EXTENSIONS OF THE CONCEPT OF NUMERICAL RANGE

DEREK JAMES BAKER

Doctor of Philosophy

University of Edinburgh

1974

CONTENTS

PREFACE

This thesis is concerned with attempts to extend the concept of numerical range. Up to the present time this work has largely been confined to extensions for the algebra of operators on Hilbert space. In a series of papers beginning in 1969 W.B. Arveson [4, §26] replaced the algebra numerical range of an operator on Hilbert space by sets of matrices and showed that these "matrix ranges" form a complete set of unitary invariants for irreducible compact operators. At about the same time S.K. Parrott [4, §26] considered a matrix range which generalised the spatial numerical range. He showed that the sequence of these ranges form a complete set of unitary invariants for compact operators with trivial reducing nullspaces. This range is closely related to the concept which we investigate in our first chapter.

It was noted by P.R. Halmos in his Hilbert space problem book [12, Page 111] that the classical spatial numerical range of a Hilbert space operator is a particular case of a multi-dimensional concept. It is a study of this range, which we call the Halmos k-numerical range, that forms the content of our first chapter. Apart from a result on the convexity of this k-range no published work seems to have appeared until 1971 when P.A. Fillmore and J.P. Williams [11] exploited the range for operators on finite dimensional spaces. We confine our study to operators on Hilbert spaces of infinite dimension. The sets possess many elegant properties. We show that the behaviour of the k-ranges characterises compactness. A simple description of the k-range of a compact normal operator allows us to deduce that the ranges form a complete set of unitary invariants for compact normal operators with trivial kernels. The full unitary invariants theorem of S.K. Parrott does not hold for the Halmos ranges. We are unable to give any general necessary and sufficient conditions for the k-range of an operator to be closed, however we give a positive result for the special case of the k-range of a compact operator. The background to the chapter is one of standard functional analysis and operator theory, the most well used result being the Spectral theorem. In the face of the same notation occuring in the established literature for two different concepts we follow F.F. Bonsall and J. Duncan [4] and use $W_k(A)$ for the Parrott matrix range of A, and we adopt $P_k(A)$ to denote the Halmos k-range of A.

We have referred to the main results in the successful theory of matrix ranges of operators on Hilbert space. The problem of whether corresponding ideas can be formulated and developed for operators between general Banach spaces has been raised by F.F. Bonsall. We are indebted to F.F. Bonsall for access to some unpublished ideas on the subject. Faced with the scarcity of structure on a completely general Banach space and the problems of developing a general theory which result, we have taken a particular example of the Banach space of summable sequences. This space possesses a readily identifiable dual and pre-dual which can be exploited. We introduce a definition and develop a theory. We show that our matrix ranges are "invariants" for compact diagonal operators and compact weighted shifts with zero kernels.

Finally chapter 3 introduces a new concept which extends the notion of the numerical range of an element of an arbitrary unital normed algebra. We have christened this extension the Williams k-numerical range. The observation arose from a description of the numerical range by J.P. Williams as an intersection of closed discs. We examine the implications for the algebra that the Williams range should be a printer for each integer k, and we consider a natural extension of the k-range to a joint concept for several elements of a unital normed algebra.

So far as the background material on numerical ranges is concerned our list of references does not include any individual papers. We have tried to attribute known results where possible to the author concerned and refer the reader to the books of F.F. Bonsall and J. Duncan [3] and [4]. These provide an invaluable exposition of the known work in this field together with an extensive list of references.

The work contained in this thesis was carried out at Edinburgh University as a research student under Professor F.F. Bonsall. I wish to record my appreciation of his good advice and guidance. He has shown a continued interest in my work and offered ideas and much constructive criticism and encouragement.

I am fortunate to have had discussions with Professors P.R. Halmos and J.P. Williams during their visits to Britain in 1973

Finally I owe a considerable debt of gratitude to my parents and friends for their understanding and encouragement.

For the past three years I have been supported by a Research Studentship from the Science Research Council.

Edinburgh July 1974

NOTATION

Throughout this thesis we adopt the following notation. An index of symbols on page 86 incorporates additional notation introduced during the text.

$\underset{\sim}{R}$, $\underset{\sim}{C}$ denote the sets of real and complex numbers respectively.

$\underset{\sim}{N}$ denotes the set of positive integers.

μ^* denotes the complex conjugate of an element $\mu \in \underset{\sim}{C}$.

$\mathrm{Arg}\,\mu$ denotes the argument of $\mu \in \underset{\sim}{C}$ $(0 \leq \mathrm{Arg}\,\mu < 2\pi \; ; \; \mathrm{Arg}\, 0 \equiv 0)$.

$\mathrm{Re}\,\mu$, $\mathrm{Im}\,\mu$ denote the real and imaginary parts respectively of $\mu \in \underset{\sim}{C}$.

$\Delta(\lambda;r)$, Δ $\Delta(\lambda;r) = \{ z \in \underset{\sim}{C} : |z - \lambda| < r \}$ $(\lambda \in \underset{\sim}{C} \; ; \; r>0)$.

$\Delta(0;1)$ is generally abbreviated to Δ .

$\widehat{K}$ denotes the polynomial convex hull of the compact set $K \subset \underset{\sim}{C}^n$.

$\overline{S}$ denotes the closure of a subset S of a topological space.

∂S denotes the boundary of S (i.e. $\partial S = \overline{S} \cap \overline{S^c}$).

$\mathrm{int}\, S$ denotes the interior of S .

$\mathrm{Ext}\, S$ denotes the set of extreme points of a subset S of a linear topological space.

Let (X,λ) be a Banach space. Let $Z \subseteq X$.

$\mathrm{Span}\, Z$ denotes the set of all finite linear combinations of members of Z .

$S_\lambda(X)$ (abbreviated to $S(X)$ when the norm on X is understood) denotes the unit shell of X (i.e. $S_\lambda(X) = \{x \in X : \lambda(x) = 1\}$).

$B(X)$ denotes the space of all bounded linear operators mapping X into X. The abbreviation of bounded linear operator to operator is used throughout.

$K(X)$ denotes the bi-ideal of compact operators in $B(X)$.

$\ker(T)$ denotes the kernel of $T \in B(X)$.

Sp(T), pSp(T) denote the spectrum and point spectrum respectively of $T \in B(X)$.

Let H be a complex Hilbert space.

dim S denotes the (Hilbert) dimension of a subspace S of H.

T^* denotes the adjoint of $T \in B(H)$.

Re T, Im T denote the real $(= (T+T^*)/2)$ and imaginary $(= (T-T^*)/2i)$ parts respectively of $T \in B(H)$.

Let A be a unital normed algebra.

$r(a), v(a)$ denote the spectral radius and numerical radius respectively of $a \in A$.

$D(A,1)$ (abbreviated to $D(A)$) denotes the set of states of A (i.e. $D(A,1) = \{ f \in A' : \|f\| = f(1) = 1 \}$).

DECLARATION

This thesis embodies the results of my own work and has been composed by myself.

signed: D J Baker

CHAPTER 1

THE HALMOS k-NUMERICAL RANGE

§1. Definition, properties and examples.

Our aim has been to study the k-range of operators on Hilbert spaces of infinite dimension. In this account we omit discussion of the special properties which result from an assumption of finite dimensionality on the underlying Hilbert space. These are well documented in the paper entitled "Some convexity theorems for matrices" by P.A. Fillmore and J.P. Williams [11]. We state and prove those properties which hold for operators on Hilbert spaces of infinite dimension. The section ends with some examples which illustrate types of behaviour characteristic for certain classes of operators.

Notation. Let H be an infinite dimensional complex Hilbert space.

$\underset{\sim}{P}_k$ denotes the set of all (orthogonal) projections of rank k.

$\underset{\sim}{0}_k$ denotes the set of all orthonormal k-tuples of elements of H.

$\mathcal{C}_1$ denotes the two sided ideal of trace class operators in B(H) and $\|.\|_1$ denotes the trace class norm on $\mathcal{C}_1$. We use the abbreviation tr(A) to mean the trace of A whenever $A \in \mathcal{C}_1$.

Definition 1. Let $k \in \underset{\sim}{N}$. The Halmos k-numerical range of an operator $A \in B(H)$ is the set of complex numbers

$$P_k(A) = \{ \frac{1}{k} \operatorname{tr}(PA) : P \in \underset{\sim}{P}_k \} .$$

The following proposition provides a description of $P_k(A)$ in terms of the set of orthonormal k-tuples of elements of H. The relationship between $P_k(A)$ and the classical numerical range

$W(A)$ becomes transparent.

PROPOSITION 2. Let $A\in B(H)$. Then for $k=1,2,\ldots$

$$P_k(A) = \{ \frac{1}{k}\sum_{i=1}^{k}(Ax_i,x_i) : (x_1,x_2,\ldots,x_k)\in \underset{\sim}{O}_k \} .$$

Proof.

Given $P\in \underset{\sim}{P}_k$, choose any orthonormal k-tuple $(x_1,x_2,\ldots,x_k)$ of elements in the range of P. Then

$tr(PA) = tr(AP) = \sum_{i=1}^{k}(APx_i,x_i) = \sum_{i=1}^{k}(Ax_i,x_i)$. Therefore we have the inclusion $\subseteq$. Conversely, given $(x_1,x_2,\ldots,x_k)\in \underset{\sim}{O}_k$ let P be the orthogonal projection onto $\mathrm{Span}\{x_1,x_2,\ldots x_k\}$. Then $\frac{1}{k}\sum_{i=1}^{k}(Ax_i,x_i) = \frac{1}{k}\sum_{i=1}^{k}(APx_i,x_i) = \frac{1}{k}tr(AP) = \frac{1}{k}tr(PA)$ which gives the inclusion $\supseteq$. $\square$

THEOREM 3. Let $A\in B(H)$, $k\in \underset{\sim}{N}$. Then

(1) $P_1(A) = W(A)$, the ordinary numerical range of A ,

$P_k(A^*) = P_k(A)^*$; $P_k(\alpha A + \beta) = \alpha P_k(A) + \beta$ $(\alpha,\beta\in \underset{\sim}{C})$.

(2) $P_k(A)$ is convex;

(3) $P_{k+1}(A) \subseteq P_k(A)$;

(4) $P_k(U^{-1} A U) = P_k(A)$ if U is unitary;

(5) $P_k(A)$ contains each arithmetic mean of k eigenvalues of A, where each eigenvalue may occur in a mean at most as many times as its multiplicity.

Proof. Parts (1) and (4) are immediate from Proposition 2.

Part (2) was first proved by C.A.Berger and we include here an argument based on a proof by Halmos [12].

Let $P,Q \in \underset{\sim}{P}_k$, $0\leq\alpha\leq 1$. Let T be the operator $QP|_{PH}$ regarded as a map from PH into QH. Let $T= WS$ be the polar decomposition of T where W is a partial isometry from PH into QH and S is a positive operator mapping PH into PH. Note that $\dim(PH) = \dim(QH) = k$.(1) ~~Therefore we may extend W to an isometry U, if necessary, of PH onto QH (choose any isometry U from PH onto QH such that $U|_{(\ker W)^\perp} = W|_{(\ker W)^\perp}$)~~. It is a consequence of the construction of the polar decomposition of T that the initial space of W equals $(\text{Range } S)^-$ and therefore we have $T = WS = US$. By the finite dimensional Spectral theorem there exists an orthonormal basis $\{x_1, x_2, \ldots x_k\}$ for PH such that $Sx_i = \alpha_i x_i$ $(i=1,2,\ldots,k)$ for some scalars $\alpha_i \geq 0$ $(i=1,2,\ldots,k)$. Let $y_i = Ux_i$ $(i=1,2,\ldots,k)$. $\{y_1, y_2, \ldots, y_k\}$ is an orthonormal basis for QH. For each $i=1,2,\ldots,k$ let Z_i be the linear span of x_i and y_i. The subspaces Z_i are pairwise orthogonal. To prove this assertion, it suffices to show that $x_i \perp y_j$ whenever $i \neq j$ since $\{x_1, x_2, \ldots, x_k\}$ and $\{y_1, y_2, \ldots, y_k\}$ are orthonormal k-tuples. However

$$(x_i, y_j) = (Px_i, Qy_j) = (QPx_i, y_j) = (Tx_i, y_j) = \alpha_i(y_i, y_j) = 0$$

whenever $i \neq j$ and therefore the assertion holds.

By the Toeplitz-Hausdorff theorem, there exist $z_i \in Z_i$, $\|z_i\|=1$ such that

$$(Az_i, z_i) = \alpha(Ax_i, x_i) + (1-\alpha)(Ay_i, y_i) \quad (i=1,2,\ldots,k).$$

$\{z_1, z_2, \ldots, z_k\} \in \underset{\sim}{0}_k$ and therefore

$$\alpha \frac{1}{k} \operatorname{tr}(PA) + (1-\alpha)\frac{1}{k} \operatorname{tr}(QA) = \frac{\alpha}{k}\sum_{i=1}^{k}(Ax_i, x_i) + \frac{(1-\alpha)}{k}\sum_{i=1}^{k}(Ay_i, y_i)$$

$$= \frac{1}{k}\sum_{i=1}^{k}(Az_i, z_i) \in P_k(A). \ \square$$

(3). Let $\lambda \in P_{k+1}(A)$, then there exists an orthonormal $k+1$-tuple $\{x_1, x_2, \ldots, x_{k+1}\}$ such that $\lambda = \frac{1}{k+1}\sum_{i=1}^{k+1}(Ax_i, x_i)$.

1. Therefore there exists a unitary mapping $U : PH \to QH$ which coincides with W on the initial space of W.

Any average μ of k numbers selected from the set $\{ (Ax_1,x_1),(Ax_2,x_2),\ldots,(Ax_{k+1},x_{k+1}) \}$ is a member of $P_k(A)$. λ can be expressed as $\lambda = \frac{1}{k+1}\sum \mu$ where the sum is taken over all possible (k+1) distinct averages μ. λ is therefore a convex combination of elements of $P_k(A)$ which again belongs to $P_k(A)$ by part (2).

(5). The (geometric) multiplicity of an eigenvalue λ of A is defined to be the (Hilbert) dimension of the kernel of $\lambda I-A$. Let $\lambda_1,\lambda_2,\ldots,\lambda_k$ be eigenvalues of A which satisfy the repetition condition. Let P be the (orthogonal) projection onto the linear span of the corresponding eigenvectors where, for repeated eigenvalues we choose orthogonal eigenvectors. With this precaution we have $P \in \underset{\sim}{P}_k$. By the triangulation theorem for finite dimensional Hilbert spaces, there exists an orthonormal basis $\{ e_1,e_2,\ldots,e_k \}$ for PH relative to which the matrix for $PA|_{PH}$ (regarding $PA|_{PH}$ as an operator mapping PH into PH) has triangular form. The eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_k$ occur down the main diagonal. Therefore

$$\lambda_i = \text{the } ii^{\text{th}} \text{ entry of the matrix for } PA|_{PH}$$

$$= (PAe_i,e_i) = (Ae_i,e_i) \qquad (i=1,2,\ldots,k).$$

Therefore $\frac{1}{k}\sum_{i=1}^{k} \lambda_i \in P_k(A)$. □

SOME EXAMPLES.

1. Projections. Let Q be a projection on H, ~~then if~~

(1). If Q has infinite dimensional kernel and range, then

$$P_k(Q) = [0,1] \quad (k=1,2,\ldots).$$

(2). If $\dim(QH) = m < \infty$, then $P_k(Q) = \begin{cases} [0,1] & k=1,2,\ldots,m \\ [0,\frac{m}{k}] & k=m+1,m+2,\ldots \end{cases}$

Proof. The k-range of Q is bounded by the norm of Q. Also $tr(PQ) \geq 0$ $(P \in \underset{\sim}{P}_k)$. Therefore $0 \leq P_k(Q) \leq 1$. For part (1) and part (2) when $1 \leq k \leq m$, select any orthonormal k-tuple $\{x_1, x_2, \ldots, x_k\}$ of elements lying in the range of Q. Then $\frac{1}{k}\sum_{i=1}^{k}(Qx_i, x_i) = 1 \in P_k(Q)$. Similarly 0 is obtained as an element of the k-range by using any k orthonormal elements from the kernel of Q. Now suppose $k > m$ and let P be the orthogonal projection onto any k-dimensional subspace of H containing QH. $PQ = Q$ and therefore $\frac{1}{k}\,tr(PQ) = \frac{1}{k}\,tr(Q) = \frac{m}{k}$. For each $P \in \underset{\sim}{P}_k$ we have $\frac{1}{k}\,tr(PQ) \leq \frac{1}{k}\,\|P\|\,\|Q\|_1 = \frac{m}{k}$ and the result follows from the convexity of the k-range. $\square$

2. Isometries and Unitaries.

(1). Let U be a unilateral shift, then

$$P_k(U) = \Delta = \{\, z \in \underset{\sim}{C} : |z| < 1 \,\} \quad (k=1,2,\ldots).$$

(2). Let W be a bilateral shift, then $P_k(W) = \Delta$ $(k=1,2,\ldots)$.

Proof. (1). $U \in B(H)$ is a unilateral shift if there exists an orthonormal basis $\{e_n\}_1^{\infty}$ of H such that $Ue_n = e_{n+1}$ $(n=1,2,\ldots)$. It is well known that $pSp(U^*) = \Delta$ and that $W(U) = \Delta$. Any complex number lying in Δ may be expressed as the average of k distinct numbers also lying in Δ. Applying Theorem 3 parts (1) and (5) we have $\Delta \subseteq P_k(U) \subseteq P_1(U) = W(U) = \Delta$ which gives the desired result.

(2). $W \in B(H)$ is a bilateral shift if there exists an orthonormal basis $\{e_n\}_{-\infty}^{+\infty}$ of H such that $We_n = e_{n+1}$ $(n = \pm 1, \pm 2, \ldots)$. We exhibit directly that $P_k(W)$ contains a closed disc of radius as close to one as we please.

Let $\theta \in \underset{\sim}{R}$, $m \in \underset{\sim}{N}$. For $j = 1,2,\ldots,k$ let

$$v_j = \frac{1}{\sqrt{m}} \left(e^{im\theta} e_{(j-1)m+1} + e^{i(m-1)\theta} e_{(j-1)m+2} + \ldots + e^{i\theta} e_{jm} \right).$$

$$\{ v_1, v_2, \ldots, v_k \} \in \underset{\sim}{O}_k .$$

$$Wv_j = \frac{1}{\sqrt{m}} \left(e^{im\theta} e_{(j-1)m+2} + e^{i(m-1)\theta} e_{(j-1)m+3} + \ldots + e^{i\theta} e_{jm+1} \right)$$

$$(Wv_j, v_j) = \frac{(m-1)}{m} e^{i\theta} \quad (j=1,2,\ldots,k).$$

Therefore $P_k(W)$ contains the closed disc with radius $\frac{(m-1)}{m}$. The desired conclusion follows using the well known fact that $P_1(W) = \Delta$. □

3. Compact Operators.

Let T be a compact self adjoint operator with trivial kernel and spectral decomposition $T = \sum_{i=1}^{\infty} \lambda_i E_i$ where the E_i $(i=1,2,\ldots)$ are mutually orthogonal rank one projections and $\lambda_i \geq \lambda_{i+1} > 0$ $(i=1,2,\ldots)$.

Then $$P_k(T) = \left(0, \frac{1}{k}\sum_{i=1}^{k} \lambda_i \right] \qquad (k=1,2,\ldots) .$$

Proof: Let $(v_1, v_2, \ldots, v_k) \in \underset{\sim}{O}_k$.

$$(Tv_j, v_j) = \sum_{i=1}^{\infty} \lambda_i (E_i v_j, v_j) = \sum_{i=1}^{\infty} \lambda_i \|E_i v_j\|^2 \quad (j=1,2,\ldots,k).$$

$$\frac{1}{k}\sum_{j=1}^{k} (Tv_j, v_j) = \sum_{i=1}^{\infty} \lambda_i \left(\frac{1}{k}\sum_{j=1}^{k} \|E_i v_j\|^2 \right) = \sum_{i=1}^{\infty} \alpha_i \lambda_i$$

where $\alpha_i = \frac{1}{k}\sum_{j=1}^{k} \|E_i v_j\|^2$. Notice that $0 \leq \alpha_i \leq \frac{1}{k}$ $(i=1,2,\ldots)$ and $\sum_{i=1}^{\infty} \alpha_i \leq 1$. The maximum value of $\sum_{i=1}^{\infty} \alpha_i \lambda_i$ is attained when $\alpha_i = \frac{1}{k}$ $(i=1,2,\ldots,k)$ and this occurs if $v_i = e_i$ $(i=1,2,\ldots,k)$. It is clear that $\sum_{i=1}^{\infty} \alpha_i \lambda_i \geq 0$ and can be made as small as we please with suitable choices of $\{ v_1, v_2, \ldots, v_k \} \in \underset{\sim}{O}_k$. Suppose $\sum_{i=1}^{\infty} \alpha_i \lambda_i = 0$ for some $\{v_1, v_2, \ldots, v_k\} \in \underset{\sim}{O}_k$, then $\alpha_i = 0$ $(i=1,2,\ldots)$ and therefore $\mathrm{Span}\{ v_1, v_2, \ldots v_k\} \subset H,^{\perp}$ a contradition. Hence $0 \notin P_k(T)$. □

Remarks. Certain features of example(3) are interesting and provide motivation for what follows. For this operator we have $\bigcap_{k=1}^{\infty} P_k(T)^{-} = \{0\}$. In the next section we show that this property characterises compact operators. The ranges determine the eigenvalues of T together with their multiplicities. We show that this is a general feature for compact normal operators. The ranges of T are half-open half-closed intervals. An inspection of the proof shows that under the assumption $\dim(\ker T) = m$ we have the same right hand end point for $P_k(T)$, however $0 \in P_k(T)$ if and only if $k \leq m$. The dimension of the kernel of an operator therefore plays a part in determining whether the range is a closed subset of the plane. We shall see that the spacing of eigenvalues relative to the origin also plays a part.

---

§2. The essential numerical range.

Theorem 3 of §1 shows that the k-ranges are nested subsets of the plane. This section deals with the problem: Describe $\bigcap_{k=1}^{\infty} P_k(A)^{-}$ $(A \in B(H))$. To provide a complete account we include a definition and some known results on the essential range of an operator. The discussion leans heavily on §34 of "Numerical Ranges II" [4]. We have selected only those results necessary in order to deduce our theorem. We have not listed individually the original papers by the authors concerned, these may be found by consulting [4].

Notation. Let $\mathfrak{A}$ denote a complex unital Banach algebra. Let $D(\mathfrak{A})$ denote the set of states of $\mathfrak{A}$. $V(\mathfrak{A}, a)$ denotes the algebra numerical range of an element $a \in \mathfrak{A}$.

Let X be an infinite dimensional complex Banach space. Let $K = K(X)$ denote the closed two sided ideal of compact operators in $B(X)$. Let $\pi : B(X) \to \frac{B(X)}{K(X)}$ be the canonical homomorphism of $B(X)$ into the Calkin algebra.

Definition 1. Let $T \in B(X)$. The essential numerical range of T $V_{ess}(T)$ is defined by

$$V_{ess}(T) = V\left(\frac{B(X)}{K(X)}, \pi(T) \right)$$

i.e. $V_{ess}(T)$ is the algebra numerical range of the canonical image of T as an element of the unital Banach algebra $\frac{B}{K}$.

The following proposition is straightforward [4;34.2].

PROPOSITION 2. Let $T \in B(X)$. Then

(1). $V_{ess}(T) = \cap \{ V(B, T+K) : K \in K(X) \}$,

(2). $V_{ess}(T) = \{ f(T) : f \in D(B(X)), f(K(X)) = \{0\} \}$.

When H is an infinite dimensional complex Hilbert space we have

COROLLARY 3. Given $A \in B(H)$,

$$W_{ess}(A) = \cap \{ W(A+K)^{-} : K \in K(H) \}.$$

Proof. $W(A)^{-} = V(B(H), A)$. □

Notation: Let M be a (closed) subspace of H, let P_M be the orthogonal projection onto M, and let $C_M(A)$ denote the compression of the operator $A \in B(H)$ to M. i.e. $C_M(A) = P_M A|_M$.

LEMMA 4. (Fillmore, Stampfli, Williams)

Let M be a closed subspace of H such that $M^{\perp}$ has finite dimension. Then $\mathrm{Wess}(A) = \mathrm{Wess}(P_M A P_M) = \mathrm{Wess}\ C_M(A)$ $(A \in B(H))$.

Proof. Write $P = P_M$. $I-P$ has finite rank and therefore

$$A - PAP = (I - P)A(I - P) + PA(I - P) + (I - P)AP \in K(H).$$

$\pi(A) = \pi(PAP)$ and therefore $\mathrm{Wess}(A) = \mathrm{Wess}(PAP)$.

Suppose $f \in D(\ B(H)\ )$ annihilates the compact operators and define $g : B(M) \to \underset{\sim}{C}$ by $g(T) = f(TP)$ $(T \in B(M)\ )$. $g(I_M) = f(P) = 1$ and $T \in K(M) \Rightarrow TP \in K(H)$, therefore g is a state on $B(M)$ which annihilates $K(M)$.

$f(PAP) = f(C_M(A)P) = g(C_M(A)) \in \mathrm{Wess}(C_M(A))$ by Proposition 2 part (2). Therefore $\mathrm{Wess}(PAP) \subseteq \mathrm{Wess}(C_M(A))$.

Finally, let $f \in D(B(M))$ and $f(K(M)) = \{0\}$. Define $g : B(H) \to \underset{\sim}{C}$ by $g(T) = f(C_M(T))$ $(T \in B(H))$. Then $g(I) = f(I_M) = 1$ and $\|C_M(T)\| \leqslant \|T\|$. Therefore $g \in D(B(H))$ and g annihilates $K(H)$ because the compression of a compact operator is compact. Hence $\mathrm{Wess}(C_M(A)) \subseteq \mathrm{Wess}(A)$.$\square$

LEMMA 5. (Anderson, Stampfli)

Let $\lambda \in \mathrm{Wess}(A)$. Then there exists a closed subspace E of H with infinite dimension, an orthonormal basis $\{e_k\}_1^{\infty}$ for E, and complex numbers λ_k such that

(1). $\lim\limits_{k \to \infty} \lambda_k = \lambda$,

(2). $C_E(A)$ has matrix representation $\mathrm{diag}\{\lambda_k\}$ relative to the orthonormal basis $\{e_k\}_1^{\infty}$, i.e. $(Ae_j, e_i) = \lambda_i \delta_{ij}$.

Proof. By Corollary 3, $\lambda \in W(A)^{-}$ and therefore there exists a unit vector $e_1 \in H$ with $|(Ae_1,e_1) - \lambda| < 1$. The proof proceeds by induction. Suppose that $\{e_1,e_2,\ldots,e_n\} \in \underset{\sim}{O}_n$ has been found such that $(Ae_j,e_i) = 0$ $(i \neq j)$ and $|(Ae_i,e_i) - \lambda| < \frac{1}{i}$ for $i,j = 1,2,\ldots,n$.

Let $M = \mathrm{Span}\{e_1,e_2,\ldots,e_n,Ae_1,Ae_2,\ldots,Ae_n,A^*e_1,A^*e_2,\ldots,A^*e_n\}^{\perp}$. $\lambda \in W\mathrm{ess}(C_M(A))$ by Lemma 4 and by Corollary 3 there exists a unit vector $e_{n+1} \in M$ such that $|(C_M(A)e_{n+1},e_{n+1}) - \lambda| < \frac{1}{n+1}$. $\{e_1,e_2,\ldots,e_{n+1}\} \in \underset{\sim}{O}_{n+1}$, $(Ae_j,e_i) = 0$ $(i,j = 1,2,\ldots,n+1;\ i \neq j)$ and $|(Ae_{n+1},e_{n+1}) - \lambda| < \frac{1}{n+1}$. Let E be the closed linear span of the infinite orthonormal set $\{e_n\}_1^{\infty}$ obtained using this procedure and let $\lambda_k = (Ae_k,e_k)$ $(k = 1,2,\ldots)$. □

We are now in a position to prove the new theorem of this section.

<u>THEOREM</u> 6. Let $A \in B(H)$.

Then $$W\mathrm{ess}(A) = \bigcap_{k=1}^{\infty} P_k(A)^{-} .$$

<u>Proof</u>.

Let $\lambda \in W\mathrm{ess}(A)$ and let E, $\{e_k\}_1^{\infty}$, $\{\lambda_k\}_1^{\infty}$ be as in Lemma 5. Let m be a fixed positive integer and define $\mu_n \in \underset{\sim}{C}$ $(n=1,2,\ldots)$ by $\mu_n = \frac{1}{n}\sum_{i=1}^{n}(Ae_i,e_i)$. $\mu_n \in P_m(A)$ for $n=m,m+1,\ldots$ by the nested property of the k-ranges. $\mu_n \to \lambda$ as $n \to \infty$, so $\lambda \in P_m(A)^{-}$ and therefore $\lambda \in \bigcap_{m=1}^{\infty} \overline{P_m(A)}$. The inclusion $W\mathrm{ess}(A) \subseteq \bigcap_{m=1}^{\infty} P_k(A)^{-}$ is therefore established.

We may suppose that $0 \in \bigcap_{k=1}^{\infty} P_k(A)^{-}$ by replacing A with $\alpha A + \beta$ for suitable $\alpha,\beta \in \underset{\sim}{C}$. The proof is completed by showing that $0 \in \bigcap_{k=1}^{\infty} P_k(A)^- \implies 0 \in W_{ess}(A)$.

For each $k \geq 1$ there exists an orthonormal k-tuple $\{e_1, e_2, \ldots, e_k\}$ such that

$$\left| \frac{1}{k} \sum_{i=1}^{k} (Ae_i, e_i) \right| < \frac{1}{k} .$$

Let f_k be the linear functional on $B(H)$ defined by

$$f_k(T) = \frac{1}{k} \sum_{i=1}^{k} (Te_i, e_i) \quad (T \in B(H)) . \qquad f_k \in D(B(H)) .$$

Let f be a weak * cluster point of the set $\{f_k : k \geq 1\}$. Then $f \in D(B(H))$ and $f(A) = 0$. Let T be a finite rank operator.

$$T = u_1 \otimes v_1 + u_2 \otimes v_2 + \ldots + u_n \otimes v_n \quad \text{for some}$$

$u_1, u_2, \ldots, u_n; v_1, v_2, \ldots, v_n \in H$. ($u \otimes v$ is the rank one operator defined by $(u \otimes v)x = (x,v)u \quad (x \in H)$).

$$f_k(T) = \sum_{j=1}^{n} \frac{1}{k} \sum_{i=1}^{k} (e_i, v_j)(u_j, e_i) \quad ,$$

$$|f_k(T)| \leq \frac{1}{k} \sum_{j=1}^{n} \|u_j\| \|v_j\| \quad .$$

Hence f vanishes on finite rank operators and therefore by continuity f vanishes on the ideal of compact operators. By Proposition 2 we have $0 = f(A) \in W_{ess}(A)$. □

<u>COROLLARY</u> 7. An operator $A \in B(H)$ is compact if and only if

$$\bigcap_{k=1}^{\infty} P_k(A)^- = \{0\}.$$

<u>Proof.</u> $W_{ess}(A) = \{0\}$ if and only if A is compact. □

<u>COROLLARY</u> 8. Let $S, T \in B(H)$. If $P_k(S) \subseteq P_k(T)^-$ $(k = 1, 2, \ldots)$ then $W_{ess}(S) \subseteq W_{ess}(T)$.

<u>Proof.</u> Immediate. □

§3. The states which generate the k-range.

The set of states on $B(H)$ is a convex weak* compact subset of the dual space of $B(H)$. A linear functional $f \in B(H)'$ is said to be a vector state if there exists an element $x \in H$ of norm one such that $f(T) = (Tx, x)$ $(T \in B(H))$. The spatial numerical range of an operator A is just the image under A of the set of vector states. The closure of the spatial range is the image under A of all states. We examine in this section some questions which arise from a consideration of the subsets of the set of all states which generate the k-range and its closure. This leads to a sufficient condition for the k-range of a compact operator to be closed.

Introductory Remarks.

Let $P, Q \in \underset{\sim}{P}_k$, then $tr([\alpha P + (1-\alpha)Q]A) = \alpha tr(PA) + (1-\alpha)tr(QA)$ and therefore $tr([\alpha P + (1-\alpha)Q]A) \in kP_k(A)$ whenever $0 \leq \alpha \leq 1$. More generally $\frac{1}{k}\, tr(TA) \in P_k(A)$ whenever T is a convex combination of members of $\underset{\sim}{P}_k$.

Given a trace class operator T, the linear functional f_T defined by $f_T(A) = tr(TA)$ $(A \in B(H))$ satisfies $\|f_T\| \leq \|T\|_1$ and $f_T(I) = tr(T)$. In particular let T be a positive trace class operator with $tr(T) = \|T\|_1 = 1$. Then $f_T \in D(B(H))$. Let $T = \sum_j \lambda_j e_j \otimes e_j$ be the spectral decomposition of T, where $\{e_j\}$ is an orthonormal sequence of elements of H.

$T \geq 0 \Rightarrow \lambda_j \geq 0$ $(j = 1, 2, \dots)$

and therefore $\sum_j \lambda_j = 1$ because $tr(T) = 1$.

$$f_T(A) = tr(TA) = tr(AT) = \sum_j (ATe_j, e_j) = \sum_j \lambda_j(Ae_j, e_j) \quad (A \in B(H)).$$

Therefore $\quad f_T = \sum_j \lambda_j \omega_{e_j}$ where ω_x ($\|x\| = 1$) denotes the vector state $\omega_x(A) = (Ax, x)$. $(A \in B(H))$.

The numerical range of A is the image under A of of the set of all states of the form f_T with T as above (more generally, the image under A of the set of all ultraweakly continuous states) since convex subsets of the plane are closed with respect to the formation of infinite convex combinations.

The following two technical lemmas provide the main step in identifying the condition on the sequence $\{\lambda_j\}$, or equivalently, the condition on the positive trace class operator T which ensures that $f_T(A)$ belongs to the k^{th} Halmos range of A.

Notation.

Let $\gamma \geq 0$. Let $[\gamma]$ denote the integral part of γ. Let m be any integer satisfying $m \geq [\gamma]+1$. Let Ω_m denote the set of m-tuples $\{\alpha_1, \alpha_2, .., \alpha_m\}$ satisfying $0 \leq \alpha_i \leq 1, \sum_{i=1}^{m} \alpha_i = \gamma$.

LEMMA 1.

Ω_m is a compact convex subset of $\underset{\sim}{R}^m$ and the set $Ext(\Omega_m)$ of extreme points of Ω_m consists of all m-tuples with $[\gamma]$ co-ordinates equal to 1, one co-ordinate equal to $\gamma - [\gamma]$ and the rest zero.

Proof. Ω_m is compact and convex as a subset of $\underset{\sim}{R}^m$ with the usual topology. Let $\underset{\sim}{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_m\} \in \Omega_m$ and suppose there exist two entries α_{i_1} , α_{i_2} ($i_1 < i_2$) with

$0 < \alpha_{i_1} < 1$, $0 < \alpha_{i_2} < 1$.

Let $\delta = \min\{ \alpha_{i_1}, \alpha_{i_2}, 1-\alpha_{i_1}, 1-\alpha_{i_2}\} > 0$.

Then $\underset{\sim}{\alpha}_1 = \{\alpha_1,\ldots,\alpha_{i_1}+\delta,\ldots,\alpha_{i_2}-\delta,\ldots,\alpha_m\} \in \Omega_m$

$\underset{\sim}{\alpha}_2 = \{\alpha_1,\ldots,\alpha_{i_1}-\delta,\ldots,\alpha_{i_2}+\delta,\ldots,\alpha_m\} \in \Omega_m$

and $\underset{\sim}{\alpha} = \frac{1}{2}(\underset{\sim}{\alpha}_1 + \underset{\sim}{\alpha}_2)$. Therefore $\underset{\sim}{\alpha}$ is not an extreme point of Ω_m . This argument holds for any member of Ω_m with at least two entries lying between 0 and 1. Therefore the set of extreme points of Ω_m is contained in the set of those m-tuples with $[\gamma]$ co-ordinates equal to one, one co-ordinate equal to $\gamma - [\gamma]$ and the rest zero. Conversely, it is clear that any such point is an extreme point of Ω_m. □

Remark. $\mathrm{Ext}(\Omega_m)$ consists of a finite set of m-tuples. $\mathrm{co}(\mathrm{Ext}(\Omega_m))$ is closed and therefore $\mathrm{co}(\mathrm{Ext}(\Omega_m)) = \Omega_m$ by the Krein-Milman theorem.

Notation. Let W_γ denote the set of sequences with precisely $[\gamma]$ entries equal to one, one entry equal to $\gamma - [\gamma]$ and the remaining entries zero. Let $\underset{\sim}{\mu} = \{\mu_n\}_1^\infty$ be a bounded sequence of complex numbers.

Let $Y_\gamma(\underset{\sim}{\mu}) = \{ \sum_{j=1}^{\infty} \lambda_j\mu_j \;:\; \sum_{j=1}^{\infty}\lambda_j = \gamma \,,\ 0 \leq \lambda_j \leq 1 \ (j=1,2,\ldots)\}$.

LEMMA 2.

$Y_\gamma(\underset{\sim}{\mu})$ is a convex subset of $\underset{\sim}{C}$ and

$$Y_\gamma(\underset{\sim}{\mu}) = \mathrm{co}\{ \sum_{j=1}^{\infty} \lambda_j\mu_j \;:\; \underset{\sim}{\lambda} = \{\lambda_j\}_1^\infty \in W_\gamma \} .$$

Proof. The Lemma will follow from two observations.

(1). $\mathrm{Ext}(\, Y_\gamma(\underset{\sim}{\mu})\,) \subseteq \{ \sum_{j=1}^{\infty} \lambda_j\mu_j \;:\; \underset{\sim}{\lambda} = \{\lambda_j\}_1^\infty \in W_\gamma \}$.

(2). Any member of $Y_\gamma(\underset{\sim}{\mu})$ may be approximated arbitrarily closely by members of $\quad co\{ \sum_{j=1}^{\infty} \lambda_j \mu_j : \underset{\sim}{\lambda} = \{\lambda_j\}_1^{\infty} \in W_\gamma \}$.

<u>Proof of (1)</u>. Let $t = \sum_{j=1}^{\infty} \beta_j \mu_j \in \mathrm{Ext}(Y_\gamma(\underset{\sim}{\mu}))$. We may assume, by making rearrangements and taking combinations if necessary that whenever $\beta_j \mu_j \neq 0$ the set $\{ k : \beta_k \mu_k \neq 0, \mu_k = \mu_j \}$ is finite, $0 < \beta_k \leq 1$, and at most one of the coefficients β_k lies strictly between 0 and 1 . With this assumption, suppose there exist coefficients β_{j_1}, β_{j_2} such that $0 < \beta_{j_1} < 1$, $0 < \beta_{j_2} < 1$ $(j_1 < j_2)$.
Let $\delta = \min\{ \beta_{j_1}, \beta_{j_2}, 1 - \beta_{j_1}, 1 - \beta_{j_2} \}$ then

$$t_1 = \beta_1 \mu_1 + \ldots + (\beta_{j_1} + \delta)\mu_{j_1} + \ldots + (\beta_{j_2} - \delta)\mu_{j_2} + \ldots \in Y_\gamma(\underset{\sim}{\mu})$$

$$t_2 = \beta_1 \mu_1 + \ldots + (\beta_{j_1} - \delta)\mu_{j_1} + \ldots + (\beta_{j_2} + \delta)\mu_{j_2} + \ldots \in Y_\gamma(\underset{\sim}{\mu})$$

$$t_2 - t_1 = 2\delta(\mu_{j_2} - \mu_{j_1}) \neq 0 \quad \text{(by assumption)}$$

$$t = (t_1 + t_2)/2 \quad .$$

Therefore t is not an extreme point. This contradiction implies that at most one β_j lies between 0 and 1 (and must equal $\gamma - [\gamma]$). Therefore we have $t = \sum_{j=1}^{\infty} \beta_j \mu_j$ for some $\underset{\sim}{\beta} \in W_\gamma$ as required .

<u>Proof of (2)</u>. Let $t = \sum_{j=1}^{\infty} \beta_j \mu_j \in Y_\gamma(\underset{\sim}{\mu})$. Define δ_n by $\delta_n = \gamma - \sum_{j=1}^{n} \beta_j$, and let $N \in \underset{\sim}{N}$ be sufficiently large so that $\delta_n \leq 1$ whenever $n \geq N$.

$\beta_1 \mu_1 + \ldots + \beta_n \mu_n + \delta_n \mu_{n+1} \in Y_\gamma(\underset{\sim}{\mu})$. If $n \geq \max\{[\gamma], N\}$ an application of Lemma 1 (with $m = n+1$) together with the remark shows that

$$\beta_1 \mu_1 + \ldots + \beta_n \mu_n + \delta_n \mu_{n+1} \quad \in \quad co\{ \sum_{j=1}^{\infty} \lambda_j \mu_j : \underset{\sim}{\lambda} = \{\lambda_j\}_1^{\infty} \in W_\gamma \} \quad .$$

Also

$$\begin{aligned} | t - (\beta_1 \mu_1 + \ldots + \beta_n \mu_n + \delta_n \mu_{n+1}) | &= | \sum_{j=n+1}^{\infty} \beta_j \mu_j - \delta_n \mu_{n+1} | \\ &\leq \|\underset{\sim}{\mu}\|_\infty \left(\sum_{j=n+1}^{\infty} \beta_j + \delta_n \right) \\ &= 2\delta_n \|\underset{\sim}{\mu}\|_\infty \to 0 \quad \text{as } n \to \infty . \end{aligned}$$

Proof of Lemma 2 .

Let $t \in Y_y(\mu)$. Either t is an interior point of $Y_y(\mu)$ or $t \in \partial Y_y(\mu)$. If t is an interior point of $Y_y(\mu)$ then it follows that t can be written as a convex combination of elements of the required form by (2). If $t \in$ $Y_\gamma(\mu)\setminus \mathrm{int}\, Y_\gamma(\mu)$ $\cancel{\partial Y_y(\mu)}$ then t is either an extreme point or the convex combination of two extreme points and an application of (1) finishes the job. $\square$

THEOREM 3. Let $A \in B(H)$. Then for each $k = 1,2,\ldots$

$$P_k(A) = \{ \operatorname{tr}(SA) : 0 \leq S \leq \tfrac{1}{k} I ,\ S \in \mathcal{C}_1 ,\ \operatorname{tr}(S) = 1 \} .$$

Proof. The inclusion $\subseteq$ is clear since if $P \in \underset{\sim}{P}_k$ we have $0 \leq \frac{1}{k}P \leq \frac{1}{k}I$, $\operatorname{tr}(\frac{1}{k}P) = 1$. Conversely, let $S \in \mathcal{C}_1$ where $0 \leq S \leq \frac{1}{k}I$, $\operatorname{tr}(S) = 1$. Let $S\mathscr{X} = \sum_j \xi_j\, e_j \otimes e_j$ be the spectral decomposition of $S\mathscr{X}$ where $\{e_j\}$ is an orthonormal sequence of elements of H . $0 \leq \xi_j \leq \frac{1}{k}$, $\sum_j \xi_j = 1$.

$$\operatorname{tr}(SA) = \operatorname{tr}(AS) = \sum_j (ASe_j,e_j) = \sum_j \xi_j(Ae_j,e_j) .$$

Applying Lemma 2 with $y = k$, $\mu = \{ \frac{1}{k} (Ae_j,e_j) \}$ we have

$$\operatorname{tr}(SA) \in \operatorname{co} \{ \tfrac{1}{k} \sum_j \lambda_j(Ae_j,e_j) : \lambda_j=1 \text{ for exactly } k \text{ distinct integers } j \} .$$

Therefore $\operatorname{tr}(SA) \in P_k(A)$. $\square$

Remarks.

(1). Theorem 3 says that the k-range is closed with respect to the formation of all infinite convex combinations of the form $\sum_{j=1}^{\infty} \lambda_j(Ax_j,x_j)$ where $\{x_j\}$ is any orthonormal sequence and $0 \leq \lambda_j \leq \frac{1}{k}$ $(j = 1,2,\ldots)$, $\sum_{j=1}^{\infty} \lambda_j = 1$. Equivalently, $P_k(A)$

is the image under A of all (ultraweakly continuous) states of the form $f = \sum_{j=1}^{\infty} \lambda_j \omega_{x_j}$ with $\{x_j\}$ and $\{\lambda_j\}$ as before.

(2). The map

$$T \mapsto f_T \quad : \quad \{ T \in \mathcal{C}_1 : 0 \leq T \leq \tfrac{1}{k} I \ , tr(T) = 1 \} \to D(B(H)) \ , \text{ where}$$

f_T is defined by $f_T(S) = tr(ST) \quad (S \in B(H))$, is an affine isometric map onto the set of all states of the form $f = \sum_j \lambda_j \omega_{x_j}$ where $\{x_j\}$, $\{\lambda_j\}$ are as in Remark 1 .

<u>Definition</u> 4. Given $k \in \mathbb{N}$, let D_k denote the set of states given by

$$D_k = \overline{co}^{w^*}\Big(\{ f_S : S \in \mathcal{C}_1 , \ 0 \leq S \leq \tfrac{1}{k} I, \ tr(S) = 1 \} \cup \{ f \in D : f(K(H)) = \{0\} \} \Big).$$

<u>Remarks</u>.

(1). D_k is a convex weak* compact subset of D, the set of all states on $B(H)$. $D_{k+1} \subseteq D_k \ (k = 1,2,\dots)$.

(2). A result of J.Dixmier [7] shows that if $f \in D$ then $f = \alpha f_T + (1-\alpha)g$, where f_T is the state associated with ~~the~~ a positive trace class operator T with $tr(T) = 1$, and g is a state which annihilates the ideal of compact operators and $0 \leq \alpha \leq 1$. Therefore $D_1 = D$, and moreover, taking the weak* closure is redundant when $k = 1$. However for higher values of k we shall see that the corresponding set with the weak* closure omitted is not in general weak* closed.

<u>PROPOSITION</u> 5.

Let $A \in B(H)$. Then $P_k(A)^- = \{ f(A) : f \in D_k \}$.

<u>Proof</u>. The set $\{ f(A) : f \in D_k \}$ is the continuous image of a

compact set and is therefore closed and it clearly contains $P_k(A)$.

Conversely, let $f \in D_k$, then there exists a net of states $\{f_\lambda : \lambda \in \Lambda\}$ such that $f_\lambda \to f$ (w*) as $\lambda \to \infty$ where

$$f_\lambda = \alpha_\lambda f_{T_\lambda} + (1-\alpha_\lambda)g_\lambda \qquad \text{for some } 0 \leq \alpha_\lambda \leq 1 \text{ , } g_\lambda \in D \text{ ,}$$

$g_\lambda(K(H)) = \{0\}$, $T_\lambda \in \mathcal{C}_1$, $0 \leq T_\lambda \leq \frac{1}{k}I$, $\mathrm{tr}(T_\lambda) = 1$, for each $\lambda \in \Lambda$.

We have $f_\lambda(A) = \alpha_\lambda f_{T_\lambda}(A) + (1-\alpha_\lambda)g_\lambda(A)$, which is the convex combination of a member of the k-range and a member of the essential range of A. By theorem 2.6 $f_\lambda(A) \in P_k(A)^-$ $(\lambda \in \Lambda)$ and hence $f(A) \in P_k(A)^-$. □

We require the following well known result which we state without proof (see for example [6 ; 4.1.2]).

<u>THEOREM</u> 6.

For each $T \in \mathcal{C}_1$, let ϕ_T be the bounded linear functional on $K(H)$ defined by $\phi_T(K) = \mathrm{tr}(KT)$ $(K \in K(H))$. The map $T \to \phi_T$ is a linear bijection of $\mathcal{C}_1$ onto the dual space of the Banach space $K(H)$. ϕ_T is hermitian (resp. positive) if and only if T is self adjoint (resp. positive).

Let $D_k | K(H)$ denote the set of restrictions to $K(H)$ of the members of D_k . With ϕ_T $(T \in \mathcal{C}_1)$ defined as in Theorem 6 we have:

LEMMA 7.

$$D_k|K(H) = \{ \phi_T : T \in \mathcal{C}_1 , 0 \leq T \leq \tfrac{1}{k}I , tr(T) \leq 1 \}.$$

Proof. Let $\phi \in D_k|K(H)$. By theorem 6, $\phi = \phi_S$ for some $S \in \mathcal{C}_1$ with $S \geq 0$. Also $\phi = g|K(H)$ for some $g \in D_k$. Let $\{f_\lambda : \lambda \in \Lambda\}$ be a net of states such that $f_\lambda \to g$ (w*) as $\lambda \to \infty$, where for each $\lambda \in \Lambda$

$$f_\lambda = \alpha_\lambda f_{T_\lambda} + (1-\alpha_\lambda)g_\lambda \quad (0 \leq \alpha_\lambda \leq 1,\ g_\lambda \in D ,\ g_\lambda(K(H)) = \{0\},$$
$$T_\lambda \in \mathcal{C}_1 ,\ 0 \leq T_\lambda \leq \tfrac{1}{k}I,\ tr(T_\lambda) = 1).$$

Given $x \in S(H)$ we have

$$(Sx,x) = tr(x \otimes x\ S) = \phi_S(x \otimes x) = g(x \otimes x) ,$$

$$f_\lambda(x \otimes x) = \alpha_\lambda f_{T_\lambda}(x \otimes x) = \alpha_\lambda tr(T_\lambda x \otimes x)$$
$$= \alpha_\lambda (T_\lambda x,x) \qquad (\lambda \in \Lambda) ,$$

So $\quad 0 \leq f_\lambda(x \otimes x) \leq 1/k \qquad (\lambda \in \Lambda)$.

Therefore $\quad 0 \leq g(x \otimes x) \leq 1/k$.

Hence $\quad 0 \leq S \leq \frac{1}{k}I$.

Let $S = \sum_{i=1}^{\infty} \lambda_i E_i$ be the spectral decomposition of S where the E_i are mutually orthogonal rank one projections. For each $n \geq 1$, let $P_n = \sum_{i=1}^{n} E_i$. $P_n \in \underset{\sim}{P}_n$.

$$\sum_{i=1}^{n} \lambda_i = tr(P_n S) = \phi_S(P_n) \leq \|g\|\|P_n\| = 1 \ (n=1,2,\ldots) \qquad -- (1).$$

Therefore $\quad tr(S) = \sum_{i=1}^{\infty} \lambda_i \leq 1$.

If the spectral decomposition of S has only finitely many terms (M say) then line (1) holds with $n=M$ and the desired conclusion is immediate.

Conversely, let $S \in \mathcal{C}_1$ be given with $0 \leq S \leq \frac{1}{k}I$, $tr(S) \leq 1$. Let $S = \sum_{i=1}^{\infty} \lambda_i E_i$ be the spectral decomposition of S where the series has been made formally infinite , if necessary, by the addition of appropriate mutually orthogonal rank one projections with zero coefficients.

Let $q_n = \sum_{i=1}^{n} \lambda_i$, $\delta_n = 1-q_n$. Define $S_n \in B(H)$ by

$$S_n = \sum_{i=1}^{n} \lambda_i E_i + (\delta_n/k) \sum_{i=n+1}^{k+n} E_i \qquad (n = 1,2,\ldots) .$$

Then $S_n \in \mathcal{B}_1$, $0 \leq S_n \leq \frac{1}{k} I$, $tr(S_n) = q_n + \delta_n = 1$.
Let g be any weak* cluster point of the set of states

$\{f_{S_n} : n=1,2,\ldots\} \subset D_k$. Then $g \in D_k$ and we claim that $g|K(H) = \phi_S$. Recall that an operator $T \in B(H)$ is compact if and only if, for each net $\{ E_\lambda : \lambda \in \Lambda \}$ of rank one mutually orthogonal projections, $tr(TE_\lambda) \to 0$ as $\lambda \to \infty$. Let $A \in K(H)$. ~~and suppose~~ $\cancel{f_{S_m} \to g \ (w^*) \text{ as } m \to \infty}$. We have

$$|g(A) - \phi_S(A)| \leq |g(A) - f_{S_m}(A)| + |f_{S_m}(A) - \phi_S(A)|$$

$$= |g(A) - f_{S_m}(A)| + |tr((S_m - S)A)|$$

$$|tr((S_m - S)A)| = |tr(A(S - S_m))|$$

$$= |tr(A[\sum_{i=m+1}^{\infty} \lambda_i E_i - (\delta_m/k) \sum_{i=m+1}^{k+m} E_i])|$$

$$= |\sum_{i=m+1}^{\infty} \lambda_i tr(AE_i) - (\delta_m/k) \sum_{i=m+1}^{k+m} tr(AE_i)|$$

$$\leq 2\delta_m \sup\{ |tr(AE_i)| : i \geq m+1 \}$$

$$\to 0 \text{ as } m \to \infty \text{ since } A \text{ is compact.}$$

Therefore $g|K(H) = \phi_S$. □

THEOREM 8. Let $A \in B(H)$ be compact. Then for each $k = 1,2,\ldots$

$$P_k(A)^- = \{ tr(TA) : T \in \mathcal{B}_1 , 0 \leq T \leq \tfrac{1}{k} I , tr(T) \leq 1 \} .$$

Proof.

Apply Proposition 5 and Lemma 7. □

Example. This description does not hold if A is not compact. Let $A = I$, the identity operator. Then $P_k(I) = \{1\}$ but $\{ tr(T) : T \in \mathcal{C}_1 , 0 \leq T \leq \frac{1}{k}I , tr(T) \leq 1 \} = [0,1]$.

Notation. Let $D_\infty = \bigcap_{k=1}^{\infty} D_k$.

PROPOSITION 9.

Let $A \in B(H)$. Then $W_{ess}(A) = \{ f(A) : f \in D_\infty \}$ and D_∞ consists of the set of all states which vanish on the ideal of compact operators.

Proof. The set of states which vanish on $K(H)$ is contained in D_∞ by definition and therefore we have the inclusion $\subseteq$.

Conversely, $f \in D_\infty \Rightarrow f(A) \in P_k(A)^-$ $(k = 1,2,\dots)$ by Proposition 5 . An application of Theorem 2.6 then shows that $f(A) \in W_{ess}(A)$. □

We complete this section with a sufficient condition for the k-range of a compact operator to be closed.

THEOREM 10.

Let $A \in B(H)$ be compact with kernel of dimension at least k. Then $P_k(A)$ is closed .

Proof. Re-stating Theorem 8 in terms of orthonormal sequences, the result says that the closure of the k-range of A consists of all sums of the form $\sum_j \lambda_j(Ax_j, x_j)$ where $\{x_j\}$ is an orthonormal sequence of elements of H , $0 \leq \lambda_j \leq 1/k$, $\sum_j \lambda_j \leq 1$.
The proof consists of showing that every such sum is a member of $P_k(A)$.

Let $t = \sum_{j=1}^{\infty} \lambda_j(Ax_j, x_j)$, $0 \leq \lambda_j \leq \frac{1}{k}$, $\sum_{j=1}^{\infty} \lambda_j = \alpha \leq 1$ be given, where $\{x_j\}$ is an orthonormal sequence of elements of H. By lemma 2, it suffices to assume that $t = \frac{1}{k}\sum_{j=1}^{\infty} \lambda_j(Ax_j, x_j)$ where $\underset{\sim}{\lambda} = \{\lambda_j\} \in W_\gamma$ and $\gamma = k\alpha$. Therefore, renumbering the x's, if necessary, it suffices to prove that

$$t = \frac{1}{k}\left[\sum_{j=1}^{[\gamma]}(Ax_j, x_j) + (\gamma - [\gamma])\,(Ax_{[\gamma]+1}, x_{[\gamma]+1})\right] \in P_k(A)$$

if $[\gamma] \neq 0$,

$$t = \gamma\,\frac{1}{k}(Ax_1, x_1) \in P_k(A) \quad \text{if} \quad [\gamma] = 0 .$$

If $\gamma = [\gamma] = k$ then $t \in P_k(A)$. So suppose $k > [\gamma] \geq 1$.

$$\text{Let} \quad t_1 = \frac{1}{k}\sum_{j=1}^{[\gamma]}(Ax_j, x_j) \quad ;$$

$$t_2 = \frac{1}{k}\left[\sum_{j=1}^{[\gamma]}(Ax_j, x_j) + (Ax_{[\gamma]+1}, x_{[\gamma]+1})\right].$$

Then $t = (1 - (\gamma-[\gamma]))t_1 + (\gamma - [\gamma])t_2$, a convex combination of t_1 and t_2. Therefore it suffices to show that

$\frac{1}{k}\sum_{j=1}^{r}(Ax_j, x_j) \in P_k(A)$ whenever r is a positive integer satisfying $r < k$.

$$H = \ker A \oplus (\ker A)^{\perp} .$$

Let $x_i = y_i \oplus z_i$ $(i=1,2,..,r)$ where for each i, $y_i \in \ker A$, $z_i \in (\ker A)^{\perp}$. $y_1, y_2, \ldots y_r$ span a subspace of $\ker A$ of dimension not exceeding r and therefore there exist at least $k-r$ orthonormal elements

$$y_{r+1}, y_{r+2}, \ldots, y_k \in \left(\operatorname{Span}\{y_1, y_2, \ldots, y_r\}\right)^{\perp \ker A} .$$

Let $x_i = y_i \oplus 0$ $(i=r+1, r+2, \ldots, k)$. Then $\{x_1, x_2, \ldots, x_k) \in \underset{\sim}{O}_k$ and $\frac{1}{k}\sum_{j=1}^{r}(Ax_j, x_j) = \frac{1}{k}\sum_{j=1}^{k}(Ax_j, x_j) \in P_k(A)$.

Finally, the remaining case is clear since

$\dim(\ker A) \geqslant k \Longrightarrow 0 \in P_k(A)$ and therefore when $[\gamma] = 0$

$t = (1-\gamma)0 + \gamma \frac{1}{k}(Ax_1, x_1) \in P_k(A)$. $\square$

COROLLARY 11.

Let $A \in B(H)$ be a compact operator on a non-separable Hilbert space H. Then $P_k(A)$ is closed for each $k = 1, 2, \ldots$.

Proof.

A compact $\Rightarrow$ A^* compact. $\overline{A^*H}$ is separable and therefore $\ker A = (A^*H)^{\perp}$ has infinite dimension. $\square$

Remarks.

(1). The condition on the dimension of the kernel of A is not in general necessary for a closed range. Necessary and sufficient conditions have been given under which the numerical range ($=P_1(.)$) is closed by J.P.Williams (unpublished note). We mention his result in the next section where we give some additional observations and examples for the special case of compact normal operators.

(2). Theorem 10. does not hold if compactness is relaxed. e.g. Let U be a unilateral shift . Then

$P_k(U) = \Delta = \{z \in \underset{\sim}{C} : |z| < 1\}$ (§1. Example 2.(1)) .

$\dim(\ker U^*) = 1$, but $P_1(U^*) = \Delta$ is open.

§4. The k-range of normal operators.

When $A \in B(H)$ is normal it is well known that the closure of the numerical range of A is the convex hull of the spectrum. In this section we obtain a description of the closure of the k-range of a normal operator and a description of the k-range itself for a compact normal operator.

Preliminary remarks.

Throughout the section a reference to the Spectral Theorem will mean the following version of that theorem (see e.g.[9] ; Page 911 Cor. 4).

Let $A \in B(H)$ be normal. Then there exists a regular positive measure space (S, Σ, μ) and a unitary map U of H onto $L^2(S,\Sigma,\mu)$ such that

$$UAx = f.Ux \qquad (x \in H) \quad \text{for some} \quad f \in L^\infty(S, \Sigma,\mu) \; .$$

Let $\phi : S \to \underset{\sim}{C}$ be a μ-measurable function. We recall that the essential range of ϕ, which we denote by $\operatorname{essran}\phi$, is the set of complex numbers λ such that

$$\mu(\phi^{-1}(V_\lambda)) > 0 \qquad \text{for every neighbourhood } V_\lambda \text{ of } \lambda \; .$$

$Sp(A) = \operatorname{essran}\phi$, where ϕ is any function in the equivalence class of functions f , since

$(\lambda - \phi)^{-1}$ is essentially bounded

$\iff$ ϕ is bounded away from λ a.e.

$\iff$ $\lambda \notin \operatorname{essran}\phi$.

Notation.

Given ~~$A \in B(H)$~~ a normal operator A , let $F_k(A)$ denote the set of all arithmetic means of k numbers selected from the spectrum of A such that in any sum isolated points of the spectrum are repeated at most as many times as their multiplicity as eigenvalues.

Let $\Delta(\lambda;r)$ denote the open disc of radius r, centre $\lambda \in \underset{\sim}{C}$.

LEMMA 1. Let $A \in B(H)$ be normal. Then $P_k(A)^{-} \supseteq F_k(A)$.

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be distinct non-isolated points of Sp(A) and let $\lambda_{r+1}, \ldots, \lambda_k$ be isolated points of Sp(A) with the repetition condition. Choose $N \in \underset{\sim}{N}$ sufficiently large so that the discs $\Delta(\lambda_i; 1/n)$ $(i=1,2,\ldots,r)$ are mutually disjoint for $n = N+1, N+2, \ldots$ and $\{\lambda_{r+1}, \ldots, \lambda_k\} \cap \bigcup_{i=1}^{r} \Delta(\lambda_i; 1/n) = \phi$ $(n \geq N+1)$. By the Spectral Theorem there exists a measure space $(S, \sum, \mu)$ such that A is unitarily equivalent to multiplication by some $f \in L^{\infty}(S, \sum, \mu)$ on $L^2(S, \sum, \mu)$. Let ψ be any function in the equivalence class of functions f . For each $n > N$ define a map $\psi_n : S \to \underset{\sim}{C}$ by

$$\psi_n(x) = \begin{cases} \lambda_i & x \in \psi^{-1}(\Delta(\lambda_i; 1/n)) \quad i=1,2,\ldots,r \\ \psi(x) & \text{otherwise} \end{cases}$$

Each ψ_n is measurable and essentially bounded. $\psi_n \to \psi$ uniformly as $n \to \infty$. Let f_n be the equivalence class defined by ψ_n , and let A_n denote the normal operator corresponding to f_n . Then $\|A_n - A\| \to 0$ as $n \to \infty$. For each $n > N$, $\lambda_1, \lambda_2, \ldots, \lambda_r$ are distinct eigenvalues of A_n since these points are members of the essential range of ψ .

Write $\lambda = \lambda_j$ $(r+1 \leq j \leq k)$. Then for some non-zero square integrable function $\phi : S \to \underset{\sim}{C}$ we have the following chain of implications .

$$\psi(x)\phi(x) = \lambda\phi(x) \qquad \text{a.e. } x \in S$$

$$\Rightarrow \quad \mu(\{x \in S : \psi(x) \neq \lambda\} \cap \{x \in S : \phi(x) \neq 0\}) = 0$$

$$\Longrightarrow \quad \mu(\ \{x\in S : \psi_n(x) \neq \lambda\ \} \cap \{x\in S : \phi(x) \neq 0\ \}\ ) = 0 \qquad (n>N)$$

$$\Longrightarrow \quad \psi_n(x)\phi(x) = \lambda\phi(x) \quad \text{a.e. } x\in S \ .$$

Conversely, if A_n $(n>N)$ has λ as an eigenvalue then the implications reverse. We conclude that the eigenvalues $\lambda_{r+1},.,\lambda_k$ are also eigenvalues for A_n with unchanged multiplicities.

For each $n>N$, let $x_{n1},x_{n2},\ldots,x_{nk}$ be a set of orthonormal eigenvectors for A_n with corresponding eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_k$ respectively. We have

$$\left| \frac{1}{k}\sum_{i=1}^{k}\lambda_i - \frac{1}{k}\sum_{i=1}^{k}(Ax_{ni},x_{ni}) \right| = \frac{1}{k}\left|\sum_{i=1}^{k}\{(A_nx_{ni},x_{ni}) - (Ax_{ni},x_{ni})\}\right|$$

$$\leqslant \| A_n - A \| \to 0 \quad \text{as } n \to \infty \ .$$

Therefore $\frac{1}{k}\sum_{i=1}^{k}\lambda_i \in P_k(A)^{-}$. Finally, given any member t of $F_k(A)$ there exists a sequence $\{t_n\}$ of points of $F_k(A)$ such that each t_n contains no repetitions of non-isolated points of the spectrum and such that $t_n \to t$ as $n \to \infty$. □

<u>Notation.</u> Let $Sp'(A)$ denote the set of accumulation points of $Sp(A)$.

<u>LEMMA</u> 2. Given $\delta>0$, $Sp'(A)$ can be covered with finitely many mutually disjoint measurable sets $E_1,E_2,\ldots,E_m$ such that

(1). Each E_i is contained in a closed disc of radius δ .

(2). $\operatorname{int}E_i \cap Sp'(A) \neq \phi$.

<u>Proof.</u> Cover $Sp(A)$ with finitely many open discs Δ of radius δ and delete any discs Δ from the list with the property that $Sp'(A)\cap \Delta = \phi$. Let $\Delta_1,\Delta_2,\ldots,\Delta_r$ be an enumeration of the remaining discs. Define $E_1 = \overline{\Delta}_1$.

(i). If $\operatorname{int}(\overline{\Delta}_2\setminus\overline{\Delta}_1) \cap Sp'(A) \neq \phi$, let $E_2 = \overline{\Delta}_2\setminus\overline{\Delta}_1$.

(ii). If $\operatorname{int}(\bar{\Delta}_2 \setminus \bar{\Delta}_1) \cap Sp'(A) = \phi$ then $Sp'(A) \cap \Delta_2 \subset \bar{\Delta}_1$ and we delete Δ_2 from the list.

If action under (i) was taken then proceed as follows. If $\operatorname{int}(\bar{\Delta}_3 \setminus \bar{\Delta}_1 \cup \bar{\Delta}_2) \cap Sp'(A) \neq \phi$ then define $E_3 = \bar{\Delta}_3 \setminus \bar{\Delta}_1 \cup \bar{\Delta}_2$. If $\operatorname{int}(\bar{\Delta}_3 \setminus \bar{\Delta}_1 \cup \bar{\Delta}_2) \cap Sp'(A) = \phi$ then $Sp'(A) \cap \Delta_3 \subseteq \bar{\Delta}_1 \cup \bar{\Delta}_2$ and we delete Δ_3 from the list.

If action was taken under (ii), renumber the remaining discs from 2 onwards and re-apply (i) & (ii).

Continuing in this way we obtain a finite number of mutually disjoint sets $E_1, E_2, \ldots, E_m$ with the desired properties. $\square$

LEMMA 3. Let $A \in B(H)$ be normal.

Then $\sup \operatorname{Re} P_k(A) \leq \sup \operatorname{Re} F_k(A) \quad (k=1,2,\ldots)$.

Proof. Apply the Spectral Theorem to give a measure space $(S, \sum, \mu)$ and an element $f \in L^\infty(S, \sum, \mu)$ such that A is unitarily equivalent to multiplication by f on $L^2(S, \sum, \mu)$. Given $\delta > 0$, let $E_1, E_2, \ldots, E_m$ be a cover of $Sp'(A)$ with the properties of Lemma 2. Let ϕ be any function in the equivalence class of functions f. Define a map $\psi : S \to \underset{\sim}{C}$ by

$$\psi(x) = \begin{cases} \mu_i & x \in \phi^{-1}(E_i) \quad i=1,2,\ldots,m \\ \phi(x) & \text{otherwise} \end{cases}$$

where $\mu_1, \mu_2, \ldots, \mu_m$ is any choice of points such that

$\mu_i \in \operatorname{int} E_i \cap Sp'(A) \quad (i=1,2,\ldots,m)$.

ψ is measurable and essentially bounded.

$|\phi(x) - \psi(x)| \leq 2\delta \quad (x \in S)$.

For each $n=1,2,\ldots$ take $\delta = 1/n$ and let $\{\psi_n\}$ be the corresponding sequence of functions. $\psi_n \to \phi$ uniformally as $n\to\infty$.

Let f_n be the equivalence class defined by ψ_n , and let A_n denote the normal operator corresponding to f_n . For each fixed $n \geq 1$, A_n has spectrum consisting of distinct eigenvalues $\mu_1, \mu_2, \ldots, \mu_m$ together with (possibly) a finite number of eigenvalues of A . The argument in Lemma 1 applies again here to show that any such eigenvalue occurs as an eigenvalue of A_n with unchanged multiplicity.

Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ denote the members of $Sp(A_n)$ written according to increasing real part i.e. $\mathrm{Re}\,\lambda_i \leq \mathrm{Re}\,\lambda_{i+1}$ $(i=1,..,r-1)$. Let $X_1, X_2, \ldots, X_r$ be the corresponding eigenspaces with orthonormal bases $\{ e_{\pi_i} : \pi_i \in \Pi_i \}$ $(i=1,2,\ldots,r)$ respectively .

Given $(e_1, e_2, \ldots, e_k) \in \underset{\sim}{O}_k$ we have , for $j=1,2,\ldots,k$

$$e_j = \sum_{i=1}^{r} \sum_{\pi_i \in \Pi_i} (e_j, e_{\pi_i}) e_{\pi_i} + h_j \quad \text{where } h_j \in (\overline{\mathrm{Span}}\{ \bigcup_{i=1}^{r} X_i \})^{\perp}$$

$$f_n e_j = \sum_{i=1}^{r} \lambda_i \sum_{\pi_i \in \Pi_i} (e_j, e_{\pi_i}) e_{\pi_i} + f_n h_j$$

$$(f_n e_j, e_j) = \sum_{i=1}^{r} \lambda_i \sum_{\pi_i \in \Pi_i} |(e_j, e_{\pi_i})|^2 + (f_n h_j, h_j) \quad \text{since eigen-}$$

spaces of A_n are reducing .

We claim that $(f_n h_j, h_j) = 0$ $(j=1,2,\ldots,k;\ n \geq 1)$.

Proof. Fix $n \geq 1$ and let $K = \mathrm{esssup}\,\psi_n$. For each $m \geq 1$ let D_m be the punched disc $D_m = \bar{\Delta}(0;K) \setminus \bigcup_{i=1}^{r} \Delta(\lambda_i; 1/m)$. For each $\lambda \in D_m$, there exists an open neighbourhood V_λ of λ such that $\mu(\psi_n^{-1}(V_\lambda)) = 0$.
Let $V_{\lambda_1}, V_{\lambda_2}, \ldots, V_{\lambda_q}$ be a finite cover of D_m .

Then $\mu(\psi_n^{-1}(D_m)) \leq \mu(\psi_n^{-1}(\bigcup_{i=1}^{q} V_{\lambda_i})) = 0$.

$$\{ x \in S : \psi_n(x) \neq \lambda_i \ (i=1,2,\ldots,r),\ |\psi_n(x)| \leq K \} = \psi_n^{-1}(\bigcup_{m=1}^{\infty} D_m) .$$

Therefore

$$\mu\{\ x\in S\ :\ \psi_n(x)\neq\lambda_i\ (i=1,2,\ldots,r)\ \}$$
$$=\ \mu\{\ x\in S\ :\ |\psi_n(x)|>K\ \}\ +\ \mu(\ \psi_n^{-1}(\bigcup_{m=1}^{\infty} D_m\ ))\ =0\ . \qquad ------(1)$$

Since $h_j \in \overline{\mathrm{Span}\{\bigcup_{i=1}^{r} X_i\}}^{\perp}$ it follows that

$$\mu\{\ \psi_n^{-1}(\lambda_i)\cap\{\ x\ :\ \xi_j(x)\neq 0\}\}=0 \qquad \text{for } i=1,2,\ldots,k$$

where ξ_j is any representative from the equivalence class h_j .
This fact together with (1) above implies that

$$\int_S \psi_n|\xi_j|^2 d\mu\ =0 \quad \text{and therefore} \quad (f_n e_j^{h_j}, e_j^{h_j})=0$$
$$(j=1,2,\ldots,k\ ;\ n\geq 1).$$

Returning to the proof of Lemma 3 , we have

$$\mathrm{Re}\ \frac{1}{k}\sum_{j=1}^{k}(f_n e_j,e_j)=\sum_{i=1}^{r}\ \sum_{\pi_i\in\Pi_i} c_{\pi_i}(\mathrm{Re}\lambda_i) \qquad -----(2)$$

where $c_{\pi_i}\ =\ \frac{1}{k}\sum_{j=1}^{k}|(e_j,e_{\pi_i})|^2$.

$$0\leq c_{\pi_i}\leq 1/k \quad , \quad \sum_{i=1}^{r}\sum_{\pi_i\in\Pi_i} c_{\pi_i}\leq 1 \quad .$$

The right hand side of (2) attains a maximum when $e_1,e_2,\ldots,e_k$ are eigenvectors with eigenvalues chosen in order of descending real parts including allowed repetitions for multiplicity. Furthermore the construction of f_n was such that this maximum belongs to Re $F_k(A)$.

Therefore we have sup Re $P_k(A_n)$ $\leq$ sup Re $F_k(A)$ $(n=1,2,\ldots)$. It follows that sup Re $P_k(A)\leq$ sup Re $F_k(A)$ $(k=1,2,\ldots)$. □

<u>THEOREM</u> 4. Let $A\in B(H)$ be normal . Then $P_k(A)^{-}=\mathrm{co}F_k(A)$.

<u>Proof</u>. co $F_k(A)\subseteq P_k(A)^{-}$ is clear from Lemma 1. co$[F_k(A)^{-}]$ is the intersection of all closed half planes containing $F_k(A)$. $\alpha A+\beta$ is normal for all $\alpha,\beta\in\mathbb{C}$. Hence it follows from Lemma 3 that

$P_k(A)^- \subseteq co[F_k(A)^-]$. Let $\lambda \in F_k(A)^-$, then there exists a sequence of arithmetic means $\frac{1}{k}\sum_{j=1}^{k}\alpha_{nj} \in F_k(A)$ $(n=1,2,\ldots)$ which converges to λ. Find a subsequence $\{n_m\}$ of the positive integers such that $\{\alpha_{n_m j}\}$ converges (to α_j say) for each $j=1,2,..,k$. $\lambda = \frac{1}{k}\sum_{j=1}^{k}\alpha_j$. $\alpha_j \in Sp(A)$ $(1 \leq j \leq k)$. If $\alpha_{j_1},\ldots,\alpha_{j_r}$ coincide and α_{j_1} is an accumulation point of the spectrum then this repetition is allowed. If α_{j_1} is isolated then for sufficiently large m, $\alpha_{n_m j_1} = \alpha_{n_m j_2} = \ldots = \alpha_{n_m j_r} = \alpha_{j_1}$ i.e. α_{j_1} is an eigenvalue of A with multiplicity at least r and so this repetition is also allowed. This argument applied to each group of coincident α_j's shows that $\lambda \in F_k(A)$ and so $F_k(A)$ is closed.

The desired conclusion follows. □

THEOREM 5. Let $A \in B(H)$ be normal and compact. Then $P_k(A)$ consists of the convex hull of the set of all arithmetic means of k eigenvalues of A where each eigenvalue may occur in a mean at most as many times as its multiplicity.

Proof. The inclusion $\supseteq$ is clear by Theorem 1.3 parts (2)&(5). Let $A = \sum_i \lambda_i e_i \otimes e_i$ be the Spectral decomposition of A, where $\{e_i\}$ is an orthonormal sequence of elements of H and

$|\mu_i| \geq |\mu_{i+1}|$ $i=1,2,\ldots$.

Let $(x_1,x_2,\ldots,x_k) \in \underset{\sim}{O}_k$ be given.

Case 1. Suppose H is separable. Let $\{f_j\}$ be an orthonormal basis for $\ker(A)$ with the convention that $f_1=0$ is an orthonormal basis when $\ker(A)=\{0\}$. We have

$$x_m = \sum_i (x_m,e_i)e_i + \sum_j (x_m,f_j)f_j$$

$$(Ax_m,x_m) = \sum_i \mu_i |(x_m,e_i)|^2$$

Therefore

$$\frac{1}{k}\sum_{m=1}^{k}(Ax_m,x_m) = \sum_i c_i\mu_i \quad \text{where} \quad c_i = \frac{1}{k}\sum_{m=1}^{k}|(x_m,e_i)|^2 .$$

Define $d_j = \frac{1}{k}\sum_{m=1}^{k}|(x_m,f_j)|^2$. We have $0 \leq c_i \leq 1/k$, $0 \leq d_j \leq 1/k$,

and $\sum_i c_i + \sum_j d_j = 1$. An application of Lemma 3.2 (with $\gamma = k$) shows that $\sum_i c_i\mu_i \in P_k(A)$.

Case 2. Suppose H is non-separable. With the notation of Case 1 we have $\frac{1}{k}\sum_{m=1}^{\infty}(Ax_m,x_m) = \sum_i c_i\mu_i$. Let $\gamma = k\sum_i c_i$.
Applying Lemma 3.2 it suffices to show that every sum of the form

(i). $\gamma\frac{1}{k}\mu_{j_1} \in P_k(A)$ if $[\gamma] = 0$,

(ii). $\frac{1}{k}(\mu_{j_1}+\dots+\mu_{j_{[\gamma]}} + (\gamma-[\gamma])\mu_{j_{[\gamma]+1}}) \in P_k(A)$ if $[\gamma]\geq 1$.

(i). $\gamma\frac{1}{k}\mu_{j_1}$ is a convex combination of 0 and $\frac{1}{k}\mu_{j_1}$ and therefore belongs to $P_k(A)$ (A has infinite dimensional kernel).

(ii). $\frac{1}{k}(\mu_{j_1}+\dots+\mu_{j_{[\gamma]}} + (\gamma-[\gamma])\mu_{j_{[\gamma]+1}})$ can be written as the convex combination

$$(1-(\gamma-[\gamma]))\frac{1}{k}\{\mu_{j_1}+\dots+\mu_{j_{[\gamma]}}\} + (\gamma-[\gamma])\frac{1}{k}\{\mu_{j_1}+\dots+\mu_{j_{[\gamma]+1}}\}.$$

It follows that (ii) belongs to $P_k(A)$. $\square$

To end this section we return to the problem of necessary and sufficient conditions for the k-range of an operator to be closed. We recall from Theorem 3.10 that the k-range of a compact operator A is closed whenever A has a kernel of dimension not less than k.

PROPOSITION 6. Let $A \in B(H)$ be normal and compact. Suppose that the spectrum of A is contained in a sector of angle $< \pi$ (i.e. for some $0 \leq \theta < 2\pi$, $Sp(A) \subseteq \{ re^{i\phi} : r \geq 0 , \theta \leq \phi < \theta+\pi \}$). Then $P_k(A)$ is closed if and only if $\dim\ker(A) \geq k$.

Proof. We have to establish the 'only if' part of the Proposition. By Theorem 5 we see that the k-range of A will also lie in the given sector. By Corollary 2.7, $P_k(A)$ closed $\Rightarrow$ $0 \in P_k(A)$. 0 is therefore an extreme point of $P_k(A)$ and by Theorem 5 must be an arithmetic mean of the allowed type. Therefore 0 is an eigenvalue of A of multiplicity at least k. □

Example. The condition $\dim \ker(A) \geqslant k$ is not in general necessary for a closed k-range when A is compact. For example, let $A \in B(H)$ be compact self-adjoint with trivial kernel and with infinite sequences of (strictly) positive and (strictly) negative eigenvalues $\{\mu_n\}$ and $\{\lambda_n\}$ respectively. ($\mu_n \geqslant \mu_{n+1}$; $\lambda_{n+1} \geqslant \lambda_n$, n=1,2,..). By Theorem 5 we have

$$P_k(A) = \left[\frac{1}{k}\sum_{j=1}^{k} \lambda_j \ , \ \frac{1}{k}\sum_{j=1}^{k} \mu_j \right] \qquad (k=1,2,\ldots) .$$

PROPOSITION 7. (J.P.Williams)

Let $A \in B(H)$. Then $W(A)$ is closed if and only if $W_{ess}(A) \subseteq W(A)$.

Proof. With the notation of §3, given a state f on B(H) Dixmier [7] has shown that

$f = \alpha f_T + (1-\alpha) g$ where f_T is the state associated with ~~the~~ a trace class operator T, $tr(T) = 1$, $0 \leqslant T \leqslant I$. g is a state which vanishes on the ideal of compact operators, and $0 \leqslant \alpha \leqslant 1$. We remarked in the introduction to §3 that

$$\{ f_T(A) : T \in \mathcal{B}_1 , tr(T)=1 , 0 \leqslant T \leqslant I \} = W(A) .$$

It follows that

$$W(A)^- = co \{ W(A) \cup W_{ess}(A) \}$$

and therefore $W(A)$ is closed if and only if $W(A) \supseteq W_{ess}(A)$. □

Example.

If $P_k(A)$ is closed then $W ess(A) \subseteq P_k(A)$ by Theorem 2.6 . The obvious generalisation of Williams' observation does not hold for the k-range when $k>1$. For example, let $A\in B(H)$ be compact and self-adjoint with trivial kernel and with infinitely many negative eigenvalues $\{\mu_n\}$ and finitely many positive eigenvalues $\{ \lambda_1, \lambda_2, \ldots, \lambda_m \}$. ($\mu_{n+1} \geq \mu_n$ $n=1,2,..$; $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$)
By Theorem 5 we have

$$P_k(A) = \begin{cases} \left[\frac{1}{k}\sum_{j=1}^{k} \mu_j \, , \, \frac{1}{k}\sum_{j=1}^{k} \lambda_j \right] & k=1,2,\ldots,m \\ \left[\frac{1}{k}\sum_{j=1}^{k} \mu_j \, , \, \frac{1}{k}\sum_{j=1}^{m} \lambda_j \right) & k=m+1,m+2,\ldots \end{cases}$$

Thus, when $k > m$, $P_k(A)$ is not closed but $0 \in P_k(A)$.

§5. Unitary invariants.

S.K.Parrott [4] generalised the spatial numerical range of an operator $A \in B(H)$ by replacing the subset of the plane with sets of matrices. More precisely, the Parrott matrix range $W_k(A)$ of A is the set of compressions of A to all k-dimensional subspaces of H. i.e. $W_k(A) = \{ PA|_{PH} : P \in \underset{\sim}{P}_k \}$.

Parrott proved that these matrix ranges form a complete set of unitary invariants for compact operators with zero reducing nullspaces [4 :§36 Theorem 9]. Given $P \in \underset{\sim}{P}_k$ then $tr(PA) = tr(PA|_{PH})$ and therefore the Halmos k-range consists of the normalised traces of the operators in the Parrott range. This gives rise to a natural question. Can the Halmos k-ranges be substituted in place of the Parrott ranges in the invariants theorem ? The answer is no in general and we give a counter-example. We give a positive result concerning the problem of which classes of operators are completely determined, up to unitary equivalence, by their k-ranges. Finally we mention another candidate generalising the classical numerical range of A, denoted by $X_k(A)$, which 'lies between' the Parrott and Halmos ranges in the following sense. Given $A,B \in B(H)$ then

$$W_k(A) = W_k(B) \Rightarrow X_k(A) = X_k(B) \Rightarrow P_k(A) = P_k(B) \quad (k=1,2,\ldots).$$

PROPOSITION 1.

Let H be a complex Hilbert space of dimension 2, and let $A,B \in B(H)$. Then $W(A) = W(B)$ if and only if A and B are unitarily equivalent.

Proof. There exists an orthonormal basis $\{e_1, e_2\}$ for H relative to which the matrix for A has upper triangular form.

Let $\begin{bmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{bmatrix}$ be the matrix of A relative to this basis.

Let S be any operator on H with matrix representation

$\begin{bmatrix} \lambda_1 & \mu \\ 0 & \lambda_2 \end{bmatrix}$ relative to an orthonormal basis $\{u_1, u_2\}$

and such that $|\alpha| = |\mu| > 0$. Then S and A are unitarily equivalent. (The map $U : H \to H$ defined by $Uu_1 = e_1$, $Uu_2 = (\mu/\alpha)e_2$ and extended linearly to H is unitary and $S = U^{-1}A\,U$.)

The implication ($\Longleftarrow$) of the Proposition is trivial. In view of the foregoing remarks the converse will be established if the numerical range determines the eigenvalues of A together with the modulus of the entry in the top right hand corner of the matrix for A (when in upper triangular form) . A calculation shows that

$$W(A) = \{\, \lambda_1|x|^2 + \lambda_2|y|^2 + \alpha y x^* \;:\; x,y \in \mathbb{C},\ |x|^2 + |y|^2 = 1 \,\}.$$

Case 1. $\lambda_1 = \lambda_2 = \lambda$. If $\alpha = 0$ then $W(A) = \{\lambda\}$. If $\alpha \neq 0$, $W(A)$ consists of the closed disc centre λ and radius $|\alpha|/2$.

Case 2. $\lambda_1 \neq \lambda_2$. If $\alpha = 0$ then $W(A)$ consists of the straight line joining λ_1 and λ_2. If $\alpha \neq 0$, $W(A)$ is an ellipse with foci λ_1, λ_2 and length of minor axis $|\alpha|$.

In each case the numerical range determines the eigenvalues of A together with $|\alpha|$. □

LEMMA 2. Let $T \in B(H)$, then $P_k(\operatorname{Re}T) = \operatorname{Re} P_k(T)$;

$$P_k(\operatorname{Im}T) = \operatorname{Im} P_k(T) \quad (k = 1,2,\ldots).$$

Proof. Straightforward. □

LEMMA 3. Let $S \in B(H)$ be compact and self-adjoint. The family of k-ranges of S determines the eigenvalues of S together with their multiplicities. In addition, if S possesses only finitely many positive or finitely many negative eigenvalues then the k-ranges determine the dimension of the kernel of S.

Proof. Apply the Spectral Theorem to write S in the form

$$S = \sum_i \lambda_i E_i - \sum_j \mu_j E_j \quad \text{where } E_i, F_j \text{ are mutually orthogonal}$$

rank one projections, and $\lambda_i \geq \lambda_{i+1} > 0$, $\mu_j \geq \mu_{j+1} > 0$.

Case 1. Suppose S has infinitely many positive and negative eigenvalues. It follows from Theorem 4.5 that

$$P_k(S) = [\, -\frac{1}{k}\sum_{i=1}^{k} \lambda_i \, , \; \frac{1}{k}\sum_{j=1}^{k} \mu_j \,] \quad (k=1,2,\ldots) .$$

Case 2. Suppose S has infinitely many negative eigenvalues and m positive eigenvalues . In each case, using Theorem 4.5 we have :

(i). If m=0 and dim(kerS)=0 then $P_k(S) = [\, -\frac{1}{k}\sum_{i=1}^{k} \lambda_i \, , \, 0 \,)$ $k=1,2,\ldots$.

(ii). If m=0, $1 \leq \dim(\ker S) < \infty$ then

$$P_k(S) = [\, -\frac{1}{k}\sum_{i=1}^{k} \lambda_i \, , \, 0 \,] \qquad 1 \leq k \leq \dim(\ker S) ,$$

$$P_k(S) = [\, -\frac{1}{k}\sum_{i=1}^{k} \lambda_i \, , \, 0 \,) \qquad k > \dim(\ker S) .$$

(iii). If m=0, dim(kerS) $= \infty$ then

$$P_k(S) = [\, -\frac{1}{k}\sum_{i=1}^{k} \lambda_i \, , \, 0 \,] \qquad k = 1,2,\ldots .$$

(iv). If $m \geq 1$, dim(kerS) $= \infty$ then

$$P_k(S) = [\, -\frac{1}{k}\sum_{i=1}^{k} \lambda_i \, , \, \frac{1}{k}\sum_{j=1}^{\min(k,m)} \mu_j \,] \qquad k=1,2,\ldots .$$

(v). If $m \geq 1$ and $\dim(\ker S) < \infty$ then

$$P_k(S) = [-\frac{1}{k}\sum_{i=1}^{k}\lambda_i \, , \, \frac{1}{k}\sum_{j=1}^{\min(k,m)}\mu_j\,] \qquad k \leq m + \dim(\ker S) \quad ,$$

$$P_k(S) = [-\frac{1}{k}\sum_{i=1}^{k}\lambda_i \, , \, \frac{1}{k}\sum_{j=1}^{\min(k,m)}\mu_j\,) \qquad k > m + \dim(\ker S) \quad .$$

The remaining case when there are finitely many negative eigenvalues is exactly similar. The desired conclusion follows immediately from these explicit expressions for the k-ranges. $\square$

<u>THEOREM</u> 4. Let $S,T \in B(H)$ be compact normal operators with zero nullspaces such that $P_k(S) = P_k(T)$ $(k = 1,2,\ldots)$.

Then S and T are unitarily equivalent.

<u>Proof</u>. Apply the Spectral theorem to write T in the form

$$T = \sum_{j\in\Lambda_1}(\lambda_j^+ + i\mu_j)E_j - \sum_{\ell\in\Lambda_2}(\lambda_\ell^- + i\eta_\ell)F_\ell + i\sum_{m\in\Lambda_3}\mu_m^+G_m - i\sum_{n\in\Lambda_4}\mu_n^-H_n$$

where each index set Λ_m may be either empty , finite, or countable. E_j, F_ℓ, G_m, H_n are mutually orthogonal rank one projections.

$$\lambda_j^+ \geq \lambda_{j+1}^+ > 0 \;,\; \lambda_\ell^- \geq \lambda_{\ell+1}^- > 0 \;,\; \mu_m^+ \geq \mu_{m+1}^+ > 0, \; \mu_n^- \geq \mu_{n+1}^- > 0 \;.$$

Apply Lemma 3 to the real part of T and invoke Lemma 2. We see that the sets $\{\lambda_j^+ : j\in\Lambda_1\}$, $\{\lambda_\ell^- : \ell\in\Lambda_2\}$ are completely determined by the k-ranges of T . Suppose $\Lambda_1 \neq \phi$, and suppose λ_1^+ occurs n_1 times. Re-order the numbers $\lambda_1^+ + i\mu_1, \lambda_2^+ + i\mu_2, \ldots,$ $\lambda_{n_1}^+ + i\mu_{n_1}$, so that the imaginary parts are decreasing relative to increasing suffix. Assume that this re-ordering has been carried out for each set of distinct λ_j^+ . By Theorem 4.5 the member of $P_1(T)$ with real part λ_1^+ (= max Re $P_1(T)$) and maximum imaginary part is $\lambda_1^+ + i\mu_1$. Therefore P_1 determines μ_1 .

Suppose $n_1>1$, then $\max \operatorname{Re} P_2(T) = \lambda_1^+$. The member of $P_2(T)$ with real part λ_1^+ and maximum imaginary part is $\lambda_1^+ + [i(\mu_1+\mu_2)/2]$. P_2 therefore determines μ_2. In this way $P_k(T)$ $(k=1,2,\dots,n_1)$ determine $\mu_1,\mu_2,\dots,\mu_{n_1}$. If $\Lambda_1=\{1,\dots,n_1\}$ we stop. If $n_1+1 \in \Lambda_1$, suppose λ_{n_1+1} occurs n_2 times , and let $\tilde{\mu}_{n_1} = \sum_{i=1}^{n_1} \mu_i$.

$P_{n_1+1}(T)$ has maximum real part $(n_1\lambda_1^+ + \lambda_{n_1+1}^+)/n_1+1$ with corresponding maximum imaginary part $(\tilde{\mu}_{n_1} + \mu_{n_1+1})/n_1+1$. P_{n_1+1} therefore determines μ_{n_1+1} . By a similar argument as for λ_1^+ we see that $P_{n_1+1},\dots,P_{n_1+n_2}$ determine $\mu_{n_1+1},\dots,\mu_{n_1+n_2}$.

In this way the k-ranges determine the set $\{ \mu_j : j\in\Lambda_1 \}$. The set $\{\eta_\ell : \ell\in\Lambda_2 \}$ is determined in a like manner from the minimum real parts of the k-ranges. All the eigenvalues of T are so far accounted for, except those which are purely imaginary.

$$\operatorname{Im} T = \sum_{j\in\Lambda_1} \mu_j E_j - \sum_{\ell\in\Lambda_2} \eta_\ell F_\ell + \sum_{m\in\Lambda_3} \mu_m^+ G_m - \sum_{n\in\Lambda_4} \mu_n^- H_n$$

Apply Proposition 3 to the imaginary part of T and invoke Lemma 2. The eigenvalues of $\operatorname{Im}T$ together with their multiplicities are determined by the k-ranges. Since the sets $\{\mu_j : j\in\Lambda_1 \}$, $\{\eta_\ell : \ell\in \Lambda_2\}$ are already fixed , the sets $\{ \mu_m^+ : m\in\Lambda_3\}$ and $\{ \mu_n^- : n\in\Lambda_4 \}$ are therefore determined.

Thus the eigenvalues of S and T together with their associated multiplicities coincide. Since S and T have zero nullspaces the map which sends eigenvectors of S to eigenvectors of T with the same eigenvalues extends linearly to a unitary map of H onto H and $U^{-1} T U = S$. $\square$

Definition 6. Given $A \in B(H)$, let $X_k(A)$ denote the subset of $\underset{\sim}{C}^k$ given by

$$X_k(A) = \{ \ (\ (Ae_1,e_1),(Ae_2,e_2),\ldots,(Ae_k,e_k) \ ) : (e_1,e_2,\ldots,e_k) \in \underset{\sim}{O}_k \ \} .$$

Remarks.

(1). This k-range was mentioned by F.F.Bonsall [1]. P.A.Fillmore and J.P.Williams [11] considered the set $X_n(A)$ when A is an operator on a finite dimensional Hilbert space of dimension n . They were concerned with the following unsolved problem. If A is a given normal $n \times n$ matrix, determine which n-tuples can serve as the diagonal of some matrix unitarily equivalent to A .

(2). It was remarked by F.F.Bonsall and J.Duncan [4 , §36.2] that the set of matrix representations relative to the natural basis for $\underset{\sim}{C}^k$ of the operators in the Parrott range $W_k(A)$ is

$$\{ \ (\alpha_{ij}) \ : \ \alpha_{ij} = (Au_j,u_i) \ ; \ i,j=1,2,\ldots,k \ ; \ (u_1,u_2,\ldots,u_k) \in \underset{\sim}{O}_k \} .$$

The following example shows that the Parrott unitary invariants theorem (as stated in the introduction of this section) does not hold for the X_k-ranges and hence does not hold for the Halmos k-ranges.

Example 7. Let H be a separable Hilbert space and let $S,T \in B(H)$ have matrix representations relative to an orthonormal basis $\{e_n\}$ of H as follows :

$$m(S) = \begin{bmatrix} 1 & 2 & 0 & & & & \\ 0 & 1 & 1 & & & 0 & \\ 0 & 0 & 1 & & & & \\ & & & 1/2 & & & \\ & 0 & & & 1/3 & & \\ & & & & & 1/4 & \\ & & & & & & \ddots \end{bmatrix}$$

$$m(T) = \begin{bmatrix} 1 & 1 & 0 & & & & \\ 0 & 1 & 2 & & & 0 & \\ 0 & 0 & 1 & & & & \\ & & & 1/2 & & & \\ & 0 & & & 1/3 & & \\ & & & & & 1/4 & \\ & & & & & & \ddots \end{bmatrix}$$

S and T are non-normal compact operators each having zero kernel. $X_k(S) = X_k(T)$ $(k=1,2,\dots)$. S and T are not unitarily equivalent.

Proof. It is plain that S and T are non-normal compact operators with zero kernels. Let $x,y \in H$,

$$x = \sum_{i=1}^{\infty} x_i e_i \quad ; \quad y = \sum_{i=1}^{\infty} y_i e_i \quad .$$

A calculation shows that

$$(Sx,x) = 2x_1^* x_2 + x_2^* x_3 + |x_1|^2 + |x_2|^2 + \sum_{n=3}^{\infty} \frac{|x_n|^2}{(n-2)}$$

$$(Ty,y) = y_1^* y_2 + 2y_2^* y_3 + |y_1|^2 + |y_2|^2 + \sum_{n=3}^{\infty} \frac{|y_n|^2}{(n-2)} \quad .$$

The map $\sigma : \{ z_1, z_2, \dots \} \mapsto \{ z_3^*, z_2^*, z_1^*, z_4^*, z_5^*, \dots \} : H \to H$ is a 1-1 correspondence between the elements of the unit shell of H. σ preserves orthogonality, i.e. $x \perp y \iff \sigma(x) \perp \sigma(y)$ $(x,y \in H)$. Also $(T\sigma(x),\sigma(x)) = (Sx,x)$ $(x \in H)$. Therefore $\chi_k(S) = \chi_k(T)$ $(k=1,2,\dots)$.

Suppose there exists a unitary operator U such that $U^* T U = S$. $TUe_1 = USe_1 = Ue_1$. Therefore $Ue_1 = \alpha e_1$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$.

$$TUe_2 = USe_2 = 2Ue_1 + Ue_2 = 2\alpha e_1 + Ue_2 \qquad \text{--------- (1)}$$

$$Ue_2 = \sum_{j=1}^{\infty} (Ue_2,e_j)e_j$$

$$TUe_2 = (Ue_2,e_1)e_1 + (Ue_2,e_2)[e_1+e_2] + (Ue_2,e_3)[2e_2+e_3] + \sum_{j=4}^{\infty} \frac{(Ue_2,e_j)}{j-2} e_j \qquad \text{--------- (2)}$$

From (1) and (2), equating coefficients of e_1 gives

$(Ue_2,e_2) = 2\alpha$ which is impossible. $\square$

PROPOSITION 8. (Fillmore [10])

A complex square $n\times n$ matrix A is unitarily equivalent to a matrix with main diagonal $(\mathrm{tr}(A),0,0,\dots,0)$ if and only if $\mathrm{tr}(A) \in W(A)$.

Proof. " only if " is trivial. For the converse we use induction on the size of the matrix. Since $\mathrm{tr}(A) \in W(A)$, there exists a unit vector x with $\mathrm{tr}(A) = (Ax,x)$. Let $\{x_1,x_2,\dots,x_n\}$ be any orthonormal basis with $x_1 = x$. Relative to this basis A has matrix $\begin{bmatrix} \mathrm{tr}(A) & B \\ C & D \end{bmatrix}$ where D is $(n-1)\times(n-1)$ and $\mathrm{tr}(D) = 0$. If $n = 2$, $D = 0$ and we are finished. Suppose $n>2$. Let $\lambda_1,\dots,\lambda_{n-1}$ be the eigenvalues of D. Then $\frac{1}{n-1}(\lambda_1+\lambda_2+\dots+\lambda_{n-1}) = \frac{1}{n-1}\mathrm{tr}(D) = 0$ Therefore $0 \in \mathrm{coSp}(D)$ and hence $0 \in W(D)$. By induction there exists a unitary matrix U such that U^*DU has main diagonal consisting of zeros.

$$V = \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix} \text{ is unitary and } V^* \begin{bmatrix} tr(A) & B \\ C & D \end{bmatrix} V = \begin{bmatrix} tr(A) & BU \\ U^*C & U^*D\,U \end{bmatrix}$$

has main diagonal $(tr(A),0,\dots,0)$. □

<u>THEOREM</u> 9. Let $A \in B(H)$. $\lambda \in P_k(A)$ if and only if $X_k(A)$ contains a vector with each co-ordinate equal to λ.

<u>Proof.</u> " if " is trivial. We may assume that $\lambda = 0 \in P_k(A)$. Then there exists $P \in \underset{\sim}{P}_k$ such that $tr(PA) = 0$.

$0 = tr(PA) = tr(C_{PH}(A))$. By Proposition 8, there exists an orthonormal basis $e_1, e_2, \dots, e_k$ of PH such that

$(C_{PH}(A)e_i, e_i) = 0 \quad (i=1,2,\dots,k)$.

But $(C_{PH}(A)e_i, e_i) = (PAe_i, e_i) = (Ae_i, e_i) \quad (i=1,2,\dots,k)$.

Therefore $\underset{\sim}{0} \in X_k(A)$. □

<u>Remark.</u> A few preliminary results on the X_k-ranges are described in [11]. In connection with our discussion the notable outstanding problem is the following.

Which operators are distinguished by the X_k-ranges and not by the Halmos ranges?

We end this section with some examples of X_k-ranges.

<u>Examples.</u>

(1). Let S be a unilateral shift of arbitrary multiplicity.

Then $X_k(S) = \Delta^k$.

<u>Proof.</u> Recall that S is an isometry with the property that there exists a wandering subspace $K \subset H$ for S such that K, SK,, S^2K, S^3K, . . . are pairwise orthogonal and

$H = K \oplus \sum_{j=1}^{\infty}{}_{\oplus}\, S^j K$. The multiplicity of S = $\dim K$.

Let $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Delta^k$. Choose any positive integer $n \geq 2$ such that $|\lambda_i| < 1 - \frac{1}{n}$ $(i = 1, 2, \ldots, k)$, and let $\theta_i = \operatorname{Arg} \lambda_i$ for $i = 1, 2, \ldots, k$. Given $x \in S(K)$ define

$$y_1 = \frac{1}{\sqrt{n}} \left(e^{in\theta_1} x + e^{i(n-1)\theta_1} Sx + \ldots + e^{i\theta_1} S^{n-1} x \right) ,$$

$$z_1 = x \quad . \qquad \|y_1\| = \|z_1\| = 1 \; .$$

A calculation shows that

$$(Sy_1, y_1) = \frac{n-1}{n} e^{i\theta_1} \quad , \quad (Sz_1, z_1) = 0 \quad .$$

Let $t = |\lambda_1| n/n-1$, then $0 \leq t < 1$. By the Toeplitz-Hausdorff Theorem there exists $u_1 \in \operatorname{Span}\{y_1, z_1\}$, $\|u_1\| = 1$, such that $(Su_1, u_1) = (1-t)0 + t \frac{(n-1)}{n} e^{i\theta_1} = \lambda_1$.

Now define y_2 and z_2 in a similar manner using $S^n x$ in place of x. The Toeplitz-Hausdorff Theorem yields $u_2 \in \operatorname{Sp}\{y_2, z_2\}$, $\|u_2\| = 1$, such that $(Su_2, u_2) = \lambda_2$. Note that $(u_1, u_2) = 0$. Continuing in this way we find $u_3, \ldots, u_k$ with the property that $(u_1, u_2, \ldots, u_k) \in \underset{\sim}{O}_k$ and $(Su_j, u_j) = \lambda_j$ $(j = 1, 2, \ldots, k)$.

Therefore $\Delta^k \subseteq X_k(S)$. $X_k(S)$ cannot contain a k-tuple having an entry with modulus one, for such an entry would be an eigenvalue of S . □

(2). Let V be a non-unitary isometry . Then $X_k(V)$ consists of all k-tuples of numbers taken from $\bar{\Delta}$ with the property that any entry of modulus one in a k-tuple may not occur more often than its multiplicity as an eigenvalue of **V** .

Proof. Let $V = U \oplus S$ be the Wold decomposition of V and let $H = H_1 \oplus H_2$ denote the corresponding decomposition of H . (U is a unitary map of H_1 onto H_1 , S is a shift on $H_2 \neq \{0\}$) .

Applying the result of example (1), $X_k(V)$ contains the specified set of k-tuples. Conversely if

$((Ve_1,e_1),(Ve_2,e_2),\ldots,(Ve_k,e_k)) \in X_k(V)$, then any component (Ve_j,e_j) having modulus one is an eigenvalue of V with eigenvector e_j . $\square$

CHAPTER 2

A MATRIX RANGE FOR OPERATORS

ON THE BANACH SPACE ℓ^1

§1. Definition and properties.

In this chapter we propose a definition for a matrix range of an operator on the Banach space of summable sequences. The results depend heavily on the exploitation of the special structure of ℓ^1, notably the presence of a readily identifiable predual and dual.

Notation.

Let c_0, ℓ^1, ℓ^∞, denote the spaces of complex sequences which converge to zero, are summable, are bounded, respectively. c_0, ℓ^1, ℓ^∞ will have their usual Banach space norms.

Let $(\ , \ ) : \ell^1 \times \ell^\infty \to \underset{\sim}{C}$ denote the sesquilinear map defined by

$$(x,y) = \sum_{j=1}^{\infty} x_j y_j^* \qquad (x = \{x_j\} \in \ell^1 \ ; \ \ y = \{y_j\} \in \ell^\infty \ ).$$

The map $\Psi : y \mapsto \Psi_y : \ell^\infty \to (\ell^1)'$ where $\Psi_y(x) = (x,y) \quad (x \in \ell^1)$ is an antilinear isometric isomorphism of ℓ^∞ onto the dual space of ℓ^1.

Let $(\ , \ )_{\underset{\sim}{C}^n} : \underset{\sim}{C}^n \times \underset{\sim}{C}^n \to \underset{\sim}{C}$ denote the sesquilinear map given by

$$(\alpha,\beta)_{\underset{\sim}{C}^n} = \sum_{i=1}^{n} \alpha_i \beta_i^* \quad (\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), \ \ \beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \underset{\sim}{C}^n).$$

The map $\Phi : \beta \mapsto \Phi_\beta : \underset{\sim}{C}^n \to (\underset{\sim}{C}^n)'$ where $\Phi_\beta(z) = (z,\beta)_{\underset{\sim}{C}^n} \quad (z \in \underset{\sim}{C}^n)$ is an antilinear isometric isomorphism of $\underset{\sim}{C}^n$ onto the dual space of $\underset{\sim}{C}^n$.

<u>Definition</u> 1. Let A be a bounded linear map of $\underset{\sim}{C}^n$ into ℓ^1, For a fixed $y \in \ell^\infty$ the map $z \mapsto \Psi_y(Az) : \underset{\sim}{C}^n \to \underset{\sim}{C}$ is a bounded linear functional on $\underset{\sim}{C}^n$. Therefore there exists a unique member of $\underset{\sim}{C}^n$ which we denote by $A^T y$, such that

$$\Phi_{A^T y}(z) = \Psi_y(Az) \quad (z \in \underset{\sim}{C}^n) .$$

The bounded linear map $y \mapsto A^T y : \ell^\infty \to \underset{\sim}{C}^n$ will be called <u>the transpose of A</u>.

<u>Notation</u>. Let $i : \ell^1 \to \ell^\infty$ denote the inclusion map of ℓ^1 in ℓ^∞. Given $A \in B(\underset{\sim}{C}^n, \ell^1)$, we abbreviate to A^* the composition of A^T and i, and so A^* is a linear map from ℓ^1 into $\underset{\sim}{C}^n$.

If $z \in \underset{\sim}{C}^n$ then the subscript ℓ^1 attached to the norm sign, $\|z\|_{\ell^1}$, indicates that $\underset{\sim}{C}^n$ is endowed with the ℓ^1-norm.

i.e. $\|z\|_{\ell^1} = \sum_{i=1}^{n} |z_i| \qquad (z = (z_1, z_2, \dots, z_n) \in \underset{\sim}{C}^n)$.

<u>Definition</u> 2. Let $\Pi_n(\ell^1)$ denote the set of isometric linear maps $A : \underset{\sim}{C}^n \to \ell^1$ (with the ℓ^1-norm on $\underset{\sim}{C}^n$) which satisfy

(1). $A^* A = I_{\underset{\sim}{C}^n}$;

(2). $A A^*$ is a contraction (i.e. $\|A A^* x\| \leq \|x\| \quad x \in \ell^1$).

Given $T \in B(\ell^1)$, <u>the n^{th} spatial matrix range for T</u> is the set $V_n(T)$ of linear operators on $\underset{\sim}{C}^n$ given by

$$V_n(T) = \{ A^* T A : A \in \Pi_n(\ell^1) \} .$$

<u>PROPOSITION</u> 3.

Let $A \in \Pi_n(\ell^1)$. Then A^* is a contraction.

<u>Proof</u>. If $A \in \Pi_n(\ell^1)$ we have

$\| A^* x \|_{\ell^1} = \|A A^* x\| \leq \|x\| \quad (x \in \ell^1)$ since A is isometric and $A A^*$

is a contraction. □

LEMMA 4. For each $n=1,2,\ldots$ there exists $A_n \in \Pi_n(\ell^1)$ such that $\{A_n A_n^*\}_1^\infty$ is a sequence of norm one projections which converge strongly to the identity map on ℓ^1.

Proof. Let A be the linear map of $\underset{\sim}{C}^n$ into ℓ^1 given by

$$Az = \{z_1, z_2, \ldots, z_n, 0, 0, \ldots\} \quad (z = (z_1, z_2, \ldots, z_n) \in \underset{\sim}{C}^n).$$

A is isometric with the ℓ^1 norm on $\underset{\sim}{C}^n$. We have

$$\Phi_{A^T y}(z) = \Psi_y(Az) = (Az, y) = \sum_{i=1}^{n} z_i y_i^* = (z, w)_{\underset{\sim}{C}^n} = \Phi_w(z)$$

for all $z = (z_1, z_2, \ldots, z_n) \in \underset{\sim}{C}^n$ where $y = \{y_i\} \in \ell^\infty$ and $w = (y_1, y_2, \ldots, y_n) \in \underset{\sim}{C}^n$.

Therefore $A^* y = A^T \circ i\,(y) = (y_1, y_2, \ldots, y_n) \quad (y = \{y_i\} \in \ell^1)$.

It is clear that $A^* A = I_{\underset{\sim}{C}^n}$ and

$$A A^* y = \{y_1, y_2, \ldots, y_n, 0, 0, \ldots\} \quad (y = \{y_i\} \in \ell^1).$$

Therefore $A \in \Pi_n(\ell^1)$. Let $P = A A^*$. The sequence $\{P_n\}$ of such projections for $n = 1,2,\ldots$ converges strongly to the identity operator on ℓ^1. □

Notation. Let $\Pi(\ell^1)$ denote the subset of $\ell^1 \times (\ell^1)'$ given by

$$\Pi(\ell^1) = \{(x,f) : \|x\| = \|f\| = f(x) = 1\}.$$

For the remainder of the chapter we use the symbol e_k to denote the sequence with 1 in the k^{th} place and zeros elsewhere.

PROPOSITION 5.

(1). The map $A \to (A1, A^*)$ is a one to one map of $\Pi_1(\ell^1)$ into $\Pi(\ell^1)$.

(2). $\Pi_1(\ell^1)$ consists of those linear maps A, of the form $A1 = \alpha_k e_k$ for some integer $k \geq 1$ and complex number α_k, $|\alpha_k| = 1$.

<u>Proof</u>. (1). Applying Proposition 3 we see that $(A1, A^*) \in \Pi(\ell^1)$ whenever $A \in \Pi_1(\ell^1)$. Since A is a linear map of $\underset{\sim}{C}$ into ℓ^1, A is completely determined by its action on 1 and therefore the map $A \to (A1, A^*)$ is 1-1 .

(2). Given $A \in \Pi_1(\ell^1)$, write $A1 = \{a_j\}_1^\infty$.

$A^*A1 = 1 \Rightarrow 1 = (1, A^*A1)_{\underset{\sim}{C}} = (1, A_\circ^T i(A1))_{\underset{\sim}{C}} = (A1, i(A1))$. .

i.e. $1 = \sum_{j=1}^{\infty} |a_j|^2$.

A isometric $\Rightarrow \sum_{j=1}^{\infty} |a_j| = 1$.

The desired conclusion follows from these two facts. □

<u>Remark.</u>

$V_1(.)$ is not a Printer. $V_1(T)$ can be very much smaller than the spatial numerical range of T . For example, let $T \in B(\ell^1)$ be a shift on ℓ^1 (i.e. $Te_n = e_{n+1}$ n=1,2,...) . Given $A \in \Pi_1(\ell^1)$, then $A1 = \alpha_k e_k$ for some $k \geq 1$ and $\alpha_k \in \underset{\sim}{C}$, $|\alpha_k| = 1$, by Proposition 5.(2) . For each $z \in \underset{\sim}{C}$, $y = \{y_j\} \in \ell^1$ we have

$$\begin{aligned}\Phi_{A^*y}(z) &= \Psi_{i(y)}(Az) = (Az, i(y)) = z\,\alpha_k\, y_k^* = (z, \alpha_k^* y_k)_{\underset{\sim}{C}} \\ &= \Phi_{\alpha_k^* y_k}(z) .\end{aligned}$$

Therefore $A^*y = \alpha_k^* y_k \qquad (y = \{y_j\} \in \ell^1)$.

$A^*T\,A1 = A^*T\,\alpha_k e_k = \alpha_k A^* e_{k+1} = 0$. With $B(\underset{\sim}{C})$ and $\underset{\sim}{C}$ identified in the usual way (i.e. $S \mapsto S1$) we therefore have $V_1(T) = \{0\}$.

Let $y = \{1, y_1, y_2, \ldots\}$, $|y_i| \leq 1$ (i=1,2,...) .

Then $(e_1, \Psi_y) \in \Pi(\ell^1)$.

$\Psi_y(Te_1) = y_1^*$. Therefore $V(T) = \{ z \in \underset{\sim}{C} : |z| \leq 1 \}$.

<u>Definition</u> 6. Let $U \in B(\ell^1)$ and let Ψ be as before. For a fixed $y \in \ell^\infty$, the map $x \mapsto \Psi_y(Ux)$ is a bounded linear functional on ℓ^1. Therefore there exists a unique element of ℓ^∞, which we denote by $U^T y$, such that

$$\Psi_{U^T y}(x) = \Psi_y(\overset{Ux}{\cancel{x}}) \quad (x \in \ell^1) .$$

The bounded linear map $y \to U^T y : \ell^\infty \to \ell^\infty$ will be called <u>the transpose</u> of U. As before write $U^* = U^T \circ i$. $U^* \in B(\ell^1, \ell^\infty)$.

<u>PROPOSITION</u> 7. Let U be an isometric linear bijection of ℓ^1 onto ℓ^1. Then U^T is an isometric linear map of ℓ^∞ into ℓ^∞.

<u>Proof</u>. Let $y \in \ell^\infty$ then

$$\begin{aligned} \|U^T y\| &= \sup \{ \, | \Psi_{U^T y}(x) | : \|x\| = 1 \, \} \quad (\Psi \text{ is isometric}) \\ &= \sup \{ \, | \Psi_y(Ux) | : \|x\| = 1 \, \} \\ &= \|\Psi_y\| = \|y\| \quad . \; \square \end{aligned}$$

<u>Notation</u>. Let $\mathcal{V}$ denote the set of all isometric linear bijections of ℓ^1 onto ℓ^1 which in addition satisfy $U^* U = i \circ I_{\ell^1}$.

<u>PROPOSITION</u> 8. $\mathcal{V}$ consists of those isometric linear bijections of ℓ^1 onto ℓ^1 which are also isometries relative to the ℓ^∞-norm on ℓ^1.

<u>Proof</u>. Let $x \in \ell^1$, and denote the supremum norm of x by $\|x\|_\infty$. We have

$$\begin{aligned} \|x\|_\infty = \|i(x)\| = \| U^* U x \| &= \| U^T i(Ux) \| \\ &= \| i(Ux) \| \qquad \text{by Proposition 7} \\ &= \| Ux \|_\infty \quad . \; \square \end{aligned}$$

Proposition 8 leads to a precise description of the members of $\mathcal{V}$.

<u>PROPOSITION</u> 9. Let $U \in \mathcal{V}$, then there exists a permutation σ of the positive integers and a sequence of complex numbers $\{\alpha_n\}_1^\infty$ of modulus one such that $Ue_k = \alpha_k e_{\sigma(k)}$ $(k=1,2,\dots)$. Conversely any $U \in B(\ell^1)$ which acts in this way on the natural basis of ℓ^1 is a member of $\mathcal{V}$.

<u>Proof</u>. Let $U \in \mathcal{V}$ and fix an integer $k\geq 1$. It follows from Proposition 8 that all components of Ue_k vanish except one, and this non-zero component has unit modulus. Therefore there exists a map σ sending the positive integers into themselves such that $Ue_k = \alpha_k e_{\sigma(k)}$ for some $\alpha_k \in \underset{\sim}{C}$, $|\alpha_k| = 1$ $(k=1,2,\dots)$.
Let n be any positive integer. There exists $x \in \ell^1$ such that $Ux = e_n$. By Proposition 8, $x = x_p e_p$ for some $p \in \underset{\sim}{N}$, $x_p \in \underset{\sim}{C}$. It follows that $n = \sigma(p)$ and therefore σ is surjective. σ is injective since if $p,q \in \underset{\sim}{N}$:

$$\sigma(p) = \sigma(q) \implies U(\alpha_p^{-1} e_p) = U(\alpha_q^{-1} e_q) \implies p = q .$$

Therefore σ is a permutation.

Conversely, suppose $U \in B(\ell^1)$ has the property that $Ue_k = \alpha_k e_{\sigma(k)}$ $(k=1,2,\dots)$ for some permutation σ of $\underset{\sim}{N}$ and complex numbers α_k, $|\alpha_k| = 1$ $(k=1,2,\dots)$. Then

$$(x , U^*y) = (Ux , i(y)) = \sum_{n=1}^{\infty} \alpha_n x_n y^*_{\sigma(n)}$$

$$= (x, i(\sum_{n=1}^{\infty} \alpha^*_n y_{\sigma(n)} e_n)) \quad (y = \{y_n\} \in \ell^1) .$$

Therefore $U^*y = i(\sum_{n=1}^{\infty} \alpha^*_n y_{\sigma(n)} e_n)$ $\quad (y = \{y_n\} \in \ell^1)$.

$$U^*Ux = \sum_{n=1}^{\infty} x_n \alpha_n U^* e_{\sigma(n)} = i(\sum_{n=1}^{\infty} |\alpha_n|^2 x_n e_n) = i(x) \quad (x = \{x_n\} \in \ell^1).$$

Therefore $U \in \mathcal{V}$. □

<u>PROPOSITION</u> 10. Let $U \in \mathcal{V}$ then $V_n(U^{-1} T U) = V_n(T)$ $(n=1,2,\dots)$.

Proof. Let $A \in \Pi_n(\ell^1)$. Then

$$(z , (UA)^T y)_{\underset{\sim}{C}^n} = (UAz , y) = (Az , U^T y)$$

$$= (z , A^T U^T y)_{\underset{\sim}{C}^n} \quad (z \in \underset{\sim}{C}^n , y \in \ell^\infty) .$$

Therefore $(UA)^T = A^T U^T$ and so $(UA)^* = (UA)^T \circ i = A^T U^T \circ i = A^T U^*$.

$$(UA)^* UA = A^T U^* U A = A^T i A = A^* A = I_{\underset{\sim}{C}^n} .$$

Let $y \in \ell^1$, then there exists $x \in \ell^1$ such that $Ux = y$, $\|x\| = \|y\|$.

$$U^* y = U^* Ux = i(x) .$$

Therefore $A^T U^* y = A^* x$. $\|A^T U^* y\|_{\ell^1} = \|A^* x\|_{\ell^1} \leq \|x\| = \|y\|$ since A^* is a contraction (Proposition 3).

Hence $\|(UA)(UA)^*\| = \| UAA^T U^* \| \leq 1$. Therefore $UA \in \Pi_n(\ell^1)$.

$$A^* U^{-1} = A^T_o i \, U^{-1} = A^T U^* \qquad (\text{since } U^* = iU^{-1})$$

Therefore $A^*(U^{-1} T U) A = (UA)^* T (UA) \in V_n(T)$.

Hence $V_n(U^{-1} T U) \subseteq V_n(T)$.

Also $V_n(T) = V_n(U(U^{-1} T U)U^{-1}) \subseteq V_n(U^{-1} T U)$.

Therefore $V_n(U^{-1} T U) = V_n(T)$ as required. □

THEOREM 11. Let $S,T \in B(\ell^1)$.

Then $V_n(S) \subseteq V_n(T) \quad (n = 1,2,\ldots) \Rightarrow \| S \| \leq \| T \|$.

Proof. Let $A_n \in \Pi_n(\ell^1)$ $(n=1,2,\ldots)$ possess the property that $\{ A_n A_n^* \}_1^\infty$ is a sequence of norm one projections which converge strongly to I_{ℓ^1} (Lemma 4). For each A_n there exists $B_n \in \Pi_n(\ell^1)$ such that $A_n^* S A_n = B_n^* T B_n$.

$$P_n S P_n = A_n B_n^* T B_n A_n^* \qquad \text{where } P_n = A_n A_n^* \quad (n=1,2,\ldots).$$

$$\| P_n S P_n\| \leq \| T\| \qquad (\text{Proposition 3}) .$$

$$P_n S P_n \to S \text{ strongly as } n \to \infty .$$

Let $\delta > 0$, then there exists $x \in \ell^1, \|x\|=1$ ~~$x \in S(\ell^1)$~~ such that $\| Sx\| > \| S\| - \delta$.

Therefore for all sufficiently large n : $\|T \| \geq \|P_n S P_n x\| > \|S\| - \delta$.

The result follows. □

§2. Some convergence properties and the inclusion theorem.

The aim of this section is to construct a relation between two operators given information about their matrix ranges. In particular, what is implied by $V_n(S) \subseteq V_n(T)$ $(n=1,2,\dots)$ for $S,T \in B(\ell^1)$?

Notation.

Let Γ denote the antilinear isometric isomorphism of ℓ^1 onto $(c_0)'$ given by $x \mapsto \Gamma_x$ where $\Gamma_x(w) = \sum_{j=1}^{\infty} w_j x_j^*$

$$(w = \{w_j\} \in c_0 \ , \ x = \{x_j\} \in \ell^1).$$

Let τ_s denote the strong operator topology on $B(\ell^1)$.

Let τ_{w^*} denote the weak* operator topology on $B(\ell^1)$ generated by the family of seminorms

$$p_{w,x}(T) = | \Gamma_{Tx}(w) | \quad (w \in c_0 \ , \ x \in \ell^1) \quad T \in B(\ell^1) .$$

Let τ_ρ denote the topology on $B(\ell^1,\ell^\infty)$ generated by the family of seminorms

$$q_{x,y}(S) = | \Psi_{Sy}(x) | \quad (x,y \in \ell^1) \quad S \in B(\ell^1,\ell^\infty).$$

Theorem 1. The unit ball of $B(\ell^1)$ is compact in the topology τ_{w^*}.

Proof. Let $D_{w,x} = \{ \lambda \in \mathbb{C} \ : \ |\lambda| \leq \|w\|\|x\| \ \}$ $\quad w \in c_0 \ , \ x \in \ell^1$.

$$Q = \prod \{ D_{w,x} : w \in c_0 \ , \ x \in \ell^1 \} .$$

Q is compact in the product topology. Given $T \in B(\ell^1)$, $\|T\| \leq 1$ let $F_T : c_0 \times \ell^1 \to \mathbb{C}$ be defined by

$F_T(w,x) = \Gamma_{Tx}(w)$ $(w \in c_0 \ , \ x \in \ell^1)$. Let Q_0 be the image of the unit ball of $B(\ell^1)$ with the weak* topology (τ_{w^*}) under the map $T \to F_T$. An arbitrary neighbourhood of $F_T \in Q$ is of the form

$$V(F_T) = \{ G \in Q : | G(w_k,x_k) - F_T(w_k,x_k) | < \epsilon \ (k=1,2,\ldots,n) \}$$

for some positive integer n , some $\epsilon > 0$, and some $w_1,w_2,\ldots,w_n \in c_0$, $x_1,x_2,\ldots,x_n \in \ell^1$.

An arbitrary neighbourhood of $T \in B(\ell^1)$ is of the form

$$U(T) = \{ S \in B(\ell^1) : | \Gamma_{Sx_k}(w_k) - \Gamma_{Tx_k}(w_k) | < \epsilon \ (k=1,2,\ldots,n) \}$$

for some $n \in \underset{\sim}{N}$, $\epsilon > 0$, $w_1,w_2,\ldots,w_n \in c_0$, $x_1,x_2,\ldots,x_n \in \ell^1$. The map $T \to F_T$ is therefore a homeomorphism of the unit ball of $B(\ell^1)$ onto Q_0 . Hence it suffices to show that Q_0 is closed in the product topology. Let $F \in \overline{Q_0}$. Then F is sesquilinear and $| F(w,x)| \leq \|w\|\|x\| \quad (w \in c_0 , x \in \ell^1)$. For a fixed $x \in \ell^1$ the map $w \mapsto F(w,x)$ is a bounded linear functional on c_0 , and therefore there exists a unique $Tx \in \ell^1$ such that $\Gamma_{Tx}(w) = F(w,x) \quad (w \in c_0)$. The map $x \mapsto Tx$ is well defined , linear , and belongs to the unit ball of $B(\ell^1)$ since

$$\begin{aligned} \|Tx\| = \|\Gamma_{Tx}\| &= \sup \{ | \Gamma_{Tx}(w) | : w \in c_0 , \|w\| \leq 1 \} \\ &= \sup \{ | F(w,x) | : w \in c_0 , \|w\| \leq 1 \} \\ &\leq \|x\| \quad (x \in \ell^1) . \end{aligned}$$

Therefore Q_0 is closed. []

Definition 2. $T \in B(\ell^1)$ is the dual of an operator on c_0 if there exists $S \in B(c_0)$ such that

$$\Gamma_x(Sw) = \Gamma_{Tx}(w) \qquad (w \in c_0 , x \in \ell^1) .$$

PROPOSITION 3. Let $\{A_n\}$ be a sequence of operators on ℓ^1 which converge to $A \in B(\ell^1)$ in the weak* operator topology, and let $T \in B(\ell^1)$ be compact and the dual of an operator on c_0 .

Then $TA_n \to TA \ (\tau_s)$ as $n \to \infty$.

Proof. $A_n \to A$ (τ_{w^*}) as $n \to \infty$ if and only if

$$\Gamma_{A_n x}(w) \to \Gamma_{Ax}(w) \quad \text{as } n \to \infty \text{ for each } w \in c_0 \ , \ x \in \ell^1 .$$

For a fixed $x \in \ell^1$, $\{ \Gamma_{A_n x} : n \geq 1 \}$ is a family of bounded linear functionals on c_0 such that $\sup_n | \Gamma_{A_n x}(w) | < \infty$ for each $w \in c_0$. By the Uniform Boundedness theorem

$\sup_n \| \Gamma_{A_n x} \| < \infty$, and therefore $\{ A_n x \}_1^\infty$ is a bounded sequence. By the compactness of T there exists a subsequence $\{ A_m x \}$ such that $TA_m x \to y \in \ell^1$ as $m \to \infty$. Suppose T is the dual of $S \in B(c_0)$. We have

$$\Gamma_y(w) = \lim_{m \to \infty} \Gamma_{TA_m x}(w) = \lim_{m \to \infty} \Gamma_{A_m x}(Sw) = \Gamma_{Ax}(Sw) = \Gamma_{TAx}(w) \quad (w \in c_0).$$

Therefore $y = TAx$.(1) It follows that $TA_n x \to TAx$ as $n \to \infty$. □

Lemma 4. Let $T_n, T \in B(\ell^1)$ $(n = 1,2,\dots)$. Then

$$T_n \to T \ (\tau_{w^*}) \text{ as } n \to \infty \implies T_n^* \to T^* \ (\tau_\rho) \text{ as } n \to \infty .$$

Proof.

$$\begin{aligned}
T_n \to T \ (\tau_{w^*}) &\iff \Gamma_{T_n x}(w) \to \Gamma_{Tx}(w) && (w \in c_0,\ x \in \ell^1) \\
&\iff (w\ ,\ T_n x) \to (w\ ,\ Tx) && (w \in c_0,\ x \in \ell^1) \\
&\implies (y\ ,\ T_n x) \to (y\ ,\ Tx) && (x,y \in \ell^1) \\
&\iff (T_n x, i(y)) \to (Tx, i(y)) && (x,y \in \ell^1) \\
&\iff \Psi_{i(y)}(T_n x) \to \Psi_{i(y)}(Tx) && (x,y \in \ell^1) \\
&\iff \Psi_{T_n^* y}(x) \to \Psi_{T^* y}(x) && (x,y \in \ell^1) \\
&\iff T_n^* \to T^* \ (\tau_\rho) && \text{as } n \to \infty . \ \square
\end{aligned}$$

Lemma 5. Let A_n, $B_n \in B(\ell^1)$, $\|A_n\| = 1$ $(n=1,2,\dots)$ satisfy

(1). $A_n \to A$ (τ_{w^*}) as $n \to \infty$;

(2). $B_n \to B$ (τ_s) as $n \to \infty$.

Then $A_n^* B_n \to A^* B$ (τ_ρ) as $n \to \infty$.

1. Every subsequence of $\{TA_n x\}$ has a subsequence that converges to TAx.

Proof.

$$| \Psi_{A_n^* B_n y}(x) - \Psi_{A^* By}(x) |$$

$$\leq | \Psi_{A_n^* B_n y}(x) - \Psi_{A_n^* By}(x) | + | \Psi_{A_n^* By}(x) - \Psi_{A^* By}(x) |$$

$$= | \Psi_{i(B_n y - By)}(A_n x) | + | \Psi_{(A_n^* - A^*)By}(x) | \qquad (x,y \in \ell^1) .$$

$| \Psi_{(A_n^* - A^*)By}(x) | \to 0$ as $n \to \infty$ by Lemma 4 ,

$\| i(B_n y - By) \| \leq \| B_n y - By \| \to 0$ as $n \to \infty$ by condition (2).□

Lemma 6. Let $A,B \in B(\underset{\sim}{C}^n, \ell^1)$, then $(BA^*)^* = iAB^*$.

Proof. Let $x \in \ell^1, y \in \ell^\infty$. We have

$$\begin{aligned} \Psi_{(BA^*)^T y}(x) &= \Psi_y(BA^* x) = \Phi_{B^T y}(A^* x) = (A^* x , B^T y)_{\underset{\sim}{C}^n} \\ &= \Phi_{A^* x} (B^T y)^* = \Phi_{A^T i(x)} (B^T y)^* \\ &= \Psi_{i(x)}(AB^T y)^* = (AB^T y , i(x))^* \\ &= (x , i(AB^T y)) = \Psi_{i(AB^T y)}(x) . \end{aligned}$$

Therefore $(BA^*)^T = iAB^T$, and so $(BA^*)^* = (BA^*)^T_o i = iAB^*$. □

THEOREM 7. Let $S,T \in B(\ell^1)$ be such that $V_n(S) \subseteq V_n(T)$ $(n=1,2,\ldots)$ and suppose that T is compact and the dual of an operator on c_0. Then iS is compact and there exists a contraction F on ℓ^1 such that $iS = F^* T F$.

Proof. Let $A_n \in \Pi_n(\ell^1)$ $(n=1,2,\ldots)$ be a sequence of operators with the property that $P_n = A_n A_n^* \to I_{\ell^1}$ (τ_s) as $n \to \infty$.(Lemma 1.4). For each $n=1,2,\ldots$ there exists $B_n \in \Pi_n(\ell^1)$ such that

$$A_n^* S A_n = B_n^* T B_n$$

Therefore $P_n^* S P_n = F_n^* T F_n$ by Lemma 6, writing $P_n = A_n A_n^*$, $F_n = B_n A_n^*$. By Proposition 1.3, $\|F_n\| \leq 1$ ($n = 1,2,\dots$).

(1) ~~Applying Theorem 1 there exists a subsequence $\{F_m\}$ of $\{F_n\}$ such that $F_m \to F$ (τ_{w*}) as $m \to \infty$~~. Proposition 3 shows that $T F_m \to T F$ (τ_s), and finally Lemma 5 shows that

$$F_m^* T F_m \to F^* T F \quad (\tau_\rho) \quad \text{as } m \to \infty .$$

Also $P_m^* S P_m = iP_m S P_m \to iS$ (τ_s) as $m \to \infty$.

Therefore $iS = F^* T F$. □

§3. Applications.

In this section we ask the question : What is implied by the condition $V_n(S) = V_n(T)$ $n=1,2,\dots$ $(S,T \in B(\ell^1))$? For two simple classes of operators, namely compact diagonal operators and compact weighted shifts the matrix ranges form a set of invariants.

Definition 1.

$T \in B(\ell^1)$ is said to be diagonal if there exists a sequence of complex numbers $\{\lambda_n\}_1^\infty$ such that $T\{\xi_n\} = \{\lambda_n \xi_n\}$ $(\{\xi_n\} \in \ell^1)$.

$T \in B(\ell^1)$ is said to be a weighted shift if there exists a sequence of complex numbers $\{\lambda_n\}_1^\infty$ such that $Te_n = \lambda_n e_{n+1}$ $(n=1,2,\dots)$.

PROPOSITION 2.

(1). Let $T \in B(\ell^1)$ be diagonal, then T is the dual of a (diagonal) operator acting on c_0.

(2). Let $T \in B(\ell^1)$ be a weighted shift, then T is the dual of a backward shift on c_0.

Proof. (1). Let $\{\lambda_n\}$ denote the bounded sequence associated with T. Define $S \in B(c_0)$ by

1. Applying Theorem 1, let F be a τ_{w*}-cluster point of $\{F_n\}$.

$S\{x_n\} = \{\lambda_n^* x_n\}$ $(\{x_n\} \in c_0)$. With Γ as before we have

$$\Gamma_x(Sw) = (Sw,x) = \sum_{n=1}^{\infty} w_n\lambda_n^*x_n^* = (w,Tx) = \Gamma_{Tx}(w) \quad (w=\{w_n\}\in c_0,\ x=\{x_n\}\in \ell^1).$$

Therefore T is the dual of S .

(2). Let $\{\lambda_n\}_1^\infty$ denote the sequehce of weights associated with T. Define $S \in B(c_0)$ by $S\{x_n\} = \{\lambda_n^* x_{n+1}\}_1^\infty$ $(\{x_n\} \in c_0)$. S is (by definition) a backward weighted shift.

$$\Gamma_x(Sw) = (Sw,x) = \sum_{n=1}^{\infty} w_{n+1}\lambda_n^*x_n^* = (w,Tx) = \Gamma_{Tx}(w) \quad (w=\{w_n\}\in c_0,\ x=\{x_n\}\in\ell^1).$$

Therefore T is the dual of S . $\square$

<u>THEOREM</u> 3. Let $S,T \in B(\ell^1)$ be compact diagonal operators with zero nullspaces such that $V_n(S) = V_n(T)$ $(n=1,2,\dots)$.

Then there exists $F \in \mathcal{U}$ such that $S = F^{-1}T F$.

<u>Proof</u>. Since S and T are compact their associated sequences converge to zero. We may assume that each sequence has terms with non-increasing modulus. (Otherwise apply Proposition 1.9 to find U , $V \in \mathcal{U}$ with the property that $U^{-1}S\,U$, $V^{-1}T\,V$ have the required form.)

Let $T\{\xi_n\} = \{\lambda_n\xi_n\}$, $S\{\xi_n\} = \{\mu_n\xi_n\}$ $(\{\xi_n\} \in \ell^1)$ where

$$|\lambda_n| \geq |\lambda_{n+1}|,\ |\mu_n| \geq |\mu_{n+1}| \quad (n=1,2,\dots).$$

Proposition 2 together with Theorem 2.7 shows that there exist $F,G \in B(\ell^1)$, $\|F\| \leq 1$, $\|G\| \leq 1$, such that $iS = F^*T\,F$, $iT = G^*S\,G$.

$$iSe_1 = i(\mu_1 e_1) = F^*T\,Fe_1 .$$

Therefore $\mu_1^* = (e_1, i(\mu_1 e_1)) = (e_1, F^*T\,Fe_1) = (Fe_1, i(TFe_1))$.

Write $Fe_1 = \{f_{1k}\}_1^\infty$, then $\mu_1 = \sum_{k=1}^{\infty}\lambda_k|f_{1k}|^2$.

Therefore $|\mu_1| \leq \sum_{k=1}^{\infty} |\lambda_k|\ |f_{1k}|^2 \leq |\lambda_1|$ (by Jensen's inequality together with $\|F\| \leq 1$). Notice that $\|S\| = |\mu_1|$, $\|T\| = |\lambda_1|$, and therefore by Theorem 1.11 we have $|\mu_1| = |\lambda_1|$.
Hence $|\lambda_1| = \sum_{k=1}^{\infty} |\lambda_k|\ |f_{1k}|^2$. Suppose $|\lambda_1|$ occurs n_1 times in the sequence $\{|\lambda_k|\}$. Since $\sum_{k=1}^{\infty} |f_{1k}|^2 \leq 1$ we must have $\sum_{k=1}^{n_1} |f_{1k}|^2 = 1$.

$$1 = \sum_{k=1}^{n_1} |f_{1k}|^2 \leq \sum_{k=1}^{n_1} |f_{1k}| \leq 1 .$$

Therefore there exists an integer $\sigma(1)$, $1 \leq \sigma(1) \leq n_1$, such that $|f_{1k}| = \delta_{k\sigma(1)}$ (Kronecker's δ) . Suppose $|\mu_1|$ occurs m_1 times in the sequence $\{|\mu_n|\}$. The foregoing argument can be applied equally well to $e_2, e_3, \ldots, e_{m_1}$ and since F is 1-1 (because S has trivial kernel) we have $m_1 \leq n_1$.

A repetition of the reasoning using $iT = G^*S\,G$ shows that $n_1 \leq m_1$, and therefore $n_1 = m_1$. Also

$Fe_k = \alpha_k\, e_{\sigma(k)}$ where $|\alpha_k| = 1$ $(k=1,2,\ldots,m_1)$ and σ is a permutation of the first m_1 positive integers.

$$i(\mu_k e_k) = iSe_k = F^*T\,Fe_k = \lambda_{\sigma(k)}\alpha_k\, F^*e_{\sigma(k)} \quad (k=1,2,\ldots,m_1) \quad -(1)$$

Let $X_1 = \text{Span}\{ e_1, e_2, \ldots, e_{n_1} \}$;

$X_1^{\infty} = \overline{\text{Span}}\{ e_{n_1+1}, e_{n_1+2}, \ldots \}$ (closure taken in ℓ^1) .

Given $x \in X_1^{\infty}$, $y \in X_1$ then $(Fx, i(y)) = (x, F^*y) = 0$ (using (1)).

Therefore $F\,X_1^{\infty} \subseteq X_1^{\infty}$.

Given $x \in \ell^1$, $y \in X_1$ then

$$(x, F^*Fy) = (Fx, i(Fy)) = \sum_{k=1}^{n_1} x_k \alpha_k y_k^* \alpha_k^* = (x, i(y)).$$

Therefore $F^*Fy = i(y)$ $(y \in X_1)$.

Write $S_1 = S|_{X_1^{\infty}}$, $T_1 = T|_{X_1^{\infty}}$. Since X_1^{∞} is invariant for F

$$iS_1 = F^*T\,F|_{X_1^\infty} = F^*|_{X_1^\infty} T_1\, F|_{X_1^\infty} \ .$$

Therefore $|\mu_{n_1+1}| = \|iS_1\| \leq \|T_1\| = |\lambda_{n_1+1}|$.

Similarly $|\lambda_{n_1+1}| \leq |\mu_{n_1+1}|$ from $iT_1 = G^*|_{X_1^\infty} S_1\, G|_{X_1^\infty}$.

Therefore $|\mu_{n_1+1}| = |\lambda_{n_1+1}|$ and the arguments may be re-applied to the next block of coinciding terms in the sequence $\{|\lambda_k|\}$. It follows that there exists a permutation σ of the positive integers and complex numbers $\{\alpha_k\}$ of modulus one such that $Fe_k = \alpha_k e_{\sigma(k)}$ $(k=1,2,\dots)$. $F \in \mathcal{U}$ by Proposition 1.9, $F^* = i\,F^{-1}$, and therefore $S = F^{-1}T\,F$. □

<u>THEOREM</u> 4. Let $S,T \in B(\ell^1)$ be compact weighted shifts with zero nullspaces such that $V_n(S) = V_n(T)$ $(n=1,2,\dots)$. Then there exists $F \in \mathcal{U}$ such that $S = F^{-1}T\,F$.

<u>Proof</u>. Let T be given by $Te_n = \lambda_n e_{n+1}$ $(n=1,2,\dots)$ where $\{\lambda_n\}$ is a bounded sequence of complex numbers. Define $U \in B(\ell^1)$ by $U\{\xi_n\} = \{\delta_n \xi_n\}$ $(\{\xi_n\} \in \ell^1)$ where $\{\delta_n\}$ is given by $\delta_1 = 1$, $\lambda_n \delta_n = |\lambda_n|\delta_{n+1}$ $(n=2,3,\dots)$. Then $U \in \mathcal{U}$ (Proposition 1.9) and $U^{-1}T\,Ue_k = U^{-1}T\delta_k e_k = \delta_k\lambda_k U^{-1}e_{k+1} = |\lambda_{k+1}|e_{k+1}$ i.e. $U^{-1}T\,U$ is a shift with weights $\{|\lambda_n|\}$. Therefore we may assume that the sequences of weights for S and T consist of positive numbers.

Let $Te_n = \lambda_n e_{n+1}$, $Se_n = \mu_n e_{n+1}$ $\lambda_n, \mu_n \geq 0$ $(n=1,2,\dots)$.

Let p (respectively q) be any integer such that $\lambda_p = \max\{\lambda_n : n \geq 1\}$ (respectively $\mu_q = \max\{\mu_n : n \geq 1\}$). By Theorem 1.11 we have $\lambda_p = \mu_q$. By Proposition 2 and Theorem 2.7 there exist $F,G \in B(\ell^1)$, $\|F\| \leq 1$, $\|G\| \leq 1$, such that $iS = F^*T\,F$, $iT = G^*S\,G$.

Let $A = \{\, j : \lambda_j = \lambda_p \,\}$, $B = \{\, j : \mu_j = \mu_q \,\}$. Given $m \in B$, we have $i(\mu_m e_{m+1}) = iSe_m = F^*T\,Fe_m$.

Therefore $\mu_m = (e_{m+1}, i(\mu_m e_{m+1})) = (Fe_{m+1}, TFe_m) = \sum_{j=1}^{\infty} \lambda_j f_{m+1\ j+1}\ f^*_{mj}$

where $Fe_k = \{f_{kj}\}_{j=1}^{\infty}$ $(k=1,2,\dots)$.

$$\mu_m \leq \sum_{j=1}^{\infty} \lambda_j |f_{m+1\, j+1}\ f_{mj}| \leq \lambda_p = \mu_q \quad \text{since } \|Fe_k\| \leq 1 \ (k=1,2,\dots).$$

It follows that $\sum_{j \in A} |f_{m+1\, j+1}\ f_{mj}| = 1$ and therefore there exists an integer $\sigma(m) \in A$ such that $|f_{m+1\ \sigma(m)+1}| = |f_{m\ \sigma(m)}| = 1$. Write $Fe_m = \alpha_m e_{\sigma(m)}$ where $|\alpha_m| = 1$. The map $\sigma : m \mapsto \sigma(m)$ of B into A is 1-1 since F has trivial kernel.

Applying the same argument using $iT = G^*S\,G$ we find a 1-1 map of A into B , and since each set is finite A and B must possess the same number of elements. Let $m \in \cancel{\kappa}^{B}$ then

$$1 = (Fe_m, Fe_m) = (e_m, F^*Fe_m) = \alpha_m^* (e_m, F^*e_{\sigma(m)}) \ .$$

F^* is a contraction , for let $y \in \ell^1$ then

$$\|F^*y\| = \|F^T i(y)\| = \|\Psi_{F^T i(y)}\| = \sup\{ |\Psi_{i(y)}(Fx)| : x \in \ell^1, \|x\| \leq 1 \}$$

$$\leq \|\Psi_{i(y)}\| \|F\| \leq \|i(y)\| \leq \|y\| \ .$$

Therefore $F^*e_{\sigma(m)} = \alpha_m^* e_m$ $(m \in \cancel{\kappa}B)$.

Let $X_1 = \text{Span}\{ e_j : j \in A \}$, $Y_1 = \text{Span}\{ e_j : j \in B \}$,

$X_1^{\infty} = \overline{\text{Span}}\{ e_j : j \in \underset{\sim}{N} \setminus A \}$, $Y_1^{\infty} = \overline{\text{Span}}\{ e_j : j \in \underset{\sim}{N} \setminus B \}$.

Given $x \in X_1^{\infty}$, $y \in Y_1$ then $(Fx , y) = (x , F^*y) = 0$ and therefore $F X_1^{\infty} \subseteq Y_1^{\infty}$.

$$iS|_{X_1^{\infty}} = F^*T\,F|_{X_1^{\infty}} = F^*T|_{Y_1^{\infty}} F|_{X_1^{\infty}} \ : \ \|iS|_{X_1^{\infty}}\| \leq \|T|_{Y_1^{\infty}}\| \quad \text{--(1)}$$

Equally well, using $iT = G^*S\,G$ we obtain $\|iT|_{Y_1^{\infty}}\| \leq \|S|_{X_1^{\infty}}\|$ --(2)

Let p_2 be the smallest integer such that $\lambda_{p_2} = \max\{ \lambda_n : n \in \underset{\sim}{N} \setminus A \}$.

Let q_2 be the smallest integer such that $\mu_{q_2} = \max\{ \mu_n : n \in \underset{\sim}{N} \setminus B \}$.

From (1) and (2) we have $\lambda_{p_2} = \mu_{q_2}$ and the foregoing arguments can be repeated for the sets

$$A_2 = \{ j : \lambda_j = \lambda_{p_2} \} , \quad B_2 = \{ j : \mu_j = \mu_{q_2} \} .$$

It follows that there exists a permutation σ of the positive integers and complex numbers $\{\alpha_k\}$ of modulus one such that

$$Fe_k = \alpha_k e_{\sigma(k)} \quad (k=1,2,\dots) .$$

$F \in \mathcal{U}$ (Proposition 1.9) , $F^* = iF^{-1}$, and therefore $S = F^{-1} T F$. □

CHAPTER 3

THE WILLIAMS k-RANGE

§1. Definition and elementary properties.

In this chapter we investigate a new concept which extends the idea of the numerical range of an element of an arbitrary complex unital normed algebra. The observation arose from a characterisation of the numerical range by J.P.Williams.

Notation.

Let A denote a complex unital normed algebra. Given $a \in A$, let $V(A,a)$ denote the numerical range of a, i.e.

$V(A,a) = \{ f(a) : f \in D(A,1) \}$ where $D(A,1)$ denotes the set of states on A.

LEMMA 1. (J.P.Williams). Given $a \in A$

$$V(A,a) = \bigcap_{z \in \underset{\sim}{C}} \{ \lambda : |z-\lambda| \leqslant \| z-a \| \} .$$

Proof. Let $\lambda \in V(A,a)$, then there exists $f \in D(A,1)$ such that $f(a) = \lambda$ and therefore

$$|z-\lambda| = |z-f(a)| = |f(z-a)| \leqslant \|z-a\| \quad (z \in \underset{\sim}{C}) .$$

Conversely, suppose $\lambda \in \underset{\sim}{C}$ satisfies $|z-\lambda| \leqslant \|z-a\| \quad (z \in \underset{\sim}{C})$.

If $a = \mu 1$ for some $\mu \in \underset{\sim}{C}$ then for any $f \in D(A,a)$

$|f(a)-\lambda| = |\mu-\lambda| \leqslant \|\mu 1-a\| = 0$. Therefore $\lambda = f(a) \in V(A,a)$.

If 1 and a are linearly independent, define f_0 on $\mathrm{Span}\{ 1, a\}$ by $f_0(\alpha + \beta a) = \alpha + \beta\lambda \quad (\alpha,\beta \in \underset{\sim}{C})$.

$|f_0(\alpha + \beta a)| = |\alpha + \beta\lambda| \leqslant \|\alpha + \beta a\|$, $f_0(1) = 1$.

Extend f_0 to $f \in D(A,1)$ by the Hahn Banach Theorem. $f(a) = \lambda$. □

Notation. Let Pol_k denote the set of all complex polynomials of degree $\leq k$. Let $\widehat{K}$ denote the polynomially convex hull of the compact set $K \subset \mathbb{C}^n$.

Definition 2. Given $a \in A$, the k^{th} (algebra) Williams range of a is the subset of $\mathbb{C}$ given by

$$J_k(A,a) = \{ \lambda \in \mathbb{C} : |p(\lambda)| \leq \|p(a)\|, p \in \mathrm{Pol}_k \} .$$

For the remainder of this section we assume that A is complete.

PROPOSITION 3. For each $a \in A$, $k \in \mathbb{N}$, $J_k(A,a)$ is a non-void polynomially convex compact subset of $\mathbb{C}$ such that

(1). $J_1(A,a) = V(A,a)$; $J_{k+1}(A,a) \subseteq J_k(A,a)$ $(k=1,2,\dots)$.

(2). $Sp(A,a) \subseteq J_k(A,a)$ $(k=1,2,\dots)$.

Proof. $V(A,a) = J_1(A,a)$ is a statement of Lemma 1 . It is clear that the Williams ranges are closed bounded nested subsets of the plane. Let $\lambda \in \widehat{J_k(A,a)}$ then

$|p(\lambda)| \leq \max \{ |p(z)| : z \in J_k(A,a) \}$ for all polynomials p .

Therefore $|p(\lambda)| \leq \|p(a)\|$ $(p \in \mathrm{Pol}_k)$ i.e. $\lambda \in J_k(A,a)$ and so $J_k(A,a)$ is polynomially convex . Finally an application of the spectral mapping theorem for polynomials shows that

$Sp(A,a) \subseteq J_k(A,a)$ $(k=1,2,\dots)$. □

Remarks.

(1). The k^{th} Williams range of $a \in A$ depends only on the subspace of A spanned by $1,a,a^2,\dots,a^k$. If B is any subalgebra of A containing 1 and a , then $J_k(B,a) = J_k(A,a)$. We shall omit reference to the underlying algebra and use the abbreviation $J_k(a)$ whenever no confusion arises.

(2). J_k is not a connected set in general when $k>1$.

Example. Let $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $q(z) = z^2 - 1 \in Pol_2$.

$$J_2(T) \subseteq \{ \lambda \in \underset{\sim}{C} : |q(\lambda)| \leq \|q(T)\| = 0 \} = \{ 1, -1 \} = Sp(T) .$$

Therefore $J_k(T) = \{1,-1\} = Sp(T)$ $(k=2,3,\dots)$.

More generally if $T \in B(H)$ where H is a complex Hilbert space of dimension n then $J_k(T) = Sp(T)$ $(k=n,n+1,\dots)$ because T satisfies its characteristic polynomial.

If a is an algebraic element of a unital normed algebra and $\lambda_1, \lambda_2, \dots, \lambda_m$ are the roots of the minimal monic polynomial (having degree n) then $J_k(a) = \{ \lambda_1, \lambda_2, \dots, \lambda_m \}$ $(k=n,n+1,\dots)$.

(3). Examples of Williams ranges are difficult to calculate from the basic definition . As a result of theory developed in §2 we see that the Williams ranges of elements of certain group algebras are amenable to calculation , and we give some examples .

PROPOSITION 4. For each $a \in A$, $\alpha \in \underset{\sim}{C}$

(1). $J_k(\alpha a) = \alpha J_k(a)$.

(2). $J_k(1+a) = 1 + J_k(a)$.

Proof. (1). Let $\alpha \neq 0$, then

$$\begin{aligned} J_k(\alpha a) &= \{ \lambda : |p(\lambda)| \leq \|p(\alpha a)\| , p \in Pol_k \} \\ &= \{ \alpha\lambda : |p(\alpha\lambda)| \leq \|p(\alpha a)\| , p \in Pol_k \} \\ &= \{ \alpha\lambda : |p_\alpha(\lambda)| \leq \|p_\alpha(a)\| , p \in Pol_k \} \end{aligned}$$

where $p_\alpha(z) = p(\alpha z)$. Therefore $J_k(\alpha a) = \alpha J_k(a)$ $(\alpha \neq 0)$.

If $\alpha = 0$, $J_k(0) \subseteq V(0) = \{0\}$.

(2).
$$\begin{aligned} J_k(1+a) &= \{ \lambda : |p(\lambda)| \leq \|p(1+a)\| , p \in Pol_k \} \\ &= \{ 1+\lambda : |p(1+\lambda)| \leq \|p(1+a)\| , p \in Pol_k \} \\ &= \{ 1+\lambda : |q(\lambda)| \leq \|q(a)\| , q \in Pol_k \} \\ &= 1 + J_k(a) . \square \end{aligned}$$

§2. Intersections, equivalent norms and states.

We first answer the outstanding problem raised by §1 , namely, describe $\bigcap_{k=1}^{\infty} J_k(a)$. The effect of calculating the Williams ranges relative to an equivalent norm is considered , and finally we identify those states which generate the k^{th} Williams range.

Throughout this section A will denote a complex unital Banach algebra.

THEOREM 1. Given $a \in A$, $\widehat{Sp(a)} = \bigcap_{k=1}^{\infty} J_k(a)$.

Proof. $\lambda \in \widehat{Sp(a)} \implies |p(\lambda)| \leq \max\{|p(z)| : z \in Sp(a)\}$
$\leq \|p(a)\|$

for all polynomials p . Therefore $\lambda \in \bigcap_{k=1}^{\infty} J_k(a)$.
Conversely , suppose $\lambda \in \bigcap_{k=1}^{\infty} J_k(a) \setminus \widehat{Sp(a)}$. Then there exists a polynomial p such that

$$|p(\lambda)| > \max\{ |p(z)| : z \in Sp(a) \} = r(p(a)) .$$

Consider the sequence of polynomials $\{q_m\}$ defined by

$$q_m(z) = \sum_{k=0}^{m} \left(\frac{p(z)}{p(\lambda)} \right)^k \qquad m = 1,2,\ldots$$

$\{q_m(a)\}$ is a convergent sequence. $q_m(\lambda) = m+1$. Therefore there exists $m_0 \in \underset{\sim}{N}$ such that $|q_{m_0}(\lambda)| > \|q_{m_0}(a)\|$ which implies that $\lambda \notin \bigcap_{k=1}^{\infty} J_k(a)$, a contradiction. □

Notation. Let N(A) denote the set of all algebra norms λ on A that are equivalent to the given norm and satisfy $\lambda(1) = 1$.

Let $J_k^{\lambda}(a)$ denote the k^{th} Williams range of $a \in A$ determined by the norm λ ($\in N(A)$) in place of the given norm .

We recall the following two well known facts [3]

(1). Given $a_1, a_2, \ldots, a_n$ mutually commuting elements of A and $\epsilon > 0$, there exists $\lambda \in N(A)$ such that

$$\lambda(a_k) < r(a_k) + \epsilon \quad (k=1,2,\ldots,n).$$

(2). Given $a \in A$, $\operatorname{co} Sp(a) = \cap \{ J_1^{\lambda}(a) : \lambda \in N(A) \}$.

<u>THEOREM</u> 2. Let $a \in A$ and let U be any open neighbourhood of $\widehat{Sp(a)}$. Then there exists a positive integer m, depending on U, such that

$$\widehat{Sp(a)} \subseteq \cap \{ J_k^{\lambda}(a) : \lambda \in N(A) \} \subset U \quad \text{whenever } k \geq m .$$

<u>Proof</u>. $\widehat{Sp(a)} \subseteq J_k^{\lambda}(a)$ $(k \in \underset{\sim}{N}, \lambda \in N(A))$ by Theorem 1.
(1) ~~If Sp(a) is convex then m=1~~. Suppose $\xi \in \operatorname{co} Sp(a) \setminus U$. Then there exists a polynomial q_ξ of degree m_ξ say, such that $|q_\xi(\xi)| > r(q_\xi(a))$. Therefore there exists $\lambda \in N(A)$ such that $\lambda(q_\xi(a)) < |q_\xi(\xi)|$. Thus $\xi \notin J_k^{\lambda}(a)$ whenever $k \geq m_\xi$.
Let U_ξ be any open neighbourhood of ξ such that $\lambda(q_\xi(a)) < |q_\xi(z)|$ $(z \in U_\xi)$. Then $\cup \{ U_\xi : \xi \in \operatorname{co} Sp(a) \setminus U \}$ is an open cover of the compact set $\operatorname{co} Sp(a) \setminus U$. Let $U_{\xi_1} \cup U_{\xi_2} \cup \ldots \cup U_{\xi_n}$ be any finite subcover. Take $m = \max \{ m_{\xi_j} : j=1,2,\ldots,n \}$. Then $z \in \operatorname{co} Sp(a) \setminus U \implies$
$z \notin \cap \{ J_k^{\lambda}(a) : \lambda \in N(A) \}$ whenever $k \geq m$.
Therefore $\cap \{ J_k^{\lambda}(a) : \lambda \in N(A) \} \subset U$ if $k \geq m$. $\Box$

<u>Notation</u>. Given $x \in S(A)$, $a \in A$, $k \in \underset{\sim}{N}$ let

$$J_k(A;a,x) = \{ z \in \underset{\sim}{C} : |p(z)| \leq \|p(a)x\|, \ p \in Pol_k \} ;$$

$$D_k(A;a,x) = \{ f \in A' : \|f\| = f(x) = 1 , \ f(a^j x) = f(ax)^j \quad j=1,2,\ldots,k \} .$$

1. If $\operatorname{co} Sp(a) \subset U$, then $m=1$.

Remarks.

(1). $J_k(A;a,x) \subseteq J_k(A,a) \quad (x \in S(A))$;

$\cup \{ J_k(A;a,x) : x \in S(A) \} = J_k(A,a)$ since $J_k(A;a,1) = J_k(A,a)$.

(2). $D_k(A;a,x)$ is a weak* compact subset of the set of all support functionals at x. $D_{k+1}(A;a,x) \subseteq D_k(A;a,x) \quad (k=1,2,\ldots)$. $D_k(A;a,x)$ ~~is~~ can be non-empty ~~in general~~ when $k>1$.

Example. Let A be the group algebra $\ell^1(\underset{\sim}{Z}_3)$ ($\underset{\sim}{Z}_3 = \{0,1,2\}$ under addition modulo 3). Given $f \in \ell^1(\underset{\sim}{Z}_3)$, write $f=(f(0),f(1),f(2))$.

$$D(A) = \{ (1,\mu_1,\mu_2) : \mu_1,\mu_2 \in \underset{\sim}{C} , |\mu_1|\le 1, |\mu_2| \le 1 \}.$$

Let $a = (1,1,0)$, $x = (0,0,1)$. A calculation shows that

$$a*x = (1,0,1) , \ a*a = (1,2,1) , \ a*a*x = (2,1,1) .$$

Let $f = (1,\mu_1,\mu_2) \in D(A)$ then

$$f(a*x)^2 = f(a*a*x) \iff \mu_2^2 + \mu_2 = 1 + \mu_1$$

Therefore $D_2(A;a,x) = \{(1, \mu_2^2+\mu_2-1 , \mu_2) : |\mu_2| \le 1 \}$.

PROPOSITION 3. Given $a \in A$, $x \in S(A)$ then

$$J_k(A,a,x) = \{ f(ax) : f \in D_k(A;a,x) \} \quad (k=1,2,\ldots) .$$

Proof. Let $A_k(x) = \mathrm{Span}\{ x, ax, a^2x, \ldots , a^kx \}$. Given $\zeta \in J_k(a,x)$ define f_ζ on $A_k(x)$ by $f_\zeta(p(a)x) = p(\zeta) \quad (p \in \mathrm{Pol}_k)$. $p_1,p_2 \in \mathrm{Pol}_k \Longrightarrow p_1-p_2 \in \mathrm{Pol}_k$, so ~~$p_1(a)=p_2(a) \Longrightarrow$~~ $p_1(a)x = p_2(a)x$ $\Rightarrow |p_1(\zeta) - p_2(\zeta)| \le \|p_1(a)x-p_2(a)x\| = 0$. Therefore f_ζ is a well defined linear functional on $A_k(x)$ and $f_\zeta(x) = 1 = \|f_\zeta\|$. Extend f_ζ to $f \in D_k(A;a,x)$ using the Hahn Banach Theorem. Conversely it is clear that $f(ax) \in J_k(A;a,x)$ whenever $f \in D_k(A;a,x)$.

COROLLARY 4. Let $a \in A$. $J_k(a) = \{ f(a) : f \in D_k(a,1) \}$.

PROPOSITION 5. Given $a \in A$,

$$\widehat{Sp(a)} = \{ f(a) : f \in D(A,1) , f(a^j) = f(a)^j \ (j=1,2,\dots) \} .$$

Proof. We claim that

$$\bigcap_{k=1}^{\infty} J_k(a) = \{ f(a) : f \in D(A,1) , f(a^j) = f(a)^j \ (j=1,2,\dots) \} .$$

For let $\lambda \in \bigcap_{k=1}^{\infty} J_k(a)$, then by Corollary 4 , for each $k \geq 1$ there exists $f_k \in D_k(A,1)$ such that $f_k(a) = \lambda$. Let f be any weak* cluster point of the set $\{f_k : k=1,2,\dots \}$. Then $f(a) = \lambda$, $f \in D(A,1)$, and $f(a^j) = f(a)^j \ (j=1,2,\dots)$. The converse is clear. An application of Theorem 1 gives the desired conclusion.$\square$

Remark. The well known fact that the generator of a monothetic algebra has polynomially convex spectrum follows immediately from Proposition 5.

These elementary observations on the states which generate J_k are applied in the next result to show how the Williams ranges of the tensor product of two elements is related to the Williams ranges of the individual elements. F.F.Bonsall and J.Duncan [4 ; §22.6] have given the following result .

Let A and B be unital normed algebras and let λ be any algebra norm on $A \otimes B$ which dominates the weak tensor product norm ω . Then

$$\mathrm{co}\{ \lambda\mu : \lambda \in V(a) , \mu \in V(b) \} \subseteq V(a \otimes b) \quad (a \in A , b \in B).$$

Examples are given where equality holds and also where strict inclusion holds .

THEOREM 6. Let A and B be unital normed algebras and let $A \otimes B$ be given any algebra norm λ which dominates the weak tensor product norm. Then

$$\{ \lambda\mu : \lambda \in J_k(a) \,,\ \mu \in J_k(b) \} \subseteq J_k(a \otimes b) \qquad (a \in A \,,\ b \in B \,;\ k \geq 2).$$

Proof. Let $p(z) = \alpha_0 z^k + \alpha_1 z^{k-1} + \dots + \alpha_k \quad (\alpha_0, \alpha_1, \dots, \alpha_k \in \underset{\sim}{C})$.

$$p(a \otimes b) = \alpha_0 a^k \otimes b^k + \alpha_1 a^{k-1} \otimes b^{k-1} + \dots + \alpha_k 1 \otimes 1 \ .$$

We note that completeness of A was not required for the definition of D_k and that Proposition 3 and Corollary 4 hold when A is an arbitrary unital normed algebra. Therefore given $\lambda \in J_k(a)$, $\mu \in J_k(b)$, there exist $f \in D_k(A;a,1)$, $g \in D_k(B;b,1)$ such that $\lambda = f(a)$, $\mu = g(b)$.

$$\begin{aligned} |p(\lambda\mu)| &= |\alpha_0 f(a)^k g(b)^k + \alpha_1 f(a)^{k-1} g(b)^{k-1} + \dots + \alpha_k| \\ &= |f(\alpha_0 a^k) g(b^k) + f(\alpha_1 a^{k-1}) g(b^{k-1}) + \dots + f(\alpha_k) g(1)| \\ &\leq \omega(\, p(a \otimes b)\,) \leq \lambda(p(a \otimes b)) \ . \end{aligned}$$

Therefore $\lambda\mu \in J_k(a \otimes b)$. □

Examples. (1) Let $A = C(E)$, $B = C(F)$ where E, F are compact Hausdorff spaces. We recall that the completion of $A \otimes B$ with the weak tensor product norm is isometrically isomorphic to $C(E \times F)$ [2 ; §42]. Suppose $a \in C(E)$, $b \in C(F)$ possess the property that $a(E)b(F) = \{ a(s)b(t) : (s,t) \in E \times F \}$ is convex. Then

$$\begin{aligned} a(E)b(F) &= Sp(a \otimes b) \subseteq J_k(a \otimes b) \subseteq J_1(a \otimes b) \\ &= \text{co}\ Sp(a \otimes b) = a(E)b(F) \ . \end{aligned}$$

$$a(E) = Sp(a) \subseteq J_k(a) \ ; \quad b(F) = Sp(b) \subseteq J_k(b) \qquad (k = 1, 2, \dots) .$$

These observations together with the theorem imply that

$$J_k(a \otimes b) = J_k(a)\ J_k(b) \quad (k = 1, 2, \dots) .$$

(2). Let $A = (\ell^1(\underset{\sim}{Z}_3), *)$ where $\underset{\sim}{Z}_3 = \{0,1,2\}$ under addition modulo 3. As before, given $f \in A$ write $f = (f(0), f(1), f(2))$.

$$D(A,1) = \{ (1,\mu_1,\mu_2) : \mu_1,\mu_2 \in \underset{\sim}{C}, |\mu_1| \leq 1, |\mu_2| \leq 1 \}.$$

Let $a = (1,1,0)$, then $J_1(a) = \{ 1+\mu_1 : |\mu_1| \leq 1 \} = \{ z \in \underset{\sim}{C} : |z-1| \leq 1 \}$.

$$a^2 = a*a = (1,2,1).$$

Let $f = (1,\mu_1,\mu_2) \in D(A,1)$ then $f(a)^2 = f(a^2)$

$$\Longleftrightarrow \quad \mu_1^2 = \mu_2 .$$

Therefore $D_2(A,a) = \{ (1,\mu,\mu^2) : |\mu| \leq 1 \}$ and hence $J_2(a) = J_1(a)$.

We recall that for any group G, the completion of $\ell^1(G) \otimes \ell^1(G)$ with the projective tensor product norm is isometrically isomorphic to $\ell^1(G \times G)$ [2;§42].

A calculation shows that under this isomorphism we have $a \otimes a = (1,1,0,1,1,0,0,0,0)$ relative to the following ordering of the elements of $\underset{\sim}{Z}_3 \times \underset{\sim}{Z}_3$:

$$(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2).$$

$$J_1(a \otimes a) = \{ z \in \underset{\sim}{C} : |z-1| \leq 3 \}.$$

$(a \otimes a)^2 = a \otimes a * a \otimes a = (1,2,1,2,4,2,1,2,1)$.

Let $f = (1,-1,0,-1,0,1,0,1,0)$. $f \in D(\ell^1(\underset{\sim}{Z}_3 \times \underset{\sim}{Z}_3))$.

$$f((a \otimes a)^2) = 1 = f(a \otimes a)^2 . \quad \text{Therefore } f \in D_2(\ell^1(Z_3 \times Z_3); a).$$

$$f(a \otimes a) = -1 ; \quad -1 \notin J_2(a)\, J_2(a).$$

Therefore $\quad J_2(a)\, J_2(a) \subsetneqq J_2(a \otimes a)$.

(3). For the element a of example 2 we have

$$J_3(a \otimes a) \subsetneqq J_2(a \otimes a) \subsetneqq J_1(a \otimes a) .$$

Suppose $-2 \in J_2(a \otimes a)$, then there exist $\mu_i \in \underset{\sim}{C}, |\mu_i| \leq 1$ $(1 \leq i \leq 8)$ such that $1+\mu_1+\mu_3+\mu_4 = -2$ ----------(1)

and $\quad (-2)^2 = 1+2\mu_1+\mu_2+2(\mu_3+2\mu_4+\mu_5)+\mu_6+2\mu_7+\mu_8$ --(2)

(1) $\Longrightarrow \mu_1 = \mu_3 = \mu_4 = -1$, and then (2) becomes

$\mu_2+2\mu_5+\mu_6+2\mu_7+\mu_8 = 11$ which cannot be satisfied.

Therefore $-2 \in J_1(a \otimes a) \setminus J_2(a \otimes a)$.

$(a \otimes a)^3 = (4,6,6,6,9,9,6,9,9)$.

Suppose $-1 \in J_3(a \otimes a)$, then there exist $\mu_i \in \mathbb{C}$, $|\mu_i| \leq 1$ $(1 \leq i \leq 8)$ such that

$$1+\mu_1+\mu_3+\mu_4 = -1 \qquad --(1)$$

$$(-1)^2 = 1+2\mu_1+\mu_2+2(\mu_3+2\mu_4+\mu_5)+\mu_6+2\mu_7+\mu_8 \qquad --(2)$$

$$(-1)^3 = 4+6(\mu_1+\mu_2+\mu_3)+9(\mu_4+\mu_5)+6\mu_6+9(\mu_7+\mu_8) \qquad --(3)$$

$6\times(2)-(3)$ gives $17 = 9\mu_4+3(\mu_5+\mu_7-\mu_8)$. The maximum real part of $\mu_5+\mu_7-\mu_8$ is 3. Therefore the real part of μ_4 cannot be less than 8/9 , but then (1) is impossible to satisfy .

Therefore $-1 \in J_2(a \otimes a) \setminus J_3(a \otimes a)$.

§3. Printers.

The notion of a printer was introduced by F.F.Bonsall and a full account is to be found in "Numerical ranges II " [4 ; §37]. A printer provides a unified concept of numerical range by selecting as defining axioms three properties enjoyed by several concrete numerical ranges. In this section we establish a definition for a printer on a subspace of a unital normed algebra and investigate the relationship between this notion of printer and the Williams ranges.

Notation. Let (A, $\|.\|$) be a unital normed algebra , and let λ be any linear norm on A equivalent to $\|.\|$ such that $\lambda(1) = 1$. Given $a \in A$, let L_a denote left multiplication by a on A , i.e. $L_a x = ax$ $(x \in A)$. Let $|.|_\lambda$ denote the operator norm on $B(A)$ determined by λ .

The map $L : a \mapsto L_a : (A, \lambda) \to (B(A), |.|_\lambda)$ is a norm-increasing monomorphism. Let B be a linear subspace of A containing 1 .

Definition 1. <u>Φ is an (algebra) printer on (B, λ)</u> if and only if $\Phi_o L^{-1}$ is a (spatial) printer on $L(B)$ in the usual sense , i.e., Φ is a mapping from B into the class of non-void subsets of $\mathbb{C}$ which satisfies :

(1). $\Phi(\alpha + \beta a) = \alpha + \beta\, \Phi(a) \quad (\alpha, \beta \in \mathbb{C} \ ; a \in B)$,

(2). $\sup|\Phi(a)| \leqslant |L_a|_\lambda \quad (a \in B)$,

(3). $\inf|\Phi(a)| \leqslant \lambda(ax) \quad (a \in B ,\ x \in S_\lambda(A))$.

Remarks.

(1). Taking $\lambda(.) = \|.\|$ we have $|L_a|_\lambda = \|a\| \ (a \in A)$ and therefore $a \mapsto V(a)$, the algebra numerical range of a , is a printer on $(A , \|.\|)$.

(2). The given algebra norm on A does not play an explicit rôle in the axioms. Since λ is equivalent to $\|.\|$ we have

$K_1\|.\| \leqslant \lambda(.) \leqslant K_2\|.\|$ for some $K_1, K_2 > 0$, and therefore

$\lambda(L_a x) = \lambda(ax) \leqslant K_2\|ax\| \leqslant K_2\|a\|\|x\| \leqslant K_2K_1^{-2}\lambda(a)\lambda(x)$.

Hence $|L_a|_\lambda \leqslant K_2K_1^{-2}\lambda(a)$.

Therefore the condition " λ is equivalent to $\|.\|$ " ensures that the map $a \mapsto L_a : (A,\lambda) \to (B(A) , |.|_\lambda)$ is bounded . Define a new norm on A by $\|a\|_\lambda = |L_a|_\lambda \ (a \in A)$. Then $(A , \|.\|_\lambda)$ is a unital normed algebra and $\|.\|_\lambda$ is equivalent to λ .

Given any algebra A with unit together with a linear norm λ on A such that (i). $\lambda(1) = 1$,

(ii). The map $L : (A, \lambda) \to (B(A), |.|_\lambda)$ is bounded;
Then A can be given a norm , namely $\|.\|_\lambda$, relative to which A is a unital normed algebra.

We require the following well known result on the (spatial) printer .

Notation. Let X be a normed linear space, L a subspace of B(X) containing I .

THEOREM 2. (4: §37.4) Let Φ be printer on L , and let $T \in L$. Then $\overline{co}\ \Phi(T) = V(B(X), T)$.

COROLLARY 3. Let Φ be a printer on (B,λ) , and let $a \in B$. Then $\overline{co}\ \Phi(a) = V(B(A), |.|_\lambda, L_a)$.

Proof. By definition $\Phi_0 L^{-1}$ is a printer on $\{ L_a : a \in A \} \subseteq B(A)$ and so by Theorem 2 ,

$$V(B(A), |.|_\lambda, L_a) = \overline{co}(\Phi_0 L^{-1}(L_a)) = \overline{co}\ \Phi(a) . \square$$

Definition 4. The statement " J_k is a printer on (B,λ) " will mean that the map

$a \mapsto J_k(B(A), |.|_\lambda, L_a)$ is a printer on (B,λ) .

PROPOSITION 5. If $\{ z \in \underset{\sim}{C} : |p(z)| \leq \lambda(p(a)x),\ p \in Pol_k \} \neq \phi$ for each $x \in S_\lambda(A)$, $a \in B$, then J_k is a printer on (B,λ).

Proof. Clearly the mapping $a \mapsto J_k(B(A), |.|_\lambda, L_a)$ satisfies axioms 1 and 2 of definition 1 and $J_k(B(A), |.|_\lambda, L_a) \neq \phi$ (Propositions 1.3 & 4) . Given $x \in S_\lambda(A)$,

$w \in \{ z \in \underset{\sim}{C} : |p(z)| \leq \lambda(p(a)x) , p \in Pol_k \} \implies |w| \leq \lambda(ax)$.

Also $\lambda(p(a)x) = \lambda(p(L_a)x) \leq |p(L_a)|_\lambda$. Therefore

$w \in J_k(B(A),|.|_\lambda,L_a)$ and so $\inf|J_k(B(A),|.|_\lambda,L_a)| \leq \lambda(ax)$

$(a \in B , x \in S_\lambda(A))$. □

Notation. With (A,λ) as before, let v_λ denote the numerical radius calculated relative to the norm $\|.\|_\lambda$ on A, i.e.

$$v_\lambda(a) = \max\{|z| : z \in V(A , \|.\|_\lambda , a) \} \quad (a \in A).$$

THEOREM 6. Let $(A,\|.\|)$ be a unital Banach algebra , let λ be a linear norm on A equivalent to $\|.\|$ with $\lambda(1) = 1$. Then if $r(a) = v_\lambda(a)$ $(a \in A)$, J_k is a printer on (A,v_λ) for each $k = 1,2,\ldots$.

Proof. The condition $r(a) = v_\lambda(a)$ for each $a \in A$ implies that A is commutative [3 ; §4.7] . Let $x \in S_{v_\lambda}(A)$, then there exists an element ϕ of the carrier space of A such that $|\phi(x)| = 1$.
For each $p \in Pol_k$ we have

$$|p(\phi(a))| = |\phi(p(a))| = |\phi(p(a)x)| \leq v_\lambda(p(a)x) .$$

Therefore $\phi(a) \in \{ z : |p(z)| \leq v_\lambda(p(a)x) , p \in Pol_k \}$.
An application of Proposition 5 (with v_λ in place of λ) gives the desired conclusion. □

Examples.

(1). Let $A = C(E)$ where E is a compact Hausdorff space . Let A have the uniform norm $\|.\|_\infty$, and take $\lambda = \|.\|_\infty$. Then $r(a) = v(a) = v_\lambda(a) = \|a\|_\infty$. This is a trivial example for which the hypotheses of the theorem are satisfied. Bonsall and Duncan [4 ; §25.9] give an example of a unital Banach algebra with the following properties. The details are not included here as they fall outside the scope of this discussion.

(2). Let T denote the closed unit disc, $A(T)$ the algebra of continuous functions on T which are analytic in the open unit disc. There exists a norm $\|.\|$ for $A(T)$ relative to which $A(T)$ is a complex unital Banach algebra such that

(i). $r(a) = v(a) \quad (a \in A(T))$,

(ii). $\|u\| = \frac{1}{2} e \quad$ where $u(z) = z \quad (z \in T)$.

Let $\lambda = v$, then λ is an algebra norm and $\|a\|_\lambda = |L_a|_\lambda = \lambda(a) \ (a \in A)$. Therefore $v_\lambda(a) \leqslant \lambda(a) = v(a)$, and since $r(a) \leqslant v_\lambda(a)$ we have $v_\lambda(a) = v(a) = r(a) \quad (a \in A)$. The hypotheses of Theorem 6 are satisfied .

<u>THEOREM</u> 7. Let $(A, \|.\|)$ be a unital Banach algebra , let λ be a linear norm on A equivalent to $\|.\|$ with $\lambda(1) = 1$. If J_k is a printer on (A,λ) for each $k=1,2,\ldots$ then $r(a) = v_\lambda(a) \quad (a \in A)$. Therefore A is commutative and the spectral radius is an algebra norm equivalent to the given norm $\|.\|$ on A .

<u>Proof</u>. Notice that $J_k(B(A), |.|_\lambda, L_a) = J_k(A, \|.\|_\lambda, a) \quad (a \in A; k \geq 1)$ and therefore Corollary 3 together with the fact that $J_k(A, \|.\|_\lambda, a)$ is compact shows us that

$$\text{co } J_k(A , \|.\|_\lambda , a) = J_1(A , \|.\|_\lambda , a) .$$

Given $a \in A$, let $\Gamma = \{ z \in \underset{\sim}{C} : |z| = v_\lambda(a) \}$. Suppose $r(a) < v_\lambda(a)$. Then $\text{co Sp}(a) \cap \Gamma = \phi$ and therefore $\bigcap_{k=1}^{\infty} J_k(a) \cap \Gamma = \phi$ (Theorem 2.1). Since the Williams ranges are nested compact subsets of the plane there exists $n \in \underset{\sim}{N}$ such that $J_n(a) \cap \Gamma = \phi$.

$z \in J_k(a) \implies |z| \leqslant v_\lambda(a)$. Therefore $\text{co } J_n(a) \cap \Gamma = \phi$ and so

$J_1(a) \cap \Gamma = \phi$ which is a contradiction. Therefore $r(a) = v_\lambda(a)$ $(a \in A)$. This condition implies that A is commutative [3 ; §4.7]. v_λ is a linear norm equivalent to $\|.\|_\lambda$ and therefore by Remark 2 v_λ is equivalent to $\|.\|$. Since the spectral radius is an algebra semi-norm the desired conclusion follows. $\square$

It is natural to ask whether the results of Theorems 6&7 are best possible or whether some strengthening would achieve general necessary and sufficient conditions that J_k should be a printer for each $k \geq 1$.

§4. Joint Williams ranges.

The notion of a k^{th} Williams range has a natural extension to a joint concept in the case of several elements of a unital normed algebra, which also extends the notion of the joint numerical range.

Notation. Let Pol_k^n denote the set of all complex polynomials in n non-commuting variables of degree $\leq k$.

Let A denote a complex unital normed algebra. Given $\underset{\sim}{a} = (a_1, a_2, \dots, a_n) \in A^n$; $\underset{\sim}{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \underset{\sim}{C}^n$; $A(\underset{\sim}{a})$ will denote the norm closed algebra generated by $1, a_1, a_2, \dots, a_n$. $I_L(A ; \underset{\sim}{a}, \underset{\sim}{\lambda})$, $I_R(A ; \underset{\sim}{a}, \underset{\sim}{\lambda})$ will denote the left, respectively right, ideals of A generated by $\lambda_1 - a_1, \dots, \lambda_n - a_n$.

Definition 1. Given $\underset{\sim}{a} \in A^n$, the k^{th} joint Williams range of $\underset{\sim}{a}$ is the subset of $\underset{\sim}{C}^n$ given by

$$J_k(A, \underset{\sim}{a}) = \{ \underset{\sim}{\lambda} \in \underset{\sim}{C}^n : |p(\underset{\sim}{\lambda})| \leq \|p(\underset{\sim}{a})\| , \quad p \in \mathrm{Pol}_k^n \} .$$

For the remainder of the section we assume that A is complete. We recall the definition of the joint spectrum of $\underset{\sim}{a} \in A^n$. The left joint spectrum $LSp(A, \underset{\sim}{a})$, respectively right joint spectrum $RSp(A, \underset{\sim}{a})$, of $\underset{\sim}{a} \in A^n$ is the set of n-tuples $\underset{\sim}{\lambda} \in \underset{\sim}{C}^n$ such that $I_L(A ; \underset{\sim}{a}, \underset{\sim}{\lambda})$ is a proper left ideal, respectively $I_R(A ; \underset{\sim}{a}, \underset{\sim}{\lambda})$ is a proper right ideal of A. The joint spectrum $Sp(A, \underset{\sim}{a})$ of $\underset{\sim}{a} \in A^n$ is the union of the left and right joint spectra.

Remark. The joint spectrum may be empty and the left and right joint spectra are in general distinct.

<u>Example</u>. Let A be the algebra of all 2×2 matrices.

Let $a_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $a_1a_2 + a_2a_1 = 1$, $a_1^2 = a_2^2 = 0$.

Clearly the equation $b_1(\lambda_1 - a_1) + b_2(\lambda_2 - a_2) = 1$ can be solved for $b_1, b_2 \in A$ for all $(\lambda_1, \lambda_2) \in \underset{\sim}{C}^2$. Therefore $LSp(a_1, a_2) = \phi$. Similarly the right joint spectrum is empty.

Let $A = B(H)$, $\dim(H) = 4$.

Let $U_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $U_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ relative to a basis

e_1, e_2, e_3, e_4 for H. $U_2U_1e_1 = e_3$, $U_1U_2e_1 = e_4$, therefore $U_2U_1 \neq U_1U_2$. $\underset{\sim}{0} \in RSp(U_1, U_2)$ since e_1 ~~is not in the range of either U_1 or U_2~~ $\notin U_1H + U_2H$.

Let $T_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}$, $T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. $T_1U_1 + T_2U_2 = I$.

Therefore $\underset{\sim}{0} \notin LSp(U_1, U_2)$.

<u>Theorem</u> 2. For each $\underset{\sim}{a} \in A^n$, $k \in \underset{\sim}{N}$, the k^{th} joint Williams range is a polynomially convex compact subset of $\underset{\sim}{C}^n$ such that

(1). $J_{k+1}(A, \underset{\sim}{a}) \subseteq J_k(A, \underset{\sim}{a})$ $(k = 1, 2, \ldots)$; $J_1(A, \underset{\sim}{a}) = V(A, \underset{\sim}{a})$;

(2). $Sp(A, \underset{\sim}{a}) \subseteq J_k(A, \underset{\sim}{a})$ $(k = 1, 2, \ldots)$.

<u>Proof</u>. It is clear that the joint Williams ranges are nested polynomially convex compact subsets of $\underset{\sim}{C}^n$. The remaining assertions will follow once the states which generate J_k have been identified. We return to the proof of Theorem 2 after this has been done.

Given $\underset{\sim}{a} \in A^n$, let $A_k(\underset{\sim}{a}) = \{ p(\underset{\sim}{a}) : p \in Pol_k^n \}$. $A_k(\underset{\sim}{a})$ is the subspace of A spanned by all words formed from $1, a_1, \ldots, a_n$ of degree $\leq k$. Given $\underset{\sim}{\lambda} \in J_k(A,\underset{\sim}{a})$ define $f_{\underset{\sim}{\lambda}} : A_k(\underset{\sim}{a}) \to \underset{\sim}{C}$ by

$$f_{\underset{\sim}{\lambda}}(p(\underset{\sim}{a})) = p(\underset{\sim}{\lambda}) \quad (p \in \mathrm{Pol}_k^n) .$$

$p_1, p_2 \in \mathrm{Pol}_k^n \implies p_1 - p_2 \in \mathrm{Pol}_k^n$. Therefore $p_1(\underset{\sim}{a}) = p_2(\underset{\sim}{a}) \implies$

$|p_1(\underset{\sim}{\lambda}) - p_2(\underset{\sim}{\lambda})| \leq \|p_1(\underset{\sim}{a}) - p_2(\underset{\sim}{a})\| = 0$, and so $f_{\underset{\sim}{\lambda}}$ is well defined.

$f_{\underset{\sim}{\lambda}}(1) = 1 = \|f_{\underset{\sim}{\lambda}}\|$. Apply the Hahn Banach theorem to extend $f_{\underset{\sim}{\lambda}}$ to a state f on A.

Conversely, suppose f is a state on A with the property that

$$f(p(\underset{\sim}{a})) = p(f(a_1), f(a_2), \ldots, f(a_n)) \quad (p \in \mathrm{Pol}_k^n) .$$

Then it is clear that $(f(a_1), f(a_2), \ldots, f(a_n)) \in J_k(A,\underset{\sim}{a})$.

<u>Notation.</u> Given $\underset{\sim}{a} \in A^n$, let $D_k(A,\underset{\sim}{a})$ denote the set of states of A with the property

$$f(p(\underset{\sim}{a})) = p(f(a_1), f(a_2), \ldots, f(a_n)) \quad (p \in \mathrm{Pol}_k^n) .$$

We have established

<u>PROPOSITION</u> 3. Given $\underset{\sim}{a} \in A^n$ then

$$J_k(A,\underset{\sim}{a}) = \{ (f(a_1), f(a_2), \ldots, f(a_n)) : f \in D_k(A,\underset{\sim}{a}) \} \quad (k \geq 1) .$$

<u>Proof of Theorem</u> 2. $J_1(A,\underset{\sim}{a}) = V(A,\underset{\sim}{a})$ is a statement of Proposition 3 when $k = 1$. Let $\underset{\sim}{\lambda} \in \mathrm{LSp}(A,\underset{\sim}{a})$, then $I_L(A;\underset{\sim}{a},\underset{\sim}{\lambda})$ is a proper left ideal of A and therefore $d(1, I_L) = 1$. By the Hahn Banach theorem there exists $f \in D(A)$ with $f(I_L) = \{0\}$.

$f(I_L) = \{0\} \implies f(a_j) = \lambda_j \quad (j = 1, 2, \ldots, n)$.

Let $a_{i_1} a_{i_2} \ldots a_{i_m}$ be any word formed from $1, a_1, a_2, \ldots, a_n$.

$$\begin{aligned} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_m} - a_{i_1} a_{i_2} \ldots a_{i_m} \\ = a_{i_1} a_{i_2} \ldots a_{i_{m-1}} (\lambda_{i_m} - a_{i_m}) + \lambda_{i_m} (\lambda_{i_1} \ldots \lambda_{i_{m-1}} - a_{i_1} \ldots a_{i_{m-1}}) . \end{aligned}$$

$a_{i_1} a_{i_2} \ldots a_{i_{m-1}} (\lambda_{i_m} - a_{i_m}) \in I_L$. It follows by induction that

$$\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_m} - a_{i_1}a_{i_2}\cdots a_{i_m} \in I_L .$$

Therefore $f(a_{i_1}a_{i_2}\cdots a_{i_m}) = f(a_{i_1})f(a_{i_2})\ \cdots\ f(a_{i_m})$ and hence $f \in D_k(A,\underset{\sim}{a})$. Therefore $Sp(A,\underset{\sim}{a}) \subseteq J_k(A,\underset{\sim}{a})$ $(k=1,2,\dots)$. □

LEMMA 4. $\bigcap_{k=1}^{\infty} D_k(A,\underset{\sim}{a}) = \{ f \in D(A) : f|A(\underset{\sim}{a}) \text{ is multiplicative } \}$.

Proof. Immediate. □

THEOREM 5. $\bigcap_{k=1}^{\infty} J_k(A,\underset{\sim}{a}) = \{ (f(a_1),f(a_2),\dots,f(a_n)) : f \in D(A) ,$
$f|A(\underset{\sim}{a}) \text{ is multiplicative } \}$.

Proof. Let $\underset{\sim}{\lambda} \in \bigcap_{k=1}^{\infty} J_k(A,\underset{\sim}{a})$, then for each $k \geq 1$, there exists $f_k \in D_k(A,\underset{\sim}{a})$ such that $f_k(a_j) = \lambda_j$ $(j=1,2,\dots,n)$. Let f be a weak* cluster point of $\{f_k : k \geq 1\}$. Then $f(a_j) = \lambda_j$ $(j=1,2,\dots,n)$ and $f \in \bigcap_{k=1}^{\infty} D_k(A,\underset{\sim}{a})$. Apply Lemma 4 . The converse is clear. □

COROLLARY 6. Let A be a commutative unital Banach algebra . Then $\widehat{Sp(A,\underset{\sim}{a})} = \{ (f(a_1),f(a_2),\dots,f(a_n)) : f \in D(A) ,$
$f|A(\underset{\sim}{a}) \text{ is multiplicative } \}$.

Proof. We show that $\widehat{Sp(A,\underset{\sim}{a})} = \bigcap_{k=1}^{\infty} J_k(A,\underset{\sim}{a})$ and then appeal to Theorem 5. $\widehat{Sp(A,\underset{\sim}{a})} \subseteq \bigcap_{k=1}^{\infty} J_k(A,\underset{\sim}{a})$ follows from Theorem 2. Let $\underset{\sim}{\lambda} \in \bigcap_{k=1}^{\infty} J_k(\underset{\sim}{a}) \setminus \widehat{Sp(\underset{\sim}{a})}$. Then there exists a polynomial p such that $|p(\underset{\sim}{\lambda})| > \max\{ |p(\phi(a_1),\phi(a_2),\dots,\phi(a_n))| : \phi \in \Phi_A \}$ where Φ_A denotes the carrier space of A.

Therefore $|p(\underset{\sim}{\lambda})| > \max\{ |\phi(p(\underset{\sim}{a}))| : \phi \in \Phi_A \} = r(p(\underset{\sim}{a}))$.

Let $$q_m(\underset{\sim}{z}) = \sum_{j=0}^{m} (p(\underset{\sim}{z})/p(\underset{\sim}{\lambda}))^j \quad (m=1,2,\dots) .$$

$\{ q_m(\underset{\sim}{a}) \}$ is a convergent sequence , however since $\underset{\sim}{\lambda} \in \bigcap_{k=1}^{\infty} J_k(\underset{\sim}{a})$ we have $m+1 = |q_m(\underset{\sim}{\lambda})| \leq \|q_m(\underset{\sim}{a})\|$ $(m=1,2,\dots)$, a contradiction. □

LEMMA 7. Given $\underset{\sim}{a} = (a_1, a_2, \ldots, a_n) \in A^n$, let f be a multiplicative state on $A(\underset{\sim}{a})$. Then the kernel of f equals the norm closed algebra generated by

$$f(a_1)-a_1,\ f(a_2)-a_2,\ \ldots,\ f(a_n)-a_n \ .$$

Proof. Let $x \in \ker f$. There exists a sequence of polynomials $\{ p_m \}$ in n non-commuting variables such that

$$\| x - p_m(f(a_1)-a_1, f(a_2)-a_2, \ldots, f(a_n)-a_n) \| \to 0 \quad \text{as } m \to \infty .$$

Write $p_m(\underset{\sim}{z}) = q_m(\underset{\sim}{z}) + \alpha_m$ where q_m has no constant term.

$f(p_m(f(a_1)-a_1, f(a_2)-a_2, \ldots, f(a_n)-a_n)) = \alpha_m \to f(x) = 0$ as $m \to \infty$.

Therefore x belongs to the norm closed algebra generated by $f(a_1)-a_1, f(a_1)-a_1, \ldots, f(a_n)-a_n$. □

We are indebted to P.Rosenthal for pointing out to us the full strength of the next result and also for an attractive direct proof which we give after Corollary 9.

THEOREM 8. Let $\underset{\sim}{a} = (a_1, a_2, \ldots, a_n) \in A^n$, $\underset{\sim}{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \underset{\sim}{C}^n$ be given. The closed left and closed right ideals of $A(\underset{\sim}{a})$ generated by $a_1-\lambda_1, a_2-\lambda_2, \ldots, a_n-\lambda_n$ coincide and equal the norm closed algebra generated by

$$a_1 - \lambda_1 \ , \ a_2 - \lambda_2 \ , \ \ldots \ , \ a_n - \lambda_n \ .$$

Proof. Suppose $I_L(A(\underset{\sim}{a}); \underset{\sim}{a}, \underset{\sim}{\lambda})$ is a proper left ideal of $A(\underset{\sim}{a})$, then by Theorem 5 there exists a multiplicative state f on $A(\underset{\sim}{a})$ such that $f(a_j) = \lambda_j$ $(j=1,2,\ldots,n)$. Lemma 7 together with the observation that $I_R(A(\underset{\sim}{a}); \underset{\sim}{a}, \underset{\sim}{\lambda}) \subseteq \ker f$ shows us that

$$I_R \cup I_L \subseteq \ker f \subseteq \overline{I_R} \cap \overline{I_L} \ .$$

Therefore $\overline{I_L} = \overline{I_R} = \ker f$. □

COROLLARY 9. Let $\underset{\sim}{a} \in A^n$ be given. Then

$$LSp(A(\underset{\sim}{a}), \underset{\sim}{a}) = RSp(A(\underset{\sim}{a}), \underset{\sim}{a}) = \bigcap_{k=1}^{\infty} J_k(A(\underset{\sim}{a}), \underset{\sim}{a}) \quad (k=1,2,\dots).$$

Proof. Since $A(\underset{\sim}{a})$ is a Banach algebra $I_L(A(\underset{\sim}{a}), \underset{\sim}{a}, \underset{\sim}{\lambda})$ is a proper left ideal of $A(\underset{\sim}{a})$ if and only if $\overline{I_L(A(\underset{\sim}{a}), \underset{\sim}{a}, \underset{\sim}{\lambda})}$ is a proper left ideal of $A(\underset{\sim}{a})$. Apply Theorems 5 and 8. □

Proof of Theorem 8.(P.Rosenthal).

Let $x \in I_L(A(\underset{\sim}{a}); \underset{\sim}{a}, \underset{\sim}{\lambda})$, then there exists $x_1, x_2, \dots, x_n \in A(\underset{\sim}{a})$ such that $x = \sum_{i=1}^{n} x_i(a_i - \lambda_i)$. For each $i=1,2,\dots,n$, there exists a sequence of polynomials $\{p_{mi}\}_1^\infty$ such that

$$\| p_{mi}(a_1-\lambda_1, \dots, a_n-\lambda_n) - x_i \| \to 0 \quad \text{as } m \to \infty.$$

Let $q_m(\underset{\sim}{z}) = \sum_{i=1}^{n} p_{mi}(\underset{\sim}{z})(z_i - \lambda_i) \qquad \underset{\sim}{z} = (z_1, z_2, \dots, z_n) \quad m=1,2,\dots$

$$\| q_m(a_1-\lambda_1, \dots, a_n-\lambda_n) - x \| \to 0 \quad \text{as } m \to \infty.$$

$$q_m(a_1-\lambda_1, \dots, a_n-\lambda_n) \in \overline{I_L} \cap \overline{I_R} \quad (m=1,2,\dots).$$

Therefore $\overline{I_L} \subseteq \overline{I_L} \cap \overline{I_R}$ and so $\overline{I_L} \subseteq \overline{I_R}$. Similarly $\overline{I_R} \subseteq \overline{I_L}$. Therefore $\overline{I_L} = \overline{I_R}$ and the result follows. □

Remark. Completeness of A is not required in Theorem 8.

To end this section we consider the effect of calculating the joint Williams range relative to an equivalent norm on the algebra. Let $J_k^\lambda(A, \underset{\sim}{a})$ denote the k^{th} joint Williams range of $\underset{\sim}{a} \in A^n$ calculated relative to the norm $\lambda \in N(A)$.

PROPOSITION 10. Given $\underset{\sim}{a} \in A^n$,

$$\bigcap \{ J_k^\lambda(A, \underset{\sim}{a}) : \lambda \in N(A) \} = \{ \underset{\sim}{z} \in \underset{\sim}{C}^n : |p(\underset{\sim}{z})| \leq r(p(\underset{\sim}{a})),\ p \in Pol_k^n \}.$$

Proof. The inclusion $\supseteq$ is clear since $r(.) \leq \lambda(.)$ for each $\lambda \in N(A)$. Conversely, let $\underset{\sim}{z} \in \bigcap \{ J_k^\lambda(\underset{\sim}{a}) : \lambda \in N(A) \}$. For each $p \in Pol_k^n$

we have $\inf \{ \lambda(p(\underset{\sim}{a})) : \lambda \in N(A) \} = r(p(\underset{\sim}{a}))$.

Therefore $|p(\underset{\sim}{z})| \leqslant r(p(\underset{\sim}{a}))$ $(p \in \mathrm{Pol}_k^n)$. □

<u>THEOREM</u> 11. Let $\underset{\sim}{a}$ be an n-tuple of generators for the unital Banach algebra A and let U be any open neighbourhood of $Sp(A,\underset{\sim}{a})$. Then there exists an integer $m \geqslant 1$ depending on U such that

$$Sp(A,\underset{\sim}{a}) \subseteq \{ \underset{\sim}{z} \in \underset{\sim}{C}^n : |p(\underset{\sim}{z})| \leqslant r(p(\underset{\sim}{a})) , p \in \mathrm{Pol}_k^n \} \subset U$$

whenever $k \geqslant m$.

<u>Proof</u>. From Corollary 9 we have

$$\bigcap_{k=1}^{\infty} J_k(\underset{\sim}{a}) = Sp(A,\underset{\sim}{a}) \subseteq \bigcap \{ J_k^{\lambda}(\underset{\sim}{a}) : \lambda \in N(A) \} .$$

If $\bigcap \{ J_1^{\lambda}(a) : \lambda \in N(A) \} = Sp(\underset{\sim}{a})$ an application of Proposition 10 with $\overset{m}{\cancel{k}} = 1$ gives the result. If $\bigcap \{ J_1^{\lambda}(\underset{\sim}{a}) : \lambda \in N(A) \} \setminus U \neq \phi$ let $\underset{\sim}{\xi} \in \bigcap \{ J_1^{\lambda}(\underset{\sim}{a}) : \lambda \in N(A) \} \setminus U$. Then there exists $k_{\underset{\sim}{\xi}} \in \underset{\sim}{N}$ such that $\underset{\sim}{\xi} \notin J_{k_{\underset{\sim}{\xi}}}(\underset{\sim}{a})$ (Corollary 9). Therefore there exists $p_{\underset{\sim}{\xi}} \in \mathrm{Pol}_{k_{\underset{\sim}{\xi}}}^n$ with $|p_{\underset{\sim}{\xi}}(\underset{\sim}{\xi})| > \|p_{\underset{\sim}{\xi}}(\underset{\sim}{a})\| \geqslant r(p_{\underset{\sim}{\xi}}(\underset{\sim}{a}))$.

i.e. $\underset{\sim}{\xi} \notin \bigcap \{ J_k^{\lambda}(a) : \lambda \in \underset{\sim}{N}(A) \}$ if $k \geqslant k_{\underset{\sim}{\xi}}$. Let $U_{\underset{\sim}{\xi}}$ be any open neighbourhood of $\underset{\sim}{\xi}$ such that $|p_{\underset{\sim}{\xi}}(\underset{\sim}{z})| > \|p_{\underset{\sim}{\xi}}(\underset{\sim}{a})\|$ $(\underset{\sim}{z} \in U_{\underset{\sim}{\xi}})$. $\bigcup \{ U_{\underset{\sim}{\xi}} : \underset{\sim}{\xi} \in \bigcap \{ J_1^{\lambda}(\underset{\sim}{a}) : \lambda \in N(A) \} \setminus U \}$ is an open cover of a compact set. Let $U_{\underset{\sim}{\xi}_1} \cup U_{\underset{\sim}{\xi}_2} \cup \ldots \cup U_{\underset{\sim}{\xi}_n}$ be a finite subcover. Take $m = \max\{ k_{\underset{\sim}{\xi}_j} : j=1,2,\ldots,n \}$. We have

$\bigcap \{ J_k^{\lambda}(\underset{\sim}{a}) : \lambda \in N(A) \} \subset U$ if $k \geqslant m$. The result follows from Proposition 10. □

§5. Future progress - A spatial concept.

We have concentrated our study so far on the Williams ranges which extend the notion of the algebra numerical range. It is also clear by analogy how to define a spatial k^{th} Williams range which extends the notion of the spatial range of an operator on a normed linear space. This section contains a definition and records a few immediate observations. It is included to indicate an open area which we think merits further investigation.

Notation. Let X be a complex normed linear space, X' its dual space, $\Pi(X)$ the subset of $X \times X'$ given by

$$\Pi(X) = \{ (x,f) \in S(X)\times S(X') : f(x) = 1 \} .$$

Given $x \in S(X)$, let $D(X,x) = \{ f \in S(X') : f(x) = 1 \}$.
Given $T \in B(X)$, $x \in S(X)$, let $V(T,x) = \{ f(Tx) : f \in D(X,x) \}$.

It was observed that the method of Lemma 1.1 of J.P.Williams could be used to exhibit $V(T,x)$ as an intersection of closed discs :

$$V(T,x) = \{ \lambda \in \underset{\sim}{C} : | \lambda - \zeta | \leq \|(T - \zeta I)x\| \quad (\zeta \in \underset{\sim}{C}) \} .$$

By analogy with the definition of the k^{th} (algebra) Williams range we make the following definition.

Definition. Given $T \in B(X)$, the k^{th} spatial Williams range of T, $SpJ_k(T)$, is the subset of the complex plane given by

$$SpJ_k(T) = \cup \{ SpJ_k(T,x) : x \in S(X) \}$$

where

$$SpJ_k(T,x) = \{ \lambda \in \underset{\sim}{C} : |p(\lambda)| \leq \|p(T)x\| \quad (p \in Pol_k) \} .$$

Remarks.

(1). The following properties of SpJ_k are immediate.

(i). $SpJ_1(T) = V(T)$; $SpJ_{k+1}(T) \subseteq SpJ_k(T)$ $(k=1,2,\dots)$,

(ii). $SpJ_k(\alpha I + \beta T) = \alpha + \beta\, SpJ_k(T)$ $(\alpha,\beta \in \underset{\sim}{C})$,

(iii). $SpJ_k(T) \subseteq J_k(T)$.

(2). If $Tu = \lambda u$ for some $u \in S(X)$ then $\lambda \in SpJ_k(T,u)$ and therefore $pSp(T) \subseteq SpJ_k(T)$. If X has finite dimension then $SpJ_k(T) = Sp(T)$ $(k \geq \dim X)$.

Problems.

(1). Is the k^{th} spatial range non-empty in general ? It is well known that $Sp(T) \subseteq \overline{V(T)}$. Is the approximate point spectrum of T contained in $\overline{SpJ_k(T)}$?

(2). It is well known that $\overline{co}\, V(T) = V(B(X),T)$. What is the relationship, if any, between the k^{th} spatial and k^{th} algebra Williams ranges when $k \geq 2$?

(3). Describe $\bigcap_{k=1}^{\infty} \overline{SpJ_k(T)}$ whenever the intersection is non-empty. If $\lambda \in \underset{\sim}{C}$ is a limit of a sequence of eigenvalues of T we have $\lambda \in \overline{SpJ_k(T)}$. If T is either a compact operator on a Banach space , or a diagonal operator on a Hilbert space then $Sp(T) = \bigcap_{k=1}^{\infty} \overline{SpJ_k(T)}$ (because for such T, $Sp(T) \subseteq \overline{SpJ_k(T)} \subseteq J_k(T)$).

INDEX OF SYMBOLS

When no page number is indicated reference should be made to pages (vi) and (vii).

CHAPTER 1

CHAPTER 2

CHAPTER 3

REFERENCES

1. F.F. Bonsall, "An inclusion theorem for the matrix and essential ranges of operators", J. London Math. Soc. 6 (1973) 329-332 .

2. F.F. Bonsall and J. Duncan, Complete normed algebras, (Springer-Verlag, 1973).

3. F.F. Bonsall and J. Duncan, Numerical ranges of operators on normed spaces and of elements of normed algebras, (London Math. Soc. Lecture Note Series 2. C.U.P. 1971).

4. F.F. Bonsall and J. Duncan, Numerical ranges II, (London Math. Soc. Lecture Note Series 10. C.U.P. 1973).

5. J. Dixmier, Les algèbres d'operateurs dans l'espace Hilbertien, (Gauthier-Villars 1968).

6. J. Dixmier, Les C*-algèbres et leurs représentations, (Deuxième édition, Gauthier-Villars, 1969).

7. J. Dixmier, "Les functionnelles linéaires sur l'ensemble des opérateurs bornés d'un espace de Hilbert", Annals of Mathematics 51 (1950) 387-408.

8. N. Dunford and J.T. Schwartz, Linear operators, Part 1. (Interscience, New York 1958).

9. N. Dunford and J.T. Schwartz, Linear operators, Part 2. (Interscience, New York 1963).

10. P.A. Fillmore, "On similarity and the diagonal of a matrix", Amer. Math. Monthly 76 (1969) 167-169.

11. P.A. Fillmore and J.P. Williams, "Some convexity theorems for matrices", Glasgow Math. Journal 12 (1971) 110-117.

12. P.R. Halmos, A Hilbert space problem book, (Van Nostrand, 1967).

13. J.R. Ringrose, Compact non-self-adjoint operators. (Van Nostrand, 1971).