APPLICATIONS OF FIELD THEORY IN SURFACES AND INTERFACES

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TO MY PARENTS

MR. AND MRS. ZUEI-TIEN LIN

AND MY WIFE GRACE HUILING YU

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DECLARATION

The work in this thesis is entirely my own, except where otherwise indicated.

ABSTRACT

The Euclidean version of field theory is defined on a d-dimensional Euclidean space; the surfaces and interfaces are (d-n)-dimensional extended objects given by imposing n constraints on the d-dimensional Euclidean space. The effective action describing the latter can be derived from the former, in the long-wavelength limit, by a semiclassical method. One can then study the renormalisability, and associated properties, of the effective action.

In Chapter I of this thesis, the connection between field theory and classical statistical mechanics is discussed; ideas of renormalisation group and differential geometry of manifolds are also covered. In chapter II, the actual derivation of effective action from scalar field theory is demonstrated by using the method of collective coordinates, which is a semiclassical method. The higher derivative geometrical interactions are also derived from the scalar field theory by using a generalised collective coordinate method. In Chapter III, the differential geometric analysis, on the (d-1)-dimensional submanifold embedded in a d-dimensional Euclidean space, is set up in order to identify those geometrically invariant interactions from Chapter II. In Chapter IV, the renormalisation of the simplest of those interactions, the contraction of the second fundamental form with the metric tensor (g.b), is studied and carried out within the framework of an ϵ expansion in $1+\epsilon$ dimensions. In Chapter V, the technique developed in Chapter IV is extended to consider a system with extra Goldstone modes.

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CHAPTER I INTRODUCTION

Parallelism and analogy have been used as very powerful tools to understand the mysteries of our universe since the time of Greek physics. Recently, the connection between quantum field theory and classical statistical mechanics has generated very fruitful results both in particle physics and many-body theories. In this introduction, we shall first discuss important phenomena occuring in statistical physics, i.e., phase transitions and critical phenomena, then the connection between these phenomena and quantum field theory will be pointed out. The ideas of renormalisation group will also be explained, especially their applications in both fields. Some differential geometry concerning manifolds will be introduced next as the tool to understand surfaces and interfaces. Finally, the physical significance of these extended objects will be discussed.

1.1. Many Scales of Length in Physics

Most of the problems in conventional physics have their own scales of length, i.e., a certain scale of length is usually associated with a physical problem. Even in the same physical system, the effects of different scales of length have little influence on each other. But there is a class of problems whose many scales of length make equal contributions in the same physical system. A simple example is given by the critical point in a fluid [Wilson 1979]: The critical point of water is specified by that temperature and pressure where the density difference between

coexisting liquid and vapour goes to zero (at 217 atmospheres, 647 K). At its critical point, fluctuations in water density could be seen from the mixture of liquid drops and gas bubbles; moreover, these fluctuations in density happen at all possible scales from single molecule up to the volume of the specimen. This phenomenon can be observed directly in fluids: as the system approaches its critical point, one of the length scales will be comparable to the wavelength of light, the fluctuations begin to scatter light strongly and the fluid turns milky. This opalescence persists even when the system is extremely close to the critical point. Problems with many scales of length often relate to phenomena near the critical region, therefore, they are the problems of critical phenomena.

In physics, the equal importance of multiple scales of length is usually associated with scale invariance. We may use the notion of the scale transformation to classify physical systems into scale-dependent and scale-invariant systems. Ordinary physics relating to some particular length scale is of course scale-dependent. Scale-invariant physics is quite different from ordinary physics. If scale-invariant physics is described by, for instance, the Hamiltonian of the system, then the effective Hamiltonian will not change even when the length scales change. Examples are: the critical point of a statistical system undergoing a second order phase transition (A discussion of the order of phase transitions will appear in the next section); the limit of collision energy very much larger than the typical hadron masses in particle physics. In

such cases, the effects of the degrees of freedom associated with all scales of length are naturally very important; many scales of length mean very many degrees of freedom in the physical system, which is then usually not solvable in the conventional sense. A mathematical method called the renormalisation group (RG), which originates from particle physics, has been generalised to deal with problems that have multiple scales of length. The essential ideas of the RG method are to incorporate the effects of the short wavelengths into the effective Hamiltonian at each step of scale transformation, i.e., the old Hamiltonian H(s) for the old degrees of freedom (s), will be transformed into H'(s') for the new degrees of freedom (s'), the number of degrees of freedom being reduced as the length scale increases. The change from H(s) to H'(s') is called a renormalisation group transformation. The method of renormalisation group is thus very closely related to the scaling behaviour of the physical systems.

Close to, but not at critical point, there is a limit to the range of all possible scales of length, called the correlation length. At separations greater than the correlation length, density correlations decrease exponentially with distances; up to the correlation length, they decay only by a power law. Therefore, regions separated by a distance greater than the correlation length are essentially independent from each other. Phase transitions of second order (now more commonly known as continuous phase transition) correspond to the divergence of the correlation length. When the method of renormalisation group is used to reduce the

number of degrees of freedom in a physical system, correlation length may also be regarded as the minimum size one can reduce to without qualitatively changing the properties of the system [Wilson and Kogut 1974]. In the case of quantum field theory, which has infinite number of degrees of freedom, the correlation length is usually the Compton wavelength of the particle with the lowest mass. This hints to us that quantum field theory corresponds to a critical region and massless field theory to a critical point.

In the following sections, we expand on these ideas and relationships.

1.2. Phase Transitions and Critical Phenomena

In equilibrium statistical mechanics, the study of phase transitions is one of the most interesting and challenging problems. Most substances in thermodynamical systems change from one distinct form or phase to another, as the temperature, pressure, or other conditions are varied. Simple examples of phase transitions are: liquid-gas transition, order-disorder transition in the magnetic system, etc. Mathematically, the task is to explain or derive the existence of phase transitions and the behaviour of the transition point from the statistical-mechanical ensembles. If we believe that the partition function of a statistical system contains all the essential information necessary for the study of phase transitions [Onsager, 1944], a phase transition point would be a singular point

of the partition function. Therefore, it is necessary to go to the thermodynamic limit to have a phase transition in the mathematical sense. Consider a statistical system specified by some thermodynamic potential; the potential cannot be expanded by the Taylor's series about the transition point because of the above-mentioned non-analiticity. Normally, the order of phase transitions corresponds to one of the particular derivative of some thermodynamic potential, which could be discontinuous or infinite. Thus following Landau [Landau and Lifshitz, 1969], the liquid-gas transition is a first order transition (but water-vapour transition at 647 degrees Kelvin and pressure of 217 atmospheres is a second order transition) since the state of the body, which may be characterised by the order parameter, changes discontinuously. In general, it is sufficient to use the value of the order parameter, if one exists, to specify the order of the phase transition [Pfeuty and Toulouse, 1977]. But what is the order parameter? We shall use a thermodynamical argument to illustrate its physical meanings. The order-disorder phase transition is actually quite a general property of many-body systems. For a system restricted to a fixed volume, the state of thermodynamic equilibrium at temperature T is that which minimises the free energy

$$F = U - TS. \tag{1.1}$$

At high temperature, the negative second term dominates, so that the minimum value of F is related to the maximum value of entropy S; therefore, it is a disordered phase at high temperature and the

order parameter is typically zero. At low temperature, the internal energy U is the dominating factor, the state with the minimum internal energy is the ordered state and the order parameter is typically non-zero. Examples of order parameters are: the homogeneous magnetisation in a ferromagnetic transition, the magnitude of the alternating magnetisation in an antiferromagnetic transition, and the difference in density between liquid and gas in the liquid-gas transition.

Near a critical point, one observes that physical quantities obey some sort of power laws. Taking the spin system as an example, the order parameter M, which is the magnetisation in this case, the specific heat C, the magnetic susceptibility χ and the correlation length ξ behave respectively as

$$
\begin{aligned}
M &\sim |t|^{\beta} \\
C &\sim \frac{1}{|t|^{\alpha}} \\
\chi &\sim \frac{1}{|t|^{\gamma}} \\
\xi &\sim \frac{1}{|t|^{\nu}}
\end{aligned}
\tag{1.2}
$$

where all critical properties are proportional to the absolute value of the reduced temperature t, $t=(T-T_c)/T_c$, raised to the power of the critical exponent α, β, γ, ν, respectively. These critical exponents are the physical quantities measured by the experimentalists. One remarkable thing about critical exponents is that they do not take any arbitrary values; they satisfy some sort of simple relations like $\alpha + 2\beta + \gamma = 2$, which are called

scaling laws. The idea of scaling was first suggested by Widom [Widom 1965], who conjectured that the equation of state near the critical point should be written as the following form

$$H = M^{\delta} f\left(\frac{t}{M^{1/\beta}}\right) \tag{1.3}$$

where H is the applied magnetic field. The equation of state (1.3) depends only on a single variable, instead of the expected two variables, M and T. We may then apply the following scale transformation to Widom's equation of state,

$$\begin{aligned} t &\longrightarrow t' = \frac{t}{k} \\ M &\longrightarrow M' = \frac{M}{k^{\beta}} \\ H &\longrightarrow H' = \frac{H}{k^{\delta\beta}} \end{aligned} \tag{1.4}$$

it is very easy to see that it is invariant. Other similar scaling functions may be written for the singular part of the free energy and for the correlation function. Using the above equations plus the equation of state, the relations among critical exponents, e.g. $\alpha + 2\beta + \gamma = 2$, can be derived. This is the reason why the relations among the critical exponents are called the scaling laws.

There is another interesting property of critical behaviours, namely, different physical systems can be assigned to a small number of classes. Specifically, the critical exponents are quite independent of the microscopic details such as the strength and the precise range of the interaction (first and second neighbours etc.)

or the structure of the lattices as in the Ising model. This is the concept of universality. As we shall see in the later sections, scaling and universality near the critical region may be explained by the renormalisation group analysis. But what characteristics of a physical system specify the class which the system belongs to? Since the critical exponents have shown a clear dependence on the number of space dimensions d and on the number of dimensions and symmetry of the order parameter n, therefore, at least d and n are required to specify the universality classes (If the range of interactions decays as a power law, e.g. dipolar forces, this may also be important). A table of universality classes is given in Wilson [Wilson 1979].

1.3. Connections Between Field Theory and Statistical Mechanics

Quantum field theory is one of the most obscure and complicated subjects in the history of science. It has been studied since the the time of the discovery of quantum mechanics, and it is still being developed nowadays. There were times that the majority of theoretical physicists believed that the concepts of quantum field theories should be abandoned altogether, but quantum field theories still survive today and are even stronger than ever. When quantum theory was discovered by Planck at the turn of the century, the essential research interests were the behaviours of atoms and radiation. Although the radiation is a "field" itself, physicists were more interested in the behaviour of the sources of the fields,

the electrons in the atoms, instead of the fields. This lead to the discovery of quantum mechanics, which deals mainly with material particles, during 1925 to 1926. The classical electromagnetic field theory was completed after the discovery of Einstein's special theory of relativity in 1905. The quantisation of electrodynamics was only partially completed by Dirac in his 1927 paper [Dirac 1927] of the mathematical treatment of the spontaneous emission of radiation from the atom. His ideas are based on the classical ideas of treating fields as oscillators. This quantum mechanical treatment of radiation provides a mechanism to explain how photons can be freely created and destroyed. Thus, fundamentally speaking, particles and fields are treated differently in quantum mechanics; a physical system composed of material particles is described by calculating the probabilities for finding each particle in any given region of space or range of velocities, but the fields are quantised into photons which can be created and destroyed.

The central new ingredient of quantum field theory is the idea that the probability wave of a material particle should also be quantised as the electromagnetic radiation has been done; this is often referred to as second quantisation. The material particles can be understood as the quanta of various fields, just as the photon is the quantum of the electromagnetic field. Based on the principles of relativity and quantum mechanics, quantum field theory adopts the philosophy that the essential reality is a set of fields. All else is derived as a consequence of the quantum dynamics of these fields. Of course, there were some immediate problems for quantum field

theory. The first one is the existence of particles with negative energy; this problem was first solved by Dirac in his "hole" theory. There may exist some unfilled "holes" in the sea of the negative energy particles, and these "holes" would behave like particles of positive energy with opposite electrical charges; they are the "antiparticles" in contrast to the particles. Dirac's "hole" theory can even offer a mechanism for the creation and annihilation of particles without using the ideas of quantum field theory. To the mind of Dirac, the "hole" theory is the result of the marriage of quantum mechanics and special relativity, therefore, it is not necessary to accept the quantum field theory in order to describe any material particles except the photon. This negative energy problem can also be solved in quantum field theory without using the ideas of unobserved particles of negative energy, and still keeps the mechanism of creation and annihilation of particles and antiparticles. The second major problem is that of infinities. This one can only be solved by the method of renormalisation, which will be discussed in the next section. Quantum field theory gave the concepts of particles new interpretations, and also changed the concepts of interactions or forces. For example, the force between two charged particles is no longer understood as the interaction of the classical electromagnetic field with the charged particles. Instead, the interaction happens as the results of the continual exchange of the quanta of the electromagnetic field, i.e. by exchanging the photons. This picture of interactions should also apply to other types of forces as the results of exchanging virtual particles. They are called virtual particles because it would

violate the conservation law of energy to create these exchanging particles as real particles; the process is allowed by the uncertainty principle of Heisenberg.

There are several equivalent formalisms of quantum field theories. The path integral formulation of Feynman's [Nash 1978] is the simplest and clearest in physical concepts among all the different formalisms of quantum field theories. The advantages of working in the path integral formulation are at least two fold: first of all, it gives a very simple and physically intuitive connection between classical and quantum theories; secondly, it also provides an extremely beautiful analogy with classical statistical mechanics [Symanzik, 1958]. It is a well known fact that a field theory defined on a d-dimensional Minkowski space will be transformed into a field theory on a d-dimensional Euclidean space by applying the Wick rotation to the original theory. This Euclidean field theory is also one of the essential tools in the rigorous approach to the construction of an interacting quantum field theory [Simon, 1975]. The connections between quantum field theory and classical statistical mechanics started to be noticed by the majority of physicists after they had realised that some of the long standing difficult problems in classical statistical mechanics could be solved by borrowing techniques developed in quantum field theories, for example, the problem of calculating the critical exponents for the second order phase transitions is connected to the control of infrared divergences in the massless field theory [Amit 1978]. The explicit connection between the two subjects are

given in the following table which is summarised from Parisi [Parisi 1980].

Quantum Field Theory	Classical Statistical Mechanics
Minkowski space	Euclidean space
$\hbar$	$\beta^{-1} = k\,T$
$\mathcal{L}$	H
Feynman factor for amplitudes: $\exp(i\mathcal{L}/\hbar)$	Boltzmann factor for probabilities : $\exp(-\beta H)$
Sum of all vacuum to vacuum diagrams : $\int D[\phi]\ \exp(i\mathcal{L}/\hbar)$	Partition function: $\int D[\phi]\ \exp(-\beta H)$
Vacuum energy	Free energy
Vacuum expectation value: $\langle 0 \mid A \mid 0 \rangle$	Statistical expectation value: $\langle A \rangle$
Quantum fluctuations	Statistical fluctuations
Mass	(Correlation length)$^{-1}$

Green functions: $\langle 0 \mid T[\phi(x)\phi(0)] \mid 0 \rangle \sim \exp im \mid t \mid$	Correlation functions: $\langle \phi(x)\phi(0) \rangle \sim \exp - m \mid x \mid$
Change of vacuum	Phase transition
Goldstone bosons	Spin waves
Decrease to zero-mass	Approach to a second order phase transition
Hamiltonian	(Logarithm of) Transfer matrix
Cutoff	e.g. (lattice spacing)$^{-1}$

1.4. Ideas of Renormalisation Group Method

Historically, the methods of renormalisation group were based on the ideas of renormalisation, which were devised in order to solve the problems of infinities in the quantum electrodynamics (QED) [Bjorken and Drell 1965]. QED is the theory which describes the interactions between electrically charged particles and the electromagnetic field, and the problems of infinities are the unphysical appearances of infinity in unobservable quantities in the

theories which describe some real physical processes. Some of the well known examples of problems of infinities in QED are: self-energy of the electron, which is produced by the emission of virtual photons and their reabsorption by the same electron; the polarisation of the vacuum by the applied electric field; and the scattering of electrons by the electric fields of atoms. Take the example of the self-energy of the electron. Since an electron would change into a photon and an electron in the process which produces the self-energy, these two particles can share the momentum of the original electron in an infinite variety of ways and there is no limit to how large the two momenta can be. Therefore, the sum over all the ways that the momentum can be shared out can and does lead to an infinite contribution to the self-energy of the electron. The cure of this problem is based on the ideas that the physically observed mass is not only just the "bare" mass, which appears in the equations for the electron, but also the infinite "self" mass, which is produced by the interactions of the electron with its own virtual photon cloud. The finiteness of the physically observed mass is the result of the cancellation of the infinite part of the bare mass with that of the self mass. Similar procedures would also apply to other physical parameters such as the charge of the electron. This method of eliminating infinities by absorbing them into the redefinitions of the physical parameters is called renormalisation.

The procedure of renormalisation also reveals a picture of the structure of matters, which is more natural and consistent with the field theory than with the mechanics. In the context of the quantum

field theory, the renormalisation effects of the photon-electron-positron cloud may be stripped away as one probes successively shorter distances. The infinities, such as bare mass, appearing in our fundamental field equations leave no physical paradox, however, since, for example, the virtual photon cloud of the electron can never be completely switched off in order to measure the bare mass of the electron. This picture of the renormalisation can hardly be realised from the mechanical point of view.

As the procedure of the renormalisation can be pictured as probing successively shorter distances (or incorporating their effects in the effective Lagrangian), there is a freedom of the choice of the initial distance to start probing. This arbitrariness of choice means that any initial value can be chosen without changing the ultimate results of the physical quantities; this also means that there is an infinite set of equivalent renormalisation procedures. This infinite set of renormalisation procedures forms what is called the renormalisation group. The idea of the renormalisation group was first suggested by Stueckelberg and Petermann [Stueckelberg and Petermann 1953], and was applied to the actual physical processes of QED by Gell-Mann and Low [Gell-Mann and Low 1954]. A group is a set of objects which satisfies certain rules such as the product of any two objects must also be in the group (the renormalisation group is, in some formulation, a semigroup since the inverse of the group element may not be defined [Pfeuty and Toulouse 1977]). The above renormalisation group procedure in the context of particle physics may also be called the "older"

version of the renormalisation group. In this original version, the inverse of the initial distance may be treated as a "cutoff" parameter, which has to go to infinity at the end of the day in order to preserve the locality, i.e. causality of the field theory. The methods of renormalisation group guarantee the validity of the theory by ensuring that it is independent of the cutoff. In other words, by allowing an arbitrary choice of the initial distance, we can make the theory independent of the cutoff parameter.

A new version of the renormalisation group was suggested by K.G.Wilson [Wilson and Kogut 1974]. His approach is a combination of the block spin picture of Kadanoff [Kadanoff 1966] and the original version of the renormalisation group of Gell-Mann and Low. In his treatment of, for example, the ferromagnetic phase transition, Kadanoff used a model consisting of a lattice of interacting spins with each spin sitting at a site. When the system approaches the critical point, the correlation length becomes very much larger than the lattice spacing; as was discussed in the previous sections, the problem of extremely large number of degrees of freedom comes in and this makes the problem too complicated and difficult to be solved by conventional methods. The way to solve a large problem is to break it into a sequence of smaller and more manageable pieces. Kadanoff, firstly, divides the lattice into blocks of spins, the blocks being square or triangular or whatever. Then, he considers the blocks to be the new basic entities and the effective interactions between the blocks are calculated. After finding the effective interactions of the new entities, he rescales the dimensions of the lattice down to

the original scale such that the lattice spacings are kept constant. Repeating this process would create a family of corresponding effective Hamiltonians with coupling constants covering different ranges of distance. In this manner, we may choose a very simple type of interacting spins to start with, such as the Ising model of the nearest neighbouring couplings. Clearly, the fundamental scale of length gets longer after each iterative operation of the block-spin transformation; and this would average out or include the short-wavelength fluctuations whose scales are shorter than the block sizes up to the correlation length. The resultant system would reflect only the long-range properties of the original Ising model, the effects of the smaller scale fluctuations all being incorporated into the effective couplings. Wilson used the block spin picture of Kadanoff as a basis to illustrate his new version of renormalisation group, where the block-spin transformation can be treated as an example of a renormalisation group transformation. In particular, it provides a very clear physical meaning for the renormalisation procedure. (The fact that the renormalisation group may be a semi-group is illustrated by the block-spin techniques, since the original spin configuration before any block-spin (renormalisation group) transformation could not be recovered as the essential information has been lost during the averaging out procedures for calculating the effective Hamiltonian for the new block spins.) Another feature of the new version of the renormalisation group is that it usually allows more than one quantity to vary (there may be many couplings constants), instead of only one (for example, the charge of the electron in QED). The many longer-range coupling

constants generated by the new version of the renormalisation group transformations may form a multi-dimensional parameter space, while the parameter in the original version can form a line only. The new version of the renormalisation group is more general than the original version; the crucial link between the two versions lies in the cutoff parameter. In the original version, the cutoff has to be eliminated at the end by letting it go to infinity (zero spacing); the cutoff parameter is always kept as a constant in the block-spin techniques, since the rescaling to the original lattice takes place at each iteration. But ideas behind the two versions are still quite similar: infinite cutoff (zero spacing) in the original version is, in fact, equivalent to the large size of the correlation length compared with the small, finite lattice spacing of the new version. In conclusion, Wilson's renormalisation group methods offer new physical ideas about reducing the degrees of freedom in a systematic way, and they can also be made to work in real space.

The principal structures in the generalised renormalisation group formalism are the following. We start from the existence of the transformation

$$H' = R(H) \tag{1.5}$$

giving the couplings of the new effective Hamiltonian in terms of the old one. At criticality, this transformation can be iterated an infinite number of times. The simplest property of such a sequence is to tend to a limit which must be a fixed point:

$$H^* = R(H^*). \tag{1.6}$$

Universality will emerge naturally if there exists an H^* which is stable to all critical perturbations (This is infrared stable since the transformation leads to successively larger distances). The fixed point must, however, be (infrared) unstable to deviations away from criticality because the correlation in units of lattice spacing (or inverse cutoff) is being decreased at each stage. For a given fixed point, it is natural to decompose interactions with respect to a basis which are eigenperturbations of the fixed point. Associated with each eigenperturbation is an eigenvalue or critical exponent. These interactions can then be classified as relevant or irrelevant according to the sign of this exponent.

1.5. Differential Geometry of Manifolds

In Chapter III, we shall use the tools of differential geometry to calculate the geometrical invariants of the (d-1)-dimensional submanifold embedded in a d-dimensional Euclidean space. It is quite helpful, therefore, to introduce the concepts of differentiable manifolds here [Auslander and MacKenzie 1977; Klingenberg 1978],

DEF (i) A topological manifold M of dimension n is a Hausdorff topological space with a countable basis such that there

exists a family of homeomorphisms $\{ u_\alpha : M_\alpha \rightarrow U_\alpha \subset R^n \}_{\alpha \in A}$ from open sets $M_\alpha \subset M$ to open sets $U_\alpha \subset R^n$ and $\bigcup_\alpha M_\alpha = M$. These homeomorphisms will usually be denoted by (u_α, M_α), and they are called coordinate systems or charts for M. The collection $(u_\alpha, M_\alpha)_{\alpha \in A}$ is called a (topological) atlas for M.

(ii) An atlas $(u_\alpha, M_\alpha)_{\alpha \in A}$ is a differentiable atlas if, for every $(\alpha, \beta) \in A \times A$, the homeomorphism $u_\beta \circ u_\alpha^{-1} : u_\alpha(M_\alpha \cap M_\beta) \rightarrow u_\beta(M_\beta \cap M_\alpha)$ is a diffeomorphisms.

(iii) Two atlases $(u_\alpha, M_\alpha)_{\alpha \in A}$ and $(u_{\alpha'}, M_{\alpha'})_{\alpha' \in A'}$ are equivalent if the union of these atlases is a differentiable atlas.

(iv) A differentiable manifold is a topological manifold together with an equivalence class of differentiable atlases.

The essential ideas of a manifold are that it is locally homeomorphic to an Euclidean space and can be represented by more than one coordinate system. Two sets are homeomorphic to each other if there exists a homeomorphism u_α ; u_α is a homeomorphism if u_α is bijective (one-to-one and onto) and both u_α and u_α^{-1} are continuous. Roughly speaking, two sets are homeomorphic if these two sets can be continuously deformed into each other. A diffeomorphism is a differentiable homeomorphism. If more than one local coordinate system were to be allowed on a manifold, then the change of coordinate systems must satisfy condition (ii) of the above definition such that the transformation of the two coordinate systems must be smooth between the intersections of the two

coordinate systems $u_\alpha(M_\alpha \cap M_\beta) \rightarrow u_\beta(M_\beta \cap M_\alpha)$, i.e., $u_\beta \circ u_\alpha^{-1}$ must be a diffeomorphism. (The field theoretic treatment in this thesis is open to criticism precisely because it does not allow for the possibility of several coordinate patches.) In this definition of the differentiable manifold, a manifold is also called a surface for the case dim m = 2. But, what we are really interested is not only in the differentiable manifold but also in the embedding of the surface in Euclidean space, which is one of the core topics of classical differential geometry [Eisenhart 1926; Sternberg 1964].

There are two pairs of concepts related to the descriptions of surfaces: a local and global properties; intrinsic and extrinsic aspects. The two pairs of concepts are, in fact, quite independent from each other. The local behaviour of a surface should be able to be specified by one local coordinate system, and may or may not involve the total behaviour of the surface, which depends on the type of the surface. The study of the surface from the local point of view uses the differential and integrational techniques based on a local coordinate system, while a "global" problem can be described as one which, in general, could not be stated locally in terms of one single coordinate system on a surface, but must necessarily involve the total behaviour of the surface. The most interesting and important aspect of the total behaviour of a surface is related to the topology of the surface. Another important object of interests in the global geometry is the fibre space which is the extension of the differentiable manifold; fibre space is, locally, a cartesian product of a base manifold times a fibre, and each fibre could carry

some sort of algebraic structures such as vector space or a Lie group etc. [Kobayashi 1963 and 1969]. When the fibre is a Lie group locally, the fibre space becomes a fibre bundle which has enormous interesting applications to the core of the current theoretical physics, i.e., gauge theories [Chouquet-Bruhat, Y. et al, 1977; Schutz 1980]. The study of global geometry on differentiable manifolds and fibre space manifests one of the most striking and beautiful achievements of modern mathematics, i.e., the unification of the geometry, topology and analysis.

The studies of the intrinsic and extrinsic properties of the surfaces are much older fields in differential geometry, although there are some more modern studies on these fields [Chern 1967]. The intrinsic properties of the surfaces are those which could be defined in terms of the tangent vectors to the surface and the first fundamental form and its derivatives. In contrast, those geometrical properties, which are related to the normal vector fields and the second fundamental form, are called the extrinsic properties of the surfaces. The extrinsic properties are usually related to the details of how the surfaces embed in Euclidean space and cannot, in general, be reduced to expressions in terms of the first fundamental form and its derivatives. We may use a model to illustrate the intrinsic properties of surfaces. Imagine a "flat" creature which lives on a surface; those creatures cannot imagine what the surface, which they live on, looks like from the "extrinsic" point of view. What the "flat" creatures could understand are those properties which relate to the measurement of the distance between two points

on the surface. These properties, such as the angle between two lines and the areas on the surface etc., are collectively called the intrinsic properties of the surface. The extrinsic properties are usually related to how the surface is curved. It turns out that not all the different curvatures defined on the surface are extrinsic: the mean curvature is extrinsic, but the Gaussian curvature is intrinsic, which is the result of the Gauss Theorema Egregium (The Principal Theorem of Gauss). The difference between the intrinsic and extrinsic properties may be illustrated by an example: take a flat piece of paper, the extrinsic properties of the piece of paper will change if the paper is rolled into a cylindrical shape, but the intrinsic properties are not changed at all. The intrinsic properties of the surface may determine the extrinsic properties of the same surface if the surface is closed and convex. This is the implication of the Cohn-Vossen theorem [Chern 1967].

1.6. Significance of Interfaces

In this last section, we introduce a mathematical formalism for the description of an m-dimensional submanifold embedded in n-dimensional Euclidean space. Apart from the special parametrisation for the description of the m-dimensional submanifold, which is relevant to the field theories of interfaces and surfaces, most of the geometrical properties of the m-dimensional submanifolds of n-dimensional Euclidean space are well investigated by mathematicians. This mathematical formalism of the m-dimensional submanifold of the

n-dimensional Euclidean space, with n and m are treated as any integer value, is very useful for its applications to field theories, especially in classical statistical mechanics. Following the conventional notations in field theory, we shall use d and n, where n is the number of constraints imposed on the d-dimensional Euclidean space in which the generalised (d-n)-dimensional submanifold is embedded. The range of physical applications of the (d-n)-dimensional manifolds are enormous [Zia 1983 a]; they include the applications in both statistical mechanics and elementary particle physics: the (d-n)-dimensional extended objects could be interfaces between two coexisting phases, critical droplets, topological excitations like strings, membranes or whatever. The statistics and dynamics of interfaces also have vast applications in the theory of metastable and unstable states [Gunton and Droz 1983]. Most of the theories related to the (d-n)-dimensional extended objects postulate a special expression for the Lagrangian or Hamiltonian of the form [Zia 1983 b]

(1.7) $$H \propto \text{(d-n)-dimensional volume,}$$

This generalised "volume" would include the surface area for the (d-n) = 2 case, and the constant of proportionality would have the dimensionality of the energy per unit volume. The models of the extended objects are usually treated as the phenomenological models for the various physical phenomena. But, these Hamiltonians may also be treated as the effective Hamiltonian derived from the long-wavelength (or low-temperature) limit of a field theory which is

defined on the full d-dimensional space. In this approach, we shall derive, in the next chapter, the effective interfacial Hamiltonian from the Landau-Ginsburg-Wilson Hamiltonian in the long-wavelength limit. The characteristic of our derivation, in the next chapter, is based on a physical system which has two coexisting phases, and the interface plays the role of dividing these two coexisting phases. We now turn to this derivation.

CHAPTER II FROM LGW MODEL TO INTERFACE MODEL

As we have mentioned in last chapter, near the critical region, scaling behaviour and universality suggested that the microscopic details of a system are not relevant, since, we are interested in the long-range behaviour. The success of solving the problems of critical phenomena depends on a good and simple model to start with. In the history of physics, the earlier attempts to propose models for the critical phenomena were collectively called the mean field theories. Examples of mean field theories are: the theory for the phase changes in fluids by van der Waals in 1873, the theory of magnetic phase transition by P. Weiss in 1907 and a general formulation of the mean field theory by L.D. Landau in 1937 [Landau 1937]. The reason why the above theories are called the mean field theories is because, in all of them, the state of any selected particle or spin of the system is determined by the average properties of all the material as a whole. One may view all the particles or spins in the system as contributing equally to the forces at every site of the system, in other words, we have assumed that the interactions in the system have infinite range and all the microscopic behaviours of the system are ignored. (In the renormalisation group approaches, microscopic behaviours are incorporated, instead of being ignored, into the renormalisation group transformation.) We shall start to discuss Landau's mean field theory and its applications to the critical phenomena. Then, a simple field theoretical model, which is based on Landau's mean

field theory, will be discussed. The derivations of the effective Hamiltonian from the field theoretical model of Landau-Ginsburg-Wilson, in the long-wavelength limit, will also be given via the methods of collective coordinates [Gervais & Neveu 1976]. Finally, possible generalisations of the effective Hamiltonian of the interface model will also be discussed.

2.1. Landau's theory of critical phenomena

We shall use a magnetic system to illustrate the ideas of Landau's mean field theory [Landau and Lifshitz, 1969]. The thermodynamics of this system is determined by the Helmholtz thermodynamic potential or free energy $A(T, M)$. Landau assumed that $A(T, M)$ is an analytic function of T and M, where T is the temperature and M is the magnetisation, even in the region near T_c. If we consider a system with symmetry

$$M \longrightarrow -M, \tag{2.1}$$

as in the case of Ising model, analyticity implies that $A(T, M)$ as

$$A(T, M) = a_0(T) + a_1(T)\, M^2 + a_2(T)\, M^4 + O(M^6). \tag{2.2}$$

Guided by the phenomenology of the Ising spin systems, the disordered state, $T > T_c$, corresponds to zero magnetisation and the ordered state, $T < T_c$, corresponds to the "spontaneous non-zero magnetisation". We may thus assume that

(2.3) $$a_1(T) \sim a\cdot(T - T_c)$$
$$a_2(T) > 0,$$

where a is a positive constant. If $T > T_c$ then $a_1(T) > 0$, the shape of the potential is trivial; if $T < T_c$, the value of $a_1(T)$ changes from positive to negative, the shape of the potential develops into a double-well potential, then the system will have to choose from either up or down values of the spontaneous magnetisation. This spontaneous magnetisation corresponds to the occurance of some sort of "ordering" phenomenon, that is to say, some symmetry have been lost. This illustrates the analogy between the second order phase transitions and spontaneous symmetry breakings in the field theory.

Rewrite A(T, M) as

(2.4) $$A(T, M) = a_0(T) + 1/2\,(T - T_c)\,M^2 + 1/4\,M^4$$

where a and $a_2(T)$ have been absorbed into the scales of $(T - T_c)$ and M (or rescale them). The Equation of State is given by the external magnetic field H,

(2.5) $$H = \frac{\partial A(T,M)}{\partial M} = (T - T_c)\,M + M^3$$

or $$H = M^3\left[\frac{(T - T_c)}{M^2} + 1\right]$$
$$= M^{\delta} f(t / M^{1/\beta})$$

where the scaling form (1.3), from Section 1.2., has been used. We may deduce that

(2.6) $$\delta = 3, \quad \beta = 1/2, \quad f(x) = x + 1.$$

This is not bad compared with three dimensional experimental results: $\delta \simeq 4.5$, $\beta \simeq 0.35$. The susceptibility χ is given by the change of the magnetisation in the presence of the applied external magnetic field,

$$\chi = \frac{\partial M}{\partial H} \quad \ldots$$

(2.7) $$\text{or} \quad \chi^{-1} = \frac{\partial H}{\partial M} = \frac{\partial^2 A(T,M)}{\partial M^2} = (T - T_c) + 3M^2$$

The magnetic susceptibility for Landau's mean field theory may be written as

(2.8) $$\chi = \frac{1}{(T - T_c) + 3M^2}.$$

At $T > T_c$ with $H = 0$ and so $M = 0$, the susceptibility can be written as

(2.9) $$\chi = (T - T_c)^{-\gamma}$$

with $\gamma = 1$. The experimental values for γ, in three dimensional space, range from 1.25 to 1.37 for different magnetic system.

If we would like to calculate the correlation length of a system in mean field theory, we need to know the correlation function first. It is possible to generalise Landau's theory to incorporate a treatment for the correlation function by allowing the order parameter to be a function of position in the system. Therefore, the general free energy can be written as [Wallace and Zia, 1977; Amit 1978]

$$(2.10) \qquad A(T,\phi) = \int d^d x \sum_{n=1}^{\infty} [\, a_n(T)\,\phi^n + b_n(T)\,(\nabla\phi)^n \,]$$

where $\phi(x)$ is the magnetisation density. Assuming only small spatial variations exist in $\phi(x)$ (i.e. $\nabla\phi$ is small) and the system is symmetric under $\phi \longrightarrow -\phi$, we may again rewrite the free energy as

$$(2.11) \qquad A(T,\phi) = \int d^d x \; [\, a_0(T) + 1/2\,\zeta\,(\nabla\phi)^2 + 1/2\,(T - T_c)\,\phi^2(x) + 1/4\,\phi^4(x) \,].$$

Effectively, the smooth spatial variations in the magnetisation give rise to an extra term proportional to $(\nabla\phi)^2$. The correlation function $G(x, y)$ is a sort of measurement of the correlation: $G(x, y)$ gives the change in $\phi(x)$ due to the change of the external field $H(y)$, since the external field at y, $H(y)$, may flip a spin at y, for example, as in the Ising model.

$$(2.12) \qquad G(x, y) = \frac{\delta M(x)}{\delta H(y)}.$$

There are two ways to get the correlation length ξ from the correlation function: The indirect way is to look at the behaviour of the correlation function at large separation,

$$G(x, y) \underset{|x-y|\to\infty}{\propto} \exp\left(-\frac{|x-y|}{\xi}\right) \tag{2.13}$$

where the correlation length ξ is the effective range of correlation and $|x-y|$ is the separation of the two points. The direct way is to examine the location of the singularity of the correlation function,

$$\xi^2 \propto -\left.\frac{\frac{d}{dq^2} G_q}{G_q}\right|_{q=0} \tag{2.14}$$

$$\text{or equivalently} \quad \xi^2 \propto \frac{\int d^d x \; x^2 G(x)}{\int d^d x \; G(x)}$$

where $G_q = \int \exp(i\, q.x) \cdot G(x)$ is the momentum representation of the correlation function which is presumed to be translational invariant, i.e. $G(x, y) \equiv G(x - y)$, for a homogeneous external field.

We shall start to calculate the correlation length ξ now. The applied external field $H(y)$ is given by

$$H(y) = \frac{\delta A}{\delta \phi(y)}$$

$$= \frac{\delta}{\delta \phi(y)} \int d^d x \left[a_0(T) + 1/2\, \zeta (\nabla \phi(x))^2 + 1/2\, (T - T_c)\, \phi^2(x) + 1/4\, \phi^4(x) \right]$$

$$= -\zeta \nabla^2 \phi(y) + (T - T_c)\,\phi(y) + \phi^3(y). \tag{2.15}$$

And since

$$\frac{\delta H(y)}{\delta \phi(x)} = [-\zeta \nabla_y^2 + (T - T_c) + 3\phi^2(y)]\,\delta^d(x-y), \tag{2.16}$$

the correlation function is

$$G(x, y) = \left[\frac{\delta H(y)}{\delta \phi(x)}\right]^{-1}. \tag{2.17}$$

Taking $\phi(y) = \phi$ as a constant, the Fourier transform for $G(x) \equiv G(x, 0)$ can be written as

$$G_q = \int d^d x \, \exp(i\, q.x)\, G(x)$$

$$= \frac{1}{\zeta q^2 + (T - T_c) + 3\phi^2}. \tag{2.18}$$

The correlation length is given by

$$\xi^2(T, \phi) = -\left.\frac{\frac{d}{dq^2} G_q}{G_q}\right|_{q=0}$$

$$= \frac{\zeta}{(T - T_c) + 3\phi^2}. \tag{2.19}$$

For $T > T_c$, $\phi = 0$ and from the definition of $\xi \propto (T - T_c)^{-\nu}$ we identify $\nu = 1/2$ in mean field theory ($\nu \simeq 0.63$ for the Ising model in three dimensions). Many other properties are calculable in this mean field approximation.

2.2. Field Theoretic Model of Landau-Ginsburg-Wison

In last section, we found out that the predicted values of Landau's theory of critical phenomena are not far from the experimental values, despite its amazing simplicity. These factors give some very strong arguments for treating Landau's theory as a starting point for further investigations. Of course, there is something wrong with Landau's theory, since this theory does not agree with experiments exactly. But what are the reasons for the discrepancies and can the theory be cured? From the assumptions of Landau's mean field theory, it seems quite plausible that the main origins of the inadequacies of Landau's theory may lie on the total neglect of the microscopic behaviour of the system. The very important phenomenon of statistical fluctuations should have been considered, before any other microscopic characteristic, since it is so universal. As was discussed in Section 1.3, there are very close connections between quantum field theory and classical statistical mechanics. A field $\phi(x)$ can be treated as a random variable or order parameter, in the classical statistical mechanical system, and the statistical mechanics of $\phi(x)$ will give the average value which can then be identified as, for instance, the mean magnetisation (mean order parameter). The fact, that the statistical fluctuations are analogous to the quantum fluctuations in the field theory, would imply that all the good old perturbation theoretical techniques can be applied to statistical mechanics and also to critical phenomena (as a massless field theory).

The physics of a statistical system is determined by the Hamiltonian of the system. The free energy in Landau's theory corresponds to the Hamiltonian of a ϕ^4-theory [Wallace and Zia, 1978], i.e.,

$$(2.20) \quad H = \int d^d x \; [\, 1/2 \, (\nabla\phi)^2 + 1/2 \, m^2\phi^2 + 1/4! \, g\phi^4 \,]$$

where the spatial dimension is taken to be d for the applications of general cases, m^2 contains the temperature dependent factors and is equivalent to $(T - T_c)$ near T_c, and $(\nabla\phi)^2$ ensures that the short-wavelength fluctuations have higher energies. The partition function or generating functional is given by

$$(2.21) \quad Z = \int D[\phi] \quad \exp(-H)$$

where $1/k_B T$ is absorbed into H in the Boltzmann factor. The presence of an external field $J(x)$ in the statistical system corresponds to adding a term to the original Hamiltonian,

$$(2.22) \quad - \int d^d x \quad J(x) \, \phi(x).$$

The prescription for calculating the averages is given by the correlation functions or n-point functions, which average the Boltzmann factor over all the configurations of the field

$$\langle \phi(x_1)\phi(x_2)\phi(x_3) \ldots\ldots \phi(x_n) \rangle$$

$$(2.23) \quad = \frac{\int D[\phi]\ \phi(x_1)\,\phi(x_2)\cdots\phi(x_n)\cdot \exp(-H)}{\int D[\phi]\ \exp.(-H)} .$$

In the above, $\int D[\phi]$ denotes the functional integral which sums over all the possible field configurations $\phi(x)$. This method used to calculate the average of a single field would give rise to the mean order parameter $\langle \phi \rangle$ of a specific system. It is also straightforward to set up a perturbation theory for the above expressions; for example, the vertex is "- g/4" and the propagator looks like " $1/ q^2 + m$ ", without "i" compared with the ordinary relativistic quantum field theory. Note also that, in our case, q is a d-dimensional vector in the d-dimensional space. The complete details of the full perturbation theory are given in Amit's book [Amit 1978].

By the principle of correspondence, we would like to recover Landau's theory of critical phenomenon, from the field theoretical model, as a zeroth order approximation. It turns out that this emerges naturally in the loop expansion, in which the functional integral is presumed to be dominated by the maximum of the integrand, as follows. We start from the partition function with the external applied field included in the Boltzmann factor

$$(2.24) \quad Z\{J\} = \int D[\phi] \quad \exp\left(-H + \int J\phi\right) .$$

The Gibbs free energy or generating functional for connected graphs can be defined as

$$(2.25)\quad \exp\left(-G\{J\}\right) = \int D[\phi]\ \exp\left(-H + \int J\phi\right) = Z\{J\}$$

$$\text{or}\quad G\{J\} = -\ln Z\{J\}\ .$$

In the saddle-point method [Mathews and Walker, 1970; Migdal 1976], the dominant contribution to $Z\{J\}$ comes from the extremum ϕ_0 of H,

$$(2.26)\quad \left.\frac{\delta H}{\delta \phi}\right|_{\phi=\phi_0} = J.$$

For a homogeneous system, the solution $\phi_0(x)$ of this equation will be independent of x. Expanding $\phi(x)$ as,

$$(2.27)\quad \phi(x) = \phi_0 + \psi(x)$$

and using the relation, $\exp(-G) = \int D[\phi]\ \exp(-H)$, we get

$$\exp(-G) = \exp\left\{-\int d^dx\left(1/2\ m^2\phi_0^2 + 1/4!\ g\phi_0^4 - J\phi\right)\right\}$$

$$(2.28)\quad \cdot \int D[\phi]\ \exp\left[-\left(\iint_{x\,y} 1/2\ \left.\frac{\delta^2 H}{\delta\phi(x)\delta\phi(y)}\right|_{\phi=\phi_0}\psi(x)\psi(y)\right.\right.$$

$$\left.\left. + O(\psi^3)\right]\right).$$

The zeroth order approximation gives rise to

$$(2.29)\quad \exp -\int d^dx\ g$$

where g is the Gibbs free energy density:

$$g = 1/2\, m^2 \phi_0^2 + 1/4!\; g \phi_0^4 - J\phi_0 . \tag{2.30}$$

The mean order-parameter $\langle \phi \rangle$ equals the magnetisation M in the spin system

$$M \equiv \langle \phi \rangle = \frac{\int D[\phi]\, \phi \exp(-H)}{\int D[\phi] \exp(-H)}$$

$$= \frac{\int D[\psi]\, (\phi_0 + \psi(x)) \exp\left[-\int d^d x\, g\right] \exp\left\{-\left[\frac{1}{2}\iint \frac{\delta^2 H}{\delta\phi\delta\phi}\psi\psi + O(\psi^3)\right]\right\}}{\int D[\psi] \exp\left[-\int d^d x\, g\right] \exp\left\{-\left[\frac{1}{2}\iint \frac{\delta^2 H}{\delta\phi\delta\phi}\psi\psi + O(\psi^3)\right]\right\}}$$

$$= \phi_0 + \frac{\int D[\psi]\, \psi(x) \exp\left\{-\left[\frac{1}{2}\int_x\int_y \frac{\delta^2 H}{\delta\phi\delta\phi}\Big|_{\phi=\phi_0}\psi\psi + O(\psi^3)\right]\right\}}{\int D[\psi] \exp\left\{-\left[\frac{1}{2}\int_x\int_y \frac{\delta^2 H}{\delta\phi\delta\phi}\Big|_{\phi=\phi_0}\psi\psi + O(\psi^3)\right]\right\}}$$

$$= \phi_0 \tag{2.31}$$

where those terms of order ψ^3 in $\int D[\psi]$ are neglected. The Legendre transform in the thermodynamics is usually written as

$$A(M) - g(h) = h M \quad \text{with} \quad h = \frac{\partial A}{\partial M} \tag{2.32}$$

where A(M) is the Helmholtz potential density (or Helmholtz free energy), g(h) is Gibbs free energy density, h is the external field and M is the magnetisation (order parameter). This Legendre

transform in thermodynamics is analogous to the following in the field theory [Taylor 1976]

$$\Gamma [\phi_c] - W [J] = -\hbar \int J \phi_c \tag{2.33}$$

with $$J = \frac{\delta \Gamma[\phi_0]}{\delta \phi_0}$$

where $\Gamma [\phi_c]$ is the generating functional of the one particle irreducible (1PI) diagram, and W [J] is the generating functional for the connected diagrams as pointed out before. Using the Legendre transform, we may obtain the free energy of Landau's theory at the zeroth order in the Feynman graph expansion,

$$\begin{aligned} A &= g + h M \\ &= 1/2\, m^2 M^2 + 1/4!\, g M^4 - h M + h M \\ &= 1/2\, (T - T_c)\, M^2 + 1/4!\, g M^4 \end{aligned} \tag{2.34}$$

with the equation of state

$$h = (T - T_c)\, M + 1/3!\, g M^3. \tag{2.35}$$

Comparing this equation with (2.5), we see that Landau's theory of critical phenomenon may also be regarded as the zeroth order of the full Landau-Ginsburg-Wilson field theoretical model, in a steepest descent or saddle point approximation.

2.3. Interfacial Hamiltonian

In this section, we shall discuss the interface model derived from the Landau-Ginsburg-Wilson model. A simple and intuitive derivation of the interfacial Hamiltonian [Wallace 1982; 1980] will be given and some of the technical details of a formal treatment [Diehl, Kroll and Wagner, 1980] will be discussed as well. As was pointed out in Section 1.6 the subject of interfaces, surfaces and strings is contained in the study of extended objects of (d-n) dimensions embedded in a d-dimensional world. The simplest example of this category is the (d-1)-dimensional interface, which can be applied to many physical systems [Zia 1983]. The characteristic of this type of system is the existence of two coexisting phases, the interface dividing these two coexisting phases. Therefore, the model which we are going to discuss is a rather general theory. We shall start from the Landau-Ginsburg-Wilson (LGW) model with a general potential $V(\phi)$

$$H = \int d^d x \; [\, 1/2 \, (\nabla\phi)^2 + V(\phi) \,] \tag{2.36}$$

where the scalar field $\phi(x)$ is the order parameter which may be the magnetisation in the spin system or the density difference of the liquid-gas system and the general potential $V(\phi)$ may, sometimes, be given in the form of the familiar ϕ^4-potential for demonstrations. If we wish to use the LGW model to describe interfaces, we then have to impose some restrictions on the general potential $V(\phi)$. Since the vacuum or the ground state in field

theory is analogous to the equilibrium state of a statistical system, if two coexisting phases in the system are desired then the general potential $V(\phi)$ must have two degenerate minima; that is to say, the potential $V(\phi)$ will somehow look like a double well potential, at least near the bottom of the potential. In the steepest descent or saddle point approach, we look for a configuration $\phi_c(z)$ to describe how the order parameter changes from one phase to another in the z-direction, which is taken to be perpendicular to the (d-1)-dimensional interface. This configuration $\phi_c(z)$ is the solution of the classical field equation (or Euler-Lagrange equation),

$$\frac{d^2}{dz^2}\phi_c(z) = \frac{\partial V}{\partial \phi_c}. \tag{2.37}$$

subject to the boundary conditions $\phi_c(z) \to \phi_{\pm}$ as $z \to \pm\infty$, where ϕ_+ and ϕ_- correspond to different phases, respectively. If the potential $V(\phi)$ is of ϕ^4-type then the solution looks like $\phi_c(z) = \sqrt{m^2/2g}\ \tanh\ m/2\ (z - z_0)$, where z_0 is a constant, and is usually called the kink solution. There has been some extensive studies of such classical solutions of field theories and their quantum meanings [Coleman 1975; Jackiw 1977]. We follow a development based on the method of collective coordinates in which perturbation theory is employed around the coordinate dependent classical configuration [Gervais and Sakita, 1975; Gervais, Jevicki and Sakita, 1975]. The meaning of the method of collective coordinates will be discussed later.

For the interfacial configuration $\phi_c(z)$, we may expand the original field $\phi(x)$ by

$$\phi(x) = \phi_c(z) + \hat{\phi}(x) \tag{2.38}$$

where $x = (z, y)$ represents the d-dimensional coordinate and y are the remaining (d-1)-dimensional coordinates. The translational invariance of the original system implies that the classical solution ϕ_c is a function of $(z - z_0)$ where z_0 represents the position of the interface, i.e., $\phi_c(z - z_0)$. By differentiating the Euler-Lagrange Equation (2.37) with respect to z_0, we obtain

$$\mathcal{M}\, \partial_{z_0} \phi_c(z) = 0 \tag{2.39}$$

where $\mathcal{M}$ is the Schrödinger operator in the perturbation theory and equals $-\nabla^2 + \left.\frac{\partial^2 V}{\partial \phi^2}\right|_{\phi=\phi_c}$. It is clear to see that $\partial_{z_0}\phi_c$ is the zero mode of the differential operator $\mathcal{M}$. This is the consequence of the expansion, in the (2.38), about a classical solution with a given position z_0. In practice, this expansion spontaneously breaks the translational invariance of the original model and $\partial_{z_0}\phi_c$ is the Goldstone mode. The field configuration corresponding to a fluctuation $\partial_{z_0}\phi_c$ with small amplitude "a" is given as

$$\begin{aligned} \phi &= \phi_c(z) + a\, \partial_{z_0}\phi_c \\ &\simeq \phi_c(z - a)\,; \end{aligned} \tag{2.40}$$

this corresponds to an (infinitesimal) shift in the interface by the

amount of a. If we denote q as the (d-1)-component wave vector perpendicular to z, we find

$$(2.41)\quad \mathcal{M} \exp(i\,q.y)\,\partial_{z_0}\phi_c(z) = q^2 \exp(i\,q.y)\,\partial_{z_0}\phi_c(z).$$

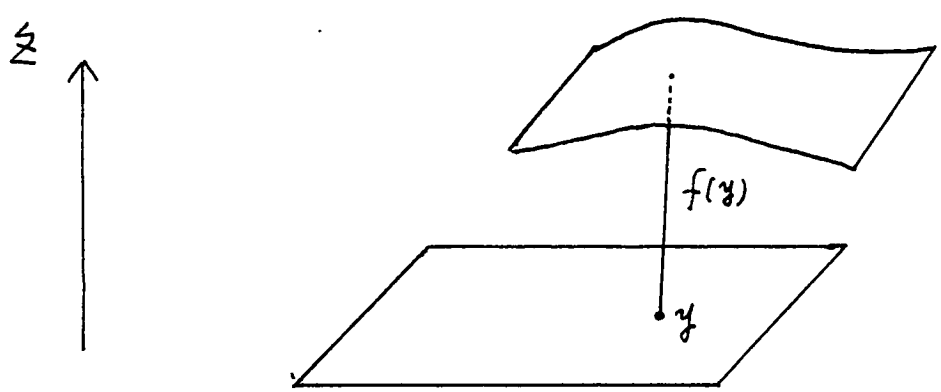
In the long-wavelength limit of $q \longrightarrow 0$, we obtain a continuous spectrum of eigenfunctions of $\mathcal{M}$ with no mass gap. The field configuration corresponding to a superposition of these modes is

$$\begin{aligned}\phi &= \phi_c(z) + \int \frac{d^{d-1}q}{(2\pi)^{d-1}}\, a(q)\cdot \exp(i\,q.y)\,\partial_{z_0}\phi_c(z) \\ &\simeq \phi_c(z - f(y)) \qquad (2.42)\end{aligned}$$

with f(y), a function of the (d-1) coordinates, is defined as

$$(2.43)\quad f(y) = \int \frac{d^{d-1}q}{(2\pi)^{d-1}}\, a(q)\, \exp(i\,q.y) + O(a^2).$$

The configuration $\phi_c(z - f(y))$ may be interpreted as a surface which is translated locally by an amount f (y) from the planar surface. Therefore, the Goldstone modes of the spontaneously broken Euclidean geometry represent the collective displacement of the general surface away from the planar surface, and the amount of displacement f(y) is called the collective coordinate.



In the long-wavelength limit, we may substitute $f(y)\partial_{z_0}\phi_c(z)$ into the expansion around the classical configuration $\phi(x) \simeq \phi_c(z) + f(y)\partial_{z_0}\phi_c(z)$ and put it back into the expansion of the Hamiltonian, (write $\hat{\phi} = f(y)\partial_{z_0}\phi_c(z)$ for convenience)

$$H(\phi) = H(\phi_c) + 1/2 \int d^d x\, \hat{\phi}(x)\, \mathcal{M}\, \hat{\phi}(x) + O(\hat{\phi}^3)$$

$$= \int d^d x\, [\, 1/2\, (\nabla\phi_c)^2 + V(\phi_c)\,]$$
$$+ 1/2 \int d^d x \int \frac{d^{d-1}q'}{(2\pi)^{d-1}}\, a(q') \cdot \exp(i q'\cdot y)$$
$$\cdot \partial_{z_0}\phi_c(z)\, \mathcal{M} \int \frac{d^{d-1}q}{(2\pi)^{d-1}}\, a(q) \cdot \exp(i q\cdot y)\, \partial_{z_0}\phi_c(z)$$
$$+ O(\hat{\phi}^3) + \cdots\cdots$$

$$= \int d^{d-1}y \int dz\, \left(\frac{d\phi_c}{dz}\right)^2$$
$$+ \int dz\, \left(\frac{d\phi_c}{dz}\right)^2 \int d^{d-1}y\, [\, 1/2\, (\nabla f)^2\,] + \cdots\cdots$$

$$= \int_{-\infty}^{\infty} dz\, \left(\frac{d\phi_c}{dz}\right)^2 \int d^{d-1}y\, [\, 1 + 1/2\, (\nabla f)^2 + \cdots\cdots] \qquad (2.44)$$

where the identity $\int 1/2\, \left(\frac{d\phi_c}{dz}\right)^2 dz = \int V(\phi_c)\, dz$ has been used.

In (2.44), it is not really easy to see what the next order terms would look like, besides, the methods are not systematic at all. A systematic method to obtain the higher order terms in f in (2.44) was given by H. W. Diehl et al., who used a version of the method of collective coordinates. The original configuration in the d-dimensional space can be written in terms of the collective

coordinate f(y) and others [Diehl, Kroll and Wagner, 1980]

$$\phi(y, z) = \phi_c(z - f(y)) + \hat{\phi}(y, z - f(y)) \tag{2.45}$$

where the kink position (or the interface position) f(y) is considered as the collective coordinate. The meaning of this expansion can be understood as a sort of canonical transformation from the original set of variables $\{\phi\}$ to a new set of variables $\{f, \hat{\phi}\}$. Since some new variables have been introduced, we impose a subsidiary condition with (2.45) so that $\hat{\phi}$ does not contain the zero frequency modes:

$$\int dz \; \phi_c'(z) \; \hat{\phi}(y, z) = 0, \tag{2.46}$$

where $\phi_c'(z)$ is the derivative of the classical solution and is the eigenfunction of the zero mode. The Hamiltonian (2.36) can be expanded by substituting (2.45) into the Hamiltonian

$$H(\phi) = H(\phi_c(z - f(y))) + \int d^d x \; \hat{\phi} \left.\frac{\delta H}{\delta \phi}\right|_{\phi=\phi_c} + O(\hat{\phi}^2) + \cdots\cdots \tag{2.47}$$

where the $\left.\frac{\delta H}{\delta \phi}\right|_{\phi=\phi_c}$ term is no longer zero again, due to the introduction of the collective coordinate transformation. In deriving the formula (2.44), $\hat{\phi}$ in (2.45) is effectively ignored. Diehl et al. examined the effects of the ignored $\hat{\phi}$'s by setting up a systematic approximation scheme in the context of the methods of collective coordinates. If we would like to sum over all the $\hat{\phi}$ -

modes, the $\hat{\phi}$ -tree diagrams are the most important and significant terms in the expansion and they contribute to the same order as H (ϕ_c). Therefore, it is necessary to include the term, which is linear in $\hat{\phi}$, in the expansion (2.47) and the source term for the $\hat{\phi}$ is nothing but $\frac{\delta H}{\delta \phi}\Big|_{\phi=\phi_c}$. From the formula (2.47), we may write

$$H = \int dz (\phi_c')^2 \int d^{d-1}y \left[1 + \tfrac{1}{2}(\nabla_y f)^2\right] + \int \hat{\phi} \frac{\delta H}{\delta \phi_c} + O(\hat{\phi}^2) \tag{2.48}$$

where the $H(\phi_c)$ is the same as in (2.44). Letting u = z - f(y), the $\hat{\phi}$ -source can be written as

$$\begin{aligned} \frac{\delta H}{\delta \phi_c} &= -\nabla^2 \phi_c + \frac{\partial V}{\partial \phi_c} \\ &= (-\nabla^2 u)\,\phi_c' - (\nabla u)^2 \phi_c'' + \frac{\partial V}{\partial \phi_c} \\ &= (-\nabla^2 u)\,\phi_c' + [1 - (\nabla u)^2]\,\phi_c'' \\ &= (\nabla^2 f)\,\phi_c' - (\nabla f)^2 \phi_c'', \end{aligned} \tag{2.49}$$

where the Euler-Lagrange equation (2.37) have been used. Using (2.46) to ensure that the zero modes are excluded from $\hat{\phi}$, the source term which consists of $(\nabla^2 f)$ has to vanish and only $(\nabla f)^2$ terms left as the sources. So the $\hat{\phi}$ -trees will contribute powers of $(\nabla f)^2$ to H. The resummation of these tree diagrams has been done in different contexts. J-L Gervais et al. [Gervais, Jevicki and Sakita, 1975] first developed a scheme of resummation of the tree diagrams for the soliton; H.W. Diehl et al. [Diehl, Kroll and Wagner 1980] have calculated the tree diagrams in this particular

case to derive the effective Hamiltonian for interfaces

$$H = \sigma \int d^{d-1}y \sqrt{1 + (\nabla f)^2} \tag{2.50}$$

where $\sigma = \int dz \, (\phi_c')^2$ is the surface tension and the integral is just the surface area of the manifold. The first two terms in the expansion in $(\nabla f)^2$ reduce to (2.44), of course.

2.4. Higher Order Corrections

The resummation of the tree diagrams in the calculation for the effective interfacial Hamiltonian is indeed quite tedious and long. The collective coordinate, in our treatment, is identified as f(y). This field f could be either the interface or kink position. The collective coordinate transformation is given by expanding around the field configuration as in equation (2.45) with the subsidiary condition (2.46). Under this transformation, the variable set describing the same system changes from $\{\phi\} \rightarrow \{f, \hat{\phi}\}$. Therefore, the partition functional, Hamiltonian (or Lagrangian) and Euler-Lagrange Equation will all have to change accordingly. After the transformation, the whole systematic perturbation theory can also be set up in terms of the fluctuating field or perturbation $\hat{\phi}$. But in the case of looking for the effective Hamiltonian of the Goldstone modes or for the interface, we are actually interested in the long wavelength (or low temperature) behaviour of the system only. The surface tension model discussed in the previous section

is the effective interfacial Hamiltonian in the long wavelength limit; it depends only on the gapless modes f describing thermal capillary waves, the average having been already performed over the modes $\hat{\phi}$ which has a gap of order $O(m^2)$. As was discussed in the previous section, the choice of the interface position f(y) as a collective coordinate has led us to the tedious resummation of tree diagrams. Since the source term for the tree diagrams comes from the first variation of the Hamiltonian $\frac{\delta H}{\delta \phi}\Big|_{\phi_c}$, in order to avoid the tedious calculation of tree diagrams, there is a natural question to ask, "Is there a better choice of the collective coordinate such that the source term derived from $\frac{\delta H}{\delta \phi_c}$ will contain only the higher derivatives of f ?". In this case, the surface tension term inside the effective Hamiltonian will appear naturally within the effective Hamiltonian $H(\phi_c)$ such that we can avoid the resummation of the tree diagrams entirely. A better choice for the collective coordinate was suggested as [Wallace 1980, 1982; Lin and Lowe 1983]

$$\phi = \phi_c\left(\frac{z - f(y)}{\sqrt{1+(\nabla f)^2}}\right) + \hat{\phi} . \tag{2.51}$$

Geometrically, this choice of the collective coordinate means that the classical configuration is not only translated but also rotated, since the added factor $1 / [1 + (\nabla f)^2]^{1/2}$ is the directional cosine for the z-y axis rotations. For f up to linear in y, we are expanding about an exact solution of the Euler-Lagrange equation. With the argument for ϕ_c in (2.51), therefore, the source term must depend on second derivatives of f (or higher). Thus the source

term is negligible in the long wavelength limit. Substituting (2.51) into the LGW Hamiltonian, the first term in the expansion of the LGW Hamiltonian, $H(\phi_c)$, can be written as

$$
\begin{aligned}
H(\phi_c) &= \int d^d x \left[\frac{1}{2} (\nabla \phi_c)^2 + V(\phi_c) \right] \\
&= \int d^d x \left[\frac{1}{2} (\nabla u)^2 \left(\phi_c'(u)\right)^2 + V(\phi_c) \right] \\
&= \int d^d x \left\{ \frac{1}{2} \left(\phi_c'(u)\right)^2 \left[\frac{1}{1+(\nabla f)^2} + \left(\nabla_y \frac{z - f(y)}{\sqrt{1+(\nabla f)^2}} \right)^2 \right] \right. \\
&\qquad \left. + V(\phi_c) \right\}
\end{aligned}
\tag{2.52}
$$

where $u = (Z - f(y)) / [1 + (\nabla f)^2]^{1/2}$. Using the identity again,

$$
\int V(\phi_c)\, du = \frac{1}{2} \int \left(\phi_c'\right)^2 du \tag{2.53}
$$

and changing the variables

$$
(\underset{\sim}{y}, z) \longrightarrow (\underset{\sim}{y}, u), \tag{2.54}
$$

in the above expression, (2.52) can be rewritten as

$$
\begin{aligned}
H(\phi_c) = \int d^{d-1} y \; & \sqrt{1+(\nabla f)^2} \\
& \cdot \left\{ \sigma_1 + \frac{\sigma_2}{2} \left[1+(\nabla f)^2\right]^{-3/2} \nabla f \cdot \nabla (\nabla f)^2 \right. \\
& \left. + \frac{\sigma_3}{8} \left[1+(\nabla f)^2\right]^{-2} \left[\nabla (\nabla f)^2 \right]^2 \right\},
\end{aligned}
\tag{2.55}
$$

where

$$\sigma_1 \equiv \int_{-\infty}^{\infty} (\phi_c')^2 \, du$$

$$\sigma_2 \equiv \int_{-\infty}^{\infty} u (\phi_c')^2 \, du \tag{2.56}$$

$$\sigma_3 \equiv \int_{-\infty}^{\infty} u^2 (\phi_c')^2 \, du .$$

The first term in the expression (2.55) is indeed the surface tension term as obtained from the tree diagram resummation in the previous section.

The better choice for the collective coordinate in (2.51) is fine, provided that we are not interested in the higher derivatives of f at all. The difficulty for the effective Hamiltonian (2.55) is that the higher order contributions are not significant since comparable terms have been neglected in the source term; also they are not geometrically invariant. We propose now an even better choice of the collective coordinate: [Lin and Lowe 1983]

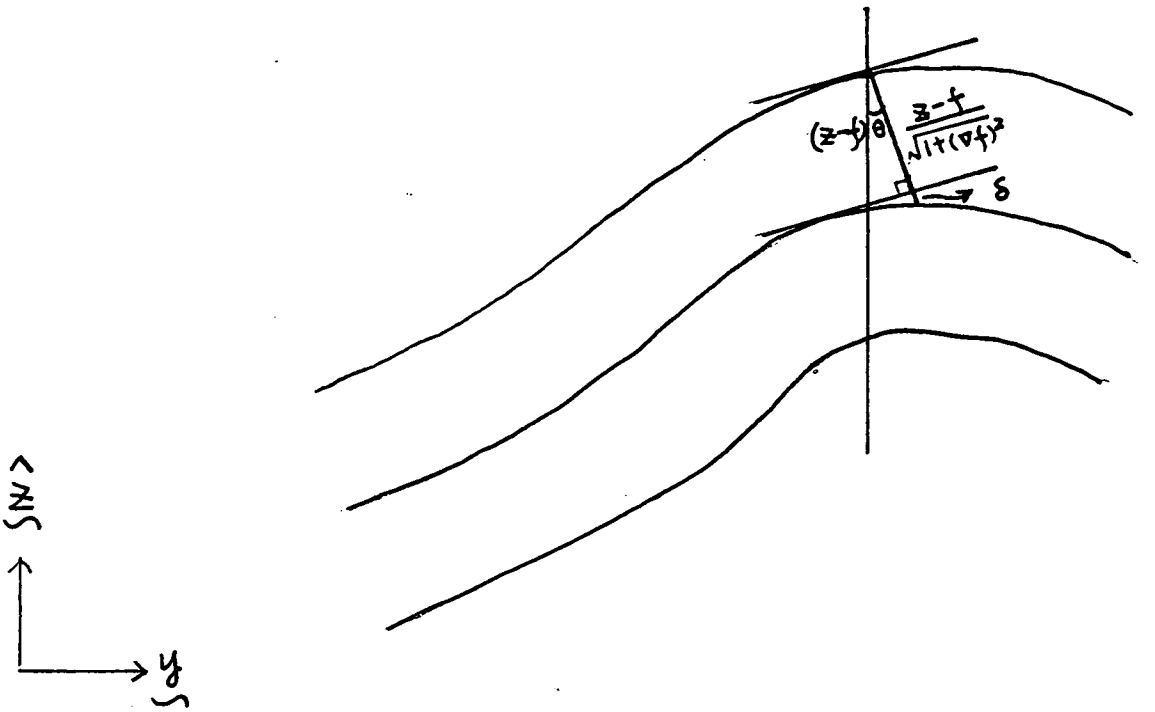
$$\phi = \phi_c(w) + \hat{\phi}\left(y, \frac{\xi - f(y)}{\sqrt{1+(\nabla f)^2}}\right) \tag{2.57}$$

with

$$w = \frac{\xi - f(y)}{\sqrt{1+(\nabla f)^2}} + \frac{1}{2} \frac{(\xi - f(y))^2}{(1+(\nabla f)^2)^{5/2}} \nabla_i f \nabla_j f \nabla_{ij} f . \tag{2.58}$$

In this choice of collective coordinate, it would imply that the transformation on the surface should include not only a translation

and rotation from a plane surface collectively but also a small distance of shift due to the deviation of the surface from the tangent plane as indicated in the following graph:



where δ is a small deviation from the tangent plane. What we have achieved in this method is the following: apart from the natural appearance of the interface or surface model, we are also exposing the higher order corrections such as the curvature R and other terms which relate to the embedding details of the interface contained in the coefficients of the metric tensor g_{ij} and that of the second fundamental forms b_{ij}.

Using this formalism, we have calculated the effective Hamiltonian with up to two more derivatives than the surface tension term. This may be viewed as higher order contribution from the higher derivative geometric invariants. As will be discussed in the

next chapter, that these terms will be identified with geometric invariants such as curvature R, $(g.b)^2$, and b^2. In the expansion of

$$H[\phi_c(u+\delta)]$$

$$\simeq H\left[\phi_c(u)+\delta\phi_c'(u)+\frac{\delta^2}{2!}\phi_c''(u)\right], \tag{2.59}$$

where

$$u=\frac{z-f(y)}{\sqrt{1+(\nabla f)^2}}, \qquad \delta=\frac{u^2}{2}\,\frac{\partial_i f\,\partial_{ij} f\,\partial_j f}{(1+(\nabla f)^2)^{3/2}}, \tag{2.60}$$

we obtain the effective Hamiltonian up to two extra derivatives

$$\begin{aligned} &H[\phi_c(w)] \\ &\simeq H[\phi_c(u)]+\int d^{d-1}y\,\sqrt{1+(\nabla f)^2}\int du\left(\delta\phi_c'+\frac{\delta^2}{2}\phi_c''\right)\frac{\delta H}{\delta\phi_c} \\ &\quad+\frac{1}{2}\int d^{d-1}y\,\sqrt{1+(\nabla f)^2}\int du\;\delta\phi_c'\left[-\nabla^2+\frac{\partial^2 V}{\partial\phi^2}\right]\delta\phi_c'. \end{aligned} \tag{2.61}$$

By using the Euler-Lagrange equation and performing partial differentiation plus integration by parts of various types (in order to throw away the total derivatives), such as

$$\int u^3\phi_c'\phi_c''=-\frac{3}{2}\int u^2(\phi_c')^2 \tag{2.62}$$

the final result for the effective Hamiltonian reads

$$H[\phi_c(w)]$$

$$= \int d^{d-1}y \sqrt{1+(\nabla f)^2}$$

$$\cdot \int du \, (\phi_c')^2 \cdot \left\{ 1 + u \frac{\partial_i f \, \partial_{ij} f \, \partial_j f}{[1+(\nabla f)^2]^{3/2}} \right.$$

$$\left. + \frac{u^2}{2} \left[\frac{\partial_{ij} f \, \partial_j f \, \partial_{ik} f \, \partial_k f}{[1+(\nabla f)^2]^2} - \frac{\partial_{kk} f \, \partial_i f \, \partial_{ij} f \, \partial_j f}{[1+(\nabla f)^2]^2} \right] \right\} \tag{2.63}$$

We shall see in the next chapter how all of these extra terms can be identified with differential geometric invariants.

CHAPTER III GEOMETRY OF THE INTERFACE AND SURFACE

In this chapter, we are going to discuss some geometrical details of surfaces. The general geometrical properties of surfaces have been discussed in Section 1.5. The origin of the interfacial Hamiltonian in the context of field theory has also been demonstrated in the previous chapter, and the physical and geometrical reasonings for the emergence of an interface model from the LGW field theoretic model are given there. The further pursuit of the higher order corrections to the simple interface model shows that the understanding of the geometrical meaning of the terms which we have played with are essential to our calculation in Section 2.4. It will be essential to set up tensor calculus on the surface and interface so that the higher derivative geometric invariants can be derived. Then, some symmetries relating to the theories in Chapter II will also be discussed, namely, the nonlinear realisation of the group of transformations for a spontaneous broken global theory. Finally, a discussion on the method of normal coordinates, which is a geometric invariant extension for the collective coordinate, due to R. Zia, will follow.

3.1. Tensor Calculus on the Surface and Interface

There are several ways to set up the geometrical analysis on a geometrical object, some of which are covariant and some not. The most natural and fashionable way to set up the analysis on the

geometrical objects is to use differential forms on manifolds. But, in order to be as close as possible to the notation used in field theory, the conventional tensor calculus techniques will be used instead. The parametrisation of the surface or interface in a Euclidean d-dimensional space can be chosen as

$$z = f(\underset{\sim}{y}), \tag{3.1}$$

where

$$\chi = (y_i, z), \quad i = 1, 2, 3, \cdots\cdots (d-1),$$

is a vector in Euclidean d-dimensional space and y_i are the remaining (d-1) dimensional coordinates. This is the geometry of the (d-1) dimensional submanifold embedded in a Euclidean d-dimensional space and the description of the submanifold (3.1) is obtained by imposing a constraint on the d-dimensional coordinate ($\underset{\sim}{y}$, z) to be ($\underset{\sim}{y}$, z = f($\underset{\sim}{y}$)). Of course, by imposing more constraints on the d-dimensional space, it would be possible to get many submanifolds with lower dimensionality. In general, this type of submanifold of a d-dimensional Euclidean (flat) space is curved. Taking the example of a three-dimensional flat space, the description of a two-dimensional embedded surface can be written as

$$z = f(x, y). \tag{3.2}$$

The geometry of this surface is determined by the coefficients of the first fundamental form (or the metric tensor) g_{ij} and those of the second fundamental form b_{ij} [Klingenberg 1978]. The first and

second fundamental forms are usually written in terms of their coefficients, respectively,

$$\mathrm{I} = g_{ij}\, dx^i\, dx^j$$

$$\mathrm{II} = b_{ij}\, dx^i\, dx^j, \tag{3.3}$$

$$i, j = 1, 2.$$

The intrinsic properties of the embedded surface are specified by g_{ij}, and the extrinsic properties are specified by b_{ij}. All the geometrical invariant properties of the embedded surface are also determined by g_{ij} and b_{ij}, and the tensor calculus on the embedded surface can be set up by the same set of coefficients $\{ g_{ij}, b_{ij} \}$.

The geometry of the two-dimensional embedded surface in the Euclidean three-dimensional space is well understood, therefore, the most obvious extension of this knowledge is the generalisation to a higher dimension d. The nineteenth century geometers did much research into this subject [Eisenhart 1926]. But, unfortunately, most of their works were done in a parametrisation of the surface such that the direct application to the field theory can not be established easily. Therefore, we have to develop our own tensor calculus of the (d-1)-dimensional embedded submanifold on the d-dimensional Euclidean space. Our approach is to find out the generalisation of the (g_{ij}, b_{ij}) of the (d-1)-dimensional embedded submanifold, then, to use the coefficients (g_{ij}, b_{ij}) in order to set up the tensor calculus of the (d-1)-dimensional submanifold.

Let the first fundamental form in the d-dimensional flat space be written as

$$\begin{aligned} I &= ds^2 \\ &= \delta_{\ell m}\, dx^{\ell}\, dx^{m}, \\ \ell, m &= 1, 2, 3, \cdots\cdots, d \end{aligned} \tag{3.4}$$

in d dimensions. Since this is a flat space, the metric tensor of the d-dimensional space, $\delta_{\ell m}$, is a unit matrix. Recall the constraint $z = f(\underset{\sim}{y})$, the first fundamental form of the d-dimensional Euclidean space becomes

$$\begin{aligned} I &= dy^{i}\, dy^{i} + (dz)^2 \\ &= dy^{i}\, dy^{i} + \left(\frac{\partial f}{\partial y^{i}}\, dy^{i}\, \frac{\partial f}{\partial y^{j}}\, dy^{j}\right) \\ &= \left(\delta_{ij} + \frac{\partial f}{\partial y^{i}}\, \frac{\partial f}{\partial y^{j}}\right) dy^{i}\, dy^{j} \\ &\equiv g_{ij}\, dy^{i}\, dy^{j} \end{aligned} \tag{3.5}$$

where

$$g_{ij} = \delta_{ij} + \frac{\partial f}{\partial y^{i}}\, \frac{\partial f}{\partial y^{j}}$$

are the coefficients of the first fundamental form (or the metric tensor) on the (d-1)-dimensional embedded submanifold. The metric tensor with two contravariant indices can be obtained through

$$g_{ij}\, g^{jm} = \delta_i^m \;; \tag{3.6}$$

therefore, the g^{ij} can be written as

$$g^{ij} = \delta^{ij} - \frac{1}{1 + \frac{\partial f}{\partial x^l}\frac{\partial f}{\partial x^l}} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} . \tag{3.7}$$

We shall proceed to find out the generalised b_{ij} of the (d-1)-dimensional embedded submanifold. As was explained in Chapter I, the extrinsic properties of the embedded surface are related to the normal vector fields and the second fundamental form, and the coefficients of the second fundamental form b_{ij} give the measure of the embedding details. This measure of the embedding details is related to the deviation of the surface from the tangent plane, and is summarised in the following theorem [Goetz 1970].

Thm The signed distance of the point $(u^1 + h^1 ,\ u^2 + h^2)$ of a surface from the tangent plane at (u^1 , u^2) equals

$$\delta = \frac{1}{2}\, b_{ij}\, h^i\, h^j \tag{3.8}$$

with an error of order higher than two relative to $\sqrt{(h^1)^2 + (h^2)^2}$.

This is a theorem in the case of two dimensions. The geometrical interpretation will give us a useful guiding principle to derive the

second fundamental form for the (d-1)-dimensional submanifold on the d-dimensional Euclidean space. In the case of the two dimensional embedded surface, the second fundamental form is given as [do Carmo 1976],

$$\text{(3.9)} \qquad \begin{aligned} \text{II} &= - d\underset{\sim}{r} \cdot d\underset{\sim}{N} \\ &= b_{ij}\, dx^i dx^j \end{aligned}$$

where $\underset{\sim}{N}$ is a unit normal and $\underset{\sim}{r}$ = (x, y, f(x,y)). The unit normal $\underset{\sim}{N}$ is given, in the two dimensional case, as

$$\text{(3.10)} \qquad \frac{1}{\sqrt{1+(\partial_x f)^2+(\partial_y f)^2}} \left(-\partial_x f, -\partial_y f, 1 \right).$$

The normalisation factor in the two dimensional case can be easily generalised to

$$\text{(3.11)} \qquad \frac{1}{\sqrt{1+(\nabla f)^2}}$$

where the gradient of f is taken in the (d-1) dimensions. The geometric meaning of b_{ij} gives the measure of the curvature of surface relative to the Euclidean space and since we are interested in the analytical surface only, this measure of embedding details could be given by the ordinary Taylor's expansion in (d-1) dimensions

$$\text{(3.12)} \qquad f(y^i + dy^i) - f(y^i),$$

$$i = 1, 2, 3, \cdots\cdots, (d-1).$$

Since the first order contribution from the expansion would be perpendicular to the normal at the (d-1)-dimensional coordinate ($\underset{\sim}{y}$), therefore, we just have to examine the second order effect

$$\left(\sum_{i=1}^{d-1} dy^i \frac{\partial}{\partial y^i}\right)^2 f(\underset{\sim}{y})$$

$$= \left(dy^1 \frac{\partial}{\partial y^1} + dy^2 \frac{\partial}{\partial y^2} + \cdots\cdots\right)^2 f(y^1, y^2, \cdots\cdots)$$

(3.13)

$$= \sum_{i,j} \frac{\partial^2 f}{\partial y^i \partial y^j} dy^i dy^j$$

where

$$i, j = 1, 2, 3, \cdots\cdots, (d-1).$$

With our choice of normalisation, the second fundamental form can be written as

(3.14) $$\text{II} = b_{ij}\, dy^i dy^j,$$

with

$$b_{ij} = \frac{1}{\sqrt{1 + (\nabla f)^2}} \frac{\partial^2 f}{\partial y^i \partial y^j},$$

where

$$i, j = 1, 2, 3, \cdots\cdots, (d-1).$$

The geometry of the (d-1)-dimensional embedded submanifold of a Euclidean d-dimensional manifold can thus be completely determined by

$$g_{ij} = \delta_{ij} + \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j}$$

(3.15)

$$g^{ij} = \delta^{ij} - \frac{\frac{\partial f}{\partial y_i} \frac{\partial f}{\partial y_j}}{1 + (\nabla f)^2}$$

and $$b_{ij} = \frac{1}{\sqrt{1 + (\nabla f)^2}} \frac{\partial^2 f}{\partial y^i \partial y^j}$$

given these fundamental expressions, we now turn to the tensor calculus in this coordinate system.

The Riemann symbols of the second kind are usually defined in terms of g_{ij}'s and their derivatives,

$$(3.16)\quad R_{mijk} = \frac{\partial}{\partial y^j} \Gamma_{ikm} - \frac{\partial}{\partial y^k} \Gamma_{ijm} + \Gamma^{\alpha}_{ij} \Gamma_{mk\alpha} - \Gamma^{\alpha}_{ik} \Gamma_{mj\alpha}$$

where

$$(3.17)\quad \Gamma_{ijk} = \frac{1}{2} \left[\frac{\partial g_{jk}}{\partial y^i} + \frac{\partial g_{ki}}{\partial y^j} - \frac{\partial g_{ij}}{\partial y^k} \right]$$

are the Christoffel symbols of the first kind, and

$$(3.18)\quad \Gamma^{\alpha}_{ij} = g^{\alpha l} \Gamma_{ijl}$$

are the Christoffel symbols of the second kind. The associated Riemann symbols of first kind are defined as

$$R^{p}_{ijk} = g^{\alpha p} R_{\alpha ijk}.$$

The Riemann symbols of the second kind can also be defined in terms of the coefficients of the second fundamental forms, b_{ij}, as

$$(3.19)\quad R_{mijk} = b_{ik} b_{jm} - b_{ij} b_{km}.$$

Although the Riemann symbols can be defined in terms of b_{ij} only, they are still related to the intrinsic properties of the (d-1)-dimensional manifold since they are also expressible in terms of the coefficients of the first fundamental form, i.e., the metric tensors, and their derivatives. The symmetry properties of the above geometrical quantities are also useful:

(i) g_{ij} and b_{ij} are symmetrical with respect to ($i \longleftrightarrow j$), i.e. $g_{ij} = g_{ji}$ and $b_{ij} = b_{ji}$.

(ii) Γ_{ijk} are symmetrical w.r.t ($i \longleftrightarrow j$), i.e. $\Gamma_{ijk} = \Gamma_{jik}$.

(iii) R_{mijk} are antisymmetric w.r.t. ($m \longleftrightarrow i$) or ($j \longleftrightarrow k$), i.e. $R_{mijk} = -R_{imjk}$ or $R_{mijk} = -R_{mikj}$.

Substituting the Christoffel symbols into (3.16), we obtain

$$
\begin{aligned}
& R_{mijk} \\
(3.20) \quad & = \frac{1}{2}\partial_j\left[\partial_i g_{km} + \partial_k g_{im} - \partial_m g_{ik}\right] \\
& \quad -\frac{1}{2}\partial_k\left[\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\right] \\
& \quad + g^{\alpha l}\Gamma_{ijl}\Gamma_{mk\alpha} - g^{\alpha l}\Gamma_{ikl}\Gamma_{mj\alpha} .
\end{aligned}
$$

Write the Christoffel symbols in terms of the metric tensors, the Riemann symbols of the second kind looks like

$$
\begin{aligned}
& R_{mijk} \\
(3.21) \quad & = \frac{1}{2}\left[\partial_{ji} g_{km} - \partial_{jm} g_{ik} - \partial_{ki} g_{jm} + \partial_{km} g_{ij}\right] \\
& \quad + \frac{1}{4} g^{\alpha l}\left[\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right]\left[\partial_m g_{k\alpha} + \partial_k g_{m\alpha} - \partial_\alpha g_{mk}\right]
\end{aligned}
$$

$$-\frac{1}{4}\, g^{\alpha\ell}\left[\partial_\ell g_{k\ell} + \partial_k g_{i\ell} - \partial_\ell g_{ik}\right] \cdot \left[\partial_m g_{j\alpha} + \partial_j g_{m\alpha} - \partial_\alpha g_{mj}\right].$$

If we intend to simplify this expression to the " $\partial_{ij} f\, \partial_{k\ell} f$ " level, there will be

$$4 \times 4 + 18 \times 4 = 88$$

terms to be calculated although most of them will cancel among each other. A straightforward but tedious calculation yields

$$R_{mijk} = \frac{1}{1+(\nabla f)^2}\left[\partial_{ik} f\, \partial_{jm} f - \partial_{ij} f\, \partial_{km} f\right]. \tag{3.22}$$

Now we shall use R_{mijk} to derive the Ricci scalar (curvature), R, using the standard procedure of deriving the curvature from the Riemann symbols

$$\begin{cases} R^{p}_{ijk} = g^{mp} R_{mijk} \\ R_{ij} = R^{k}_{ijk} \\ R = g^{ij} R_{ij}. \end{cases} \tag{3.23}$$

Therefore, the curvature scalar is

$$R = \frac{1}{1+(\nabla f)^2}\left(\partial_{ij} f\, \partial_{ij} f - \partial_{ii} f\, \partial_{jj} f\right) + \frac{1}{\left[1+(\nabla f)^2\right]^2}\left(\partial_{kk} f\, \partial_i f\, \partial_{ij} f\, \partial_j f - \partial_{ik} f\, \partial_{im} f\, \partial_m f\, \partial_k f\right). \tag{3.24}$$

By contracting two second fundamental forms, b^2 can also be obtained

$$b^2 = b^{ij}\, b_{ij}$$

$$= \frac{\partial_{ij} f\, \partial_{ij} f}{1+(\nabla f)^2} - 2\,\frac{\partial_{ik} f\, \partial_{im} f\, \partial_m f\, \partial_k f}{[\,1+(\nabla f)^2\,]^2} \tag{3.25}$$

$$+ \frac{[\,\partial_i f\, \partial_{ij} f\, \partial_j f\,]^2}{[\,1+(\nabla f)^2\,]^3} \,.$$

The (g.b) is

$$g\cdot b = g^{ij}\, b_{ij}$$

$$= \frac{\partial_{ii} f}{\sqrt{1+(\nabla f)^2}} - \frac{\partial_i f\, \partial_{ij} f\, \partial_j f}{[\,1+(\nabla f)^2\,]^{3/2}} \,, \tag{3.26}$$

so $(\,g.b\,)^2$ is written as

$$(g\cdot b)^2$$

$$= \frac{\partial_{ii} f\, \partial_{jj} f}{[\,1+(\nabla f)^2\,]} - 2\,\frac{\partial_i f\, \partial_j f\, \partial_{ij} f\, \partial_{kk} f}{[\,1+(\nabla f)^2\,]^2} \tag{3.27}$$

$$+ \frac{\partial_i f\, \partial_{ij} f\, \partial_j f\, \partial_k f\, \partial_{km} f\, \partial_m f}{[\,1+(\nabla f)^2\,]^3} \,.$$

The above geometrical quantities give us the identity

$$R + (g\cdot b)^2 = b^2 \,, \tag{3.28}$$

which is very useful to simplify the geometrical quantities or transform them to some more presentable forms. We have found a great deal of applications to the calculation involved with the method of collective coordinate.

We turn now to the application of the above formalism to the interpretation of the interfacial Hamiltonian of Chapter II.

3.2. Symmetry of the Interface Model

The discussion of the tensor calculus of surfaces in the previous section gives us a geometrical description of surfaces and interfaces. The study of symmetry properties of the interface are also essential to our understanding of the geometrical properties of the interface model. With the results of the previous section and using integration by parts such as

$$\int d^{d-1}y \sqrt{1+(\nabla f)^2}\ \frac{\partial_{ij} f\, \partial_{ij} f - \partial_{ii} f\, \partial_{jj} f}{[1+(\nabla f)^2]} \tag{3.29}$$

$$= \int d^{d-1}y \sqrt{1+(\nabla f)^2}\ \frac{\partial_{ij} f\, \partial_j f\, \partial_{ik} f\, \partial_k f - \partial_{kk} f\, \partial_i f\, \partial_{ij} f\, \partial_j f}{[1+(\nabla f)^2]^2}$$

we can identify the appearance of (g.b) and R terms in the effective Hamiltonian (2.63) [Lin and Lowe 1983]

$$H_{eff.} = \int d^{d-1}y \sqrt{1+(\nabla f)^2}\left(\frac{1}{T} + C\, g\cdot b + d\, R\right) \tag{3.30}$$

where

$$g\cdot b = [1+(\nabla f)^2]^{-3/2}\, \partial_i f\, \partial_{ij} f\, \partial_j f$$

$$R = [1+(\nabla f)^2]^{-1}\,[\partial_{ij} f\, \partial_{ij} f - \partial_{ii} f\, \partial_{jj} f]$$

$$\frac{1}{T} = \int d\xi \left[\frac{d\phi_c(\xi)}{d\xi}\right]^2$$

$$C = \int d\xi\ \xi \left[\frac{d\phi_c(\xi)}{d\xi}\right]^2$$

$$d = \int d\xi \; \xi^2 \left[\frac{d\Phi_c(\xi)}{d\xi} \right]^2 .$$

There is a further symmetry aspect. As the effective Hamiltonian can be derived from the Landau-Ginsburg-Wilson model in d-dimensional Euclidean space, it is argued that, in the spirit of Section 2.3., the classical solution breaks translational symmetry in the z axis and rotational symmetry in the z-y plane in our case [Wallace 1980 and 1982]. This is a special case of the property that f(y) carries a nonlinear realisation of the original full symmetry group of the LGW model, namely the Euclidean group E(d) of rotations and translations in d dimensions. That is to say, the Goldstone field f(y) carries the original large symmetry group with the transformations corresponding to spontaneously broken symmetries acting nonlinearly on the field. This is analogous to the use of the nonlinear sigma model to describe the interaction of the Goldstone modes of a global symmetry spontaneously broken, for example, from O(n) to O(n-1). Mathematically, a realisation of a group G is an association (map) between any element g of G and a transformation T(g) of some space M in such a way that the group properties are preserved:

(i) $T(e) = I$, the identity transformation.

(ii) $T(g^{-1}) = [T(g)]^{-1}$

(iii) $T(g).T(h) = T(gh)$.

It is clear that a realisation of a group is more general than a representation of a group. In fact, when M is a vector space and every T(g) is a linear transformation then the realisation is called

a representation [Schutz 1980]. The corresponding nonlinear transformations on f are given by [Wallace 1980 and 1982],

$$\begin{cases} f \rightarrow f + a \\ f \rightarrow f - \theta_i (f \partial_i f + y_i) + O(\theta^2) \end{cases} \tag{3.31}$$

for translation by "a" in the z axis and rotation by some infinitesimal angle $\underset{\sim}{\theta}$ in the (z-y) plane. It is easy to check that terms appearing in the effective Hamiltonian are invariant under the above transformations up to a total derivative. The first derivative of the transformed field f' reads

$$\begin{aligned} \partial_i f' &= \partial_i f - \partial_i [\theta_j (f \partial_j f + y_j)] \\ &= \partial_i f - \theta_j (\partial_i f \partial_j f + f \partial_{ij} f + \delta_{ij}). \end{aligned} \tag{3.32}$$

By the help of integration by parts, we arrive at the expected result, for example,

$$\int d^{d-1} y \sqrt{1 + (\nabla f')^2} = \int d^{d-1} y \sqrt{1 + (\nabla f)^2}. \tag{3.33}$$

It should also be pointed out that there is only one single field f acting as a Goldstone field for the breaking of the symmetry from E(d) to E(d-1) [Wallace 1980]. This is in contrast to the general theory of nonlinear realisations in field theory, which requires one Goldstone field for each spontaneously broken generator of an internal symmetry group. This happens in the case of the nonlinear sigma model with a spontaneously broken global symmetry from O(n)

to O(n-1). In our case, the original action of E(d) on the d-dimensional coordinates $(z, \underset{\sim}{y})$ is already a representation on the coset space of the Euclidean group factored by rotations.

Apart from the nonlinear realisations of symmetry groups, there are some other symmetry properties possessed by our system. For a magnetic system as discussed in Section 2.3, we know that the Hamiltonian of the LGW model H is symmetric with respect to

(3.34) $$\phi \longleftrightarrow -\phi .$$

By imposing nonvanishing value to the field configuration $\phi_c(z)$ in the boundary value condition, the nontrivial solution of the Euler-Lagrange equation in that system must be an odd function of z, i.e.

$$\phi_c(z) = -\phi_c(-z).$$

The consequence of the odd property of the field configuration ϕ_c is that no odd number of f fields contributions in the effective Hamiltonian H(f). Terms with odd number of f fields, such as (g.b), do not then appear in the effective Hamiltonian H(f). In summary,

(3.35) $$\left.\begin{aligned} H(\phi_c) &= H(-\phi_c) \\ \phi_c(z) &= -\phi_c(-z) \end{aligned}\right\} \Longrightarrow H(f) = H(-f).$$

However, the LGW model for fluids, for example, where ϕ_c represents

the fluid density has no such symmetry properties and a (g.b) term must be allowed in the effective surface model.

3.3. Method of Normal Coordinates

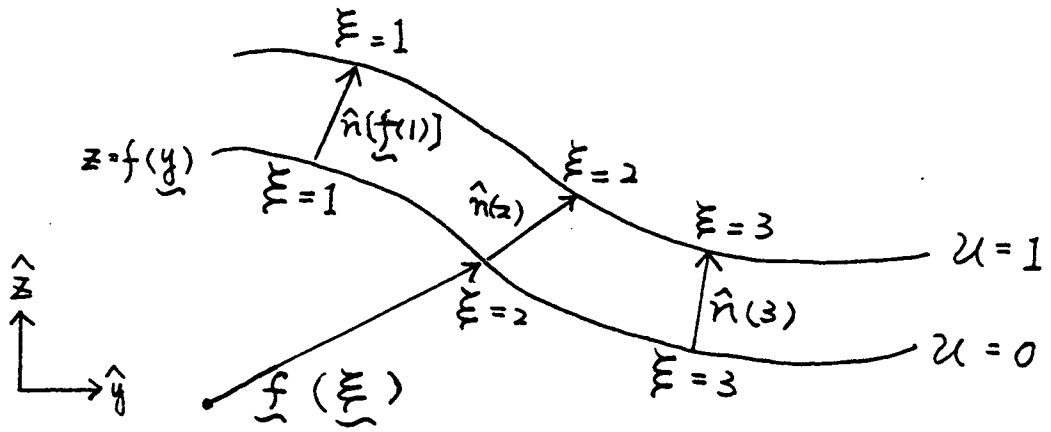
In our method of generalised collective coordinate of Section 2.4., the choice of collective coordinate represents some sort of transformation on a surface given by $z=f(\underset{\sim}{y})$. The transformation includes a translation and rotation from a plane surface collectively plus the distance due to the deviation of the surface from the tangent plane. R.K.P. Zia rewrites our choice of collective coordinate (2.62) as the first two terms in an expansion of $u(\underset{\sim}{y}, z)$ for small $(z - f)$ [Zia 1984]

$$(3.36)\quad u = (z-f)\, g^{-1/2} + \frac{1}{2}(z-f)^2 g^{-5/2} \partial_i f \, \partial_{ij} f \, \partial_j f$$

where

$$g = [\,1 + (\nabla f)^2\,].$$

He then argues that the second term would be the first correction to a region of the surface where $\nabla^2 f \neq 0$, since the competition between $(z - f)$ and the radius of curvature is responsible for the correction. He suggests that we may regard "u" as a curvilinear coordinate, which is associated with each f, where the "u=constant" surfaces are equally spaced along the whole surface as shown in the figure. The curvilinear coordinate "u" is called a normal coordinate [Zia 1983].



The d dimensional space ($\underset{\sim}{y}$, z) is covered by the coordinates ($\underset{\sim}{\xi}$, u) and $\underset{\sim}{\xi}$ is chosen to be $\underset{\sim}{y}$ for the sake of simplicity. The original d dimensional coordinate can be written as

$$\underset{\sim}{\chi} = (\underset{\sim}{y}, z) = \underset{\sim}{f}(\underset{\sim}{\xi}) + u\,\hat{n}(\underset{\sim}{\xi}) \tag{3.37}$$

where $\underset{\sim}{f}$ is a d-dimensional vector instead of f which is a scalar function. The two d dimensional vectors $\underset{\sim}{f}$ and $\hat{n}$ are given by

$$\begin{cases} \underset{\sim}{f}(\underset{\sim}{\xi}) = (\underset{\sim}{\xi}, f(\underset{\sim}{\xi})) \\ \hat{n}(\underset{\sim}{\xi}) = \dfrac{(-\nabla f, 1)}{\sqrt{g}} \end{cases} \tag{3.38}$$

where both $\hat{n}$ and u depend on f. The vector $\underset{\sim}{f}(\underset{\sim}{\xi})$ is a vector of the point "$\underset{\sim}{\xi}$" on the interface relative to the origin of the space. If we denote the components of $\underset{\sim}{f}$ and $\underset{\sim}{\xi}$ as

$$f^{\mu}, \quad \mu = 1, 2, 3, \cdots\cdots, d \tag{3.39}$$

$$\xi^a, \quad a = 1, 2, 3, \ldots\ldots, (d-1)$$

derivatives with respect to ξ^a will be written as a subscript with a comma, e.g. $f_{,a}$ are the components of ∇f.

By using the above general formalism, we may start to look for the effective Hamiltonian in this formalism. First the Jacobian of the transformation is given by

(3. 40) $$|\det T|$$

where $$T^{\mu}_{\alpha} \equiv \frac{\partial x^{\mu}}{\partial \xi^{\alpha}}$$

$$= \begin{cases} f^{\mu}_{,a} + u\, n^{\mu}_{,a} & \alpha = a = 1, 2, \ldots (d-1) \\ n^{\mu} & \alpha = d. \end{cases}$$

The measure of the normal's changes can be expanded in terms of the $f^{\mu}_{,a}$'s

(3. 41) $$n^{\mu}_{,a} = - k^{b}_{a}\, f^{\mu}_{,a}$$

where k^{b}_{a} are the components of the extrinsic curvature K. The metric for (ξ , u) space is given as

(3. 42) $$G_{\alpha\beta} = T^{\mu}_{\alpha}\, \delta_{\mu\nu}\, T^{\nu}_{\beta}$$

$$= \begin{cases} (\delta^{c}_{a} - u k^{c}_{a})\, g_{ce}\, (\delta^{e}_{b} - u k^{e}_{b}) & \alpha, \beta = a, b \\ 1 & \alpha = \beta = d \\ 0 & \text{Otherwise} \end{cases}$$

where

$$g_{ab} = \underset{\sim}{f}_{,a} \cdot \underset{\sim}{f}_{,b} = \delta_{ab} + f_{,a} f_{,b} \tag{3.43}$$

is the metric on the original $z = f(\underset{\sim}{y})$ surface. Substituting (3.41) into (3.40) and using (3.43), the Jacobian of the transformation is given by

$$\left| \det \mathbb{T} \right| \equiv \sqrt{G} = \det (\mathbb{1} - u\,\mathbb{K}) \sqrt{g} \tag{3.44}$$

We now expand (3.44) in power of u

$$\left| \det \mathbb{T} \right| = \sqrt{g} \left[1 - u\, \mathrm{Tr}\, \mathbb{K} + \tfrac{1}{2} u^2 R + \cdots\cdots \right] \tag{3.45}$$

where

$$R \equiv (\mathrm{Tr}\, \mathbb{K})^2 - (\mathrm{Tr}\, \mathbb{K}^2)$$

is the curvature scalar. If we write the Hamiltonian as

$$H[\phi_e(u)] = \int d\, \underset{\sim}{\xi}\, du \left| \det \mathbb{T} \right| [\phi_e'(u)]^2 \tag{3.46}$$

which can be expanded by the help of (3.45). Therefore, the effective Hamiltonian is given by

$$H[\phi_e(u)] = \int d^{d-1} \xi\, \sigma_0 \sqrt{g} \cdot (1 - \Delta\, \mathrm{Tr}\, \mathbb{K} + \tfrac{1}{2} w^2 R + \cdots\cdots), \tag{3.47}$$

where

$$\Delta \equiv \sigma_0^{-1} \int du\, u[\phi_c'(u)]^2$$

$$w \equiv \left\{ \sigma_0^{-1} \int du\, u^2 [\phi_c'(u)]^2 \right\}^{1/2}.$$

This gives the same result as our previous calculation. Although the two method are equivalent in physical and geometrical pictures, the beauty and power of the method of normal coordinates certainly makes our treatment of the effective Hamiltonian rather ugly.

CHAPTER IV Correction to Scaling

In the field theoretic model of surface tension, the higher derivative interactions beyond the surface tension term take the form of geometric invariants such as (g.b), curvature, $(g.b)^2$, etc, in the long wavelength limit. The forms of those geometric invariant interactions are derived in the previous chapter by using a generalised method of collective coordinates and the geometric identifications are made by differential geometric techniques. The next interesting question is how to renormalise those geometric invariants and what the scaling properties of those invariants are. Our approach to the renormalisation analysis is based on two other calculations: the ϵ -expansion for the interface model in $d=1+\epsilon$ dimensions [Wallace and Zia 1979], which is analogous to the renormalisation of the nonlinear sigma model in $2+\epsilon$ dimensions [Brezin and Zinn-Justin 1976a, b]; and the effective potential techniques [Forster and Gabriunas 1981]. The renormalisation of the higher derivative geometric invariants, such as the curvature term, proves to be too complicated to be done; besides, they may be phenomenologically irrelevant anyway. This leaves the simplest of the geometric invariants, the contraction of the coefficients of the second fundamental form b_{ij} with those of the metric tensor g_{ij} , i.e., (g.b). The problem then is to perform a perturbative analysis of the resulting effective Hamiltonian in $d=1+\epsilon$ bulk dimensions, where the higher geometric invariant interactions appearing within the resulting effective Hamiltonian are generally treated as small perturbations to the surface tension term. Before

we proceed to perform the perturbative calculation for (g.b), we shall discuss the nonlinear sigma model in $2+\epsilon$ dimensions and the analogous analysis of the interface in $1+\epsilon$ dimensions. The renormalisation and scaling properties of (g.b) in $1+\epsilon$ dimensions will be performed up to one loop level in the context of the perturbation analysis. The result shows that this type of interaction is indeed irrelevant to the leading long distance behaviour in low dimensions; it is just a correction to scaling [Lin and Lowe 1983]. In higher dimensions, it is possibly a relevant operator, but such a regime is beyond the control of the perturbative one loop calculation described in this chapter.

4.1. Nonlinear Sigma Model

There is a long history of the study of the linear and nonlinear sigma model in physics. The sigma model was proposed by Gell-Mann and Lévy [Gell-Mann and Lévy 1960] in the context of chiral symmetry breaking in elementary particle physics. They suggested it as a field theoretic model which realises chiral symmetry and partial conservation of the axial current. The Lagrangian is written as

$$\mathcal{L} = \mathcal{L}_S + c\,\sigma \tag{4.1}$$

$$\mathcal{L}_S = \bar{\psi}\left[i\not{\partial} + g(\sigma + i\,\underset{\sim}{\pi}\cdot\underset{\sim}{\tau}\,\gamma_5)\right]\psi + \frac{1}{2}\left[(\partial\underset{\sim}{\pi})^2 + (\partial\sigma)^2\right] - \frac{\mu^2}{2}(\sigma^2 + \underset{\sim}{\pi}^2) - \frac{\lambda}{4!}(\sigma^2 + \underset{\sim}{\pi}^2)^2, \tag{4.2}$$

where ψ is a fermionic isodoublet field of zero mass, $\underset{\sim}{\pi}$ is a triplet of pseudoscalar pions, and σ is a scalar field. The symmetric part of the Lagrangian, $\mathcal{L}_s$, is invariant under the SU(2)xSU(2) chiral group. It is a well known fact that SU(2)xSU(2) is isomorphic to the O(4) group, and the (1/2, 1/2) representation of SU(2)xSU(2) group, $(\sigma, \underset{\sim}{\pi})$, will transform as a vector under O(4). If the fermion fields are omitted, the Lagrangian becomes

$$\mathcal{L} = \mathcal{L}_s + c\,\sigma \tag{4.3}$$

$$\mathcal{L}_s = \frac{1}{2}(\partial \underset{\sim}{\phi})^2 - \frac{\mu^2}{2}\,\underset{\sim}{\phi}^2 - \frac{\lambda}{4!}(\underset{\sim}{\phi}^2)^2, \tag{4.4}$$

where

$$\underset{\sim}{\phi} \equiv \begin{pmatrix} \sigma \\ \underset{\sim}{\pi} \end{pmatrix} \tag{4.5}$$

is the compact notation for a multiplet of four fields transforming according to the vector representation of the symmetry group O(4). If the symmetry is broken from the larger group O(4) to the little group O(3), 6 - 3 = 3 Goldstone bosons corresponding to the spontaneous symmetry breaking will be generated. These three Goldstone bosons are usually identified as pions. In the renormalisation of this spontaneous symmetry breaking scheme, it seems reasonable to start with the renormalised symmetric theory with $\mu^2 > 0$ and then continue to the region of $\mu^2 < 0$. This procedure would generate a transition through a singular point for c = 0 in (4.3). Therefore, it is necessary to keep $c \neq 0$ as a small

breaking term in (4.3) to circumvent the singularity. In the limit of vanishing c, the theory is spontaneous broken without any renormalisation problem.

As was discussed in the previous chapter, a realisation of a group is more general than a representation of a group. If one chooses a space on which the group of transformation acts in such a way that the space is not necessary a linear vector space, one obtains a realisation of a group. We can choose a constraint condition to specify a realisation of group O(4), for example, the nonlinear realisation of the chiral group on the manifold given by

$$\underset{\sim}{\phi}^2 = \sigma^2(x) + \underset{\sim}{\pi}^2(x) = v^2, \tag{4.6}$$

where v is a constant value. The nonlinearly realised Lagrangian for the O(4) group is written as

$$\mathcal{L}_{NL\sigma} = \frac{1}{2}(\partial \underset{\sim}{\phi})^2. \tag{4.7}$$

Writing σ in terms of $\underset{\sim}{\pi}$, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}(\partial \underset{\sim}{\pi})^2 + \frac{1}{2}\frac{(\underset{\sim}{\pi}\cdot\partial\underset{\sim}{\pi})^2}{\sqrt{v^2-\underset{\sim}{\pi}^2}}. \tag{4.8}$$

From the above Lagrangian, it is straightforward to see that the chiral symmetry is realised in the Goldstone mode. The composite field $\sigma = \sqrt{v^2 - \underset{\sim}{\pi}^2}$ has a nonvanishing value, i.e. it is massive, and the $\underset{\sim}{\pi}$ stand for three massless fields. In the long wavelength,

low energy limit, the nonlinear sigma model is a good approximation.

In statistical mechanics, the Hamiltonian of the classical Heisenberg model with an O(n) symmetric group is given as [Brézin and Zinn-Justin 1976b]

$$H = -\sum_{i,j} K_{ij}\, \underset{\sim}{S}_i \cdot \underset{\sim}{S}_j \tag{4.9}$$

where the $\underset{\sim}{s}_i$ are unit n-component vectors associated with the sites i of a periodic d-dimensional lattice, K_{ij} is a short range positive translationally invariant interaction. The critical properties of the classical Heisenberg model can also be described by a continuous field theoretic model called the linear sigma model,

$$H = \int d^d x \left[\frac{1}{2}(\nabla \underset{\sim}{\phi})^2 + \frac{1}{2} m_0^2 \underset{\sim}{\phi}^2 + \frac{g_0}{4!}(\underset{\sim}{\phi}^2)^2\right]. \tag{4.10}$$

Brézin and Zinn-Justin showed that, in the long distance limit, the classical Heisenberg model is equivalent to the nonlinear sigma model [Brézin and Zinn-Justin 1976b]. The Euclidean action of the nonlinear sigma model again looks like

$$\mathcal{L} = \int d^d x \, \frac{1}{2}\left[(\nabla \sqrt{1-\underset{\sim}{\pi}^2})^2 + (\nabla \underset{\sim}{\pi})^2\right]. \tag{4.11}$$

Renormalisation of this nonlinear sigma model in $2+\epsilon$ dimensions has been carried out by the same authors [Brézin and Zinn-Justin 1976a; B, Z-J and Le Guillou 1976]. They use a

generating functional for Green's functions in Euclidean space, defined by:

$$\mathcal{Z} = \int d\sigma\, d^{n-1}\underset{\sim}{\pi} \prod \delta(\sigma^2 + \underset{\sim}{\pi}^2 - 1) \cdot \exp\left[-\frac{\Lambda^{d-2}}{2T} \int d^d x \left\{ (\nabla\sigma)^2 + \sum_{i=1}^{n-1} (\nabla \pi^i)^2 + \underset{\sim}{J} \cdot \underset{\sim}{\pi} \right\} \right], \tag{4.12}$$

where T is a dimensionless coupling constant and a regularisation, which preserves the O(n) symmetry, must be introduced in (4.12). By integrating out the delta function within the generating functional, the 2N-point interaction vertices are obtained by expanding the $[\nabla (1 - \underset{\sim}{\pi}^2)^{1/2}]^2$. It turns out that only field strength and coupling constant renormalisation are needed, so that the renormalised Lagrangian is

$$\mathcal{L}_R = \frac{\mu^{d-2}}{2 Z_1 T_R} \int d^d x \left\{ Z (\nabla \underset{\sim}{\pi}_R)^2 + [\nabla (1 - Z \underset{\sim}{\pi}_R^2)^{1/2}]^2 \right\}, \tag{4.13}$$

where the parameter μ fixes the scale of the renormalised theory. The lattice spacing or dimensional regularisation may be used to regularise the theory. The infrared divergence generated from the pion propagator, $1 / p^2$, can also be controlled by introducing an external source term, h, coupled linearly to the σ field. [Brézin, Zinn-Justin, and Le Guillou 1976]. The source term h plays the role of the magnetic field. The essential role of this term lies in the observation that the expansion of

$$h \int d^2 x \left[1 - \underset{\sim}{\pi}^2(x) \right]^{1/2} \tag{4.14}$$

in powers of $\underset{\sim}{\pi}^2$ generates a mass for the pion. The generating functional of the Green's function, thus, reads as

$$(4.15)\quad \mathcal{Z}(\underset{\sim}{J}, h) = \int \prod_x \frac{d\underset{\sim}{\pi}}{(1-\underset{\sim}{\pi}^2)^{1/2}} \exp\left\{\frac{1}{T_R}\left[-\mathcal{L} + \int \underset{\sim}{J}(x)\cdot\underset{\sim}{\pi}(x) d^2x\right]\right\}$$

with

$$(4.16)\quad \mathcal{L}(\underset{\sim}{\pi}, h) = \int d^2x \left[\frac{1}{2}(\nabla\underset{\sim}{\pi})^2 + \frac{1}{2}\frac{(\underset{\sim}{\pi}\cdot\nabla\underset{\sim}{\pi})^2}{(1-\underset{\sim}{\pi}^2)} - h(x)(1-\underset{\sim}{\pi}^2)^{1/2}\right]$$

For the lattice regularisation, the vertex functions of the σ and $\underset{\sim}{\pi}$ fields satisfy the renormalisation group equation

$$(4.17)\quad \Gamma_R^{(N)}(T_R, \mu) = \mathcal{Z}^{N/2}\, \Gamma^{(N)}(T, \Lambda).$$

The ordinary differential form of the renormalisation group equation is obtained by differentiation with respect to Λ, at fixed T_R and μ,

$$(4.18)\quad \left\{\Lambda\frac{\partial}{\partial\Lambda} + W(T)\frac{\partial}{\partial T} - \frac{N}{2}\zeta(T)\right\}\Gamma^{(N)}(T, \Lambda) = 0,$$

with

$$(4.19)\quad \zeta(T) = -\Lambda\left.\frac{\partial \ln \mathcal{Z}}{\partial\Lambda}\right|_R,$$

$$(4.20)\quad W(T) = \Lambda\left.\frac{\partial T}{\partial\Lambda}\right|_R.$$

The renormalisation group equation may also be written in terms of the renormalised vertex function $\Gamma_R^{(N)}$,

$$(4.21)\quad \left\{ \mu \frac{\partial}{\partial \mu} + W(T_R) \frac{\partial}{\partial T_R} - \frac{N}{2} \zeta(T_R) \right\} \Gamma_R^{(N)}(T_R, \mu) = 0$$

If one deals with external σ lines, it is easier to calculate the connected Green's function, $G^{(N)}$, instead of the vertex function, $\Gamma^{(N)}$. The equation reads

$$(4.22)\quad \left\{ \Lambda \frac{\partial}{\partial \Lambda} + W(T) \frac{\partial}{\partial T} + \frac{N}{2} \zeta(T) \right\} G^{(N)}(T, \Lambda) = 0.$$

The form of $W(T_R)$ at one loop in perturbation theory,

$$(4.23)\quad W(T_R) = \epsilon T_R - \frac{(n-2)}{2\pi} T_R^2 + O(T_R^3),$$

is crucial to the physical interpretation of the theory since it controls the flow of the effective coupling $T_R(\mu)$ at momentum scale μ according to

$$(4.24)\quad \mu \frac{d T_R}{d \mu} = W(T_R).$$

In two dimensions ($\epsilon \longrightarrow 0$), the theory is asymptotically free, with $T_R(\mu) \sim (\ln \mu)^{-1} \longrightarrow 0$ as $\mu \longrightarrow \infty$. In $2+\epsilon$ dimensions, the fixed point at $T_R = 0$ is infrared stable. The new fixed point $T_R^* = 2\pi / (n-2)\epsilon + O(\epsilon^2)$ is ultraviolet stable, i.e. infrared

unstable. Thus deviations of T_R from T_R^* are relevant in the sense of Section 1.5, so that T_R^* can be interpreted as a critical temperature, in fact as the phase transition temperature corresponding to the restoration of the O(n) symmetry. The key concept is that the linear and nonlinear sigma models are in the same [O(n)] universality class; the former is simple to renormalise near four dimensions, the latter near two. In the next section, we shall discuss an analogous interpretation of the renormalisation of the surface tension term in $1+\epsilon$ dimensions in terms of the Ising universality class.

4.2. Interface Model In $1+\epsilon$ Dimensions

The observation, made in the previous chapter, that the capillary waves transform as a nonlinear realisation of the Euclidean group of d dimensions gives us an interpretation of the capillary waves, as the Goldstone modes whose fluctuations lower to zero the critical temperature as $d \longrightarrow 1^+$. This is analogous to the use of the nonlinear sigma model to describe the interaction of the Goldstone modes (spin waves) arising from the spontaneous symmetry breaking of a global symmetry, such as the O(n) group, which was discussed in the previous section. The difference between these two systems, interface model and nonlinear sigma model, lies in the type of symmetry which they possess: the Ising model has a discrete internal symmetry, $\phi \longrightarrow -\phi$, which supports an interface model; the (continuous) internal symmetry of the

nonlinear sigma model is O(n).

The renormalisation group calculations of the interface model in $1+\epsilon$ dimensions follow very closely with the calculations of the nonlinear sigma model in $2+\epsilon$ dimensions. We start from the effective Hamiltonian of the field f in the form

$$H(f) = \frac{1}{T}\int d^{d-1}x \left\{ \left[1+(\nabla f)^2\right]^{1/2} + \frac{1}{2}m^2 f^2 \right\}, \tag{4.25}$$

with

$$\frac{1}{T} = \frac{\sigma}{k_B T}, \tag{4.26}$$

where the first term is the surface area of the interface, σ is the interfacial energy per unit area at zero temperature. The mass term in H is introduced as an infrared regulator to control the infrared problem. The generating functional of the Green's function can be written as usual

$$Z = \int \mathcal{D}f \quad e^{-H(f)}. \tag{4.27}$$

In order to set up the perturbation theory, it is necessary to expand the square root term inside the effective Hamiltonian. This leads to an infinite number of interaction vertices:

$$\begin{aligned} H &= \frac{1}{T}\int d^{d-1}x \sqrt{1+(\nabla f)^2} + \frac{1}{2}m^2/T\, f^2 \\ &= \frac{1}{T}\int d^{d-1}x \left\{ 1 + \frac{1}{2}(\nabla f)^2 - \frac{1}{8}((\nabla f)^2)^2 \right. \\ &\qquad \left. + \frac{1}{16}((\nabla f)^2)^3 + \cdots + \frac{1}{2}m^2 f^2 \right\} \end{aligned} \tag{4.28}$$

This gives the propagator

$$(4.29) \qquad \frac{T}{(q^2+m^2)}$$

and the interaction vertices are

$$\text{Four Points:} \qquad -\frac{1}{8T}\left((\nabla f)^2\right)^2$$

$$(4.30) \quad \text{Six Points:} \qquad \frac{1}{16T}\left((\nabla f)^2\right)^3$$

$$\text{Eight Points:} \qquad -\frac{5}{128T}\left((\nabla f)^2\right)^4$$

etc. The correlation functions (generating functional for connected diagrams) of the field f can also be written down. The ultraviolet and infrared properties for the interface model are controlled by the dimension of T. The effective Hamiltonian H is dimensionless in the power counting, $T = K^{-(d-1)}$ where K is an inverse length, or momentum. Thus H is naively nonrenormalisable for $d > 1$; the large momentum behaviour of the system cannot be controlled in a straightforward perturbative expansion in T. This problem can be solved perturbatively in $1 + \epsilon$ dimensions by using the renormalisation group techniques. Wallace and Zia have calculated the two and four-point vertex functions to two loops in a dimensional regularisation scheme. They found that the renormalisation required is a coupling constant renormalisation

$$T \equiv K^{-(d-1)} t_0$$

$$= K^{-(d-1)} \left\{ t + t^2 (d-1)^{-1} + t^3 \left[(d-1)^{-2} + \frac{1}{4} (d-1)^{-1} \right] + O(t^4) \right\}, \tag{4.31}$$

and a mass renormalisation

$$\frac{m^2}{T} = \frac{m_R^2 K^{d-1}}{t} \tag{4.32}$$

where t is the dimensionless renormalised coupling constant. They argued that it is not necessary to have a field strength (wave function) renormalisation, to all orders in perturbation theory. This is the consequence of the Ward identities and is also based on the interpretation that f represents a length.

The renormalisation group equation for all vertex functions Γ_R is thus written as

$$\left\{ K \frac{\partial}{\partial K} + \beta(t) \frac{\partial}{\partial t} + \gamma_1(t) m_R^2 \frac{\partial}{\partial m_R^2} \right\} \Gamma_R = 0, \tag{4.33}$$

where the beta function and the anomalous dimension are given as

$$\beta(t) = (d-1) t - t^2 - \frac{1}{2} t^3 + \cdots\cdots$$

$$\gamma_1(t) = -(d-1) + \frac{\beta}{t}. \tag{4.34}$$

There are two fixed points related to the beta function: the infrared stable fixed point, t = 0, which controls the low momentum

behaviour of H, and the ultraviolet stable one,

$$t_c = (d-1) - \frac{1}{2}(d-1)^2 + O[(d-1)^3], \tag{4.35}$$

which is the effective coupling for the high momentum behaviour.

The essential differences between the interface model in $1+\epsilon$ dimensions and the nonlinear sigma model in $2+\epsilon$ dimensions are: (1) The nonlinear sigma model requires field strength and coupling constant renormalisation; the interface model requires coupling constant and mass renormalisation. (2) An external source term (a magnetic field) coupling linearly to the sigma field is used to control the infrared divergence in the nonlinear sigma model; the interface model employs a mass regulator to control the problem explicitly (It can be interpreted in the fluid case as the effect of gravity).

Forster and Gabriunas carried out the calculations of the interface model in $1 + \epsilon$ dimensions up to four loops [Forster and Gabriunas 1981]. They used a method of effective potential instead of calculating the n-point vertex functions directly. The generating functional for the vertex functions (Gibbs free energy functional) is given as

$$\Gamma\{\Phi\} \equiv \int dx\, h(x)\, \Phi(x) - \ln \int \mathcal{D}f \exp\left(-H + \int dx\, h(x) f(x)\right), \tag{4.36}$$

which generates the vertex functions as the derivatives with respect to $\underset{\sim}{\Phi}(x) = \langle f(x) \rangle$. In order to cure the divergences of the theory, they introduce a field redefinition

$$f = \underset{\sim}{M} \cdot \underset{\sim}{\chi} + \phi(x) \tag{4.37}$$

where $\underset{\sim}{M}$ is a constant vector. This enables one to study the divergent parts of the n-point vertex functions. It follows that the free energy functional can be rewritten as

$$\Gamma\{\underset{\sim}{\Phi}\} = \int d^{d-1}x \left[\frac{m^2}{2T} \underset{\sim}{\Phi}^2 + \Gamma_0(M) \right], \tag{4.38}$$

where

$$-\int d^{d-1}x \, \Gamma_0(M) = \ln \int \mathcal{D}\phi \, \exp(-H'), \tag{4.39}$$

with

$$H'(\phi; m, \underset{\sim}{M}) = \frac{1}{T} \int d^{d-1}x \left\{ \left[1 + (\nabla\phi + \underset{\sim}{M})^2 \right]^{\frac{1}{2}} + \frac{1}{2} m^2 \phi^2 \right\}. \tag{4.40}$$

The beauty of the effective potential method is that once the effective potential is renormalised, the whole theory is also renormalised.

4.3. Renormalisation of (g.b)

By employing the techniques developed in the renormalisation of the interface model in $1 + \epsilon$ dimensions, we may proceed to perform

a perturbative analysis of the effective Hamiltonian which includes the higher derivative interactions. In this section, we shall demonstrate the renormalisation of the simplest of the higher derivative interactions, i.e. (g.b), to one loop. The result shows that this type of interaction remains in low dimensions is irrelevant, in the sense of the renormalisation group flow, for the leading critical behaviour; it is only a correction to scaling [Lin and Lowe 1983]. However, in higher dimensions, it can still be relevant. We extend the effective potential techniques, developed by Forster and Gabriunas, to study the following effective Hamiltonian:

$$H_{eff}(f) = \int d^{d-1}y \left\{ [1+(\nabla f)^2]^{1/2} \left(\frac{1}{T} + c\, g\cdot b \right] + \frac{1}{2} \tilde{\mu}^2 f^2 \right\}, \tag{4.41}$$

$$= H_0 + H_I$$

with

$$H_0 = \frac{1}{T} \int \sqrt{1+(\nabla f)^2} + \frac{1}{2} \tilde{\mu}^2 f^2$$

$$H_1 = c \int g\cdot b$$

where c is a small parameter, and the mass term is introduced as a infrared regulator to control the infrared problem as usual. The contraction of the coefficients of the second fundamental form with those of the metric tensor, g.b, is given as

$$g\cdot b = \frac{\partial f}{\partial y^i} \frac{\partial^2 f}{\partial y^i \partial y^j} \frac{\partial f}{\partial y^j} [1+(\nabla f)^2]^{-3/2} \tag{4.42}$$

Since c is treated as a small parameter, it implies that c (g.b) is treated as a small perturbation to the surface tension term. The idea behind this treatment is that we intend to demonstrate that

g.b $[1 + (\nabla f)^2]^{1/2}$ is multiplicatively renormalisable. This means that we consider the diagrams with only one vertex coming from the expansion of g.b $[1 + (\nabla f)^2]^{1/2}$ in powers of f; they are the terms which are linear in c. In their treatment of the method of effective potential, Forster and Gabriunas demonstrated that, in order to control the divergent parts of the vertex functions, it is only necessary to renormalise the effective potential (free energy) for certain field configurations. The divergent parts of the n-point vertex functions in our system involve one insertion from g.b $[1 + (\nabla f)^2]^{1/2}$ and any number of insertions from the $[1 + (\nabla f)^2]^{1/2}$ (surface tension) part of the Hamiltonian. We introduce, in order to cope with the extra derivative of f, a field configuration of f involving terms linear and quadratic in y (i.e. with first and second derivatives only),

$$f(y) = N_i y_i + \tfrac{1}{2} M_{ij} y_i y_j + \eta(y) \tag{4.43}$$

and work to first order in M and to any power of $\underset{\sim}{N}$. The above argument and other conventional considerations give us a set of criteria for the expansion of the generating functional of the vertex functions (Gibbs free energy functional) in powers of $(1+\underset{\sim}{N}^2)$:

(i) Throw away terms which are of orders higher than M^1.

(ii) Terms which are linear in η do not contribute to the one particle irreducible diagrams.

(iii) Total derivatives can be ignored.

(iv) Terms which are of the orders higher than $O(\eta^2)$ do not contribute to one loop diagrams.

We are now in a position to carry out the perturbation theory for the higher derivative interactions. By substituting the configuration of f, (4.43), into the effective potential, we first arrive at

$$\left[1+(\nabla f)^2\right]^{1/2} \simeq \sqrt{1+\underset{\sim}{N}^2} + \frac{1}{2}(1+\underset{\sim}{N}^2)^{-1/2} \frac{1}{\delta_{ij}+N_i N_j}\partial_i\eta\,\partial_j\eta$$

(4.44)

$$-(1+\underset{\sim}{N}^2)^{-3/2} M_{ki} y_i \left\{\frac{1}{2}N_k \frac{1}{\delta_{rs}+N_r N_s} + N_r \frac{1}{\delta_{ks}+N_k N_s}\right\}\partial_r\eta\,\partial_s\eta,$$

and

$$\left[1+(\nabla f)^2\right]^{-1} f_i f_j f_{ij}$$

$$\simeq (1+\underset{\sim}{N}^2)^{-1}\left\{N_i M_{ij} N_j + M_{ij}\partial_i\eta\,\partial_j\eta + 2N_i\partial_{ij}\eta\,\partial_j\eta\right\}$$

(4.45)

$$-(1+\underset{\sim}{N}^2)^{-2}\left\{4N_s\partial_s\eta M_{ji} N_i \partial_j\eta + 2N_s\partial_s\eta N_i N_j \partial_{ij}\eta\right\}$$

$$-(1+\underset{\sim}{N}^2)^{-2} N_i N_j M_{ij}\left\{\frac{1}{\delta_{rs}+N_r N_s} - 3\frac{N_r N_s}{1+\underset{\sim}{N}^2}\right\}\partial_r\eta\,\partial_s\eta.$$

The propagator for our perturbation theory can be derived from (4.44) as follows:

$$\frac{1}{2T}(1+\underset{\sim}{N}^2)^{-1/2} \frac{1}{\delta_{ij}+N_i N_j}\partial_i\eta\,\partial_j\eta + \frac{1}{2T}\mu^2\eta^2$$

(4.46)

$$= \frac{(1+\underset{\sim}{N}^2)^{-1/2}}{2T}\eta\left\{q_i \frac{1}{\delta_{ij}+N_i N_j} q_j + \mu^2\right\}\eta$$

$$= \frac{(1+\underset{\sim}{N}^2)^{-1/2}}{2T}\eta\left\{\bar{q}^2+\mu^2\right\}\eta,$$

where we have used the partial integration and plane waves as the eigenfunctions. The propagator can be written as

$$(4.47)\quad T\,\frac{(1+\underset{\sim}{N}^2)^{1/2}}{\tilde{q}^2+\mu^2}$$

with

$$(4.48)\quad \tilde{q}_i = \left(\frac{1}{\delta_{ij}+N_iN_j}\right)^{1/2} q_j$$

$$\tilde{q}^2 = q_i\,\frac{1}{\delta_{ij}+N_iN_j}\,q_j .$$

There are several types of vertices of interaction involved for the contractions of diagrams which contribute up to one loop level. The vertices from H_I are:

(4.49)

(i) $$(1+\underset{\sim}{N}^2)^{-1} M_{ij}\,\partial_i\eta\,\partial_j\eta$$

(ii) $$2(1+\underset{\sim}{N}^2)^{-1} N_i\,\partial_{ij}\eta\,\partial_j\eta$$

(iii) $$-4(1+\underset{\sim}{N}^2)^{-2} N_i\,M_{ij}\,N_s\,\partial_s\eta\,\partial_j\eta$$

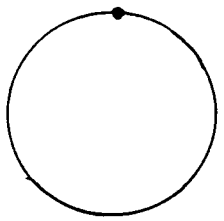
(iv) $$-2(1+\underset{\sim}{N}^2)^{-2} N_iN_jN_s\,\partial_s\eta\,\partial_{ij}\eta$$

(v) $$-(1+\underset{\sim}{N}^2)^{-2} N_i\,M_{ij}\,N_j\left\{\frac{1}{\delta_{rs}+N_rN_s}-3\frac{N_rN_s}{1+\underset{\sim}{N}^2}\right\}\cdot\partial_r\eta\,\partial_s\eta$$

By connecting η legs together we form the one loop contributions. Since we are working in the effective Hamiltonian to first power in M and to any power of $\underset{\sim}{N}$, there are still two types of diagrams to be done. If we denote the vertex from H_o as "." and vertices from

H_I as "x", the self contraction of a vertex from H_o is

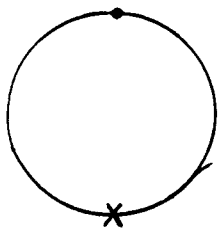
(4.50)



$$\text{"}\cdot\text{"}: -(1+\underset{\sim}{N}^2)^{-3/2} M_{ki} y_i \Big\{ \tfrac{1}{2} N_k \frac{1}{\delta_{rs}+N_r N_s} + N_r \frac{1}{\delta_{ks}+N_k N_s} \Big\} \partial_r \eta \partial_s \eta ;$$

and the contractions from one "." and one "x" are:

(4.51)



$$\text{"}\cdot\text{"}: -(1+\underset{\sim}{N}^2)^{-3/2} M_{ki} y_i \Big\{ \tfrac{1}{2} N_k \frac{1}{\delta_{rs}+N_r N_s} + N_r \frac{1}{\delta_{ks}+N_k N_s} \Big\} \partial_r \eta \partial_s \eta$$

$$\text{"X"}: 2\,(1+\underset{\sim}{N}^2)^{-1} N_l \partial_{lm} \eta \partial_m \eta$$

and

$$-2(1+\underset{\sim}{N}^2)^{-2} N_l \partial_l \eta\, N_m N_n \partial_{mn} \eta .$$

There are a few tricks to employ in order to calculate the dimensionally regularised integrals. We need some identities in order to do the dimensionally regularised integrals:

(4.52)
$$\begin{cases} \det \sqrt{A} = \sqrt{\det A} \\ \det(\delta_{ij} + N_i N_j) = 1 + \underset{\sim}{N}^2 \\ \det\left(\dfrac{1}{\delta_{ij}+N_i N_j}\right)^{1/2} = \dfrac{1}{\sqrt{1+\underset{\sim}{N}^2}} . \end{cases}$$

The calculations of the integrals present some interesting technical aspects. The contributions from part (ii) and (iv) of (4.49) vanish because they both involve integration of odd functions. For the diagrams of (4.50) and (4.51), the vertices involved in the calculation depend linearly on y_i. The integrals of these vertices

can be performed in momentum space with y_i interpreted as a Fourier transformation of a derivative of a momentum conservation delta function

$$y_i = i \int \frac{d^{d-1} p}{(2\pi)^{d-1}} e^{ip \cdot y} \frac{\partial \delta(p)}{\partial p_i} . \tag{4.53}$$

This trick shows that the integral calculated in (4.50) is, in fact, a total derivative only. These techniques would also be useful for future calculations which relate to the higher derivative interactions. The contribution for (4.51) is also involved with both coordinates and momenta. We also need the momentum representation of the Green function

$$\begin{aligned} G(x,y) &= \langle \eta(x) \eta(y) \rangle \\ &= (1+\underset{\sim}{N}^2)^{1/2} \int d^{d-1} p \; \frac{e^{ip(x-y)}}{\tilde{p}^2 + \mu^2} , \end{aligned} \tag{4.54}$$

and

$$(\delta_{lr} + N_l N_r)(\delta_{rs} + N_r N_s)^{-1} = \delta_{ls} ,$$

and using relations, such as

$$\int d^{d-1} p \; \frac{\partial}{\partial p_i} \left(\frac{p_r p_l p_m p_s p_n}{(\tilde{p}^2 + \mu^2)^2} \right) = 0 , \tag{4.55}$$

in order to transform (by using integration by parts) the contractions of (4.51) into shapes on which the technique of dimensional regularisation can perform.

After a long and tedious calculation, we obtain the

final contributions from the divergent parts of the dimensionally regularised integrals

$$\frac{1}{4}\left[\frac{-\mu^{\epsilon}}{\epsilon} T N_i M_{ij} N_j + 4\frac{\mu^{\epsilon}}{\epsilon} T N_i M_{ij} N_j + \frac{\mu^{\epsilon}}{\epsilon} T \frac{N_i M_{ij} N_j}{1+\underset{\sim}{N}^2} - 3\frac{\mu^{\epsilon} T}{\epsilon}\underset{\sim}{N}^2 \frac{N_i M_{ij} N_j}{1+\underset{\sim}{N}^2}\right] \tag{4.56}$$

$$= \frac{\mu^{\epsilon} T}{\epsilon} \frac{N_i M_{ij} N_j}{1+\underset{\sim}{N}^2}$$

We obtain the divergent part of the effective functional (free energy functional) as

$$\Gamma(N,M) = \int\left\{\frac{1}{T}\sqrt{1+\underset{\sim}{N}^2} + C^b \frac{N_i M_{ij} N_j}{1+\underset{\sim}{N}^2} + \frac{\mu^{\epsilon} T}{\epsilon} C^b \frac{N_i M_{ij} N_j}{1+\underset{\sim}{N}^2}\right\}. \tag{4.57}$$

As discussed in the previous section, Wallace and Zia have demonstrated that the surface tension term of the Hamiltonian is perturbatively renormalisable in an epsilon expansion context for $\epsilon = (d-1)$ with a coupling constant (T) renormalisation.

$$T = Z\, K^{-\epsilon} t\,, \tag{4.58}$$

where t is introduced as a dimensionless renormalised coupling constant and K is a renormalisation mass scale. Now we require an additional renormalisation to make the system finite. A simple multiplicative renormalisation of the coefficient c can be written down as in the coupling constant renormalisation.

$$C^R = Z_c^{-1} C^b \tag{4.59}$$

where

(4.60) $$\Sigma_c = \left(1 - \frac{t}{\epsilon}\right),$$

and c^R is a renormalised coefficient. The renormalisation group flow of the renormalised c^R, for temperatures close to the critical value, is thus obtained as

(4.61) $$C^R \sim K^{1-\epsilon+O(\epsilon^2)}.$$

Here the K' arises from the extra derivative in (g.b) and the correction of order ϵ from the one loop renormalisation above. The long distance behaviour is obtained in the limit $K \to 0$. When the power of K is positive, the insertion is therefore irrelevant. This is indeed the case for small ϵ but there is clearly a warning here that for larger ϵ this term may be relevant. If this is the case, the $1+\epsilon$ expansion is inappropriate for generic critical behaviour.

The picture of a long distance behaviour of the effective Hamiltonian (4.41) governed by the fixed point of the surface tension term in the Hamiltonian modified by corrections due to the higher derivative geometric invariants is, therefore, confirmed to be stable under renormalisation group flows for small ϵ.

CHAPTER V ONE-LOOP CALCULATION IN SYSTEMS WITH ADDITIONAL GOLDSTONE MODES

The renormalisation calculations of (g.b), concerned with the (d-1)-dimensional fluctuating interface and surface, have been completed in the last chapter. In this chapter, we shall start to investigate the effective Hamiltonian for (d-n)-dimensional generalised "interfaces" or "strings". This effective Hamiltonian governs the interactions of the Goldstone modes, in the long wavelength limit, due to the spontaneous breaking of the spatial (Euclidean) symmetries by a solution of the Euler-Lagrange equations. The difference between the (d-1)- and (d-n)-dimensional models lies in the dimension of the Euler-Lagrange equation: the dimension of the solution for the (d-1)-dimensional case is one; the solution describing a flat (d-n)-dimensional surface depends on n of the d coordinates of the system. In fact, the (d-1)-dimensional interface model can be regarded as a special case of the (d-n)-dimensional model [Lowe and Wallace 1980]. This flat (d-n)-dimensional interface obtained from the classical solution will fluctuate into the remaining n dimensions in d bulk dimensions. The effective Hamiltonian of the (d-n)-dimensional interface or "string" usually takes a form which is proportional to the hypersurface area of the "string" (or the generalised (d-n)-dimensional volume)

$$H_{eff} \propto \int d^{d-n}y \sqrt{\det g_{ij}} \tag{5.1}$$

where g_{ij} is the metric of the generalised interface or string. In Section 5.1, we shall derive this effective Hamiltonian [Lowe 1982] by using a semiclassical method similar to the generalised collective coordinate method of Chapter II. We shall also consider the possibility of a (d-n)-dimensional surface with an extra O(2) Goldstone mode which arises in addition to the breaking of the spatial symmetry. In the long distance limit, the effective Hamiltonian, which represents the nonlinear sigma model for the O(2) Goldstone mode, is defined on a curved surface given by the interface's position. The renormalisation calculations for the effective Hamiltonian of a (d-1)-dimensional interface carrying an extra O(2) Goldstone mode have been carried out, in the same spirit as in Chapter IV, up to one loop: the one-loop calculations will be discussed in Section 5.2.

5.1. Effective Hamiltonians for Strings

The string solution to the Euler-Lagrange equation of a system is a n-dimensional solution with $n > 1$. We shall start to investigate the possibility of constructing a string solution for the scalar field theory, such as the LGW model in Chapter II. For the classical solution of a scalar field system, $\phi_c(x)$, the energy will be given by the Hamiltonian

$$H = \frac{1}{2}\int d^n x \left(\nabla \phi_c(x)\right)^2 + \int d^n x \, U(\phi_c(x)). \tag{5.2}$$
$$\equiv H_1 + H_2$$

Under a scale transformation

$$X \longrightarrow aX, \tag{5.3}$$

the Hamiltonian is transformed into

$$H(a) = a^{2-n} H_1 + a^{-n} H_2. \tag{5.4}$$

This Hamiltonian must be stable under variations of the would-be classical solution,

$$\left.\frac{\delta H(a)}{\delta a}\right|_{a=1} = 0, \tag{5.5}$$

i.e., the variations of the Hamiltonian with respect to the scale change must be zero. This leads to the equation

$$(n-2) H_1 + n H_2 = 0. \tag{5.6}$$

The only possible solution for positive integer n to this equation is $n = 1$. Therefore, it is impossible to construct a string solution ($n \geq 2$) for the scalar field theory. This result is called Derrick's theorem [Coleman 1975].

One possibility for a scalar field theory to possess a higher dimensional solution is by coupling with some vector field. A specific example is the vortex solution of the abelian Higgs model

$$H = \int d^d x \left[\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2} |D_\mu \phi|^2 + \frac{1}{4} \lambda \left(|\phi|^2 - \frac{m^2}{\lambda} \right)^2 \right], \tag{5.7}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$
$$D_\mu \phi = (\partial_\mu - ie A_\mu) \phi, \tag{5.8}$$

(this is the Euclidean version with complex scalar field ϕ (x)). The field equations (Euler-Lagrange equations) are

$$\partial_\nu F_{\mu\nu} = e j_\mu$$
$$D_\mu D_\mu \phi = -\lambda \left(|\phi|^2 - \frac{m^2}{\lambda} \right) \phi, \tag{5.9}$$

with

$$j_\mu = i [\phi^* D_\mu \phi - \phi (D_\mu \phi)^*]$$
$$= i (\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*) - 2 e A_\mu \phi^* \phi. \tag{5.10}$$

The above equations are gauge invariant under the U(1) group of phase transformations. The vortex solutions are constructed as [Nielsen and Olesen 1973]

$$\phi(\underset{\sim}{x}) = \phi_c(r)\, e^{i\chi(\theta)} \tag{5.11}$$

$$A_\mu(\underset{\sim}{x}) = \frac{i}{2e|\phi|^2} (\phi \nabla \phi^* - \phi^* \nabla \phi),$$

with

(5.12) $$r = \sqrt{x_1^2 + x_2^2}\ , \quad \tan\theta = \frac{x_2}{x_1}\ .$$

By substituting $\phi(\underset{\sim}{x})$ into $A_\mu(\underset{\sim}{x})$, we may rewrite the (d-2)-dimensional "string" solution in a d bulk dimensional system as

$$\phi(\underset{\sim}{x}) = \phi_c(r)\, e^{i\alpha(\theta)}$$

(5.13) $$A_\mu(\underset{\sim}{x}) = A_c(r)\,\hat{\theta}$$

where

$$\hat{\theta} = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Following the notation introduced by D.J. Wallace [Wallace 1980], the above solutions have the general form

(5.14) $$\phi_c(\zeta_a)$$

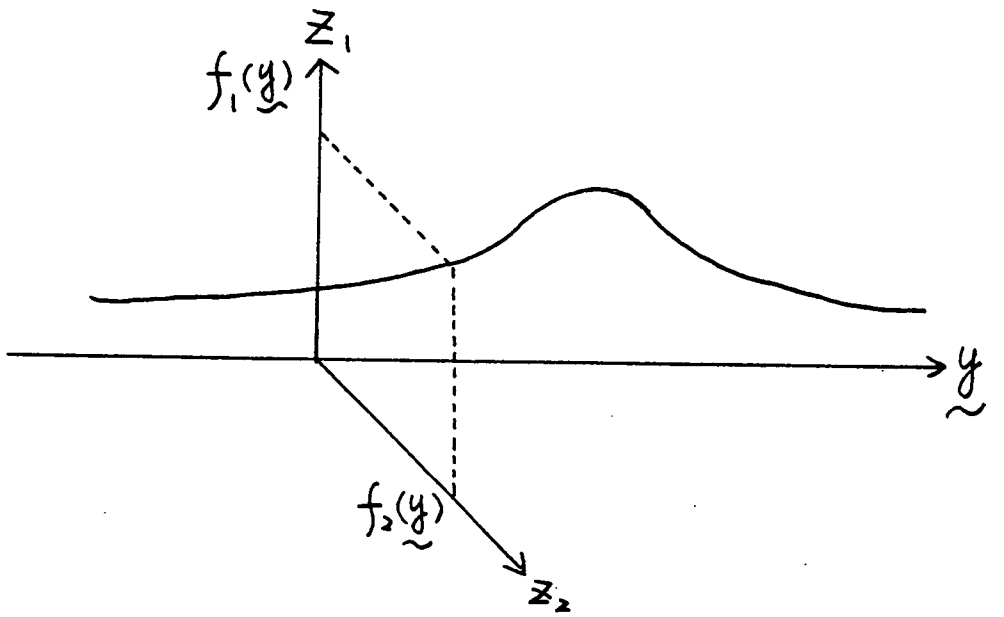
where a = 1, 2, ...,n and all indices which specify the components of the field are suppressed. These stable classical solutions would give a description of a planar (d-n)-dimensional reference hyperplane, at say $z_a = 0$. The fluctuations of the surface from this reference hyperplane are given by

(5.15) $$\zeta_a = f_a(\underset{\sim}{y})$$

where $$\underset{\sim}{x} = (y_i, \zeta_a)$$

with $$i = 1, 2, \cdots\cdots (d-n)$$
$$a = 1, 2, \cdots\cdots, n .$$

A figure is given in the following for a (d-n)-dimensional "string" fluctuating into the remaining n dimensions.



The descriptions of "strings" in this way transforms this problem into a similar situation to our (d-1)-dimensional interface model. We are interested in looking for the effective Hamiltonian, in the long distance limit, of the modified field configuration which must be related to the original solution by the symmetry carried by the system. Again, $f(\underset{\sim}{y})$ carry nonlinear realisations of the original full Euclidean symmetry group. The modified field configuration is given as [Lowe 1982],

(5.16) $$\phi_c' (\underset{\sim}{z}_a'),$$

where
$$\underset{\sim}{z}_a' = \frac{\underset{\sim}{z}_a - f_b(\underset{\sim}{y})}{\sqrt{M_{ab}}}$$

with
$$M_{ab} = \delta_{ab} + \frac{\partial f_a}{\partial y_i} \frac{\partial f_b}{\partial y_i}$$

The leading order contribution from the effective Hamiltonian for the modified field configuration, $\phi_c'(z_a')$, involves a change of variables only

$$(5.17)\quad (\underset{\sim}{y}, \underset{\sim}{z}_b) \longrightarrow (\underset{\sim}{y}, \underset{\sim}{z}_a').$$

This would give rise to a Jacobian which relates to the change of variables

$$(5.18)\quad \begin{aligned} d\underset{\sim}{z}_b &= \left(\frac{\delta_{ab}}{\sqrt{M_{ab}}}\right)^{-1} d\underset{\sim}{z}_a' \\ &= \sqrt{\det g_{ij}}\; d\underset{\sim}{z}_a', \end{aligned}$$

with

$$(5.19)\quad g_{ij} = \delta_{ij} + \frac{\partial f_a}{\partial y_i}\frac{\partial f_a}{\partial y_j}.$$

We, thus, obtain a surface tension type model again

$$(5.20)\quad H_{eff}(f) = \sigma \int d^{d-n}y \sqrt{\det g_{ij}}.$$

It is argued that it is possible for strings to possess other types of Goldstone modes than those which arise from the breaking of the spatial symmetry [Lowe 1982]. In the n = 1 case, there is a system which may allow a one-dimensional solution with an extra O(2) symmetry [Lajzerowicz and Neiz 1979; Lawrie and Lowe 1981]

$$(5.21)\quad \begin{aligned} H = \int d^d x \Big[&\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}|\nabla\psi|^2 + \tfrac{h}{2}|\psi|^2 \\ &+ \frac{\lambda}{4}\left(\phi^2 + |\psi|^2 - \frac{\gamma}{\lambda}\right)^2\Big] \end{aligned}$$

where ϕ is a real scalar field and ψ is a complex field. For $r \gg h$ there is a stable generalised interface-type solution

$$\begin{aligned} \phi &\propto \phi_c(z) \\ \psi &\propto \psi_c(z) e^{i\alpha} \end{aligned} \tag{5.22}$$

where α is an arbitrary phase factor. This field configuration breaks not only the Euclidean spatial symmetry but also the O(2) symmetry. In the n = 2 case, a string with an associated O(2) Goldstone mode occurs for the following modified Abelian Higgs model

$$\begin{aligned} H = \int d^d x \Big[&\tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} + \tfrac{1}{2} |D_\mu \phi|^2 + \tfrac{\lambda}{4} (m^2/\lambda - |\phi|^2)^2 \\ &+ \tfrac{1}{2} |\partial_\mu \sigma|^2 + \tfrac{h}{2} \sigma^2 + \tfrac{\mu}{4} \sigma^4 + \beta (m^2/\lambda - |\phi|^2) |\sigma|^2 \Big], \end{aligned} \tag{5.23}$$

where σ is an additional complex scalar field. For small h and large μ, there is a stable string solution

$$\begin{aligned} \phi(x) &= \tilde{\phi}_c(r) e^{i\omega} \\ A(x) &= \tilde{A}_c(r) \hat{\theta} \\ \sigma(x) &= \sigma_c(r) e^{i\alpha}, \end{aligned} \tag{5.24}$$

which still represents a (d-2)-dimensional string in a d-dimensional system but with an associated O(2) Goldstone mode.

In the general case, the effective Hamiltonian takes the following form [Lowe 1982]

$$H_{eff} = \sigma \int d^{d-n} y \sqrt{\det g_{ij}}\; G\left[\partial_i \theta(\underset{\sim}{y}) g_{ij}^{-1} \partial_j \theta(\underset{\sim}{y})\right], \tag{5.25}$$

where G is a complicated function which depends on the details of the potential and the modified field configurations. In this model, we see the expected form for a "free" O(2) Goldstone mode (θ) in a curved surface of metric g_{ij}. The arbitrariness of the function G leads us to consider a simplified model in which the reference hyperplane is held flat and fixed and the fluctuations from the reference plane are described by $\theta(\underset{\sim}{y})$. The unknown $(\nabla\theta)^2$ dependence inside the G function is written in the form, in the n=1 case,

$$(5.26)\quad H_{eff} = \sigma\int d^{d-1}y\,\sqrt{1+(\nabla\theta)^2} + \sum_{i=-\infty}^{\infty} C_i \int d^{d-1}y\,[1+(\nabla\theta)^2]^{a+i}$$

which is anyway the natural form which emerges from (5.25), where a is half-odd integer.

The remaining section in this chapter is concerned with a study of the renormalisation of the operator insertions of $[1+(\nabla\theta)^2]^a$, such as appear in (5.26).

5.2. One Loop Calculations

We shall start to set up the calculations for the effective Hamiltonian of the (d-1)-dimensional generalised interface or "string" with extra O(2) symmetry. The technique involved in this section is a simplified version of the method of effective potential developed in last chapter. The one loop

calculation for our effective Hamiltonian will be done in this section, and the techniques developed in the one loop calculations are also applicable to the higher loop calculations. The Hamiltonian of interest reads

$$(5.27)\quad H = \frac{1}{T}\int d^{d-1}y\sqrt{1+(\nabla\theta)^2} + C\int d^{d-1}y\left[1+(\nabla\theta)^2\right]^a + \frac{1}{2}m^2\theta^2$$

where "c" is treated as a small parameter and "a" is some rational number, that is, the higher order contribution in the effective Hamiltonian, $[\ 1 + (\nabla\theta)^2\ ]^a$, is also treated as a small perturbation in the sense of Section 4.3 (in the hope that it will be multiplicatively renormalisable). The mass term is also introduced as a infrared regulator to control the infrared problem as was done in Chapter IV. Since the renormalisability of the effective potential would also ensure the renormalisability of the whole theory, once again, we are interested in the generating functional of the vertex functions (Gibbs free energy functional)

$$\begin{aligned}(5.28)\quad &\Gamma[\theta(\underset{\sim}{y})] \\ &= \Sigma\int d^{d-1}y_1\, d^{d-1}y_2 \cdots d^{d-1}y_n \\ &\quad \times \Gamma^{(n)}(y_1, y_2 \cdots y_n)\,\theta(\underset{\sim}{y}_1)\,\theta(\underset{\sim}{y}_2)\cdots\theta(\underset{\sim}{y}_n).\end{aligned}$$

Since (5.27) involves with first derivative of the field $\theta(\underset{\sim}{y})$ only, it is only necessary to introduce a linear term in the field configuration redefinition

$$(5.29)\quad \theta(\underset{\sim}{y}) = N_k y_k + \eta(\underset{\sim}{y})$$

(5.30)
$$\partial_i \theta(\underset{\sim}{y}) = N_i + \partial_i \eta(\underset{\sim}{y}).$$

The idea is, of course, to transform the original generating functional for the n-point vertex functions into a new one such that interacting vertices from the surface tension part of the Hamiltonian are already included. The transformation looks like

(5.31)
$$\Gamma[\theta(\underset{\sim}{y})] \longrightarrow \Gamma[N, \eta(\underset{\sim}{y})].$$

There are two ways to expand the above Hamiltonian (5.27) with respect to the field configuration redefinition (5.30). The first one is similar to the method in Chapter IV, the field configuration (5.30) is substituted into the effective Hamiltonian (5.27) which is then expanded in powers of $(1 + \underset{\sim}{N}^2)$ with one particle reducible diagrams discarded. It is also possible to expand the effective Hamiltonian around N_i directly; the expansion would be simplified greatly since there is no higher derivatives in the Hamiltonian and the contribution from field configuration $\partial_i \eta(\underset{\sim}{y})$ can be easily separated from $H(\underset{\sim}{N}^2)$. Of course, these two schemes are equivalent to each other and they have been checked to be valid up to fourth order of the expansion.

We therefore write the effective Hamiltonian as

(5.32)
$$\begin{aligned} H[\partial_i \theta(\underset{\sim}{y})] &= H[N_i + \partial_i \eta(\underset{\sim}{y})] \\ &= H(N_i) + \int d^{d-1}y\, \partial_i \eta(\underset{\sim}{y}) \frac{\delta H}{\delta(\partial_i \theta)}\Big|_{\partial_i \theta = N_i} \\ &\quad + \tfrac{1}{2}\int d^{d-1}x\, d^{d-1}y\, \partial_i \eta(\underset{\sim}{x}) \partial_j \eta(\underset{\sim}{y}) \frac{\delta^2 H}{\delta(\partial_i \theta)\,\delta(\partial_j \theta)}\Big|_{\substack{\partial_i \theta = N_i \\ \partial_j \theta = N_j}} \end{aligned}$$

where the functional differentiation of $H = H_0 + H_I$ should be done separately for the free and perturbative parts. For the free part, we have

$$\frac{\delta H_0}{\delta N_i} = \frac{1}{T}(1+\underset{\sim}{N}^2)^{-1/2} N_i$$

(5.33)

$$\frac{\delta^2 H_0}{\delta N_j \delta N_i} = \frac{1}{T} \frac{(1+\underset{\sim}{N}^2)^{-1/2}}{(\delta_{ij} + N_i N_j)} .$$

The propagator is obtained with the help of the mass regulator,

$$(5.34) \quad T \frac{(1+\underset{\sim}{N}^2)^{1/2}}{\tilde{q}^2 + \mu^2} \quad \text{with} \quad \tilde{q}^2 = q_i (\delta_{ij} + N_i N_j)^{-1} q_j .$$

The perturbation from the interacting vertex, up to one loop level, is written as

$$(5.35) \quad \frac{1}{2} \frac{\delta^2 H_I}{\delta N_i \delta N_j} = C a (1+\underset{\sim}{N}^2)^{a-1} \left\{ \delta_{ij} + 2(a-1) \frac{N_i N_j}{1+\underset{\sim}{N}^2} \right\} .$$

In our formalism of effective potential, there is only one diagram, up to one loop and to first order in c. The result reads

$$\Gamma \sim \frac{1}{T} \sqrt{1+\underset{\sim}{N}^2} + C (1+\underset{\sim}{N}^2)^a$$

(5.36)

$$+ C T \frac{\mu^\epsilon}{\epsilon} 2a(a-1) \left[(1+\underset{\sim}{N}^2)^a - (1+\underset{\sim}{N}^2)^{a+1} \right]$$

where a dimensional regularisation has been introduced as in the calculations of Section 4.3.

We observe that the dimensional regularisation of the power "a"

term in the original effective Hamiltonian has generated a term with power "a+1". In order to renormalise the system with a power "a" perturbation term in a multiplicative way, we must introduce two other perturbative terms with powers of "a-1" and "a+1", respectively, since "a-1" term would generate terms with power "a-1" and "a" and "a+1" term would give "a+1" and "a+2". Of course, the "a-1" term would then need "a-2", etc. In theory then, an infinite number of terms with rational number power varying from "a-n" to "a+n" are needed in order to make the system finite. For the sake of simplicity, we shall restrict ourselves to three terms only to illustrate how the idea of multiplicative renormalisation works. We write the three term effective Hamiltonian as

$$
\begin{aligned}
H_{eff} = \frac{1}{T}\int & \sqrt{1+(\nabla\theta)^2} \\
& + C_0 \int [1+(\nabla\theta)^2]^{a-1} \\
& + C_1 \int [1+(\nabla\theta)^2]^{a} \\
& + C_2 \int [1+(\nabla\theta)^2]^{a+1} .
\end{aligned}
\tag{5.37}
$$

The generating functional of vertex functions is also modified accordingly,

$$
\Gamma \sim \frac{1}{T}\sqrt{1+\underset{\sim}{N}^2} + \mathbb{N}\left(\mathbb{1} + \frac{T\mu^{\epsilon}}{\epsilon}\mathbb{A}\right)\mathbb{C}^{B} \tag{5.38}
$$

where $\mathbb{A}$, $\mathbb{1}$ are matrices and $\mathbb{N}$, $\mathbb{C}$ are row and column vectors. They are given by the following:

$$
\mathbb{N} = \left[\cdots, (1+\underset{\sim}{N}^2)^{a-1}, (1+\underset{\sim}{N}^2)^{a}, (1+\underset{\sim}{N}^2)^{a+1}, \cdots\right]
$$

$$
\mathbb{C}^{B} = \begin{pmatrix} \vdots \\ C_0^{B} \\ C_1^{B} \\ C_2^{B} \\ \vdots \end{pmatrix} .
$$

$\mathbb{1}$ = unit matrix

(5.39)
$$\mathbb{A} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ & A_0 & & 0 & & \\ & -A_0 & A_1 & & & \\ 0 & & -A_1 & A_2 & & \\ & & & -A_2 & & \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{with} \quad \begin{cases} A_0 = 2(a-1)(a-2) \\ A_1 = 2a(a-1) \\ A_2 = 2(a+1)a \\ \vdots \end{cases}$$

We introduce a simple multiplicative renormalisation scheme in order to make the system finite

(5.40)
$$\mathbb{C}^B = \mathbb{Z}_c \, \mathbb{C}^R$$

where the generalised coupling constant renormalisation is described by a matrix equation. The quantities in the equation are given as

(5.41)
$$\mathbb{C}^R = \begin{pmatrix} \vdots \\ C_0^R \\ C_1^R \\ C_2^R \\ \vdots \end{pmatrix} \qquad \mathbb{Z}_c = \mathbb{1} - \frac{t}{\epsilon} \mathbb{A}$$

with
$$T = \frac{t}{k^\epsilon} Z_t .$$

The finiteness, up to one loop level, of the system after the renormalisation is thus very easy to verify.

An extended attempt has been made to extend these calculations to two loops, to confirm this picture of matrix-multiplicative renormalisation. The technique is similar to the one loop but much more demanding technically: fourth order in η must be retained in

H, there are four generic types of diagrams and each involves complicated tensorial algebra. This calculation has been carried through to the point where $1/\epsilon$ divergent terms have been isolated and identified. As one might anticipate, it is clear that the matrix-multiplicative renormalisation must be extended to include mixing between $(1+\underset{\sim}{N}^2)^{\alpha}$ and $(1+\underset{\sim}{N}^2)^{\alpha+2}$. What we have been unable to verify in the time available is whether the coefficients of the divergences are consistent with the one loop renormalisation. The structure of the renormalisation at higher order remains therefore an open question.

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