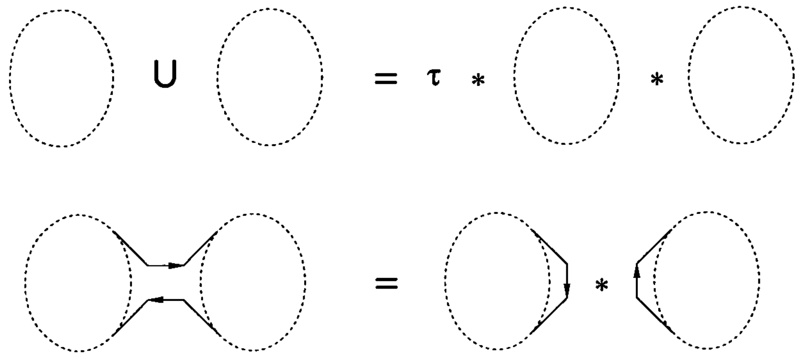


Figure 3.11: The pictorial representation of lemma 3.5.6.

Proof: To prove this result we need merely observe that, as t_1 is the separating transition at v, so $\Gamma * t_1 = \Gamma_1 \cup \Gamma_2$. And similarly, as t_2 and t_3 are not separating, we have that $\Gamma * t_2 = \Gamma * t_3 = \Gamma_1 = \Gamma_2$. So, by Lemma 3.5.6 we have

$$Q(\Gamma * t_1, A_{1,2}, \tau) = \tau Q(\Gamma_1, A_1, \tau) Q(\Gamma_2, A_2, \tau)$$

and

$$Q(\Gamma * t_2, A_{1,2}, \tau) = Q(\Gamma * t_3, A_{1,2}, \tau) = Q(\Gamma_1, A_1, \tau) Q(\Gamma_2, A_2, \tau)$$

But since clearly $A/v = A_{1,2}$ the result follows by substituting these identities into equation (3.1). □

Consider the following example. Let A assign 1 to each coherent transition and 0 otherwise. Then propositions 3.5.1 and 3.5.2 are just the rules of the Martin polynomial. So we have that $Q(\Gamma, A, \tau) = m'(\Gamma; \tau + 1)$. From this observation we may make the following immediate deductions.

Proposition 3.5.3. *Suppose Γ is a 2-regular digraph on $n > 0$ vertices. Then*

1. *If $m'(\Gamma; \tau+1) = \sum_{j \geq 1} a_j \tau^{j-1}$ then the coefficient a_j is the number of circuit decompositions of Γ of size j.*

2. $m'(\Gamma, 0) = 0$

3. $m'(\Gamma, 1) =$ *the number of Eulerian circuits of* Γ.

4. $m'(\Gamma, 2) = 2^n$

5. $m'(\Gamma_1 = \Gamma_2; \tau) = m'(\Gamma_1; \tau) m'(\Gamma_2; \tau)$

6. $m'(\Gamma_1 \cup \Gamma_2; \tau) = (\tau - 1) m'(\Gamma_1; \tau) m'(\Gamma_2; \tau)$

Proof: The first result follows immediately from definition 3.5.5 and the above observation that $Q(\Gamma, A, \tau) = m'(\Gamma; \tau + 1)$, with result 3 following immediately from result 1. Result 4 follows from result 1, the observation therefore that $m'(\Gamma; 1)$ is the total number of circuit decompositions and the observation in Note 5 that this is 2^n. Results 5 and 6 are just particular cases of Lemma 3.5.6. Result 2 follows from the fact that as Γ has more than one vertex, so by considering the rules given in definition 3.5.2 we can see that, in the inductive process that defines $m'(\Gamma; \tau)$ the last step but one always involves multiplying by τ (as the last step but one reduces a graph with one vertex to a graph consisting of disjoint copies of L). □

This example motivates the following result.

Theorem 3.5.7. *Let Γ be a 2-regular digraph with a given circuit decomposition $\mathcal{C}$. Let $A_{\mathcal{C}}$ be the weighting function given by:*

$$A_{\mathcal{C}}(t(v)) = \begin{cases} \alpha & \text{if } t(v) = \mathcal{C}(v) \\ \beta & \text{if } t(v) \neq \mathcal{C}(v) \text{ but } t(v) \text{ is still coherent} \\ 0 & \text{if } t(v) \text{ is incoherent} \end{cases}$$

for each $t \in T(\Gamma)$. Then $Q(\Gamma, A_{\mathcal{C}}, \tau) = p(\Gamma, \mathcal{C}, \alpha, \beta, \tau)$.

Proof: Propositions 3.5.1 and 3.5.2 applied with weight function $A_{\mathcal{C}}$ are the equations given in Figure 3.9 and so give the result. □

We shall now connect this with dual circuit decompositions. Remember that a dual circuit decomposition is the circuit decomposition which has the other coherent transition at each vertex to that in the given circuit decomposition.

Immediately we get the following result:

Corollary 3.5.8. *If $\mathcal{C}'$ is the dual decomposition to the circuit decomposition $\mathcal{C}$ then $p(\Gamma, \mathcal{C}', \alpha, \beta, \tau) = p(\Gamma, \mathcal{C}, \beta, \alpha, \tau)$.*

Proof: By inspection of the formula in Theorem 3.5.7 □

Definition 3.5.9. Consider the Adapted Martin polynomial of some graph Γ together with a circuit decomposition $\mathcal{C}$. This is a finite sum of the form

$$\sum_{i,j,k} \alpha^i \beta^j \tau^k$$

for integers i, j and k where $i + j = |V_\Gamma|$, the number of vertices of Γ. Define

$[\alpha]_{p,\tau}$ = the value of k when i takes its maximum value, and

$[\beta]_{p,\tau}$ = the value of k when j takes its maximum value

Corollary 3.5.10. *In the polynomial $p(\Gamma, \mathcal{C}, \alpha, \beta, \tau)$ defined above, $[\alpha]_{p,\tau}$ and $[\beta]_{p,\tau}$ are well-defined, $[\alpha]_{p,\tau} + 1 = |\mathcal{C}|$ and $[\beta]_{p,\tau} + 1 = |\mathcal{C}'|$ where $\mathcal{C}'$ is the dual circuit decomposition to $\mathcal{C}$.*

Proof: Notice that the maximum value of i (j) is $|V_\Gamma|$ ($|V_\Gamma|$) and, by Theorem 3.5.7, this occurs for only one circuit, namely $\mathcal{C}$ ($\mathcal{C}'$). Hence $[\alpha]_{p,\tau}$, $[\beta]_{p,\tau}$ are well-defined. Moreover, from Theorem 3.5.7 we see that when $t = \mathcal{C}$, we have

$$\left(\prod_{v \in V_\Gamma} A_{\mathcal{C}}(t(v))\right) \tau^{|\mathcal{C}|-1} = \alpha^{|V_\Gamma|}\tau^{|\mathcal{C}|-1}$$

so by definition $[\alpha]_{p,\tau} = |\mathcal{C}| - 1$. Similarly $[\beta]_{p,\tau} = |\mathcal{C}'| - 1$ □

This means the following result, the result we need for future discussion, is now obvious.

Corollary 3.5.11. *A ribboned graph $R = (\Gamma, \{\mathcal{C}, \mathcal{C}'\})$ is balanced if and only if $[\alpha]_{p,\tau} = [\beta]_{p,\tau}$ where p is the adapted Martin polynomial on Γ and $\mathcal{C}$.*

3.6 From digraphs to ribbon graphs

In this section we shall present an algorithm that produces all the different balanced ribbon graphs for a given digraph Γ.

So, let Γ be a 2-regular digraph and let $\mathcal{C}(\Gamma)$ be the set of all circuit decompositions of Γ.

Step 1 For each circuit decomposition $\mathcal{C} \in \mathcal{C}(\Gamma)$ calculate the adapted Martin polynomial $p(\Gamma, \mathcal{C}, \alpha, \beta, \tau)$ and from it using corollary 3.5.11 identify if circuit decomposition $\mathcal{C}$ is balanced. Let $\mathcal{B}(\Gamma)$ be the set of all balanced circuit decompositions of Γ.

Step 2 For each balanced circuit decomposition $\mathcal{C} \in \mathcal{B}(\Gamma)$ calculate the ribbon graph $R = (\Gamma, \{\mathcal{C}, \mathcal{C}'\})$. The set of all balanced ribbon graphs is $\mathcal{S}_\Gamma$.

Step 3 Factor $\mathcal{S}_\Gamma$ by the action of $\mathrm{Aut}(\Gamma)$ to give the set $\mathcal{S}_\Gamma \ \mathrm{Aut}(\Gamma)$ of all different ribboned graphs with Γ as their underlying graphs.

See 3.14 for a flow-chart for this algorithm.

Note that this method immediately gives an approximation for the number of balanced ribbon graphs with underlying graph Γ. This is because the size of $\mathcal{S}_\Gamma$ cannot be larger than the total number of ribbon graphs. But this figure is simply the total number of pairs of circuit decomposition, and as there are $2^{|V_\Gamma|}$ circuit decompositions and they are in pairs, there at $2^{(|V_\Gamma|-1)}$ ribbon graphs. So the maximum number of balanced ribbon graphs is $2^{(|V_\Gamma|-1)}$ and if the total number of different ribboned graphs with underlying graph Γ is given by $\mathcal{N}_\Gamma$ we have the immediate result:

Corollary 3.6.1.

$$\mathcal{N}_\Gamma \leq \frac{2^{(|V_\Gamma|-1)}}{|Aut(\Gamma)|}$$

and for any even number $2n$ there exits a graph Γ_{2n} that satisfies

$$\mathcal{N}_{\Gamma_{2n}} = \frac{2^{(2n-1)}}{|Aut(\Gamma_{2n})|}$$

Proof:

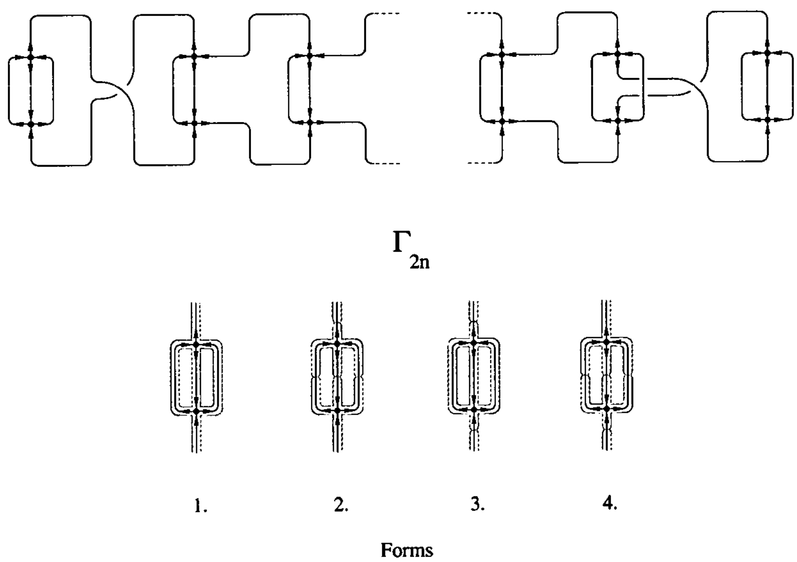
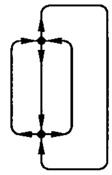


Figure 3.12: An example of a graph Γ_{2n} on $2n$ vertices that achieves the given bound for $\mathcal{N}_\Gamma$

The above discussion shows why this is an upper bound. The graph Γ_{2n} shown in Figure 3.12 is the required graph for $n > 1$. The graph on 2 vertices is shown in Figure 3.13. To prove that this is the case, we apply the Adapted Martin polynomial with some circuit decomposition C to each pair of vertices, working from the right in the diagram. It is easy to see that the for any circuit, the transition at the right-hand two vertices takes one of the four forms 1, 2, 3 and 4 shown below the graph in Figure 3.12. So we apply the Adapted Martin polynomial to each one in turn. The results are shown below:



Γ_2

Figure 3.13: An example of a graph Γ_2 on 2 vertices that achieves the given bound for $\mathcal{N}_\Gamma$

Forms 1 and 3 $p(\Gamma_{2n}, \mathcal{C}, \alpha, \beta, \tau) = [(\alpha^2+\beta^2)\tau + 2\alpha\beta]p(\Gamma_{2(n-1)}, \tilde{\mathcal{C}}, \alpha, \beta, \tau)$

Forms 2 and 4 $p(\Gamma_{2n}, \mathcal{C}, \alpha, \beta, \tau) = [2\alpha\beta\tau + (\alpha^2+\beta^2)]p(\Gamma_{2(n-1)}, \tilde{\mathcal{C}}, \alpha, \beta, \tau)$

where $\tilde{\mathcal{C}}$ is the circuit decomposition of $\Gamma_{2(n-1)}$ gotten from $\mathcal{C}$ after the last two vertices have been removed from Γ_{2n}. So, by iteration, we find that the Adapted Martin polynomial of Γ_{2n} with some circuit decomposition $\mathcal{C}$ has the form

$$[(\alpha^2+\beta^2)\tau + 2\alpha\beta]^k[2\alpha\beta\tau + (\alpha^2+\beta^2)]^{n-k}$$

for some integer k depending on $\mathcal{C}$. So thus $[\alpha]_{p,\tau} = k = [\beta]_{p,\tau}$ and so the circuit decomposition is balanced. But $\mathcal{C}$ was any circuit decomposition, hence any circuit decomposition is balanced, and so the inequality is an equality in this case. Hence result. □

Note 9. Further to above theorem, it is interesting to note that the graph Γ_{2n} (with $n > 1$) shown in Figure 3.12 has a trivial automorphism group (i.e. $\text{Aut}(\Gamma_{2n}) = \{\text{Id}\}$). To see this, notice that there only three vertices with parallel edges leaving them (i.e. only three vertices that send both out-going edges to the same vertex). So any automorphism of the graph must permute these three vertices. But by consideration of the other edges connected to these vertices it can be seen that the only map that does not lead to a contradiction is the identity map. Hence for $n > 1$ we have that

$$\mathcal{N}_{\Gamma_{2n}} = 2^{2n-1}$$

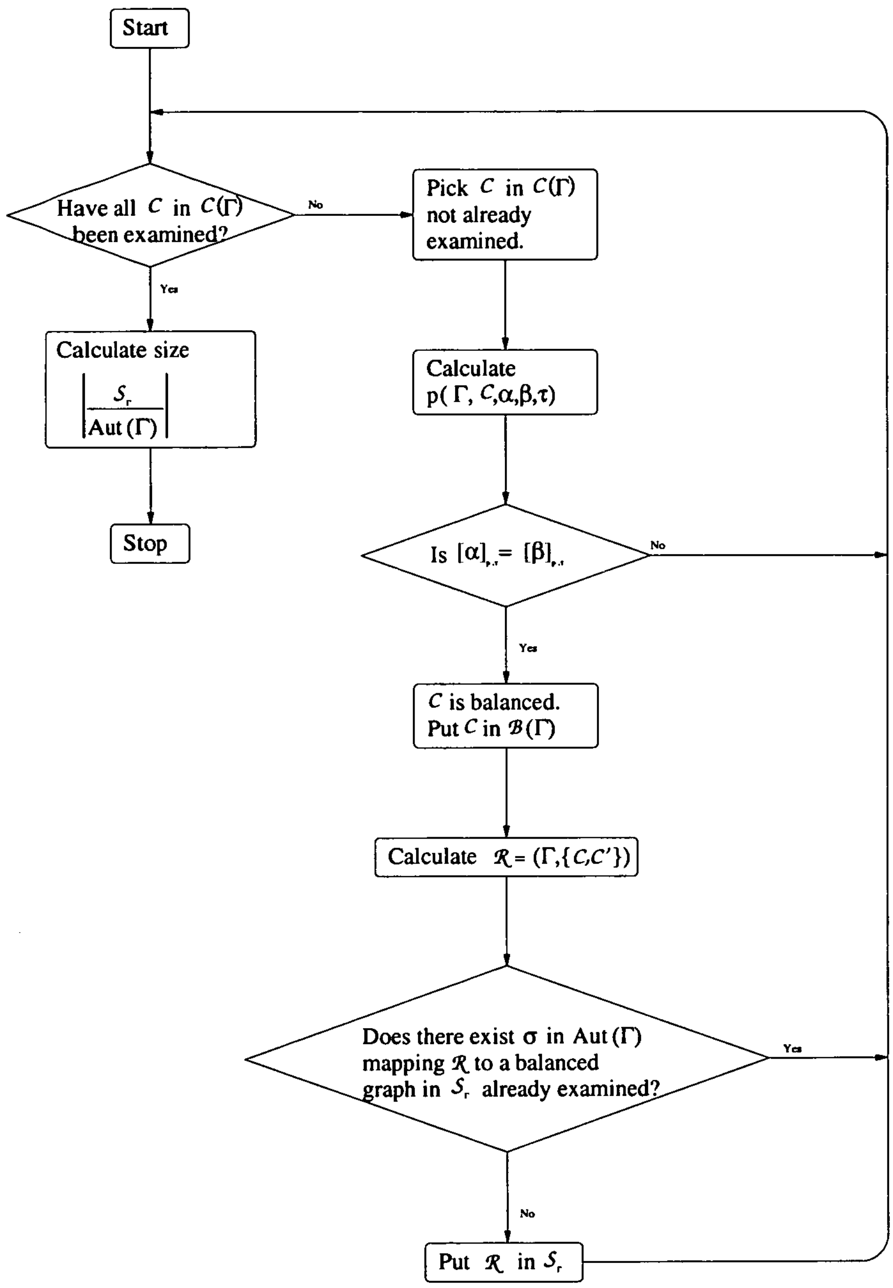


Figure 3.14: From 2-regular digraphs to ribbon graphs. An algorithm.

Chapter 4

On the construction of minimal vector field

It is the intention in this chapter to discuss results concerning the topology of graphs embedded on surfaces. We will give a topological definition for a ribboned graph and show that this is equivalent to the graphical definition given in the previous chapter. We will then show how to construct a minimal vector field from a 2-regular digraph, thus paving the way for the classification and combinatorial results that will follow in later chapters.

We shall again start with some basic definitions. However, these definitions themselves rely on the definitions given in previous chapters.

4.1 On defining ribbonings of graphs

In this section we shall discuss a topological definition of a ribboned graph and its construction. To do this, we need to consider the following definitions concerning topological descriptions of graphs.

Definition 4.1.1 (Thickened graphs and thickened edges). Let $\mathcal{V}$ be a set of discs, called *thickened vertices*. Let $\mathcal{E}$ be a set consisting of pairs (e, f), where e is a surface diffeomorphic to $I \times I$ (where I is the unit interval) and f is a diffeomorphism $f : I \times I \to e$, i.e.

$$\mathcal{E} = \{(e, f) : f \text{ is a diffeomorphism of } I \times I \text{ into } e\}$$

The set $\mathcal{E}$ is called the set of *thickened edges*. We form a *thickened graph* by gluing each end of an edge $(e, f) \in \mathcal{E}$ to some disc $v \in \mathcal{V}$, as follows.

Let $(e, f) \in \mathcal{E}$ be a thickened edge. Then the diffeomorphism f allows us to distinguish the *ends* of e as $f(0, t)$ and $f(1, t)$. We can then identify one end of e

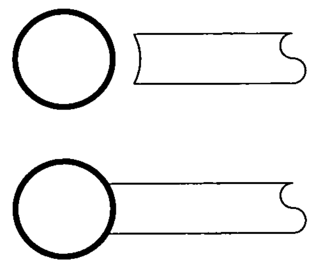


Figure 4.1: Gluing a thickened edge to a thickened vertex.

with an arc of the boundary of a thickened vertex v, as in figure 4.1. Figure 4.2 shows a thickened vertex with four edges glued to it. The *thickened graph* is the

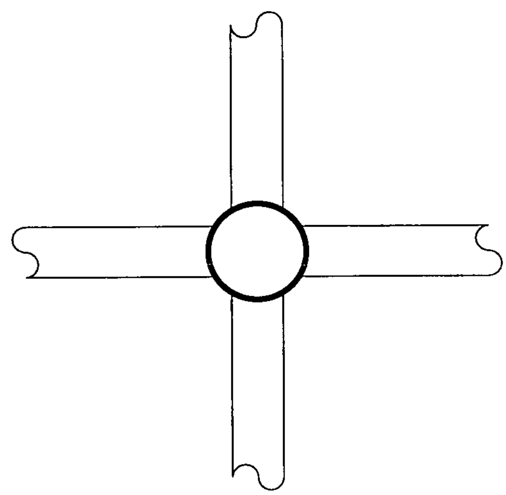


Figure 4.2: A thickened vertex glued to 4 thickened edges.

ordered pair $(\mathcal{V}, \mathcal{E})$.

The *underlying graph* of a thickened graph is the graph $\Gamma = (V, E, \iota, \tau)$ that we get by retracting each disc in $\mathcal{V}$ to a point, and each edge in $\mathcal{E}$ to a line. Indeed, if $(e, f) \in \mathcal{E}$ then one way to view this retraction is retracting the map $f : I \times I \to e$ to a map from $I \times \{p\}$ for some point p. Thus the retracted edge does indeed connect to retracted vertices. An edge e of Γ has end points v_1 and v_2 if and only if the ends of the thickened edge corresponding to e are glued to the discs corresponding to v_1 and v_2.

The purpose of specifying the diffeomorphism f is to allow us to give a direction to e. We shall say that if $f : (r, t) \to f(r, t)$ then e has direction corresponding to increasing r. We can define the initial and terminal vertices of a thickened edge e as the vertices glued to $f(0, t)$ and $f(1, t)$ respectively and define ι and τ for the underlying graph similarly.

We shall usually suppress the diffeomorphism f, and write a thickened edge (e, f) as just e. See figure 4.3 for an example of a thickened graph.

All ribboned graphs that follow will be assumed to be 2-regular and directed, i.e. their underlying graphs will be 2-regular directed graphs.

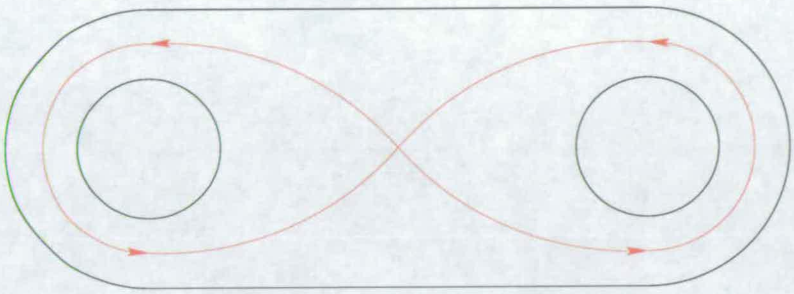


Figure 4.3: An example of a thickened graph with underlying graph Γ

Thus a thickened graph is a 2-manifold with boundary. We will now define a ribboning of a graph as a thickened graph that retracts to a given graph.

Definition 4.1.2 (Topological Ribboning of a graph Γ). A thickened graph $\mathcal{R}$ is a ribboning of Γ if and only if Γ is the underlying graph of $\mathcal{R}$.

Notice that this definition is equivalent to the following.

Definition 4.1.3. A *topological ribboning* $\mathcal{R}$ of a graph Γ (equivalently, a *topological ribboned graph*) is a 2-dimensional manifold with boundary such that the following properties hold.

1. Γ is a sub-manifold of $\mathcal{R}^o = \mathcal{R} \backslash \partial\mathcal{R}$. Equivalently, Γ is a sub-manifold of $\mathcal{R}$ such that $\Gamma \cap \partial\mathcal{R} = \emptyset$

2. Γ is a deformation retract of $\mathcal{R}$.

Note that it is an immediate consequence of this definition that for any topological ribboning $\mathcal{R}$ of some graph Γ, $\mathcal{R} - \Gamma$ is the disjoint union of annuli and Möbius bands.

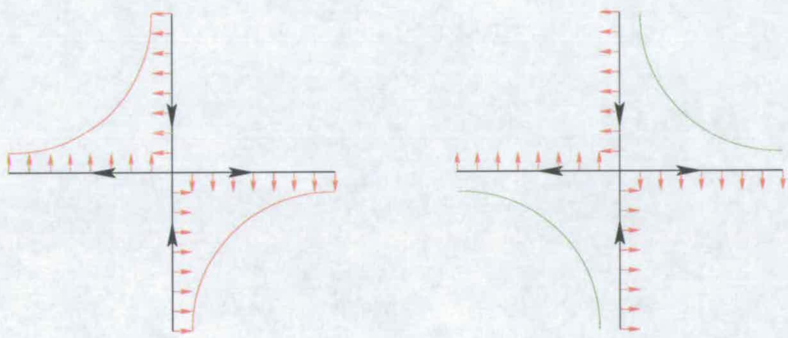
The following two propositions show that any topological ribboned graph is also a ribboned graph in the sense of the previous chapter, and vice versa.

Proposition 4.1.1. *Let $\mathcal{R}$ be an oriented topological ribboning of the graph Γ. Then $\partial\mathcal{R}$ can be partitioned into two sets, $\mathcal{C}$ and $\mathcal{C}'$ such that each corresponds to a circuit decomposition of Γ and together they form a dual pair.*

Proof: Let $\mathcal{A}$ be a connected component of $\mathcal{R} - \Gamma$. It is clear that, as $\mathcal{R}$ is oriented, $\mathcal{A}$ is an annulus. Hence it has precisely two boundary circuits, one of which is a circuit of Γ. Let c be the circuit of Γ and let $\partial\mathcal{A}$ be the other circuit.

$\mathcal{A}$ is oriented as it inherits the orientation of $\mathcal{R}$. c is oriented as it is a circuit of the 2-regular digraph Γ so we can talk about the normal to the tangent of c. However, we have to make a choice about the direction of this normal. We shall make a convention that the normal to the tangent at c points to the right of the direction along c. Thus the tangent to c points into $\mathcal{A}$ and the tangent to $\partial\mathcal{A}$ points out. Thus we can classify the boundary circuits of $\mathcal{R}$ according to the direction of this tangent. Let $\mathcal{C}$ be the set of boundary circuits for which the tangent points in and $\mathcal{C}'$ be the set of remaining boundary circuits.

We know need to show that these two sets are a dual pair of circuits decompositions of Γ.



Pairing of edges generated at c
by choice of normal to circuit c.

Pairing of edges generated at c'
by choice of normal to circuit c.

Figure 4.4: How the direction of the normal on a circuit c corresponds to the local orientation of a vertex v.

Now, this choice of direction is closely tied to the orientation of any vertex v that is in c. We can see this by considering figure 4.4. Here we see that by considering the chosen normal direction as being that to the right of the direction given to an edge, we have an orientation at v. Indeed, if we make the same choice of normal direction for each $c \in \mathcal{C}$ (which we may as Σ is oriented) we can use this to define a local orientation system OR_Γ. But we note that this local orientation system pairs up edges where the perpendicular direction points towards the circuit being defined, as figure 4.4 shows. Thus we can see that $\mathcal{C}$ is indeed a circuit decomposition and moreover, it contains c.

Similarly $\mathcal{C}'$ is also a circuit decomposition.

This also shows that they are a dual pair. $\square$

Proposition 4.1.2. *Given two circuit decompositions of* Γ *that form a dual pair then there exists an oriented 2-dimensional manifold with boundary that is a ribboning of* Γ *with the given circuit decompositions as the boundary components.*

Moreover, this manifold will be unique up to the action of a diffeomorphism that induces the identity on homology.

Proof: To construct $\mathcal{R}$ we take one annulus for each element c of $\mathcal{C}$ and glue it onto the circuit of Γ represented by c. We repeat this for each element of $\mathcal{C}'$. Then by Definition 4.1.2 $\mathcal{R}$ constructed in this way is a topological ribboning of Γ. The uniqueness comes from the fact that any two such constructions will be the same up to an action of a diffeomorphism that is the identity on Γ, and hence acts only on the annuli. Hence it is the identity on homology. □

Thus we may conclude

Lemma 4.1.4. *Every oriented topological ribboned graph defines a graphical ribboned graph and vice versa.*

Proof: Given a topological ribboned graph $\mathcal{R}$, proposition 4.1.1 shows that we may partition $\partial\mathcal{R}$ into the dual pair $\{\mathcal{C}, \mathcal{C}'\}$. Thus we can define a graphical ribboned graph $R(\mathcal{R}) = (\Gamma_{\mathcal{R}}, \{\mathcal{C}, \mathcal{C}'\})$, where $\Gamma_{\mathcal{R}}$ is the underlying graph of $\mathcal{R}$.

Conversely, given a graphical ribboned graph $R = (\Gamma, \{\mathcal{C}, \mathcal{C}'\})$ proposition 4.1.2 shows how we may construct a unique topological ribboned graph $\mathcal{R}$. □

4.2 Topological operations on Ribboned Graphs

There are two topological operations that we may define on a ribboned (or a thickened) graph, namely edge-twisting an edge $e \in \mathcal{E}$, written ET_e and vertex flipping a vertex $v \in \mathcal{V}$, written VF_v. We shall define both of these in this section. It should be noted that neither of these operations affect the underlying graph, i.e. if $\mathcal{R}$ is a ribboning of Γ then $ET_e\mathcal{R}$ for $e \in \mathcal{E}_\Gamma$ and $VF_v\mathcal{R}$ for any $v \in \mathcal{V}_\Gamma$ are both ribbonings of Γ.

Definition 4.2.1 (Edge Twist). The operator ET_e is defined on the thickened edge $e \in \mathcal{E}$ to be the map that cuts e in two, puts a half twist in one half of e and then glues the two edges back together. So ET_e puts a half twist in the edge e.

Figure 4.5 shows how a ribboned edge is twisted by this operator.

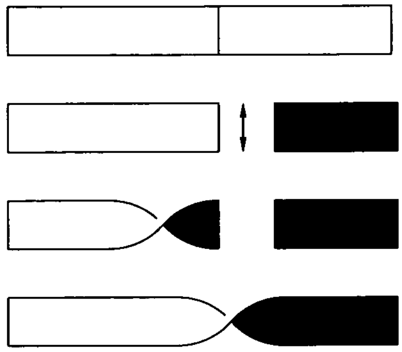


Figure 4.5: The action of the operator ET_e.

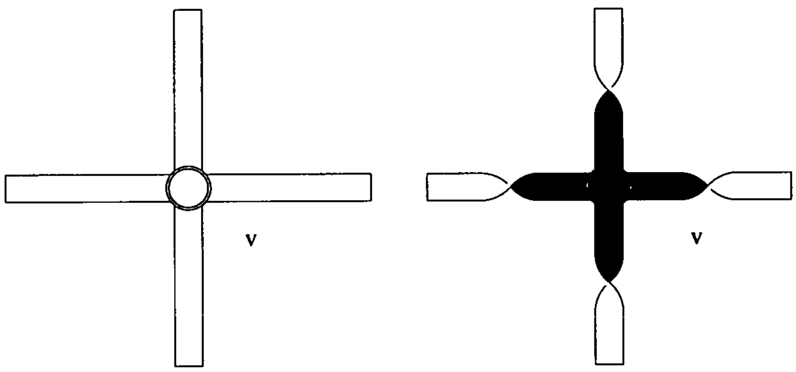


Figure 4.6: The action of the operator VF_v.

Definition 4.2.2 (Vertex Flip). The operator VF_v is defined on the vertex $v \in \mathcal{V}_\Gamma$ in terms of the thickened edges at v. Formally, let $e_1^i, e_2^i, e_1^o, e_2^o$ be the four edges that have v as either the initial or final vertex. Then $VF_v = ET_{e_1^i} \circ ET_{e_2^i} \circ ET_{e_1^o} \circ ET_{e_2^o}$.

See figure 4.6 for the effect of the operator VF_v on the edges surrounding the vertex v

It is easy to see that for any $e \in \mathcal{E}$, ET_e is an involution, i.e. $ET_e\mathcal{R} \neq \mathcal{R}$ in general, but $ET_e^2\mathcal{R} = \mathcal{R}$. Equally, for any $v \in \mathcal{V}$ VF_v is an involution.

Clearly for any thickened edge e, if $\mathcal{R}$ is oriented then $ET_e\mathcal{R}$ will not be orientable. However, in the next lemma we shall show that, for any $v \in \mathcal{V}_\Gamma$, VF_v preserves orientability (i.e. $VF_v\mathcal{R}$ is orientable if and only if $\mathcal{R}$ is orientable).

Lemma 4.2.3. *Let $\mathcal{R}$ be a ribboning of some graph Γ. Then, for any $v \in \mathcal{V}_\Gamma$, $VF_v\mathcal{R}$ is orientable if and only if $\mathcal{R}$ is orientable.*

Proof: Observe that, with regard to the definition of a ribboned graph in terms of local orientation systems, the operator VF_v interchanges or_v and $\overline{or_v}$.

So $\mathcal{R}$ is orientable if and only if it defines a local orientation system, as Lemma 4.1.4 shows. Thus if $\mathcal{R}$ is orientable, then there is a local orientation system $OR_{\mathcal{R}} = \{or_w : w \in V\}$ defined by it, where V is the vertex set for the underlying graph of $\mathcal{R}$. Then $VF_v\mathcal{R}$ is defined by the orientation system $\{or_w : w \neq v\} \cup \{\overline{or_v}\}$ hence is orientable. The converse is similar. □

Definition 4.2.4. The set of ribbonings of a graph Γ is defined to be $\mathcal{R}(\Gamma)$

We define a special subset of all ribbonings.

Definition 4.2.5 (Balanced ribbonings). The set $\Sigma(\Gamma)$ is defined to be the set of ribbonings of a graph Γ such that dual pair of circuit decompositions have the same cardinality. That is

$$\Sigma(\Gamma) = \{(\Gamma, \{\mathcal{C}, \mathcal{C}'\}) : |\mathcal{C}| = |\mathcal{C}'|\}$$

We call this set the set of *balanced* (or *gluable*) *ribbonings.* This will be important later when we attempt to construct minimal vector fields from ribboned graphs.

4.3 Construction results for ribbonings of graphs

We shall now discuss some construction results concerning ribbonings of graphs. In this section we shall show that any circuit of a graph Γ can be a boundary circuit of a ribboning of Γ. We shall also give a construction showing that for any surface with two or more boundary components there exists a graph such that the surface can be considered as a ribboning of that graph.

Theorem 4.3.1. *Let Γ be a 2-regular digraph with vertex set V and edge set E. A necessary and sufficient condition for a subset $B \subset E$ to be a boundary circuit of some ribboning of Γ is that B is a circuit, i.e. a boundary circuit of an element of $\mathcal{R}(\Gamma)$ is a circuit of Γ and conversely, given a circuit of Γ there exists a ribboning of the graph with the given circuit as a boundary circuit.*

Proof: That it is a necessary condition is clear. A boundary circuit clearly retracts to a circuit.

That it is a sufficient condition is as follows. By induction, we construct a circuit decomposition of Γ that contains the circuit B. We can do this as deleting the edges contained in B leaves us with a graph with strictly fewer edges that Γ, and so the induction follows. Then we construct the dual decomposition as in Proposition 4.1.1. This gives us a pair of circuit decompositions which, by

Proposition 4.1.2, form a ribboning of Γ in which they are the boundary circuits. But one of these boundary circuits is, by construction, B. $\square$

Theorem 4.3.2. *Let $\delta \geq 2$. Then, for any surface $\Sigma_{g,\delta}$ with genus g and δ discs removed, there exists a graph Γ which can be embedded in Σ in such a way as to make $\Sigma \in \mathcal{R}(\Gamma)$, i.e. Σ is the ribboned graph of Γ.*

Proof: By demonstration. We shall write down the graph with the required property.

First we note that given an enumeration of V we can uniquely describe a graph using a matrix $M_\Gamma = (a_{i,j})$, where $a_{i,j}$ is equal to the number of edges e with $\iota(e) = v_i$ and $\tau(e) = v_j$, and $a_{i,i}$ equals the number of single-edge loops at e. This matrix is the *incidence matrix* of Γ and will be discussed in greater detail in chapter 7.

So to write down the graph Γ, we need only give the matrix and show how to embed it in Σ. The matrix $M_\Gamma = (a_{i,j})$ is given below.

Let $n = 2g - 2 + \delta$

$$a_{1,j} = \begin{cases} 2 \; j = 2 \\ 0 \; j \neq 2 \end{cases}$$

$$a_{2,j} = \begin{cases} 1 \; j = 1,4 \\ 0 \text{ otherwise} \end{cases}$$

for i $\neq 1, 2, 2g-2, 2g-1$

$$a_{i,j} = \begin{cases} 2 & i \equiv 0 \bmod 4 \text{ and } j = i - 1 \\ & i \equiv 1 \bmod 4 \text{ and } j = i + 1 \\ 1 & i \equiv 2 \bmod 4 \text{ and } j = i \pm 2 \\ & i \equiv 3 \bmod 4 \text{ and } j = i \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

if $n = 2g$ then

$$a_{2g-1,j} = \begin{cases} 2 & 2g \equiv 2 \bmod 4 \text{ and } j = 2g \\ 1 & 2g \equiv 0 \bmod 4 \text{ and } j = 2g, 2g-3 \\ 0 & \text{otherwise} \end{cases}$$

$$a_{2g,j} = \begin{cases} 2 & 2g \equiv 0 \bmod 4 \text{ and } j = 2g-1 \\ 1 & 2g \equiv 2 \bmod 4 \text{ and } j = 2g-2, 2g-1 \\ 0 & \text{otherwise} \end{cases}$$

if $n > 2g$ then

$$a_{2g-1,j} = \begin{cases} 2 & 2g \equiv 2 \bmod 4 \text{ and } j = 2g, 2g+1 \\ 1 & 2g \equiv 0 \bmod 4 \text{ and } j = 2g, 2g-3 \\ 0 & \text{otherwise} \end{cases}$$

$$a_{2g,j} = \begin{cases} 1 & 2g \equiv 0 \bmod 4 \text{ and } j = 2g \pm 1 \\ & 2g \equiv 2 \bmod 4 \text{ and } j = 2g-2, 2g-1 \\ 0 & \text{otherwise} \end{cases}$$

$$a_{2g+1,j} = \begin{cases} 1 & j = 2g+2 \text{ and } n \geq 2g+2 \text{ or } j = 2g+1 \text{ and } n = 2g+1 \\ & 2g \equiv 0 \bmod 4 \text{ and } j = 2g-1 \\ & 2g \equiv 2 \bmod 4 \text{ and } j = 2g \\ 0 & \text{otherwise} \end{cases}$$

and $\forall n \geq k \geq 2g+2$

$$a_{k,j} = \begin{cases} 1 & k \neq n \text{ and } j = k \pm 1 \\ & k = j = n \\ 0 & \text{otherwise} \end{cases}$$

This gives a matrix which represents the required graph Γ. For sake of clarity, three examples are given in figures 4.7, 4.8 and 4.9. The first example shows the graph when $\delta = 2$ (the minimum value it can take) and $g > 0$. The second example shows the graph when $g = 0$ (the minimum value it can take) and $\delta > 2$. The last example shows the graph in a particular case, namely $g = 2$ and $\delta = 4$.

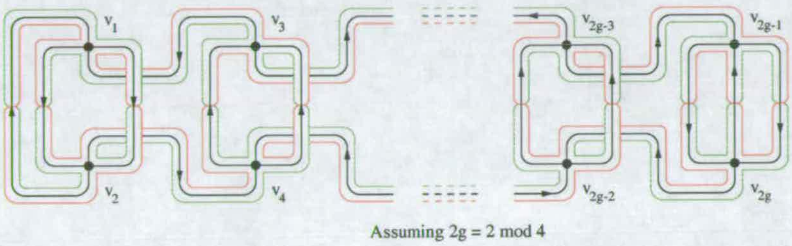
To produce the required ribboning, it may be necessary to flip some or all of the vertices $v_2, v_4, v_6, \ldots, v_{2g}$. □

4.4 On oriented ribbonings of a graph

Now that we have shown that any surface is a ribboning of some graph, and defined an operation on ribbonings of a particular graph, we can show that any two oriented ribbonings of the same graph differ only by a finite number of vertex flips.

Theorem 4.4.1. *Suppose $\mathcal{R}_1$ and $\mathcal{R}_2$ are two oriented ribbonings of the same graph Γ. Then there exists $v_1, \ldots, v_r \in V$ such that $VT_{v_1} \circ \cdots \circ VT_{v_r}\mathcal{R}_1 = \mathcal{R}_2$.*

Proof: We know from Lemma 4.1.4 that an oriented ribboning of Γ defines a local orientation system. Let OR_i be the local orientation system defined by $\mathcal{R}_i$. Then there exists $v_1, \ldots, v_r \in V$ such that the local orientation systems differ at each v_i. However, as there are only two choices of orientation at each v_i, it is clear that if OR_1 and OR_2 differ at v_i then $VT_{v_i}OR_1$ and OR_2 agree there. Thus $VT_{v_1} \circ \cdots \circ VT_{v_r}OR_1 = OR_2$. Hence result. □



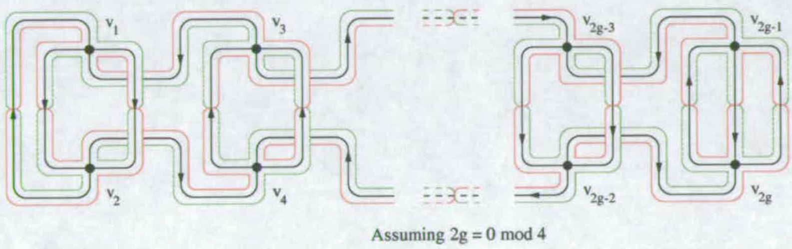


Figure 4.7: First example: $\delta = 2$ and $g > 0$

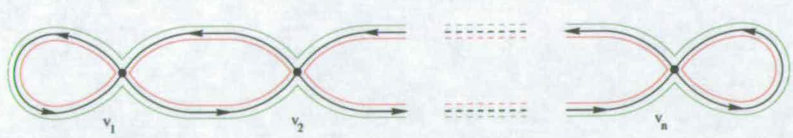


Figure 4.8: Second example: $\delta > 2$ and $g = 0$

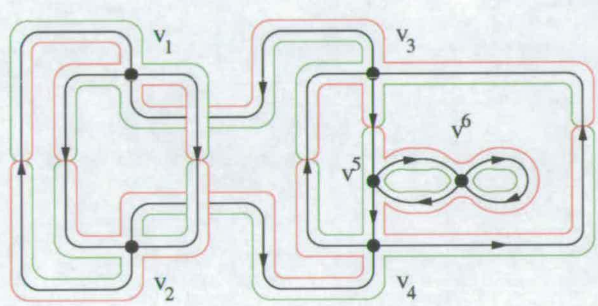


Figure 4.9: Third example: $\delta = 4$ and $g = 2$

Chapter 5

On invariants and the classification of equivalence classes of minimal vector fields

The intention of this chapter is to discuss various invariants of minimal vector fields. We will show that we may combine some of these to form a complete set of invariants in the sense that they completely classify minimal vector fields up to the equivalence defined in chapter 1. In doing so, we will pull together all the work of the previous chapters to produce the classification result.

The classification is contained within the following three theorems, which will be proved in the course of this chapter. Although all the graphical and topological concepts required by these theorems have been defined in previous chapters, how they relate to minimal vector fields has not yet been made clear.

Theorem 1. For each minimal vector field X there exists a unique (up to diffeomorphism) balanced ribboned graph (R_X, τ_X) defined in terms of X (i.e. there is a function between the set of minimal vector fields and the set of balanced ribboned graphs).

Theorem 2. For each balanced ribboned graph $\mathcal{B} = (R, \tau)$ there exists a smooth vector field $X_{\mathcal{B}}$ such that the balanced ribboned graph of X as given by Theorem 1 is $\mathcal{B}$ (i.e. the function defined in Theorem 1 is surjective).

Theorem 3. $X \sim Y$ if and only if $(R_X, \tau_X) \approx (R_Y, \tau_Y)$. That is, X is equivalent to Y in the sense defined in chapter 1 if and only if the balanced ribboned graph of X is equal to the balanced ribboned graph of Y (i.e. the function defined in Theorem 1 is well-defined and injective).

So, the aim of this chapter is to prove that there is a natural bijection between set of equivalence classes of minimal vector fields on Σ and the set of balanced ribboned graphs. This is done by restricting to a surface of genus g. In this case the two sets are both finite and we shall prove that they have same number of elements.

5.1 Graphical invariants of a minimal vector field X and the proof of Theorem 1

We now define how the above graph theoretic definitions relate to a given vector field. Let Σ be a smooth, closed, compact, oriented surface with genus $g > 1$.

Definition 5.1.1 (Graph of a minimal vector field). The graph of a minimal vector field is the graph $\Gamma_X = (V, E, \iota, \tau)$ where V is the set of zeros of X and E is the set of flowlines γ of X that satisfy

$$\lim_{t \to \pm\infty} \gamma(t) \in V$$

It is necessary to prove that $E \neq \emptyset$. However, X is a minimal vector field and hence by Theorem 1.5.1 and Definition 1.5.2 its flowlines can be categorized into three distinct types, namely:

1. Zeros of X.

2. Periodic cycles of X.

3. $\gamma : \mathbb{R} \to \Sigma$ such that $X(\lim_{t \to \pm\infty} \gamma(t)) = 0$.

It is clear then that V is the set of flowlines of the first type and E is the set of flowlines of the third type. ι and τ are defined as follows. For e, a flow-line of X in E

$$i(e) = \lim_{t \to -\infty} e(t)$$
$$\tau(e) = \lim_{t \to +\infty} e(t)$$

Now, as the zeros of X are all saddle points (as X is minimal and so pseudo-minimal), so for any $v \in V$ there are two flowlines of the third type that terminate at v (i.e. satisfy $\tau(e) = v$). Thus, as no flowline can terminate at more than one vertex, E is non-empty and there are twice as many edges as there are vertices.

We shall note in passing that

Lemma 5.1.2. *$X \sim Y$ implies Γ_X is isomorphic to Γ_Y.*

although the proof of this fact will be contained with the proof of Theorem 3, which occurs towards the end of this chapter.

Definition 5.1.3 (Circuit decompositions of X). Theorem 1.3.10 and Definition 1.5.2 give that Γ_X is a 2-regular digraph on $2g-2$ vertices, where g is the genus of Σ. Moreover, as the vector field is smooth, we can see that it is embedded on Σ in such a way that all vertices have the form shown in Figure 3.1 and thus have a local orientation. Thus, as Γ_X is embedded as a set on an oriented surface Σ we have a local orientation system OR_{Γ_X} inherited from the embedding. As a local orientation system (and its dual) together define a dual pair of circuit decompositions we can see that this defines the *circuit decompositions* of X, $\{\mathcal{C}_X, \mathcal{C}'_X\}$.

That the circuit decompositions of X are indeed invariants of the equivalence class of X is one part of Theorem 3. Hence the proof of this fact will be delayed until later in this chapter.

Definition 5.1.4 (Ribboned graph of a vector field). The ribboned graph of a minimal vector field X is the graphical ribboned graph

$$R_X = (\Gamma_X, \{\mathcal{C}, \mathcal{C}'\})$$

where Γ_X is the graph of X defined above in Definition 5.1.1, and $\mathcal{C}$ and $\mathcal{C}'$ are the circuit decompositions defined by the local orientation system, as given in Lemma 3.3.2.

Definition 5.1.5 (Topological ribboning of a graph of a vector field). Suppose R_X is the ribboned graph of a vector field. Then by Lemma 4.1.4 we can construct a 2-manifold with boundary, $\mathcal{R}_X$ that is the topological ribboning equivalent to R_X. We call $\mathcal{R}_X$ the *topological ribboning of* Γ_X.

Now consider $\Sigma - \Gamma_X$. This is topologically identical to $\Sigma - \mathcal{R}_X$, in the sense that each component of $\Sigma - \Gamma_X$ is diffeomorphic to a component of $\Sigma - \mathcal{R}_X$ and vice versa.

Proposition 5.1.1. *$\Sigma - \Gamma_X$ consists of a finite number of annuli. Each annulus is diffeomorphic to the annulus, $\mathcal{A}$ in $\mathbb{R}^2$ given by*

$$\mathcal{A} = \{(x, y) : 1 < x^2 + y^2 < 2\}.$$

Moreover, it is possible to choose this diffeomorphism in such a way that the flowlines of $X|_{\mathcal{A}}$ are concentric circles, centered at the origin.

Proof: That the boundary components are oriented circles is obvious (they are closed cycles of X). Now, let A be a connected component of $\Sigma - \Gamma_X$ and let A have δ_A boundary components. Then it is easy to see that;

1. X does not vanish on A.
2. A is orientable.
3. $\delta_A > 0, g_A \geq 0$.

Point 1. and Corollary 1.1.10 together imply that $\chi(A) = 0$.
Point 2. implies that A has a genus (g_A say).
Thus $\chi(A) = 2 - 2g_A - \delta_A = 0$, and so this, together with point 3. implies then that $g_A = 0$ and $\delta_A = 2$. Hence A is a 2-punctured sphere i.e. an annulus.

Now, by the definition of Γ_X we know that those flowlines that do not make up Γ_X are periodic. So $\mathcal{A}$ is diffeomorphic to an annulus containing only periodic cycles. This is clearly diffeomorphic to A via a diffeomorphism that sends the flowlines of $X|_{\mathcal{A}}$ to concentric circles centered at the origin. □

Corollary 5.1.6. *δ is even.*

Proof: $\Sigma - \mathcal{R}_X$ is the union of finitely many annuli. Each annulus has 2 boundary circuits and the total number of boundary circuits is δ. Hence δ is even. □

In order to define τ_X, the pairing of R_X given by X we first need the following lemma.

Lemma 5.1.7. *Suppose $\mathcal{A}$ is a connected component of $\Sigma - \Gamma_X$ and c is a boundary circuit of $\mathcal{A}$ such that $c \in C$. Then $\mathcal{A}$ has only two boundary circuits and the other one is some $c' \in C'$.*

Before we begin the proof, we shall note that the point of the above lemma is that every connected component of $\Sigma - \Gamma_X$ contains two circuits, one in each of a dual pair of circuit decompositions. So every connected component pairs up a circuit in one circuit decomposition with a circuit in the dual decomposition.

Proof: Notice that this proof follows the style of a previous proof, namely proposition 4.1.1.

As $\Sigma - \Gamma_X$ is the disjoint union of annuli (by proposition 5.1.1) it is clear that $\mathcal{A}$, as a connected component of $\Sigma - \Gamma_X$ has precisely two boundary circuits. Let these be c and c'. We are given that $c \in \mathcal{C} = E/\sim_{OR_{\Gamma_X}}$, so we only need to prove that $c' \in \mathcal{C}' = E/\sim_{\overline{OR}_{\Gamma_X}}$.

Consider $\mathcal{A}$. $\mathcal{A}$ is oriented as it inherits the orientation of Σ. c is oriented as it is a circuit of the 2-regular digraph Γ_X so we can talk about the normal to the tangent of c. However, we have to make a choice about the direction of this normal. We shall make a convention that the normal to the tangent at c points into $\mathcal{A}$ at c. Notice that this same direction will point out of $\mathcal{A}$ at c'.

As before, this choice of direction is closely tied to the orientation of any vertex v that is in c. In Figure 4.4 we see that by considering the chosen normal direction as being that to the right of the direction given to an edge, we have an orientation at v. Indeed, if we make the same choice of normal direction for each $c \in \mathcal{C}$ (which we may as Σ is oriented) we just recover the local orientation system OR_{Γ_X}. But we note that we can now define this local orientation system as that which pairs up edges where the perpendicular direction points towards the circuit being defined.

We now consider the circuit c'. On this circuit the edges are paired up because the perpendicular direction points away from the circuit being defined. But by considering all such c' we can construct a circuit decomposition dual to $\mathcal{C}$ that contains c'. But by definition that circuit decomposition is $\mathcal{C}'$. □

This lemma immediately allows us to conclude the following.

Corollary 5.1.8. *$|\mathcal{C}| = |\mathcal{C}'|$, i.e. R_X is balanced.*

Proof: Lemma 5.1.7 shows that to each element of $\mathcal{C}$ there is a unique element of $\mathcal{C}'$ and vice versa. □

In fact, Lemma 5.1.7 does more than just show that R_X is a balancable graph (i.e. one for which we may define a pairing $\tau : \mathcal{C} \to \mathcal{C}'$), it provides an explicit pairing τ_X, as the next definition shows.

Definition 5.1.9 (Pairing of a ribboned graph). We define $\tau_X : \mathcal{C} \to \mathcal{C}'$ as follows: $\tau_X c = c' \Leftrightarrow \exists\ \mathcal{A}$ a connected component of $\Sigma - \Gamma_X$ such that $\partial\mathcal{A} = \{c, c'\}$ (i.e. the boundary circuits of $\mathcal{A}$ are c and c').

Moreover, as the above discussion shows, this definition is independent of the choice of vector field within the equivalence class containing X and hence unique. This proves Theorem 1

This last comment allows to identify one last graphical invariant of X.

Definition 5.1.10. Given a minimal vector field X, we can define the *balanced ribboned graph* of X, $\mathcal{B}_X = (R_X, \tau_X)$, where R_X and τ_X are respectively the ribboned graph of X and the pairing of R_X inherited from X.

5.2 Homological Invariants of X

So far, we have discussed only results concerning the graph of a minimal vector field. Whilst it should be noted that this will eventually contributed to a complete classification of the class of minimal vector fields there are other invariants that can be mentioned. We shall discuss these in this and the next few sections.

We have already discussed and defined Γ_X, the graph of the minimal vector field X. We know that this is an embedded graph and that the embedding equips Γ_X with a dual pair of circuit decompositions $\{\mathcal{C}, \mathcal{C}'\}$. We shall now consider the complement of Γ_X in Σ.

Recall that $\Sigma - \Gamma_X$ consists of disjoint annuli. Let the number of these annuli be a_{Γ_X}. Notice that these annuli are glued to the circuits of $\mathcal{C}$ and $\mathcal{C}'$ via the pairing τ already discussed. We proceed by considering these annuli, together with the edges of the embedded graph Γ_X.

First, note that in chapter 2, we showed that the first homology class of the pair (Σ, V) is given by

$$H_1(\Sigma, V) =< a_1, b_1, \ldots, a_g, b_g, f_1, \ldots, f_{2g-3} > \cong \mathbb{Z}^{4g-3}$$

Now each edge e_i of Γ_X is embedded on Σ, so it is a representative of a class in $H_1(\Sigma, V)$. So we have a map

$$\phi : H_1(\Gamma_X, V) \to H_1(\Sigma, V)$$

Proposition 5.2.1. *The rank of this map ϕ satisfies*

$$rank\ \phi + a_{\Gamma_X} = 4g - 3$$

Proof: To prove this, consider the following relative homology sequence of the triple (Σ, Γ_X, V) with $\mathbb{Z}$-coefficients.

$$\begin{aligned} \ldots \longrightarrow H_2(\Gamma_X, V) \longrightarrow H_2(\Sigma, V) \xrightarrow{\alpha} H_2(\Sigma, \Gamma_X) \\ \xrightarrow{\beta} H_1(\Gamma_X, V) \xrightarrow{\phi} H_1(\Sigma, V) \xrightarrow{\Gamma_X} H_1(\Sigma, \Gamma_X) \xrightarrow{\delta} 0 \end{aligned}$$

Now, we know that the dimension of $H_1(\Sigma, V)$ is $4g - 3$. Also, it is clear that the dimension of $H_1(\Gamma_X, V)$ is $4g - 4$ and that $H_2(\Gamma_X, V)$ has dimension 0.

Notice also that the dimensions of both $H_1(\Sigma, \Gamma_X)$ and $H_2(\Sigma, \Gamma_X)$ are a_{Γ_X}, as $\Sigma - \Gamma_X$ is the disjoint union of a_{Γ_X} annuli.

So the above exact sequence reduces to

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}^{a_{\Gamma_X}} \xrightarrow{\beta} \mathbb{Z}^{4g-4} \xrightarrow{\phi} \mathbb{Z}^{4g-3} \xrightarrow{\gamma} \mathbb{Z}^{a_{\Gamma_X}} \xrightarrow{\delta} 0$$

From this it is easy to see that the proof is now by diagram chasing. In the above exact sequence, we know that

$$\text{rank } \phi + \text{nullity } \phi = 4g - 4$$

But by exactness nullity ϕ = rank β. Now

$$\text{rank } \beta + \text{nullity } \beta = a_{\Gamma_X}$$

and as nullity β = rank $\alpha = 1$ we have that rank $\beta = a_{\Gamma_X} - 1$. Hence

$$\begin{aligned} \text{rank } \phi + a_{\Gamma_X} &= 4g - 4 + a_{\Gamma_X} - \text{nullity } \phi \\ &= 4g - 4 + a_{\Gamma_X} - \text{rank } \beta \\ &= 4g - 4 + a_{\Gamma_X} - a_{\Gamma_X} + 1 \\ &= 4g - 3 \end{aligned}$$

as required. □

Notice it is possible to recast this result using matrices as follows.

Definition 5.2.1 (Homology matrix of the embedding of Γ_X). Under the action of the map ϕ, e_i, the i-th edge of Γ_X has homology class

$$[e_i] = \lambda_1^i a_1 + \mu_1^i b_1 + \cdots + \lambda_g^i a_g + \mu_g^i b_g + \nu_1^i f_1 + \cdots + \nu_{2g-3}^i f_{2g-3}$$

where $H_1(\Sigma, V) =< a_1, b_1, \ldots, a_g, b_g, f_1, \ldots, f_{2g-3} >$ and λ_j^i, μ_j^i and ν_j^i are integers.

Define m_j^i as follows

$$\begin{aligned} m_{2j-1}^i &= \lambda_j^i && \text{for } 1 \leq j \leq g \\ m_{2j}^i &= \mu_j^i && \text{for } 1 \leq j \leq g \\ m_j^i &= \nu_{j-2g}^i && \text{for } 2g - 1 \leq j \leq 4g - 3 \end{aligned}$$

Define M_{Γ_X} to be the matrix that has entries m_j^i. Then M_{Γ_X} is the *homology matrix of the embedding of* Γ_X

There are two comments that can be made about this matrix. The first is obvious, namely that M_{Γ_X} is the matrix of the map ϕ. The second is contained within the following proposition.

Proposition 5.2.2. *Suppose M is the matrix of a minimal vector field X and N is the matrix of a minimal vector field Y. Then $X \sim Y$ implies that there exists a $4g-3$ square matrix $\Psi \in Aut(H_1(\Sigma, V))$ and a $4g-4$ square matrix S in the group of permutation matrices (i.e. the matrix group of the set of permutations of $4g-4$ points) such that*

$$\Psi M S = N$$

Proof: $X \sim Y$ implies there exists a homeomorphism f of Σ that sends the flowlines of X to the flowlines of Y. As the flowlines define the homology classes of Γ_X and Γ_Y this is all we shall need.

Now, chapter 2 showed that $\text{Aut}(H_1(\Sigma, V))$, the group of automorphisms of $H_1(\Sigma, V)$ that are induced by a diffeomorphism of Σ, is a matrix group of $4g-3$ square matrices. But as f is an orientation-preserving homeomorphism so it is isotopic to a product of Dehn twists by Lemma 2.2.3. So f induces a matrix Ψ in this group that sends the homology classes of Γ_X to the homology classes of Γ_Y.

To complete the proof, we need only observe that by writing the homology classes of the edges of Γ_X as a matrix we are labeling them. It is possible that the labeling of the edges of Γ_Y given by N are not that same as the labeling of the edges of $\Gamma_X = \Gamma_Y$ given by M. However, they will be the same up to right multiplication by a permutation matrix S, as this will just permute the labels of the edges of the graph. Hence result. □

We can thus conclude that M_{Γ_X} is an invariant of X up to the equivalences given.

However, it is not a complete invariant. It is possible for there to be two minimal vector fields X and Y such that $\Psi M_{\Gamma_X} S = M_{\Gamma_Y}$ but $X \nsim Y$, as Figure 5.1 shows. In this example, the two vector fields have the same graph Γ but different ribboned graphs, Ψ is the identity matrix and S is the matrix that sends f_2 to $f_2 - f_3$. Now, consider the annuli themselves. By proposition 5.1.1 we know that $X|_A$ consists of concentric circles. Hence the vector field on A defines a unique homology class in $H_1(\Sigma)$. So we may unambiguously talk of the homology class of an annulus. So we have a map ψ given by

$$\psi : H_1(\Gamma_X) \to H_1(\Sigma)$$

The rank of this map is governed by a_Γ, k and g as the following proposition shows.

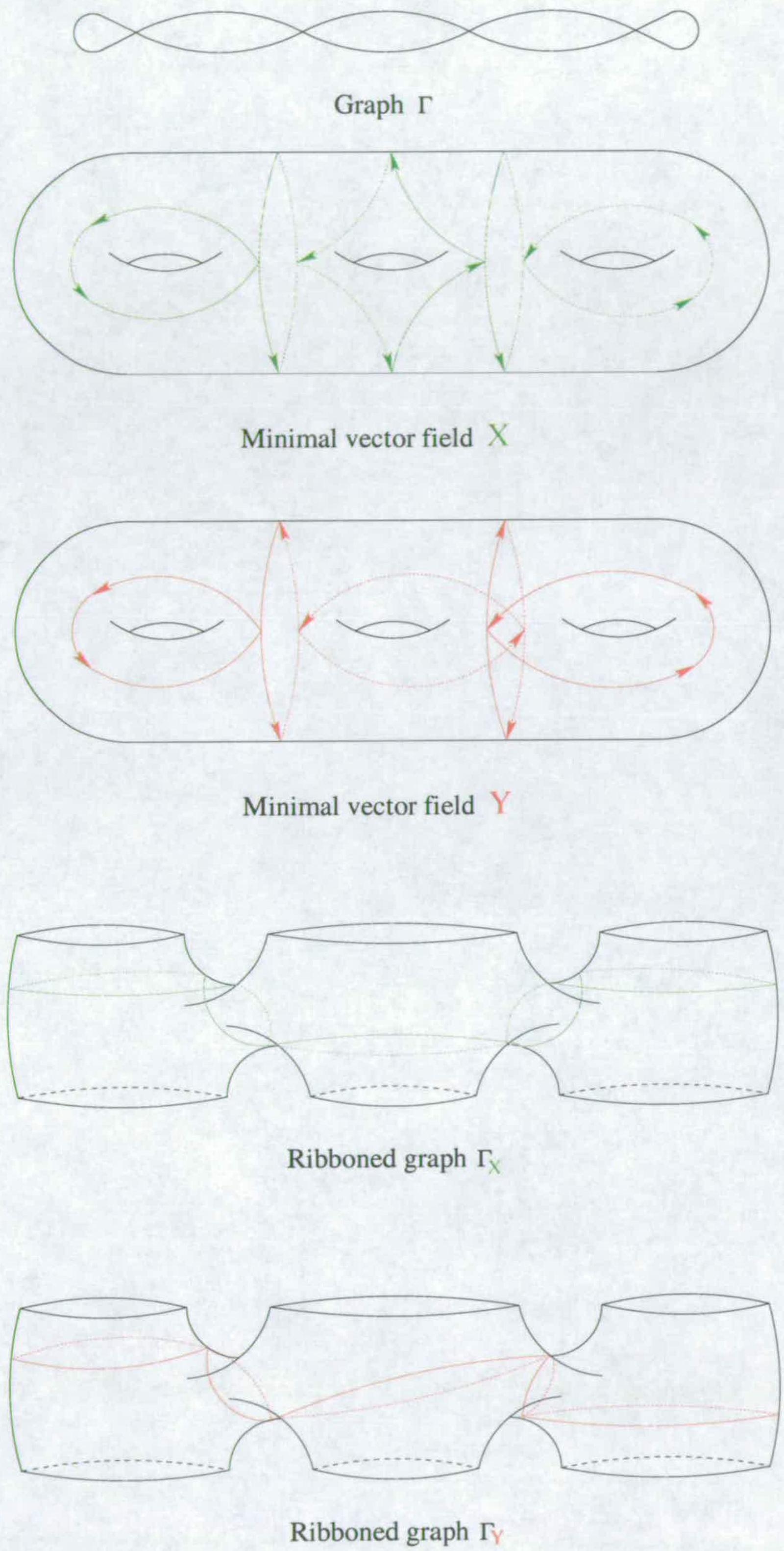


Figure 5.1: An example showing that M_{Γ_X} is not a complete invariant of X.

Proposition 5.2.3.

$$\mathit{rank}\ \psi + a_{\Gamma_X} = 2g + k - 1$$

Proof: Consider the following diagram of exact sequence.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\alpha"] & \mathbb{Z}^{a_{\Gamma_X}} \arrow[r, "\beta"] & \mathbb{Z}^{4g-4} \arrow[r, "\phi"] & \mathbb{Z}^{4g-3} \\
0 \arrow[r] & H_2(\Sigma) \arrow[r, "\epsilon"] \arrow[u] & H_2(\Sigma, \Gamma_X) \arrow[r, "\nu"] \arrow[u] & H_1(\Gamma_X) \arrow[r, "\psi"] \arrow[u] & H_1(\Sigma) \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r, "\gamma"] & H_1(\Sigma, \Gamma_X) \arrow[r, "\delta"] & 0 & & \\
\arrow[r, "\mu"] & H_1(\Sigma, \Gamma_X) \arrow[r, "\sigma"] \arrow[u] & H_0(\Gamma_X) \arrow[r, "\pi"] & H_0(\Sigma) \arrow[r, "\tau"] & 0
\end{tikzcd}$$

From the observations made above, this is clearly

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\alpha"] & \mathbb{Z}^{a_{\Gamma_X}} \arrow[r, "\beta"] & \mathbb{Z}^{4g-4} \arrow[r, "\phi"] & \mathbb{Z}^{4g-3} \arrow[r, "\gamma"] & {} \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\epsilon"] \arrow[u] & \mathbb{Z}^{a_{\Gamma_X}} \arrow[r, "\nu"] \arrow[u] & H_1(\Gamma_X) \arrow[r, "\psi"] \arrow[u] & \mathbb{Z}^{2g} \arrow[r, "\mu"] \arrow[u] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r, "\gamma"] & \mathbb{Z}^{a_{\Gamma_X}} \arrow[r, "\delta"] & 0 & & \\
\arrow[r, "\mu"] & \mathbb{Z}^{a_{\Gamma_X}} \arrow[r, "\sigma"] \arrow[u] & \mathbb{Z}^k \arrow[r, "\pi"] & \mathbb{Z} \arrow[r, "\tau"] & 0
\end{tikzcd}$$

The proof is then immediate by diagram chasing. □

5.3 Proof of Theorem 2

Recall that Theorem 2 states that for each balanced ribboned graph there exists a smooth vector field.

Proof: The proof is by construction. Given a balanced ribboned graph (R, τ), where $R = (\Gamma_X, \{C, C'\})$ we use Lemma 4.1.4 to construct $\mathcal{E}$, an oriented manifold with boundary (i.e. the topological ribboned graph). We then construct Σ from $\mathcal{R}$ by gluing up the boundary circuits of $\mathcal{R}$ (which correspond to the circuits of R contained in C and C') in pairs by the rule $c \in C$ and $c' \in C'$ are glued together if and only if $\tau c = c'$. This constructs Σ.

We now need to construct X on Σ. But this is just an extension of the vector field given by the graph Γ_X to the annuli that were glued to Γ_X to form $\mathcal{R}$. It is done in such a way that the vector field on those annuli consists of closed compact flow-lines. We can then smooth the vector field to find the required vector field X. □

5.4 Definitions of equivalence

In this section we restate some previously given equivalence definitions. Recall the following from chapter 1.

Definition 5.4.1 (Vector field equivalence). Let X and Y be vector fields on Σ. Then X and Y are equivalent if and only if the corresponding flows f_X, f_Y are equivalent, i.e. there exists a homeomorphism of Σ which maps each flowline of f_X to a flowline of f_Y and preserves orientation of orbits.

This equivalence merely claims that the two vector fields are equivalent if the flowlines 'look' the same, i.e. are smoothly equivalent. We say that $X \sim Y$

Definition 5.4.2. Let (R_X, τ_X) and (R_Y, τ_Y) be two balanced ribboned graphs. Then they are equivalent (and we write $(R_X, \tau_X) \approx (R_Y, \tau_Y)$) if and only if there exists $\sigma \in$Aut(Γ_X) such that $\sigma(\mathcal{C}_X) = \mathcal{C}_Y$ i.e. $R_X \sim_{\Gamma_X} R_Y$ and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}_X & \xrightarrow{\tau_X} & \mathcal{C}'_X \\ \sigma\big\downarrow & & \big\downarrow\sigma \\ \mathcal{C}_Y & \xrightarrow{\tau_Y} & \mathcal{C}'_Y \end{array}$$

i.e. $\tau_Y\sigma = \sigma\tau_X$. Note that if σ instead maps $\mathcal{C}_X$ to $\mathcal{C}'_Y$ then we can compose with τ_Y to produce the necessary σ from $\mathcal{C}_X$ to $\mathcal{C}_Y$.

5.5 The Proof of Theorem 3

Recall that Theorem 3 states that $X \sim Y$ if and only if $(R_X, \tau_X) \approx (R_Y, \tau_Y)$

Proof: Suppose $X \sim Y$. Then by Definition 5.4.1 there a homeomorphism f of Σ such that maps every flowline of X to a flowline of Y, preserving the orientation.

So consider f. As Γ_X, Γ_Y, the graphs of X and Y respectively, are submanifolds of Σ, so f sends Γ_X to Γ_Y. So it induces an isomorphism of graphs that we shall call $f|_\Gamma$. Moreover f sends $\mathcal{C}_X$ to $\mathcal{C}_Y$ and $\mathcal{C}'_X$ to $\mathcal{C}'_Y$. Also, for a curve $c \in \mathcal{C}_X$, $(\tau_Y \circ f)(c)$ is that curve in $\mathcal{C}'_Y$ that is glued to $f(c)$. But by definition of τ_X as the pairing of the balanced ribboned graph of X, the curve glued to $f(c)$ is $(f \circ \tau_X)(c)$. So $\tau_Y \circ f = f \circ \tau_X$. So $f|_\Gamma \in$Aut(Γ) such that $\tau_Y \circ f_\Gamma = f_\Gamma \circ \tau_X$ and so $(R_X, \tau_X) \approx (R_Y, \tau_Y)$ via f as required.

Conversely suppose $(R_X, \tau_X) \approx (R_Y, \tau_Y)$, that is, there exists $\sigma \in \mathrm{Aut}(\Gamma)$ such that $\sigma\mathcal{C}_X = \mathcal{C}_Y$ and $\tau_Y \circ \sigma = \sigma \circ \tau_X$.

We claim that $\sigma \in \mathrm{Aut}(\Gamma)$ implies that there exits a homeomorphism f_σ of Σ such that f_σ induces σ. For $\sigma\mathcal{C}_X = \mathcal{C}_Y$ implies $\sigma\mathcal{C}'_X = \mathcal{C}'_Y$, hence we have an automorphism that sends R_X to R_Y. Now we can construct $\mathcal{R}_X$ and $\mathcal{R}_Y$ using Proposition 4.1.2. Using this construction and the automorphism σ we can then construct a homeomorphism $f_\sigma : \mathcal{R}_X \to \mathcal{R}_Y$ such that $f_\sigma|_\Gamma = \sigma$. The construction of f_σ away from Γ is simply that the annulus glued to a circuit $c \in \mathcal{C}_X$ is mapped to the annulus glued to the circuit $\sigma c \in \mathcal{C}_Y$ etc. That this is a homeomorphism is then clear. Moreover it is clear that f maps every flowline of X to a flowline of Y of the same type (as in periodic orbits go to periodic orbits, fixed points to fixed points etc.) and the orientation is preserved.

Thus, by definition 5.4.1 $X \sim Y$ $\square$

The following corollary is thus immediate.

Corollary 5.5.1. *Let $\Re$ be the set of all balanced ribboned graphs on $2g-2$ vertices. Then the number of different equivalence classes of minimal vector fields on Σ_g is equal to the number of equivalence classes of the relation $\approx$ in $\Re$, i.e.* $|\Re/\approx|$.

Chapter 6

On the Martin polynomial of a 2-regular digraph.

In chapter 3, we defined a polynomial of graphs, the Martin polynomial, first defined in [18]. However, previously this has only been used as a means to an end. In this chapter, we shall discuss the Martin polynomial as an end in itself. We shall aim to show that it is a polynomial that encodes a large amount of data of a 2-regular digraph. We shall also discuss the problem of deciding when a given polynomial is the Martin polynomial of some graph.

We begin by recalling a few basic definitions.

6.1 Basic definitions

Recall that a directed graph (or digraph) is a quadruple $\Gamma = (V, E, \iota, \tau)$ consisting of a set V of vertices, a set E that is projected onto a set of ordered pairs of elements of V and two maps, ι and τ, the initial and terminal maps respectively.

Recall the definition of the Directed Martin Polynomial (or the Martin Polynomial, when there is no confusion) from chapter 3, Definition 3.5.2.

Definition 6.1.1 (Directed Martin Polynomial). The *Directed Martin Polynomial* of a 2-regular digraph Γ is a polynomial $m'(\Gamma; \tau)$ defined on Γ as follows:

1. If v is not a cut-vertex of Γ, then $m'(\Gamma; \tau) = m'(\Gamma_1, \tau) + m'(\Gamma_2, \tau)$, where Γ_1, Γ_2 are the results of the two possible splittings at v.

2. If v is a cut-vertex of Γ and Γ_1, Γ_2 are the components of Γ formed by this cut, then $m'(\Gamma; \tau) = \tau m'(\Gamma_1, \tau) m'(\Gamma_2, \tau)$

3. If L is a free loop (i.e. the connected graph on no vertices) then $m'(L, \tau) = 1$.

The following lemma of Las Vergnas [15] (also quoted by Jaeger [14] as proposition 2) gives us a second formulation for the Martin polynomial which will suffice for most of what follows.

Lemma 6.1.2 (Las Vergnas 1983). *For a 2-regular digraph Γ let $f_r(\Gamma)$ be the number of circuit decompositions of Γ with r circuits. Then*

$$m'(\Gamma; \tau + 1) = \sum_{r \geq 0} f_{r+1}(\Gamma)\tau^r$$

Definition 6.1.3. Given a polynomial $p(\tau)$ we say that the graph Γ *expresses* $p(\tau)$ if

$$m'(\Gamma; \tau) = p(\tau)$$

In this case Γ is *an expression of $p(\tau)$ as a Martin polynomial.*

Note that Γ need not be a unique expression of $p(\tau)$. For example, the two graphs shown in Figure 6.2 both have Martin polynomial $4\tau^2$ but are clearly non-isomorphic graphs.

Notice that in the definition of the Martin polynomial we have two rules, 1 and 2 such that the first can only be used on a non-cut vertex, and the second can only be used on a cut vertex. However, if we also extend the Martin polynomial by adding the following rule:

If Γ_1 and Γ_2 are two distinct graphs and $\Gamma_1 \cup \Gamma_2$ are the disjoint union of them, then we shall expand the above definition of the Martin polynomial by introducing a second variable σ and added the rule:

$$m'(\Gamma_1 \cup \Gamma_2; \tau, \sigma) = \sigma m'(\Gamma_1; \tau, \sigma) m'(\Gamma_2, \tau, \sigma)$$

So we can apply the first rule to all vertices. However, as we have shown in proposition 3.5.3, part 6, this only makes sense if $\sigma = \tau - 1$. So the extra variable σ is dependent on τ and the Martin polynomial will be shown to be truly a polynomial in one variable.

In chapter 3 we showed that the Martin polynomial is a transition polynomial. Previous work by Ellis-Monaghan [6] showed that the undirected Martin polynomial is a translation of a Hopf map. As a consequence of this she was able to construct an iterative relation for the undirected Martin polynomial on an expanded class of graphs. However the proof involved Hopf algebras. We given an analogous iterative result here for the directed Martin polynomial, which we have proved in a much more direct way. We then use this iterative result to obtain a series of results directly analogous to those obtained by Ellis-Monaghan for the undirected Martin polynomial.

It can be shown that the directed Martin polynomial is also a translation of a Hopf map, in a directly analogous way to the method used by Ellis-Monaghan. However, we do not do this here.

6.2 An iterative formula for the Martin polynomial

By this we mean that we have the following result; for all τ,

$$m'(\Gamma;\tau) = \frac{\tau-1}{4}\sum_{A\in U(\Gamma)} m'(\Gamma|_A;\frac{\tau+1}{2})m'(\Gamma|_{A^c};\frac{\tau+1}{2})$$

where $U(\Gamma)$ will be defined.

This result is an analogue of a result on the undirected Martin polynomial, proved by Ellis-Monaghan ([6]) using Hopf algebras.

Definition 6.2.1. Define a polynomial K to be

$$K(\Gamma;\tau) = \sum_{n\geq 1} f_n(\Gamma)\tau^n$$

where $f_n(\Gamma)$ is (as above) the number of circuit decompositions of Γ of size n.

By convention we define $K(E;\tau) = 1$ where $E = (\emptyset, \emptyset)$, the empty graph. This convention allows for the following:

$$f_0(\Gamma) = \begin{cases} 1 & \Gamma = E \\ 0 & \Gamma \neq E \end{cases}$$

So the above definition can now be extended to

$$K(\Gamma;\tau) = \sum_{n\geq 0} f_n(\Gamma)\tau^n$$

Before we continue, we shall need the following well-known claim on multiplying polynomials.

Claim 6.2.2. *For any two power series we have*

$$\left(\sum_{r\geq 0} b_r\tau^r\right)\left(\sum_{s\geq 0} c_s\tau^s\right) = \sum_{n\geq 0}\tau^n\sum_{j=0}^{n} b_{n-j}c_j$$

Proof: Simply compare the coefficients of τ^n. □

This polynomial, whilst not having the explicit recursive definition of the Martin polynomial, has two important properties, as the following lemma and theorem now show.

Lemma 6.2.3. *Suppose $\Gamma_1 \cup \Gamma_2$ is the disjoint union of two distinct graphs. Then*

$$K(\Gamma_1 \cup \Gamma_2; \tau) = K(\Gamma_1; \tau) K(\Gamma_2; \tau)$$

Proof: The key to this proof is seeing that any circuit decomposition $\mathcal{C}$ of size n on $\Gamma_1 \cup \Gamma_2$ splits into two circuit decompositions $\mathcal{C}_1$ and $\mathcal{C}_2$ on Γ_1 and Γ_2 respectively of total size n. Moreover, given two such circuit decompositions $\mathcal{C}_i$ on Γ_i $(i = 1, 2)$ we have a circuit decomposition on $\Gamma_1 \cup \Gamma_2$. Thus for any n.

$$f_n(\Gamma_1 \cup \Gamma_2) = \sum_{r=0}^{n} f_r(\Gamma_1) f_{n-r}(\Gamma_2)$$

The proof then proceeds as follows.

$$\begin{aligned}
RHS &= \left(\sum_{r\geq 0} f_r(\Gamma_1)\tau^r\right)\left(\sum_{s\geq 0} f_s(\Gamma_2)\tau^s\right) \\
&= \sum_{n\geq 0}\sum_{r=0}^{n} f_r(\Gamma_1) f_{n-r}(\Gamma_2)\tau^n \\
&= \sum_{n\geq 0} f_n(\Gamma_1 \cup \Gamma_2)\tau^n \\
&= LHS
\end{aligned}$$

as required. Claim 6.2.2 was used at the second step. □

Given $A \subset E_\Gamma$, define $\Gamma|_A = (V, A, \iota|_A, \tau|_A)$.

Theorem 4.

$$K(\Gamma; 2\tau) = \sum_{A\in U(\Gamma)} K(\Gamma|_A; \tau) K(\Gamma|_{A^c}; \tau)$$

where $U(\Gamma) = \{A \subset E_\Gamma : \Gamma|_A$ and $\Gamma|_{A^c}$ are both Eulerian digraphs $\}$.

Proof: From Definition 6.2.1 we have that the right hand side of the above equation is

$$\begin{aligned}
RHS &= \sum_{A\in U(\Gamma)} \left(\sum_{n\geq 0} f_n(\Gamma|_A)\tau^n\right)\left(\sum_{r\geq 0} f_r(\Gamma|_{A^c})\tau^r\right) \\
&= \sum_{A\in U(\Gamma)} \sum_{n\geq 0} \tau^n \sum_{r=1}^{n} f_{n-r}(\Gamma|_A) f_r(\Gamma|_{A^c}) \\
&= \sum_{n\geq 0} \tau^n \sum_{A\in U(\Gamma)} \sum_{r=0}^{n} f_{n-r}(\Gamma|_A) f_r(\Gamma|_{A^c})
\end{aligned}$$

where we have used claim 6.2.2 at the second step. So, if it can be shown that

$$\sum_{A \in U(\Gamma)} \sum_{r=0}^{n} f_{n-r}(\Gamma|_A) f_r(\Gamma|_{A^c}) = 2^n f_n(\Gamma)$$

then we would be done.

The key point here, however, is observing that what we are counting is the size of the set

$$\begin{aligned} &\{(A, \mathcal{C}) : \mathcal{C} \text{ is a circuit decomposition of } \Gamma \text{ such that } |\mathcal{C}| = n \text{ and } A \subset \mathcal{C}\} \\ = &\{(A, \mathcal{C}) : A \text{ is a set of circuits of } \Gamma \text{ and } \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \text{ where} \\ &\mathcal{C}_1 \text{ is a circuit decomposition of } \Gamma|_A \text{ and} \\ &\mathcal{C}_2 \text{ is a circuit decomposition of } \Gamma|_{A^c}\} \end{aligned}$$

It is clear that both of the above descriptions describe the same set and that

$$|\{(A, \mathcal{C}) : \mathcal{C} \text{ is a circuit decomposition of } \Gamma,\ |\mathcal{C}| = n \text{ and } A \subset \mathcal{C}\}| = 2^n f_n(\Gamma)$$

We can then notice that

$$\begin{aligned} \sum_{A \in U(\Gamma)} \sum_{r=0}^{n} f_{n-r}(\Gamma|_A) f_r(\Gamma|_{A^c}) = \\ &|\{(A, \mathcal{C}) : A \text{ is a set of circuits of } \Gamma \text{ and } \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \\ &\quad \text{where } \mathcal{C}_1 \text{ is a circuit decomposition of } \Gamma|_A \text{ and} \\ &\quad\quad \mathcal{C}_2 \text{ is a circuit decomposition of } \Gamma|_{A^c}\}| \end{aligned}$$

which, from above

$$\begin{aligned} &=|\{(A, \mathcal{C}) : \mathcal{C} \text{ is a circuit decomposition of } \Gamma \text{ of size } n \text{ and } A \subset \mathcal{C}\}| \\ &=2^n f_n(\Gamma) \end{aligned}$$

□

It would be a fair question at this point to ask if K is related to m' in any way. It is, as the following lemma shows.

Lemma 6.2.4.

$$K(\Gamma; \tau) = \tau m'(\Gamma; \tau + 1)$$

Proof: Simply compare the definition of K given in Definition 6.2.1 and Lemma 6.1.2. □

Corollary 6.2.5.

$$m'(\Gamma_1 \cup \Gamma_2; \tau) = (\tau - 1)m'(\Gamma_1; \tau)m'(\Gamma_2; \tau)$$

Proof: Simply substitute Lemma 6.2.4 into Lemma 6.2.3. □

Corollary 6.2.6.

$$m'(\Gamma; \tau) = \frac{\tau - 1}{4} \sum_{A \in U(\Gamma)} m'(\Gamma|_A; \frac{\tau + 1}{2})m'(\Gamma|_{A^c}; \frac{\tau + 1}{2})$$

Proof: Simply substitute Lemma 6.2.4 into Theorem 4 □

Definition 6.2.7. Recall we defined E to be the empty 2-regular digraph, so $K(E; \tau) = 1$. Then by Lemma 6.2.4 we see that

$$m'(E; \tau) = \frac{1}{\tau - 1}$$

and the following is an immediate consequence of Corollary 6.2.6

Corollary 6.2.8.

$$m'(\Gamma; \tau) = m'(\Gamma; \frac{\tau + 1}{2}) + \frac{\tau - 1}{4} \sum_{A \in U(\Gamma) A \neq E, \Gamma} m'(\Gamma|_A; \frac{\tau + 1}{2})m'(\Gamma|_{A^c}; \frac{\tau + 1}{2})$$

With corollaries 6.2.6 and 6.2.8 as tools we can construct many new results on the Martin polynomials of 2-regular digraphs. To do this, however, we need to recall the following results:

Lemma 6.2.9. *For any 2-regular digraph $\Gamma = (V_\Gamma, E_\Gamma)$,*

$$m'(\Gamma; 2) = 2^n$$

where $n = |V_\Gamma|$, the number of vertices in Γ.

Proof: From Lemma 6.1.2 we can see that $m'(\Gamma; 2) = \sum_{r \geq 0} f_r(\Gamma)$, i.e. that $m'(\Gamma; \tau)$ is equal to the total number of circuit decompositions of Γ. But as a circuit decomposition requires choosing one of two possible local orientations at each vertex, there are precisely 2^n such circuit decompositions. Hence result. □

Lemma 6.2.10. *[18] For any 2-regular digraph $\Gamma = (V_\Gamma, E_\Gamma)$,*

$$m'(\Gamma; -1) = (-1)^n(-2)^\lambda$$

where $n = |V_\Gamma|$, the number of vertices in Γ and λ is the number of anti-circuits in Γ.

Note that an *anti-circuit* is define to be a circuit for which the directions alternate, i.e. an element of the circuit decomposition consisting of the incoherent transitions.

Proof: This is a lemma of Martin [18], although also quoted by Jaeger [14] as proposition 3. □

Proposition 6.2.1. *Let $\nu(A)$ = the number of 2-regular vertices in $\Gamma|A$ where $A \in U(\Gamma)$ and Γ is a 2-regular digraph. Then*

$$m'(\Gamma;3) = \frac{1}{2}\sum_{A\in U(\Gamma)} 2^{\nu(\Gamma|_A)+\nu(\Gamma|_{A^c})}$$

This is in fact a special case of the more general Proposition 6.2.3, below.

Definition 6.2.11. Define $\mathcal{E}_{2^k}(\Gamma)$, the set of *Eulerian edge 2^k-partitions of* Γ, as

$$\mathcal{E}_{2^k}(\Gamma) = \{a = (a_1, \ldots, a_{2^k}) : \text{ for each } a_i\ \Gamma|_{a_i} \in U(\Gamma) \bigcup_{i=1}^{2^k} a_i = V_\Gamma\}$$

and let $N(a) = \sum_{i=1}^{2^k} \nu(a_i)$

Then

Proposition 6.2.2.

$$m'(\Gamma; 1+2^k) = 2^{-k} \sum_{a\in\mathcal{E}_{2^k}(\Gamma)} 2^{N(a)}$$

Proof: The proof is by induction on k. Firstly, note that the case when $k = 0$ is covered by Lemma 6.2.9. Now, suppose that the statement is true for k. By Corollary 6.2.6 we have

$$\begin{aligned}
m'(\Gamma; 1+2^{k+1}) &= 2^{k-1} \sum_{A\in U(\Gamma)} m'(\Gamma|_A; 1+2^k) m'(\Gamma|_{A^c}; 1+2^k) \\
&= 2^{k-1} \sum_{A\in U(\Gamma)} \left(2^{-k} \sum_{b\in\mathcal{E}_{2^k}(\Gamma|_A)} 2^{N(b)}\right)\left(2^{-k} \sum_{c\in\mathcal{E}_{2^k}(\Gamma|_{A^c})} 2^{N(c)}\right) \\
&= 2^{k-1} \sum_{A\in U(\Gamma)} 2^{-2k} \sum_{(b,c)\in\mathcal{E}_{2^k}(\Gamma|_A)\times\mathcal{E}_{2^k}(\Gamma|_{A^c})} 2^{N(b)+N(c)}
\end{aligned}$$

Note, however, that choosing $A \in U(\Gamma)$ and then choosing (b, c) is the same as choosing $a \in \mathcal{E}_{2^{k+1}}(\Gamma)$ and splitting it into two sets, b, c as we can identify $(b,c) \in \mathcal{E}_{2^k}(\Gamma|_A) \times \mathcal{E}_{2^k}(\Gamma|_{A^c})$ with $a \in \mathcal{E}_{2^{k+1}}(\Gamma)$ via the identification

$$a_i = \begin{cases} b_i & \text{for } i = 1, \ldots, 2^k \\ c_{i-2^k} & \text{for } i = 2^k+1, \ldots, 2^{k+1}. \end{cases}$$

Under this identification $N(a) = N(b) + N(c)$, and so

$$m'(\Gamma; 1 + 2^{k+1}) = 2^{-(k+1)} \sum_{a \in \mathcal{E}_{2^{k+1}}(\Gamma)} 2^{N(a)}$$

as required. □

Definition 6.2.12. Define the set $\mathcal{V}_{2^k}(\Gamma)$, the set of *cyclic edge 2^k-partitions of Γ*, as

$$\mathcal{V}_{2^k}(\Gamma) = \{a = (a_1, \ldots, a_{2^k}) : \text{for each } a_i\ \Gamma|_{a_i} \text{ is a disjoint union of cycles and } \bigcup_{i=1}^{2^k} a_i = V_\Gamma\}$$

Now, for some $a \in V_{2^k(\Gamma)}$ define $q(a) = \sum_{i=1}^{2^k} K(a_i)$ where $K(a_i) =$ the number of connected components of $\Gamma|_{a_i}$. Then

Proposition 6.2.3.

$$m'(\Gamma; 1 - 2^k) = -2^{-k} \sum_{a \in \mathcal{V}_{2^k}(\Gamma)} (-1)^{q(a)}$$

Proof: Again, the proof is by induction on k.

We first need to consider one exceptional case, when Γ is a single circuit (i.e. the vertex set of Γ is empty). Then $m'(\Gamma; \tau) = 1$ and for any k $|\mathcal{V}_{2^k}(\Gamma)| = 2^k$ and for all $a \in \mathcal{V}_{2^k}(\Gamma)$, $q(a) = 1$. Hence the statement is true.

For non-trivial Γ, the base step (i.e. $k = 0$) is vacuously true and the rest of the proof is analogous to the proof of proposition 6.2.2, the induction step being almost identical. □

At this point we may make an observation based on this proposition and Lemma 6.2.10 as follows.

Lemma 6.2.13. *Let $D(\Gamma) = \{A \subset E_\Gamma : \Gamma|_A$ and $\Gamma|_{A^c}$ are disjoint union of cycles$\}$ and define $\kappa(A) =$ the total number of connected components in $\Gamma|_A$ and $\Gamma|_{A^c}$. Then*

$$(-1)^n(-2)^\lambda = \sum_{A \in D(\Gamma)} (-1)^{\kappa(A)}$$

Proof: Notice that $D(\Gamma)$ is just $\mathcal{V}_2(\Gamma)$ and $\kappa(A)$ is just $q((A, A^c))$. Then this is just the equating of Lemma 6.2.10 and proposition 6.2.3. □

Proposition 6.2.4.

$$m'(\Gamma; 1-2^k) = 2^{-k} \sum_{a \in \mathcal{E}_{2^k}(\Gamma)} (-1)^{N(a)}(-2)^{\Lambda(a)-1}$$

where $\lambda(\Gamma)$ is the number of anti-circuits of Γ and $\Lambda(a) = \sum_{i=1}^{2^k} \lambda(\Gamma|_A)$.

Proof: The proof is again by induction on k and again is very similar to the proof of 6.2.2. In this case, the base step is covered by Lemma 6.2.10. □

6.3 The Martin polynomial of an incidence matrix

In this section we will define the Martin polynomial on the incidence matrix of a graph. We will do this by constructing the equivalent recursive rules for a matrix. But in order to retain many of the characteristics of the original Martin polynomial we will be forced to introduce certain restrictions. These restrictions will all have counter-parts in the graphical case, including the rule, previously observed in chapter 3, proposition 3.5.3, part 6, that the disjoint union of two graphs Γ_1 and Γ_2 has Martin polynomial $(\tau - 1)m'(\Gamma_1; \tau)m'(\Gamma_2; \tau)$.

Definition 6.3.1. An *incidence matrix* of a graph Γ is an $n \times n$ matrix M_Γ defined so that the $(i, j)-$th entry $M_{i,j}$ is the number of edges of Γ of the form (v_i, v_j). So for a 2−regular digraph, the sum of each row and column of the incidence matrix is 2.

Conversely, any matrix with non-negative integer entries such that the sum of each row and column is 2 is the incidence matrix of some 2−regular digraph.

It is clear that if we relabel the vertex set, provided we also consistently relabel the edge set, we change the incidence matrix without changing the underlying graph. Thus the incidence matrix is defined only up to the following equivalence.

$$M \sim N \text{ if and only if there exists } S \in \text{Sym}(n) \text{ such that } S^{-1}MS = N.$$

where $\text{Sym}(n)$ is the symmetric group on n elements.

Definition 6.3.2. We may now define the Martin polynomial, $m' \in \mathbb{Z}[\tau, \sigma]$, of the incident matrix of a 2−regular digraph Γ. The polynomial is defined on variables τ, σ in a recursive manner using the following rules, in which the initial matrix is an $n \times n$ incidence matrix.

1.

$$m'\left(\begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{bmatrix}, \tau, \sigma\right) = \begin{cases} \tau\sigma\, m'(A,\tau,\sigma) & \text{if } A \neq \emptyset \\ \tau & \text{otherwise} \end{cases}$$

2.

$$m'\left(\begin{bmatrix} 1 & e_i \\ e_j^T & A \end{bmatrix}, \tau, \sigma\right) = \tau\, m'\left(A + E_{i,j}, \tau, \sigma\right)$$

3.

$$m'\left(\begin{bmatrix} 0 & e_i + e_j \\ e_k^T + e_l^T & A \end{bmatrix}, \tau, \sigma\right) = m'\left(A + E_{i,k} + E_{j,l}, \tau, \sigma\right) + m'\left(A + E_{i,l} + E_{j,k}, \tau, \sigma\right)$$

4.

$$m'\left(\begin{bmatrix} 0 & 2e_i \\ e_k^T + e_l^T & A \end{bmatrix}, \tau, \sigma\right) = 2\, m'\left(A + E_{i,k} + E_{i,l}, \tau, \sigma\right)$$

5.

$$m'\left(\begin{bmatrix} 0 & e_i + e_j \\ 2e_k^T & A \end{bmatrix}, \tau, \sigma\right) = 2\, m'\left(A + E_{i,k} + E_{j,k}, \tau, \sigma\right)$$

6.

$$m'\left(\begin{bmatrix} 0 & 2e_i \\ 2e_k^T & A \end{bmatrix}, \tau, \sigma\right) = 2\, m'\left(A + 2E_{i,k}, \tau, \sigma\right)$$

where e_i is the $n-1$ length row matrix with $j-th$ entry, $\delta_{i,j}$, the Kronecker delta, and $E_{i,j}$ is the $n-1$ square matrix with k,l-th entry $\delta_{i,k}\delta_{j,l}$ and A is a $n-1$ square matrix with non-negative integer coefficients.

Lemma 6.3.3. *$M \sim N$ implies $m'(M,\tau,\sigma) = m'(N,\tau,\sigma)$ if and only if $\sigma = \tau - 1$.*

Proof:

We shall prove the only if direction by an example. Consider the graph on 5 vertices shown in Figure 6.1. Two possible incidence matrices for this graph are

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

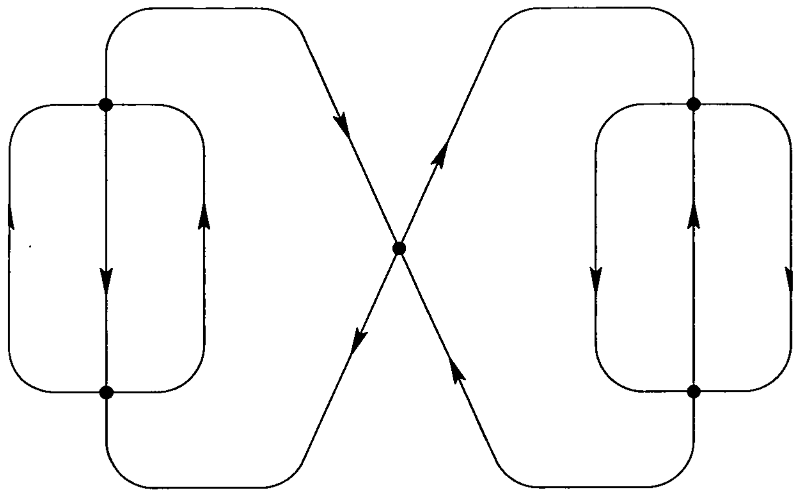


Figure 6.1: The graph mentioned in Lemma 6.3.3

Now we can calculate the Martin polynomial for each of the above matrices.

$$
\begin{aligned}
m'\left(\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \tau, \sigma\right) &= 2m'\left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \tau, \sigma\right) && \text{(by rule 5)} \\
&= 2\tau m'\left(\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \tau, \sigma\right) && \text{(by rule 2)} \\
&= 4\tau m'\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tau, \sigma\right) && \text{(by rule 4)} \\
&= 4\tau^2 m'\left(\begin{bmatrix} 2 \end{bmatrix}, \tau, \sigma\right) && \text{(by rule 2)} \\
&= 4\tau^3 && \text{(by rule 1)}
\end{aligned}
$$

and

$$
\begin{aligned}
m'\left(\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \tau, \sigma\right) &= m'\left(\begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \tau, \sigma\right) \\
&\quad + m'\left(\begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}, \tau, \sigma\right) && \text{(by rule 3)} \\
&= 2m'\left(\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \tau, \sigma\right) \\
&\quad + 2m'\left(\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \tau, \sigma\right) && \text{(by rules 4 and 6)} \\
&= 4m'\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tau, \sigma\right) \\
&\quad + 4m'\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \tau, \sigma\right) && \text{(by rule 4)} \\
&= 4\tau m'\left(\left[2\right], \tau, \sigma\right) \\
&\quad + 4\tau\sigma m'\left(\left[2\right], \tau, \sigma\right) && \text{(by rules 2 and 1)} \\
&= 4\tau^2\sigma + 4\tau^2 && \text{(by rule 1)}
\end{aligned}
$$

Now, as these are equivalent matrices, we require that the Martin polynomials coincide, i.e. that

$$4\tau^3 = 4\tau^2\sigma + 4\tau^2$$

that is,

$$\sigma = \tau - 1$$

as required.

The proof for the other direction relies on the fact that if $\sigma = \tau - 1$ then the above rules correspond to the graphical rules for the Martin polynomial given in chapter 3, defined using relations on graphs. So, if $M(\Gamma)$ is the incidence matrix of a labeled 2−regular digraph Γ then

$$m'(M(\Gamma), \tau) = m'(\Gamma; \tau)$$

and that if Γ_1 and Γ_2 are two labeled 2−regular digraphs that differ only in the labeling of the vertices, then

$$m'(\Gamma_1, \tau) = m'(\Gamma_2, \tau)$$

i.e. the Martin polynomials coincide for different labelings of the same graph. So the Martin polynomial is defined for unlabeled 2-regular digraphs.

Thus if $M \sim N$ then M, N are incidence matrices of two graphs that are merely two different labelings of some graph Γ and we have

$$m'(M(\Gamma), \tau) = m'(\Gamma; \tau) = m'(N(\Gamma), \tau)$$

as required. $\square$

Whilst this is of interest, it should be noted that the Martin polynomial on its own cannot tell us much about the dual circuit decompositions of a given graph. Indeed, it is possible for two distinct graphs, with different balanced circuit decompositions, to have the same Martin polynomial, as the following example shows.

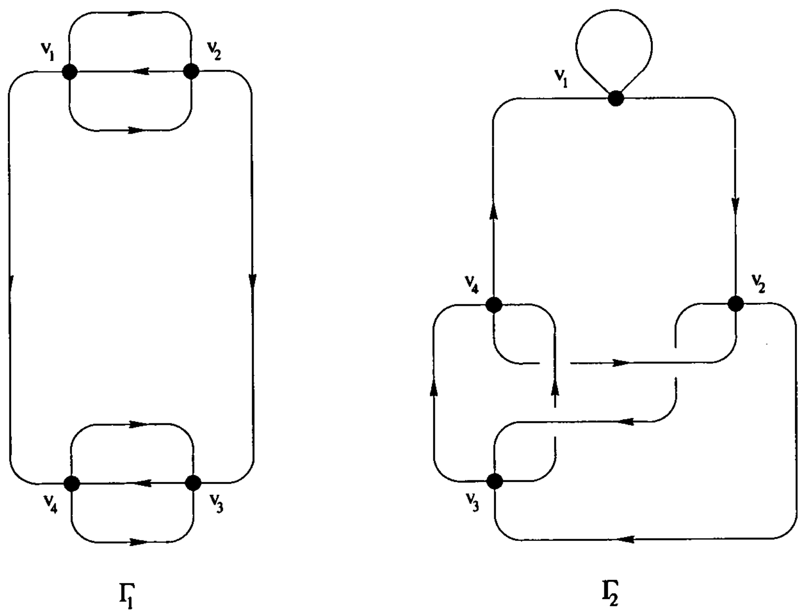


Figure 6.2: The graphs Γ_1 and Γ_2 for example 6.3.4

Example 6.3.4. The two graphs Γ_1 and Γ_2 are shown in Figure 6.2. The incident matrices are, respectively

$$\begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

and so it is easy to see that the Martin polynomial for each graph is

$$m'(\Gamma_i, \tau) = 4\tau^2$$

Now, we may deduce from the Martin polynomial that for both graphs there are 4 circuit decompositions of size 3, 8 of size 2 and 4 of size 1.

However, in Γ_1 every circuit decomposition is balanced (i.e. has the same number of components as its dual decomposition) but in Γ_2 only those circuits of size 2 are balanced.

Hence we can see that the Martin polynomial does not contain enough information to distinguish balanced or dual decompositions. □

6.4 An axiomatic discussion of the Martin polynomial

In this section we shall prove that the Martin polynomial is the only polynomial of graphs that satisfies a particular set of conditions. We begin with some definitions of notation used in this section.

Definition 6.4.1. The graph $\times$ used in the following theorem indicates a vertex within some graph Γ and $||$ and $=$ are the results of the two possible splittings at that vertex.

$\cup$ indicates disjoint union.

The graph $\Gamma_1 \times \Gamma_2$ is such that one of the possible splittings at the vertex marked by $\times$ splits the graph into two disjoint components Γ_1 and Γ_2.

To define the graph $\propto \Gamma$ suppose $\Gamma = (V, E, \iota, \tau)$, where $V = \{v_1, \dots, v_n\}$ and $e_j \in E$. Then $\propto \Gamma = (V \cup \{v_0\}, E \backslash \{e_j\} \cup \{(\iota e_j, v_0), (v_0, v_0), (v_0, \tau e_j)\}, \iota', \tau')$ where ι' and τ' are the obvious extensions of ι and τ.

The graph $\Gamma_1 = \Gamma_2$ is a graph such that there exists two edges e_i, e_2 such that the removal of these edges separates the graph into two distinct subgraphs, Γ_1 and Γ_2. Moreover, one edge has direction from Γ_1 to Γ_2 and the other has the other direction. Two examples of such a graph is shown in Figure 6.3.

Recall, the graph L is the graph on a single loop, with no vertices, i.e. $L = (\emptyset, \{\emptyset\})$. This is strictly different from the empty graph $E = (\emptyset, \emptyset)$.

These definitions are made more explicit by Figure 6.4.

Theorem 6.4.2. *If $p(\Gamma)$ is a polynomial on graphs in variables $\alpha, \beta, \mu, \delta, \tau, \phi$ that satisfies the following properties*

Property 1 $p(\times) = \alpha p(||) + \beta p(=)$

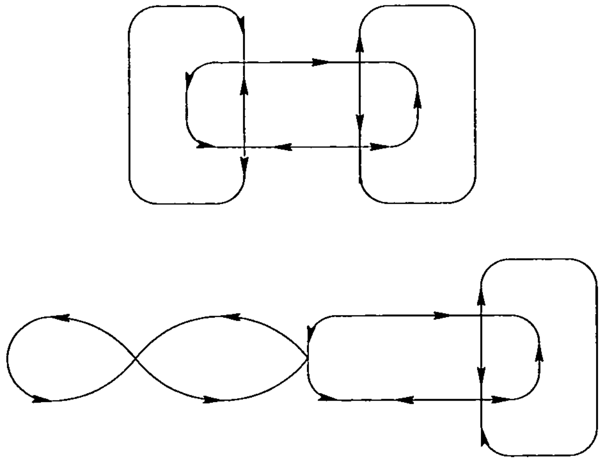


Figure 6.3: Two examples of a graph that can be written as $\Gamma_1 = \Gamma_2$.

Property 2 $p(\Gamma_1 \cup \Gamma_2) = \mu p(\Gamma_1)p(\Gamma_2)$

Property 3 $p(\propto \Gamma) = \delta p(\Gamma)$

Property 4 $p(L) = 1$

Property 5 $p(\Gamma_1 \times \Gamma_2) = \tau p(\Gamma_1)p(\Gamma_2)$

Property 6 $p(\Gamma_1 = \Gamma_2) = \phi p(\Gamma_1)p(\Gamma_2)$

then $\alpha = \beta, \tau = \delta = \alpha(\mu + 1)$, $\phi = 1$ *and so* $p(\Gamma, \alpha, \beta, \mu, \delta, \tau, \phi) = \alpha^{|V_\Gamma|} m'(\Gamma, \tau)$. *The pictorial representation of these properties is given in Figure 6.4.*

Proof: We will first show that $\alpha = \beta$. But as the edges at the vertex in question are unlabeled this property must be symmetric. Hence $\alpha = \beta$. An example of this is shown in Figure 6.5.

We will now show that $\tau = \delta = \alpha(\mu + 1)$ Consider Figure 6.6. This shows a graph Γ containing a single vertex loop. However, we can equally view this as two graphs Γ_1 and Γ_2 joined by a cut point, where Γ_1 is a loop without vertices, i.e. $p(\Gamma_1) = 1$. Thus we have three potential ways of evaluating our polynomial on this graph, i.e. we can apply property 3, apply property 5 or we can use property 1.

In the first case we get that

$$p(\Gamma_1 \times \Gamma_2) = \delta p(\Gamma_2)$$

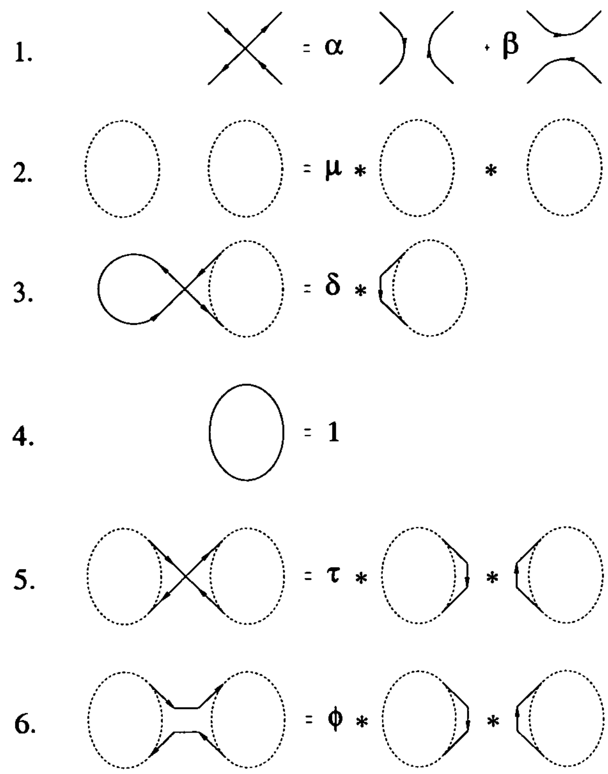


Figure 6.4: Pictorial axioms for the general polynomial $p(\Gamma, \alpha, \beta, \mu, \delta, \tau, \phi)$

In the second case we get that

$$\begin{aligned} p(\Gamma_1 \times \Gamma_2) &= \tau p(\Gamma_1)p(\Gamma_2) \\ &= \tau p(\Gamma_2) \end{aligned}$$

using property 4 at the last step. Thus we conclude that $\delta = \tau$.

And in the third case we get:

$$\begin{aligned} p(\Gamma_1 \times \Gamma_2) &= \alpha p(\Gamma_1 \cup \Gamma_2) + \alpha p(\Gamma_1 = \Gamma_2) \\ &= \alpha \mu p(\Gamma_2) + \alpha p(\Gamma_2) \\ &= \alpha(\mu + 1)p(\Gamma_2) \end{aligned}$$

using property 4 and the fact that $\Gamma_1 = \Gamma_2$ is just Γ_2 in this case. Thus $\tau = \delta = \alpha(\mu + 1)$.

Finally, we will prove that $\phi = 1$. Consider the graph shown in Figure 6.7. In this case we consider the situation with $\Gamma_1 \times \Gamma_2$ and evaluate in two ways, firstly using properties 1 and 6 and secondly using property 5. From this we get that

$$\begin{aligned} \tau p(\Gamma_1)p(\Gamma_2) &= \alpha p(\Gamma_1 \cup \Gamma_2) + \alpha p(\Gamma_1 = \Gamma_2) \\ &= \alpha \mu p(\Gamma_1)p(\Gamma_2) + \alpha \phi p(\Gamma_1)p(\Gamma_2) \end{aligned}$$

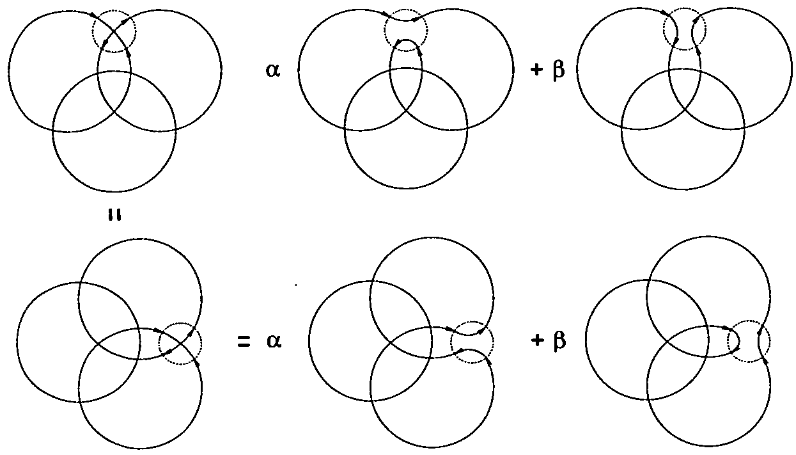


Figure 6.5: Demonstration why $\alpha = \beta$

But as $\tau = \alpha(\mu + 1)$ we must conclude that $\phi = 1$.

So we have shown that $\alpha = \beta$, $\tau = \delta = \alpha(\mu + 1)$ and $\phi = 1$. Now, notice that every time we apply an operation to remove a vertex, we multiply by α. Thus we can conclude that

$$\begin{aligned} p(\Gamma, \alpha, \beta, \mu, \delta, \tau, \phi) &= p(\Gamma, \alpha, \alpha, \mu, \alpha(\mu + 1), \alpha(\mu + 1), 1) \\ &= \alpha^{|V_\Gamma|} p(\Gamma, 1, 1, \tau - 1, \tau, \tau, 1) \end{aligned}$$

But by considering the properties of the Martin polynomial given in definition 6.1.1, Lemma 6.2.5 and chapter 3, proposition 3.5.3, section 5 we see that $p(\Gamma, 1, 1, \tau - 1, \tau, \tau, 1) = m'(\Gamma, \tau)$ as required. □

Thus we can now list the following as properties of the Martin polynomial.

Property 1 $p(\times) = p(||) + p(=)$

Property 2 $p(\Gamma_1 \cup \Gamma_2) = (\tau - 1)p(\Gamma_1)p(\Gamma_2)$

Property 3 $p(\propto \Gamma) = \tau p(\Gamma)$

Property 4 $p(L) = 1$

Property 5 $p(\Gamma_1 \times \Gamma_2) = \tau p(\Gamma_1)p(\Gamma_2)$

Property 6 $p(\Gamma_1 = \Gamma_2) = p(\Gamma_1)p(\Gamma_2)$

6.5 Characterising Martin polynomials?

In this section we will discuss the problem of finding a graph to express a given polynomial $p(\tau)$ as a Martin polynomial. We shall aim to show that this is a

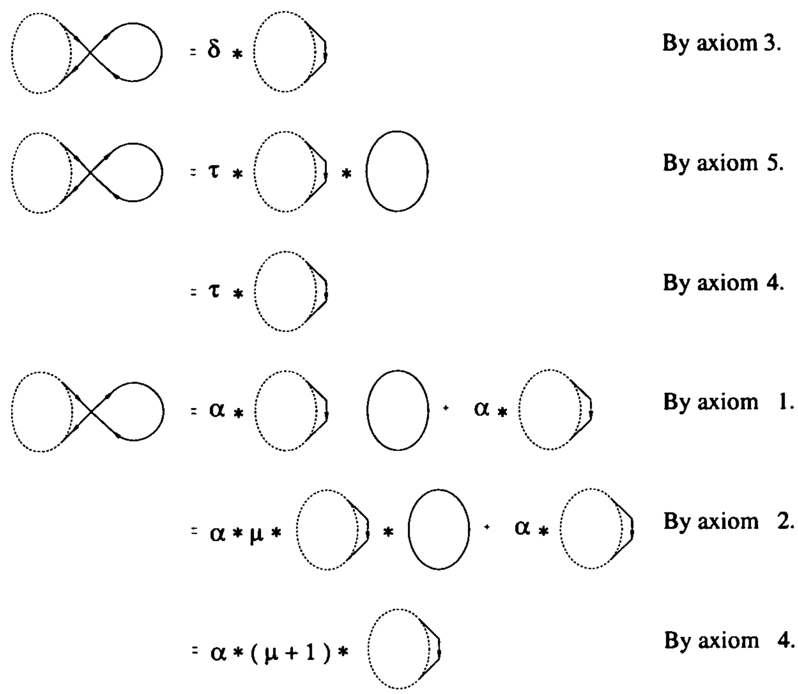


Figure 6.6: Pictorial equations showing $\tau = \delta = \alpha(\mu + 1)$.

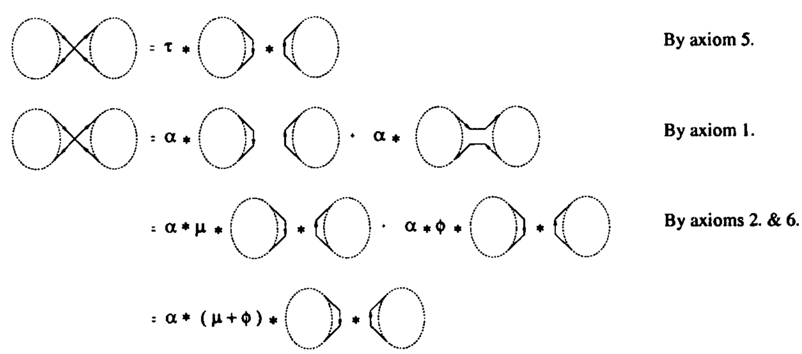


Figure 6.7: Pictorial equations showing $\phi = 1$.

difficult problem. After Lemma 6.2.4 we can see that this problem is equivalent to the graphical problem of constructing a 2-regular digraph that has f_j circuit decompositions of size j. But this is known to be a hard problem in graph theory.

The problem here is that even if we restrict the degree of a Martin polynomial the number of vertices required to express it has no bounds. For example, consider the following lemma.

Lemma 6.5.1. *For any n there exists a polynomial p of degree 1 and 2-regular digraph Γ on n vertices such that $p(\tau) = m'(\Gamma; \tau)$.*

Proof: Consider the operation on graphs shown in Figure 6.8. By considering the properties of the Martin polynomial discussed above, we can see that if we act on a vertex of a graph using the above operation, then we increase the number of vertices by one and double the Martin polynomial. Thus by $n-1$ application of the operation to the single vertex graph we obtain a graph Γ with n vertices and Martin polynomial $2^{n-1}\tau$, the polynomial of degree 1 as required.

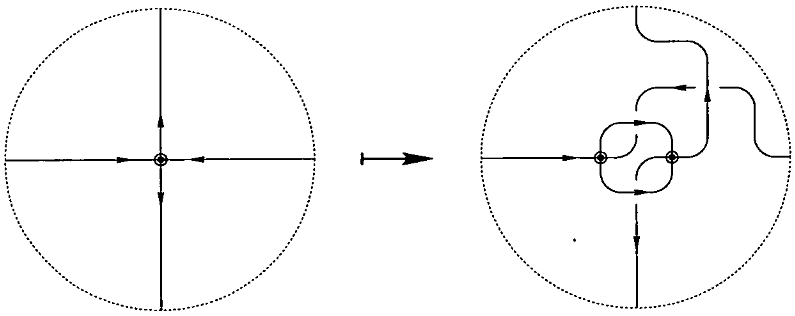


Figure 6.8: Operation on a graph Γ to double the Martin polynomial $m'(\Gamma;\tau)$.

□

So we can conclude that the degree of a Martin polynomial is no guide to the number of vertices of the graph required to express it. However, as we shall now show, it is always true that if we know that $p(\tau)$ is a Martin polynomial of a graph Γ, then we know that Γ has $\log_2(p(2))$ vertices.

So, we shall now consider a set of polynomials that has the Martin polynomials as a subset. Consider the following results on the Martin polynomial:

Lemma 6.5.2. *For Γ a 2-regular digraph on n vertices ($n > 0$), with λ anti-circuits the following are true:*

1. $m'(\Gamma;2) = 2^n$
2. $m'(\Gamma;1) = f_1(\Gamma)$
3. $m'(\Gamma;0) = 0$
4. $m'(\Gamma;-1) = (-1)^n(-2)^{\lambda-1}$

where, as before $f_1(\Gamma)$ is the number of Eulerian circuits of Γ.

Proof: Condition 1 is just Lemma 6.2.9. Similarly condition 4 is just Lemma 6.2.10.

Similarly, condition 2 is proven by putting $\tau = 0$ in Lemma 6.1.2.

To show that condition 3 is true, consider the definition of the Martin polynomial given in Definition 6.1.1. This shows that the Martin polynomial of the graph with one vertex is τ. But as the Martin polynomial is defined iteratively, we know that for any graph Γ on $n \geq 1$ vertices, the process that defines the polynomial will reduce Γ to a single vertex graph. Hence the Martin polynomial of such Γ will have no constant term. The exception to this occurs when Γ has no vertices in the first place, in which case the Martin polynomial is 1. But this case has been excluded from this lemma. □

Corollary 6.5.3. *The equivalent conditions for $K(\Gamma;\tau)$ are;*

1. $K'(\Gamma;1) = 2^n$
2. $K'(\Gamma;0) = 0$
3. $K'(\Gamma;-1) = 0$
4. $K'(\Gamma;-2) = (-1)^n(-2)^\lambda$

Proof: All these conditions following from the fact that $K(\Gamma;\tau) = \tau m'(\Gamma;\tau+1)$ and Lemma 6.5.2. □

It is now possible to construct the set of all polynomials with integer coefficients that satisfy the conditions shown in 6.5.2. However, we will initially construct the set of all polynomials with integer coefficients that satisfy the conditions shown in Corollary 6.5.3 as these are initially easier. Given this set, we will then be able to construct the set satisfying the conditions of Lemma 6.5.2.

Lemma 6.5.4. *Suppose $p(\tau)$ is a polynomial of degree n with integer coefficients with roots $1, 0, -1, -2$. Then*

$$p(\tau) \in P_n =< p_r(\tau) : r = 4, \ldots, n >$$

where

$$\begin{aligned} p_r(\tau) &= \tau^r + ((-2)^r - 3(-1)^r - 1)\frac{\tau^3}{6} \\ &+ ((-1)^r + 1)\frac{\tau^2}{2} \\ &+ (3(-1)^r - 1 + (-2)^{r-1})\frac{\tau}{3} \end{aligned}$$

Proof: The proof is by induction on n. For the base step, suppose $n = 4$. Then we need to show that any polynomial of degree 4 with roots $1, 0, -1, -2$ and with integer coefficients is an integer multiple of $p_4(\tau) = \tau^4 + 2\tau^3 - \tau^2 - 2\tau$. But

$$\begin{aligned} p_4(\tau) &= \tau^4 + 2\tau^3 - \tau^2 - 2\tau \\ &= \tau(\tau-1)(\tau+1)(\tau+2) \end{aligned}$$

so as the degree of p is 4, we can see that the base step is clear.

Now for the induction step. Suppose that the result is true for any $r < n$ and suppose $p(\tau)$ satisfies the conditions of the lemma with degree n and the coefficient of τ^n is $\alpha \in \mathbb{Z}$. Then as $p_n(1) = p_n(0) = p_n(-1) = p_n(-2) = 0$ we can see that $p(\tau) - \alpha p_n(\tau)$ is a polynomial of degree strictly less than n satisfying all the conditions of the lemma. Thus by induction $p(\tau) - \alpha p_n(\tau) \in P_{n-1}$, i.e.

$$p(\tau) - \alpha p_n(\tau) = \sum_{r=4}^{n-1} \alpha_r p_r(\tau)$$

for integers α_r. Hence

$$p(\tau) = \alpha p_n(\tau) + \sum_{r=4}^{n-1} \alpha_r p_r(\tau) \in P_n$$

as required. □

Corollary 6.5.5. *Suppose $p(\tau)$ is a polynomial with non-negative integer coefficients such that*

1. $p(1) = 2^n$
2. $p(0) = 0$
3. $p(-1) = 0$
4. $p(-2) = (-1)^n(-2)^\lambda$.

Then

$$p(\tau) = q(\tau) + \sum_{r=4}^{n} \alpha_r p_r(\tau)$$

where

$$q(\tau) = (2^n - (-1)^{n+\lambda}2^\lambda)\frac{\tau^3}{6} + 2^{n-1}\tau^2 + (2^{n+1} + (-1)^{n+\lambda}2^\lambda)\frac{\tau}{6}$$

where α_r are integers and $p_r(\tau)$ is defined in the previous lemma.

Proof: It is easy to see that $q(\tau)$ satisfies the 4 conditions given above. Thus $(p - q)(\tau)$ has roots $1, 0, -1, -2$ and so is in P_n. Then the result follows from Lemma 6.5.4. The coefficients α_r are necessarily integers and are chosen so that the coefficients of p are non-negative. □

Notice that the conditions used in Corollary 6.5.5 are precisely those conditions that Corollary 6.5.3 says are necessary for a polynomial to be the polynomial $K(\Gamma;\tau)$ for some graph Γ.

The analogous lemma for the Martin polynomial, proven in a directly analogous way is

Lemma 6.5.6. *Suppose we have a polynomial $p(\tau)$ satisfying the following conditions, for some integers n and λ:*

1. $p(2) = 2^n$
2. $p(0) = 0$
3. $p(-1) = (-1)^n(-2)^{\lambda-1}$

Then

$$p(\tau) = 2^{\lambda-1}\tau^{n-(\lambda-1)} + \sum_{j=3}^{n-\lambda+1} \alpha_j p_j(\tau)$$

where α_j are integers and

$$p_j(\tau) = \frac{1}{3}(3\tau^j - (2^{j-1} + (-1)^j)\tau^2 + (-2^{j-1} + 2(-1)^j)\tau)$$

So we now have a list of possible candidates for the polynomial $m'(\Gamma;\tau)$. We can further subdivide this list as follows. Let the set of all polynomials constructed in this way be $\mathcal{M}$. Let $\mathcal{M}_n = \{p \in \mathcal{M} : p(2) = 2^n\}$ Then

$$\mathcal{M} = \bigcup_{n\geq 0} \mathcal{M}_n$$

The list of $\mathcal{M}_n$ for $n \leq 5$ found using this method is given in table 6.1. The list of equivalent K polynomials is not given.

However, whilst we have shown that this list must contain all the Martin polynomials, the converse statement is not true. There are polynomials in this list for which no graph Γ exists such that $m'(\Gamma;\tau)$ is the polynomial. For example, the polynomial $\tau^4 + 2\tau^2 + 4\tau$ is one such polynomial as the following lemma now shows.

Lemma 6.5.7. *There does not exist a 2-regular digraph Γ such that $m'(\Gamma;\tau) = \tau^4 + 2\tau^2 + 4\tau = p(\tau)$*

n	$m'(\Gamma;\tau)$
1	τ
2	τ^2
	2τ
3	τ^3
	$\tau^2 + 2\tau$
	$2\tau^2$
	4τ
4	τ^4
	$3\tau^2 + 2\tau$
	$\tau^3 + 2\tau^2$
	$2\tau^3$
	$\tau^3 + \tau^2 + 2\tau$
	$2\tau^2 + 4\tau$
	$4\tau^2$
	8τ
5	τ^5
	$3\tau^3 + 2\tau^2$
	$2\tau^3 + 3\tau^2 + 2\tau$
	$\tau^3 + 4\tau^2 + 4\tau$
	$5\tau^2 + 6\tau$
	$\tau^4 + 2\tau^3$
	$\tau^4 + \tau^3 + \tau^2 + 2\tau$
	$\tau^4 + 2\tau^2 + 4\tau$
	$2\tau^4$
	$\tau^4 + \tau^3 + 2\tau^2$
	$\tau^4 + 3\tau^2 + 2\tau$
	$2\tau^3 + 4\tau^2$
	$6\tau^2 + 4\tau$
	$\tau^3 + 5\tau^2 + 2\tau$
	$4\tau^3$
	$3\tau^3 + \tau^2 + 2\tau$
	$2\tau^3 + 2\tau^2 + 4\tau$
	$\tau^3 + 3\tau^2 + 6\tau$
	$4\tau^2 + 8\tau$
	$8\tau^2$
	16τ

Table 6.1: The list of all possible Martin polynomials expressible by graphs with less than or equal to 5 vertices.

Proof: Observe that $p(2) = 2^5$ so if Γ exists then Γ has precisely 5 vertices. Suppose for a contradiction that Γ exists.

Recall from the properties discussed in Theorem 6.4.2 that a Martin polynomial can be constructed from Martin polynomials of smaller graphs in a finite number of ways. Moreover, all but one of these methods of building a Martin polynomial do so by multiplying Martin polynomials. But we can easily see that in this particular case, as the coefficient of τ is non-zero that in order for this to be a Martin polynomial it must be constructed from previous Martin polynomials by the means of property 1 alone. Recall also that in the case of property 1 the graphs which are "added" to make Γ have precisely one less vertex. So we now consider possible pairs of Martin polynomials q_1, q_2 such that $q_1(2) = q_2(2) = 2^4$. But it is easy to see that in this case the only possible option is $\{q_1, q_2\} = \{\tau^4, 2\tau^2 + 4\tau\}$.

Now the Martin polynomial $2\tau^2 + 4\tau$ has a unique expression (i.e. there is a unique graph Γ_1 such that $m'(\Gamma_1; \tau) = 2\tau^2 + 4\tau$) and there are precisely three possible graphs $\Gamma_2^1, \Gamma_2^2, \Gamma_2^3$ expressing τ^4. All these graphs are shown in Figure 6.9. However, inspection shows that it is impossible to construct Γ out of any pair Γ_1, Γ_2^i. Hence result. □

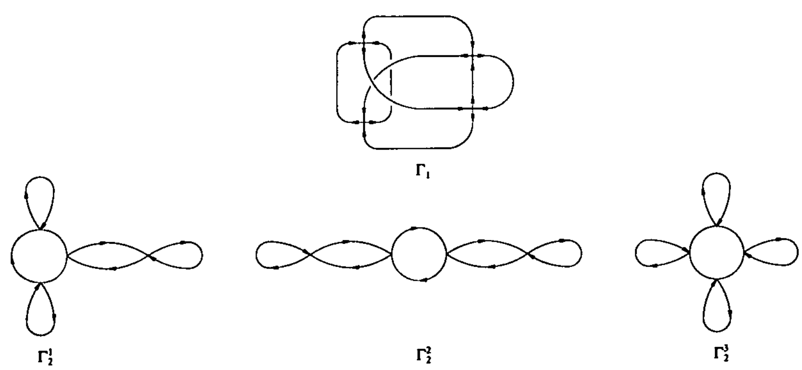


Figure 6.9: The graphs $\Gamma_1, \Gamma_2^1, \Gamma_2^2, \Gamma_2^3$ used in Lemma 6.5.7.

Thus we can see that although our set is necessary, it is not sufficient. Moreover, although we do have other results which could further allow us to restrict the size of the set, for example the results found using the inductive properties of the Martin polynomial, in practice it would appear that as each of these newer results is dependent on results already included, or requires detailed information about the graph concerned, we cannot further restrict our list of possible Martin polynomials.

However, we can use this set to make the following observations:

Claim 6.5.8. *For any n there is a unique Martin polynomial p_n such that $p_n(2) = 2^n$ and $p_n(-1) = (-1)^n(-2)^{n-1}$.*

For any $n > 2$ there is a unique Martin polynomial q_n such that $q_n(2) = 2^n$ and $q_n(-1) = (-1)^n(-2)^{n-2}$

Proof: $p_n(\tau) = 2^{n-1}\tau$. $q_n(\tau) = 2^{n-2}\tau^2$. That these are Martin polynomials follows from respectively $n-1$ and $n-2$ applications of the operation shown in Figure 6.8 to the unique graphs that express the polynomials τ and τ^2.

Uniqueness follows by considering that both p_n and q_n must be in $\mathcal{M}_n$. Now recall that for a polynomial p to be in $\mathcal{M}_n$ means that it can be expressed in the form

$$p(\tau) = 2^{\lambda-1}\tau^{n-(\lambda-1)} + \sum_{j=3}^{n-\lambda+1} \alpha_j p_j(\tau)$$

where α_j are integers and

$$p_j(\tau) = \frac{1}{3}(3\tau^j - (2^{j-1} + (-1)^j)\tau^2 + (-2^{j-1} + 2(-1)^j)\tau)$$

But for $\lambda > n-2$ the sum is empty and hence the polynomial is unique. □

One other option available is to consider the polynomials constructed using the axioms for the Martin polynomial given in 6.1.1. By considering these as rules for building polynomials we have the following set

Definition 6.5.9. The set $\mathcal{P}$ is defined to be the smallest set of polynomials such that

1. $\tau \in \mathcal{P}$.
2. If $p_1, p_2 \in \mathcal{P}$ such that $p_i(-1) = (-1)^{n-1}(-2)^{\lambda_i}$ and $|\lambda_1 - \lambda_2| \leq 1$ then $p_1 + p_2 \in \mathcal{P}$.
3. If $p, q \in \mathcal{P}$ then $\tau p(\tau)q(\tau) \in \mathcal{P}$.

Then we have the following lemma:

Lemma 6.5.10. *Suppose Γ is a 2-regular digraph. Then $m'(\Gamma;\tau) \in \mathcal{P}$.*

Proof: The proof is by induction on n, the number of vertices of Γ. For if $n = 1$ then $m'(\Gamma;\tau) = \tau$. By axioms 2 and 3 of the definition of the Martin polynomial this is a Martin polynomial. But $\tau \in \mathcal{P}$, by rule 1.

Now suppose that the Martin polynomial for any 2-regular digraph on $r < n$ vertices is an element of $\mathcal{P}$. Let Γ be a 2-regular digraph on n vertices. If Γ has a cut point, then let Γ_1 and Γ_2 be the two 2-regular digraphs on r and $n-1-r$ vertices, respectively, for some $0 \leq r < n$ that are derived from Γ by removing the cut point. Then by axiom 2 $m'(\Gamma;\tau) = \tau m'(\Gamma_1;\tau)m'(\Gamma_2;\tau)$. But by rule 3 $\tau m'(\Gamma_1;\tau)m'(\Gamma_2;\tau) \in \mathcal{P}$ as required.

Now, if Γ does not have a cut point we can only use axiom 1. Suppose Γ_1 and Γ_2 are the two graphs formed by the application of axiom 1. Then we know that $m'(\Gamma_i;\tau) \in \mathcal{P}$ by induction. Now suppose further that $m'(\Gamma_i;-1) = (-1)^{n-1}(-2)^{\lambda_i-1}$. Then by axiom 1 we know that $m'(\Gamma_1;\tau)+m'(\Gamma_2;\tau)$ is a Martin polynomial, so we can conclude that

$$m'(\Gamma_1;-1) + m'(\Gamma_2;-1) = (-1)^n(-2)^\lambda$$

for some λ. And so

$$(-1)^{n-1}(-2)^{\lambda_1-1} + (-1)^{n-1}(-2)^{\lambda_2-1} = (-1)^n(-2)^{\lambda-1}$$

hence $\lambda_1 - \lambda_2 = -1, 0, 1$ as required. So, by rules 3 $m'(\Gamma;\tau) \in \mathcal{P}$ as required.

Moreover, if $\lambda_1 < \lambda_2$ then $\lambda = \lambda_1$ and if $\lambda_1 = \lambda_2$ then $\lambda = \lambda_1 + 1$. □

So we now have another possible way of constructing a set which must contain all the Martin polynomials. This can clearly be seen to be of the form

$$\mathcal{P} = \bigcup_{n\leq 0} \mathcal{P}_n$$

where $P_n = \{p \in \mathcal{P} : p(2) = 2^n\}$.

So, in principle we now have two different ways of constructing potential Martin polynomials. In fact, as the following lemma shows, this new method is the more restrictive of the two.

Lemma 6.5.11. $\mathcal{P} \subsetneq \mathcal{M}$.

Proof: We will show by induction that $\mathcal{P}_n \subset \mathcal{M}_n$. Firstly, notice that $\mathcal{M}_1 = \{\tau\} = \mathcal{P}_1$.

Now suppose that for all $k < n$, $\mathcal{P}_k \subset \mathcal{M}_k$ and suppose $p \in \mathcal{P}_n$. Then there are two cases.

Case one: $p(\tau) = \tau q(\tau)$ for some $q \in \mathcal{P}_{n-1} \subset \mathcal{M}_{n-1}$. But in this case

$$\begin{aligned}
p(2) &= 2q(2) \\
&= 2\,2^{n-1} \\
&= 2^n \\
p(0) &= 0\,q(0) \\
&= 0 \\
p(-1) &= -q(-1) \\
&= -(-1)^{n-1}(-2)^{\lambda-1} \\
&= (-1)^n(-2)^{\lambda-1}
\end{aligned}$$

as $q \in \mathcal{M}_{n-1}$. So $p \in \mathcal{M}_n$.

Case two: $p(\tau) = q_1(\tau) + q_2(\tau)$ for some $q_i \in \mathcal{P}_{n-1} \subset \mathcal{M}_{n-1}$. But then in this case

$$\begin{aligned}
p(2) &= q_1(2) + q_2(2) \\
&= 2^{n-1} + 2^{n-1} \\
&= 2^n \\
p(0) &= q_1(0) + q_2(0) \\
&= 0 + 0 \\
&= 0 \\
p(-1) &= q_1(-1) + q_2(-1) \\
&= (-1)^{n-1}(-2)^{\lambda_1-1} + (-1)^{n-1}(-2)^{\lambda_2-1} \\
&= (-1)^n(-2)^{\lambda}
\end{aligned}$$

as $q_i \in \mathcal{M}_{n-1}$, where the last is by the definition of $\mathcal{P}_n$. Thus $p \in \mathcal{M}_n$ as required.

However the converse does not hold. Consider the polynomial

$$p(\tau) = \tau^4 + 4\tau^3 + \tau^2 + 6\tau$$

This is clearly in $\mathcal{M}_6$ as $p(2) = 2^6$, $p(-1) = -8 = (-1)^6(-2)^3$, so $\lambda = 4$. Also clear is that $p(0) = 0$. But $p(\tau) \notin \mathcal{P}$, for, if it was, then we would have $p_1, p_2 \in \mathcal{P}_5 = \mathcal{M}_5$ such that $p = p_1 + p_2$. Then either $\lambda_1 = \lambda_2 = 3$ or $\lambda_1 = \lambda + 1 = 5$, but by inspection of the polynomials listed in table 6.2 this doesn't happen. Hence $p(\tau)$ is not in $\mathcal{P}_6$. □

Table 6.2 shows P_n for $n \leq 5$.

So, in principle we would appear to be in a better position. However, by inspection we observe that, for $n \leq 5$, $\mathcal{M}_n = \mathcal{P}_n$, so the counter-example for $\mathcal{M}_n$ is

n	λ	$m'(\Gamma;\tau)$
1	1	τ
2	1	τ^2
	2	2τ
3	1	τ^3
		$\tau^2+2\tau$
	2	$2\tau^2$
	3	4τ
4	1	τ^4
		$3\tau^2+2\tau$
		$\tau^3+2\tau^2$
	2	$2\tau^3$
		$\tau^3+\tau^2+2\tau$
		$2\tau^2+4\tau$
	3	$4\tau^2$
	4	8τ
5	1	τ^5
		$3\tau^3+2\tau^2$
		$2\tau^3+3\tau^2+2\tau$
		$\tau^3+4\tau^2+4\tau$
		$5\tau^2+6\tau$
		$\tau^4+2\tau^3$
		$\tau^4+\tau^3+\tau^2+2\tau$
		$\tau^4+2\tau^2+4\tau$
	2	$2\tau^4$
		$\tau^4+\tau^3+2\tau^2$
		$\tau^4+3\tau^2+2\tau$
		$2\tau^3+4\tau^2$
		$6\tau^2+4\tau$
		$\tau^3+5\tau^2+2\tau$
	3	$4\tau^3$
		$3\tau^3+\tau^2+2\tau$
		$2\tau^3+2\tau^2+4\tau$
		$\tau^3+3\tau^2+6\tau$
		$4\tau^2+8\tau$
	4	$8\tau^2$
	5	16τ

Table 6.2: The list of all polynomials in the set $\mathcal{P}_n$ for $n \leq 5$.

still a counter-example for the conditions defining $\mathcal{P}_n$ being both necessary and sufficient.

At this stage, we can observe that the only true statement we can currently make about when a polynomial is a Martin polynomial is that a polynomial is a Martin polynomial if and only if there exists a graph expressing it.

Chapter 7

Counting minimal vector fields on a surface.

In this chapter the intention is to produce some combinatorial results concerning the structures constructed in previous chapters. We shall attempt to answer questions such as "How many 2-regular digraphs are there on n vertices?". In many cases, it will prove easier to calculate values if we label some part of the structure. In such cases it is clear that the number of unlabelled structures is the number of orbits of some group acting on the labelling in such a way as to preserve the structure whilst changing the labelling. For example, when enumerating 2-regular digraphs one can first label each of the vertices $v_1, \ldots, v_n$ and then consider all such labelled digraphs. Then we can act on a labelled digraph via the permutation group on n letters, $\mathrm{Sym}(n)$. The unlabelled graphs are then just the orbits of the labelled graphs with this action.

An example of this is given in Figure 7.1 In this figure we see the three different

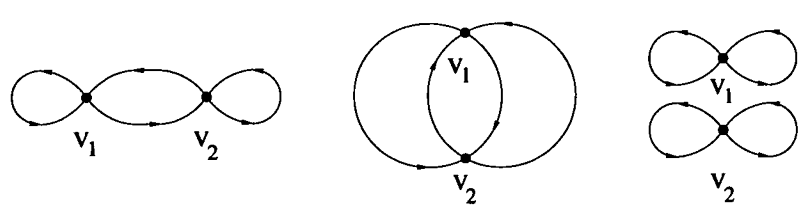


Figure 7.1: The three 2-regular digraphs on 2 vertices.

graphs on 2 vertices. For each graph we could obtain a potentially different graph by swapping the labellings of the vertices. However, in all three cases $\mathrm{Aut}(\Gamma) = C_2$, so there exists an automorphism of the graph that swaps the labellings, so this will not produce a different graph in this case.

For this reason we shall need the following lemma, usually known, incorrectly, as Burnside's lemma.

Lemma 7.0.12. *If the group G acts on the set S with action $\circ$ then the number of orbits r of this action is given by*

$$r = \frac{1}{|G|}\sum_{g\in G} \mathit{fix}(g)$$

where $\mathit{fix}(g) = |\{s \in S : g \circ s = s\}|$, i.e. the number of points in S fixed by g.

Proof: This is a standard undergraduate result and as such the proof may be found in many text books. The one that follows here is adapted from the proof given in [3].

Consider the set

$$X = \{(g,s) \in G \times S : g \circ s = s\}$$

We shall calculate $|X|$ in two ways. Firstly, counting by $g \in G$ it is

$$|X| = \sum_{g\in G} |\{s \in S : g \circ s = s\}| \tag{7.1}$$

$$= \sum_{g\in G} \mathrm{fix}(g) \tag{7.2}$$

Secondly, by counting by $s \in S$ it is

$$|X| = \sum_{s\in S} |\{g \in G : g \circ s = s\}|$$

$$= \sum_{s\in S} |\mathrm{Stab}(s)|$$

where $\mathrm{Stab}(s)$ is the stabilizer of s. But by the Orbit-Stabilizer theorem we know that $|\mathrm{Stab}(s)||\mathrm{Orbit}(s)| = |G|$ for all $s \in S$. Let the orbits of the action of G on S be $S_1, \ldots, S_r$ and for each orbit S_i let s_i be a representative element, i.e. $s_i \in S_i$. Then for each $s \in S_i$, $|\mathrm{Orbit}(s)| = |\mathrm{Orbit}(s_i)|$. Hence

$$|X| = \sum_{s\in S} \frac{|G|}{|\mathrm{Orbit}(s)|} \tag{7.3}$$

$$= |G| \sum_{i=1}^{r} \sum_{s_i \in S_i} \frac{1}{|\mathrm{Orbit}(s_i)|} \tag{7.4}$$

$$= |G|\, r \tag{7.5}$$

Hence equating Equations 7.2 and 7.5 and dividing through by $|G|$ gives the result. □

One further note concerns notation. We define

$$(2n-1)!! = (2n-1)\cdot(2n-3)\cdot\cdots\cdot 3\cdot 1 = \frac{(2n)!}{2^n n!}$$

7.1 How many labelled 2-regular digraphs are there on n vertices?

In this section we shall answer this question in two separate ways, for, when we ask this question we must first decide how we are going to label a 2-regular digraph. We could label the edges, with the labelling on the vertices deriving from this labelling, or vice versa. We shall consider both methods as they shall both give new combinatorial results.

7.1.1 How many 2-regular digraphs are there on n labelled vertices?

We seek to count 2-regular digraphs on n vertices, labelled by vertices. Thus such a graph has vertex set $V = \{v_1, \ldots, v_n\}$ and edge set E, labelled by V. So an edge e with initial and terminal vertices v_i and v_j will have label (i, j). For this reason if a graph has two edges with the same initial and terminal vertices then both edges will receive the same label.

To do this, we shall use the *incidence matrix* of a graph Γ. Recall Definition 6.3.1:

Definition 7.1.1. An *incidence matrix* of a graph Γ is an $n \times n$ matrix M_Γ defined so that the $(i, j)-$th entry $M_{i,j}$ is the number of edges of Γ of the form (i, j). So for a $2-$regular digraph, the sum of each row and column of the incidence matrix is 2.

Conversely, any matrix with non-negative integer entries such that the sum of each row and column is 2 is the incidence matrix of some $2-$regular digraph, as the matrix gives us a means of constructing the graph by connecting up each pair of vertices v_i, v_j with m_{ij} edges.

It is clear that if we relabel the vertex set, provided we also consistently relabel the edge set, we change the incidence matrix without changing the underlying, unlabelled, graph. Thus the incidence matrix of an unlabelled graph is defined only up to the following equivalence.

$$M \sim N \text{ if and only if there exists } S \in \text{ Sym}(n) \text{ such that } S^{-1}MS = N.$$

where $\text{Sym}(n)$ is the symmetric group on n elements.

So, counting 2-regular digraphs with labelled vertices is precisely the same as counting incidence matrices up to this equivalence. We shall first count incidence matrices directly, then apply Burnside's Lemma (lemma 7.0.12).

Definition 7.1.2. A *permutation matrix* on n points is an $n \times n$ matrix P with entries $0, 1$ such that the row and column sum is 1 for every row and column.

Now, observe that as an incidence matrix is any matrix that has row and column sum 2, so a matrix is an incidence matrix if and only if it is the sum of two permutation matrices. So define the set $\mathcal{P}_n$ to be the set of all matrices that are the sum of two permutation matrices. Our problem is now calculating

$$a_n = |\mathcal{P}_n|$$

Theorem 7.1.3. *a_n satisfies the following difference equation.*

$$a_n = n^2 a_{n-1} - \frac{n(n-1)^2}{2} a_{n-2} \tag{7.6}$$

and $a_0 = a_1 = 1$.

Proof: Firstly observe that by convention there is only one non-empty graph with no vertices, namely the single loop. Also, there is clearly only one 2-regular digraph with one vertex, and that has incidence matrix $[2]$. Hence the values of the initial conditions are correct.

Let b_{n-1} denote the cardinality of the set of $n \times n$ matrices in $\mathcal{P}_n$ such that the first two entries are 1.

Then, by considering the possibilities for the first row we observe that

$$a_n = n a_{n-1} + \binom{n}{2} b_{n-1} \tag{7.7}$$

Now, the matrices in $\mathcal{P}_n$ consist of two types, according to what happens in the first column. We have those of type I, where both the ones are in the same row, and those of type II, where the ones are in two different rows. Counting the number of both types gives

$$b_{n-1} = (n-1)a_{n-2} + (n-1)(n-2)b_{n-2} \tag{7.8}$$

To derive this last equation we proceed as follows. We shall aim to construct a matrix of type II and see what choices we have to make in doing so. We begin by putting the two 1s in the first two columns in the first row. We no longer have any choice over the entries for the remainder of the top row, they must all be zero. The matrix is of the form shown below:

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \end{bmatrix}$$

Now, we can put a single 1 somewhere in the remaining $n-1$ places in the first column, likewise in the second. If we decide to put them in the same row, then we have no choice about the entries for the rest of that row, leaving us with a $n-2 \times n-2$ block. At this stage the entries for this remaining block are unrestricted, i.e. it is a $n-2$ square matrix of type I. As we had a choice of $n-1$ possible rows in which to put our pair of 1s, we see that there is a contribution of $(n-1)a_{n-2}$ to b_{n-1} and our matrix is equivalent to the one shown:

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & & \\ 0 & 0 & & & \end{bmatrix}$$

So all that remains is to consider what happens if we put our two 1s in different rows. Notice that we have a $(n-1)(n-2)$ choices about how to distribute these 1s in this case. Any choice we make however is equivalent to putting them in the top two rows we can, i.e. rows 2 and 3. So we have a matrix of the form

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & & & \\ 0 & 1 & & & \\ \vdots & \vdots & & & \end{bmatrix}$$

But in terms of the choices we are left with, this is equivalent to the transpose of a matrix of type II, i.e. a matrix with a 1 in the top two entries of the first column. So the number of choices we have left to make is b_{n-2}. So the contribution from putting the two 1s in different rows is $(n-1)(n-2)b_{n-2}$.

Hence we have two Equations, 7.7 and 7.8 in a_n and b_n for various different values of n. We can now solve these equations for a_n by substitution, as follows:

Firstly, note that Equation 7.7 can be rewritten as

$$a_n - na_{n-1} = \frac{n(n-1)}{2}b_{n-1}$$

which implies that

$$\frac{(n-1)(n-2)}{2}b_{n-2} = a_{n-1} - (n-1)a_{n-2} \tag{7.9}$$

We can now substitute Equation 7.9 into Equation 7.8 to get that

$$\begin{aligned} b_{n-1} &= (n-1)a_{n-2} + 2a_{n-1} - 2(n-1)a_{n-2} && (7.10) \\ &= 2a_{n-1} - (n-1)a_{n-2} && (7.11) \end{aligned}$$

And if we substitute Equation 7.10 into Equation 7.7 we get the desired result. □

Theorem 7.1.4. *The unique solution to Equation 7.6 is*

$$a_n = \frac{n!}{2^n} \sum_{r=0}^{n} \binom{n}{r} (2r-1)!! \tag{7.12}$$

Proof: Just substitute Equation 7.12 in Equation 7.6. □

Whilst this proof is true, it is not exactly enlightening. We now give a non-rigorous argument for interest showing how the solution may be derived.

The key idea is to express Equation 7.6 in terms of a differential equation. Then a series expansion of the solution to this differential equation will give the required result. We begin by defining a function f as followings:

$$f(x) = \sum_{n \geq 0} \frac{a_n}{n!^2} x^n$$

Then Equation 7.6 gives that f satisfies the following differential equation:

$$2(1-x)f'(x) = (2-x)f(x)$$

with initial condition $f(0) = 1$. This gives that

$$f(x) = \left(\frac{e^x}{1-x}\right)^{\frac{1}{2}}$$

which can then be expanded to give the above result.

Table 7.1 shows the first 25 values of a_n.

So we have now counted incidence matrices. To get the number of 2-regular digraphs we apply lemma 7.0.12 to this result. This method of calculation returns the first three values as 1,3 and 8. For $n = 1$ this is trivial. For $n = 2$ the set of matrices can be written as $\{2I, I+J, 2J\}$ where I is the 2×2 identity matrix and J is the other 2×2 permutation matrix. In this case the action of the permutation group is trivial, so the result is 3 as claimed. For $n = 3$ the result is only slightly harder to prove. In fact, if we label the 3×3 permutation matrices as I, the identity, T_1, T_2, T_3, the three matrices corresponding to each of the three single transpositions and R, R^{-1} corresponding to the two remaining elements then we see that the 8 incidence matrices are $2I, 2T_1, 2R, I+T_1, I+R, T_1+T_2, T_1+R, R+R^{-1}$.

n	a_n
1	1
2	3
3	21
4	282
5	6210
6	202410
7	9135630
8	545007960
9	41514583320
10	3930730108200
11	452785322266200
12	62347376347779600
13	10112899541133589200
14	1908371363842760216400
15	414517594539154672566000
16	102681435747106627787376000
17	28772944645196614863048048000
18	9055359650665478876752602576000
19	3180421710272322693959227638192000
20	1239478835770026698838614159977440000
21	533252395391438018873200088469644640000
22	252081447537135601618562725529257444320000
23	130383002914395989243171450112555145979040000
24	73500396649726353004992119083172037827383680000
25	44998990285095319505569239986172126591065712000000

Table 7.1: The first 25 values of a_n

7.1.2 How many 2-regular digraphs are there with $2n$ labelled edges?

We seek to count 2-regular digraphs on n vertices, labelled by edges. So such a graph has vertex set $V = \{v = (\{e_i, e_j\}, \{e_k, e_l\})\}$ and edge set $E = \{e_1, \ldots, e_{2n}\}$ where e_i, e_j are the edges with v as the initial vertex and e_k, e_l are the edges with v as the terminal vertex.

To do this, we shall construct all such graphs from their circuit decompositions. The idea is that we start with a set of circuits and from it construct the 2-regular digraphs that have the given set of circuits as a circuit decomposition.

Definition 7.1.5 (Partition). A partition of a number p is a set of integers $p_1, \ldots, p_l$ such that $\sum_{i=1}^{l} p_i = p$. However, as there may be more than one occurance of p_i for any i we shall use the notation that a partition is

$$\underline{r} = (r_1^{n_1}, \ldots, r_k^{n_k})$$

where $r_1 < r_2 < \cdots < r_k$ and r_i appears n_i times in the partition. Thus $\sum_{i=1}^{k} n_i r_i = p$ So, for example, under this notation the two partitions of 2 are (1^2) and (2^1), whilst the three partitions of 3 are $(1^3), (1^1, 2^1), (3^1)$

Define $P(p)$ to be the set of partitions of p

So, we start with a set of circuits, of total length $2n$. This is equivalent to taking a partition $\underline{r} = (r_1^{n_1}, \ldots, r_k^{n_k})$ of a set E of $2n$ edges and forming the edges of each part into a single circuit. Call the resulting partitioned set of edges $E(\underline{r})$.

We then *label* the edges of this set of circuits according to the following definition.

Definition 7.1.6 (Labelling). A labelling of a set $E(\underline{r})$ of $2n$ edges is a bijection $\mathcal{L} : E(\underline{r}) \to \{1, \ldots, 2n\}$.

We now have a set $E(\underline{r})$ of circuits of lengths given by the partition $\underline{r}$ and labelled by the labelling $\mathcal{L}$.

One step remains to form the 2-regular digraph. We need only to *glue* up the vertices of the circuits to make each vertex 2-regular. But such a gluing is a pairing of the vertices of $E(\underline{r})$. So we have the following definition

Definition 7.1.7. A gluing map for the set of labelled circuits $E(\underline{r})$ is a map

$$\tau : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$$

Notice that as the vertices of the resulting graph must be 2-regular, we immediately see that for any edge e, $\tau e \neq e$ and $\tau^2 e = e$. Thus τ is a free involution of $E(\underline{r})$.

In order to construct the graph given a gluing map, we glue two vertices together if they are the terminal vertices of edges e, f such that $\mathcal{L}^{-1}\tau\mathcal{L}(e) = f$

So, all the information needed to form a 2-regular digraph from a circuit decomposition is a triple $(E(\underline{r}), \mathcal{L}, \tau)$ as defined above.

As the labellings of the above are independent, the total number of such triples is given by

$$\begin{aligned} \text{No. of triples} &= \text{No. partitions of } 2n \\ &\times \text{ No. labellings of a set of } 2n \text{ points} \\ &\times \text{ No. of free involutions of a set of } 2n \text{ points} \end{aligned}$$

But it is clear that the number of labellings of a set of $2n$ points is $(2n)!$. The number of free involutions of a set of $2n$ points is given by the following lemma

Lemma 7.1.8. *The number of free involutions of a set of $2n$ points is*

$$(2n-1)!! = \frac{(2n)!}{2^n n!}$$

Proof: Notice that a free involution is a pairing of the points of the set. So, we can pick 2 points from the set and say those are paired, then pick another 2 from the remaining set and keep going till we have paired up all points in the set. The number of ways of doing this is

$$\begin{aligned} \binom{2n}{2}\binom{2n-2}{2}\cdots\binom{2}{2} &= \frac{(2n)!}{2!(2(n-1))!}\frac{(2(n-1))!}{2!(2(n-2))!}\cdots\frac{2!}{2!0!} \\ &= \frac{(2n)!}{2^n} \end{aligned}$$

But this method of picking points to pair assumes an ordering of the couples. As there is no such ordering we must divide the above answer by $n!$, which gives the desired result. $\square$

So, if we define the number of partitions of a set of p points to be $P(p)$ then we conclude that

Lemma 7.1.9. *The number of triples is*

$$P(2n) \times (2n)! \times (2n-1)!!$$

7.1.2.1 Consider different triples

We may now consider the question of different triples.

The first question that arises is "What does it mean for two triples to be the same?" We shall answer this by stating that two triples are the same if they define the same labelled digraph with given circuit decomposition. So it is clear that two triples $(E(\underline{r}_1), \mathcal{L}_1, \tau_1)$ and $(E(\underline{r}_2), \mathcal{L}_2, \tau_2)$ are different if $\underline{r}_1 \neq \underline{r}_2$. So we shall fix the partition $\underline{r} = (r_1^{n_1}, \ldots, r_k^{n_k})$.

Claim 7.1.10. *We claim that given the partition, the group of symmetries of the partitioned set is*

$$G(\underline{r}) = \bigoplus_{i=1}^{k} C_{r_i} \ wr \ Sym(n_i)$$

where A wr B is the wreath product of B with A.

Proof: It is clear that, as we have labelled all the $2n$ edges any symmetry of our partitioned set can be taken to be an element of the symmetric group on $2n$ points. So suppose we have such a symmetry g. Consider what g may do.

On each cycle g can only cause the cycle to rotate. Hence, when restricted to a cycle of length r_i, g is just an element of the cyclic group C_{r_i}. However, if we have more than one cycle of length r_i g may interchange them. Thus, for each i, g can be any element of the group of symmetries of n_i copies of cycles of r_i. But this is just C_{r_i} wr $\mathrm{Sym}(n_i)$. We can then form the product of all such groups to find the group of symmetries to be $G(\underline{r})$ as claimed. □

We shall now define the action of the group $G(\underline{r})$ on a triple.

Definition 7.1.11. Two triples $(E(\underline{r}), \mathcal{L}_1, \tau_1)$ and $(E(\underline{r}), \mathcal{L}_2, \tau_2)$ are equivalent if there is an element $g \in G(\underline{r})$ that sends one to the other.

We shall now discuss the implications of this definition.

It is clear that, as mentioned above, we can view $G(\underline{r})$ as a subgroup of the symmetric group on $2n$ points, in which case the action of the group on $\{1, \ldots, 2n\}$ is obvious.

Firstly, we define the action of the group on a labelling $\mathcal{L}$. Recall $\mathcal{L}$ is a bijection between $E(\underline{r})$ and the set $\{1, \ldots, 2n\}$. We define the action of the group on $\mathcal{L}$ as the action on $\{1, \ldots, 2n\}$, i.e. for $e \in E(\underline{r})$, $(g \circ \mathcal{L})(e) = g\mathcal{L}(e)$.

Secondly, to define the action of the group on a free involution τ recall that τ is a permutation of $\{1, \ldots, 2n\}$, hence an element of the symmetric group on $2n$

points. Thus we define the group action as conjugacy, i.e. $g \circ \tau = g^{-1}\tau g$. Notice that as we glue the terminal vertices of two edges together if and only if $\mathcal{L}^{-1}\tau\mathcal{L}$ interchanges them, these two actions agree, i.e. $\mathcal{L}^{-1}(g \circ \tau)\mathcal{L} = (g \circ \mathcal{L})^{-1}\tau(g \circ \mathcal{L})$. So, two triples $(E(\underline{r}), \mathcal{L}_1, \tau_1)$ and $(E(\underline{r}), \mathcal{L}_2, \tau_2)$ are the same if there is an element $g \in G(\underline{r})$ such that

$$g \circ (E(\underline{r}), \mathcal{L}_1, \tau_1) = (E(\underline{r}), g\mathcal{L}_1, g^{-1}\tau_1 g) \tag{7.13}$$
$$= (E(\underline{r}), \mathcal{L}_2, \tau_2) \tag{7.14}$$

So, the number of different 2-regular digraphs labelled by edges with a marked circuit decomposition is (using Lemma 7.0.12)

$$\sum_{\underline{r} \in P(2n)} \frac{1}{|G(\underline{r})|} \sum_{g \in G(\underline{r})} \mathrm{fix}_{(\mathcal{L},\tau)}(g)$$

where

$$\mathrm{fix}_{(\mathcal{L},\tau)}(g) = |\{(\mathcal{L}, \tau) : g\mathcal{L} = L, g^{-1}\tau g = \tau\}|$$

Fix $\mathcal{L}$ as a labelling. Then the set of all labels is $\{\sigma\mathcal{L} : \sigma \in \mathrm{Sym}(2n)\}$ and for any $\sigma \in \mathrm{Sym}(2n)$, $(\sigma\mathcal{L}, \tau)$ is the same as $(\mathcal{L}, \sigma^{-1}\tau\sigma)$. So we conclude that:

$$\begin{aligned}
\mathrm{fix}_{(\mathcal{L},\tau)}(g) &= |\{(\mathcal{S}, \tau) : g\mathcal{S} = \mathcal{S}, g^{-1}\tau g = \tau\}| \\
&= |\{(\sigma\mathcal{L}, \tau) : g\sigma\mathcal{L} = \sigma\mathcal{L}, g^{-1}\tau g = \tau\}| \\
&= |\mathrm{Sym}(2n)| \times |\{(\mathcal{L}, \tau) : g^{-1}\tau g = \tau\}| \\
&= (2n)! \times |\{(\mathcal{L}, \tau) : g^{-1}\tau g = \tau\}| \\
&= (2n)! \times |\{\tau : g^{-1}\tau g = \tau\}|
\end{aligned}$$

so if we define $\mathrm{fix}_\tau(g) = |\{\tau : g^{-1}\tau g = \tau\}|$ then we can conclude that:

Lemma 7.1.12. *The number of different 2-regular digraphs labelled by edges, (where two different labellings give two different graphs) with a marked circuit decomposition is*

$$\sum_{\underline{r} \in P(2n)} \frac{(2n)!}{|G(\underline{r})|} \sum_{g \in G(\underline{r})} \mathit{fix}_\tau(g)$$

Now, it is easy to see that this last result will give $(2n)!$ times the number of labelled graphs with a marked circuit decomposition, as it will regard two different labellings of the same labelled graph as different. So, we can further improve this result by dividing by $(2n)!$. To clarify this result, we shall make the following definition.

Definition 7.1.13.

$$b_n(r_1^{n_1},\dots,r_k^{n_k}) = \left(\prod_{i=1}^{k} r_i n_i!\right)^{-1} \sum_{g\in\bigoplus_{i=1}^{k} C_{r_i} \text{ wr Sym}(n_i)} \text{fix}_\tau(g)$$

Then our improved result is:

Lemma 7.1.14. *The number of different 2-regular digraphs labelled by edges with a marked circuit decomposition is*

$$\sum_{\underline{r}\in P(2n)} \frac{1}{|G(\underline{r})|} \sum_{g\in G(\underline{r})} \mathit{fix}_\tau(g) = \sum_{\underline{r}=(r_1^{n_1},\dots,r_k^{n_k})} \frac{1}{\prod_{i=1}^{k} r_i n_i!} \sum_{g\in G(\underline{r})} \mathit{fix}_\tau(g)$$
$$= \sum_{\underline{r}=(r_1^{n_1},\dots,r_k^{n_k})} b_n(r_1^{n_1},\dots,r_k^{n_k})$$

where the equality is clear, because $G((r_1^{n_1},\dots,r_k^{n_k})) = \bigoplus_{i=1}^{k} C_{r_i}$ wr $\text{Sym}(n_i)$

Although actually calculating values with this formula is not simple, calculations for the first 3 values give 1, 3 and 8, which agree with the results in the previous section.

Example 7.1.15. n=1 This is the smallest possible case. Recall that Lemma 7.1.9 gives that there are 4 triples. This gives us our first, albeit rather crude, estimate for the number of 2-regular digraphs with a marked circuit decomposition on a single vertex.

Now, $P(2) = \{1^2, 2^1\}$ and for either partition $G(\underline{r}) = C_2 =< (12) >$. Also, the only free involution of a set of two points is the transition (12) so we have that for either $g \in C_2$, $\text{fix}(g) = 1$ so

$$\sum_{\underline{r}\in P(2n)} \frac{1}{|G(\underline{r})|} \sum_{g\in G(\underline{r})} \text{fix}_\tau(g) = \sum_{\underline{r}\in P(2)} \frac{1}{2}(1+1)$$
$$= 2$$

so in this case, the count given by assuming that every triple corresponds to a graph is the same as that given by the more precise count given by assuming that every 2-regular digraph has a unique circuit decomposition, i.e. using Lemma 7.1.12. However, this is assumption is clearly only an approximation. If we instead assume that two different labellings of the same graph should be counted as the same graph, then we use the count given by Lemma 7.1.14, which is a clear improvement. The 4 graphs of Lemma 7.1.12 are shown in Figure 7.2. It is easy to see, in this figure, the 2 graphs of Lemma 7.1.14 as to get these, we simply ignore the labelling in this case. □

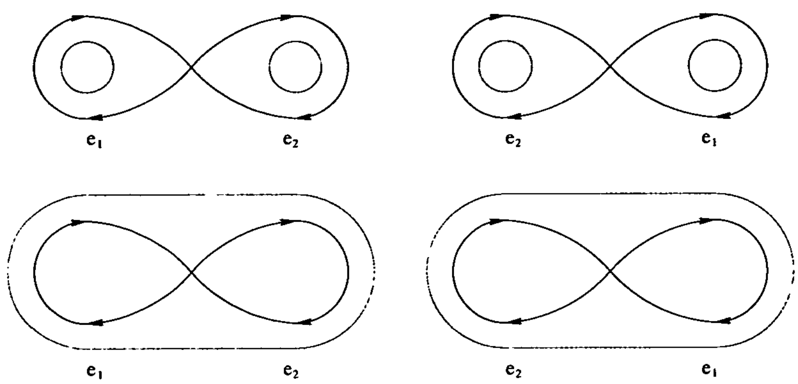


Figure 7.2: The 4 different 2-regular digraphs labelled by edges as given by Lemma 7.1.12

Example 7.1.16. $\underline{r} = 2n^1$ In this case we are counting the number of different labelled graphs with a single circuit of length $2n$. But such a circuit will be Eulerian, by definition. However, we have previously shown that any 2-regular digraph has an Eulerian circuit. Thus we can see that the number of connected graphs with a marked Eulerian circuit is

$$\frac{1}{|G(\underline{r})|} \sum_{g \in G(\underline{r})} \mathrm{fix}_\tau(g)$$

Now, in this case $|G(\underline{r})| = |C_{2n}| = 2n$ so the above becomes

$$b_n = b_n(2n^1) = \frac{1}{2n} \sum_{g \in C_{2n}} \mathrm{fix}_\tau(g)$$

Calculations for $n = 1, 2, 3$ give that b_n is $1, 2, 5$ respectively.

We shall now give an explicit formula for this equation, due to Bar-Natan. □

7.1.2.2 An explicit formula for the number of graphs with labelled Eulerian circuit

Theorem 7.1.17 ([1]). *The number of different free involutions on a set of $2n$ points (and hence the number of different graphs with labelled Eulerian circuit) is given by the formula*

$$\frac{1}{2n} \sum_{k|2n} \varphi(\frac{2n}{k}) u(k, 2n)$$

where

$$u(k, 2n) = \begin{cases} \frac{k!}{2^{\frac{k}{2}}(\frac{k}{2})!}(\frac{2n}{k})^{\frac{k}{2}} & \text{if } k \nmid n \\ \sum_{j \geq 0, k-j \text{ even}}^{k} \binom{k}{j} \frac{(k-j)!}{2^{\frac{k-j}{2}}(\frac{k-j}{2})!}(\frac{2n}{k})^{\frac{k-j}{2}} & \text{if } k | n \end{cases}$$

and $\varphi(r)$ is the Euler-φ function.

Proof: It is an immediate consequence of Lemma 7.0.12 that the number of orbits of the action of the group $G(\underline{(}2n)) = C_{2n}$ on the set of free involutions of a set of $2n$ points, b_n is given by

$$\begin{aligned} b_n &= \frac{1}{2n} \sum_{h \in G} \text{fix}_\tau(h) \\ &= \frac{1}{2n} \sum_{k=0}^{2n} \text{fix}_\tau(g^k) \end{aligned}$$

where $C_{2n} =< g >$.

Now, it is a further consequence of the results on cycle index discussed in [9], pp35-41 that this result is

$$b_n = \frac{1}{2n} \sum_{k|n} \varphi(\frac{2n}{k}) \text{ fix}_\tau(g^k)$$

It remains to prove that

$$\text{fix}_\tau(g^k) = u(k, 2n) = \begin{cases} \frac{k!}{2^{\frac{k}{2}} (\frac{k}{2})!} d^{\frac{k}{2}} & \text{if } k \nmid n \\ \sum_{j \geq 0} \binom{k}{2j} \frac{(2j)!}{2^j j!} d^{\frac{k-j}{2}} & \text{if } k|n \end{cases}$$

We proceed as follows. Suppose τ is fixed by g^k. Let $dk = 2n$. Let V, the set of $2n$ points, be written as

$$V = \{p_0, p_1, p_2, \ldots, p_{2n}\}$$

Let the action of $C_{2n} =< g : g^{2n} = 1 >$ on V be $gp_i = p_{i+1}$. Then we may write $p_i = g^i p_0$ and so V may be written as

$$V = \bigcup_{j=0}^{d-1} V_j$$

where

$$V_j = \{g^{jk+i} p_0 : 0 \geq i < k\}$$

Thus we see that

$$V = \bigcup_{j=0}^{d-1} g^{jk} V_0$$

The idea of this proof is that τ is defined entirely by its action on V_0. We may see this as follows.

Suppose $p_i = g^i p_0 \in V_0$. Then for all j, $g^{jk+i} p_0 \in V_j$. So

$$\begin{aligned}\tau g^{jk+i} p_0 &= (g^{jk} \tau g^{-jk}) g^{jk+i} p_0 \\ &= g^{jk} \tau g^i p_0\end{aligned}$$

that is

$$\tau p_{jk+i} = g^{jk} \tau p_i$$

so the action of τ on p_i defines the action of τ on all p_{jk+i}.

Notice also that, if $\tau p_i = p_r$ then there exists suitable j, t such that $r = jk + t$ and so

$$\begin{aligned} & & \tau g^i p_0 &= g^{jk+t} p_0 \\ &\text{so} & g^{-jk} \tau g^i p_0 &= g^t p_0 \\ &\text{that is} & \tau : g^i p_0 &\to g^{jk+t} p_0 \\ &\text{which implies} & \tau : g^{-jk+i} p_0 &\to g^t p_0 \end{aligned}$$

that is, the action of τ is given by pairing up p_i and p_t within V_0 together with a choice of V_j.

Now, the question arises of when p_i can be paired with itself in this manner, i.e. can we select some value of j such that τ maps p_i to p_{jk+i} and vice versa. But as τ is an involution, if such a j exists then τ takes p_{jk+i} to p_i. But we have seen above that τ takes p_{-jk+i} to p_i. So $p_{-jk+i} = p_{jk+i}$, that is $g^{-jk+i} = g^{jk+i}$. Thus $g^{-jk} = g^{jk}$, i.e. $g^{2jk} = g^{2n}$ =id. So p_i can only be paired with itself if k divides n and in this case we have no choice over V_j, that is p_i is paired with p_{nk+i}.

So, if k does not divide n then to define the action of τ we simply pair up all the points of V_0 and pick a j for each pairing. Thus there are precisely

$$(k-1)!!\ d^{\frac{k}{2}}$$

ways of doing this. Notice that this does not work if k is not even. However observe that if k is odd then d must be even. Thus $d = 2r$ and so $2n = 2rk$ and hence k divides n. Thus if k does not divide n we may conclude that k is even.

Now, if k does divide n then we may have up to k points of V_0 paired with themselves. If we have j pairs of distinct points in V_0 then the number of such τ fixed by g^k is precisely

$$\binom{k}{2j} (2j-1)!!\ d^j$$

and thus the number of τ fixed by g^k in general is the sum of this over all j, i.e.

$$\sum_{j\geq 0} \binom{k}{2j} (2j-1)!!\, d^j$$

as required. □

This last result is well-known, and the set of free involutions of a set of $2n$ points is well-studied. They are usually studied up to the action of C_{2n} given above and as such are referred to as *chord diagrams.* See for example [1].

7.2 Given a 4-regular graph on n vertices, how many 2-regular digraphs can be constructed from it?

The following is [6], corollary 5.5.

Theorem 7.2.1. *For G a 4-regular graph with n vertices, the number of Eulerian orientations $F(G)$ of G (i.e. the number of 2-regular digraphs for which G is the underlying graph) is*

$$F(G) = \frac{1}{2^n} \sum_{A \in U(G)} 3^{N(A)}$$

where $N(A)$ is the number of 4-regular vertices of G and $U(G)$ is the set defined in Theorem 4 in chapter 6.

The proof is given in [6], page 345. It uses an inductive relationship on the undirected Martin polynomial, similar to that given in chapter 6.

7.3 Given a 2-regular digraph, how many circuit decompositions are there?

If we begin by considering the circuit decompositions to be labelled then we immediately see that there is a one to one relationship between the local orientation systems of labelled graphs and the circuit decompositions of the same. This is because, as shown in chapter 3 a circuit decomposition both defines and is defined by a local orientation system. But the number of local orientation systems is just the number of ways of making n independent choices, when each choice is between two options, i.e. 2^n. So, the number of circuit decompositions of a labelled graph on n vertices is just 2^n.

To find the number of circuit decompositions of an unlabelled graph, we note that this is simply the number of orbits of the action of $\mathrm{Aut}(\Gamma)$ on the labelled graph Γ. So we can apply Lemma 7.0.12 to this result. However, for now we shall continue with the graphs labelled.

Now, before we can proceed to the question of how many of these circuit decompositions are balanced, we should recall that in order to construct a ribboned graph we need a dual pair of circuit decompositions. Thus on a labelled graph, there are $\frac{2^n}{2} = 2^{n-1}$ such labelled ribboned graphs. We may now ask how many of these are balanced.

7.4 How many minimal vector fields can be constructed from a given balanced 2-regular digraph?

Recall that a minimal vector field is equivalent to a balanced 2-regular digraph R with a pairing $\tau : \mathcal{C} \to \mathcal{C}'$, where $|\mathcal{C}| = \kappa$ So in the case of vector fields with labelled graphs, the question is equivalent to asking how many pairings there are of $\mathcal{C}$ with $\mathcal{C}'$. But this can easily be seen to be $\kappa!$.

So, we now have a way of calculating the number of minimal vector fields X with a labelled graph Γ. And thus, to calculate the number of minimal vector fields without such a restriction we need only apply Lemma 7.0.12 using the group $\mathrm{Aut}(\Gamma)$.

Performing this calculation in the case when $g = 2$ gives that there are 6 minimal vector fields. These can be constructed using the methods given in chapters 3 and 4 and are shown in figure 7.3. The underlying graph of the ribboned graph is shown here in red, whilst typical periodic cycles are shown in blue.

7.5 Given a graph G, when does there exist $\Gamma(G)$ such that the automorphism group of $\Gamma(G)$ is G?

In this section we shall prove that for any finitely generated group G there exists a 2-regular digraph Γ such that the automorphism group of Γ is isomorphic to G.

Theorem 7.5.1. *For any finite group G there exists $\Gamma(G)$ a 2-regular connected digraph such that $Aut(\Gamma(G)) = G$.*

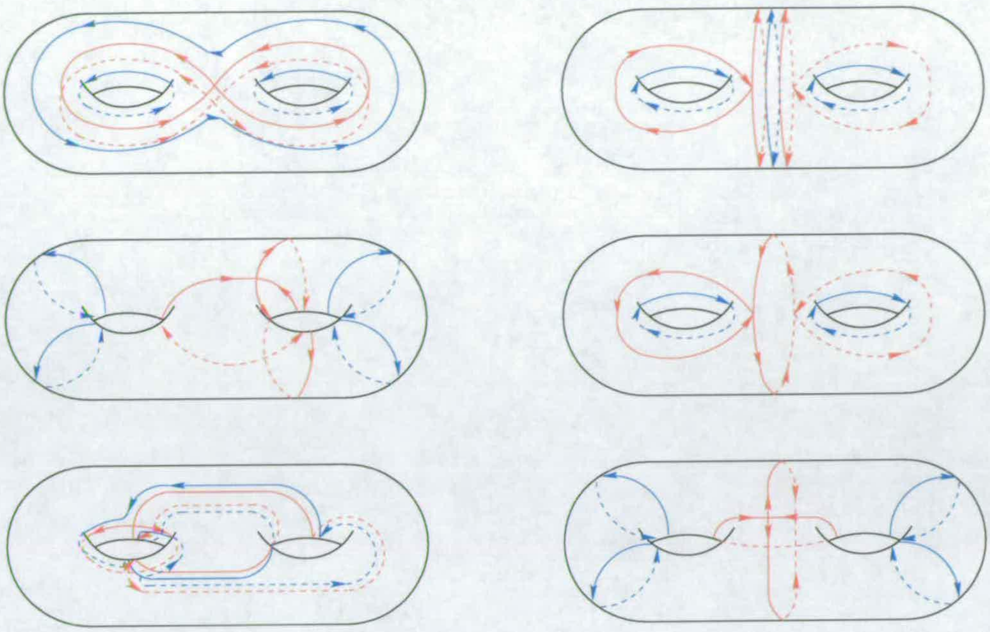


Figure 7.3: The 6 minimal vector fields on a surface with genus 2.

The idea behind the proof of this theorem is to construct a Cayley digraph Γ' of G with respect to a set of r generators S of G. This will be an r-regular digraph. We then replace each edge of Γ' by a "labelled" edge and each vertex by an "r-cube" so that the resulting graph $\Gamma(G)$ is 2-regular. Theorem 7.5.1 will then be a consequence of properties of r-cubes and a lemma about Cayley graphs.

Notice that if $r = 2$ then we need only label the edges using loops (as discussed below) and we're done. If $r = 1$ then there is only one edge leaving each vertex, so we don't need to label it with loops. So we can "bud" a loop onto the Cayley graph of G, as shown in Figure 7.4, to make $\Gamma(G)$, a 2-regular digraph as required. Thus from now on we shall assume that $r > 2$.

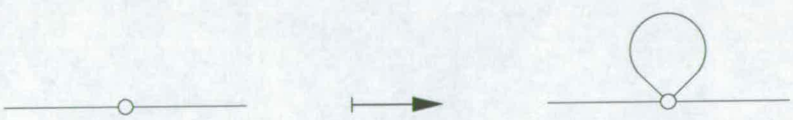


Figure 7.4: How to bud loops to the vertices of a 1-regular digraph to make them 2-regular

The following discussion of Cayley graphs owes much to [3]

Definition 7.5.2 (Cayley graphs). Let $S = \{s_1, \ldots, s_r\}$ be a set of r elements that generate the finite group G and no element is the inverse of another element in S. The *Cayley graph* $\Gamma(G; S)$ of G with respect to S is the digraph defined with vertex set $V = G$ and the edge set E defined as

$$E = \{(g, sg) : s \in S \text{ and } g \in G\}$$

Notice that this is more usually called the *Cayley digraph* with the Cayley graph being the underlying graph. As we are not interested in undirected graphs, when we talk of the Cayley graph, we say be referring to the digraph defined above.

Notice that we may regard the elements of S as labelling the edges, i.e. the edges $(g, s_i g)$ has label s_i, or, for brevity, i. Notice that each vertex g has r edges coming in of the form $(s_i^{-1}g, g)$, one for each of the r labels, and r edges coming out $(g, s_i g)$, again one for each of the r labels, hence $\Gamma(G; S)$ is an r-regular digraph.

Now, we have the following lemmas:

Lemma 7.5.3. *S generates G implies that $\Gamma(G; S)$ is connected.*

Proof: As S generates G, so any element $g \in G$ can be written as a product of elements of S and their inverses. But this product can then be used to define a path from the identity to g. Hence $\Gamma(G; S)$ is connected. □

Lemma 7.5.4. *For each $g \in G$, the map $\rho_g : x \to xg$ is an automorphism of $\Gamma(G; S)$.*

Proof: This is clear, as if (x, sx) is an edge, then so is $(x\rho_g, sx\rho_g) = (xg, sxg)$. □

Notice that the permutations ρ_g comprise a permutation group isomorphic to G. So there is an action of G on the vertices of $\Gamma(G; S)$. Notice that this is a transitive action as for any $g, h \in G$, $\rho_{g^{-1}h}$ sends g to h. As we have used G to denote the vertices of the Cayley graph, we shall denote the permutation group by $\rho(G)$.

We also have the following lemma, which will be key in the proof of Theorem 7.5.1.

Lemma 7.5.5. *Any automorphism of $\Gamma(G; S)$ which preserves the labels on the edges belongs to $\rho(G)$.*

Proof: Let f be a label-preserving automorphism. Then, as all elements of $\rho(G)$ are also label-preserving we can compose f with $\rho_{1f^{-1}}$ to obtain an automorphism fixing the vertex corresponding to the identity of G. Thus we may assume f fixes a vertex of $\Gamma(G; S)$. Now, for each $s_i \in S$ there exists a unique edge with label i and initial vertex the identity, namely (id, s_i) and a unique edge with label i and terminal vertex the identity, i.e. (s_i^{-1}, id). So f fixes all vertices s_i and s_i^{-1}. But we can iterate this result, which leads to the conclusion that f fixes all G and so f is the identity. Hence result. □

So we have constructed an r-regular digraph such that the group G is the group of automorphisms that preserve the labels on the edges of this graph. We now want to adapt this construction so that the graph is 2-regular and the edges are "labelled" in such a way as to force any automorphism to necessarily fix the labelling.

This latter requirement is easy. If an edge $(g, s_i g)$ has labelling i then we replace it by an "edge" consisting of i loops, as shown in Figure 7.5.

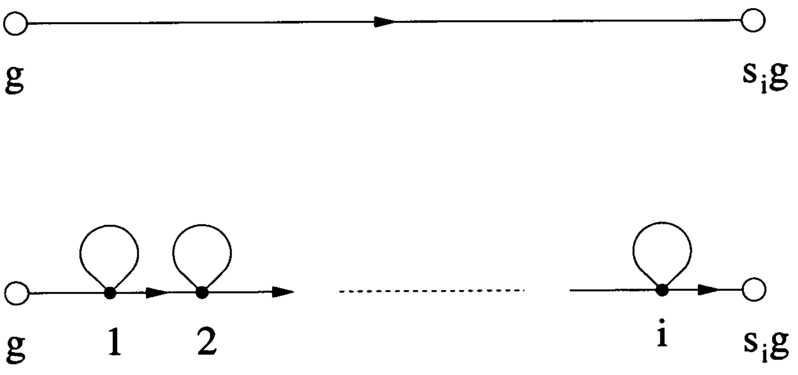


Figure 7.5: The "edge" with label i that replaces the edge $(g, s_i g)$

So, all we need now do is to construct an r-cube Γ_0 with the properties that

1. Γ_0 has r initial vertices and r terminal vertices.

2. Aside from these initial and terminal vertices, all vertices of Γ_0 are 2-regular.

3. Any automorphism of Γ_0 that fixes the initial and terminal vertices necessarily fixes Γ_0

An r-cube is merely a 2-regular subgraph that we will put in place of each of the r-regular vertices. It will consist of r primary "edges", each of which will have a single vertex in common with each of the others. Each of these primary "edges" will also have a certain number of loops attached to it in order to distinguish it from any other primary edges. The reason for this latter condition is to satisfy the restriction above that an automorphism that fixes the initial and terminal

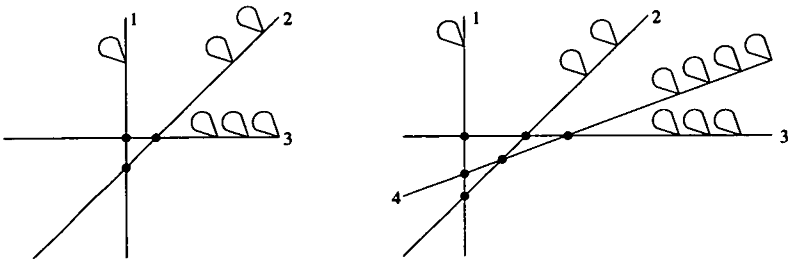


Figure 7.6: A 3-cube and a 4-cube.

vertices will fix the entire r-cube. Figure 7.6 shows a 3-cube and a 4-cube for clarity.

We will construct an r-cube labelled by an element $g \in G$.

Definition 7.5.6 (An r-cube, Γ_0). We define $\Gamma_0 = (V_0, E_0)$ where

$$\begin{aligned} V_0 =& \{v_{(j,0,g)} : j = 1, \ldots, r\} \\ & \cup \{v_{(\{i,j\},g)} : i \neq j = 1, \ldots, r\} \\ & \cup \{v_{(j,1,g)} : j = 1, \ldots, r\} \end{aligned}$$

and

$$\begin{aligned} E_0 =& \{(v_{(j,0,g)}, v_{(\{1,j\},g)}) : j = 2, \ldots, r\} \cup \{(v_{(1,0,g)}, v_{(\{1,2\},g)})\} \\ & \cup \{(v_{(\{i,j\},g)}, v_{(\{i+1,j\},g)}) : 2 \leq i+1 < j\} \\ & \cup \{(v_{(\{j-1,j\})}, v_{(\{j+1,j\},g)}) : 1 < j < r\} \\ & \cup \{(v_{(\{i,j\},g)}, v_{(\{i+1,j\},g)}) : j < i \leq r-1\} \\ & \cup \{(v_{(\{r,j\},g)}, v_{(j,1,g)}) : j < r\} \cup \{(v_{(\{r-1,r\},g)}, v_{(r,1,g)})\} \end{aligned}$$

The j-th line segment of Γ_0, i.e. the link between all vertices where j appears, is shown in Figure 7.7

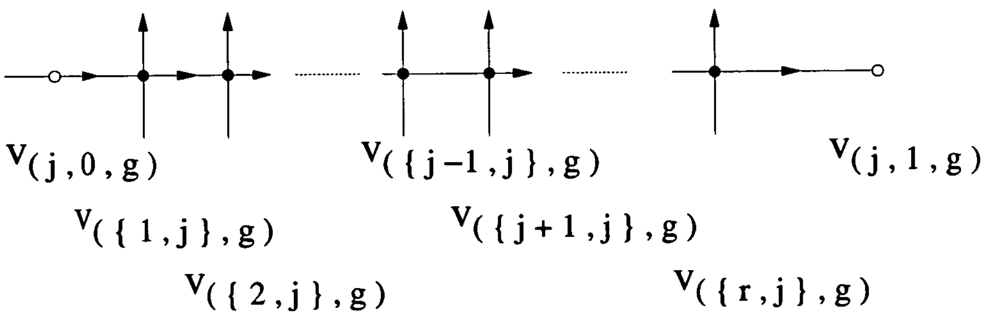


Figure 7.7: The j-th line segment of the r-cube labelled by g.

Now, it is easy to see that such a gadget has r initial vertices, namely $v_{(j,0,g)}$ for $j = 1, \ldots, r$ and r terminal vertices, namely $v_{(j,1,g)}$ for $j = 1, \ldots, r$. Moreover, it is clear that aside from these vertices all vertices of Γ_0 are 2-regular. However, to prove the last condition, i.e. that any automorphism of Γ_0 that fixes these initial and terminal vertices necessarily fixes Γ_0 we require the following lemma.

Lemma 7.5.7. *Let α be some fixed automorphism of Γ_0. Define $P(\lambda)$ to be the set*

$$P(\lambda) = \{v_{(\{j,k\},g)} : k \neq j \leq \lambda, 1 \leq j \leq g\}$$

Then α fixes $P(\lambda)$ pointwise (i.e. $\alpha v = v$ for any $v \in P(\lambda)$) implies α fixes $P(\lambda+1)$ pointwise.

Proof: Suppose α fixes $P(\lambda)$ pointwise. Then we need to prove that for all $j \neq \lambda + 1$, α fixes $v_{(\{j,\lambda+1\},g)}$. So consider all edges with $v_{(\{j,\lambda\},g)}$ as the initial vertex, where $j \neq \lambda$. These form the set

$$E' = \{(v_{(\{j,\lambda\},g)}, v_{(\{j+1,\lambda\},g)}), (v_{(\{j,\lambda\},g)}, v_{(\{j,\lambda+1\},g)})\}$$

But as α fixes $v_{(\{j,\lambda\},g)}$ so $\alpha E' = E'$ and as α fixes $v_{(\{j+1,\lambda\},g)}$ so α fixes $v_{(\{j,\lambda+1\},g)}$. However, this still leaves $v_{(\{\lambda,\lambda+1\},g)}$. But the same logic still works using

$$E' = \{(v_{(\{\lambda-1,\lambda\},g)}, v_{(\{\lambda-1,\lambda+1\},g)}), (v_{(\{\lambda-1,\lambda\},g)}, v_{(\{\lambda,\lambda+1\},g)})\}$$

as we've just shown that α fixes $v_{(\{\lambda-1,\lambda+1\},g)}$. Hence α fixes $P(\lambda+1)$ pointwise.
□

We now have all we need to prove the required results, namely

Lemma 7.5.8. *Suppose α is an automorphism of Γ_0 that fixes $v_{(j,0,g)}$ for $j = 1, \ldots, r$. Then α is the identity automorphism.*

Proof: For all $j \neq 1$ we have that $(v_{(j,0,g)}, v_{(\{1,j\},g)})$ is the unique edge leaving $v_{(j,0,g)}$ and that $(v_{(1,0,g)}, v_{(\{1,2\},g)})$ is the unique edge leaving $v_{(1,0,g)}$. So as α fixes $v_{(j,0,g)}$ it follows that α fixes $v_{(\{1,j\},g)}$ and $v_{(\{1,2\},g)}$ for all $j \neq 1$. Thus $P(1)$ is fixed by α pointwise. And so, by Lemma 7.5.7, we have that for all j $P(j)$ is fixed by α pointwise. Thus for all $i \neq j$ α fixes $v_{(\{i,j\},g)}$. Hence α fixes Γ_0 and is the identity automorphism. □

So we now have all we need to prove Theorem 7.5.1. The proof proceeds as follows.

Proof: Define $\Gamma(G) = (V_G, E_G)$ as follows:

$$\begin{aligned} V_G = &\{v_{(j,0,g)} : j = 1, \ldots, r, g \in G\} \\ &\cup\{v_{(\{j,t\},g)} : j = 1, \ldots, r, t \leq j, g \in G\} \\ &\cup\{v_{(i,j,\text{loop},g)} : i \neq j, g \in G\} \end{aligned}$$

and

$$\begin{aligned} E_G = &\{(v_{(j,0,g)}, v_{(\{1,j\},g)}) : j = 2, \ldots, r\} \cup \{(v_{(1,0,g)}, v_{(\{1,2\},g)})\} \\ &\cup\{(v_{(\{i,j\},g)}, v_{(\{i+1,j\},g)}) : 2 \leq i + 1 < j\} \\ &\cup\{(v_{(\{j-1,j)\}}, v_{(\{j+1,j\},g)}) : 1 < j < r\} \\ &\cup\{(v_{(\{i,j\},g)}, v_{(\{i+1,j\},g)}) : j < i \leq r - 1\} \\ &\cup\{(v_{(\{r,j\},g)}, v_{(j,1,g)}) : j < r\} \cup \{(v_{(\{r-1,r\},g)}, v_{(r,1,g)})\} \\ &\cup\{(v_{(j,t,\text{loop},g)}, v_{(j,t+1,\text{loop},g)}) : 1 \leq t < j - 1\} \\ &\cup\{(v_{(j,j,\text{loop},g)}, v_{(j,0,gs_j)})\} \\ &\cup\{(v_{(j,t,\text{loop},g)}, v_{(j,t,\text{loop},g)}) : 1 \leq t \leq j\} \end{aligned}$$

$\Gamma(G)$ is formed by first constructing the Cayley graph $\Gamma(G; S)$ for some set S of r-generating elements of G. The edges of $\Gamma(G; S)$ are labelled by replacing them with the extended edges shown in Figure 7.5 with the vertex $v_{(j,0,g)}$ added on the end in a similar form. Notice that these last vertices are distinguished by the fact that any vertex of this form has an edge to an r-regular vertex and they are the only such vertices. Thus they are distinguished as potential initial vertices of the r-cubes. Moreover, it is clear that any automorphism of this graph must fix the labelled edges, hence, by Lemma 7.5.5 is an element of the group G.

Now, observe that each r-regular vertex is labelled by an element g of G. We now replace each r-regular vertex with the r-cube labelled by g. It is clear that the resulting graph is now $\Gamma(G)$ as defined above. Moreover, any automorphism of the graph that fixed a particular r-regular vertex must now fix the initial vertices of the r-cube that replaces that vertex. And so, by Lemma 7.5.8 such an automorphism fixes all the r-cube. The resulting extended j-th line segment of the r-cube in shown in figure 7.8.

We can now put these two results together. It is clear that any automorphism of $\Gamma(G)$ must preserve the labelling of the edges. It is now also clear that if it fixes any vertex of the form $v_{(j,0,g)}$ then it fixes the r-cube labelled by g. Thus any automorphism of $\Gamma(G)$ is in G.

It is obvious that any element of G is an automorphism of $\Gamma(G)$. Thus $\mathrm{Aut}(\Gamma(G))$ $= G$ as required. □

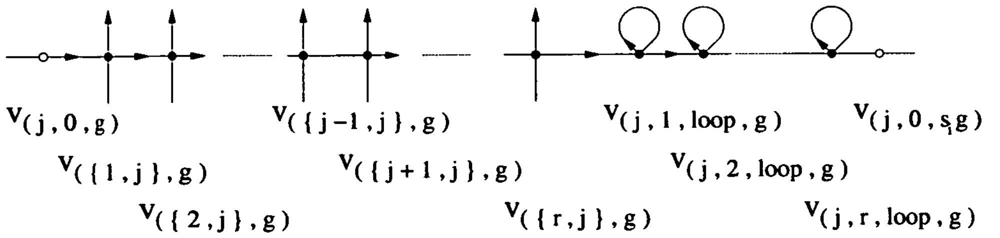


Figure 7.8: The extended j-th line segment of the r-cube labelled by g, with labelling loops.

Corollary 7.5.9. *The number of vertices in* $\Gamma(G)$ *is* $|G|r^2$.

Proof: Simply count them. Each r-cube has $r(r-1)/2$ vertices. Each labelled edge has $j+1$ vertices (including the initial vertex of an r-cube as part of the preceding labelled edge). So the total number of vertices in each r-cube and its preceding labelled edges is

$$\frac{r(r-1)}{2} + \sum_{j=1}^{r} j + 1 = \frac{r(r-1)}{2} + \frac{r(r+1)}{2} = r^2$$

Moreover, as there are $|G|$ vertices in the Cayley graph of G so there are $|G|$ r-cubes in $\Gamma(G)$. Thus

$$|V_{\Gamma(G)}| = |G|r^2$$

as required. □

However, there is no reason to believe that this figure is optimal. There may well be other ways of constructing the r-cube and labelling the edges that result in fewer vertices being used. For example, the method of labelling the edges used here amounts to adding as many copies of a graph on one vertex as required. As there is only one such graph and only one way it may be adapted the number of vertices required to label the r-th edge is r. However, we could use graphs on more vertices to label an edge. Equally there may be ways of simplifying the r-cube.

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