WEIGHT FUNCTIONS ON THE TORUS

AND

THE APPROXIMATION PROPERTY IN BANACH SPACES

by

James Reid

Presented for the degree of

Doctor of Philosophy in Mathematics

at the

University of Edinburgh

September, 1977

CONTENTS

PREFACE

This thesis has been composed by myself and the work in it is claimed as original except where mention is made to the contrary.

ACKNOWLEDGEMENTS

I am especially grateful to my supervisor, Dr. A.M. Davie, for his guidance, advice and encouragement throughout this work. I also wish to thank Professor F.F. Bonsall for the interest he has taken in this work and I am grateful to members of staff and research students in the Mathematics Department at Edinburgh University for several interesting conversations. My thanks are due to Mrs. Alison Johnson for the careful typing of this thesis, and to the University of Edinburgh for financial support in the form of a Chalmers Research Scholarship.

Finally I owe a real debt of gratitude to my parents, my wife and my friends.

This thesis is divided into two distinct and independent parts.

Part 1 concerns the Approximation Property (a.p.) and Radon Nikodym Property (RNP) in Banach Spaces.

In Chapter 1 we outline the importance of the a.p. and produce examples of Banach Spaces without the a.p. by modifying a construction due to Szankowski. These spaces are closed subspaces of ℓp direct sums of finite dimensional ℓq spaces $(1 \leq q < p < \infty)$, so with $p \leq 2$ we obtain Banach spaces of cotype 2 without the a.p. - this was unknown.

In Chapter 2 we discuss the RNP proving in Theorem 2.9 the characterisation in terms of dentable subsets due to Rieffel and Huff (among others), of Banach spaces with the RNP. In theorem 2.18 we prove that dual spaces with the a.p. and RNP have the metric approximation property, obtaining as corollaries results of Grothendieck. We introduce p- nuclear and p- integral maps between Banach spaces E and F and prove in theorem 2.26 that, if E^* has the RNP, all p-integral maps are p-nuclear, and in theorem 2.29 that, if F has the RNP all integral maps are nuclear. This extends work of Grothendieck, Perrson and Pietsch.

Part 2 concerns the prediction theory of doubly stationary processes.

In Chapter 3 we outline the basic prediction theory, and state, for the absolutely continuous case, Helson and Lowdenslager's characterisation, for a weight function w and an irrational α, of a process as type 1,2 or 3. We give an example of a process of type 2, for all irrational α.

In Chapter 4 we obtain in Theorem 4.10 an exact analogue of Helson and Szegö's result, viz. that the past and future of a process are at positive angle if and only if $d\mu = w d\sigma$, $w = \exp(u + \tilde{v})$, where u, v are real L^∞ functions with $\|v\|_\infty < \frac{\pi}{2}$.

We introduce a class of functions - BMO(α) functions, analogous to BMO functions, and prove BMO(α) is the dual of $H^1(\sigma)$ and $\{u + \tilde{v} : u, v \in L^\infty(\sigma)\} = \text{BMO}(\alpha)$ in Theorems 4.19 and 4.20.

PART 1.

CHAPTER 1.

The Approximation Problem in Banach Spaces

The chief object of this chapter is to produce examples of Banach spaces which do not have the approximation property. These examples are in fact closed subspaces of ℓp - direct sums of finite dimensional ℓq - spaces where $p > q \geq 1$ $(p \neq \infty)$, and so in the case where $p \leq 2$ we obtain Banach spaces of cotype 2 which do not have the approximation property. This was unknown.

If X and E are Banach spaces, $B(X,E)$ will denote the space of bounded linear maps from X into E. If $X = E$ we use the notation $B(E)$.

A linear map $T : X \to E$ is said to be <u>compact</u> if the closure of the image of the unit ball of X under T i.e. $T(\text{Ball } X)^-$ is a compact subset of E. Since a compact subset is always bounded, such a map is necessarily bounded, and denoting by $K(X,E)$ the set of all such maps, ($K(E)$ if $X = E$) we have $K(X,E) \subseteq B(X,E)$.

A bounded linear map T is said to be <u>finite rank</u> if the image space of T is finite dimensional. If $T \in B(X,E)$ is finite rank, $T(\text{Ball } X)^-$ is a closed and bounded subset of the finite dimensional space TX, so is compact, and T is therefore compact.

The set $K(X,E)$ is in fact a closed subspace of $B(X,E)$, and if $F(X,E)$ ($F(E)$ if $X = E$) denotes the closure in $B(X,E)$ of the finite rank maps, we have $F(X,E) \subseteq K(X,E)$ and it is a natural question as to whether equality occurs.

<u>Definition 1.1</u>: [5] A Banach space E is said to have the <u>Approximation Property (a.p.)</u> if for each $\varepsilon > 0$, for each compact subset K of E, there is a $T \in F(E)$ such that $\|Te - e\| < \varepsilon$, for all $e \in K$.

Put another way, in the topology of uniform convergence on compacts, the identity operator is in the closure of the finite rank operators on E.

The crucial result is the following which is in Grothendieck's memoir.[5]

<u>Theorem 1.2</u>: E has the a.p. iff $F(X,E) = K(X,E)$ for all Banach spaces X.

Most of the naturally occurring Banach spaces have the a.p.: all the ℓ^p, L^p spaces with $1 \leq p \leq \infty$, $C(K)$ where K is a compact hausdorff space, the disc algebra $A(D)$, the space of compact operators on a Hilbert space. It is unknown whether H^∞ or $B(H)$ (= space of all bounded operators on a Hilbert space H) have the approximation property.

For many years it was unknown whether there existed a Banach space which did not have the a.p. In [4] Enflo produced such a space. In [1] A.M. Davie produced an elegant simplification of Enflo's result, and in fact showed that for $2 < p \leq \infty$ there is a closed subspace of ℓ^p failing the a.p.

Szankowski [15] presented an example of a Banach lattice (i.e. a Banach space also having a lattice structure) failing the a.p. We shall show that, by suitably modifying his construction, we can produce closed subspaces of ℓ^p - direct sums of finite dimensional ℓ^q spaces which fail the a.p. These give, for the case where $2 \geq p > q \geq 1$, Banach spaces of cotype 2 failing the a.p.

The procedure is as follows: we construct a Banach space E, a compact subset K of E and a linear functional β on $B(E)$ such that $\beta(I) = 1$ where I is the identity, $\beta(T) = 0$ for all finite rank operators T.

Also $|\beta(T)| \leq C \sup\{ \|Te\| : e \in K \}$ for all bounded operators T on E, where C is a constant. If E had the approximation property, then I could be approximated arbitrarily closely on K by finite rank operators and this would necessitate $\beta(I) = 0$, a contradiction.

<u>Notation</u>: $I = \{-1,1\}^{\mathbb{N}}$ is the Cantor group equipped with the natural product measure, λ. I_n denotes $\{-1,1\}^n$, and π_n is the natural projection of I onto I_n, π_{mn} (with $m \geq n$) the natural projection of I_m onto I_n.

For $u \in I_n$, let $Zu = \pi_n^{-1}(u) \subseteq I$. χ_u is the characteristic function of this set. For each $n \in N$, $\mathcal{B}_n$ denotes the (finite) algebra of subsets of I generated by the sets Zu, with $u \in I_n$.

By $\bar{u}$ we mean $\bar{u} = (u_1, \ldots, u_{n-1}, -u_n)$ if $u = (u_1, \ldots, u_{n-1}, u_n) \in I_n$.

If $A \in \mathcal{B}_n$ let ${}^nA^- = \cup \{Z\bar{u} : Zu \subset A\}$.

<u>Lemma 1.3</u>: For $\alpha \geq 3$, for all n sufficiently large (depends on α) there is a partition $\mathcal{O}_n \subseteq \mathcal{B}_n$ of I (i.e. $I = \cup\{A : A \in \mathcal{O}_n$, A pairwise disjoint elements of $\mathcal{B}_n\}$) such that

(a) $\frac{1}{2^\alpha} n^{-\alpha} \leq \lambda(A) \leq n^{-\alpha}$ if $A \in \mathcal{O}_n$ (1)

(b) $\lambda({}^nA^- \cap B) \leq 5\, n^{-\alpha} m^{-\alpha}$ if $A \in \mathcal{O}_n$, $B \in \mathcal{O}_m$ (2)

(A modification of Szankowski's lemma in [15].)

<u>Proof</u>: To prove the lemma we require the following combinatorial lemma first.

Lemma 1.4: Let $X = \{1,2, \dots N\}$, $m < \sqrt{N}$ and $K < m/4 \log m$. If μ is the usual counting measure on X i.e. if $Y \subseteq X$, $\mu(Y) = \text{card } Y/N$, then there is a family $\{\mathcal{D}k\}_{k=1}^{K}$ of partitions of X so that

(a) $m^{-1} \leq \mu(A) \leq 2m^{-1}$ for all $A \in \bigcup_{k=1}^{K} \mathcal{D}k$, (3)

(b) $\mu(A \cap B) \leq 5m^{-2}$ if $A, B \in \bigcup_{k=1}^{K} \mathcal{D}k$, $A \neq B$. (4)

Proof: For large enough N, the prime number theorem ensures the existence of K distinct primes, $p_1 \dots p_K$, lying between $\frac{1}{2} m$ and m.

$$\text{Let } A_{jk} = \{n \in X : n \equiv j \pmod{p_k}\}$$

$$\mathcal{D}k = \{A_{jk}\}_{j=1}^{p_k} .$$

$$\text{Then } \left[\frac{N}{p_k}\right] \leq \text{card } A_{jk} \leq \left[\frac{N}{p_k}\right] + 1 \text{ for each } k$$

(where $[t]$ denotes the integer part of t) and so

$$\frac{1}{N}\left[\frac{N}{p_k}\right] \leq \mu(A_{jk}) \leq \frac{1}{N}\left(\left[\frac{N}{p_k}\right] + 1\right).$$

Since $N/p_k \geq N/m$ and $N/p_k \leq 2N/m$ we obtain (3).

Since, for $k \neq l$, $A_{jk} \cap A_{jl}$ is a coset mod $p_k p_l$,

$$\left[\frac{N}{p_k p_l}\right] \leq \text{card } (A_{jk} \cap A_{jl}) \leq \left[\frac{N}{p_k p_l}\right] + 1 ,$$

and, using $m < \sqrt{N}$, $m/2 < p_1 < \dots p_k < m$, we obtain (4).

Proof: (of lemma 1.3)

Put $I = \prod_{j=0}^{\infty} X_j$ where $X_j = \{-1,1\}^{2^j}$

Let $\rho_j : I \to X_j$ denote the natural projection.

Take $N = 2^{2^{j-1}}$ and $m = m_j = (2^{j+1})^{\alpha}$ in lemma 1.4 , we obtain $K = 2.2^j$ partitions $\mathcal{D}_k^{(j)}$, $k = 1, \dots 2.2^j$, of the set X_{j-1} so that (3) and (4) are satisfied. (This will work for all j sufficiently large.)

The partitions $\mathfrak{A}_n$ are defined as follows, for $n = 2^j , 2^j + 1 , \dots 2^{j+1} - 1$,

$$\mathfrak{A}_n = \{\rho_{j-1}^{-1}(D) \cap r_n^{-1}(1) : D \in \mathcal{D}_{n+1-2^j}^{(j)}\} \quad \cup$$

$$\{\rho_{j-1}^{-1}(E) \cap r_n^{-1}(-1) : E \in \mathcal{D}_{n+1}^{(j)}\} \qquad (5)$$

$r_n(t)$ is the nth Rademacher function on I , defined by $r_n(t) = t_n$ where $t = (t_k)_{k=1}^{\infty} \in I$.

We first note that $\mathfrak{A}_n \subseteq \mathcal{B}_n$. This means that the sets in $\mathfrak{A}_n$ depend only on the first n co-ordinates.

From (5) it is clear that the sets depend only on the co-ordinates $2^{j-1} , 2^{j-1} + 1 , \dots 2^j - 1$ and n , which are all less than or equal to n .

In what follows $2^j \leq n < 2^{j+1}$, $2^i \leq m < 2^{i+1}$

$A = \rho_{j-1}^{-1}(D) \cap r_n^{-1}(\varepsilon)$ $\qquad B = \rho_{i-1}^{-1}(E) \cap r_m^{-1}(\eta)$

where $D \in \mathcal{D}^{j}_{n+1-(\varepsilon+1)2^{j-1}}$, $E \in \mathcal{D}^{i}_{m+1-(\eta+1)2^{i-1}}$,

ε and η are ± 1 .

Now $2^{-\alpha} n^{-\alpha} \leq m_j^{-1} \leq \mu(D) \leq 2m_j^{-1} \leq 2(2^{j+1})^{-\alpha} \leq 2n^{-\alpha}$, using (3).

It is clear that $\lambda(\rho_{j-1}^{-1}(D)) = \mu(D)$.

Since $\rho_{j-1}^{-1}(D)$ depends only on the co-ordinates 2^{j-1} $2^{j-1}, 2^{j-1}+1 \dots 2^j-1$ all $< n$ and $r_n^{-1}(\varepsilon)$ depends only on the nth co-ordinate, we have $\lambda(A) = \lambda(\rho_{j-1}^{-1}(D)) \cdot \lambda(r_n^{-1}(\varepsilon)) = \mu(D) \cdot 2^{-1}$

Thus we obtain (1).

Lastly we obtain (2).

Suppose for a start, that $j \neq i$.

Then $\rho_{j-1}^{-1}(D)$ and $\rho_{i-1}^{-1}(E)$ depend on disjoint sets of co-ordinates and so

$$\begin{aligned}\lambda(A \cap B) &\leq \lambda(\rho_{j-1}^{-1}(D) \cap \rho_{i-1}^{-1}(E)) \\ &= \lambda(\rho_{j-1}^{-1}(D))\ \lambda\ (\rho_{i-1}^{-1}(E)) \\ &= 4\ \lambda(A)\ \lambda(B)\ ,\end{aligned}$$

and applying (1) gives (2) .

Now suppose $j = i$ and that $m = n$, $\varepsilon = \eta$, so ${}^{n}A^{-} \subseteq r_n^{-1}(\varepsilon)$ $B \subseteq r_n^{-1}(-\varepsilon)$ and therefore ${}^{n}A^{-} \cap B = \phi$. Otherwise D and E belong to different partitions $\mathcal{D}_n^{(j)}$, $\mathcal{D}_m^{(j)}$ respectively, so using (4),

$$\lambda({}^{n}A^{-} \cap B) \leq \mu(D \cap E) \leq 5m_j^{-2} \leq 5n^{-2\alpha} .$$

Szankowski's example was constructed as a certain subspace of functions defined on the Cantor Group I. We obtain our examples by modifying the definition of norm.

Before we define our Banach space and our functional β , we set up a little more machinery.

If $\mathcal{G}_n$ = collection of all subsets of $\{1,2, \dots n\}$ let $\mathcal{G} = \bigcup_n \mathcal{G}_n$. For $G \in \mathcal{G}$ define the Walsh function $W_G(t)$ on I by $W_G(t) = \prod_{k \in G} r_k(t)$ where the r_k are the Rademacher functions defined previously. These functions are characters on I with its natural group structure i.e.

$$W_G(t)\ W_G(u) = W_G(tu) \qquad t, u \in I .$$

We construct our example in a series of steps : for ease we take $q = 1$, $p > 1$.

Step 1: With $\mathfrak{A}_n$ a partition as in lemma 1.3, and $\{c_n\}$ a sequence of strictly positive numbers which will be chosen later, for any function f measurable on I define

$$\|f\|_p = \left[\sum_n c_n \sum_{A \in \mathfrak{A}_n} \left(\sum_{Z_u \subset A} \frac{\left| Z_u \int f \right|}{\lambda(A)} \right)^p \right]^{1/p} \qquad (6)$$

provided this is finite. Call the Banach space obtained using this norm E_p.

Step 2: We define a linear functional β_n on the bounded operators on E_p by

$$\beta_n(T) = 2^{-n} \sum_{G \in \mathcal{G}_n} (W_G , TW_G) .$$

$$\left(\text{By } (f,g) \text{ we mean } \int f\, g\, d\lambda; \text{ so } (W_G , TW_G) = \int W_G(t)\, (TW_G)(t)\, d\lambda . \right)$$

If $u \in I_m$, $G \in \mathcal{G}_n$ with $m \geq n$, we notice W_G is constant on Z_u. Denoting this constant value by $W_G(u)$, we have

$$\beta_n(T) = 2^{-n} \sum_{G \in \mathcal{G}_n} \sum_{u \in I_n} W_G(u) \int_{Z_u} TW_G \, d\lambda$$

$$= \sum_{u \in I_n} \int_{Z_u} T\, \psi_u \, d\lambda ,$$

where $\psi_u = 2^{-n} \sum_{G \in \mathcal{G}_n} W_G(u)\, W_G$.

For $v \in I_n$, we have

$$\psi_u(v) = 2^{-n} \sum_{G \in \mathcal{G}_n} W_G(u)\, W_G(v) = 2^{-n} \sum_{G \in \mathcal{G}_n} W_G(uv)$$

$$= \begin{cases} 1 & \text{if } u = v , \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\psi_u = \chi_u$ and

$$\beta_n(T) = \sum_{u\varepsilon I_n} \int_{Z_u} T \ \chi_u \, d\lambda .$$

We obtain

$$\beta_n(T) - \beta_{n+1}(T) = \sum_{u\varepsilon I_{n+1}} \int_{Z_u} T \chi_{\bar{u}} \, d\lambda . \tag{7}$$

Lemma 1.5: We have

$$|\beta_n(T) - \beta_{n+1}(T)| \leq \sum_{A\varepsilon \mathfrak{A}_{n+1}} \max \{ \sum_{Z_u \subset A} | \int_{Z_u} Tf | : f \text{ is}$$

$$\mathcal{B}_n \text{ measurable and } |f| = \chi n_{\bar{B}} \text{ for some } B\varepsilon \mathfrak{A}_n \} . \tag{8}$$

Proof: From (7) we have

$$\beta_n(T) - \beta_{n+1}(T) = \sum_{u\varepsilon I_{n+1}} \int_{Z_u} T \chi_{\bar{u}} \, d\lambda$$

$$= \sum_{A\varepsilon \mathfrak{A}_n} \sum_{Z_u \subset A} \int_{Z_u} T \chi_{\bar{u}} \, d\lambda .$$

Consider $\displaystyle\sum_{Z_u \subset A} \int_{Z_u} T \chi_{\bar{u}} \, d\lambda$. (9)

Denoting $t_{uv} = \displaystyle\int_{Z_u} T \chi_{\bar{v}} \, d\lambda$ and enumerating the u's we obtain a matrix t_{uv} and (9) is just the trace $\sum t_{uu}$ of this matrix.

Given a matrix $(t_{uv})^m_{u,v=1}$ (square) $\exists$ $\varepsilon = (\varepsilon_u)^m_{u=1}$ with $\varepsilon_u = \pm 1$ such that

$$\sum_{u=1}^{m} | \sum_{v=1}^{m} t_{uv} \varepsilon_v | \geq \sum_{u=1}^{m} t_{uu} .$$

In fact we will demonstrate the existence of an ε such that

$$\sum_{u=1}^{m} \varepsilon_u \sum_{v=1}^{m} t_{uv} \varepsilon_v \geqslant \sum_{u=1}^{m} t_{uu} .$$

To show this, we will prove that for a matrix (b_{uv}), we can choose ε such that

$$\sum_{v<u} b_{uv} \varepsilon_u \varepsilon_v \geq 0 \qquad (10)$$

and then put $b_{uv} = t_{uv} + t_{vu}$ for $u > v$. (b_{uv}, t_{uv} are real.)

For (10) we use induction. If it is true for matrices of order m, consider $(b_{uv})_{u,v=1}^{m+1}$

$$\sum_{1 \leq u < v \leq m+1} \varepsilon_u \varepsilon_v b_{uv} = \sum_{1 \leq v < u \leq m} \varepsilon_u \varepsilon_v b_{uv} + \varepsilon_{m+1} \sum_{v=1}^{m} \varepsilon_v b_{m+1,v} .$$

Choose $\varepsilon_1 \ldots \varepsilon_m$ by the inductive hypothesis and ε_{m+1} = sign of $\sum_{v=1}^{m} \varepsilon_v b_{m+1,v}$. Now, applying this to (9), we obtain

$$\sum_{Z_u \subset A} \int_{Z_u} T \chi_{\bar{u}} \, d\lambda \leq \sum_{Z_u \subset A} \left| \int_{Z_u} T \left(\sum \varepsilon_v \chi_{\bar{v}}\right) d\lambda \right|$$

$$\leq \max \sum_{Z_u \subset A} \left| \int_{Z_u} T f \, d\lambda \right|$$

where the maximum is taken over f, measurable with respect to $\mathcal{B}_n$, and such that $|f| = \chi_{n_B^-}$ for some $B \in \mathfrak{A}_n$.

Thus we obtain

$$\left|\beta_n(T) - \beta_{n+1}(T)\right| \leq \sum_{A \in \mathfrak{A}_n} \max \left\{ \sum_{Z_u \subset A} \left| \int_{Z_u} T f \right| : |f| = \chi_{n_B^-} \text{ for some } B \in \mathfrak{A}_n \right\} .$$

Step 3 : We have from (8)

$$\left|\beta_n(T) - \beta_{n+1}(T)\right| \leq \sum_{A \in \mathfrak{A}_n} \max \left\{ \sum_{Z_u \subset A} \left| \int_{Z_u} T f \right| : |f| = \chi_{n_B^-} \right\}$$

$$\leq \sum_{A \in \mathfrak{A}_n} \lambda(A) \max_{\mathfrak{F}_n} \sum_{Z_u \subset A} \frac{\left| \int_{Z_u} T f \right|}{\lambda(A)} ,$$

where $\mathcal{F}_n = \{f : f$ is $\mathcal{B}_n$ measurable and $|f| = \chi_{{}^nA^-}$ for some $A \in \mathcal{O}\!\mathcal{l}_n\}$

If M_n = number of elements in the partition $\mathcal{O}\!\mathcal{l}_n$

$$|\beta_n(T) - \beta_{n+1}(T)| \leq M_n \max_{\mathcal{A}_n} \lambda(A) \max_{\mathcal{F}_n} \left[\sum_m c_m \sum_{A \in \mathcal{O}\!\mathcal{l}_m} \left(\frac{\sum_{Z_u} \left|\int_{Z_u} T f\right|}{\lambda(A)}\right)^p\right]^{1/p} \frac{1}{C_n^{1/p}}$$

$$= M_n \max \lambda(A) \frac{1}{C_n^{1/p}} \max_{f \in \mathcal{F}_n} \|Tf\|_p \qquad (11)$$

Step 4: Fix $p > 1$ and choose an integer $\alpha_p \geq 3$ and numbers γ_p, δ_p such that

$$\gamma_p > \alpha_p + 1, \qquad (12)$$

$$\frac{\gamma_p}{p} < \alpha_p - 1, \qquad (13)$$

$$\frac{\gamma_p}{p} - \alpha_p < -\delta_p < -1 . \qquad (14)$$

(All of these are possible).

Step 5:

$$\max\{\|f\|_p : f \in \mathcal{F}_n\} \leq \left[\sum_m c_m \sum_{B \in \mathcal{O}\!\mathcal{l}_m} \left(\frac{\lambda({}^nA^- \cap B)}{\lambda(B)}\right)^p\right]^{1/p}$$

Using lemma 1.3 with $\alpha = \alpha_p$ we obtain the above

$\leq [\sum_m c_m M_m]^{1/p} n^{-\alpha p} \times$ constant.

Put $c_m = m^{-\gamma p}$, and since $M_m \leq 2^{\alpha p} m^{\alpha p}$, $\sum c_m M_m$ converges using (12). With δ_p as above, define

$$K = \bigcup_{n=1}^{\infty} \{n^{\alpha p - \delta p + 1} f : f \in \mathcal{F}_n\} \cup \{0\} . \qquad (15)$$

By (12), (13) and (14) K is a sequence which converges to zero and so is compact.

Also $|\beta_n(T) - \beta_{n+1}(T)| \leq$ const. $n^{\frac{\gamma_p}{p} - \alpha_p + \delta_p - 1} \max_{f \in K} \|Tf\|_p$.

If $\varepsilon_p = \frac{\gamma_p}{p} - \alpha_p + \delta_p - 1$, $\sum n^{\varepsilon_p} < +\infty$ since $\varepsilon_p < -1$.

Therefore $\beta(T) = \lim_{n\to\infty} \beta_n(T)$ exists for all bounded operators T on Ep and

$$|\beta(T)| \leq \text{const } (1 + \sum n^{\varepsilon_p}) \max_{f \in K} \|Tf\|_p .$$

Step 6: Clearly $\beta(Id) = 1$ where Id = identity operator on Ep. We show $\beta(T) = 0$, for all finite rank operators T.

It is sufficient to show $\beta(T) = 0$ for all rank one operators T.

Suppose $Tg = Q(g)f$ where $Q \in Ep^*$, $f \in Ep$.

$$\beta_n(T) = 2^{-n} \sum (TW_G , W_G)$$
$$= 2^{-n} \sum Q(W_G) (f , W_G) ,$$

and so to show $\beta_n(T) \to 0$ as $n \to \infty$ we need only show $(f , W_G) \to 0$ as the number of elements in G becomes large. This is obviously true for bounded f , and these f are dense in Ep .

We have constructed then $\forall p > 1$, a space Ep failing the a.p. We show now how to represent Ep as a closed subspace of a ℓ_p - direct sum of ℓ_1 - spaces.

Define a map from $Ep \to \bigoplus_{\ell p} \ell^1_A$ by

$$f \longrightarrow \left\{ \left(\frac{C_n^{1/p} \int_{Z_u} f}{\lambda(A)} \right)_{Z_u \subset A} \right\}_{A \in \mathfrak{A}_n \quad n \in N} \in \bigoplus_{\ell p} \ell^1{}_A \qquad (16)$$

The definition of the norm in Ep ensures this map is an isometry onto a closed subspace of an ℓp - direct sum of finite dimensional ℓ_1 spaces.

For $1 < p \leq 2$ this supplies an example of a space of cotype 2 failing the a.p.

Definition 1.6: Let $\{\varepsilon_j\}$ be a sequence of independent, identically distributed random variables such that

$$\text{Prob}\,[\varepsilon_j = +1] = \text{Prob}\,[\varepsilon_j = -1] = 1/2 \quad \forall_j .$$

A Banach space E is of cotype (resp. type) 2 if there is a constant C such that $\mathcal{E}(\|\sum_{j=1}^{n} \varepsilon_j x_j\|) \geq C(\sum_{j=1}^{n} \|x_j\|^2)^{1/2}$ for all (resp $\leq$) $x_1, \ldots x_n \in E$, for all $n \in N$.

(Here $\mathcal{E}$ denotes expectation.)

This property is clearly preserved on passing to subspaces. To show Ep is of cotype 2 it is sufficient to show that ℓ^1 is of cotype 2 and also that if $1 < p \leq 2$, ℓ^p - direct sums of cotype 2 spaces are also of cotype 2. These results are well known.

Proposition 1.7: ℓ^1 is of cotype 2

Proof: let $x_j = \sum_{n=1}^{\infty} x_{jn} e_n \quad 1 \leq j \leq k$ where the e_n are the usual unit vectors in ℓ^1.

Khinchin's inequality states that there is a $C > 0$ such that

$$C^{-1}(\sum_{j=1}^{k} |a_j|^2)^{1/2} \leq \mathcal{E}\,(|\sum_{j=1}^{k} \varepsilon_j a_j|) \leq C(\sum_{j=1}^{k} |a_j|^2)^{1/2}$$

for all real nos $a_1 \ldots a_k$, all $k \in N$.

So using Khinchin's inequality, we have

$$(\sum_{j=1}^{k} (\sum_{n=1}^{\infty} |x_{jn}|)^2)^{1/2} \leq C \;\mathcal{E}\;(|\sum_{j=1}^{k} \varepsilon_j \sum_{n} |x_{jn}|)$$

$$\leq C \sum_{n} \mathcal{E}\,(\left|\sum_{j=1}^{k} \varepsilon_j |x_{jn}|\right|)$$

$$\leq C^2 \sum_n (\sum_j |x_{jn}|^2)^{1/2} \text{ again by Khinchin's inequality}$$

$$\leq C^3 \mathcal{E} (\sum_n |\sum \varepsilon_j x_{jn}|), \text{by one}$$

last use of Khinchin's inequality.

An easy argument shows that the dual of a type 2 space is always cotype 2. Therefore ℓ^p for $1 < p \leq 2$ is always cotype 2.

Proposition 1.8: If E_n are Banach spaces such that there is a $C > 0$ such that

$$(\sum_j \|e_{jn}\|^2)^{1/2} \leq C \mathcal{E} (\|\sum_j \varepsilon_j e_{jn}\|) \{ e_{jn} \in E_n, n \in N \} \text{ and}$$

$1 \leq p \leq 2$, then $\bigoplus_{\ell^p} E_n$ is of cotype 2.

Proof: Since ℓp is of cotype 2 for $1 \leq p \leq 2$,

$$(\sum_j (\sum_n \|e_{jn}\|^p)^{2/p})^{1/2} \leq Ap \mathcal{E} (\sum_n | \sum_j \varepsilon_j \|e_{jn}\| |^p)^{1/p}$$

$$\leq Ap \sum_n \mathcal{E} (| \sum_j \varepsilon_j \|e_{jn}\| |^p) ^{1/p}$$

Using Khinchin's inequality the above is

$$\leq Ap\ Bp \sum_n (\sum_j \|e_{jn}\|^2)^{p/2} \ ^{1/p}$$

$$\leq Ap\ Bp\ C \sum_n \mathcal{E} (\|\sum_j \varepsilon_j e_{jn}\|^p) \ ^{1/p}$$

$$= Ap\ Bp\ C \mathcal{E} (\sum_n \| \sum_j \varepsilon_j e_{jn}\|^p) \ ^{1/p}$$

and so we have the result.

Remarks: (1) For ease we restricted attention to the case $q = 1$, $p > 1$. We can obtain spaces $E\ p, q$ which are closed subspaces of ℓ^p - direct sums of finite dimensional ℓ^q spaces with $p > q \geq 1$ by defining

$$\|f\|_{p,q} = \left[\sum_n C_n \sum_{A\in\mathcal{O}\!\iota_n} \left\{ \frac{\left(\sum_{Z_u\subset A} \left| \int_{Z_u} f \right|^q\right)^{1/q}}{\lambda(A)} \right\}^p \right]^{1/p}$$

where the $C_n > 0$ are numbers chosen later.

As before

$$|\beta_n(T) - \beta_{n+1}(T)| \le \sum_{A\in\mathcal{O}\!\iota_n} \max \left\{ \sum_{Z_u\subset A} \left| \int_{Z_u} Tf \right| : f \in \mathcal{F}_n \right\}$$

$$\le M_n \max \lambda(A) \max \frac{\left(\sum_{Z_u\subset A} \left|\int_{Z_u} Tf \right|^q\right)^{1/q} 2^{n/q^1}}{\lambda(A)}$$

$$\le \text{constant} \times \left[\sum_n C_n \sum_{A\in\ \ n} \left(\sum_{Z_u} \left| \int_{Z_u} Tf \right|^q \right)^{p/q} \right]^{1/p} \frac{2^{n/q^1}}{C_n^{1/p}}$$

(where $\frac{1}{q} + \frac{1}{q^1} = 1$)

$$\le \text{constant} \max_{f\in\mathcal{F}_n} \|Tf\|_{p,q} \frac{2^{n/q^1}}{C_n^{1/p}} .$$

Then a careful choice of C_n as in step 4 yields the result in the same way as before.

(2) It is unknown whether every closed subspace of ℓ^p $(1 \le p < 2)$ has the approximation property.

(3) Szankowski obtains his Banach lattice without the a.p. by taking as his norm

$$\|f\| = \sup_n \max_{A\in\mathcal{O}\!\iota_n} \lambda(A)^{-1} \int_A |f| \, d\lambda .$$

CHAPTER 2

The Radon Nikodym Property

For ease in this chapter, we shall consider only real Banach spaces, and all scalar measures and functions will take real values.

$(X, \sum, \mu)$ is a finite measure space i.e. X is a set, $\sum$ a σ- algebra of subsets of X and μ a finite positive measure defined on $\sum$. If m is another finite measure on X, but possibly taking both positive and negative values, we say m is absolutely continuous with respect to μ if $m(A) = 0$ for all $A \in \sum$ for which $\mu(A) = 0$. If this is the case, we write $m << \mu$.

If Q is a real valued function on X, integrable with respect to μ then

$$m(A) = \int_A Q(x)\, d\mu \quad , \quad A \in \sum \qquad (1)$$

defines a finite measure on X, absolutely continuous with respect to μ. The crux of the Radon Nikodym Theorem is that all finite measures absolutely continuous with respect to μ must arise as in (1).

Theorem (Radon Nikodym): With $(X, \sum, \mu)$ as above, and m a finite measure, absolutely continuous with respect to μ, then there is $Q \in L^1(X, \sum, \mu)$ such that (1) holds
(We need only require, in fact, that μ be σ - finite.)

Our first object in this chapter will be to obtain a Radon Nikodym theorem for measures taking their values in a Banach space. This is possible only in certain spaces, those with the Radon Nikodym property. We shall then obtain results concerning the approximation property and p - integral and p - nuclear operators and spaces with the Radon Nikodym property.

Vector Valued Measures and Strongly Measurable Functions:

We first set up the necessary machinery to discuss the theorem. A fuller account of the following material is available in Dunford and Schwartz, Volume 1.

Throughout X will be a set, $\sum$ a σ - algebra of subsets of X and E a real Banach space.

Definition 2.1: A vector valued measure $m : \sum \to E$ is a set function taking values in the real Banach space E such that m is countably additive i.e. if $\{A_n\}_{n=1}^{\infty}$ is a sequence of disjoint subsets in $\sum$ then

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n) . \qquad (2)$$

We restrict attention to those vector measures which are of finite (bounded) variation.

Definition 2.2: The variation $\|m\|$ of a vector measure m is the positive measure defined by

$$\|m\|(A) = \sup \sum_{i=1}^{n} \|m(A_i)\|$$

where the supremum is taken over all finite partitions of the set $A \in \sum$ into disjoint subsets $A_1, \ldots, A_n \in \sum$.

m is of finite (or bounded) variation if $\|m\|(X) < +\infty$. We also use the terminology 'finite' .

A null set of m is simply a null set of the measure $\|m\|$.

In a fairly obvious way we can set up a theory of integration of scalar valued functions defined on X with respect to a vector measure m defined on X . We commence with simple functions.

A simple function f has the form $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ where $\alpha_i \in \underset{\sim}{R}$, and the χ_{A_i} are characteristic functions of the disjoint sets $A_i \in \sum$.

We define $\int_X f \, dm = \sum_{i=1}^{n} \alpha_i \, m(A_i)$ (3)

We can obtain easily the usual properties of the integral for simple functions. We now extend the notion of integrability.

We will say f is measurable if it is measurable with respect to the space $(X, \sum, \|m\|)$.

<u>Definition 2.3</u>: A scalar valued measurable function f is said to be integrable with respect to m if there is a sequence $\{f_n\}$ of simple functions such that

(i) $f_n(x) \to f(x)$ pointwise m almost everywhere (i.e. $\|m\|$ almost everywhere)

(ii) $\int |f_n - f_m| \, d\|m\|$ is Cauchy.

From (ii), $\| \int f_n dm - \int f_m dm \| \leq \int |f_n - f_m| \, d\|m\|$ so $\{ \int f_n dm \}$ is Cauchy and convergent to an element of E.

Define $\int_X f \, dm = \lim_n \int_X f_n \, dm$.

We can show the above definition is independent of the choice of simple functions, and an integration theory is readily obtainable.

We turn our attention to vector valued functions. μ is a finite positive measure on X. A vector valued function $f : X \to E$ is said to be <u>simple</u> if $f = \sum_{i=1}^{n} e_i \chi_{A_i}$ where $e_i \in E$ and the χ_{A_i} are characteristic functions of disjoint subsets $A_i \in \sum$.

Definition 2.4: A function $f : X \to E$ is said to be strongly measurable with respect to μ , if it is pointwise μ- almost everywhere limit of simple functions.

If f is simple, $f = \sum_{i=1}^{n} e_i \chi_{A_i}$ we define

$\int_X f \, d\mu = \sum_{i=1}^{n} e_i \; \mu(A_i)$. We extend by

Definition 2.5: A function $f : X \to E$ is strongly integrable (Bochner integrable) with respect to μ if there is a sequence $\{f_n\}$ of simple functions such that

(i) $f_n(x) \to f(x)$ μ- almost everywhere (so f is measurable)

(ii) $\{ \int \|f_m - f_n\| \, d\mu \}$ is Cauchy.

Define $\int_X f(x) \, d\mu(x) = \lim_{n\to\infty} \int_X f_n \, d\mu$ which exists by (ii) since E is a Banach space.

Again we can prove the definition is independent of the choice of simple functions.

$L^1_E \, (X, \sum, \mu) = L^1_E(\mu)$ will denote the set of all Bochner integrable functions. It is a Banach space.

If $A \in \sum$, $f \in L^1_E(\mu)$ we define

$$\int_A f \, d\mu = \int_X f \chi_A \, d\mu \; .$$

Define $m(A) = \int_A f \, d\mu$ for some $f \in L^1_E \, (\mu)$. (*)

Then m is a countably additive vector measure of bounded variation.

For if $A = A_1 \cup \ldots \cup A_n$ is a partition of $A \in \sum$ into disjoint subsets $A_1 \ldots A_n \in \sum$, then

$$\sum_{i=1}^{n} \|m(A_i)\| \le \int_{\cup A_i} \|f\| \, d\mu \le \int_X \|f\| \; d\mu(x) \; .$$

Thus $\|m\|$ exists. Moreover $m(A) = 0$ for any $A \in \sum$ for which $\mu(A) = 0$.

Definition 2.6: m is μ- continuous if $m(A) = 0$ for all A with $\mu(A) = 0$, $A \in \sum$.

The natural question is whether every E-valued finite μ- continuous m must arise as in $(*)$, through some $f \in L^1_E(\mu)$. The answer is 'no' as the following example demonstrates: the example is well known.

Example: Let $X = [0,1]$, $\sum$ be the σ- algebra of Borel measurable subsets of $[0,1]$ and μ be Lebesgue measure on $[0,1]$. E is the real Banach space $L^1[0,1]$.

Define $m : \sum \to E$ by

$$m(A) = \chi_A \text{ for each } A \in \sum , \text{ where}$$

χ_A is the characteristic function of the set A. Then m is a vector measure and since $\|m(A)\| = \mu(A)$ for each $A \in \sum$, m is finite and μ- continuous.

There is, however, no $Q \in L^1_E(\mu)$ such that $m(A) = \int_A Q \, d\mu$, for each $A \in \sum$.

Let us suppose there were and that $\{Q_n\}$ is a sequence of simple functions such that $Q_n \to Q$ pointwise almost everywhere and $\int \|Q_n - Q\| \, d\mu < 2^{-n-1}$ for each n .

We may assume $Q_n(S) = \sum_{j=1}^{k_n} \frac{\psi_{nj}}{\mu(A_{nj})} \chi_{A_{nj}}(S)$

where the A_{nj} are disjoint and $\mu(A_{nj}) \leq 2^{-n}$. Each $\psi_{nj} \in L^1 [0,1]$.

Consider $Q_n^1 (S) = \sum_{j=1}^{k_n} \frac{m(A_{nj})}{\mu(A_{nj})} \chi_{A_{nj}}(S)$.

Then $\int \|Q_n^1 - Q_n\| \, d\mu \leq \sum_{j=1}^{k_n} \|m(A_{nj}) - \psi_{nj}\|$.

Now $\|m(A_{nj}) - \psi_{nj}\| = \| \int_{A_{nj}} Q \, d\mu - \int_{A_{nj}} Q_n d\mu \| \leq \int_{A_{nj}} \|Q - Q_n\| \, d\mu$.

Therefore $\int \|Q_n^{\,1} - Q_n\|\, d\mu \leq \sum_{j=1}^{k_n} \int_{A_{nj}} \|Q - Q_n\|\, d\mu \leq \int_X \|Q - Q_n\|\, d\mu \leq 2^{-n-1}$.

So $\int \|Q_n^{\,1} - Q\|\, d\mu \leq 2^{-n}$, i.e. $Q_n^{\,1} \to Q$ in $L^1_E(\mu)$. (4)

Let A be a set such that, for some $n \geq 2$,

$$\mu(A \cap A_{nj}) = \frac{1}{2}\mu(A_{nj}) \quad \text{for } j = 1,2, \ldots k_n .$$

Then $\left\|\chi_A - \int_A \sum_{j=1}^{k_n} \frac{\chi_{A_{nj}}}{\mu(A_{nj})} \chi_{A_{nj}}(S)\, ds\right\|_{L^1} = \left\|\chi_A - \frac{1}{2}\right\|_{L^1} = \frac{1}{2}$.

But (4) gives the above $\leq 2^{-n}$, a contradiction. There is, therefore, no $Q \in L^1_E(\mu)$ such that $m(A) = \int_A Q\, d\mu$.

<u>The Radon Nikodym Property</u>:

We shall now try to discover for which Banach spaces, an analogue of the Radon Nikodym theorem is possible.

<u>Definition 2.7</u>: A Banach space E is said to have the <u>Radon Nikodym property (RNP)</u> if and only if for any finite, positive measure space $(X, \sum, \mu)$ and any E- valued, finite, μ- continuous measure m on $\sum$, there exists a $Q \in L^1_E(\mu)$ such that

$$m(A) = \int_A Q(x)\, d\mu(x) , \quad A \in \sum .$$

Phillips [12] had shown that all reflexive spaces have the RNP, although he did not state the result in this way. In an attempt to generalise Phillips' result, Rieffel [13] [14] introduced a geometric concept - dentability - and established a link between dentability of subsets of a Banach space and the Radon Nikodym property.

Definition 2.8: A subset D of a Banach space E is dentable if for each $\varepsilon > 0$, there is a $d \in D$ such that $d \notin \overline{CO}\,[D \setminus B_\varepsilon(d)]$.

Here $\overline{CO}\,(F)$ denotes the closed convex hull of a set F, and $B_\varepsilon(d) = \{e \in E : \|e-d\| < \varepsilon\}$.

We then have the following theorem:-

Theorem 2.9: A Banach space E has the RNP if, and only if, every bounded subset D of E is dentable.

Rieffel [14], who introduced the concept, proved that if every bounded subset D of E is dentable, then E has the RNP. The other implication was proved in a succession of papers by other authors including Maynard, [8] Davis and Phelps [2], Huff[6].

We shall present a proof of this theorem which is a merger of the proofs of Rieffel [13], [14] and Huff [6].

Definition 2.10: With $(X, \sum, \mu)$ a positive measure space and m a finite vector measure which is μ- continuous, $A \in \sum$ with $\mu(A) > 0$ define the set $R(A) \subseteq E$ (the range of A) by

$$R(A) = \left\{ \frac{m(B)}{\mu(B)} : B \subseteq A \text{ and } 0 < \mu(B) < \infty \right\}$$

Definition 2.11: Call a subset $A \in \sum$, with $\mu(A) > 0$, (e, ε) - pure if $R(A) \subseteq B_\varepsilon(e)$ ($\varepsilon > 0$ and $e \in E$).

Lemma 2.12: [13,14] $(X, \sum, \mu)$ as before. m is a finite, μ- continuous E- valued measure, where E is a Banach space in which every bounded subset is dentable. Let $\varepsilon > 0$, $A \in \sum$ with $\mu(A) > 0$. There is a subset $B \subseteq A$ with $\mu(B) > 0$ and an $e \in E$ such that B is (e, ε) - pure.

Proof: We show first that there is a $B \subseteq A$ with $\mu(B) > 0$, such that $R(B)$ is a bounded, and so dentable, subset of E. The following argument will be used on several occasions, and is used extensively by Rieffel.

If $R(A)$ is not bounded, let $K = \|m\|(A)/\mu(A)$.

Let k_1 = smallest integer ≥ 2 such that there is a $B_1 \subseteq A$ with $\mu(B_1) \geq \frac{1}{k_1}$

and $\dfrac{\|m(B_1)\|}{\mu(B_1)} > 2K$.

Letting $A_1 = A \setminus B_1$ decide whether $R(A_1)$ is bounded. If it is, stop.

If not choose k_2 = smallest integer $\geq k_1$ such that there is a $B_2 \subseteq A \setminus B_1$ with $\mu(B_2) \geq \frac{1}{k_2}$ and $\dfrac{\|m(B_2)\|}{\mu(B_2)} > 2K$.

Continuing the process, we either stop at some stage, or else obtain a sequence of non-decreasing integers $\{k_i\}$, a sequence $\{B_i\}$ of disjoint subsets of A with $\mu(B_i) \geq \frac{1}{k_i}$, with the property that if $C \subseteq A \setminus \bigcup_{i=1}^{n} B_i$ and $\mu(C) > 0$ and $\dfrac{\|m(C)\|}{\mu(C)} > 2K$, then $\mu(C) < \dfrac{1}{k_n - 1}$.

Let $B = A \setminus \bigcup_{i=1}^{\infty} B_i$. If $C \subseteq A \setminus \bigcup_{i=1}^{\infty} B_i$ and $\mu(C) > 0$ with $\dfrac{\|m(C)\|}{\mu(C)} > 2K$ then we have $\mu(C) < \dfrac{1}{k_i - 1}$, for all i . Since $(X, \sum, \mu)$ is a finite measure space $k_i \to \infty$ as $i \to \infty$, so that $\mu(C) = 0$, a contradiction.

Lastly we show $\mu(B) > 0$. If $\mu(B) = 0$ then $m(B) = 0$.

Thus $\frac{m(A)}{\mu(A)} = \sum_{i=1}^{\infty} \frac{m(B_i)}{\mu(B)}$ and we obtain

$$\frac{\|m\|(A)}{\mu(A)} \geq \sum_{i=1}^{\infty} \frac{\|m(B_i)\|}{\mu(A)}$$

$$= \sum_{i=1}^{\infty} \frac{\|m(B_i)\|}{\mu(B_i)} \frac{\mu(B_i)}{\mu(A)}$$

$$> 2K \sum_{i=1}^{\infty} \frac{\mu(B_i)}{\mu(A_i)} = 2K \text{ , a contradiction.}$$

We may as well assume therefore that $R(A) = D$ is bounded and so dentable.

Let $\varepsilon > 0$. $\exists\, d = \frac{m(B_o)}{\mu(B_o)} \notin \overline{CO} \left[D \setminus B_\varepsilon(d) \right]$.

Consider $R(B_o)$. If $R(B_o)$ is (d, ε) pure, stop .

If not, let k_1 = smallest integer ≥ 2 such that there is an $A_1 \subseteq B_o$ with $\mu(A_1) \geq \frac{1}{k_1}$ and $\frac{m(A_1)}{\mu(A_1)} \notin B_\varepsilon(d)$

but $\frac{m(A_1)}{\mu(A_1)} \in D \setminus B_\varepsilon(d)$.

Consider $B_1 = B_0 \setminus A_1$. If this is (d,ε) pure, stop, if not, continue the process.

As before we obtain a non-decreasing sequence of integers $\{k_i\}$, a sequence $\{A_i\}$ of disjoint subsets of A with $\mu(A_i) \geq \frac{1}{k_i}$, and if $C \subseteq B_0 \setminus \bigcup_{i=1}^{n} A_i$ with $\mu(C) > 0$ and $\frac{m(C)}{\mu(C)} \notin B_\varepsilon(d)$, then $\mu(C) < \frac{1}{k_{n-1}}$.

Consider $B = B_0 \setminus \bigcup_{i=1}^{\infty} A_i$. If $C \subseteq B_0 \setminus \bigcup_{i=1}^{\infty} A_i$,

and $\mu(C) > 0$ with $\frac{m(C)}{\mu(C)} \notin B_\varepsilon(d)$, then $\mu(C) \leq \frac{1}{k_i - 1}$, for all i ,

and so $\mu(C) = 0$ as before, a contradiction.

Also $\mu(B) > 0$. If not $\mu(B_0) = \sum_{i=1}^{\infty} \mu(A_i)$

$$\text{and } d = \frac{m(B_0)}{\mu(B_0)} = \sum_{i=1}^{\infty} \frac{m(A_i)}{\mu(A_i)} \frac{\mu(A_i)}{\mu(B_0)}$$

$$\varepsilon\ \overline{CO}\ \left[D \setminus B_\varepsilon(d)\right], \text{ a contradiction.}$$

Lemma 2.13: $(X, \sum, \mu)$ m and E as before. Given $\varepsilon > 0$, $\exists$ sequence $\{A_i\}$ of disjoint subsets of $\sum$ and $\{e_i\} \subseteq E$ such that $X = \bigcup_{i=1}^{\infty} A_i$ and each A_i is (e_i, ε) pure.

Proof: Using lemma 2.12, let k_1 = smallest integer $\geq$ 2 such that there is an $A_1 \subseteq X$ with $\mu(A_1) \geq \frac{1}{k_1}$ and A_1 is (e_1, ε) pure

for some $e_1 \varepsilon E$.

We use the same procedure as before to obtain a sequence of non-decreasing integers $\{k_i\}$, a sequence $\{A_i\}$ of disjoint subsets of X with $\mu(A_i) \geq \frac{1}{k_i}$ and if $C \subseteq X \setminus \bigcup_{i=1}^{n} A_i$ satisfies

$\mu(C) > 0$, R(C) is (e,ε) pure for some e, then $\mu(C) < \frac{1}{k_{n-1}}$.

Let $B = X \setminus \bigcup_{i=1}^{\infty} A_i$. B has measure zero. If not, there is a $C \subseteq B$ with $\mu(C) > 0$ and R(C) (e, ε) pure for some e. Then $\mu(C) < \frac{1}{k_{i-1}}$ for all i , and so has measure zero.

Adjoin B to A_1 , and we have the required decomposition.

Proof: (of theorem 2.9) ([7], [13], [14])

Suppose first that every bounded subset of E is dentable, that $(X, \sum, \mu)$ is a finite positive measure space and m is a finite μ- continuous E- valued measure.

Let Π = collection of all partitions π of X into disjoint subsets $A_1, \dots A_n$ each of positive μ- measure. This set is partially ordered in an obvious way.

For a given π define $Q_\pi = \sum_{A \varepsilon \pi} \frac{m(A)}{\mu(A)} \chi_A$.

Q_π is an integrable simple function. With $\varepsilon > 0$ given, we shall show the existence of a $\pi_o \varepsilon \Pi$ such that if $\pi \geq \pi_o$,

$$\int \|Q_\pi - Q_{\pi_o}\| \, d\mu < \varepsilon .$$

Fix $\varepsilon > 0$ and decompose $X = \bigcup_{i=1}^{\infty} A_i$ as in lemma 2.13 in which each A_i is $(e_i, \varepsilon/6\, \mu(X))$ pure. Because $\|m\|$ is absolutely continuous with respect to μ , given $\varepsilon > 0$, $\exists\, \delta > 0$ such that if $\mu(B) < \delta$, $B \varepsilon \sum$, then $\|m\|\,(B) < \varepsilon/3$.

Since μ is finite there is an $n \varepsilon \underset{\sim}{N}$ such that $B = X\backslash \bigcup_{i=1}^{n} A_i$ satisfies $\mu(B) < \delta$.

Let $\pi_o = \{A_1, \dots, A_n, B\}$

$$Q_{\pi_o} = \sum_{i=1}^{n} \frac{m(A_i)}{\mu(A_i)} \chi_{A_i} + \frac{m(B)}{\mu(B)} \chi_B .$$

Suppose $\pi \geq \pi_o$. Then

$$\int \|Q_\pi - Q_{\pi_o} d\mu\| \leq \int_{\bigcup_{i=1}^{n} A_i} \|Q_\pi - Q_{\pi_o}\| d\mu + \int_B \|Q_\pi\| d\mu + \int_B \|Q_{\pi_o}\| d\mu$$

$$\leq \int_{\bigcup_{i=1}^{n} A_i} \|Q_\pi - Q_{\pi_o}\| d\mu + \varepsilon/3 + \varepsilon/3 .$$

Now $\int_{A_i} \|Q_\pi - Q_{\pi o}\| d\mu \leq \int_{A_i} \|Q_\pi - e_i\| d\mu + \int_{A_i} \|e_i - Q_{\pi o}\| d\mu$.

$$Q_{\pi o} = \frac{m(A_i)}{\mu(A_i)}, \text{ so } \left\| e_i - \frac{m(A_i)}{\mu(A_i)} \right\| < \frac{\varepsilon}{6\,\mu(X)} .$$

Also $Q_\pi = \sum_{j=1}^{k} \frac{m(A_{ij})}{\mu(A_{ij})} \chi_{A_{ij}}$ on A_i with $A_i = \sum_{j=1}^{k} A_{ij}$,

so $\int_{A_i} \left\| e_i - \sum_{j=1}^{k} \frac{m(A_{ij})}{\mu(A_{ij})} \chi_{A_{ij}} \right\| d\mu \leq \frac{\varepsilon}{6\,\mu(X)} \sum_{j=1}^{k} \mu(A_{ij})$.

Thus $\int_X \|Q_\pi - Q_{\pi o}\| d\mu \leq \varepsilon$ if $\pi \geq \pi_o$.

The net $\{Q_\pi\}$ is Cauchy therefore and so $\exists\, Q \in L^1_E(\mu)$ with

$\int_A Q\, d\mu = \lim_\pi \int_A Q_\pi d_\mu$

Clearly $m(A) = \int_A Q\, d\mu$. $\forall A \in \sum$.

Let us suppose now there is a subset D of E which is not dentable.

There is an $\varepsilon > 0$, therefore, such that

$$d \in \overline{CO}\ [D \setminus B_\varepsilon(d)] \text{ for each } d \in D .$$

We shall construct a vector measure m, a positive measure μ , both on $[0,1[$ such that $\nexists\, Q$ with $m(A) = \int_A Q\, d\mu$ even though m and μ are finite and m is μ- continuous.

Choose some $d \in D$ such that $d \in \overline{CO}\ [D \setminus B_\varepsilon(d)]$. There are $d_j \in D$ with $\|d_j - d\| \geq \varepsilon$, and α_j such that $0 < \alpha_j < \frac{1}{2}$ and $\sum \alpha_j = 1$, with $\|d - \sum_j \alpha_j d_j\| < \frac{1}{2}$.

Consider each d_j. There are d_{ji} with $\|d_{ji} - d_j\| \geq \varepsilon$ and $0 < \alpha_{ji} < \frac{1}{2^2}$ with $\sum_i \alpha_{ji} = 1$ and $\|d_j - \sum_i \alpha_{ji} d_{ji}\| < \frac{1}{2^2}$.

Continue this process. At the nth step we have $d_{i_1 \ldots i_n}$ with $\|d_{i_1 \ldots i_n} - d_{i_1 \ldots i_{n-1}}\| \geq \varepsilon$ and $0 < \alpha_{i_1 \ldots i_n} < \frac{1}{2^n}$ with

$\sum_{i_n} \alpha_{i_1 \ldots i_n} = 1$ and $\|d_{i_1 \ldots i_{n-1}} - \sum_{i_n} \alpha_{i_1 \ldots i_n} d_{i_1 \ldots i_n}\| < \frac{1}{2^n}$.

We now construct a sequence $\{\pi_n\}_{n=0}^{\infty}$ of partitions of $[0,1[$.

Let $\pi_0 = \{ [0,1[\}$

$\pi_1 = \{I_j\}$ where $[0,1[= \bigcup_j I_j$

Each $I_j = [a_j, b_j[$ with $a_0 = 0$ $b_j - a_j = \alpha_j$

$\pi_2 = \{I_{ji}\}$ where each $I_j = \bigcup_j I_{ji}$
with measure of $I_{ji} = \alpha_{ji}$ times the measure of I_j.

$\pi_n = \{I_{i_1 \ldots i_n}\}$, $I_{i_1 \ldots i_{n-1}} = \bigcup_{i_n} I_{i_1 \ldots i_n}$
with measure of $I_{i_1 \ldots i_n} = \alpha_{i_1 \ldots i_n}$ times the measure of $I_{i_1 \ldots i_{n-1}}$.

Define simple functions $\{Q_n\}_{n=0}^{\infty}$ as follows:

$$Q_0 = d\, \chi\, [0,1[$$

$$Q_1 = \sum_j d_j \chi_{I_j}.$$

$$Q_n = \sum_{i_1 \ldots i_n} d_{i_1 \ldots i_n} \chi_{I_{i_1 \ldots i_n}} \quad \text{etc.}$$

The smallest σ- algebra containing $\bigcup_n \pi_n$ is the σ- algebra of Borel subsets of $[0,1[$. This is $\sum$. μ is just Lebesgue measure on $[0,1[$.

$\forall\ A \in \sum$ we define

$$m(A) = \lim_{n\to\infty} \int_A Q_n d\mu \ .$$

That this is reasonable follows from the following estimate:-

$$\left\| \int_{I_{i_1 \ldots i_n}} Q_n - \int_{I_{i_1 \ldots i_n}} Q_{n+1} \right\| \leq \| d_{i_1 \ldots i_n} - \sum \alpha_{i_1 \ldots i_n i_{n+1}} d_{i_1 \ldots i_n i_{n+1}} \| \ \mu\,(I_{i_1 \ldots i_n})$$

$$\leq \frac{1}{2^{n+1}} \mu(I_{i_1 \ldots i_n})$$

By decomposing $I_{i_1 \ldots i_n}$ and telescoping we obtain

$$\left\| \int_{I_{i_1 \ldots i_n}} Q_n - \int_{I_{i_1 \ldots i_n}} Q_{n+k}\, d\mu \right\| \leq \frac{1}{2^n} \mu\,(I_{i_1 \ldots i_n}) \text{ for all } k \ .$$

This guarantees the existence of $\lim_{k\to\infty} \int_{I_{i_1 \ldots i_n}} Q_k\, d\mu$ for any n .

$m(A)$ exists therefore for all A in the algebra generated by the partitions $\{\pi_n\}$, and $\|m(A)\| \leq K\,\mu(A)$, since this holds for $\int_A Q_n\, d\mu$ with n sufficiently large. By lemma IV.8.8 in [3] $m(A)$ exists for all $A \in \sum$ and $\|m(A)\| \leq K\mu(A)$ for all such A .

Thus m is finite and μ- continuous.

Suppose there is a $Q \in L^1_E(m)$ such that $m(A) = \int_A Q\, d\mu$ for all $A \in \sum$:

The sequence $\psi_n = \sum_{i_1 \dots i_n} \frac{m(I_{i_1 \dots i_n})}{\mu(I_{i_1 \dots i_n})} \chi_{I_{i_1 \dots i_n}}$

tends to Q in L^1 norm .

Now $$\int \|\psi_n - Q_n\| \, d\mu = \sum_{i_1 \dots i_n} \left\| \frac{m(I_{i_1 \dots i_n})}{\mu(I_{i_1 \dots i_n})} - d_{i_1 \dots i_n} \right\| \mu \, (I_{i_1 \dots i_n}) \quad (*)$$

But $\|m(I_{i_1 \dots i_n}) - \mu(I_{i_1 \dots i_n}) \, d_{i_1 \dots i_n}\|$

$$= \Big\| \lim_k \int_{I_{i_1 \dots i_n}} Q_k \, d\mu - \int_{I_{i_1 \dots i_n}} Q_n \, d\mu \Big\|$$

$$\leq \mu(I_{i_1 \dots i_n}) \left(\frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} \dots \right)$$

$$= \mu(I_{i_1 \dots i_n}) \frac{1}{2^n} .$$

Thus $(*) \leq \frac{1}{2^n} \sum_{i_1 \dots i_n} \mu(I_{i_1 \dots i_n}) = 2^{-n}$.

Thus $\{ \int \|Q_n - Q_m\| d\mu\}$ is Cauchy.

This is however false for

$$\int \|Q_n - Q_{n+1}\| d\mu = \sum_{i_1 \dots i_{n+1}} \| d_{i_1 \dots i_n} - d_{i_1 \dots i_{n+1}} \| \, \mu(I_{i_1 \dots i_{n+1}})$$

$$\geq \varepsilon , \text{ for all } n \in N .$$

Thus Banach spaces with the RNP have been fully characterised. As well as reflexive spaces, all Banach spaces isomorphic to separable dual spaces have the RNP. (It is clear that it is a property preserved by isomorphisms.) Also every closed subspace of a Banach space with the RNP has the RNP.

So important examples of Banach spaces with the RNP are

(1) all ℓ_p spaces with $1 \leq p < \infty$,

(2) all L^p spaces with $1 < p < \infty$.

Also all closed subspaces of these Banach spaces.

Important examples of Banach spaces without the RNP are $L^1[0,1]$, $C[0,1]$, $L^\infty[0,1]$, c_o , ℓ^∞.

We should remarks that it suffices in Theorem 2.9 to have every closed, bounded, convex set dentable. This follows from Rieffel's result that a bounded subset of a Banach space is dentable if, and only if, its closed convex hull is dentable.

The RNP is linked to another geometric property of Banach spaces, namely the Krein Milman Property. A Banach space has the Krein Milman Property if every closed, bounded, convex set is the closed, convex hull of its extreme points. Lindenstrauss has shown that the RNP implies the KMP and in [7] Huff and Morris show that, for dual spaces, the two properties are equivalent. It is unknown whether there exists a Banach space with the KMP, failing the RNP. In [11] Phelps gives a proof of Lindenstrauss' result and links the RNP with other geometric concepts in Banach spaces.

The Radon Nikodym Property and Approximation Property

As well as the a.p. Grothendieck also introduced the metric and bounded approximation properties. (m.a.p. and b.a.p. respectively).

Definition 2.14: [5] A Banach space E has the metric approximation property (bounded approximation property) if for each compact subset

$K \subseteq E$, for each $\varepsilon > 0$ there is a $T \varepsilon B(E)$ of finite rank such that $\|T\| \leq 1$ ($\|T\| \leq$ same constant M) with $\|Tk - k\| < \varepsilon$ for each $K \varepsilon K$,

Grothendieck showed that for reflexive spaces the a.p. implies the m.a.p. We will obtain a generalization of his result. In proving this it is helpful to have a little tensor product notation.

If E and F are two Banach spaces, $E \otimes F$ will denote the algebraic tensor product of E and F , and $\sum_{i=1}^{n} e_i \otimes f_i$ will denote a typical element of this space. $E \otimes F$ is a linear space and we can norm this space in a number of different ways. We shall be interested in two norms.

The projective tensor norm, λ , is defined an $E \otimes F$ by

$$\lambda(u) = \inf \sum_{i=1}^{n} \|e_i\| \|f_i\|$$

where the infimum is taken over all respresentations $u = \sum_{i=1}^{n} e_i \otimes f_i$ of the element $u \varepsilon E \otimes F$.

The injective tensor norm, p , is the norm defined by

$$p\left(\sum_{i=1}^{n} e_i \otimes f_i \right) = \sup \left| \sum_{i=1}^{n} \phi(e_i) \psi(f_i) \right| \quad \phi \varepsilon \text{ Ball } E^* , \psi \varepsilon \text{ Ball } F^*$$

where Ball E^*, Ball F^* denote the closed unit balls of E^*, F^* respectively.

These in fact are norms (so called cross-norms) on $E \otimes F$. $E \otimes F$ is not necessarily complete with respect to either of these norms. Completing $E \otimes F$ in the usual fashion with respect to λ we obtain a space denoted by $E \hat{\otimes} F$ - the projective tensor product of E and F. The space $E \check{\otimes} F$ obtained by completing $E \otimes F$ with respect to p is called the injective tensor product of E and F.

Let us consider the space $E \check{\otimes} E^*$. A typical element of $E \otimes E^*$ has the form $\sum_{i=1}^{n} e_i \otimes \phi_i$, $e_i \in E$, $\phi_i \in E^*$.

We define an operator T on E by

$$Te = \sum_{i=1}^{n} \phi_i(e)\, e_i \quad ;$$

$$\text{Now} \quad \|Te\| = \| \sum_{i=1}^{n} \phi_i(e)\, e_i \|$$

$$= \sup_{\phi \in \text{Ball } E^*} \left| \sum_{i=1}^{n} \phi_i(e)\, \phi(e_i) \right| .$$

$$\text{So} \quad \|T\| = \sup_{e \in \text{Ball } E \;\; \phi \in \text{Ball } E^*} \left| \sum_{i=1}^{n} \phi(e_i)\, \phi_i(e) \right|$$

$$= \sup_{\psi \in \text{Ball } E^* \;\; \phi \in \text{Ball } E^*} \left| \sum_{i=1}^{n} \phi(e_i)\, \psi(\phi_i) \right|$$

$$= p\left(\sum_{i=1}^{n} e_i \otimes \phi_i \right) .$$

In this way $E \check{\otimes} E^*$ can be identified as the closure of the finite rank operators in $B(E)$, namely $F(E)$.

We shall now present a proof of the fact that for a dual space with the RNP, the a.p. and m.a.p. are equivalent.

<u>Proposition 2.15</u>: Let X be a compact Hausdorff space and E a Banach space. Then $C(X) \check{\otimes} E = C(X,E)$ = the Banach space of all E- valued continuous functions on X. (This result is well-known.)

<u>Proof</u>: Define $T : C(X) \otimes E \longrightarrow C(X,E)$ by

$$T\left(\sum_{i=1}^{n} f_i \otimes e_i \right)(x) = \sum_{i=1}^{n} f_i(x)\, e_i$$

T is a well defined linear map.

Now $\|T(\sum_{i=1}^{n} f_i \otimes e_i)\|_\infty = \sup_{x \in X} \|\sum_{i=1}^{n} f_i(x)\, e_i\|$

$$= \sup_{x \in X} \sup_{\phi \in \text{Ball } E^*} |\sum_{i=1}^{n} f_i(x)\, \phi(e_i)|$$

$$= \sup_{\psi \in \text{Ball } C(X)^*\ \phi \in \text{Ball } E^*} |\sum_{i=1}^{n} \psi(f_i)\, \phi(e_i)|$$

$$= p\left(\sum_{i=1}^{n} f_i \otimes e_i\right).$$

Thus T is an isometry on $C(X) \otimes E$ and so extends by continuity to $C(X) \check{\otimes} E$ which is therefore contained in $C(X, E)$. We show $C(X) \check{\otimes} E$ is dense in $C(X, E)$, and since $C(X) \check{\otimes} E$ is closed, the result follows.

If $f \in C(X, E)$, $f(X)$ is compact.

Therefore given $\varepsilon > 0$, there are open balls $B_\varepsilon(e_1) \ldots B_\varepsilon(e_n)$ $e_i \in E$ covering $f(X)$.

Let $U_i = f^{-1}(B_\varepsilon(e_i))$, an open subset of X. Choose a partition of unity $\{\phi_i\}$ subordinate to $\{U_i\}$.

So each ϕ_i is continuous, support $\phi_i \subseteq U_i$ and $\sum_{i=1}^{n} \phi_i \equiv 1$.

Then $\|f(x) - \sum_{i=1}^{n} \phi_i(x) e_i\| = \|\sum_{i=1}^{n} \phi_i(x)(f(x) - e_i)\|$

$$< \varepsilon \text{ for all } x \in X.$$

Theorem 2.16: The dual of $C(X, E)$ is the set of all bounded, regular, Borel E^*- valued measures on the compact Hausdorff space X.

Proof: Given m, E^*- valued, bounded, regular and Borel, define $\psi_m(f) = \int_X \langle dm, f \rangle \quad (f \in C(X,E)).$ (5)

By (5) we mean an integral defined first for simple functions on X as follows:

If $f = \sum_{i=1}^{n} e_i \chi_{A_i}$, A_i disjoint Borel subsets of X, $e_i \in E$,

define $\int_X \langle dm, f \rangle = \sum_{i=1}^{n} \langle m(A_i), e_i \rangle$ - this can be easily extended to continuous functions and we obtain

$$\|\psi m\| \leq \|m\| .$$

Suppose, conversely, that $\psi \in C(X,E)^*$. ψ is a continuous linear functional on $C(X) \check{\otimes} E$. Fix $e \in E$ and identify $C(X) \otimes \{e\}$ with $C(X)$.

$\psi_e(f) = \psi(f \otimes e)$ is a well defined continuous linear functional on $C(X)$, and clearly $\|\psi_e\| \leq \|\psi\| \ \|e\|$.

By the Riesz Representation Theorem $\exists\, m_e$ a bounded, regular, Borel measure on X such that

$$\psi_e(f) = \int_X f(x)\, dm_e(x) \quad (f \in C(X))$$

and $\|m_e\| = \|\psi_e\| \leq \|\psi\| \ \|e\|$.

The map $e \longrightarrow m_e$ is linear and for each Borel set $A \subseteq X$ $e \longrightarrow m_e(A)$ is linear.

Define $m(A) \in E^*$ by $m(A)e = m_e(A)$ for each Borel set A.

Linearity is clear and $\|m(A)\| \leq \|\psi\|$ clearly.

If $A_1, \ldots A_n$ are disjoint Borel sets,

$$\begin{aligned} m(A_1 \cup \ldots \cup A_n)\, e &= m_e(A_1 \cup \ldots \cup A_n) \\ &= m_e(A_1) + \ldots m_e(A_n) \\ &= (m(A_1) + \ldots + m(A_n))\, e \end{aligned}$$

$$\begin{aligned} \text{and } m(A \setminus B)e &= m_e(A \setminus B) \\ &= m_e(A) - m_e(B) \\ &= (m(A) - m(B))e . \end{aligned}$$

Thus m is a finitely additive, E^*- valued set function.

We shall now show that $\sum_{i=1}^{n} \|m(A_i)\| \leq \|\psi\|$ for all disjoint Borel sets $A_1, \ldots A_n$, all $n \in N$.

Fix n . Let $\varepsilon > 0$. Choose $e_i \in$ Ball E such that $m_{e_i}(A_i) = m(A_i)\, e_i > \|m(A_i)\| - \varepsilon/2n$ $i=1, \ldots n$.

Each m_e is regular. Let m_e^+ , m_e^- be the positive and negative parts of m_e respectively. Using the regularity, we can find disjoint closed sets F_i, G_i contained in A_i and open sets V_i, W_i containing A_i such that $F_i \subseteq V_i$, $G_i \subseteq W_i$.

$$m_{e_i}^+(F_i) > m_{e_i}^+(A_i) - \varepsilon/2n ,$$

$$m_{e_i}^-(F_i) = 0 ,$$

$$m_{e_i}^-(G_i) > m_{e_i}^-(A_i) - \varepsilon/2n ,$$

$$m_{e_i}^+(G_i) = 0 ,$$

$$m_{e_i}^+(V_i \setminus F_i) < \varepsilon/2n , \quad m_{e_i}^-(V_i \setminus F_i) < \varepsilon/2n ,$$

$$m_{e_i}^+(W_i \setminus G_i) < \varepsilon/2n , \quad m_{e_i}^-(W_i \setminus G_i) < \varepsilon/2n .$$

We finally choose disjoint open sets O_i, U_i s.t. $F_i \subseteq O_i$, $G_i \subseteq U_i$, $O_i \subseteq V_i$, $U_i \subseteq W_i$; $O_1 \ldots O_n$, $U_1 \ldots U_n$ are all disjoint.

Using Urysohn's lemma we define continuous functions ψ_i and Θ_i such that

$$\psi_i = 1 \text{ on } F_i ; \ \psi_i \text{ vanishes off } O_i ,$$

$$\Theta_i = 1 \text{ on } G_i ; \ \Theta_i \text{ vanishes off } U_i ,$$

$$\psi_i \subseteq [0,1] , \quad \Theta_i \subseteq [0,1] .$$

Then $\sum_{i=1}^{n} \|m(A_i)\| < \sum_{i=1}^{n} m_{e_i}(A_i) + \varepsilon/2$

$$= \sum_{i=1}^{n} (m_{e_i}^{+}(A_i) - m_{e_i}^{-}(A_i)) + \varepsilon/2$$

$$< \sum_{i=1}^{n} (m_{e_i}^{+}(F_i) - m_{e_i}^{-}(G_i)) + \varepsilon$$

$$< \sum_{i=1}^{n} \int_{F_i \cup G_i} (\psi_i + \theta_i)\, dm_{e_i} + \varepsilon$$

$$< \sum_{i=1}^{n} \int_X (\psi_i + \theta_i)\, dm_{e_i} + \sum_{i=1}^{n} \int_{D_i \setminus F_i} |\psi|\, dm_{e_i}^{+} + dm_{e_i}^{-}$$

$$+ \sum_{i=1}^{n} \int_{U_i \setminus G_i} |\theta_i|\, dm_{e_i}^{-} + \varepsilon$$

$$< |\psi(\sum_{i=1}^{n} (\theta_i + \psi_i) e_i)| + 2\varepsilon .$$

Now $\sup_{x \in X} \|\sum_{i=1}^{n} (\theta_i(x) + \psi_i(x)) e_i\| \leq 1$ by the disjointness of the open sets and the fact $\|e_i\| \leq 1$.

Thus we obtain

$$\sum_{i=1}^{n} \|m(A_i)\| \leq \|\psi\| + 2\varepsilon \text{ and } \varepsilon \text{ was arbitrary.}$$

So $\sum_{i=1}^{\infty} m(A_i)$ exists for disjoint Borel sets A_i.

$$m(\bigcup_{i=1}^{\infty} A_i)\, e = m_e(\bigcup_{i=1}^{\infty} A_i)$$

$$= \sum_{i=1}^{\infty} m_e(A_i) \quad (m_e \text{ countably additive})$$

$$= \sum_{i=1}^{\infty} m(A_i)e \quad \text{for all } e \in E .$$

Therefore $m(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} m(A_i)$.

It also follows that $\|m\| \leq \|\psi\|$.

Our last task is to justify regularity. By regularity, in this context, we mean that the positive Borel measure $\|m\|$ is regular. (Existence has been justified by the previous step.)

It suffices to show that, given a Borel set A and an $\varepsilon > 0$, there is a compact $K \subseteq A$ such that $\|m\|(A) < \|m\|(K) + \varepsilon$.

Given $\varepsilon > 0$, there is a partition of A into disjoint Borel subsets $A_1 \ldots A_n$ such that

$$\|m\|(A) < \sum_{i=1}^{n} \|m(A_i)\| + \varepsilon/3 .$$

There are $e_i \varepsilon$ Ball E such that

$$\|m(A_i)\| < m(A_i)e_i + \frac{\varepsilon}{3n} \quad i=1, \ldots n .$$

Denoting by $|m_e|$ the variation of the real measure m_e we have

$$\|m\|(A) < \sum_{i=1}^{n} |m_{e_i}| (A_i) + \frac{2\varepsilon}{3} .$$

Each m_{e_i} being regular, there is a compact $K_i \subseteq A_i$ such that $|m_{e_i}|(A_i) < |m_{e_i}|(K_i) + \frac{\varepsilon}{3n}$ for each $i = 1,2 \ldots n$.

$$\text{So } \|m\|(A) < \sum_{i=1}^{n} |m_{e_i}|(K_i) + \varepsilon$$

$$< \sum_{i=1}^{n} \|m\|(K_i) + \varepsilon .$$

The set $K = K_1 U \ldots UK_n$ is compact, the K_i disjoint, so $K \subseteq A$ and

$$\|m\|(A) < \|m\|(K) + \varepsilon .$$

Lastly if $\sum_{i=1}^{n} f_i \otimes e_i \in C(X) \otimes E$ then

$$\psi\left(\sum_{i=1}^{n} f_i \otimes e_i\right) = \sum_{i=1}^{n} \int f_i \, dm_{e_i} = \sum_{i=1}^{n} \int \langle dm, f_i \otimes e_i \rangle$$

$$= \int \langle dm, \sum_{i=1}^{n} f_i \otimes e_i \rangle .$$

We extend to all of $C(X) \overset{\vee}{\otimes} E$ and so the dual space of $C(X,E)$ is the set of Borel E^*- valued measures.

Now let us suppose that E^* has the RNP.

<u>Lemma 2.17</u>: If $\psi \in (C(X,E)^*$ with $\|\psi\| < 1$, there are $e_i^* \in E^*$, $\mu_i \in C(X)^*$, such that $\sum_{i=1}^{\infty} \|e_i^*\| \, \|\mu_i\| \leq 1$ and $\psi = \sum_{i=1}^{\infty} e_i^* \otimes \mu_i$.

<u>Proof</u>: By theorem 2.16 we may write

$$\psi(f) = \int_X \langle dm, f \rangle .$$

Let $\mu = \|m\|$, then m is μ- continuous and, since E^* has the RNP, there is a $\phi \in L^1_{E^*}(\mu)$ such that $dm = \phi \, d\mu$ and so

$$\psi(f) = \int_X \langle \phi(x) , f(x) \rangle \, d\mu(x) .$$

Fix $\varepsilon > 0$. There is a simple function

$\Theta(x) = \sum_{i=1} e_i^* \chi_{A_i}$ (A_i disjoint Borel sets, $e_i^* \in E^*$)

such that $\int \|\phi - \Theta\| \, d\mu < \varepsilon$.

Let $\mu_i = \mu/A_i$.

Consider $\sum_{i=1}^{n} e_i^* \otimes \mu_i \in E^* \hat{\otimes} C(Y)^*$.

$$\left| \int_X \langle \phi(x), f(x) \rangle \, d\mu(x) - \sum_{i=1}^{n} \int_X \langle e_i^*, f(x) \rangle \, d\mu_i(x) \right|$$

$$\leq \left| \int \langle \phi(x) - \Theta(x), f(x) \rangle \, d\mu(x) \right|$$

$$\leq \varepsilon \, \|f\| . \tag{6}$$

Define $T: E^* \hat{\otimes} C(X)^* \longrightarrow C(X,E)^*$ as follows.

For $\sum_{i=1}^{n} e_i^* \otimes \mu_i$, define

$$T\left(\sum_{i=1}^{n} e_i^* \otimes \mu_i \right)(f) = \sum_{i=1}^{n} \int_X \langle e_i^*, f(x) \rangle \, d\mu_i(x) .$$

T is linear and the norm of T is less than or equal to $\sum_{i=1}^{n} \|e_i^*\| \, \|\mu_i\|$. Taking the infimum over all such representations we obtain $T(\sum e_i^* \otimes \mu_i) \leq \lambda(\sum_{i=1}^{n} e_i^* \otimes \mu_i)$.

Extend by continuity to all of $C(X)^* \otimes E^*$.

Now by (6),

$$\left| \psi(f) - \frac{1}{1+\varepsilon} \sum_{i=1}^{n} \int_X \langle e_i^*, f(x) \rangle \, d\mu_i(x) \right| \tag{7}$$

$$\leq \varepsilon \, \|f\| + \left| 1 - \frac{1}{1+\varepsilon} \right| \sum_{i=1}^{n} \|e_i^*\| \, \|\mu_i\| \, \|f\|$$

But
$$\sum_{i=1}^{n} \|e_i^*\| \, \|\mu_i\| = \int \|\Theta\| \, d\mu$$
$$\leq \int \|\phi - \Theta\| \, d\mu + \int \|\phi\| \, d\mu$$
$$< \varepsilon + 1 .$$

So (7) is less than $2 \varepsilon \|f\|$.

Thus T, which maps $C(X)^* \hat{\otimes} E^*$ into $C(X,E)^*$, maps the unit ball of $C(X)^* \hat{\otimes} E^*$ to a dense subset of the unit ball of $C(X,E)^*$. By the argument of the open mapping theorem, the map T is such that the image of Ball $(E^* \otimes C(X)^*)$ contains the open ball in $C(X,E)^*$.

Let X = Ball (E^{**}) with the weak* topology - X is a compact Hausdorff space. If $\sum_{i=1}^{n} e_i^* \otimes e_i \in E^* \otimes E$, defining

$$\left(\sum_{i=1}^{n} e_i^* \otimes e_i\right)(x) = \sum_{i=1}^{n} x(e_i^*)e_i \quad x \in X$$

we obtain a continuous function from X to E, and the sup norm of this function is equal to the injective tensor norm of $\sum_{i=1}^{n} e_i^* \otimes e_i$. Thus $E^* \check{\otimes} E$ embeds isometrically in $C(X,E)$. So if ψ is a continuous linear functional on $E^* \check{\otimes} E$ with $\|\psi\| < 1$, we may extend ψ to $C(X,E)$ without increase of norm by the Hahn Banach Theorem.

Theorem 2.18: For a dual space with the Radon Nikodym property, a.p. is equivalent to m.a.p.

Proof: For any Banach space m.a.p. implies a.p. To prove the converse, it suffices to show that the identity operator I is in the strong closure of Ball $(E \check{\otimes} E^*)$ in $B(E^*)$.

Let ψ be a linear functional on $B(E^*)$ with $|\psi(T)| < 1$ $T \in$ Ball $(E \check{\otimes} E^*)$; we show $|\psi(I)| \leq 1$ (ψ is continuous in the strong operator topology.)

By the definition of the strong operator topology ψ has the form

$$\psi(T) = \sum_{j=1}^{n} \tau_j(T\theta_j) \quad (\tau_j \in E^{**},\ \theta_j \in E^*).$$

By lemma 2.17, using $E^* \check{\otimes} E \subseteq C(X,E)$, we have

$$\psi(f) = \sum_{i=1}^{\infty} \int_X \langle e_i^*, f(x) \rangle \, d\mu_i(x) \quad \text{with} \quad \sum_{i=1}^{\infty} \|e_i^*\| \, \|\mu_i\| \leq 1 .$$

This holds for all $f \in E^* \check{\otimes} E$.

$$\text{So} \quad \psi(e \otimes e^*) = \sum_{i=1}^{\infty} \int \langle e_i^*, e^*(x)e \rangle \, d\mu_i(x)$$

$$= \sum_{i=1}^{\infty} e_i^*(e) \, \phi_i(e^*) \tag{8}$$

where $\phi_i \in E^{**}$ is defined by $\phi_i(e^*) = \int_X e^*(x) \, d\mu_i(x)$

Notice that $\sum_{i=1}^{\infty} \|\phi_i\| \, \|e_i^*\| \leq 1$, since $\|\phi_i\| \leq \|\mu_i\|$.

$\sum_{j=1}^{n} \tau_j(T\Theta_j) - \sum_{i=1}^{\infty} \phi_i(Te_i^*) = 0$ for all $T \in E \otimes E^*$ by extending (8) by linearity.

Since E^* has the a.p. this holds also for $T = I$.

To see this put

$\beta(T) = \sum_{j=1}^{n} \tau_j(T\Theta_j) - \sum_{i=1}^{\infty} \phi_i(Te_i^*)$. β is a functional on $B(E^*)$ that annihilates the finite rank operators.

Since $\sum_{i=1}^{\infty} \|e_i^*\| \, \|\phi_i\| \leq 1$, we can find a sequence $\lambda_i > 0$ such that $\lambda_i \to 0$ and $\sum_{i=1}^{\infty} \frac{\|e_i^*\| \, \|\phi_i\|}{\lambda_i} < +\infty$.

Define $K = \left\{ \Theta_j : j=1, \dots n \right\} \cup \left\{ \frac{e_i^*}{\|e_i^*\|} \lambda_i \right\}$.

K is a sequence which tends to zero and so is compact.

Also $|\beta(T)| \leq (\sum_{j=1}^{n} \|\tau_j\|) \sup_j \|T\theta_j\| + \sum_{i=1}^{\infty} \frac{\|\phi_i\| \|e_i*\|}{\lambda_i} \left\| T \frac{\lambda_i e_i*}{\|e_i*\|} \right\|$;

so $|\beta(T)| \leq$ constant x $\sup_K \|Tk\|$.

So $\beta(I) = 0$, as I can be approximated arbitrarily closely on K by finite rank operators.

So $|\psi(I)| = |\sum_{i=1}^{\infty} \phi_i(e_i*)| \leq \sum_{i=1}^{\infty} \|\phi_i\| \|e_i*\| \leq 1$.

So E^* has the metric approximation property.

As corollaries of this result we obtain

Corollary 2.19: [5] For E reflexive, a.p. $\Leftrightarrow$ m.a.p.

Proof: $E = (E^*)^*$ and being reflexive has the RNP.

Corollary 2.20: [5] If a Banach space F is isomorphic to a separable dual space, a.p. $\Leftrightarrow$ b.a.p. In particular, if F is a separable dual space a.p. $\Leftrightarrow$ m.a.p.

Proof: Separable dual spaces have the RNP. The b.a.p. is preserved by isomorphisms.

To put theorem 2.18 into perspective, it is unknown whether every dual space with the a.p. has the m.a.p.

p- nuclear, p- integral maps and the RNP:

Grothendieck [5] introduced special classes of bounded linear maps between two Banach spaces E and F, the so-called integral and nuclear maps. Every nuclear map is automatically integral, but

Grothendieck proved that an integral map $T : E \to F$ is nuclear provided one of the following four conditions holds

(1) E reflexive

(2) E^* separable

(3) F reflexive

(4) F a separable dual space.

Following Grothendieck's work Perrson and Pietsch introduced generalisations of these classes - p- integral and p- nuclear maps (Grothendieck's maps being the case $p=1$) and obtained similar theorems [9] [10] . Using the RNP we shall generalise both these pieces of work.

Definition 2.21: A linear map $T:E \to F$ (E, F Banach spaces) is said to be p-nuclear ($1 \leq p < \infty$) if T has a representation

$$Te = \sum_{n=1}^{\infty} \langle e_n^*, e \rangle f_n \left(e_n^* \in E^*, \; f_n \in F \right)$$

such that

$$\left(\sum_{n=1}^{\infty} \|e_n^*\|^p \right)^{1/p} < +\infty \quad \text{and} \quad \sup_{\|f^*\| \leq 1} \left(\sum_{n=1}^{\infty} |\langle f_n, f^* \rangle|^{p^1} \right)^{1/p^1} < +\infty$$

where $\frac{1}{p} + \frac{1}{p^1} = 1$. The collection of all such maps is denoted $N_p(E,F)$.

Proposition 2.22: [9,10] (i) With a norm defined by

$$N_p(T) = \inf \left(\sum_{n=1}^{\infty} \|e_n^*\|^p \right)^{1/p} \sup_{\|f^*\| \leq 1} \left(\sum_{n=1}^{\infty} |\langle f_n, f^* \rangle|^{p^1} \right)^{1/p^1} ,$$

the infimum being taken over all possible representations, $N_p^{\frown}(E,F)$ is a Banach space and $\|T\| \leq N_p(T)$.

(ii) If D,G are Banach spaces and $R : D \to E$, $S : F \to G$ are bounded, $T : E \to F$ p- nuclear, then $TR : D \to F$, $ST : E \to G$ are p- nuclear and $N_p(TR) \leq N_p(T) \|R\|$, $N_p(ST) \leq \|S\| N_p(T)$.

Proof: (i) For example

$$\|T\| = \sup_{e \in \text{Ball } E \; f^* \in \text{Ball } F^*} | \langle T_e, f \rangle |$$

$$= \sup_{e \in \text{Ball } E \; f^* \in \text{Ball } F^*} | \sum_{n=1}^{\infty} \langle e_n^*, e \rangle \langle f_n, f^* \rangle |$$

$$\leq (\sum_{n=1}^{\infty} \|e_n^*\|^p)^{1/p} \sup_{\text{Ball } F^*} (\sum_{n=1}^{\infty} | \langle f^*, f_n \rangle |^{p^1})^{1/p^1} .$$

Taking the infimum over all possible representations we obtain $\|T\| \leq N_p(T)$.

The other parts are proved similarly.

If in the definition of p-nuclear we interchange the roles of p and p^1 i.e. we require $Te = \sum_{n=1}^{\infty} \langle e_n^*, e \rangle f_n$ with

$\sup_{\|e\| \leq 1} (\sum_{n=1}^{\infty} | \langle e_n^*, e \rangle |^{p^1})^{1/p^1} < + \infty$ and $(\sum_{n=1}^{\infty} \|f_n\|^p)^{1/p} < + \infty$, we obtain a class $N^p(E,F)$ with a norm

$$N^p(T) = \inf \sup_{\|e\| < 1} (\sum_{n=1} | \langle e_1 \, e_n^* \rangle |^{p^1})^{1/p^1} (\sum_{n=1}^{\infty} \|f_n\|^p)^{1/p} .$$

This class is also a Banach space. In the case $p = 1$ the two classes are identical - these are the nuclear operators from E into F .

All finite rank maps from E to F are in the classes $N_p(E,F)$, $N^p(E,F)$; further in the norms $N_p(T)$, $N^p(T)$ the finite rank maps are dense in $N_p(E,F)$, $N^p(E,F)$ respectively.

If μ is a positive measure on a compact Hausdorff space X, then the set $L^p_{E*}(\mu)$ is the set of all $E*$- valued strongly measurable functions ϕ such that $\int \|\phi\|^p \, d\mu < +\infty$, and is a Banach space.

We may define a bounded linear map $T : E \to L^p(\mu)$ by

$$(T_e)(x) = \langle e, \phi(x) \rangle \quad \text{if} \quad \phi \in L^p_{E*}(\mu). \tag{9}$$

<u>Theorem 2.23</u>: [9,] For $1 \leq p < \infty$, we have natural embeddings

$$N^p(E, L^p(\mu)) \subseteq L^p_{E*}(\mu) \subseteq N_p(E, L^p(\mu))$$

each of norm ≤ 1 and such that the map $T : E \to L^p(\mu)$ and function $\phi \in L^p_{E*}(\mu)$ correspond as in (9).

<u>Proof</u>: Let $Te = \sum_{n=1}^{N} \langle e_n^*, e \rangle f_n$ be a finite rank map in $N^p(E, L^p(\mu))$ where each $f_n \in L^p(\mu)$ such that

$$\sup_{\|e\| \leq 1} \left(\sum_{n=1}^{N} |\langle e_n^*, e \rangle|^{p^1} \right)^{1/p^1} \left(\sum_{n=1}^{N} \|f_n\|^p \right)^{1/p} < N_p(T) + \varepsilon \quad \text{where } \varepsilon > 0 .$$

Consider $\phi_T(x) = \sum_{n=1}^{N} e_n^* f_n(x)$. This is a strongly measurable function and

$$\int \|\phi_T(x)\|^p \, d\mu \leq \int \sup_{\|e\| \leq 1} \left| \sum_{n=1}^{N} \langle e_n^*, e \rangle f_n(x) \right|^p d\mu(x)$$

$$\leq \sup_{\|e\| \leq 1} \left(\sum_{n=1}^{N} |\langle e_n^*, e \rangle|^{p^1} \right)^{p/p^1} \int \sum_{n=1}^{N} |f_n(x)|^p \, d\mu(x)$$

$$\leq \sup_{\|e\| \leq 1} \left(\sum_{n=1}^{N} |\langle e_n^*, e \rangle|^{p^1} \right)^{p/p^1} \left(\sum_{n=1}^{N} \|f_n\|^p \right).$$

We obtain $\|\phi_T\| \leq N_p(T)$ for the finite rank maps and we can extend by continuity to all of $N^p(E,F)$.

Suppose now ϕ is a simple function in $L^p_{E^*}(\mu)$. (By a straightforward density argument we can extend to all of $L_E{}^{*p}(\mu)$.)

Suppose $\phi = \sum_{n=1}^{N} e_n{}^* \chi A_n \quad e_n{}^* \in E^*$ and the A_n disjoint measurable sets .

$$Te = \sum_{n=1}^{n} \langle e_n{}^*, e \rangle \chi_{A_n} \in L^p(\mu) .$$

Moreover writing

$$Te = \sum_{n=1}^{N} \langle e_n{}^*, e \rangle \mu(A_n)^{1/p} . \quad \mu(A_n)^{-1/p} \chi_{A_n}$$

we obtain , by Hölder's inequality,

$$\left(\sum_{n=1}^{N} \|e_n{}^*\|^p{}_\mu (A_n) \right)^{1/p} \sup_{\substack{g \in L^{p^1}(\mu) \\ \|g\| \leq 1}} \left(\sum_{n=1}^{N} \left| \int g \chi_{A_n} \right|^{p^1} \mu(A_n)^{\frac{-p^1}{p}} \right)^{1/p^1}$$

$$\leq \left(\sum_{n=1}^{N} \|e_n{}^*\|^p \mu(A_n) \right)^{1/p} \sup_{\substack{g \in L^{p^1}(\mu) \\ \|g\| \leq 1}} \left(\sum_{n=1}^{N} \left(\int_{A_n} |g|^{p^1} d\mu \right) \mu(A_n)^{\frac{p^1}{p}} . \mu(A_n)^{\frac{-p^1}{p}} \right)^{1/p^1}$$

$$\leq \left(\sum_{n=1}^{N} \|e_n{}^*\|^p \mu(A_n) \right)^{1/p} = \|\phi\| .$$

<u>Definition 2.24</u>: A <u>p- integral</u> map $T : E \to F$ where $1 \leq p < \infty$ is characterised by the fact that it has a factorisation

$$E \xrightarrow{P} C(X) \xrightarrow{I} L^p(\mu) \xrightarrow{Q} F$$

where X = Ball E^* in the weak * topology, μ is a positive measure on X , I is the identity and $\|P\|$, $\|Q\| \leq 1$.

The set $Ip(E,F)$ of all p- integral maps from E into F is a Banach space equipped with the norm

$Ip(T) = \inf \mu(X)^{1/p}$, the infimum being taken over all such factorisations. Notice $\|T\| \leq Ip(T)$.

<u>Proposition 2.25</u>: [10] $Np(E,F) \subseteq Ip(E,F)$ for all Banach spaces E and F, and $Ip(T) \leq Np(T)$.

<u>Proof</u>: We begin by showing each $T \in Np(E,F)$ has a factorisation

$$E \xrightarrow[P]{} \ell^\infty \xrightarrow[D]{} \ell^p \xrightarrow[Q]{} F \quad \text{for each } \varepsilon > 0$$

where $\|P\|$, $\|Q\| \leq 1$ and D is a diagonal operator with $D(\{\alpha_n\}) = \{\lambda_n \alpha_n\}$ with $(\sum_{n=1}^{\infty} |\lambda_n|^p)^{1/p} < N_p(T) + \varepsilon$.

Fix $\varepsilon > 0$ and choose a representation

$$Te = \sum_{n=1}^{\infty} \langle e_n^*, e \rangle f_n$$

with $(\sum_{n=1}^{\infty} \|e_n^*\|^p)^{1/p} < N_p(T) + \varepsilon$ and

$$\sup_{\|f^*\| \leq 1} (\sum_{n=1}^{\infty} |\langle f^*, f_n \rangle|^{p^1})^{1/p^1} \leq 1 .$$

Define $Pe = \dfrac{\langle e_n^*, e \rangle}{\|e_n^*\|}$, $P : E \to \ell^\infty$ and $\|P\| < 1$.

Define $D : \ell^\infty \to \ell^p$ by $D(\{\alpha_n\}) = \{\lambda_n \alpha_n\}$,

where $\lambda_n = \|e_n^*\|$, $\|D\| \leq (\sum |\lambda_n|^p)^{1/p}$.

Define $Q(\{\alpha_n\}) = \sum_{n=1}^{\infty} \alpha_n f_n$; $Q : \ell^p \to F$

and
$$\|Q(\{\alpha_n\})\| = \sup_{\|f^*\| \leq 1} |\sum_{n=1}^{\infty} \alpha_n \langle f_n, f^* \rangle|$$

$$\leq (\sum |\alpha_n|^p)^{1/p} \sup_{\|f^*\| \leq 1} (\sum_{n=1}^{\infty} |\langle f^*, f_n \rangle|^{p^1})^{1/p^1}$$

$$\leq \|\{\alpha_n\}\|_{\ell^p} \text{ as required.}$$

It only remains to show that D is integral and $I_p(D) < N_p(T) + \varepsilon$. For the composition of a bounded $S : D \to E$ and p- integral $T : E \to F$ is p- integral with $Ip(TS) \leq \|S\| \, I_p(T)$, and the composition of a p- integral $T : E \to F$ and bounded $R : F \to G$ is p- integral and $I_p(RT) \leq \|R\| \, I_p(T)$.

D is p- integral. ℓ^∞ can be identified with $C(X)$ for some compact Hausdorff X . To $\phi \in (\ell^\infty)^*$ defined by $\phi(\{\alpha_n\}) = \sum_{n+1}^{\infty} \alpha_n |\lambda_n|^p$ corresponds a positive measure μ on X and the identity map from $C(X)$ to $L^p(\mu)$ is p- integral. Thus D is p- integral and $I_p(D) \leq N_p(T) + \varepsilon$. Therefore T is p-integral and $I_p(T) \leq N_p(T) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $I_p(T) \leq N_p(T)$.

We shall now obtain some results going the other way.

Theorem 2.26: Let E and F be Banach spaces and suppose E^* has the RNP. Then every p- integral map from E to F is p-nuclear and

$$I_p(T) = N_p(T) .$$

Proof: Suppose $T : E \to F$ is p- integral. Then given $\varepsilon > 0$, there is a factorisation

$$E \xrightarrow[P]{} C(X) \xrightarrow[I]{} L^p(\mu) \xrightarrow[Q]{} F$$

such that $\|P\| \leq 1$, $\|Q\| \leq 1$, I is the identity map from $C(X) \to L^p(\mu)$ and $\mu(X)^{1/p} < I_p(T) + \varepsilon$.

For each A, a μ- measurable subset of X, let us define m(A) as follows:-

$$\langle m(A), e \rangle = \int_A (Pe)(x) \, d\mu(x) .$$

$m(A) : E \to R$ is linear and

$$\sup_{e \in \text{Ball } E} |\langle m(A), e \rangle| \le \int_A d\mu(x) = \mu(A) .$$

So $m(A)$ is a continuous linear functional on E. We shall show m is a finite, μ - continuous E^* - valued measure.

Finite additivity is clear. Let $\{A_i\}_{i=1}^{\infty}$ be disjoint μ- measurable subsets of X . We wish to show $m(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} m(A_i)$.

It will be enough to show $m(\bigcup_{i=n+1}^{\infty} A_i) \to 0$ in norm as $n \to \infty$.

$$\text{But } |\langle e, m(\bigcup_{i=n+1}^{\infty} A_i) \rangle| = \left| \int_{\bigcup_{i=n+1}^{\infty} A_i} (P_e)(x) \, d\mu(x) \right|$$

$$\le \|e\| \int_{\bigcup_{i=n+1}^{\infty} A_i} d\mu(x)$$

$$= \|e\| \, \mu(\bigcup_{i=n+1}^{\infty} A_i) \to 0$$

as $n \to \infty$, since μ is a finite measure.

Since $\|m(A)\| \le \mu(A)$ for all A, m is a finite, μ- continuous measure.

Since E^* has the RNP, there is a $\phi : X \to E^*$ μ- strongly measurable such that

$$m(A) = \int_A \phi(x) \, d\mu(x) \quad \text{for each measurable } A .$$

$$\text{Then } \langle m(A), e \rangle = \int_A \langle e, \phi(x) \rangle \, d\mu(x) \quad \text{for each measurable } A.$$

So $(IP_e)(x) = \langle e, \phi(x) \rangle$ μ - almost everywhere.

ϕ is μ- strongly measurable and $\|\phi\| \le 1$ μ p.p. This follows from the fact that $\|m(A)\| \leq \mu(A)$ for all μ measurable A.

By theorem 2.23, IP is p- nuclear and $N_p(IP) \leq (\int \|\phi(x)\|^p \, d\mu)^{1/p}$

$$\leq \mu(X)^{1/p} \leq I_p(T) + \varepsilon .$$

$T = QIP$ is p-nuclear and proposition 2.25 and the fact $\varepsilon > 0$ was arbitrary allows us to conclude $N_p(T) \leq I_p(T)$.

Corollary 2.27: [9] If E is reflexive, every p- integral map from E to F is nuclear.

Proof: E^*, being reflexive, has the RNP.

Corollary 2.28: [9] If E^* is separable, every p- integral map is p- nuclear.

Proof: E^*, being separable, has the RNP.

Theorem 2.29: If E and F are Banach spaces and F has the RNP, then every integral map $T : E \to F$ is nuclear.

Proof: T has the usual factorisation

$$E \xrightarrow{p} C(X) \xrightarrow{I} L^1(\mu) \xrightarrow{Q} F$$

where $\varepsilon + I(T) = I_1(T) + \varepsilon > \mu(X)$.

Define m on the μ- measurable subsets of X by $m(A) = Q\chi_A$ where χ_A = characteristic function of the set A.

Then $\|m(A)\| \leq \mu(A)$ for all A. (10)

Also $\|m(\bigcup_{i=n+1}^{\infty} A_i)\| \leq \mu(\bigcup_{i=n+1}^{\infty} A_i) \to 0$ as $n \to \infty$

where the A_i are disjoint μ- measurable subsets of X.

Therefore $\quad m(\bigcup_{i=1}^{\infty} A_i) \quad = \quad \sum_{i=1}^{\infty} m(A_i)$.

m is clearly μ- continuous and of bounded variation , using (10).

Since F has the RNP, there is a μ- strongly measurable $\phi : X \to F$ such that

$$m(A) = \int_A \phi(x)\, d\mu(x) \quad \text{for each measurable } A.$$

Moreover, since $\|m(A)\| \leq \mu(A)$ for all A, $\|\phi\| \leq 1$ μ- almost everywhere.

Now $Qg = \int g(x)\, \phi(x)\, d\mu(x)$ for each $g \in L^1(\mu)$. It clearly suffices to prove this for simple functions.

If $g = \sum_{i=1}^{n} c_i \chi_{A_i}$ $(c_i \in R$, A_i disjoint sets$)$ we have

$$Q\,(\sum_{i=1}^{\infty} c_i \chi_{A_i}) = \sum_{i=1}^{n} c_i\, m(A_i)$$

$$= \sum_{i=1}^{n} c_i \int_{A_i} \phi(x)\, d\mu(x)$$

$$= \int (\sum_{i=1}^{n} c_i \chi_{A_i} \phi(x))\, d\mu(x)$$

$$= \int g(x)\, \phi(x)\, d\mu(x) .$$

In particular for each $g \in C(X)$

$$QIg = \int g(x)\, \phi(x)\, d\mu(x) .$$

QI is nuclear. Choose a sequence of uniformly bounded simple functions $\{\phi_n\}$ which tend to ϕ pointwise almost everywhere and in $L^1{}_F(\mu)$.

Denote by Q_{ϕ_n} the operator from $L^1(\mu)$ to F corresponding to ϕ_n .

$$\text{If } \phi_n = \sum_{k=1}^{m} f_k^{(n)} \chi_{A_k}{}^{(n)} \quad \left(f_k^{(n)} \varepsilon\ F,\ A_k^{(n)} \text{ disjoint} \right)$$

$$Q_{\phi_n} Ig = \sum_{k=1}^{m} f_k^{(n)} \int g(x)\ \chi_{A_k}{}^{(n)}\ d\mu(x).$$

So $Q_{\phi_n} I$ is nuclear and

$$N(Q_{\phi_n} I) \leq \sum_{k=1}^{m} \|f_k^{(n)}\|\ \mu(A_k^{(n)})$$

$$= \int_X \|\phi_n\|\ d\mu \ .$$

Since $\|Q_{\phi_n} Ig - Q\,Ig\| \leq \|g\|_\infty \int \|\phi_n - \phi\|\ d\mu$,

$$Q_{\phi_n} I \to Q_\phi I \quad \text{in norm.}$$

Lastly we show the sequence $\{Q_{\phi_n}\}$ is Cauchy in nuclear norm.

We may write

$$\phi_n - \phi_m = \sum_{k=1}^{p} f_k\ \chi_{A_k} \quad \left(f_k \,\varepsilon\, F,\ A_k \text{ disjoint}\right).$$

Then $N(Q_{\phi_n} I - Q_{\phi_m} I) \leq \sum_{k=1}^{p} \|f_k\|\ \mu(A_k)$

$$= \int \|\phi_n - \phi_m\|\ d\mu \ .$$

Therefore QI is nuclear with nuclear norm $\leq \mu(X) \leq I(T) + \varepsilon$.
By proposition 2.11 $T = QIP$ is nuclear and $N(T) \leq I(T) + \varepsilon$,
ε arbitrary. So $N(T) \leq I(T)$.

Corollary 2.30: [5] If either F is reflexive or is a separable dual space, then every integral map T from E to F is nuclear.

Proof: F has the RNP.

PART TWO

CHAPTER 3

Prediction Theory of Doubly Stationary Processes

In this chapter we shall outline the basic Prediction Theory of doubly stationary processes, as given by Helson and Lowdenslager in [21] and [22]. We state their characterisation, for the absolutely continuous case of processes as type 1, 2 or 3, and then obtain an example of a process which is of type 2 for all irrational α .

Lurking in the background throughout what follows will be a probability space $(\Omega, \sum, P)$, Ω is a set , $\sum$ a σ- algebra of subsets of Ω and P is a positive measure defined on $\sum$ such that $P(\Omega) = 1$, a probability measure. A random variable on Ω is a complex valued function defined on Ω , measurable with respect to the probability measure P .

By the expectation of a random variable X , we mean the integral $\int_\Omega X(\omega)\, dP(\omega)$ provided this exists i.e. provided

$\int_\Omega |X(\omega)|\, dP(\omega) < + \infty$. The expectation is denoted $\mathcal{E}(X)$. A square summable random variable X is one for which

$$\mathcal{E}(|X|^2) = \int_\Omega |X(\omega)|^2\, dP(\omega) < + \infty .$$

Definition 3.1: A doubly stationary stochastic (random) process (function, sequence) is a sequence $\{ X_{mn} : (m,n) \in \mathbb{Z} \times \mathbb{Z}\}$ of square summable random variables defined on Ω which satisfy

(i) $\mathcal{E}(X_{mn}) = 0$ for all $(m,n) \in \mathbb{Z} \times \mathbb{Z}$

(ii) $\mathcal{E}(X_{mn}\,\overline{X}_{k\ell})$ is a function of the differences $m-k$, $n-\ell$ only. (1)

Thus $\rho(m,n) = \mathcal{E}(X_{mn}\,\overline{X}_{oo}) = \mathcal{E}(X_{m+k,\,n+\ell}\,\overline{X}_{k\ell})$

$$(k,\ell)\,\varepsilon\;\; \underset{\sim}{Z} \times \underset{\sim}{Z} \tag{2}$$

gives a well-defined function of $\underset{\sim}{Z} \times \underset{\sim}{Z}$ to $\underset{\sim}{C}$, and this function is positive definite in the sense that if $a_1, \ldots a_k$ are complex numbers, and $(m_1,n_1), \ldots \ldots (m_k,n_k)$ elements of $\underset{\sim}{Z} \times \underset{\sim}{Z}$ then

$$\sum_{i,j=1}^{k} a_i\,\overline{a}_j\;\rho(m_i - m_j,\, n_i - n_j) \;\geq\; 0. \tag{3}$$

This follows since we have

$$0 \leq \mathcal{E}\left(\left| \sum_{i=1}^{k} a_i X_{m_i\,n_i} \right|^2 \right) = \sum_{i,j=1}^{k} a_i\,\overline{a}_j\,\mathcal{E}(X_{m_in_i}\,\overline{X}_{m_jn_j})$$

$$= \sum_{i,j=1}^{k} a_i\,\overline{a}_j\;\rho(m_i - m_j,\, n_i - n_j).$$

In the sense of harmonic analysis, $\underset{\sim}{Z} \times \underset{\sim}{Z}$ is the dual group of the Torus group T^2, where $T^2 = \{(e^{ix}, e^{iy}) : 0 \leq x < 2\pi,\; 0 \leq y < 2\pi\}$.

The Herglotz-Bochner-Weil Theorem allows us to deduce the existence of a finite, positive, regular Borel measure μ on the Torus such that

$$\rho(m,n) = \int_{T^2} e^{i(mx+ny)}\,d\mu(x,y), \left((m,n)\;\varepsilon\;\; \underset{\sim}{Z} \times \underset{\sim}{Z}\right) \tag{4}$$

Before discussing prediction problems concerned with such a process we need to establish some idea of 'past' and 'future' for such a process. This is tantamount to imposing an order on $\underset{\sim}{Z} \times \underset{\sim}{Z}$, the dual of the Torus. There are many different order relations we could impose, but we shall be interested in the following type.

Fix some irrational number $\alpha \in R$ and define $(m,n) \geq_{\alpha} (0,0)$ if and only if $m+n\alpha \geq 0$ for $(m,n) \in Z \times Z$. Where it is clear which α we are referring to, we shall often drop the α to obtain $(m,n) \geq (0,0)$.

For each α, this gives a well-defined, archimedean ordering of the lattice points in the plane, which divides them into two disjoint semi-groups, one being the positive elements (the positive half-plane) i.e. the set $\{(m,n) : (m,n) \geq_{\alpha} (0,0)\}$, the other the negative elements $\{(m,n) : (m,n) <_{\alpha} (0,0)\}$.

A typical prediction problem is the following - knowing values of the process $\{X_{mn}\}$ in the past, say at some fixed $\omega \in \Omega$, can we predict the value at some point $(0,0)$, say. That is given $\{X_{mn}(\omega) : (m,n) < (0,0)\}$ can we predict $X_{oo}(\omega)$?

In general the predicted value $\tilde{X}_{oo}(\omega)$ will be some function of the values $X_{mn}(\omega)$ where $(m,n) < (0,0)$. We immediately face the problem of deciding which is the 'best' predicted value. If we took, for example, $|X_{oo}(\omega) - \tilde{X}_{oo}(\omega)|$ it is clear that, as this differs with differing values of ω, it is not a good indicator of the quality of the predicted value $\tilde{X}_{oo}(\omega)$.

In a probability theory context, the quality of the predicted value as an estimate of $X_{oo}(\omega)$ can only be evaluated by averaging in some sense over all $\omega \in \Omega$, and the usual method used is that of 'least squares', so we consider $\mathcal{E}(|X_{oo} - \tilde{X}_{oo}|^2)$.

We shall restrict also to the case where $\tilde{X}_{oo}$ is a finite linear combination of the X_{mn} with $(m,n) < (0,0)$. The justification for this is twofold. In practical applications such combinations are easily handled. More significantly, for one of the most important examples, namely those processes with normal distributions, the best predicted value is the best linear prediction.

So we can now formulate our prediction problem as : minimise

$$\mathcal{E}\left(\left|X_{oo} - \sum_{(m,n) < (0,0)} a_{mn} X_{mn}\right|^2\right) \quad \text{over all finite}$$

linear combinations of X_{mn} with $(m,n) < (0,0)$.

Since $\mathcal{E}\left(\left|X_{oo} - \sum_{(m,n) < (0,0)} a_{mn} X_{mn}\right|^2\right)$

$$= \mathcal{E}(X_{oo}\overline{X}_{oo}) - \sum a_{mn}\,\mathcal{E}(X_{mn}\overline{X}_{oo}) - \sum \overline{a}_{mn}\,\mathcal{E}(X_{oo}\overline{X}_{mn}) + \sum a_{mn}\overline{a}_{rs}\,\mathcal{E}(X_{mn}\overline{X}_{rs})$$

$$= \rho(0,0) - \sum a_{mn}\,\rho(m,n) - \sum \bar{a}_{mn}\,\bar{\rho}(m,n) + \sum a_{mn}\,\bar{a}_{rs}\,\rho(m-r,\ n-s),$$

we obtain, using (4),

$$= \int_{T^2} \left|1 - \sum_{(m,n) < (0,0)} a_{mn}\, e^{i(mx + ny)}\right|^2 d\mu(x,y) . \qquad (5)$$

Thus the problem of approximating X_{oo} by finite linear combinations of X_{mn} with $(m,n) < (0,0)$ is equivalent to minimising the integral (5).

This problem, a generalisation of the one variable problem of Szegö, was solved by Helson and Lowdenslager in [21] . In what follows σ will denote normalised Lebesgue measure on the Torus i.e. $d\sigma = \frac{1}{4\pi^2}\,dx\,dy$, $L^p(\sigma)$, for $1 \leq p \leq \infty$, will denote the usual L^p class on the Torus, and for the finite, positive, Borel measure μ on the torus we have the usual Lebesgue decomposition:

$$d\mu = \omega\, d\sigma + d\mu_s ,$$

where $\omega \geq 0$ is in $L^1(\sigma)$ - a weight function on the Torus - and $d\mu_s$ is singular with respect to Lebesgue measure.

Theorem 3.2: [21] Let μ be a finite, positive Borel measure on the Torus with Lebesgue decomposition $d\mu = \omega \, d\sigma + d\mu_s$.

Then
$$\exp\left(\int \log \omega \, d\sigma\right) = \inf \int \left|1 - \sum_{(m,n) < (0,0)} a_{mn} e^{i(mx+ny)}\right|^2 d\mu(x,y) \tag{6}$$

where the infimum is taken over all finite sums of the form

$$\sum_{(m,n) < (0,0)} a_{mn} e^{i(mx+ny)} .$$

The left side of (6) is to be interpreted as zero in the case that $\int \log \omega \, d\sigma = -\infty$.

Helson and Lowdenslager [21] [22] (for other accounts see Helson [20], Rudin [28] and Gamelin [18]) proceed to obtain many results analogous to the one variable case. In so doing they introduce the idea of a 'generalised analytic function' on the Torus.

A function is defined to be analytic if all its Fourier coefficients a_{mn} given by

$$a_{mn} = \int_{T^2} f \, e^{-i(mx+ny)} d\sigma , \quad (m,n) \in \underset{\sim}{Z} \times \underset{\sim}{Z} ,$$

vanish off the half-plane $(m,n) \underset{\alpha}{\geq} (0,0)$. Notice the dependence on α.

The algebra $A_\alpha = \{ f : f$ is continuous on T^2 and f is analytic$\}$ replaces in this context the disc algebra. We can define Hardy spaces $H^p_\alpha(\sigma)$ by

$$H^p_\alpha(\sigma) = L^p(\sigma) \text{ closure of the algebra } A_\alpha \text{ if } 1 \leq p < \infty .$$

If $p = \infty$, $H^{\infty}_{\alpha}(\sigma)$ is the weak* closure of A_{α} in $L^{\infty}(\sigma)$. (In situations where it will lead to no confusion we shall often drop the suffix α.)

A function $f \in H^p(\sigma)$ is <u>inner</u> if $|f| = 1$ almost everywhere on the Torus; f is <u>outer</u> if

$$\int \log |f| \, d\sigma = \log \left| \int f \, d\sigma \right| .$$

$f \in H^p_{\alpha}(\sigma)$ is outer if, and only if, the set $f A_{\alpha} = \{fg : g \in A_{\alpha}\}$ is $L^p(\sigma)$ dense in $H^p_{\alpha}(\sigma)$ $(1 \leq p < \infty)$.

If $f \in H^1(\sigma)$ satisfies $\int f \, d\sigma \neq 0$ there are $g, h \in H^2_{\alpha}(\sigma)$ such that g is outer and h inner and $f = gh$. Helson and Lowdenslager also obtain a variant of the F. and M. Riesz theorem.

However a fundamental difference arises in the case of the Torus. In the one variable situation, where the problem was to decide whether 1 is in the closed span of the set $\{e^{ik\theta} : k < 0\}$ in $L^2(\mu)$ where μ is a finite positive measure on the circle, if 1 is in this closed span it is clear that $e^{i\theta}$ is in the closed span of $\{e^{ik\theta} : k < 1\}$ and so in the closed span of $\{e^{ik\theta} : k < 0\}$.

Translated into terms of a singly stationary process $\{X_k : k \in Z\}$ it means X_1 can be predicted exactly from $\{X_k : k < 0\}$ if X_o can, and the same is then true for any X_n with $n > 0$. This follows because, having predicted one value, there is an obvious 'next' point to predict.

In the case of the Torus, there is no 'next' point to predict, so we may face the situation where although X_{oo} can be predicted exactly from $\{X_{k\ell} : (k,\ell) < (0,0)\}$ X_{mn} cannot, for any $(m,n) > (0,0)$.

To analyse this situation we shall form an analogue of the Wold decomposition.

A closed subspace M of $L^2(\mu)$ where μ is a finite, positive measure on T^2 will be called <u>invariant</u> if $e^{i(mx+ny)} f \in M$ whenever $f \in M$ and $(m,n) \in Z \times Z$. $L^2(\mu)$ being a Hilbert space, there is a projection $P : L^2(\mu) \to M$ and we have

<u>Lemma 3.3</u>: [22] : P has the form $Pf = \chi f$ where $\chi \in L^2(\mu)$ takes only the values 0 and 1, $\chi = P1$.

<u>Proof</u>: Put $\chi = P1 \in M$.

$$\int (\chi - |\chi|^2)\, e^{i(mx+ny)} d\mu = \int e^{i(mx+ny)} \chi(1 - \bar{\chi})\, d\mu .$$

$e^{i(mx+ny)}\chi \in M$, $1 - \bar{\chi} \in M^{\perp}$; so the above is zero for all $(m,n) \in \underset{\sim}{Z} \times \underset{\sim}{Z}$.

Thus $\chi - |\chi|^2 = 0$ μ-almost everywhere. χ takes the values 0 and 1 almost everywhere.

Let us define $\wp_{mn}$ = closed span of the set $\{e^{i(kx+\ell y)} : (k,\ell) < (m,n)\}$ in the space $L^2(\mu)$. The closed subspace $H_3 = \cap \wp_{mn}$ (the intersection being over all $(m,n) \in \underset{\sim}{Z} \times \underset{\sim}{Z}$) is an invariant subspace which we shall call the <u>remote past</u> of the process $\{X_{mn}\}$. The corresponding projection function will be denoted χ_3.

If $1 \notin \wp_{oo}$, let Y_{oo} be the part of 1 orthogonal to $\wp_{oo}$. $1 = Y_{oo} + Z_{oo}$ where $Y_{oo} \in \wp_{oo}^{\perp}$, $Z_{oo} \in \wp_{oo}$

We may define $Y_{mn} = e^{i(mx+ny)} Y_{oo}$, and it is clear that Y_{mn} is the part of $e^{i(mx+my)}$ orthogonal to $\wp_{mn}$. The set $\{Y_{mn} : (m,n) \in \underset{\sim}{Z} \times \underset{\sim}{Z}\}$ is an orthogonal set in $L^2(\mu)$ and its closed linear span H_1 is a closed invariant subspace. χ_1 is its projection function.

The orthogonal complement of $H_1 \oplus H_3$ may not be zero - it constitutes a third closed invariant subspace H_2, with projection function χ_2.

So $L^2(\mu) = H_1 \oplus H_2 \oplus H_3$ and $\chi_1 + \chi_2 + \chi_3 = 1$, $\chi_j \chi_k = 0$ almost everywhere $(j \neq k)$. A process in which only one of the summands is non-zero is said to be <u>pure</u> and of type 1,2,3 depending on whether H_1, H_2 or H_3 is non-zero. More picturesque names are : type 1 - innovation process, type 2: evanescent process, type 3 - deterministic process.

If $d\mu_j = \chi_j d\mu$, since

$$\int (\chi_j e^{i(mx+ny)}) (\overline{\chi}_j e^{-i(kx+\ell y)}) d\mu = \int e^{i[m-k)x+(n-\ell)y]} d\mu_j ,$$

it is clear $\{\chi_j e^{i(mx+ny)}\}$ is stationary. $(j = 1,2,3)$.

<u>Theorem 3.4</u> [22] : $\{ \chi_j e^{i(mx+ny)}\}$ is purely of type j.

Thus each process decomposes into three pure orthogonal subprocesses. However the decomposition theorem supplies no information about finding the measures $d\mu_j = \chi_j d\mu$; nor does it tell us how to find the subspaces H_1, H_2, H_3 ; nor does it explain how the $d\mu_j$ are related to the usual Lebesgue decomposition of the measure μ. These questions were considered by Helson and Lowdenslager in their second paper [22], where they focussed attention on the case of measures absolutely continuous with respect to Lebesgue measure. The justification for this is the following result.

<u>Theorem 3.5</u> [22] : If χ , χ^1 are functions in $L^2(\mu)$ satisfying $\chi = 1$ almost everywhere $(d\sigma)$, $\chi^1 = 1$ almost everywhere $(d\mu_s)$, $\chi . \chi^1 \equiv 0$, then $\chi \wp_{mn}$, $\chi^1 \wp_{mn}$ are closed subspaces of $\wp_{mn}$ for all $(m,n) \in \mathbb{Z} \times \mathbb{Z}$.

<u>Proof</u>: It is sufficient to consider the case $(m,n) = (0,0)$. Suppose $f \in L^2(\mu)$ is orthogonal to $\wp_{oo}$. Then for all $(m,n) < (0,0)$

$$\int e^{i(mx+ny)} \, \overline{f} \, d\mu = 0$$

The measure $\overline{f}\, d\mu$ is therefore of analytic type, that is, its Fourier coefficients vanish off a half plane. By the variant of the F. and M. Riesz Theorem proved in [21], the same is true of the absolutely continuous and singular parts separately.

Thus $\chi f \perp \wp_{oo}$, $\chi^1 f \perp \wp_{oo}$ and equivalently $f \perp \chi \wp_{oo}$, $f \perp \chi^1 \wp_{oo}$. Thus $\chi \wp_{oo}$, $\chi^1 \wp_{oo} \subseteq \wp_{oo}$.

Since $\wp_{oo} = \chi \wp_{oo} + \chi^1 \wp_{oo}$ and the summands are mutually orthogonal we obtain the desired conclusion.

Thus questions about the second order properties of the process, can be resolved into questions about the absolutely continuous and singular parts separately.

<u>Cauchy Measures and the Absolutely Continuous case</u>:

The main result of Helson and Lowdenslager in their second paper was to recognise the crucial role played by a certain class of measures in classifying absolutely continuous processes as type 1,2 or 3.

Under the action $t \longrightarrow (e^{-it}, e^{-i\alpha t})$ $\underset{\sim}{R}$ embeds isomorphically as a dense subgroup of the Torus. The Cauchy measures μ_r $(0 < r < 1)$ live on this line and have the form

$$d\mu_r(t) = \frac{y\ dt}{\pi(t^2 + y^2)} \qquad r = e^{-y}\ (0 < r < 1)$$

Indeed we obtain a whole family of lines by the action $t \longrightarrow (e^{i(x-t)}, e^{i(y-\alpha t)})$ as (x,y) runs over the Torus.

For a measurable function f on T^2 we may form the convolution

$$(\mu_r * f)\ (x,y) = \int_R f(x - t, y - \alpha t)\ d\mu_r(t)\ .$$

(For ease, from now on, when discussing the Torus we shall often replace (e^{ix}, e^{iy}) by (x,y).)

It is clear that this convolution is finite if and only if the function $f_{xy}(t), = f(x - t, y - \alpha t)$ is in L^1 of the Cauchy measure $\left(\frac{1}{\pi} \frac{dt}{1 + t^2} \right)$ on $\underset{\sim}{R}$.

The result obtained by Helson and Lowdenslager is the following:

Theorem 3.6: [22] With a fixed order (i.e. fixed α) and $\omega \geq 0$ an integrable weight function on the Torus, the process $\{ e^{i(mx+ny)} \}$ is pure.

It is of type 1 if $\int \log \omega\ d\sigma > -\infty$, (7)

of type 2 if (7) fails but

$$(\mu_r * \log \omega)\ (x,y) > -\infty \text{ almost everywhere }, \quad (0 < r < 1) \qquad (8)$$

of type 3 if (8) fails when necessarily

$$(\mu_r * \log \omega)\ (x,y) = -\infty \text{ almost everywhere }. \quad (0 < r < 1) \qquad (9)$$

(Equivalently of type 2 if

$$\log \omega(x-t,\ y-\alpha t) \ \varepsilon \ L^1\left(\frac{dt}{1+t^2}\right) \text{ almost everywhere}, \qquad (8^1)$$

of type 3 if $\log\ (x-t,\ y-\alpha t) \not\varepsilon \ L^1\left(\frac{dt}{1+t^2}\right)$ almost everywhere) (9^1)

For absolutely continuous measures a complete classification has been obtained, therefore.

In [25] Muhly classifies those measures μ for which $\cap \wp_{mn} = H_3 = \{0\}$. They are these measures μ for which

(1) μ is quasi-invariant i.e. under the action $(x,y) \longrightarrow (x-t,\ y-\alpha t)$ the null sets of μ are preserved.

(2) Defining $\mu_t(A) = \mu\ (A - (t,\alpha t))$, A a Borel subset of T^2, and defining

$$\Theta(t,x,y) = \frac{d\mu_t}{d\mu}(x,y), \text{ for almost all } (x,y),$$

$$\log\ \Theta(t,x,y) \ \varepsilon \ L^1\left(\frac{dt}{1+t^2}\right).$$

This naturally agrees with Helson and Lowdenslager in that if $d\mu = \omega d\sigma$ (1) means $\omega \neq 0$ almost everywhere and in (2)

$$\Theta(t,x,y) = \omega\ (x-t,y-\alpha t) \ / \ \omega(x,y) .$$

So $\cap \wp_{mn} = \{0\}$ if and only if $\log \omega(x-t,y-\alpha t) \ \varepsilon \ L^1\left(\frac{dt}{1+t^2}\right)$

for almost all (x,y), as before.

The type of a process depends, even in the absolutely continuous case, on which α we are considering. Probabilistic considerations would turn our attention to those processes which are of a fixed type for all directions α .

The condition $\log \omega \in L^1(\sigma)$ is, of course, α- independent, so type 1 processes are α- independent. Helson and Lowdenslager produce an example of a process which is type 2 for some fixed α. It is a natural question as to whether there exists a process which is type 2, for all irrational α.

We shall now construct an example of a weight function $\omega \in L^\infty$ which gives a type 2 process for each α.

In view of (7) and (8^1) ω will also have to satisfy

(a) $\int \log \omega \, d\sigma = -\infty$

(b) For each irrational α, we have

$$\int \frac{\log \omega(\theta + t, \phi + \alpha t)dt}{1+t^2} > -\infty \quad \text{for almost all } (\theta,\phi) \in T^2 .$$

Step 1: For ease of calculation, the function will be constructed on the unit square $[0,1] \times [0,1]$.

Define $\omega(x,y) = \exp(-f(x,y))$,

where $f(x,y) = x^{-3}$ if $x^2 < y < 2x^2$,

$= 0$ elsewhere.

We have $\omega \in L^\infty$ clearly.

Since $\int_0^1 \int_{x^2}^{2x^2} f(x,y)dx\,dy = \int_0^1 x^{-1}\,dx = \infty$, the function is not integrable over the square, so $\log \omega \notin L'$. If Γ is any straight line segment in the square, the integral of f along Γ is finite.

We must ensure, therefore, that for each irrational α

$$\int \frac{f(\theta+t, \phi + \alpha t)}{1 + t^2}\,dt < +\infty \quad \text{for almost all } (\theta,\phi) \in [0,1]^2 .$$

It clearly suffices to show that for each irrational α ,

$\int \frac{f(t, \phi + \alpha t)}{1 + t^2}$ is finite, for almost all $\phi \in [0,1]$.

Step 2: Fix α , and choose sequences of integers $\{p_k\}$, $\{q_k\}$ such that

$$\left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_k^2} \quad (q_{k+1} > q_k \geq 0)$$ (See e.g. Hardy and Wright [19])

The sequence of functions $\{g_k\}$ defined by

$g_k(\phi) = \int_{-q_k}^{q_k} \frac{f(t,\phi + \alpha t)dt}{1 + t^2}$ is an increasing sequence of measurable functions. If the sequence converges pointwise, the limit is clearly $\int \frac{f(t, \phi + \alpha t)dt}{1 + t^2}$. It suffices to show that, except for ϕ in a set of measure zero, the functions converge to a function $g(\phi)$ which is finite almost everywhere.

Set $E_{kn} = \left\{ \phi : \int_{-q_k}^{q_k} \frac{f(t, \phi + \alpha t)dt}{1 + t^2} > n \right\}$,

$E = \left\{ \phi : \int_R \frac{f(t,\phi + \alpha t)dt}{1 + t^2} = \infty \right\}$.

$$\text{Then } E = \bigcap_{n=1}^{\infty} \left(\bigcup_{k=1}^{\infty} E_{kn} \right) . \qquad (10)$$

For, if $\phi \in E$, then $g_k(\phi) \to \infty$. So for each $n \in \mathbb{N}$ there is a k_n such that $g_{k_n}(\phi) > n$, so for each $n \in \mathbb{N}$ $\phi \in \bigcup_{k=1}^{\infty} E_{kn}$.

Conversely, if ϕ is in the right hand side of (10), for each n, there is a k_n such that $g_{k_n}(\phi) \geq n$.

Since $\{g_n\}$ is an increasing sequence $g_k(\phi) \geq n$ for all $k \geq k_n$. So $g(\phi) \geq n$, for each n. Hence $\phi \in E$.

Since $E_{kn} \subseteq E_{k+1,n}$, if m = Lebesgue measure on $[0,1]$,

$$m\left(\bigcup_{k=1}^{\infty} E_{kn}\right) = \lim_{k\to\infty} m(E_{kn}).$$

Thus $$m(E) \leq \inf_{n \in N} \left(\lim_{k\to\infty} m(E_{kn})\right). \qquad (11)$$

We must estimate $m(E_{kn})$.

Step 3: We shall first estimate $\displaystyle\int_{-q_k}^{q_k} \frac{f(t, \phi + \alpha t)dt}{1 + t^2}$ in terms of k and ϕ. As t runs from $-q_k$ to q_k, we trace out on the unit square straight line segments with slope α.

There are $\leq 2(p_k + q_k) + 1$ straight line segments crossing the unit square which comprise the above integral. We suppose now α is positive (a similar analysis can be carried out for the case $\alpha < 0$).

We split the straight line segments into two sets, those emanating from the x-axis and those emanating from the y-axis.

If we denote by $\{t\}$ the fractional part of t where t is a real number, a line emanating from the x-axis corresponds to $\{\alpha t + \phi\} = 0$; one emanating from the y-axis to $\{t\} = 0$.

Consider firstly these lines coming from the y-axis i.e. corresponding to $\{t\} = 0$ in the internal $[-q_k, q_k]$. There are $\leq 2q_k$ such lines given by $t=j$, $j\varepsilon \{-q_k, \ldots\ldots 0,1 \ldots\ldots q_k -1\}$.

If e_j, f_j are the x- co-ordinates of the points of intersection of the jth line segment with the curves $y=x^2$, $y=2x^2$ then

$$e_j^2 = \alpha e_j + c_j , \tag{12}$$

$$2f_j^2 = \alpha f_j + c_j . \tag{13}$$

c_j = intersection on the y-axis

= $\{j\alpha + \phi\}$.

Now $$\int_{f_j}^{e_j} x^{-3}\, dx = \frac{1}{2}\left(-\frac{1}{e_j^2} + \frac{1}{f_j^2}\right) \le \frac{1}{2f_j^2} \le \frac{1}{c_j}$$

since $\alpha > 0$, $f_j \le e_j$ and (13) holds.

So this part of the integral is

$$\le \sum_{j=0}^{q_k-1} \frac{1}{1 + j^2} \frac{1}{\{j\alpha + \phi\}} + \sum_{j=-q_k}^{-1} \frac{1}{1 + (j+1)^2} \frac{1}{\{j\alpha + \phi\}} . \tag{14}$$

Consider now those lines emanating from the x-axis corresponding to $\{\alpha t + \phi\} = 0$, $t \in [-q_k, q_k]$; denote by x_j the x- co-ordinate of the intersection of the jth line, ($j \in \{-p_k, \ldots.. -1, 0, \ldots.. p_k, p_k+1\}$ with the axis.

e_j , f_j , which are the x- co-ordinates of the points of intersection of the jth line with the curves $y=x^2$, $y=2x^2$ respectively, are given by

$$e_j = \frac{\alpha - \sqrt{\alpha^2 - 4\alpha x_j}}{2} , \tag{15}$$

i.e. one solution of $x^2 = \alpha x - \alpha x_j$, (16)

and $$f_j = \frac{\alpha - \sqrt{\alpha^2 - 8\alpha x_j}}{4} , \tag{17}$$

i.e. one solution of $2x^2 = \alpha x - \alpha x_j$. (18)

(Since $\alpha > 0$, e_j, f_j are the smaller roots of (16),(18) respectively.)

If e_j^1, f_j^1 are the other roots of (16), (18) we have

$$\int_{e_j}^{f_j} x^{-3}dx = \frac{1}{2}\left(\frac{1}{e_j^2} - \frac{1}{f_j^2}\right)$$

$$= \frac{1}{2}\left[\frac{{e_j^1}^2}{(e_je_j^1)^2} - \frac{{f_j^1}^2}{(f_jf_j^1)^2}\right]. \qquad (19)$$

Now $e_je_j^1 = \alpha x_j$, $f_jf_j^1 = \frac{\alpha x_j}{2}$, so we have

$$(19) = \frac{1}{2\alpha^2 x_j^2}\left[(e_j^1)^2 - 4(f_j^1)^2\right]$$

$$= \frac{1}{2\alpha^2 x_j^2}\left[\frac{(\alpha + \sqrt{\alpha^2 - 4\alpha x_j})^2}{4} - 4\frac{(\alpha + \sqrt{\alpha^2 - 8\alpha x_j})^2}{16}\right]$$

$$= \frac{1}{8\alpha^2 x_j^2}\left[2\alpha\sqrt{\alpha^2 - 4\alpha x_j} - 2\alpha\sqrt{\alpha^2 - 8\alpha x_j} + 4\alpha x_j\right]$$

$$= \frac{1}{8\alpha^2 x_j^2}\left[2\alpha^2\left(\sqrt{1 - \frac{4x_j}{\alpha}} - \sqrt{1 - \frac{8x_j}{\alpha}}\right) + 4\alpha x_j\right]. \qquad (20)$$

For $t \in [0,\frac{1}{2}]$,

$$\frac{\sqrt{1-t} - \sqrt{1-2t}}{t} = \frac{1}{\sqrt{1-t} + \sqrt{1-2t}} \leq \sqrt{2}.$$

So $(20) \leq \frac{1}{8\alpha^2 x_j^2}\left[4\alpha x_j + \sqrt{2}\, 2\alpha^2 \frac{4\alpha x_j}{\alpha^2}\right]$

$$= \frac{1}{8\alpha x_j}(4 + 8\sqrt{2}).$$

Now, certain lines may cut the curves more than once, within the unit square. For those the second part is estimated using

$$\int_{f_j^1}^{e_j^1} x^{-3}dx = \frac{1}{2}\left(\frac{1}{{f_j^1}^2} - \frac{1}{{e_j^1}^2}\right)$$

$$= \frac{1}{2\alpha^2 x_j^2}\left[4 f_j^2 - e_j^2\right]$$ and estimating

as we did previously we obtain

$$\int_{f_j^1}^{e_j^1} x^{-3}dx \leq \frac{1}{8\alpha x_j}(4 + 8\sqrt{2}) .$$

Not every line intersects both curves. Those with no intersections present no problem. Some lines cut only the curve $y = x^2$ twice at points whose x- co-ordinates are $e_j, e_j^1, e_j < e_j^1$.

Then $$\int_{e_j}^{e_j^1} x^{-3}dx = \frac{1}{2}\left(\frac{1}{e_j^2} - \frac{1}{(e_j^1)^2}\right)$$

$$= \frac{1}{2(e_j e_j^1)^2}\left((e_j^1)^2 - e_j^2\right)$$

$$= \frac{1}{8\alpha^2 x_j^2}\left((\alpha + \sqrt{\alpha^2 - 4\alpha x_j})^2 - (\alpha - \sqrt{\alpha^2 - 4\alpha x_j})^2\right)$$

$$= \frac{\alpha\sqrt{\alpha^2 - 4\alpha x_j}}{2\alpha^2 x_j^2} . \qquad (21)$$

In this situation, since the curve $y = 2x^2$ is not intersected, $4x_j \leq \alpha < 8x_j$ so that (21) is less than or equal to

$$\frac{2\sqrt{2}}{\alpha x_j} .$$

The values x_j are given by the y- co-ordinate being zero i.e.

$$\alpha t + \phi = j \quad \text{where} \quad j \in \{-p_k, \ldots\ldots\ p_k\}$$

so that $x_j = \{ \frac{1}{\alpha} (j - \phi)\}$.

Therefore this part of the integral is

(by putting $D(\alpha) = \frac{(4 + 8\sqrt{2})}{4\alpha} + \frac{2\sqrt{2})}{\alpha}$

$$\leq D(\alpha) \left[\sum_{j=1}^{p_k - 1} \frac{1}{1 + \frac{1}{\alpha^2} (j-\phi)^2} \frac{1}{\left\{\frac{1}{\alpha} (j-\phi)\right\}} + \frac{1}{\left\{\frac{-\phi}{\alpha}\right\}} \right.$$

$$\left. + \sum_{j=-p_k}^{-1} \frac{1}{1 + \frac{1}{\alpha^2}(j+1-\phi)^2} \frac{1}{\left\{\frac{1}{\alpha} (j-\phi)\right\}} \right] \qquad (22)$$

$$= D(\alpha) \ h_k(\phi) .$$

Since $m(E_{kn}) = m \left[\phi : \int_{-q_k}^{q_k} \frac{f(t, \alpha t + \phi)dt}{1 + t^2} > n \right]$

$$\leq m \left[\phi : \sum_{j=0}^{q_k - 1} \frac{1}{1 + j^2} \frac{1}{\{j\alpha + \phi\}} + \sum_{j=-q_k}^{-1} \frac{1}{1 + (j+1)^2} \frac{1}{\{j\alpha+\phi\}} \right.$$

$$+ D(\alpha) \left[\sum_{j=1}^{p_k - 1} \frac{1}{1 + \frac{1}{\alpha^2} (j-\phi)^2} \frac{1}{\{\frac{1}{\alpha} (j-\phi)\}} + \frac{1}{\{\frac{-\phi}{\alpha}\}} \right.$$

$$\left.\left. + \sum_{j=-p_k}^{-1} \frac{1}{1 + \frac{1}{\alpha^2} (j+1-\phi)^2} \frac{1}{\{\frac{1}{\alpha} (j-\phi)\}} \right] > n \right]$$

$$\leq m \left[\phi : \sum_{j=0}^{q_k - 1} \frac{1}{1 + j^2} \frac{1}{\{j\alpha+\phi\}} + \sum_{j=-q_k}^{-1} \frac{1}{1 + (j+1)^2} \frac{1}{\{j\alpha+\phi\}} > \frac{n}{2} \right]$$

$$+ m\left[\phi : \left(\sum_{j=1}^{p_k} \frac{1}{1+\frac{1}{\alpha 2}(j-\phi)^2}\,\frac{1}{\{\frac{1}{\alpha}(j-\phi)\}} + \sum_{j=-p_k}^{-1} \frac{1}{1+\frac{1}{\alpha^2}(j+1-\phi)^2}\,\frac{1}{\{(j-\phi)\frac{1}{\alpha}\}} + 1/\left\{-\frac{\phi}{\alpha}\right\}\right) > \frac{n}{2D(\alpha)}\right] \qquad (23)$$

$$= m(F_{kn}) + m(G_{kn}),$$

we need only estimate $m(F_{kn})$ and $m(G_{kn})$.

Consider first $m(G_{kn})$. We shall isolate these points where $\{\frac{1}{\alpha}(j-\phi)\} = 0$ and omit a set T_j of small measure δ_j to the left of each of these points.

Then $m(G_{kn}) \le \sum_j \delta_j + \frac{2D(\alpha)}{n} \int_{[0,1] \setminus \cup T_j} h_k(\phi)\, d\phi$.

Since $m\left[\phi \in [0,1] \setminus \cup T_j : h_k(\phi) > \frac{n}{2D(\alpha)}\right]$

$$\le \frac{2D(\alpha)}{n} \int_{[0,1] \setminus \cup T_j} h_k(\phi)\, d\phi .$$

so $m(G_{kn}) \le \sum_j \delta_j + \frac{2D(\alpha)}{n} \int_{[0,1] \setminus \cup T_j} h_k(\phi)\, d\phi$

$$\le \sum_j \delta_j + \frac{2D(\alpha)}{n}\left[\sum_{j=1}^{p_k} \int_{[0,1] \setminus T_j} \frac{1}{1+\frac{1}{\alpha^2}(j-\phi)^2}\,\frac{1}{\{\frac{1}{\alpha}(j-\phi)\}}\, d\phi\right.$$

$$\left. + \int_{[0,1] \setminus T_0} \frac{1}{\{\frac{-\phi}{\alpha}\}}\, d\phi + \sum_{j=-p_k}^{-1} \int_{[0,1] \setminus T_j} \frac{1}{1+\frac{1}{\alpha^2}(j+1-\phi)^2}\,\frac{1}{\{\frac{1}{\alpha}(j-\phi)\}}\, d\phi\right]$$

$$\le \sum_r \delta_r + \frac{\alpha^2+1}{\alpha^2} \cdot \frac{D(\alpha)}{n}\, 2 \int_{[-p_k, p_k] \setminus \cup J_r^1} \frac{1}{1+\frac{\mu^2}{\alpha^2}}\, \frac{d\mu}{\{\frac{\mu}{a}\}} \qquad (24)$$

where J_r^1 is an interval of length $\delta_r = \frac{1}{nr^2}$ to the right of

$u = r\alpha$.

So (24) gives $m(G_{kn}^{\cdot}) \leq 2 \sum_{r=1}^{\infty} \frac{1}{nr^2} + \frac{\alpha^2+1}{\alpha^2} \frac{D(\alpha)}{n} 2 \sum_r \int_{\delta_r}^{1} \frac{1}{1+r^2} \frac{dt}{t}$

$$= \frac{2}{n} \sum_{r=1}^{\infty} \frac{1}{r^2} + 2D(\alpha) \frac{\alpha^2+1}{\alpha^2} \frac{\log n}{n} \sum_{r=1}^{\infty} \frac{1}{1+r^2} + \frac{2D(\alpha)}{n} \frac{\alpha^2+1}{\alpha^2} \times \sum_{r=1}^{\infty} \frac{2 \log r}{1+r^2} .$$

(we have allowed $k \to \infty$.) (25)

We shall estimate $m(F_{kn})$ similarly:

for ease we shall consider

$$m \left[\phi : \sum_{j=-q_k}^{q_k} \frac{1}{1 + \frac{1}{\alpha^2}(j\alpha+\phi)^2} \frac{1}{\{j\alpha+\phi\}} > \frac{n}{2} \right]$$ which is

comparable with $m(F_{kn})$.

Omitting as previously a set of small measure to the right of the points where $\{j\alpha + \phi\} = 0$, and integrating over the remainder we obtain

$$m(F_{kn}) \leq \frac{4}{r} \sum_{r=1}^{\infty} \frac{1}{r^2} + \frac{4}{n} \sum_{r=1}^{\infty} \int_{\delta_r}^{1} \frac{1}{1+r^2} \frac{dt}{t}$$

$$\leq \frac{4}{n} \sum_{r=1}^{\infty} \frac{1}{r^2} + \frac{4}{n} \sum_{r=1}^{\infty} \frac{2 \log r}{1+r^2} + \frac{\log n}{n} \sum_{r=1}^{\infty} \frac{1}{1+r^2} . \quad (26)$$

(we have allowed $k \to \infty$.)

Letting $n \to \infty$ in (25) and (26) we have

$$\inf_{n \in N} \left(\lim_{k \to \infty} m(E_{kn}) \right) = 0 .$$

The previous arguments hold for $\alpha > 0$. A similar argument covers the case $\alpha < 0$.

So we have shown that for each irrational α

$$\int \frac{f(\theta+t, \phi+\alpha t)}{1+t^2} dt < +\infty \quad \text{for almost all} \quad (\theta, \phi) \in [0,1]^2 .$$

Thus $\omega(x,y) = \exp(-f(x,y))$ is the required weight function.

Some mention should be made of the case where α is rational. The order relation has to be defined somewhat differently, for there are infinitely many points on a line of slope α through the origin.

Example: Define $(m,n) > (0,0)$ if either $m > 0$, or if $m = 0$, $n > 0$. (The lexicographic ordering). This order relation is not archimedean.

Suppose we can predict X_{00} exactly. Then stationarity implies we may predict X_{01} (the 'next' point) exactly, and then all points $\{X_{0k} : k > 0\}$. It is not, however, clear that we may predict any further. So again we seem to be in a type 1, type 2 situation. It is not clear what analytic condition on the weight function ω would correspond to a type 2 process, because, for example the map $t \longrightarrow (e^{-it}, e^{-\alpha it})$ is not a dense embedding of R into T^2. This problem does not seem to have been treated anywhere.

CHAPTER 4

The Helson Szegö Problem and related topics

A number of other prediction problems have been considered over the years. In [23] Helson and Szegö considered the following problem for a singly stationary process $\{X_n\}$ with associated measure μ on the circle T.

Let $\mathcal{P}$ = closed span of $\{e^{ik\theta} : k < 0\}$ in $L^2(\mu)$.

Let $\mathcal{F}$ = closed span of $\{e^{ik\theta} : k \geq 0\}$ in $L^2(\mu)$.

If M, N are closed subspaces of a Hilbert space H, we define $\rho(M,N)$ = the cosine of the angle between M and N as

$$\rho(M,N) = \sup \{|(m,n)| : m \in M, \|m\| \leq 1, n \in N \|n\| \leq 1\} \quad (1)$$

where $(,)$ denotes the scalar product in H.

Clearly $0 \leq \rho \leq 1$. The subspaces are orthogonal if $\rho = 0$, and, if $\rho < 1$, the subspaces are said to be at a positive angle.

Helson and Szegö asked the question: for which measures μ are $\mathcal{P}$ and $\mathcal{F}$ at positive angle? Here $\mathcal{P}$ is considered the 'past' of the process and $\mathcal{F}$ its 'future'. The solution they obtained was as follows.

Theorem 4.1 [23] : $\rho(\mathcal{P}, \mathcal{F}) < 1$ if, and only if, the measure μ is absolutely continuous and in the Lebesgue decomposition $d\mu = \frac{\omega d\theta}{2\pi}$ where $\omega \geq 0$, ω may be expressed as $\omega = \exp(u + \tilde{v})$ where u, v are real valued L^∞ functions on the circle with $\|v\|_\infty < \frac{\pi}{2}$, and by $\tilde{v}$ we mean the harmonic conjugate of v.

By the harmonic conjugate of a function $v \in L^2$ of the circle, we mean the unique function $\tilde{v}$ such that $\int \tilde{v} d\theta = 0$, and $v + i\tilde{v}$ is in H^2 on the circle.

In this section our object is to obtain an analogue of this theorem for the case of a doubly stationary process. This problem was considered by Ohno [26] , but he makes unnecessary assumptions about the weight function ω. We shall present an exact analogue of the Helson Szegö result.

α will be a fixed irrational number, and the order relation on $\underset{\sim}{Z} \times \underset{\sim}{Z}$ will be that imposed by α . μ , a finite positive Borel measure on the Torus, will have the usual Lebesgue decomposition

$$d\mu = \omega \, d\sigma + d\mu_s \quad \text{as in chapter 3 .}$$

A_α , $H^p_\alpha(\sigma)$, $L^p(\sigma)$ will also be as in chapter 3. $C(T^2)$ will denote the continuous complex valued functions on the Torus, T^2 .

We define closed subspaces $\mathcal{P}_\alpha$, $\mathcal{F}_\alpha$ of $L^2(\mu)$ as follows:

$$\mathcal{P}_\alpha = \text{closed span of } \{e^{i(mx+ny)} : (m,n) \underset{\alpha}{<} (0,0)\} \text{ in } L^2(\mu)$$

$$\mathcal{F}_\alpha = \text{closed span of } \{e^{i(mx+ny)} : (m,n) \underset{\alpha}{\geq} (0,0)\} \text{ in } L^2(\mu) .$$

Let $\rho_\alpha = \rho(\mathcal{P}_\alpha , \mathcal{F}_\alpha)$ where ρ is defined as in (1).

We now ask for what measures μ is $\rho_\alpha < 1$?

<u>The Conjugate Function</u>:

Suppose $f \in C(T^2)$ has the form $f(e^{ix} , e^{iy}) = \sum a_{mn} e^{i(mx+ny)}$ where the sum is finite i.e. suppose f is a trigonometric polynomial.

Consider the function $\tilde{f} : T^2 \longrightarrow C$ defined by

$$\tilde{f}(e^{ix} , e^{iy}) = i \sum_{(m,n) \underset{\alpha}{<} (0,0)} a_{mn} e^{i(mx+ny)} - i \sum_{(m,n) \underset{\alpha}{\geq} (0,0)} a_{mn} e^{i(mx+ny)} \qquad (2)$$

$$\text{Then } \int \tilde{f} \, d\sigma = 0 \text{ and} \qquad (3)$$

$$(f + i\tilde{f})(e^{ix} , e^{iy}) = a_{00} + 2 \sum_{(m,n) \underset{\alpha}{>} (0,0)} a_{mn} e^{i(mx+ny)} .$$

This is an analytic trigonometric polynomial i.e. its Fourier coefficients vanish off the half plane $(m,n) \underset{\alpha}{\geq} (0,0)$, and we have defined a linear map of the trigonometric polynomials into themselves given by $f \longrightarrow \tilde{f}$.

<u>Theorem 4.2</u>: [28] Let $1 < p < \infty$. There is a constant A_p such that $\|\tilde{f}\|_p \leq A_p \|f\|_p$ holds for every trigonometric polynomial f . Here $\|f\|_p$ denotes the norm of f as an element of $L^p(\sigma)$.

The map $f \longrightarrow \tilde{f}$ can therefore be extended to a bounded linear map of $L^p(\sigma)$ to itself.

So $f \longrightarrow f + i\tilde{f}$ maps $L^p(\sigma)$ onto $H^p_\alpha(\sigma)$. The function $\tilde{f}$ is called the conjugate of f .

We also obtain (compare [29] , page 254).

<u>Theorem 4.3</u>: If f is a real-valued measurable function in $L^\infty(\sigma)$ with $\|f\|_\infty \leq 1$, then for $0 < k < \frac{\pi}{2}$ there is an $N_k > 0$ such that

$$\int \exp\,(k\,|\tilde{f}|\,)\,d\sigma \;\leq\; N_k < +\infty\,. \tag{4}$$

<u>Proof</u>: Suppose first that f is a real-valued trigonometric polynomial. Then $\exp\,(k\,(\tilde{f} - if))$ is in A_α for $0 < k < \frac{\pi}{2}$ and so

$$\int \exp\,(k\,\tilde{f} - ikf)\,d\sigma \;=\; \exp\,(-ik \int f d\sigma)\,, \tag{5}$$

using the fact that σ is multiplicative on A_α (i.e. $\int f\,g\,d\,\sigma = \int f\,d\,\sigma \int g\,d\,\sigma$ for $f, g \in A_\alpha$) and $\int \tilde{f}\,d\,\sigma = 0$. (3)

Taking real parts in (5) we obtain

$$\int \cos k\,f\;\exp k\,\tilde{f}\,d\,\sigma \;=\; \cos\,(k \int f\,d\,\sigma)\,.$$

Similarly $\int \cos k\, f \exp(-k\tilde{}\,)\, d\sigma = \cos(k \int f\, d\sigma)$,

$\exp(k\, |\tilde{f}|\,) \leq \exp(k\tilde{f}) + \exp(-k\tilde{f})$ for $0 < k < \frac{\pi}{2}$

since f is real-valued .

Since $|f| \leq 1$ and f is real-valued we obtain $\cos kf \geq \cos k$ almost everywhere and also $\cos(k \int f\, d\sigma) \leq 1$ so that

$$\cos k \int \exp(k\, |\tilde{f}|)\, d\sigma \leq \int \cos kf \exp(k\tilde{f})\, d\sigma + \int \cos kf \exp(-k\tilde{f})\, d\sigma$$

$$\leq 2 .$$

With $N_k = 2/\cos k$ which for $0 < k < \frac{\pi}{2}$ is finite, we obtain the result for real valued trigonometric polynomials.

For a real valued f in $L^{\infty}(\sigma)$, choose a sequence $\{f_n\}$ of real valued trigonometric polynomials such that $f_n \to f$ pointwise almost everywhere and $|f_n| \leq 1$. We may also assume, by restricting to a subsequence if necessary, that $\tilde{f}_n \to \tilde{f}$ pointwise almost everywhere.

Then using Fatou's lemma

$$\int \exp(k\, |\tilde{f}|)\, d\sigma \leq \liminf_n \int \exp(k\, |f_n|\, d\sigma) \leq N_k .$$

Proposition 4.4: If f is real and measurable and for some $0 < \varepsilon < \frac{\pi}{2}$, $|f| \leq \frac{\pi}{2} - \varepsilon$, then $\exp(-\tilde{f} + if) \in H^1_{\alpha}(\sigma)$.

Proof: By theorem 4.3 , $\exp(-\tilde{f} + if) \in L^1(\sigma)$. As in theorem 4.3 let $\{f_n\}$ be a sequence of real trigonometric polynomials such that

$|f_n| \leq \frac{\pi}{2} - \varepsilon$ for all n ,

$f_n \to f$ pointwise almost everywhere ,

$\tilde{f}_n \to \tilde{f}$ pointwise almost everywhere .

Let $g \in A_\alpha$ satisfy $\int g \, d\sigma = 0$.

For each n,

$$\int g \exp(-\tilde{f}_n + if_n) \, d\sigma = 0 .$$

Also $g \exp(-\tilde{f}_n + if_n) \to g \exp(-\tilde{f} + if)$

pointwise almost everywhere

and $\int |g| \exp(-\tilde{f}_n) \, d\sigma \to \int |g| \exp(-\tilde{f}) \, d\sigma$.

Therefore $\int g \exp(-\tilde{f}_n + if_n) \, d\sigma \to \int g \exp(-\tilde{f} + if) \, d\sigma = 0$.

Since this holds for each $g \in A_\alpha$, $\exp(-\tilde{f} + if)$ is contained in $H^1_\alpha(\sigma)$.

<u>Lemma 4.5</u>: If $f \in H^1_\alpha(\sigma)$ and $\mathrm{Re}\, f > 0$, then f is an outer function.

<u>Proof</u>: We recall [21] that a function f is outer if and only if the closure of the set $\{fg : g \in A_\alpha\}$ is $L^1(\sigma)$ dense in $H^1_\alpha(\sigma)$.

Let $g \in A_\alpha$. We shall show how to approximate g by a sequence of elements of the form fh with $h \in A_\alpha$.

The sequence $\{f + \frac{1}{n}\}$ converges to f in $L^1(\sigma)$ and also pointwise almost everywhere.

$$\int \left| \frac{f}{f + \frac{1}{n}} g - g \right| d\sigma \leq \|g\|_\infty \int \left| \frac{f}{f + \frac{1}{n}} - 1 \right| d\sigma$$

$$\to 0 \text{ as } n \to \infty .$$

by the Lebesgue dominated convergence theorem.

Choose $f_k \in A_\alpha$ such that $f_k \to f$ in $L^1(\sigma)$ and $\mathrm{Re}\, f_k \geq 0$.

Then $\int \left| \frac{fg}{f + \frac{1}{n}} - \frac{fg}{f_k + \frac{1}{n}} \right| d\sigma \leq \|g\|_\infty \int \frac{|f|\ |f_k - f|}{\left|f + \frac{1}{n}\right| \left|f_k + \frac{1}{n}\right|} d\sigma$

$\rightarrow 0$ as $k \rightarrow \infty$, clearly.

Thus given $g \in A_\alpha$, which is $L^1(\sigma)$ dense in $H^1_\alpha(\sigma)$, we may obtain an element $h \in A_\alpha$ $(h = \frac{g}{f_k + \frac{1}{n}}$ for some k and $n)$ which is arbitrarily close in $L^1(\sigma)$ to g .

Lemma 4.6: (cf [18]) If $f \in A_\alpha$, $\text{Re } f > 0$, then $\log f \in A_\alpha$.

Proof: We shall need the following result about a commutative, semi-simple Banach algebra A with an identity:

Let $a \in A$, F be a function analytic in a region of the complex plane containing the spectrum of a ; then there is a unique element $b \in A$ such that

$\hat{b}(\sigma) = F(\hat{a}(\sigma))$ for all complex homomorphisms σ of A . Here $\hat{a}$ denotes the Gelfand transform of $a \in A$. (See [18] chapter 2 for example).

Since each $(e^{ix}, e^{iy}) \in T^2$ determines a complex homomorphism : $f \in A_\alpha \longrightarrow f(e^{ix}, e^{iy})$, A_α is semi-simple. The spectrum of those elements of A_α with real part greater than zero is contained in the half plane $\text{Re}(Z) > 0$, where $F(Z) = \log Z$ is analytic.

Consequently $F(f) = \log f \in A_\alpha$.

Lemma 4.7: [cf.26] If $f \in H^1_\alpha(\sigma)$ with $\text{Re } f > 0$, then $\log f \in H^1_\alpha(\sigma)$.

Proof: We prove first that if $f \in H^1_\alpha(\sigma)$ and $\text{Re } f \geq \varepsilon > 0$ then $\log f \in H^1_\alpha(\sigma)$.

By restricting to a subsequence where necessary, choose a sequence $\{f_n\}$ in A_α with

$$\operatorname{Re} f_n \geq \frac{\varepsilon}{2} \quad \text{for each } n ,$$

$$f_n \longrightarrow f \quad \text{pointwise almost everywhere}$$

and
$$\int | f_n - f | d\sigma \to 0 .$$

Since $\frac{d}{dx}(\log x) = \frac{1}{x}$,

$$\frac{\left|\log |f_n| - \log |f|\right|}{| f_n - f |} \leq \frac{\left|\log |f_n| - \log |f|\right|}{\left| |f_n| - |f| \right|} \leq \frac{\varepsilon}{2}^{-1}.$$

Thus $\{\log |f_n|\}$ converges to $\log |f|$ in $L^1(\sigma)$.

$\arg f_n \to \arg f$ almost everywhere since $f_n \to f$ almost everywhere.

Since $|\arg f_n| < \frac{\pi}{2}$ $(\operatorname{Re} f_n \geq \frac{\varepsilon}{2})$, the Lebesgue bounded convergence theorem ensures that $\{\arg f_n\}$ converges to $\arg f$ in $L^1(\sigma)$.

Thus $\{\log f_n\}$ converges to $\log f$ in $L^1(\sigma)$. By lemma 4.6 each $\log f_n \in A_\alpha$ so $\log f \in H^1{}_\alpha(\sigma)$.

Suppose now $\operatorname{Re} f > 0$. $\{f + \frac{1}{n}\}$ converges to f in $L^1(\sigma)$ and pointwise almost everywhere. Also each $\log (f + \frac{1}{n})$ is in $H^1{}_\alpha(\sigma)$.

Lemma 4.5 shows $\{f + \frac{1}{n}\}$, f are all outer functions.

$$\int \left| \log |f + \tfrac{1}{n}| - \log |f| \right| d\sigma \leq \int \left(\log \left|f + \tfrac{1}{n}\right| - \log |f|\right) d\sigma$$

since $\operatorname{Re} f > 0$

$$= \log \left| \int (f + \tfrac{1}{n})\, d\sigma \right| - \log \left| \int f\, d\sigma \right|$$ since these are outer functions. (see lemma 4.5)

Since $\int (f + \frac{1}{n})\, d\sigma \to \int f\, d\sigma$ and $\operatorname{Re} f > 0$ the above tends to zero as $n \to \infty$.

Now $\arg (f + \frac{1}{n}) \to \arg f$ pointwise almost everywhere, and $\{\arg (f + \frac{1}{n})\}$ is bounded. Using the bounded convergence theorem again, we obtain the convergence of $\{\log (f + \frac{1}{n})\}$ to $\log f$ in $L^1(\sigma)$. Each $\log (f + \frac{1}{n})$ being in $H^1_\alpha(\sigma)$, so also is $\log f$.

Solution of the Problem

The object is to characterise those measures μ for which $\rho_\alpha = \rho(\mathcal{P}_\alpha, \mathcal{J}_\alpha) < 1$.

We have $d\mu = \omega\, d\sigma + d\mu_s$ where $\omega \geq 0$ in $L^1(\sigma)$ and μ_s is singular. We conclude immediately $\log \omega \in L^1(\sigma)$ for otherwise

$1 \in \mathcal{P}_\alpha$ and $\rho_\alpha = 1$.

We can conclude however that $\omega^{-1} \in L^1(\sigma)$. The justification for this is the following result.

Proposition 4.8: $$\inf \int |1 + F + G|^2\, d\mu = \left(\int \omega^{-1}\, d\sigma\right)^{-1} \qquad (6)$$

where the infimum is taken over all $F \in \mathcal{P}_\alpha$ and $G \in \mathcal{J}_\alpha$ such that $\int G\, d\sigma = 0$, if the infimum is positive. If the infimum is zero $\omega^{-1} \notin L^1(\sigma)$.

Proof: In (6) we may as well consider the infimum over expressions of the form

$$1 + \sum_{(m,n) < (0,0)} a_{mn}\, e^{i(mx+ny)} + \sum_{(m,n) > (0,0)} a_{mn}\, e^{i(mx+ny)}$$

where the sums are finite.

The collection of all such finite sums is a convex set K whose closure in $L^2(\mu)$ is also convex.

If $0 \notin$ closure of K, in other words, if the infimum in (6) is strictly positive and equal to δ, say, there is a unique element $1 + H$ in the closure of K such that $\int |1 + H|^2 \, d\mu = \delta$.

For each $\lambda \in \underset{\sim}{C}$, $1 + H + \lambda e^{i(mx+ny)} \in K$ for all $(m,n) \in \underset{\sim}{Z} \times \underset{\sim}{Z}$, except $(0,0)$.

So $\int | 1 + H + \lambda e^{i(mx+ny)} |^2 \, d\mu \geq \int | 1 + H |^2 \, d\mu$ for all $\lambda \in \underset{\sim}{C}$

and we can conclude that

$$\int (1 + H) \, e^{i(mx+ny)} \, d\mu = 0 \quad \text{for all } (m,n) \in \underset{\sim}{Z} \times \underset{\sim}{Z} \text{ except } (0,0) .$$

Also $\int (1 + H) \, d\mu = \int (1 + H)(1 + \overline{H}) \, d\mu = \int | 1 + H |^2 \, d\mu = \delta$.

Therefore the measure $(1 + H) \, d\mu - \delta d\sigma$ annihilates all continuous functions on the Torus and is the zero measure. $(1 + H)d\mu$ is, therefore, a constant multiple of Lebesgue measure and $(1 + H)\omega = \delta$ almost everywhere.

Therefore $\int \omega^{-1} \, d\sigma = \frac{1}{\delta} \int (1 + H) \, d\sigma = \frac{1}{\delta}$;

so $\delta = (\int \omega^{-1} \, d\sigma)^{-1}$.

If $\delta = 0$, consider $(\omega + \varepsilon)$ in place of ω.

Then $\inf \int |1 + F + G|^2 (\omega + \varepsilon) d\sigma + d\mu_s \quad (= (\int (\omega + \varepsilon)^{-1} \, d\sigma)^{-1})$

$$\leq \inf \int |1 + F + G|^2 \, d\mu + \inf \varepsilon \int |1 + F + G|^2 \, d\sigma$$

$$= \varepsilon .$$

So $\int (\omega + \varepsilon)^{-1} \, d\sigma \geq \frac{1}{\varepsilon}$.

$\int \omega^{-1} \, d\sigma \geq \frac{1}{\varepsilon}$ for all positive ε.

So $\omega^{-1} \notin L^1(\sigma)$.

Proposition 4.7 allows us to conclude that $\omega^{-1} \in L^1(\sigma)$ (7)

We shall show also that μ cannot have a singular part. Theorem 3.5 of Chapter 3 shows that if χ , $\chi^1 \in L^2(\mu)$ satisfy

$$\chi = 1 \text{ almost everywhere } (d\sigma),$$
$$\chi^1 = 1 \text{ almost everywhere } (d\mu_s),$$
$$\chi \cdot \chi^1 \equiv 0 ,$$

then $\chi\mathcal{P}_\alpha$, $\chi^1\mathcal{P}_\alpha$ are closed subspaces of $\mathcal{P}_\alpha$. A similar argument will show that $\chi\mathcal{J}_\alpha$, $\chi^1\mathcal{J}_\alpha$ are closed subspaces of $\mathcal{J}_\alpha$.

$$\chi^1 \in \mathcal{J}_\alpha \text{ since } 1 \in \mathcal{J}_\alpha .$$

But
$$\inf \int |\chi^1 - \chi^1 p_\alpha|^2 \, d\mu = \inf \int |1 - p_\alpha|^2 \, d\mu_s$$
$$= 0 ,$$

where the infimum is taken over $p_\alpha \in \mathcal{P}_\alpha$.

Therefore $\chi^1 \in \mathcal{P}_\alpha$ and $\rho_\alpha = 1$.

We may suppose, therefore, that μ is absolutely continuous and $d\mu = \omega d\sigma$.

$\log \omega$ being summable, by theorem 3 of [21] we may find an outer function h in $H^1_\alpha(\sigma)$ such that $\omega = |h|$. We define ϕ by $\omega = h\, e^{-i\phi}$.

<u>Proposition 4.9</u>: $\rho_\alpha < 1$ if and only if there is an $\varepsilon > 0$ and a $g \in H^\infty_\alpha(\sigma)$ such that

$$|g| \geq \varepsilon \text{ almost everywhere } (d\sigma) , \tag{8}$$

and
$$|\arg gh| \leq \frac{\pi}{2} - \varepsilon \text{ almost everywhere } (d\sigma). \tag{9}$$

<u>Proof</u>: There is an outer function k in $H^2_\alpha(\sigma)$ such that $h = k^2$ and $\omega = |k|^2$ - this may be concluded from [21] .

$$\rho_\alpha = \sup \left\{ \left| \int f \bar{g} k^2 e^{-i\phi} \, d\sigma \right| : \begin{array}{l} f \in \text{Ball}\, \mathcal{J}_\alpha \\ g \in \text{Ball}\, \mathcal{P}_\alpha \end{array} \right\}$$

$$= \sup \left\{ \left| \int f \bar{g} k^2 e^{-i\phi} d\,d\sigma \right| : f, g \text{ as above but restricted to finite sums} \right\}. \qquad (10)$$

Since k is outer in $H^2_\alpha(\sigma)$, as we allow f to vary, the elements fk run over a dense subset of the unit ball of $H^2_\alpha(\sigma)$, and the elements $\bar{g}k$ run over a dense subset of the unit ball of those functions in $H^2_\alpha(\sigma)$ whose Fourier coefficients vanish at the origin.

Their product, therefore, ranges over a dense subset of the closed unit ball of the subspace.

$$H^1_0(\sigma) = \{ f \in H^1(\sigma) : \int f \, d\sigma = 0 \}$$

(To avoid cumbersome notation we shall omit the α's).

(10) therefore represents ρ_α as the norm of a bounded linear functional on $H^1_0(\sigma)$. The dual of $H^1_0(\sigma)$ is $L^\infty(\sigma)/H^\infty(\sigma)$ and so $1 > \rho_\alpha = \inf \{ \| e^{-i\phi} - g \|_\infty :: g \in H^\infty(\sigma) \}$ (11)

Let $\delta > 0$ satisfy $1 > \rho_\alpha + \delta$. There is a $g \in H^\infty(\sigma)$ such that (g depends on δ)

$\rho_\alpha + \delta \geq \| e^{-i\phi} - g \|_\infty \geq 1 - |g(x,y)|$ almost everywhere

so $|g| \geq 1 - (\rho_\alpha + \delta)$ almost everywhere.

The cosine rule gives for $C = |\arg e^{-i\phi} - \arg g|$

$$(\rho_\alpha + \delta)^2 \geq 1 + |g|^2 - 2|g| \cos C$$

so that
$$2|g| \cos C \geq |g|^2 + 1 - (\rho_\alpha + \delta)^2 \geq |g|^2$$

$$\cos C \geq \frac{|g|}{2} \geq \frac{1 - (\rho_\alpha + \delta)}{2}.$$

We can therefore choose an $\varepsilon > 0$ so that $|g| > \varepsilon$ almost everywhere and $|\phi + \arg g| \leq \frac{\pi}{2} - \varepsilon$ almost everywhere.

The steps in this argument can clearly be reversed, ensuring that (8) and (9) are both necessary and sufficient for the two subspaces to be at positive angle.

Theorem 4.10: $\rho_\alpha < 1$ if and only if μ is absolutely continuous, $d\mu = \omega \, d\sigma$, and ω may be written as $\omega = \exp(u + \tilde{v})$ where u, v are real $L^\infty(\sigma)$ functions with $\|v\|_\infty < \frac{\pi}{2}$.

Proof (a) Suppose $\rho_\alpha < 1$.

Since $\omega^{-1} \in L^1(\sigma)$ we obtain, with g and ε as in Prop.4.9,

$$|\arg gh| \leq \frac{\pi}{2} - \varepsilon \qquad \text{a.e.} \tag{12}$$

$$|gh| \geq \varepsilon|h| > 0 \qquad \text{a.e.}$$

Therefore $\operatorname{Re} gh > 0$ a.e. Since $gh \in H^1(\sigma)$ by lemma 4.7 $\log gh \in H^1(\sigma)$.

Now $\log gh = \log|gh| + i \arg gh$

Let $v = \arg gh$. v is in $L^\infty(\sigma)$, real and $\|v\|_\infty < \frac{\pi}{2}$ by (12).

Put $u = -\log|g| \in L^\infty(\sigma)$.

Then $\omega = \dfrac{|gh|}{|g|} = \exp(-\tilde{v} + u)$ as required.

(b) Conversely suppose $\omega = \exp(u + \tilde{v})$ with u,v as in the statement of the theorem.

Put $h = \exp(u + i\tilde{u}) \exp(\tilde{v} - iv)$. h is in $H^1(\sigma)$ and is in fact outer. (Theorem 6 in [22])

Put $g = \exp(-u - i\tilde{u}) \in H^\infty(\sigma)$.

$|\arg gh| = |v| < \frac{\pi}{2} - \varepsilon$ a.e. where we choose $0 < \varepsilon < \frac{\pi}{2}$ such that $|g| \geq \varepsilon$ a.e. also. The condition of proposition 4.9 is therefore satisfied and the two subspaces are at a positive angle.

We have obtained an exact analogue of Helson and Szegö's result. A further characterisation of the weight functions ω for which the subspaces are at positive angle is as follows (compare [26] , [16])

Theorem 4.11: $d\mu = \omega \, d\sigma$. $\rho_\alpha < 1$ if and only if there is $\varepsilon > 0$, a $g \in H^\infty(\sigma)$ invertible in $H^\infty(\sigma)$, such that $|\arg gh| < \frac{\pi}{2} - \varepsilon$ a.e. $(d\sigma)$.

Proof: If such a g exists, with possibly a smaller value of ε, we obtain $|g| \geq \varepsilon$ a.e. and $|\arg gh| \leq \frac{\pi}{2} - \varepsilon$ a.e. By Prop. 4.9 $\mathfrak{F}_\alpha$, $\mathcal{P}_\alpha$ are at positive angle.

Suppose conversely that the two subspaces are at a positive angle. By Prop. 4.9 there is a $g \in H^\infty$, $\varepsilon > 0$, such that $|g| \geq \varepsilon$ a.e. and

$$|\arg gh| \leq \frac{\pi}{2} - \varepsilon \quad \text{a.e.}$$

In these circumstances we have seen that $\operatorname{Re} gh > 0$ a.e. and lemma 4.5 implies gh which is in $H^1(\sigma)$ is outer. h being outer, so also is g.

Thus $g \in H^\infty(\sigma)$ is outer and satisfies $|g| > \varepsilon$ a.e. There is $g^1 \in L^\infty(\sigma)$ such that $gg^1 = 1$ a.e.

g being outer, there is a sequence $\{p_n\}$ of analytic trigonometric polynomials such that $g\, p_n \to 1$ in $L^2(\sigma)$.

$$\text{So} \int ||g\,p_n - gg^1||^2\,d\sigma = \int |g|^2\ |p_n - g^1|^2\,d\sigma$$
$$\geq \varepsilon^2 \int |p_n - g^1|^2\,d\sigma\,.$$

Thus $g^1 \in H^2(\sigma) \cap L^\infty(\sigma)$ and so $g^1 \in H^\infty(\sigma)$.

L^2 - Boundedness of the Conjugate Map

We have seen that the conjugate map $f \longrightarrow \tilde{f}$ extends to a bounded linear map of $L^2(\sigma)$ into itself. It is a natural question to ask for what weight functions ω is the conjugate map a bounded map of $L^2(\omega\ d\ \sigma) = \{f : \int |f|^2\ \omega\ d\ \sigma < +\infty\} = L^2(\omega)$ into itself? It would of course be sufficient to show that this is a bounded map of trigonometric polynomials into themselves, for these are dense in $L^2(\omega)$ for any weight function ω .

If f is a trigonometric polynomial, and $\tilde{f}$ is its conjugate, then the map $f \longrightarrow f + i\tilde{f}$ is bounded if and only if the map $f \longrightarrow \tilde{f}$ is a bounded map of the trigonometric polynomials into themselves. In all this, the norm is $\|f\| = (\int |f|^2\ \omega\ d\ \sigma)^{\frac{1}{2}}$. To establish boundedness of the map $f \longrightarrow f + i\tilde{f}$ it suffices to establish whether the spaces $\mathcal{P}_\alpha, \mathcal{F}_\alpha$ corresponding to ω are at a positive angle.

To see this in the case of M,N closed subspaces of a Hilbert space H, we need to show that if $\rho(M,N) < 1$ and m,n are elements of M,N respectively

$$\|m\| \leq C\|m+n\| \quad \text{where } C > 0 \text{ is a constant} \tag{13}$$

Let $\rho = \rho(M,N) < 1$. Then

$$\|m + n\|^2 = \|m\|^2 + \|n\|^2 + 2\,\mathrm{Re}\,(m,n)$$
$$\geq \|m\|^2 + \|n\|^2 - 2\,|\mathrm{Re}\,(m,n)|$$
$$\geq \|m\|^2 + \|n\|^2 - 2\rho\|m\|\,\|n\|$$
$$= (1-\rho^2)\,\|m\|^2 + (\|n\| - \rho\|m\|)^2$$
$$\geq (1-\rho^2)\,\|m\|^2 \quad \text{so (13) follows.}$$

If (13) holds with $\|m\| = \|n\| = 1$ we have

$$\frac{1}{C^2} \leq \|m-n\|^2 = 2(1 - \mathrm{Re}(m,n))$$

so $\mathrm{Re}\,(m,n) \leq 1 - \dfrac{1}{2C^2} = \rho < 1$.

So if $(m,n) = re^{i\theta}$ where $\|m\| = \|n\| = 1$.

$$(e^{-i\theta}\,m,n) = r = \mathrm{Re}\,(e^{-i\theta}\,m,n)$$
$$\leq \rho$$

So $\sup\limits_{m \,\varepsilon\, \mathrm{Ball}\, M,\; n \,\varepsilon\, \mathrm{Ball}\, N} |(m,n)| \leq \rho < 1$ as required

Therefore we obtain:

<u>Theorem 4.12</u>: The map $f \longrightarrow \tilde{f}$ is a bounded map of $L^2(\omega)$ to itself where $\omega \geq 0$ is in $L^1(\sigma)$ if and only if there are real $L^\infty(\sigma)$ functions u,v such that $\|v\|_\infty < \frac{\pi}{2}$ and $\omega = \exp(u + \tilde{v})$.

The Space $BMO(\alpha)$:

The class of functions $\{u + \tilde{v} : u,v \text{ real } L^\infty(\sigma) \text{ functions}\}$, a subset of which occurs in the solution of the Helson Szegö problem, forms, on the circle T and real line $\underset{\sim}{R}$, the class of functions of Bounded Mean Oscillation, introduced by John and Nirenberg in [24].

Definition 4.13: A function ϕ, measurable on the line, is a function of Bounded Mean Oscillation (BMO) if there is a $K > 0$ such that for all intervals $J \subseteq \mathbb{R}$

$$\frac{1}{|J|} \int_J | \phi(t) - \phi_J| \leq K .$$

Here $|J|$ denotes the length of the interval J and

$$\phi_J = \frac{1}{|J|} \int_J \phi(t)\, dt .$$

Fefferman and Stein [17] studied these functions extensively, proved that BMO is the dual of H^1, and gave the above mentioned characterisation of functions of bounded mean oscillation.

Our object now is to examine a class of functions on the Torus which will play an analogous role to that of BMO functions on the circle or the line.

Let ϕ be a (real or complex) measurable function on the Torus.

Fix some irrational α.

For almost all (x,y), we shall define $\phi_{xy}(t)$ by $\phi_{xy}(t) = \phi(x-t, y-\alpha t)$, $t \in \mathbb{R}$.

Our analogue of BMO will be defined by requiring that all the functions ϕ_{xy} be in BMO in a uniform sense.

Definition 4.14: With ϕ a measurable function on the Torus, say $\phi \in \text{BMO}(\alpha)$ if

$$\operatorname*{ess\,sup}_{(x,y)} \sup_{|J| < \infty} \frac{1}{|J|} \int_J |\phi_{xy}(t) - \phi_{xyJ}|\,dt < +\infty$$

Here J is an interval in $\mathbb{R}$, $|J|$ is its length and

$$\phi_{xyJ} = \frac{1}{|J|} \int_J \phi(x-t, y-\alpha t)\,dt .$$

Let $\|\phi_{xy}\|_* = \sup_{|J| < \infty} \frac{1}{|J|} \int_J |\phi_{xy}(t) - \phi_{xyJ}|dt$.

$\|\phi_{xy}\|_*$ is a well defined measurable function on the Torus, and since BMO is translation invariant, $\|\phi_{xy}\|_*$ is constant on lines of slope α .

By Corollary VII.7.4 in [18] $\|\phi_{xy}\|_*$ is constant almost everywhere on the Torus.

A result of Fefferman and Stein ([17] page 141) immediately implies that for almost all $(x,y) \varepsilon T^2$

$$\phi_{xy} \varepsilon L^1 \left(\frac{dt}{1+t^2} \right) .$$

As we have seen, there are functions on the Torus which are in $L^1 \left(\frac{dt}{1+t^2} \right)$ on almost all lines, but are not in $L^1(\sigma)$. We shall now prove that this cannot happen for $BMO(\alpha)$ functions.

<u>Proposition 4.15</u>: Let ϕ be a real-valued $BMO(\alpha)$ function. Then $\phi^+ = \max(\phi,0)$, $\phi^- = \max(-\phi,0)$ are in $BMO(\alpha)$, and so therefore is $|\phi| = \phi^+ + \phi^-$.

<u>Proof</u>: Fix an (x,y) and an interval J . Suppose first that $\phi_{xyJ} \geqslant 0$.

Then $|\phi_{xy}{}^+(t) - \phi_{xyJ}| \leq |\phi_{xy}(t) - \phi_{xyJ}|$ for all $t \varepsilon J$,

so $\frac{1}{|J|} \int |\phi_{xy}{}^+(t) - \phi_{xyJ}|dt \leq \frac{1}{|J|} \int |\phi_{xy}(t) - \phi_{xyJ}|dt \leq \|\phi_{xy}\|_*$.

If $\phi_{xyJ} < 0$, then clearly

$$|\phi_{xy}{}^+(t)| \leq |\phi_{xy}(t) - \phi_{xyJ}| \text{ for all } t \varepsilon J .$$

and so $\frac{1}{|J|}\int_J |\phi_{xy}{}^{+}(t)|dt \leq \frac{1}{|J|}\int |\phi_{xy}(t) - \phi_{xyJ}|dt \leq \|\phi_{xy}\|_*$.

It follows easily that $\phi^+ \varepsilon$ BMO(α) , and so ϕ^- and $|\phi|$ are in BMO(α) .

<u>Theorem 4.16</u>: Let $\phi \varepsilon$ BMO(α). There is a $k > 0$ such that $\exp(k\ |\phi|) \varepsilon L^1(\sigma)$. In particular, $\phi \varepsilon L^p(\sigma)$ for $1 \leq p < \infty$.

<u>Proof</u>: We shall suppose that ϕ is real-valued. The complex case may be deduced by examining the real and imaginary parts separately.

By proposition 4.15 $\psi = |\phi| \varepsilon$ BMO(α) .

Define $\psi^n = \min(n , \psi)$.

Then $0 \leq \psi^n \leq n$,

$\exp(k\ \psi^n) \leq \exp(k\ \psi^{n+1})$ for each n , $k > 0$.

Also $\exp(\pm\ k\ \psi^n)$ is in $L^1(\sigma)$ for all n , all $k > 0$.

We shall prove the existence of a $k > 0$ such that there exists $D > 0$ for which

$$1 \leq \left(\frac{1}{2T}\int_{-T}^{T} \exp(k\ \psi^n(x-t, y-\alpha t)\ )dt\right)\left(\frac{1}{2T}\int_{-T}^{T} \exp(-k\ \psi^n(x-t,y-\alpha t)\ )dt\right) \leq D \qquad (14)$$

for all $T > 0$, all n , and almost all $(x,y) \varepsilon T^2$.

By a remark in Helson's paper (page 20 , [20]) as $T \to \infty$ in (14) we obtain

$$\int \exp(k\ \psi^n)d\sigma \quad \int \exp(-k\ \psi^n)d\sigma \leq D. \text{ for all } n .$$

By Lebesgue's bounded convergence theorem

$$\int \exp(-k\ \psi^n)d\sigma \to \int \exp(-k\ \psi)d\sigma \quad \text{as } n \to \infty .$$

For all sufficiently large n, therefore,

$$\int \exp(k\,\psi^n)d\sigma \leq \frac{2D}{\int \exp(-k\,\psi)d\sigma}.$$

By Fatou's lemma $\int \exp(k\,\psi)d\sigma$ exists.

It only remains to prove (14).

We shall choose k and C later, C depends on (x,y) and T.

$$\left(\frac{1}{2T}\int_{-T}^{T} \exp(k\,\psi^n)dt\right)\left(\frac{1}{2T}\int_{-T}^{T} \exp(-k\,\psi^n)dt\right)$$

$$= \left(\frac{1}{2T}\int_{-T}^{T} \exp(k\,\psi^n - kC)dt\right)\left(\frac{1}{2T}\int_{-T}^{T} \exp(-k\,\psi^n + kC)dt\right)$$

$$\leq \left(\frac{1}{2T}\int_{-T}^{T} \exp(k\,|\psi^n - C|)dt\right)^2. \tag{15}$$

Consider now ψ^n_{xy} on $J = [-T,T]$. Suppose firstly that $\psi_{xyJ} \leq n$.

Then $|\psi^n_{xy}(t) - \psi_{xyJ}| \leq |\psi_{xy}(t) - \psi_{xyJ}|$.

Putting $C = \psi_{xyJ}$, we obtain (15)

$$\leq \left(\frac{1}{2T}\int_{-T}^{T} \exp(k\,|\psi_{xy}(t) - \psi_{xyJ}|)dt\right)^2. \tag{16}$$

If $\psi_{xyJ} > n$, then

$|\psi^n_{xy}(t) - n| \leq |\psi_{xy}(t) - \psi_{xyJ}|$; so with $C = n$ we again obtain (15) $\leq$ (16).

But by a result of John and Nirenberg [24] we may choose a $k > 0$, such that for almost all (x,y), all $T > 0$

$$\left(\frac{1}{2T} \int_{-T}^{T} \exp(k \, |\psi_{xy}(t) - \psi_{xyJ}|) dt \right)^2 \leq D \text{ where } D > 0 .$$

Therefore $\int \exp(k\,|\phi|) d\sigma$ exists and in particular $\phi \in L^p(\sigma)$ for $1 \leq p < \infty$.

<u>Theorem 4.17</u>: For $\phi \in BMO(\alpha)$ define $\|\phi\|_*$ by $\|\phi\|_* = \text{ess sup} \|\phi_{xy}\|_* + \left| \int \phi \, d\sigma \right|$.

Then $BMO(\alpha)$ is a Banach space with respect to this norm.

<u>Proof</u>: If $\phi, \psi \in BMO(\alpha)$, $\lambda \in C$ we have

$$\|\phi + \psi\|_* \leq \|\phi\|_* + \|\psi\|_*$$

$$\|\lambda\phi\|_* = |\lambda| \, \|\phi\|_* .$$

If $\|\phi\|_* = 0$, then for almost all (x,y) $\|\phi_{xy}\|_* = 0$. By [18] Corollary VII.7.4 , since ϕ is then constant on lines , it is constant almost everywhere. The vanishing of $\int \phi \, d\sigma$ ensures $\phi = 0$ almost everywhere.

$BMO(\alpha)$ is therefore a normed linear space.

$$\frac{1}{2T} \int_{-T}^{T} | \phi(x-t, y-\alpha t)| dt \leq \frac{1}{2T} \int_{-T}^{T} |\phi(x-t,y-\alpha t) - \phi_{xyJ}| dt + \left| \frac{1}{2T} \int_{-T}^{T} \phi(x-t,y-\alpha t) dt \right|$$

$$\leq \underset{(x,y)}{\text{ess sup}} \|\phi_{xy}\|_* + \left| \frac{1}{2T} \int_{-T}^{T} \phi(x-t,y-\alpha t) dt \right| ;$$

letting $T \to \infty$, we obtain, using Helson's result,

$$\int |\phi| \, d\sigma \leq \|\phi\|_* .$$

A Cauchy sequence in $BMO(\alpha)$ is therefore Cauchy in $L^1(\sigma)$.

Let $\{\phi_n\}$ be such a Cauchy sequence. Then given $\varepsilon > 0$, there is an n_o such that,

$$\text{if } m,n \geq n_o, \quad \|\phi_n - \phi_m\|_* < \varepsilon \ .$$

There is a $\phi \in L^1(\sigma)$ such that $\phi_n \to \phi$ in $L^1(\sigma)$ and some subsequence $\{\phi_{n_k}\}$ converges to ϕ pointwise almost everywhere.

Using Fatou's lemma we obtain

$$\frac{1}{2T} \int_{-T}^{T} |(\phi - \phi_m) - (\phi - \phi_m)_J| dt < \varepsilon \quad \text{for all } m \geq n_o$$

all $T > 0$, almost all (x,y). We can deduce easily that $\phi \in BMO(\alpha)$ and $\|\phi_n - \phi\|_* \to 0$ as $n \to \infty$.

It is immediate that all bounded, measurable functions on the Torus are in $BMO(\alpha)$. We shall now prove that the conjugate of an $L^\infty(\sigma)$ function is in $BMO(\alpha)$.

<u>Theorem 4.18</u>: The map $f \longrightarrow \tilde{f}$ is a bounded map of $L^\infty(\sigma)$ into $BMO(\alpha)$.

<u>Proof</u>: Let $f \in L^\infty(\sigma)$. Then by theorem 4.2 $\tilde{f}$ certainly exists and is in $L^p(\sigma)$ for $1 \leq p < \infty$.

Let us choose a sequence of trigonometric polynomials $\{f_n\}$ such that

$f_n \longrightarrow f$ pointwise almost everywhere,

$\|f_n\|_\infty \leq \|f\|_\infty$ for all n,

$\tilde{f}_n \longrightarrow \tilde{f}$ pointwise almost everywhere.

Define $(T_\alpha g)(x,y) = \text{P.V.} \frac{1}{\pi} \int \frac{g(x-t, y-\alpha t)dt}{t}$, for trigonometric polynomials

g, where by P.V. we mean the Cauchy Principal Value of the integral.

We shall show this is certainly well defined for trigonometric polynomials g, and in fact gives the conjugate function of g.

It is enough to show that

$$\begin{aligned} T_\alpha e^{i(mx+ny)} &= i\, e^{i(mx+ny)} && \text{if } (m+n\alpha) < 0, \\ &= 0 && \text{if } m = n = 0, \\ &= -i\, e^{i(mx+ny)} && \text{if } (m+n\alpha) > 0. \end{aligned}$$

The case $m = n = 0$ is clear.

Now $\text{P.V.} \frac{1}{\pi} \int e^{i(mx+ny)} \frac{e^{-i(m+n\alpha)t}}{t} dt$

$$= e^{i(mx+ny)} \;\text{P.V.} \frac{1}{\pi} \int \frac{e^{-i(m+n\alpha)t}}{t} dt .$$

If $m + n\alpha < 0$, $\frac{1}{\pi} \text{P.V.} \int \frac{e^{-i(m+n\alpha)t}}{t} dt = \frac{1}{\pi} \text{P.V.} \int \frac{e^{iu}}{u} du = i$.

If $m + n\alpha > 0$, $\frac{1}{\pi} \text{P.V.} \int \frac{e^{-i(m+n\alpha)t}}{t} dt = -\frac{1}{\pi} \text{P.V.} \int \frac{e^{iu}}{u} du = -i$,

by the usual complex variable argument. So T_α agrees with the conjugate map on trigonometric polynomials.

In [17] Fefferman and Stein prove that the Hilbert Transform is a bounded map of $L^\infty \to BMO$.

Since $(T_\alpha g)(x-t, y-\alpha t) = \frac{1}{\pi} \text{P.V.} \int \frac{g(x-t-s,\, y-\alpha t-\alpha s)}{s} ds$

= Hilbert transform of the function $g_{xy}(t)$, we have that

$$\frac{1}{|J|} \int_J |(T_\alpha f_n)(x-t, y-\alpha t) - (T_\alpha f_n)_{xyJ}| \, dt \le A \, \|f_n\|_*$$

for all J, a subinterval of R, almost all (x,y) and $n \in N$.

So

$$\frac{1}{|J|} \int_J | \tilde{f}_n(x-t, y-\alpha t) - \tilde{f}_{n\,xyJ}| \, dt \leq A \|f_n\|_\infty \leq A \|f\|_\infty .$$

Letting $n \to \infty$ and using the fact that $\tilde{f}_n \to \tilde{f}$ pointwise almost everywhere, and Fatou's lemma, we obtain

$$\frac{1}{|J|} \int_J | \tilde{f}(x-t, y-\alpha t) - \tilde{f}_{xyJ}| \, dt \leq A \|f\|_\infty$$

Also $\left| \int \tilde{f} \, d\sigma \right| \leq (\int |\tilde{f}|^2 d\sigma)^{\frac{1}{2}} \leq (\int |f|^2 d\sigma)^{\frac{1}{2}} \leq \|f\|_\infty$.

So the map $f \longrightarrow \tilde{f}$ is a bounded map of $L^\infty(\sigma)$ into $BMO(\alpha)$.

We shall now prove that $BMO(\alpha)$ is the dual of $H^1(\sigma)$ in the following sense.

Let $H^1_R(\sigma) = \{g : g = \text{Re} f , \ f \in H^1(\sigma)\}$. $H^1_R(\sigma)$ is a real linear space, and is in fact a Banach space equipped with the norm

$$\|g\|_{H^1} = \|g + i\tilde{g}\|_1 \quad (\tilde{g} \text{ is in } L^1(\sigma) \text{ since } g = \text{Re} f , \ f \in H^1(\sigma)) .$$

We shall prove that there is a constant $C > 0$ such that, if ϕ is a real valued function in $BMO(\alpha)$ and g is continuous and in $H^1_R(\sigma)$,

$$\left| \int g \, \phi \, d\sigma \right| \leq C \|g + i\tilde{g}\|_1 \|\phi\|_* .$$

Then since the above set of $g \in H^1_R(\sigma)$ is dense in $H^1_R(\sigma)$ the functional defined by

$$\lambda(g) = \int g \, \phi \, d\sigma$$

extends by continuity to all of $H^1_R(\sigma)$ and $\|\lambda\| \leq C \|\phi\|_*$.

Conversely every continuous linear functional λ on $H^1_R(\sigma)$ gives rise to a $\phi \in BMO(\alpha)$ such that

$$\lambda(g) = \int g\,\phi\,d\sigma \quad \text{for all } g \in L^2_R(\sigma) \text{ say}$$

(which is dense in $H^1_R(\sigma)$). We shall prove this second claim first.

Theorem 4.19: (compare [17] Theorem 3) There is a constant $A > 0$, such that, given λ, a continuous linear functional on $H^1_R(\sigma)$, there is a $\phi \in$ real $BMO(\alpha)$ such that

$$\lambda(g) = \int g\,\phi\,d\sigma \quad \text{for all } g \in L^2_R(\sigma)$$

and $\|\phi\|_* \leq A\,\|\lambda\|$.

Proof: Let $B = L^1_R(\sigma) \oplus L^1_R(\sigma)$ normed by

$$\|(g,h)\| = \|g\|_1 + \|h\|_1 .$$

Let S = subspace of B for which $h = \tilde{g}$. S is a closed subspace of B - this follows from the completeness of $H^1(\sigma)$.

Any continuous linear functional λ on $H^1_R(\sigma)$ can be identified with a corresponding functional on S, since the norms on S and $H^1_R(\sigma)$ are clearly equivalent. So, by the Hahn Banach Theorem λ extends, without increase of norm, to a continuous linear functional on B.

The dual space of B is equivalent to $L^\infty_R(\sigma) \oplus L^\infty_R(\sigma)$ so there are $u, v \in L^\infty_R(\sigma)$ such that

$$\lambda(g) = \int gu\,d\sigma + \int \tilde{g}v\,d\sigma$$

and there is a $B > 0$ independent of λ, u and v such that $\|u\|_\infty \leq B\|\lambda\|$, $\|v\|_\infty \leq B\|\lambda\|$.

Now if $g \in L^2_R(\sigma)$, since v is also in $L^2_R(\sigma)$,

$$\int \tilde{g}v\,d\sigma = -\int g\tilde{v}\,d\sigma \quad \text{(for example, compare the Fourier Series);}$$

$$\text{so} \quad \lambda(g) = \int g(u-\tilde{v})\,d\sigma .$$

By theorem 4.18, $u - \tilde{v} \in BMO(\alpha)$ and $\|u-\tilde{v}\|_* \leq A\,\|\lambda\|$.

We shall now prove that each $\phi \in$ real $BMO(\alpha)$ gives rise to a bounded linear functional on $H^1_R(\sigma)$.

We shall first discuss a certain technique which allows us to use Fefferman and Stein's result on the duality of H^1 and BMO .

If $f \in C(T^2)$

$$\frac{T}{\pi} \int \frac{f(x-t,\ y-\alpha t)}{T^2 + t^2}\, dt \longrightarrow \int f\, d\sigma \quad \text{as } T \to \infty ,$$

for almost all (x,y) .

For, if f is a trigonometric polynomial, say $f = \sum a_{mn} e^{i(mx+ny)}$, then

$$\frac{T}{\pi} \int \frac{f(x-t,y-\alpha t)}{T^2 + t^2}\, dt = \sum a_{mn} e^{i(mx+ny)} \frac{T}{\pi} \int \frac{e^{-i(m+n\alpha)t}}{T^2 + t^2}\, dt . \qquad (17)$$

Let us examine $\frac{T}{\pi} \int \frac{e^{i\lambda t}}{T^2 + t^2}\, dt$ for $\lambda \in R$

Setting $t = Tu$ we obtain

$$\begin{aligned} \frac{T}{\pi} \int \frac{e^{i\lambda t}}{T^2 + t^2}\, dt &= \frac{1}{\pi} \int \frac{e^{i\lambda Tu}}{1 + u^2}\, du \\ &= e^{-|\lambda|\, T} \quad \text{if } \lambda \neq 0 \\ &= 1 \quad \text{if } \lambda = 0 \end{aligned}$$

Then, as $T \longrightarrow \infty$ in (17) , we obtain $a_{oo} = \int f\, d\sigma$ as required.

By approximating f uniformly by trigonometric polynomials we obtain the result for continuous f .

More generally, if $g \in L^1(\sigma)$, we can define g_T on T^2 , which is also in $L^1(\sigma)$, by

$$g_T(x,y) = \frac{T}{\pi} \int \frac{g(x-t,\ y-\alpha t)}{T^2 + t^2}\, dt .$$

Also $\int |g_T| d\sigma \leq \frac{T}{\pi} \int \frac{dt}{T^2 + t^2} \int |g| \, d\sigma = \int |g| \, d\sigma$.

We show $g_T \longrightarrow \int g \, d\sigma$ in $L^1(\sigma)$ as $T \to \infty$. Fix $\varepsilon > 0$ and choose $f \in C(T^2)$ such that $\|f-g\|_1 < \varepsilon$.

$$\text{Then } \|g_T - \int g \, d\sigma\|_1 \leq \|g_T - f_T\|_1 + \|f_T - \int f \, d\sigma\|_1 + \| \int f \, d\sigma - \int g \, d\sigma\|_1$$

$$\leq 2 \|g-f\|_1 + \|f_T - \int f \, d\sigma\|_1 \quad .$$

Now $\|f_T\|_\infty \leq \|f\|_\infty$ so the right hand term tends to zero as $T \to \infty$, by the bounded convergence theorem. Since $\varepsilon > 0$ was arbitrary we obtain

$$\lim_{T\to\infty} \|g_T - \int g \, d\sigma\|_1 = 0 \quad .$$

There is, therefore, a sequence $\{T_n\}$ such that $0 < T_1 < T_2 < \dots , \; T_n \to \infty$

and $T_n \int \frac{g(x-t, y-\alpha t)}{T_n^2 + t^2} \, dt \longrightarrow \int g \, d\sigma$ pointwise a.e.

The subsequence depends on the g , of course.

<u>Theorem 4.20</u>: Each $\phi \in$ real $BMO(\alpha)$ gives rise to a continuous linear functional λ on $H^1_R(\sigma)$, which is defined firstly for those $g \in H^1_R(\sigma)$ such that g and $\tilde{g}$ are continuous (a dense subset of $H^1_R(\sigma)$) by

$\lambda(g) = \int g \, \phi \, d\sigma$, and there is a $C > 0$ independent of λ and ϕ such that $\|\lambda\| \leq C\|\phi\|_*$. It then extends by continuity to all of $H^1_R(\sigma)$.

<u>Proof</u>: We show first that for $f \in A_\alpha$ and $\phi \in$ real $BMO(\alpha)$,

$$\left| \int f \, \phi \, d\sigma \right| \leq C\|f\|_1 \, \|\phi\|_* \, . \qquad (*)$$

We then define λ on the dense subset of $H^1_R(\sigma)$, consisting of those g such that $g, \tilde{g}$ are continuous, by

$$\lambda(g) = \int g \phi \, d\sigma = \mathrm{Re} \int (g + i\tilde{g}) \phi \, d\sigma .$$

Now
$$|\lambda(g)| = \left| \mathrm{Re} \int (g + i\tilde{g}) \phi \, d\sigma \right|$$
$$\leq \left| \int (g + i\tilde{g}) \phi \, d\sigma \right|$$
$$\leq C \, \|g + i\tilde{g}\|_1 \, \|\phi\|_* , \quad \text{by } (*) .$$

This is the required result. We now prove (*).

$\phi \in BMO(\alpha)$ and so is in $L^1(\sigma)$. Since $f \in A_\alpha$, $\int f \phi \, d\sigma$ exists.

Fixing f, ϕ choose a sequence $\{T_n\}$ such that

$$\frac{T_n}{\pi} \int \frac{f(x-t, y-\alpha t) \, \phi(x-t, y-\alpha t) dt}{T_n^2 + t^2} \longrightarrow \int f \phi \, d\sigma ,$$

for almost all (x,y) .

This integral is equal to

$$\frac{T_n}{\pi} \int \frac{f(x-t, y-\alpha t)}{(t + i T_n)^2} \quad \frac{(t + i T_n)}{\overline{(t - i T_n)}} \quad \phi(x-y, y-\alpha t) dt .$$

Now (see [18] , Chapter VII, section 7) for almost all (x,y) ,

$\dfrac{f(x-t, y-\alpha t)}{(t + i T_n)^2} \in H^1(dt)$.

We shall show, that for almost all (x,y), $\phi(x-t, y-\alpha t) \dfrac{(t + i T_n)}{\overline{(t - i T_n)}}$ is in BMO and its BMO norm $\leq K \|\phi_{xy}\|_* + K|\phi_{xy2T_n}|$.

K is a constant independent of ϕ and

$$\phi_{xy2T_n} = \frac{1}{2T_n} \int_{-T_n}^{T_n} \phi(x-t, y-\alpha t) dt .$$

Fefferman and Stein's result [17] on the duality of H^1 and BMO gives

$$\left| \frac{T_n}{\pi} \int \frac{f(x-t,\ y-t)\ \phi\ (x-t,\ y-t)dt}{T_n^2 + t^2} \right| \leq C^1 \frac{T_n}{\pi} \int \frac{|f(x-t,\ y-\alpha t)|}{T_n^2 + t^2}\, dt \quad X$$

$$\left(K \|\phi_{xy}\|_* + K \left| \frac{1}{2T_n} \int_{-T_n}^{T_n} \phi(x-t,\ y-t)dt \right| \right)$$

for almost all (x,y) .

Now letting $T_n \to \infty$ in the above, and using either Helson's result ([20] , page 20) or the elementary estimate

$$\left| \frac{1}{2T_n} \int_{-T_n}^{T_n} \phi(x-t,\ y-\alpha t)dt \right| \leq \frac{T_n}{\pi} \int \frac{|\phi(x-t,\ y-t|}{T_n^2 + t^2} dt$$

$$\to \int |\phi| d\sigma \quad \text{as} \quad T_n \to \infty \ ,$$

and $\int |\phi| d\sigma \leq \|\phi\|_*$,

we obtain the result we are seeking, namely,

$$\left| \int f\ \phi\ d\sigma \right| \leq K\ C^1 \|f\|_1 \ (\text{ess sup}\ \|\phi_{xy}\|_* + \left| \int \phi\ d\sigma \right|)$$

$$= C\ \|f\|_1\ \|\phi\|_* \ .$$

Corollary 4.21: Any $\phi \in BMO(\alpha)$ may be written as $\phi = u + \tilde{v}$ where $u, v \in L^\infty(\sigma)$ and there is a $B > 0$ such that we can choose u and v to satisfy

$$\|u\|_\infty \leq B \|\phi\|_* \ , \quad \|v\|_\infty \leq B\|\phi\|_* \ .$$

Proof: Follows immediately from theorems 4.19 and 4.20 .

We are only left to justify the statement that $\frac{(t + i T)}{(t - i T)} \phi(x-t, y-\alpha t)$ is in BMO and obtain the estimate of its BMO norm.

Notice first that the map $\phi(t) \longrightarrow \phi(at)$, $a > 0$, is an isometry of BMO onto itself. It is sufficient therefore to prove $\frac{u+i}{u-i} \psi(u)$ is in BMO if ψ is, where $\psi(u) = \phi(x-uT, y-\alpha uT)$.

We will obtain the estimate of its norm as

$$\leq K (\|\psi\|_* + |\psi_J|) \quad \text{where} \quad \psi_J = \frac{1}{2} \int_{-1}^{1} \psi(u)\, du .$$

Then reversing the process, we obtain BMO norm of

$$\frac{t + i T}{t - i T} \phi(x-t, y-\alpha t) \leq K \left(\|\phi_{xy}\|_* + \left| \frac{1}{2} \int_{-1}^{1} \psi(u)\, du \right| \right) .$$

$$\left| \frac{1}{2} \int_{-1}^{1} \psi(u)\, du \right| = \left| \frac{1}{2T} \int_{-T}^{T} \phi(x-t, y-\alpha t)\, dt \right|$$

So BMO norm $\leq K (\|\phi_{xy}\|_* + |\phi_{xy2T}|)$ as required.

To obtain our result, we require first two lemmas.

<u>Lemma 4.22</u>: [27] Let $\phi \in$ BMO, I and I_r be two concentric intervals with I_r r times the length of I (which has length 1) .

If $r > 1$, $|\phi_I - \phi_{I_r}| \leq 3(1 + \frac{\log r}{\log 2}) \|\phi\|_*$

If $r < 1$, $|\phi_I - \phi_{I_r}| \leq 3(1 + \frac{\log r^{-1}}{\log 2}) \|\phi\|_*$

<u>Proof</u>: We take $r > 1$, the other case is proved similarly.

Consider first $r = 2^s$ where s is an integer. Setting $\phi_s = \phi_{I_{2^s}}$, we obtain

$$|\phi_s - \phi_{s-1}| = \frac{1}{|I_{2^{s-1}}|} \int_{I_{2^{s-1}}} |\phi_s - \phi_{s-1}| dx$$

$$\leq \frac{1}{|I_{2^{s-1}}|} \int_{I_{2^{s-1}}} (|\phi_s - \phi| + |\phi - \phi_{s-1}|) dx$$

$$\leq \frac{2}{|I_{2^s}|} \int_{I_{2^s}} |\phi_s - \phi| dx + \frac{1}{|I_{2^{s-1}}|} \int_{I_{2^{s-1}}} |\phi - \phi_{s-1}| dx$$

$$\leq 2\|\phi\|_* + \|\phi\|_* = 3\|\phi\|_* .$$

Then $|\phi_s - \phi_o| \leq \sum_{k=1}^{s} |\phi_k - \phi_{k-1}| < 3s \|\phi\|_*$.

Suppose now $2^s \leq r < 2^{s+1}$.

Clearly $|\phi_{I_r} - \phi_I| \leq |\phi_{I_r} - \phi_s| + |\phi_s - \phi_o|$

$$\leq |\phi_{I_r} - \phi_s| + 3 \frac{\log 2^s}{\log 2} \|\phi\|_* .$$

Also $|\phi_{I_r} - \phi_s| \leq \frac{1}{|I_{2^s}|} \int_{I_{2^s}} |\phi_{I_r} - \phi| dx + \frac{1}{|I_{2^s}|} \int |\phi - \phi_s| dx$

$$\leq \frac{2}{|I_r|} \int_{I_r} |\phi_{I_r} - \phi| dx + \frac{1}{|I_{2^s}|} \int |\phi - \phi_s| dx$$

$$\leq 3\|\phi\|_* .$$

Thus $|\phi_{I_r} - \phi_I| \leq 3(1 + \frac{\log r}{\log 2}) \|\phi\|_*$.

<u>Lemma 4.23</u>: Let $\phi \in$ BMO. I_1 and I_2 are intervals of length 1 . If I_1 and I_2 are disjoint and the distance between their midpoints is r $(r > 1)$

then $|\phi_{I_1} - \phi_{I_2}| \leq \left[6 (1 + \frac{\log r}{\log 2}) + 2 \right] \|\phi\|_*$.

If I_1 and I_2 intersect then $|\phi_{I_1} - \phi_{I_2}| \le 2\|\phi\|_*$.

Proof: Let I_{1r} (resp. I_{2r}) be intervals concentric with $I_1(I_2)$ whose length is r times the length of I_1 (I_2).

I_{1r} and I_{2r} are adjacent.

Then $|\phi_{I_{1r}} - \phi_{I_{2r}}| \le |\phi_{I_{1r}} - \phi_{I_{1r} \cup I_{2r}}| + |\phi_{I_{1r} \cup I_{2r}} - \phi_{I_{2r}}|$

$$\le \frac{1}{|I_{1r}|} \int_{|I_{1r}|} |\phi - \phi_{I_{1r} \cup I_{2r}}| dx + \frac{1}{|I_{2r}|} \int_{I_{2r}} |\phi - \phi_{I_{1r} \cup I_{2r}}| dx .$$

Using $|I_{1r}| = |I_{2r}| = \frac{1}{2} |I_{r1} \cup I_{r2}|$, the above is

$$\le \frac{2}{|I_{r1} \cup I_{r2}|} \int_{I_{r1} \cup I_{r2}} |\phi - \phi_{I_{1r} \cup I_{2r}}| dx$$

$$\le 2\|\phi\|_* . \tag{18}$$

Using lemma 4.22 and (18) we have

$$|\phi_{I_1} - \phi_{I_2}| \le |\phi_{I_1} - \phi_{I_{1r}}| + |\phi_{I_{1r}} - \phi_{I_{2r}}| + |\phi_{I_{2r}} - \phi_{I_2}|$$

$$\le \left[6(1 + \frac{\log r}{\log 2}) + 2 \right] \|\phi\|_* .$$

If I_1 and I_2 overlap a proof similar to that of (18) gives the result.

Theorem 4.24: Let $\phi \in BMO$. $f(t) = \frac{1}{t-i}$. Then $f\phi \in BMO$ and $\|f\phi\|_* \le K|\phi_I| + K\|\phi\|_*$ where K is a constant and $I = [-\frac{1}{2}, \frac{1}{2}]$. (In theorem 4.20 $I = [-1,1]$ but lemma 4.22 shows this makes no difference).

Proof: J is an interval of length $|J|$.

Let us first suppose $J = [a, a + |J|]$, where a is nearer to zero than $a + |J|$. This is no restriction – the same arguments would work in the other case.

$$\frac{1}{|J|} \int_J |f\,\phi - f_J\,\phi_J|\,dx \le \frac{1}{|J|}\int_J |\phi - \phi_J|\,|f|\,dx + \frac{|\phi_J|}{|J|}\int_J |f - f_J|\,dx$$

$$\le \|\phi\|_* + \frac{|\phi_J|}{|J|}\int_J |f - f_J|\,dx ,$$

since $|f| \le 1$ everywhere.

Suppose now $|J| > 1$, and J^1 is an interval of length 1 concentric with J.

$$|\phi_J| \le |\phi_{J^1}| + 3\left(1 + \frac{\log |J|}{\log 2}\right) \|\phi\|_* , \tag{19}$$

using lemma 4.22.

$$|\phi_{J^1}| \le |\phi_I| + \left[6\left(\frac{1 + \log \left|a + \frac{|J|}{2}\right|}{\log 2} \right) + 2 \right] \|\phi\|_* , \tag{20}$$

$$\left(\text{if } a + \frac{|J|}{2} > 1 \right)$$

$$|\phi_{J^1}| \le |\phi_I| + 2\,\|\phi\|_* \quad \left(\text{if } a + \frac{|J|}{2} \le 1\right) . \tag{21}$$

Both inequalities are obtained using lemma 4.23.

Now $\displaystyle\int_J |f - f_J|\,dx = \int_a^{a+|J|} |f - f_J|\,dx$.

Either $|J| > |a|$ or $|J| \le |a|$.

If $|a| \ge |J|$, then $a > 0$ and we obtain

$$\int_a^{a+|J|} |f - f_J|dx \le 2\int^{a+|J|} |f|dx \le 2\int^{a+|J|} \frac{1}{x}\,dx$$

$$= 2 \log \frac{a + |J|}{a} .$$

Now using (19) and (20) we obtain

$$\frac{|\phi_J|}{|J|} \int_J |f - f_J|dx \le \left\{ |\phi_I| + 11\|\phi\|_* + 3\frac{\log |J|}{\log 2} \|\phi\|_* \right.$$

$$\left. + 6 \frac{\log \frac{a + |J|}{2}}{\log 2} \|\phi\|_* \right\} \frac{1}{|J|} \int_J |f - f_J|dx$$

$$\le 2|\phi_J| + 22 \|\phi\|_* +$$

$$\left\{ 3\frac{\log |J|}{\log 2} + 6 \frac{\log \frac{a + |J|}{2}}{\log 2} \right\} \frac{\|\phi\|_*}{|J|} \int_J |f - f_J|dx . \qquad (22)$$

In the case $a \ge |J|$ the above is

$$\le 2|\phi_J| + 22 \|\phi\|_* + \left\{ 3\frac{\log a}{\log 2} + 6 \frac{\log \frac{3a}{2}}{\log 2} \right\} \|\phi\|_* \frac{2}{a} ,$$

using $\log \frac{a + |J|}{a} \le \frac{|J|}{a}$, since $\frac{|J|}{a} \le 1$, the above is

$$\le 2|\phi_J| + K \|\phi\|_* \quad \text{where } K \text{ is a constant.}$$

(using $\frac{\log t}{t} \le e^{-1}$ for $t \ge 1$.)

Suppose now $|J| > |a|$. We first consider $0 < a < |J|$.

Then $\frac{1}{|J|} \int_J |f - f_J|dx \le \frac{2}{|J|} \int_J |f|dx$

$$\le \frac{2}{|J|} \left\{ \int_{\frac{1}{2}}^{a+|J|} |f|dx + \int_a^{\frac{1}{2}} |f|dx \right\} .$$

(Interpret the second integral as zero if $a \ge \frac{1}{2}$)

$$\leq \frac{2}{|J|} \log 2|J| + \frac{2}{|J|} .$$

Inserting into (22) and using $|J| \geq 1$ we obtain (22) $\leq 2|\phi_I| + K \|\phi\|_*$.

If $|J| > |a|$, but $a < 0 < a + |J|$,

$$\frac{2}{|J|} \int_a^{a+|J|} |f| dx \leq \frac{2}{|J|} \left\{ \int_a^{-\frac{1}{2}} + \int_{-\frac{1}{2}}^{\frac{1}{2}} + \int_{\frac{1}{2}}^{a+|J|} \right\} |f| dx$$

$$\leq \frac{2}{|J|} \log 2|J| + \frac{2}{|J|} + \frac{2}{|J|} \log 2|J| ,$$

so again we obtain the required result.

Suppose now $|J| \leq 1$. If $a \geq |J|$ the same estimate as before will work, for the last part of (22). The last part provides no problem if $a \leq |J|$ also.

The problem arises with $\frac{\log |J|^{-1}}{\log 2} \frac{1}{|J|} \int_J |f - f_J| dx$.

We shall estimate $\frac{1}{|J|} \int_J |f - f_J| dx$. Notice that if $x, c \in R$ $|f(x) - f(c)| \leq |x - c|$. Let c be the midpoint of the interval J.

Then $\frac{1}{|J|} \int_J |f - f_J| dx \leq \frac{1}{|J|} \int_J |f(x) - f(c)| dx + |f(c) - f_J|$

$$\leq \frac{2}{|J|} \int_J |f(x) - f(c)| dx$$

$$\leq 2 \sup_{|x-c| \leq \frac{|J|}{2}} |f(x) - f(c)|$$

$$\leq |J| ,$$

Now, since we are considering $|J| \leq 1$,

$$\log |J|^{-1} \frac{1}{|J|} \int |f - f_J| dx \leq \sup_{|J| \leq 1} |J| \log |J|^{-1} \leq e^{-1} .$$

Thus in all cases we obtain an estimate of (22) as $\leq K (|\phi_I| + \|\phi\|_*)$ where K is a constant independent of ϕ.

Since $\frac{t + i}{t - i} \phi(t) = \left(1 + \frac{2i}{t - i}\right) \phi(t)$, the former function is in BMO if $\frac{\phi(t)}{t - i}$ is, with the appropriate condition on the norm.

The space $BMO(\alpha)$ thus plays an analogous role to that of BMO on the real line or the circle.

REFERENCES

PART 1 : A.P. IN BANACH SPACES

1. A.M. Davie: "The Approximation Property for Banach Spaces" Bull. London Math. Soc. 5 (1973), 261-266.

2. W.J. Davis and R.R. Phelps: "The Radon Nikodym Property and Dentable Subsets in Banach Spaces" Proc. Amer. Math. Soc. 45 (1974).

3. Dunford and Schwartz: "Linear Operators", Part 1 Interscience, New York.

4. P. Enflo: "A Counterexample to the Approximation Property in Banach Spaces" Acta. Math. 130(1973), 309-317.

5. A. Grothendieck: "Produits Tensorieles Topologiques et Espaces Nucleaires" Memoirs Amer. Math. Soc. No.16 (1955).

6. R.E. Huff: "Dentability and the Radon Nikodym Property" Duke Math J. 41(1974) 111-114.

7. R.E. Huff and P.D. Morris: "Dual Spaces with the Krein Milman Property have the Radon Nikodym Property". Proc. Amer. Math. Soc. 49(1975), 104-108.

8. H.B. Maynard: "A Geometrical Characterisation of Banach Spaces with the Radon Nikodym Property" Trans. Amer. Math. Soc. 185 (1973), 493-500.

9. A. Persson: "On Some Properties of P-Nuclear and P-Integral Operators". Studia Math. 33 (1969), 213-222.

10. A. Persson and A. Pietsch: "P-Nukleare und P-Integrale Abbildungen in Banachraumen" Studia Math. 33(1969), 19-62.

11. R.R. Phelps: "Dentability in Banach Spaces" J. Functional Analysis 17(1974), 78-90.

12. R.S. Phillips: "On Weakly Compact Subsets of a Banach Space" Amer. J. Math 65(1943), 108-136.

13. M.A. Rieffel: "The Radon Nikodym Theorem for the Bochner Integral" Trans. Amer. Math. Soc. 131(1968), 466-487.

14. M.A. Rieffel: "Dentable Subsets of Banach Spaces, with Applications to a Radon Nikodym Theorem" Proc. Conf. Functional Analysis, Thompson Book Co. Washington D.C. (1967), 71-77.

15. A. Szankowski: "A Banach Lattice without the Approximation Property" Notes of a lecture given at the 1975 L.M.S. Symposium on Convexity, Durham.

PART 2 : WEIGHT FUNCTIONS ON THE TORUS

16. A. Devinatz: "Toeplitz Operators on H^2- spaces" Trans. Amer. Math. Soc. 112(1964), 304-317.

17. C.L. Fefferman and E.M. Stein: "H^p- spaces of Several Variables" Acta Math. 129(1972) 137-193.

18. T.W. Gamelin: "Uniform Algebras", Prentice Hall, 1969.

19. G.H. Hardy and E.M. Wright: "An Introduction to the Theory of Numbers". Oxford, 1954.

20. H. Helson: "Analyticity on Compact Abelian Groups" in "Algebras in Analysis" edited by J.H. Williamson, Academic Press, 1975, 2-62.

21. H. Helson and D. Lowdenslager: "Prediction Theory and Fourier Series in Several Variables" Acta Math 99 (1958) 165-202.

22. H. Helson and D. Lowdenslager: "Prediction Theory and Fourier Series in Several Variables II" Acta Math 106 (1961), 175-213.

23. H. Helson and G. Szegö: "A Problem in Prediction Theory". Annali di Mathematica 51 (1960) 107-138.

24. F. John and L. Nirenberg: "On Functions of Bounded Mean Oscillation". Comm. on Pure and Applied Maths 14(1961) 415-426.

25. P.S. Muhly: "The Distant Future" Indiana Univ. Math J. 24(1974) 149-159.

26. Y. Ohno: "Remarks on Helson Szegö Problems" Tohoku Math J. 18(1966) 54-59.

27. H.M. Reimann and T. Rychener: "Funktionen Beschränkter Mittlerer Oszillation" Springer Lecture Notes 487.

28. W. Rudin: "Fourier Analysis on Groups", Interscience, 1962.

29. A. Zymund: "Trigonometric Series" Cambridge.