

**Theory and applications
of
Lattice fermionic regularisations**

Tien Dũng Kieu

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**Department of Physics
University of Edinburgh
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*To my parents, my teachers
and to my high-sea companions —the Boat People.*

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問字は一時の恥
問わぬは来代の恥

*Tou wa ichiji no haji,
towanu wa matsudai no haji*

To ask may be a moment's shame, but not to ask
and remain ignorant is a lifelong shame

(Taken from the opening of a book on Kendo)

Abstract

Non-perturbative lattice regularisation and the lattice definition of quantum field theories are described with emphasis on the problem of species doubling for lattice fermions. In particular, the formulation of the multi-species staggered fermions is presented. It is demonstrated that two distinct flavour interpretations of staggered fermions are equivalent in the continuum.

Using lattice fermionic regularisations, the abelian and non-abelian chiral anomalies –the quantum-induced breaking of the symmetries– are derived and their relationship with the doubling phenomenon is clarified. The extra species are generated to cancel the anomalies. To reproduce the anomalies, these doublers are given mass of the order of the cut-off. The abelian anomaly can also be recovered by identifying the lattice axial current whose associated symmetry is broken. This is illustrated in a calculation involving a current of minimal form in the coordinate-space interpretation for staggered fermions. Furthermore, on the basis of the chiral anomalies, it is argued that the no-go theorem, that of the impossibility of avoiding the doubling problem, can be generalised to cover a wider class of regulators. Progress on the study of the chiral Schwinger model using Wilson fermions is reported and further speculations on a possible role of the lattice in the quantisation of anomalous gauge theories are made.

Phenomenological applications of the non-perturbative lattice methods are illustrated in the last chapter. The hypothesis of the operator product expansion is employed throughout to avoid a direct confrontation with the problem of lattice chiral fermions and thus only QCD effects are considered. To further simplify the quenched lattice QCD simulations the effective chiral lagrangian is also used. Results for proton decay in grand unified theories and the nucleon wavefunctions employing Wilson fermions are obtained on an $8^3 \times 16$ lattice at $\beta = 5.7$. A large decay rate for the proton is obtained tending to rule out the minimal grand unified models. In the evaluation of the matrix elements for the weak interactions of hadrons, staggered fermions are employed. An approximate method for the construction of lattice four-quark operators is outlined.

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Chapter 1

Introduction

*“Be discrete, do it on the lattice;
be indiscrete, do it continuously.”*

The first two chapters of this thesis we study the formulation of lattice fermions and the infamous problem of species doubling, the very root of which, the chiral anomalies, is the topic of chapter 3. Apart from the theoretical importance, we also illustrate the side of lattice fermionic regularisations by two phenomenological applications in the last chapter. We shall begin, however, with the motivation and necessity for studying Lattice Gauge Theories (LGT) in general.

1.1 Motivation for Lattice Gauge Theories

Present day elementary particle physics and its successes are based on the gauge invariance principle [Weyl, 1922; Yang and Mills, 1954]. Along the same line with the relativity principle, the gauge principle expresses the freedom at *each* space-time point to choose arbitrary reference frames, might they be used for measuring real space-time distances or some internal (albeit sometimes unobservable) attributes of the system. After all, the physics should not be dependent on the relative ‘positioning’ of our apparatus provided we measure the *same* attribute¹. The principle has provided us with the underlying and unifying framework to describe the various interactions in nature and to progress towards

¹Measurements of different aspects, such as the wave or particle nature of a quantum object, could lead to apparent paradox, see for example [Feynman *et al.*, 1965].

their ultimate unification, thus strengthening our view of a simple and harmonious universe. The complexity we observe, and to which our very existence is due, may be only the manifestation of simple theories as in the crude example of the generation of chaos from simple non-linear deterministic equations. Let us take the example of the hunt for a decaying proton, which establishes the non-conservation of baryonic charge, to illustrate our trust in gauge theories, more of this in chapter 4. On the other hand, the conservation of electric charge is never in doubt since there is a gauge symmetry for the latter but none for the former.

The quantisation of theories with appropriate gauge symmetries yields the Quantum Field Theories (QFT) for various interactions. Examples are the still-troublesome quantum gravity or, with better luck and successes, the theory of strong and electroweak interactions, the so-called Glashow-Salam-Weinberg [Glashow, 1961; Salam, 1968; Weinberg, 1967] or standard model. To make the connection to the lattice we will adopt the path integral quantisation formalism [Feynman and Hibbs, 1965] instead of the operator-valued distribution or canonical quantisation.

In the perturbative treatment of QFT, however, there exist ultraviolet divergences as singularities at the high-energy end of the integration range. If they can be removed sensibly, the theory is called renormalisable, otherwise it is non-renormalisable as is the case of quantum gravity due to the presence of a negative mass-dimension coupling. The singularities in renormalisable theories are absorbed into the redefinitions of the original parameters and fields, or a finite extension of the set, by the renormalisation procedure. This is done by giving a value, a meaning, in general a constraint, to certain quantities at a given scale, *the subtraction point*. Being arbitrary and having no physical significance, the physics should be independent of this scale as is compactly embodied in the renormalisation group (RG) equation [Stueckelberg and Petermann, 1953]. Apart from the elementary quantities, composite operators, singular since they are products of distributions at same space-time point, are also renormalised. We will come back to this at a later point; it suffices to mention that to introduce composite operators into the action we need a complete set, which is closed under the renormalisation mixing, to couple to external sources with appropriate transformation properties. The topic of RG merits a book on its own [Amit,

1984; Collins, 1985], here we briefly sketch one of its applications, the dimensional transmutation; another one relating to the operator product expansion will be mentioned in chapter 4.

When the coupling is dimensionless as in quantum chromodynamics (QCD) and in the chiral limit where there is no mass parameter, the RG invariance of the physics requires a dependence of the coupling on the subtraction point. This relationship allows us to trade a dimensionless quantity for a dimensionful scale. The scale, in turn, is specified by an experimental fact, the proton's mass, say. The theory, in effect, possesses no free parameter to be fiddled with; it is determined uniquely by the gauge group and its representations.

Another related property of QCD, and non-abelian theory in general, which has favoured QCD as the theory for strong interaction, is asymptotic freedom [Politzer, 1973; Gross and Wilczek, 1973]. Asymptotically, the effective coupling constant, which embodies the effect of charge (anti-)screening, gets weaker as the colour charges come closer, a fact that has been observed in deep inelastic scattering [Bjorken, 1967] and also presumably explains the confinement of quarks and gluons.

To arrive at these results, however, we have to introduce a regulator ² to isolate and manipulate the singularities, a process called regularisation. Among the available regulators, lattices stand out prominently.

Even though lattices break the Poincare invariance, which it is hoped will be recovered on removal of the cut-off ³, the gauge symmetry can be maintained at non-zero lattice spacing [Wilson, 1974]. As the formulation of gravity on the lattice is outside the scope of this work we only concern ourselves with internal gauge groups from now on. The discretisation of space-time, furthermore, allows the employment of non-perturbative techniques which are vital as more often than not perturbation theory fails to give satisfactory answers. Consequently, lattice calculations have covered topics ranging from the fundamental properties of the strong interacting theory of QCD (to name a few: confinement [Creutz, 1983], hadronic mass spectrum [Montvay, 1987a], deconfinement at fi-

²Renormalisation can be carried out without the cut-off [Zimmermann, 1970] but this is, however, intrinsically perturbative in nature.

³This is proved in chapter 3 to be the case if locality is observed [Jolicœur *et al.*, 1987].

nite temperature [Pendleton, 1987], topological objects [Smit and Vink, 1987]... to various phenomenological applications (weak matrix elements [Daniel *et al.*, 1987], the proton's lifetime [Bowler *et al.*, 1988], ...). Results from lattice QCD, in particular, will be the decisive tests for the theory through the property of dimensional transmutation. Within present computer power and various approximation schemes there is strong support for the theory so far. Also the lattice approach seems to be the only hope at the moment for the study of the Higgs theory and its phase diagram [Jersak, 1985] which have so far evaded perturbative techniques, in particular the question concerning the triviality of the model [Montvay, 1987b].

From the theoretical point of view, the lattice itself plays an intrinsically fundamental role in the formulation of path integrals. Furthermore, as continuum QFT are usually ill-defined so, constructively, they should and could be regarded as the limits of certain lattice theories at certain *critical* points [Glimm and Jaffe, 1981]. The seemingly artificial renormalisation method of dealing with the infinities has a better footing now in the statistical mechanics theory with gauge symmetry. This view of QFT is not without its problems. A satisfactory lattice theory should have the same weak-coupling properties, beside others, as in some continuum regularisation. However, there is a problem with the fermions on the lattice, the study of which forms one of the objectives of the thesis. Before moving on to this problem in chapters 2 and 3 let us recall the formulation and general properties of LGT in the rest of this chapter.

1.2 Lattice Gauge Theories

1.2.1 Formulation

We will follow the approach of Wilson [Wilson, 1974] using path integral quantisation to formulate LGT. For another approach using Hamiltonian formalism, where the time direction is not discretised, we refer to the reference [Kogut and Susskind, 1975]. The two approaches are connected via the transfer matrix formalism. In the former, the Minkowski space is first Wick rotated to be Euclidean. In going to Euclidean space we have assumed that corresponding results in Minkowski space can be deduced subsequently. This, however, is still

unclear at the non-perturbative level. All the dimensions are then discretised by a regular hypercubic lattice. Alternatively, one can replaced the continuum by a random lattice [Christ *et al.*, 1982] which we will not deal with here though occasional remarks will be made to illustrate and compare with its interesting properties. The generating functional is well-defined as the partition function, in statistical mechanics language, over a (continuous) ensemble of configurations of the dynamical fields $\{\phi_i\}$

$$\mathcal{Z} = \int \prod_i d\phi_i \exp \{-\mathcal{S}[\{\phi_i\}]\}. \quad (1.1)$$

Depending upon their nature, the dynamical fields are associated with lattice sites or links and the lattice action of a configuration of the fields $\mathcal{S}[\{\phi_i\}]$ is then obtained accordingly.

Scalar and fermion fields

The degrees of freedom of these fields are put on lattice sites and the continuum derivatives in the action are approximated by appropriate finite differences. For example, a straightforward transcription, in even d -dimensional Euclidean space, of the free Dirac action using central differences results in the so-called naive fermion action

$$\mathcal{S}_f = (a^d)^2 \sum_{x,y} \bar{\psi}(x) D(x,y) \psi(y), \quad (1.2)$$

where

$$D(x,y) = \sum_{\mu} \frac{\gamma_{\mu}}{2a} \{\delta_{x+a\hat{\mu},y} - \delta_{x-a\hat{\mu},y}\} + m\delta_{x,y}. \quad (1.3)$$

We have adopted the convention that all the fields are dimensionful as is the lattice spacing a . Furthermore, an hermitean definition of the gamma matrices,

$$\begin{cases} \{\gamma_{\mu}, \gamma_{\nu}\} = 2\delta_{\mu\nu} \\ \gamma_5 = \gamma_5^{\dagger} = \begin{cases} i\gamma_1 \cdots \gamma_d, & d = 4k + 2, \\ \gamma_1 \cdots \gamma_d, & d = 4k, \end{cases} \end{cases} \quad (1.4)$$

is used. It should be mentioned, however, that there exists, as presented in the next chapter, an interesting variation for fermions [Kogut and Susskind, 1975] where the components of the field are scattered on neighbouring sites.

The Pauli exclusion principle for the fermionic fields is manifest in the path integral formulation by treating the fields as Grassmann variables

$$\{\psi(x), \psi(y)\} = \{\psi(x), \bar{\psi}(y)\} = \{\bar{\psi}(x), \bar{\psi}(y)\} = 0. \quad (1.5)$$

Because of the departure from the Minkowski signature, we also have to modify the hermitean conjugation together with the anti-commutation of the gamma matrices (1.4). Paying due care to the time-like index, we adopt an heuristic rule for the operation [Smit, 1986]:

1. Applying Minkowski-space rule, i.e. *assuming*

$$\bar{\psi} = \psi^\dagger \gamma_4, \quad (1.6)$$

to work out the result for space-like indices, for example

$$(\bar{\psi}_1 \gamma_m \psi_2)^\dagger = -\bar{\psi}_2 \gamma_m \psi_1, \quad m = 1, 2, 3.$$

2. the time-like indices then follow from covariance, that is in the above example

$$(\bar{\psi}_1 \gamma_4 \psi_2)^\dagger = -\bar{\psi}_2 \gamma_4 \psi_1.$$

With this new definition the fields ψ and $\bar{\psi}$ are independent as the tentative definition (1.6) is no longer applicable in Euclidean space; it serves only as a rule for the space-like part. In fact, it is a simple exercise to show that there exists no matrix Γ such that

$$\bar{\psi} = \psi^\dagger \Gamma$$

by expanding Γ in the basis of the Clifford algebra formed by (1.4). It is also clear that the central differences used in (1.3) guarantees the hermiticity of the action, which is necessary for its probability-related interpretation in numerical simulations –see later.

The appearance of the first order derivative in the fermion action leads to a very fundamental problem of LGT, that of species doubling. In the simplest way, this can be seen in the Fourier transform of the inverse propagator (1.3),

$$\begin{aligned} \tilde{D}(p, q) &\stackrel{def}{=} (a^d)^2 \sum_{x, y} e^{-ipx + iqy} D(x, y), \\ &= \left\{ \sum_{\mu} \frac{i\gamma_{\mu}}{a} \sin(ap_{\mu}) + m \right\} \delta(p - q). \end{aligned} \quad (1.7)$$

The expression exhibits near the 2^d points

$$\bar{p}_\mu = 0 \text{ or } \frac{\pi}{a}$$

the 2^d particle dispersion relations with degenerate mass m and physical momenta $(p - \bar{p})_\mu$. Thus there are more species than one would naively expect. We will study this issue in depth in a later chapter.

To incorporate the gauge interaction we now introduce a gauge invariant formulation on the lattice.

Gauge fields

Apart from the obvious method of putting the gauge fields onto the sites and using finite differences [Patrasciou *et al.*, 1981], we can exploit the nature of such fields as the *connection* in some internal space. The fields, playing similar role to the Christoffel symbol in a curved space, dictate the rotation necessary for an internal reference frame as it is transported from position x to x' along some path $x = x(\xi)$

$$U_p(x, x') = \mathcal{P} \exp \left\{ ig \int_x^{x'} d\xi_\mu A_\mu \right\}, \quad (1.8)$$

which is an element of the gauge group $G = SU(N_c)$. As usual

$$A_\mu(x) \stackrel{def}{=} \sum_b A_\mu^b(x) T^b$$

takes value in the Lie algebra \mathcal{G} of the group in some representation and \mathcal{P} is the path-ordering operator. Since we require the invariance to be local, that is, we are free to internally transform the matter fields at x ,

$$\psi(x) \rightarrow V(x)\psi(x), \quad (1.9)$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}(x)V^\dagger(x), \quad V(x) \in G,$$

the transport operator (1.8) is now unique up to a transformation

$$U_p(x, x') \rightarrow V(x)U_p(x, x')V^\dagger(x'). \quad (1.10)$$

In terms of the gauge potential this requirement is

$$A_\mu(x) \rightarrow V(x)A_\mu(x)V^\dagger(x) - \frac{i}{g}V(x)\partial_\mu V^\dagger(x),$$

in which the inhomogeneous term characterises the ‘non-tensorial’ property of connections. It is then natural to work with and associate the gauge transport operators to lattice links

$$U_p(x, x + a\hat{\mu}) \stackrel{def}{=} U_\mu(x).$$

Furthermore, dealing directly with gauge groups instead of Lie algebras allows the study of local but *discrete* symmetry on the lattice and subsequent use of duality transformations, for example [Cardy, 1980].

The lattice covariant Dirac operator for naive fermions can now be obtained as

$$D(x, y) = \sum_\mu \frac{\gamma_\mu}{2a} \{U_\mu(x)\delta_{x+a\hat{\mu}, y} - U_\mu^\dagger(y)\delta_{x-a\hat{\mu}, y}\}, \quad (1.11)$$

which yields the continuum form provided the gauge links are parameterised in the obvious way

$$U_\mu(x) = \exp \left\{ igaA_\mu(x + \frac{a}{2}\hat{\mu}) \right\}. \quad (1.12)$$

The dynamics of the links is given by the action constructed from gauge invariant objects on the lattice. These are the Wilson loops, traces of products of gauge links around closed loops, of which the smallest is around a plaquette

$$S_g = -\frac{\beta}{4} \sum_x \sum_{\mu \neq \nu} \frac{1}{N_c} \text{tr} \{U_\mu(x)U_\nu(x + a\hat{\mu})U_\mu^\dagger(x + a\hat{\nu})U_\nu^\dagger(x)\}, \quad \beta = \frac{2N_c}{g^2}. \quad (1.13)$$

From the parameterisation (1.12) we see that the trace over a plaquette can be written in exponential form whose argument is, in the limit $a \rightarrow 0$, a closed path integral

$$ig \oint dx_\mu A_\mu(x).$$

We can convert this into a surface integral by the non-abelian Stokes theorem [Szczyzny, 1987]

$$ig \int d\sigma^{\mu\nu} F_{\mu\nu}(x) + (d\sigma)^2 + \dots,$$

which is traceless due to a property of the gauge group $SU(N_c)$. The field strength $F_{\mu\nu}$ has its usual definition

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig[A_\mu, A_\nu]$$

for a non-abelian gauge group. It is clear that in the continuum limit the only non-vanishing contribution to the gauge action (1.13) comes from the lowest order term of the square of the surface integrals, giving

$$S_g \sim \int d^4x \operatorname{tr} \{F_{\mu\nu} F_{\mu\nu}\}, \quad (1.14)$$

apart from a constant which can be absorbed into the normalisation of the partition function.

1.2.2 Properties

In the previous section we have already mentioned duality and discrete gauge groups. In fact, LGT have benefited greatly from the connection with statistical physics as it is possible to borrow ideas and methods from the latter to study QFT. In particular, the method of strong coupling expansion and concepts of phase transitions and critical phenomena will be illustrated here. The weak coupling expansion, an intrinsic method in any regulator, is used in subsequent chapters to address the important question of the validity of lattice regularisations. Or rather, from the point of view of the lattice definition of QFT, it is used to critically review the conventional way of quantising field theories. Furthermore, on the lattice the expansion also assumes extra roles [Hasenfratz and Hasenfratz, 1985]; in particular, it paradoxically relates non-perturbative lattice results to the continuum, more of this later in chapter 4.

The strong coupling expansion in statistical language is just the high temperature expansion which is well-known and natural on the lattice. In this regime, the confinement property of lattice Yang-Mills theories can be shown by applying the expansion to a Wilson loop. For a loop of size R (space) \times T (time) it can be thought of as a pair of heavy quark sources $q\bar{q}$ being created from the vacuum, separated to a distance $r = Ra$ for a time $t = Ta$ before their final annihilation [Cheng and Li, 1984]. Subsequently, it can be shown that for large time

$$\lim_{t \rightarrow \infty} \langle W(R, T) \rangle = e^{-V(r)t},$$

where $V(r)$ is the inter-quark potential. The measure in the generating function is defined as a product of the invariant Haar measures of the compact gauge

group, thus no gauge fixing is necessary on the lattice. For example,

$$\begin{aligned}
\int dU_\mu &= 1, \\
\int dU_\mu U_\mu &= \int dU_\mu U_\mu^\dagger = 0, \\
\int dU_\mu U_\mu^{ij} U_\mu^{\dagger kl} &= \frac{1}{3} \delta_{il} \delta_{jk},
\end{aligned}
\tag{1.15}$$

for $U_\mu \in SU(3)$. We can then expand the exponential function in the expectation value expression for an observable,

$$\langle \mathcal{O} \rangle = \mathcal{Z}^{-1} \int \prod dU_\mu \mathcal{O} e^{-S_g},$$

and perform the integration to obtain a series in $\beta \sim 1/g^2$. Up to the lowest term of this strong coupling expansion for the Wilson loop we get

$$\langle W(R, T) \rangle \sim \left(\frac{1}{g^2} \right)^{RT}, \quad T \gg R \rightarrow \infty,$$

which is called the area law and exhibits the confinement because the potential is linearly rising

$$V(r) = \sigma r, \tag{1.16}$$

$$\sigma = \frac{\ln g^2}{a^2}.$$

There would be no confinement had we, instead, the perimeter law leading to

$$V(r) \sim 1/r.$$

To see if the confinement persists in the continuum theory we have to make sure that there is no phase transition in the passage to this limit, a notion yet to be defined. The area law can be used as an order parameter to identify the phases. The appearance of gauge invariant order parameters such as the plaquette, mass gap, etc... is essential in gauge theory. It is apparent from the integration rules (1.15) that, without gauge fixing, only gauge invariant objects yield non-vanishing expectation values. This expresses the impossibility of spontaneous breaking of a local symmetry [Elitzur, 1975], since a rotation of a local variable does not require the changing of an infinite number of degrees of freedom. We should mention that with the inclusion of dynamical quarks, due to the effect of quark screening, the potential measured from the Wilson loop is no longer inter-quark but rather inter-meson.

1.2.3 The continuum limit

Consider an observable \mathcal{O} with mass dimension $\dim\mathcal{O}$ which by dimensional analysis we can write

$$\mathcal{O} = a^{-\dim\mathcal{O}} f(g, a).$$

The requirement of a finite and cut-off independent value of the observable in *the continuum limit*, as $a \rightarrow 0$, forces a relationship

$$g = g(a) \rightarrow g^* \tag{1.17}$$

such that at the same time

$$f(g(a), a) \rightarrow 0$$

and \mathcal{O} tends to a finite constant. The critical point g^* corresponds to a continuous phase transition where the correlation length in terms of the spacing a , $\xi = (ma)^{-1}$, diverges. In renormalisation language [Wilson and Kogut, 1974], to get the long distance behaviour from the original local action we have to average out the short distance degrees of freedom. The subsequent tuning of the parameters is determined if we want to keep the physics constant. Near the critical point large scale fluctuations could wash away the underlying lattice structure and thus rotational symmetry might be recovered. The definition of the point itself, however, is intrinsically dependent on the observable considered. For a meaningful continuum theory, the function (1.17) should yield finite limits for *all* observables.

Perturbatively, the dependence of the bare coupling on the lattice spacing is similar to the way the renormalised coupling g_R depends on the subtraction point μ

$$\begin{aligned} a \frac{dg}{da} &= -\beta(g), \\ &= \beta_0 g^3 + \beta_1 g^5 + \mathcal{O}(g^7), \end{aligned} \tag{1.18}$$

where $\beta(g)$ is the RG function and

$$\begin{aligned} \beta_0 &= \frac{1}{16\pi^2} \left\{ \frac{11}{3} N_c - \frac{2}{3} N_f \right\}, \\ \beta_1 &= \frac{1}{(16\pi^2)^2} \left\{ \frac{102}{3} N_c - \frac{38}{3} N_f \right\}. \end{aligned} \tag{1.19}$$

These first two coefficients are scheme-independent and N_f is the number of flavours.

From the enforcement of physics

$$0 = a \frac{d\mathcal{O}}{da} = a \frac{\partial \mathcal{O}}{\partial a} + \left(a \frac{dg}{da} \right) \frac{\partial \mathcal{O}}{\partial g},$$

we see that the critical points are the zeros of the beta function (1.18) thus the name RG *fixed points*. Even though the form of the beta function is scheme dependent, its zeros should be unique for the universality of the continuum limit. For $SU(N_c)$ gauge theory the fixed point is

$$g^* = 0.$$

We can solve equation (1.18) to get

$$a = \Lambda_{\text{latt}}^{-1} (\beta_0 g^2)^{\beta_1/2\beta_0^2} \exp \{-1/2\beta_0 g^2\} \{1 + O(g^2)\} \quad (1.20)$$

with the integration constant Λ_{latt} . This cut-off independent parameter sets the mass scale for the scheme

$$\mathcal{O} \rightarrow \text{constant } \Lambda_{\text{latt}}^{\dim \mathcal{O}}; \quad g, a \rightarrow 0 \quad (1.21)$$

Perturbation theory can relate the scales for different renormalisation schemes, in particular lattice results to the continuum. But perturbation theory alone cannot yield information about the essential singularity at vanishing bare coupling and that of the constant in equation (1.21). This necessitates the use of non-perturbative techniques.

Knowing the continuum limit is in the weak coupling regime one can try to extrapolate the physics from the strong coupling end, see for example [Kogut *et al.*, 1979], in the hope that no phase transition will be encountered along the way. However, a better and more direct approach is the use of numerical simulations on the lattice. The extraction of relevant physics is done in both the continuum and thermodynamic limits at the same time. Onset of the continuum behaviours is indicated by the property of *asymptotic scaling* expressed in equations (1.20) and (1.21) and can be studied by the method of Monte Carlo RG [Bowler *et al.*, 1985].

1.2.4 Numerical simulations

We want to evaluate the Green's function⁴ on the lattice

$$\langle O|T A_I(x)f(x',\dots)|O\rangle = \mathcal{Z}^{-1} \int \prod_{\text{links}} dU_\mu \prod_{\text{sites}} d\bar{\psi}d\psi A_I(x)f(x',\dots) e^{-S_g-S_f}. \quad (1.22)$$

Physical masses and matrix elements can then be measured if we consider the zero momentum correlators at asymptotically large time

$$a^3 \sum_{\vec{x}} \langle A_I(\vec{x},t)f(x',\dots)\rangle \sim \sqrt{Z_A} e^{-m_A t} \langle A(\vec{p}=0)|f(x',\dots)|O\rangle, \quad (1.23)$$

$|A_I(\vec{p})\rangle$ is the state with the lightest mass m_A ; more massive states are suppressed in this limit. Z_A can be found from the correlator of A with itself

$$a^3 \sum_{\vec{x}} \langle A_I^\dagger(x)A_J(0)\rangle \sim Z_A \delta_{IJ} e^{-m_A t}. \quad (1.24)$$

Due to the Grassmann nature of the fermionic variables the form (1.22) is not yet suitable for numerical considerations. We can, however, integrate out the degrees of freedom, since the action is bilinear in the Grassmann fields, according to the rules [Ramond, 1981]

$$\begin{aligned} \int d\bar{\psi}d\psi &= 0, \\ \int d\bar{\psi}d\psi \psi &= \int d\bar{\psi}d\psi \bar{\psi} = 0, \\ \int d\bar{\psi}d\psi \bar{\psi}\psi &= -\int d\bar{\psi}d\psi \psi\bar{\psi} = 1. \end{aligned} \quad (1.25)$$

That is, c.f. Wick's theorem,

$$\int d\bar{\psi}d\psi \psi_{i_1} \bar{\psi}_{j_1} \cdots \psi_{i_n} \bar{\psi}_{j_n} e^{-S_f} = \sum_P (-1)^P D_{i_{P_1}j_{P_1}}^{-1} \cdots D_{i_{P_n}j_{P_n}}^{-1} \det D. \quad (1.26)$$

The expectation value is now evaluated as the average over an ensemble of N gauge configurations $\{U_\mu\}$,

$$\bar{f} = \frac{1}{N} \sum^N f(\{U_\mu\}), \quad (1.27)$$

provided the dynamical fields are generated in such a way that the probability to get a configuration $\{U_\mu\}$ is proportional to

$$e^{-S_g(\{U_\mu\})} [\det D]^{N_f}.$$

⁴For a discussion of the ambiguities in the T -product, see, for example [Pokorski, 1987].

It is apparent that a probability interpretation is possible only if $\det D$ is real and the gauge group is now in a reducible representation as a direct sum of N_f fundamental representations. The statistical error for large N and statistically independent configurations decreases, however, notoriously slowly as $1/\sqrt{N}$. Besides, there are systematic errors from the finite size of the system and from the boundary conditions used.

Furthermore, evaluation of the fermionic determinant is time-consuming, due to its extremely non-local nature, in comparison with that of the columns of inverses of sparse matrices in (1.26). The *quenched* or *valence approximation* amounts to the replacing the determinant by unity, that is as if the number of flavours is zero. However, it is hoped that such ignorance of internal fermion loops will not alter the leading order effect, qualitatively at least. The inclusion of dynamical fermions requires more sophisticated algorithms, e.g. the molecular dynamics [Polonyi and Wyld, 1983], Langevin [Batrouni *et al.*, 1985] or hybrid [Duane *et al.*, 1987] methods. All the numerical results in this thesis have been obtained in the quenched approximation nevertheless.

Chapter 2

More on Lattice Fermionic Regularisations

“In this chapter we turn to a subject that is still not completely understood, the lattice formulation of fermionic fields.”

The quotation above was written four years ago [Creutz, 1983] and is still applicable today, even with the fast pace in the field of LGT. The problem is the doubling phenomenon touched upon in the last chapter. It is the most important problem of LGT in particular, and of the lattice formulation QFT in general.

With the freedom available on the lattice, we can avoid the doubling problem by introducing some irrelevant term that vanishes as the cut-off is removed. An illustration of this is Wilson’s formulation of lattice fermions [Wilson, 1977]. However, the cost to be paid is further symmetry breaking. In fact, there exists a no-go theorem [Nielsen and Ninomiya, 1981a, b and c] which states the impossibility of a solution for this problem. Some aspects of the theorem will be discussed in this chapter.

Away from the theorem and Wilson fermions, emphasis, however, is placed on another popular lattice fermion formulation, the multi-species staggered fermions, and the associated symmetry group. The work on the uniqueness of the flavour interpretation for the fermions has been presented in a publication [Daniel and Kieu, 1986]. Some related materials have also appeared elsewhere [Kieu, 1987a].

The chapter ends with some remarks on attempts to evade the doubling problem.

2.1 The fermion doubling problem revisited

Having pointed out in the last chapter that extra fermionic species appear as spurious poles of the propagator, we now proceed to obtain a mathematical representation for those fields [Sharatchandra *et al.*, 1981]. The action of the free theory in momentum space is

$$\mathcal{S} = \int_p \bar{\psi}(p) \left\{ \sum_{\mu} \frac{i\gamma_{\mu}}{a} \sin ap_{\mu} + m \right\} \psi(p), \quad (2.1)$$

where we have defined, in accordance with (1.7), the Fourier transforms of the fermionic fields as

$$\bar{\psi}(p) = a^d \sum_x e^{-ipx} \psi(x), \quad (2.2)$$

$$\psi(p) = a^d \sum_x e^{ipx} \bar{\psi}(x).$$

The integration domain over the first Brillouin zone $[-\frac{\pi}{a}, \frac{\pi}{a}]^d$,

$$\int_p \stackrel{def}{=} \int \frac{d^d p}{(2\pi)^d}, \quad (2.4)$$

can be partitioned into 2^d hypercubes

$$\begin{aligned} p &= k + \frac{\pi}{a} g \pmod{\frac{2\pi}{a}}, \\ k &\in \left[-\frac{\pi}{2a}, \frac{\pi}{2a}\right]^d = \mathcal{C}, \\ g &\in \{0, 1\}^d. \end{aligned} \quad (2.5)$$

With the use of the identities

$$\gamma_{\mu} \sin(ak_{\mu} + \pi g) = (-1)^{g_{\mu}} \gamma_{\mu} \sin ak_{\mu}$$

and

$$\begin{aligned} (-1)^{g_{\mu}} \gamma_{\mu} &= [(i\gamma_1 \gamma_5)^{g_1} \cdots (i\gamma_d \gamma_5)^{g_d}]^{\dagger} \gamma_{\mu} [\cdots], \\ &\stackrel{def}{=} M_g^{\dagger} \gamma_{\mu} M_g, \end{aligned} \quad (2.6)$$

we can rewrite (2.1) as the sum of 2^d independent species corresponding to different g 's

$$\mathcal{S} = \sum_g \int_k^{\mathcal{C}} \bar{q}^g(k) \left\{ \sum_{\mu} \frac{i\gamma_{\mu}}{a} \sin ak_{\mu} + m \right\} q^g(k), \quad (2.7)$$

the subscript \mathcal{C} denotes the restriction of the integration variables to the domain (2.5). The species interpretation is given by

$$\begin{aligned}\tilde{q}^g(k) &= M_g \tilde{\psi}(k + \frac{\pi}{a}g), \\ \bar{\tilde{q}}^g(k) &= \bar{\tilde{\psi}}(k + \frac{\pi}{a}g) M_g^\dagger, \\ M_g &= (i\gamma_1\gamma_5)^{g_1} \cdots (i\gamma_d\gamma_5)^{g_d}.\end{aligned}\tag{2.8}$$

Note that the mass term is not affected since

$$M_g M_g^\dagger = M_g^\dagger M_g = 1.$$

The propagator for each piece in (2.7) now has only one pole at $k_\mu = 0$ and thus contains only one species. In other words, in each newly partitioned sector the set of $\{(-1)^{g_\mu}\gamma_\mu\}$ matrices forms an irreducible representation of the Clifford algebra. The particle identification can thus be imposed in each sector.

If one takes the transformation

$$\begin{aligned}\psi(x) &\rightarrow e^{i\epsilon\gamma_5}\psi(x), \\ \bar{\psi}(x) &\rightarrow \bar{\psi}(x)e^{i\epsilon\gamma_5}\end{aligned}\tag{2.9}$$

to be the global $U_A(1)$ axial transformation on the lattice, different species behave differently with respect to the chiral property. That is, the species characterised by g is transformed infinitesimally as

$$\begin{aligned}\delta\tilde{q}^g(k) &= i\epsilon (M_g^\dagger \gamma_5 M_g) \tilde{q}^g(k), \\ &= [(-1)^{\sum_\mu g_\mu}] i\epsilon \gamma_5 \tilde{q}^g(k).\end{aligned}\tag{2.10}$$

From which its chiral charge can be deduced right away

$$Q_{\text{chiral}}^g = (-1)^{\sum_\mu g_\mu}.\tag{2.11}$$

The total charge, however, sums up to zero,

$$Q_{\text{chiral}}^{\text{total}} = \sum_\mu Q_{\text{chiral}}^g = 0,\tag{2.12}$$

in agreement with the fact that (2.9) is an invariance of the naive fermion action.

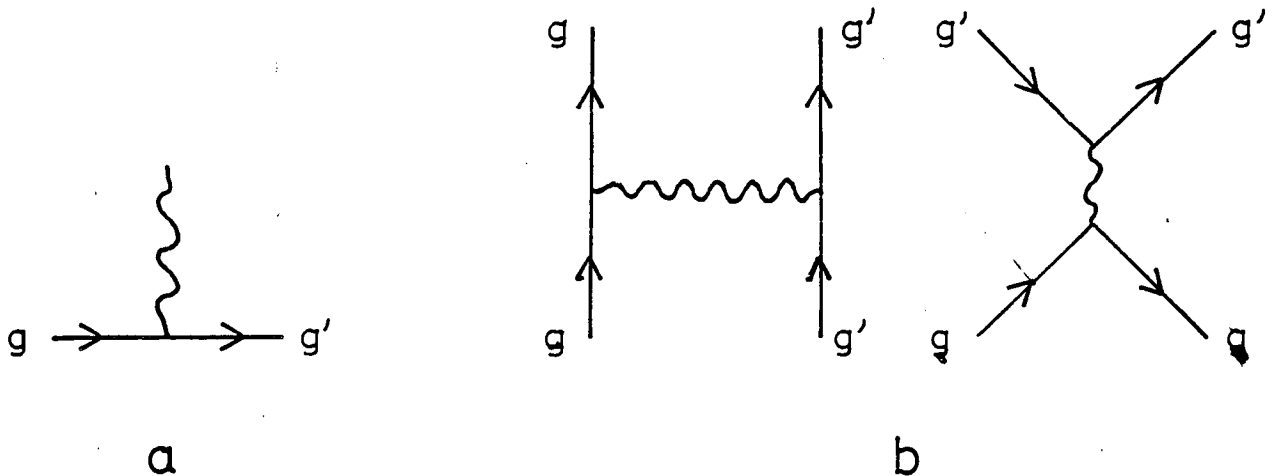


Figure 2.1: Even though species doublers cannot contribute in (1a), their presence is manifest in (1b).

The doubling remains a problem with interacting theories as the doubling transformations (2.8) are the elements of a diagonal subgroup of the larger symmetry group $U_L(2^{d/2}) \otimes U_R(2^{d/2})$ [Kawamoto-Smit, 1981], which survives the introduction of interactions. In fact, extra species can be avoided in the free theory only to reappear as soon as the interaction is switched on, for an explicit example see [Dagotto *et al.*, 1986]. This issue is raised again in the no-go theorem section later.

Graphically speaking, even though, this doubling phenomena cannot result in the process of fig. 2.1a¹ but the species doublers can still take parts in the processes depicted in fig. 2.1b.

2.2 Wilson fermions

One way to remove the doublers, namely the species characterised by $g \neq (0, \dots, 0)$ in (2.8), is to send their propagators to zero as the cut-off is removed. To realise this decoupling task, we can introduce an irrelevant term, which van-

¹This is the result of the momentum conservation at the vertex (modulo $\frac{2\pi}{a}$) as the momentum of the gauge field, either internal or external, in the limit $a \rightarrow 0$ can never be of order $O(1/a)$.

ishes formally with the lattice spacing, by exploiting the intrinsic ambiguity of the regularisation procedure. Such freedom has also been used, for instance, to improve the approach of the lattice action to the continuum limit. Irrelevant as it is, the term has to break the doubling symmetry (2.8) and can violate other physical properties. As the chiral group contains the doubling transformations, it is broken, a first sign of the incompatibility between chiral invariance and no doubling.

The Wilson fermion formulation [Wilson, 1977] amounts to adding the term

$$D^W(x, y) = -\frac{r}{2a} \sum_{\mu} \left\{ U_{\mu}(x) \delta_{x+a\hat{\mu}, y} + U_{\mu}^{\dagger}(y) \delta_{x-a\hat{\mu}, y} - 2\delta_{x, y} \right\} \quad (2.13)$$

to the naive fermion operator (1.3). As can be seen in momentum space for the free case,

$$D^W(p, q) = -\frac{r}{a} \sum_{\mu} (1 - \cos ap_{\mu}) \delta(p - q), \quad (2.14)$$

the effect of this term is to give additional mass of order $O(1/a)$ to the doublers,

$$m + 2\frac{r}{a} \left(\sum_{\mu} g_{\mu} \right). \quad (2.15)$$

Thus in the continuum limit all but one species are decoupled and the r -dependence can then be absorbed by renormalisation.

The explicit breaking of the chiral symmetry, however, creates some difficulties. First of all there is the fine-tuning problem. That is, even with zero bare mass there still arises a mass-dimension counterterm proportional to $1/a$ due to the lack of a symmetry to prevent it. Consequently, for a vanishing renormalised mass the bare mass has to be tuned appropriately. This poses a problem for numerical simulation. Fortunately, the *ad hoc* tuning can be achieved in QCD when the lattice counterpart of the pion becomes massless. At this critical value of the bare mass, the chiral symmetry is said to be restored albeit in the Goldstone realisation [Bochichio *et al.*, 1985].

Most important of all, without the chiral symmetry, Wilson fermions cannot accommodate Weyl particles nor can they be used for the investigation of dynamical breaking of the symmetry. The perturbative mixing, furthermore, is worsened in this formulation. Nevertheless, owing to their simple and straightforward interpretation and the way of embedding the lattice symmetry in the



Figure 2.2: One-loop diagrams giving rise to the $1/a$ mass counterterm for Wilson fermions (gauge fixing is necessary here and in all WCPT).

continuum group, Wilson fermions have been employed extensively [Bowler *et al.*, 1983; Bernard *et al.*, 1985]. An example is found in chapter 4.

2.3 Staggered fermions

We now move on to another interesting formulation of lattice fermions. These fermions are sometimes called Kogut-Susskind or Susskind fermions [Kogut and Susskind, 1975; Susskind, 1977]. We, however, adhere to the name ‘staggered’ to distinguish with another formulation, equivalent but only at the free level, to be assigned with the name Dirac-Kahler [Becher and Joos, 1982].

Scalar fields on the lattice are seen to be associated with the sites; vector gauge fields with the links (as two points are necessary to define a vector); and rank-two tensor fields with the plaquettes. Perhaps with fermions fields of several components, we should spread them over neighbouring sites, keeping up the geometrical association, instead of just on a single site. In QCD, this treatment of the quarks on a coarser scale than the gluons seems appropriate. It is perhaps important to include the quarks at long wavelengths, where the gluons are frozen out by the large glueball mass, but less so at short wavelengths, where gluon contribution dominate [Wilczek, 1987]. Nevertheless, to justify the spreading –the staggering– we start from the ‘well-founded’ naive fermions.

There are two approaches from the naive lattice operator (1.3). Sharatchandra *et al.* [1981] used a projection operator to restrict the Grassmann measure; the fields are projected onto a one-dimensional space. Only one component for each of $\bar{\psi}, \psi$ thus survives at each site. However, we will follow the approach of

spin diagonalisation [Kawamoto and Smit, 1981].

Being a symmetry of the naive action the doubling transformations (2.8) commute with the lattice Dirac operator. In coordinate space these transformations become

$$\begin{aligned}\psi(x) &\rightarrow H_g(x)\psi(x), \\ \bar{\psi}(x) &\rightarrow \bar{\psi}(x)H_g^\dagger(x),\end{aligned}\tag{2.16}$$

where

$$\begin{aligned}H_g(x) &= [(-1)^{x_1/a}i\gamma_1\gamma_5]^{g_1} \dots [(-1)^{x_d/a}i\gamma_d\gamma_5]^{g_d}, \\ &\stackrel{def}{=} [h_1(x)]^{g_1} \dots [h_d(x)]^{g_d}.\end{aligned}\tag{2.17}$$

The hermitean matrices h_μ form a representation of the Clifford algebra,

$$\{h_\mu(x), h_\nu(x)\} = 2\delta_{\mu\nu},$$

which spans the spinor space of $2^{d/2} \times 2^{d/2}$ matrices. The spinor part of $D_{\text{naive}}(x, y)$ thus commutes with these matrices and, as a result, is proportional to the identity in some unitarily equivalent representation by Schur's lemma (for example see [Elliot and Dawber, 1979]). One basis for such representation is given via the unitary Kawamoto-Smit transformations

$$\Gamma_{x/a} = \gamma_1^{x_1/a} \dots \gamma_d^{x_d/a}\tag{2.18}$$

which transform $h_\mu(x)$ as

$$\Gamma_{x/a}^\dagger h_\mu(x) \Gamma_{x/a} = i\gamma_\mu\gamma_5.$$

In this diagonalising basis,

$$\begin{aligned}\phi(x) &= \Gamma_{x/a}^\dagger \psi(x), \\ \bar{\phi}(x) &= \bar{\psi}(x) \Gamma_{x/a},\end{aligned}\tag{2.19}$$

the naive lattice Dirac operator becomes

$$\left[\Gamma_{x/a}^\dagger D(x, y) \Gamma_{y/a} \right]_{\alpha\beta} = \sum_\mu \frac{\alpha_\mu(x)}{2a} \{ \delta_{x+a\hat{\mu}, y} - \delta_{x-a\hat{\mu}, y} + m \} \delta_{\alpha\beta},\tag{2.20}$$

where α, β are the spinor indices and

$$\begin{aligned}\alpha_\mu(x) &= \frac{1}{2^{d/2}} \text{tr} \left\{ \Gamma_{x/a}^\dagger \gamma_\mu \Gamma_{x/a \pm \hat{\mu}} \right\}, \\ &= (-1)^{[x_1 + \dots + x_{\mu-1}]/a}.\end{aligned}\tag{2.21}$$

Note that the transformation (2.18) is not unique but satisfies the constraint [Banks and Windey, 1982]

$$\begin{aligned}\prod_{\text{around plaquette}} \Gamma_{x/a}^\dagger \gamma_\mu \Gamma_{x/a \pm \hat{\mu}} &= \gamma_\mu \gamma_\nu \gamma_\mu \gamma_\nu, \mu \neq \nu, \\ &= -1.\end{aligned}\tag{2.22}$$

The action then contains $2^{d/2}$ pieces bilinear and replicated in $\phi_\alpha(x)$, $\alpha = 1, \dots, 2^{d/2}$. The desired action for one-component Grassmann fields χ and $\bar{\chi}$ at each site is next obtained by discarding all but one of the copies

$$\begin{aligned}\mathcal{S} &= (a^d)^2 \sum_{x,y} \bar{\chi}(x) D^s(x,y) \chi(y), \\ D^s &= \sum_\mu \frac{\alpha_\mu(x)}{2a} \{ \delta_{x+a\hat{\mu},y} - \delta_{x-a\hat{\mu},y} \} + m.\end{aligned}\tag{2.23}$$

This action describes $2^{d/2}$ degenerate species with mass m , i.e. $2^{-d/2}$ times the number of naive fermions, as the number of degrees of freedom at the site is reduced by $2^{d/2}$ -fold. The species can be identified either in coordinate space [Gliozzi, 1982; Kluberg-Stern *et al.*, 1983] or in momentum space [Sharatchandra *et al.*, 1981; Van den Doel and Smit, 1983; Golterman and Smit, 1984a].

2.3.1 Flavour interpretation in coordinate space

We first partition the lattice into hypercubes

$$\begin{aligned}x_\mu &= r_\mu + a\eta_\mu, \\ \frac{r_\mu}{2a} &\in Z, \\ \eta_\mu &= 0 \text{ or } 1\end{aligned}\tag{2.24}$$

and introduce the notation

$$\begin{aligned}\chi_\eta(r) &= \frac{1}{2^{d/2}}\chi(r + a\eta), \\ \bar{\chi}_\eta(r) &= \frac{1}{2^{d/2}}\bar{\chi}(r + a\eta).\end{aligned}\tag{2.25}$$

We also introduce a matrix definition [Kieu, 1987a]

$$\left(\overline{\Gamma_s \otimes T_f}\right)_{\eta\eta'} = \frac{1}{2^{d/2}}\text{tr} \left\{ \Gamma_\eta^\dagger \Gamma_s \Gamma_{\eta'} \Gamma_f \right\},\tag{2.26}$$

where

$$T_f = \Gamma_f^* = \Gamma_f^t.$$

This notation resembles a direct product of a spinor matrix Γ_s and a flavour matrix T_f . Indeed, it can be shown to be so. That is,

$$\overline{\Gamma_s \otimes T_f \Gamma_{s'} \otimes T_{f'}} = \overline{\Gamma_s \Gamma_{s'} \otimes T_f T_{f'}}\tag{2.27}$$

and the trace also behaves as

$$\text{tr} \left\{ \overline{\Gamma_s \otimes T_f} \right\} = (\text{tr} \Gamma_s) (\text{tr} T_f).\tag{2.28}$$

To arrive at these results, the orthogonality identity

$$\sum_A \Gamma_A^{\alpha a} \Gamma_A^{\dagger b \beta} = 2^{d/2} \delta_{\alpha\beta} \delta_{ab}$$

has been employed. Also from this identity we can unitarily transform $\overline{\Gamma_s \otimes T_f}$ into $\Gamma_s \otimes T_f$,

$$\left(\Gamma_s \otimes T_f\right)^{\alpha a, \beta b} = \sum_{\eta\eta'} \left(\frac{1}{2^{d/4}} \Gamma_\eta^{\alpha a} \right) \left(\overline{\Gamma_s \otimes T_f}\right)_{\eta\eta'} \left(\frac{1}{2^{d/4}} \Gamma_{\eta'}^{\dagger b \beta} \right),\tag{2.29}$$

since, from (2.27) and (2.28), they both form some representations of the direct product of the two Clifford algebras and the algebra is known to have unique representation in even-dimensional space.

Consequently, the staggered fermion operator (2.23) can be written as

$$\begin{aligned}D^s(r, r') &= \frac{1}{4a} \sum_\mu \left\{ \gamma_\mu \otimes \mathbf{1} [\delta_{r+2a\hat{\mu}, r'} - \delta_{r-2a\hat{\mu}, r'}] \right. \\ &\quad \left. + \gamma_5 \otimes t_\mu t_5 [\delta_{r+2a\hat{\mu}, r'} + \delta_{r-2a\hat{\mu}, r'} - \delta_{r, r'}] \right\} \\ &\quad + m \mathbf{1} \otimes \mathbf{1}\end{aligned}\tag{2.30}$$

in the basis of, from (2.29),

$$q^{\alpha\alpha}(r) = \frac{1}{2^{d/4}} \sum_{\eta} \Gamma_{\eta}^{\alpha\alpha} \chi_{\eta}(r), \quad (2.31)$$

$$\bar{q}^{\alpha\alpha}(r) = \frac{1}{2^{d/4}} \sum_{\eta} \bar{\chi}_{\eta}(r) \Gamma_{\eta}^{\dagger\alpha\alpha}.$$

Alternatively, in the basis of $(\bar{\chi}_{\eta}, \chi_{\eta})$, we can replace $\Gamma_s \otimes T_f$ by $\overline{\Gamma_s \otimes T_f}$ in the expression (2.30). As the lattice spacing increases by two fold, it can be seen that now we have $2^{d/2}$ flavours, clearly characterised by the roman index of (2.31).

Note also that there is what seems to be a flavour-dependent ‘mass’ term. Because of this dependence there is, beside the vector symmetry generated by $1 \otimes 1$, some remnant of the continuous axial symmetry. The generator for this $U(1)$ axial invariance is

$$\gamma_5 \otimes t_5.$$

In terms of $\bar{\chi}$ and χ fields the two symmetries manifest as

$$\begin{aligned} \chi(x) &\rightarrow e^{i\alpha} \chi(x), \\ \bar{\chi}(x) &\rightarrow e^{i\alpha} \bar{\chi}(x), \end{aligned} \quad (2.32)$$

for the vector invariance; and

$$\begin{aligned} \chi(x) &\rightarrow e^{i\alpha\epsilon(x)} \chi(x), \\ \bar{\chi}(x) &\rightarrow e^{i\alpha\epsilon(x)} \bar{\chi}(x), \end{aligned} \quad (2.33)$$

for the axial. They are the surviving transformations of the chiral $U_L(2^{d/2}) \otimes U_R(2^{d/2})$ group that act trivially on the spinor space of naive fermions. This is expected because we have discarded all but one of the original components. Also as a result of the spreading of the components over a hypercube the lattice transformations intermingle with the flavour group in this interpretation. The situation, as will be seen, also arises for the interpretation in momentum space.

Gauge fields can be introduced either at the level after the interaction, i.e. eq. (2.30), or at the original level of eq. (2.23). In the first case, the gauge links extend over two lattice spacings. The fermions are named Dirac-Kahler [Mitra

and Wiesz, 1983; Napoly, 1983], which also suffer the fine tuning disease as the fine structure of the hypercube is erased. This kind of fermions can thus be regarded as belonging to the same class as Wilson fermions. In fact, there exists a lattice action interpolating the two fermions [Verstegen, 1984]. We call this class of lattice fermions the *doubler-decoupling class* as the ‘mass’ of the doublers becomes infinitely heavy in the continuum limit.

On the other hand, introducing gauge links directly in the eq. (2.23) results in the quite distinct staggered lattice fermions. It is easily seen that there are d link variables per $2^{d/2}$ flavours in the Dirac-Kahler versus $d \times 2^{d/2}$ ($2^{d/2}$ is the number of vertices of a hypercube) link variables for the same number of flavours in the latter. The important distinction, however, is that for staggered fermions there is a single lattice spacing shift invariance [Golterman and Smit, 1984a], which manifests itself in the q -basis of equation (2.31) as discrete flavour symmetry in the continuum limit. Together with the symmetry (2.33), they are enough to prevent the mass-counterterm, a clear advantage.

The basis (2.31) may now be made gauge covariant

$$\begin{aligned} Q^{\alpha a}(r) &= \frac{1}{2^{d/4}} \sum_{\eta} \Gamma_{\eta}^{\alpha a} \mathcal{U}_{\eta}(r) \chi_{\eta}(r), \\ \bar{Q}^{\alpha a}(r) &= \frac{1}{2^{d/4}} \sum_{\eta} \bar{\chi}_{\eta}(r) \Gamma_{\eta}^{\dagger \alpha a} \mathcal{U}_{\eta}^{\dagger}(r), \end{aligned} \tag{2.34}$$

where

$$\mathcal{U}_{\eta}(r) = [U_1(r)]^{\eta_1} \cdots [U_d(r + a(\eta_1 + \cdots + \eta_{d-1}))]^{\eta_d}$$

is the product of link variables $U_{\mu}(r)$ along a definite path going from r to $r + a\eta$.

2.3.2 Flavour interpretation in momentum space

This identification, not surprisingly, is similar to the species identification of the naive fermions in the first section of the chapter owing to the close link between the two kinds of fermions.

We split the first Brillouin zone as in (2.5) but with g being replaced by $A \in \{0, 1\}^d$ for a distinction. The notations

$$\tilde{\phi}_A(k) = \tilde{\chi}(k + \frac{\pi}{a}A),$$

(2.35)

$$\bar{\phi}_A(k) = \bar{\chi}(k + \frac{\pi}{a}A)$$

are then introduced. At the same time, the staggered fermion lattice operator (2.23) is Fourier transformed into

$$\bar{D}^s(p, q) = (2\pi)^d \sum_{\mu} \frac{i}{a} \sin ap_{\mu} \delta(p - q + \frac{\pi}{a}\bar{\mu}) + (2\pi)^d \delta(p - q)m. \quad (2.36)$$

The argument of the first (periodic) delta function contains a term proportional to

$$\bar{\mu} = \sum_{\nu=1}^{(\mu-1)} \hat{\nu}$$

for the phase factor $\alpha_{\mu}(x)$ in (2.23) can be written as

$$\alpha_{\mu}(x) = \exp \left\{ i \frac{\pi}{a} \bar{\mu} \cdot x \right\}.$$

From the partitioning of p and q

$$\begin{aligned} p &= k + \frac{\pi}{a}A, \\ q &= l + \frac{\pi}{a}B, \end{aligned}$$

we can regard the operator as a matrix with indices A and B [Golterman and Smit, 1984a]. Following the notation of the last section, the operator can be rewritten as [Daniel and Sheard, 1987]

$$D_{AB}^s(k, l) = (2\pi)^d \delta(k - l) \left\{ \sum_{\mu} \frac{i}{a} \sin ap_{\mu} \overline{\gamma_{\mu}} \otimes \overline{\mathbf{1}}_{AB} + m \overline{\mathbf{1}} \otimes \overline{\mathbf{1}}_{AB} \right\}. \quad (2.37)$$

Once again signs of the spin-flavour structure appear. In fact, it can be shown that [Daniel and Kieu, 1986] there exists an unitary relationship with the ordinary direct product

$$(\Gamma_s \otimes T_f)^{\alpha a, \beta b} = \sum_{A, B} V_A^{\alpha a} (\overline{\Gamma_s \otimes T_f})_{AB} V_B^{* \beta b}, \quad (2.38)$$

$$V_A^{\alpha a} = \sum_B \left[\frac{(-1)^{A \cdot B}}{2^{d/2}} \right] \left[\frac{1}{2^{d/4}} \Gamma_B^{\alpha a} \right].$$

Thus to separate the spinor index from the flavour, we further rotate our basis

$$\begin{aligned} \tilde{\psi}^{\alpha a}(k) &= \sum_A V_A^{\alpha a} \bar{\phi}_A(k), \\ \bar{\tilde{\psi}}^{\alpha a}(k) &= \sum_A \bar{\phi}_A(k) V_A^{* \alpha a}. \end{aligned} \quad (2.39)$$

In this basis we replace $\overline{\Gamma_s \otimes T_f}$ by $\Gamma_s \otimes T_f$ in the flavour symmetric operator (2.37). Note that our definition of (2.39) differs from the one in [Golterman and Smit, 1984a] by a rotation in flavour space. This is acceptable as flavour symmetry is clearly exhibited.

Combining with eq. (2.29) we can relate $\overline{\Gamma_s \otimes T_f}$ to $\overline{\Gamma_s \otimes T_f}$ by another unitary transform

$$(\overline{\Gamma_s \otimes T_f})_{AB} = \sum_{\eta, \eta'} \left[\frac{(-1)^{A \cdot \eta}}{2^{d/2}} \right] (\overline{\Gamma_s \otimes T_f})_{\eta\eta'} \left[\frac{(-1)^{B \cdot \eta'}}{2^{d/2}} \right]. \quad (2.40)$$

There are now two seemingly distinct interpretations for the staggered formulation. The one in momentum space preserves the flavour symmetry whereas in coordinate space it is broken up to order $O(a)$. With interactions switched on, whether the broken flavour symmetry in coordinate space interpretation is restored in the continuum limit is a non-trivial question [Jolicœur *et al.*, 1986]. The task can be made easy by the proof of the equivalence of the interpretations [Daniel and Kieu, 1986].

Directly from (2.31) and (2.39), we have

$$\begin{aligned} \tilde{\psi}(k) &= T(k)\tilde{q}(k), \\ \bar{\tilde{\psi}}(k) &= \bar{\tilde{q}}(k)T^\dagger(k). \end{aligned} \quad (2.41)$$

The unitary transformation $T(k)$ is

$$\begin{aligned} T^{\alpha a, \beta b}(k) &= \frac{1}{2^{d/2}} \sum_A \Gamma_A^{\alpha a} \Gamma_A^{\beta b} e^{-iak \cdot A}, \\ &= \exp \left\{ \frac{ia}{2} \sum_\mu k_\mu (-\mathbf{1} \otimes \mathbf{1} + \gamma_\mu \gamma_5 \otimes t_\mu t_5) \right\}^{\alpha a, \beta b}, \end{aligned} \quad (2.42)$$

which reduces to

$$\rightarrow (\mathbf{1} \otimes \mathbf{1})^{\alpha a, \beta b} \text{ as } a \rightarrow 0.$$

The uniqueness of the flavour interpretation is thus established in the continuum limit and is evidently valid even in the general interacting case. This uniqueness is up to a unitary rotation in flavour space as with mass degeneracy any linear combination of flavours works equally well.

To make full use of the equivalence (2.42) we list the symmetry group of staggered fermions next.

2.3.3 The symmetry group

In addition to the two continuous symmetries (2.32) and (2.33) (for massless fermions), there are the following discrete symmetries which are the lattice remnant of the continuum group [Golterman and Smit, 1984a; Jolicoeur *et al.*, 1986].

1. Shift by one lattice spacing

$$\begin{aligned}\chi(\mathbf{x}) &\rightarrow \zeta_\mu(\mathbf{x})\chi(\mathbf{x} + a\hat{\mu}), \\ \bar{\chi}(\mathbf{x}) &\rightarrow \zeta_\mu(\mathbf{x})\bar{\chi}(\mathbf{x} + a\hat{\mu}), \\ U_\nu(\mathbf{x}) &\rightarrow U_\nu(\mathbf{x} + a\hat{\mu}),\end{aligned}\tag{2.43}$$

where

$$\zeta_\mu(\mathbf{x}) = \frac{1}{2^{d/2}} \text{tr} \left\{ \Gamma_{\mathbf{x}/a}^\dagger \Gamma_{\mathbf{x}/a + \hat{\mu}} \Gamma_{\hat{\mu}}^\dagger \right\}.$$

2. Charge conjugation

$$\begin{aligned}\chi(\mathbf{x}) &\rightarrow \bar{\chi}(\mathbf{x})^T, \\ \bar{\chi}(\mathbf{x}) &\rightarrow \chi(\mathbf{x})^T, \\ U_\mu(\mathbf{x}) &\rightarrow U_\mu^*(\mathbf{x}).\end{aligned}\tag{2.44}$$

3. Rotation (notation taken from [Daniel, 1987])

$$\begin{aligned}\chi(\mathbf{x}) &\rightarrow \rho(R^{-1}(S)\mathbf{x})\chi(R^{-1}(S)\mathbf{x}), \\ \bar{\chi}(\mathbf{x}) &\rightarrow \rho(R^{-1}(S)\mathbf{x})\bar{\chi}(R^{-1}(S)\mathbf{x}), \\ U(\mathbf{x}, \mathbf{y}) &\rightarrow U(R^{-1}(S)\mathbf{x}, R^{-1}(S)\mathbf{y}).\end{aligned}\tag{2.45}$$

The rotation R is an element of the hypercubic group $W(d) \subset O(d)$ and S is the covering element belonging to $\text{Pin}(d)$, whereas

$$\rho(R^{-1}(S)\mathbf{x}) = \frac{1}{2^{d/2}} \text{tr} \left\{ S \Gamma_{R^{-1}(S)\mathbf{x}/a} S^\dagger \Gamma_{\mathbf{x}/a}^\dagger \right\}.$$

4. Axial reversal

$$\begin{aligned}\chi(x) &\rightarrow (-1)^{x_\mu} \chi(I_\mu x), \\ \bar{\chi}(x) &\rightarrow (-1)^{x_\mu} \bar{\chi}(I_\mu x)\end{aligned}\tag{2.46}$$

and

$$U(x, y) \rightarrow U(I_\mu x, I_\mu y).$$

Representations of the group transformations in the bases (\bar{q}, q) and $(\bar{\psi}, \psi)$ are readily derived from the definitions of the bases. The representation theory for this group has been studied extensively [Golterman and Smit, 1985; Golterman, 1986a and b; Kilcup and Sharpe, 1987; Joos and Schaefer, 1987]. The decomposition of the continuum representation with respect to the lattice's is particularly important for the connection between lattice and continuum operators.

2.3.4 Applications of the equivalence of the interpretations

The coordinate interpretation is (quasi) local, a definite advantage in numerical simulations. On the other hand, the one in momentum space is easily adopted in perturbative calculations –the Feynman rules are readily derived– and exhibits more symmetry, flavour in particular. The discrepancy in flavour symmetry manifestation is understood as follows. Symmetry is, of course, independent of the interpretation. But what is called a flavour symmetry in momentum space identification cannot be named flavour in the other. This is apparent from the transformation (2.42). In fact, in the basis $\bar{\chi}(x)$ and $\chi(x)$, this symmetry is non-local [Jolicœur *et al.*, 1986].

It is known that [Golterman and Smit, 1984a] flavour is preserved to all orders in a by the shift invariance, which is translated in the basis $\bar{\psi}(k), \psi(k)$ into

$$\sim \mathbf{1} \otimes t_\mu.$$

Thus, having to commute with a basis of the Clifford algebra in flavour space, that part of the staggered fermion operator (2.37) can only be proportional to

the identity. Furthermore, this shift invariance together with the continuous chiral remnant are enough to prevent the appearance of a mass counterterm.

These properties must also be the case for the local fields $\bar{q}(r), q(r)$ in the continuum, where there is an equivalence. The restoration of flavour symmetry in the coordinate space interpretation thus becomes transparent. Numerical simulations have, in fact, confirmed this; meson operators with different flavour structures have been found to have the same mass within 10% at $\beta = 6.0$ and within 5% at $\beta = 6.15$ [Bowler *et al.*, 1987a and b]. This fact will be exploited to full extent in chapter 4.

In addition, for both interpretations the enlargement to the continuum symmetry group is then realised in the same way. Consequently, the classification of lattice operators according to spin, flavour and other quantum numbers also is the same in both cases. Except for the clear restoration of flavour and CPT invariance, the restoration of the Lorentz group is crucially dependent on locality [Jolicoeur *et al.*, 1987] as will be mentioned in the next chapter.

In constructing lattice hadronic operators, the covariant definition of (2.34), however, could lead to some problems. Fixing a point, as in the definition, destroys the manifestation of the important shift symmetry (2.43). Furthermore, the existence of non-minimal gauge links in the gauge invariance enforcement results in greater statistical fluctuations. In both numerical and perturbative calculations this leads to bad statistics and large corrections respectively. To avoid these we can use the free field definition of $\bar{q}(r), q(r)$ to construct quasi-local lattice operators. Gauge invariance is then enforced in a minimal manner, averaging over directions to ensure the maximum symmetry.

2.4 The no-go theorem

In all the lattice fermion formulations presented above either chiral symmetry has to be given up or doubling occurs. These are, ‘unfortunately’, not isolated incidents. Nielsen and Ninomiya [1981a,b and c] proved that it is, indeed, a generic problem for a general class of lattice fermions. In this section we present the (weak) no-go theorem on the lattice. A stronger version of the theorem connected to the chiral anomalies will be argued to be applicable for a wider

class of regulators in the next chapter.

Under the assumptions of bilinearity in the fermionic fields, locality, hermiticity (i.e. reflection positivity in Euclidean space), translational and chiral invariance, a lattice fermionic action necessarily displays doubling. We adopt here a proof in the Euclidean formulation due to Karsten [1981].

Consider a general form of the lattice operator

$$\sum_{\mu} i\gamma_{\mu} F_{\mu}(x, y). \quad (2.47)$$

Translational invariance requires

$$F_{\mu}(x, y) = F_{\mu}(x - y). \quad (2.48)$$

Whereas hermiticity insists

$$F_{\mu}(x) = F_{\mu}(-x)^*, \quad (2.49)$$

that is, the Fourier transform

$$\tilde{F}_{\mu}(k) = a^d \sum_x e^{-ikx} F_{\mu}(x) \quad (2.50)$$

is real. Furthermore, locality implies $F_{\mu}(x)$ vanishes fast enough for large x , in other words, $\tilde{F}_{\mu}(k)$ is smooth enough. Around its zero \bar{k}_g we can write, by Taylor expansion,

$$\tilde{F}_{\mu}(k) \sim F_{\mu\nu}^g(k - \bar{k}_g) + O(k^2), \quad (2.51)$$

the form of which admits a particle interpretation provided

$$\det F|_{k=\bar{k}_g} \neq 0. \quad (2.52)$$

The matrix $F_{\mu\nu}$ can then be decomposed into a product of an orthogonal matrix and a symmetric positive matrix,

$$F = OP, \quad (2.53)$$

of which the orthogonal matrix O amounts to a similarity transformation on the gamma matrices,

$$\gamma_{\mu} O_{\mu\nu}^g = S_g^{-1} \gamma_{\nu} S_g, \quad (2.54)$$

and P can be absorbed into a redefinition of the momenta,

$$l_\mu = P_{\mu\nu}^g (k - \bar{k}_g)_\nu. \quad (2.55)$$

In connection with the naive fermion case, we have

$$\begin{aligned} O_{\mu\nu}^g &\rightarrow (-1)^{g_\mu} \delta_{\mu\nu}, \\ S_g &\rightarrow M_g, \\ P_{\mu\nu}^g &\rightarrow \delta_{\mu\nu}. \end{aligned} \quad (2.56)$$

Accordingly, we can find the chiral charge for the general case

$$\begin{aligned} Q_{\text{chiral}}^g &= \text{tr} (S_g^{-1} \gamma_5 S_g \gamma_5) / 2^{d/2}, \\ &= \det O, \\ &= \text{sign}\{\det F\}. \end{aligned} \quad (2.57)$$

This charge turns out to be the index of the vector fields $\tilde{F}_\mu(k)$ at the zero \bar{k}_g . From the Poincare-Hopf index theorem, the sum of these indices is equal to the Euler characteristic of the manifold. As the manifold in question is a d -dimensional torus, the characteristic vanishes,

$$\chi_E(T^d) = 0. \quad (2.58)$$

In other words, the doubling is such that the net chirality is zero. An immediate consequence is that we cannot put a single Weyl fermion, such as the neutrino, onto these lattices. Recently, there are claims of extensions of the theorem where the requirements of hermiticity [Gross *et al.*, 1987] or locality [Pelissetto, 1987] are dropped in turn.

A comment on the proof of these no-go theorems seems appropriate here. One assumption that is always implicitly taken is eq. (2.52), which says that around every zero of the function $\tilde{F}_\mu(k)$ there is a decent interpretation of a relativistic particle. This requirement excludes the situation depicted in figure 2.3 where there is an inflection point at $p = 0$, i.e. $\left. \frac{dF}{dp} \right|_{p=0} = 0$. Lattice fermions with such behaviour can be constructed, for an example see [Dagotto *et al.*, 1986], although physical interpretation for the zero is not clear. See also [Quinn and Weinstein, 1986]. However, the doubling reappears when gauge fields are introduced in an invariant way, that is, when interactions of figure 2.1b are allowed. On the

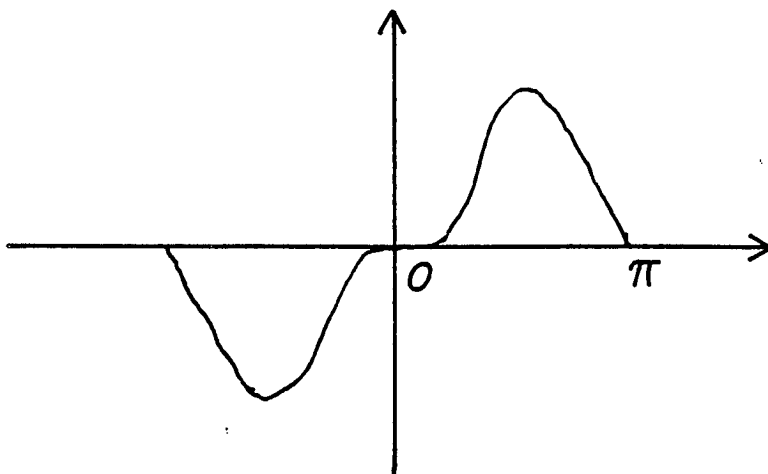


Figure 2.3: A situation excluded by an implicit assumption of the no-go theorem.

other hand, this also may be anticipated on the basis of the chiral anomalies. It is suspected that [Sen, private communication], nevertheless, by only allowing non-gauge interactions with scalar fields, via Yukawa coupling, a *single* Weyl particle can be put on the lattice. Note that this approach has also been taken on the random lattice [Perantonis and Wheeler, 1987]; thus a satisfactory chiral gauge formulation on random lattice is thus still left unanswered.

2.5 Concluding remarks

In this chapter we have further exposed the problem of fermionic species doubling on the lattice. It manifests itself as the loss of chiral symmetry in the Wilson fermion formulation. As a result, certain complications arise. Notable are the fine-tuning problem and the mixing of chirally different operators at the quantum (loop) level. Thus, the approach to the continuum limit is also affected. More serious are the impossibilities of studying a single chiral fermion and dynamical breaking of symmetry in this formulation.

The staggered formulation with one Grassmann component per site, on the other hand, is computationally simpler and suffers no tuning problem. Furthermore, the study of dynamical breaking of chiral symmetry is accessible, since part of the group is retained. Complexity, however, is shifted into the flavour interpretation –as there are $2^{d/2}$ species– and, consequently, the formulation of lattice operators. The seemingly different interpretation schemes are equivalent, up to a unitary flavour rotation, in the continuum limit. Advantages of different schemes can thus be exploited in practice. Chapter 4 demonstrates one example of this. Lifting the degeneracy, apart from loosing this freedom, we also have to face the necessary tuning of mass ratios [Golterman and Smit, 1984b].

The doubling phenomenon and its various manifestations are not confined to the lattice formulations presented here. According to the no-go theorem, it is, in fact, inevitable for a wider class of lattice fermions. A simple proof of the theorem which captures all the essentials due to Karsten is presented. It is also pointed out how to avoid the doubling but, perhaps, at a cost of Lorentz non-covariance.

Attempts to sidestep the no-go theorem by breaking the given assumptions, even though they seem to work well at the free level, have been unsatisfactory in one way or another when gauge interactions are added. As might be anticipated, this is because of, ultimately, the quantum-induced chiral anomalies.

Chapter 3

Chiral anomalies and Lattice fermionic regularisations

*“If the doors of perception were cleansed,
everything would appear to man as it is, infinite”*

William Blake

QFTs, Quantum Theories of point-like objects, usually suffer from infinities either in the ultraviolet or in the infrared regime. Regularisation and renormalisation are then needed to handle the situation. This chapter deals with a consequence of the ultraviolet regularisation, the anomalous breaking of symmetries, and particularly the lattice regulator. We begin the chapter with a short review of the enormous subject of chiral anomalies. Then it is illustrated how such anomalies emerge as the lattice regulator is removed. Of particular interest is the intimate connection with the species doubling problem presented in the previous chapter. Progress on the study of the Chiral Schwinger model [Jackiw and Rajaraman, 1985] using Wilson fermions is reported next. To end the chapter, we speculate on the possible link of the doubling phenomenon with recent proposals on the quantisation of chiral gauge theories and with the ‘exotic’ and fashionable topic of Berry’s phase. The view that, perhaps, they are only different facets of the same diamond is put forward.

Some of the work in this chapter has been published [Kieu, 1987a,b and c]; that on the Chiral Schwinger model with D. Sen and S.-S. Xue is in preparation.

3.1 A brief account of the chiral anomalies

Symmetries are of vital importance for physical theories; in fact, they are built on the symmetry principle. Even though more often than not the symmetries are broken, they still retain their fundamental role. Symmetries can be broken explicitly or spontaneously for global symmetries, or by the Higgs mechanism for gauge symmetries. They can also be broken anomalously. Furthermore, two or more ways of breaking a symmetry can also occur simultaneously.

A regularisation procedure has to violate some physical law, i.e. some symmetry at the classical level, for otherwise we would have a well-regulated, finite theory with all the desired symmetries and the need for regularisation itself would have become obsolete. Good regulators should maintain as many physical properties as possible. When the regulator is removed, some symmetries may not recover or may only recover at the expenses of others. Such breaking is called anomalous; a quantum-induced effect since only at the quantum (or, perturbatively, loop) level do we need regulators. The freedom to shift the anomaly around from one symmetry group to another comes from the freedom to add local monomial counterterms to the lagrangian. We will come back to this shortly.

Anomalies can be associated with large, homotopically non-trivial transformations which cannot be arrived at by successive infinitesimal transformation from the group identity. A typical example is the Witten anomaly [Witten, 1982] of $SU(2)$ gauge theories. Anomalies also arise from infinitesimal transformations as gravitational and chiral anomalies, among many excellent reviews see for example [Bardeen and Zumino, 1984; Jackiw, 1984; Alvarez-Gaume, 1985; Ginsparg, 1985; Fujikawa, 1986] and [Treiman *et al.*, 1972, Treiman *et al.*, 1985]. Anomalies of the latter type are further classified as coming from global or local transformations which are respectively named abelian and non-abelian anomalies.

3.1.1 Abelian anomaly

Historically, the abelian anomaly of the global axial symmetry group $U_A(1)$ was discovered from the triangle diagrams of figure 3.1 where one of the vertices is the

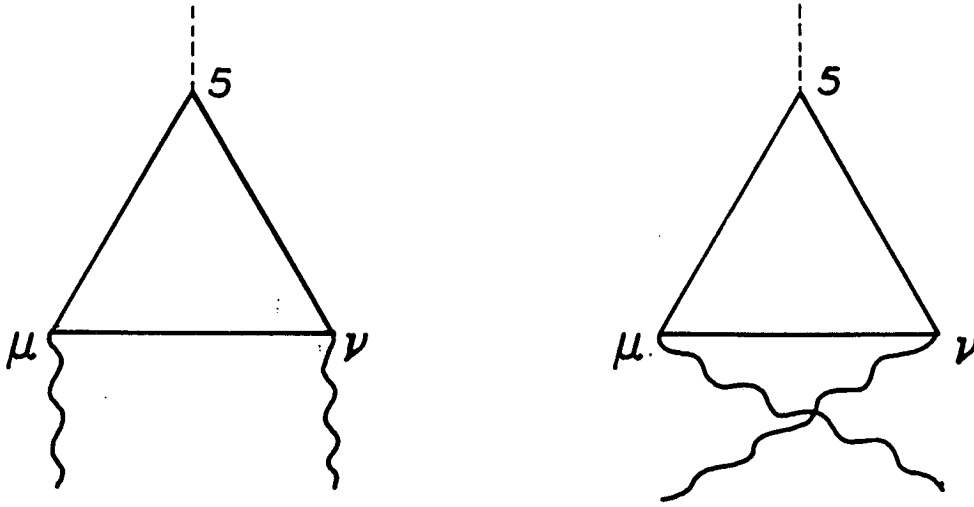


Figure 3.1: The anomalous triangle diagrams.

non-gauged chiral current. The underlying gauge group is taken to be vector-like as in QED or QCD. The diagrams are superficially linearly divergent so, even though it turns out that they yield a finite result, the result is dependent on how one routes the loop momenta. That is, by imposing different renormalisation conditions we can shift the anomaly into the vector (gauged) group. Demanding that the gauge group be conserved, the axial symmetry is thus anomalously broken and entails some physical effects (see later).

In the language of the path integral, this abelian anomaly is expressed in the Ward-Takahashi (WT) identities associated with a change of fermionic variables,

$$\psi(x) \rightarrow e^{i\gamma_5 \epsilon(x)} \psi(x), \quad (3.1)$$

in the generating functional with the action

$$\mathcal{S} = \int d^4x \bar{\psi} \gamma_\mu (\partial + igA_\mu) \psi(x) \quad (3.2)$$

where $A_\mu(x)$ is the (non)-abelian background gauge field. The WT identities express the invariance of the quantum action functional $W[A]$,

$$e^{-W[A]} \stackrel{\text{def}}{=} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-\mathcal{S}}, \quad (3.3)$$

under the change (3.1) of the integration variables

$$\left. \frac{\delta W[A]}{\delta \epsilon(x)} \right|_{\epsilon(x)=0} = 0. \quad (3.4)$$

On one hand, the change in the action is the divergence of the Noether current

$$J_\mu^5(x) \stackrel{def}{=} \bar{\psi} \gamma_\mu \gamma_5 \psi(x). \quad (3.5)$$

However, this current is not conserved because the Jacobian of the fermionic transformation is non-trivial; it is dependent on the background fields $A_\mu(x)$ giving [Fujikawa, 1979; 1980a and b], in 4- d ,

$$\partial_\mu J_\mu^5(x) = \frac{g^2}{(4\pi)^2} \epsilon_{\mu\nu\lambda\sigma} \text{tr}[F_{\mu\nu} F_{\lambda\sigma}]. \quad (3.6)$$

Owing to the non-renormalisation theorem of the anomaly [Adler and Bardeen, 1969], the form of (3.6) is not modified by higher order corrections when the gauge fields gain some dynamical content. This is because the gauge group is vector-like and thus higher-order effects do not interfere with the chiral symmetry. Renormalisation of the composite axial vector current (3.5) (see chapter 4 for a general discussion) however requires infinite subtraction.

3.1.2 Non-abelian anomaly

The non-abelian anomaly is associated with axial or chiral gauge symmetry. The two anomalies are closely related but differ from each other in many ways. It suffices to consider theories in which the gauge fields couple only to left-handed currents,

$$\mathcal{S} = \int d^4x \bar{\psi} \gamma_\mu (\partial_\mu + ig A_\mu P_L) \psi(x) \quad (3.7)$$

where

$$P_{L,R} \stackrel{def}{=} \frac{1}{2} (1 \mp \gamma_5). \quad (3.8)$$

The *consistent* anomaly is defined to be the variation of the effective action, the quantum action functional above, with \mathcal{S} of (3.7) under an infinitesimal gauge transformation

$$\delta A_\mu(x) = D_\mu \epsilon(x).$$

Here, D is the covariant derivative and

$$\epsilon(x) = \sum_b \epsilon(x)_b T_b.$$

Combining the gauge transformation with a change of fermionic variables

$$\begin{aligned} \psi(x) &\rightarrow e^{i\epsilon(x)P_L} \psi(x), \\ \bar{\psi}(x) &\rightarrow \bar{\psi}(x) e^{i\epsilon(x)P_R}, \end{aligned} \tag{3.9}$$

it can be seen that the covariant divergence of the consistent current

$$J_{\mu L}^b(x) \stackrel{def}{=} \left. \frac{\delta W[A]}{\delta A_\mu^b(x)} \right|_{A_\mu=0} \tag{3.10}$$

is equated to the effect of the Jacobian of the change of variables; once again see the method of Fujikawa quoted above. That is,

$$\begin{aligned} [D_\mu J_{\mu L}]^b(x) &= \left. \frac{\delta W[A]}{\delta \epsilon^b(x)} \right|_{\epsilon(x)=0}, \\ &= G^b(x), \\ &= -\frac{g^2}{24\pi^2} \text{tr} \left[T^b \partial_\mu \epsilon_{\mu\nu\lambda\sigma} (A_\nu \partial_\lambda A_\sigma + \frac{1}{2} A_\nu A_\lambda A_\sigma) \right]. \end{aligned} \tag{3.11}$$

The form of the last expression is also preserved due to an extension of the non-renormalisation theorem to the non-abelian case [Bardeen, 1972].

The current (3.10) being defined as the functional derivative of an effective action satisfies the integrability condition where the order of the functional differentiations can be exchanged. A consequence of this is that the consistent anomaly (3.11) satisfies the Wess-Zumino (WZ) condition [Wess and Zumino, 1971]. The consistency condition follows directly from the application of the generators of the gauge transformations

$$X^b(x) \stackrel{def}{=} \left[D_\mu \frac{\delta}{\delta A_\mu} \right]^b, \tag{3.12}$$

which satisfies the Lie algebra of the gauge group, to the effective action because, by definition,

$$X^b(x) W[A] \stackrel{def}{=} G^b(x). \tag{3.13}$$

The constraint from the Lie algebra,

$$X^a(x)G^b(x) - X^b(x)G^a(x) = f^{abc}G^c(x)\delta(x - y), \quad (3.14)$$

determines the anomaly uniquely, up to an overall normalisation, owing to the non-linear term of the covariant derivative in (3.12). In expression (3.11) we have quoted only the minimal form containing the abnormal parity terms, i.e. constructed with the epsilon tensors. All the normal parity terms can be absorbed by adding to the action local counterterms. We refer to [Christos, 1987] for a discussion of the general abnormal parity solutions of the WZ condition (3.14).

Also, the definition of the consistent current implies that it does not transform covariantly under the gauge group. This distinguishes the consistent anomaly from the *covariant* anomaly which is the covariant divergence of a covariantly transformed current,

$$\begin{aligned} [D_\mu \hat{J}_{\mu L}(x)]^a &= \frac{g^2}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} \text{tr}[T^a F_{\mu\nu} F_{\lambda\sigma}], \\ &\stackrel{\text{def}}{=} \hat{G}^a(x), \end{aligned} \quad (3.15)$$

where

$$\hat{J}_{\mu L}^a(x) \stackrel{\text{def}}{=} \bar{\psi} \gamma_\mu P_L T^a \psi(x). \quad (3.16)$$

The covariant anomaly does not satisfy the consistency condition (3.14), as a factor of 2 on the rhs would be required. It has been proved that [Bardeen and Zumino, 1984] the two kinds of currents are related, however, by a local polynomial of the gauge fields. The relationship is clearly manifest in the definition of the effective action [Banerjee *et al.*, 1986]

$$W[A] = \int_0^g de \int d^4x A_\mu^a(x) \hat{J}_{\mu L}^a(x, e). \quad (3.17)$$

Here $\hat{J}_{\mu L}^a(x, e)$ is the current defined in some regularisation and thus depends on the coupling e , which is to be integrated over. The lattice Wilson fermion regularisation, say, can be written in the form (3.17) but the covariance of the current is broken because of the Wilson term.

3.1.3 Some physical consequences of chiral anomalies

Keeping the principle of gauge invariance, we try to preserve the gauge symmetry of a theory whenever an anomaly appears. This can be done by shifting the anomalous breaking into other symmetries with appropriate counterterms. Inevitably, observable effects thus emerge as physical consequences. It is widely agreed that such chiral anomalies are responsible for the $\pi^0 \rightarrow \gamma\gamma$ decay [Bell and Jackiw, 1969] provided three colours of quarks are taken into account; the resolution of the $U(1)$ problem [Jackiw and Rebbi, 1976] where the symmetry is broken spontaneously and anomalously so that the existence of massless particles is not required; magnetic monopole induced baryon-number violating decay [Christ and Jackiw, 1980] which is due to a topological argument and should not be confused with the baryon decay in GUTs (see chapter 4).

Take the example of the anomalous non-dynamical flavour chiral currents of QCD associated with the pion decay. The effective low-energy action of the mesons –see chapter 4 for a review of the chiral lagrangian approach– should explicitly contain the anomalous interaction vertices. The WZ term which satisfies equation (3.11) for finite chiral transformations [Wess and Zumino, 1971] is a possible candidate. Being non-local in terms of the gauge fields alone, the WZ term can be expressed locally with the help of pseudoscalar fields, which are to be identified as the pion fields. In general, it is quite difficult to show that this term always appears as a result of the spontaneous breaking of chiral symmetry.

It is also proposed that the chiral anomalies may be responsible for the dynamical generation of a vector boson mass in place of the Higgs mechanism [Farhi and Jackiw, 1982]. This anomaly mechanism arising from ultraviolet effects is very much different from the latter which is a result of infrared instabilities. Finite temperature properties can thus distinguish the two, as short-distance behaviour should be insensitive to temperature. However, dynamical generation of mass seems to be confined to $2-d$ theories so far.

On the other hand, anomalies also impose constraints on the construction of physical theories. The bound-state anomaly matching conditions [tHooft, 1980], for instance, have to be observed when the confinement mechanism is invoked. Furthermore, we also want to keep the gauge invariance by arranging that the gauge symmetry anomaly is cancelled to avoid mathematical inconsistencies.

The inconsistencies arise because from the equation of motion when dynamics of the gauge fields is introduced,

$$D_\mu F_{\mu\nu}(x) = \hat{J}_{\nu L}(x),$$

yielding

$$\begin{aligned} D_\nu \hat{J}_{\nu L}(x) &= D_\nu D_\mu F_{\mu\nu}(x), \\ &= [F_{\nu\mu}, F_{\mu\nu}](x), \\ &= 0, \end{aligned} \tag{3.18}$$

which is in clear contradiction with the anomalous WT identity (3.15). On the other hand, in a different manifestation, presumably first-class constraints, i.e. the Gauss' law, are no longer closed under commutation upon quantisation. They thus become second-class. However, the cross over is a quantum effect so they cannot be classified as second-class before quantisation.

The requirement of anomaly-free theories then emerges naturally [Georgi and Glashow, 1972]. This happens for vector-like theories where fermions of both chiralities belong to the same representation of the gauge group and cancel the anomalies. For chiral gauge theories the gauge group itself can be anomaly-free such as the 'safe' groups of $SU(2)$, all symplectic groups and all orthogonal groups except $SO(6) \approx SU(4)$. When the group is not safe, like $SU(N)$ with $N > 2$ or $SU(2) \times U(1)$, we have to look for safe representations where the anomalies are cancelled. This is the origin of the quark-lepton duality in the Standard model and limitations on possible GUTs—see chapter 4 for an example—, supersymmetric and superstring theories.

Because of the direct connection between the consistent and covariant currents the anomaly cancellation mechanism is the same for both kinds of anomalies.

In the last section of the chapter we will discuss some 'new' proposals on how to consistently quantise anomalous chiral theories in the light of the results from lattice fermionic regularisations.

3.2 Abelian anomaly on the lattice

We first consider the case of naive lattice fermions of (1.3) with gauge links inserted in an invariant way. Taking the transformation (2.9) to be the $U_A(1)$ transformation on the lattice, which assumes the continuum form as in expression (3.1), it can be shown that there is no anomaly. This is because on the lattice the fermionic measure is well-defined, with respect to the ultraviolet behaviour at least, and thus so is the unitary transformation from the Heisenberg picture to the interaction picture. Consequently, the Jacobians associated with symmetry transformations are trivial, i.e. c -numbers independent of the gauge fields, since in the interaction picture a “plane wave” basis can be used to perform the path integral [Fujikawa, 1984]. This result is expected on the basis that the transformations constitute a symmetry of the naive action and thus the total chiral charge is zero (equations (2.11) and (2.12)).

In fact, the corresponding lattice axial current,

$$J_\mu^5(x)_{\text{naive}} \stackrel{\text{def}}{=} \frac{1}{2} \{ \bar{\psi}(x) \gamma_\mu \gamma_5 U_\mu(x) \psi(x + a\hat{\mu}) + \text{h.c.} \}, \quad (3.19)$$

can be shown directly to be conserved, up to the breaking effect of an explicit mass,

$$\begin{aligned} \langle \Delta_\mu J_\mu^5(x) \rangle &\stackrel{\text{def}}{=} \mathcal{Z}_\psi^{-1} \int_\psi e^{-S} \frac{1}{a} \sum_\mu \{ J_\mu^5(x) - J_\mu^5(x - a\hat{\mu}) \}, \\ &= \langle \bar{\psi}(x) \gamma_\mu \gamma_5 \frac{1}{2a} \{ U_\mu(x) \psi(x + a\hat{\mu}) - U_\mu^\dagger(x - a\hat{\mu}) \psi(x - a\hat{\mu}) \} + \text{h.c.} \rangle, \\ &= \text{tr} \{ [\gamma_5 D, (D + m)^{-1}](x, x) \}, \\ &= \text{tr} \{ \gamma_5 \{ D, (D + m)^{-1} \}(x, x) \}, \\ &= 2m \langle \bar{\psi}(x) \gamma_5 \psi(x) \rangle, \end{aligned} \quad (3.20)$$

where

$$\mathcal{Z}_\psi \stackrel{\text{def}}{=} \int_\psi e^{-S}.$$

The definitions of the gamma matrices are given in chapter 1.

To prove that one species individually, indeed, gives the correct abelian anomaly we consider the species corresponding to the momentum region, say,

$$\mathcal{C} \stackrel{\text{def}}{=} \left[-\frac{\pi}{2a}, \frac{\pi}{2a} \right] \times \left[-\frac{\pi}{2a}, \frac{\pi}{2a} \right], \quad (3.21)$$

which has a chiral charge of +1 according to (2.11) in 2- d . We will work explicitly in 2- d and massless theories from now on but the result is expected to be readily extendable to higher dimensions.

Isolating the species corresponds to restricting the momentum p to the sub-region \mathcal{C} , from the second last equation of (3.20),

$$\langle \Delta_\mu J_\mu^5(x) \rangle_{\mathcal{C}} \stackrel{def}{=} \int_{q,k} \int_p^{\mathcal{C}} e^{ikx} tr \left\{ \gamma_5 \left[\bar{D}(q+k,p) \bar{D}^{-1}(p,q) + \bar{D}^{-1}(q+k,p) \bar{D}(p,q) \right] \right\}, \quad (3.22)$$

with the superscript \mathcal{C} denoting the restriction of the domain of integration. For the gauge fields, the Fourier transform definition

$$\tilde{A}_\mu(k) \stackrel{def}{=} a^2 \sum_x e^{-ikx - iak_\mu/2} A_\mu(x + \frac{a}{2} \hat{\mu}) \quad (3.23)$$

is adopted.

This method is equivalent to the restriction of the loop momenta of the triangle diagrams [Karsten and Smit, 1981]. They found that (alternatively, it can be shown from (3.22) above by using the method in the non-abelian section later) each species possesses an anomaly term with the correct cohomological form and with an overall sign which depends on its chiral charge,

$$\langle \Delta_\mu J_\mu^5(x) \rangle_{\mathcal{C}} = \frac{g}{\pi} \epsilon_{\mu\nu} \partial_\mu A_\nu(x). \quad (3.24)$$

In all, the doublers are generated in such a way as to cancel the would-be anomaly. Thus to recover it, the cancellation has to be removed. For the abelian anomaly, there are two ways applying to two distinct classes of lattice fermions: the doubler-decoupling class and the chiral-'invariant' class.

3.2.1 The doubler-decoupling class

In section 2.2 we saw that Wilson fermions, and their relatives, get rid of the doublers by sending their masses to infinity in the continuum limit. However, it is instructive [Smit, 1986] to decouple the doublers only after taking the continuum limit. That is, we only let ^(*)

$$\tilde{r} \stackrel{def}{=} r/a \rightarrow \infty$$

(*) See also [Karsten and Smit, 1981].

in the mass expression for the doublers (2.15) after $a \rightarrow 0$. In a vector gauge theory, most of the effects of such infinitely heavy particles on the remaining light ones can be absorbed into the renormalisations of various quantities. Some of the effects, however, cannot and need not be absorbed in renormalisation; they are related to the anomalies in the WT identities. In fact, the Wilson and Dirac-Kähler fermions can reproduce the abelian anomaly in this way [Karsten and Smit, 1981; Kerler, 1981; Seiler and Stamatescu, 1982a and b; Fujikawa, 1984; Versteegen, 1984; Bodwin and Kovacs, 1987b].

From the symmetry point of view, the chiral invariance is broken by the doubler mass term, the Wilson term, and this is precisely how the anomaly is recovered afterwards.

3.2.2 The chiral-‘invariant’ class

We have seen that because of the freedom/ambiguity on the lattice –which exists in any regulator, in fact– we can modify the continuum theory with apparently irrelevant terms, the Wilson term for example. The obvious restriction is that the lattice theory should reduce to the expected continuum form in the naive limit. The irrelevant term enables us to recover the anomaly for the class of fermions of the last section.

This freedom/ambiguity also enables us to modify the lattice axial transformation (2.9), which is desired to be anomalous, instead of the lattice action as before. That is, we can modify the lattice definition of the axial transformation such that all the species transform the *same* way and the anomalous effect is not cancelled among them. As only the anomalous transformation is to be modified, it is hoped that any residual effects, apart from the ones related to the anomaly, are irrelevant, i.e. can be hidden away in renormalisation.

The observation is particularly useful for the chiral-‘invariant’ class of lattice fermions: naive and staggered fermions for instance, which are invariant only under the transformations of (2.9). Even though the fermions are multi-species as a consequence of the no-go theorem of section 2.4, they can reproduce the abelian anomaly provided the cancellation is lifted. We will see in section 3.4 and the concluding section that, since the notion of species is definable for these

fermions, a necessary condition for the anomaly is that this newly defined axial symmetry is broken in the regularisation.

To realise this, the interpretation of lattice operators is vitally important. The necessary identification of relevant continuum operators can be facilitated in the strong coupling regime [Groot *et al.*, 1984], by the use of auxiliary regulators, or Zimmermann's normal product algorithm [Fujikawa, 1984] or in the framework of the species interpretation which is known to be valid. On the other hand, the reproduction of the anomaly can be used to check the validity of the interpretation, usually obtained in the free case, when interaction is turned on.

In the framework of the species interpretation (2.8) for naive fermions we see that under the infinitesimal axial transformation defined as [Sharatchandra *et al.*, 1981]

$$\delta\psi(x) = i\epsilon\gamma_5 \frac{1}{2^d} \sum_{\{\xi_\mu=\pm 1, \forall \mu\}} \psi(x + a\xi) \quad (3.25)$$

all the species $\tilde{q}^g(k)$ transform alike. This is because the translations of the fields on the rhs of (3.25) change the phases of the species under the transformation and the chiral charge can thus be defined to be +1 uniformly for all the species. That is, the total chiral charge is no longer zero but equal to the number of species.

Note that even though the transformations (3.25) do not exponentiate to form a group, there is an associated current

$$J_\mu^5(x)_{\text{STW}} \stackrel{\text{def}}{=} \frac{1}{2^d} \bar{\psi}(x) \gamma_\mu \gamma_5 \sum_{\{\xi_\nu=\pm 1, \forall \nu\}} \psi(x + a\hat{\mu} + a\xi), \quad (3.26)$$

in which gauge links are to be inserted in some appropriate way, e.g. averaged over different shortest paths. Consequently, it was shown that the abelian anomaly could be recovered in the usual form multiplied by the number of species. Of course there still are the problems with the doubling and with the interpretation of the naive axial transformations which remain conserved.

A comment on a recent work [Alonso *et al.*, 1987] seems appropriate here. They introduced a lattice fermion scheme that is chirally-'invariant' but violates reflection positivity. It is then claimed that there is no doubling as, although the associated current is anomaly-free, an alternative current can be constructed to give the abelian anomaly. In the light of the argument presented here in

general, and in the example of the naive fermion case in particular, it is seen that the alternative current should correspond to an alternative axial transformation which is not a symmetry of the action. The authors seem to be unaware of this fact which is not a special property of their scheme but is rather general. Correspondingly, we expect that the fermion scheme either still suffers from the doubling, as in the naive or staggered case, or from other serious problems. Our argument will be further strengthened in a discussion of the generalisation of the no-go theorem later.

For the staggered fermion formulation, due to its close link with the naive one, the current (3.26) can be carried over by spin diagonalisation. The correct anomaly with $2^{d/2}$ flavours was then subsequently derived [Sharatchandra *et al.*, 1981]. On the other hand, the staggered fermions themselves also admit an elegant flavour interpretation in coordinate space, see chapter 2, which has been employed extensively in numerical simulations. It is, therefore, important and interesting to investigate the $U(1)$ axial anomaly in the framework of this interpretation.

We restrict ourselves to the case of $d = 2$ as it illustrates the interesting features of the calculation without the complexity of the mathematics in higher dimensions. The calculation can be extended readily to $d = 4$ and to non-abelian gauge fields.

In the context of the coordinate-space interpretation, the staggered fermion counterpart of the current (3.26) assumes the form

$$\begin{aligned}
J_{\mu}^5(r)_{\text{STW}} = \frac{1}{8} \bar{q}(r) \quad & \{ \gamma_{\mu} \gamma_5 \otimes \mathbf{1} \quad [2q(r + 2a\hat{\mu}) + 2q(r) \\
& + q(r + 2a\hat{\mu} + 2a\hat{\nu}) + q(r + 2a\hat{\mu} - 2a\hat{\nu}) \\
& + q(r + 2a\hat{\nu}) + q(r - 2a\hat{\nu})] \\
& + \sigma_{\mu\nu} \otimes t_{\nu} t_5 \quad [q(r + 2a\hat{\mu} + 2a\hat{\nu}) - q(r + 2a\hat{\mu} - 2a\hat{\nu}) \\
& + q(r + 2a\hat{\nu}) - q(r - 2a\hat{\nu})] \} , \quad \mu \neq \nu .
\end{aligned}
\tag{3.27}$$

Clearly, this current is not of minimal form as it contains extra Dirac-flavour structure in comparison with the continuum axial current, although the correction is of order a so it vanishes in the limit $a \rightarrow 0$. When interactions are switched on, furthermore, the current does not have a simple expression.

On the other hand, the first term on the right hand side of (3.27) is of familiar form for the lattice chiral current corresponding to a particular split-point definition, which has just the required Dirac-flavour structure. Thus we propose the gauge invariant current, up to renormalisation, (*)

$$J_\mu^5(r) \stackrel{def}{=} \frac{1}{2} \bar{Q}(r) \gamma_\mu \gamma_5 \otimes 1 U_\mu(r) U_\mu(r + a\hat{\mu}) Q(r + 2a\hat{\mu}) + \text{h.c.} \quad (3.28)$$

with $Q(r)$ is the covariant quark fields defined in (2.34). Note that we can define a lattice operator in several ways which will reduce to the same continuum form naively; however, they are not all equivalent. In particular, to reproduce the anomaly additional criteria can be supplemented. The current (3.28) is chosen on the basis of locality, gauge invariance and, at least in the free theory, it corresponds to the correct assignment of the chiral charge to the species. Such an assignment should hold here, as will be confirmed by the calculation below, for the interaction does not affect the chiral properties. A further point here is that we use the point-split form. This is necessary to remove the ambiguity associated with the product of fields at the same space-time point on the lattice, which cuts off the short distance structure such as the anomaly or the notion of species. We mention in passing that in the block-variable formulation of Susskind fermions, namely the Dirac-Kahler theory, and its generalisation, a current of this form but with gauge links defined on the block lattice has also been employed [Gockeler, 1984; Versteegen, 1984].

In a background (external) gauge field we want to evaluate the continuum limit of the vacuum expectation value of the lattice divergence

$$\Delta_\mu J_\mu^5(r) = \frac{1}{2a} \sum_\mu \{ J_\mu^5(r) - J_\mu^5(r - 2a\hat{\mu}) \}. \quad (3.29)$$

To this end, we need a perturbative expression for the staggered fermion propagator in the basis $(\bar{\chi}_\eta, \chi_\eta)$ from the action (2.30) but with gauge links inserted,

$$\begin{aligned} \mathcal{S} &= a^2 \sum_{\mu, x} \alpha_\mu(x) \bar{\chi}(x) \frac{1}{2a} [U_\mu(x) \chi(x + a\hat{\mu}) - U_\mu^\dagger(x - a\hat{\mu}) \chi(x - a\hat{\mu})], \\ &\stackrel{def}{=} [(2a)^2]^2 \sum_{r, r', \eta, \eta'} \bar{\chi}_\eta(r) [G^{-1}(r, r')]_{\eta\eta'} \chi_{\eta'}(r'). \end{aligned} \quad (3.30)$$

We refer to equations (2.24) and (2.25) for the relationships among x_μ, r_μ, η_μ and the definitions of $\chi_\eta(r)$.

(*) The current $\bar{Q}(r) \gamma_\mu \gamma_5 \otimes 1 Q(r)$ can also be used to derive the anomaly.



Using the identities

$$\chi_\eta(r \pm 2a\hat{\mu}\eta_\mu) = \eta_\mu \chi_\eta(r \pm 2a\hat{\mu}) + (1 - \eta_\mu) \chi_\eta(r)$$

we can rewrite the operator in (3.30) as

$$\begin{aligned} [G^{-1}(r, r')]_{\eta\eta'} &= \frac{1}{2^{d/2}} \text{tr} \left(\Gamma_\eta^\dagger \gamma_\mu \Gamma_{\eta'} \right) \times \frac{1}{2a} \left\{ U_\mu(r + a\eta) \eta_\mu \delta_{r+2a\hat{\mu}, r'} \right. \\ &\quad \left. - U_\mu^\dagger(r + a\eta - a\hat{\mu}) \eta'_\mu \delta_{r-2a\hat{\mu}, r'} \right. \\ &\quad \left. + [U_\mu(r + a\eta) \eta'_\mu - U_\mu^\dagger(r + a\eta - a\hat{\mu}) \eta_\mu] \delta_{r, r'} \right\}. \end{aligned} \quad (3.31)$$

We assume that A_μ is slowly varying and $agA_\mu \ll 1$ as usual in the anomaly derivation on the lattice. We can then, after expanding

$$U_\mu(r + a\eta - a\hat{\mu}) = 1 + igaA_\mu(r + a\eta - a\hat{\mu}/2) + \dots,$$

Taylor expand the gauge field $A_\mu(r + a\eta - a\hat{\mu}/2)$ around r up to the necessary orders in a to obtain

$$\begin{aligned} G^{-1}(r, r') &= \frac{1}{2a} \int_p e^{ip(r-r')} \left\{ [\overline{\gamma_\mu \otimes \mathbf{1}} i \sin 2ap_\mu + \overline{\gamma_5 \otimes t_\mu t_5} (\cos 2ap_\mu - 1)] \right. \\ &\quad \left. + igaA_\mu(r) [\overline{\gamma_\mu \otimes \mathbf{1}} (\cos 2ap_\mu + 1) + \overline{\gamma_5 \otimes t_\mu t_5} i \sin 2ap_\mu] \right\} \\ &\quad + \text{higher order terms in } a, \\ &\stackrel{\text{def}}{=} \frac{1}{2a} \int_p e^{ip(r-r')} \left\{ S_0^{-1}(pa) + igaA_\mu(r) Q_\mu(pa) \right\} \\ &\quad + \text{higher order terms in } a. \end{aligned} \quad (3.32)$$

Here we have concealed in ‘higher order terms’ $O(g^2)$ and derivative terms, with appropriate powers of a , coming from the Taylor expansion around r .

This is equivalent to the Taylor expansion in momentum space of the *full* interacting vertex in al where l is the gauge field momentum. The connection between the expansion of A_μ around r and that of the full vertex in momentum space, follows since the Fourier transforms of $(a\partial)^n A_\mu(r)$, integer n , are $(al)^n \tilde{A}_\mu(l)$. Such an expansion in al is allowed for slowly varying A_μ , which means that the gauge field only has low momentum components. That is, the Fourier transform is only non-vanishing for $|al| \ll 1$ which is always satisfied for finite l as $a \rightarrow 0$. This assumption of sufficiently soft external momenta ensures that there is no excitation of one species into another. The vertex $Q_\mu(pa)$ is then

arrived at as the lowest order of the expansion and all the other terms, containing explicit powers of a (from powers of al), are then represented by the 'higher order terms'. On the other hand, we can see that the expansion is permissible for the anomaly derivation since there is no gauge field propagator contribution to the internal loops, i.e. we only have internal fermion loops up to one-loop order taking into account that the anomaly is one-loop effect.

The first term on the rhs of equation (3.32) is the free inverse propagator denoted by $G_0^{-1}(r, r')$, and the rest of the terms are denoted by $V(r, r')$. In the derivation of this equation, the anti-commutators of the gamma matrices together with the representation of η_μ

$$\eta_\mu = \frac{1}{2}\{1 - (-1)^{\eta_\mu}\}$$

enable us to cast various structures into the matrix notation of (2.26) For example,

$$\begin{aligned} \frac{1}{2^{d/2}} \text{tr} \left\{ \Gamma_\eta^\dagger \gamma_\mu \gamma_5 \Gamma_{\eta'} \right\} (-1)^{\eta_\nu} &= \frac{1}{2^{d/2}} \text{tr} \left\{ \Gamma_\eta^\dagger \gamma_\mu \gamma_5 (\gamma_5 \gamma_\nu) \Gamma_{\eta'} (\gamma_\nu \gamma_5) \right\} \\ &= \overline{[\gamma_\mu \gamma_\nu \otimes t_5 t_\nu]_{\eta\eta'}}. \end{aligned}$$

In the series expansion of the full propagator

$$G(r, r') = \left\{ G_0 \sum_{n=0}^{\infty} (-V G_0)^n \right\} (r, r'), \quad (3.33)$$

as will be seen later, only the first few terms contribute to the chiral anomaly. In fact, we only require two terms; so from (3.32) and (3.33) we get

$$\begin{aligned} G(r, r') &= 2a \int_p e^{ip(r-r')} \frac{S_0^{-1}(pa)}{\text{den}(pa)} \\ &\quad - 2iga^2 \int_p \int_k e^{ipr+k(r-r')} \bar{A}_\mu(p) \frac{S_0^{-1}(pa+ka) Q_\mu(ka) S_0^{-1}(ka)}{\text{den}(pa+ka) \text{den}(ka)} \\ &\quad + \text{higher order terms in } a, \end{aligned} \quad (3.34)$$

with

$$\text{den}(pa) \stackrel{\text{def}}{=} -4 \sum_\sigma \sin^2 ap_\sigma, \quad (3.35)$$

and the gauge field momenta are now in the range $\left[-\frac{\pi}{2a}, \frac{\pi}{2a}\right]$.

Coming back to the lattice divergence (3.29) of the axial current (3.28) we rewrite the former expression as

$$\begin{aligned}\langle \Delta_\mu J_\mu^5(r) \rangle &= \frac{1}{4a} \langle \bar{Q}(r) \gamma_\mu \gamma_5 \otimes \mathbf{1} \{ U_\mu(r) U_\mu(r + a\hat{\mu}) Q(r + 2a\hat{\mu}) \\ &\quad - U_\mu^\dagger(r - a\hat{\mu}) U_\mu^\dagger(r - 2a\hat{\mu}) Q(r - 2a\hat{\mu}) \} + \text{h.c.} \rangle, \\ &= -\text{tr} \{ [K, G](r, r) \};\end{aligned}\quad (3.36)$$

where we have defined

$$[K(r, r')]_{\eta\eta'} \stackrel{\text{def}}{=} \frac{1}{4a} \frac{1}{2^{d/2}} \text{tr} (\Gamma_\eta^\dagger \gamma_\mu \gamma_5 \Gamma_{\eta'}) \{ W_{\eta\eta'}(r) \delta_{r+2a\hat{\mu}, r'} - W_{\eta'\eta}^\dagger(r') \delta_{r-2a\hat{\mu}, r} \}, \quad (3.37)$$

and, from the covariant definitions of the quark fields (2.34),

$$W_{\eta\eta'}(r) \stackrel{\text{def}}{=} U_\eta^\dagger(r) U_\mu(r) U_\mu(r + a\hat{\mu}) U_{\eta'}(r + 2a\hat{\mu}). \quad (3.38)$$

Taylor expansion of the gauge links around r up to the necessary orders in a , as in the derivation of the inverse propagator before, yields

$$\begin{aligned}K(r, r') &= \frac{1}{2} \int_k e^{ik(r-r')} \{ \overline{\gamma_\mu \gamma_5 \otimes \mathbf{1}} \frac{i}{a} \sin 2ak_\mu \\ &\quad + ig A_\mu(r) [\overline{\gamma_\mu \gamma_5 \otimes \mathbf{1}} 2 \cos 2ak_\mu + \overline{\sigma_{\nu\mu} \otimes t_\mu t_5} i \sin 2ak_\nu] \} \\ &\quad + \text{higher order terms in } a.\end{aligned}\quad (3.39)$$

Here, once again, the higher order terms also contain various partial derivatives of $A_\mu(r)$ with appropriate factors of a .

In the commutator of (3.36) the first term (the second term) on the rhs of (3.39) combines with the second term (the first term) of the propagator (3.34) to yield the only non-vanishing contributions as $a \rightarrow 0$. We thus have, from the cyclic property of taking the trace,

$$\begin{aligned}\langle \Delta_\mu J_\mu^5(r) \rangle &= \\ &= \int_p \int_k e^{ipr} ig a^2 \bar{A}_\mu(p) \text{tr} \left\{ \overline{\gamma_\lambda \gamma_5 \otimes \mathbf{1}} \frac{i}{a} [\sin 2a(k_\lambda + p_\lambda) - \sin 2ak_\lambda] \right. \\ &\quad \times \frac{S_0^{-1}(ka + pa) Q_\mu(ka) S_0^{-1}(ka)}{\text{den}(pa + ka) \text{den}(ka)} \\ &\quad \left. + \frac{1}{a} \left[\frac{S_0^{-1}(pa + ka)}{\text{den}(pa + ka)} - \frac{S_0^{-1}(ka)}{\text{den}(ka)} \right] \left[\overline{\gamma_\mu \gamma_5 \otimes \mathbf{1}} 2 \cos 2ak_\mu + \overline{\sigma_{\nu\mu} \otimes t_\mu t_5} i \sin 2ak_\nu \right] \right\} \\ &\quad + \text{irrelevant terms.}\end{aligned}\quad (3.40)$$

The external gauge field momentum ap is to be expanded in and the internal momentum is to be rescaled $k \rightarrow k/2a$. Then in the continuum limit the integration over p gives the partial derivative of $A_\mu(r)$

$$\lim_{a \rightarrow 0} \langle \Delta_\mu J_\mu^5(r) \rangle = \langle \partial_\mu J_\mu^5(r) \rangle = g \partial_\lambda A_\mu(r) I_{\lambda\mu}, \quad (3.41)$$

where, after calculating the various traces of gamma matrices,

$$I_{\lambda\mu} = \frac{\epsilon_{\lambda\mu}}{\pi^2} \int_{-\pi}^{\pi} d^2k \left\{ \cos k_\lambda \left[\sin k_\mu \frac{\partial}{\partial k_\mu} \left(\frac{1}{\text{den}(k/2)} \right) + \frac{1}{2} \frac{\cos k_\mu + 1}{\text{den}(k/2)} \right] \right. \\ \left. + \cos k_\mu \frac{\partial}{\partial k_\lambda} \left(\frac{\sin k_\lambda}{\text{den}(k/2)} \right) - \frac{\sin k_\lambda}{2} \frac{\partial}{\partial k_\lambda} \left(\frac{\cos k_\mu - 1}{\text{den}(k/2)} \right) \right\}, \quad (3.42)$$

note that there is no summation over λ or μ . To arrive at the above expression, trivially vanishing integrals have been omitted, i.e. those whose integrands are odd functions.

Direct integration by parts in (3.42) would naively yield a vanishing result. However, such an operation is illegitimate since the integrands on the rhs are singular at the origin, except for the last term which has a removable singularity there. We thus partition the integration domain as shown in figure 3.2. In the region excluding the origin we can integrate by parts and some of the surface terms survive; Taylor expansions of the integrands in the inner region, on the other hand, cancel each other:

$$I_{\lambda\mu} = \frac{\epsilon_{\lambda\mu}}{\pi^2} \left\{ \int_{-\epsilon}^{\epsilon} dk_\lambda \frac{\cos k_\lambda \sin k_\mu}{\text{den}(k/2)} \Big|_{k_\mu=\epsilon}^{k_\mu=-\epsilon} + \int (\mu \leftrightarrow \lambda) \right. \\ \left. + \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} dx dy \left[x \frac{2x}{(r^2 + y^2)^2} - \frac{1}{r^2 + y^2} - \frac{r^2 - y^2}{(r^2 + y^2)^2} \right] \right\}, \\ = \frac{\epsilon_{\lambda\mu}}{\pi^2} 4 \int_{-\epsilon}^{\epsilon} dk_\mu \frac{\epsilon}{\epsilon^2 + k_\mu^2}, \\ = 2 \frac{\epsilon_{\lambda\mu}}{\pi}. \quad (3.43)$$

We thus obtain the correct anomaly for a theory containing two flavours.

To complete the proof we now show that this is the only non-vanishing contribution by a power counting argument. This is expected as the anomaly is a quantum effect induced by the singularities only.

Generically, a term of $K(r, r')$ can be written as

$$\int_{p,q,l} e^{i(px - qr')} \kappa(p, q, l) \delta(p - q - \sum l),$$

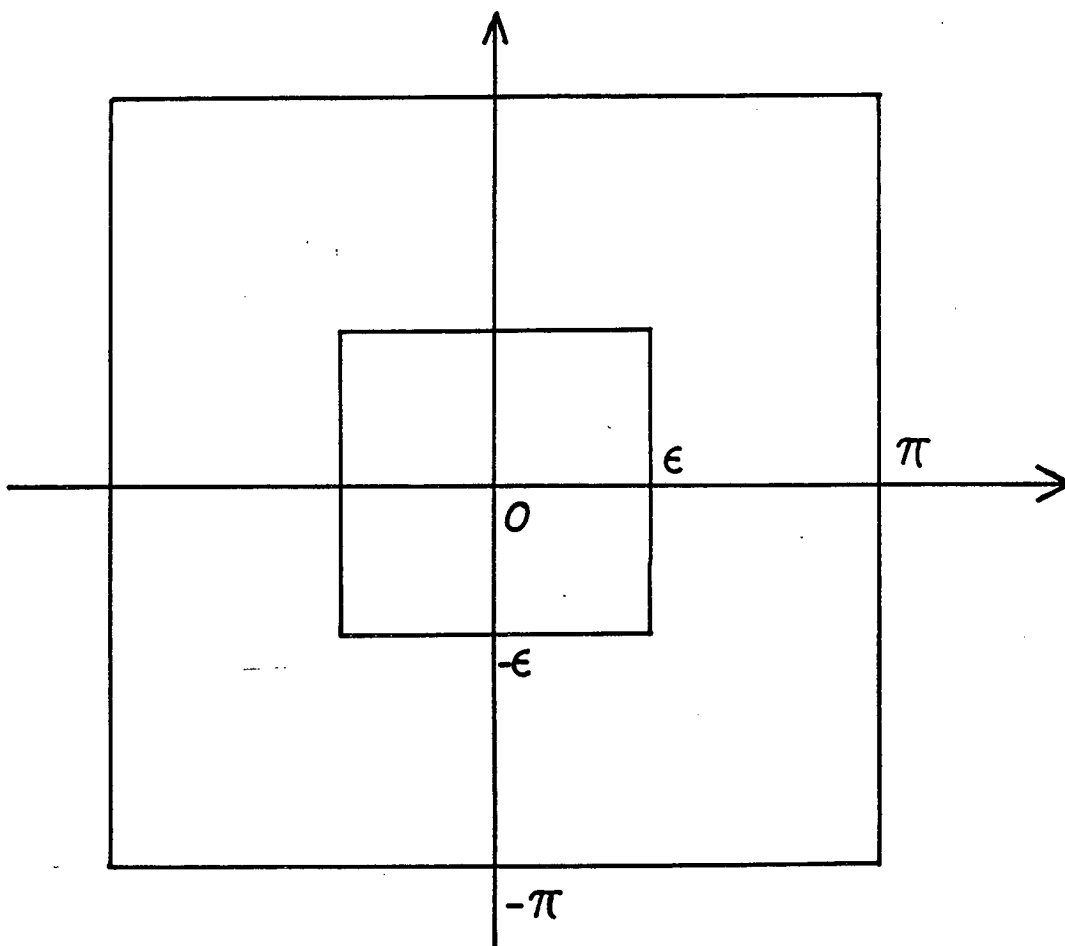


Figure 3.2: Partitioning of the integration domain

and a typical term of $G(r, r')$ as

$$\int_{m,n,l'} e^{i(mx-nr')} g(m, n, l') \delta(m - n - \sum l'),$$

where l and l' are the gauge field momenta in which we can Taylor expand. The first term of the commutator in the trace

$$\text{tr}\{[K, G](r, x)\} = \text{tr} \left\{ a^2 \sum_y [K(r, r')G(y, x) - G(r, r')K(y, x)] \right\}$$

is then

$$\begin{aligned} &\sim \int_{p,l,l',n} e^{ir(p-n)} \kappa(p, p - \sum l, l) g(p - \sum l, n, l') \delta(p - n - \sum l - \sum l'), \\ &\sim \int_{p,l,l'} e^{ir(\sum l + \sum l')} \kappa(p, p - \sum l, l) g(p - \sum l, p - \sum l - \sum l', l'). \end{aligned} \quad (3.44)$$

Similarly, the other term of the commutator is

$$\sim \int_{p,l,l'} e^{ir(\sum l + \sum l')} g(p + \sum l', p, l') \kappa(p, p - \sum l, l). \quad (3.45)$$

Then the commutator of the first term on the rhs of equation (3.39) and the free propagator $G_0(r, r')$ do not contribute to the divergence, equation (3.36), as can be seen by setting l and l' to be zero in the last two expressions and from the cyclic property of taking the trace. We also need to show that the trace of the commutator of the second terms of (3.34) and (3.39) vanishes in the limit and so do the ones involving the 'higher order terms'. This can be seen once again from equations (3.44) and (3.45). From the Taylor expansions of the functions g, κ in $\sum la$ and $\sum l'a$, we gain extra factors of a as only higher derivative terms survive in the trace of the commutator. Thus all the other contributions are, at worst, of order $O(a)$.

We note that for the power counting argument it is necessary to introduce, after rescaling of the internal momentum, an infrared regulator which is to be taken to zero after the limit $a \rightarrow 0$. As the infrared behaviour is the same on an infinite lattice as in the continuum, the discarding of 'higher order terms' here is in agreement with the vanishing in the continuum, in 2 dimensions, of corresponding diagrams with more gauge fields. For a proof of this see [Bodwin and Kovacs, 1987b] for example.

Furthermore, the anomaly result (3.43) can now be shown to be independent of the choice of path of gauge links going from $r + a\eta$ to $r' + a\eta'$. The difference

between two such paths amounts to the difference of a closed path product of gauge links from unity, that is $O(a^2)$, i.e. $\sim a^2 g F_{\mu\nu}$. Such a term will be absorbed into 'higher order terms' of $K(r, r')$ which, in turn, has just been demonstrated to have no effect on the anomaly.

Recently, Oshima [Oshima, 1987] proved that the reduced version of staggered fermions, which has only half of the number of flavours [Kluberg-Stern *et al.*, 1983], could also reproduce the anomaly.

Finally, the study of the non-abelian anomaly on the lattice, the topic of the next section, could benefit from such investigations of the abelian anomaly. In particular, we speculate that a suitable generalisation of the current (3.28) could be used to study the covariant anomaly of Dirac-Kahler fermions.

3.3 Non-abelian anomaly on the lattice

The study of the non-abelian anomaly is a natural extension of the previous discussion. Furthermore, the anomaly is a necessary precursor of the attempts to put chiral gauge theories onto the lattice [Hands and Carpenter, 1986; Eichten and Preskill, 1986; Smit, 1986; Aoki, 1987][†]. At the present it is still unclear whether any of these schemes will work and there are only indirect methods, as presented in chapter 4, to study non-perturbative effects *not* related to the chiral gauge group directly. However, a direct study of the theory is desirable for, to name a few, the Higgs sector and its phase diagrams and the spontaneous breaking of symmetries via the realisation of the massless spectrum. Apart from these, it is also of intrinsic interest, due to the natural role of the lattice regulator in the formulation of Euclidean path integrals, to 'satisfactorily' set up a lattice theory with the same weak-coupling properties as in the continuum.

Several authors [Coste *et al.*, 1986 and 1987; Aoki, 1987] have recently derived the consistent anomaly for Wilson and Dirac-Kahler fermions. Here we illustrate how the anomaly is obtained for the doubler-decoupling class of lattice fermions, in general, by isolating one species.

[†] See also [Swift, 1984].

3.3.1 The method of restriction to one species

We will demonstrate that the relationship of the species doubling and the abelian anomaly is also applicable to the case of the non-abelian anomaly. More specifically, by isolating one chiral fermionic species of the naive fermions with chiral gauge interaction, the consistent anomaly is reproduced. Here, again, the chiral symmetry is broken explicitly by the restriction to just one species as, otherwise, the chiral gauge transformation transforms one species into others. Other species can be shown to yield similar results where the overall sign depends on the chiral charge. Thus, since the total charge is known to be zero, it explains why there is no anomaly even though this is not the case for each species. Consequently, in parallel with the abelian anomaly case, we thus see how Wilson and Dirac-Kahler fermions yield the correct non-abelian anomaly by decoupling the doublers.

Only two-dimensional theories are presented as they already capture the essential features and can be readily extended to higher dimensions¹. Likewise, the scenario is also expected to be applicable to the covariant anomaly, whose connection with the consistent anomaly was mentioned in the first section of the chapter. The results have been reported elsewhere [Kieu, 1987b and c].

With chiral gauge interaction for the naive fermion operator (1.11) can be written in the form

$$D(x, y) = \sum_{\mu} \gamma_{\mu} \frac{1}{2a} \left\{ [U_{\mu}(x)P_L + P_R] \delta_{x+a\hat{\mu}, y} - [U_{\mu}^{\dagger}(y)P_L + P_R] \delta_{x-a\hat{\mu}, y} \right\}. \quad (3.46)$$

Here

$$P_{L,R} = \frac{1}{2}(1 \mp \gamma_5) \quad (3.47)$$

are the chirality projectors. Recall that the consistent non-abelian anomaly is defined to be the variation of the effective action $W[U]$,

$$\exp \{-W[U]\} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S}, \quad (3.48)$$

under an infinitesimal gauge transformation, which on the lattice has the form

$$U_{\mu}(x) \rightarrow V(x)U_{\mu}(x)V^{\dagger}(x+a\hat{\mu}), \quad (3.49)$$

¹Four dimensional theories are being studied in collaboration with S.-S. Xue.

where

$$\left. \begin{aligned} U_\mu(x) &= \exp \left\{ i g a A_\mu^b(x + \frac{a}{2} \hat{\mu}) t^b \right\}, \\ V(x) &= \mathbf{1} + i \epsilon^b(x) t^b \end{aligned} \right\} \in SU(N) \quad (3.50)$$

and t^b 's are the generators of the gauge group in the fundamental representation, say. As before, we can treat the gauge field as a background field. This follows as there is no gauge propagator contribution, i.e. we only have internal fermion loops up to one-loop order taking into account that the anomaly is a one-loop effect.

It can be seen, however, directly from the invariance of the functional integration measure with respect to the change of the Grassmann variables

$$\begin{cases} \psi(x) & \rightarrow [V^\dagger(x)P_L + P_R]\psi(x), \\ \bar{\psi}(x) & \rightarrow \bar{\psi}(x)[V(x)P_R + P_L], \end{cases} \quad (3.51)$$

that the lattice effective action is invariant under (3.49) and thus is anomaly-free.

Corresponding to the isolation of one species, we now denote by \mathcal{S}_1 the part of the action (3.46) in momentum space where both momenta of the fermion fields are restricted to the subregion \mathcal{C} of (3.21). The vacuum expectation value of the variation of \mathcal{S}_1 under the gauge transformation of the *background* gauge field (3.49) is then

$$\begin{aligned} \langle \Delta \mathcal{S}_1 \rangle &= \\ & \left\langle \int_{p,q}^{\mathcal{C}} \int_{k,l} \bar{\psi}(p) \left\{ [\tilde{V}(k)P_R + P_L] \tilde{D}(p-k, q-l) [\tilde{V}^\dagger(l)P_L + P_R] - \tilde{D}(p, q) \right\} \psi(q) \right\rangle, \end{aligned} \quad (3.52)$$

with the superscript \mathcal{C} denoting the restriction of the domain of integration.

From the Fourier transform of the infinitesimal gauge transformation and up to the lowest order in ϵ , this variation of the one-species piece of the action becomes

$$\begin{aligned} \langle \Delta \mathcal{S}_1 \rangle &= i \left\langle \int_{p,q}^{\mathcal{C}} \int_k \bar{\psi}(p) \left\{ \tilde{\epsilon}(k) P_R \tilde{D}(p-k, q) - \tilde{D}(p, q-k) \tilde{\epsilon}(-k) P_L \right\} \psi(q) \right\rangle, \\ &= -i Tr \int_{p,q}^{\mathcal{C}} \int_k \tilde{\epsilon}(k) \left\{ P_R \tilde{D}(p-k, q) \tilde{D}^{-1}(q, p) - P_L \tilde{D}^{-1}(q, p) \tilde{D}(p, q+k) \right\}, \end{aligned} \quad (3.53)$$

where Tr is the trace over both spinor and internal indices. The notation tr is reserved for the trace separately over either spinor or internal indices according

to the context. This expectation value would have vanished, i.e. there would have been no anomaly had the momenta p and q not been restricted. In fact, because of the identity

$$\int_p \tilde{D}(q, p) \tilde{D}^{-1}(p, q') = (2\pi)^2 \delta(q - q'),$$

the rhs of eq. (3.53) would have been proportional to $tr \gamma_5 = 0$ otherwise. In other words, the gauge transformation mixes up different domains of the momentum space and, as a result, the gauge symmetry is automatically and explicitly broken when a particular domain is chosen.

Neglecting the possible local counterterms, we concentrate on the anomaly and thus consider only terms linear in the gauge field $A_\mu(x)$. We then get for the first term on the rhs of (3.53), upon expanding $U_\mu(x)$ up to terms of order $O(g)$ and inverting $\tilde{D}(p, q)$, the expression

$$\frac{ig}{a} Tr \left\{ \gamma_\nu \gamma_\mu P_L \int_{p,q}^{\bar{c}} \int_k (2\pi)^2 [\delta(p - q) - \delta(p - q - ka)] \tilde{\epsilon}(k) \tilde{A}_\mu(-k) \times \frac{\cos(p_\mu - ak_\mu/2) \sin p_\nu}{d(p)} \right\}. \quad (3.54)$$

The momenta p, q have been rescaled to be dimensionless and the following notations introduced

$$\begin{aligned} \bar{c} &= \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \\ d(p) &= -\sum_\sigma \sin^2 p_\sigma. \end{aligned} \quad (3.55)$$

Upon integrating over q , the second delta function of eq. (3.54) requires that

$$-\frac{\pi}{2} + p_\mu \leq ak_\mu < \frac{\pi}{2} + p_\mu \pmod{2\pi}. \quad (3.56)$$

Such an integration domain for a pair of variables (p_μ, ak_μ) is depicted in figure 3.3. We can interchange the order of integration for each pair to get

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dp_\mu \int_{-\frac{\pi}{2a} + \frac{p_\mu}{a}}^{\frac{\pi}{2a} + \frac{p_\mu}{a}} dk_\mu = \int_{-\frac{\pi}{2}}^0 dk_\mu \int_{-\frac{\pi}{2}}^{\frac{\pi}{2} + ak_\mu} dp_\mu + \int_0^{\frac{\pi}{2}} dk_\mu \int_{-\frac{\pi}{2} + ak_\mu}^{\frac{\pi}{2}} dp_\mu. \quad (3.57)$$

To make the step rigorous one can introduce an infrared regulator. The gauge field is assumed to be varying slowly enough so that it has only low momentum component; that is, the Fourier transform is only non-vanishing for $|ak| \ll 1$ to

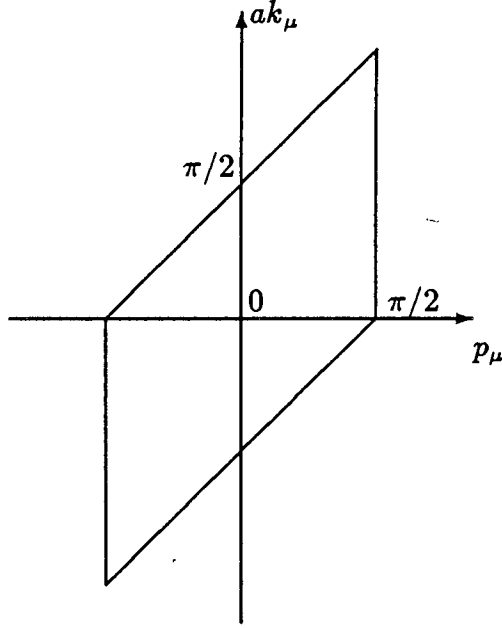


Figure 3.3: The integration domain for a pair of variables (p_μ, ak_μ)

prevent one species from being excited to another. As we are only concerned with infinitesimal gauge transformations, this behaviour is maintained throughout. Taylor expansion in ak is then permitted and gives the result

$$\begin{aligned}
& ig \text{tr} \{ \gamma_\nu \gamma_\mu P_L \} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dk_\mu}{(2\pi)^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dp_\mu}{(2\pi)^2} \\
& \text{tr} \left\{ \int_0^{\frac{\pi}{2}} dk_\lambda k_\lambda \bar{\epsilon}(k) \bar{A}_\mu(-k) \frac{\cos p_\mu \sin p_\nu}{d(p)} \Big|_{p_\lambda = -\frac{\pi}{2}} \right. \\
& \left. - \int_{-\frac{\pi}{2}}^0 dk_\lambda k_\lambda \bar{\epsilon}(k) \bar{A}_\mu(-k) \frac{\cos p_\mu \sin p_\nu}{d(p)} \Big|_{p_\lambda = \frac{\pi}{2}} \right\} \\
& + O(a)
\end{aligned} \tag{3.58}$$

for the expression (3.54). Care has been taken with respect to the expansion when ak also appears in the limits of the integrals, as pointed out in (3.57). This results in the surface terms of equation (3.58). Evaluating the trace over spinor indices and taking the continuum limit, we get the final expression for the first term of eq.(3.53), modulo possible local counter terms,

$$\frac{1}{2} \frac{ig}{4\pi} \epsilon_{\nu\mu} \int \frac{d^2 k}{(2\pi)^2} \text{tr} \left\{ \bar{\epsilon}(k) \left[ik_\nu \bar{A}_\mu(-k) \right] \right\}. \tag{3.59}$$

The numerical factor comes from an elementary integral over p . Similar calculations for the other term of the variation of the one-species part of the action yields, in total, the consistent non-abelian anomaly

$$-\frac{ig}{4\pi}\epsilon_{\nu\mu}\int d^2x\text{tr}\{\epsilon(x)\partial_\nu A_\mu(x)\}. \quad (3.60)$$

Instead of restricting both momenta p and q of eq.(3.53) to be in \mathcal{C} we can obtain the same result by just restricting one momentum variable, q say. Consequently, the second term there will not contribute to the anomaly. For the first term, we get, in addition to expression (3.59), another similar contribution coming from the integration of p outside \mathcal{C} . This kind of restriction is particularly useful; it allows us to partition the action just into 4 pieces corresponding to 4 regions in momentum space of the variable q , say. We can then show that each piece possesses the anomaly term with a sign depending on the chiral charge corresponding to the domain. In total the anomaly is thus cancelled for the full action.

To demonstrate that the species decoupling will recover the anomaly, we relate the lattice regularisation, Wilson say, to some continuum regulator. The difference between the two regulators can be shown to vanish as $a \rightarrow 0$ for expressions like that of (3.54). For heuristic reasons, as in the discussion of the abelian anomaly, the limit of infinitely heavy mass of the doublers is to be taken after the limit $a \rightarrow 0$. In the continuum regularisation, however, it is seen directly that such a limit will send these expressions to zero. The effect of the doublers on the anomaly is thus decoupled except some manifestation which, hopefully, should be removed by local counterterms. We have, in the end, only one contribution coming from the surviving fermion, which remains massless.

In the calculations above we have ignored terms other than the ones linear in the gauge field. To prove that the neglected terms only amount to local counterterms in general, as repeatedly mentioned, we move on to the next section.

3.3.2 Local counterterms

We present here a theorem by the Saclay group [Jolicœur *et al.*, 1987] which says that there is no anomaly for global compact groups, e.g. flavour or rotational

groups. That is, any local solution of the Wess-Zumino consistency condition associated with a compact group, is the variation of a local counterterm.

Define the anomaly to be a variation of the effective action under some transformation g

$$\mathcal{A}[B, g] \stackrel{def}{=} W[B^g] - W[B] \quad (3.61)$$

where B is some field and only consider the anomaly when it is a local functional of B

$$\mathcal{A} = \int d^d x P(B(x), g). \quad (3.62)$$

The rhs of the last expression is a polynomial in the field $B(x)$ and its derivatives.

The Wess-Zumino condition (3.14) can be written in terms of the group transformations g and g' as

$$\mathcal{A}[B, gg'] = \mathcal{A}[B^g, g'] + \mathcal{A}[B, g], \quad (3.63)$$

which can be integrated with respect to g' giving

$$\mathcal{A}[B, g] = \mathcal{R}[B^g] - \mathcal{R}[B] \quad (3.64)$$

with

$$\mathcal{R}[B] \stackrel{def}{=} - \int d^d x \int d\mu(g) P(B(x), g). \quad (3.65)$$

The integration is possible as there exists a normalised integration measure $d\mu(g)$ which is left-invariant for the compact group.

Since the functional $\mathcal{R}[B]$ of (3.65) is local and is thus an admissible counterterm, the 'anomaly' $\mathcal{A}[B]$ can be expressed, from (3.64), as the variation of a counterterm. In other words, there is no anomaly as the effective action can always be modified legitimately to absorb any non-invariance.

The proof above can be naturally rephrased in the language of cohomology, the study of quantities that can be expressed as some variations of others.

It is important to emphasize that the assumption of compactness in the theorem does not cover local gauge group, which is, formally, the product of a compact group at every space-time point. Also the $U(1)$ -axial global group,

which gives rise to the abelian anomaly, is excluded as the anomaly can be obtained from a gauge transformation of an axial gauge field coupled to the axial current [Jolicoeur *et al.*, 1987].

From this result, it can be shown that, as no other symmetry is anomalously broken and as the cohomological form of the consistent anomaly is uniquely determined by the Wess-Zumino condition up to an overall normalisation constant, it is only necessary to check the normalisation of the cohomological term. All other terms which appear in the variation of the effective action, under a chiral gauge transformation, are removable by local counterterms.

We have shown above that the desired normalisation is obtained for one species. Consequently, to complete the proof for the doubler-decoupling lattice fermions, it suffices to check that the variations of the lattice effective actions, under various global transformations, conform with the given assumptions of the theorem. The calculation details are quite technical, we refer the reader to the original work [Jolicoeur *et al.*, 1987].

In general, the form of the counterterms, however, depends on the exact details of the regularisation employed, i.e. on the details of the irrelevant term which induces the decoupling. This can be seen from our discussion on the decoupling previously. Those irrelevant terms constitute the doubler mass terms which contribute to the variation of the effective action in the form of extra terms with the mass term in the numerator. In the infinite mass limit, the cohomological term (3.54) is sent to zero as before, but the extra terms will survive and can be removed by adding local counterterms to the original action. Later we will argue for the production of such local counterterms from the Wilson term in the lattice chiral Schwinger model.

An important spin-off of the no-anomaly theorem is the result that the Euclidean invariance, and basically any global invariance –except the axial– can be recovered from lattice gauge theories in the continuum limit provided locality is preserved. Thus there are incentives to stick with local LGTs.

To end this section on the non-abelian anomaly we comment on the situation for the chiral-‘invariant’ lattice fermions. First of all, the chiral gauge interaction cannot be incorporated into the staggered fermion formulation. Second, the cancellation of the non-abelian anomaly for the naive fermions cannot be lifted

as in the case of the abelian anomaly. The consistent anomaly reflects the invariance of the effective action under a gauge transformation of the gauge fields and this cannot be reconciled with the previous trick of a lattice axial transformation which acts on the fermionic fields and shifts their arguments.

3.4 The no-go theorem revisited

Near the end of chapter 2 a no-go theorem on the lattice with certain assumptions was discussed. All the attempts to avoid the doubling of fermionic species by breaking those assumptions, while preserving chiral symmetries, have been hitherto unsatisfactory when gauge interactions are introduced. In fact, recently there is a claim of an extension of the theorem where the assumption of hermiticity is dropped [Gross *et al.*, 1987a].

The proofs for these no-go theorems rely crucially on the notion of species and on the identification of chiral charge directly. Indirectly behind these concepts is the demand for the appearance of the chiral anomalies, which cannot exist if the total chiral charge vanishes. This explains why troubles only appear with gauge interactions, which switch on the anomalies, but not in the free theories or non-gauge interactions, e.g. four-point interactions. Exposing this underlying argument, the no-go theorem is thus generalised to cover a wider class of regulators. The seeds of the arguments can be found in [Nielsen and Ninomiya, 1981c].

In a regularisation if the chiral symmetries are to be preserved, the notion of species is definable² and survives the continuum limit –the regulator-removal process–, that is, the chiral anomalies are to be recovered for each species, then there is doubling of fermionic species.

Intuitively, it is difficult to understand how a well-regulated theory which is chirally invariant can give rise to the anomalies. That is, unless the regulator-removal process is carefully defined so that some peculiar mechanism can break the invariances. On such a regulator with such a limit, however, a necessary condition is that the notion of species has to be intrinsically ambiguous. And

²Technically, the action is required to be bilinear, which also implies the existence of the conserved Noether currents.

the generalised no-go theorem is no longer applicable. This seems to be the case for non-local LGTs, random lattices or random regulating fields.

With the randomness built in, due to the lack of translational invariance, the concept of momentum and, for that matter, the concept of particle via some kind of dispersion relation breaks down. The concept is only restored when averages over some ensemble of random lattices is taken; this, however, obscures the continuum limit. Thus if there is a *fundamental* cut-off of space-time, random or fractal structures are possible candidates. I should make a comment, however, that the proposal that chiral symmetries on random lattice are broken spontaneously in the continuum limit is not, in a sense, satisfactory. In order to avoid the embarrassing massless excitations, the chiral symmetries need to be broken explicitly [Espiru *et al.*, 1986; Gross *et al.*, 1987].

For non-local LGTs like the SLAC fermions [Drell *et al.*, 1976] and the Rebbi's fermions [Rebbi, 1987] there are also problems [Karsten and Smit, 1978 and 1979; Ninomiya and Tan, 1984; Bodwin and Kovacs, 1987a and b; Camprostrini *et al.*, 1987; Pelissetto, 1987]. As the notion of species, introduced in the free case, needs to be extended in the perturbation theories, either the chiral anomalies cannot be obtained, or Lorentz invariance is broken, or spurious ghost states appear³. However, the abelian anomaly is claimed to be restorable [Hernandez and Mawhinney, 1987] by some definition of the axial current⁴.

Once again, this should lead to some revision of the continuum limit, e.g. for SLAC fermions see [Rabin, 1981; Quinn and Weinstein, 1986].

On the other hand, acceptance of the inevitability of the doubling could lead to a critical review of the method of quantising anomalous gauge theories in a consistent way. See the end of this chapter for some speculative remarks. In the next section we report on the progress of a study of a simple chiral gauge theory, the chiral Schwinger model. It is precisely the explicit chiral breaking that realises the consistency of the theory.

³In the formulation of Rebbi, in particular, these ghost states are due to the contributions from the zeros of the propagator as there are infinite differences across the Brillouin zone boundaries.

⁴I have some reservation on the method of calculation employed there, nevertheless.

3.5 On the lattice chiral Schwinger model

The chiral Schwinger model is a two-dimensional theory in which the left-handed current is coupled to dynamical gauge field of an abelian gauge group, in Euclidean space,

$$S = \int d^2x \left\{ -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\psi} i \gamma_\mu [\partial_\mu + g A_\mu P_L] \psi \right\}. \quad (3.66)$$

This is a soluble model in which the fermionic determinant can be evaluated exactly to yield an effective action [Jackiw and Rajaraman, 1985]

$$\mathcal{S}_{eff}(A) = \int d^2x \left\{ -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{g^2}{8\pi} A_\mu \left[a \delta_{\mu\nu} - (\delta_{\mu\alpha} + \epsilon_{\mu\alpha}) \frac{\partial_\alpha \partial_\beta}{\square} (\delta_{\beta\nu} - \epsilon_{\beta\nu}) \right] A_\nu \right\}, \quad (3.67)$$

where a is some arbitrary parameter. Based on this arbitrariness, it has been shown that the theory, despite the loss of gauge invariance due to the gauge anomaly associated with the chiral coupling, can be quantised consistently and unitarity is obtained.

The very existence of this parameter a , however, has sparked off controversy [Das, 1985; Hagen, 1985; Jackiw and Rajaraman, 1985] and the model has attracted much attention owing to this arbitrariness and its consequences, [Banerjee, 1986; Harada *et al.*, 1986] to quote just two to compare with our method later.

In general, it is argued that the arbitrariness originates from the anomaly; the gauge symmetry is lost upon quantisation so that the appearance of the symmetry-breaking a -term need not be prevented. It could either appear as a counterterm or, from the regularisation point of view, come from the ambiguity available on regulators.

We propose a study of the model using lattice as a regulator. From the discussion of the last section, the chiral symmetry is necessarily broken to avoid the doubling phenomenon. The Wilson fermion method seems to be appropriate here; the ambiguity mentioned above is in the arbitrariness of the Wilson parameter r and in the way the Wilson term is gauged. Keeping the hermiticity, we use, for the fermionic counterpart of the chiral Schwinger action, the lattice

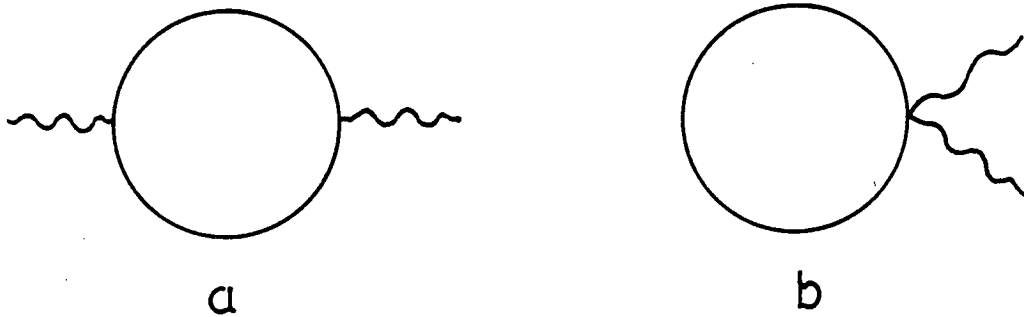


Figure 3.4: Diagrams for the evaluation of the fermionic determinant in 2 dimensions.

action which has the form (3.46) plus the Wilson operator

$$W(x, y) = \frac{r}{2a} \sum_{\mu} \{ U_{\mu}(x) \delta_{x+a\hat{\mu}, y} + U_{\mu}^{\dagger}(y) \delta_{x-a\hat{\mu}, y} - 2\delta_{x, y} \}. \quad (3.68)$$

In this way, the gauge field couples to the right handed current in the Wilson term. This coupling was introduced *ad hoc* in continuum calculations [Banerjee, 1986; Harada *et al.*, 1986] and was shown to be crucial in obtaining the desired result.

In this study we hope to achieve two points. Firstly, to reproduce the arbitrary parameter a , which should not be mistaken for the lattice spacing, as a function of r . Secondly, to demonstrate the necessity for such term –to avoid doubling, ultimately– as opposed to mere admissibility as indicated by the continuum calculations.

To evaluate the fermionic determinant we have to calculate, on the lattice, the diagrams of figure 3.4. All other diagrams with more external gauge fields can be shown to be vanish. After using the Feynman parametrisation to combine the denominators, we shift the integration variables, which is a valid operation on the lattice, in such a way that when the integration domain is splitted as in figure 3.2 and the expressions are Taylor expanded, there is no linear momentum term left in the denominator. Near the origin of the integration domain, we thus effectively have a simple cut-off regularised expression which contributes to the second term on the rhs of (3.67). For the rest of the integration domain outside

the origin we can put the external gauge field momenta to zero in the integrand as there is no longer any infrared problem to worry about. If there were gauge invariance, as in the Schwinger model, the integrand could be written as a total derivative as a direct result of the Ward identities. The surface term resulted on the boundary surrounding the origin would contribute to the rest of the second term of (3.67) whereas the surface term on the edges of the Brillouin zone would be cancelled due to the periodicity. However, as there is no gauge invariance, on top of this we will have some explicit dependence on the r -parameter. This is how the arbitrary term of (3.67) appears.

We are trying to obtain a closed form for the dependence of the parameter a on r in terms of integrals of trigonometric functions. These integrals are to be evaluated numerically to see whether the parameter can take any value or is confined in some finite interval as r covers the positive real axis.

3.6 Speculative ending remarks

In this chapter we have tried to investigate chiral anomalies in the framework of lattice fermionic regularisations. The relationship between the phenomenon of species doubling and the anomalies, abelian and non-abelian, is clarified. Each species individually gives the correct anomalies but if the symmetries are to be preserved on the lattice then there are cancellations among the species. It thus can be seen how the doubler-decoupling class of lattice fermions, which includes Wilson and Dirac-Kahler, can reproduce the anomalies and forms a valid class of regulators. This class does not contain some unorthodox lattice fermions like SLAC fermions, Rebbi fermions or some theories involving random fields or lattices.

In the case of staggered fermions it is still not known how to incorporate a chiral gauge interaction. Nevertheless, for such a class of multi-species lattice fermions, a lattice $U(1)_A$ chiral transformation can be defined, owing to the regularisation ambiguity, in such away that all the species transform the same way. The abelian anomaly is thus recovered as the transformation is not a symmetry of the action and the cancellation is removed. In either way of removing the anomaly cancellation, the species interpretation plays an important role. In the

framework of the coordinate-space interpretation, the derivation makes use of a minimal form of the chiral current, which contains just the required spin-flavour structure and assumes the familiar point-split definition.

All calculations have been carried out in two dimensions, but we expect that corresponding anomaly results will hold for higher dimensions. The point-split form is required even on the lattice, which cuts off the short-distance structure, to remove the ambiguity associated with the product of fields at the same space-time point. In the framework of the path integral, the lattice approach presented in this chapter provides an alternative derivation of the chiral anomalies to that of Fujikawa.

We have also presented arguments for the generalisation of the no-go theorem to be valid not only on the lattice but for a wider class of regulators. The vital point of the arguments is the reproduction of the chiral anomalies; in short, chiral-invariant regularisations are in contradiction with the anomalies so that chiral partners are generated; that is, unless the notion of species becomes ambiguous in the approach to the continuum. These apparently are the cases of non-local LGTs, random lattices or random fields; however, they are so far still unsatisfactory in one way or another.

Furthermore, in lattice gravitational theories, it is still far from clear what restriction the conformal anomalies may impose on the doubling problem [Fujikawa, 1984].

On the other hand, accepting the doubling phenomenon, we may be in agreement with some recent proposals on the quantisation of anomalous gauge theories.

We also reported on the progress of the study of the chiral Schwinger model using Wilson fermions. It is hoped that by doing so, the existence of the controversial arbitrary term, resulting from the lack of a preventing symmetry, is not only admissible but also necessary to avoid the doubling.

The rest of this chapter is as a biased review of the literature and speculative ideas are put forward.

In section 1 we pointed out the inconsistencies associated with the anomalous breaking of gauge symmetries. Thus either some anomaly cancellation must

be at work or the conventional quantisation procedure has to be modified to accommodate these theories. Fundamentally, the theories are kept anomaly-free. Some authors [Faddeev and Shatashvili, 1986] proposed to add a Wess-Zumino type term to the action to cancel the anomaly. As the term is introduced by hand, even though it is satisfactory in $2-d$ where it is equivalent to the effect of anomaly-cancelling fermions being decoupled, care must be taken in 4 dimensions. In fact, a recent calculation [Levy, 1987] claims that this does not lead to the desirable result, for a certain Jacobi identity is not satisfied in $4-d$. This might be avoided if the WZ term is generated dynamically as in the approach of [Harada and Tsutsui, 1987] where the Faddeev-Popov gauge fixing procedure of anomalous gauge theories needs to be revised. It is claimed that the WZ term is manifest through the gauge degrees of freedom. Basically, the theory is anomaly free due to the integration over the gauge orbits. On one hand, however, this approach in the path integral is so far purely symbolic. On the other hand, the lattice approach is naturally associated with the path integral and, furthermore, it seems that, at least for a certain class of lattices, lattice chiral gauge theories are intrinsically anomaly-free (due to doubling).

This leads to the suggestion that there is an intimate relationship between the WZ term and the doubling on the lattice (as they both are deeply connected with the anomalies). Perhaps fermion doubling and the WZ term are just manifestations of the same thing. After all, the WZ term can be written in many forms; non-locally in gauge fields alone, an example is given in [Hwang, 1987], or locally with the appearance of some scalar fields.

Interestingly, such a term could modify the commutators of composite operators (and/or) elementary fields. It certainly changes those of the Gauss law.

A related topic is the Berry's phase picture of the anomaly, see for example [Nelson and Alvarez-Gaume, 1985]. The phase, which embodies the anomaly effects, could be regarded as the connection in some internal 'curved' space. Consequently, should the usual covariant derivative then be modified to incorporate the new connection, uniquely for anomalous gauge theories? If so, is the additional piece of the new covariant derivative just another mask of the WZ term? Interestingly, this idea of modifying the covariant derivative has also appeared [Thompson and Zhang, 1987] as an action principle.

Yet another way of cancelling the anomalies has also been proposed [d'Hoker and Farhi, 1984a and b]. A fermion in the anomaly-free doublet is decoupled; its chirally-invariant mass generated by Yukawa coupling with scalar fields is sent to infinity together with the coupling. Decoupling in this way, there is a residual effect in the form of a WZ type term of the Higgs fields, whose main function is to keep the theory free of anomalies. Once again, the idea can be applied to lattices to decouple the doublers, see for further details [Smit, 1986]. The lattice approach is particularly suitable here for a non-perturbative treatment of the resulting strongly coupled Higgs sector. If this is realised, perhaps another role of the scalar fields would be revealed. Analytical and numerical studies of this, with the inclusion of dynamical fermions, deserve more attention.

Chapter 4

Phenomenological applications of lattice QCD

“The age in which we live is the age in which we are discovering the fundamental laws of nature, and that day will never come again. It is very exciting, it is marvellous, but this excitement will have to go.”

Feynman, 1967.

QCD is universally believed to be the theory of the strong interaction. However, it is precisely the confinement property that obstructs the road to phenomenology by perturbative means; so the lattice can be employed here to bridge the gap by providing sound and attractive non-perturbative methods. Even though there are still problems with chiral fermions and there is the practical limit of existing computing power, one can, with due care, extract leading-order behaviour in the framework of certain approximations.

To avoid a direct confrontation with the problem of lattice chiral fermions we adopt the low energy effective theory for energy well below the scale where the chiral gauge group is broken down. As the chiral vector bosons become heavy in this regime, the effective operators are products of physical currents at nearby space-time points. The hypothesis of the Operator Product Expansion (OPE) [Wilson, 1969] is then employed to separate out the perturbative part, in the evaluation of the so-called Wilson coefficients, and the non-perturbative part, the matrix elements of certain composite local operators. The calculation of the latter is further approximated on a finite-size lattice with quenched fermions and in the framework of the phenomenological lagrangian. Also questionable is the evaluation at one-loop level of the perturbative matching factors relating lattice results to observable facts. Although unfortunate, as it is at present these

approximations do not arise from any theoretical limitation. In fact, apart from the problem of the ‘notorious’ chiral fermions, given enough time and computer power we could, from first principles, obtain the numbers to (in)validate the theories.

The above comments are illustrate in greater detail in the examples of proton decay and the nucleon wavefunction [Bowler *et al.*, 1988] in section 2, and the weak matrix elements [Daniel *et al.*, 1987] in section 3. Wilson fermions are employed for the former study and staggered fermions for the latter. Unifying aspects of such studies are gathered in section 1 where brief accounts of the OPE and the phenomenological lagrangian are given. Being the results of two collaborations, there is some overlapping of the material of this chapter with that in [Daniel, 1987].

4.1 Introduction to phenomenological methods

4.1.1 Operator product expansion

The right-handed neutrino is yet to be found, so it is possible that chiral gauge theories, either the standard model or some Grand Unified Theory (GUT), could be the laws of nature. At well below the energy scale of the typical mass of the chiral vector bosons, which is presumably acquired via the Higgs mechanism, the transition amplitude for the process $I \rightarrow F$, at lowest order in perturbation theory, is given by

$$\mathcal{M}_{IF} = \int d^4x D(x^2, M^2) \langle F | T J^\mu(x) J_\mu^\dagger(0) | I \rangle. \quad (4.1)$$

$D(x^2, M^2)$ is the massive vector boson propagator and the J 's are the relevant currents for the process. The dominant contribution to the integral comes only from a space-time region of scale $x \sim 1/M$, which is small for large M . To deal with such a short distance product of operators, Wilson [1969] postulated an asymptotic expansion, the OPE,

$$A(x)B(y) \sim \sum_n C^{(n)}(x-y) \mathcal{O}_n\left(\frac{x+y}{2}\right) \text{ as } x \rightarrow y. \quad (4.2)$$

The expansion is taken to be valid in the weak sense, i.e. when sandwiched between physical states. Although it is true in free-field theories and can be

proved in perturbation theory for interacting fields [Zimmerman, 1970], the validity of the OPE remains a postulate otherwise. However, one can expect that to a good approximation the coefficients can be calculated in perturbation theory –if the theory is asymptotically free– while the non-perturbative effects are accounted for by the matrix elements of the local operators. Long-distance and short-distance physics are thus separated. It is worth mentioning that, furthermore, the coefficients reflect the symmetries of the lagrangian even though they may not be symmetries of the Hilbert space [Bernard *et al.*, 1975]. This proves useful for the study of spontaneously broken symmetry. However, here we only use the expansion in QCD.

For an asymptotically scale invariant theory, the Wilson coefficients can be seen to contain all the short distance singularities

$$C^{(n)}(x) \sim x^{d_n - d_A - d_B}, \quad (4.3)$$

where the d 's are the scale dimensions of appropriate operators, equal to the canonical dimensions plus the anomalous dimensions γ 's. When the scale invariance is broken, as in QCD, equation (4.3) is modified by logarithmic terms. Either way, dominant contributions come from the composite operators with the lowest scale dimensions.

The Green's functions of these composite operators require renormalisation, as do the coefficients. The same scale μ can be introduced for both of them. In practice, however, to avoid large logarithms in the perturbative estimation of the coefficients it is necessary to choose $\mu \sim \frac{1}{|x|}$. On the other hand, the connection between lattice and continuum operators in the perturbation context, as will be seen later, 'dictates' the scale μ in terms of the lattice spacing. Due to the difficulty in fixing the lattice spacing, one can evaluate the coefficients at $\mu \sim \frac{1}{|x|}$, where the calculation is reliable, then use the RG to change the scale. This RG improved perturbation amounts to summing up the leading logarithmic contributions to all orders in perturbation theory –the Leading Logarithmic Approximation (LLA) [Collins, 1985]– when the RG functions are calculated up to one loop.

The desired RG equation for the coefficients can be derived from those for the renormalised operators in the OPE. It can be solved to yield the result [Pokorski,

$$C^{(n)}(x^2, g, \mu) \sim C^{(n)}(x^2, g(t), e^t \mu) \left[\frac{g(0)}{g(t)} \right]^{\frac{[\gamma_A + \gamma_B - \gamma_n]}{\beta_0}} \quad (4.4)$$

with

$$g(0) \stackrel{def}{=} g|_{\mu}.$$

All of this is to be modified for the RG mixing with operators of dimension equal or less than that of the composite operators appearing in the OPE.

Before moving on, we mention in passing that the OPE can be extended to the light cone region, which is relevant in the analysis of certain high-energy processes. The leading singularities now come from operators with the lowest twist, the difference between the canonical dimension and the spin. There are, however, infinitely many operators of the same twist since differentiation raises the spin and the dimension of an operator by one unit simultaneously.

4.1.2 Renormalisation of composite operators

The next step is to identify the lattice operators corresponding to the composite operators on the rhs of the OPE expression. The matching factors connecting operators on the lattice and in some continuum regularisation scheme used in the evaluation of the Wilson coefficients, are to be calculated perturbatively due to asymptotic freedom. Thus perturbation theory, paradoxically, realises the physical interpretation of non-perturbative lattice results. Particularly, in the Wilson fermion formulation, subtraction of extra operators with different chiral behaviour is necessary.

To this end, we have to evaluate Green's functions with the operator \mathcal{O}_n inserted. In the generating functional language, we introduce sources Δ_n coupled to the operators \mathcal{O}_n to obtain the Green's functions in the usual manner from the functional derivative of

$$W[J, \Delta] = \int d\phi \exp \left\{ -S - \int (J\phi + \Delta\mathcal{O}) \right\}. \quad (4.5)$$

The counterterms already present in the lagrangian for the Green's functions involving only elementary fields ϕ are not sufficient to eliminate the new divergences associated with composite operators. In fact, it may require the subtraction of divergences having the structure of composite operators different from

\mathcal{O}_n itself. That is, we have to introduce in (4.5) a complete set of composite operators, with appropriate sources, allowed by the symmetry and of dimensions less than or equal that of \mathcal{O}_n . Such a mixing of Green's functions can be written in operator form, in the weak sense,

$$\begin{aligned}\mathcal{O}_n(\Lambda) &= M_{nm} \mathcal{O}_m^{\text{tree}}, \\ M_{nm} &= 0 \text{ for } \dim \mathcal{O}_n < \dim \mathcal{O}_m.\end{aligned}\tag{4.6}$$

The lhs of (4.6) is the perturbatively calculated bare operator, hence the dependence on the regulator cut-off Λ . All the singularities as $\Lambda \rightarrow 0$ are contained in $M_{nm}(\Lambda)$. A particular renormalisation scheme at a scale μ then gives

$$[\mathcal{O}_i]_\mu = Z_{ij}^{-1} \mathcal{O}_j(\Lambda).\tag{4.7}$$

As the tree operators are regularisation independent, equation (4.6) can be used to relate bare operators on the lattice and in some continuum regulator

$$\mathcal{O}_i^{\text{cont}}(\Lambda) = R_{ij} \mathcal{O}_j^{\text{latt}}(a)\tag{4.8}$$

for asymptotically free theory at large enough cut-off. The c-number matrix R_{ij} depends on the symmetries of the bare lagrangians in the two regularisation schemes. Note that since we are dealing with ultraviolet singularities both regulators give the same dependence on infrared cut-off, if required, and on external momenta. In this way the dependence will drop out in equation (4.8).

Combining now with equation (4.7) we thus make the correspondence between the lattice operators, whose matrix elements are to be measured, and the renormalised continuum operators used in the OPE. Such a relationship puts a constraint on the quantity $a\mu$ in order to avoid large corrections to the leading order perturbative term.

4.1.3 Phenomenological lagrangian approach

Matrix elements of lattice operators could be calculated directly from first principles, see chapter 1 for a discussion. In practice, however, we can reduce the task by using the so-called phenomenological lagrangian, so that only simpler matrix elements are needed.

The low energy physical states of the confining QCD theory are bound states of the elementary entities, quarks and gluons, of the underlying lagrangian. The effective lagrangian approach then arises as a systematic method for isolating those composite fields and studying their interaction in terms of a finite number of phenomenological parameters, up to a given energy scale. This is better than the method of current algebra even though the reliability of the perturbative loop expansion is still unclear [Gasser and Leutwyler, 1984 and 1985].

The lagrangian is obtained as the most general one consistent with the symmetry of the underlying theory, the chiral symmetry $G = SU_L(N_f) \otimes SU_R(N_f)$ [Weinberg, 1979]. Expanded to a given order of momenta for low energy processes, the only fields appearing are the phenomenological fields. As the chiral symmetry is spontaneously broken, i.e. realised in the Goldstone mode, non-linear representation of the group is needed to bridge the manifest difference between the lagrangian and Hilbert space symmetry properties.

Consider a general matrix in the representation (N_f, \bar{N}_f) of $SU_L(N_f) \otimes SU_R(N_f)$

$$\Sigma \rightarrow U_R \Sigma U_L^\dagger, \quad (4.9)$$

where $U_{L,R}$ are in the fundamental representation of $SU(N_f)$

$$U_{L,R} = \exp \left\{ -i \alpha_{L,R}^a \lambda^a \right\}. \quad (4.10)$$

The generators λ^a are normalised as $\text{tr}(\lambda^a \lambda^b) = \frac{1}{2} \delta^{ab}$. The spontaneously broken axial transformations correspond to $\alpha_L = -\alpha_R$ and the unbroken vector transformations correspond to $\alpha_L = \alpha_R$, which are realised linearly. At each space-time point we can rotate the vacuum by an axial transformation

$$A(x) = \exp \left\{ \frac{i}{f} \pi^a(x) \lambda^a \right\} \quad (4.11)$$

in such a way that the field $\Sigma(x)$ transforms like a linear representation of $SU_V(N_f)$ under the *full* chiral group. That is,

$$\tilde{\Sigma}(x) \stackrel{\text{def}}{=} A(x) \Sigma A(x) \rightarrow V(x) \tilde{\Sigma} V^\dagger(x), \quad (4.12)$$

which implies an equation to specify $V(x)$ uniquely

$$A(x) \rightarrow V(x) A(x) U_R^\dagger = U_L A(x) V^\dagger(x). \quad (4.13)$$

Thus, a global transformation of G is realised as a gauge transformation of $SU_V(N_f)$ on $\tilde{\Sigma}$ and a non-linear transformation on $\pi^a(x)$. In particular, $A(x)$ transforms linearly under $SU_V(N_f)$ global transformations.

The non-linearly transformed axial scalar fields $\pi^a(x)$ represent the degrees of freedom of the Goldstone boson sector and should be interpreted as pion fields [Weinberg, 1968; Coleman *et al.*, 1969]. This interpretation is permissible since although different definitions of the fields lead to different off-shell matrix elements, they should all give the same results on the mass-shell. The phenomenological lagrangian can now be constructed for the meson fields $\pi^a(x)$ ¹ the other fields $\tilde{\Sigma}$ and the baryonic fields. In the purely mesonic theory, the baryonic fields, however, could be regarded as solitons of the theory [Witten, 1983].

Due to the non-zero quark masses, the chiral symmetry is also broken explicitly (but softly). The mass term can be incorporated into the effective theory of the soft pion limit as a non-leading effect, that is, the lagrangian is regarded as an expansion in mass. This reproduces the PCAC-like relationships between quark and meson masses.

The effective theory should also reflect the anomalous breaking of the assumed chiral symmetry. In other words, the anomalous Ward identities have to be satisfied at the *tree level*. We refer to elsewhere, for example [Pokorski, 1987], for such incorporation of the $U_A(1)$ anomaly and the non-abelian anomaly, via the Wess-Zumino term.

With the chiral lagrangian, different matrix elements of a composite operator of the fundamental fields can be related [Maiani *et al.*, 1987]. One replaces the operator by a linear combination of those of the phenomenological fields which possess the same chiral properties up to appropriate order in momenta and mass, to take into account the effect of symmetry breaking. Sandwiching this linear combination between enough different physical states will then enable us to obtain relationships among $\langle F|\mathcal{O}|I\rangle$ with different F 's and I 's. To this end, the LSZ reduction is employed in conjunction with the phenomenological

¹Or equivalently, in terms of

$$U(x) \stackrel{def}{=} A^\dagger(x)A^\dagger(x) \rightarrow U_R U(x) U_L^\dagger.$$

lagrangian.

In the last subsection we saw that it is necessary to match the lattice QCD operators onto perturbative calculations at small enough lattice spacing. On the other hand, at long distance, we also want to match non-perturbative lattice results onto the chiral lagrangian, to calculate the phenomenological parameters. That is, the chiral lagrangian should be able to tell us how to extrapolate from the lattice world with moderately light pions to the real world with very light pions. Note that with Wilson fermions this is quite non-trivial as certain effects have to be subtracted to correct for the chiral behaviour.

In practice, it is far from the ideal situation above; lattice results are far from the chiral limit where there are dominant finite-size and (critical) slowing down effects. Conversely, the chiral lagrangian is assumed *a priori* so that we can replace the lattice calculations of somewhat difficult matrix elements by easier, albeit off-shell, ones.

4.2 Measurements of baryonic operators

4.2.1 The relevant matrix elements

The overall normalisation f_N for the quark distribution amplitude in the proton, which is useful for hard exclusive processes, can be obtained from the matrix elements of the lowest twist three-quark operator between the proton and the vacuum. It is convenient to work with the operator, in the spinorial component δ ,

$$\hat{f}_\delta^\nu = \epsilon^{ijk} [u^i C \gamma^\nu \gamma_5 d^j] u_\delta^k \quad (4.14)$$

from which a suitable correlator can be constructed to remove higher twist components of the matrix elements.

Matrix elements of other three-quark operators that govern the short-distance properties of the wavefunction, instead of the light-cone properties above, are also of special interest. In particular there are those of the lowest dimension operators contributing to the decay of the proton in Grand Unified Theories

(GUTs). Such theories stem from the desire to unify all the forces in nature, except gravitation, in the framework of gauge theories.

Take the theory with the simplest simple gauge group, $SU(5)$, of minimum rank that can accommodate the standard model as a low energy effective theory. The unification of the strong and electroweak forces is now embodied in the single coupling. Gauge fields are in the **24** (adjoint) representation and two multiplets of Higgs bosons are required in the minimal model, the **24**-plet and the **5**-plet, to break

$$SU(5) \rightarrow SU(3) \otimes SU(2) \otimes U(1)$$

at the GUT scale of 10^{15} GeV and

$$SU(3) \otimes SU(2) \otimes U(1) \rightarrow SU(3) \otimes U(1)$$

at the electroweak scale, respectively. The fermions are grouped into $\bar{\mathbf{5}} \oplus \mathbf{10}$ for each generation to satisfy the anomaly-free requirement for the chiral gauge group. As quarks and leptons are grouped together in the same irreducible representation, baryon and lepton number are no longer conserved separately. This prediction of proton decay is a feature of most, but not all, GUTs.

The baryon number violating effective interaction is governed by the short-distance products of the currents, for one generation,

$$\begin{aligned} J_{\mu}^X &= \epsilon_{ijk} \bar{u}_{kL}^c \gamma_{\mu} u_{jL} + \bar{d}_i \gamma_{\mu} e^+, \\ J_{\mu}^Y &= \epsilon_{ijk} \bar{u}_{kL}^c \gamma_{\mu} d_{jL} - \bar{u}_{iL} \gamma_{\mu} e_L^+ + \bar{d}_{iR} \gamma_{\mu} \nu_R^c, \end{aligned}$$

which couple to the X and Y bosons respectively. We ignore the contribution from the Higgs sector and the effect of flavour mixing [Buras *et al.*, 1978; Mohapatra, 1986]. The OPE for the decay process $p \rightarrow \pi^0 e^+$ (the best candidate for experimental detection) yields the composite operators [Wise *et al.*, 1981]

$$\begin{aligned} Q_1 &= \epsilon^{ijk} (\bar{u}_R^i C d_R^j) (\bar{e}_L^+ u_L^k), \\ Q_2 &= \epsilon^{ijk} (\bar{u}_L^i C d_L^j) (\bar{e}_R^+ u_R^k), \\ Q_3 &= \epsilon^{ijk} (\bar{u}_L^i C d_L^j) (\bar{e}_L^+ u_L^k), \\ Q_4 &= \epsilon^{ijk} (\bar{u}_R^i C d_R^j) (\bar{e}_R^+ u_R^k), \end{aligned} \tag{4.15}$$

all of which have the same anomalous dimension of -4 [Abbot and Wise, 1980]. The Wilson coefficients are given by (4.4), with g the strong coupling as the electroweak correction is neglected, and thus contain all the model dependence.

The QCD effects of (4.15) are derivable from the two three-quark operators

$$(\mathcal{O}_\alpha)^\delta = \epsilon_{ijk}(\bar{u}_{iR} C d_{jR}) u_{kL}^\delta, \quad (4.16)$$

$$(\mathcal{O}_\beta)^\delta = \epsilon_{ijk}(\bar{u}_{iL} C d_{jL}) u_{kL}^\delta,$$

where α and β are just labels for the operators and δ is the spinor index. The chiral lagrangian approach in the soft pion limit relates the matrix elements of (4.16) between the proton and the neutral pion to

$$\langle 0 | (\mathcal{O}_\alpha)^\delta | p \rangle = \alpha p_L^\delta, \quad (4.17)$$

$$\langle 0 | (\mathcal{O}_\beta)^\delta | p \rangle = \beta p_L^\delta,$$

where p^δ is a component of the proton spinor. Strictly speaking, the validity of taking the soft pion limit is questionable in that the pion typically carries of the order of half the momentum of the decaying proton.

When the decay is mediated purely by the exchange of superheavy gauge bosons, i.e. no Higgs exchange as in the case with minimal GUTs, the interaction vertices must involve both left and right handed fields. Thus the decay rate depends only on the non-perturbative parameter α above. For $SU(5)$ it is [Brodsky *et al.*, 1984]

$$\Gamma(p \rightarrow \pi^0 e^+) = \frac{5\pi}{4} |\alpha|^2 g_{\text{GUT}}^2 A^2 (1 + g_A^2) \frac{m_p}{f_\pi^2 M_X^4} \left\{ 1 - \frac{m_\pi^2}{m_p^2} \right\}^2, \quad (4.18)$$

$$g_A \approx 1.25$$

and $A \approx 3$ is the short-distance enhancement factor relating, in the LLA, the Wilson coefficients renormalised at the GUT scale and a lower scale.

4.2.2 Perturbative corrections

The one-loop perturbative corrections for the Green's functions

$$\langle \mathcal{O}_{\alpha\beta} \bar{u}_{R,L} \bar{d}_{R,L} \bar{u}_L \rangle$$

have been calculated [Richards *et al.*, 1987] in both the Pauli-Villars regularisation with cut-off Q and for lattice Wilson fermions with spacing a . In the Pauli-Villars regularisation, diagrams of figure 4.1 give

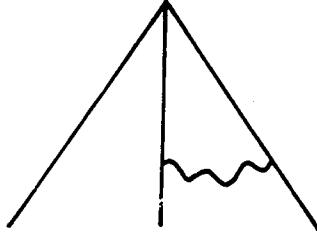


Figure 4.1: A typical diagram evaluated for the one-loop renormalisation of three-quark operators.

$$\mathcal{O}_{\alpha,\beta}^{\text{PV}}(Q) = \mathcal{O}_{\alpha,\beta}^{\text{tree}} \left\{ 1 - \frac{g^2}{4\pi} \gamma_{\alpha,\beta} \ln \left(\frac{Q}{\kappa} \right) \right\}, \quad (4.19)$$

where κ is the infrared regulating mass and the γ 's are the anomalous dimensions

$$\gamma_\alpha = \gamma_\beta \stackrel{\text{def}}{=} \gamma = -4.$$

Note that both \mathcal{O}_α and \mathcal{O}_β transform into themselves and there are no finite corrections in (4.19). On the other hand, due to the explicit breaking of chiral symmetry on the lattice, \mathcal{O}_α and \mathcal{O}_β mix among themselves and with a third operator \mathcal{O}_γ

$$(\mathcal{O}_\gamma)_\delta = \epsilon_{ijk} (u_i \mathbf{C} \gamma_\rho \gamma_5 d_j) (\gamma_\rho u_{kR})_\delta. \quad (4.20)$$

Additional diagrams of figure 4.2 are also required for the self-energy corrections to the quark external lines. However, as $\mathcal{O}_{\alpha,\beta}^{\text{latt}}$ are local –i.e. no gauge link is needed in their definitions to maintain gauge invariance– there are no diagrams with gluon lines emerging from the operators themselves, a feature that would be typically present in lattice perturbative calculations. It was found that

$$\begin{aligned} \mathcal{O}_{\alpha,\beta}^{\text{latt}}(a) &= \mathcal{O}_{\alpha,\beta}^{\text{tree}} \left[1 + \frac{g^2}{(4\pi)^2} (\gamma \ln a\kappa + C_1^L) \right] \\ &+ \mathcal{O}_{\beta,\alpha}^{\text{tree}} \left[\frac{g^2}{(4\pi)^2} C_2^L \right] \\ &\pm \mathcal{O}_\gamma^{\text{tree}} \left[\frac{g^2}{(4\pi)^2} C_3^L \right]. \end{aligned} \quad (4.21)$$

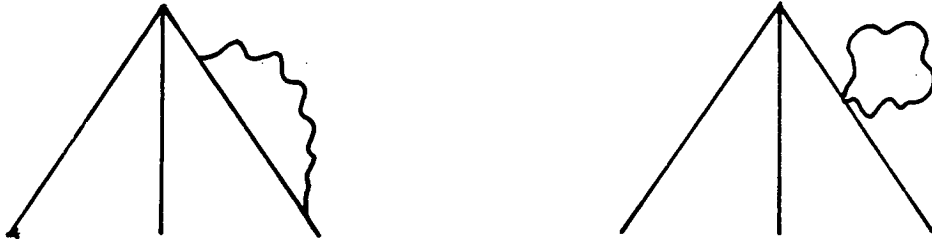


Figure 4.2: Contributions from the self-energy corrections to the external quark line.

Both the Pauli-Villars and lattice calculations were carried out at zero external momenta and with the same infrared cut-off κ . At the value $r = 1$ for the Wilson parameter the finite corrections are, after the subtraction of the singular term in (4.21),

$$\begin{aligned}
 C_1^L &= 37.9, \\
 C_2^L &= -3.2 \\
 C_3^L &= -0.8.
 \end{aligned}
 \tag{4.22}$$

Equations (4.19) and (4.21) can be rewritten as

$$\begin{aligned}
 \mathcal{O}_{\alpha,\beta}^{\text{PV}}(Q) &= \mathcal{O}_{\alpha,\beta}^{\text{latt}}(a) \left[1 - \frac{g^2}{(4\pi)^2} (\gamma \ln aQ + C_1^L) \right] \\
 &\quad - \mathcal{O}_{\beta,\alpha}^{\text{latt}}(a) \left[\frac{g^2}{(4\pi)^2} C_2^L \right] \\
 &\quad \mp \mathcal{O}_{\gamma}^{\text{latt}}(a) \left[\frac{g^2}{(4\pi)^2} C_3^L \right].
 \end{aligned}
 \tag{4.23}$$

where g is the tree (zeroth order) coupling. Since C_1^L is relatively large the mixing represented by other terms of (4.23) can be neglected provided none of those operators are anomalously enhanced (as is confirmed numerically).

There is the question of what to choose for the renormalisation scale μ in the *MOM* (momentum subtraction scheme) of the continuum [Hasenfratz and

Hasenfratz, 1980]

$$[\mathcal{O}_{\alpha,\beta}^{\text{PV}}]_{\text{MOM}}(\mu) \approx \mathcal{O}_{\alpha,\beta}^{\text{latt}}(a) \left[1 - \frac{g^2}{(4\pi)^2} (\gamma \ln a\mu + C) \right]. \quad (4.24)$$

The question of what value to take for g in (4.24) can only be answered by performing higher-loop calculations –as the scale parameter Λ cannot be determined by one-loop calculations. However, we have chosen in (4.24) to the lowest order

$$g_{\text{MOM}}(\mu) = g_{\text{latt}}(a), \quad (4.25)$$

Assuming the scaling behaviour, it is natural to take

$$\mu a = \frac{\Lambda^{\text{MOM}}}{\Lambda^{\text{latt}}} \approx 100$$

so as to maintain (4.25) beyond this order. Alternatively, one could have adopted other values for μ , but with the above value it is hoped that higher order corrections to (4.24) are negligible.

Similarly, one-loop corrections of the operator (4.14) only to the leading twist part yield

$$\begin{aligned} [\hat{f}^{\text{PV}}]_{\text{MOM}}(\mu) &\approx \hat{f}^{\text{latt}}(a) \left\{ 1 - \frac{g^2}{(4\pi)^2} [\gamma_0 \ln a\mu + d] \right\}, \quad (4.26) \\ \gamma_0 &= 4/3, \\ d &= 34.28. \end{aligned}$$

In all, the corrections in passing from the lattice measurements to the continuum are potentially quite large.

4.2.3 Numerical results with $r = 1$ Wilson fermions

The results in this subsection have appeared in the literature [Bowler *et al.*, 1988]. To extract the non-perturbative parameters α, β on the lattice, we measure the correlators

$$C_{\alpha,\beta}(t) = \sum_{\vec{x}} \langle \mathcal{O}_{\alpha,\beta}^\dagger(\vec{x}, t) \mathcal{O}_{\alpha,\beta}(0, t) \rangle$$

which is expected to behave asymptotically as

$$C_\alpha(t) \sim |\alpha|^2 e^{-m_p t}, \quad (4.27)$$

$$C_\beta(t) \sim |\beta|^2 e^{-m_p t},$$

for $t \rightarrow \infty$. The correlator used to determine f_N has a more complicated structure, being built from the operator \hat{f} so as to remove the non-leading twist components. At large time it is fitted to

$$C_f(t) \sim \frac{9}{4} m_p^2 |f_N|^2 e^{-m_p t}. \quad (4.28)$$

All measurements were performed using gauge configurations generated in the quenched approximation on an $8^3 \times 16$ lattice at $\beta_g = 5.7$. The Metropolis algorithm [Metropolis *et al.*, 1953] had been used to generate the configurations for an earlier study [Bowler *et al.*, 1984], where the critical hopping parameter $\kappa = 1/2ma$ was found to be 0.1695(7). Successive configurations were separated by 1200 MC sweeps to ensure minimal correlations. Among those, 32 configurations were selected for the present study. Wilson quark propagators were calculated in 4 byte real arithmetic using a relaxation method on the ICL Distributed Array Processors (DAPs) at Edinburgh University. The DAPs had 64×64 processors with 4 Kbits of local memory each. Calculations were done at three different quark masses corresponding to $\kappa = 0.1525, 0.1575$ and 0.1625 with periodic boundary conditions in the spatial directions and Dirichlet boundary conditions in the time direction. The source for the propagator was placed at the third time slice.

The correlators were fitted to the forms of (4.27) and (4.28), but with an additional exponential, by minimizing the following function

$$\sum_t \left[\frac{\Delta(t) - C(t)}{|\Delta(t)| + |C(t)|} \right]^2$$

where Δ and C are the measured and the fitted values of the correlators respectively. The sum is over the range of time slices 5-15. The measured values of the correlators on those time slices are tabulated in tables 4.1, 4.2 and 4.3. To get an indication of the quality of the fits, the range of time slices to which the correlators were fitted was varied. Errors were estimated by the ‘‘Jack-knife’’

Time-slice	Bare Quark Mass		
n_4	m_1	m_2	m_3
5	$(2.60 \pm 0.07) \times 10^{-4}$	$(3.54 \pm 0.10) \times 10^{-4}$	$(4.64 \pm 0.14) \times 10^{-4}$
6	$(1.77 \pm 0.07) \times 10^{-5}$	$(2.80 \pm 0.12) \times 10^{-5}$	$(4.18 \pm 0.18) \times 10^{-5}$
7	$(1.72 \pm 0.10) \times 10^{-6}$	$(3.39 \pm 0.21) \times 10^{-6}$	$(6.26 \pm 0.46) \times 10^{-6}$
8	$(2.28 \pm 0.16) \times 10^{-7}$	$(5.86 \pm 0.51) \times 10^{-7}$	$(1.38 \pm 0.15) \times 10^{-6}$
9	$(3.68 \pm 0.36) \times 10^{-8}$	$(1.24 \pm 0.14) \times 10^{-7}$	$(3.65 \pm 0.49) \times 10^{-7}$
10	$(6.53 \pm 0.81) \times 10^{-9}$	$(2.79 \pm 0.42) \times 10^{-8}$	$(9.92 \pm 1.73) \times 10^{-8}$
11	$(1.16 \pm 0.17) \times 10^{-9}$	$(6.25 \pm 1.15) \times 10^{-9}$	$(2.63 \pm 0.67) \times 10^{-8}$
12	$(2.03 \pm 0.34) \times 10^{-10}$	$(1.40 \pm 0.31) \times 10^{-10}$	$(7.30 \pm 2.45) \times 10^{-9}$
13	$(3.78 \pm 0.74) \times 10^{-11}$	$(3.28 \pm 0.88) \times 10^{-10}$	$(1.81 \pm 0.92) \times 10^{-9}$
14	$(7.67 \pm 1.91) \times 10^{-12}$	$(9.08 \pm 3.18) \times 10^{-11}$	$(7.85 \pm 5.31) \times 10^{-10}$
15	$(1.68 \pm 0.64) \times 10^{-12}$	$(2.98 \pm 1.50) \times 10^{-11}$	$(3.99 \pm 3.53) \times 10^{-10}$

Table 4.1: 32-configuration averages of correlator $C_\alpha(t)$.

Time-slice	Bare Quark Mass		
n_4	m_1	m_2	m_3
5	$(3.09 \pm 0.10) \times 10^{-4}$	$(4.29 \pm 0.15) \times 10^{-4}$	$(5.77 \pm 0.20) \times 10^{-4}$
6	$(2.02 \pm 0.09) \times 10^{-5}$	$(3.23 \pm 0.16) \times 10^{-5}$	$(4.90 \pm 0.24) \times 10^{-5}$
7	$(1.84 \pm 0.11) \times 10^{-6}$	$(3.60 \pm 0.25) \times 10^{-6}$	$(6.66 \pm 0.50) \times 10^{-6}$
8	$(2.27 \pm 0.18) \times 10^{-7}$	$(5.73 \pm 0.56) \times 10^{-7}$	$(1.39 \pm 0.18) \times 10^{-6}$
9	$(3.52 \pm 0.37) \times 10^{-8}$	$(1.16 \pm 0.16) \times 10^{-7}$	$(3.88 \pm 0.67) \times 10^{-7}$
10	$(6.29 \pm 0.83) \times 10^{-9}$	$(2.72 \pm 0.43) \times 10^{-8}$	$(1.22 \pm 0.25) \times 10^{-7}$
11	$(1.14 \pm 0.17) \times 10^{-9}$	$(6.30 \pm 1.07) \times 10^{-9}$	$(3.31 \pm 0.87) \times 10^{-8}$
12	$(2.09 \pm 0.35) \times 10^{-10}$	$(1.50 \pm 0.31) \times 10^{-9}$	$(9.06 \pm 2.81) \times 10^{-9}$
13	$(3.90 \pm 0.75) \times 10^{-11}$	$(3.64 \pm 0.93) \times 10^{-10}$	$(3.34 \pm 1.43) \times 10^{-9}$
14	$(7.65 \pm 1.94) \times 10^{-12}$	$(8.75 \pm 3.17) \times 10^{-11}$	$(8.65 \pm 8.01) \times 10^{-10}$
15	$(1.47 \pm 0.50) \times 10^{-12}$	$(2.31 \pm 1.09) \times 10^{-11}$	$(3.62 \pm 4.25) \times 10^{-10}$

Table 4.2: 32-configuration averages of correlator $C_\beta(t)$.

Time-slice	Bare Quark Mass		
n_4	m_1	m_2	m_3
5	$(7.58 \pm 0.23) \times 10^{-4}$	$(9.82 \pm 0.33) \times 10^{-4}$	$(1.23 \pm 0.44) \times 10^{-3}$
6	$(4.37 \pm 0.21) \times 10^{-5}$	$(6.26 \pm 0.33) \times 10^{-5}$	$(8.33 \pm 0.46) \times 10^{-5}$
7	$(3.73 \pm 0.27) \times 10^{-6}$	$(6.34 \pm 0.55) \times 10^{-6}$	$(9.67 \pm 1.06) \times 10^{-6}$
8	$(4.39 \pm 0.41) \times 10^{-7}$	$(9.34 \pm 1.14) \times 10^{-7}$	$(1.75 \pm 0.29) \times 10^{-6}$
9	$(6.63 \pm 0.75) \times 10^{-8}$	$(1.81 \pm 0.27) \times 10^{-7}$	$(4.14 \pm 0.87) \times 10^{-7}$
10	$(1.15 \pm 0.16) \times 10^{-8}$	$(3.98 \pm 0.71) \times 10^{-8}$	$(1.05 \pm 0.27) \times 10^{-7}$
11	$(2.07 \pm 0.30) \times 10^{-9}$	$(9.14 \pm 1.74) \times 10^{-9}$	$(2.98 \pm 0.81) \times 10^{-8}$
12	$(3.57 \pm 0.59) \times 10^{-10}$	$(2.01 \pm 0.44) \times 10^{-9}$	$(9.97 \pm 3.75) \times 10^{-9}$
13	$(6.66 \pm 1.29) \times 10^{-11}$	$(5.13 \pm 1.40) \times 10^{-10}$	$(3.91 \pm 1.92) \times 10^{-9}$
14	$(1.37 \pm 0.36) \times 10^{-11}$	$(1.53 \pm 0.57) \times 10^{-10}$	$(1.87 \pm 1.05) \times 10^{-9}$
15	$(3.24 \pm 1.11) \times 10^{-12}$	$(4.96 \pm 2.34) \times 10^{-11}$	$(8.36 \pm 5.58) \times 10^{-10}$

Table 4.3: 32-configuration averages of correlator $C_f(t)$.

procedure [Gottlieb *et al.*, 1986]: fits were performed on the 32 ensembles of data, each obtained by eliminating one configuration at a time.

From the correlators an ‘effective mass’,

$$M(t) = \ln \left[\frac{C(t-1)}{C(t)} \right], \quad (4.29)$$

was also calculated. The leading exponential decay is exposed as a plateau against time. Such graphs for $C_{\alpha,f}$ are plotted in figures 4.3 and 4.4 respectively.

In general, we see that at small quark mass values the stable regions are shortened in accordance with the critical slowing down as the finite-size effects begin to take over. This is particularly severe for C_f . However, with the caveat that consistent fits at the lowest quark mass are obtained only over a limited range of time slices, measurements of $|f_N|$ at three quark masses are shown in figure 4.5; $|\alpha|$ and $|\beta|$ in 4.6 and 4.7.

Finally we have also measured \mathcal{O}_γ of (4.20) for a few configurations to ensure that it is not anomalously large. In fact, it turned out to be the same order of magnitude as the others so that neglecting of the mixing in equation (4.24) is justified.

Linear extrapolation to the chiral limit then yields

$$\begin{aligned} |\alpha|a^3 &\approx 2.6 \times 10^{-2}, \\ |\beta|a^3 &\approx 2.1 \times 10^{-2}, \\ |f_N|a^2 &\approx 1.5 \times 10^{-2}. \end{aligned} \quad (4.30)$$

However, at $\beta_g = 5.7$ there is no unique value of the lattice spacing for Wilson fermions. This is due to the lack of asymptotic scaling [Bowler *et al.*, 1984]; if the lattice spacing is chosen so as to ensure the correct value for the ρ meson mass then the proton mass is overestimated. Since, ultimately, our aim is to set a lower bound on the lifetime of the proton, the estimate corresponding to the largest value of a was used. This gives the correct value for the proton mass at

$$a^{-1} = 0.85 \pm 0.08 \text{ GeV}. \quad (4.31)$$

From equations (4.24) and (4.27) the continuum values of the parameters at renormalisation scale $\mu \approx 100a^{-1} \approx 85 \text{ GeV}$ turn out to be

$$|\alpha|^{\text{cont}} \approx 1.3 \times 10^{-2} \text{ GeV}^3,$$

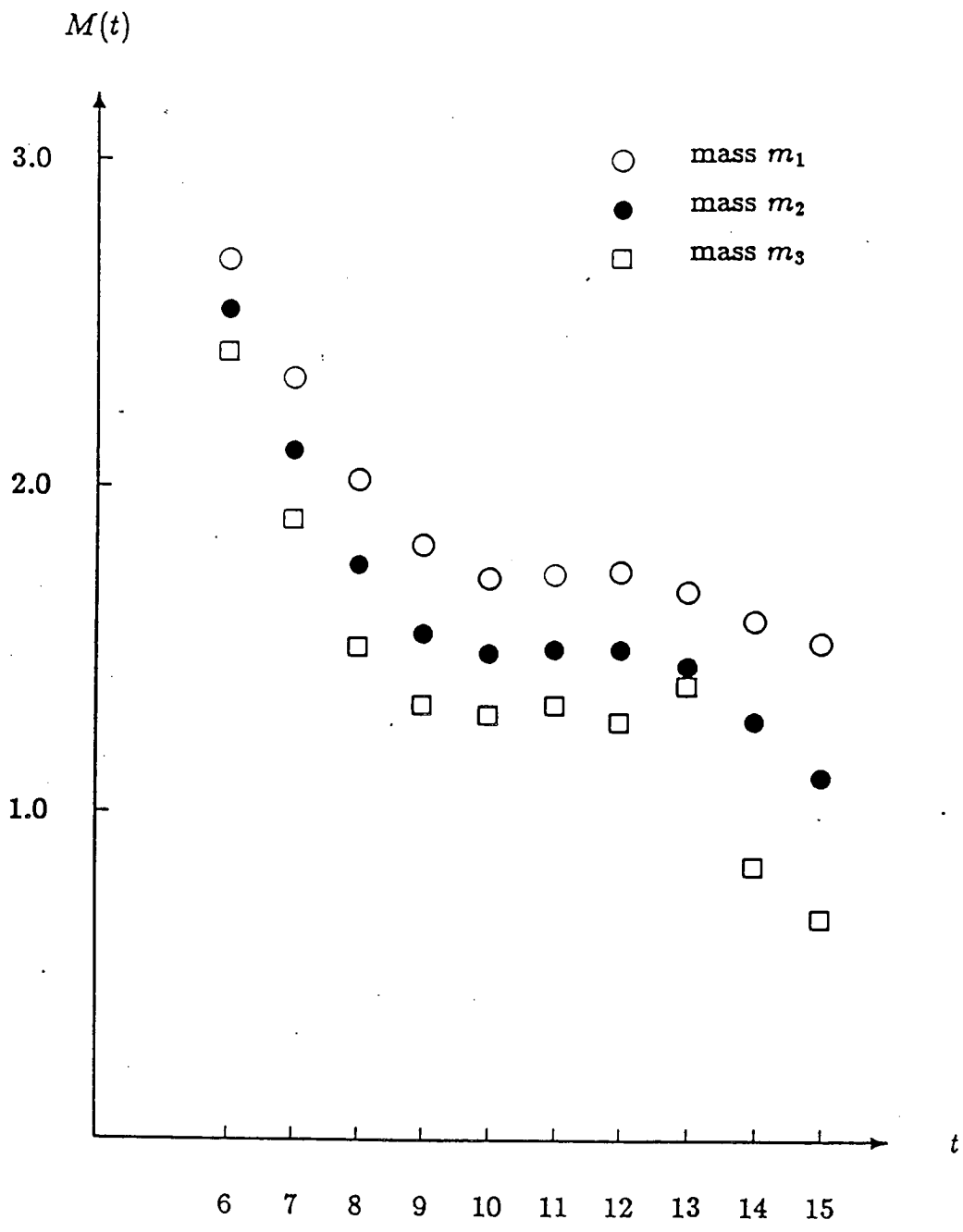


Figure 4.3: A plot of the effective mass $M(t)$ versus the time slice for C_α .

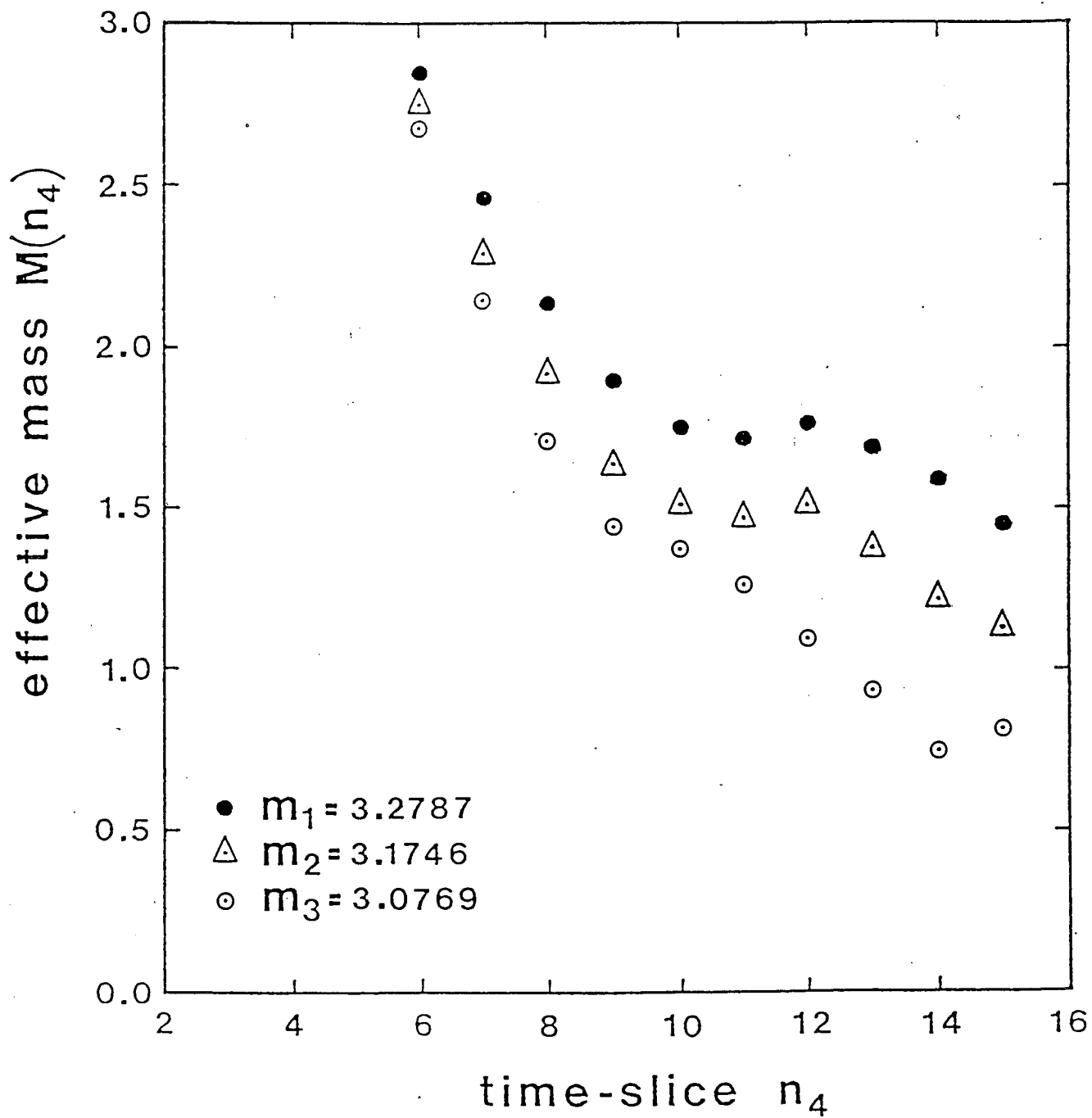


Figure 4.4: A plot of the effective mass $M(t)$ versus the time slice for C_f .

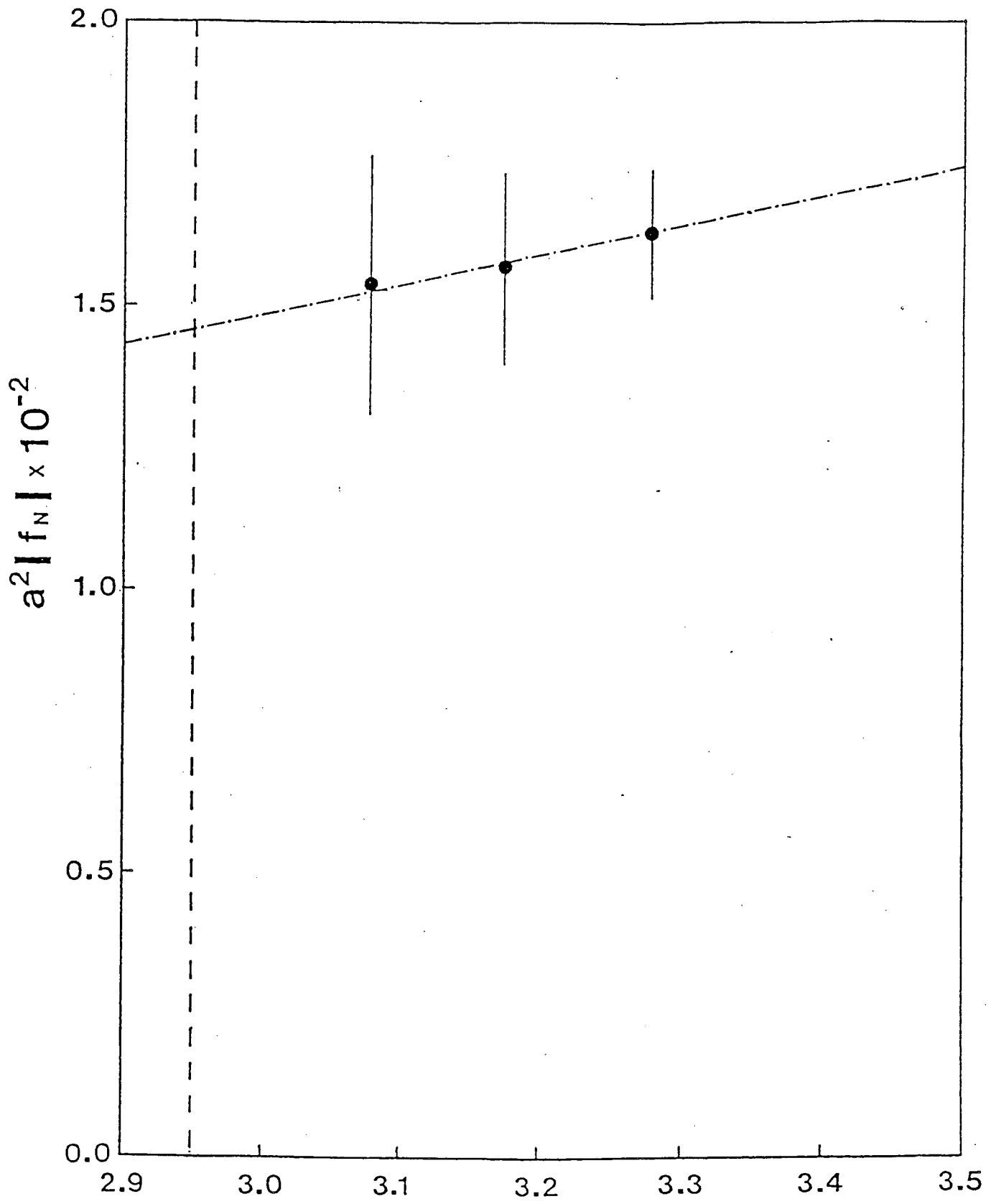


Figure 4.5: Measurements of $|f_N|$ at the three values of the quark mass.

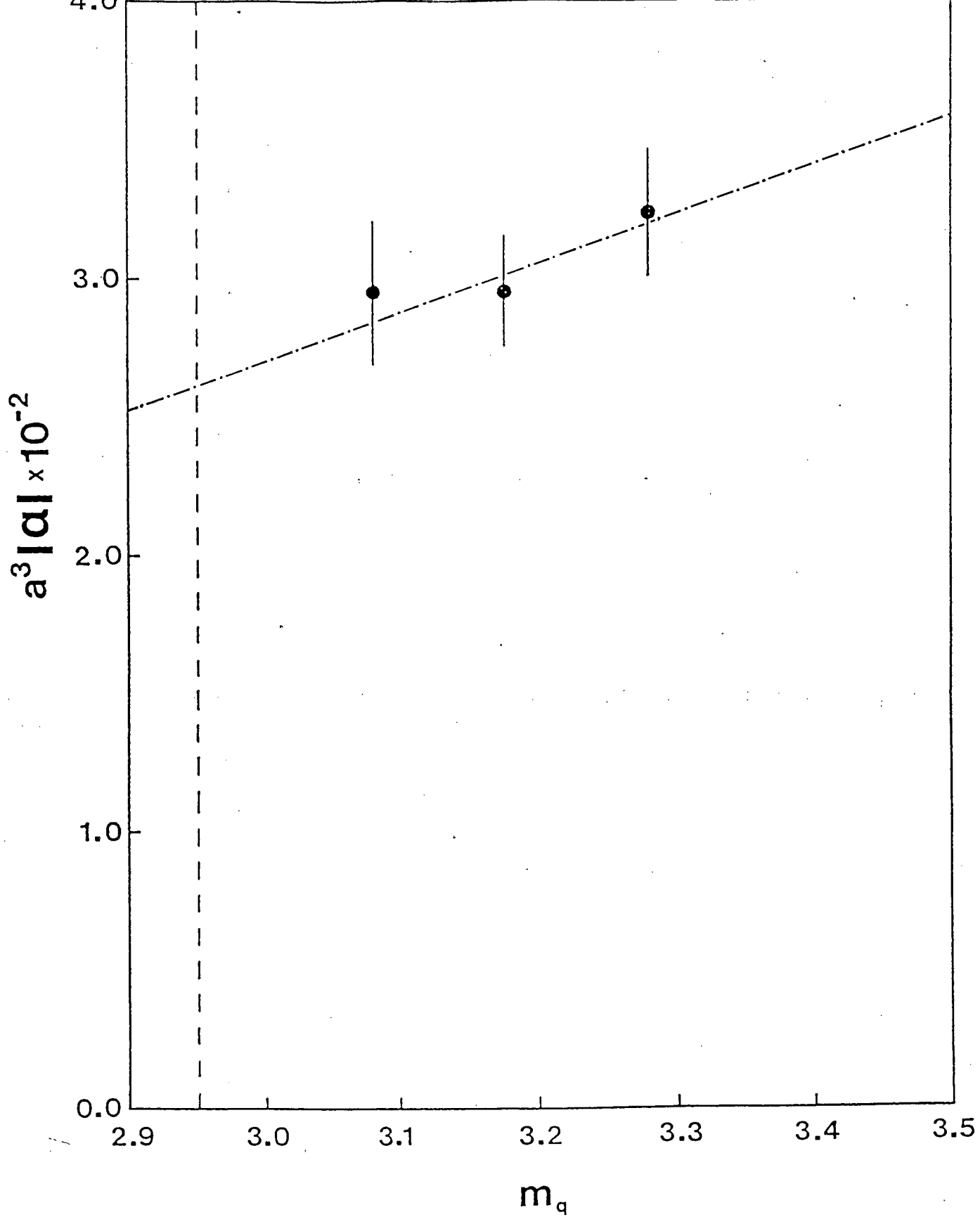


Figure 4.6: Measurements of $|\alpha|$ at the three values of the quark mass.

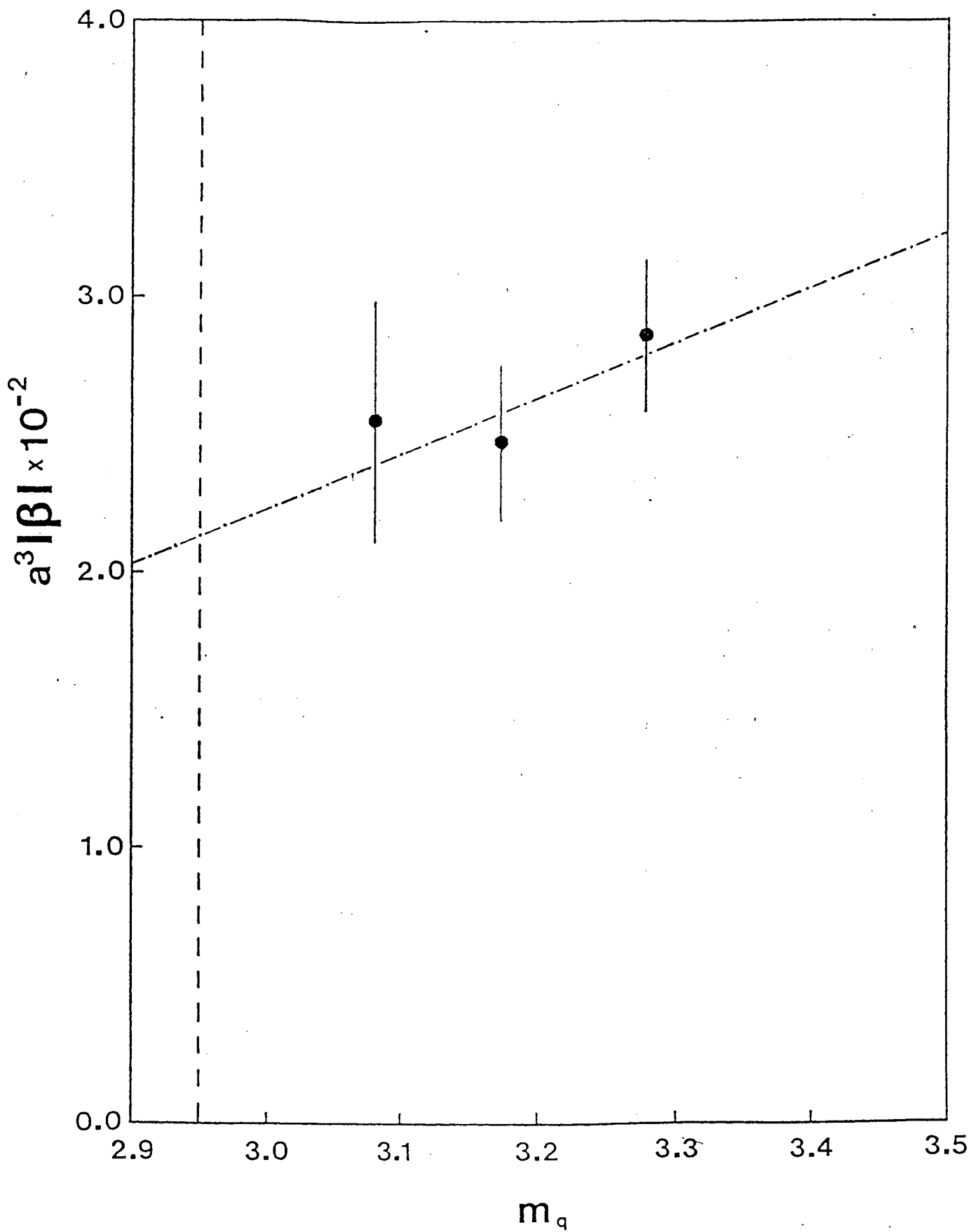


Figure 4.7: Measurements of $|\beta|$ at the three values of the quark mass.

(4.32)

$$|\beta|^{\text{cont}} \approx 1.0 \times 10^{-2} \text{ GeV}^3,$$

and

$$|f_N|^{\text{cont}} \approx 6.6 \times 10^{-3} \text{ GeV}^2. \quad (4.33)$$

Results of (4.32) are consistent with other estimates obtained by QCD sum rules or bag models [Brodsky *et al.*, 1984]. However, a recent lattice calculation [Hara *et al.*, 1986] on a $16^3 \times 48$ lattice with spacing $a^{-1} \approx 1.8 \text{ GeV}$, but with an improved action, found a larger value for $\alpha \approx 0.029 \text{ GeV}^3$. This number is subjected to a reduction of 20% to yield the continuum value at a renormalisation scale of 180 GeV. Though the discrepancy with our measurements seems quite large, it should be pointed out that neither of the simulations are in the asymptotic scaling regime. Thus the final results are somewhat arbitrary.

Our value of $|f_N|$ in (4.33) is rather larger than the sum rule estimate [King and Sachrajda, 1986] but a discrepancy of only 20% is perhaps acceptable in view of the large systematic errors that also afflict sum rule calculations.

We now come to the implications of our results for the proton decay rate. Setting α as in (4.32), $\Lambda^{MS} = 100 \text{ MeV}$ and truncating the enhancement factor A at $\mu = 85 \text{ GeV}$, we find from (4.18)

$$\tau(p \rightarrow \pi^0 e^+) \approx 5.4 \times 10^{31} \left(\frac{M_X}{10^{15} \text{ GeV}} \right)^4 \text{ years}. \quad (4.34)$$

The experimental limit [Blewitt *et al.*, 1986]

$$\tau(p \rightarrow \pi^0 e^+) > 3.1 \times 10^{32} \text{ years}$$

then implies that

$$M_X > 1.5 \times 10^{15} \text{ GeV}, \quad (4.35)$$

whereas estimates based upon RG analysis [Langacker, 1986] suggest

$$M_X \approx 1.3 \times 10^{14} \times (1.5)^{\pm 1} \text{ GeV}.$$

Remember that equation (4.35) is based on the chiral lagrangian approach, which is only valid in the soft pion limit, whereas the π^0 in the proton decay typically takes half of the proton energy. Nevertheless, the approach is systematic

and should at least reveal the leading behaviour unless there is some (group theoretical) suppression of the $\langle 0|qqq|p\rangle$ matrix element in the chiral limit. There is, in fact, no evidence for such suppression [Brodsky, 1984]. We thus can say that our lattice results rule out $SU(5)$ as the Grand Unified Theory.

4.3 Weak matrix elements

The success of the Standard Model is unprecedented but even in the low energy regime –low in comparison with the GUT scale– there still are phenomena yet to be accounted for. In particular there are the non-leptonic strangeness changing processes: CP violation and the approximate $\Delta I = 1/2$ rule. We concern ourselves with the later and try to tackle the task of non-perturbative calculation on the lattice of relevant matrix elements using staggered fermions. This project is still continuing and some results have been published [Daniel *et al.*, 1987].

4.3.1 Introduction to the $\Delta I = 1/2$ problem

We present here a brief review of this long-standing problem and its present status. See [Eeg, 1987] and also [Altarelli and Maiani, 1974; Shifman *et al.*, 1977; Minkowski 1979; Cheng 1987] for further details.

For flavour changing processes, the mass of the W boson is much higher than the characteristic hadronic scale thus the vector boson field can be effectively integrated out. In analogy with the integration of the X and Y vector bosons of the $SU(5)$ GUT of the previous section, all the decoupling effects can be absorbed by renormalisation [Appelquist and Carrazone, 1975]. Also analogously, the OPE of the currents give the composite operators

$$\mathcal{O}_{\pm} = \frac{1}{2}[(\bar{s}\gamma_{\mu}P_L d)(\bar{u}\gamma_{\mu}P_L u) \pm (\bar{s}\gamma_{\mu}P_L u)(\bar{u}\gamma_{\mu}P_L d)] - \{u \rightarrow c\}, \quad (4.36)$$

where \mathcal{O}_+ is $\Delta I = 1/2$ under the isospin group, and \mathcal{O}_- is a mixture of $\Delta I = 1/2$ and $3/2$. There are corresponding Wilson coefficients $C_{\pm}(g^2, m_W/\mu)$ which are the same in the absence of strong interaction, i.e. when $g = 0$. If we further integrate out the charm quark, a less reliable approximation than the decoupling of the W boson, we get a different set of operators

$$\mathcal{O}_1 = (\bar{s}\gamma_{\mu L} d)(\bar{u}\gamma_{\mu L} u)$$

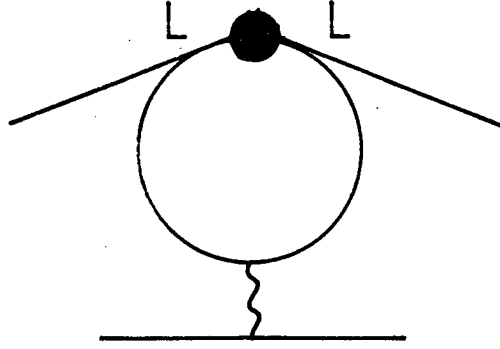


Figure 4.8: QCD-induced penguin diagrams.

$$\begin{aligned}
& -(\bar{s}\gamma_{\mu L}u)(\bar{u}\gamma_{\mu L}d), & 8_f, \Delta I = 1/2 \\
\mathcal{O}_2 = & (\bar{s}\gamma_{\mu L}d)[(\bar{u}\gamma_{\mu L}u) + 2(\bar{d}\gamma_{\mu L}d) + 2(\bar{s}\gamma_{\mu L}s)] \\
& +(\bar{s}\gamma_{\mu L}u)(\bar{u}\gamma_{\mu L}d), & 8_d, \Delta I = 1/2 \\
\mathcal{O}_3 = & (\bar{s}\gamma_{\mu L}d)[(\bar{u}\gamma_{\mu L}u) + 2(\bar{d}\gamma_{\mu L}d) - 3(\bar{s}\gamma_{\mu L}s)] \\
& +(\bar{s}\gamma_{\mu L}u)(\bar{u}\gamma_{\mu L}d), & 27, \Delta I = 1/2 \\
\mathcal{O}_4 = & (\bar{s}\gamma_{\mu L}d)[(\bar{u}\gamma_{\mu L}u) - (\bar{d}\gamma_{\mu L}d)] \\
& +(\bar{s}\gamma_{\mu L}u)(\bar{u}\gamma_{\mu L}d), & 27, \Delta I = 3/2 \\
\mathcal{O}_5 = & 4(\bar{s}\gamma_{\mu L}t_n d)[(\bar{u}\gamma_{\mu R}t_n u) + (\bar{d}\gamma_{\mu R}t_n d) + (\bar{s}\gamma_{\mu R}t_n s)], & 8, \Delta I = 1/2 \\
\mathcal{O}_6 = & (\bar{s}\gamma_{\mu L}d)[(\bar{u}\gamma_{\mu R}u) + (\bar{d}\gamma_{\mu R}d) + (\bar{s}\gamma_{\mu R}s)], & 8, \Delta I = 1/2.
\end{aligned} \tag{4.37}$$

The representations of those operators under flavour $SU(3)$ and $SU(2)$ are also given. \mathcal{O}_1 to \mathcal{O}_4 are already contained in (4.36) (but without the charm term). On the other hand, \mathcal{O}_5 and \mathcal{O}_6 are pure QCD effects, generated through the penguin diagrams of figure 4.8. Note that they have the new chiral LR structure and mediate $\Delta I = 1/2$ transitions only. At this level we have the effective Hamiltonian

$$\mathcal{H}_{\text{eff}}^{|\Delta S|=1} = \frac{g_W^2}{2M_W^2} \sin \theta_c \cos \theta_c \sum_{n=1}^6 C^{(n)}(g^2, M_W/\mu, m_c/\mu)[\mathcal{O}_n]_{\mu}. \tag{4.38}$$

Consider the processes $K \rightarrow 2\pi$. The amplitudes for these decays can be

parameterised as

$$\begin{aligned}
\mathcal{M}(K^+ \rightarrow \pi^+\pi^0) &= \frac{\sqrt{3}}{2}A_2e^{i\delta_2}, \\
\mathcal{M}(K^0 \rightarrow \pi^+\pi^-) &= \sqrt{\frac{2}{3}}A_0e^{i\delta_0} + \sqrt{\frac{1}{3}}A_2e^{i\delta_2}, \\
\mathcal{M}(K^0 \rightarrow \pi^0\pi^0) &= \sqrt{\frac{1}{3}}A_0e^{i\delta_0} - \sqrt{\frac{2}{3}}A_2e^{i\delta_2}.
\end{aligned}
\tag{4.39}$$

Here $A_0(A_2)$ is the amplitude for an $I = 0(I = 2)$ final state obtained from the $I = 1/2$ initial state, and $\delta_0(\delta_2)$ is the corresponding phase shift. The enhancement of the $\Delta I = 1/2$ decay over that of $\Delta I = 3/2$ can be expressed as the ratio

$$\frac{A_0[\Delta I = 1/2]}{A_2[\Delta I = 3/2]} = 22.2(1).
\tag{4.40}$$

From the effective interactions of (4.38), assuming no strong interaction apart from the effect of binding the asymptotic states, we get the result

$$\frac{A_0}{A_2} = \frac{5}{4\sqrt{2}} \approx 0.9,
\tag{4.41}$$

quite far away from the observed value. Switching on QCD, the Wilson coefficients can be calculated in the LLA to give

$$C^{(n)}(g^2, m_W/\mu, m_c/\mu) = \left[\frac{g(\mu)}{g(m_W)} \right]^{-\frac{(\gamma_n)_0}{\beta_0}} C^{(n)}, \text{ for } n = 1, \dots, 4
\tag{4.42}$$

and

$$C^{(5,6)} \approx \frac{g^2}{(4\pi)^2} \ln \left(\frac{m_c^2}{\mu^2} \right) \approx 10^{-1}.
\tag{4.43}$$

The small value of the last two coefficients is expected for they would vanish if it were not for the strong interaction. At the scale of m_W we have $C^{(1)} = -\frac{1}{2}$, $C^{(2)} = \frac{1}{10}$, $C^{(3)} = \frac{1}{15}$ and $C^{(4)} = \frac{1}{3}$. Thus there is some enhancement in the right direction as \mathcal{O}_1 is purely $\Delta I = 1/2$. Note that in the expressions (4.42) the dependence on how the heavy quarks are decoupled is manifest in the number of flavours used in the beta function coefficient β_0 . It is suggested [Minkowski, 1979] that an effective number of flavours between 3 and 6 can be used for the whole range from m_W down to 1GeV. However, the dominant effect is expected to come from the matrix elements of the operators themselves.

In the framework of the vacuum insertion approximation, all the operators except \mathcal{O}_5 and \mathcal{O}_6 have matrix elements of roughly the same order of magnitude. The ratio of (4.40) under the QCD corrected Wilson coefficients, at $\mu \approx 1\text{GeV}$, is found to be only in the range 2 to 5, depending on the precise choice of the renormalisation point and on how the heavy quarks are decoupled. The new operators $\mathcal{O}_{5,6}$, because of their new chiral structure, might have enhanced matrix elements. It is argued [Vainshtein, 1977] that the effect is due to the quark mass in the denominator associated with the loop as the W -boson is contracted to a point. It is also worth pointing out that these new operators generate amplitudes that do not conform with the chiral lagrangian approach [Dupont and Pham, 1984]. The solution to this problem is to take into account of the so-called anomalous commutator terms, see for example [Donoghue, 1984]. Unfortunately, the net result is a suppression of the newly gained enhancement. Alternatively, one can argue that [Bernard *et al.*, 1987] the lowest order chiral perturbation theory is not required to be valid in the kaon regime, and thus the chiral constraint is ignored.

However, the difficult task of evaluating the matrix elements $\langle \pi\pi|\mathcal{O}_n|K\rangle$ can be simplified by chiral perturbation theory. Up to the leading order, i.e. with non-zero quark mass and momenta at lowest order, there are only two independent (8,1) operators and only one (27,1) operator in the representations of $SU(3)_L \otimes SU(3)_R$ [Bernard *et al.*, 1985; Sharpe, 1985; Maiani *et al.*, 1987]. Thus there are only two independent matrix elements of the effective Hamiltonian of the $\Delta I = 1/2$ transitions. From the chiral lagrangian the on-shell $\Delta I = 1/2$ transition amplitude $\langle \pi\pi|\mathcal{H}_{eff}|K\rangle$ can be related to the off-shell $\langle \pi|\mathcal{H}_{eff}|K\rangle$ and $\langle 0|\mathcal{H}_{eff}|K\rangle$ via

$$\begin{aligned}\langle \pi\pi|\mathcal{H}_{eff}|K\rangle &= a \frac{m_K^2 - m_\pi^2}{f} + \dots, \\ \langle \pi|\mathcal{H}_{eff}|K\rangle &= b \frac{-m_K^2}{f} - a(p_K \cdot p_\pi) + \dots, \\ \langle 0|\mathcal{H}_{eff}|K\rangle &= b(m_K^2 - m_\pi^2) + \dots.\end{aligned}\tag{4.44}$$

The parameters a and b include all the Wilson coefficients, normalisations, etc... The problem is thus reduced to the calculation of the last two matrix elements. The appearance of the $K \rightarrow 0$ matrix element can be interpreted as the required subtraction of the renormalisation effects due to the mixing with two-quarks operator [Bernard *et al.*, 1985] for off-shell matrix elements.

Various non-perturbative attempts have been extensively studied to resolve the problem: $1/N$ expansion, QCD sum rules, bag model, Skyrme model, harmonic oscillator model, instanton effect, etc... None has so far given a satisfactory answer. Before moving onto the discussion of how one would tackle it on the lattice we comment on the renormalisation point dependence of the calculation. The matrix elements of the operators and the Wilson coefficients are manifestly dependent on the renormalisation scale μ in such a way that the physical amplitudes are independent of it. Performing model dependent calculations of the matrix elements we encounter a problem because μ is not a parameter. Then it is not clear what scale should be chosen for the coefficients $C^{(n)}$, since the model result for the matrix elements need not coincide with the true value for any μ . Furthermore, if in some calculations the scale is estimated at a typical hadronic scale, such as in sum rules, the reliability of the LLA based on the RG for the coefficients is questionable at the scale deep inside the confinement region. That is, the Wilson coefficients can be estimated reliably only for $\mu \gg \Lambda_{QCD}$ (and only in the chiral limit, see [Maiani *et al.*, 1987]).

4.3.2 Weak matrix elements on the lattice

Lattice calculation of these matrix elements directly from the underlying theory of strong interactions is most appealing. Taking the more difficult matrix elements $\langle \pi | \mathcal{O}_n | K \rangle$, on the lattice they can be extracted from the correlator

$$C_n(t_\pi, t_K) = \sum_{\vec{x}_\pi, \vec{x}_K} \langle \pi(x_\pi) \mathcal{O}_n K(x_K) \rangle \quad (4.45)$$

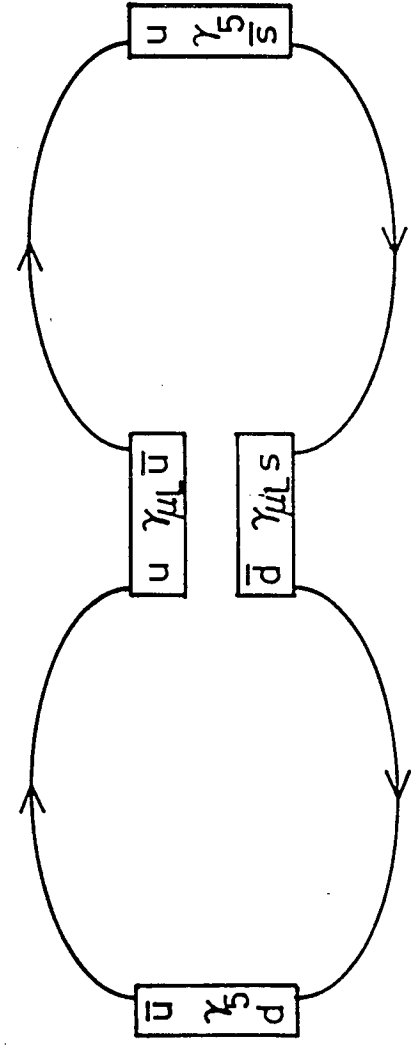
in the usual manner. Here \mathcal{O}_n , π and K are lattice operators with the required quantum numbers. Asymptotically, we expect C_n to decay as

$$C_n(t_\pi, t_K) \sim \sqrt{Z_\pi Z_K} \mathcal{A}_n e^{-m_\pi |t_\pi| - m_K |t_K|} \quad (4.46)$$

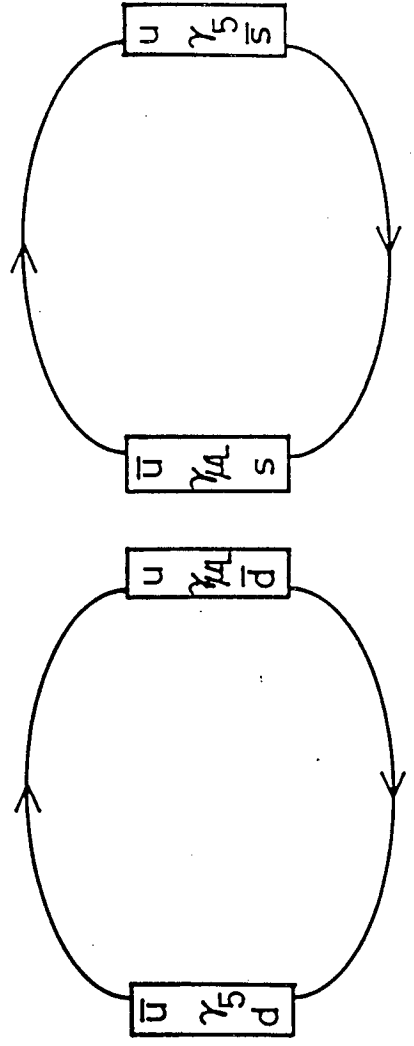
where \mathcal{A}_n is the matrix element of interest.

Wick contraction of quark propagators in the correlator (4.45) gives two types of graphical representations, the eight and eye diagrams. Such a contraction for \mathcal{O}_1 is depicted in figure 4.9. Note that in the approximation of vacuum insertion the eye diagrams are completely neglected, as the method relies on factorising the four-quark operator into two two-quark operators before the Wick

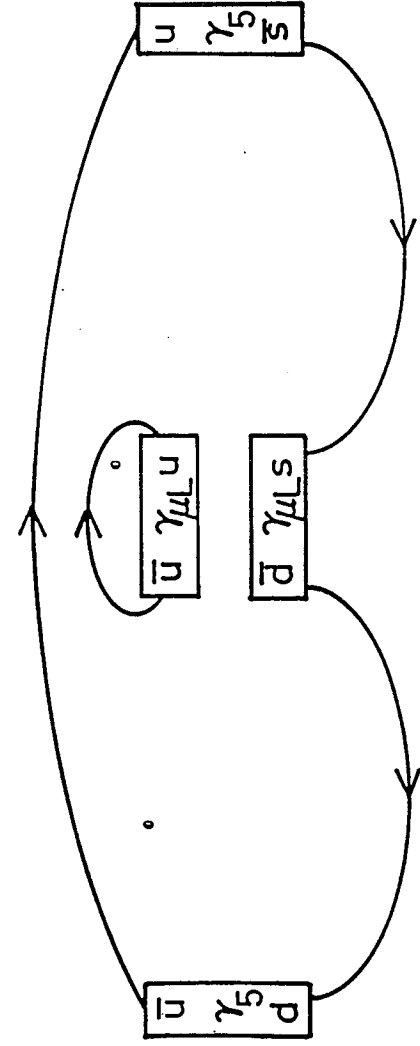
(c)



(a)



(d)



(b)

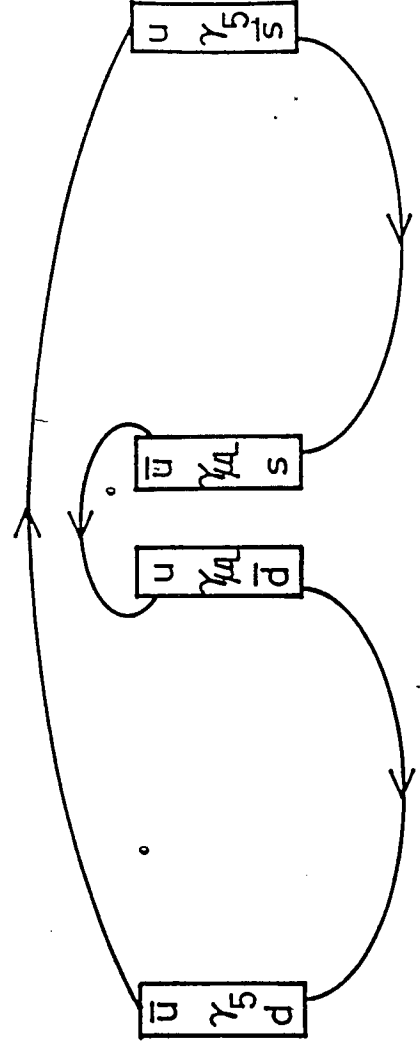


Figure 4.9: Graphs obtained in the Wick expansion of $C_1(t_\pi, t_K)$. Lines correspond to quark propagators in a background gauge configuration, and loops are traced around.

contraction. The “eyes” therefore deserve particular attention on the lattice. However, evaluation of such graphs is more difficult than that of the “eights” for quark propagators connecting arbitrary points x_π and x_K are required and thus the number of quark propagator calculations naively equal to the number of lattice sites. This will lead us to a later discussion of certain method devised especially for the eye graph.

There are some investigations of the lattice calculations using Wilson fermions [Brower *et al.*, 1984; Bernard, 1984; Bernard *et al.*, 1985; Bernard *et al.*, 1986a and b; Bernard *et al.*, 1987] and [Cabbibo *et al.*, 1984; Martinelli, 1984; Maiani *et al.*, 1987]. Because of the “bad” chiral property of Wilson fermions, i.e. the symmetry is broken by a hard term, severe ultraviolet divergences appear in the limit of vanishing lattice spacing. Unlike the $\Delta I = 3/2$ case, this is manifest as mixings with lower dimension and chirally-wrong operators beside the mixings with operators of the same dimension with coefficients only logarithmically dependent on a which can be treated perturbatively. These lower dimension operators have coefficients proportional to inverse powers of a and become increasingly important in the continuum limit. It is still unclear whether perturbative treatment is adequate, or whether non-perturbative subtraction needs to be called upon [Bernard *et al.*, 1987], or worst still whether the $K \rightarrow \pi\pi$ matrix elements have to be computed directly [Maiani *et al.*, 1987]. In the next section we present an alternative approach using staggered fermions which is free of this problem. But there are other new problems, although a continuous chiral symmetry is gained.

4.3.3 Staggered fermion approach to weak matrix elements

The main difficulties in this approach are associated with the nontrivial interpretations, see chapter two, and complicated mixing of the lattice and flavour groups due to the thinning out of the fermionic degrees of freedom. Adopting the quasi-local construction of lattice operators, it seems to require the generation of propagators from many sources, in fact, sixteen corresponding to the same number of vertices of a hypercube. Consequently up to four links have to be inserted to maintain gauge invariance. Not only are such objects time-consuming and lead to greater statistical fluctuations in Monte Carlo simulations, but perturba-

tive corrections might also be more significant than for Wilson fermions [Daniel and Sheard, 1987]. There is another but related problem concerning the flavour degrees of freedom. At non-zero lattice spacing the staggered action contains four flavours but has only a discrete flavour symmetry not the full continuum flavour group –that is, in the coordinate interpretation.

Nevertheless, the staggered fermion approach is attractive due to its chiral behaviour and should provide an alternative and independent comparison with lattice results from Wilson fermions. There are ways and means and approximations to deal with the difficulties raised above.

One way to deal with the flavour complication has been proposed in [Kilcup and Sharpe, 1987; Sharpe *et al.*, 1987]. For each continuum flavour there is one staggered species, leading to a direct correspondence between continuum and lattice chiral symmetry. The lattice matrix elements, as a result, satisfy certain Ward identities analogous to those in the continuum. The appearance of more “Susskind flavours” than continuum flavours, at a ratio of four to one, should result only in overcounting and can be kept track of in the quenched approximation.

We propose here a method [Daniel *et al.*, 1987] again with three species of staggered fermions –if the charm is integrated out– to reduce as low as possible the number of sources from which propagators to all other lattice sites are to be computed. The approximation is based on the restoration of Susskind flavour symmetry, which is observed numerically to a good degree at current coupling values ($\beta > 6$). Basically, in this regime, the flavour symmetry is exploited to project out the right spinorial structures of the four-quark operators. For the eye graphs further complications arise, however.

Eight graphs

Assuming the charm quark decouples we introduce three staggered fermion species corresponding to the u , d and s quarks with masses m_u , m_d , m_s respectively. That is, in the basis (q, \bar{q}) of the coordinate interpretation, the action now contains three different fields of expression (2.31) each of which has four Susskind flavour degrees of freedom.

The meson operators can be taken to be

$$\pi_\eta(r) = \bar{u}(r)\gamma_5 \otimes t_\eta d(r), \quad (4.47)$$

$$K_\eta(r) = \bar{s}(r)\gamma_5 \otimes t_\eta u(r).$$

Note that only $\pi_5(K_5)$ -i.e. $t_\eta = t_5$ - is the lattice Goldstone boson in the sense that it corresponds to the spontaneously broken lattice axial symmetry $U(1)_\epsilon$ of (2.33). However, in the continuum limit all other meson fields in (4.46) are equivalent as the Susskind flavour is restored.

The four-fermion operators \mathcal{O}_n in general require up to four-link operators in terms of $(\bar{\chi}, \chi)$. However, making the assumption of flavour restoration, we can exploit the equivalence of different meson operators to project out the desired spinorial structure of the four fermion operators from

$$\bar{\chi}_d \chi_u(0) \bar{\chi}_u \chi_s(0) = \sum_{\theta, \theta'} [\bar{d}\gamma_\theta \otimes t_\theta u](0) [\bar{u}\gamma_{\theta'} \otimes t_{\theta'} s](0), \quad (4.48)$$

a single local source at one corner of the hypercube. For example, the two-trace eight graph of figure 4.9(a) corresponds to the expression

$$\sum_{\vec{r}_\pi, \vec{r}_K} \sum_{\theta, \theta'} \text{tr} \{ S_u(0, r_\pi) \gamma_5 \otimes t_\eta S_d(r_\pi, 0) \gamma_\theta \otimes t_\theta \} \\ \text{tr} \{ S_s(0, r_K) \gamma_5 \otimes t_{\eta'} S_u(r_K, 0) \gamma_{\theta'} \otimes t_{\theta'} \}. \quad (4.49)$$

Here, S_q is the quark propagator and the trace would project out $\theta = \eta$ and $\theta' = \eta'$. Thus by choosing $\eta(\eta')$, i.e. the corners of the hypercube at $r_{\pi(K)}$ where the quark propagators emanating from the hypercube at the origin end, we can fix the structures of the four quark operators situated at the origin hypercube. The one-trace type of the eight graphs (figure 4.9(c)) can be brought into the two-trace form by a Fierz transformation to apply the projection method here.

Eye graphs

We can immediately see that this projection will not work for the eye graphs (figure 4.9(b,d)) as there are not enough degrees of freedom, in either the one-trace or two-trace forms, at the meson positions to 'tune' θ and θ' . This is because both the mesons are on the same branch: they are connected directly

by a quark propagator. This fact, in turn, increases the number of quark propagators to be computed to the order of the volume as $S_u(r_\pi, r_K)$ are now required.

We propose to overcome these problems simultaneously by combining the projection method with the technique of exponentiation, also named Extended Source Propagators (ESP) [Bernard, 1984; Kilcup *et al.*, 1985]. We now choose the external mesons to be, say, π_5 and K_5 , the lattice Goldstone bosons, as the flavour freedom is not of much use here. For the operators at the origin we start with

$$\sum_{\theta} [\bar{d}\gamma_{\theta} \otimes t_{\theta}u](0)[\bar{u}\gamma_{\eta} \otimes t_{\eta'}s](0), \quad (4.50)$$

which is non-local. The expression of the single trace eye diagram of figure 4.9(b) thus has the form

$$\sum_{\bar{r}_\pi, \bar{r}_K, \theta} \text{tr} \{ S_u(r_K, r_\pi) \gamma_5 \otimes t_5 S_d(r_\pi, 0) \gamma_{\theta} \otimes t_{\theta} S_u(0, 0) \gamma_{\eta} \otimes t_{\eta'} S_s(0, r_K) \gamma_5 \otimes t_5 \}. \quad (4.51)$$

As the quark propagators become diagonal in Susskind flavour near the continuum limit, the trace picks out the $\theta = \eta'$ contribution.

In terms of the χ -field propagators, $G_q(x, x')$, equation (4.51) becomes

$$C(t_\pi, t_K) = \text{const.} \sum_{\xi, \xi'} \text{tr}(\gamma_{\xi}^{\dagger} \gamma_{\eta} \gamma_{\xi'} \gamma_{\eta'}^{\dagger}) \text{tr}_c \{ G_u(0, \xi) U(\xi, \xi') G_{\pi, K}(\xi', 0) \} \quad (4.52)$$

where $U(\xi, \xi')$ is the symmetric combination of gauge links between corners ξ and ξ' of the hypercube at the origin, and

$$G_{\pi, K}(x, 0 | t_\pi, t_K) = \sum_{\bar{r}_\pi, \bar{r}_K, \zeta, \zeta'} G_s(x, r_K + \zeta) \epsilon(\zeta) G_u(r_K + \zeta, r_\pi + \zeta', 0) \epsilon(\zeta') G_d(r_\pi + \zeta') \quad (4.53)$$

with $\epsilon(x) = (-1)^{x_1 + \dots + x_4}$ arising from the choice of meson fields in (4.47). Equation (4.53) may be regarded as the propagation of a staggered fermion in the presence of a pion and a kaon source. We need two extra inversions for each pair of (t_π, t_K) on top of the inversion for the staggered field propagator from the origin to every lattice site. That is, we have to solve

$$[(\mathcal{D} + m_u)G_{\pi(t_\pi)}](x, 0) = \delta_{x_4, t_\pi} \epsilon(x) G_d(x, 0), \quad (4.54)$$

and then

$$[(\mathcal{D} + m_s)G_{\pi(t_\pi),K(t_K)}](x, 0) = \delta_{x_4, t_K} \epsilon(x) G_{\pi(t_\pi)}(x, 0). \quad (4.55)$$

In these equations, $(\mathcal{D} + m_q)$ is the staggered fermion operator in the χ -basis, so that these inversions require minor modifications of the standard programs used in hadron mass calculations.

Asymptotic behaviour of the form (4.46) can be used to fit the data for those time slices far enough from the sources and the boundary. However, there are two time parameters for the two mesons, so this still demands a fairly large number of quark propagator calculations of order of the square of the lattice size in the time direction. As we only want to estimate the amplitude \mathcal{A}_n , we can sum the correlators (4.46) over those appropriate time slices before the fittings. That is, we can sum equations (4.54) and (4.55) over the time slices. Thus only two extra inversions are needed.

It is important to emphasize that our method is only valid in the quenched approximation. With dynamical fermions it is not possible to keep track of the extra Susskind species for each physical flavour. It might then be necessary to associate Susskind flavours with the mass degeneracy lifted with physical flavours. Apart from the fine tuning required, see chapter two, the freedom to project out spin structures would be lost.

There are other suggestions to construct appropriate lattice staggered fermion operators to deal with the eight and eye diagrams [Sharpe *et al.*, 1987]. All of these, though more complicated and different from the method mention above, share the main assumption used here, namely that of the restoration (to a good degree) of Susskind flavour symmetry. Hadron mass calculations from Edinburgh configurations [Bowler *et al.*, 1987a and b] indicate that, on the basis of the masses of the pseudoscalar mesons, Susskind flavour symmetry is restored to the order of 10% at $\beta = 6.0$, and to the order of 3-4% at $\beta = 6.15$.

A numerical project based on the method outlined above is under way, using a $16^3 \times 24$ lattice with β values in the range 6.0-6.3. Staggered fermion propagators from a point source for several quark mass values are borrowed from the hadron mass calculation project. Perturbative corrections have so far been carried out for two-quark operators only [Daniel and Sheard, 1987]. The case of four-quark operators is much more complicated due to the mixing. However, in our trial run

we intend to measure first some simple operators which have minimum mixings and thus cut down the perturbative calculations.

4.4 Concluding remarks

In this last chapter of the thesis we have presented two typical examples of how lattice QCD can be used to address phenomenological problems. On the lattice, to avoid the problem with chiral fermions, the OPE hypothesis is used to reduce the chiral gauge theories to the domain of QCD, which is a vector gauge theory. The expansion further factorises the properties into non-perturbative pieces contained in the matrix elements of certain local composite operators to be measured, and perturbative pieces absorbed into the Wilson coefficients. Perturbative calculations are also necessary, paradoxically, to make sense out of the non-perturbative numbers. Also on the lattice, due to the loss of Euclidean symmetry, and of chiral symmetry for Wilson fermions, lattice operators can mix with non-covariant and/or chirally-wrong operators. In some cases it seems that these effects have to be subtracted in a non-perturbative way.

Assuming that lattice simulations are in the regime where the chiral lagrangian approach is valid, we can relate the required matrix elements to simpler ones. Up to date most of the calculations here and elsewhere have to be approximated in one way or another; the biggest draw back of all is the quenched approximation.

Ideally simulations in the future should be carried out with dynamical fermion algorithms on larger lattices with more statistics and particularly with 'small enough' quark masses and coupling.

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