COMPETITION IN AN EVOLVING STOCHASTIC MARKET

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ABSTRACT

> In an efficient market all identical goods must have only one price

So states the aptly named *law of one price*. In the real world, however, one may easily verify that identical products are often sold for different prices. This thesis develops an extension of the Bertrand model in economics to include spatially localised competition to explain this price variation, which is then studied through simulation methods and theoretical analysis.

Our model studies the effect that local heterogeneities in the environment experienced by sellers have on successful pricing strategies. Taking inspiration from models of evolutionary dynamics, we define the fitness of a seller and evolve seller prices through selection and mutation.

We find three distinct steady states in our model related to the probability that a seller experiences competition for a buyer, mediated by the number of bankrupt sites in the system. When competition-free sales are unlikely, the system collapses on to a single price. If temporary monopoly situations do exist sellers can accumulate capital and variation in prices is stable. In this scenario, sellers spontaneously separate into two classes: cheap sellers – requiring sales to every potential buyer; and expensive sellers – requiring only occasional sales. Finally, we find an intermediate regime in which there is a single highly favoured price in the system which oscillates between high and low extrema.

We study the properties of these steady states in detail, building a picture of how globally uncompetitive sellers can nonetheless survive if competition is strictly local. We show how the system builds up correlations, leading to niches for expensive sellers. These niches change the nature of the competition and allow for long-term survival of uncompetitive sellers.

Not all expensive prices are equally likely in the steady state and we analyse why (and where) peaks in the price distribution appear. We can do this exactly for the early time dynamics of the model and extend the argument more qualitatively to the steady state. This latter analysis allows us to predict, for an observed steady distribution, the minimum price an expensive seller should charge to guarantee profit.

The oscillatory 'steady state' is qualitatively reminiscent of boom and bust cycles in the global market. We study methods to suppress the oscillations and suggest ways of avoiding catastrophic crashes in the global economy – without negatively affecting the ability of outliers to make large profits.

DECLARATION

This work has been composed by myself and has not been submitted for any other degree or professional qualification. Except where otherwise stated, the work reported herein was completed by me. The model presented in this thesis was originally reported in [48]. Some of the original work described in this thesis has also been reported in [49, 50].

Lawrence Mitchell

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If you have read this far hoping to spot yourself, and have yet to be mentioned, I'm sorry. The omission is not deliberate.

Having reached the end of the beginning, I hope you make it, still interested, to the beginning of the end.

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1 INTRODUCTION

How should I price my produce? A question mulled over by shopkeepers for millennia. If the seller enjoys a monopoly, the answer is straightforward – minimise production costs and (assuming constant demand) charge as much as consumers are willing to pay. When competition and choice between shopkeepers is introduced, the problem becomes significantly harder.

When we observe a marketplace, we see many different solutions to the problem. Sellers adopt different pricing strategies for the same product. Explaining the observed pricing strategies has troubled economic modellers since at least the 19th century. The solutions of such models also specify which pricing strategies are good ones (within the model framework) – answering the question posed at the beginning of this section.

1.1 WHY SHOULD A PHYSICIST STUDY A PROBLEM IN ECONOMICS?

A cursory glance over the field of economic modelling reveals many proposed models to address questions of price setting. Why, therefore, should we want to add yet another model? To answer this question, we need to understand a bit about the existing models. The 19th century contributions of Cournot [17] and Bertrand [7] both conclude that competition between sellers results in both charging the same price.

Subsequent study in this field has revolved around the evidence that the predictions of Bertrand and Cournot are not realised in real-world scenarios. When we look at sellers in the wild, even when they supply the same product, prices can differ widely. Economists have therefore constructed models which demonstrate some of these features. As in physics, the holy grail of any such modelling effort is an exact solution to the model. This allows fullest study of its behaviour and provides the greatest insight. To arrive at these exact solutions, economists must make a number of assumptions about the behaviour of the entities involved – for example, that all participants are completely rational and in possession of complete information about the marketplace. Critically, these assumptions are used even when they are subsequently found to be a bad match for evidence [9].

In this thesis, we construct a model to study price setting that requires far fewer assumptions about human behaviour. For instance, we require that participants in our model are only boundedly rational and are also only in possession of a small amount of information about the marketplace. Furthermore, we try to include as few preconceptions as possible about the market – for example, we do not directly specify how participants might select for a good price. That is, our model tries to strip out as much as necessary while still exhibiting non-trivial behaviour.

Our inspiration for this study comes from the recent successes in the field of econophysics (see [10, 43] for recent introductory texts). The field is a relatively new one, the term itself being a scant fifteen or so years old. Initial forays into economics were inspired by the bright lights of the stock market. If one is able to predict pricing fluctuations to somewhat better than 50% accuracy, then profits should be easy to obtain. Access to large datasets and computing resources allowed quantitative descriptions of stock market returns to be developed and tested. Physicists, principally those with a background in statistical mechanics, applied their analysis techniques to the data available on stock market fluctuations. Characterising the distribution of returns accurately is key to making good predictions about the likely movement of markets. The development of the *minority game* [14], an evolutionary model for a game in which the minority strategy choice wins, gave further impetus: in competition among stock market traders, the minority choice will win. This suggests it might be possible to model the herd behaviour of traders with some kind of minority game and use this knowledge in our own trading [28, 33, 34]. At the very least, the arrival of econophysics has resulted in a useful updating [11] of the venerable Black-Scholes option pricing formula [8].

One area untouched by such studies is that of price-competition. Work exists characterising the distribution of firm sizes and proposing plausible dynamics that would result in this distribution (see de Wit [19] for a recent review), but focus on the details of competition has been left in the hands of economists. Computational models of such competition are then typically very specific to a particular setting (for example, modelling price setting in a French fish market [36]). Interestingly, simple game theoretic models (of which economic models of price setting are an example) have been studied using computational techniques – to incorporate elements such as spatial heterogeneity – in other fields to address open questions such as how altruistic and cooperative behaviour can develop in an evolutionary setting.

This thesis applies computational methods to the problem of price competition. Our approach is not specific to a particular marketplace, instead we take a simple, well understood, model and extend it using ideas from computational game theory. Casual observation shows that shops are often spatially disperse. Further, except when purchasing expensive items (and often even then), consumers are never able or willing to acquire complete information about the marketplace. Thus, our simple view of a marketplace is of a spatially separate set of firms and around them a cloud of consumers that are also spatially localised and only able to see a small number of sellers. Now we can again ask 'what is my best pricing strategy?' and further, is there a specific best strategy, or do a variety of prices necessarily exist? It is these question which this thesis makes an attempt at answering.

Our modelling method wins in some areas over existing economic models, and loses in others. As already mentioned, we try to make as few axiomatic assumptions about human behaviour as possible. Additionally, we attempt not to cloud the construction of the model with our intuition as to how the market 'should' evolve. In doing so we lose specificity and will likely not be able to take away quantitative predictions, however, we will be able to see if the assumptions of economists are needed to produce results that qualitatively mimic the real world.

1.2 THESIS OUTLINE

This short introduction should allow the reader to see why the question of price setting is an interesting one. Chapter 2 provides a more detailed overview of existing concepts and models. We introduce and review game theoretic concepts and some simple economic models of price competition. This extended scene-setting should allow the reader to see in more detail where our model fits and provide motivation for our subsequent choices.

Chapter 3 makes these motivations concrete and introduces our model in detail, here we flag up the differences from previous work and why they are important. Following this introduction, we proceed swiftly onto results of our model. These data are presented initially in two separate blocks. In chapter 4 we detail the behaviour of our model under discrete time dynamics, in chapter 6 we study the behaviour with continuous time dynamics.

Our model is set up with a fixed spatial structure and most of the results we present are obtained when this structure is a one-dimensional ring. In chapter 5 we motivate the reasons for restricting ourselves to such a case, demonstrating that the qualitative picture our results paint is independent of the spatial structure our model is based on.

In chapter 6 we find an interesting 'steady state' of the model dynamics not observed in the discrete time formulation. We observe system-wide oscillatory behaviour. We propose that the cycles we see are perhaps analogous to boom and bust cycles in the global economy and, topically, study ways of suppressing this behaviour in chapter 7.

We might hope that our model is simple enough to be amenable to exact analysis, rather than merely computational study. It turns out that the full details are too complex, however, in chapter 4 we are able to analyse the early time dynamics. Subsequently, in chapter 8, we sketch how one might determine some steady state properties exactly by considering a simplified mean field version of our model.

Finally, in chapter 9, we summarise our results and explore some of the general conclusions of our model. Having studied the simple model in great detail, we propose some areas of future work that could build on our results. The steady state of our model could be seen as a playing field in which one could test the efficacy of complicated price-setting strategies: we outline one possible such scheme.

BACKGROUND & MOTIVATION

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The model presented later in this thesis draws inspiration from a range of different fields: game theory, evolutionary biology, ecology and economics. In order that the reader is not lost when encountering the model, this chapter outlines the problems we will address and the manner in which each of these somewhat disjoint subject areas helps. This will take the form of a whistle-stop tour through the development of game theory and its application in economics and evolutionary biology. Along the way we will note some common criticisms of the models so constructed and look to ecological and physical systems to think how we might address these.

The problem we address comes from economics. How should an individual seller choose their price in a competitive market? Consider a case where sellers offer an identical product to many buyers. If buyers choose randomly between sellers, we can easily imagine how sellers can make sales at high prices. If buyers are discerning in their choices, we might imagine that sellers charging high prices would not make sales. Evidence from price comparison studies [5, 6, 39] shows that even if buyers are discerning sellers can survive offering different prices.

The problem of price-setting has been the subject of economic models for over 100 years (for example the models of Cournot (1838) and Bertrand (1883)) and the subject of traders for presumably much longer. These early attempts idealise the situation to that of two sellers competing for a single consumer. We discuss Cournot's and Bertrand's models in detail in section 2.1 and onwards. The key outcome of their analyses is that sellers should end up agreeing on a single price. When we observe prices in the real world, however, competing sellers seldom agree on a single price. Instead, we see some distribution of prices for a single product. This phenomenon is termed *price dispersion* in the economic literature and many modelling efforts have been expended in attempting to understand it. We shall summarise three models representative of the approach taken in section 2.4. This thesis also constructs a model to address the phenomenon of price dispersion, although the approach we take is very different to that considered in the economic literature.

Before we delve in to our proposed model in detail in the next chapter, some extended scene-setting is required. We begin with the 19^{th} century models of sales of Cournot and Bertrand. We then move on and summarise the main theories from the early 20^{th} century developments of game theory. This mathematical formulation of games allows a systematic treatment of the exactly solveable models of competition we subsequently describe. The theory is not without its flaws which we also touch upon. The narrative thread then moves on to three representative game theoretic models that demonstrate price dispersion which we examine critically. Our exposition ends with some more in-depth discussion of why game theoretic models might be flawed in their approach to the analysis of real marketplaces.

2.1 MODELS OF COMPETITION

A short word on nomenclature used in the following section. We use *equilibrium* in its economic, rather than physical sense. In this context an equilibrium state of a game is a set of strategies for which no single participant finds it in their interest to change strategy. Note that the equilibrium state may not be uniquely defined: multiple different sets of strategies may have this same equilibrium property.

2.1.1 *Monopoly pricing and perfect competition*

In order to best discuss models of sales in which competition takes place we need some benchmarks which set the limiting behaviour: upper and lower bounds on the price sellers charge. These are obtained by looking at the behaviour of a monopolist and the behaviour of sellers in perfect competition. The case of the monopolist is straightforward. Consider a seller choosing a price $p > 0$ for a product. There is a demand for this product at the given price, $D(p)$ with $\frac{dD(p)}{dp} < 0$. That is, as the price of the product increases, so the demand decreases. This seller's profit will be given by

$$\pi(p) = pD(p) - c(D(p)). \tag{2.1}$$

Where $c(x)$ is the cost to the seller of producing a quantity x of the product. $\frac{dc(x)}{dx} > 0$, cost of production increases with the amount of product. An example profit curve is shown in figure 2.1, maximising the profit is equivalent to setting $\frac{d\pi}{dp} = 0$. We shall denote the monopoly price by p^m and the supply level by $q^m \equiv D(p^m)$.

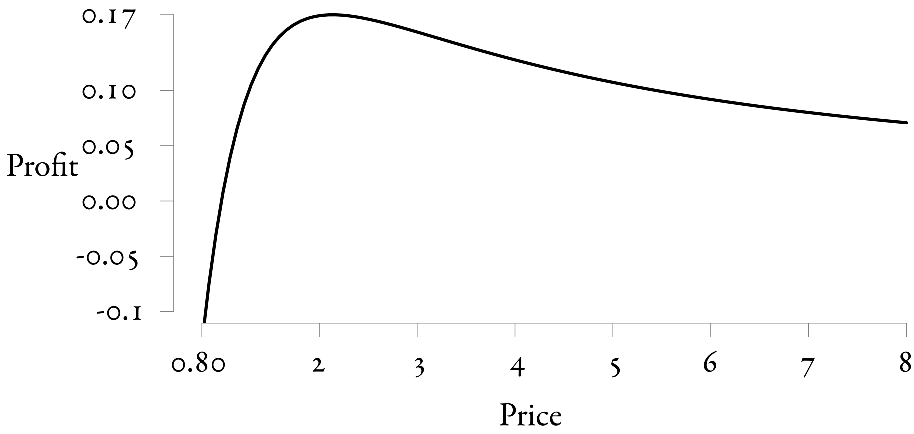

Figure 2.1 *Profit curve for a price-setting monopolist with* $D(p) = \frac{2}{p^2+1}$ *and* $c(x) = \sqrt{x}$. *The maximum profit is obtained for* $p^m \approx 2.14$

At the other end of the scale from a monopolist is the pricing and supply structure obtained under *perfect competition*. In this regime, the assumption is one of many buyers and sellers. So many, that no single seller can affect the market: all individual sellers are assigned a price for their goods and their choices of supply level will not change that price. Buyers have perfect information about all firms: they will always choose the cheapest. In the simple examples we shall consider, the perfectly competitive limit is one in which the price matches the cost. This limit also corresponds to the highest supply level. This is easily seen by recalling that buyer demand is a decreasing function of the price. The competitive price is the lowest price buyers will experience and so the demand will be higher at this price at than all others. We shall denote the perfect competition supply level by q^p and the concomitant price by p^p.

2.1.1.1 *Monopoly pricing in the presence of interactions between consumers*

Gordon et al. [24] demonstrate how the addition of interactions between consumers in a spatial region can change the best strategy of a monopolist. In their model, buyers have some internal preference for buying and are additionally influenced by the buying choice of their neighbours. This model can be analysed by considering it to be like an Ising spin model, with the binary buyer choice mapping onto up and down spins. Gordon et al. show that if the buyers are fully connected (*i.e.*, they all influence each other), the monopolist's price exhibits a first order phase transition if the interaction strength is high enough. In this case, the monopolist's price jumps from a low solution to a high solution (the former attracting a large number of buyers, the latter a small number) as the mean preference to pay decreases or the production costs increase.

2.1.2 *Cournot competition [17]*

Cournot competition addresses the issue of which price will be chosen in a marketplace in which sellers may alter the quantity they supply, but not their price. The price is fixed by some external third party once all competing sellers have chosen their supply levels.

This is a model of competition between $N > 1$ sellers. For simplicity, we shall consider the $N = 2$ case, though the arguments easily generalise to higher N. All sellers offer an identical product. At the beginning of a sales day sellers decide simultaneously how much they will produce. Once the total supply level has been determined, the market price is set according to some function known to all sellers. Each seller has a cost function that is dependent on its production quantity. Let the quantity produced by seller one (two) be q_1 (q_2). The profit of seller one is then given by

$$\pi_1(q_1, q_2) = P(q_1 + q_2)q_1 - c_1(q_1). \tag{2.2}$$

Where $P(\cdot)$ is the externally determined market price and $c_1(\cdot)$ is the cost to seller one of producing its specified output level. The model has $P'(x) < 0$ and $c'(x) > 0$: as supply increases prices drop and as production increases so do costs. Seller two has an equivalent profit obtained by switching the indices in equation 2.2.

Cournot's analysis is carried out by finding a pair of functions for sellers one and two which provide the best response to their conjecture about their opponent's choice of supply level. For any fixed q_2 there is a value of q_1 which maximises seller one's profit. Equally, for seller two, at fixed q_1 there is a q_2 which maximises profit. To find the equilibrium supply level, we maximise the profits of the two firms by setting $\partial_{q_i}\pi_i = 0$. With known cost functions and pricing function, we obtain two equations for the two unknowns q_1 and q_2. These may be solved for the equilibrium supply levels, q_1^* and q_2^*. If the cost functions are identical for both firms, $q_1^* = q_2^*$.

The resulting equilibrium performs better (for the consumer) than a monopoly. Specifically, $q_1^* + q_2^* > q^m$ and hence $P(q_1^* + q_2^*) < p^m$: there is a larger supply than in the monopoly case and the price of goods is hence lower. To see this, consider an example with $P(q) = a - q$ and $c(q) = bq^2$, $a, b > 0$. The monopoly supply level is given by

$$q = \frac{c'(q) - P(q)}{P'(q)} = \frac{2bq - a + q}{-1}. \tag{2.3}$$

Solving for q we find

$$q^m = \frac{a}{2(1+b)}. \tag{2.4}$$

The Cournot supply level is given by

$$\begin{aligned} q_1 + q_2 &= \frac{\partial_{q_1} c(q_1) - P(q_1 + q_2)}{\partial_{q_1} P(q_1 + q_2)} + \frac{\partial_{q_2} c(q_2) - P(q_1 + q_2)}{\partial_{q_2} P(q_1 + q_2)} \\ &= 2a - 2(q_1 + q_2)(1 + 2b) \end{aligned} \tag{2.5}$$

Solving for $q_1 + q_2$ we find

$$q_1 + q_2 = \frac{a}{(b + \frac{3}{2})} \tag{2.6}$$

and so

$$q_1 + q_2 - q^m = \frac{a(1+2b)}{4b^2 + 10b + 6} > 0 \tag{2.7}$$

as claimed. This form of competition does not bring supply levels all the way to those of perfect competition: $q_1^* + q_2^* < q^p$ and $p(q_1^* + q_2^*) > p^p$. It is possible to show that as the number of firms participating in Cournot competition becomes large, the supply level and hence price tend to those of perfect competition. The perfect competition limit is reached in the limit of an infinite number of sellers.

2.1.3 *Bertrand competition [7]*

Almost fifty years after Cournot's work on firm competition, Bertrand developed an alternative theory in a lengthy critique of Cournot's work. Instead of allowing firms to choose their supply and having their price set exogenously, Bertrand's scenario is one in which market demand is set externally and sellers may choose their price freely. Similar to our analysis of Cournot competition the model is defined for $N > 1$ sellers, we consider the $N = 2$ case for simplicity.

Let the price chosen by seller one (two) be p_1 (p_2). These prices are chosen simultaneously with $p_i \in [0, \infty)$. The profit of seller one is given by

$$\pi_1(p_1, p_2) = p_1 D(p_1, p_2) - c_1(D(p_1, p_2)). \tag{2.8}$$

Where $D(x, y)$ is the market demand for goods offered at a price x when the alternate price is y. $c_1(\cdot)$ is the cost of producing the demanded product. The profit of seller two is also given by equation 2.8 but with the indices swapped.

In Bertrand's original model all sellers have the same unit cost: $c_i(x) = cx$. Sellers compete for the total demand $D \equiv D(p^m)$ experienced by a monopolist with price p^m. Buyers choose the cheapest possible seller and thus the lowest-priced seller obtains all the demand. If the two sellers choose the same price, the demand is split equally between them. The demand function is thus discontinuous and given by

$$D(p_1, p_2) = \begin{cases} D & p_1 < p_2 \\ \frac{D}{2} & p_1 = p_2 \\ 0 & p_1 > p_2. \end{cases} \tag{2.9}$$

The Bertrand equilibrium has both sellers charging $p_1^* = p_2^* = c$. It is easiest to see this if we approach the price setting problem a seller faces in steps. Consider seller two with a price $p_2 = c + \delta$. The first seller maximises their profit considering p_2 as fixed. To do so, they pick a price $p_1 = p_2 - \epsilon$, making profit $\pi_1 = (\delta - \epsilon)D$. Seller two notices this and reduces their price to $p_2 = p_1 - \epsilon$ making profit $\pi_2 = (\delta - 2\epsilon)D$. This back and forth continues until further price reductions result in negative profits. The end result is both sellers adopting $p_1 = p_2 = c$. Now any further reduction in price, despite gaining the entire market demand, results in negative profit. Now, both sellers are assumed to be completely rational. They also know that their opponent is completely rational. Given these statements, the only rational price to choose is the final one $p_i^* = c$. Both sellers therefore open their doors on the first morning having chosen the marginal cost price.

Thus, in the Bertrand equilibrium, all firms earn zero profits, despite a monopolist being able to make positive profits. The model results in perfect competition for the case of two sellers (unlike Cournot competition where this limit is only reached for an infinite number of sellers).

2.1.4 *A short critique of the two models*

Although these simple models of sales capture some essential aspects of competition, by their very nature they are unable to model all subtleties of real-world competition[1]. In order to investigate the phenomenon of price dispersion, we must add some additional complexity somewhere. Obviously, this is not the first time these criticisms have been raised, numerous models of competition have been proposed since Cournot and Bertrand to explain empirical phenomena better. We shall continue our exposition of models of sales in section 2.4 but first we consider a diversion into the mathematical theory of games that we may place the models we present on a firm theoretical footing.

2.2 GAME THEORY

The development of game theory was prompted by the wish to answer a simple question that crops up in many parts of everyday life. von Neumann [74] states the problem succinctly in his first work on game theory:

> “ n players, $S_1, S_2, \ldots, S_n$, play a specified game $\mathcal{G}$. How should a particular participant, S_m, play to obtain the most profitable outcome for themselves?[2]

Originally, the games considered were ‘simple’ with well-defined strategies, such as poker, bridge and chess. However, we shall see that the framework developed lends itself equally well to simple economic models of sales. The concepts, rather than the mathematical details of their development, are the important message of this section. The latter are presented for fullness of exposition.

[1] To take one simple example, it is assumed that sellers can produce goods on demand: should they make no sales, they make no losses. Unless sellers can correctly predict their demand in advance, this seems a bad assumption for product that is perishable (bread, for instance)

[2] Translated from the original by the current author: ‘n Spieler, $S_1, S_2, \ldots, S_n$, spielen ein gegebenes Gesellshaftsspiel $\mathcal{G}$. Wie muß einer dieser Spieler, S_m, spielen, um dabei ein möglichst günstiges Resultat zu erzielen?’

2.2.1 *The Minimax Theorem*

The key concept developed in von Neumann's paper [74] was the *Minimax Theorem*. Consider a two-player game with S_1 playing a strategy x and S_2 a strategy y. The payoffs to S_1 and S_2 are $g(x, y)$ and $-g(x, y)$ respectively. This is an example of a *zero sum game*: any gains to S_1 are offset by losses to S_2 and vice versa. If S_1 chooses a specific x, then her payoff depends on what choice S_2 makes. However, the payoff (P_1) will definitely obey the inequality:

$$P_1 \geqslant \min_y g(x, y). \tag{2.10}$$

Where $\min_y g$ denotes the minimum value of g with respect to changes in y. S_1 can now maximise the right hand side of equation 2.10 with respect to changes in her strategy guaranteeing a payoff:

$$P_1 \geqslant \max_x \min_y g(x, y). \tag{2.11}$$

That is, S_1 can choose an x such to maximise her minimum payoff. Equally S_2 can, independently of S_1, choose a y that guarantees a payoff

$$P_2 \leqslant \min_y \max_x g(x, y) \tag{2.12}$$

this minimises the maximum loss that S_2 will experience. In general, the following inequality holds

$$\max_x \min_y g(x, y) \leqslant \min_y \max_x g(x, y). \tag{2.13}$$

It is easy to construct an example where the inequality holds strictly, for example

$$g(x, y) = \begin{array}{c c} & \begin{array}{cc} y=1 & y=2 \end{array} \\ \begin{array}{c} x=1 \\ x=2 \end{array} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array} \tag{2.14}$$

results in $\max_x \min_y g = -1$ and $\min_x \max_y g = 1$. The equality only holds in equation 2.13 if there is a saddle point in $g(x, y)$ [75].

Although it is not always possible to satisfy the equality in equation 2.13, we can construct a *mixed strategy* – a probability distribution over *pure* strategies – with expected payoff $h(\vec{\xi},\vec{\eta}) = \sum_{x,y} g(x,y)\xi_x\eta_y$ such that

$$\max_{\vec{\xi}} \min_{\vec{\eta}} h(\vec{\xi},\vec{\eta}) = \min_{\vec{\xi}} \max_{\vec{\eta}} h(\vec{\xi},\vec{\eta}) \tag{2.15}$$

where the components of $\vec{\eta}$ and $\vec{\xi}$ give the probability of playing an individual or *pure* strategy. For example, for rock-paper-scissors a pure strategy would be one of rock, paper or scissors; a mixed strategy would be 'play rock with probability $\frac{1}{2}$, paper with probability $\frac{1}{3}$ and scissors with probability $\frac{1}{6}$'. The equality in equation 2.15 is the minimax theorem, proven in [74]

> For every two person, zero sum game with a finite number of strategies, there is a mixed strategy for each player and an expected payoff P such that, given S_2's strategy S_1's best possible payoff is P and given S_1's strategy S_2's best possible payoff is $-P$.

For our simple two-strategy game the components of $\vec{\xi}$ and $\vec{\eta}$ are easy to find [75]

$$\xi_1 = \frac{g(2,2) - g(2,1)}{g(1,1) + g(2,2) - g(1,2) - g(2,1)} \tag{2.16a}$$

$$\xi_2 = 1 - \xi_1 \tag{2.16b}$$

and

$$\eta_1 = \frac{g(2,2) - g(1,2)}{g(1,1) + g(2,2) - g(1,2) - g(2,1)} \tag{2.17a}$$

$$\eta_2 = 1 - \eta_1 \tag{2.17b}$$

If either g or h is antisymmetric (that is $g(x,y) = -g(y,x)$ or $h(\vec{\xi},\vec{\eta}) = -h(\vec{\eta},\vec{\xi})$) the game is *fair* and the payoff to both players is zero. This is not always the case: consider the game shown in equation 2.18. The best strategies in this game are mixed and have $\vec{\xi} = \vec{\eta} = (\frac{2}{5}, \frac{3}{5})$. The expected payoff to S_1 is $P = \frac{1}{5} \neq 0$.

$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \end{array}. \tag{2.18}$$

We can now see how this applies to the models of sales presented above.

The Bertrand equilibrium when competition is on price is exactly that strategy arrived at by applying the minimax concept. Seller one chooses a price which maximises their minimum payoff, irrespective of the choice of the opponent and vice versa. We have glossed over a slight complication in this case: the minimax theorem as it stands applies only to zero sum games. The model of sales is not zero sum (both the two sellers and the buyer are assumed to gain some payoff when a transaction is carried out). For such games, a different solution concept, developed in the early 1950s [52–55], is necessary. We shall describe this solution concept in the next section.

2.2.2 *The Nash equilibrium*

Consider a game played by N players each of whom has a finite number of strategies to choose from. The set of strategies adopted by the participants is a Nash equilibrium if no *single* player can do better by changing their strategy unilaterally. That is, in a Nash equilibrium, if I choose to change my strategy and all other $N-1$ players keep the same strategy as before, I will *at best* perform as well as previously.

This concept is perhaps best illustrated by a simple example. Consider a two-player game with two strategies, C and D. The payoffs of S_1's choice against S_2 are encoded in the following payoff matrix:

$$M = \begin{array}{c c} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 5 & 0 \\ 9 & 1 \end{pmatrix} \end{array}. \tag{2.19}$$

The strategy choices of S_1 and S_2 are indicated by the row and column labels respectively. The entries in the matrix give the payoff to S_1 in each case. The payoffs to player two are given by the transpose of M. Thus, if S_1 chooses C and S_2 D, S_1 receives payoff zero and S_2 payoff nine. The Nash equilibrium for this game is that both players adopt the strategy D.

To see this, consider starting position where both participants play C, if S_1 keeps her strategy, a change from C to D by S_2 increases his payoff from five to nine. Now S_2 cannot increase his payoff any further and is thus disinclined to change strategy. In this situation, S_1 increases her payoff by switching from C to D: her payoff changes from zero to one. Now no further changes are possible without the cooperation of the two players, any change by a single player in their strategy is downhill in payoff space.

This raises an interesting 'paradox'. The rational choice of participants in this game seems to an outside party to be highly irrational. Naïvely we expect the C state to be the correct choice of rational players, but are forced to conclude that this is not the case. This is an oft-raised criticism [29, 64, 65] of game theoretic results, namely that real participants do not behave in the manner predicted by analysis. See for example the experiments performed by Flood [22] and the meta-analysis of experiments by Sally [60] showing that the assumptions of rational self-interest are not always good ones.

2.2.3 *Mixed strategy equilibria*

The game shown in equation 2.19 has a Nash equilibrium in pure strategies. That is, the equilibrium state is to choose a single strategy. Such a state does not always exist. Some games have a Nash equilibrium in which multiple different pure strategies feature. These mixed strategies, which we encountered in section 2.2.1, take the form of a probability distribution over pure strategies. Nash proved that while not every game has a pure strategy equilibrium, every game with a finite number of strategies does at least have a mixed strategy equilibrium [54]. A classic example of such a game is rock, paper, scissors.

$$\begin{array}{c} \\ \mathrm{R} \\ \mathrm{P} \\ \mathrm{S} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{R} & \mathrm{P} & \mathrm{S} \end{array} \\ \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \end{array}. \tag{2.20}$$

There is no pure strategy equilibrium in this game, since no single strategy dominates the other two. Instead, there is a mixed strategy equilibrium in which each strategy is played with equal probability of $\frac{1}{3}$. This maximises the payoff against a rational opponent.

A number of problems have been raised with the concept of mixed strategies. Primarily that players of a game will rarely have a perfect random number generator available to them (which would allow correct mixing of pure strategies). Thus, the interpretation of a mixed strategy is somewhat difficult. One proposal is to think of a mixed strategy as an expectation of the possible strategies an opponent will play. An alternative appealing interpretation is that provided by evolutionary game theory which we expand upon in the following section. Developed in the 1960s and 70s primarily by evolutionary biologists [30, 41, 46], it applies the concepts of game theory to an evolutionary setting.

2.3 EVOLUTIONARY GAME THEORY

There are a number of criticisms of the assumptions behind the solution strategies of Nash and von Neumann when they are applied to experiments with people. Amongst these is the need for participants in a game to be aware of the form and outcomes of the game and be consciously maximising their return. This perfectly rational and informed behaviour is then used to explain why participants choose Nash equilibria as their strategies. Experiments show that real participants often do not make the same choices that game theory predicts: there is often a higher level of cooperation [22, 29, 60, 63]. This suggests that the solution concepts applied are not those that participants actually use.

Furthermore, if we wish to use the models of game theory to explain the development of stable strategies in, for example, biological systems, the Nash solution concept is not a good one. For example, we might wish to describe the behaviour of animals competing for the attention of a mate. The simplest game theoretic model that captures this is the Hawk-Dove or Chicken game [45, 46]. In this game participants either escalate a conflict (hawks) or back away (doves) The payoff matrix is given in equation 2.21 with $C > V$

$$\begin{array}{cc} & \begin{array}{cc} \text{H} & \text{D} \end{array} \\ \begin{array}{c} \text{H} \\ \text{D} \end{array} & \begin{pmatrix} \frac{V-C}{2} & V \\ 0 & \frac{V}{2} \end{pmatrix} \end{array}. \tag{2.21}$$

If we wish to use such a game to model animal behaviour, we need to come up with a solution concept that does not require the participants to act rationally and with perfect information. Evolutionary game theory provides this. The idea is to consider how different strategies will fare under selection in an infinite well-mixed population of players. Rather than being concerned with the equilibrium strategy, we consider the dynamics of strategies under replicator equations.

As an example, consider the Hawk-Dove game. We can think of the payoffs to players as corresponding to fitness gains in a population of individual animals. A hawk encountering a dove in competition for breeding grounds will get a positive payoff and will produce a larger number of offspring: it will have a higher fitness gain. With a few simple assumptions we can write down how the proportion of hawks in a population will change.

Let the initial fraction of hawk strategies be p and let the base fitness of competitors who have not played be w_0. If individuals pair off at random and engage in a single contest the fitness of hawks will be

$$w_h = w_0 + pE(H, H) + (1 - p)E(H, D) \tag{2.22}$$

and that of doves

$$w_d = w_0 + pE(D, H) + (1 - p)E(D, D). \tag{2.23}$$

Where $E(X, Y)$ is the fitness gain to a player of X when they encounter a player of Y. Individuals now reproduce asexually in proportion to their fitness and so the new proportion of hawks in the population will be

$$p' = p\frac{w_h}{pw_h + (1 - p)w_d}. \tag{2.24}$$

The stable strategies are given by the fixed points of equation 2.24. The criteria for stability define an *evolutionarily stable strategy* or ESS.

2.3.1 *Evolutionarily stable strategies*

The concept of evolutionary stability is not a new one: a population is stable if it is resistant to invasion by a mutant purely through natural selection. That is, the mutant has a lower Darwinian fitness than the population. The ESS is then simply a strategy choice which performs as well or better against itself than any mutant while the mutant performs less well against itself. More formally, denote the expected fitness gain of strategy A against strategy B by $E(A, B)$ then A is an ESS if

$$E(A, A) > E(B, A) \text{ or} \tag{2.25}$$

$$E(A, A) = E(B, A) \text{ and } E(A, B) > E(B, B) \tag{2.26}$$

2.3.2 *Mixed strategies in evolutionary game theory*

Evolutionary game theory allows us to define mixed strategies in a much more natural way than our previous definition (see section 2.2.3). Recall the requirement that participants in the game have to draw from a specified probability distribution if they are to correctly follow a mixed strategy. This is much simplified in evolutionary game theory. For example, consider an allele that only allows pure strategy phenotypic expression. If the ESS is a mixed strategy, what happens to the population? In the case of a game with two pure strategies, the stable state of the population is one in which each genetic type appears in proportion to the frequency its strategy is played in the mixed ESS. For example, if the ESS is 'play A 60% and B 40%', the population will have (in its stable state) 60% of type A players and 40% type B players.

2.4 A RETURN TO MODELS OF SALES

Over the years many models of competition and sales have been developed. Prior to Nash's work, these were all analysed in a rather *ad hoc* manner. Nash's general framework, however, makes analysis easier and allows us to precisely define equilibria in the models. In the context of marketplace games, the Nash equilibrium is the state in which all sellers make the same profit[3] and no firm may improve on their situation by varying their strategy without some other firms also varying strategies. That is, if the sellers can adjust their prices, the Nash equilibrium is a set of prices such that any seller changing their price will make a negative profit.

The reason for the proliferation of models is the wish to explain how price setting in marketplaces works. We shall give a brief survey of some representative models and enumerate their successes and failings.

We have already described the price-setting models of Cournot and Bertrand. With an introduction to game theory now behind us, we now go on to consider further models of sales. These models typically concentrate on the empirical observation that in real markets there is no single price for homogeneous products, but rather dispersion of prices. We shall summarise a few of these models and contrast them. One critical thread running through the section is that these models all assume completely rational be-

[3] This is conventionally set to be zero

haviour on the part of participants. Additionally, sellers will always have perfect information of the market. These are quite restrictive assumptions that ideally we would like to relax such that we more closely mimic the real world.

The models we present in the next few sections, and indeed, most analytically tractable models of price dispersion, do not take into account any explicit spatial structure. Search costs imply that travel to sellers to determine prices is not free, however, there is no attempt to factor in different search costs for near and far sellers (for example). The systems considered are all well-mixed. Should a buyer choose a seller at random, they sample uniformly from the global set of sellers, rather than some (local) subset.

Studies of consumer behaviour show that such models are likely too simple to capture real decision making processes. Buyer search costs are not perfectly understood but there is evidence to suggest multiple different factors play a part, including, but not limited to, distance [47], loyalty to a store [51] and buyer personality [23]. We do not claim to address all of these problems in our model, however, the varying of search cost with distance is easily added to mean-field descriptions through the addition of an explicit spatial structure.

2.4.1 *Bargains and Ripoffs (Salop and Stiglitz [61])*

This model introduces an extra element of complexity into the price setting problem in the form of a search cost for buyers. Salop & Stiglitz consider the case of a large number of buyers L and a number of sellers N all offering an identical product. Each seller has a price p_i $(i = 1, \ldots, N)$ and a location l_i. Buyers have complete knowledge of the vector prices $\vec{p}$ and the vector of locations $\vec{l}$, but do not know the mapping between prices and locations. That is, buyers know the cheapest price, but not where to find it. There is a maximum price buyers are willing to pay for the product, this is the monopolist's price p^m.

To gain knowledge of the mapping between prices and locations, a buyer can pay a search cost c_i. Salop & Stiglitz group buyers into two sets, the first set have search cost c_1 (there are αL of these); the second have search cost $c_2 > c_1$ (there are $(1 - \alpha)L$ of these). If buyers pay to search, they obtain the mapping between prices and locations and will buy at the minimum offered price. If they do not search, buyers will pick a random seller. Buyers behave rationally and thus will only pay to search if $p_{min} + c_i < \langle p \rangle$, where $\langle p \rangle = N^{-1} \sum_{i=1}^{N} p_i$ is the expected price of a randomly chosen seller. Furthermore, a buyer will only choose to go shopping *at all* if the total expected cost is less than or equal to p^m.

Sellers also know the vector of prices charged in the system and additionally know the distribution of buyer search costs. These two pieces of information allow them to predict how many buyers will pay to search the market. This information allows a seller i to know its expected demand $D(p_i|\vec{p})$. Every seller attempts to maximise profits by modifying p_i. In doing so, it considers all other prices in the system as fixed (Nash-like price setting) but considers the search *rule* of buyers fixed, rather than the search decision. That is, a seller calculates, when modifying its price, the new number of individuals searching and how this affects its demand. The cost of supplying buyers demanding an amount q is given by $c(q) = d + \nu(q)$ where d is a fixed cost and $\nu(q)$ a variable cost with $\nu'(q) > 0$.

As long as profits are positive, new sellers enter the marketplace (increasing N). This ensures that in an equilibrium state all sellers have equal, zero, profit. Due to the fixed cost independent of demand, this state occurs at finite N. Salop *&* Stiglitz show that the only possible equilibrium states for this market exhibit either a single price or two distinct prices. In the former case, the equilibrium price is either the monopoly price p^m or the competitive price p^p (the latter depends on the search costs of buyers).

The two price equilibrium has some fraction of firms charging a low price, and the others a high price. This must be done in such a way that only consumers paying c_1 find it worthwhile to become informed: the price distribution obeys the inequality $p_{min} + c_1 < \langle p \rangle \leqslant p_{min} + c_2$. In this case, the low price is the competitive price $p^l = p^p$; the high price is bounded above by p^m, $p^l < p^h \leqslant p^m$.

A final 'equilibrium' state the model realises is a limit cycle between the competitive price p^p and some high price p^L. $p^L \approx p^p + c$ if N is large and $c_1 = c_2 = c$. In this regime, an individual seller can achieve positive profits through some change in price. However, when this happens it will be beneficial for a different seller to lower their price to capture buyers by inducing them to pay search costs. Dependent on the exact dynamics of price adjustment, seller prices will then oscillate between p^p and p^L. Salop & Stiglitz are coy on what form this oscillation might take, merely noting that it might appear. The driver for the cycle is a seller deciding to change their price by a large amount to obtain short-term profits. The rest of the market reacts to this and a price war ensues that reduces $\langle p \rangle$ again. In section 6.5 we shall show that our model carries out mean price oscillations that qualitatively capture this picture.

2.4.2 *The theory of sales (Salop and Stiglitz [62])*

This model of Salop & Stiglitz attempts to capture an idealised version of pricing whereby sellers have a product they either sell at an advertised price or at a temporary (unadvertised) discount. Their model consists of buyers who live for two turns needing to use one unit of product per turn. The buyers can obtain these two units in one of two ways. They visit a seller and buy two units in their first turn, paying a storage cost δ to keep the surplus. On the second turn they do not buy anything. Alternatively, the buyer can only buy a single unit in each turn. The buyer must pay a transaction cost c to visit sellers in the second round: we can think of this as a search cost of some kind. Finally, the buyers have a maximum price they are willing to pay, p^m.

Buyers pick a seller at random and thus there is a price $\hat{p}$ at which they are indifferent to buying either two units or one. If the seller has a price $p < \hat{p}$ the buyer will buy two units, if not, only one. This price is given by

$$\hat{p} + \delta = \langle p \rangle + c \tag{2.27}$$

where $\langle p \rangle$ is the expected price of a randomly chosen seller. The paper establishes that in market equilibrium there are either two prices offered – one low price (at which sellers sell two units to 'young' buyers) and one high price – or a single price. The high price is $p^h = p^m$ (the monopoly price), the low price is $p^l = \frac{p^m - (\delta + c)}{2}$. If $c = 0$ and $\delta \geqslant \frac{p^m}{3}$ the only equilibrium is a single price one, with $p = p^m$.

This model again tries to capture how buyers might behave in the absence of complete information. It demonstrates that should buyers be unable to map their knowledge of prices onto locations, price dispersion can result. Note again that the model posits that buyers have a large amount of information at their disposal (the complete price distribution) and that all participants act perfectly rationally. We have already seen (section 2.3) that this assumption of rationality, the so-called *homo economicus*, is likely a bad one in the context of real people.

2.4.3 *A model of sales (Varian [73])*

Varian considers the case of sellers pricing to some informed and some uninformed consumers in which sellers can vary their price in time by choosing each turn a price from some probability distribution. This is an attempt to characterise the phenomenon of sales whereby sellers occasionally offer goods at a cheap price to attract more customers.

Varian considers a system in which there are I informed and M uninformed buyers. There are a total of N sellers. Buyers, if they are uninformed, visit a seller at random. Each seller therefore attracts $U = M/N$ uninformed buyers. Sellers choose a price on a daily basis from a probability distribution $f(p)$. $f(p)$ is the pricing strategy for these sellers: they follow Nash-like price setting in that they maximise their profits conditioned on the known pricing strategies of all other sellers and knowing the number of informed and uninformed buyers. There is no *a priori* reason to assume that each seller should have the same pricing strategy, however, Varian's model chooses all $f(p)$'s equal. Buyers have some maximum price they are willing to pay, this sets the monopolist's price p^m. Sellers have a cost function for supplying goods to the buyers, $c(q)$ which is non-decreasing in q. The minimum price sellers will charge is thus given by the average cost of supplying $I + U$ buyers (the maximum a seller can expect to attract): $p^p = \frac{c(I+U)}{I+U}$.

Varian goes on to show that if $N \to \infty$ and $c(q) = k \; \forall q$, prices will be distributed according to

$$f(p) = \begin{cases} \frac{1}{p(1-p/p^m)} & p \in (p^p, p^m) \\ 0 & \text{otherwise.} \end{cases} \tag{2.28}$$

A plot of this is shown in figure 2.2.

Again, we note the common criticism of these models: namely, sellers must act perfectly rationally and be endowed with complete information of the market. It would perhaps be preferable to construct a model that does

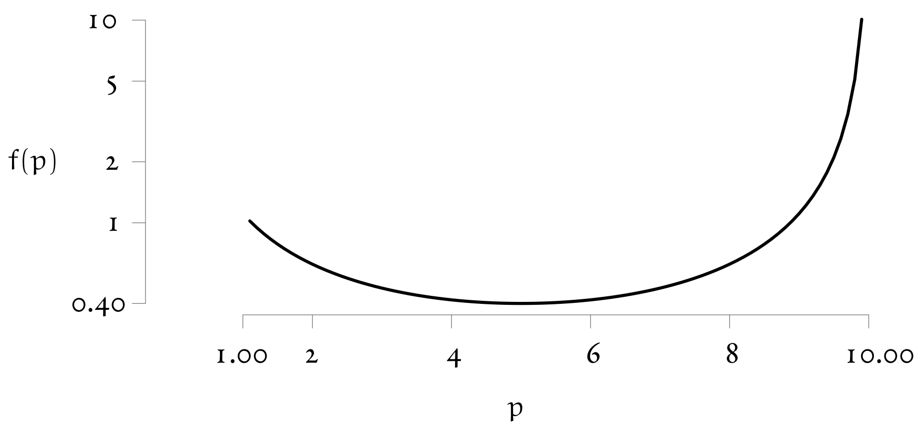

Figure 2.2 *Seller price distribution in Varian's* Model of Sales. *This is equation 2.28 with* $p^p = 1$ *and* $p^m = 10$. *There is a logarithmic divergence at the upper value, indicating that as the number of sellers becomes large, the probability of charging anything other than the monopolist price goes to zero.*

not put such strict requirements on sellers. Furthermore, the log-divergence in the price distribution (equation 2.28), indicates that in large systems, the model behaves as if there were only a single price.

2.4.4 *A short summary*

These three models are by no means the only proposed methods of reaching an equilibrium state with price dispersion. Burdett and Judd [12] propose a model in which buyer search samples a subset of all prices, but all buyers are equivalent, and show that price dispersion can result. With the advent of readily available computing resources, more complicated models have also been studied that allow for evolution of optimal strategies [36, 44] using genetic algorithms. Such models are typically studied under the banner of agent-based computational economics (see [70] for a review). Typically these models require detailed specification of individual players and study the behaviour of groups of individuals. A criticism of this approach is the specificity of the models: each new market must be modelled differently and it is difficult to draw generic conclusions from the model. In developing our model we shall suggest that there is a happy middle ground to be found. It is possible to construct a model that is simple and yet does not rely on complete rationality of individuals to produce price dispersion.

All the models we have described so far are mean field-like in their construction. Sellers are aware of the entire distribution of competitor prices; buyers are aware of the all the sellers. When buyers choose at random, they pick from a well-mixed population, such that the mean price they observe is the global mean. Equally, when they search, they can search exhaustively and thus will pick the globally cheapest price.

One feature that these models throw away by design is the ability to differentiate between participants purely on account of their spatial position in the system. That is, there is no concept of different sellers competing against different opponents due to spatial locality. We shall now consider some simple models where such an approach throws away vital information about the system. In these models, microscopic simulation of the dynamics on a spatial playing field produces behaviour very different from that suggested by mean field analyses. In light of these results, we argue that in the case of price competition a mean field approach may well not be ideal.

2.5.1 *The prisoner's dilemma*

One simple game theoretic model that has been studied in detail in a spatial setting is the prisoner's dilemma. The zero-dimensional version of this model was first studied by the RAND cooperation in the 1950s as a simple model of the arms race between the USA and USSR, it first appears in a now unfamiliar form in Flood [22]. The formulation in terms of two prisoners is due to Tucker [42].

Two criminals are caught at the scene of a crime and are each, independently[4], offered a deal. They may choose either to implicate their partner (*defection*), or remain silent (*cooperation*). The maximum possible sentence for this crime is ten years. The payoff in the game corresponds to a reduction in the length of incarceration. A typical payoff matrix for the game is shown in equation 2.29. So, for example, if one player defects and the other cooperates, the first gets a prison sentence of $10 - T = 0$ years, the second a

[4]That the offers are independent is crucial: the game setup assumes that the prisoners cannot communicate and arrange to cooperate for mutual benefit

sentence of $10 - S = 10$ years.

$$\begin{array}{cc} & \begin{array}{cc} \text{C} & \text{D} \end{array} \\ \begin{array}{c} \text{C} \\ \text{D} \end{array} & \begin{pmatrix} R = 9 & S = 0 \\ T = 10 & P = 6 \end{pmatrix}. \end{array} \tag{2.29}$$

Here C and D are respectively cooperate and defect. R is the reward for cooperating, T the temptation to defect, P the punishment for mutual defection and S the sucker's payoff for cooperating in the face of defection. The general form of the game has

$$T > R > P > S \tag{2.30}$$

and

$$R > \frac{T + S}{2}. \tag{2.31}$$

The latter requirement is to ensure that the mutual cooperation state is more beneficial than playing a mixed strategy of C and D each with equal probability. With these two requirements satisfied, the Nash equilibrium is for both players to defect. This is not an optimal solution, were they to cooperate, both prisoners would receive a short (one year) prison sentence as opposed to the longer (four year) sentence they receive for mutual defection.

In an analogy with the models of price competition we have presented, if sellers were able to cooperate (by agreeing to charge some high price) they would obtain a much higher payoff than mutual defection (price undercutting until the competitive price is reached). Proposed extensions to the prisoner's dilemma allowing cooperative strategies to flourish suggest methods whereby the same could occur in price setting games.

2.5.1.1 *The prisoner's dilemma and altruism*

The prisoner's dilemma can be seen as a conceptualisation of the problem facing altruists (cooperators) in evolution. Darwinian selection only acts at the individual level and so altruistic behaviour which reduces individual fitness (but increases group fitness) cannot be selected for. In the context of the prisoner's dilemma the problem is easy to see. Consider a population of individuals composed entirely of defectors, a single mutant cooperator will perform less well and will be selected against. Similarly, in a population of cooperators, a single defector performs better and will be preferentially selected for in an evolutionary setting. We note that recent work suggests that the prisoner's dilemma is not the most appropriate model in some circumstances (see for example [16, 77]). For the purposes of our exposition, however, it is a simple and useful choice.

We might argue that this is a moot point: that altruism in humans requires a high degree of reasoning not captured by the prisoner's dilemma. If, however, we assume that animals are not in possession of such reasoning ability the problem of cooperation is still extant. A proposed answer to this problem comes in the iterated prisoner's dilemma [2]. If a game is played multiple times against the same opponent then defection is no longer always a dominant strategy: the repeated interactions can support cooperative outcomes [1]. This model does not provide the whole story, see for example [77] and [20] for short reviews of some models for the generation of cooperative behaviour. Our interest in it here is as an inspiration for our marketplace model: repeated competition events allowing for richer behaviour than a one-off interaction.

The iterated prisoner's dilemma came to prominence in the early 1980s with a computer tournament organised by Axelrod and Hamilton [2]. Axelrod invited submission of 'players' – algorithms for choosing whether to defect or cooperate – to a computer tournament in which said players would compete pairwise in an iterated prisoner's dilemma game.

Players competed in this pairwise manner with the number of rounds determined by a parameter w setting the probability of meeting again. That is, two players would play for one round with probability one, for two with probability w, for three with probability w^2 and so forth. Players were aware of the history of moves and could use this information to choose a best response in the next round. All strategies submitted to the tournament played each other in a round-robin fashion and were subsequently ranked according to their average score. In this non-evolved tournament, the best performing strategy was one which demonstrated some elements of altruistic behaviour. Named tit-for-tat, the idea was among the simplest submitted: the strategy would cooperate on first meeting a new opponent and in subsequent rounds simply play the opponent's previous move. Axelrod showed [2] that tit-for-tat was an evolutionary stable strategy (*i.e.*, uninvadable by a mutant) if

$$w \geqslant \frac{T - R}{T - P} \text{ and}$$
$$w \geqslant \frac{T - R}{R - S}. \tag{2.32}$$

Why does tit-for-tat demonstrate altruism? The claim is that tit-for-tat never makes the first escalatory move in a conflict, and furthermore is easily willing to forgive, signs of reciprocal altruism. Clearly, however, such a strategy does worse against a population entirely consisting of defectors than another defector, due to losing in the first round. Tit-for-tat does not explain how a single altruist can evolve to stability in a hostile population. Further, real life situations rarely consist solely of pairwise interaction and punishment at this level.

Nowak and May [57, 58] propose a solution to this problem. They carry out a simulation of the prisoner's dilemma on a square lattice. Individual sites of the lattice adopt a single strategy which they play against all eight neighbours (figure 2.3). The game is played multiple times and sites change their strategy at the beginning of each round. An evolutionary update rule is used: sites compare the score of their neighbours and themselves and adopt the strategy which obtained the highest score in the previous round.

The steady state obtained in this model is dependent on the relative values of the payoffs. If the system consists purely of cooperators, a single defector will have the best score and will locally replicate. However, once multiple defectors sit in a cluster together, they do not necessarily perform better than cooperators. For instance, consider the configuration of sites shown in figure 2.3. As the figure shows, the payoff of a player in the game is dependent on its neighbourhood. It is not a given that an all-defecting state will dominate. In fact, choosing $S = P = 0$, $R = 1$ and $T = b > 1$, Nowak & May find three distinct regions. Clusters of defectors shrink for $b < 1.8$, clusters of cooperators shrink for $b > 2$ and cooperators and defectors coexist for $1.8 < b < 2$ [58]. Thus it seems, at least for this system, altruism can be a viable strategy: although the level of altruists is not particularly high, around 32% of the population. Further, altruists are still highly invadeable by cheaters, it is just cheaters doing badly against one-another that stops them invading entirely.

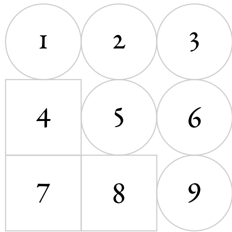

Figure 2.3 *Diagram showing nine sites in a spatial prisoner's dilemma game. Circles are defectors, squares are cooperators. The remaining sites in the system are all cooperators. The best performing defectors are at sites one and nine: they get payoff* $3P + 6T$. *Sites two, three and six get payoff* $4P + 5T$, *site five gets payoff* $6P + 3T$. *The cooperators at sites four and eight get* $3S + 6R$ *while site seven gets* $S + 8R$. *If* $S + 8R > 3P + 6T$ *then site five will switch to a cooperating state, similarly for sites one and nine.*

2.5.2 *Spatial patterning in ecological systems*

In observations of natural systems, we often observe spatial clustering and patterning of individuals within a reasonably homogeneous background, see [15] for one particularly striking example. In the following short example [81], we consider a system that undergoes no evolution of strategies, just birth and death of individuals. This model also demonstrates how a mean-field analysis can produce 'incorrect' results.

Consider a system of diffusing particles with number density $n(x,t)$, diffusion constant D, dieing at rate μ and reproducing asexually at rate λ. If n is large we might write down a continuum approximation, the advection-diffusion-reaction equation:

$$\partial_t n = D\nabla^2 n + (\lambda - \mu)n. \tag{2.33}$$

If the initial population of particles is uniform everywhere and with $\lambda = \mu$ then $n(x,t) = n(x,0)$. That is, the population density remains constant in space. Microscopic simulations of such a process show that this does not occur. Instead, typical results show strong clustering of particles. This occurs because death acts everywhere uniformly, but birth of particles is local, introducing correlations. If the diffusion is slow, then the random-walking of particles will not destroy all the correlations in their positions before another reproductive event occurs, leading to patchiness. The noise term ignored in the mean field model becomes important in specifying behaviour in individual realisations of the system. We will carry this idea into our model of a marketplace: allowing some players to withdraw from the game temporarily will lead to fluctuations in the prices buyers observe (even if they can search the entire market).

2.6 MOTIVATION FOR CONSIDERING SPATIAL MARKETPLACE MODELS

As both the examples of a section 2.5.2 (spatial birth-death processes) and section 2.5.1 (a spatial prisoner's dilemma game) show, if there is an element of spatial locality to a process along with reproduction and death, explicitly modelling the system and solving the continuum mean-field equations can lead to widely different results. With this in mind, especially given the influences on economic theory that evolutionary game theory has had, it seems somewhat surprising that there is a lack of literature dealing with spatial analogs of mean-field-like marketplace games.

This is perhaps a result of preferring to consider much more complicated virtual-world models in simulation studies, the simple models having already been solved. However, as the short examples above show, the details of microscopic simulation – which are typically more life-like than mean-field assumptions – need not be irrelevant. The physicist's mentality, that the large-system limit is the interesting case strikes a happy medium here. We can think of the simple game-theoretic models as corresponding to mean-field analyses: in these cases, the system is essentially zero-dimensional and the mean field limit is valid (since there is no space for spatial structure to exist in). At the other end of the scale are complicated agent-based simulations with a large number of parameters that must be obtained by calibration against data (for example models of disease spread such as foot and mouth [35] and SARS [21]). The aim of these models is to provide realistic quantitative predictions and guidance for policy makers.

Such approaches, by design, do not explore the full phase space of the models. Rather, they suggest the likely best course of action in an outbreak. Due to the modelling methods, it is not always possible to claim with any certainty that the demonstrated predictions are the 'best' result: 'what if?' questions are hard to answer.

Given these issues, it seems useful to be able to construct models which interpolate between the two regimes: agent-based models which have only a small set of adjustable parameters. In this way, we are able to study how spatially structured models deviate from mean field results in a manner which allows understanding of why they do so. Clearly, there are drawbacks to this approach. The results we get will not likely provide testable quantitative predictions: there are too many factors that will have been thrown away. However, we may find that we can make qualitative claims about more complicated systems. Instead of guessing at parameters which *may* influence the behaviour of a system, we can construct a model that studies if one *particular* parameter changes the behaviour.

2.7 SUMMARY

To sum up, there are many theoretical models of seller competition, each adapted to a particular interpretation of marketplace dynamics and studying slightly different situations. The ideas of informed and uninformed consumers are a step towards full spatial separation in the models and demonstrate that it is possible for multiple different prices to exist in the equilibrium state. All these models share the Nash equilibrium feature of equal profits amongst all firms (conventionally this is set to zero in analysis). This is clearly a drawback if we wish to explain price-setting in the real world where profits can vary between sellers.

We have also seen that mean-field models can produce strikingly different results when spatial heterogeneity is added: local correlations become important. Given results of this kind in other fields, we suggest that the addition of explicit spatial structure to zero-dimensional marketplace models presents an interesting line of enquiry. Such a model should attempt to bridge the gap between the simple game theoretic analyses and complicated multi-parameter 'computer worlds'. In the next chapter we present our model in detail, drawing on the ideas presented here.

AN EVOLVING PRICE-SETTING MARKET

3

A common criticism running through the models of price setting we described in chapter 2 was the need for perfectly rational participants to obtain the equilibrium state. A further issue, raised in section 2.5, was the mean field like property of the models: in the presence of flucutations, such an analysis may not provide the whole picture. In this chapter we construct a simple model of price setting to address these concerns.

3.1 OVERVIEW

Our model is a simple spatial extension to Bertrand's model of competition. We consider N sellers all producing an identical product. There are many buyers, each aware of just two sellers. As in Bertrand competition, sellers compete on price: a buyer will choose the cheaper of two sellers. The set of links created by the pairwise competition between sellers for individual buyers is encoded in a competition network. Sellers are not able to vary their production quantities: they always produce enough to satisfy all potential buyers. Poorly performing sellers are removed from the system and replaced with new sellers. These new sellers pick a new price to compete with.

3.2 SPECIFICS OF THE MODEL

3.2.1 *The competition network*

Most of the results we consider will be for sellers arranged on a regular network – we will show that consideration of different networks produces the same results given our choice of dynamics. The regular network we choose is an n-dimensional square lattice. Sellers are placed on the sites of the lattice and compete pairwise for buyers (represented by the bonds of the lattice). Rather than making sites at the boundary special, we use periodic boundary conditions. Figure 3.1 shows the network structure for the two-dimensional case.

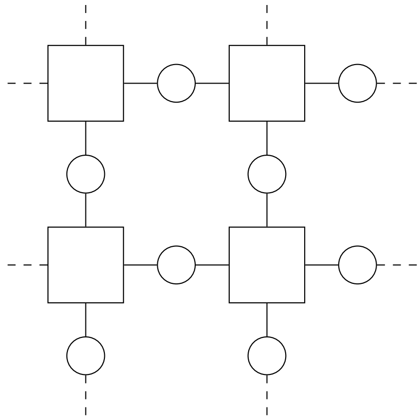

Figure 3.1 *Schematic of buyer/seller spatial structure in the two dimensional case. Sellers are shown as squares, buyers as circles; links are shown as lines with implied periodic boundary conditions*

3.2.2 *The characteristics of buyers and sellers*

Sellers exhibit a single external feature which attracts buyers (or not): we shall call this the price. This deserves some elaboration. In a real world situation, buyers do not only consider the price of goods when they go shopping [23, 51]. Factors such as the levels of service, quality of product and many more play a rôle. However, this study does not profess to be an in-depth analysis of buyer behaviour. We therefore assume that buyers are all reasonably alike and thus will settle on a similar metric when evaluating their shopping options. Sellers may adjust a number of different internal factors in production, but this translates to a change in a single variable as far as their attractiveness to buyers is concerned. In the model of Bertrand competition this is the price and we retain the nomenclature here.

Buyers are constrained to visit a small subset of all N sellers. The motivation behind this choice is that, for many products, a buyer does not exhaustively visit every seller in town before making a choice. The cost of searching the space of sellers is too high. We therefore choose a very simple search cost. For a finite number of sellers the search cost to a given consumer is zero, for all others the search cost is infinite. Within this finite number of sellers, a buyer will always visit the cheapest seller; should sellers exhibit identical prices, a buyer will choose randomly between them. Buyers have a fixed demand which we set to one unit.

Sellers are assumed to know the maximum the number of buyers they can attract. They produce enough to satisfy the demand of all potential buyers, with a production cost of one per unit. Thus, a seller with three potential buyers will have an overhead of three units. This sets the competitive Bertrand price to $p_0 = 1$ in the equivalent pairwise competition game. Note that sellers have a fixed cost and so a seller with $p = p_0$ must sell to *all* its potential buyers to avoid making a loss.

3.2.3 *Identifying poorly performing sellers*

Our sellers do not perform Nash-like price-setting (choosing a price by assuming rationality of all participants). Instead, they choose a price and keep it until they are identified as performing badly. When this occurs they leave the game temporarily. Upon reentry they pick a new price. The details of the reentry and price-picking step are described in section 3.2.4, first we must decide how to pick out poorly performing sellers.

The metric we choose to define poorly performing sellers is the capital. This is the sum of a seller's profits (or losses), the capital update is detailed in section 3.3. If the capital of a seller is negative we consider the seller to be performing poorly and remove it from the system. Note that this is the only form of seller removal in the system. Unlike other evolutionary models, we do not impose an exogenous death rate on sellers. The removal of sellers emerges purely from the dynamical rules.

3.2.4 *Reentry of sellers*

Bankrupt sellers in the system do not pay any overheads and are not visible to buyers. At these vacant sites, a new seller can enter the game with capital set to zero and a new price. This reentry occurs with probability $\gamma \in (0, 1]$, set by the initial conditions. Note particularly, for $\gamma < 1$ a site may remain vacant for multiple selling rounds. Those sellers in competition with this vacant site will be in a monopoly situation: the buyers will have no choice in their shopping destination.

When the new seller chooses a new price, they do not pick from the distribution set by the initial conditions. Instead, we allow a newly entered seller to observe the distribution of prices exhibited by all non-bankrupt sellers. Again, we choose an evolutionary approach when picking a new price. The new seller chooses uniformly at random from the set of non-bankrupt sellers (*i.e.*, they choose a price from the recently observed price distribution). The seller's new price is obtained by copying this random price with some noise.

As an example, consider a vacant site i. A new seller arrives at this site with probability γ and takes a price $p_i(t+1) = p_{j\neq i}(t) + \eta$ where j is chosen uniformly at random from all non-bankrupt sellers in the system and η is a noise term. The details of this update scheme are given below in section 3.3.1.

We can see that this update scheme makes no strong assumptions on which strategies might be good ones. It only assumes that surviving strategies are better than ones that are not (this seems reasonable). Unlike Nash-like behaviour in which sellers would update their price to maximise profits, choosing a price from the set of all live sellers is equivalent to minimising the probability of bankruptcy. We can think of this a bit like a probabilistic minimax procedure, minimising the maximum loss.

Although this goes against the ideas espoused in game theoretic settings of *homo economicus*, a completely rationally acting participant in a game, this is not necessarily a bad thing. Said metaphor has been shown in a number of real world experiments to be a bad fit to actual human behaviour. Even in cases where the game is small enough and simple enough that players do have perfect information, they typically do not behave as game theorists would predict (*c.f.* section 2.3 and Flood [22]). Indeed, some economists suggest a need to move away from *homo economicus* to a more human participant in models [40, 71].

3.2.5 *Problems facing a rational price-setter*

A further reason for not implementing Nash-like behaviour in our sellers is the stochastic nature of competition. At any one time, some number of a seller's competitors will be bankrupt. The profit-maximising strategy for a seller will vary according to how many bankrupt opponents it encounters. This makes the strategy optimisation problem hard. Consider for example the case depicted in figure 3.2. If seller one and buyers A and C do not exist, the problem reduces to the Bertrand case (assuming that both sellers are always active). However, if seller one and buyer A start trading, the optimisation problem seller two must solve is significantly harder. Consider

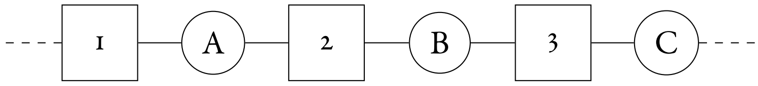

Figure 3.2 *Part of a one-dimensional lattice, sellers shown as orange squares, buyers as blue circles. This figure illustrates the optimisation problem facing a seller when choosing a price: see text for details*

a situation with the prices of sellers being given by $P_1 = 4$, $P_3 = 2$, with a cost of production $D = 1$ per buyer. The profit/loss made by seller two is $\Delta c = \theta(P_1 - P_2)P_2 + \theta(P_3 - P_2)P_2 - 2D$. There are multiple strategies which allow for the same profit, however, the maximum profit is not made (as one might initially assume) by undercutting both sellers, but rather (in this case) by only undercutting seller one. The former strategy, with $P_2 = P_3 - \epsilon$ gives $\Delta c = 2 - 2\epsilon$: the latter has $P_2 = P_1 - \epsilon$ and $\Delta c = 2 - \epsilon$. Figure 3.3 shows the profit-loss curve for different strategy choices by seller two.

Figure 3.3 *Profit (Δc) as function of price for seller two with $P_1 = 4$, $P_3 = 2$, $D = 1$ for the three firm situation described in text and illustrated in figure 3.2*

The problem becomes even more difficult when $\gamma \neq 1$, in this case, the best (profit-maximising) strategy can only be determined *post facto*. A seller cannot predict through rationality assumptions when its competitors are bankrupt, since the state of bankruptcy is not a strategy choice, and thus cannot choose the best response strategy to a dead competitor (namely to charge as much as possible). We might think that, given that an opponent will sometimes be dead, to maximise the expected profit a seller should set $p = \infty$. This is not a viable strategy. An opponent's demise is in part related to whether they are able to outcompete us. With a price of $p = \infty$, the opponent will likely never be bankrupt, since they may charge $p < \infty$ and make a large, finite profit. Note that a seller cannot react to an opponent's bankruptcy by changing price: Nash price-setting takes the strategy of an opponent as fixed and maximises profit subject to this constraint. Sellers are not able to observe the bankruptcy state of their opponents until after they have made their strategy choice, they cannot therefore adjust accordingly.

3.2.5.1 *A simplified Nash equilibrium analysis*

If we make the standard economic assumptions of a well-mixed system, rationality and perfect information it is possible to find the Nash equilibrium price distribution. We do so here to later compare the result with our spatially separated model. In our model, individual sellers are constrained to a single, fixed price (pure strategies in the parlance of game theory). However, we know from section 2.3.1 that the mixed strategy Nash equilibrium, if it exists, will be exhibited in the ensemble average of the population of pure strategies. Thus we posit that the distribution of prices observed in the system will resemble the Nash equilibrium of the non-spatial game. We now wish to find the Nash equilibrium strategy.

We assume there is a distribution of prices $f(p)$ and impose that $f(p)$ is defined on the interval $(-\infty, \infty)$. In a Nash equilibrium, all prices have the same payoff. A seller with $p = 0$ has payoff $-d$ and so we include all other prices with payoff $-d$ in our distribution. The payoff to a seller with price p is given by

$$\pi(p) = p \int_p^\infty f(p')dp' - d. \tag{3.1}$$

This must be equal to $\pi(0)$ for all p and so

$$\int_p^\infty pf(p')dp' - d = 0 \int_0^\infty f(p')dp' - d. \tag{3.2}$$

$f(p)$ is a probability distribution and so the integral on the right hand side of equation 3.2 if finite. We thus find that

$$p \int_p^\infty f(p')dp' = 0. \tag{3.3}$$

If $p < 0$, the left hand side of equation 3.3 is negative, ruling out any negative prices in our equilibrium. If $p > 0$ the relation only holds if the integral is zero: this is only possible if $f(p)$ is a δ-function at the origin. Hence the Nash equilibrium price distribution is

$$f(p) = \delta(p). \tag{3.4}$$

Note how this is different from the Bertrand equilibrium, for which we have $f(p) = \delta(p-1)$. This is due to the constant fixed cost in our model. In Bertrand's model, sellers can produce on demand and pay no costs if they experience no demand. In our model, sellers experience costs even if they have no demand. The difference leads to the discrepancy in the two Nash equilibria.

3.2.6 *The mutation term*

One part of the model we need to be careful with is the choice of noise term. The size of the noise controls how related new sellers are to their parent seller. When the mutation is small, offspring are very close in strategy choice to their parent; when the mutation is large, the parental strategy is only weakly correlated with the child strategy. In the former case, fitness-based selection advantages are passed from parent to child, while in the latter they are not. Clearly then, if we want our model to provide us with meaningful evolutionary results, we must choose a sensible value for the noise parameter. In terms of evolutionary behaviour, and interpreting the noise term as a mutation, we think of the noise being a small effect. We shall adopt this approach here. We take the noise, η, to be distributed uniformly at random in a small interval centered on zero: $\eta \in [-\frac{\Delta}{2}, \frac{\Delta}{2}]$. Δ is a free parameter which must be set in the initial conditions. We find that a small value of Δ requires $\Delta \ll p_0 = 1$.

3.3 IMPLEMENTATION OF THE MODEL

We have described the individual components of our model and given an overview of how they fit together, but have yet to specify any details of the update scheme for the dynamics. We do so in this section.

First, we enumerate the properties of buyers and sellers. There are N sellers labelled $i = 1, \dots, N$. Each seller has an unvarying price, $p_i \in (0, p_{max})$[1], capital c_i and a binary state variable a_i indicating if the seller is alive. If a_i

[1] We only restrict prices to a finite interval in the initial conditions of the system. Once the simulation starts, there is no upper limit on the price of a seller.

is true the seller is alive, if false the seller is bankrupt. There are N_b buyers, this number varies with the specific competition network we consider, each with unit demand. Sellers each produce enough to satisfy their entire local demand at a cost of one per unit. Denote the demand a seller could experience by d_i. After a round of selling, if the seller attracts k_i buyers, its capital is $c_i(t+1) = c_i(t) + p_i k_i - d_i$.

3.3.1 *Update schemes*

We now consider how to update our model system. We have broadly two choices: discrete time (synchronous) updates or continuous time (asynchronous) updates. In a synchronous update scheme, everything in the simulation happens in lockstep and all participants have an equal number of trading opportunities. In an asynchronous scheme, participants are updated stochastically and will not always experience an equal number of trading opportunities. We shall detail the update algorithms for both methods in turn.

3.3.1.1 *Synchronous updates*

Under synchronous updates, each seller always experiences exactly the same steps in the same order. The update scheme for the whole system is detailed in algorithm 1. We can see how this corresponds to synchronous updates. Every seller and buyer is updated before moving on to the next stage of the dynamics.

Note that under these dynamics, sellers are never able to make a profit with $p_i < p_0$. Sellers are aware of this, and so for synchronous updates we restrict seller prices to $p_i \geqslant p_0$. We assume that sellers will never charge a price that is guaranteed to make a loss. To set this up in our system, we set a lower bound on the initial prices: $p_i(t=0) \in [p_0, p_{max})$. In the price update step we ensure that we never set $p_i < p_0$ by updating prices according to algorithm 2.

3.3.1.2 *Asynchronous updates*

Asynchronous updates are equally simple to implement, however, we must put more care into deciding in exactly which manner we make the algorithm asynchronous. We choose to separate the timescales for selling and

for $i = 1$ to N **do**
  $c_i \leftarrow c_i - d_i$ *Seller* i *pays overhead*
end for
for $i = 1$ to N_b **do**
  Buyer i visits cheapest live seller (j)
  $c_j \leftarrow c_j + p_j$ *Seller* j *makes a sale*
end for
for $i = 1$ to N **do**
  if $c_i < 0$ **then**
    $a_i \leftarrow$ **false**
  end if
end for
for $i = 1$ to N **do**
  if a_i is **false** and with probability γ **then**
    $c_i \leftarrow 0$
    SYNCHRONOUS-PRICE-UPDATE(i) *See algorithm 2*
    $a_i \leftarrow$ **true**
  end if
end for

Algorithm 1 *The dynamics of the system under synchronous updating*

$j \leftarrow r$ j *a random live seller*
repeat
  $p_i \leftarrow p_j + \eta$ $\eta \in [-\frac{\Delta}{2}, \frac{\Delta}{2}]$ *a noise term*
until $p_i \geqslant p_0$

Algorithm 2 SYNCHRONOUS-PRICE-UPDATE(i): *Price update algorithm to ensure that* $p_i \geqslant p_0$

bankruptcy. We would expect the former activity to take place much more rapidly than the latter in a real firm and our asynchronous update scheme models this intuition. To do this we make the buying and selling part of algorithm 1 asynchronous, but carry out reentry to the market in a synchronous manner. The update algorithm for our choice of asynchronous dynamics is shown in algorithm 3.

Unlike the synchronous update case, under asynchronous updates, sellers are able to make a profit with $p_i < p_0$. Sellers are aware of this and so the price update stage just ensures that $p_i > 0$. The algorithm for performing asynchronous price updates is shown in algorithm 4.

for $i = 1$ to N **do**
  $j \leftarrow r$ *j is a seller chosen at random*
  $c_j \leftarrow c_j - d_j$
end for
for $i = 1$ to N_b **do**
  $j \leftarrow r$ *j is a buyer chosen at random*
  Buyer j visits cheapest live seller (k)
  $c_k \leftarrow c_k + p_k$ *Seller k makes a sale*
end for
for $i = 1$ to N **do**
  if $c_i < 0$ **then**
    $a_i \leftarrow$ **false**
  end if
end for
for $i = 1$ to N **do**
  if a_i is **false** and with probability γ **then**
    $c_i \leftarrow 0$
    ASYNCHRONOUS-PRICE-UPDATE(i) *See algorithm 4*
    $a_i \leftarrow$ **true**
  end if
end for

Algorithm 3 *The dynamics of the system under asynchronous updates*

$j \leftarrow r$ *j a random live seller*
repeat
  $p_i \leftarrow p_j + \eta$
until $p_i > 0$

Algorithm 4 ASYNCHRONOUS-PRICE-UPDATE(i): *Price update algorithm to ensure that* $p_i > 0$

3.4 ANALOGY TO A MODEL ECOLOGY

Our system, although couched in terms of a simple marketplace, has an appealing interpretation as a model for a simple ecology. We can think of the sellers as stationary organisms (plants) competing for a finite resource (the buyers) which they can store. Plants compete locally but reproduce via seed dispersal (global reproduction). The phenotype exhibited by the plants corresponds to the pricing strategy of sellers. The 'price' of a plant tells us how good it is at both gathering and storing the resource. A low price corresponds to a plant that is excellent at acquiring resources but poor at storage and vice versa.

Note that this is not a strong analogy, we make no claim to be modelling anything close to a real plant. To take but one example, organisms spring fully-fledged from seed: this seems an unlikely state of affairs in a forest. In the remainder of this thesis we shall predominantly use the economic parlance for our system, occasionally (where it makes most sense) we shall draw parallels with the ecological model.

3.5 DIFFERENCES FROM EXISTING MODELLING TECHNIQUES

It is worth pointing out a few key ways in which our model differs from existing models. The idea of an evolutionary approach to finding steady states in toy economic systems is not new (see section 2.3), however, we diverge from standard evolutionary models in that death (bankruptcy in our system) is not an *a priori* assumption of the system. A standard evolutionary model would have bankruptcy occurring as a point process in the system with rates possibly dependent on the strategy choice and seller capital. In our model we only track the capital of a seller and remove the seller deterministically when it drops below an externally specified level. Thus, two externally identical sellers (they exhibit the same strategy) will not necessarily experience the same death rate. The model also differs from spatial prisoner's dilemma games [32, 57, 59] in which both competition *and* strategy copying occurs locally. In our system, competition is local but strategy copying is global.

In the next chapters we present results from model simulations demonstrating that this simple economic system with only one real free parameter (the reentry probability) exhibits non-trivial steady state behaviour. Including, but not limited to, dispersion of prices in the marketplace.

4 SYNCHRONOUS TIME SIMULATION RESULTS

Our first foray into the results of the model considers the synchronous time implementation. We choose this implementation first since it is easiest to perform some validation of the model: it is easy to specify the expected steady state behaviour in the $\gamma = 0$ and $\gamma = 1$ limits and we can check these against our simulation. Here, we consider the one-dimensional case – this is to make analysis of the situation simpler. In chapter 5 we will show how the results extend to higher dimensions.

The chapter has a somewhat exploratory motif. We first show that our model does indeed admit price dispersion in the steady state, one of the main aims in its construction. Our next step is to show explicitly how dispersed prices emerge in the early time dynamics. This section also demonstrates that exact analysis of the steady state will be very difficult. Having derived the early time properties, we build a more qualitative picture of the steady state's properties. Finally we show when the dispersed price steady state breaks down and how the system behaviour changes over this transition.

4.1 DOES OUR IMPLEMENTATION WORK?

We begin discussion of the simulation results with some validation of the implementation. If we set $\gamma = 1$, we know sellers are always in competition with all their neighbours. This means that a high priced strategy will always be uncompetitive. Therefore we expect the $\gamma = 1$ steady state price distribution to be a δ-function at $p = p_0 = 1$ (modulo small deviations due to noise in the system). If we obtain a different steady state distribution, we know that the implementation of the model is somehow incorrect.

In the $\gamma = 0$ limit, we should find a steady state determined only by the initial conditions. In this case, once a seller finds itself in a profit-making position, that will never change. The spatial distribution of sellers and their prices will thus be fixed after the first round: seller capital will either remain constant (at $c = 0$) or rise by a fixed amount every round.

Simple tests show that these two limiting cases are indeed achieved. With $N = 10^5$, $\gamma = 1$ and $\Delta = 0.08$, the steady state price distribution is essentially a δ-function at $p = 1$ as shown in figure 4.1. The step observed in the distribution is due to an implementation detail. Recall that sellers never adopt a strategy that is guaranteed to make a loss, *i.e.*, they always have a price $p \geqslant p_0 = 1$. In our simulation, this is carried out by performing price updates according to algorithm 2. This update scheme produces the step we observe in the price distribution. Reducing the size of the noise term (decreasing Δ) results in a corresponding decrease in the width of the step. For $\Delta \to 0$ the price distribution becomes a true δ-function at $p = 1$ (figure 4.1).

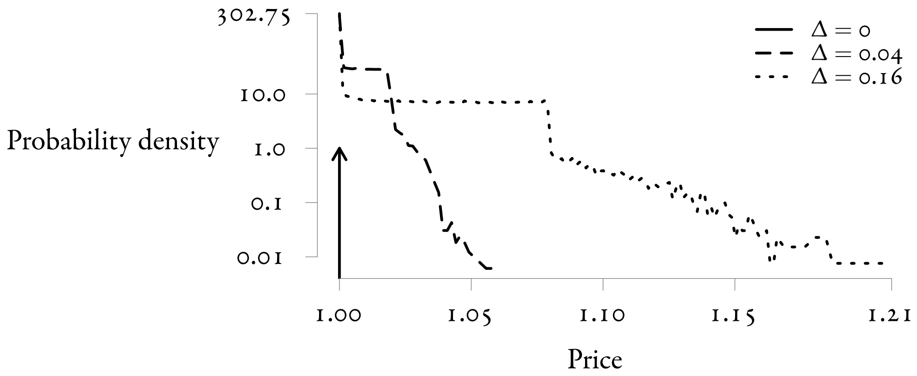

Figure 4.1 *Probability density function of the steady state price distribution for* $\gamma = 1$, $N = 10^5$ *with* Δ *as specified. The step observed above* $p = 1$ *is explained in the text*

4.2 EVIDENCE OF A NON-TRIVIAL STEADY STATE

We recall the primary aim in constructing this model, namely to construct a system which exhibits price dispersion in its steady state by considering a large number of homogeneous participants. The $\gamma = 1$ steady state does not show price dispersion (other than due to mutation noise); the $\gamma = 0$ state does show price dispersion, but we have argued that this is only due to the initial conditions.

Simulation of the model with $\gamma = 0.5$ shows a steady state which does admit price dispersion (figure 4.2). Moreover, the distribution of prices we obtain is highly non-trivial, it is evidently not a simple distribution such as an exponential, Gaussian, or Boltzmann-like. The origins of this complicated structure are not immediately obvious. This confirms our suspicion that spatial heterogeneities could lead to price dispersion. Having demonstrated

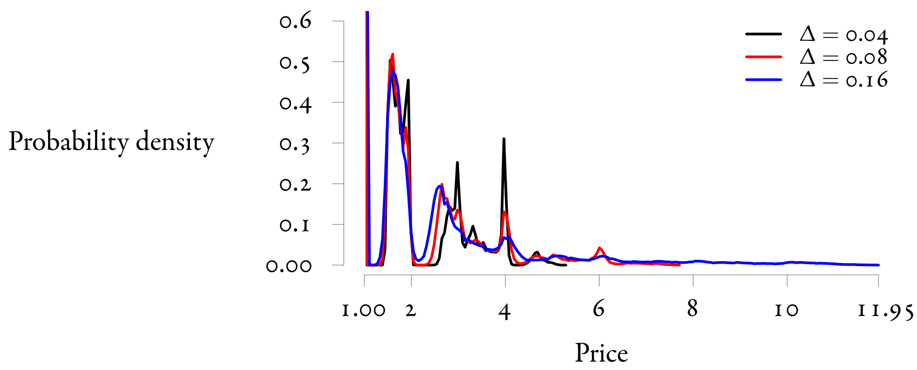

Figure 4.2 *Steady state price distribution for* $\gamma = 0.5$, $\Delta = 0.04, 0.08$ *and* 0.16, $N = 10^5$. *The basic structure remains unchanged as* Δ *changes but the peaks broaden with* Δ. *The peak at* $p = 1$ *extends significantly above the axis to around twelve, but is cut off for clarity. Note how at very high prices the number of sellers can go to zero. There are only a small number of sellers with these prices and so fluctuations can kill all of them; once gone, these prices will not reappear due to selection pressures*

that high prices are sustainable, we wish to understand the nature of the steady state we obtain.

We first show how high prices are able to flourish at all and subsequently make a more qualitative argument for the steady state persistence of high-priced sellers.

4.3 An Analysis of the Early Time Behaviour

In addition to the appearance of a wide range of prices in the steady state for $\gamma = 0.5$, we also observe that there are definite favoured prices in the distribution. The distribution is generally decreasing with increasing price, but there are occasional peaks to be seen. The position of these peaks is of particular interest and merits further study.

As figure 4.2 shows, favoured prices appear, broadly, at integer multiples of the minimum price ($p = 1$), this is especially noticeable for $p > 3$. There is a peak close to $p = 2$ but it is not nicely centered at $p = 2$. We explain the cause of this spreading in section 4.4. Our first aim is to explain how such high-priced peaks can arise at all. To do this we look at the early-time dynamics of the price distribution (figure 4.3). These distributions show an almost immediate appearance of a peak in the distribution at $p = 1$ and $p = 2$ and subsequently at $p = 4$ and $p = 6$. Why does this occur?

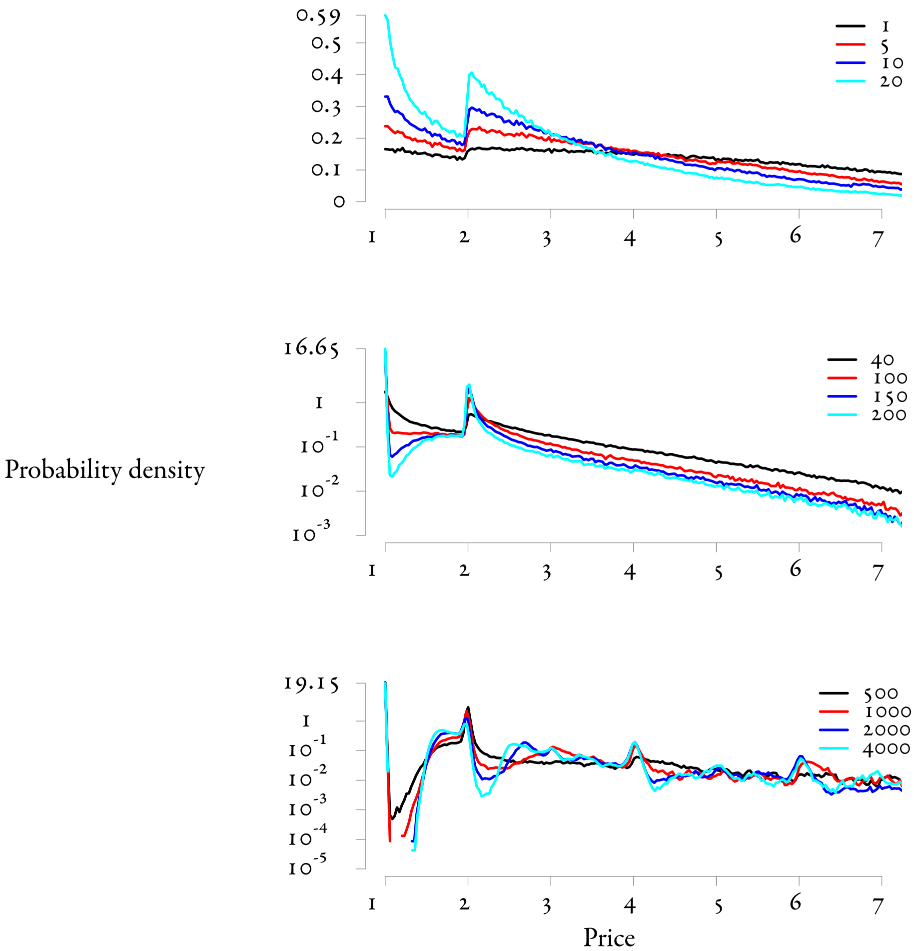

Figure 4.3 *Early time price distributions showing symmetry-breaking behaviour in price structure. The three figures show price distributions after the indicated number of timesteps in a simulation with* $N = 10^6$, $\Delta = 0.08$, $\gamma = 0.5$. *Note the almost immediate emergence of a favoured price at* $p = 1$ *and* $p = 2$ *and later appearance of peaks at* $p = 4$ *and* $p = 6$

4.3.1 *Symmetry breaking*

Recall that in this one-dimensional model, sellers can only sell to zero, one or two buyers in any given round. Consider the first round of the simulation. All sellers initially have zero capital. The cost of production, d_i, is two units and so, to avoid bankruptcy ($\Delta c \geqslant 0$), a seller must make sales totalling at least two units.

Now consider two sellers with prices $p_1 = 2 - \epsilon$ and $p_2 = 2$. To avoid bankruptcy in the first round the first seller must make two sales (making profit $\Delta c = 2 - 2\epsilon$), the second only one sale (making profit $\Delta c = 0$). The probability of making a sale is decreasing with increasing price and so there will be a discontinuous change in the survival probability as the price of a seller changes from $p = 2 - \epsilon$ to $p = 2$.

Since only surviving sellers have their prices copied during the reentry phase, this asymmetry is reinforced in new sellers leading to an increase in sellers with $p = 2$ relative to those with $p = 2 - \epsilon$. This early-time behaviour explains the distributions we see in the first few rounds, but not why the peak at $p = 2$ spreads. This latter phenomenon is examined in section 4.4, first we make the argument for early-time favoured prices more quantitative.

4.3.2 *Analysis for the first round of the game*

In the first round of the game the prices of sellers are spatially uncorrelated. The probability that a randomly chosen seller with price p outcompetes a neighbouring seller (and thus makes a sale) is therefore just

$$f_1(p) = \int_p^{p_{max}} P_0(x)dx. \tag{4.1}$$

Where $P_0(p)$ is the distribution of prices set by the initial conditions

$$P_0(p) = \begin{cases} \frac{1}{p_{max} - p_0} & p \in [p_0, p_{max}] \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

The profit of a seller making k sales is simply given by (recall the overhead $d = 2$)

$$\Delta c = -2 + kp \quad k = 0, 1, 2. \tag{4.3}$$

All sellers initially have $c = 0$ and so to avoid bankruptcy a seller with $p < 2$ must sell twice, a seller with $p \geqslant 2$ only once. The survival probability of a seller with price p is thus

$$p_{s,1}(p) = \begin{cases} f_1(p)^2 & p < 2 \\ f_1(p)(2 - f_1(p)) & \text{otherwise.} \end{cases} \tag{4.4}$$

This function is plotted in figure 4.4, we see immediately that sellers with $p = 2 - \epsilon$ are significantly less likely to survive than those with $p = 2$.

We now attempt to derive an exact expression for the price distribution at the end of the first round of sales. Subsequent rounds quickly become

Figure 4.4 *The survival probability of sellers as a function of price in the first round of the game with* $p_0 = 1$, $p_{max} = 10$. *Note the discontinuous jump at* $p = 2$ *accounting for the early time increase in the fraction of sellers at that price*

intractable, as explained in section 4.3.4.

Denote the probability distribution of live seller prices after t rounds by $P_t(p)$. The initial conditions set $P_0(p)$ (equation 4.2) and this allows us to obtain the survival probability given in equation 4.4. Ignoring the mutation term in the strategy copying stage (equivalent to setting $\Delta = 0$), we can now write down $P_1(p)$, the price distribution at the end of the first round

$$P_1(p) = P_0(p) - \underbrace{\left[1 - p_{s,1}(p)\right] P_0(p)}_{\text{Loss from bankruptcy}} + \underbrace{\gamma p_{s,1}(p)\left[1 - p_{s,1}(p)\right] P_0(p)^2}_{\text{Gain due to reentry}}. \quad (4.5)$$

Simplifying, we find (the denominator just ensures correct normalisation)

$$P_1(p) = \frac{P_0(p) p_{s,1}(p)\left[1 + \gamma\left[1 - p_{s,1}(p)\right] P_0(p)\right]}{\int_{p_0}^{p_{max}} P_1(x) dx}. \quad (4.6)$$

This has an explicit solution

$$P_1(p) = \begin{cases} \frac{5(p-10)^2(729-\gamma(p-19)(p-1))}{1741095+42246\gamma} & p_0 \leqslant p < 2 \\ \frac{5(10-p)(p+8)(729+\gamma(p-1)^2)}{1741095+42246\gamma} & 2 \leqslant p \leqslant p_{max} \\ 0 & \text{otherwise.} \end{cases} \quad (4.7)$$

Figure 4.5 shows this theoretical result with the equivalent distribution obtained from simulation. We see an excellent, but not perfect, agreement between the two. In particular, there is a noticeable discrepancy between

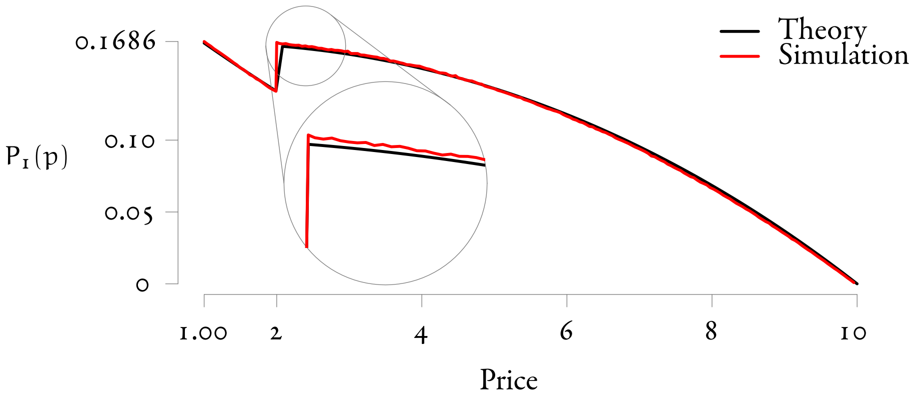

Figure 4.5 *Theoretical and simulated distributions for the price of live sellers at the end of the first round of simulation. The theoretical distribution is given by equation 4.7 with* $p_0 = 1$, $p_{max} = 10$ *and* $\gamma = 0.5$*. The distribution from simulation is obtained with 100 realisations of a system with* $N = 10^6$, $\Delta = 0$, $\gamma = 0.5$, $p_0 = 1$ *and* $p_{max} = 10$*. The distribution from simulation is a normalised histogram of observed prices with 200 equal width bins. Inset shows region around* $p = 2$ *where the deviation between theory and simulation is most noticeable.*

theory and experiment close to $p = 2$. To check if this difference is an actual discrepancy or just noise in the simulation result, we perform a goodness of fit test.

Our null hypothesis for this test is that the discrepancy we observe is purely due to noise in the simulation results. In other words, we claim that the observed empirical distribution consists of data drawn from the theoretical distribution given by equation 4.7. The test statistic we use is the Kolmogorov-Smirnov distance statistic [37]

$$D_n = \sup_x |F_n(x) - C(x)| \tag{4.8}$$

where $F_n(x)$ is the empirical cumulative distribution function of n datapoints and $C(x)$ is the hypothesised cumulative distribution function

$$C(x) = \int_{p_0}^{x} P_1(p)dp. \tag{4.9}$$

Just as our dataset is a set of random variables drawn from some distribution, D_n is also a random variable from some distribution. Remarkably, if $F_n(x)$ is drawn from $C(x)$ (*i.e.*, if the null hypothesis is true), the distribution of D_n is *independent* of the choice of $C(x)$[1] in the limit of large n [18]

$$\lim_{n\to\infty} P(\sqrt{n}D_n \leqslant t) = 1 - 2\sum_{k=1}^{\infty}(-1)^{k-1}e^{-2k^2t^2} = \vartheta_4(0, e^{-2t^2}). \qquad (4.10)$$

Where $\vartheta_4(u, q)$ is an elliptic theta function [25, equation 8.180].

We can now use equation 4.10 to calculate the probability of observing a particular value for D_n given the null hypothesis. This is the p-value of the test: the probability of observing a fit as bad as we did if the null hypothesis is true. In this particular case, for which the dataset contains approximately 10^8 points, the p-value is less than 10^{-10}. We should therefore reject the null hypothesis that the observed discrepancy between theory and simulation is purely due to noise.

We note, however, that the theoretical distribution does follow the empirical distribution rather well. It is likely that we only notice a difference with very large empirical datasets. We now consider *how* large the dataset needs to be before we can reject our null hypothesis. To do this, we note that the p-value is a random variable whose distribution is uniform on $[0, 1]$ if the null hypothesis is true, and strongly peaked at zero if it is false [18].

We have a single large dataset (S) which produces the empirical distribution we saw in figure 4.5. From this, we construct many small datasets (s_i) by sampling uniformly at random from S with replacement. For each of these datasets s_i we can calculate the goodness of fit under our null hypothesis and obtain a p-value for the fit. We repeat this process many times and construct a distribution of the p-values so obtained. Figure 4.6 shows the distribution of p-values obtained when constructing 10^5 artificial datasets with sizes of 10^3, 10^4 and 10^5. When the small dataset contains 10^3 points, there is no evidence that we should reject the null hypothesis (the distribution of p-values is flat). For 10^4, the distribution shows we should weakly reject the null hypothesis. For 10^5, however, the distribution is strongly peaked at zero indicating that our null hypothesis should indeed be rejected.

4.3.2.1 *A further caveat*

In writing equation 4.6 we have assumed that the joint survival probability of two sellers, i and j, is uncorrelated, *i.e.*, $p_{s,1}(p_i, p_j) = p_{s,1}(p_i)p_{s,1}(p_j)$. This is

[1] This result is only true if $C(x)$ is continuous, which is the case here

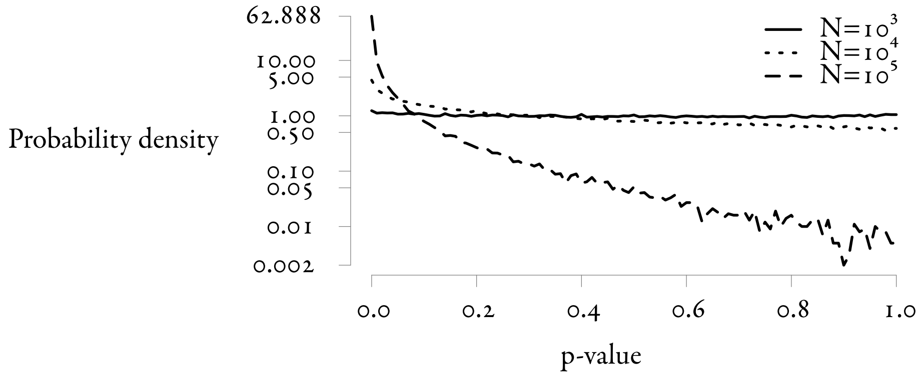

Figure 4.6 *Distribution of p-values for the null hypothesis that the system price distribution is drawn from equation 4.7 for different dataset sizes (*$N = 10^3$, 10^4 *and* 10^5*). The empirical datasets each have* $p_0 = 10$, $p_{max} = 10$, $\gamma = 0.5$ *and* $\Delta = 0$*. The theoretical distribution has* $\gamma = 0.5$, $p_0 = 1$ *and* $p_{max} = 10$*. Were the data drawn from the null hypothesis, the distribution would be uniform. We therefore reject the null hypothesis except when* $N = 10^3$

generally true, unless the two sellers are neighbours. Consider, without loss of generality, the case where $p_i < p_j$. If the two sellers are not neighbours the survival probability is

$$p_{s,1}(p_i, p_j) = \begin{cases} f(p_i)^2 f(p_j)^2 & p_i < p_j < 2 \\ f(p_i)^2 f(p_j) & p_i < 2 \leqslant p_j \\ f(p_i) f(p_j) & 2 \leqslant p_i < p_j. \end{cases} \tag{4.11}$$

This is easily seen to be equal to $p_{s,1}(p_i)p_{s,1}(p_j)$, *i.e.*, uncorrelated. If the two sellers *are*, however, neighbours, then the joint survival probability is

$$p_{s,1}(p_i, p_j) = \begin{cases} 0 & p_i < p_j < 2 \\ f(p_i) f(p_j) & p_i < 2 \leqslant p_j \\ f(p_j) & 2 \leqslant p_i < p_j. \end{cases} \tag{4.12}$$

This is not the same as equation 4.11. A similar argument leads to the conclusion that the three site joint survival probability is also correlated, for three neighbouring sellers $p_{s,1}(p_1, p_2, p_3) \neq p_{s,1}(p_1, p_2)p_{s,1}(p_3)$. Our closure scheme of choosing uncorrelated survival probabilities is evidently quite good, but not exact and we suggest that this correlation may be the cause of discrepancy between theory and experiment noted in the previous section.

4.3.3 *Explanation of peaks at prices* $p > 2$

The single round survival probability decreases monotonically above $p = 2$ and so the analysis of the previous section can only explain the development of a peak at $p = 2$ in the price distribution. To explain the peaks at higher prices we need to consider the survival probability over multiple rounds. Just as a seller with $p \geqslant 2$ only needs a single sale in the first round to survive, a seller with $p \geqslant 4$ only needs a single sale in the first round to survive until the third round. Denoting the probability of outcompeting a neighbour in the second round by $f_2(p)$, the probability that a seller of price p survives both the first and second rounds is

$$\begin{cases} f_1(p)^2 f_2(p)^2 & p < 2 \\ f_1(p)(2 - f_1(p))f_2(p)(2 - f_2(p)) & 2 \leqslant p < 4 \\ f_1(p)(2 - f_1(p)) & \text{otherwise.} \end{cases} \tag{4.13}$$

We do not have an explicit expression for $f_2(p)$, but it will not be equal to unity for all prices. There will thus be a discontinuous step in the two-round survival probability at $p = 4$. A similar situation will arise for $p = 3$ in the three-round survival probability and so forth. This translates to further symmetry breaking in the price distribution at $p = 4$ and so on, leading to favoured prices as observed in simulation (figure 4.3).

4.3.4 *Analysis for the second round of the game*

If we wish to derive an exact expression for the price distribution after two rounds, we need to write down the survival probability in the second round. Unfortunately, this is not possible since the dynamics induce correlations which we cannot ignore. Our analysis of section 4.3.2 assumes that the system is uncorrelated in space, *i.e.*, the probability of outcompeting a neighbour is merely a function of the system price distribution. This is no longer the case in round two. In addition, we have just shown that the survival probability in round two can depend on the number of sales made in round one. We now show that these effects are not negligible, making analysis of the system in rounds other than the first very difficult. To do this, we construct the second round price distribution assuming there are no correlations and show that it deviates significantly from the empirical result.

Ignoring correlations in the system is equivalent to saying that the survival of a seller in round two depends only on the fraction of live sellers and the distribution of live seller prices: not on the history. Although we have seen that the expression (equation 4.6) we derived for the first round price distribution is not quite exact, it is our best effort and a very reasonable fit to the data. We use this result to calculate the distribution of prices in the second round.

Using equation 4.6 we can calculate the fraction of sellers alive at the beginning of round two:

$$\rho_2 = \gamma + (1-\gamma)\underbrace{\int_{p_0}^{p_{max}} P_0(p)p_{s,1}(p)dp}_{\text{mean survival probability}}. \tag{4.14}$$

The probability of making a single sale is then given by

$$f_2(p) = \underbrace{(1-\rho_2)}_{\text{neighbour bankrupt}} + \underbrace{\rho_2\int_{p}^{p_{max}} P_1(x)dx}_{\text{neighbour more expensive}} \tag{4.15}$$

and the survival probability of a seller with price p is (analogously with equation 4.4)

$$p_{s,2}(p) = \begin{cases} f_2(p)^2 & p < 2 \\ f_2(p)(2-f_2(p)) & \text{otherwise.} \end{cases} \tag{4.16}$$

To find the live seller price distribution, we need to consider both sellers that started the round alive, and those that began the round bankrupt

$$\begin{aligned} P_2(p) = {} & \overbrace{\rho_2\Big[\underbrace{p_{s,2}(p)P_1(p)}_{\text{Surviving}} + \underbrace{\gamma\big[1-p_{s,2}(p)P_1(p)\big]p_{s,2}(p)P_1(p)}_{\text{Reentry}}\Big]}^{\text{Alive from round one}} \\ & + \underbrace{(1-\rho_2)\gamma p_{s,2}(p)P_1(p).}_{\text{Reentry, bankrupt from round one}} \end{aligned} \tag{4.17}$$

Simplifying and normalising appropriately we find the live seller price distribution at the end of round two, assuming no correlations, is given by

$$P_2(p) = \frac{p_{s,2}(p)P_1(p)\left[\rho_2 + \gamma - \rho_2\gamma p_{s,2}(p)P_1(p)\right]}{\int_{p_0}^{p_{max}} P_2(x)dx} \tag{4.18}$$

To compare this result with simulation, we look at the cumulative distribution function (CDF)

$$C_2(p) = \int_{p_0}^{p} P_2(x)dx. \tag{4.19}$$

We do not give the full expression here, it is about four pages long and offers no real insight.

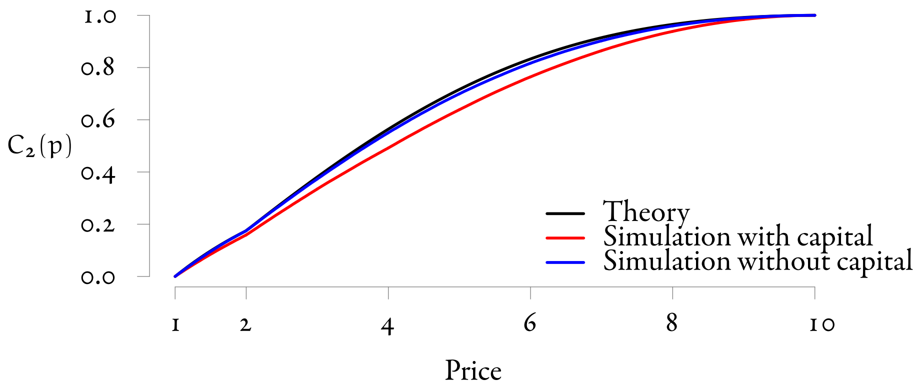

Figure 4.7 *Comparison of theoretical and empirical* CDFs *for live seller prices. The theoretical curve (black) is equation 4.19 with* $\gamma = 0.5$. *The two empirical curves have* $N = 10^6$, $\Delta = 0$, $\gamma = 0.5$. *The red empirical curve has sellers accumulating capital, the blue curve has no accumulation of capital in an attempt to remove correlations. Both theory and simulation have* $p_0 = 1$, $p_{max} = 10$. *The theoretical* CDF *does follow the shape of the empirical* CDFs*, especially the case of no capital accumulation. However, we can reject (again using a Kolmogorov-Smirnov test) the hypothesis that the data were drawn from the theoretical distribution in both cases.*

Figure 4.7 compares the theoretical CDF given in equation 4.19 with two obtained from simulation. One simulation result is for the completely correlated case, the other for a case in which we artificially set capital to zero at the end of round one, removing any round-to-round correlation there. This latter simulation should match the theoretical result in equation 4.19 if there are no correlations in the system other than those induced by accumulation of capital. Both simulations deviate significantly from the theoretical result, the no-capital one less so, indicating that both capital accumulation and the bankruptcy and reentry process induce correlations in the system which our analysis has not captured.

The problem of capital accumulation and its effects on survival probability are addressed in part in chapter 8. In short, we show how it is possible to explicitly construct history-dependent survival paths if the price distribution is known and the paths are not too long. However, this method only treats the case of a spatially uncorrelated system. As figure 4.7 shows the spatial correlations are non-negligible and so, since the analysis is sufficiently complicated even for a δ-function price distribution, we do not find it worthwhile to attempt the same analysis here.

4.4 STEADY STATE SURVIVAL OF EXPENSIVE SELLERS

4.4.1 *What is an expensive seller?*

In the following sections, we shall refer to sellers as being either *cheap*, or *expensive*. Our definitions of these two sets are determined from the observed steady state price distribution. For $\gamma = 1$ we have already seen (figure 4.1) that the price distribution has a single peak at $p = 1$, the highest observed prices in this case are $p \approx 1 + \Delta$. We therefore define any seller with a price $p \leqslant 1 + \Delta$ as cheap. Conversely, any seller with a price $p > 1 + \Delta$ is, by definition, expensive. We note also that there is typically a gap in the price distribution between cheap and expensive sellers (see figure 4.2), making the distinction more obvious.

4.4.2 *Lifetimes of expensive sellers*

The analysis of the section 4.3 allows us to see how expensive sellers can survive in the early rounds of a simulation, but how do they do so in the steady state? The key factor here is competition-free sales. If an expensive seller finds itself in a situation without competition it is able to make a profit and subsequently survive those situations in which it cannot attract buyers. We must now ask how this can occur in a predictable, long-termed manner. It is of little use to the survival of an expensive seller if it gets lucky in one round and makes a sale but then never does so again and subsequently quickly goes bankrupt.

Of course, it may be that all expensive sellers are short-lived but that there is enough rebirth of expensive sellers to continue the steady state. Figure 4.8 shows that the expected lifetime is approximately constant for prices between $p = 2.5$ and $p = 6$, but significantly higher for $p \approx 1$. Very cheap sellers are therefore copied a larger number of times in their lifetime than expensive sellers and we would expect cheap sellers to dominate in the long time limit. This argument suggests that the expected lifetime of sellers is not the only factor in determining steady state behaviour.

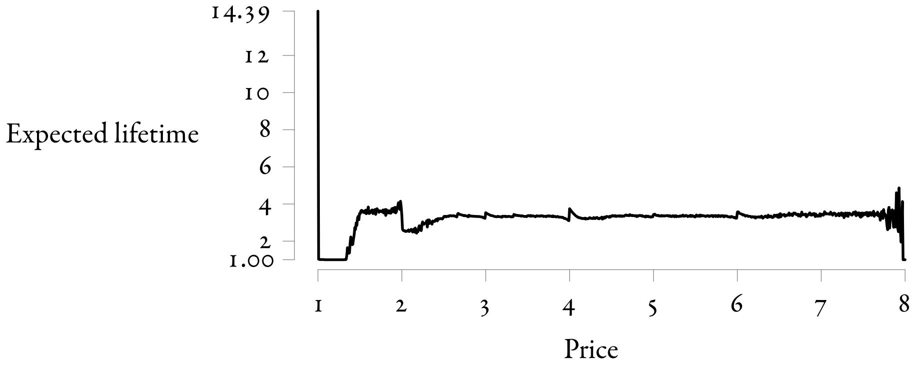

Figure 4.8 *Expected lifetime of sellers as a function of price in the steady state.* $N = 10^5$, $\gamma = 0.5$, $\Delta = 0.08$. *Note the peak at* $p = 1$, *giving an expected lifetime of around fourteen. Most other prices have an expected lifetime close to three rounds.*

We now study the distribution of the age of sellers at bankruptcy, dividing sellers into one of two groups: cheap ($p < 1 + \Delta$) and expensive sellers ($p > 1 + \Delta$). The complementary cumulative distribution functions for the age at bankruptcy of expensive and cheap sellers are shown in figure 4.9. Some expensive sellers survive for the entire lifetime of the system.

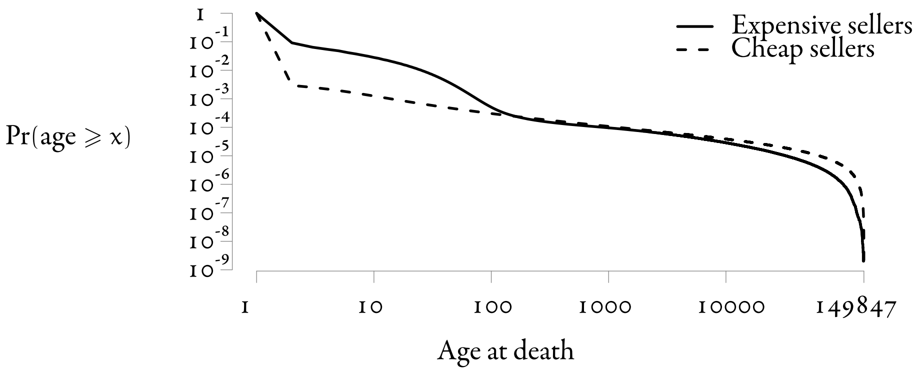

Figure 4.9 *Complementary cumulative probability distribution* $\Pr(\text{age} \geqslant x)$ *for age at bankruptcy of expensive sellers (*$p > 1 + \Delta$*, solid) and cheap sellers (*$p < 1+\Delta$*, dashed) in steady state with* $\gamma = 0.5$, $N = 10^5$, $\Delta = 0.08$. *Maximum possible age in the system is 150000, we see that some sellers almost reach this age, indicating that they survive essentially indefinitely. Note the long tail of ages for both cheap and expensive sellers indicating that sellers do not (all) die according to a single point process with fixed rate*

For a seller with $p = 2$, this would correspond to having one expected sale per round. A quick calculation shows that this is highly unlikely if the steady state we observe is uncorrelated in space. Consider the empirical price distribution obtained with $\gamma = 0.5$, $N = 10^5$, $\Delta = 0.08$. In this system we observe that the fraction of live sellers with a price $p < 2$ is $\alpha \approx 0.75$ and the density of live sellers is $\rho \approx 0.79$. Assuming the system is well-mixed, we can write down the expected change in capital for a seller with $p = 2$

$$\begin{aligned}\Delta c = \underbrace{-d}_{\text{overhead}} + \underbrace{2p\left[(1-\rho)^2 + 2\rho(1-\rho)(1-\alpha) + \rho^2(1-\alpha)^2\right]}_{\text{two sales}} \\ + \underbrace{p\left[2\rho(1-\rho)\alpha + 2\rho^2\alpha(1-\alpha)\right]}_{\text{one sale}} \\ = 2p - 2\alpha\rho p - d\end{aligned} \tag{4.20}$$

Putting in the numbers, we find $\Delta c \approx -0.37$. This expected change is not enough to survive indefinitely and there must therefore be some correlation in the system which allows for survival of these 'uncompetitive' sellers.

4.4.3 *Niche construction and competition-free sales*

With the knowledge that an expensive seller cannot survive in the system for a long time if it is well-mixed, we look at the neighbourhoods of long-lived expensive seller to find out how they survive. Essentially, expensive sellers must appear in positions in the system which increase the probability of making a sale. To do this, they must sit in positions that allow for competition-free sales.

We quickly notice a repeating pattern in the neighbourhood of long-lived expensive sellers. Almost all such sellers have next nearest neighbours with a price very close to p_0. This scenario is sketched in figure 4.10.

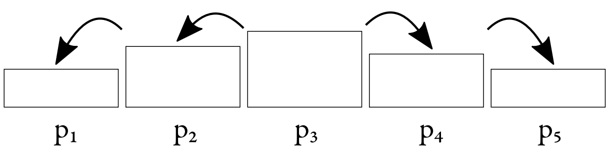

Figure 4.10 *Five site structure observed around long-lived expensive sellers. Heights of boxes schematically indicate the relative sizes of seller prices. Arrows show which seller a buyer visits. When the outermost sellers have prices very close to p_0, the inner sellers will make at most a single sale*

4.4.3.1 *A simplified example for a seller with* $p = 2$

To see how the structure shown in figure 4.10 changes the survival probability of expensive sellers we consider a simple example. Assume there is a single expensive seller in the system with $p = 2$ and that all other sellers have $p < 2$. Now consider how many sales this expensive seller gets if it is in the central position in the structure shown in figure 4.10. The two neighbouring sellers only ever make a single sale, and (since they have a price $p < 2$) will go bankrupt. The survival probability of the expensive seller then becomes a function of the reentry probability γ rather than the live site density. It will make either zero, one or two sales with probabilities determined by γ:

1. no sales ($\Delta c = -2$) occur with probability γ^2;
2. one sale ($\Delta c = 0$) occurs with probability $2\gamma(1 - \gamma)$;
3. two sales ($\Delta c = 2$) occur with probability $(1 - \gamma)^2$.

The capital of this expensive thus carries out a random walk with step probabilities set by γ. If $\gamma > 0.5$, this walk is biased towards the origin and the expected lifetime of the seller is finite. Otherwise the expected lifetime of the seller is infinite (assuming that the sellers at p_1 and p_2 survive).

4.4.4 *Steady state survival of sellers with* $p < 2$

This simple explanation for $p = 2$ is not the whole story. It fails to explain how sellers with a price below $p = 2$ can survive. To treat this case, we must look at the actual system price distribution, rather than positing a system that consists of a single expensive seller and many cheap sellers. There are two ways a long-lived expensive seller can make a profit: the neighbouring site is vacant, or the neighbouring seller is more expensive. Figure 4.11 shows an example of such a case. In this example we have an expensive seller with $p \approx 1.7$ surviving for more than ten thousand rounds. The figure shows the distribution of prices in the neighbouring sites experienced by the seller: note how some of the time, the neighbours appear with a price larger than 1.7.

For a known price distribution, we can write down the expected change in capital of a seller with price $p < 2$ in the five site situation depicted in

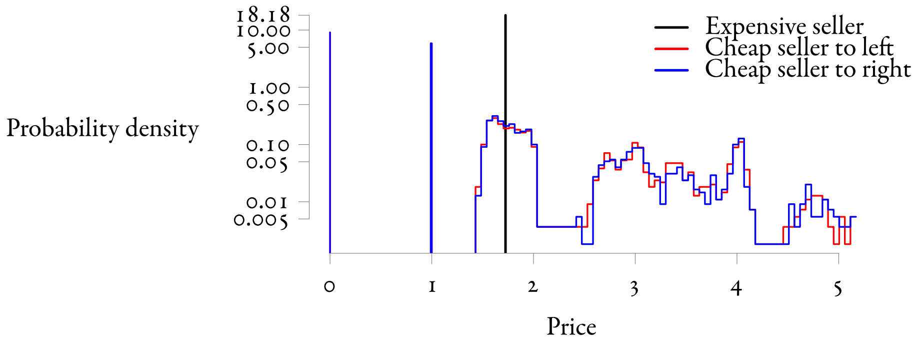

Figure 4.11 *Price distributions over 10000 rounds of sellers near to a long-lived expensive seller (this one has* $p = 1.725$ *black line). The next-nearest neighbours both have price* $p = 1$ *for the duration (not shown). The left and right neighbours have a distribution that samples the global steady state distribution (red and blue lines). Note that a dead seller is indicated as having a price* $p = 0$*. The probability that the expensive seller made a sale to the right during the 10000 rounds was 0.625, sales to the left were made with probability 0.622. This means the expected profit of the expensive seller was 0.15 per round leading to continued survival.* $N = 10^5$, $\gamma = 0.5$, $\Delta = 0.08$

figure 4.10.

$$\Delta c(p) = \underbrace{-d}_{\text{overhead}} + \underbrace{2p\left[(1-\gamma)^2 + 2\gamma(1-\gamma)\phi(p) + \gamma^2\phi(p)^2\right]}_{\text{two sales}} + \underbrace{p\left[2\gamma(1-\gamma)(1-\phi(p)) + 2\gamma^2\phi(p)(1-\phi(p))\right]}_{\text{one sale}}. \quad (4.21)$$

Where

$$\phi(p) = \int_p^{p_{max}} P_\infty(x)dx \quad (4.22)$$

is the probability a seller of price p outcompetes a newly entered seller whose price is drawn from the steady state live price distribution $P_\infty(p)$. Equation 4.21 can be simplified significantly giving

$$\Delta c(p) = 2p\left[1 + \gamma(\phi(p) - 1)\right] - d. \quad (4.23)$$

Any seller in this situation with a price p such that their expected capital change is negative will go bankrupt. By setting $\Delta c(p) = 0$ we can find the minimum price which allows survival in such a situation (recalling that $d = 2$):

$$p_{min} = \frac{1}{\left[1 + \gamma(\phi(p_{min}) - 1)\right]}. \quad (4.24)$$

We cannot derive an expression for $\phi(p)$ analytically (see section 4.3.4), however, we know what the distribution is from simulation. With this information we can now perform a *post hoc* analysis of the empirical price distribution and self-consistently solve for the gap between cheap sellers with $p \approx 1$ and expensive ones. Substituting the empirically observed price distribution into equation 4.24 we find the expected winnings of long-lived sellers – *i.e.*, sellers in the situation depicted in figure 4.10. This result is shown in figure 4.12. It is evident that the minimum surviving expensive price approximately corresponds to the price at which expected winnings of long-lived sellers become positive.

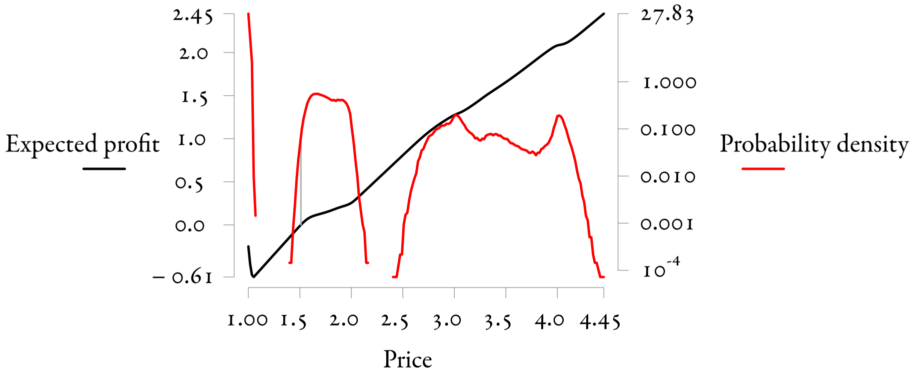

Figure 4.12 *Expected profit (left axis, black) of long-lived sellers as a function of price in the steady state with $\gamma = 0.5$. Data obtained by solving equation 4.25 using an empirically determined steady state distribution (shown (right axis, red), with $N = 10^5$, $\Delta = 0.08$, $\gamma = 0.5$). The grey vertical line marks the minimum profitable price as predicted by our theory. We see that the lower limit of expensive prices ($p \approx 1.5$) matches the point at which the expected profit becomes negative reasonably well. However, the expected profit does not predict the structure at higher prices.*

4.4.5 *Sellers with* $p > 2$

The analysis we have carried out for survival of sellers with $p \leqslant 2$ needs some modification for higher priced sellers. The situation is more complicated in this case. A situation like the one depicted in figure 4.10 is still possible, but if the central seller has $p_3 > 2$, sellers can arrive at sites two or four with $2 \leqslant p_{2,4} < p_3$ and survive for as long as seller three survives: they have one guaranteed sale per round. Sellers with $p > 2$ are thus not as stable in the structure as those with $p \leqslant 2$, but they can still survive for a long time. They are also able to survive in less favourable situations. For example, a seller with $p > 2$ can survive even if only one neighbour is occasionally bankrupt. Such a seller makes fewer sales, but they can still survive as long as they have a price

$$p^* = \frac{2}{1 + \gamma(2\phi(p^*) - 1)}. \tag{4.25}$$

We can also construct larger structures which display this same stability. Consider a long line of sellers with prices p_i ($i = 1, \ldots, n$). If $p_1 = p_0 = p_n$, then we can construct a system with $p_{n-2} \geqslant p^*$, $p_3 = \frac{2}{2-\gamma\phi(p_3)}$ and p_i increasing between p_3 and p_{n-2} which is only invadeable by a seller at site $n-1$. Such structures allow for enough copying of expensive sellers that they persist in the steady state. Figure 4.13 shows an example structure with these properties observed in simulation data.

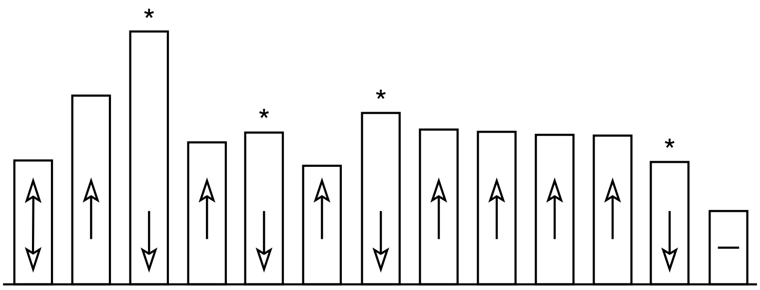

Figure 4.13 *Diagram showing an example expensive structure observed in simulation (recorded after 30000 rounds,* $N = 10^5$*,* $\gamma = 0.5$*,* $\Delta = 0.08$*). Each box is a seller, prices are proportional to the height of a box. Arrows within each box indicate the long-term sign of* Δc*. The leftmost seller's* Δc *is dependent on the environment to its left (not recorded), the rightmost seller has* $p = p_0$ *and thus keeps a constant (zero) capital. We can see how the structure is only invadeable in four of the twelve sites (marked with stars) allowing for long-term stability of expensive sellers*

4.5 VARYING γ

So far we have studied three different choices for γ, the only real free parameter in our model. Our analysis of the survival of expensive sellers at $\gamma = 0.5$ suggests that there will be some value of $\gamma \equiv \gamma_c$ above which expensive sellers do not appear in the steady state. In the next sections, we study how the steady state changes as we vary γ. We first show that the transition between a steady state with expensive sellers and one without (hereafter referred to as the *expensive state* and *Bertrand state* respectively) occurs at $\gamma_c < 1$. We then study the behaviour of the system for small γ and subsequently study the nature of the transition.

Note that the transition only happens at $\gamma < 1$ if prices are restricted to some finite maximum. To see this, consider a system composed entirely of cheap sellers, except for a single expensive seller with $p = p^*$. We know we can put this seller in a structure which allows for non-negative profits, as long as its price is high enough. Equation 4.24 tells us that the expected change in capital of this seller is zero if

$$p^* = \frac{1}{1-\gamma} \tag{4.26}$$

since $\phi(p^*) = 0$ by construction. For any value of $\gamma < 1$ we can therefore choose a p^* to ensure expected survival of the expensive seller. Although, in our simulations, we do not restrict the maximum price in the dynamics, we do set a finite maximum in the initial conditions. This will lead to $\gamma_c < 1$.

4.5.1 *A transition in steady state behaviour*

In order to see if a transition in the steady state occurs, we need to define something that looks like an order parameter. Since we are interested in the difference between the Bertrand state and the expensive state, the obvious choice is to measure the fraction of expensive sellers in the system, ρ_{exp}. In the Bertrand state this is zero, rising to some non-zero value in the expensive state. We use a slight modification of our previous definition of cheap and expensive sellers. In order that we do not pick up noise due to occasional cheap sellers getting a price $p = 1 + \Delta$, we choose to regard sellers with $p < 1.2$ as cheap and all others as expensive. Note that this will not have any real effect on our categorisation of sellers, since (as seen in figure 4.12) there is a gap in the price distribution between cheap and expensive sellers.

Figure 4.14 shows the variation in the fraction of expensive sellers as a function of γ, demonstrating that expensive sellers die out well before $\gamma = 1$. Were this an example of a phase transition as observed in typical, physical non-equilibrium systems [31], we would most likely expect that the order parameter decays to zero according to a power-law (when $\gamma < \gamma_c$)

$$\rho_{\text{exp}} \sim (\gamma_c - \gamma)^{\beta}. \tag{4.27}$$

We would then estimate a value for the scaling exponent β and find the

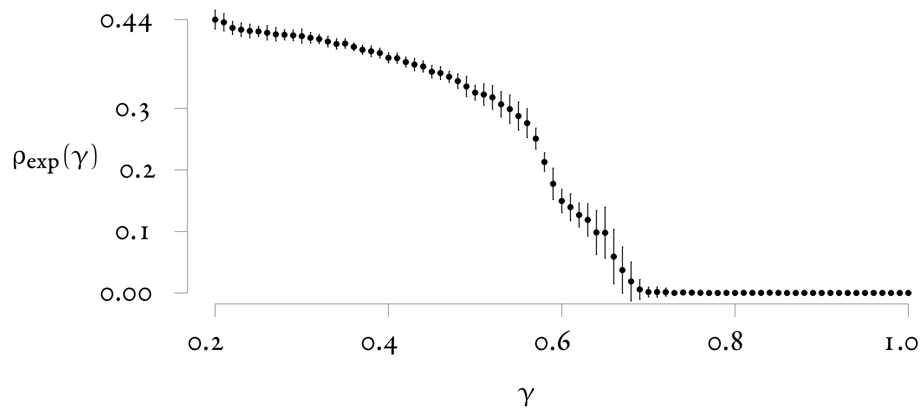

Figure 4.14 *Fraction of expensive sellers, $\rho_{\exp}(\gamma)$, before rebirth as a function of γ. Note sharp rise around $\gamma_c = 0.7$. $N = 2 \times 10^4$, $\Delta = 0.08$. Each data point is the fraction of live expensive sellers observed in a simulation after the steady state has been reached (10^5 rounds of the game), each such point is then averaged over forty realisations of the same initial conditions. Errorbars show standard error in this mean value over the forty realisations*

value of γ_c. This approach does not work here. We note a kink in the decay of the order parameter near $\gamma = 0.6$. This corresponds to disappearance of expensive sellers with $p < 2$ as we show in section 4.5.3.1. Our order parameter therefore goes through a number of transitions rather than just one, the particular point at which it becomes zero is due to our definition of expensive sellers.

4.5.2 *The expensive steady state as γ varies*

In the region of parameter space with $\gamma < \gamma_c$, expensive sellers exist in the steady state. In this region the large-scale structure of the distribution is reasonably unchanged. There are peaks in the distribution (the one at $p = 4$ is especially noticeable) and unfavoured prices in between: for example, between $p = 1$ and $p = 2$ (figure 4.15).

The most noticeable difference is the lower limit of expensive prices, which decreases with decreasing γ. This may easily be explained by considering the effect of reducing γ on the minimum price with which an expensive seller can expect survival (equation 4.24). Figure 4.16 shows the minimum profitable price as calculated using equation 4.24 and the minimum observed expensive price in simulation data. The former is calculated using an empirical steady state distribution obtained with $N = 10^5$, $\Delta = 0.08$ and $\gamma = 0.5$, the latter by carrying out simulations at different γ values and measuring the

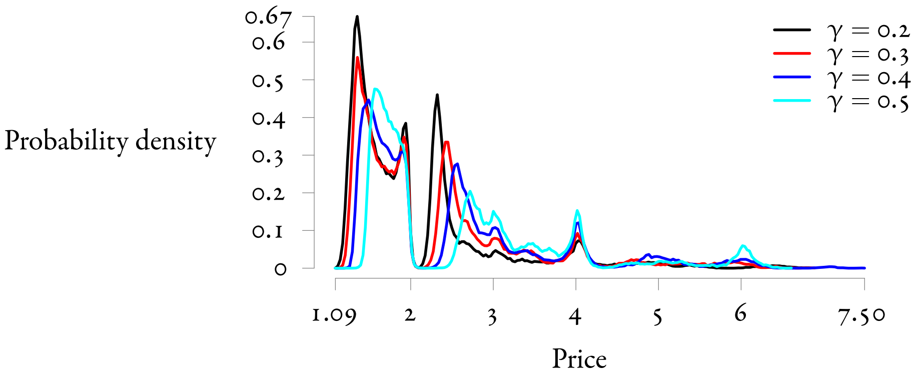

Figure 4.15 *Comparison of the expensive steady state price distribution as γ varies. Shown are distributions with $\gamma = 0.2, 0.3, 0.4, 0.5$. $N = 10^5$, $\Delta = 0.08$. The peak at $p = 1$ has been suppressed (no difference occurs here). Note how the lower limit of expensive prices ($p \approx 1.3$) decreases with decreasing γ in line with the analysis of section 4.4.4, but the overall structure is generally unchanged*

smallest expensive price.

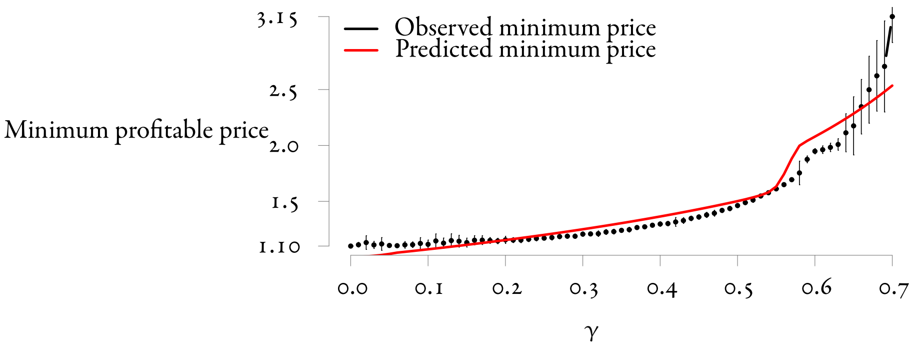

Figure 4.16 *Minimum profitable price as a function of γ found by solving equation 4.25 with a given (fixed) price distribution observed at $\gamma = 0.5$, $N = 2 \times 10^4$, $\Delta = 0.08$. Also shown is the minimum expensive price ($p > 1 + \Delta$) observed in simulation data in the steady state (errorbars show standard error in the mean over 40 realisations of the same initial conditions). Up to around $\gamma = 0.6$ the two curves show reasonable agreement. At higher values, the imposed price distribution is no longer a good fit to the observed one and the agreement breaks down*

4.5.3 *The Bertrand steady state*

If $\gamma > \gamma_c$ then the steady state we observe is the Bertrand one. This is expected since the high birth rate means expensive sellers are unable to make enough competition-free sales to survive. We now study the transition between expensive and Bertrand steady states more closely. We first wish to as-certain exactly where the transition occurs. Figure 4.14 showed a transition around $\gamma = 0.7$. We now carry out the same experiment for different system sizes which shows that this value is not universal. Figure 4.17 shows how the fraction of expensive sellers approaches zero for different system sizes. We notice that with increasing system size, the value of γ_c we observe increases. This is not completely unexpected, near the transition point the number of

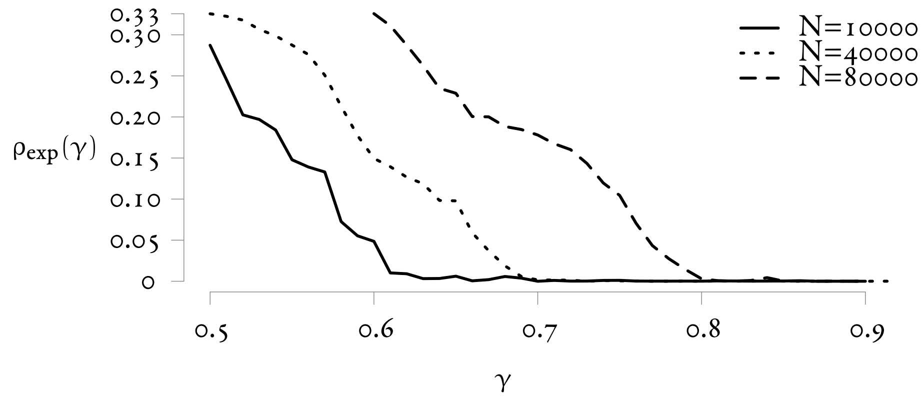

Figure 4.17 *Fraction of expensive sellers as a function of γ for different system sizes illustrating the change in the point at which expensive sites become unviable. Shown are systems of size* 10^4, 2×10^4 *and* 8×10^4, $\Delta = 0.08$

expensive sellers drops to $\mathcal{O}(1)$ and so stochastic fluctuations in the system will lead to disappearance of expensive sellers. These fluctuations play the strongest role in smaller systems. Since expensive sellers will not reappear spontaneously from a Bertrand state, the small system size simulations may underestimate the value of γ_c. Further, as we show in the next section, the last expensive sellers to disappear are those with the highest prices. These are the smallest in number and may well not establish themselves at all in small systems, leading to an underestimate of γ_c. Our results show that this transition is not just a function of γ but of the form of the price distribution at very high prices, and is thus affected significantly by the system size.

4.5.3.1 *The price distribution during disappearance of expensive sellers*

We might think that as γ increases, the first expensive sellers to go would be very high-priced ones. This is not the case. Recall our analysis for the minimum profitable expensive price. This price increases with γ. This allows us to explain the kink that appears in figure 4.14 at $\gamma = 0.6$. Notice that the expensive price distribution is separated (broadly) into 'bands' of prices (*e.g.*, figure 4.2 and figure 4.15). The high-priced part of the distribution remains unchanged as γ increases, but expensive sellers with $p < 2$ become less profitable and eventually disappear (figure 4.18).

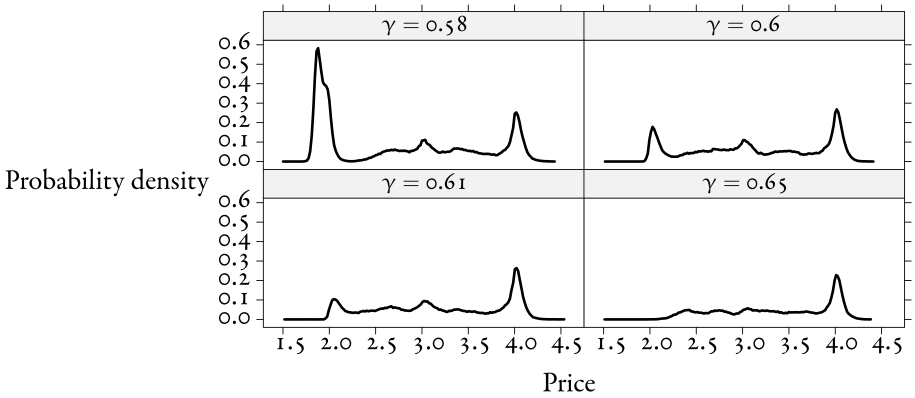

Figure 4.18 *Steady state price distributions with* $\gamma = 0.58$, 0.6, 0.61 *and* 0.65. *Note disappearance of band of expensive sellers with* $p < 2$ *at* $\gamma = 0.6$. $N = 2 \times 10^4$, $\Delta = 0.08$. *Each distribution is composed of seller prices from forty realisations of the specified initial conditions.* $p < 1.5$ *prices suppressed for clarity since no change is observed in this region*

This disappearance maps onto the kink observed in figure 4.14 between $\gamma = 0.55$ and $\gamma = 0.6$. The minimum expensive price moves through the band of expensive sellers with $p < 2$ and so the fraction of expensive sellers drops quite quickly (the gaps are filled by cheap sellers). Once the $p < 2$ band is completely uncompetitive, the next peak in the distribution is not until $p = 3$. Until the minimum profitable price reaches this level, the fraction of expensive sellers drops only slowly with γ. There are a series of such transitions in the system as particular bands of expensive prices become unprofitable with increasing γ. Figure 4.18 shows a sequence of steady state price distributions for values of γ near the disappearance of the $p < 2$ band.

The value of γ at which these transitions occur is dependent on the fraction of sellers at higher prices (this follows from the arguments of section 4.4). Fluctuations in the steady state will have a large effect for small systems and less of an effect for larger ones. This effect will be particularly noticeable in the high price region of the price distribution, since the absolute number of sellers with these prices is quite small. For a particular system size, however, the mechanism is always the same. Figure 4.18 shows the behaviour of the price distribution for a system with $N = 2 \times 10^4$ near to the loss of the $p = 2$

band ($\gamma \approx 0.6$). Figure 4.19 shows the same event for $N = 8 \times 10^4$, although this time the $p = 2$ band disappears at $\gamma \approx 0.66$. Note how the distributions look the same and the $p = 2$ sellers disappear in the same way in both cases, despite the different values of γ.

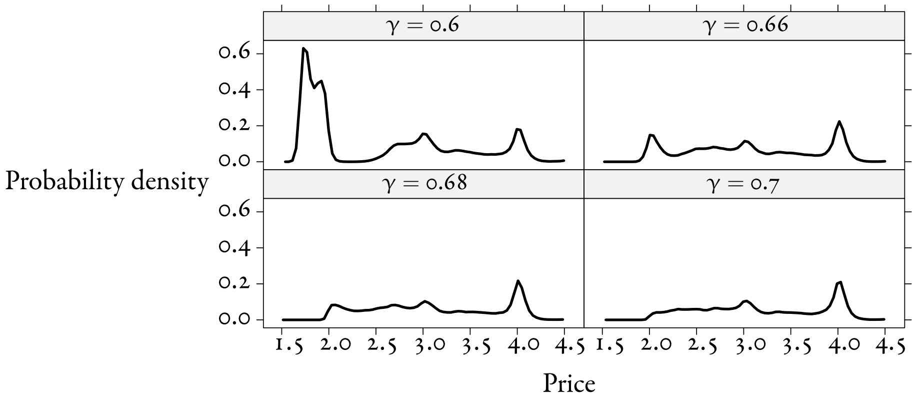

Figure 4.19 *Steady state price distributions with* $\gamma = 0.6, 0.66, 0.68$ *and* 0.7. *Note the disappearance of sellers with* $p < 2$ *around* $\gamma = 0.68$. $N = 8 \times 10^4$, $\Delta = 0.08$. *Each distribution is composed of seller prices from fifty realisations of the specified initial conditions. Compare with figure 4.18. Distributions only shown for* $p \in [1.5, 4.5]$ *for clarity (the change is minimal outwith this region)*

4.5.4 *Invasion of the Bertrand state by a high-priced seller*

We have argued that the expensive state disappears at high enough γ due in part to selective pressures (high prices are uncompetitive) and our choice of initial conditions (setting a maximum initial price leads to bankruptcy of all high-priced sellers when $\gamma < 1$). We have already argued that for any value of $\gamma < 1$, a seller with a high enough price can invade (section 4.5). We now show that this invasion point is independent of system size (for a fixed invading price). We also show that the steady state distribution obtained in this manner has the same structure as that obtained previously.

We generate a known Bertrand state (with $\gamma = 1$) and use this as the initial price distribution for further simulation. With a low probability ($\sim N^{-1}$) at each rebirth event we introduce an expensive seller with either $p = 2$ or $p = 4$ (instead of copying from the system). So approximately one expensive seller is introduced into the game per round. The dynamics are otherwise unchanged.

For low γ, we expect that these expensive sellers will invade the system:

if they appear in the stable structures described in section 4.4.3 they will survive and proliferate. At high enough γ this will no longer be possible. With our particular choice of invading prices, we expect the invasion threshold to be given by (see section 4.4, equation 4.24)

$$\gamma_c = \frac{p^* - 1}{p^*(1 - \phi(p^*))} \tag{4.28}$$

where $p^* = 4$ and $\phi(p^*) = 0$ by construction, giving $\gamma_c = 0.75$.

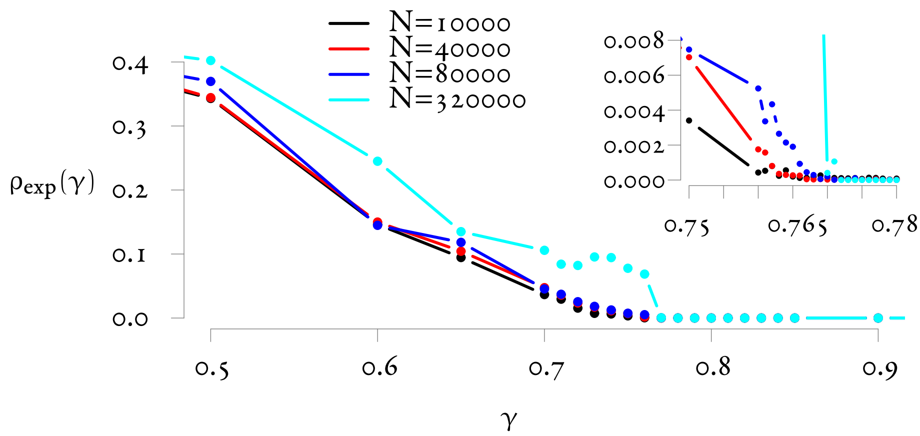

Figure 4.20 *Fraction of expensive sellers in long-time limit as a function of γ. The system is initially in a Bertrand state and expensive sellers are seeded with a low ($p \sim N^{-1}$) probability per round. For low values of γ, we see that the steady state contains expensive sellers, for high values it does not. The transition point ($\gamma \approx 0.77$) is independent of the system size for $N = 10^4$ to $N = 3.2 \times 10^5$. Inset shows detail close to transition point*

When we carry out the simulations, we find the transition occurs at $\gamma \approx 0.77$ (figure 4.20), reasonably in line with our prediction. Interestingly, once high-priced sellers have invaded, the steady state reached in the system is very similar to that obtained previously (figure 4.21) indicating that the state we obtain is reasonably independent of initial conditions. Some small differences are noticeable: the price band at $p < 2$ has a slightly different structure, this may be due to the continued seeding at $p = 2$. Further, very high prices do not appear in the invaded state. This latter fact is due to the selection pressures against mutations upward in price: the highest price in the initial conditions in this invasion mechanism is four, as opposed to ten in the case of our previous results.

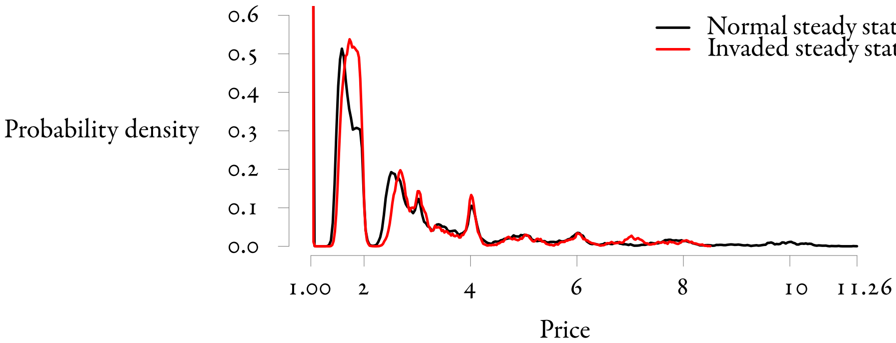

Figure 4.21 *Steady state price distributions obtained under normal simulation dynamics (*$N = 5 \times 10^5$*,* $\Delta = 0.08$*,* $\gamma = 0.5$*) and from invasion of a Bertrand state by seeding with sellers of price* $p = 2$ *and* $p = 4$ *(*$N = 3.2 \times 10^5$*,* $\Delta = 0.08$*,* $\gamma = 0.5$*). The overall structure is essentially unchanged, except at very high prices and in the* $p = 2$ *price band. Note that the peak at* $p = 1$ *is cut off for clarity*

4.5.4.1 *Resistance of the Bertrand state to invasion*

Note that despite demonstrating that at low γ values, the Bertrand state may be invaded by a few expensive sellers, it still exhibits the properties of an evolutionary stable state. Specifically, mutation and selection alone (as they are defined in the dynamics of the game) are unable to reintroduce expensive sellers once they have died out. This is due to the large gap in the price distribution between the upper price of the Bertrand state and the lower price of the $p < 2$ band of expensive sellers. Many successive mutations increasing the price are required to bridge this gap and the intermediate prices are all highly suppressed by the selection process. Thus, although the Bertrand state is not always able to resist invasion from an outside source, internally it is stable.

4.6 ARE EXPENSIVE SELLERS BENEFICIAL?

We now consider the benefits to the system as a whole of incorporating expensive sellers into the steady state. The two obvious metrics that come to mind here are the fraction of buyers that are able to satisfy their demand for goods (the efficiency of resource exploitation) and linked to this, the fraction of live sellers. If all buyers are satisfied, then we expect there to be a larger fraction of live sellers than if not. Since increasing γ will *ceteris paribus* increase the fraction of live sellers, we compare results for different values of γ in two ways. Firstly, we consider the live site density *before* reentry has taken place. This allows us to compare like-for-like simulations with different γ values. Secondly, we carry out simulations in both the observed steady state and an artificially enforced Bertrand steady state. This allows us to compare the relative success of the two steady states for a fixed value of γ.

4.6.1 *Comparison of natural and Bertrand steady states*

Due to the finite demand in the system, and its locality (buyers cannot wander around if they find all their known sellers are bankrupt), not all buyer demand will necessarily be exploited. We can consider the amount of unsatisfied demand as a metric of how efficiently the system is exploiting the environment.

Does the development of a non-Bertrand state at low γ lead to better use of buyer demand? We might imagine that it would: if all sellers charge $p = 1$, there is only enough demand in the system to support half of them. This occurs if every other site in the lattice is occupied. We can thus easily put an upper bound on the number of active sellers after the rebirth stage, it is just $0.5(1 + \gamma)$. In contrast, if high prices are allowed, the system contains enough demand to support almost all sellers. If the system were in the state described in section 4.4.5, we could have a system in which all but three sellers survive. This state is easily invadeable, but it demonstrates that the system can support a larger number of sellers if prices are high.

Now, both of the above scenarios utilise all the available demand. However, the probability that the system is perfectly correlated (required in the Bertrand case) is vanishingly small. Further, rebirth events and the stochastic nature of choosing between two equally-priced sellers mean that were such a correlated structure to appear, it would soon be destroyed. This will lead to gaps appearing in the correlated structure with potentially two adjacent vacant sites. The buyer in between these two sites will not find anywhere to shop: some demand will be wasted. In contrast, a system containing higher prices does not need to attain a perfectly correlated state in order to exploit the available demand fully. We would therefore expect that for low values of γ the non-Bertrand state will use more of the available demand and additionally have a higher steady state site density than the Bertrand state.

This expectation is borne out in simulation. Since the Bertrand state is not the attractor for low γ we obtain the Bertrand results by artificially

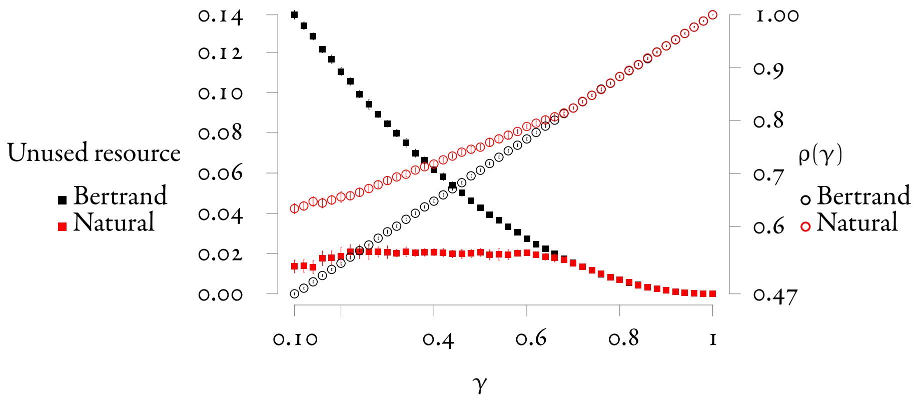

Figure 4.22 *Comparison of resource usage for enforced Bertrand and naturally obtained steady states as a function of rebirth probability. Shown are the unused resource (left axis) and fraction of live sellers after reentry (right axis). Errorbars show standard error in the mean over forty realisations of a system with* $N = 5 \times 10^4$, $\Delta = 0.08$.

restricting the allowed prices to $p \in [1, 1 + \Delta)$. Ignoring the $\gamma = 0$ case, for which the result is determined by the initial conditions, we see markedly different behaviour in the Bertrand and non-Bertrand states (figure 4.22). The figure also shows the cross-over of the natural state to the Bertrand one (the two resource usage curves lie on top of one-another). We might wonder if somehow above this cross-over, the Bertrand state becomes better at exploiting resource than the non-Bertrand one. To answer this question we first obtain a non-Bertrand steady state system (note that the state does not vary appreciably for $\gamma < \gamma_c$) and take the observed price distribution. Now we simulate with a higher value of γ but always sampling from our non-Bertrand steady state distribution, rather than the system. In this way, we impose a non-Bertrand state on the system up to $\gamma = 1$.

Figure 4.23 shows the resource usage and live site density of the system in a enforced expensive steady state and the natural steady state. The non-Bertrand state always utilises more resource than the natural one, but this difference becomes negligible at high γ when the dominant factor in the live site density is the reentry probability, rather than the price distribution. Our conclusion then, is that the natural state is more efficient than the Bertrand state at low γ, when the existence of expensive sellers has a large effect on the live site density. This becomes less important at higher γ and the Bertrand state becomes the attractor. Interestingly, this occurs where the straight line extrapolation of the expensive live site density crosses the Bertrand live site

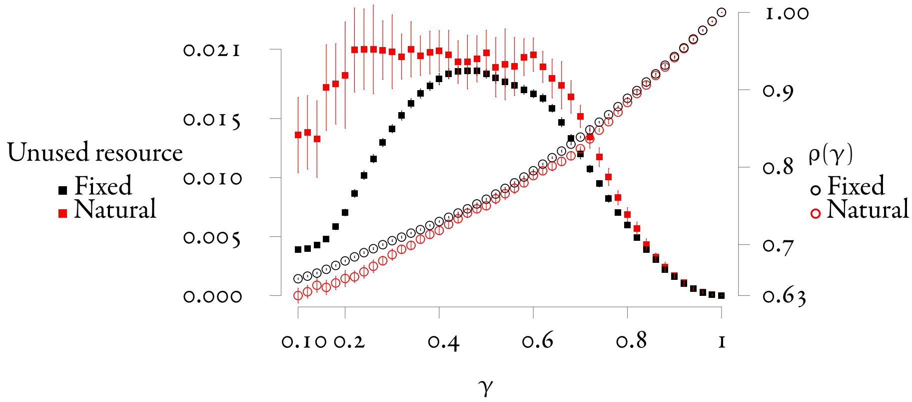

Figure 4.23 *Comparison of resource usage (left axis) and live seller fraction (right axis) for the natural steady state and a fixed low-γ steady state. For the latter, a steady state distribution was obtained for $\gamma = 0.5$ and subsequent simulations sampled from that distribution when choosing new prices during seller rebirth. Errorbars show standard error in the mean over forty realisations of a system with $N = 5 \times 10^4$, $\Delta = 0.08$. We see that the natural state does almost, but not quite as well as the fixed $\gamma = 0.5$ state. At low γ correlations build up in the system allowing for better resource usage than the live seller fraction would suggest*

density.

4.7 SUMMARY

We have presented results for a stochastic model market under evolutionary dynamics. Since the model is difficult to study analytically due to the high level of correlations in the system, we have predominantly studied the behaviour through simulation.

Our model shows that adding spatial heterogeneity to a Bertrand game can produce dispersion of prices. The model has two different steady states which are chosen depending on how full it is and can be controlled by varying the probability of bankrupt sellers reentering the market, γ.

The Bertrand steady state, which appears at high γ, is essentially the same as for the original Bertrand model: there is a single price which is set by the cost of production. At low γ, the steady state has a highly non-trivial structure supporting a variety of prices above the Bertrand price. We have shown that this state increases the overall level of buyer satisfaction (there is an increased density of live sellers) over the Bertrand state. The transition between the two states occurs in stages as bands of expensive prices become unprofitable.

GENERAL COMPETITION NETWORKS

5

In this short chapter, we extend our study of the previous chapter to a variety of different competition networks. The results we have seen so far have all been for a system on a one-dimensional ring. One-dimensional equilibrium systems in physics are somewhat special – they do not allow for a first order phase transition (although driven systems do allow for such transitions) – we might therefore wonder if our results are specific to our particular choice of competition network. The following results will show that this is not the case. With our choice of dynamics, particularly the linear scaling between number of buyers and overhead size, the qualitative picture is independent of the competition network chosen.

There are an infinite number of different networks we could study the behaviour of our model on. We restrict ourselves to just three types. Firstly, square lattices in more than one dimension. Latterly, two forms of random networks. Both the random networks we choose have the *small world* property. That is, the average distance between any two nodes in the network grows very much slower than the number of nodes in the network. A small world network is appealing since evidence suggests that social networks display this property [56]. See, for example, the classic work of Travers and Milgram [72]. If our competition network is a small world network, then we can think of buyers as clustered together, with the occasional link between clusters.

A further empirical observation that we might wish to include is that the size distribution of firms is heavy-tailed. The exact form of the tail is quibbled over, but typically authors settle on either a log-normal [13, 67] or power-law distribution [3, 27] or some combination of the two, see de Wit [19] for a review. In our model, we can think of the number of potential buyers a seller has as a proxy for its size. The size of a seller is then given by its degree – the number of of links it has – in the competition network. To model a heavy-tailed seller size distribution, we just need a heavy-tailed network degree distribution.

Since power-law networks are easy to construct, and we are not too concerned with the exact form of the tail, we choose to use a power-law competition network in our study.

Note that in this section we do not change the connectivity of the buyers. Competition between sellers always occurs in a pairwise fashion. It is likely that changing the connectivity of buyers will have a larger effect than the seller connectivity. If all buyers are infinitely connected, they will always visit the globally cheapest seller and the system will collapse onto the Bertrand result. If buyers only see a single seller, each seller is in a monopoly and the initial conditions will set the price distribution. We would expect there to be a transition between the behaviour we observe in pairwise competition and fully connected competition as a function of the connectivity, but without studying such a system in detail cannot say if it happens at finite connectivity or not.

5.1 HIGH-DIMENSIONAL SQUARE LATTICES

In this section, we study the behaviour of the system on square lattices of dimension two, three and four. In these systems, each seller has $2d$ competitors, where d is the dimensionality of the system. The overhead payment is hence also $2d$ and the Bertrand price is, as before, $p = p_0 = 1$.

Simulations with $\gamma < 1$ show that the system price distribution reaches a steady state. The qualitative structure of this distribution is unchanged from our results in one dimension (figure 5.1). The majority of sellers have a price $p = 1$, but there is price dispersion allowing for prices up to around $p = 10$ in the steady state. Additionally, there are favoured peaks in the distribution as we saw previously. Note how the higher dimensionality (corresponding to more potential buyers) stretches out the structure from the one-dimensional lattice (and adds more). The peaks at $p = 8$ in the 4-D, $p = 6$ in the 3-D, $p = 4$ in the 2-D and $p = 2$ in the 1-D distributions all correspond to survival with a single sale per round. In the two, three and four-dimensional distributions there is also a visible peak corresponding to two sales per round (at $p = 2$, $p = 3$ and $p = 4$ respectively) – this peak appears at $p = 1$ in 1-D. Notice the similarity in the price distributions either side of these peaks: a large drop as the price increases slightly (these features are marked with boxes in the figure). At high values of γ, the distribution, as seen previously, collapses to the Bertrand state in which all sellers charge $p = 1$ (modulo

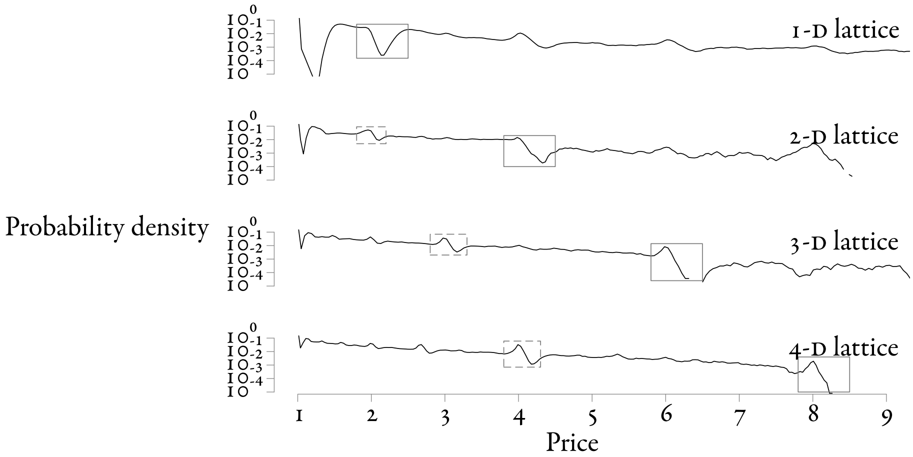

Figure 5.1 *Steady state price distributions for one* (N = 10^5)*, two* (N = 316^2)*, three* (N = 50^3) *and four-dimensional* (N = 20^4) *square lattices with* $\gamma = 0.5$*,* $\Delta = 0.08$*. Note the similar structure in all four distributions. Boxes show common features in each distribution due to survival of sellers making one (solid) and two (dashed) sales per round*

noise).

These results demonstrate that our system does not behave in a special manner in one dimension. On a fixed, high-dimensional, square lattice the features of the price distribution and system steady state are essentially unchanged.

5.2 SMALL-WORLD RANDOM NETWORKS

If the competition takes place on a fixed, but high-dimensional, square lattice, the model results are reasonably unchanged. Is the same still true if the system is simulated on a fixed, but no longer spatially localised, random network? To answer this question, we start with a lattice-based competition network and use the rewiring method of Watts and Strogatz [76] to construct a partially random network. In this scheme, we consider each link in the lattice in turn and with some fixed probability (q) pick up one end of the link and move it to a randomly chosen seller. There are a few restrictions involved here. We disallow both single-link loops (*i.e.*, both ends of the link cannot point to the same seller) and also links which duplicate an existing link in the competition network. Figure 5.2 shows a diagram of two attempted relinking events in a 2-D lattice. One allowed event (marked with a blue dot) and one disallowed event (red dot). The latter is disallowed since the link would duplicate an existing link in the network.

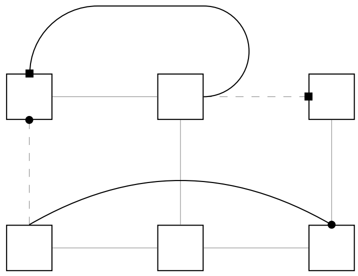

Figure 5.2 *Two relinking events in construction of a Watts-Strogatz small-world network. The marked end of the dashed link is picked up and placed somewhere else in the system. The circle indicates an allowed move, the square a disallowed one (see text for details)*

By tuning the parameter q, we can change the fraction of links that are rewired. If $q = 0$, we keep our original lattice, if $q = 1$ we end up with a network similar to an Erdős-Rényi random network – the two networks are not quite the same in this case due to the restriction of no self-loops and double links [56].

Here we show results for a three-dimensional lattice relinked with probability $q = 0.1$ and $q = 1$. The former preserves most of the square-lattice structure, the latter does not. Both values of q preserve the mean number of buyers per seller (since the construction does not remove or add links) at six.

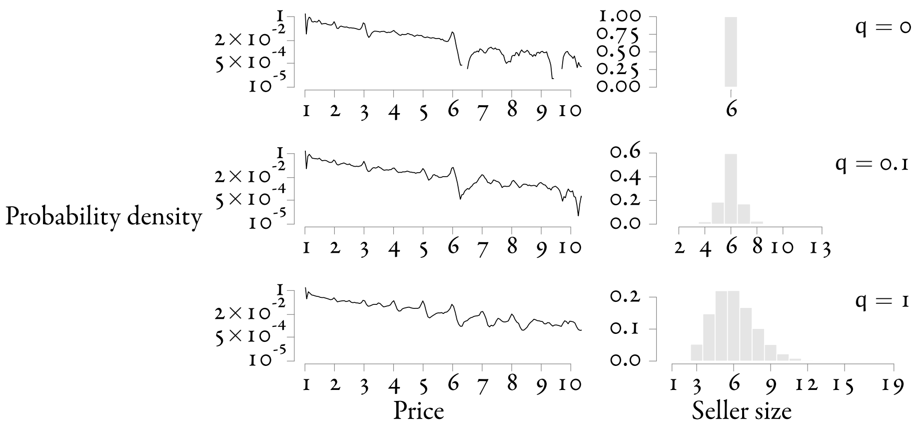

Figure 5.3 *Comparison of steady state price distributions obtained on a Watts-Strogatz competition network (left figures). The original network is a three-dimensional square lattice, the networks are constructed with relinking probability* $q = 0$*,* $q = 0.1$ *and* $q = 1$ *as indicated. Histograms on the right show the size distribution (number of potential buyers) of sellers.* $N = 50^3$, $\gamma = 0.5$, $\Delta = 0.08$

Figure 5.3 compares the steady state price distribution for a 3-D square lattice with $q = 0$, $q = 0.1$ and $q = 1$. Again, we find that the qualitative form of the steady state price distribution is unchanged by the modification of the competition network. If anything, rather than suppressing high prices, the high-priced peaks are more pronounced at high q. Note how the square lattice has a peak in the price distribution at $p = 6$, one sale per round, the $q = 0.1$ system has a peak at $p = 6$ but also peaks at $p = 5$ and 7 not present for $q = 0$. Looking at the distribution of seller sizes, we see that these peaks are also due to sellers making a single sale per round. The same feature is seen in the system with $q = 1$, the peaks in the price distribution at $p = 4$, 5, 6, 7 and 8 all correspond to sellers making a single sale per round. These sizes are the most probable in the size distribution, and so the peaks in the price distribution are more prominent than others.

5.3 POWER-LAW NETWORKS

Finally, we look at the behaviour of the system when the competition network has a heavy tail. As mentioned previously, the motivation for this is that real firms appear to have a heavy-tailed size distribution. We construct the competition network using the method of preferential attachment [4, 66, 82]. In particular, we follow the exactly linear attachment method analysed by Krapivsky et al. [38]. In this scheme, sellers are added to the system one at a time each with one link, the free end of this link is attached to an existing seller with probability proportional to the number of links it already has. Figure 5.4 shows a small example network constructed in this manner. The probability of finding a seller with k potential buyers is

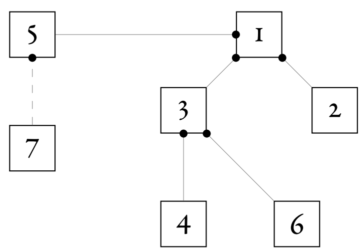

Figure 5.4 *Construction of a network via preferential attachment. Dots indicate the free ends of links that have been attached to existing sellers. Numbers indicate the order in which sellers appeared in the system. The most recently attached link is dashed*

therefore given by

$$n(k) = \frac{4}{k(k+1)(k+2)}. \tag{5.1}$$

The probability that we find a seller with k or more potential buyers is

$$N(k) = \sum_{j=k}^{\infty} n(j) = \frac{2}{k(1+k)}. \tag{5.2}$$

Knowing the exact size distribution is convenient because it allows us to test if our network construction algorithm is performing correctly.

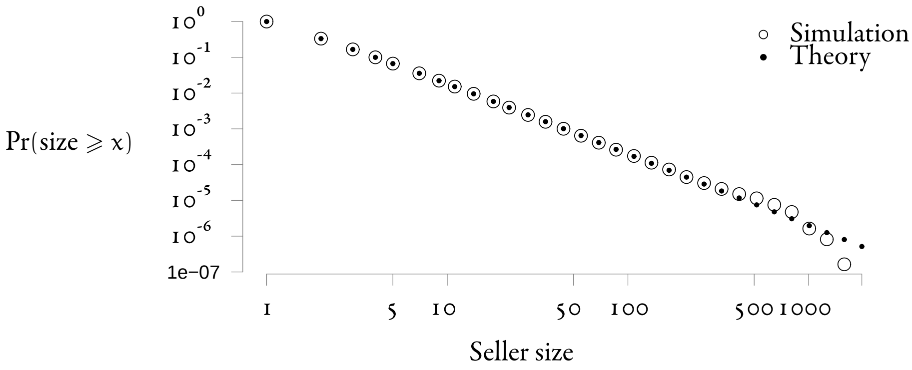

Figure 5.5 *Comparison of predicted and observed complementary cumulative seller size distributions of the power law competition network*

Figure 5.5 shows the observed seller size distribution when we construct a power law competition network and the predicted distribution. Notice the only slight deviation is in tail of the distribution, checking the statistical significance of this deviation requires a bit of work. We cannot use typical non-parametric tests for continuous distributions since neither the empirical nor the theoretical distribution are continuous. We can, however, use a Smirnov transformation to transform our discontinuous distribution to the uniform distribution on $[0, 1]$ [26].

Denote the discontinuous theoretical cumulative distribution function by $F(x) = 1 - N(x)$ (N given above by equation 5.2). Since our data are discrete, there will be n_{x_i} data points with value x_i. Order the data such that $x_1 < x_2 < \ldots < x_n$. We now generate n new random variables U_i. n_{x_1} of these new data points are distributed uniformly at random in $[0, F(x_1)]$, n_{x_2} uniformly at random in $[F(x_2), F(x_1)]$ and so on. If the null hypothesis is true (*i.e.*, if the experimental data are drawn from $F(x)$) then U_i will be distributed uniformly at random between zero and one. This allows us to perform a Kolmogorov-Smirnov goodness of fit test on our data obtaining a p-value of 1, indicating that the deviations are statistically insignificant.

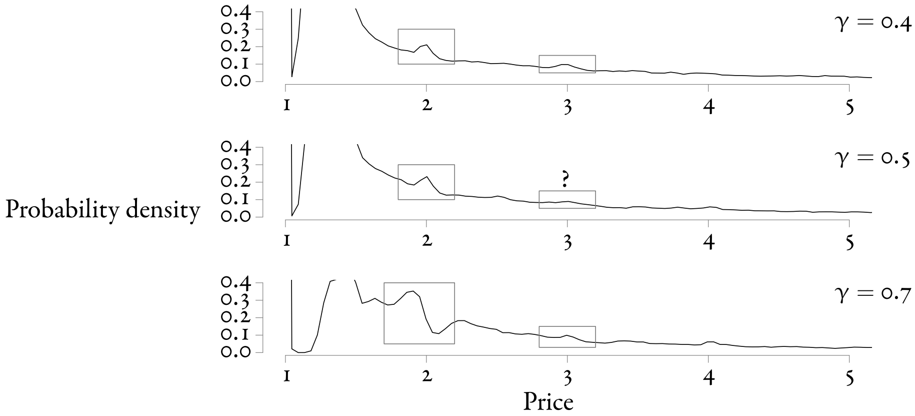

Figure 5.6 *Comparison of the steady state price distribution obtained on a power law competition network for different values of* γ*.* $N = 50^3$, $\Delta = 0.08$, γ *as indicated. Notice the weak peaks in the distribution at* $p = 2$ *(and sometimes* $p = 3$*) which we expect when considering the size distribution of sellers. Boxes indicate plausible peaks in the price distribution due to sellers of size two and three. The y-axis has been truncated at 0.4 for clarity of these features*

Having satisfied ourselves that the size distribution we obtain is correct, we now look at the steady state behaviour of the system. The previous section on Watts-Strogatz networks should give us an idea of what to expect. We still expect a steady state admitting expensive sellers but the structure of the price distribution should change somewhat. Specifically, the most common seller size is one, hence we expect a large peak in the distribution at $p = 1$. Sellers of size two and three are also reasonably common, so we expect peaks in the price distribution at $p = 2$ and $p = 3$ corresponding to these sellers making one sale per round. Larger sizes are not so widespread and so high-priced peaks will be washed out. Figure 5.6 shows the observed steady state price distributions for different values of γ. As expected, expensive sellers are able to persist in the steady state and some of the features we predicted are observable (small peaks at $p = 2$ and maybe $p = 3$).

5.4 SUMMARY

In this chapter we have briefly studied the behaviour of our synchronously updated model on a variety of different competition networks. The takehome message is that our choice of dynamics mean that the general form of the steady state is unaffected by the competition network. Expensive sellers continue to exist for some values of the reentry probability. The preferred prices in the steady state are, in part, determined by the seller size distribution. If sellers of size x appear with a high probability, we are likely to see a preferred price $p = x$ appear in the steady state which allows those sellers to survive making just a single sale in each round.

6 ASYNCHRONOUS TIME SIMULATION RESULTS

Our analysis of the model in chapter 4 talked extensively in terms of the discrete rounds of the game. Further, the explanations for stability are turn-based. We might think that the favoured prices appearing in the low-γ steady state are artifacts of the discrete nature of the update scheme.

In this chapter we show that the results persist even when the update scheme is not discretised into rounds as before. Rather the discrete nature of the sales that can be made is enough to ensure favoured peaks. In addition, we find a further 'steady state' in the system, not observed in the synchronous time results, that exhibits oscillations between expensive and cheap prices. This state bears at least a superficial resemblance to the limit pricing oscillations described by Salop and Stiglitz [61] (see section 2.4.1).

6.1 VALIDATION OF THE IMPLEMENTATION

As before, we first test that the model behaves as expected in certain limiting cases where we can predict the steady state behaviour. We then move on to more interesting cases. This is, however, not as simple as the synchronous time case (section 4.1). Consider first the $\gamma = 1$ state.

The number of buyers a shop attracts is drawn from a binomial distribution with mean 2, and so a seller could be very lucky and attract N buyers per round, allowing a price of $p = \frac{2}{N}$. In an infinite system, the minimum viable price is thus $p = 0$, rather than $p = 1$. Equally, a seller may not always pay the overhead, allowing survival even with no sales. On the flip side of this coin, a seller may pay multiple overheads: meaning that even if it attracts the expected number of consumers, it still makes a loss.

Due to these complications, when $\gamma = 1$, survival is not just as simple as outcompeting neighbours by charging $p = 1$. Nonetheless, we expect in the steady state to find no expensive sellers: they may survive for a while due to the noise, but eventually should be outcompeted and die off.

We might also expect that the price is driven below $p = 1$. The expected profit of $p = 1$ sellers is still zero, and so they are unlikely to survive being outcompeted by cheaper sellers. If the price drops significantly below $p = 1$, such that sellers must make more than five or so sales in order to survive when paying a single overhead, we should be able to predict the mean steady state density (prior to reentry).

We assume that no seller makes enough sales to survive paying a single overhead, and thus the probability of survival is just the probability that a seller pays zero overheads. This probability is given by a binomial distribution, so the probability of survival is $p_s = \left(\frac{N-1}{N}\right)^N$ which in the large N limit gives $p_s = e^{-1}$. This will not be quite correct, since it is possible for sellers with $p < 1$ to make enough sales to survive, however, it gives us an idea of what to expect.

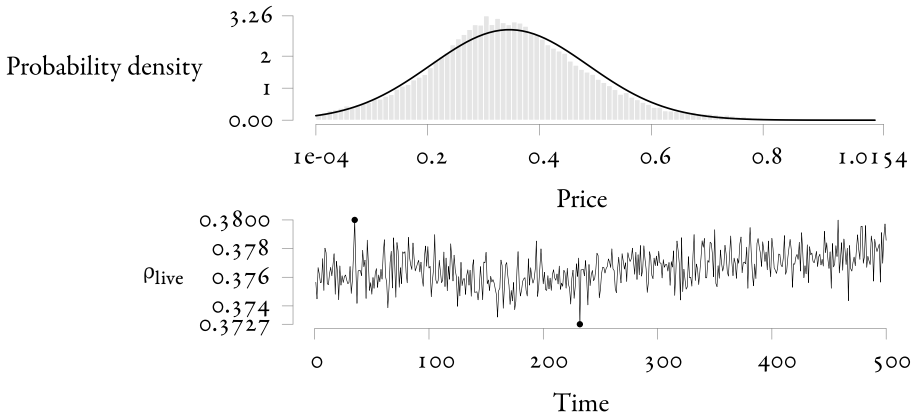

Figure 6.1 *Distribution of prices in the steady state and maximum likelihood Gaussian fit (upper figure) and time series of the live seller fraction prior to rebirth (ρ_{live}, lower figure) with $\gamma = 1$, $N = 10^5$ and $\Delta = 0.08$. The predicted live seller fraction if no seller makes a profit is $e^{-1} \approx 0.368$, somewhat lower than the steady state fraction we obtain. The price distribution appears Gaussian, however we can reject this hypothesis with high confidence, suffice it to note that the distribution is single-peaked*

Figure 6.1 shows the live seller fraction for the $\gamma = 1$ state. We see that the steady state fraction is close to (but somewhat above) the value $e^{-1} \approx 0.368$ predicted if live sellers are only those that pay zero overheads. This allows us to be reasonably confident that our simulation is doing what it should be.

6.2 LOW γ STEADY STATE

As before, we begin with a study of the system steady state for low γ. Recall (section 4.2) that under synchronous updates this region of parameter space contains expensive sellers. Our reasoning for the stability of the state relied somewhat on the idea of discrete selling rounds, we might therefore wonder if the steady state structure we observe is only stable due to the discretised nature of updates.

Simulations of a system undergoing asynchronous updates show that this is not the case. For a similar range of parameters, the system exhibits a steady state containing expensive sellers. The structure of the steady state price distribution is visually similar to that observed in the synchronous case (figure 6.2) especially at high prices, although this does not stand up to a statistical test. Differences are especially noticeable in low γ regime.

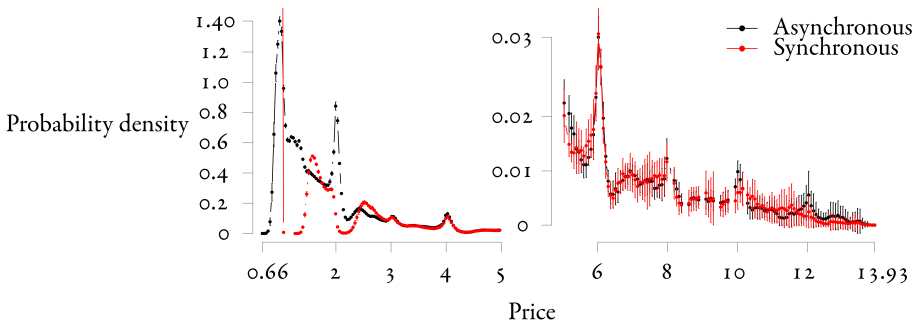

Figure 6.2 *Comparison of steady states obtained under asynchronous (black) and synchronous (red) dynamics. Both simulations have* $N = 5 \times 10^5$, $\gamma = 0.5$, $\Delta = 0.08$. *The asynchronous price distribution is smoother (unfavoured prices are not as sharply suppressed) but the main features of peaks and troughs remain in approximately the same positions. Error bars show standard error in the mean over ten realisations of the same system: note that the structure observed in between large peaks does not appear to be due to noise*

As before, favoured prices appear at approximately integer multiples of the expected break-even price ($p = 1$). We saw how this feature appears in the early time synchronous dynamics in section 4.3.2. We now perform a similar analysis for the asynchronous dynamics to show that the effect persists, albeit in modified form.

6.2.1 *The first round price distribution under asynchronous updates*

Similar to our results for the synchronous time simulation, we can carry out an exact analysis for the first round of the game since there are not yet any spatial correlations in the system. The methodology is essentially unchanged from our exposition in section 4.3.2 although the details are somewhat more involved. We wish to write down the survival probability of a seller as a function of price. With this information, and knowing the initial conditions, we can then write down the probability distribution of live seller prices at the end of the first round. The tricky step comes in writing down the survival probability. We can no longer state with certainty how many potential buyers a seller has, nor how many sales it must make to survive. We can only write down probability distributions for these events.

Denote the system size by N, each round, N sellers are chosen at random to pay the overhead. The probability that a seller pays k overheads is given by

$$d_N(k) = \binom{N}{k} N^{-k} \left(\frac{N-1}{N}\right)^{N-k}. \tag{6.1}$$

In the one-dimensional system there are also N buyers in total, and so N random buyers are selected to go shopping. Each seller has two potential buyers and so the probability that a seller attracts k buyers is

$$b_N(k) = \binom{N}{k} (2N)^{-k} \left(\frac{N-2}{N}\right)^{N-k}. \tag{6.2}$$

This is *not* the probability that a seller makes k sales, merely the probability that k buyers 'walk in the door' and look at a seller's price.

Finally, we need to write down the probability that a seller makes at least k sales (we can then easily obtain the survival probability). This poses more problems than in the synchronous time case, since a seller may attract the same buyer multiple times. The probability that a seller sells twice to the same buyer is the same as the probability of selling once: we cannot treat the buyers as independent in this case. With some thought, we can write down the probability of making at least k sales given we attracted a certain number m of buyers, $s_N(k|m)$. The probability of making at least k sales is then just the average of $s_N(k|m)$ over the distribution $b_N(m)$.

Every seller has two distinct potential buyers which we shall denote by L and R. Since a seller's opponents do not change their prices in the course of a single round, the probability of making a certain number of sales will only be a function of $f_1(p)$ and $f_1(p)^2$, where $f_1(p)$ is given by equation 4.1. To make the exposition clearer we first derive $s_N(k|m)$ for the particular case of $m = 5$ and then generalise.

There are $2^5 = 32$ possible ways of getting five buyers, corresponding to different numbers of L and R visits. The different combinations are shown, with their corresponding frequency (given by a binomial coefficient) in table 6.1. From this table, we can easily see that $s_N(k = 1|5)$ is just the probability

$5L\,(1)$	$4L + R\,(5)$	$3L + 2R\,(10)$
$5R\,(1)$	$L + 4R\,(5)$	$2L + 3R\,(10)$

Table 6.1 *Possible combinations of* L *and* R *buyers, and the number of different ways they can occur, for* $m = 5$

of outcompeting a random seller $f_1(p)$ (given by equation 4.1). Similarly for $k = 2$ and $k = 3$. For $k = 4$ the probability changes somewhat. We find the probability of making four or more sales is

$$s_N(4|5) = 2^{-5} f(p)\big[12 + 20 f(p)\big]. \tag{6.3}$$

The combinations with three L buyers and two R buyers (and vice versa) require we sell to both L and R (probability $f(p)^2$), the other combinations only require selling to L or R (probability $f(p)$). A similar analysis allows us to write down $s_N(5|5)$. All sales probabilities are shown in table 6.2.

k	$s_N(k\|5)$
0	1
1	$f(p)$
2	$f(p)$
3	$f(p)$
4	$2^{-5} f(p)\big[12 + 20 f(p)\big]$
5	$2^{-5} f(p)\big[2 + 30 f(p)\big]$
6 or more	0

Table 6.2 *The probability that a seller with price* p *makes at least* k *sales, given they have attracted five buyers*

A bit of thought allows us to write down $s_N(k|m)$ for general k and m. If $k \leqslant \lceil \frac{m}{2} \rceil$, the the combination of L and R visits will always be such that the seller only needs sales to one or the other buyer and hence

$$s_N(k|m) = f(p) \qquad k \leqslant \left\lceil \frac{m}{2} \right\rceil. \tag{6.4}$$

If $k > \lceil \frac{m}{2} \rceil$ then the seller might have to sell to R and L. If the L (or by symmetry R) buyer arrives at least k times, then we need only sell to it, if not, we must sell to both. We can count the number of times each of these events occurs by summing binomial coefficients and so

$$s_N(k|m) = \frac{f(p) 2 \sum_{i=k}^{m} \binom{m}{i}}{2^m} + \frac{f(p)^2 \left[2^m - 2 \sum_{i=k}^{m} \binom{m}{i}\right]}{2^m} \qquad k > \left\lceil \frac{m}{2} \right\rceil. \tag{6.5}$$

We can now write down the form of $s_N(k|m)\ \forall k, m$:

$$s_N(k|m) = \begin{cases} 1 & k = 0 \\ 0 & k > m \\ f(p) & k \leqslant \lceil \frac{m}{2} \rceil \\ \frac{f(p)}{2^m} \Bigg[\underbrace{2 \sum_{i=k}^{m} \binom{m}{i}}_{\text{sales to L or R}} + \underbrace{f(p) \left(2^m - 2 \sum_{i=k}^{m} \binom{m}{i}\right)}_{\text{sales to L and R}} \Bigg] & \text{otherwise.} \end{cases} \tag{6.6}$$

The probability of making k or more sales is then simply

$$s_N(k) = \sum_{i=k}^{N} b_n(i) s_N(k|i) \tag{6.7}$$

and the probability of survival is

$$p_{s,1}(p) = \sum_{k=0}^{N} d_N(k) s_N \left(\left\lceil \frac{2k}{p} \right\rceil \right). \tag{6.8}$$

Finally, we can find the live seller price distribution at the end of the first round (again assuming no correlations in the joint survival probability)

$$P_1(p) = \frac{P_0(p) p_{s,1}(p) (1 + \gamma(1 - p_{s,1}(p)) P_0(p))}{\int_{p_0}^{p_{max}} P_1(x) dx}. \tag{6.9}$$

This is exactly the same as equation 4.6, but with a different expression for the survival probability. Unlike in the synchronous case, our result for $p_{s,1}(p)$ means that the solution does not have a simple analytic form. We can, however, evaluate it numerically for small N[1] and compare the resulting

[1]The resulting distribution does not change noticeably with $N \in [50, 10^3]$

distribution with that obtained from simulation (figure 6.3).

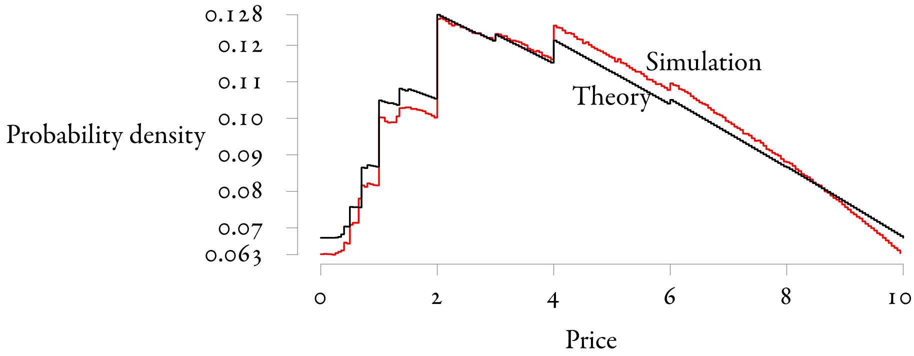

Figure 6.3 *Comparison of theoretical (black) and empirical (red) live seller price distributions at the end of the first round. Both are histograms normalised such that the area underneath is unity. The theoretical distribution has the same overall shape as the empirical one, but gives lower survival probabilities for mid-ranged prices. Empirical distribution obtained from simulation with one hundred realisations of a system with* $N = 10^6$, $\Delta = 0$, $\gamma = 0.5$, $p_0 = 0$ *and* $p_{max} = 10$. *The theoretical distribution has survival probability obtained from by substituting equation 6.8 into equation 6.5 with* $\gamma = 0.5$, $p_0 = 0$, $p_{max} = 10$ *and* $N = 200$

The result is not the same quality of fit to the data that we saw previously for the synchronous case in section 4.3.2, although the two distributions do have a qualitatively similar shape, indicating that our analysis is in the right direction. This may be because we have ignored the correlations in the two site joint survival probability mentioned in section 4.3.2.1. Since selling events can occur multiple times at the same site in a single round, ignoring the correlation in the joint survival probability is likely a worse assumption than for the synchronous time analysis.

Notice how, in comparison to the synchronous result (figure 4.5), the distribution has many more peaks in it, this occurs due to the distribution of overhead payments a seller might make.

6.2.2 *Differences in the steady state relative to synchronous dynamics*

The most noticeable difference between asynchronous and synchronous steady states is the lack of a sharp peak at $p = 1$ in the asynchronous price distribution. In the synchronous time system, the minimum allowed price is $p = 1$ and such a seller (with expected capital of zero) is unlikely to be out-competed giving a long lifetime. In the asynchronous system, the minimum allowed price is $p > 0$ and so a seller with $p = 1$ (still with expected capital of zero) is much more likely to be outcompeted. Hence, less probability mass appears at $p = 1$ and the peak in the distribution spreads out (similarly to how higher-priced peaks also spread).

Note that for prices $p \gtrapprox 3$ the synchronous and asynchronous steady states look reasonably similar. For low prices there is a large difference. We therefore wonder if the low price difference is due only to the different boundary conditions imposed at the low edge. Recall for synchronous dynamics we have a hard boundary at $p = 1$ while for asynchronous dynamics it is at $p = 0$.

This hypothesis is easily tested: we simply simulate synchronous dynamics with a $p > 0$ rather than $p \geqslant 1$ requirement. This gives us the price distribution shown in figure 6.4. As this figure shows, the removal of the

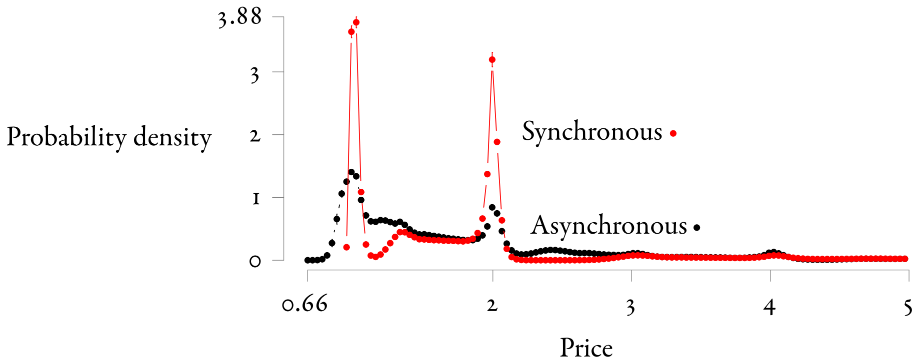

Figure 6.4 *Price distribution for synchronous dynamics with a requirement only that* $p > 0$ *rather than* $p \geqslant 1$ *(red). Although the distribution is slightly different from that seen previously* (e.g., *figure 4.2), the distribution still does not match up with that observed under asynchronous dynamics (black). Prices above* $p = 5$ *suppressed for clarity, the agreement between distributions in this region is good.* $N = 5 \times 10^5$, $\gamma = 0.5$, $\Delta = 0.08$. *Errorbars show standard error in the mean over ten realisations of the initial conditions*

sharp boundary does make a difference to the price distribution, but the synchronous and asynchronous cases are still noticeably different.

6.3 EXPENSIVE SELLERS IN THE STEADY STATE

Clearly, expensive sellers can exist in the steady state under asynchronous dynamics: the distribution we get is not simply an artifact of the choice of update algorithm. In a similar manner to the synchronous case, there must be some mechanism by which expensive sellers can persist. We look at the distribution of ages at bankruptcy for a clue (figure 6.5) and find that some expensive sellers can survive for the entire lifetime of the system as was the case in the synchronous simulation.

Comparing the lifetime distribution for the asynchronous system with that of the synchronous system (see figure 4.9) we note that cheap sellers do not typically survive as long in the asynchronous case. This effect is due to the different minimum price boundary. In the synchronous system, a seller with a price $p = 1 + \epsilon$ has an expected profit of around ϵ which is all but guaranteed since the probability that a neighbour sets up with a price $p' < 1 + \epsilon$ is very small. In the asynchronous system, prices less than $p = 1$ are allowed and so a seller with $p = 1 + \epsilon$ accrues only a very small amount of capital and is much more likely to be undercut. This reduces the expected lifetime of cheap sellers relative to that observed in synchronous dynamics.

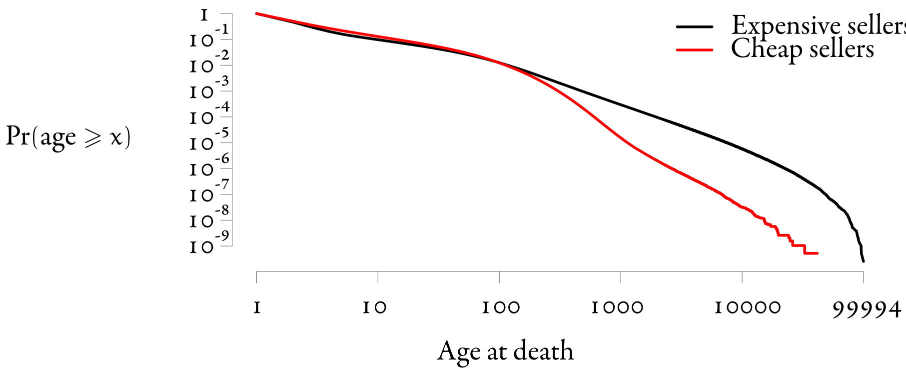

Figure 6.5 *Complementary cumulative probability distribution* $\Pr(\text{age} \geqslant x)$ *of the age at bankruptcy of expensive (*$p > 1 + \Delta$*, black) sellers and cheap sellers (*$p < 1 + \Delta$*, red) in the steady state with* $N = 5 \times 10^5$, $\gamma = 0.5$, $\Delta = 0.08$*. The maximum possible lifetime in the system is* 10^5*. Both expensive and cheap sellers have a heavy-tailed age distribution, the cheap sellers less so, in line with our expectation that* $p = 1$ *sellers should not be very long-lived*

Following the results of section 4.4.3 we look for a common pattern in the neighbourhood of long-lived expensive sellers to see if there is some theme linking them. It happens that the survival of expensive sellers occurs in essentially the same manner as that described in section 4.4.3. Long-lived expensive sellers survive in niches created by correlated structures of cheap sellers. An examination of the neighbourhoods of long-lived expensive sellers confirms the appearance of structures essentially identical to those shown in figures 4.10 & 4.13. Note that these structures are not quite as stable as the equivalent setup under synchronous dynamics. Even if a seller sets up next to a more expensive opponent, it cannot guarantee a sale, only an expected sale. Equally though, a seller might set up next to a cheaper opponent but avoid bankruptcy by avoiding overhead payments.

6.4 ARE THE EXPENSIVE SELLERS BENEFICIAL?

Now we move on and study how the addition of expensive sellers alters the efficiency of resource exploitation. As figure 6.6 shows, the unsatisfied demand follows a curve broadly similar to that of the discrete time model. The main difference is in the value at which the Bertrand curve diverges from the natural steady state and the larger overall variance in the results. This latter feature is due to the additional type of steady state dynamics in the asynchronous model: namely system-wide oscillations (see section 6.5).

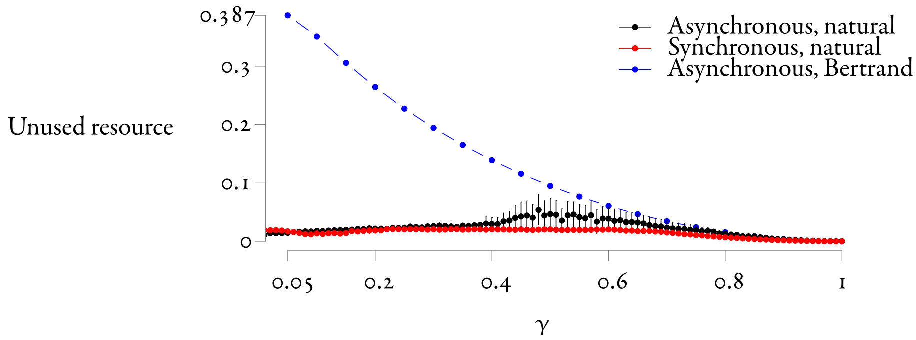

Figure 6.6 *Fraction of unsatisfied customers in natural (black) and enforced Bertrand steady state (blue) for an asynchronously updated system. Also shown is the fraction of unsatisfied customers in the natural steady state under synchronous dynamics (red). Errorbars show standard error in the mean over forty realisations.* $\Delta = 0.08$, $N = 2 \times 10^4$ *for the natural steady states and* $N = 5 \times 10^4$ *for the cheap steady state. The large errorbars in the asynchronous case occur in the region of the parameter space in which oscillations are observed (see section 6.5)*

6.4.1 *Comparison of synchronous and asynchronous steady states*

We now compare the relative successes of sellers in the steady states obtained under synchronous and asynchronous updating. We have already observed the similarities in the fraction of unsatisfied buyers in figure 6.6. For low γ (and $\gamma \rightarrow 1$) there is no real difference in the efficiency of resource usage. For intermediate values of γ, the system performs worse with asynchronous dynamics (although still better than the enforced Bertrand state).

We now consider two further surrogates for success in the system, the global mean capital and the global mean price. These should typically follow one another quite closely, but may give us slightly different information about the system. The previous results tell us how much of the available resource is being used by sellers. These measures will tell us how much profit the buyers are being squeezed for.

Figure 6.7 compares the mean system capital as a function of γ for the two cases of synchronous and asynchronous updating. For low γ the system does significantly better under synchronous updates. However, as the high-priced steady state disappears under synchronous updates (between $\gamma = 0.5$ and $\gamma = 0.7$), the mean system capital decreases almost to zero. For the same parameter values, the system now performs better under asynchronous updates. So, although the system always satisfies more buyers under syn-

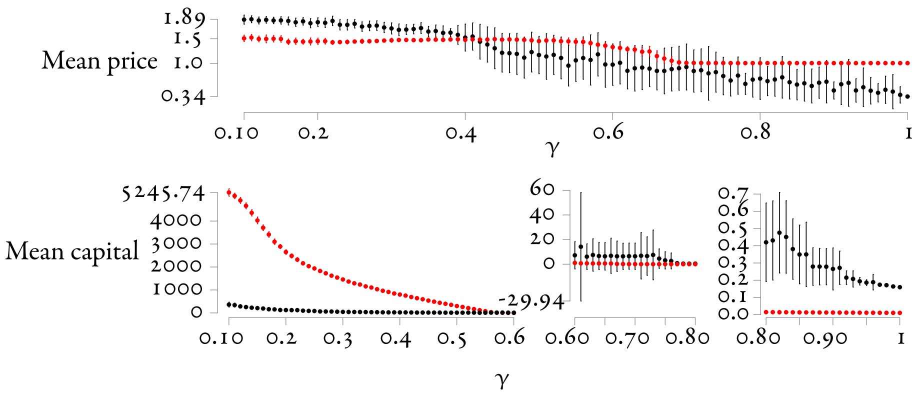

Figure 6.7 *Comparison of the mean price (top) and mean capital (bottom) in systems undergoing synchronous (red) and asynchronous (black) updates as a function of* γ*. Errorbars show standard error in the mean over forty realisations. The measurement is taken after fifty thousand timesteps.* $N = 2 \times 10^4$, $\Delta = 0.08$. *Notice how the mean capital in the synchronous simulation is typically higher than the asynchronous case until expensive sellers disappear*

chronous updates, the large oscillations seen in the system state under asynchronous updates allow for more efficient exploitation of the buyers that are attracted. Essentially, asynchronous updates allow expensive sellers to persist at higher γ values than the synchronous updating scheme. We now ask why this should be. In the preceding sections we have already made passing reference to the idea that the asynchronously updated system enters an oscillatory state at high γ: something not observed in the synchronous simulations. We now study these oscillations in more detail and show how they allow for survival of expensive sellers at high γ values.

6.5 OSCILLATIONS: SURVIVAL THROUGH EXTINCTION EVENTS

We now look more closely at the region of parameter space in which the asynchronously and synchronously updated systems diverge most noticeably. We find that here, where the synchronous system collapses onto a δ-function price distribution, the asynchronous updates allow for oscillatory behaviour in the price distribution. The system mean price repeatably oscillates between $p \approx 2.5$ and $p \approx 0.8$. A typical timeseries of the system mean price in such a state is shown in figure 6.8.

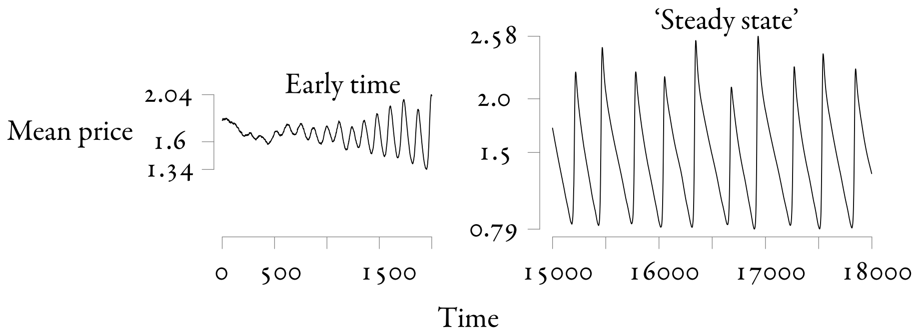

Figure 6.8 *Typical oscillations in mean price. Shown are the initial small oscillations while the system reaches the oscillatory steady state and the late 'steady state' oscillations. Note that the x-axis is not contiguous.* $N = 10^5$, $\gamma = 0.75$, $\Delta = 0.08$

We note a few key features of this oscillation. Firstly, the minimum in the mean price is less than the break even price of $p = 1$. This implies that the majority of sellers will fail to survive: we can see this somewhat like an extinction event. Secondly, the period and amplitude are irregular: although in isolation each oscillatory period looks very similar to another, the height of the peaks and troughs changes. Finally, the oscillation is not symmetric: the price increase occurs much faster than the price decrease.

What is the cause of these system-wide swings in the optimal price? The price distribution cannot have turned into a δ-function whose peak moves through the period: the price increase happens too quickly to be accounted for by mutation alone. For example, the above timeseries has some price increases of $\Delta p = 1.5$ occurring in around thirty simulation rounds. The minimum number of mutation events needed to effect this change in the mean price is around forty (with $\Delta = 0.08$). Rather than looking at the mean price for clues, we consider the change in the system price distribution over the course of a cycle. The modal price changes in sync with the mean price fluctuations, however the support of the distribution remains largely constant. Even in the lowest points of the cycle, some high priced sellers remain, and vice versa. Upswings are nucleated by high priced sellers and happen very suddenly while downturns are more gradual and appear to happen due to competition and selection favouring lower prices. This is illustrated in the sequence of price distributions in figure 6.9

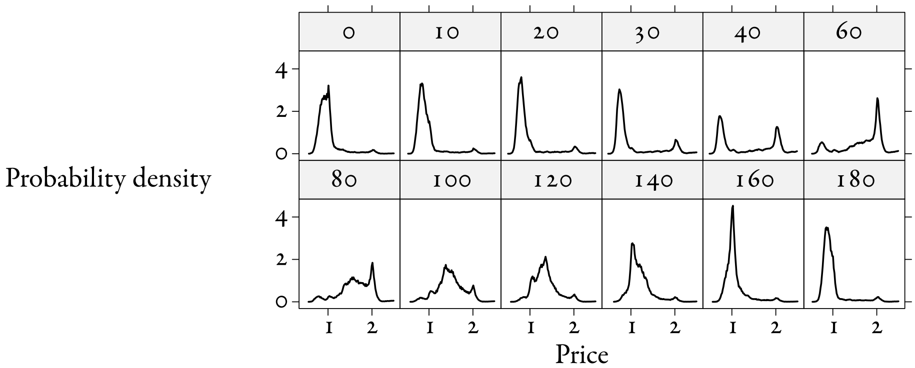

Figure 6.9 *A sequence showing the change in the price distribution of a system undergoing oscillatory behaviour. Labels in each subfigure indicate time elapsed since first figure. Price distribution truncated at* $P = 2.5$ *for clarity (some sellers have prices as high as* $P = 8$*). Note the nucleation of the high priced peak which then migrates slowly to lower prices.* $N = 10^5$, $\gamma = 0.75$, $\Delta = 0.08$

We now look at these nucleation events more closely. We label each seller uniquely at the beginning of a simulation and copy this label along with the price onto a reentered seller. This allows us to define *franchises*: sets of sellers with a common ancestor. We now look at the statistics of these franchises through oscillation events. High-priced franchises typically only contain a handful of sellers during the low-price phase of the oscillation (this is to be expected). During the high-price phase, the majority of the sellers in the system belong to a handful of expensive franchises. Figure 6.10 shows an example of this. We track the size and mean price of a single expensive franchise through six oscillatory periods. In one of these periods, this franchise accounts for almost half the sellers in the system.

The picture here is of a few high-priced sellers appearing in a favourable local environment and building up a large capital. These sellers are then able to persist through the period of price decrease (they have enough capital to

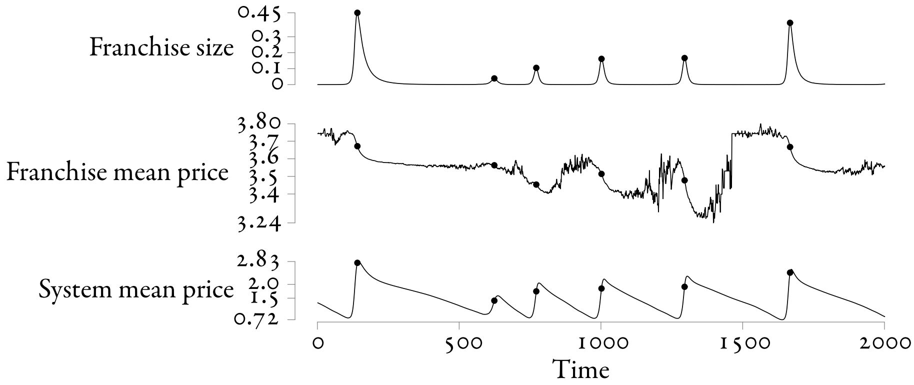

Figure 6.10 *Time evolution of an expensive franchise. Shown top to bottom are the size of the franchise as a fraction of system size, the mean price of the franchise and the mean price of the system. Dots mark the position of the maximum franchise size through each cycle.* $N = 10^5$, $\Delta = 0.08$, $\gamma = 0.75$. *Note how the increases in the mean price are strongly positively correlated with increases in franchise size*

survive for many tens of rounds without a single sale). When the system reaches the bottom of the cycle these sellers are preferentially chosen for reproduction over the many cheap sellers (which are now bankrupt). They proliferate rapidly, causing the mean price to rise and the fraction of live sellers to increase. Eventually the fraction of live sellers becomes large again and now expensive sellers no longer compete with empty spaces, but rather other sellers. Now the dominant strategy is to undercut competitors, rather than sell to captive consumers. In this situation, we find a combination of Bertrand competition and favouring of certain prices pushes the mean price down. Again, at the bottom of the cycle, the system overshoots which switches the fittest strategy from a low-priced to a high-priced one. The cycle begins again.

The gradual downward trend in the mean price once a peak is reached has a different cause to the price increase. The change is due both to sellers switching franchises and also adaptation within a franchise. This is evident in two ways. When looking at the rate of change in the mean price there are two definite regimes. When the mean price is roughly above $p = 2$ the rate of change is fast, it then becomes smaller and constant until the bottom of the cycle is reached (figure 6.11). The former phase is characterised by sellers jumping ship from very high-priced franchises which are no longer profitable to lower-priced ones. The latter phase has Bertrand competition

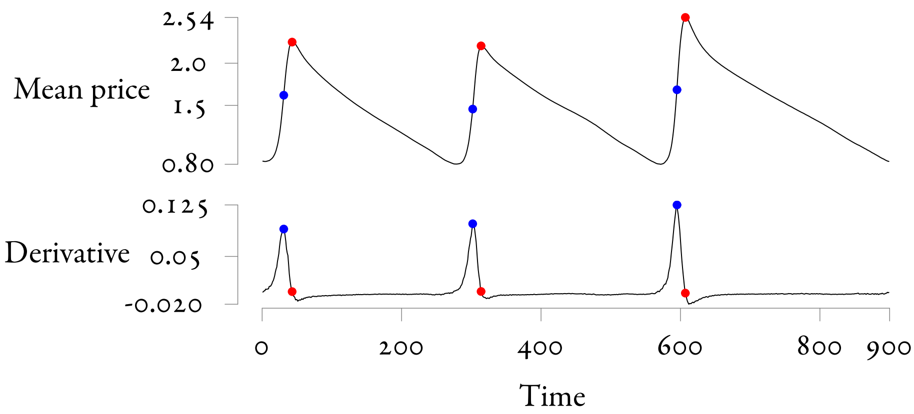

Figure 6.11 *Change in mean price and its derivative through three oscillatory periods. Note the approximately constant value of the derivative once the mean price drops below* $p \approx 2$. *Maxima of the mean price and its derivative are marked on each timeseries by red and blue dots respectively*

within franchises driving the price down slowly. We see this in the timeseries of price distributions (figure 6.9). In the high price phase a peak at around $p = 2$ appears. Once this is established, a peak forms at $p \approx 1.5$ and the higher-priced sellers disappear. Now no more lower-priced peaks form in the distribution, but rather the single peak decreases in price.

We may wonder why very low prices do not re-establish themselves directly once the system saturates with high-priced sellers. This is simply due to the choice of copying dynamic: there are only a few low-priced sellers and so they are only occasionally copied. Additionally, the low-priced sellers that are in existence have a price $p < 1$ which is not long-term viable, hence any new low-priced sellers that do appear die almost immediately. This can be seen nicely in the price distribution, although the mean price decreases, the probability mass in the low end of the price distribution also falls.

The long-term stability of the oscillatory behaviour is dependent on the survival of a very small number of expensive sellers. These sellers are able to survive through the extinction events since they have a large store of capital. This capital is not always fully replenished when the expensive sellers thrive and it is therefore possible that the few expensive sellers eventually become bankrupt. Should no expensive sellers survive through an extinction event, the oscillations reduce in amplitude. Once expensive prices disappear, mutation will not bring them back. We still see some oscillatory behaviour – although at much lower prices and over a smaller range – since the system still does prefer more expensive sellers over less expensive ones when it is empty.

6.5.1 *Evolutionary interpretation*

We can view the oscillations in an evolutionary light and the description becomes very simple. In the low γ case, the fitness landscape is static and the system approaches it and eventually reaches a steady state. For high γ, this does not occur. Rather, the fitness landscape presents a moving target. In the high-priced state, a fit strategy is a low-priced one, and the system moves towards this high fitness state. Upon reaching the low-price state, however, the landscape changes: due to the different death rate a high-priced strategy is fitter. The system is continuously chasing its own tail. This behaviour of extinction and subsequent recovery would appear to have a very similar driver to the cyclical events observed in a two-dimensional Daisyworld [79, 80].

We note that this changing of the fitness landscape happens as the effective death rate changes (measured as the number of individuals dieing per individual per round). However, imposing a global death rate equal to the one observed in an oscillating system does not produce anything even close to similar results. The death rate for individual sellers cannot be described by a global mean. In fact, even discretising the death rate into a price-dependent one will not work. This is perhaps a key point in our model. Unlike in other evolutionary models, the death of entities does not happen with some externally determined rate (as in a point process), but rather emerges from the dynamics. It therefore does not make sense to talk about a death rate per se. The probability of death of a particular seller is not only a function of its price, but also the seller's age and local environment.

6.5.2 *Aperiodicity*

One feature of the oscillatory behaviour that immediately stands out is the lack of a fixed period. There is a modal period but some significant deviation around it (figure 6.12).

This aperiodicity is interesting since it suggests that the period of the oscillation is not some simple function of the system parameters. When we look at the details of the oscillatory cycle we see the cause of this stochasticity

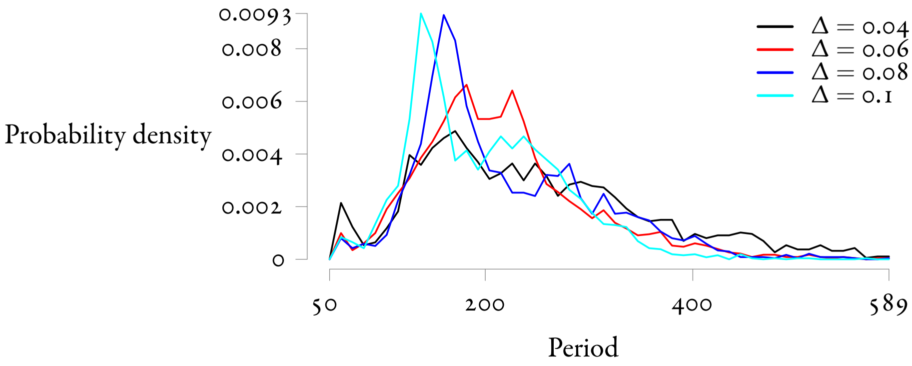

Figure 6.12 *Distribution of oscillatory periods for different values of Δ as indicated in the legend. Note the appearance of a modal period but also a large deviation around it.* $\gamma = 0.75$, $N = 5 \times 10^4$

in the period. The downward trajectory of the mean price happens at an approximately constant rate: competition between sellers forces the price down. The length of this phase is mediated by the size of the mutation term. The minimum of the cycle occurs at an approximately constant price, the maximum does not.

At the bottom of a cycle, most cheap sellers do not survive and the few expensive franchises can proliferate. We can see this when correlating the price oscillations with the fraction of expensive sellers alive at the end of a round (before reentry). Obviously this is at a peak at the top of the price oscillation, however, once the mean price drops below $p = 1$ (an unsustainable mean level), the relative fraction of expensive sellers starts to increase. While the overall level of live sellers is low, this produces a positive feedback and the increase accelerates.

As shown in figure 6.10, the size of an expensive franchise at the peak of an oscillatory cycle is not constant. If the franchise is lucky, it will increase in size before other expensive franchises near the cycle minimum. Since the probability of selecting a franchise for price copying is proportional to its size, the franchise will continue to grow (preferentially over other expensive franchises). The mean price at the top of the cycle is therefore dependent on both which franchises get lucky and also their prices. Rather than being a function of the system parameters, it is a random variable. The price decrease occurs at an approximately constant rate and so the period of the oscillation is a function of the height of the cycle maximum. Hence, the period is also a random variable.

6.6 SUMMARY

We have studied our model under asynchronous updates finding that the complex structure of the low γ steady state remains, despite an initial analysis of the synchronous updating scheme suggesting the result might be an artifact of the discrete simulation rounds. We have further shown that the asynchronous dynamics allow for an intermediate oscillatory state to exist between the low γ expensive steady state and high γ Bertrand state. This region of the phase space is not seen under under synchronous dynamics.

In the next chapter we look at a few different ways of reducing the amplitude of these oscillations. Drawing an analogy to the global economic system, subsystems of which undergo oscillatory boom and bust cycles, we consider how to reduce the likelihood of a global crash (analogous to reducing the likelihood of mass extinction in our model).

7 COUNTERACTING BOOM AND BUST

The oscillations we see in the asynchronous time model are broadly reminiscent of the large-scale cycles observed in parts of the global economy. Equally, in an ecological interpretation, we might see them as repeated extinction events and subsequent recovery. Certainly in the field of economics, many policy makers would be keen to avoid the crashes (if not the upturns) associated with these cycles. In this chapter we consider two methods of suppressing (with varying degrees of efficacy) the oscillatory cycles exhibited in our model and suggest that similar methods may be applicable to real-world systems.

7.1 A MODEL OF LOOSELY-COUPLED ISLANDS

The first method of stabilisation we propose is motivated by the observation that the period of the oscillations is not constant. If we take the average price of two separate systems, both oscillating independently, it is unlikely that they will remain in phase with one another for any length of time. Although the individual systems will go through boom and bust cycles, the effect on the *global* system will likely be a reduction in the amplitude of oscillations. If we divide the system into many such islands, we might hope that the global oscillatory mode is almost completely suppressed. As it stands, this setup is no more desirable than previously, individual islands oscillate and boom and bust still continues. However, consider what might happen if we couple the islands in some way. In a non-stochastic system, any small amount of coupling will eventually bring oscillations into phase. In a noisy system, this may not be the case. Initially any oscillations are likely to develop out of phase, some weak coupling may stabilise the globally observed system fluctuations. Furthermore, recall that occasionally the high-priced oscillations in the system could collapse (when the number of high-priced seed sellers dropped to zero). If we couple many islands together, we can import strategies from outside any particular island which may mitigate this latter effect, leading to an increased global mean price.

7.1.1 *Choice of coupling*

Country economies are coupled across the world primarily through global trade. Separate ecosystems are coupled through migration events. In our system, there is no easy way to carry out the former coupling choice: sellers do not interact by trading goods at all and shoe-horning such an effect into the dynamics would feel rather false. Instead, we can think of migration between islands as similar to how global stock markets respond to news from other markets. The interpretation here is that sellers observe the strategy choices that sellers in other markets use and occasionally follow suit.

In this model, sellers predominantly copy strategies from within their own island, sometimes they copy from somewhere in the whole system. To increase the coupling between islands, we increase the probability that sellers copy from outwith their own island.

In addition, rather than copying from anywhere in the system at the same time, we can think of how islands might affect dispersion of strategies. Inside a single ecosystem, dispersion is fast, over long distances between ecosystems, dispersion is slower. We can model this by choosing to copy within an island from the currently observed strategy distribution, but observe a delayed picture of the rest of the system. We shall consider the two cases in turn.

7.2 EQUAL TIME MIGRATION COUPLING

The update scheme we employ for migration-based island coupling is as follows. In the initialisation stage, randomly-sized contiguous chunks of the system are marked as particular islands (figure 7.1 shows a cartoon of this).

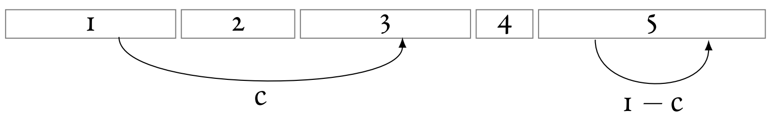

Figure 7.1 *Cartoon showing division of one-dimensional system into islands. This particular system has five islands (labelled). Also shown are two copying events: an inter-island copy (occurring with probability* c*) and an intra-island copy (probability* 1 − c*)*

Since we only consider a one-dimensional system, the algorithm for this division is simple, it is shown in algorithm 5. In essence, we specify minimum and maximum island sizes and generate sizes uniformly at random within these bounds.

$t \leftarrow 0$ *How many sellers have we marked?*
$i \leftarrow 1$ *Which island are we in?*
$c_{min} \leftarrow 10^4$ *Minimum island size*
$c_{max} \leftarrow 5 \times 10^4$ *Maximum island size*
while $t < N$ **do**
  $c \leftarrow r$ r *an integer uniformly distributed in* $[c_{min}, c_{max}]$
  for $k = t$ to $c + t$ **do**
    $s_k \leftarrow i$ *Assign the* k^{th} *seller to island* i
  end for
  $t \leftarrow t + c$
  $i \leftarrow i + 1$
end while

Algorithm 5 GENERATE-ISLANDS(): *Algorithm to divide a one-dimensional system into randomly sized islands*

The updates then proceed as described in section 3.3. The difference comes when we reach the rebirth stage. With probability $1 - c$, the rebirth event chooses a random seller from within the seller's island. With probability c, the random seller is chosen from the whole system (*including* the island the seller is in). $c = 0$ corresponds to completely uncoupled islands, $c = 1$ to a system without islands. By tuning the value of c we modify the strength of migration effects in the copying phase.

7.2.1 *Non-oscillating regime*

The steady state regime should be unaffected by the addition of islands, coupled or not. Each island should independently reach the steady state distribution we have previously encountered. In the steady state, any coupling should make no difference as sampling either from an individual island or the entire system should pick the same distribution. As long as the islands are not so small that stochastic fluctuations kill the high-priced sellers, we therefore expect that the global price distribution will be the same irrespective of coupling strength or number of islands.

Results from such a simulation confirm our expectations. Both in the synchronously (figure 7.2) and asynchronously updated systems (figure 7.3), the value of the coupling constant makes no difference to the form of the steady state distribution.

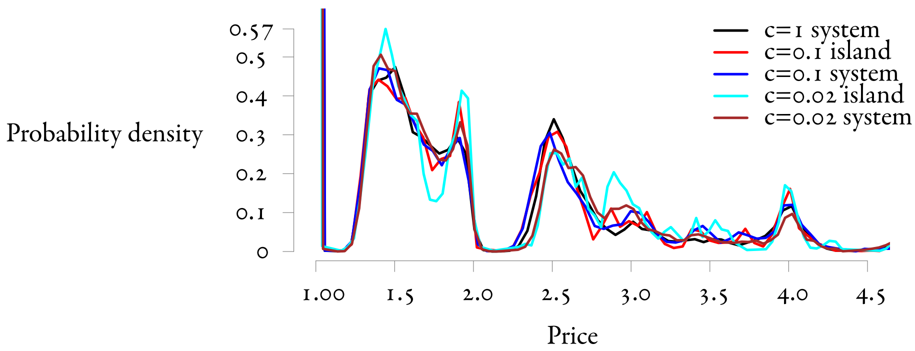

Figure 7.2 *Steady state price distributions for systems with coupling constants of* $c = 0.02$, $c = 0.1$ *and* $c = 1$, $N = 10^5$, $\gamma = 0.35$ *and* $\Delta = 0.08$ *under synchronous dynamics. Shown in each case are the global steady state distribution and a typical distribution of prices in a single island. As we can see, the distributions are essentially identical (modulo finite size fluctuations). Peak at* $p = 1$ *cut off for clarity*

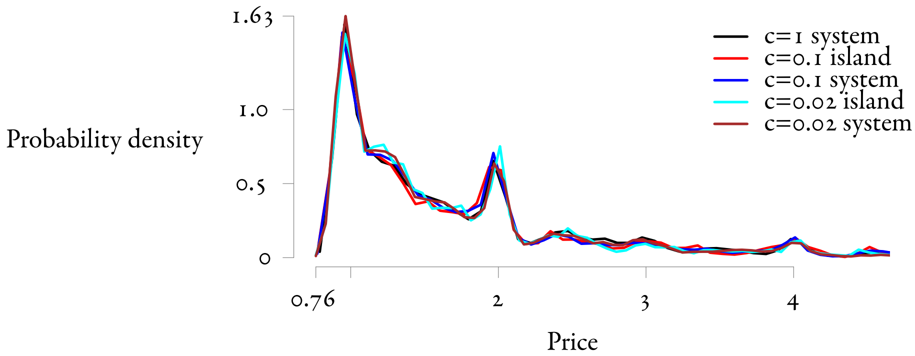

Figure 7.3 *Steady state price distributions for systems with coupling constants of* $c = 0.02$, $c = 0.1$ *and* $c = 1$, $N = 10^5$, $\gamma = 0.35$ *under asynchronous dynamics. Shown in each case are the global steady state distribution and a typical distribution of prices in a single island. As we can see, the distributions are essentially identical (modulo finite size fluctuations)*

7.2.2 *Coupled islands in the oscillating regime*

In the continuous time simulations, we have already seen that we observe oscillations in the mean price of the system for large values of the birth rate γ. We now study how these global oscillations change in a system composed of coupled islands. If there is no coupling, we expect each island to enter an oscillatory mode: the dynamics are unchanged from the previous results. The effect on the system mean price is, however, likely to be much less noticeable. Recall that the oscillations do not have a constant period but rather some distribution of periods around a typical value. Due to this, even if two islands were to begin oscillating in-phase, they should soon drift out of phase producing a partial cancellation in the effect of the oscillations on the global mean price. Further, recall that the reason for the varied period is the varying amplitude of oscillations which should further reduce the fluctuations in the global price.

Now, in a noiseless system, if the distribution of natural frequencies is not too large, any coupling between elements will eventually lead to in-phase behaviour (see for example [69] and [78]). With the noisy dynamics we have, it is not clear that this will occur. If the coupling is only weak it may not be enough to overcome the noise and the islands would continue to oscillate independently. As the coupling strength increases we expect that eventually the system will oscillate in phase and behave as if there were no separate islands at all.

Another way of looking at this is to ask if we can change the path of the system through phase space by altering the sampled price distribution. We know that a single oscillating system follows a set path through the phase space. Expensive sellers cannot invade the system during the downturn because the system is too full and cheap sellers cannot invade quickly because there are not enough of them. If we change these proportions (by sometimes copying from outwith a single island) will the path through the oscillatory cycle change? We expect weak coupling to make little difference because it will not noticeably change the price distribution used to pick a new strategy. Conversely, strong coupling will likely change the observed distribution which may well have an effect.

7.2.3 *A note on nomenclature*

In the following sections we shall refer both to averages of quantities in the system at a specific point in the simulation *and* the time-averaged values of these quantities. To avoid confusion, in this brief interlude, we shall introduce some notation to differentiate clearly between the two.

Consider an observable quantity in the system that varies from seller to seller, denote this by A_i $(i = 1, 2, \ldots, N)$. This quantity may be time-dependent (for example, it might be the capital of a seller). We use the notation $A_i(t)$ to indicate the value of the quantity A at site i and time t. Typically we are interested in the average value of these quantities, for example, the mean price in the system. This is an average over all sellers at a single point in time. We shall denote the n^{th} single-time moment of a quantity A by

$$\langle A(t)^n \rangle_s \equiv \frac{\sum_{i=1}^{N} A_i(t)^n}{N}. \tag{7.1}$$

So, for example, the mean system price (which varies in time) is written as $\langle p(t) \rangle_s$. The standard deviation about this mean, measuring the spread in the mean price *at a single point in time* is given by

$$\sigma_s(p(t)) = \sqrt{\langle p(t)^2 \rangle_s - \langle p(t) \rangle_s^2}. \tag{7.2}$$

As mentioned, these system averages may be time-dependent. We can therefore also define a time-average of such global quantities as follows. Consider a system-averaged quantity $\langle A(t) \rangle_s$ between two points t_0 and $t_0 + T$. The n^{th} moment of this timeseries is given by

$$\langle A^n \rangle_{T,t_0} \equiv \frac{\sum_{t=t_0}^{t_0+T} \langle A(t) \rangle_s^n}{T}. \tag{7.3}$$

We can hence also define the standard deviation of the timeseries

$$\sigma_{T,t_0}(A) = \sqrt{\langle A^2 \rangle_{T,t_0} - \langle A \rangle_{T,t_0}^2}. \tag{7.4}$$

Figure 7.4 shows these two different averaging schemes pictorially.

The final piece of notation we need is for the correlation of two timeseries. Consider two observable quantities $A_i(t)$ and $B_i(t)$ that we observe in simulation from $t = t_0$ to $t = t_0 + T$. Their timeseries are given by $\langle A(t) \rangle_s$ and $\langle B(t) \rangle_s$ respectively. The correlation of the two timeseries, at a lag τ is given by

$$C_{A,B}(\tau) = \frac{\left\langle \left[\langle A(t) \rangle_s - \langle A \rangle_{T,t_0} \right] \left[\langle B(t+\tau) \rangle_s - \langle B \rangle_{T,t_0} \right] \right\rangle_{T,t_0}}{\sigma_{T,t_0}(A) \sigma_{T,t_0}(B)}. \tag{7.5}$$

7.2.4 *Quantifying oscillation size*

In order to characterise the size of oscillations in an island system, we first look at the completely coupled case ($c = 1$). We measure the variance in the timeseries of the mean system price, $\sigma_{c=1}^2 \equiv \sigma_{T,0}(p)^2$, for this system. This variance is due to oscillations around the long-time mean value and is

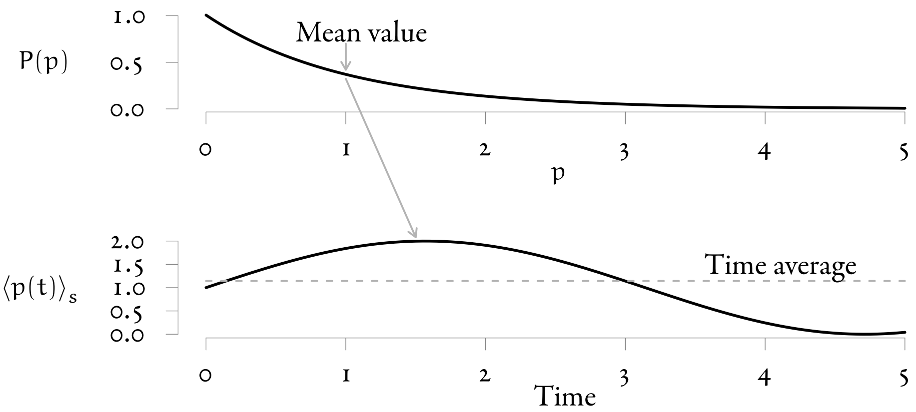

Figure 7.4 *Schematic showing the two methods of producing averages in the system. The upper figure shows the distribution of system prices at a particular point in time, with indicated mean value (we denote this $\langle p(t) \rangle_s$, see equation 7.1). This mean value varies through time as shown in the lower figure. The time average of the mean price is shown as a dashed line on the lower figure (we denote this $\langle p \rangle_{T,t_o}$, see equation 7.3)*

thus a surrogate for the amplitude of oscillations. We now vary the coupling strength and measure $\sigma(c) \equiv \frac{\sigma_{T,0}(p)^2}{\sigma^2_{c=1}}$ for each of these simulations in turn. For small coupling, as previously argued, we do not expect in-phase oscillations and thus this fractional variance should be less than one. For large coupling, we expect islands to migrate into phase, which should lead to a fractional variance around one. As figure 7.5 shows, this does indeed occur. Small coupling gives rise to low variance; when $c \gtrapprox 0.15$ the system behaves as if the islands were completely coupled.

We see then, that a system composed of a number of weakly coupled islands exhibits smaller system-wide fluctuations than a strongly coupled system. We should now check that this is a result of our hypothesis that the weakly coupled islands oscillate but do not migrate into phase. To confirm this hypothesis, we look at the autocorrelation of individual island timeseries and the cross-correlation between island timeseries and the system timeseries. If the individual islands are oscillating strongly, there should be a peak in the autocorrelation at the period of the oscillations. The cross-correlation between two islands should have a peak of almost one at zero lag if the two islands are in phase. Similarly for the cross-correlation between an island and the entire system.

Figure 7.6 shows these different timeseries correlations for a weakly coupled system with $c = 0.01$. The results are as expected. The cross-correlation

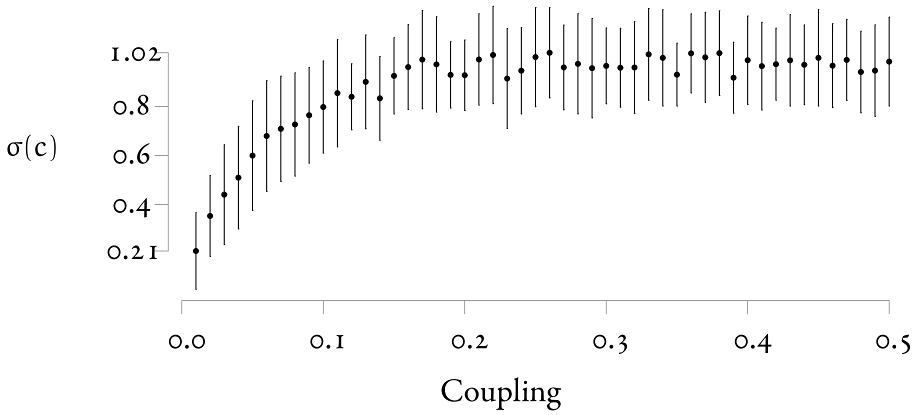

Figure 7.5 *Fractional variance in the global mean price of a system,* $\sigma(c)$*, as a function of the coupling constant. The variance is measured relative to a system with* $c = 1$ *and so a value around one indicates that the partially coupled system is exhibiting fluctuations of the same size as the fully coupled one. All systems have* $N = 10^5$*,* $\Delta = 0.08$*,* $\gamma = 0.75$*. For each value of* c *the simulation is repeated one hundred times, errorbars show standard error in the mean value*

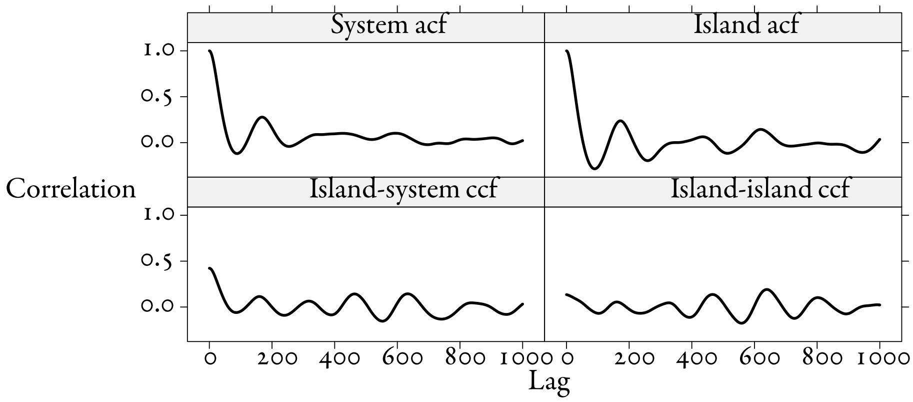

Figure 7.6 *Correlation in the timeseries of the mean system price with weakly coupled islands. Shown are the autocorrelation of the global mean price and an individual island price and the cross-correlation between an island price and the global price and two separate islands. Note that the autocorrelation peaks in the global mean price are smaller than for the individual island (indicating weaker correlation) and the cross-correlation shows no strong peak at a lag of zero. Compare this with figure 7.7.* $N = 10^5$, $\Delta = 0.08$, $\gamma = 0.75$, $c = 0.01$

between different timeseries never rises above one half, indicating that different islands are indeed out of phase. The island autocorrelation shows strong peaks at the oscillatory period.

Conversely, if the coupling constant is large, we have seen that the fluctuations are as large as in the completely coupled case. This indicates that all islands are oscillating in phase and with the same amplitude. We therefore expect that the timeseries correlations should all be identical: the cross-correlation between an island the the global system should be the same as the autocorrelation of the system price. As shown in figure 7.7, this does indeed occur. The various different measured correlations are essentially identical.

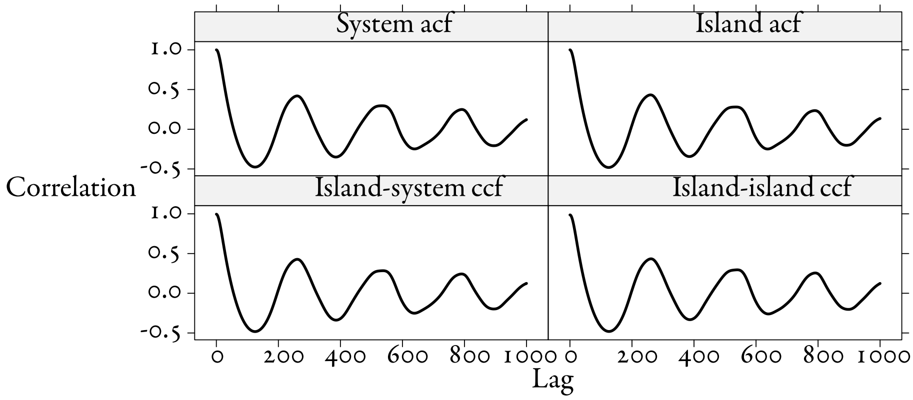

Figure 7.7 *Correlation in the timeseries of the mean system price in a strongly coupled system. Shown are the autocorrelation of the global mean price and an individual island price and the cross-correlation between an island price and the global price and two separate islands. Note that all four curves are essentially identical, indicating that individual islands oscillate in phase with one another.* $N = 10^5$, $\Delta = 0.08$, $\gamma = 0.75$, $c = 0.3$

7.2.5 *Summary*

We have demonstrated one possible method of reducing the severity of oscillatory cycles on the global system. If the system is composed of a large number of loosely coupled regions, the globally observed mean price does not fluctuate as much as that observed in each region. As the coupling strength increases, the oscillatory cycles of each region migrate into phase with one-another. In the latter regime, extinction events in which all expensive sellers become bankrupt are more likely to happen, resulting in a global crash in prices. We may draw a weak analogy here to the global economic system: in previous eras, coupling between different economies was relatively weak and disastrous occurrences in one economy would not necessarily migrate to others. In modern times, coupling is much stronger and crashes in one economy are much more likely to bring down others. This might lead us to suggest, though not particularly seriously, that comparative economic isolation might be a plausible method of avoiding global recession.

In this section we consider the islands to be spatially distinct entities. As a result, information about prices takes time to propagate between islands. In our ecological analog, we can think of strategy-copying as similar to seed dispersal. Intra-island dispersal is fast, but inter-island dispersal is slow (seeds must be carried across vast arid plains, or such-like).

In our model, we implement this delayed copying as follows. In addition to their current price, sellers also remember their price from some (specified) number of rounds in the past. When a new seller enters the market they either copy a price from within their island, or from the entire system. In the former case, they pick up the current strategy; in the latter, they get the historical strategy.

These dynamics have no effect on the system for parameter choices that exhibit a steady state price distribution. Copying from some time in the past makes does not alter the distribution of copied prices. The steady state remains unchanged. In the oscillating regime, delayed copying may have an effect. In a weakly coupled system, adding a time delay is unlikely to change the dynamics. In this case, islands oscillate in an uncorrelated manner: adding a time lag to the copied price will not change the lack of correlation. For large values of the coupling constant, we expect that there may well be some change in behaviour. Without a delay, the islands oscillate in phase. If there is a delay, the sampling distribution of prices will be composed of the island price distribution and the system price distribution from some time in the past. So, for example, if an island is at the bottom of a cycle but a new seller copies from outwith the island, it may pick a price from the top of the cycle. If this occurs frequently enough, the path through phase space may be altered.

7.3.1 *Fixing the oscillatory period*

Our first step is to look at the behaviour of the system in the fully coupled case. In this case, we find the system exhibits oscillatory behaviour but the period of the oscillations is fixed. Instead of the oscillations having some natural (system-determined) frequency, we find that the period is simply equal to the copying delay. So, for example, if the copying delay is one hundred rounds, the system oscillates with a period of one hundred rounds.

Figure 7.8 shows the autocorrelation of the system mean price for various

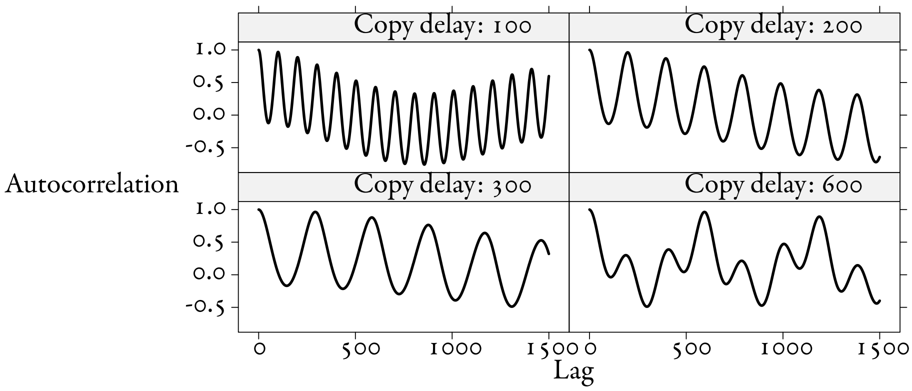

Figure 7.8 *Autocorrelation of the mean system price in oscillating systems with fixed time-delayed copying.* $N = 10^5$, $\Delta = 0.08$, $\gamma = 0.75$, $c = 1$. *Copy delay indicated in figure. There is a very noticeable correlation on the length of the copying delay. Additionally, the system exhibits further slow oscillations with a period of about ten times the copying delay*

copying delays. Each of these shows a very definite peak at the delay length, indicating that the period of the oscillating has been fixed. This is further confirmation of our belief that in the oscillating state, there is only one possible path through the phase space. The oscillations develop with a period such that the copied (time delayed) price distribution matches the environmental (current) price distribution. This way, the copied price distribution is always optimal.

We can see this most clearly by looking at the price distribution over the period of such an oscillatory cycle. In figure 7.9 we show the price distribution over the course of one period for a system with a copying delay of one hundred rounds. The overall dynamics of the oscillation are unchanged from the picture we described previously in section 6.5 (see figure 6.9), but the price distribution now repeats almost exactly every one hundred rounds.

7.3.2 *Does the system have a memory?*

This behaviour raises an interesting question. Does the system know the period of the oscillatory cycle? In other words, if we look at a single time snapshot of the system price distribution, can we tell the period of oscillation? The pictures in figure 7.9 suggest that it might be possible. However, the price distribution does not tell the whole story. The price distribution is not the only factor in deciding the period of the oscillations. The distri-

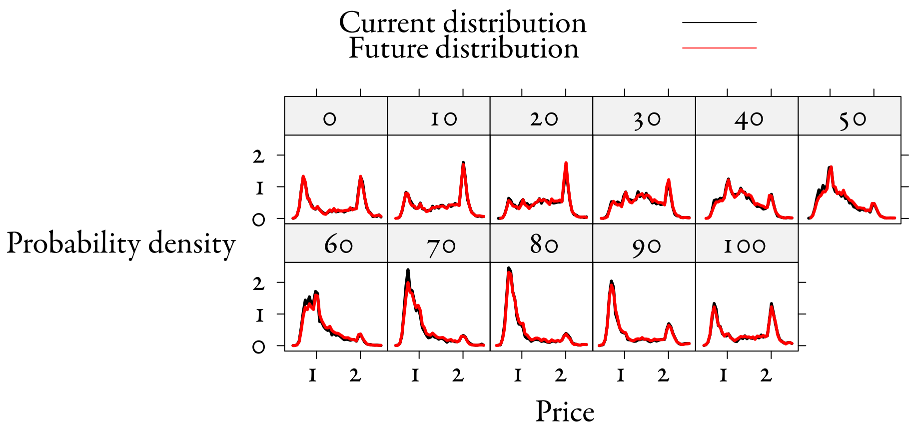

Figure 7.9 *Sequence of price distributions through a cycle with price copying at a delay of one hundred rounds. Each subfigure shows the price distribution at a particular point in time (black) and the distribution one hundred rounds later (red). Subfigure numbers indicate the number of rounds elapsed since the first figure. At every point in time, distributions separated by one hundred rounds are essentially identical. The dynamics of the oscillation are still the same as the system with no delay.* $N = 10^5$, $\gamma = 0.75$, $\Delta = 0.08$, $c = 1$

bution of capital is also important, as is the position of sellers. So, although the price distribution repeats every period, the capital distribution does not. This difference is enough to ensure that the oscillation with fixed period (due to delayed copying) does not continue in the same manner once the delay in copying is removed. The system does not know what period it should oscillate with if given only the information from a single point in time.

This is easily demonstrated by setting up a system with a delay in the copying stage and letting it reach an oscillatory state. We then switch off the copying delay and see what happens. If there is a system memory, the system should continue to oscillate with the previous (fixed) period and drift slowly away to the natural frequency. Conversely, if the system immediately switches from the fixed period to a natural oscillatory state, we can conclude that there is no system memory and the price distribution does not tell us about the period of the oscillation. Figure 7.10 shows the time-series of the mean system price in such a simulation. As soon as the delay in copying prices is switched off the system changes from a fixed oscillatory period to the previously observed 'natural' oscillation. This demonstrates that the price distribution alone does not contain enough information to fix the period of oscillations. Despite this, it is nonetheless interesting that with a delay in copying, the system is able to find the 'correct' oscillatory pe-

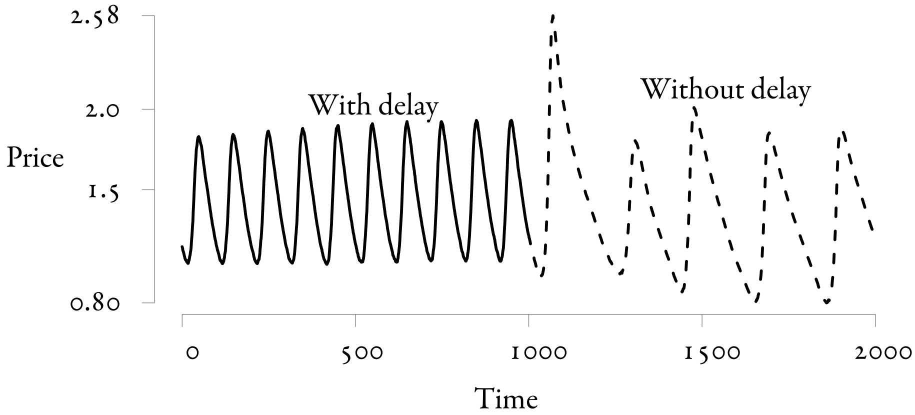

Figure 7.10 *Change in the oscillatory period of a system with time-delayed copying when the time delay (of one hundred rounds) is switched off. Shown are the mean price for 5000 timesteps with time-delayed copying (solid) and the subsequent 5000 timesteps when the time-delay was switched off (dashed). Immediately following the switch away from delayed copying, the period changes noticeably.* $\gamma = 0.8$, $N = 3 \times 10^4$, $\Delta = 0.08$.

riod. No individual seller has any knowledge of the delay length, and so this spontaneous arrangement to fix up the system price distribution to match the sampled distribution is somewhat surprising.

7.3.3 *Time delayed migration only between islands*

We now turn to the question we posed at the beginning of the previous section before studying the fully-coupled case. What happens if strategies come from different islands with a delay but without a delay from within any one island?

We have just seen that if the islands are fully coupled, the system adopts an oscillatory cycle whose period matches the time delay. Does this still occur if the coupling decreases? The results presented in section 7.3.2 show that the oscillatory period is not built in to the price distribution. We expect the effect of only taking time-delayed prices from different islands to be similar. There will be a competition between the natural frequency of oscillations of each individual island and the frequency imposed by time-delayed copying between islands. It may be that these two effects can serve to stabilise the system away from its oscillatory behaviour.

In performing this simulation, we must ensure that individually, all the islands are large enough that their steady state in the uncoupled case is an oscillatory one. If the islands are too small, the system quickly collapses to a state similar to that observed for $\gamma = 1$: all sellers are cheap with $p < 1$. This behaviour is easy to avoid, we just specify a minimum size for an individual island that is sufficiently high.

With this caveat out of the way we now look at the behaviour of the system when inter-island copying picks strategies from the past and intra-island copying picks from the present. When we do this we find little change from the behaviour we observed previously without any time delay. The price of individual islands oscillates with a period set by the delay time. For high values of the coupling constant, the islands still end up in phase with one another so that there is no conflict between the historical distribution, seen when inter-island price copying takes place, and the current price distribution. For low values of the coupling the story is again similar to the system without a delay. Each island fluctuates individually with a period given by the delay, but the coupling is weak enough that islands do not migrate into phase.

As suspected, the addition of a delay makes no difference to the uncorrelated islands of the weakly coupled system, other than fixing the period of each individual island. For the strongly coupled case, we postulated that competition between the current and historical price distributions might serve to stabilise the system slightly against oscillations. This appears not to be the case. Instead, the coupling is strong enough that the islands all migrate into phase, at which point there is no difference between historical and current price distributions.

7.4 COPYING WITH A RANDOM DELAY

We now study a change to the dynamics that allows for a steady state with expensive sellers in regions of the parameter space that otherwise give oscillations. Recall that the oscillations occur when prices are drawn from the current time distribution because the evolutionary dynamics favour the oscillatory state. The system chases its own tail through phase space as the fitness landscape change. In an oscillating state, if we change the sampled price distribution sufficiently, we will be able to negate this tail-chasing effect and counteract the oscillatory behaviour.

The question is now just how to change the sampled distribution. Obviously if we pick new prices from an externally set distribution the system will tend towards a state given by the distribution. Instead we change the distribution by modifying the delayed-copying idea. Instead of remembering a single price from some fixed time in the past, sellers remember a set of historical prices. For example, each seller might keep track of its price in the last ten rounds. When a bankrupt seller reenters the system, it chooses a random seller to copy from as before and then chooses uniformly at random from that seller's historical prices.

This price copying scheme produces a steady state price distribution that is very similar to the state for low γ. Figure 7.11 compares steady state distributions for a system with $\gamma = 0.5$ without delayed copying and the distribution obtained from a system with $\gamma = 0.75$ with randomised copying and a history of two hundred rounds. Also shown in this figure is the time averaged price distribution of an oscillating system with $\gamma = 0.75$. We see that the stabilised distribution and the steady state distribution are broadly similar. The time-averaged oscillating distribution does not match up closely with either. As an analogy to a real marketplace, we can think of the strategy

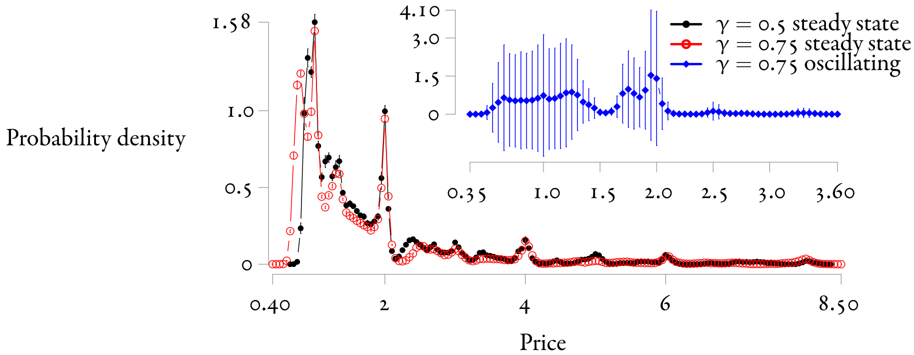

Figure 7.11 *Comparison of the steady state price distribution obtained with* $\gamma = 0.5$ *and the distribution obtained with randomised strategy copying with* $\gamma = 0.75$. *Small errorbars show standard error in the mean in a time average over five hundred rounds. Inset shows the average price distribution observed over an oscillatory cycle with* $\gamma = 0.75$, *the errorbars are the standard error in the mean of each point over the cycle. All three distributions have* $N = 10^5$, $\Delta = 0.08$ *and* $c = 1$

copying step as being equivalent to reacting to news events. If individuals react to news from some random point in the recent past (rather than the current time) the panic undercutting that occurs in the oscillatory cycle is avoided.

Having shown that it is possible to avoid entry into an oscillatory state, we now ask two further questions. How much history must each seller hold on to to suppress the oscillations, and does the suppression of oscillations decrease the profits that can be made by single sellers? We know that the oscillations are driven by a few high-priced sellers with large capital stocks. Clearly, the repeated crashes and recovery phases are beneficial to them. Does the suppression of these phases remove the possibility of making enormous capital gains?

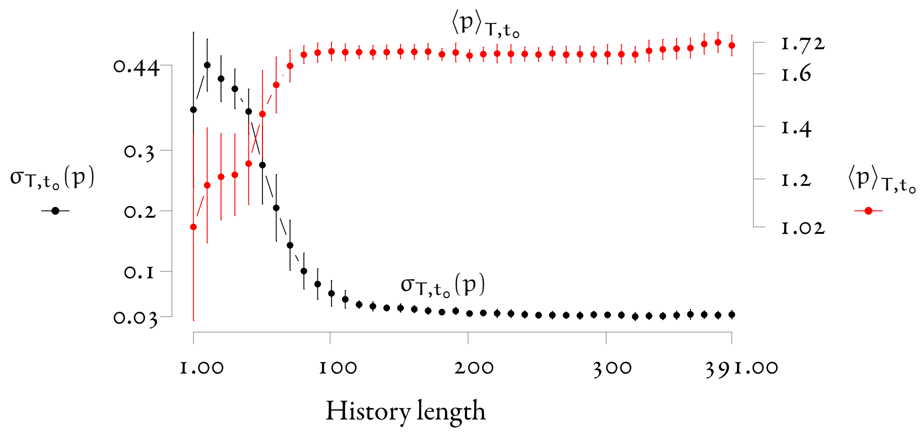

Figure 7.12 *Standard deviation in the timeseries of the mean system price,* $\sigma_{T,t_0}(p)$ *(given by equation 7.4) as a function of history size with* $T = 25000$. *Errorbars show standard error in the mean over forty realisations. Also shown is the time-averaged mean system price,* $\langle p\rangle_{T,t_0}$. *The simulations all have* $\gamma = 0.75$, $N = 5 \times 10^4$, $\Delta = 0.08$. *Note how once the history length reaches approximately one hundred rounds, the mean price stabilises and the standard deviation drops almost to zero, indicating that the system is no longer in an oscillatory state*

Figure 7.12 shows that for a system with $N = 5 \times 10^5$ once the history length reaches around one hundred rounds, the oscillatory state has been suppressed. Further increases in history length neither affect the mean system price nor the size of the fluctuations. The steady state we reach in this system remains unchanged as the history length increases past one hundred. This stabilisation may be beneficial for the global system, but is it also better for anomalous individuals? To study this, we look at what happens to the mean capital in the system (and also its variance) as a function of the delay period. A large mean capital is obviously a good thing: it means the system is wealthier as a whole. A large variance means that the system contains a few sellers with large capital reserves. Since capital is bounded below by zero, if the variance is large compared to the mean this is a sign of a few sellers with a large capital. These are the sellers that would have driven the oscillations in the un-stabilised system.

We would like to be able to strike a happy medium in which the risk of large crashes is removed while still allowing individuals to prosper. Our data show (figure 7.13) that in this system, the best option is to hold on to a long history. The mean capital in the system shows a weak positive correlation

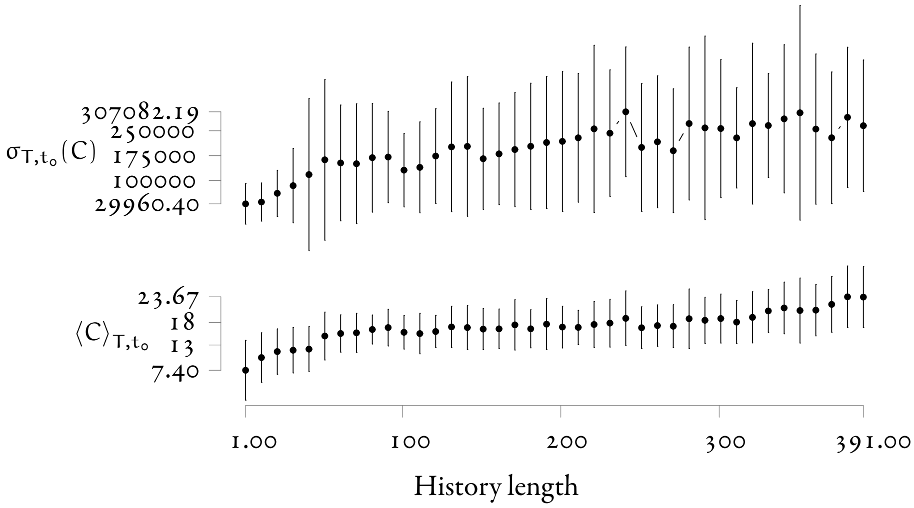

Figure 7.13 *Mean system capital,* $\langle C \rangle_{T,t_0}$*, (lower figure) and the standard deviation in the mean system capital,* $\sigma_{T,t_0}(C)$*, (upper figure). Error bars show standard error in the mean over forty realisations of the initial conditions,* $N = 5 \times 10^4$, $\gamma = 0.75$, $c = 1$, $\Delta = 0.08$. *Note the slight upward trend in the mean capital. but no significant change in the standard deviation (indicating that changing the history length does not affect the profitability of anomalous sellers)*

with increasing history length. Indicating that if we are optimising for overall prosperity, longer histories are better. Additionally, there is no significant peak in the standard deviation of system capital as a function of the history length. Increasing the history does not negatively affect the very prosperous sellers.

7.5 SUMMARY

In this chapter we have studied two extensions to our simple model that can lead to partial suppression of the oscillatory state when viewed on a global scale. By separating the system into independent islands and coupling the islands through exchange of strategies, we can reduce the size of global oscillations. If the coupling between islands is weak, they do not synchronise their movements through price space. These out of phase oscillations partially cancel in the system price and reduce the size of global fluctuations. If the islands are strongly coupled, they migrate into phase with one another and the system behaves as previously. We suggested, with tongue firmly in cheek, that decreasing coupling between the economies of the world might have a beneficial effect on global economic stability.

We also postulated that if price copying looked at a fixed point in the past (introducing a delay in the propagation of information from different islands), the system might suppress oscillations. This is not the case, although we did see that it was possible to fix the period of oscillations by fixing the copying delay.

Finally, we showed it was possible to suppress oscillations in the system completely by randomising the point in history a seller would copy from. This had a positive effect on the mean system wealth, without negatively affecting the few particularly wealthy sellers that prosper and drive the oscillatory behaviour. This suggests a further, semi-serious, method of avoiding economic meltdown. Crashes in our system occur when sellers keep reacting to one-another's price changes by undercutting each other. If we randomise the history, this price undercutting no longer takes place. Clearly then, to avoid stock market crashes, all we need do is isolate traders from the news of market fluctuations by feeding them data from a random point in the past. In our system at least, this has only a positive effect on every participant.

A SIMPLE MEAN FIELD MODEL

8

In section 4.3 we constructed an analysis to derive the price distribution after a single round of updates. We showed that this result was almost exact (except for ignoring pairwise spatial correlations) but argued that extending the analysis to further rounds of the game would prove too difficult. The reason for this statement was the creation of both spatial correlations (when one seller outcompetes another it changes the probability of survival in the following round) and capital induced correlations. Sellers are able to accumulate capital and so we cannot know their survival probability in any given round unless we know their history. In this chapter we sketch how one might approach the capital induced correlations in an analysis of the model.

Our scope here is somewhat restricted. We are not attempting to derive the steady state price distribution, and so, for simplicity of exposition, we consider a system with a specified δ-function price distribution. Furthermore, we destroy any spatial correlations in our simulations and do not attempt to treat them at all in our analysis. Our aim, under these restrictions, is to derive the steady state live seller density, $\rho_\infty(\gamma)$, observed in simulation.

The simulation results are obtained as follows. We set the initial price distribution to $P_0(p) = \delta(p-1)$ and consider a system without mutation ($\Delta = 0$). The steady state price distribution is thus $P_\infty(p) = \delta(p-1)$. We remove spatial correlations in the system by delocalising buyers. Every time a buyer is chosen to go shopping it chooses, uniformly at random, two sellers in the system and visits one of them (as described in section 3.3.1). Recall that if two sellers have the same price, the buyer chooses randomly between them. Figure 8.1 shows how $\rho_\infty(\gamma)$ behaves under these dynamics in the simulation.

To find ρ_∞ analytically we need to find the survival probability of a seller, this latter will depend on ρ. Denote the survival probability of a seller by $p_s(\rho)$. If we can find p_s then we can find ρ_∞ by solving the equation

$$\rho = \underbrace{\rho p_s(\rho)}_{\text{surviving}} + \underbrace{\gamma(1-\rho p_s(\rho))}_{\text{reentry}}. \tag{8.1}$$

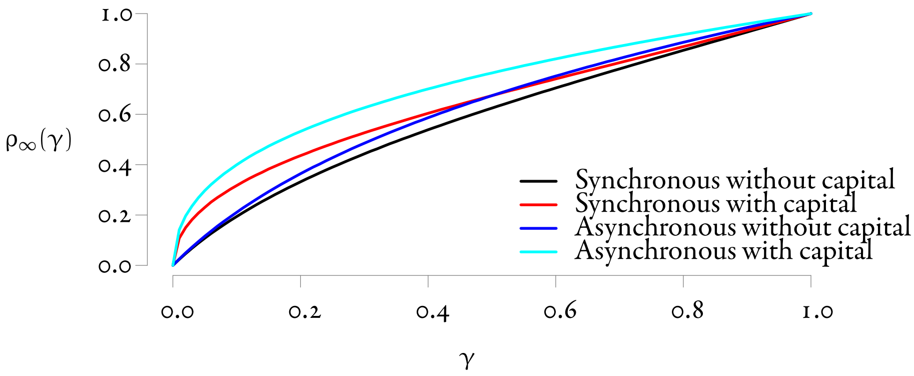

Figure 8.1 *Steady state live site densities ($\rho_\infty(\gamma)$) for systems in which spatial correlations are destroyed (method described in main text). The initial price distribution is a δ-function at* $p = 1$, $N = 10^4$ *and* $\Delta = 0$

The left hand side of this equation is the current live site density, the right hand side is the live site density at the end of the round. The steady state live site density is just given by the fixed point of this equation. Once we have obtained an expression for p_s finding ρ should be straightforward.

Before we consider how to treat capital accumulation we consider a related, but simpler, problem. By setting the capital of sellers to zero at the beginning of every round we can effectively ignore any capital induced correlations in our analysis. We carry out this analysis first as a sanity check of our method.

8.1 ERASING HISTORY, SELLERS WITH NO CAPITAL ACCUMULATION

We treat the simplest case first. If we do not allow any accumulation of capital, survival probability depends only on performance in the current round. As we shall see, this simplifies the analysis tremendously.

8.1.1 *The synchronously updated system*

We build up the survival probability in stages. The first question to answer is how many buyers a seller must sell to in order to survive. We know that every seller pays an overhead of two. This requires that a seller to sell to two or more buyers per round in order to survive. Each of the buyers in the system randomly chooses two sellers to visit, the probability that a buyer chooses a particular seller is therefore $\frac{2}{N}$. The probability of attracting k buyers is therefore given by

$$f_N(k) = \binom{N}{k} \left(\frac{N-2}{N}\right)^{N-k} \left(\frac{2}{N}\right)^{-k}. \tag{8.2}$$

Note that in subsequent steps we will ignore the probability that a single buyer chooses the same seller twice (which would look like two buyers but only result in a single sale). If the system is sufficiently large this is a reasonable assumption (it occurs with frequency $\mathcal{O}(N^{-2})$).

Now we need to calculate the probability of selling to j of the k buyers we have attracted. A sale either occurs if our competitor is dead (this happens with probability $1-\rho$), or if the opponent is alive but we outcompete them (this happens with probability $\frac{\rho}{2}$ since all sellers have the same price). Hence we sell to a buyer with probability $1-\frac{\rho}{2}$. The probability of selling to j of k total buyers is therefore given by another binomial distribution:

$$g(j|k) = \binom{k}{j} \left(1-\frac{\rho}{2}\right)^{j} \left(\frac{\rho}{2}\right)^{k-j}. \tag{8.3}$$

We can now write down the probability of selling exactly n times. It is the expectation value of $g(n|k)$ over the distribution $f_N(k)$.

$$e_N(n) = \sum_{k=n}^{N} f_N(k) g(n|k). \tag{8.4}$$

We are interested in the large N limit of this expression, taking $N \to \infty$ in equation 8.2 gives (see section A.1)

$$\lim_{N\to\infty} f_N(k) = \frac{2^k}{e^2\Gamma(1+k)} \tag{8.5}$$

and so the probability of selling to exactly n buyers in the infinite system is

$$e(n) = \sum_{k=n}^{\infty} \frac{2^k}{e^2\Gamma(1+k)} g(n|k) = \frac{(2-\rho)^n e^{\rho-2}}{\Gamma(1+n)}. \tag{8.6}$$

We must sell some number of buyers and so

$$\sum_{n=0}^{\infty} e(n) = e^{\rho-2} \sum_{n=0}^{\infty} \frac{(2-\rho)^n}{\Gamma(1+n)} = 1. \tag{8.7}$$

Finally, using equations 8.6 and 8.7, we can write down the probability of selling to n or more buyers

$$s(n, \rho) = 1 - \sum_{k=0}^{n-1} e(k). \tag{8.8}$$

So, the survival probability for the discrete time system with no accumulation of capital is given by

$$p_s(\rho) = s(2, \rho). \tag{8.9}$$

The sum may be carried out exactly (see section A.2) giving

$$p_s(\rho) = 1 + e^{\rho-2}(\rho - 3). \tag{8.10}$$

We can now plug this result back into equation 8.1 and solve the resulting transcendental equation numerically to find the steady state live site density. As shown in figure 8.2, the agreement between simulation and theory is excellent.

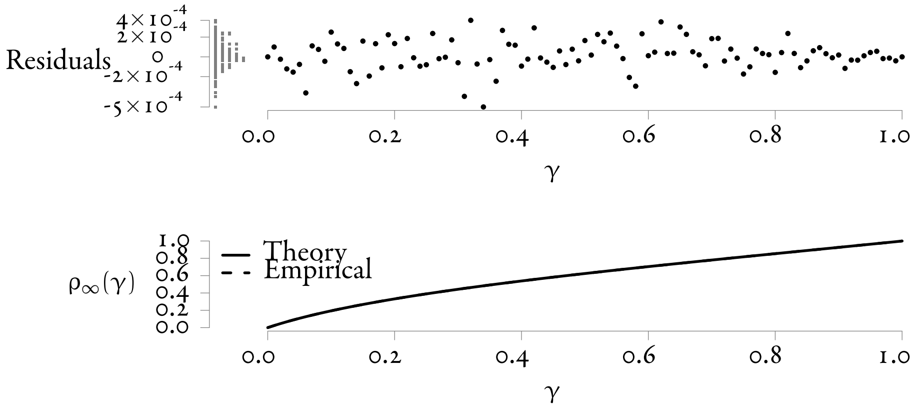

Figure 8.2 *Comparison of the live site density predicted by mean field analysis and simulation results for a synchronously updated system without capital accumulation. The upper figure shows the residuals of the fit, along with their marginal distribution. The lower figure shows the predicted steady state density along with empirical results. Theoretical and empirical results appear as a single line since they lie almost on top of one another: the fit has* $R^2 = 0.9999$. *The simulation has* $N = 10^4$, $\Delta = 0$

8.1.2 *The asynchronously updated system*

This case is marginally more complicated than the previous one. The complication arises because the number of overhead payments a seller makes is not fixed. Sellers are randomly chosen to pay an overhead and so may need to make any number of sales between zero and N in order to survive. The probability that a seller pays k overheads is given by a binomial distribution

$$c_N(k) = \binom{N}{k} \left(\frac{N-1}{N}\right)^{N-k} N^{-k}. \tag{8.11}$$

with large N limit

$$c(k) = \frac{1}{e\Gamma(1+k)}. \tag{8.12}$$

The survival probability of a seller paying k overheads is the probability that they sell to at least $2k$ buyers. The total survival probability is thus

$$p_s(\rho) = \sum_{k=0}^{\infty} c(k) s(2k, \rho). \tag{8.13}$$

We can write $s(k, \rho)$ explicitly as (see section A.2)

$$s(k, \rho) = 1 - \frac{\Gamma(k, 2-\rho)}{\Gamma(k)}, \tag{8.14}$$

however, the expression for p_s has no explicit form. We just have

$$p_s(\rho) = \sum_{k=0}^{\infty} \frac{1 - \frac{\Gamma(2k, 2-\rho)}{\Gamma(2k)}}{e\Gamma(1+k)}. \tag{8.15}$$

For a given value of ρ we can evaluate this sum numerically and use the result to find the steady state live site density by iterating equation 8.1. Again, the agreement between theory and simulation is excellent, as shown in figure 8.3.

8.2 ADDING CAPITAL ACCUMULATION BACK IN

Now that we have derived the live site density for a system in which no capital accumulation is allowed, we extend the method to treat the case where capital can be stored. This allows sellers to survive in situations which would previously have resulted in bankruptcy. To incorporate capital accumulation

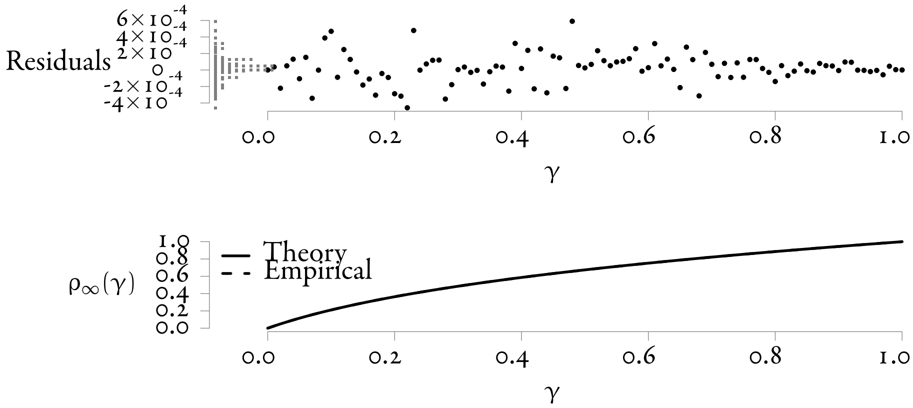

Figure 8.3 *Comparison of the live site density predicted by mean field analysis and simulation results for an asynchronously updated system without capital accumulation. The upper figure shows the residuals of the fit, along with their marginal distribution. The lower figure shows the predicted steady state density along with empirical results. Theoretical and empirical results appear as a single line since they lie almost on top of one another: the fit has* $R^2 = 0.9999$. *The simulation has* $N = 10^4$, $\Delta = 0$

into the survival probabilities, we must explicitly enumerate all possible seller histories that allow for survival. We treat the synchronously updated system first and then the asynchronous system.

8.2.1 *Capital accumulation under synchronous updates*

Under synchronous updates, every seller pays one overhead per round. If we know the age of a seller surviving at the end of the current round, we can enumerate the different histories it could have experienced to reach its present state. For example, consider a seller of age two. If it made zero sales in the current round, it must have made four or more sales in the previous round. If it made a single sale in the current round, it must have made three or more sales in the previous round. Making two or more sales in the current round is taken care of when we treat sellers of age one. There are therefore two possible survival paths for a seller of age two. We use the notation $k \to l$ to denote such a path. This should be read as

> k or more sales in the previous round and exactly l sales in the current round.

For our example above, a seller of age two has two survival paths: $4 \to 0$ and $3 \to 1$. A seller of age one has a single survival path: 2.

These survival paths can be thought of as similar to Brownian bridges. Each path of length n is a bridge from $(0, 0)$ to $(n, 0)$ constrained to lie above the x-axis and on or below the triangle given by the endpoints and the point $(1, 2(n-1))$. Figure 8.4 shows these diagrams for all paths of length three.

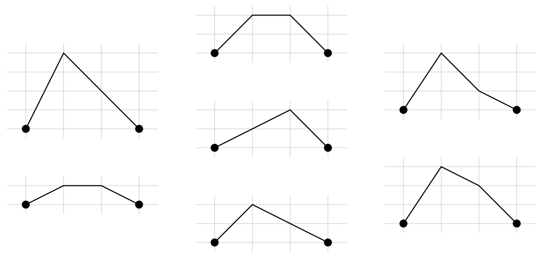

Figure 8.4 *Diagrams showing all survival paths of length three. The gradient of the line indicates the profit (loss) made during each step. All other paths either result in bankruptcy of a seller or are subsumed by shorter survival paths: if a seller reaches zero capital at an intermediate point, their survival can be described by a shorter survival path.*

The number of paths of length n is given by Sloane [68], A006013

$$a(n+1) = \binom{3n+1}{n} \frac{1}{n+1}. \tag{8.16}$$

It is tedious, but relatively trivial, to now enumerate the paths which allow survival. We now need to work out the probability of each path. Since we are in the steady state, and are ignoring correlations, the probability of a particular path is just the product of the probabilities of the individual steps. Furthermore, the individual step probabilities are time-independent: they only depend on ρ which is constant in the steady state. So, for example, the survival probability associated with the path $5 \to 1 \to 0$ is

$$P(5 \to 1 \to 0) = s(5, \rho)e(1)e(0) \tag{8.17}$$

with $s(n, \rho)$ given by equation 8.8 and $e(n)$ by equation 8.6. The total survival probability considering all paths up to length n is just the sum over the individual path survival probabilities and so

$$p_s(\rho, n) = \sum_{\{\text{paths}\}} P(\text{path}). \tag{8.18}$$

We are unable to find $p_s(\rho, \infty)$ but can get a reasonable approximation to survival probability by truncating the expression at some finite maximum path length. The survival probability of a path decreases with increasing path length so the main contributions to p_s come from short paths. We set the maximum path length to be thirteen and use the expression we obtain for p_s (given explicitly in section A.3) to find the steady state density by solving equation 8.1. The result of this analysis is shown in figure 8.5.

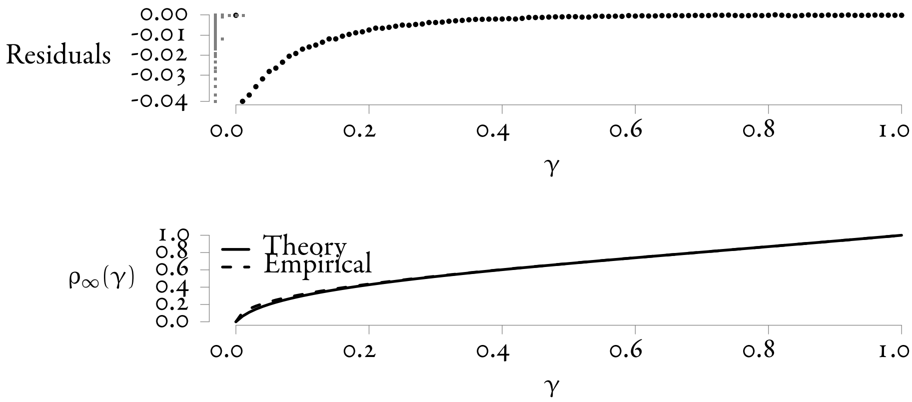

Figure 8.5 *Comparison of the live site density predicted by mean field analysis and simulation results for a synchronously updated system with capital accumulation. The theoretical analysis considers contributions to the survival probability of paths up to thirteen steps in length. Upper figure shows the residuals of the fit, lower figure shows the theoretical and empirical results. The* R^2 *goodness of fit measure is* 0.81*, however, the residuals are not normally distributed, indicating that we cannot trust this number overly. Note how the fit is worst at low* γ *for which longer survival paths have more weight in the survival probability. The simulation has* $N = 10^4$, $\Delta = 0$

As expected, truncating the history at a finite length gives an underestimate for the survival probability, and hence the live site density. This is most noticeable at small values of ρ (low γ): the probability of making a large number of sales in a single round increases with a decrease in ρ and hence the truncation at finite path length becomes more noticeable. We can see this particularly clearly if we plot the difference between theory and experiment for an increasing history size. As figure 8.6 shows, at high γ there is little to be gained from adding to the path length, whereas for low γ increasing the history length produces a noticeable improvement.

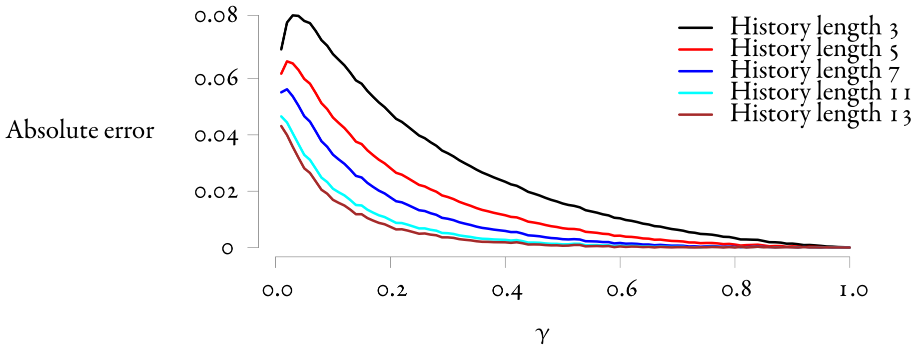

Figure 8.6 *Comparison of absolute errors between theory and simulation for different values of the maximum history length. The errors are largest at small* γ *(corresponding to small* ρ*) and decrease with increasing history length. The simulation is carried out with synchronous updates and has* $N = 10^4$, $\Delta = 0$

8.2.2 *Treating the asynchronous system*

Finally, we carry out a similar analysis for the asynchronously updated system. In this case, the number of overhead payments a seller makes is a random variable, it is consequently not quite as simple to enumerate the survival paths. We start as before by building up the survival probability slowly. For a history of length one (*i.e.*, no accumulation of capital) the survival probability is just given by equation 8.15.

Now consider a seller surviving for two rounds. This seller will have made i overhead payments in the previous round and j in the current round. The distribution of these overhead payments is binomial and given by $c(i)$ (equation 8.12). To ensure survival, this seller must have sold to k buyers in the current round ($k = 0, 1, \dots, 2j - 1$) and $2(i+j) - k$ buyers in the previous round. Hence, the survival probability for paths of length two is

$$p_{s,2}(\rho) = \sum_{i=0}^{\infty} c(i) \sum_{j=0}^{\infty} c(j) \sum_{k=0}^{2j-1} s(2(i+j) - k, \rho) e(k) \qquad (8.19)$$

and the total survival probability becomes

$$p_s(\rho) = p_{s,1}(\rho) + p_{s,2}(\rho). \qquad (8.20)$$

The method easily extends to longer survival paths. When we come to solving for the steady state value of ρ, however, we encounter a problem. There is no closed-form expression for $p_s(\rho)$ and numerical solutions of the problem with infinite sums fail. To avoid this, we notice that the probability of making more than around ten sales is almost zero. We therefore choose to truncate our survival probability by assuming that the probability of making more than ten sales is exactly zero. With this approximation scheme, the two round survival probability can be written as

$$p_{s,2}(\rho) \approx \sum_{i=0}^{5} c(i) \sum_{j=0}^{5} c(j) \sum_{k=0}^{2j-1} s(2(i+j)-k, \rho) e(k). \tag{8.21}$$

Consideration of all survival paths up to length four gives a result which is computationally tractable and allows us to find the steady state density. We compare the analytic and empirical results for this case in figure 8.7. The analytic result does poorly at low ρ, this is because our approximation that the probability of making many sales is small becomes worse as ρ decreases.

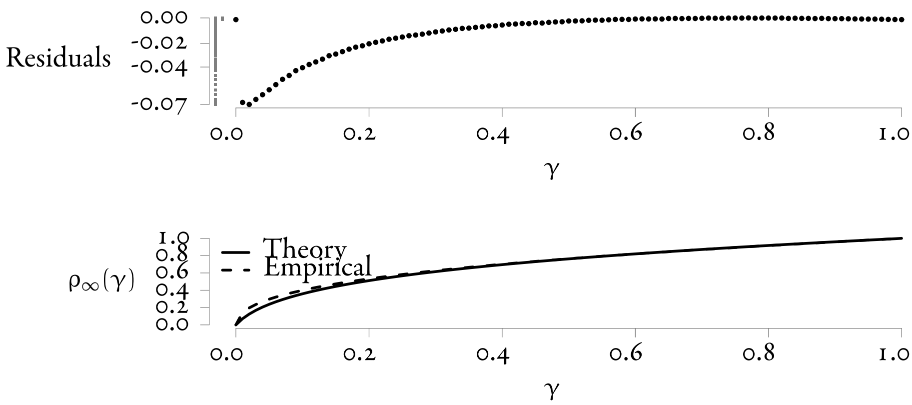

Figure 8.7 *Comparison of the live site density predicted by mean field analysis and simulation results for an asynchronously updated system with capital accumulation. The theoretical analysis considers contributions to the survival probability of paths up to four steps in length. Upper figure shows the residuals of the fit, lower figure the theoretical and empirical results. The* R^2 *goodness of fit measure is* 0.089*. Note how the fit is worst at low* γ *for which longer survival paths have more weight in the survival probability. The simulation has* $N = 10^4$*,* $\Delta = 0$

8.3 ANALYSING THE CORRELATED SYSTEM

Finally, we attempt to find the live site density in the spatially correlated system. Under synchronous updates we will always attract zero, one, or two buyers and hence will either end the round bankrupt or with zero capital. We do not therefore need to consider capital accumulation. Survival of a seller can happen in one of three ways:

1. Both neighbours are dead.

2. One neighbour is alive, but we beat it anyway.

3. Neither neighbour is dead, but we beat them both.

Let d denote the probability that a neighbour is alive. We survive with probability (assuming that left and right neighbours have equal probabilities of being dead)

$$p_s(d) = \frac{d^2}{4} + 2d\frac{(1-d)}{2} + (1-d)^2 = \frac{(d-2)^2}{4}. \tag{8.22}$$

Now we need to find an expression for d. To do this, let us assume that the mean field model is correct, in which case, d is given by the solution to equation 8.1 with p_s replaced by the discrete time zero capital mean field result (equation 8.10). We thus close our system of equations and can arrive at a steady state value for the live site density by solving for ρ in the following equation:

$$\rho = \rho\frac{(\rho_{MF}(\gamma) - 2)^2}{4} + \gamma\left(1 - \rho\frac{\rho_{MF}(\gamma) - 2)^2}{4}\right). \tag{8.23}$$

Where $\rho_{MF}(\gamma)$ is given by the mean field solution previously derived in section 8.1.1. We solve this equation numerically as before, and compare with empirical results. The result is shown in figure 8.8. As previously, we have good agreement at large γ and worsening agreement as γ decreases. This is because the mean field solution slightly overestimates the probability that a neighbouring site will be occupied, and this error is largest at low γ. This occurs because in the correlated system, if we survive, we must have beaten our neighbours. They are then surely bankrupt. The probability that a neighbour is dead is then $1-\gamma$, not $1-\rho_{MF}$. The latter is always larger than the former, underestimating our seller's survival probability.

8.4 SUMMARY

We have sketched how it is possible to analyse the steady state live site density in a system without any spatial correlations. If we do not allow for accumulation of capital, these results are exact. If capital accumulation is allowed,

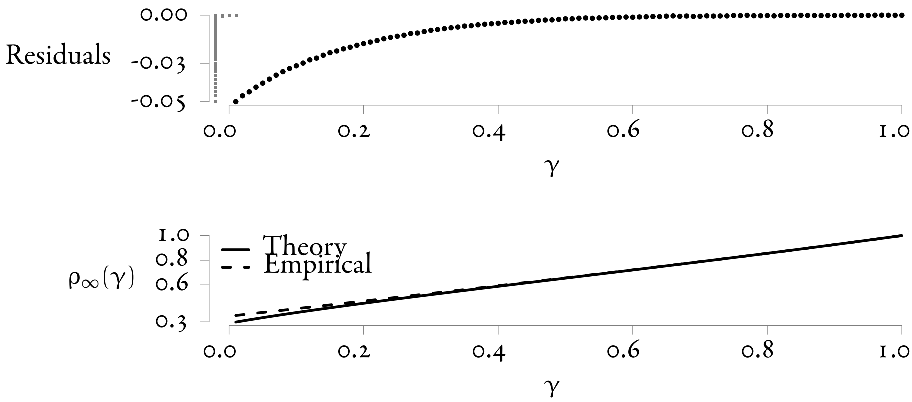

Figure 8.8 *Comparison of the live site density predicted by mean field-like analysis and simulation results for a synchronously updated system with spatial correlations. The theoretical result uses the spatially uncorrelated live site density to solve for the correlated live site density. Upper figure shows the residuals of the fit, lower figure the theoretical and empirical results. The* R^2 *goodness of fit measure is* -15. *Note how the fit is worst at low* γ *for which the uncorrelated result overestimates the probability that a neighbouring site will be occupied. The simulation has* $N = 10^4$, $\Delta = 0$

we must explicitly consider a sum over seller histories to find the steady state result. Using this latter method, it should be possible, though likely very long-winded, to extend our analysis of the general model presented in sections 4.3 and 6.2.1 past the first round of simulation. Our analysis here has been restricted to a single-peaked δ-function price distribution, though this has primarily been for ease of exposition. The extension to a general (time-independent) price distribution should be reasonably trivial but long-winded. In essence we just need to make the probability of outcompeting a neighbour a function of price and update the survival probability appropriately.

CONCLUSIONS & OUTLOOK

9

9.1 SUMMARY

In this thesis we introduced a simple spatial extension to a well-understood competition game. As well as recovering the original results in an appropriate limit, the resulting model demonstrates suprisingly rich behaviour not observed in the original.

One aim in its construction was to produce a model simple enough to be amenable to exact analysis. This aim met with only partial success: in chapter 8 we presented one method for addressing the time-like correlations in the dynamics due to capital accumulation. In section 4.3 we showed that it was possible to derive the system price distribution in the absence of spatial correlations. These latter correlations are the larger part of the mismatch between mean field and empirical results and we have not been able to address them in our analysis.

Subsequently, we explored the behaviour of the model through simulation. The explanations for the minutiae of the results eluded us, however, we were able to construct, and confirm, hypotheses for the behaviour of the steady state. Niche construction and competition-free sales were a common theme in the observed steady state behaviour of the model. The steady state behaviour itself broadly mimics observed pricing strategies in real marketplaces. Sellers could be classified into two groups, cheap and expensive. The former have a strategy that requires they sell to every buyer walking in the door, the latter only require that they sell to some of the potential buyers. This appears qualitatively similar to the different pricing strategies adopted by, say, Harrods and Tesco.

In chapter 6 we found the qualitative behaviour of the steady state was independent of two distinct choices of the update scheme. Furthermore, we showed that the choice of asynchronous dynamics allowed for a system-wide oscillatory cycle not seen in the previous results. We subsequently studied methods of stabilising the system against these oscillations, demonstrating that randomisation in the strategy copying step served to suppress them entirely. We also noted that a system composed of independently oscillating parts, each only weakly coupled to the others, showed significantly smaller oscillations on a global scale. In analogy with global economy, we suggested that reducing the coupling between nations might serve to increase global stability.

9.2 OPEN QUESTIONS

Looking forward, we note that our model does not attempt to capture either the nuances of price-setting in the real world or the vagaries of consumer choices and loyalty. This was by design, we wanted to construct a model simple enough to understand in detail. The model does, however, provide a framework in which one could study the efficacy of strategic price-setting. We know how well sellers perform if they pick new strategies using our evolutionary dynamics. Is it possible for an individual with a complex price-setting strategy to outperform the evolved sellers? We might imagine that an intelligent price-setter would be able to exploit local variation in the system for higher profits than the existing sellers.

The game-theoretic solution concepts presented in chapter 2 might provide the necessary inspiration to construct such intelligent strategies. Sticking with the evolutionary theme, the strategy switching of the minority game [14] might, with suitable modification, also lend itself to our model. In this scheme, sellers might have a portfolio of prices, rather than a single price. One such price is offered in the marketplace, and all other prices are internally scored as if they had been played against the observed competition. The sellers may then choose to swap strategies into and out of their portfolio and exhibit a different price to buyers. We might imagine that these dynamics could allow for 'predatory' sellers: switching between a loss-making price to kill off competition and a high-profit price when the competition has disappeared. The addition of possible loss-leading strategies would certainly be appealing: such techniques are employed regularly in the real world.

A further line of enquiry would be to study a variation of the model that allowed for economies of scale. In our study of the model on general networks in chapter 5, we found that results were unaffected by the choice of network. This was due to our choice of a linear relationship between the number of potential buyers and the expected overhead. Under these circumstances, a seller with the potential to attract many buyers is not advantaged over a seller capable of attracting only a few. Economic theory suggests that a U-shaped cost curve is a more realistic choice: there is some optimal size which minimises the cost per unit sold. Our model essentially has no economy of scale. To add such a process to the model one would have to choose a suitable cost curve. In addition, rather than having the dynamics play out on a static competition network, one should construct some rewiring dynamics that allow for sellers to change the number of potential buyers. It would be interesting to see if the system then arranges itself in a way that places sellers at the optimum size for their cost curves.

Finally, recall our common criticism of many of the simple models of price competition presented in chapter 2: that the assumptions of rationality and perfect competition are bad. Our model produces many qualitatively similar results to those derived with these assumptions, and yet did not make them. It might therefore be possible to use the results of our model to update, with suitable recourse to empirical data, the assumptions necessary for economic modelling.

To conclude, this model is unable to fully answer the question an individual shopkeeper would find most pertinent: 'what price should I sell my goods for?'. Under certain restrictions we can, however, give a probability distribution from which the shopkeeper should choose. With further study we hope that models such as presented in this thesis will be able to give more specific answers to the question. Even in its current form, our model results might allow us to make some qualitative recommendations – not to shopkeepers, but keepers of the global economy – on possible policy choices.

❧

EXACT RESULTS FOR THE MEAN FIELD MODEL

A

A.1 EVALUATING THE LARGE n LIMIT OF EQUATION 8.2

We wish to find the $N \to \infty$ expression for equation 8.2

$$f_N(k) = \binom{2N}{k} \left(\frac{N-1}{N}\right)^{2N-k} N^{-k}. \tag{A.1}$$

First, we consider the term in brackets, rewriting it as $1 - 1/N$ and Taylor-expanding for large N we find

$$\begin{aligned}\left(1-\frac{1}{N}\right)^{2N-k} &= 1 - \frac{2N}{N} + \frac{(2N)^2}{2!N^2} - \frac{(2N)^3}{3!N^3} + \ldots + \mathcal{O}(N^{-1}) \\ &= 1 - 2 + \frac{2^2}{2!} - \frac{2^3}{3!} + \ldots + \mathcal{O}(N^{-1}) = e^{-2} + \mathcal{O}(N^{-1}). \end{aligned} \tag{A.2}$$

The other two terms may be rewritten, expanding the binomial coefficient to give

$$\begin{aligned}\binom{2N}{k} N^{-k} &= \frac{(2N)(2N-1)(2N-2)\ldots(2N-k+1)}{k!N^k} \\ &= \frac{(2N)^k}{k!N^k} + \mathcal{O}(N^{-1}) = \frac{2^k}{k!} + \mathcal{O}(N^{-1}). \end{aligned} \tag{A.3}$$

Hence

$$\lim_{N\to\infty} f_N(k) = \frac{2^k}{e^2\Gamma(1+k)} \tag{A.4}$$

as stated in the main text.

A.2 EVALUATING EQUATION 8.9

To find the survival probability quoted in equation 8.10, we need to evaluate $e(0)$ and $e(1)$ explicitly. This is straightforward to do in the general case: from equation 8.6 and substituting in we have

$$e(n) = \sum_{k=n}^{\infty} \frac{2^k}{\Gamma(1+k)e^2} \binom{k}{n} \left(1-\frac{\rho}{2}\right)^n \left(\frac{\rho}{2}\right)^{k-n}. \tag{A.5}$$

Factoring out the constant terms and expanding the binomial coefficient we have

$$e(n) = \frac{2^n}{\Gamma(1+n)e^2}\left(1 - \frac{\rho}{2}\right)^n \sum_{k=n}^{\infty} \frac{\rho^{k-n}}{\Gamma(1+k-n)}. \tag{A.6}$$

Relabelling in the sum $k \to k - n$ we see that the sum is just e^ρ giving

$$e(n) = \frac{2^n e^\rho}{\Gamma(1+n)e^2}\left(1 - \frac{\rho}{2}\right)^n. \tag{A.7}$$

Hence

$$e(0) = e^{\rho-2} \tag{A.8}$$

and

$$e(1) = (2-\rho)e^{\rho-2}. \tag{A.9}$$

We can also write down an explicit form for $s(k, \rho)$:

$$\sum_{k=0}^{n-1} e(k) = e^{\rho-2} \sum_{k=0}^{n-1} \frac{(2-\rho)^k}{\Gamma(1+k)} = \frac{\Gamma(n, 2-\rho)}{\Gamma(n)} \tag{A.10}$$

since [25, equation 8.352]

$$\Gamma(n, 2-\rho) = \Gamma(n)e^{\rho-2} \sum_{k=0}^{n-1} \frac{(2-\rho)^k}{\Gamma(1+k)} \tag{A.11}$$

and so

$$s(k, \rho) = 1 - \frac{\Gamma(k, 2-\rho)}{\Gamma(k)}. \tag{A.12}$$

Combining equations A.7 and A.8 and simplifying we arrive at the result quoted in equation 8.10

$$p_s(\rho) = 1 - e^{\rho-2} - (2-\rho)e^{\rho-2} = 1 + e^{\rho-2}(\rho - 3) \tag{A.13}$$

A.3 SURVIVAL PROBABILITY WITH CAPITAL ACCUMULATION

In section 8.2.1 we make use of the survival probability as calculated with a maximum history length of thirteen but do not quote it explicitly. The full expression is shown in equation A.14.

$$
\begin{aligned}
p_s(\rho) = 1 + e^{13(\rho-2)} \\
(-3691886014466748940695337754448363 52+ \\
371450559148108898817297636256645 1200\rho- \\
1816949104295252657686083728257843 2000\rho^2+ \\
5746618827658680902808199535924346 8800\rho^3- \\
1318571729966825532161416826029735 93600\rho^4+ \\
2334652108199416124668247695384038 60480\rho^5- \\
3312833568621087436516685675997036 54400\rho^6+ \\
3862679811173481207233102491730575 36000\rho^7- \\
3764975000379478346133011578050256 89600\rho^8+ \\
3104967131177019377462613770535657 47200\rho^9- \\
2184848128703312663081418966902107 34080\rho^{10}+ \\
1319154096327903448719465615190376 44800\rho^{11}- \\
6857125281895049209856748198343321 6000\rho^{12}+ \\
3073085950792982552770950674784901 1200\rho^{13}- \\
1186909120200193510388968292588759 0400\rho^{14}+ \\
3941883046210043099275279576000675 840\rho^{15}- \\
1121025648644893922781269673737929 600\rho^{16}+ \\
2712388238830219849154312061232800 00\rho^{17}- \\
5532204941683842263399736884347840 0\rho^{18}+ \\
9389522961414066974583788538476800\rho^{19}- \\
1302365045063219251155162794294160\rho^{20}+ \\
143854161248628991706304228857200\rho^{21}- \\
12172551841656985119298486126000\rho^{22}+ \\
741137794006061028377582980800\rho^{23}- \\
28911415492716247626737068975\rho^{24}+ \\
542800770374370512771595361\rho^{25}) \quad \text{(A.14)}
\end{aligned}
$$

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