Generalisations of an inequality of Hardy under polynomial changes of variables

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Abstract

Hardy's inequality is a basic inequality of the theory of the Hardy spaces. In this thesis, we outline the development of the Hardy spaces from their complex analytic roots to their real variable interpretation of the 1960's and 1970's and we then prove a generalisation of the classical Hardy's inequality in the context of R^n under polynomial changes of variables. Central to our proof of this generalisation, is a weighted restriction theorem on polynomial curves, which is global in a certain sense. We also give a simpler proof which only works in 2 dimensions.

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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Chapter 1

Introduction

1.1 The classical Hardy spaces H^p

The starting point of the theory for Hardy spaces H^p is the following theorem of Hardy [H], established in 1915.

Theorem 1.1.1 *If f is an analytic function in the interior of the unit disc and p is a positive number, then*

$$\mu_p(f;r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta, \tag{1.1}$$

defined for $0 \le r < 1$, is an increasing function of r.

Later in 1922, F. Riesz [R1] proved Theorem 1.1.1 using subharmonic functions. Proofs of Theorem 1.1.1,using subharmonic functions, can be found in [GR] or in [Ru]. To this result can also be added the result, due to the maximum modulus theorem, that

$$\mu_\infty(f;r) = \sup_{0\le\theta<2\pi} |f(re^{i\theta})| \tag{1.2}$$

is an increasing function of r, for $r \in [0,1)$. In view of Theorem 1.1.1 and that $\mu_\infty(f;r)$ is an increasing function of r, we can make the following definition of Hardy spaces as they were introduced by F. Riesz in [R1].

Definition 1.1.2 For $0 < p \le \infty$ we define H^p to be the space of functions f, analytic in the interior of the unit disc, such that

$$\|f\|_{H^p} \equiv \sup_{0\le r<1} \mu_p(f;r) < \infty.$$

We note that H^∞ is the space of bounded analytic functions on the unit disc. Two obvious considerations are that for $0 < p' < p < \infty$, we have $H^\infty \subset H^p \subset H^{p'}$ and that due to Theorem 1.1.1 and the corresponding statement for $\mu_\infty(f;r)$, the supremums in Definition 1.1.2 can be replaced by limits as $r \to 1$. For $1 \le p \le \infty$, it can also be seen that H^p is a Banach space under the $\|\cdot\|_{H^p}$ norm. The triangle inequality is not satisfied by $\|\cdot\|_{H^p}$, for $p < 1$.

A natural question that was posed and answered by F. Riesz [R2] in 1923, concerns the existence of a limit function on the boundary of the unit disc, associated to each function in H^p. To be more precise we have the following theorem.

Theorem 1.1.3 *For $f \in H^p$, $0 < p < \infty$:*

1. *$f(z)$ has a non-tangential limit almost everywhere on the boundary of the unit disc. In addition, if we denote by $f(e^{i\theta})$ that limit, the function $\theta \mapsto f(e^{i\theta})$ belongs to $L^p([0, 2\pi])$.*

2. *The functions $f_r(\theta) = f(re^{i\theta})$ converge in the $L^p([0, 2\pi])$ norm to the function $f(e^{i\theta})$ as $r \to 1$.*

3. *$\|f\|_{H^p} = \lim_{r\to 1} \mu_p(f; r) = \|f(e^{i\cdot})\|_{L^p([0,2\pi])}$.*

The way this theorem was proved, is the following. First of all, it is observed that Theorem 1.1.3 holds for $p = 2$. This is because if $f \in H^2$ has the power series expansion

$$f(z) = \sum_{j=0}^{\infty} a_j z^j,$$

then it follows from the Parseval formula that

$$\mu_2(f; r) = \sum_{j=0}^{\infty} |a_j|^2 r^{2j}$$

and from the boundedness of $\mu_2(f; r)$ follows the convergence of the series $\sum |a_j|^2$. Therefore, due to the Riesz-Fischer theorem, there is an $L^2([0, 2\pi])$ function $f(e^{i\theta})$, whose Fourier series is $\sum_{j\geq 0} a_j e^{ij\theta}$. Thus, due to a theorem by Fatou [Fa], the Poisson integral

$$\sum_{j\geq 0} a_j r^j e^{ij\theta} = f(re^{i\theta})$$

converges to $f(e^{i\theta})$ almost everywhere as $(r, \theta) \to (1, \theta)$ non-tangentially. This proves the first assertion of Theorem 1.1.3 for $p = 2$. For the second assertion, still for $p = 2$, we have from the Parseval formula that

$$\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta}) - f(re^{i\theta})|^2 d\theta = \sum_{j=1}^{\infty} |a_j|^2 (1 - r^j)^2.$$

This sum and hence also the integral tend to zero as $r \to 1$, proving the second assertion of Theorem 1.1.3 for $p = 2$. We already see from this proof that in fact the H^2 functions are the Poisson integrals of their boundary values (which are precisely the functions in L^2, whose Fourier coefficients a_j vanish for $j < 0$). This method of proof could have also been generalised to all $1 < p < \infty$ (see e.g. [GR]). F. Riesz was able to prove Theorem 1.1.3 for $0 < p < \infty$ after he had proved the following factorisation theorem.

Theorem 1.1.4 *For each function $f(z)$ in H^p there is a factorisation in two factors: $f(z) = g(z)h(z)$, such that for $|z| < 1$, $h(z)$ is analytic and bounded, while $g(z)$ belongs to H^p and is nowhere zero in $|z| < 1$.*

In fact $h(z)$ is given by

$$h(z) = z^m \prod_{k=1}^{\infty} |\alpha_k| \frac{1 - \frac{z}{\alpha_k}}{1 - \bar{\alpha}_k z},$$

where $\alpha_1, \alpha_2, \ldots$ are the nonzero zeros of $f(z)$ listed with multiplicity and m is the order of the zero of $f(z)$ at 0. This is called the Blaschke product (the analogous factorisation theorem for H^∞ functions is contained in Blaschke [Bl]) and $h(z)$ is analytic and satisfies

$$|h(z)| \leq 1. \tag{1.3}$$

We omit the proof of Theorem 1.1.4 here, we just mention that it makes use of Theorem 1.1.1 together with properties of the Blaschke product, which rely on the distribution of the zeros of H^p functions. A proof of Theorem 1.1.4 can be found in [R2].

From the factorisation theorem it follows that we can write $f(z) = g(z)h(z) = [\gamma(z)]^{2/p}h(z)$, where γ is analytic, since $g(z)$ is nowhere zero. Both $\gamma(z)$ and $h(z)$ belong to H^2, so it immediately follows that $f(z)$ has a non-tangential limit, say $f(e^{i\theta})$, almost everywhere on the boundary of the unit disc. The way to prove

$$\int_0^{2\pi} |f(e^{i\theta}) - f(re^{i\theta})|^p d\theta \to 0 \tag{1.4}$$

as $r \to 1$, is to first prove that, as $r \to 1$,

$$\int_M |f(re^{i\theta})|^p d\theta \to \int_M |f(e^{i\theta})|^p d\theta \tag{1.5}$$

for any measurable set M in $\mathbb{T}$. To see (1.5) we have

$$\begin{aligned} \left| \int_M |f(re^{i\theta})|^p - |f(e^{i\theta})|^p d\theta \right| &= \left| \int_M |h(re^{i\theta})|^p |\gamma(re^{i\theta})|^2 - |h(e^{i\theta})|^p |\gamma(e^{i\theta})|^2 d\theta \right| \\ &\leq \int_M \left| |h(re^{i\theta})|^p - |h(e^{i\theta})|^p \right| |\gamma(e^{i\theta})|^2 d\theta \\ &+ \int_M |h(re^{i\theta})|^p \left| |\gamma(re^{i\theta})|^2 - |\gamma(e^{i\theta})|^2 \right| d\theta. \end{aligned}$$

Because of (1.3), the integrand of the first integral on the right side of the inequality is bounded above by $2|\gamma(e^{i\theta})|^2$, and so it tends to zero because of the Lebesgue dominated convergence theorem. Also the second integral is less than $\int_M \left||\gamma(re^{i\theta})|^2 - |\gamma(e^{i\theta})|^2\right| d\theta$ which in turn is less than $\int_M \left|\gamma(re^{i\theta})^2 - \gamma(e^{i\theta})^2\right| d\theta = \int_M |\gamma(re^{i\theta}) - \gamma(e^{i\theta})||\gamma(re^{i\theta}) + \gamma(e^{i\theta})| d\theta$. Using the Cauchy-Schwarz inequality we see that this last integral is

$$\begin{aligned} &\leq \left(\int_M |\gamma(re^{i\theta}) - \gamma(e^{i\theta})|^2 d\theta\right)^{1/2} \left(\int_M |\gamma(re^{i\theta}) + \gamma(e^{i\theta})|^2 d\theta\right)^{1/2} \\ &\leq 2\|\gamma\|_{H^2} \left(\int_M |\gamma(re^{i\theta}) - \gamma(e^{i\theta})|^2 d\theta\right)^{1/2} \to 0. \end{aligned}$$

Finally, to show (1.4) from (1.5), a theorem of Egoroff (see [E]) is used, according to which since $f(re^{i\theta})$ tends to $f(e^{i\theta})$ a.e., we can choose a set μ of arbitrarily small measure such that on the complement of this set the convergence is uniform. We can thus choose μ such that

$$\int_\mu |f(e^{i\theta}) - f(re^{i\theta})|^p d\theta < \epsilon$$

and because the convergence on μ^c is uniform,

$$\int_{\mu^c} |f(e^{i\theta}) - f(re^{i\theta})|^p d\theta < \epsilon$$

for r sufficiently near to 1, thus showing (1.4). The third assertion of Theorem 1.1.3 follows from (1.5) and so the proof of Theorem 1.1.3 is complete.

An immediate and important corollary of Theorem 1.1.3 is the following.

Corollary 1.1.5 *Every function $f \in H^p$, $1 \leq p \leq \infty$, is the Poisson and the Cauchy integral of its boundary function $f(e^{i\theta})$.*

The proof of this corollary can be found in [R2], [GR] or [Ru].

In 1929 Fichtenholz [Fi] obtained the following characterisation of H^1.

Theorem 1.1.6 *A function $f(z)$ analytic in the unit disc D, is represented by its Poisson and its Cauchy integral if and only if it satisfies*

$$\sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})| d\theta \leq M.$$

By $f(z)$ being represented by its Poisson and its Cauchy integral, we mean that $f(z)$ has a limit function on the unit circle and $f(z)$ is equal to the Poisson and the Cauchy integral of that limit function. Furthermore, similarly to Theorem 1.1.6 we have the characterisation of the H^p spaces for $1 \leq p \leq \infty$, that an analytic function f is in H^p in the sense of Definition 1.1.2 if and only if it is the Poisson and the Cauchy integral of an $L^p(\mathbb{T})$ function. Because of Theorem 1.1.3 we can associate each H^p with a subset of $L^p(\mathbb{T})$. In this way we can define $H^p(\mathbb{T})$, as the limit functions of functions in H^p, which we will denote as $H^p(D)$ where needed to avoid confusion. So, because of the a.e. convergence and the Poisson integral representation of functions in $H^p(D)$, we see that to each function in $H^p(D)$ corresponds a unique function in $H^p(\mathbb{T})$ and vice versa. Another characterisation of $H^p(\mathbb{T})$ for $1 \leq p \leq \infty$, that one could easily derive at this stage, is that a function is in $H^p(\mathbb{T})$ if and only if it is in $L^p(\mathbb{T})$ and its negative Fourier coefficents vanish. We can also equip $H^p(\mathbb{T})$ for $1 \leq p \leq \infty$ with a norm, that will just be the $L^p(\mathbb{T})$ norm, which is incidently equal to the $H^p(D)$ norm of the corresponding function in $H^p(D)$. The considerations in this paragraph, follow from what we have already proved and some other results on Poisson integrals. For further details on these one can see [GR], chapters I.1 and I.3. We only talked about these various characterisations for $1 \leq p \leq \infty$, because for $p < 1$ neither the Poisson integral nor the Fourier transform make sense in general.

In the 1930's various authors considered spaces of functions on the upper half-plane, similar to the H^p spaces on the unit disc introduced by F. Riesz, mainly using a conformal mapping from the unit disc to the upper half-plane. The first to consider such spaces were Hille and Tamarkin, [HT1] and [HT2], for $1 \leq p < \infty$. In [HT2], they defined the class $H^p(\mathbb{R}^2_+)$ in the following way.

Definition 1.1.7 A function $f(z)$, analytic in the half-plane $y > 0$, is said to be in $H^p(\mathbb{R}^2_+)$ if

$$\int_{-\infty}^{\infty} |f(x+iy)|^p dx \leq M^p,$$

where M only depends on f and p.

They then went on to prove the analogue of Theorem 1.1.3 for functions in $H^p(\mathbb{R}^2_+)$, $1 \leq p < \infty$.

Theorem 1.1.8 (i) *A function $f(z) \in H^p(\mathbb{R}^2_+)$, $1 \leq p < \infty$, has a limit function $f(x) \in L^p(\mathbb{R})$ to which it tends to almost everywhere non-tangentially.*

(ii) *Any $f(z) \in H^p(\mathbb{R}^2_+)$, $1 \leq p < \infty$, is represented by its Cauchy and Poisson integrals.*

(iii) *Any $f(z) \in H^p(\mathbb{R}^2_+)$, $1 \leq p < \infty$, tends to its limit function $f(x)$ in the L^p norm,*

$$\int_{-\infty}^{\infty} |f(x+iy) - f(x)|^p dx \to 0 \quad \text{as} \quad y \to 0.$$

Moreover, as $y \downarrow 0$,

$$\int_{-\infty}^{\infty} |f(x+iy)|^p dx \uparrow \int_{-\infty}^{\infty} |f(x)|^p dx.$$

(iv) *If $f(x) \in L^p$ and $f(z)$ is the Poisson integral of f and $f(z)$ is analytic for $y > 0$, then $f(z) \in H^p(\mathbb{R}^2_+)$ and therefore is represented by its Poisson and Cauchy integrals.*

The proof of this theorem relies on a conformal mapping from the unit disc in order to prove the a.e. convergence and additionally the Poisson integral representation for the norm convergence. This is the reason why the restriction $1 \leq p < \infty$ is required. Also, due to the unbounded nature of the underlying space $\mathbb{R}^2_+$, a decay result as z tends to infinity is needed, which is also proved using the Cauchy representation of $f(z)$. For a proof of Theorem 1.1.8, see [HT2]. The same authors, in [HT1], also showed the following, analogous to $H^p(D)$, characterisation of $H^p(\mathbb{R}^2_+)$.

Theorem 1.1.9 *Let $f(x) \in L^p(\mathbb{R})$, $1 \leq p < \infty$, be such that its Fourier transform $\widehat{f}$ is in some $L^q(\mathbb{R})$, $1 \leq q \leq \infty$. A necessary and sufficient condition that $f(x)$ is a limit-function of a function $f(z)$ which is analytic in the half-plane $\mathrm{Im}(z) > 0$ and which is represented by its Cauchy and Poisson integral, is that $\widehat{f}(u)$ vanish for $u < 0$.*

The following characterisation of $H^p(\mathbb{R}^2_+)$ in terms of the conjugate function $\widetilde{g}$ of g given by

$$\widetilde{g}(x) = \frac{1}{\pi} \text{ P.V.} \int_{-\infty}^{\infty} \frac{g(t)}{x-t} dt,$$

is also contained in [HT2]. First let us define the Poisson and conjugate Poisson integrals of g on the upper half-plane $\mathbb{R}^2_+$, by

$$P(z;g) = \frac{1}{\pi} \int_{-\infty}^{\infty} g(t) \frac{y}{(t-x)^2 + y^2} dt,$$

and

$$\widetilde{P}(z;g) = \frac{1}{\pi} \int_{-\infty}^{\infty} g(t) \frac{t-x}{(t-x)^2 + y^2} dt,$$

respectively.

Theorem 1.1.10 *Let $g(x)$ and $\widetilde{g}(x) \in L^p(\mathbb{R})$, $1 \leq p < \infty$. The function $f(z) = P(z;g) + i\widetilde{P}(z;g)$ is analytic in the half-plane $y > 0$ and is representable by its Cauchy and Poisson integrals. Furthermore its limit function $f(x)$ is such that $f(x) = g(x) + i\widetilde{g}(x)$.*

Conversely if $f(z)$ is analytic in the half-plane $y > 0$ and is representable by its Cauchy and Poisson integrals with the limit function $f(x) \in L^p(\mathbb{R})$, $1 \leq p < \infty$, then $\mathrm{Im} f(x) = \widetilde{\mathrm{Re} f}(x)$.

Of course the result in M. Riesz [RM] guarantees that $\widetilde{g}(x) \in L^p$ if $g \in L^p$, if $1 < p < \infty$.

The proofs of Theorems 1.1.9 and 1.1.10 both rely on the Cauchy and Poisson integral representations and their invariance under conformal mappings from the unit disc.

Thus from Theorems 1.1.6 to 1.1.10, we see that from very early on, different ways were considered to characterise H^p spaces on the unit disc or half-plane, mainly for $1 \leq p < \infty$. The alternative characterisations of H^p spaces do not have equivalents for $p < 1$. For all these reasons, the reasoning of Hille and Tamarkin does not extend to $p < 1$. The analogous properties in Theorem 1.1.3 for functions in $H^p(\mathbb{R}^2_+)$ for $0 < p < \infty$, were derived in one stroke in Kryloff [K]. In [K] are contained the exact analogues of Theorems 1.1.1, 1.1.3 and 1.1.4 for $f \in H^p(\mathbb{R}^2_+)$, $0 < p < \infty$. The method of proof follows the same lines as the proof of Theorem 1.1.3 and uses conformal mapping from the unit disc and some properties of subharmonic functions.

In the meantime Hardy and Littlewood in [HL2] introduced and proved the boundedness of the Hardy-Littlewood maximal function, which they used for proving a L^p-boundedness result, for $0 < p \leq \infty$, for the non-tangential maximal function of an H^p-function of the unit disc. Their method of proof, reducing the problem by majorising a certain subharmonic function by a harmonic function and then proving the equivalent result for the harmonic function using the Hardy-Littlewood maximal function, would in time be used in more general settings. Initially the attempt was to generalise H^p-spaces in the context of functions of several complex variables. In this setting no decomposition exists of the type of Theorem 1.1.4. Nevertheless, using the procedure of [HL2], Theorem 1.1.3 was generalised by Rauch [Ra] to functions analytic in the solid unit hypersphere

$$B_{2n} : r^2 \equiv |z_1|^2 + \ldots + |z_n|^2 < 1.$$

That is writing an analytic function of n complex variables, $f(z_1, \ldots, z_n)$ as $f(r, P)$ where

$$P \in S_{2n-1} : |z_1|^2 + \ldots + |z_n|^2 = 1,$$

we have the following theorems.

Theorem 1.1.11 *If f is an analytic function in B_{2n} and for some $p > 0$ satisfies*

$$\int_{S_{2n-1}} |f(r,P)|^p dV_P \leq C^p, \quad r < 1, \tag{1.6}$$

where dV_P is the volume element on S_{2n-1} at P, then

$$\int_{S_{2n-1}} (\sup_{0 \leq r < 1} |f(r,P)|)^p dV_P \leq \alpha_n C^p.$$

Theorem 1.1.12 *Under the same hypothesis as Theorem 1.1.11, there exists an $L^p(S_{2n-1})$ function $f(P)$ such that*

$$\lim_{r\to 1}\int_{S_{2n-1}} |f(r,P)-f(P)|^p dV_P = 0.$$

Theorem 1.1.12 follows from Theorem 1.1.11 by a remark in Zygmund [Z2], that under the same hypothesis, $f(r,P)$ has a pointwise limit, $f(P)$, almost everywhere. Then, convergence can be majorised according to Theorem 1.1.11 and hence imply mean convergence. In Zygmund [Z1] there is an equivalent theorem for the polycylinder

$$|z_1|<1, |z_2|<1, \ldots, |z_n|<1.$$

Theorem 1.1.13 *If for $0<p\le\infty$, the analytic function $f(z_1,\ldots,z_n)$ satisfies*

$$\int_0^{2\pi}\cdots\int_0^{2\pi} |f(r_1e^{ix_1},\ldots,r_ne^{ix_n})|^p dx_1\ldots dx_n < M^p \tag{1.7}$$

for $r_1,\ldots,r_n<1$, then there is a function $f(e^{ix_1},\ldots,e^{ix_n})$, such that

$$\lim_{r_1,\ldots,r_n\to 1}\int_0^{2\pi}\cdots\int_0^{2\pi} |f(r_1e^{ix_1},\ldots,r_ne^{ix_n})-f(e^{ix_1},\ldots,e^{ix_n})|^p dx_1\ldots dx_n = 0.$$

Theorem 1.1.13 follows from the following theorem, also in [Z1], on the a.e. existence of a limit function $f(e^{ix_1},\ldots,e^{ix_n})$ and an L^p, $0<p\le\infty$, boundedness result for the non-tangential maximal function of $f(z_1,\ldots z_n)$.

Theorem 1.1.14 *Let $f(z_1,\ldots,z_n)$ be analytic in*

$$|z_1|<1, |z_2|<1, \ldots, |z_n|<1$$

and let

$$\int_0^{2\pi}\cdots\int_0^{2\pi} \log^+|f(r_1e^{ix_1},\ldots,r_ne^{ix_n})| \{\log^+\log^+|f(r_1e^{ix_1},\ldots,r_ne^{ix_n})|\}^{n-1}dx_1\ldots dx_n < M \tag{1.8}$$

for $r_1,\ldots,r_n<1$. Then, a.e. in

$$0\le x_1<2\pi,\quad 0\le x_2<2\pi,\quad \ldots,\quad 0\le x_n<2\pi,$$

the limit $f(e^{ix_1},\ldots,e^{ix_n})$ of $f(z_1,\ldots,z_n)$ exists, as $(z_1,\ldots,z_n)$ tends to $(e^{ix_1},\ldots,e^{ix_n})$ non-tangentially.

Of course if f satisfies (1.7), then it also satisfies (1.8). Theorem 1.1.13 was already known in the cases $1<p$, see Bergman and Marcinkiewicz [BM] and Bers [Be], where also a Poisson integral representation is obtained for functions

satisfying (1.7) for $1 \leq p$. In the case $1 < p$, the result is valid for functions harmonic in each complex variable separately. Theorem 1.1.14 is proved by first proving a weaker form of it where the convergence of f along non-tangential paths is replaced by the boundedness of f along such paths.

The theory of H^p spaces on $\mathbb{R}^n$ began in Stein and Weiss [SW1]. To define the H^p spaces in this context, they used the vector-valued function $F(X, y) = (u(X, y), V(X, y))$, with $X = (x_1, x_2, \ldots, x_n)$ and $V = (v_1, v_2, \ldots, v_n)$, the $u, v_1, v_2, \ldots, v_n$ being real-valued harmonic functions on $\mathbb{R}^{n+1}_+$ satisfying the generalised Cauchy-Riemann equations:

$$\frac{\partial u}{\partial y} + \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i} = 0,$$

$$\frac{\partial u}{\partial x_i} = \frac{\partial v_i}{\partial y}, \quad i = 1, \ldots, n, \tag{1.9}$$

$$\frac{\partial v_i}{\partial x_j} = \frac{\partial v_j}{\partial x_i}, \quad i \neq j, 1 \leq i, j \leq n.$$

These were introduced by Horváth [Ho]. The functions $v_1, \ldots, v_n$ are called the conjugates of u. The H^p spaces, $p > 0$, were then defined as the classes of systems of conjugate harmonic functions, $F(X, y)$, satisfying

$$\mathfrak{M}_p(y; F) = \left(\int_{\mathbb{R}^n} |F(X, y)|^p dX\right)^{1/p} \leq A < \infty, \tag{1.10}$$

for $0 < y < \infty$, where $|F(X, y)| = (u^2 + v_1^2 + v_2^2 + \ldots + v_n^2)^{1/2}$. Stein and Weiss were able to prove a.e. non-tangential convergence and norm convergence to the boundary of $\mathbb{R}^{n+1}_+$ for $p \geq (n-1)/n$ and $p > (n-1)/n$ respectively. This, they were able to do because they proved that $|F|^p$ is subharmonic for $p \geq (n-1)/n$. Then using the L^p boundedness of the Hardy-Littlewood maximal function, they proved the following L^p boundedness of the non-tangential maximal function of a harmonic function. From now on we denote by $\Gamma_\alpha(X) \subset \mathbb{R}^{n+1}_+$, the conical region with vertex X, of all points (Z, y) satisfying $|X - Z| < \alpha y$.

Theorem 1.1.15 *Suppose that $m(X, y)$ is harmonic in $\mathbb{R}^{n+1}_+$ and, for $q \geq 1$,*

$$\int_{\mathbb{R}^n} |m(X, y)|^q dX \leq C^q < \infty \tag{1.11}$$

unifomly for $y > 0$. Let

$$m^*_\alpha(X) = \sup_{(Z,y)\in\Gamma_\alpha(X)} |m(Z, y)|.$$

Then,

1. *if* $q > 1$, $m^*_\alpha(X) \in L^q(\mathbb{R}^n)$ *and* $\|m^*_\alpha\|_q \leq AC$, *where* A *depends only on* α, q *and* n*;*

2. *if* $q = 1$, $m^*_\alpha(X) < \infty$ *almost everywhere.*

Analogous results, for other contexts in which H^p spaces had been considered, can be found in [HL2], [Ra] and [Z1]. In all the contexts though, the proof relies on the use of the Poisson integral representation for a harmonic function satisfying (1.11) and the majorisation of the non-tangential maximal function by the Hardy-Littlewood maximal function. An analogous result could have been proven for our F instead of a harmonic function and for $q \geq (n-1)/n$. This could be derived from Theorem 1.1.15 by majorising $|F|^{(n-1)/n}$ by a harmonic function.

In order to prove the non-tangential convergence and hence, using Theorem 1.1.15, also the norm convergence, Stein and Weiss used the following theorem of Calderón [Ca1].

Theorem 1.1.16 *Let* $w(X, y)$ *be harmonic in* $\mathbb{R}^{n+1}_+$. *Suppose that for a measurable set* $S \subset \mathbb{R}^n$

$$|w(X, y)| \leq M < \infty$$

for $(Z, y) \in \Gamma_\alpha(X)$, X *in* S. *Then, for almost every* X *in* S, $\lim w(Z, y)$ *exists, as* (Z, y) *tends to* $(X, 0)$ *non-tangentially.*

For a more detailed discussion of this theorem, one can see Zygmund [Z3] Chapters XIV and XVII or Stein [S1] Chapter VII. So in this context the analogue to Theorem 1.1.3 can be stated as follows:

Theorem 1.1.17 *Suppose* $F(X, y)$ *is in* H^p, $p \geq (n-1)/n$, *then*

$$\lim F(Z, y) = F(X, 0),$$

where (Z, y) *tends to* $(X, 0)$ *non-tangentially, exists for almost every* X *in* $\mathbb{R}^n$. *In case* $p > (n-1)/n$, $F(X, 0)$ *is also the limit in the* L^p *norm of* $F(X, y)$.

Stein and Weiss also proved that for $p \geq (n-1)/n$, $\mathfrak{M}_p(y; F)$ increases as $y \to 0$. Thus, we can define the H^p norm as

$$\|F\|_p = \sup_{y>0} \mathfrak{M}_p(y; F) = \lim_{y \to 0} \mathfrak{M}_p(y; F).$$

In case $p > (n-1)/n$, a consequence of Theorem 1.1.17 is that

$$\|F\|_p = \left(\int_{\mathbb{R}^n} |F(X, 0)|^p dX \right)^{1/p}.$$

Since it can be seen that for $p \geq (n-1)/n$ the set of equations (1.9) uniquely determine $v_1, v_2, \ldots, v_n$ given u (see [SW1]), we can identify our F with the boundary value of u, thus identifying H^p with a subset of $\mathrm{Re}L^p(\mathbb{R}^n)$, $p \geq (n-1)/n$. In defining the H^p spaces using (1.9) and (1.10) it is not essential to have the requirement that the $u, v_1, v_2, \ldots, v_n$ are real-valued. Thus, we can simply extend the definition to complex-valued u and indeed this is what we will call $H^p(\mathbb{R}^n)$ from now on. When we just consider real-valued u, we will refer to $\mathrm{Re}H^p(\mathbb{R}^n)$. Going from real-valued u to complex-valued u of course does not change matters much when we are considering norms. Comparing the definition of H^p of Stein and Weiss [SW1] in the case $n = 1$, with that of Hille and Tamarkin [HT2] or Kryloff [K], we see that in the sense of [SW1] the functions in $\mathrm{Re}H^p(\mathbb{R})$ are the real parts of the functions of $H^p(\mathbb{R})$ taken in the sense of [HT2] or [K]. Again, one could derive the various results for H^p in the sense of [HT2] or [K], knowing the same results for H^p in the sense of [SW1], and vice versa.

It is a result of Horváth [Ho], for $1 < p < \infty$, that if $u, v_1, v_2, \ldots, v_n$ satisfy (1.9) and (1.10), and given that u is the Poisson integral of $f \in L^p(\mathbb{R}^n)$ and each of the v_j is the Poisson integral of an $f_j \in L^p(\mathbb{R}^n)$, then $f_j = R_j(f)$, where R_j are the M. Riesz transforms defined by convolution with the kernel $c_n x_j/|x|^{n+1}$ and c_n is a certain function of n only. The converse also holds. That is using the same notation, if f and all the $R_j(f)$ are in $L^p(\mathbb{R}^n)$, then their Poisson integrals satisfy (1.9) and (1.10). The Riesz transforms are bounded on $L^p(\mathbb{R}^n)$, for $1 < p < \infty$, so $H^p(\mathbb{R}^n)$ is actually identical to $L^p(\mathbb{R}^n)$ and similarly $\mathrm{Re}H^p(\mathbb{R}^n)$ is identical to $\mathrm{Re}L^p(\mathbb{R}^n)$. In addition the L^p-norm of f is equivalent to the H^p-norm of F. For $p = 1$ this no longer holds and H^1 is a proper subset of L^1 and many times it is used as a substitute for L^1, in the sense that various results that do not hold for L^1, still hold for H^1. A way of describing $H^1(\mathbb{R}^n)$ in terms of the Riesz transforms was given in Stein [S1] p. 221. That is, the space $H^1(\mathbb{R}^n)$ is naturally isomorphic with the space of $L^1(\mathbb{R}^n)$ functions f which have the property that $R_j(f) \in L^1(\mathbb{R}^n)$, $j = 1, \ldots, n$. The H^1-norm is then equivalent with $\|f\|_1 + \sum_{j=1}^n \|R_j(f)\|_1$. This is a consequence of Theorem 1.1.17.

1.2 The space of functions of bounded mean oscillation (BMO) and its relation to H^1

One of the results that played a significant role in the development of the theory of H^p spaces, was the identification of the dual of H^1 with the space of functions of bounded mean oscillation (BMO). The space BMO was introduced first by John and Nirenberg [JN] in the context of partial differential equations. We have the following definition.

Definition 1.2.1 Let f be in $L^1_{\text{loc}}(\mathbb{R}^n)$. Then f is of *bounded mean oscillation* ($f \in BMO$) if

$$\sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx = \|f\|_* < \infty, \tag{1.12}$$

where the supremum is over all finite cubes in $\mathbb{R}^n$, $|\cdot|$ denotes the Lebesgue measure and $f_Q = (1/|Q|) \int_Q f(x)dx$ (the mean value of f over Q).

We define $\|f\|_*$ to be the BMO norm of f. This is not quite a norm since an a.e. constant function would have norm equal to zero, but we think of two functions that differ by a constant to coincide as functions in BMO. Under this convention, BMO is a Banach space under the norm $\|\cdot\|_*$. As far as examples of BMO functions are concerned, we have that $L^\infty \subset BMO$ and the typical example of an unbounded $BMO(\mathbb{R})$ function is $\log|x|$. In fact we will obtain, as a consequence of a generalisation of Hardy's inequality that we will prove in Chapter 2, further examples of BMO functions.

Both H^1 and BMO serve as substitute spaces for L^1 and L^∞ respectively. For example, we have that classical Calderón-Zygmund singular integral operators are bounded from H^1 to L^1 and from L^∞ to BMO. The identification of the dual of H^1 with BMO ties these facts together. This identification was announced in C. Fefferman [F2] and proved in Fefferman and Stein [FS]. They use an appropriate dense subspace H^p_0 of H^p (see [S1] p. 225). If $f \in H^p_0$, then in particular, f is bounded and rapidly decreasing at infinity.

Theorem 1.2.2 *The dual of H^1 is BMO, in the following sense.*

1. *Suppose $\phi \in BMO$. Then the linear functional $f \to \int_{\mathbb{R}_n} f(x)\phi(x)dx$, initially defined for $f \in H^p_0$, has a bounded extension to H^1.*

2. *Conversely, every continuous linear functional on H^1 arises as in 1. with a unique element ϕ of BMO.*

1.3 Real Hardy spaces

There were two significant steps that started the development of the theory of Hardy spaces using only real variable techniques. The first one was the identification of the dual of H^1 as BMO, which we discussed in Section 1.2. The second significant step was the theorem of Burkholder, Gundy and Silverstein in [BGS], that in the classical situation of an analytic function $F = f + ig$, the property $F \in H^p$, $0 < p < \infty$, is equivalent with the non-tangential maximal function of

f belonging to L^p. This was proved in one dimension in [BGS] using Brownian motion. The direction that if $F \in H^p$ then the non-tangential maximal function of f is in L^p was already known for H^p in various settings, starting with the result of Hardy and Littlewood [HL2]. So the procedure used in [HL2], [Ra], [Z1] and [SW1], already went some way in freeing the theory of H^p spaces from its complex analytic roots by succeeding in obtaining the a.e. and norm convergence results by using the L^p-boundedness of the non-tangential maximal function instead of using Blaschke products and conformal maps. After [BGS] though, the question that arose was whether the role of the Poisson kernel was essential or just incidental.

The answer to this question was given in Fefferman and Stein [FS], where they proved the equivalence of the following four properties. Let $u(x,t)$ be a (complex-valued) harmonic function on $\mathbb{R}^{n+1}_+$.

1. $f = \lim_{t\to 0} u(\cdot, t)$ in the sense of tempered distributions, for some $u \in H^p$.

2. $\sup_{t>0} |f * \phi_t(x)| \in L^p$, where $\phi_t(x) = t^{-n}\phi(t^{-1}x)$, for each function ϕ in the Schwartz class $\mathcal{S}$.

3. $\sup_{t>0} |f * \phi_t(x)| \in L^p$ for one such $\phi \in \mathcal{S}$.

4. $\sup_{|x-y|<t} |u(y,t)| \in L^p$.

The way Fefferman and Stein defined H^p is the same as in [SW1] (see (1.9) and (1.10)) for $(n-1)/n < p < \infty$ and, after Calderón and Zygmund [CZ] and Stein and Weiss [SW2], they were able to extend this definition for any $p > 0$, by considering more general systems of conjugate harmonic functions. It is appropriate to talk about H^p spaces as spaces of distributions for $p < 1$. For $p \geq 1$ it does not make any difference. The equivalence of 1. and 4. is an n-dimensional extention of the theorem of Burkholder, Gundy and Silverstein, which shows that H^p arises naturally as a space of harmonic functions, free from notions of conjugacy. They proved the equivalence of 1. and 4. using the Lusin S-function, defined by

$$(Su)(x) = \left(\int_{\Gamma_\alpha(x)} |\nabla u(x',t)|^2 t^{1-n} dx' dt \right)^{\frac{1}{2}}.$$

In fact they used the following theorem. Let us recall the non-tangential maximal function u^* defined by

$$u^*(x) = \sup_{(x',t)\in\Gamma_\beta(x)} |u(x',t)|.$$

Theorem 1.3.1 *$u^* \in L^p(\mathbb{R}^n)$ if and only if $S(u) \in L^p(\mathbb{R}^n)$ and $u(x,t) \to 0$, as $t \to \infty$. Moreover, $\|u^*\|_p \sim \|S(u)\|_p$, $0 < p < \infty$.*

The direction that 1. implies 4. was already partially known and the generalisation follows the existing method of harmonic majorisation of a certain subharmonic function. For the opposite direction, they construct the system of conjugate harmonic functions and use Theorem 1.3.1. A consequence of the proof is also that $\|u^*\|_p \sim \|u\|_{H^p}$. The equivalence between 1. and 4. and Theorem 1.3.1, imply the following equivalent characterisations of H^p, $0 < p < \infty$ (see also Stein [S1]).

Corollary 1.3.2 *Let $u(x,t)$ be harmonic in $\mathbb{R}^{n+1}_+$. Then $u \in H^p$, $0 < p < \infty$, if and only if $S(u) \in L^p(\mathbb{R}^n)$, and $u(x,t) \to 0$, as $t \to \infty$.*

An alternative characterisation, in terms of the function u^+ defined by $u^+(x) = sup_{t>0}|u(x,t)|$, is the following.

Corollary 1.3.3 *Let $u(x,t)$ be harmonic in $\mathbb{R}^{n+1}_+$. Then $u \in H^p$, $0 < p < \infty$, if and only if $u^+ \in L^p(\mathbb{R}^n)$. Moreover $\|u\|_{H^p} \sim \|u^+\|_p$.*

Finally, there is a characterisation of H^p in terms of the radial analogue of the Lusin S function, the g function defined by

$$g(u)(x) = \left(\int_0^\infty |\nabla u(x,t)|^2 t dt\right)^{\frac{1}{2}}.$$

Corollary 1.3.4 *Let $u(x,t)$ be harmonic in $\mathbb{R}^{n+1}_+$. Then $u \in H^p$, $0 < p < \infty$, if and only if $g(u) \in L^p(\mathbb{R}^n)$, and $u(x,t) \to 0$, as $t \to \infty$.*

These characterisations were considered prior to Fefferman and Stein [FS], by Calderón [Ca2].

The equivalence of 1. and 4. with 2. and 3. brings out the real-variable character of H^p. These properties show that the Poisson integral plays no special role in the definition of H^p spaces, but H^p spaces arise from regularising distributions with approximate identities.

The real-variable nature of H^p is also evident from the atomic decomposition characterisation of H^p, $0 < p \leq 1$. Let us first explain what we mean by the atomic decomposition characterisation of H^p. We shall first define a *p-atom.*

Definition 1.3.5 A *p-atom,* $0 < p \leq 1$, associated to a cube Q is a measurable function a on $\mathbb{R}^n$ satisfying the following three properties:

(P1) $\text{supp}(a) \subseteq Q$,

(P2) $\|a\|_\infty \leq \frac{1}{|Q|^{1/p}}$ where $|\cdot|$ denotes the Lebesgue measure, (P3) $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$, for all multi-indices α of order $|\alpha| \leq [n(1/p-1)]$, the integer part of $n(1/p-1)$.

We can now define the atomic Hardy space H^p_{at} and the H^p_{at}-norm on $\mathbb{R}^n$.

Definition 1.3.6 A distribution f is in $H^p_{at}(\mathbb{R}^n)$, $0 < p \leq 1$, if it can be represented in the form $f = \sum_{i=1}^\infty \lambda_i a_i$ (the atomic decomposition), where each a_i is a p-atom, the convergence is taken in the sense of distributions and $\sum_{i=1}^\infty |\lambda_i|^p < \infty$. The $H^p_{at}(\mathbb{R}^n)$-norm is defined by

$$\|f\|_{H^p_{at}(\mathbb{R}^n)} = \inf\left\{ \left(\sum_{i=1}^\infty |\lambda_i|^p\right)^{1/p} : f = \sum_{i=1}^\infty \lambda_i a_i \right\},$$

where the infimum is taken over all possible atomic decompositions.

In the 1970's after the very important paper of Fefferman and Stein [FS], it was realised that the H^p-spaces admit an atomic decomposition, that is

$$H^p(\mathbb{R}^n) = H^p_{at}(\mathbb{R}^n),$$

and the H^p and H^p_{at} norms are equivalent. For $p = 1$, C. Fefferman showed that this is equivalent to the duality of H^1 and BMO (Theorem 1.2.2). This can be seen by showing that $(H^p_{at}(\mathbb{R}^n))^* = BMO$ and using the Hahn-Banach theorem. For general $0 < p \leq 1$, the atomic decomposition representation was obtained constructively by Coifman [Co] for $n = 1$ and Latter [L] for general n (see also [LU]).

The atomic decomposition characterisation of H^p allows us to define Hardy spaces in more general settings; for instance, on spaces of *homogeneous type*. These are topological spaces endowed with a Borel measure μ and a quasi-metric d. Then Definitions 1.3.5 and 1.3.6 can be used to define the H^p spaces, with the Lebesgue measure $|\cdot|$ substituted by μ and the Euclidean metric substituted by d. An extensive list of spaces of homogeneous type together with definitions of Hardy spaces in these settings can be found in Coifman and Weiss [CW].

Chapter 2

Generalisation of Hardy's inequality under polynomial changes of variables

2.1 Statement of the main result

The classical Hardy's inequality first appeared in 1927 in Hardy and Littlewood [HL2] in the context of the theory of Fourier series. It states that

$$\sum_{r=0}^{\infty} \frac{|\widehat{f}(r)|}{1+r} \leq C\|f\|_{L^1(\mathbb{T})}, \tag{2.1}$$

for $f \in H^1(\mathbb{T})$, where $H^1(\mathbb{T})$ is the classical Hardy space, discussed in Section 1.1. Inequality (2.1) has an analogue for $f \in H^1(\mathbb{R}^n)$;

$$\int_{\mathbb{R}^n} \frac{|\widehat{f}(x)|}{|x|^n} dx \leq C_n \|f\|_{H^1(\mathbb{R}^n)} . \tag{2.2}$$

This has an easy proof using the atomic decomposition characterisation of $H^1(\mathbb{R}^n)$ and Plancherel's theorem. As mentioned in Chapter 1, the Hardy space $H^1(\mathbb{R}^n)$ often serves as a substitute (for $L^1(\mathbb{R}^n)$) endpoint space. For example (2.2) can be regarded as an endpoint inequality for the family of inequalities

$$\int_{\mathbb{R}^n} \frac{|\widehat{f}(x)|^p}{|x|^{(2-p)n}} dx \leq C_p \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p \leq 2, \tag{2.3}$$

and it was actually in this context that (2.1) was first proved. Note that (2.2) (and also (2.1)) is clearly false for general $L^1(\mathbb{R}^n)$ functions f since $\widehat{f}(x)$ may decay very slowly to zero as $|x| \to \infty$.

In this chapter we investigate the behaviour of inequality (2.2) under polynomial changes of variables. To be precise we prove the following theorem.

Theorem 2.1.1 *Let $P : \mathbb{R}^m \to \mathbb{R}^n$ be an arbitrary polynomial mapping of degree* $\mathbf{d} = (d_1, \ldots, d_n)$*, with $P(0) = 0$. Then*

$$\int_{\mathbb{R}^m} \frac{|\widehat{f}(P(x))|}{|x|^m} dx \leq C_{m,n,\mathbf{d}} \|f\|_{H^1(\mathbb{R}^n)}, \tag{2.4}$$

where $C_{m,n,\mathbf{d}}$ only depends on m, n and $\mathbf{d}$, and not on the coefficients of P. Here $P = (P_1, \ldots, P_n)$ where the components P_i are real-valued polynomials on $\mathbb{R}^m$ of degree d_i.

Due to the duality of H^1 and BMO (see Section 1.2), Theorem 2.1.1 implies that

$$b(y) := \int_{\mathbb{R}^m} e^{iP(x)\cdot y} \frac{dx}{|x|^m}$$

is in $BMO(\mathbb{R}^n)$ with a uniformly bounded (independent of the coefficients of P) BMO norm. To be more precise, one should insert an L^∞ normalised $a(x)$ in the definition of b so that the integral converges.

In 1990 Stein and Wainger [SWa] proved using analytic number theory that for a polynomial $P : \mathbf{Z} \to \mathbf{Z}$

$$\sum_{r=-\infty}^{\infty} \frac{|\widehat{f}(P(r))|}{1+|r|} \leq C \|f\|_{H^1(\mathbb{T})}. \tag{2.5}$$

In fact they proved the equivalent statement that

$$\sum_{r=-\infty}^{\infty} \frac{e^{iP(r)\theta}}{1+|r|} \in BMO(\mathbb{T}).$$

Furthermore they proved similar results for certain polynomials on $\mathbf{Z}^n$ leading to functions being in $BMO(\mathbb{T}^n)$. Our Theorem 2.1.1 considers more general polynomial mappings but in the much easier setting of Euclidean spaces. Due to the continuous nature of our theorem, our method of proof is not related to the method of proof in the discreet case.

Inequality (2.4) can be regarded as a restriction inequality for the Fourier transform of H^1 functions. As an example, consider the mapping $P(t) = (t, t^2)$. Then (2.4) takes the form

$$\int_{-\infty}^{\infty} \frac{|\widehat{f}(t,t^2)|}{|t|} dt \leq C \|f\|_{H^1(\mathbb{R}^2)}, \tag{2.6}$$

which can be regarded as a global restriction theorem to the parabola for functions in $H^1(\mathbb{R}^2)$. In fact to prove (2.4), we use a sharp "global" L^2-restriction theorem for polynomial curves, which is proved in Section 2.8. Sharp L^2-restriction estimates will play the role of Plancherel's theorem in the standard proof of (2.2) via the atomic decomposition. To illustrate this we give a quick proof of the easier analogue of (2.6) where $H^1(\mathbb{R}^2)$ is replaced by the parabolic Hardy space $H^1_{par}(\mathbb{R}^2)$, defined with respect to parabolic dilations,

$$\delta \circ x = (\delta x_1, \delta^2 x_2).$$

Using the atomic decomposition of $H^1_{par}(\mathbb{R}^2)$, it suffices to prove the bound

$$\int_{-\infty}^{\infty} \frac{|\widehat{a}_Q(t,t^2)|}{|t|} dt \leq C, \tag{2.7}$$

uniformly for all atoms a_Q which are defined with respect to a "parabolic cube" Q with dimensions $r \times r^2$, say. That is

$$\text{supp}\,(a_Q) \subseteq Q, \quad \int_{\mathbb{R}^2} a_Q = 0 \quad \text{and} \quad \|a_Q\|_\infty \leq \frac{1}{|Q|}. \tag{2.8}$$

By translation-invariance we may assume that Q is centred at the origin. To prove (2.7), we first split the integration by

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{|\widehat{a}_Q(t,t^2)|}{|t|} dt &= \int_{|t|\leq 1/r} \frac{|\widehat{a}_Q(t,t^2)|}{|t|} dt + \int_{|t|\geq 1/r} \frac{|\widehat{a}_Q(t,t^2)|}{|t|} dt \\ &= I + II. \end{aligned}$$

Using (2.8) we have the pointwise estimate

$$\begin{aligned} |\widehat{a}_Q(t,t^2)| &= \left| \int_{\mathbb{R}} \int_{\mathbb{R}} a_Q(x,y)(e^{i(xt+yt^2)} - 1) dx dy \right| \\ &\leq \int\int_Q |a_Q(x,y)|(|xt| + |yt^2|) dx dy \\ &\leq r|t| + r^2|t^2| \end{aligned}$$

and hence

$$I = \int_{|t|\leq 1/r} \frac{|\widehat{a}_Q(t,t^2)|}{|t|} dt \leq \int_{|t|\leq 1/r} [r|t| + (r|t|)^2] \frac{dt}{|t|} \leq C.$$

For II, using the Cauchy-Schwarz inequality we see that

$$\begin{aligned} \int_{|t|\geq 1/r} \frac{|\widehat{a}_Q(t,t^2)|}{|t|} dt &\leq \int_{\mathbb{R}} \left(|\widehat{a}_Q(t,t^2)|^2 dt\right)^{1/2} \left(\int_{|t|\geq 1/r} \frac{1}{|t|^2} dt \right)^{1/2} \\ &\leq C \|a_Q\|_{L^{6/5}(\mathbb{R}^2)} r^{1/2} \\ &\leq C \frac{1}{|Q|^{1/6}} r^{1/2} \leq C, \end{aligned}$$

concluding the proof of (2.7). The second inequality uses the well-known sharp L^2-restriction theorem for the parabola due to C. Fefferman and Stein (see [F1]), and the penultimate inequality uses (2.8).

Of course Theorem 2.1.1 shows that (2.6) holds for isotropic $H^1(\mathbb{R}^2)$ where the cubes which arise in the atomic decomposition are standard dilates $(r \times r)$ of the unit cube. Again matters are reduced to proving (2.7), uniformly for isotropic atoms. The proof is slightly more involved because of the mixed homogeneities; isotropic, $\delta \circ x = (\delta x_1, \delta x_2)$, versus parabolic, $\delta \circ x = (\delta x_1, \delta^2 x_2)$. In general an arbitrary polynomial map $P : \mathbb{R}^m \to \mathbb{R}^n$ will have many competing homogeneities and the main difficulty will be to separate the various homogeneities. In this sense Theorem 2.1.1 is reminiscent of Stein's result that $\log|P(x)|$ is in $BMO(\mathbb{R}^n)$ for any real-valued polynomial P on $\mathbb{R}^n$, see [S2].

In Section 2.2 we will outline the strategy for the proof of Theorem 2.1.1, establishing the basic reductions. In Section 2.3 we prove a few lemmas on the behaviour of polynomials of a single variable. In Section 2.4 we prove the one dimensional version of (2.4) which we will need for the induction argument of Section 2.6. Section 2.6 contains the main body of the proof of Theorem 2.1.1 which will depend on two further estimates. The first of these is proved in Section 2.7 and the second is the restriction inequality alluded to above and is proved in Section 2.8.

Notation: For the rest of this thesis we denote by $\beta \lesssim \gamma$ or $\beta = O(\gamma)$ that there exists a constant $C = C_{m,n,\mathbf{d}}$ only depending on the degree $\mathbf{d}$ and the dimensions m, n, such that $|\beta| \leq C|\gamma|$. Let $\beta \sim \gamma$ mean that $\beta \lesssim \gamma \lesssim \beta$. Also, when we say that A is sufficiently large, we mean that there exists a constant $K(\mathbf{d})$ only depending on the degree such that $A > K(\mathbf{d})$.

2.2 Preliminary reductions

We shall be using the atomic decomposition characterisation of $H^1(\mathbb{R}^n)$, see Section 1.3. Let us recall the definition of a 1-atom.

Definition 2.2.1 A 1-*atom* (or *atom*) associated to a cube Q is a function a on $\mathbb{R}^n$ satisfying the following three properties:

(P1) $\operatorname{supp}(a) \subseteq Q$,

(P2) $\|a\|_\infty \leq \frac{1}{|Q|}$ where $|\cdot|$ denotes the Lebesgue measure,

(P3) $\int_{\mathbb{R}^n} a = 0$.

Let us also recall the definition of the atomic space H^1, according to Definition 1.3.6.

Definition 2.2.2

$$H^1_{at}(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : f = \sum_{i=1}^{\infty} \lambda_i a_i, \sum_{i=1}^{\infty} |\lambda_i| < \infty\}$$

and

$$\|f\|_{H^1_{at}(\mathbb{R}^n)} = \inf\{\sum_{i=1}^{\infty} |\lambda_i| : f = \sum_{i=1}^{\infty} \lambda_i a_i\},$$

where the infimum is taken over all possible atomic decompositions of f.

In Section 1.3, we observed that $H^1(\mathbb{R}^n) \cong H^1_{at}(\mathbb{R}^n)$.

It is straightforward to see that the proof of (2.4) reduces to proving

$$\int_{R^m} \frac{|\widehat{a}(P(x))|}{|x|^m} dx \leq C_{\mathbf{d}} \, , \tag{2.9}$$

for an arbitrary atom a with $C_{\mathbf{d}}$ depending only on the degree of P.

Now let $P : \mathbb{R}^m \to \mathbb{R}^n$ be a polynomial of degree $\mathbf{d} = (d_1, \dots, d_n)$, and such that $P(0) = 0$. Then, by using polar coordinates $x = |x|x' = rx'$, we write

$$\begin{aligned} P(x) &= \left(\sum_{1 \leq |\alpha| \leq d_1} \lambda_\alpha^{(1)} x^\alpha, \dots, \sum_{1 \leq |\alpha| \leq d_n} \lambda_\alpha^{(n)} x^\alpha \right) \\ &= \left(\sum_{k=1}^{d_1} r^k \sum_{|\alpha|=k} \lambda_\alpha^{(1)} x'^\alpha, \dots, \sum_{k=1}^{d_n} r^k \sum_{|\alpha|=k} \lambda_\alpha^{(n)} x'^\alpha \right) . \end{aligned}$$

Then, writing

$$b_k^{(j)}(x') = \sum_{|\alpha|=k} \lambda_\alpha^{(j)} x'^\alpha \, ,$$

we have

$$\int_{\mathbb{R}^m} \frac{|\widehat{a}(P(x))|}{|x|^m} dx = \int_{S^{m-1}} \int_{r=0}^{\infty} \frac{1}{r} \left| \widehat{a} \left(\sum_{k=1}^{d_1} b_k^{(1)}(x') r^k, \dots, \sum_{k=1}^{d_n} b_k^{(n)}(x') r^k \right) \right| dr dx' .$$

This implies that if we knew (2.9) for $P : \mathbb{R}^+ \to \mathbb{R}^n$ with C not depending on the coefficients, then (2.9) holds for general $P : \mathbb{R}^m \to \mathbb{R}^n$. Thus proving (2.9) reduces to proving

$$\int_0^{\infty} \frac{|\widehat{a}(P(t))|}{t} dt \leq C_{\mathbf{d}} \, , \tag{2.10}$$

uniformly for all atoms a, with $C_{\mathbf{d}}$ only depending on $\mathbf{d}$.

For the rest of the thesis we concentrate on proving (2.10). By using Lemmas 2.3.1 and 2.3.3 below, we will be able to restrict the integration in (2.10) to an interval on which each polynomial "looks" like a monomial. We then use a procedure, similar to the one in the introduction, of splitting the integration near and away from the origin 0. For the part near the origin we use an induction argument which is carried out in Section 2.4. For the part away from the origin we use the Cauchy-Schwarz inequality together with an L^2-restriction theorem for polynomial curves. In order to use the restriction theorem though we must have that all the monomials that the polynomials look like are distinct. To make sure that we are in this situation, we use the fact that the integral in (2.10) stays invariant under rotations. That is, we may replace $P(t)$ with $RP(t)$ where R is a rotation. Actually considering only rotations is sufficient only in 2 dimensions. For higher dimensions we will need to consider a more general class of transformations which we will call "almost" rotations. In Section 2.5, a procedure will be described, using "almost" rotations, which will allow us to reduce ourselves to the situation where P behaves like various distinct monomials, putting us in a position to employ our restriction theorem for polynomial curves. The proof of the restriction result will be carried out in Section 2.8.

2.3 Analysis of polynomials of a single variable

In this section we concentrate on the analysis of the behaviour of polynomials of a single variable. To prove (2.10) we require a lemma which describes the splitting of the domain of integration into a number of intervals, some of which we call *gaps* and others *dyadic intervals*. This will be explained later in this section. We start by quoting Lemma 2.5 of [CRW] and give its proof for completeness. We then prove a generalisation. After we have established this we will proceed to a number of results that will be needed in Section 2.7.

Lemma 2.3.1 *Let $t_1, \ldots, t_d$ be the complex roots of a polynomial*

$$R(t) = \sum_{m=0}^{d} r_m t^m = r_d \prod_{m=1}^{d} (t - t_m)$$

of degree d, ordered so that $|t_1| \leq |t_2| \leq \ldots \leq |t_d|$. Then there exist positive constants $K(d)$ and $\epsilon(d)$ such that if $A > K(d)$ and t satisfies $A|t_k| < t < A^{-1}|t_{k+1}|$, for some $0 \leq k \leq d$ (let $t_0 = 0$ and $t_{d+1} = \infty$), then

a) $R(t) \sim r_k t^k$,

b) $\left|\frac{R'(t)}{R(t)}\right| \geq \frac{\epsilon(d)}{t}$ for $k \geq 1$,

c) $|R(t)|$ is strictly increasing on $[A|t_k|, A^{-1}|t_{k+1}|]$.

REMARK. Strictly speaking the lemma in [CRW] only shows that $R(t) \sim c_k t^k$, where $c_k = r_d t_{k+1} \ldots t_d$. However it was shown in [FW] that $r_k \sim r_d t_{k+1} \ldots t_d$ if $K(d)$ is large enough.

PROOF. To prove part *a)* we write $R(t) = r_d \prod_{m=1}^{d}(t - t_m)$. Since $A|t_k| < t < A^{-1}|t_{k+1}|$,

$$\left(1 - \frac{1}{A}\right)|t| \leq |t - t_m| \leq \left(1 + \frac{1}{A}\right)|t|$$

for $1 \leq m \leq k$, and

$$\left(1 - \frac{1}{A}\right)|t_m| \leq |t - t_m| \leq \left(1 + \frac{1}{A}\right)|t_m|$$

for $k + 1 \leq m \leq d$. In short, we have the following two relations that are going to be used extensively,

$$t - t_m \sim t \quad \text{for} \quad 1 \leq m \leq k \tag{2.11}$$

and

$$t - t_m \sim t_m \quad \text{for} \quad k + 1 \leq m \leq d. \tag{2.12}$$

Substituting these inequalities in the expansion for $R(t)$ in terms of its roots shows that on the interval $[A|t_k|, A^{-1}|t_{k+1}|]$, $R(t) \sim c_k t^k$ where $c_k = r_d t_{k+1} \ldots t_d$. This together with the remark after the statement of Lemma 2.3.1 proves part *a)*.

For part *b)*, first observe that

$$\frac{R'(t)}{R(t)} = \sum_{m=1}^{n} \frac{1}{t - t_m}, \tag{2.13}$$

so

$$\left|\frac{R'(t)}{R(t)}\right| \geq \left|\sum_{m=1}^{k} \frac{1}{t - t_m}\right| - \sum_{m=k+1}^{n} \frac{1}{|t - t_m|} \geq \left|\sum_{m=1}^{k} \frac{1}{t - t_m}\right| - \frac{n - k}{(A - 1)t}, \tag{2.14}$$

since $|t_m| \geq At$ if $m \geq k + 1$ and $A|t_k| < t < A^{-1}|t_{k+1}|$. For $m \leq k$, consider

$$\text{Re}\frac{1}{t - t_m} = \frac{t - \text{Re}\, t_m}{|t - t_m|^2} > \frac{\left(1 - \frac{1}{A}\right)t}{\left(1 + \frac{1}{A}\right)^2 t^2} = \frac{\left(1 - \frac{1}{A}\right)}{\left(1 + \frac{1}{A}\right)^2}\frac{1}{t},$$

since $t > A|t_m|$. Therefore

$$\left|\frac{R'(t)}{R(t)}\right| \geq \left(k\frac{\left(1 - \frac{1}{A}\right)}{\left(1 + \frac{1}{A}\right)^2} - \frac{n - k}{A - 1}\right)\frac{1}{t}.$$

If A is sufficiently large, the coefficient of $1/t$ is positive, which implies that $|tR'(t)/R(t)|$ is bounded below by an absolute constant. That proves part *b)*.

To prove part *c)* we notice that we have in fact shown that

$$\frac{R'(t)}{R(t)} = \text{Re}\frac{R'(t)}{R(t)} > 0.$$

That is $\log|R(t)|$ is increasing on $[A|t_k|, A^{-1}|t_{k+1}|]$, which implies that $|R(t)|$ is increasing on $[A|t_k|, A^{-1}|t_{k+1}|]$. This completes the proof of Lemma 2.3.1.

We shall pause now and consider some of the consequences of Lemma 2.3.1. For a polynomial whose roots are ordered by $|t_1| \leq |t_2| \leq \ldots |t_d|$ we consider a *dyadic interval* $[A^{-1}|t_k|, A|t_k|]$ associated to each root t_k, whose logarithmic measure is bounded above by $2\log A$. These intervals are harmless for our problem since on them we can just use the trivial bound $\|\widehat{a}\|_\infty \leq \|a\|_1 \leq 1$, giving a contribution to (2.10) which is $\lesssim 1$. In what follows we denote these dyadic intervals by D_k, $1 \leq k \leq d$. The complement of the union of the dyadic intervals is a disjoint union of possibly very long intervals which we call *gaps*. It is on the gaps that we focus our attention. According to Lemma 2.3.1, on the gaps the polynomial "looks" like a monomial and in particular if there is a gap between $|t_1|$ and $|t_2|$ it looks like t, if there is a gap between $|t_2|$ and $|t_3|$ it looks like t^2 and so on. Of course some roots might not be seperated enough to guarantee the existence of a gap "between" the roots. The significance of a polynomial looking like a monomial is that if we can prove something for a monomial, we can hope we can mimic the proof on a gap and prove the same for a polynomial. In what follows we denote the gaps, $[A|t_k|, A^{-1}|t_{k+1}|]$ by G_k, $0 \leq k \leq d$. The number of gaps is bounded by a number which only depends on the degree of the polynomial.

Part *b)* of Lemma 2.3.1 says that on the interval $[A|t_k|, A^{-1}|t_{k+1}|]$, the first derivative of the polynomial behaves like that of a monomial (it is one power lower). We extend this to certain higher derivatives. To accomplish this we will need the following formula.

Lemma 2.3.2 *Let $R(t)$ be a polynomial of degree d and let $t_1, \ldots, t_d$ be its complex roots. Then for any $r \geq 1$,*

$$\frac{R^{(r)}}{R}(t) = \sum_{1\leq l_1 \neq \ldots \neq l_r \leq d} \prod_{i=1}^{r} \frac{1}{t - t_{l_i}}. \tag{2.15}$$

PROOF. The proof of Lemma 2.3.2 is by induction on r. The statement for $r = 1$ is equation (2.13). We now assume that (2.15) holds for $r = m - 1$ and we turn to showing (2.15) for $r = m$. We have

$$\begin{aligned} \left(\frac{R^{(m-1)}}{R}\right)' &= \frac{RR^{(m)} - R^{(m-1)}R'}{R^2} \\ &= \frac{R^{(m)}}{R} - \frac{R^{(m-1)}}{R}\frac{R'}{R} \end{aligned}$$

which implies that

$$\frac{R^{(m)}}{R} = \left(\frac{R^{(m-1)}}{R}\right)' + \frac{R^{(m-1)}}{R}\frac{R'}{R}.$$

We now use the induction hypothesis to obtain

$$
\begin{aligned}
\frac{R^{(m)}}{R}(t) &= \left(\sum_{1\le l_1\neq\ldots\neq l_{m-1}\le d}\prod_{i=1}^{m-1}\frac{1}{t-t_{l_i}}\right)' \\
&\quad + \left(\sum_{1\le l_1\neq\ldots\neq l_{m-1}\le d}\prod_{i=1}^{m-1}\frac{1}{t-t_{l_i}}\right)\sum_{l=1}^{d}\frac{1}{t-t_l} \\
&= \left(\sum_{1\le l_1\neq\ldots\neq l_{m-1}\le d}\prod_{i=1}^{m-1}\frac{1}{t-t_{l_i}}\right)\sum_{l=1}^{d}\frac{1}{t-t_l} \\
&\quad - \sum_{1\le l_1\neq\ldots\neq l_{m-1}\le d}\sum_{q=1}^{m-1}\prod_{i=1}^{q-1}\left(\frac{1}{t-t_{l_i}}\right)\frac{1}{(t-t_{l_q})^2}\prod_{i=q+1}^{m-1}\left(\frac{1}{t-t_{l_i}}\right) \\
&= \sum_{1\le l_1\neq\ldots\neq l_{m-1}\le d}\left(\sum_{l=1}^{d}\frac{1}{t-t_l}\prod_{i=1}^{m-1}\frac{1}{t-t_{l_i}}\right. \\
&\quad \left. -\sum_{q=1}^{m-1}\prod_{i=1}^{q-1}\left(\frac{1}{t-t_{l_i}}\right)\frac{1}{(t-t_{l_q})^2}\prod_{i=q+1}^{m-1}\left(\frac{1}{t-t_{l_i}}\right)\right) \\
&= \sum_{1\le l_1\neq\ldots\neq l_m\le d}\prod_{i=1}^{m}\frac{1}{t-t_{l_i}},
\end{aligned}
$$

which concludes the proof of Lemma 2.3.2.

We are now in a position to extend part *b)* of Lemma 2.3.1 to higher derivatives.

Lemma 2.3.3 *Using the notation of Lemma 2.3.1, there exist constants $\epsilon_1(d)$ and $\epsilon_2(d)$ such that if t satisfies $A|t_k| < t < A^{-1}|t_{k+1}|$, for A sufficiently large and some $0 \le k \le d$, then for any $0 \le r \le k$,*

$$
\frac{\epsilon_1(d)}{t^r} \ge \left|\frac{R^{(r)}(t)}{R(t)}\right| \ge \frac{\epsilon_2(d)}{t^r}.
$$

PROOF. The upper bound follows immediately from (2.15), (2.11) and (2.12), in fact for $0 \le r \le d$. Thus we concentrate on the lower bound. We use (2.15) from Lemma 2.3.2 to write

$$
\frac{R^{(r)}}{R}(t) = \sum_{1\le l_1\neq\ldots\neq l_r\le d}\prod_{i=1}^{r}\frac{1}{t-t_{l_i}}.
$$

By the triangle inequality we have

$$
\begin{aligned}
\left|\frac{R^{(r)}}{R}(t)\right| &\ge \left|\sum_{1\le l_1\neq\ldots\neq l_r\le k}\prod_{i=1}^{r}\frac{1}{t-t_{l_i}}\right| - \sum_{q=1}^{r}\left|\sum_{\substack{1\le l_1\neq\ldots\neq l_r\le d\\ k+1\le l_q\le d}}\prod_{i=1}^{r}\frac{1}{t-t_{l_i}}\right| \\
&=: \qquad \mathrm{I} \qquad - \qquad \sum_{q=1}^{r}\mathrm{II}_q \;.
\end{aligned}
$$

For the II_q's we use the same argument as in (2.14) to bound each term from above by $\mathrm{O}(A^{-1}t^{-r})$, since for each q the corresponding l_q satisfies $k+1 \leq l_q \leq d$. For I we have

$$\begin{aligned}
\mathrm{I} &= \left| \sum_{1 \leq l_1 \neq \ldots \neq l_r \leq k} \prod_{i=1}^{r} \frac{1}{t - t_{l_i}} \right| \\
&\geq \mathrm{Re} \left(\sum_{1 \leq l_1 \neq \ldots \neq l_r \leq k} \prod_{i=1}^{r} \frac{t - \bar{t}_{l_i}}{|t - t_{l_i}|^2} \right) \\
&= \sum_{1 \leq l_1 \neq \ldots \neq l_r \leq k} \frac{t^r \pm \mathrm{Re}(\sum_{i=1}^{r} \bar{t}_{l_i}) t^{r-1} \pm \ldots \pm \mathrm{Re} \prod_{i=1}^{r} \bar{t}_{l_i}}{\prod_{i=1}^{r} |t - t_{l_i}|^2}.
\end{aligned}$$

We note that unless $r \leq k$ the sum in I is empty. Hence

$$\begin{aligned}
\left| \frac{R^{(r)}}{R}(t) \right| &\geq \sum_{1 \leq l_1 \neq \ldots \neq l_r \leq k} \frac{t^r \pm \mathrm{Re}(\sum_{i=1}^{r} \bar{t}_{l_i}) t^{r-1} \pm \ldots \pm \mathrm{Re} \prod_{i=1}^{r} \bar{t}_{l_i}}{\prod_{i=1}^{r} |t - t_{l_i}|^2} - \mathrm{O}(A^{-1}t^{-r}) \\
&\gtrsim \frac{1}{t^r},
\end{aligned}$$

since for each $l_i \leq k$, $|t_{l_i}| \leq A^{-1}t$. This ends the proof of Lemma 2.3.3.

We formally record the estimate derived near the end of the above proof in the following lemma.

Lemma 2.3.4 *Let $\alpha \in \mathrm{N}$, $\alpha = \mathrm{O}(1)$ and L any index set such that $\sharp(L) = \mathrm{O}(1)$. Consider any arbitrary set of complex numbers $\{t_{l,i}\}_{\substack{1 \leq i \leq \alpha \\ l \in L}}$ satisfying $|t_{l,i}| \leq A^{-1}t$ for some $t > 0$. Then, for sufficiently large A,*

$$\sum_{l \in L} \prod_{i=1}^{\alpha} \frac{1}{t - t_{l,i}} \sim \frac{1}{t^\alpha}.$$

PROOF. The bounds from above are trivial and so we concentrate on the lower bounds.

$$\begin{aligned}
\left| \sum_{l \in L} \prod_{i=1}^{\alpha} \frac{1}{t - t_{l,i}} \right| &\geq \mathrm{Re} \sum_{l \in L} \prod_{i=1}^{\alpha} \frac{1}{t - t_{l,i}} \geq \sum_{l \in L} \mathrm{Re} \prod_{i=1}^{\alpha} \frac{1}{t - t_{l,i}} \\
&= \sum_{l \in L} \mathrm{Re} \prod_{i=1}^{\alpha} \frac{t - \bar{t}_{l,i}}{|t - t_{l,i}|^2} \\
&\geq C \sum_{l \in L} \frac{1}{t^{2\alpha}} \mathrm{Re} \left[t^\alpha - \left(\sum_{i=1}^{\alpha} \bar{t}_{l,i} \right) t^{\alpha-1} + \ldots + (-1)^\alpha \prod_{i=1}^{\alpha} \bar{t}_{l,i} \right],
\end{aligned}$$

with C an absolute constant, since $|t - t_{l,i}| \leq 2t$ for sufficiently large A and all i,l. Finally, again since each $|t_{l,i}| \leq A^{-1}t$, the last expression is greater than

$$C \sum_{l \in L} \frac{1}{t^{2\alpha}} \left(t^\alpha - \frac{D}{A} t^\alpha \right) \gtrsim \frac{1}{t^\alpha},$$

where D is an absolute constant, thus completing the proof of Lemma 2.3.4.

The last lemma of this section is about the difference of two α-fold products as considered in Lemma 2.3.4.

Lemma 2.3.5 *Let* $\{t_{l,i}\}_{\substack{1\le i\le\alpha \\ l\in L}}$ *and* $t > 0$ *be as in Lemma 2.3.4, but with* $L = \{1,2\}$. *Then, for* A *sufficiently large,*

$$\prod_{i=1}^{\alpha}\frac{1}{t-t_{1,i}} - \prod_{i=1}^{\alpha}\frac{1}{t-t_{2,i}} = \mathrm{O}\left(\frac{1}{At^{\alpha}}\right).$$

PROOF.

$$\begin{aligned}
&\left|\prod_{i=1}^{\alpha}\frac{1}{t-t_{1,i}} - \prod_{i=1}^{\alpha}\frac{1}{t-t_{2,i}}\right| \\
= &\left|\frac{\prod_{i=1}^{\alpha}(t-t_{2,i}) - \prod_{i=1}^{\alpha}(t-t_{1,i})}{\prod_{i=1}^{\alpha}(t-t_{1,i})\prod_{i=1}^{\alpha}(t-t_{2,i})}\right| \\
\le &\frac{\left|\left(\sum_{i=1}^{\alpha}t_{1,i} - \sum_{i=1}^{\alpha}t_{2,i}\right)t^{\alpha-1} + \ldots + (-1)^{\alpha}\left(\prod_{i=1}^{\alpha}t_{2,i} - \prod_{i=1}^{\alpha}t_{1,i}\right)\right|}{\left(1-\frac{1}{A}\right)t^{2\alpha}} \\
\le &\frac{\frac{1}{A}t^{\alpha}}{\left(1-\frac{1}{A}\right)t^{2\alpha}} \le \frac{C}{At^{\alpha}},
\end{aligned}$$

for C an absolute constant and for sufficiently large A. This completes the proof of Lemma 2.3.5.

2.4 An inductive step

In this section we prove (2.10) for $n = 1$ and an inductive step which is needed in the induction argument described in Section 2.6.

Theorem 2.4.1 *Let* $P : \mathbb{R}^+ \to \mathbb{R}$ *be a polynomial of degree* d, *with* $P(0) = 0$. *Then*

$$\int_0^{\infty}\frac{|\widehat{a}(P(r))|}{r}dr \le C_d\,, \tag{2.16}$$

uniformly for all atoms a *on* $\mathbb{R}$.

PROOF. We apply Lemma 2.3.1 to the polynomial P. According to the discussion after the proof of Lemma 2.3.1, $\mathbb{R}^+$ can be decomposed into O(1) gaps $\{G_k\}$ and dyadic intervals $\{D_k\}$ with respect to P and so

$$\int_0^{\infty}\frac{|\widehat{a}(P(r))|}{r}dr = \sum_k\int_{D_k}\frac{|\widehat{a}(P(r))|}{r}dr + \sum_k\int_{G_k}\frac{|\widehat{a}(P(r))|}{r}dr\,.$$

On the dyadic intervals D_k we use $\|\widehat{a}\|_{\infty} \le 1$ to obtain

$$\int_{D_k}\frac{|\widehat{a}(P(r))|}{r}dr \le \int_{a_k}^{b_k}\frac{1}{r}dr \le \log A(d)\,. \tag{2.17}$$

By part *c)* of Lemma 2.3.1 we can make the change of variables $u = P(r)$ on the gaps G_k to obtain

$$\begin{aligned}\int_{G_k} \frac{|\widehat{a}(P(r))|}{r} dr &= \int_{P(G_k)} \frac{|\widehat{a}(u)|}{r} \frac{du}{|P'(r)|} = \int_{P(G_k)} \frac{|\widehat{a}(u)|}{|u|} \left| \frac{P(r)}{rP'(r)} \right| du \\ &\leq \epsilon(d) \int_0^\infty \frac{|\widehat{a}(u)|}{u} du \leq C\epsilon(d) , \end{aligned} \tag{2.18}$$

where we have used part *b)* of Lemma 2.3.1, together with the one-dimensional classical Hardy's inequality (2.2). So combining (2.17) and (2.18) we obtain (2.16).

We now show how inequality (2.10) for $P(t) = (P_1(t), \dots, P_n(t))$ implies inequality (2.10) for $P(t) = (P_1(t), \dots, P_n(t), 0)$ in one higher dimension. This observation will be needed for the induction argument in the main proof of (2.10).

Lemma 2.4.2 *Let $P(t) = (P_1(t), \dots, P_n(t))$ be a polynomial curve in $\mathbb{R}^n$ with degree $\mathbf{d} = (d_1, \dots d_n)$. Suppose that*

$$\int_0^\infty \frac{|\widehat{a}(P_1(t), \dots, P_n(t))|}{t} dt \lesssim 1$$

holds uniformly for all atoms a on $\mathbb{R}^n$. Then

$$\int_0^\infty \frac{|\widehat{a}(P_1(t), \dots, P_n(t), 0)|}{t} dt \lesssim 1$$

holds uniformly for all atoms a on $\mathbb{R}^{n+1}$.

PROOF. Let a be an atom on $\mathbb{R}^{n+1}$. With $x = (x_1, \dots, x_n)$, we write

$$\begin{aligned} &\int_0^\infty \frac{|\widehat{a}(P_1(t), \dots, P_n(t), 0)|}{t} dt \\ &= \int_0^\infty \left| \int_{\mathbb{R}^{n+1}} a(x, x_{n+1}) e^{-2\pi i x \cdot (P_1(t), \dots, P_n(t))} dx dx_{n+1} \right| \frac{dt}{t} \\ &= \int_0^\infty \left| \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} a(x, x_{n+1}) dx_{n+1} \right) e^{-2\pi i x \cdot (P_1(t), \dots, P_n(t))} dx \right| \frac{dt}{t} \\ &= \int_0^\infty |\widehat{b}(P_1(t), \dots, P_n(t)| \frac{dt}{t} \end{aligned}$$

where $b(x) = \int_{\mathbb{R}} a(x, x_{n+1}) dx_{n+1}$. Hence it suffices to show that $b(x)$ is an atom on $\mathbb{R}^n$. That is b has to satisfy properties (P1), (P2) and (P3) of Definition 2.2.1 with respect to some cube in $\mathbb{R}^n$. Suppose the cube associated to a is $Q = Q' \times [C, D]$, where Q and Q' are $n+1$ and n-dimensional cubes respectively. Then $b(x)$ is supported on the projection of the $(n+1)$-dimensional cube Q which is the n-dimensional cube Q'. For the L^∞ norm property we have

$$\begin{aligned}
||b||_\infty = \left\|\int_{\mathbb{R}} a(\cdot, x_{n+1})dx_{n+1}\right\|_\infty &= \sup_{x\in\mathbb{R}^n}\left|\int_{\mathbb{R}} a(x, x_{n+1})dx_{n+1}\right| \\
&\leq \sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}} |a(x, x_{n+1})|dx_{n+1} \\
&\leq \int_C^D \frac{1}{|Q|}dx_{n+1} \\
&= \frac{1}{|Q'|}.
\end{aligned}$$

Also the cancellation property for b is satisfied since

$$\int_{\mathbb{R}^n} b(x)dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}} a(x, x_{n+1})dx_{n+1}dx = \int_{\mathbb{R}^{n+1}} a(x, x_{n+1})dxdx_{n+1} = 0.$$

So $b(x)$ is indeed an n-dimensional atom and the lemma is proved.

2.5 Reduction to distinct monomials

Let the mapping $P : \mathbb{R}^+ \to \mathbb{R}^n$ in (2.10) be given by $P(t) = (P_1(t), \dots, P_n(t))$. Then for each P_i we have a corresponding splitting of $\mathbb{R}^+$ into gaps and dyadic intervals (following the discussion after Lemma 2.3.1). We then take S to be the union of the n-fold intersections of the various gaps corresponding to the P_i's, which is a union of disjoint intervals and whose complement S^c is an O(1) union of dyadic intervals. We then split the domain of integration in (2.10) into S and S^c. On S^c we use the fact that $||\widehat{a}||_\infty \leq ||a||_1 \leq 1$ to obtain

$$\int_{S^c} \frac{|\widehat{a}(P(t))|}{t}dt \leq \int_{S^c} \frac{1}{t}dt \lesssim 1 .$$

For S we have

$$\int_S \frac{|\widehat{a}(P(t))|}{t}dt = \sum_\alpha \int_{I_\alpha} \frac{|\widehat{a}(P(t))|}{t}dt$$

where the number of intervals I_α is bounded by a constant $C_{\mathbf{d}}$, only depending on $\mathbf{d}$. Hence it suffices to obtain a bound

$$\int_{I_\alpha} \frac{|\widehat{a}(P(t)|}{t}dt \lesssim 1 \tag{2.19}$$

for each α where on I_α the components of $P(t) = (P_1(t), \dots, P_n(t))$ look like various monomials according to Lemma 2.3.1. Specifically if $P_i(t) = \sum_{m=1}^{d_i} p_{i,m}t^m$, then on I_α,

$$P_i(t) \sim p_{i,j_i}t^{j_i},$$

for some j_i. The main ingredient in the proof of (2.19) is to employ a sharp restriction theorem for polynomial curves on an interval where the components behave like monomials; however this theorem requires the monomials to be distinct. This is resolved in this section by using a sequence of "almost" rotations which will transform the polynomials in such a way as to guarantee that the j_i's are distinct. Furthermore, when an atom is transformed under an "almost" rotation, the properties (P1), (P2), (P3) defining an atom are still essentially satisfied.

Definition 2.5.1 A linear transformation on $\mathbb{R}^n$ given by an $n \times n$ matrix $A = (a_{i,j})$ is called an *almost* rotation if $\det A \sim [\max |a_{i,j}|]^n$.

For an almost rotation A and an arbitrary atom a, we set $a_A(x) = \det A\, a(A^T x)$ and observe that properties (P1), (P2), (P3) are essentially satisfied by a_A.

For the cancellation property (P3), clearly $\int_{\mathbb{R}^n} a_A = 0$. Let Q be the cube associated to a (by translation-invariance we may assume that Q is centred at the origin) and Q' be the smallest cube containing $(A^{-1})^T Q$. We note that

$$\text{supp } a_A \subseteq (A^{-1})^T Q \subseteq Q'.$$

Furthermore we claim that

$$\|a_A\|_\infty \lesssim \frac{1}{|Q'|}$$

which is essentially the L^∞ property (P2). This follows from

$$\|a_A\|_\infty \leq \det A \|a\|_\infty \leq \frac{\det A}{|Q|} \lesssim \frac{1}{|Q'|}$$

which holds if

$$|Q'| \det A \lesssim |Q|. \tag{2.20}$$

The sidelength of Q' is essentially equal to the sidelength of Q times $\|A^{-1}\|$ since

$$L' \sim \sup_{v \in Q} |(A^{-1})^T v| \sim \|A^{-1}\| L$$

where L' is the sidelength of Q' and L is the sidelength of Q. Hence (2.20) will be satisfied if

$$\det A \|A^{-1}\|^n \lesssim 1 \tag{2.21}$$

holds for any almost rotation A. Let $m = \max |a_{i,j}|$ where $A = (a_{i,j})$ and note that (2.21) can be rewritten as

$$\|A^{-1}\| \lesssim m^{-1}.$$

However

$$A^{-1} = \frac{1}{\det A} \text{Adj}(A)$$

where the classical adjoint matrix $\mathrm{Adj}(A) = ((-1)^{i+j}\det(A_{j,i}))$ is defined with respect to the cofactors of A, $(-1)^{i+j}\det(A_{i,j})$, where $A_{i,j}$ is the $(n-1)\times(n-1)$ matrix formed by deleting the ith row and the jth column of A. For instance

$$\det(A_{1,1}) = \sum_{\pi} (-1)^{\mathrm{sgn}\,\pi} a_{2,\pi(2)} \cdots a_{n,\pi(n)}$$

where the sum is taken over all permutations π of $\{2,3,\ldots,n\}$. Hence $\det(A_{1,1}) \lesssim m^{n-1}$ and similarly $\det(A_{i,j}) \lesssim m^{n-1}$ for all entries in $\mathrm{Adj}(A)$ and so

$$\|A^{-1}\| \sim \max\left|\frac{1}{\det A}\det A_{i,j}\right| \lesssim \frac{m^{n-1}}{m^n} = m^{-1}$$

whenever A is an almost rotation, establishing (2.21) and hence (2.20). Thus for any almost rotation A, since

$$\begin{aligned} \widehat{a}(P(t)) &= \widehat{a}(A^{-1}AP(t)) \\ &= \widehat{a}_A(AP(t)) \end{aligned} \tag{2.22}$$

and a_A satisfies the properties (P1), (P2) and (P3) (essentially), we may replace $P(t)$ in (2.19) with $AP(t)$.

The idea is that we will proceed in several steps, each involving an almost rotation, so that in the end we will have substituted the polynomial P in (2.19) by another polynomial $\tilde{P} = (\tilde{P}_1, \ldots, \tilde{P}_n)$ and reduced ourselves to a subinterval of I_α on which the $\tilde{P}_i$'s look like distinct monomials. At each step we shall be using the following consequence of part *a)* of Lemma 2.3.1: if a polynomial does not contain a t^a term in its expansion, then it can never look like t^a on any of its gaps, unless the polynomial identically vanishes. Since the proof of (2.19) will be carried out in Section 2.6 by induction in n, Lemma 2.4.2 shows that we may suppose that no component P_i of P is identically zero.

The r'th step in the procedure will be as follows. By the previous $r-1$ steps we will have reduced ourselves to the following situation. We have a polynomial $P(t) = (P_1(t), \ldots, P_{L_{r-1}}(t), P_{L_{r-1}+1}(t), \ldots, P_n(t))$, on a gap I, so that on I,

$$P_i(t) \sim p_{i,j_i} t^{j_i} \quad \text{for} \quad 1 \le i \le n, \tag{2.23}$$

with all the j_i, for $1 \le i \le L_{r-1}$, distinct and none of the P_i, with $L_{r-1}+1 \le i \le n$, containing any $t^{j_1}, t^{j_2}, \ldots, t^{j_{L_{r-1}}}$ term. Assume $L_{r-1} < n$. For the first step we simply have $L_0 = 0$ and no condition on the P_i, $1 \le i \le n$. We then, using (2.22), replace P by $\tilde{P} = AP$, where A is an almost rotation given by an $n \times n$ matrix of the form

$$A = \left(\begin{array}{c|c} I_{L_{r-1}} & 0 \\ \hline 0 & B \end{array}\right). \tag{2.24}$$

Here $I_{L_{r-1}}$ is the $L_{r-1} \times L_{r-1}$ identity matrix and B is an $(n - L_{r-1}) \times (n - L_{r-1})$ matrix. Note that under A, the first L_{r-1} polynomials $P_1, \dots P_{L_{r-1}}$ remain unchanged, so

$$\tilde{P}_i = P_i \quad \text{for} \quad 1 \le i \le L_{r-1},$$

and none of the $\tilde{P}_{L_{r-1}+1}, \dots, \tilde{P}_n$ contain any $t^{j_1}, t^{j_2}, \dots, t^{j_{L_{r-1}}}$ terms. In addition to this we will choose B in (2.24) in such a way that an additional $M_r \ge 1$ polynomials, say $P_{i_1}, \dots, P_{i_{M_r}}$, out of the $P_{L_{r-1}+1}, \dots, P_n$, will remain unchanged. Recall that on I,

$$P_{i_m}(t) \sim p_{i_m, j_{i_m}} t^{j_{i_m}} \quad \text{for} \quad 1 \le m \le M_r.$$

The polynomials $P_{i_1}, \dots, P_{i_{M_r}}$ will be chosen so that all the j_{i_m}, $1 \le m \le M_r$, are distinct from each other and are of course also distinct from all the j_i with $1 \le i \le L_{r-1}$, because of Lemma 2.3.1, as discussed above. Moreover, the remaining components of $\tilde{P}$ which will be linear combinations of the $P_{L_{r-1}+1}, \dots, P_n$ will not contain a $t^{j_{i_m}}$ term for any $1 \le m \le M_r$. We can then reorder the $\tilde{P}_i$ so that the first $L_{r-1} + M_r$ of the $\tilde{P}_i$ are the ones that were left unchanged by A and on I satisfy

$$\tilde{P}_i(t) \sim p_{i,j_i} t^{j_i} \quad \text{for} \quad 1 \le i \le L_{r-1} + M_r, \tag{2.25}$$

with all the j_i, for $1 \le i \le L_{r-1}+M_r$, distinct, and none of the $\tilde{P}_i$, with $L_{r-1}+M_r+1 \le i \le n$, containing a $t^{j_1}, t^{j_2}, \dots, t^{j_{L_{r-1}+M_r}}$ term. We now subdivide I further into gaps and dyadic intervals with respect to the last $n - L_{r-1} - M_r$ polynomials in $\tilde{P}$, the new dyadic intervals being harmless and the first $L_{r-1}+M_r$ polynomials still satisfying (2.25) on the new gaps. Hence concentrating on one of the new gaps $I' \subseteq I$ and setting $L_r = L_{r-1} + M_r$, we have for $\tilde{P} = (\tilde{P}_1, \dots, \tilde{P}_{L_r}, \tilde{P}_{L_r+1}, \dots, \tilde{P}_n)$,

$$\tilde{P}_i(t) \sim p_{i,j_i} t^{j_i}, \quad 1 \le i \le L_r$$

on I' where as before the j_i are distinct. Furthermore for $L_r \le i \le n$,

$$\tilde{P}_i(t) \sim \tilde{p}_{i,\tilde{j}_i} t^{\tilde{j}_i}$$

on I' and $\tilde{p}_{i,j_k} = 0$ for $1 \le k \le L_r$, putting us in the right position to go onto the $(r+1)$'th step.

It now remains to explicitly construct the almost rotation A in (2.24) with the desired properties described above. It suffices to explicitly determine the $(n - L) \times (n - L)$ matrix B (in what follows $L = L_{r-1} < n$). Recall that on I, each

$$P_i(t) \sim p_{i,j_i} t^{j_i}, \quad 1 \le i \le n,$$

where $P_i(t) = \sum_{m=1}^{d_i} p_{i,m} t^m$. In order to construct the matrix B, we will use the following $(n-L) \times (n-L)$ array:

$$\begin{array}{ccccc}
|p_{L+1,j_{L+1}}| & |p_{L+2,j_{L+1}}| & |p_{L+3,j_{L+1}}| & \cdots & |p_{n,j_{L+1}}| \\
|p_{L+1,j_{L+2}}| & |p_{L+2,j_{L+2}}| & |p_{L+3,j_{L+2}}| & \cdots & |p_{n,j_{L+2}}| \\
\vdots & \vdots & \vdots & & \vdots \\
|p_{L+1,j_{k_1}}| & |p_{L+2,j_{k_1}}| & |p_{L+3,j_{k_1}}| & \cdots & |p_{n,j_{k_1}}| \\
\vdots & \vdots & \vdots & & \vdots \\
|p_{L+1,j_{k_2}}| & |p_{L+2,j_{k_2}}| & |p_{L+3,j_{k_2}}| & \cdots & |p_{n,j_{k_2}}| \\
\vdots & \vdots & \vdots & & \vdots \\
|p_{L+1,j_n}| & |p_{L+2,j_n}| & |p_{L+3,j_n}| & \cdots & |p_{n,j_n}|
\end{array}$$

Using the exponents $\{j_k\}_{k=1}^n$, which are going to be chosen appropriately later, we have the following claim.

Claim. There exists a sequence $(k_m) \subseteq \mathbb{N}$ of length $M \leq n-L$, where the $\{k_m\}$ are distinct and $L+1 \leq k_m \leq n$, such that

$$|p_{k_{m+1},j_{k_m}}| = \max_{L+1 \leq i \leq n}\{|p_{i,j_{k_m}}|\}, \quad 1 \leq m \leq M-1, \tag{2.26}$$

and

$$|p_{k_1,j_{k_M}}| \geq \frac{1}{K} \max_{L+1 \leq i \leq n}\{|p_{i,j_{k_M}}|\} \tag{2.27}$$

for $K = 2(n-L)^{n-L}$. In the case $M = 1$, only (2.27) holds.

In fact we will construct a sequence that, instead of (2.27), satisfies the stronger condition

$$|p_{k_1,j_{k_M}}| = \max_{L+1 \leq i \leq n}\{|p_{i,j_{k_M}}|\}.$$

However once we have established the existence of a sequence satisfying (2.26) and (2.27), we will consider the shortest possible sequence and the constant $\frac{1}{K}$ in (2.27) will be convenient as will become clear later. We can visualise the proof of our claim in terms of picking out a certain sequence of entries from the above array. To construct the alleged sequence, we first pick any k_1 between $L+1$ and n. We then look at the k_1'th row of the array and we pick an element of that row, $|p_{k_2,j_{k_1}}|$ say, satisfying $|p_{k_2,j_{k_1}}| = \max_{1 \leq i \leq n}\{|p_{i,j_{k_1}}|\}$ for some $L+1 \leq k_2 \leq n$ (if there are many possible values for k_2 pick the smallest one). We then look at the k_2'th row of the array and pick an element of that row, $|p_{k_3,j_{k_2}}|$ say, satisfying $|p_{k_3,j_{k_2}}| = \max_{1 \leq i \leq n}\{|p_{i,j_{k_2}}|\}$ for some $L+1 \leq k_3 \leq n$. Continuing this procedure, we form a sequence $|p_{k_2,j_{k_1}}|, |p_{k_3,j_{k_2}}|, |p_{k_4,j_{k_3}}|, \ldots$, whose elements satisfy (2.26). Since there are only $n-L$ possible maximal elements in the array to choose from, there will be some m_0 and m_0' with $1 \leq m_0 < m_0' \leq n-L+1$, such that $\{k_{m_0+j}\}_{j=0}^{m_0'-m_0-1}$ are distinct and $k_{m_0} = k_{m_0'}$. We can then form the sequence $\{k_{m_0}, k_{m_0+1}, \ldots, k_{m_0'-1}\}$ which has length $1 \leq M = m_0' - m_0 \leq n-L$. If we rename $m_0 = 1, m_0 + 1 = 2, \ldots, m_0' - 1 = M$, we form a sequence $\{k_1, \ldots, k_M\}$ that satisfies (2.26) and (2.27), consequently establishing the claim.

Having established the existence of a sequence satisfying (2.26) and (2.27), we can consider a shortest such sequence. The length M of that shortest sequence is equal to M_r in the above discussion. A property of this shortest sequence is that

$$j_{k_{m_1}} \neq j_{k_{m_2}} \quad \text{for} \quad 1 \leq m_1 < m_2 \leq M. \tag{2.28}$$

This is because in the array above, any two rows k_{m_1} and k_{m_2} with $j_{k_{m_1}} = j_{k_{m_2}}$ are identical.

We use this shortest sequence to form the following M vectors:

$$\begin{aligned}
\mathbf{p}_{j_{k_1}} &= (p_{L+1,j_{k_1}}, \ldots, p_{n,j_{k_1}}) \\
\mathbf{p}_{j_{k_2}} &= (p_{L+1,j_{k_2}}, \ldots, p_{n,j_{k_2}}) \\
&\vdots \\
\mathbf{p}_{j_{k_M}} &= (p_{L+1,j_{k_M}}, \ldots, p_{n,j_{k_M}}).
\end{aligned}$$

We can now find $n - L - M$ vectors

$$\begin{aligned}
\mathbf{a}_1 &= (a_{L+1,1}, a_{L+2,1}, \ldots, a_{n,1}) \\
\mathbf{a}_2 &= (a_{L+1,2}, a_{L+2,2}, \ldots, a_{n,2}) \\
&\vdots \\
\mathbf{a}_{n-L-M} &= (a_{L+1,n-L-M}, a_{L+2,n-L-M}, \ldots, a_{n,n-L-M}),
\end{aligned}$$

so that each $\mathbf{a}_l$ is perpendicular to all of the $\{\mathbf{p}_{j_{k_1}}, \mathbf{p}_{j_{k_2}}, \ldots, \mathbf{p}_{j_{k_M}}\}$. In addition we require that for each row vector $\mathbf{a}_l$, all the components except for the k_1'th, k_2'th, $\ldots$, k_M'th plus one more component, equal to zero. In fact we require that the additional component have value equal to 1. The extra nonzero component, has to be chosen so that it is in a different position for each $\mathbf{a}_l$, $1 \leq l \leq n - L - M$. We note that there are enough positions left for this, since out of the $n - L$ total positions, M are taken by the k_m's and hence there are exactly $n - L - M$ positions left, same as the number of $\mathbf{a}_l$'s. To see that we can choose the vectors $\mathbf{a}_l$, $1 \leq l \leq n - L - M$, with these additional restrictions and so that they remain perpendicular to all of the $\{\mathbf{p}_{j_{k_1}}, \mathbf{p}_{j_{k_2}}, \ldots, \mathbf{p}_{j_{k_M}}\}$, we consider a generic $\mathbf{a}_l$ which has the form

$$\mathbf{a}_l = (\ldots, a_{k_1,l}, \ldots, a_{k_2,l}, \ldots, 1, \ldots, a_{k_3,l}, \ldots, a_{k_m,l}, \ldots), \tag{2.29}$$

the remaining entries being 0. The statement that $\mathbf{a}_l$ is perpendicular to all the $\mathbf{p}_{j_{k_m}}$, $1 \leq m \leq M$, is equivalent to the matrix equation

$$\begin{pmatrix} p_{k_1,j_{k_1}} & p_{k_2,j_{k_1}} & \cdots & p_{k_M,j_{k_1}} \\ p_{k_1,j_{k_2}} & p_{k_2,j_{k_2}} & \cdots & p_{k_M,j_{k_2}} \\ \vdots & \vdots & & \vdots \\ p_{k_1,j_{k_M}} & p_{k_2,j_{k_M}} & \cdots & p_{k_M,j_{k_M}} \end{pmatrix} \begin{pmatrix} a_{k_1,l} \\ a_{k_2,l} \\ \vdots \\ a_{k_M,l} \end{pmatrix} = - \begin{pmatrix} p_{l',j_{k_1}} \\ p_{l',j_{k_2}} \\ \vdots \\ p_{l',j_{k_M}} \end{pmatrix}, \tag{2.30}$$

where l' is some number between $L+1$ and n, not equal to any of the k_m for $1 \leq m \leq M$.

We can find the required vector $\mathbf{a}_l$ if we can solve (2.30) for $(a_{k_1,l}, a_{k_2,l}, \ldots, a_{k_M,l})$. In particular, this is possible if the determinant of the matrix in (2.30) does not vanish. This is actually guaranteed by the fact that we are considering the shortest sequence (k_m) satisfying the claim above. In fact the determinant of the matrix

$$C = \begin{pmatrix} p_{k_1,j_{k_1}} & p_{k_2,j_{k_1}} & \cdots & p_{k_M,j_{k_1}} \\ p_{k_1,j_{k_2}} & p_{k_2,j_{k_2}} & \cdots & p_{k_M,j_{k_2}} \\ \vdots & \vdots & & \vdots \\ p_{k_1,j_{k_M}} & p_{k_2,j_{k_M}} & \cdots & p_{k_M,j_{k_M}} \end{pmatrix},$$

satisfies

$$\frac{1}{2}|p_{k_2,j_{k_1}} p_{k_3,j_{k_2}} \cdots p_{k_1,j_{k_M}}| \leq |\det C| \leq \frac{3}{2}|p_{k_2,j_{k_1}} p_{k_3,j_{k_2}} \cdots p_{k_1,j_{k_M}}|, \tag{2.31}$$

which in turn is a consequence of the fact that for each row of C

$$|p_{k_{m+1},j_{k_m}}| \geq K \max_{1 \leq i \leq m} \{|p_{k_i,j_{k_m}}|\} \quad \text{for} \quad 1 \leq m \leq M-1 \tag{2.32}$$

and

$$K|p_{k_1,j_{k_M}}| \geq \max_{L+1 \leq i \leq n} \{|p_{i,j_{k_M}}|\} \geq K \max_{2 \leq i \leq M} \{|p_{k_i,j_{k_M}}|\}, \tag{2.33}$$

with $K = 2(n-L)^{n-L}$. This can be seen by expanding

$$\det C = \sum_{\sigma \in S_M} (-1)^{\operatorname{sgn} \sigma} \prod_{i=1}^{M} p_{k_{\sigma(i)},j_{k_i}}$$

and observing that because of (2.32) and (2.33), the dominant term of the sum is the product corresponding to the permutation $\sigma = (123\ldots M)$. The truth of (2.32) and (2.33) is a result of (k_m) being the shortest sequence satisfying the claim above. This is because if for some m in the range $1 \leq m \leq M-1$, there was an $i \leq m$ such that

$$\max_{L+1 \leq i \leq n} \{|p_{i,j_{k_m}}|\} = |p_{k_{m+1},j_{k_m}}| \leq K|p_{k_i,j_{k_m}}|,$$

then a strictly shorter sequence satisfying the claim would be the sequence k_i, $k_{i+1}, \ldots, k_m$. Also if there is an i in $2 \leq i \leq M$ such that

$$\max_{L+1 \leq i \leq n} \{|p_{i,j_{k_M}}|\} \leq K |p_{k_i, j_{k_M}}|,$$

then a strictly shorter sequence satisfying the claim would be the sequence k_i, $k_{i+1}, \ldots, k_M$.

Having now established what the $\mathbf{a}_l$'s are, and denoting by $\mathbf{e}_m$ the row vector with the value 1 at the k_m'th position and 0's in all the other positions, we form the $(n-L) \times (n-L)$ matrix

$$B = \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_M \\ \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{n-L-M} \end{pmatrix}. \tag{2.34}$$

This is the desired matrix B in (2.24). By the form of B it is easy to see that under the transformation A, the polynomials $P_{k_1}, \ldots, P_{k_M}$ are left unaltered whereas the $\tilde{P}_i$ in the range $L+1 \leq i \leq n$ with $i \neq k_m$ for any $1 \leq m \leq M$, do not contain a $t^{j_{k_m}}$ term for any $1 \leq m \leq M$.

Finally, we have to make sure that the matrix A is an almost rotation. By direct computation we see that $\det A = 1$. Therefore we need to show that the maximum entry of B and hence also of A is ~ 1. In other words it suffices to show that all the components appearing in $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-L-M}$ are $\lesssim 1$. For every l in $1 \leq l \leq n-L-M$ and every m in the range $1 \leq m \leq M$ we have

$$\mathbf{a}_l \cdot \mathbf{p}_{j_{k_m}} = 0. \tag{2.35}$$

For any fixed l we first look at

$$\mathbf{a}_l \cdot \mathbf{p}_{j_{k_M}} = 0.$$

This equation written out explicitly is

$$a_{k_1,l} p_{k_1, j_{k_M}} + p_{l', j_{k_M}} + \sum_{2 \leq m \leq M} a_{k_m,l} p_{k_m, j_{k_M}} = 0.$$

Therefore, using (2.27) and (2.33) appropriately, we have

$$\begin{aligned} 0 &= \Big|a_{k_1,l}p_{k_1,j_{k_M}} + p_{l',j_{k_M}} + \sum_{2\le m\le M} a_{k_m,l}p_{k_m,j_{k_M}}\Big| \\ &\ge |a_{k_1,l}p_{k_1,j_{k_M}}| - |p_{l',j_{k_M}}| - \sum_{2\le m\le M} |a_{k_m,l}p_{k_m,j_{k_M}}| \\ &\ge |a_{k_1,l}||p_{k_1,j_{k_M}}| - K|p_{k_1,j_{k_M}}| - |p_{k_1,j_{k_M}}| \sum_{2\le m\le M} |a_{k_m,l}| \\ &= \Big(|a_{k_1,l}| - K - \sum_{2\le m\le M} |a_{k_m,l}|\Big)|p_{k_1,j_{k_M}}|. \end{aligned}$$

This implies that

$$|a_{k_1,l}| \le M \max\{|a_{k_2,l}|, |a_{k_3,l}|, \dots, |a_{k_M,l}|, K\}. \tag{2.36}$$

We then look at

$$\mathbf{a}_l \cdot \mathbf{p}_{j_{k_1}} = 0.$$

This equation written out explicitly is

$$a_{k_2,l}p_{k_2,j_{k_1}} + p_{l',j_{k_1}} + \sum_{\substack{1\le m\le M\\ m\ne 2}} a_{k_m,l}p_{k_m,j_{k_1}} = 0.$$

Therefore, using (2.26) and (2.32) appropriately, we have

$$\begin{aligned} 0 &= \Big|a_{k_2,l}p_{k_2,j_{k_1}} + p_{l',j_{k_1}} + \sum_{\substack{1\le m\le M\\ m\ne 2}} a_{k_m,l}p_{k_m,j_{k_1}}\Big| \\ &\ge |a_{k_2,l}p_{k_2,j_{k_1}}| - |p_{l',j_{k_1}}| - \sum_{\substack{1\le m\le M\\ m\ne 2}} |a_{k_m,l}p_{k_m,j_{k_1}}| \\ &\ge |a_{k_2,l}||p_{k_2,j_{k_1}}| - |p_{k_2,j_{k_1}}| - \frac{1}{K}|p_{k_2,j_{k_1}}||a_{k_1,l}| - |p_{k_2,j_{k_1}}| \sum_{3\le m\le M} |a_{k_m,l}| \\ &= \Big(|a_{k_2,l}| - 1 - \frac{1}{K}|a_{k_1,l}| - \sum_{3\le m\le M} |a_{k_m,l}|\Big)|p_{k_2,j_{k_1}}|. \end{aligned}$$

This implies that

$$|a_{k_2,l}| \le M \max\{\frac{1}{K}|a_{k_1,l}|, |a_{k_3,l}|, \dots, |a_{k_M,l}|, 1\}.$$

Similarly, by looking at

$$\mathbf{a}_l \cdot \mathbf{p}_{j_{k_2}} = 0$$

and performing the same argument, we obtain

$$|a_{k_3,l}| \le M \max\{\frac{1}{K}|a_{k_1,l}|, \frac{1}{K}|a_{k_2,l}|, |a_{k_4,l}|, \dots, |a_{k_M,l}|, 1\}.$$

In general, using the an appropriate equation from (2.35), we obtain

$$|a_{k_m,l}| \leq M \max\{\frac{1}{K}|a_{k_1,l}|, \frac{1}{K}|a_{k_2,l}|, \dots, \frac{1}{K}|a_{k_{m-1},l}|, |a_{k_{m+1},l}|, \dots, |a_{k_M,l}|, 1\} \tag{2.37}$$

for any m in the range $2 \leq m \leq M$. The substitute equation for $m = 1$ is (2.36). We can now deduce that $|a_{k_m,l}| \leq M^{M+1}K \lesssim 1$ for all $1 \leq m \leq M$ and $1 \leq l \leq n - L - M$.

Let us suppose that for some m_1, $|a_{k_{m_1},l}| > M^{M+1}K \geq MK$. By (2.36) and (2.37), this implies that there is an $m_2 \neq m_1$ such that

$$|a_{k_{m_1},l}| \leq \frac{M}{K}|a_{k_{m_2},l}| \quad \text{if} \quad m_2 < m_1, \tag{2.38}$$

or

$$|a_{k_{m_1},l}| \leq M|a_{k_{m_2},l}| \quad \text{if} \quad m_2 > m_1. \tag{2.39}$$

The second inequality is clearly weaker than the first and even in the case of (2.39), we see that

$$|a_{k_{m_2},l}| \geq \frac{1}{M}|a_{k_{m_1},l}| > M^M K \geq MK.$$

This in turn implies, because of (2.36) and (2.37), that there is an $m_3 \neq m_2$ such that

$$|a_{k_{m_2},l}| \leq \frac{M}{K}|a_{k_{m_3},l}| \quad \text{if} \quad m_3 < m_2, \tag{2.40}$$

or

$$|a_{k_{m_2},l}| \leq M|a_{k_{m_3},l}| \quad \text{if} \quad m_3 > m_2. \tag{2.41}$$

Again the second inequality is weaker than the first and even in the case of (2.41), we can see that

$$|a_{k_{m_3},l}| \geq \frac{1}{M}|a_{k_{m_2},l}| > M^{M-1}K \geq MK.$$

Now m_3 could be equal to m_1 in which case we stop and consider the sequence m_1, m_2, m_3. If $m_3 \neq m_1$, then we just continue the same procedure and form a sequence (m_q), until we arrive at an m_{q_0} equal to m_1 where we stop. First we note that this is guaranteed since the m_q's can only take a finite number of values between 1 and M and secondly we note that for all m_q in the sequence,

$$|a_{k_{m_q},l}| > MK,$$

since for every extra element we pick up an extra constant $\frac{1}{M}$, but we have at most $M + 1$ elements in the sequence. The crucial point is that no element is equal to the previous one and the last element is equal to the first, and so there must be at least two consequtive elements, say $m_{q'}$ and $m_{q'+1}$, that satisfy $m_{q'+1} < m_{q'}$. For these elements we then have

$$|a_{k_{m_{q'}},l}| \leq \frac{M}{K}|a_{k_{m_{q'+1}},l}|.$$

This implies that

$$|a_{k_{m_1},l}| \leq \frac{M^M}{K}|a_{k_{m_{q_0}},l}| = \frac{M^M}{K}|a_{k_{m_1},l}|,$$

which is a contradiction since $K = 2(n-L)^{n-L}$. In this way, we have shown that $a_{k_m,l} \lesssim 1$ for all $1 \leq l \leq n-L-M$ and $1 \leq m \leq M$. Hence the maximum entry of A is ~ 1, which implies that A is an almost rotation.

This completes the description of how the r'th step in the procedure is performed. Since $0 = L_0 < L_1 < \ldots$, there is an $L_k = n$, $k \leq n$, and so after k steps, the above procedure reduces the proof of (2.19) to establishing

$$\int_I \frac{|\widehat{a}(P(t))|}{t} dt \lesssim 1 \tag{2.42}$$

for an "atom" a supported in a cube Q, centred at the origin, such that $\int a = 0$ and $\|a\|_\infty \lesssim |Q|^{-1}$. Furthermore, each component of $P(t) = (P_1(t), \ldots, P_n(t))$ satisfies

$$P_i(t) \sim p_{i,j_i} t^{j_i} \tag{2.43}$$

on I where the exponents $\{j_i\}$ are distinct and nonzero. In fact

$$I \subseteq \cap_i [A|t_{i,j_i}|, A^{-1}|t_{i,j_i+1}|]$$

and so the conclusions of Lemmas 2.3.1, 2.3.3, 2.3.4 and 2.3.5 hold for each P_i on I if A is chosen large enough. In the following sections we will make use of the following functions, for $1 \leq \mu \leq n$,

$$L_{P_1 \ldots P_\mu}(t) = \det(P'(t), P''(t), \ldots, P^{(\mu)}(t)), \tag{2.44}$$

where $P(t) = (P_1(t), \ldots, P_\mu(t))$.

2.6 The main line of the proof

We prove (2.42) by induction on n, the number of components of $P(t)$. First of all we split the integration interval $I = [B, D]$ in (2.42) at $\lambda = (|Q| \prod_i |p_{i,j_i}|)^{-\frac{1}{\sum_i j_i}}$ and write

$$\int_B^D \frac{|\widehat{a}(P(t))|}{t} dt = \int_B^\lambda \frac{|\widehat{a}(P(t))|}{t} dt + \int_\lambda^D \frac{|\widehat{a}(P(t))|}{t} dt\,. \tag{2.45}$$

Now for the part of the above integral near the origin we have for any $1 \le k \le n$,

$$\begin{aligned}
&\int_B^\lambda \frac{|\widehat{a}(P_1(t),\dots,P_n(t))|}{t}dt \\
&\le \int_B^\lambda \frac{|\widehat{a}(P_1(t),\dots,P_n(t)) - \widehat{a}(P_1(t),\dots,P_{k-1}(t),0,P_{k+1}(t),\dots,P_n(t))|}{t}dt \\
&+ \int_B^\lambda \frac{|\widehat{a}(P_1(t),\dots,P_{k-1}(t),0,P_{k+1}(t),\dots,P_n(t))|}{t}dt\,.
\end{aligned}$$

The last integral is O(1) by Lemma 2.4.2 and the induction hypothesis. For the first term of the right hand side we have

$$\begin{aligned}
&|\widehat{a}(P_1(t),\dots,P_n(t)) - \widehat{a}(P_1(t),\dots,P_{k-1}(t),0,P_{k+1}(t),\dots,P_n(t))| \\
&= \left|\int_{\mathbb{R}^n} a(x)\left(e^{-2\pi i x\cdot(P_1(t),\dots,P_n(t))} - e^{-2\pi i x\cdot(P_1(t),\dots,P_{k-1}(t),0,P_{k+1}(t),\dots,P_n(t))}\right)dx\right| \\
&\le 2\pi\int_Q |a(x)||x_k P_k(t)|dx \;\le\; 2\pi|P_k(t)||Q|^{1/n}\int_Q |a(x)|dx \\
&\lesssim |P_k(t)||Q|^{1/n}\,.
\end{aligned}$$

Now we use the fact that on I, $P_k(t) \sim p_{k,j_k}t^{j_k}$, and so

$$|\widehat{a}(P_1(t),\dots,P_n(t)) - \widehat{a}(P_1(t),\dots,P_{k-1}(t),0,P_{k+1}(t),\dots,P_n(t))| \lesssim p_{k,j_k}t^{j_k}|Q|^{1/n}\,.$$

Therefore

$$\begin{aligned}
&\int_B^\lambda \frac{|\widehat{a}(P_1(t),\dots,P_n(t)) - \widehat{a}(P_1(t),\dots,P_{k-1}(t),0,P_{k+1}(t),\dots,P_n(t))|}{t}dt \\
&\lesssim p_{k,j_k}|Q|^{1/n}\int_B^\lambda t^{j_k-1}dt \;\lesssim\; |Q|^{1/n}|p_{k,j_k}|\frac{1}{(|Q|\prod_i |p_{i,j_i}|)^{\frac{j_k}{\sum_i j_i}}} := A_k,
\end{aligned}$$

and this is valid for any $1 \le k \le n$. We can choose k so that the final expression above is bounded above by 1. In fact

$$\prod_{k=1}^n |A_k| = \prod_{k=1}^n |Q|^{1/n}|p_{k,j_k}|\frac{1}{(|Q|\prod_i |p_{i,j_i}|)^{\frac{j_k}{\sum_i j_i}}} = 1$$

and so if all the terms in this product were strictly greater than 1 then the product would be strictly greater than 1. Hence there is a k such that

$$|A_k| = |Q|^{1/n}|p_{k,j_k}|\frac{1}{(|Q|\prod_i |p_{i,j_i}|)^{\frac{j_k}{\sum_i j_i}}} \le 1$$

and this completes the proof for the part of the integral near the origin.

For the part of (2.45) away from the origin we make use of one of the functions defined in (2.44),

$$L_{P_1\dots P_n}(t) = \det(P'(t), P''(t), \dots, P^{(n)}(t))$$

where $P(t) = (P_1(t), \dots, P_n(t))$. If each $P_i(t) = p_{i,j_i} t^{j_i}$, then

$$L_{P_1 \dots P_n}(t) \sim \prod_{i=1}^{n} p_{i,j_i} t^{\sum_{i=1}^{n} j_i - \frac{n(n+1)}{2}}. \tag{2.46}$$

However only (2.43) holds but Proposition 2.7.1 below will show that (2.46) still holds on $I = [B, D]$. Then by the Cauchy-Schwarz inequality

$$\begin{aligned} \int_{\lambda}^{D} \frac{|\widehat{a}(P(t))|}{t} dt &= \int_{\lambda}^{D} \frac{|\widehat{a}(P(t))| (L_{P_1 \dots P_n}(t))^{1/n(n+1)}}{t (L_{P_1 \dots P_n}(t))^{1/n(n+1)}} dt \\ &\leq \left(\int_{\lambda}^{D} \frac{1}{t^2 (L_{P_1 \dots P_n}(t))^{2/n(n+1)}} dt \right)^{\frac{1}{2}} \left(\int_{\lambda}^{D} |\widehat{a}(P(t))|^2 (L_{P_1 \dots P_n}(t))^{2/n(n+1)} dt \right)^{\frac{1}{2}}. \end{aligned}$$

If (2.46) holds, then the first term of the product is bounded above by

$$\left(\int_{(\prod_{i=1}^{n} |p_{i,j_i}| |Q|)^{-\frac{1}{\sum_i j_i}}}^{D} \frac{1}{(\prod_{i=1}^{n} |p_{i,j_i}|)^{\frac{2}{n(n+1)}} t^{\frac{2}{n(n+1)} \sum_i j_i + 1}} dt \right)^{1/2} \leq |Q|^{1/n(n+1)}. \tag{2.47}$$

For the second term we use a weighted restriction theorem mentioned in the introduction (and proved as Theorem 2.8.1 below) to bound this from above by

$$\|a\|_{\frac{n(n+1)}{n(n+1)-1}} \lesssim \frac{1}{|Q|^{1/n(n+1)}}, \tag{2.48}$$

where in the last inequality we have used the fact that $\|a\|_p \lesssim |Q|^{-1/p'}$ which follows from $\|a\|_\infty \lesssim |Q|^{-1}$. Finally combining (2.47) and (2.48) we get the desirable bound which is independent of the coefficients of P and only depends on the degree $\mathbf{d}$. It remains to establish (2.46) and the weighted restriction theorem for polynomial curves.

2.7 Bounding $L_{P_1 \dots P_n}(t)$

Proposition 2.7.1 *Let $L_{P_1 \dots P_n}(t)$ be defined as above and $I = [B, D]$ the interval in (2.42). Recall that for $t \in I$, $P_i(t) \sim p_{i,j_i} t^{j_i}$ and $0 < j_1 < j_2 < \dots < j_n$. Then for $t \in I$*

$$L_{P_1 \dots P_n}(t) \sim \left(\prod_{i=1}^{n} p_{i,j_i} \right) t^{\sum_{i=1}^{n} j_i - \frac{n(n+1)}{2}}.$$

PROOF. First, let us denote by d_i the degree of the polynomial P_i, by σ a permutation of $\{1, \dots, n\}$, by $t_{i,k}$ the (complex) roots of P_i ordered so that $|t_{i,k_1}| \leq |t_{i,k_2}|$ if $k_1 \leq k_2$ Then by expanding the determinant $L_{P_1 \dots P_n}$

$$\frac{L_{P_1 \dots P_n}}{P_1 \cdots P_n}(t) = \sum_{\sigma \text{ even}} \frac{P_1^{(\sigma(1))} \cdots P_n^{(\sigma(n))}}{P_1 \cdots P_n}(t) - \sum_{\sigma \text{ odd}} \frac{P_1^{(\sigma(1))} \cdots P_n^{(\sigma(n))}}{P_1 \cdots P_n}(t).$$

We then use Lemma 2.3.2 to express the derivatives of polynomials in terms of the roots $t_{i,k}$. Thus

$$\begin{aligned}
&\frac{L_{P_1\ldots P_n}}{P_1\ldots P_n}(t) \\
&= \sum_{\sigma\text{ even}} \prod_{i=1}^{n} \sum_{1\le k_1\neq\ldots\neq k_{\sigma(i)}\le d_i} \prod_{l=1}^{\sigma(i)} \frac{1}{t-t_{i,k_l}} - \sum_{\sigma\text{ odd}} \prod_{i=1}^{n} \sum_{1\le k_1\neq\ldots\neq k_{\sigma(i)}\le d_i} \prod_{l=1}^{\sigma(i)} \frac{1}{t-t_{i,k_l}} \\
&= \sum_{\sigma\text{ even}} \prod_{i=1}^{n} \sum_{1\le k_1\neq\ldots\neq k_{\sigma(i)}\le j_i} \prod_{l=1}^{\sigma(i)} \frac{1}{t-t_{i,k_l}} - \sum_{\sigma\text{ odd}} \prod_{i=1}^{n} \sum_{1\le k_1\neq\ldots\neq k_{\sigma(i)}\le j_i} \prod_{l=1}^{\sigma(i)} \frac{1}{t-t_{i,k_l}} \\
&\quad + \mathrm{O}\left(\frac{1}{At^{\frac{n(n+1)}{2}}}\right).
\end{aligned}$$

When $\sigma(i) > j_i$, the sum over $k_1 \neq \ldots \neq k_{\sigma(i)}$ is empty and interpreted as zero. We then proceed to interchange the order of the middle product and sum. That is we can express $L_{P_1\ldots P_n}(t)/P_1(t)\cdots P_n(t)$ as a difference,

$$\begin{aligned}
&\frac{L_{P_1\ldots P_n}}{P_1\cdots P_n}(t) \\
&= \sum_{E_+} \prod_{i=1}^{n} \prod_{l=1}^{\sigma(i)} \frac{1}{t-t_{i,k_{i,l}}} - \sum_{E_-} \prod_{i=1}^{n} \prod_{l=1}^{\sigma(i)} \frac{1}{t-t_{i,k_{i,l}}} + \mathrm{O}\left(\frac{1}{At^{\frac{n(n+1)}{2}}}\right), \quad (2.49)
\end{aligned}$$

where

$$E_+ = \{(k_{i,1},\ldots,k_{i,\sigma(i)}) : 1 \le k_{i,1} \neq \ldots \neq k_{i,\sigma(i)} \le j_i,\ 1 \le i \le n,\ \sigma \text{ even}\}$$

and

$$E_- = \{(k_{i,1},\ldots,k_{i,\sigma(i)}) : 1 \le k_{i,1} \neq \ldots \neq k_{i,\sigma(i)} \le j_i,\ 1 \le i \le n,\ \sigma \text{ odd}\}.$$

We observe that both sums in (2.49) are sums of $\frac{n(n+1)}{2}$-fold products. This allows us to use Lemma 2.3.5 to compare a term from E_+ with a term from E_-, creating an error $\mathrm{O}(A^{-1}t^{-\frac{n(n+1)}{2}})$. Hence if $\sharp E_+ \neq \sharp E_-$, we have

$$\frac{L_{P_1\ldots P_n}}{P_1\cdots P_n}(t) = \pm \sum_{S} \prod_{i=1}^{n} \prod_{l=1}^{\sigma(i)} \frac{1}{t-t_{i,k_{i,l}}} + \mathrm{O}(A^{-1}t^{-\frac{n(n+1)}{2}})$$

where either S is a nonempty subset of E_+ (if $\sharp E_+ > \sharp E_-$) or a nonempty subset of E_- (if $\sharp E_- > \sharp E_+$). Now Lemma 2.3.4 can be employed to obtain the desired bounds for $L_{P_1\ldots P_n}(t)$. So it only remains to verify $\sharp E_+ \neq \sharp E_-$. This is done by counting. We recall the fact that the inverse of an even permutation is an even permutation and likewise for odd permutations. So

$$
\begin{aligned}
&\sharp E_+ - \sharp E_- \\
&= \prod_{r=1}^{n} r! \left[\sum_{\sigma \text{ even}} \binom{j_1}{\sigma(1)} \binom{j_2}{\sigma(2)} \cdots \binom{j_n}{\sigma(n)} \right. \\
&\quad \left. - \sum_{\sigma \text{ odd}} \binom{j_1}{\sigma(1)} \cdots \binom{j_n}{\sigma(n)} \right] \\
&= \prod_{r=1}^{n} r! \left[\sum_{\sigma \text{ even}} \binom{j_{\sigma^{-1}(1)}}{\sigma(\sigma^{-1}(1))} \binom{j_{\sigma^{-1}(2)}}{\sigma(\sigma^{-1}(2))} \cdots \binom{j_{\sigma^{-1}(n)}}{\sigma(\sigma^{-1}(n))} \right. \\
&\quad \left. - \sum_{\sigma \text{ odd}} \binom{j_{\sigma^{-1}(1)}}{\sigma(\sigma^{-1}(1))} \cdots \binom{j_{\sigma^{-1}(n)}}{\sigma(\sigma^{-1}(n))} \right] \\
&= \prod_{r=1}^{n} r! \left[\sum_{\sigma \text{ even}} \binom{j_{\sigma(1)}}{1} \binom{j_{\sigma(2)}}{2} \cdots \binom{j_{\sigma(n)}}{n} \right. \\
&\quad \left. - \sum_{\sigma \text{ odd}} \binom{j_{\sigma(1)}}{1} \cdots \binom{j_{\sigma(n)}}{n} \right] \\
&= \left[\sum_{\sigma \text{ even}} j_{\sigma(1)} j_{\sigma(2)} (j_{\sigma(2)} - 1) j_{\sigma(3)} (j_{\sigma(3)} - 1)(j_{\sigma(3)} - 2) \quad \cdots \right. \\
&\quad \cdots \quad j_{\sigma(n)} (j_{\sigma(n)} - 1) \ldots (j_{\sigma(n)} - n + 1) \\
&\quad - \sum_{\sigma \text{ odd}} j_{\sigma(1)} j_{\sigma(2)} (j_{\sigma(2)} - 1) j_{\sigma(3)} (j_{\sigma(3)} - 1)(j_{\sigma(3)} - 2) \quad \cdots \\
&\quad \left. \cdots \quad j_{\sigma(n)} (j_{\sigma(n)} - 1) \ldots (j_{\sigma(n)} - n + 1) \right] \\
&= \begin{vmatrix} j_1 & j_2 & \cdots & j_n \\ j_1(j_1 - 1) & j_2(j_2 - 1) & \cdots & j_n(j_n - 1) \\ \vdots & \vdots & & \vdots \\ j_1 \ldots (j_1 - n + 1) & j_2 \ldots (j_2 - n + 1) & \cdots & j_n \ldots (j_n - n + 1) \end{vmatrix}.
\end{aligned}
$$

Then by expanding the products and performing row operations the determinant above is equal to

$$
\begin{vmatrix} j_1 & j_2 & \cdots & j_n \\ j_1^2 & j_2^2 & \cdots & j_n^2 \\ \vdots & \vdots & & \vdots \\ j_1^n & j_2^n & \cdots & j_n^n \end{vmatrix} = \prod_{i=1}^{n} j_i \begin{vmatrix} 1 & \cdots & 1 \\ j_1 & \cdots & j_n \\ \vdots & & \vdots \\ j_1^{n-1} & \cdots & j_n^{n-1} \end{vmatrix}.
$$

The last determinant is a Vandermonde determinant and so the last expression is equal to

$$
\prod_{i=1}^{n} j_i \prod_{1 \le l < k \le n} (j_k - j_l).
$$

This is nonzero since $j_k \neq j_l$ for all $1 \leq l < k \leq n$ and $j_i > 0$ for all $1 \leq i \leq n$. This completes the proof of Proposition 2.7.1.

REMARK. It is easy to see that Proposition 2.7.1 still holds with $P_1, \dots P_n$ replaced by any $P_{\xi(1)}, \dots, P_{\xi(\mu)}$ with $1 \leq \mu \leq n$ and ξ a one-to-one function from $1, \dots, \mu$ to $1, \dots, n$.

2.8 A weighted restriction theorem

According to the discussion in Section 2.6, in order to establish (2.42), it now only remains to prove the following restriction theorem. Let us recall that we have reduced ourselves to an interval I which is inside the j_i'th gap of each polynomial P_i for $1 \leq i \leq n$ so that $P_i(t) \sim p_{i,j_i} t^{j_i}$ on I and $j_1 < j_2 < \dots < j_n$.

Theorem 2.8.1 *Let $P(t) = (P_1(t), \dots, P_n(t))$ and let us write $L(t) = L_{P_1 \dots P_n}(t)$. Then for all $f \in L^p(\mathbb{R}^n)$,*

$$\left(\int_I |\widehat{f}(P(t))|^q L(t)^{2/n(n+1)} dt \right)^{\frac{1}{q}} \lesssim \|f\|_p, \tag{2.50}$$

with $\frac{1}{q} = \frac{n(n+1)}{2} \frac{1}{p'}$ and $p < \frac{n(2+n)}{n(2+n)-2}$.

Similar restriction theorems have been established in [Ch], [DM1], [DM2] and [D]. We note that the condition that all the j_i's are distinct is crucial for the proof of Theorem 2.8.1. Reducing ourselves to exactly this situation was the matter of Section 2.5. We will first state and prove some results that are required for the proof of Theorem 2.8.1 (this will be carried out in Section 2.8.1) and subsequently we will give the proof of Theorem 2.8.1 in Section 2.8.2.

2.8.1 Preliminary results

Proposition 2.8.2 *With*

$$J_{P_1 \dots P_n}(t_1, \dots, t_n) = \begin{vmatrix} P_1'(t_1) & \cdots & P_1'(t_n) \\ \vdots & & \vdots \\ P_n'(t_1) & \cdots & P_n'(t_n) \end{vmatrix},$$

the Jacobian of the mapping $t \mapsto x(t) = (x_1(t), \dots, x_n(t))$, where

$$x_k(t) = \sum_{i=1}^{n} P_k(t_i),$$

$1 \leq k \leq n$ *and* $t = (t_1, \dots t_n)$*, the following lower bound holds for* $0 \leq t_1 \leq t_2 \leq \dots \leq t_n$ *and for* $[t_1, t_n] \subseteq I$*:*

$$\begin{aligned} J_{P_1 \dots P_n}(t_1, \dots, t_n) &\gtrsim \left(\prod_{i=1}^{n} p_{i,j_i} \right) t_1^{j_1-1} t_2^{j_2-2} \dots t_n^{j_n-n} \prod_{1 \leq k < l \leq n} (t_l - t_k) \\ &\gtrsim \prod_{i=1}^{n} L_{P_1 \dots P_n}(t_i)^{1/n} \prod_{1 \leq k < l \leq n} (t_l - t_k) . \end{aligned}$$

The proof will be carried out in several steps. We start by establishing the second inequality first. In view of Proposition 2.7.1, it suffices to show the inequality

$$t_1^{j_1-1} t_2^{j_2-2} \dots t_n^{j_n-n} \geq \prod_{i=1}^{n} t_i^{\frac{1}{n}(\sum_{k=1}^{n} j_k) - \frac{n+1}{2}} . \tag{2.51}$$

Using the fact that $0 \leq t_1 \leq t_2 \leq \dots \leq t_n$ and $j_1 < j_2 < \dots < j_n$, we have

$$\begin{aligned} & t_1^{j_1-1} t_2^{j_2-2} \dots t_n^{j_n-n} \\ = \ & t_1^{\frac{1}{n}(\sum_{k=1}^{n} j_k) - \frac{n+1}{2}} t_1^{\frac{1}{n}[(n-1)j_1 - \sum_{k=2}^{n} j_k] + \frac{n+1}{2} - 1} t_2^{j_2-2} \dots t_n^{j_n-n} \\ \geq \ & t_1^{\frac{1}{n}(\sum_{k=1}^{n} j_k) - \frac{n+1}{2}} t_2^{\frac{1}{n}[(n-1)j_1 - \sum_{k=2}^{n} j_k] + \frac{n+1}{2} - 1} t_2^{j_2-2} \dots t_n^{j_n-n} \\ = \ & t_1^{\frac{1}{n}(\sum_{k=1}^{n} j_k) - \frac{n+1}{2}} t_2^{\frac{1}{n}[(n-1)\sum_{k=1}^{2} j_k - \sum_{k=3}^{n} j_k] + \frac{n+1}{2} - (1+2)} t_3^{j_3-3} \dots t_n^{j_n-n} \\ = \ & \prod_{i=1}^{2} t_i^{\frac{1}{n}(\sum_{k=1}^{n} j_k) - \frac{n+1}{2}} t_2^{\frac{1}{n}[(n-2)\sum_{k=1}^{2} j_k - 2\sum_{k=3}^{n} j_k] + 2\left(\frac{n+1}{2}\right) - (1+2)} t_3^{j_3-3} \dots t_n^{j_n-n} \\ \geq \ & \prod_{i=1}^{2} t_i^{\frac{1}{n}(\sum_{k=1}^{n} j_k) - \frac{n+1}{2}} t_3^{\frac{1}{n}[(n-2)\sum_{k=1}^{2} j_k - 2\sum_{k=3}^{n} j_k] + 2\left(\frac{n+1}{2}\right) - (1+2)} t_3^{j_3-3} \dots t_n^{j_n-n} \\ = \ & \prod_{i=1}^{2} t_i^{\frac{1}{n}(\sum_{k=1}^{n} j_k) - \frac{n+1}{2}} t_3^{\frac{1}{n}[(n-2)\sum_{k=1}^{3} j_k - 2\sum_{k=3}^{n} j_k] + 2\left(\frac{n+1}{2}\right) - (1+2+3)} t_4^{j_4-4} \dots t_n^{j_n-n} \\ = \ & \prod_{i=1}^{3} t_i^{\frac{1}{n}(\sum_{k=1}^{n} j_k) - \frac{n+1}{2}} t_3^{\frac{1}{n}[(n-3)\sum_{k=1}^{3} j_k - 3\sum_{k=3}^{n} j_k] + 3\left(\frac{n+1}{2}\right) - (1+2+3)} t_4^{j_4-4} \dots t_n^{j_n-n} \\ \geq \ & \prod_{i=1}^{3} t_i^{\frac{1}{n}(\sum_{k=1}^{n} j_k) - \frac{n+1}{2}} t_4^{\frac{1}{n}[(n-3)\sum_{k=1}^{3} j_k - 3\sum_{k=3}^{n} j_k] + 3\left(\frac{n+1}{2}\right) - (1+2+3)} t_4^{j_4-4} \dots t_n^{j_n-n} . \end{aligned}$$

Thus by repeating the argument we obtain

$$
\begin{aligned}
& t_1^{j_1-1} t_2^{j_2-2} \dots t_n^{j_n-n} \\
& \vdots \\
\geq\ & \prod_{i=1}^{n-1} t_i^{\frac{1}{n}(\sum_{k=1}^{n} j_k)-\frac{n+1}{2}} t_n^{\frac{1}{n}[\sum_{k=1}^{n-1} j_k-(n-1)j_n]+(n-1)(\frac{n+1}{2})-\sum_{k=1}^{n-1} k} t_n^{j_n-n} \\
=\ & \prod_{i=1}^{n-1} t_i^{\frac{1}{n}(\sum_{k=1}^{n} j_k)-\frac{n+1}{2}} t_n^{\frac{1}{n}[\sum_{k=1}^{n} j_k]+(n-1)(\frac{n+1}{2})-\sum_{k=1}^{n} k} \\
=\ & \prod_{i=1}^{n} t_i^{\frac{1}{n}(\sum_{k=1}^{n} j_k)-\frac{n+1}{2}} t_n^{\frac{n(n+1)}{2}-\sum_{k=1}^{n} k} \\
=\ & \prod_{i=1}^{n} t_i^{\frac{1}{n}(\sum_{k=1}^{n} j_k)-\frac{n+1}{2}},
\end{aligned}
$$

thus establishing (2.51) and hence the second inequality of Proposition 2.8.2. For the first inequality of Proposition 2.8.2, we will express $J_{P_1,\dots,P_n}(t_1,\dots,t_n)$ in terms of the $L_{P_1,\dots,P_m}$'s, $1 \leq m \leq n$, for $t_1 \leq \dots \leq t_n$ and $[t_1,t_n] \subseteq I$. This will be accomplished by the following two lemmas.

Lemma 2.8.3 *Let $f_i = \frac{g_i'}{g_1'}$, $1 \leq i \leq n$, and assume that g_i and f_i are differentiable functions in $[t_1,t_n]$ for all i. Then*

$$
\begin{vmatrix} g_1'(t_1) & \cdots & g_1'(t_n) \\ \vdots & & \vdots \\ g_n'(t_1) & \cdots & g_n'(t_n) \end{vmatrix}
= \prod_{i=1}^{n} g_1'(t_i) \int_{t_1}^{t_2} dx_1 \dots \int_{t_{n-1}}^{t_n} dx_{n-1} \begin{vmatrix} f_2'(x_1) & \cdots & f_2'(x_{n-1}) \\ \vdots & & \vdots \\ f_n'(x_1) & \cdots & f_n'(x_{n-1}) \end{vmatrix}. \tag{2.52}
$$

PROOF. By factoring $g_1'(t_i)$ out of every column we write

$$
\begin{vmatrix} g_1'(t_1) & \cdots & g_1'(t_n) \\ \vdots & & \vdots \\ g_n'(t_1) & \cdots & g_n'(t_n) \end{vmatrix} = \prod_{i=1}^{n} g_1'(t_i) \begin{vmatrix} 1 & \cdots & 1 \\ f_2(t_1) & \cdots & f_2(t_n) \\ \vdots & & \vdots \\ f_n(t_1) & \cdots & f_n(t_n) \end{vmatrix}.
$$

Then by conducting column operations the determinant involving the f_i's is equal to

$$
\begin{vmatrix} 1 & 0 & \cdots & 0 \\ f_2(t_1) & f_2(t_2)-f_2(t_1) & \cdots & f_2(t_n)-f_2(t_1) \\ \vdots & \vdots & & \vdots \\ f_n(t_1) & f_n(t_2)-f_n(t_1) & \cdots & f_n(t_n)-f_n(t_1) \end{vmatrix}
$$

$$= \int_{t_1}^{t_2} dx_1 \dots \int_{t_1}^{t_n} dx_{n-1} \begin{vmatrix} f_2'(x_1) & \cdots & f_2'(x_{n-1}) \\ \vdots & & \vdots \\ f_n'(x_1) & \cdots & f_n'(x_{n-1}) \end{vmatrix}.$$

For fixed $x_1, x_2, \dots, x_{n-1}$ except x_l and x_m with $1 \le l < m \le n-1$, consider

$$I_k := \int_{t_k}^{t_{k+1}} dx_l \int_{t_k}^{t_{k+1}} dx_m \begin{vmatrix} f_2'(x_1) & \cdots & f_2'(x_{n-1}) \\ \vdots & & \vdots \\ f_n'(x_1) & \cdots & f_n'(x_{n-1}) \end{vmatrix}.$$

By interchanging the lth with the mth column we have

$$\begin{aligned} & I_k \\ = & \int_{t_k}^{t_{k+1}} dx_l \int_{t_k}^{t_{k+1}} dx_m \begin{vmatrix} f_2'(x_1) & \cdots & f_2'(x_l) & \cdots & f_2'(x_m) & \cdots & f_2'(x_{n-1}) \\ \vdots & & \vdots & & \vdots & & \vdots \\ f_n'(x_1) & \cdots & f_n'(x_l) & \cdots & f_n'(x_m) & \cdots & f_n'(x_{n-1}) \end{vmatrix} \\ = & -\int_{t_k}^{t_{k+1}} dx_l \int_{t_k}^{t_{k+1}} dx_m \begin{vmatrix} f_2'(x_1) & \cdots & f_2'(x_m) & \cdots & f_2'(x_l) & \cdots & f_2'(x_{n-1}) \\ \vdots & & \vdots & & \vdots & & \vdots \\ f_n'(x_1) & \cdots & f_n'(x_m) & \cdots & f_n'(x_l) & \cdots & f_n'(x_{n-1}) \end{vmatrix} \\ = & -\int_{t_k}^{t_{k+1}} dx_m \int_{t_k}^{t_{k+1}} dx_l \begin{vmatrix} f_2'(x_1) & \cdots & f_2'(x_l) & \cdots & f_2'(x_m) & \cdots & f_2'(x_{n-1}) \\ \vdots & & \vdots & & \vdots & & \vdots \\ f_n'(x_1) & \cdots & f_n'(x_l) & \cdots & f_n'(x_m) & \cdots & f_n'(x_{n-1}) \end{vmatrix}, \end{aligned}$$

the last equality follows by changing the variables of integration. Thus $I_k = -I_k$ and so $I_k = 0$. So finally

$$\begin{aligned} & \prod_{i=1}^{n} g_1'(t_i) \int_{t_1}^{t_2} dx_1 \dots \int_{t_1}^{t_n} dx_{n-1} \begin{vmatrix} f_2'(x_1) & \cdots & f_2'(x_{n-1}) \\ \vdots & & \vdots \\ f_n'(x_1) & \cdots & f_n'(x_{n-1}) \end{vmatrix} \\ = & \prod_{i=1}^{n} g_1'(t_i) \int_{t_1}^{t_2} dx_1 \dots \int_{t_{n-1}}^{t_n} dx_{n-1} \begin{vmatrix} f_2'(x_1) & \cdots & f_2'(x_{n-1}) \\ \vdots & & \vdots \\ f_n'(x_1) & \cdots & f_n'(x_{n-1}) \end{vmatrix}, \end{aligned}$$

concluding the proof of Lemma 2.8.3.

We aim to use Lemma 2.8.3 inductively to obtain an expression of $J_{P_1 \dots P_n}$ in terms of the $L_{P_1 \dots P_m}$'s with $1 \le m \le n$. We shall do this by the following lemma.

Lemma 2.8.4

$$\left(\frac{L_{P_1 \dots P_m Q}}{L_{P_1 \dots P_m R}} \right)' = \frac{L_{P_1 \dots P_m RQ} L_{P_1 \dots P_m}}{L^2_{P_1 \dots P_m R}} \tag{2.53}$$

PROOF. The proof will be by induction on m. The statement is true for $m = 0$ because

$$\left(\frac{L_Q}{L_R} \right)' = \frac{L_Q' L_R - L_R' L_Q}{L_R^2} = \frac{Q''R' - R''Q'}{L_R^2} = \frac{L_{RQ}}{L_R^2}.$$

Now suppose the statement is true for $m = k-1$. Then for $m = k$ we have,

$$
\begin{aligned}
\left(\frac{L_{P_1\dots P_kQ}}{L_{P_1\dots P_kR}}\right)' &= \frac{L'_{P_1\dots P_kQ}L_{P_1\dots P_kR} - L'_{P_1\dots P_kR}L_{P_1\dots P_kQ}}{L^2_{P_1\dots P_kR}} \\
&= \frac{1}{L^2_{P_1\dots P_kR}}\left(\begin{vmatrix} P_1' & \cdots & P_1^{(k)} & P_1^{(k+2)} \\ \vdots & & \vdots & \vdots \\ P_k' & \cdots & P_k^{(k)} & P_k^{(k+2)} \\ Q' & \cdots & Q^{(k)} & Q^{(k+2)} \end{vmatrix} L_{P_1\dots P_kR}\right. \\
&\quad \left. - \begin{vmatrix} P_1' & \cdots & P_1^{(k)} & P_1^{(k+2)} \\ \vdots & & \vdots & \vdots \\ P_k' & \cdots & P_k^{(k)} & P_k^{(k+2)} \\ R' & \cdots & R^{(k)} & R^{(k+2)} \end{vmatrix} L_{P_1\dots P_kQ}\right).
\end{aligned}
$$

This equation can be written in terms of the L's by expanding the determinants using the last column:

$$
\begin{aligned}
\left(\frac{L_{P_1\dots P_kQ}}{L_{P_1\dots P_kR}}\right)' &= \frac{1}{L^2_{P_1\dots P_kR}}[(L_{P_1\dots P_k}Q^{(k+2)}L_{P_1\dots P_kR} \\
&\quad -L_{P_1\dots P_{k-1}Q}P_k^{(k+2)}L_{P_1\dots P_kR} \\
&\quad +L_{P_1\dots P_{k-2}P_kQ}P_{k-1}^{(k+2)}L_{P_1\dots P_kR} \\
&\quad \vdots \\
&\quad (-1)^kL_{P_2\dots P_kQ}P_1^{(k+2)}L_{P_1\dots P_kR}) \\
&\quad -(L_{P_1\dots P_k}R^{(k+2)}L_{P_1\dots P_kQ} \\
&\quad -L_{P_1\dots P_{k-1}R}P_k^{(k+2)}L_{P_1\dots P_kQ} \\
&\quad +L_{P_1\dots P_{k-2}P_kR}P_{k-1}^{(k+2)}L_{P_1\dots P_kQ} \\
&\quad \vdots \\
&\quad (-1)^kL_{P_2\dots P_kQ}P_1^{(k+2)}L_{P_1\dots P_kQ})],
\end{aligned}
$$

so grouping the terms appropriately we obtain

$$
\begin{aligned}
\left(\frac{L_{P_1\dots P_kQ}}{L_{P_1\dots P_kR}}\right)' &= \frac{1}{L^2_{P_1\dots P_kR}}(L_{P_1\dots P_k}Q^{(k+2)}L_{P_1\dots P_kR} - L_{P_1\dots P_k}R^{(k+2)}L_{P_1\dots P_kQ} \\
&\quad -L_{P_1\dots P_{k-1}Q}P_k^{(k+2)}L_{P_1\dots P_kR} + L_{P_1\dots P_{k-1}R}P_k^{(k+2)}L_{P_1\dots P_kQ} \\
&\quad +L_{P_1\dots P_{k-2}P_kQ}P_{k-1}^{(k+2)}L_{P_1\dots P_kR} - L_{P_1\dots P_{k-2}P_kR}P_{k-1}^{(k+2)}L_{P_1\dots P_kQ} \\
&\quad \vdots \\
&\quad (-1)^kL_{P_2\dots P_kQ}P_1^{(k+2)}L_{P_1\dots P_kR} - (-1)^kL_{P_2\dots P_kQ}P_1^{(k+2)}L_{P_1\dots P_kQ}).
\end{aligned} \tag{2.54}
$$

All the terms in (2.54), except the first two, can be combined in pairs. We make the claim,

$$
-L_{P_1\dots P_{k-1}Q}L_{P_1\dots P_kR} + L_{P_1\dots P_{k-1}R}L_{P_1\dots P_kQ} = L_{P_1\dots P_k}L_{P_1\dots P_{k-1}RQ}, \tag{2.55}
$$

with similar claims for the rest of the pairs in (2.54). If the claim is true then by substituting (2.55) in (2.54) we obtain an expansion for $L_{P_1\ldots P_k RQ}$ using the last column. This would then complete the proof of Lemma 2.8.4. To show (2.55) we use the induction hypothesis to obtain

$$\begin{aligned}
L_{P_1\ldots P_{k-1}R}L_{P_1\ldots P_k Q} &= L^2_{P_1\ldots P_k}L_{P_1\ldots P_{k-1}R}\left(\frac{L_{P_1\ldots P_{k-1}Q}}{L_{P_1\ldots P_k}}\right)' \\
&= L^2_{P_1\ldots P_k}L_{P_1\ldots P_{k-1}R}\left(\frac{L_{P_1\ldots P_{k-1}Q}}{L_{P_1\ldots P_{k-1}R}}\frac{L_{P_1\ldots P_{k-1}R}}{L_{P_1\ldots P_k}}\right)' \\
&= L^2_{P_1\ldots P_k-1R}L_{P_1\ldots P_k}\left(\frac{L_{P_1\ldots P_{k-1}Q}}{L_{P_1\ldots P_{k-1}R}}\right)' \\
&\quad + L^2_{P_1\ldots P_k}L_{P_1\ldots P_{k-1}Q}\left(\frac{L_{P_1\ldots P_{k-1}R}}{L_{P_1\ldots P_k}}\right)' \\
&= L_{P_1\ldots P_k}L_{P_1\ldots P_{k-1}RQ} + L_{P_1\ldots P_{k-1}Q}L_{P_1\ldots P_k R}.
\end{aligned}$$

This proves the claim in (2.55) and consequently Lemma 2.8.4.

We are now in a position to express $J_{P_1\ldots P_n}(t_1,\ldots,t_n)$ in terms of the $L_{P_1\ldots P_m}$'s with $1\le m\le n$, for $t_1<\ldots<t_n$ and $[t_1,t_n]\subseteq I$. Let us define inductively in k, $1\le k\le n$,

$$F_{i,1}=\frac{P_i'}{P_1'},\ F_{i,k}=\frac{F'_{i,k-1}}{F'_{k,k-1}},$$

for i in $k\le i\le n$. Then by repeated applications of Lemma 2.8.3 we obtain

$$\begin{aligned}
&\begin{vmatrix} P_1'(t_1) & \cdots & P_1'(t_n) \\ \vdots & & \vdots \\ P_n'(t_1) & \cdots & P_n'(t_n)\end{vmatrix} \\
&= \prod_{i=1}^{n} P_1'(t_i)\int_{t_1}^{t_2} dx_{1,1}\ldots\int_{t_{n-1}}^{t_n} dx_{n-1,1}\prod_{i=1}^{n-1}F'_{2,1}(x_{i,1})\int_{x_{1,1}}^{x_{2,1}} dx_{1,2}\ldots \\
&\int_{x_{n-2,1}}^{x_{n-1,1}} dx_{n-2,2}\prod_{i=1}^{n-2}F'_{3,2}(x_{i,2})\int_{x_{1,2}}^{x_{2,2}} dx_{1,3}\ldots\int_{x_{n-3,2}}^{x_{n-2,2}} dx_{n-3,3}\quad\ldots \\
&\ldots\prod_{i=1}^{2}F'_{n-1,n-2}(x_{i,n-2})\int_{x_{1,n-2}}^{x_{2,n-2}} dx_{1,n-1}F'_{n,n-1}(x_{1,n-1}),
\end{aligned}\tag{2.56}$$

where in the applications of Lemma 2.8.3 we make sure that the $F_{i,k}$ are differentiable. In fact, using Lemma 2.8.4, one can show that

$$F'_{i,k}=\left(\frac{L_{P_1\ldots P_{k-1}P_i}}{L_{P_1\ldots P_k}}\right)'=\frac{L_{P_1\ldots P_{k-1}}L_{P_1\ldots P_k P_i}}{L^2_{P_1\ldots P_k}}\tag{2.57}$$

for $k\le i\le n$. This would then imply that the $F_{i,k}$ are differentiable on $[t_1,tn]$ by Proposition 2.7.1. We prove (2.57) by induction on k. For $k=1$

$$F'_{i,1}=\left(\frac{L_{P_i}}{L_{P_1}}\right)'=\frac{L_{P_1P_i}}{L^2_{P_1}}.$$

If (2.57) is true for $k = m-1$ then for $k = m$ we have

$$\begin{aligned} F'_{i,m} &= \left(\frac{F'_{i,m-1}}{F'_{m,m-1}}\right)' = \left(\frac{L_{P_1\dots P_{m-2}}L_{P_1\dots P_{m-1}P_i}/L^2_{P_1\dots P_{m-1}}}{L_{P_1\dots P_{m-2}}L_{P_1\dots P_m}/L^2_{P_1\dots P_{m-1}}}\right)' \\ &= \left(\frac{L_{P_1\dots P_{m-1}P_i}}{L_{P_1\dots P_m}}\right)' = \frac{L_{P_1\dots P_{m-1}}L_{P_1\dots P_m P_i}}{L^2_{P_1\dots P_m}}, \end{aligned}$$

where the last inequality follows by Lemma 2.8.4. This completes the proof of (2.57). We are now in a position to substitute (2.57) into (2.56) to express $J_{P_1\dots P_n}(t_1,\dots,t_n)$ in terms of $L_{P_1\dots P_m}$'s with $1 \le m \le n$. Precisely

$$\begin{aligned} &J_{P_1\dots P_n}(t_1,\dots,t_n) \qquad (2.58)\\ &= \prod_{i=1}^{n} L_{P_1}(t_i)\int_{t_1}^{t_2} dx_{1,1}\dots\int_{t_{n-1}}^{t_n} dx_{n-1,1}\prod_{i=1}^{n-1}\frac{L_{P_1P_2}}{L^2_{P_1}}(x_{i,1})\int_{x_{1,1}}^{x_{2,1}} dx_{1,2}\dots \\ &\int_{x_{n-2,1}}^{x_{n-1,1}} dx_{n-2,2}\prod_{i=1}^{n-2}\frac{L_{P_1}L_{P_1P_2P_3}}{L^2_{P_1P_2}}(x_{i,2})\int_{x_{1,2}}^{x_{2,2}} dx_{1,3}\dots\int_{x_{n-3,2}}^{x_{n-2,2}} dx_{n-3,3}\quad\dots \\ &\dots\int_{x_{1,n-3}}^{x_{2,n-3}} dx_{1,n-2}\int_{x_{2,n-3}}^{x_{3,n-3}} dx_{2,n-2}\prod_{i=1}^{2}\frac{L_{P_1\dots P_{n-3}}L_{P_1\dots P_{n-1}}}{L^2_{P_1\dots P_{n-2}}}(x_{i,n-2}) \\ &\int_{x_{1,n-2}}^{x_{2,n-2}} dx_{1,n-1}\frac{L_{P_1\dots P_{n-2}}L_{P_1\dots P_n}}{L^2_{P_1\dots P_{n-1}}}(x_{1,n-1}). \qquad (2.59) \end{aligned}$$

To complete the proof of Proposition 2.8.2, we will need to make use of the consequence of the remark after Proposition 2.7.1, that on the interval I each of the $L_{P_1\dots P_m}$ is either positive or negative. Hence on I we have

$$\begin{aligned} |J_{P_1\dots P_n}(t_1,\dots,t_n)| &= \prod_{i=1}^{n}|L_{P_1}(t_i)|\int_{t_1}^{t_2} dx_{1,1}\dots\int_{t_{n-1}}^{t_n} dx_{n-1,1}\prod_{i=1}^{n-1}\left|\frac{L_{P_1P_2}}{L^2_{P_1}}(x_{i,1})\right| \\ &\int_{x_{1,1}}^{x_{2,1}} dx_{1,2}\dots\int_{x_{n-2,1}}^{x_{n-1,1}} dx_{n-2,2}\prod_{i=1}^{n-2}\left|\frac{L_{P_1}L_{P_1P_2P_3}}{L^2_{P_1P_2}}(x_{i,2})\right|\quad\dots \\ &\dots\int_{x_{1,n-2}}^{x_{2,n-2}} dx_{1,n-1}\left|\frac{L_{P_1\dots P_{n-2}}L_{P_1\dots P_n}}{L^2_{P_1\dots P_{n-1}}}(x_{1,n-1})\right|, \qquad (2.60) \end{aligned}$$

so we can substitute the estimate from Proposition 2.7.1 to obtain

$$\begin{aligned} J_{P_1\dots P_n}(t_1,\dots,t_n) &\gtrsim \prod_{i=1}^{n} p_{i,j_i}\prod_{i=1}^{n} t_i^{j_1-1}\int_{t_1}^{t_2} dx_{1,1}\dots\int_{t_{n-1}}^{t_n} dx_{n-1,1}\prod_{i=1}^{n-1} x_{i,1}^{j_2-j_1-1} \\ &\int_{x_{1,1}}^{x_{2,1}} dx_{1,2}\dots\int_{x_{n-2,1}}^{x_{n-1,1}} dx_{n-2,2}\prod_{i=1}^{n-2} x_{i,2}^{j_3-j_2-1}\quad\dots \\ &\dots\int_{x_{1,n-2}}^{x_{2,n-2}} dx_{1,n-1}x_{1,n-1}^{j_n-j_{n-1}-1}. \qquad (2.61) \end{aligned}$$

We finally need to bound from below the multiple integral in (2.61). This will be done through the following two lemmas.

Lemma 2.8.5 *With* $s_1 \le s_2 \le \ldots \le s_m$,

$$\int_{s_1}^{s_2} dy_{1,1} \ldots \int_{s_{m-1}}^{s_m} dy_{m-1,1} \int_{y_{1,1}}^{y_{2,1}} dy_{1,2} \ldots \int_{y_{m-2,1}}^{y_{m-1,1}} dy_{m-2,2} \quad \ldots$$

$$\int_{y_{1,m-2}}^{y_{2,m-2}} dy_{1,m-1}$$

$$= \prod_{1\le q\le m} ((q-1)!)^{-1} \prod_{1\le k<l\le m} (s_l - s_k). \tag{2.62}$$

PROOF. We will prove Lemma 2.8.5 by induction on m. For $m = 2$ we just have $s_2 - s_1 = \int_{s_1}^{s_2} dy_{1,1}$. Assuming then (2.62) for $m = p-1$, we can use the Vandermonde determinant to write

$$\prod_{1\le k<l\le p} (s_l - s_k) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ s_1 & s_2 & \cdots & s_p \\ s_1^2 & s_2^2 & \cdots & s_p^2 \\ \vdots & \vdots & & \vdots \\ s_1^{p-1} & s_2^{p-1} & \cdots & s_p^{p-1} \end{vmatrix}.$$

Then subtracting the first column from the second column, the second from the third and so on, we have

$$\begin{aligned}
\prod_{1\le k<l\le p} (s_l - s_k) &= \begin{vmatrix} 1 & 0 & \cdots & 0 \\ s_1 & s_2 - s_1 & \cdots & s_p - s_{p-1} \\ s_1^2 & s_2^2 - s_1^2 & \cdots & s_p^2 - s_{p-1}^2 \\ \vdots & \vdots & & \vdots \\ s_1^{p-1} & s_2^{p-1} - s_1^{p-1} & \cdots & s_p^{p-1} - s_{p-1}^{p-1} \end{vmatrix} \\
&= \begin{vmatrix} s_2 - s_1 & \cdots & s_p - s_{p-1} \\ s_2^2 - s_1^2 & \cdots & s_p^2 - s_{p-1}^2 \\ \vdots & & \vdots \\ s_2^{p-1} - s_1^{p-1} & \cdots & s_p^{p-1} - s_{p-1}^{p-1} \end{vmatrix} \\
&= \begin{vmatrix} \int_{s_1}^{s_2} dy_{1,1} & \cdots & \int_{s_{p-1}}^{s_p} dy_{p-1,1} \\ 2\int_{s_1}^{s_2} y_{1,1} dy_{1,1} & \cdots & 2\int_{s_{p-1}}^{s_p} y_{p-1,1} dy_{p-1,1} \\ \vdots & & \vdots \\ (p-1)\int_{s_1}^{s_2} y_{1,1}^{p-2} dy_{1,1} & \cdots & (p-1)\int_{s_{p-1}}^{s_p} y_{p-1,1}^{p-2} dy_{p-1,1} \end{vmatrix},
\end{aligned}$$

leading to

$$\prod_{1\le k<l\le p}(s_l-s_k) = (p-1)!\int_{s_1}^{s_2}dy_{1,1}\dots\int_{s_{p-1}}^{s_p}dy_{p-1,1}\begin{vmatrix}1 & \cdots & 1\\ y_{1,1} & \cdots & y_{p-1,1}\\ \vdots & & \vdots\\ y_{1,1}^{p-2} & \cdots & y_{p-1,1}^{p-2}\end{vmatrix}$$

$$= (p-1)!\int_{s_1}^{s_2}dy_{1,1}\dots\int_{s_{p-1}}^{s_p}dy_{p-1,1}\prod_{1\le k'<l'\le p-1}(y_{l',1}-y_{k',1})$$

$$= (p-1)!\int_{s_1}^{s_2}dy_{1,1}\dots\int_{s_{p-1}}^{s_p}dy_{p-1,1}\prod_{q=1}^{p-1}(q-1)!$$

$$\int_{y_{1,1}}^{y_{2,1}}dy_{1,2}\dots\int_{y_{p-2,1}}^{y_{p-1,1}}dy_{p-2,2}\quad\dots\quad\int_{y_{1,p-2}}^{y_{2,p-2}}dy_{1,p-1},$$

proving (2.62) for $p=m$ and completing the proof of Lemma 2.8.5.

To estimate the integral in (2.61) it will be useful to have in mind the following diagram of the ranges of the various variables in (2.61):

$$\begin{array}{ccccccccccc} t_1 & & t_2 & & t_3 & & \dots & & t_{n-1} & & t_n\\ & x_{1,1} & & x_{2,1} & & x_{3,1} & & \dots & & x_{n-1,1} & \\ & & x_{1,2} & & x_{2,2} & & \dots & & x_{n-2,2} & & \\ & & & & & \vdots & & & & & \\ & & & & & x_{1,n-1} & & & & & \end{array}$$

First we have the following lemma.

Lemma 2.8.6 *With $t_1\le t_2\le\ldots\le t_n$ and $\alpha_{i,j}\in\mathbb{N}$, $1\le j\le n$ and $1\le i\le n-j+1$, we have*

$$\prod_{i=1}^{n}t_i^{\alpha_{i,i}}\int_{t_1}^{t_2}dx_{1,1}\dots\int_{t_{n-1}}^{t_n}dx_{n-1,1}\prod_{i=1}^{n-1}x_{i,1}^{\alpha_{i,2}}\int_{x_{1,1}}^{x_{2,1}}dx_{1,2}\dots\int_{x_{n-2,1}}^{x_{n-1,1}}dx_{n-2,2}$$

$$\prod_{i=1}^{n-2}x_{i,2}^{\alpha_{i,3}}\quad\dots\quad\int_{x_{1,n-2}}^{x_{2,n-2}}dx_{1,n-1}x_{1,n-1}^{\alpha_{1,n}}$$

$$\gtrsim\prod_{i=1}^{n}t_i^{A_i}\prod_{1\le k<l\le n}(t_l-t_k), \tag{2.63}$$

where $A_i=\sum_{r=1}^{i}\alpha_{i-r+1,r}$ and the constant involved in the $\gtrsim$ sign only depends on the $\alpha_{i,j}$ and n.

PROOF. In the proof of this lemma, it is worth having in mind the following diagram, similar to the one above, which shows not only the ranges of the variables, but also the powers that they are raised to in the integrands in (2.63).

$$\begin{array}{ccccccccccc} t_1^{\alpha_{1,1}} & & t_2^{\alpha_{2,1}} & & t_3^{\alpha_{3,1}} & & \cdots & & t_{n-1}^{\alpha_{n-1,1}} & & t_n^{\alpha_{n,1}} \\ & x_{1,1}^{\alpha_{1,2}} & & x_{2,1}^{\alpha_{2,2}} & & x_{3,1}^{\alpha_{3,2}} & & \cdots & & x_{n-1,1}^{\alpha_{n-1,2}} & \\ & & x_{1,2}^{\alpha_{1,3}} & & x_{2,2}^{\alpha_{2,3}} & & \cdots & & x_{n-2,2}^{\alpha_{n-2,3}} & & \\ & & & & & \vdots & & & & & \\ & & & & & x_{1,n-1}^{\alpha_{1,n}} & & & & & \end{array}$$

We prove Lemma 2.8.6 by induction on n. For $n = 2$, (2.63) becomes

$$t_1^{\alpha_{1,1}} t_2^{\alpha_{2,1}} \int_{t_1}^{t_2} x_{1,1}^{\alpha_{1,2}} dx_{1,1} \gtrsim t_1^{\alpha_{1,1}} t_2^{\alpha_{2,1}+\alpha_{1,2}} (t_2 - t_1),$$

which is equivalent to

$$\int_{t_1}^{t_2} x_{1,1}^{\alpha_{1,2}} dx_{1,1} \gtrsim t_2^{\alpha_{1,2}} (t_2 - t_1). \tag{2.64}$$

In the case that $t_2 < At_1$ for a sufficiently large A, we have

$$\int_{t_1}^{t_2} x_{1,1}^{\alpha_{1,2}} dx_{1,1} \geq t_1^{\alpha_{1,2}} \int_{t_1}^{t_2} dx_{1,1} \gtrsim t_2^{\alpha_{1,2}} (t_2 - t_1).$$

Also, in the opposite case $t_2 \geq At_1$,

$$\int_{t_1}^{t_2} x_{1,1}^{\alpha_{1,2}} dx_{1,1} \sim t_2^{\alpha_{1,2}+1} - t_1^{\alpha_{1,2}+1} \gtrsim t_2^{\alpha_{1,2}+1} \geq t_2^{\alpha_{1,2}} (t_2 - t_1),$$

establishing (2.64) and hence proving (2.63) for $n = 2$. Now assuming (2.63) for $n = p - 1$, we have

$$\begin{aligned} &\prod_{i=1}^{p} t_i^{\alpha_{i,1}} \int_{t_1}^{t_2} dx_{1,1} \cdots \int_{t_{p-1}}^{t_p} dx_{p-1,1} \prod_{i=1}^{p-1} x_{i,1}^{\alpha_{i,2}} \int_{x_{1,1}}^{x_{2,1}} dx_{1,2} \cdots \int_{x_{p-2,1}}^{x_{p-1,1}} dx_{p-2,2} \\ &\prod_{i=1}^{p-2} x_{i,2}^{\alpha_{i,3}} \quad \cdots \quad \int_{x_{1,p-2}}^{x_{2,p-2}} dx_{1,p-1} x_{1,p-1}^{\alpha_{1,p}} \\ \gtrsim &\prod_{i=1}^{p} t_i^{\alpha_{i,1}} \int_{t_1}^{t_2} dx_{1,1} \cdots \int_{t_{p-1}}^{t_p} dx_{p-1,1} \prod_{i=1}^{p-1} x_{i,1}^{B_i} \prod_{1 \leq k < l \leq p-1} (x_{l,1} - x_{k,1}), \end{aligned} \tag{2.65}$$

where $B_i = \sum_{r=1}^{i} \alpha_{i-r+1,r+1}$. Lemma 2.8.6 would then be proved if we showed that

$$\begin{aligned} &\int_{t_1}^{t_2} dx_{1,1} \cdots \int_{t_{p-1}}^{t_p} dx_{p-1,1} \prod_{i=1}^{p-1} x_{i,1}^{B_i} \prod_{1 \leq k < l \leq p-1} (x_{l,1} - x_{k,1}) \\ \gtrsim &\prod_{i=2}^{p} t_i^{B_{i-1}} \int_{t_1}^{t_2} dx_{1,1} \cdots \int_{t_{p-1}}^{t_p} dx_{p-1,1} \prod_{1 \leq k < l \leq p-1} (x_{l,1} - x_{k,1}), \end{aligned} \tag{2.66}$$

because of Lemma 2.8.5 and because

$$B_{i-1} + \alpha_{i,1} = \sum_{r=1}^{i-1} \alpha_{i-r,r+1} + \alpha_{i,1} = \sum_{r=0}^{i-1} \alpha_{i-r,r+1} = A_i.$$

Inequality (2.66) essentially asserts that we can take the product of the monomials out of all the integrals evaluating them each time at the highest endpoint. We show (2.66) using an iterative procedure of which we describe the q'th step. After $q-1$ steps we will have shown that

$$\begin{aligned} &\int_{t_1}^{t_2} dx_{1,1} \dots \int_{t_{p-1}}^{t_p} dx_{p-1,1} \prod_{i=1}^{p-1} x_{i,1}^{B_i} \prod_{1\le k<l\le p-1} (x_{l,1} - x_{k,1}) \\ \gtrsim\ &\prod_{i=2}^{q} t_i^{B_{i-1}} \int_{t_1}^{t_2} dx_{1,1} \dots \int_{t_{p-1}}^{t_p} dx_{p-1,1} \prod_{i=q}^{p-1} x_{i,1}^{B_i} \prod_{1\le k<l\le p-1} (x_{l,1} - x_{k,1}). \end{aligned}$$

Concentrating now on the $dx_{q,1}$ integration, we have

$$\begin{aligned} &\int_{t_q}^{t_{q+1}} dx_{q,1} \prod_{i=q}^{p-1} x_{i,1}^{B_i} \prod_{1\le k<l\le p-1} (x_{l,1} - x_{k,1}) \\ =\ &\prod_{i=q+1}^{p-1} x_{i,1}^{B_i} \prod_{\substack{1\le k<l\le p-1 \\ k,l\neq q}} (x_{l,1} - x_{k,1}) \int_{t_q}^{t_{q+1}} dx_{q,1} x_{q,1}^{B_q} \\ &\prod_{1\le k<q} (x_{q,1} - x_{k,1}) \prod_{q<l\le p-1} (x_{l,1} - x_{q,1}). \end{aligned}$$

In the case where $t_{q+1} \le A t_q$ for A sufficiently large, we only have

$$\begin{aligned} &\prod_{i=q+1}^{p-1} x_{i,1}^{B_i} \prod_{\substack{1\le k<l\le p-1 \\ k,l\neq q}} (x_{l,1} - x_{k,1}) \cdot \int_{t_q}^{t_{q+1}} dx_{q,1} x_{q,1}^{B_q} \\ &\prod_{1\le k<q} (x_{q,1} - x_{k,1}) \prod_{q<l\le p-1} (x_{l,1} - x_{q,1}) \\ \gtrsim\ & t_{q+1}^{B_q} \int_{t_q}^{t_{q+1}} dx_{q,1} \prod_{i=q+1}^{p-1} x_{i,1}^{B_i} \prod_{1\le k<l\le p-1} (x_{l,1} - x_{k,1}), \end{aligned}$$

putting us in the right position for the $(q+1)$'th step. In the opposite case $t_{q+1} > At_q$ we have

$$\prod_{i=q+1}^{p-1} x_{i,1}^{B_i} \prod_{\substack{1\le k<l\le p-1\\ k,l\neq q}} (x_{l,1}-x_{k,1}) \int_{t_q}^{t_{q+1}} dx_{q,1} x_{q,1}^{B_q}$$

$$\prod_{1\le k<q} (x_{q,1}-x_{k,1}) \prod_{q<l\le p-1} (x_{l,1}-x_{q,1})$$

$$\gtrsim \prod_{i=q+1}^{p-1} x_{i,1}^{B_i} \prod_{\substack{1\le k<l\le p-1\\ k,l\neq q}} (x_{l,1}-x_{k,1}) \int_{\sqrt{A}t_q}^{t_{q+1}/\sqrt{A}} dx_{q,1} x_{q,1}^{B_q}$$

$$\prod_{1\le k<q} (x_{q,1}-x_{k,1}) \prod_{q<l\le p-1} (x_{l,1}-x_{q,1})$$

$$\gtrsim \prod_{i=q+1}^{p-1} x_{i,1}^{B_i} \prod_{\substack{1\le k<l\le p-1\\ k,l\neq q}} (x_{l,1}-x_{k,1}) \int_{\sqrt{A}t_q}^{t_{q+1}/\sqrt{A}} dx_{q,1} x_{q,1}^{B_q+q-1} \prod_{q<l\le p-1} x_{l,1}$$

$$\gtrsim \prod_{i=q+1}^{p-1} x_{i,1}^{B_i} \prod_{\substack{1\le k<l\le p-1\\ k,l\neq q}} (x_{l,1}-x_{k,1}) \prod_{q<l\le p-1} x_{l,1} \quad t_{q+1}^{B_q+q}$$

$$\gtrsim t_{q+1}^{B_q} \int_{t_q}^{t_{q+1}} dx_{q,1} \prod_{i=q+1}^{p-1} x_{i,1}^{B_i} t_{q+1}^{q-1} \prod_{\substack{1\le k<l\le p-1\\ k,l\neq q}} (x_{l,1}-x_{k,1}) \prod_{q<l\le p-1} x_{l,1}$$

$$\gtrsim t_{q+1}^{B_q} \int_{t_q}^{t_{q+1}} dx_{q,1} \prod_{i=q+1}^{p-1} x_{i,1}^{B_i} \prod_{1\le k<l\le p-1} (x_{l,1}-x_{k,1}),$$

again putting us in the right position for the $(q+1)$'th step. This iterative procedure will finish after $p-1$ steps, proving (2.66) and hence completing the proof of Lemma 2.8.6.

For the integral in (2.61) that we want to estimate, we have $\alpha_{k,l} = j_l - j_{l-1} - 1$. Thus

$$A_i = \sum_{r=1}^{i} \alpha_{i-r+1,r} = \sum_{r=1}^{i} j_r - j_{r-1} - 1 = j_i - i.$$

So from (2.61) and Lemma 2.8.6, we have

$$J_{P_1\ldots P_n}(t_1,\ldots,t_n) \gtrsim \prod_{i=1}^{n} p_{i,j_i} \prod_{i=1}^{n} t_i^{j_i-i} \prod_{1\le k<l\le n} (t_l - t_k),$$

completing the proof of Proposition 2.8.2.

REMARK. An analogous estimate to Proposition 2.8.2 holds for $P_1,\ldots P_n$ replaced by any $P_{\xi(1)},\ldots,P_{\xi(\mu)}$ with $1\le\mu\le n$ and ξ a one-to-one function from $1,\ldots,\mu$ to $1,\ldots,n$.

In the proof of Theorem 2.8.1 we will perform the change of variables $t \mapsto x(t)$, $t = (t_1, \dots t_n)$, where $x_k(t) = \sum_{i=1}^{n} P_k(t_i)$, $1 \le k \le n$. The following lemma will allow us to perform this change of variables.

Lemma 2.8.7 *If $s_i', s_i'' \in I$ with I as above and $P = (P_1, \dots, P_n)$, $s_1' < \dots < s_n'$, $s_1'' < \dots < s_n''$, and*

$$\sum_{i=1}^{n} P(s_i') = \sum_{i=1}^{n} P(s_i'') , \tag{2.67}$$

then $s_i' = s_i''$ for all $1 \le i \le n$.

PROOF. The proof of this lemma makes use of Proposition 2.8.2 which is also used directly in the proof of Theorem 2.8.1.Let us assume first that for any $1 \le i, j \le n$, $s_i' \ne s_j''$. The equation

$$\sum_{i=1}^{n} P(s_i') = \sum_{i=1}^{n} P(s_i'') ,$$

can be rewritten as

$$\sum_{k=1}^{2n} \epsilon_k P(s_k) = 0 ,$$

where each s_k is one of the s_i' or the s_i'' such that $s_1 < \dots < s_{2n}$ and $\epsilon_k = 1$ if $s_k \in \{s_1', \dots, s_n'\}$ and $\epsilon_k = -1$ if $s_k \in \{s_1'', \dots, s_n''\}$. We observe that $\sum_{k=1}^{2n} \epsilon_k = 0$. Let $\alpha_l = \sum_{k=1}^{l} \epsilon_k$. Then α_l has at most $n-1$ changes of sign. Thus

$$0 = \sum_{k=1}^{2n} \epsilon_k P(s_k) = \sum_{k=1}^{2n-1} \alpha_k (P(s_k) - P(s_{k+1})) = \int_{s_1}^{s_{2n}} \phi(s) P'(s) ds$$

with $\phi(s)$ a step function. Let $\cup_{l=1}^{\mu} I_l$ be a partition of $[s_1, s_{2n}]$ into intervals on which ϕ is single-signed. Note that $\mu \le n$ and

$$0 = \sum_{l=1}^{\mu} \int_{I_l} \phi(s) P'(s) ds. \tag{2.68}$$

Hence we have

$$\begin{vmatrix} \int_{I_1} |\phi(s)| P_1'(s) ds & \cdots & \int_{I_1} |\phi(s)| P_\mu'(s) ds \\ \vdots & & \vdots \\ \int_{I_\mu} |\phi(s)| P_1'(s) ds & \cdots & \int_{I_\mu} |\phi(s)| P_\mu'(s) ds \end{vmatrix} = 0.$$

This in turn implies

$$\int_{u_1 \in I_1} \cdots \int_{u_\mu \in I_\mu} |\phi(u_1)| \dots |\phi(u_\mu)| \begin{vmatrix} P_1'(u_1) & \cdots & P_\mu'(u_1) \\ \vdots & & \vdots \\ P_1'(u_\mu) & \cdots & P_\mu'(u_\mu) \end{vmatrix} du_1 \dots du_\mu = 0. \tag{2.69}$$

But by the remark after Proposition 2.8.2 we have that

$$\begin{vmatrix} P_1'(u_1) & \cdots & P_\mu'(u_1) \\ \vdots & & \vdots \\ P_1'(u_\mu) & \cdots & P_\mu'(u_\mu) \end{vmatrix} \gtrsim \prod_{i=1}^{\mu} p_{i,j_i} \prod_{i=1}^{\mu} u_i^{j_i-i} \prod_{1\le k<l\le\mu} (u_l - u_k), \tag{2.70}$$

which implies that

$$\begin{vmatrix} P_1'(u_1) & \cdots & P_\mu'(u_1) \\ \vdots & & \vdots \\ P_1'(u_\mu) & \cdots & P_\mu'(u_\mu) \end{vmatrix} = J_{P_1\dots P_\mu}(u_1,\dots,u_\mu)$$

is single signed and because of (2.69)

$$J_{P_1\dots P_\mu}(u_1,\dots,u_\mu) \equiv 0.$$

This then contradicts (2.70). If we have that at least some $s_i' \neq s_j''$ for some $1 \le i, j \le n$, but there are some $s_i' = s_j''$, we can still obtain a contradiction by cancelling the corresponding $P(s_i')$'s and $P(s_j'')$'s from either side of (2.67) and then considering a smaller number of equations. This leaves us with the case that for each s_i' there is a s_j'' such that $s_i' = s_j''$. Recalling though that $s_1' < \ldots < s_n'$ and $s_1'' < \ldots < s_n''$, one can realise that the only way this can happen is if $i = j$ for all $1 \le i \le n$. This completes the proof of Lemma 2.8.7.

2.8.2 Proof of the restriction theorem

We now conclude with the proof of Theorem 2.8.1, by making use of the results of the previous section.

To prove Theorem 2.8.1 we see by duality that it suffices to show

$$\|\widehat{gd\sigma}\|_{p'} \lesssim \|g\|_{q'(d\omega)}\,, \tag{2.71}$$

where

$$d\sigma(\phi) = \int_I \phi(P(s))L(s)^\alpha ds$$

and

$$d\omega(\phi) = \int_I \phi(s)L(s)^\alpha ds\,,$$

with $\alpha = \frac{2}{n(n+1)}$. Now with $gd\sigma * \ldots * gd\sigma$ denoting the n-fold convolution of $gd\sigma$ with itself we have

$$\|\widehat{gd\sigma}\|_{p'}^n = \|\widehat{gd\sigma}^n\|_{p'/n} = \|gd\sigma \widehat{* \ldots *} gd\sigma\|_{p'/n} \le \|gd\sigma * \ldots * gd\sigma\|_r\,, \tag{2.72}$$

where $nr' = p'$ by the Hausdorff-Young inequality. Note that because $p < \frac{n(n+2)}{n(n+2)-2}$, we have $1 \le r \le 2$. Now

$$gd\sigma * \ldots * gd\sigma(\phi) = \int_{I^n} \phi\left(\sum_{i=1}^n P(t_i)\right) \prod_{i=1}^n g(t_i)L(t_i)^\alpha dt\,,$$

where $t = (t_1, \ldots, t_n)$. For $\pi \in S_n$ a permutation of $\{1, \ldots, n\}$ and writing $x = (x_1, \ldots, x_n)$,

$$\begin{aligned} gd\sigma * \ldots * gd\sigma(\phi) &= \sum_{\pi \in S_n} \int_{\{t_{\pi(1)} < \ldots < t_{\pi(n)}\} \cap I^n} \phi\left(\sum_{i=1}^n P(t_i)\right) \prod_{i=1}^n g(t_i) L(t_i)^\alpha dt \\ &= \sum_{\pi \in S_n} \int_{D_\pi} \phi(x) \prod_{i=1}^n g(t_i) L(t_i)^\alpha \frac{1}{|J(t)|} dx , \end{aligned}$$

where in the second inequality we perform the change of variables

$$x_k = \sum_{i=1}^n P_k(t_i)$$

separately on each region $t_{\pi(1)} < \ldots t_{\pi(n)}$, and which is well defined in each region $t_{\pi(1)} < \ldots < t_{\pi(n)}$ by Lemma 2.8.7 (note the slight abuse of notation). D_π is the image of the region $\{t_{\pi(1)} < \ldots < t_{\pi(n)}\} \cap I^n$ under this transformation and $J(t) = J_{P_1 \ldots P_n}(t)$ is the Jacobian of the transformation. Hence

$$gd\sigma * \ldots * gd\sigma = \sum_{\pi \in S_n} \prod_{i=1}^n g(t_i) L(t_i)^\alpha \frac{1}{|J(t)|} \chi_{D_\pi} .$$

Therefore

$$\begin{aligned} \|gd\sigma * \ldots * gd\sigma\|_r &\leq \sum_{\pi \in S_n} \| \prod_{i=1}^n g(t_i) L(t_i)^\alpha \frac{1}{|J(t)|} \chi_{D_\pi} \|_r \\ &= \sum_{\pi \in S_n} \left(\int_{\{t_{\pi(1)} < \ldots < t_{\pi(n)}\} \cap I^n} \prod_{i=1}^n |g(t_i)|^r L(t_i)^{r\alpha} \frac{1}{|J(t)|^{r-1}} dt \right)^{\frac{1}{r}} , \end{aligned}$$

by changing variables back. From the estimate for the Jacobian in Proposition 2.8.2 it follows that

$$\begin{aligned} &\|gd\sigma * \ldots * gd\sigma\|_r \\ &\leq \sum_{\pi \in S_n} \left(\int_{\{t_{\pi(1)} < \ldots < t_{\pi(n)}\} \cap I^n} \prod_{i=1}^n |g(t_i)|^r L(t_i)^{r\alpha - \frac{r-1}{n}} \prod_{k<l} (t_l - t_k)^{1-r} dt \right)^{\frac{1}{r}} . \end{aligned}$$

Finally we will need to use a result of M. Christ which is Proposition 2.2 in [Ch]. Let us state the result as it appears in [Ch].

Proposition 2.8.8 *If* $0 \leq \gamma$ *then*

$$\int \prod_{i=1}^n f(x_i) \prod_{i<j\leq n} |x_i - x_j|^{-\gamma} dx_1 \ldots dx_n \leq C \|f\|_p^n,$$

for all f, *if and only if* $\gamma < 2/n$, $1 \leq p < n$ *and* $p^{-1} + \gamma(n-1)/2 = 1$.

We need to use this proposition with $\gamma = r - 1$. One can easily check that $r - 1 < 2/n$ since $nr' = p'$ and $p < \frac{n(n+2)}{n(n+2)-2}$. Using Proposition 2.8.8, we obtain

$$\|gd\sigma * \ldots * gd\sigma\|_r \lesssim \left(\int (|g(t)|^r L(t)^{\frac{1}{n}+r(\alpha-\frac{1}{n})})^{\tilde{p}} dt \right)^{\frac{n}{\tilde{p}r}} ,$$

where

$$\frac{1}{\tilde{p}} + (r-1)\frac{n-1}{2} = 1. \tag{2.73}$$

By (2.71) and (2.72) we see that the required relations for (2.71) to hold are

$$\tilde{p}r = q' \quad \text{and} \quad \frac{\tilde{p}}{n} + r\tilde{p}\left(\frac{2}{n(n+1)} - \frac{1}{n}\right) = \frac{2}{n(n+1)} = \alpha\,.$$

This can be verified by algebraic calculations, using (2.73), $nr' = p'$ and $\frac{1}{q} = \frac{n(n+1)}{2}\frac{1}{p'}$.

Chapter 3

An alternative method for the two-dimensional case

In this chapter we provide an alternative method of proving Theorem 2.1.1, which only works for polynomials $P : \mathbb{R}^m \to \mathbb{R}^2$.

3.1 Preliminary reductions

By following the same arguments as in Section 2.2 and the beginning of Section 2.5, we reduce ourselves to proving (2.19) for $P : \mathbb{R}^+ \to \mathbb{R}^n$ for each α where on I_α the components of $P(t) = (P_1(t), \dots, P_n(t))$ look like various monomials according to Lemma 2.3.1. The method we use in this chapter though only works for $n = 2$ and this is the case that we concentrate on. Therefore, (2.19) becomes

$$\int_{I_\alpha} \frac{|\widehat{a}(P_1(t), P_2(t))|}{t} dt \lesssim 1. \tag{3.1}$$

In particular if $P_i(t) = \sum_{m=1}^{d_i} p_{i,m} t^m$, then on I_α,

$$P_i(t) \sim p_{i,j_i} t^{j_i},$$

for some $1 \leq j_i \leq d_i$, $i = 1, 2$. We still wish to reduce ourselves to the case where $j_1 \neq j_2$. This of course can be done as in Section 2.5 by using almost rotations, but in this section we show that it can be done by using actual rotations. Let us recall that a rotation is represented by a matrix of the form

$$\frac{1}{\sqrt{C^2 + D^2}} \begin{pmatrix} D & -C \\ C & D \end{pmatrix}$$

and that atoms are invariant under rotations (see Section 2.5). If we already have that $j_1 \neq j_2$, then we have nothing to do. If $j_1 = j_2 = l$, then we are in the situation where, on I_α,

$$P_1(t) \sim p_{1,l} t^l \quad \text{and} \quad P_2(t) \sim p_{2,l} t^l.$$

We are trying to create two new polynomials $\tilde{P}_1$ and $\tilde{P}_2$ given by

$$\begin{pmatrix} \tilde{P}_1 \\ \tilde{P}_2 \end{pmatrix} = \frac{1}{\sqrt{C^2+D^2}} \begin{pmatrix} D & -C \\ C & D \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix},$$

so that $\tilde{P}_1$ will not contain a t^l term and hence will not look like t^l in any of its gaps by the remark after Lemma 2.3.1 and I_α will be inside the t^l'th gap of $\tilde{P}_2$. We then subdivide I_α further into gaps and dyadic intervals according to $\tilde{P}_1$ and on these new gaps $\tilde{P}_1$ and $\tilde{P}_2$ look like distinct monomials. In order for $\tilde{P}_1$ not to have a t^l term, we need to set

$$D = p_{2,l} \quad \text{and} \quad C = p_{1,l}.$$

Therefore, it remains to prove that I_α will be inside the t^l'th gap of $\tilde{P}_2 = p_{1,l}P_1 + p_{2,l}P_2$; that is, if $t_{1,m}$, $t_{2,m}$ are the $m'th$ roots of P_1, P_2 respectively and w_m are the roots of $\tilde{P}_2 = p_{1,l}P_1 + p_{2,l}P_2$ ordered so that $|w_{m_1}| \leq |w_{m_2}|$ for $m_1 < m_2$, then we have

$$|w_l| \leq \max(|t_{1,l}|, |t_{2,l}|) \tag{3.2}$$

$$\text{and} \qquad |w_{l+1}| \geq \min(|t_{1,l+1}|, |t_{2,l+1}|). \tag{3.3}$$

So let us consider first the case where the degree of one of the P_1, P_2 is strictly less than the degree of the other, that is w.l.o.g. $d_2 < d_1$. Using the notation we have so far established, we have

$$P_1 = p_{1,d_1} \prod_{m=1}^{d_1} (t - t_{1,m}) \quad \text{and} \quad P_2 = p_{2,d_2} \prod_{m=1}^{d_2} (t - t_{2,m}).$$

Let us suppose that we are in a gap where both P_1 and P_2 look like t^l. Then

$$\begin{aligned} \tilde{P}_2 \quad &= \quad p_{1,d_1}(-1)^{d_1-l} \left(\sum_{j_1<\dots<j_{d_1-l}} t_{1,j_1} \dots t_{1,j_{d_1-l}} \right) P_1(t) \\ &\quad + p_{2,d_2}(-1)^{d_2-l} \left(\sum_{j_1<\dots<j_{d_2-l}} t_{2,j_1} \dots t_{2,j_{d_2-l}} \right) P_2(t) \end{aligned} \tag{3.4}$$

and at the same time we can express $\tilde{P}_2$ in terms of its roots w_m as

$$\tilde{P}_2(t) = p_{1,d_1}^2(-1)^{d_1-l} \left(\sum_{j_1<\dots<j_{d_1-l}} t_{1,j_1} \dots t_{1,j_{d_1-l}} \right) \prod_{m=1}^{d_1} (t - w_m). \tag{3.5}$$

Let us recall that being in the "t^l" gap for both P_1 and P_2 means that for A sufficiently large, $A|t_{1,l}| < A^{-1}|t_{1,l+1}|$, $A|t_{1,l}| < A^{-1}|t_{2,l+1}|$, $A|t_{2,l}| < A^{-1}|t_{1,l+1}|$ and $A|t_{2,l}| < A^{-1}|t_{2,l+1}|$. We first show (3.3).

By comparing the t^l coefficients from the right sides of (3.4) and (3.5), we obtain

$$p_{1,d_1}^2 \left(\sum_{j_1<\ldots<j_{d_1-l}} t_{1,j_1} \ldots t_{1,j_{d_1-l}} \right) \left(\sum_{j_1<\ldots<j_{d_1-l}} w_{j_1} \ldots w_{j_{d_1-l}} \right)$$
$$= p_{1,d_1}^2 \left(\sum_{j_1<\ldots<j_{d_1-l}} t_{1,j_1} \ldots t_{1,j_{d_1-l}} \right)^2 + p_{2,d_2}^2 \left(\sum_{j_1<\ldots<j_{d_2-l}} t_{2,j_1} \ldots t_{2,j_{d_2-l}} \right)^2 ,$$

which implies that

$$w_{l+1} \ldots w_{d_1}$$
$$\gtrsim \sum_{j_1<\ldots<j_{d_1-l}} w_{j_1} \ldots w_{j_{d_1-l}}$$
$$= \frac{p_{1,d_1}^2 \left(\sum_{j_1<\ldots<j_{d_1-l}} t_{1,j_1} \ldots t_{1,j_{d_1-l}} \right)^2 + p_{2,d_2}^2 \left(\sum_{j_1<\ldots<j_{d_2-l}} t_{2,j_1} \ldots t_{2,j_{d_2-l}} \right)^2}{\left| p_{1,d_1}^2 \sum_{j_1<\ldots<j_{d_1-l}} t_{1,j_1} \ldots t_{1,j_{d_1-l}} \right|}.$$

Next we observe that there exists an integer $k > l$ such that $|w_{l+1}| \sim |w_k| < A^{-2}|w_{k+1}|$ (in the case that $k = d_1$, take $|w_{k+1}| = \infty$). Then, comparing the t^k coefficients from the right sides of (3.4) and (3.5), we have

$$p_{1,d_1}^2 \left(\sum_{j_1<\ldots<j_{d_1-l}} t_{1,j_1} \ldots t_{1,j_{d_1-l}} \right) \left(\sum_{j_1<\ldots<j_{d_1-k}} w_{j_1} \ldots w_{j_{d_1-k}} \right)$$
$$= p_{1,d_1}^2 \left(\sum_{j_1<\ldots<j_{d_1-l}} t_{1,j_1} \ldots t_{1,j_{d_1-l}} \right) \left(\sum_{j_1<\ldots<j_{d_1-k}} t_{1,j_1} \ldots t_{1,j_{d_1-k}} \right)$$
$$+ p_{2,d_2}^2 \left(\sum_{j_1<\ldots<j_{d_2-l}} t_{2,j_1} \ldots t_{2,j_{d_2-l}} \right) \left(\sum_{j_1<\ldots<j_{d_2-k}} t_{2,j_1} \ldots t_{2,j_{d_2-k}} \right)$$

(note that if $k \geq d_2$ the last term in the above expresion vanishes). Thus

$$w_{k+1} \ldots w_{d_1}$$
$$\lesssim \sum_{j_1<\ldots<j_{d_1-k}} w_{j_1} \ldots w_{j_{d_1-k}}$$
$$= \frac{p_{1,d_1}^2 \left(\sum_{j_1<\ldots<j_{d_1-l}} t_{1,j_1} \ldots t_{1,j_{d_1-l}} \right) \left(\sum_{j_1<\ldots<j_{d_1-k}} t_{1,j_1} \ldots t_{1,j_{d_1-k}} \right)}{p_{1,d_1}^2 \sum_{j_1<\ldots<j_{d_1-l}} t_{1,j_1} \ldots t_{1,j_{d_1-l}}}$$
$$+ \frac{p_{2,d_2}^2 \left(\sum_{j_1<\ldots<j_{d_2-l}} t_{2,j_1} \ldots t_{2,j_{d_2-l}} \right) \left(\sum_{j_1<\ldots<j_{d_2-k}} t_{2,j_1} \ldots t_{2,j_{d_2-k}} \right)}{p_{1,d_1}^2 \sum_{j_1<\ldots<j_{d_1-l}} t_{1,j_1} \ldots t_{1,j_{d_1-l}}}.$$

Hence,

$$\begin{aligned}
& w_{l+1}^{k-l} \\
\sim\ & w_{l+1}\dots w_k \\
\gtrsim\ & \left[p_{1,d_1}^2\left(\sum_{j_1<\dots<j_{d_1-l}} t_{1,j_1}\dots t_{1,j_{d_1-l}}\right)^2 + p_{2,d_2}^2\left(\sum_{j_1<\dots<j_{d_2-l}} t_{2,j_1}\dots t_{2,j_{d_2-l}}\right)^2\right] \Big/ \\
& \left[p_{1,d_1}^2\left(\sum_{j_1<\dots<j_{d_1-l}} t_{1,j_1}\dots t_{1,j_{d_1-l}}\right)\left(\sum_{j_1<\dots<j_{d_1-k}} t_{1,j_1}\dots t_{1,j_{d_1-k}}\right)\right. \\
& \left. +\ p_{2,d_2}^2\left(\sum_{j_1<\dots<j_{d_2-l}} t_{2,j_1}\dots t_{2,j_{d_2-l}}\right)\left(\sum_{j_1<\dots<j_{d_2-k}} t_{2,j_1}\dots t_{2,j_{d_2-k}}\right)\right].
\end{aligned}$$

We now have two cases. For case (i) we have

$$\begin{aligned}
& p_{1,d_1}^2\left(\sum_{j_1<\dots<j_{d_1-l}} t_{1,j_1}\dots t_{1,j_{d_1-l}}\right)\left(\sum_{j_1<\dots<j_{d_1-k}} t_{1,j_1}\dots t_{1,j_{d_1-k}}\right) \\
\geq\ & p_{2,d_2}^2\left(\sum_{j_1<\dots<j_{d_2-l}} t_{2,j_1}\dots t_{2,j_{d_2-l}}\right)\left(\sum_{j_1<\dots<j_{d_2-k}} t_{2,j_1}\dots t_{2,j_{d_2-k}}\right).
\end{aligned}$$

In this case

$$\begin{aligned}
& w_{l+1}^{k-l} \\
\gtrsim\ & \frac{p_{1,d_1}^2\left(\sum_{j_1<\dots<j_{d_1-l}} t_{1,j_1}\dots t_{1,j_{d_1-l}}\right)^2}{p_{1,d_1}^2\left(\sum_{j_1<\dots<j_{d_1-l}} t_{1,j_1}\dots t_{1,j_{d_1-l}}\right)\left(\sum_{j_1<\dots<j_{d_1-k}} t_{1,j_1}\dots t_{1,j_{d_1-k}}\right)} \\
=\ & \frac{\sum_{j_1<\dots<j_{d_1-l}} t_{1,j_1}\dots t_{1,j_{d_1-l}}}{\sum_{j_1<\dots<j_{d_1-k}} t_{1,j_1}\dots t_{1,j_{d_1-k}}} \\
\gtrsim\ & \frac{t_{1,l+1}\dots t_{1,d_1}}{t_{1,k+1}\dots t_{1,d_1}} \\
\gtrsim\ & t_{1,l+1}^{k-l},
\end{aligned}$$

proving (3.3) for case (i). Case (ii) is the case where

$$\begin{aligned}
& p_{2,d_2}^2\left(\sum_{j_1<\dots<j_{d_2-l}} t_{2,j_1}\dots t_{2,j_{d_2-l}}\right)\left(\sum_{j_1<\dots<j_{d_2-k}} t_{2,j_1}\dots t_{2,j_{d_2-k}}\right) \\
\geq\ & p_{1,d_1}^2\left(\sum_{j_1<\dots<j_{d_1-l}} t_{1,j_1}\dots t_{1,j_{d_1-l}}\right)\left(\sum_{j_1<\dots<j_{d_1-k}} t_{1,j_1}\dots t_{1,j_{d_1-k}}\right)
\end{aligned}$$

and it is easy to see that in this case, following the same argument we obtain

$$w_{l+1} \gtrsim t_{2,l+1},$$

proving (3.3) in case (ii).

We turn now to showing (3.2). For this we observe that there exists an integer p with $0 \leq p \leq l-1$ such that $A^2|w_p| < |w_{p+1}| \sim |w_l|$. Then, by equating the "t^p" coefficients of both equivalent expressions for $\tilde{P}_2$, we have

$$\begin{aligned}
& w_{p+1} \cdots w_{d_1} \\
\sim\ & \sum_{j_1 < \ldots < j_{d_1 - p}} w_{j_1} \cdots w_{j_{d_1-p}} \\
=\ & \Bigg| \frac{p_{1,d_1}^2 \left(\sum_{j_1<\ldots<j_{d_1-l}} t_{1,j_1} \cdots t_{1,j_{d_1-l}}\right) \left(\sum_{j_1<\ldots<j_{d_1-p}} t_{1,j_1} \cdots t_{1,j_{d_1-p}}\right)}{p_{1,d_1}^2 \sum_{j_1<\ldots<j_{d_1-l}} t_{1,j_1} \cdots t_{1,j_{d_1-l}}} \\
& + \frac{p_{1,d_1}^2 \left(\sum_{j_1<\ldots<j_{d_1-l}} t_{2,j_1} \cdots t_{2,j_{d_1-l}}\right) \left(\sum_{j_1<\ldots<j_{d_1-p}} t_{2,j_1} \cdots t_{2,j_{d_1-p}}\right)}{p_{1,d_1}^2 \sum_{j_1<\ldots<j_{d_1-l}} t_{2,j_1} \cdots t_{2,j_{d_1-l}}} \Bigg| .
\end{aligned}$$

This implies that

$$\begin{aligned}
& w_l^{l-p} \\
\lesssim\ & \left| \frac{p_{1,d_1}^2 \left(\sum_{j_1<\ldots<j_{d_1-l}} t_{1,j_1} \cdots t_{1,j_{d_1-l}}\right) \left(\sum_{j_1<\ldots<j_{d_1-p}} t_{1,j_1} \cdots t_{1,j_{d_1-p}}\right)}{w_{l+1} \cdots w_{d_1} p_{1,d_1}^2 \sum_{j_1<\ldots<j_{d_1-l}} t_{1,j_1} \cdots t_{1,j_{d_1-l}}} \right| \\
& + \left| \frac{p_{1,d_1}^2 \left(\sum_{j_1<\ldots<j_{d_1-l}} t_{2,j_1} \cdots t_{2,j_{d_1-l}}\right) \left(\sum_{j_1<\ldots<j_{d_1-p}} t_{2,j_1} \cdots t_{2,j_{d_1-p}}\right)}{w_{l+1} \cdots w_{d_1} p_{1,d_1}^2 \sum_{j_1<\ldots<j_{d_1-l}} t_{2,j_1} \cdots t_{2,j_{d_1-l}}} \right| \\
\leq\ & \left| \frac{\sum_{j_1<\ldots<j_{d_1-p}} t_{1,j_1} \cdots t_{1,j_{d_1-p}}}{\sum_{j_1<\ldots<j_{d_1-l}} t_{1,j_1} \cdots t_{1,j_{d_1-l}}} \right| \\
& + \left| \frac{\sum_{j_1<\ldots<j_{d_1-p}} t_{2,j_1} \cdots t_{2,j_{d_1-p}}}{\sum_{j_1<\ldots<j_{d_1-l}} t_{2,j_1} \cdots t_{2,j_{d_1-l}}} \right| \\
\lesssim\ & \left| \frac{t_{1,p+1} \cdots t_{1,d_1}}{t_{1,l+1} \cdots t_{1,d_1}} \right| + \left| \frac{t_{2,p+1} \cdots t_{2,d_1}}{t_{2,l+1} \cdots t_{2,d_1}} \right| \\
\leq\ & |t_{1,l}|^{l-p} + |t_{2,l}|^{l-p}.
\end{aligned}$$

It then follows that

$$\begin{aligned}
w_l &\lesssim (|t_{1,l}|^{l-p} + |t_{2,l}|^{l-p})^{\frac{1}{l-p}} \\
&\lesssim \max(|t_{1,l}|, |t_{2,l}|),
\end{aligned}$$

completing the proof of (3.2).

We still have to consider the case where $d_1 = d_2 = d$. In this case

$$\begin{aligned}
\tilde{P}_2(t) &= p_{1,d}(-1)^{d-l} \sum_{j_1<\ldots<j_{d-l}} t_{j_1} \cdots t_{j_{d-l}} P_1(t) \\
&\quad + p_{2,d}(-1)^{d-l} \sum_{j_1<\ldots<j_{d-l}} t_{1,j_1} \cdots t_{1,j_{d-l}} P_2(t) \qquad (3.6)
\end{aligned}$$

and also in terms of the roots w_m of $\tilde{P}_2$,

$$\tilde{P}_2(t) = (-1)^{d-l}\left(p_{1,d}^2 \sum_{j_1<\ldots<j_{d-l}} t_{1,j_1}\ldots t_{1,j_{d-l}} + p_{2,d}^2 \sum_{j_1<\ldots<j_{d-l}} t_{2,j_1}\ldots t_{2,j_{d-l}}\right)\prod_{m=1}^{d}(t-w_m). \tag{3.7}$$

Let us recall that we still have $A|t_{1,l}| < A^{-1}|t_{1,l+1}|$, $A|t_{1,l}| < A^{-1}|t_{2,l+1}|$, $A|t_{2,l}| < A^{-1}|t_{1,l+1}|$ and $A|t_{2,l}| < A^{-1}|t_{2,l+1}|$ for sufficiently large A. Now by equating the "t^l" coefficients from the right sides of (3.6) and (3.7) we have

$$\begin{aligned}&\left(p_{1,d}^2 \sum_{j_1<\ldots<j_{d-l}} t_{1,j_1}\ldots t_{1,j_{d-l}} + p_{2,d}^2 \sum_{j_1<\ldots<j_{d-l}} t_{2,j_1}\ldots t_{2,j_{d-l}}\right)\sum_{j_1<\ldots<j_{d-l}} w_{j_1}\ldots w_{j_{d-l}}\\ &= p_{1,d}^2\left(\sum_{j_1<\ldots<j_{d-l}} t_{1,j_1}\ldots t_{1,j_{d-l}}\right)^2 + p_{2,d}^2\left(\sum_{j_1<\ldots<j_{d-l}} t_{2,j_1}\ldots t_{2,j_{d-l}}\right)^2.\end{aligned}$$

This implies that

$$\begin{aligned}&w_{l+1}\ldots w_d\\ &\gtrsim \sum_{j_1<\ldots<j_{d_l}} w_{j_1}\ldots w_{j_{d-l}}\\ &= \frac{p_{1,d}^2\left(\sum_{j_1<\ldots<j_{d-l}} t_{1,j_1}\ldots t_{1,j_{d-l}}\right)^2 + p_{2,d}^2\left(\sum_{j_1<\ldots<j_{d-l}} t_{2,j_1}\ldots t_{2,j_{d-l}}\right)^2}{p_{1,d}^2\sum_{j_1<\ldots<j_{d-l}} t_{1,j_1}\ldots t_{1,j_{d-l}} + p_{2,d}^2\sum_{j_1<\ldots<j_{d-l}} t_{2,j_1}\ldots t_{2,j_{d-l}}}.\end{aligned}$$

As before there exists an integer $k > l$ such that $|w_{l+1}| \sim |w_k| < A^{-2}|w_{k+1}|$ (again take $|w_{d+1}| = \infty$). Then equating the "t^k" coefficients from the right sides of (3.6) and (3.7) we obtain

$$\begin{aligned}&w_{k+1}\ldots w_d\\ &\sim \sum_{j_1<\ldots<j_{d-k}} w_{j_1}\ldots w_{j_{d-k}}\\ &= \frac{p_{1,d}^2\left(\sum_{j_1<\ldots<j_{d-l}} t_{1,j_1}\ldots t_{1,j_{d-l}}\right)\left(\sum_{j_1<\ldots<j_{d-k}} t_{1,j_1}\ldots t_{1,j_{d-k}}\right)}{p_{1,d}^2\sum_{j_1<\ldots<j_{d-l}} t_{1,j_1}\ldots t_{1,j_{d-l}} + p_{2,d}^2\sum_{j_1<\ldots<j_{d-l}} t_{2,j_1}\ldots t_{2,j_{d-l}}}\\ &\quad + \frac{p_{2,d}^2\left(\sum_{j_1<\ldots<j_{d-l}} t_{2,j_1}\ldots t_{2,j_{d-l}}\right)\left(\sum_{j_1<\ldots<j_{d-k}} t_{2,j_1}\ldots t_{2,j_{d-k}}\right)}{p_{1,d}^2\sum_{j_1<\ldots<j_{d-l}} t_{1,j_1}\ldots t_{1,j_{d-l}} + p_{2,d}^2\sum_{j_1<\ldots<j_{d-l}} t_{2,j_1}\ldots t_{2,j_{d-l}}}.\end{aligned}$$

Therefore,

$$
\begin{aligned}
& w_{l+1}^{k-l} \\
\sim\ & w_{l+1} \dots w_k \\
\gtrsim\ & \left[p_{1,d}^2 \left(\sum_{j_1 < \dots < j_{d-l}} t_{1,j_1} \dots t_{1,j_{d-l}} \right)^2 + p_{2,d}^2 \left(\sum_{j_1 < \dots < j_{d-l}} t_{2,j_1} \dots t_{2,j_{d-l}} \right)^2 \right] \Bigg/ \\
& \left[p_{1,d}^2 \left(\sum_{j_1 < \dots < j_{d-l}} t_{1,j_1} \dots t_{1,j_{d-l}} \right) \left(\sum_{j_1 < \dots < j_{d-k}} t_{1,j_1} \dots t_{1,j_{d-k}} \right) \right. \\
& \left. + p_{2,d}^2 \left(\sum_{j_1 < \dots < j_{d-l}} t_{2,j_1} \dots t_{2,j_{d-l}} \right) \left(\sum_{j_1 < \dots < j_{d-k}} t_{2,j_1} \dots t_{2,j_{d-k}} \right) \right].
\end{aligned}
$$

We then have two cases. For case (i), we have

$$
\begin{aligned}
& \left| p_{1,d}^2 \left(\sum_{j_1 < \dots < j_{d-l}} t_{1,j_1} \dots t_{1,j_{d-l}} \right) \left(\sum_{j_1 < \dots < j_{d-k}} t_{1,j_1} \dots t_{1,j_{d-k}} \right) \right| \\
\geq\ & \left| p_{2,d}^2 \left(\sum_{j_1 < \dots < j_{d-l}} t_{2,j_1} \dots t_{2,j_{d-l}} \right) \left(\sum_{j_1 < \dots < j_{d-k}} t_{2,j_1} \dots t_{2,j_{d-k}} \right) \right|.
\end{aligned}
$$

Then,

$$
\begin{aligned}
w_{l+1}^{k-l} &\gtrsim \frac{\sum_{j_1 < \dots < j_{d-l}} t_{1,j_1} \dots t_{1,j_{d-l}}}{\sum_{j_1 < \dots < j_{d-k}} t_{1,j_1} \dots t_{1,j_{d-k}}} \\
&\gtrsim \frac{t_{1,l+1} \dots t_{1,d}}{t_{1,k+1} \dots t_{1,d}} \\
&= t_{1,l+1} \dots t_{1,k} \\
&\sim t_{1,l+1}^{k-l},
\end{aligned}
$$

proving (3.3) for case (i). For the opposite case

$$
\begin{aligned}
& \left| p_{1,d}^2 \left(\sum_{j_1 < \dots < j_{d-l}} t_{1,j_1} \dots t_{1,j_{d-l}} \right) \left(\sum_{j_1 < \dots < j_{d-k}} t_{1,j_1} \dots t_{1,j_{d-k}} \right) \right| \\
\leq\ & \left| p_{2,d}^2 \left(\sum_{j_1 < \dots < j_{d-l}} t_{2,j_1} \dots t_{2,j_{d-l}} \right) \left(\sum_{j_1 < \dots < j_{d-k}} t_{2,j_1} \dots t_{2,j_{d-k}} \right) \right|,
\end{aligned}
$$

a completely identical argument shows that $w_{l+1} \gtrsim t_{2,l+1}$, hence completing the proof of (3.3).

It now remains to show (3.2) for the case where $d_1 = d_2 = d$. There exists a

p, $0 \le p \le l-1$ such that $A^2|w_p| < |w_{p+1}| \sim |w_l|$ (take $|w_0| = 0$). Then

$$\begin{aligned}
& w_{p+1} \dots w_d \\
\sim & \sum_{j_1 < \dots < j_{d-p}} w_{j_1} \dots w_{j_{d-p}} \\
= & \frac{p_{1,d}^2 \left(\sum_{j_1<\dots<j_{d-l}} t_{1,j_1} \dots t_{1,j_{n-l}}\right) \left(\sum_{j_1<\dots<j_{d-p}} t_{1,j_1} \dots t_{1,j_{n-p}}\right)}{p_{1,d}^2 \sum_{j_1<\dots<j_{d-l}} t_{1,j_1} \dots t_{1,j_{n-l}} + p_{2,d}^2 \sum_{j_1<\dots<j_{d-l}} t_{2,j_1} \dots t_{2,j_{n-l}}} \\
& + \frac{p_{2,d}^2 \left(\sum_{j_1<\dots<j_{d-l}} t_{2,j_1} \dots t_{2,j_{n-l}}\right) \left(\sum_{j_1<\dots<j_{d-p}} t_{2,j_1} \dots t_{2,j_{n-p}}\right)}{p_{1,d}^2 \sum_{j_1<\dots<j_{d-l}} t_{1,j_1} \dots t_{1,j_{n-l}} + p_{2,d}^2 \sum_{j_1<\dots<j_{d-l}} t_{2,j_1} \dots t_{2,j_{n-l}}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
& w_l^{l-p} \\
\lesssim & \left| \frac{p_{1,d}^2 \left(\sum_{j_1<\dots<j_{d-l}} t_{1,j_1} \dots t_{1,j_{n-l}}\right) \left(\sum_{j_1<\dots<j_{d-p}} t_{1,j_1} \dots t_{1,j_{n-p}}\right)}{w_{l+1} \dots w_d \left(p_{1,d}^2 \sum_{j_1<\dots<j_{d-l}} t_{1,j_1} \dots t_{1,j_{n-l}} + p_{2,d}^2 \sum_{j_1<\dots<j_{d-l}} t_{2,j_1} \dots t_{2,j_{n-l}}\right)} \right. \\
& \left. + \frac{p_{2,d}^2 \left(\sum_{j_1<\dots<j_{d-l}} t_{2,j_1} \dots t_{2,j_{n-l}}\right) \left(\sum_{j_1<\dots<j_{d-p}} t_{2,j_1} \dots t_{2,j_{n-p}}\right)}{w_{l+1} \dots w_d \left(p_{1,d}^2 \sum_{j_1<\dots<j_{d-l}} t_{1,j_1} \dots t_{1,j_{n-l}} + p_{2,d}^2 \sum_{j_1<\dots<j_{d-l}} t_{2,j_1} \dots t_{2,j_{n-l}}\right)} \right| \\
\le & \left| \frac{p_{1,d}^2 \left(\sum_{j_1<\dots<j_{d-l}} t_{1,j_1} \dots t_{1,j_{n-l}}\right) \left(\sum_{j_1<\dots<j_{d-p}} t_{1,j_1} \dots t_{1,j_{n-p}}\right)}{p_{1,d}^2 \left(\sum_{j_1<\dots<j_{d-l}} t_{1,j_1} \dots t_{1,j_{n-l}}\right)^2} \right| \\
& + \left| \frac{p_{2,d}^2 \left(\sum_{j_1<\dots<j_{d-l}} t_{2,j_1} \dots t_{2,j_{n-l}}\right) \left(\sum_{j_1<\dots<j_{d-p}} t_{2,j_1} \dots t_{2,j_{n-p}}\right)}{p_{2,d}^2 \left(\sum_{j_1<\dots<j_{d-l}} t_{2,j_1} \dots t_{2,j_{n-l}}\right)^2} \right| \\
= & \left| \frac{\sum_{j_1 \dots j_{d-p}} t_{1,j_1} \dots t_{1,j_{d-p}}}{\sum_{j_1 \dots j_{d-l}} t_{1,j_1} \dots t_{1,j_{d-l}}} \right| + \left| \frac{\sum_{j_1 \dots j_{d-p}} t_{2,j_1} \dots t_{2,j_{d-p}}}{\sum_{j_1 \dots j_{d-l}} t_{2,j_1} \dots t_{2,j_{d-l}}} \right| \\
\lesssim & \left| \frac{t_{1,p+1} \dots t_{1,d}}{t_{1,l+1} \dots t_{1,d}} \right| + \left| \frac{t_{2,p+1} \dots t_{2,d}}{t_{2,l+1} \dots t_{2,d}} \right| \\
\le & |t_{1,l}|^{l-p} + |t_{2,l}|^{l-p}
\end{aligned}$$

and from this follows that

$$|w_l| \lesssim \left(|t_{1,l}|^{l-p} + |t_{2,l}|^{l-p}\right)^{\frac{1}{l-p}} \lesssim \max(|t_{1,l}|, |t_{2,l}|),$$

thus completing the proof of (3.2).

By applying this rotation in the same manner as in Section 2.5, we reduce ourselves to establishing

$$\int_I \frac{|\widehat{a}(P_1(t), P_2(t))|}{t} dt \lesssim 1 \tag{3.8}$$

for an atom a supported in a cube Q, centred at the origin, such that $\int a = 0$ and $||a||_\infty \leq |Q|^{-1}$. Furthermore, P_1, P_2 satisfy

$$P_i(t) \sim p_{i,j_i} t^{j_i}$$

for $i = 1, 2$, on I where the exponents j_1, j_2 are distinct and nonzero. In fact

$$I \subseteq [A|t_{1,j_1}|, A^{-1}|t_{1,j_1+1}|] \cap [A|t_{2,j_2}|, A^{-1}|t_{2,j_2+1}|]$$

and so the conclusions of Lemmas 2.3.1, 2.3.3, 2.3.4 and 2.3.5 hold for each P_i on I if A is chosen large enough. We will use these lemmas in what follows, since we do not have a special proof in two dimensions. For the same reasons we will also just make use of Proposition 2.7.1. We thus proceed as in Section 2.6, to split the integral in 3.8 near and away from the origin. For the part near the origin we again don't have a special proof. For the part away from the origin we do and that is what we concetrate on. We therefore have to establish

$$\int_\lambda^D \frac{|\widehat{a}(P_1(t), P_2(t))|}{t} dt \lesssim 1, \tag{3.9}$$

with $\lambda = (|Q||p_{1,j_1}||p_{2,j_2}|)^{-\frac{1}{j_1+j_2}}$ (for the reason that we choose this value for λ, see Section 2.6). By applying the Cauchy-Schwarz inequality, we have

$$\int_\lambda^D \frac{|\widehat{a}(P_1(t), P_2(t))|}{t} dt \leq \left(\int_\lambda^D \frac{1}{t^2} dt\right)^{\frac{1}{2}} \left(\int_\lambda^D |\widehat{a}(P_1(t), P_2(t))|^2 dt\right)^{\frac{1}{2}}.$$

The first term in the product is bounded above by

$$\left(\int_\lambda^D \frac{1}{t^2} dt\right)^{\frac{1}{2}} \leq (|Q||p_{1,j_1}||p_{2,j_2}|)^{\frac{1}{2(j_1+j_2)}}.$$

Hence we would be finished if we proved the following theorem which is effectively a L^2-restriction theorem for atoms.

Theorem 3.1.1 *For an atom a and P_1, P_2, I as above,*

$$\left(\int_I |\widehat{a}(P_1(t), P_2(t))|^2 dt\right)^{\frac{1}{2}} \lesssim (|Q||p_{1,j_1}||p_{2,j_2}|)^{-\frac{1}{2(j_1+j_2)}}.$$

The proof of this theorem is carried out in the next section.

3.2 Proof of a restriction theorem for $\mathbb{R}^2$-atoms

The proof will be different for the case where one of the j_1, j_2 is equal to 1 to the case where both j_1, j_2 are strictly greater than 1. Of course in both cases we have that $j_1 \neq j_2$. In Section 3.2.1 we deal with the latter case whereas in Section 3.2.2 we deal with the former case.

3.2.1 The case $2 \leq j_1 \neq j_2$

We start by defining the measure σ by

$$\sigma(\phi) = \int_I \phi\left(\frac{P_1(t)}{p_{1,j_1}}, \frac{P_2(t)}{p_{2,j_2}}\right) dt \tag{3.10}$$

and the quasimetric ρ by

$$\rho(x,y) = |x|^{\frac{1}{j_1}} + |y|^{\frac{1}{j_2}}. \tag{3.11}$$

We will require the following two preliminary results.

Proposition 3.2.1 *With $2 \leq j_1 \neq j_2$ and σ, ρ, defined as in (3.10) and (3.11), we have*

$$\widehat{\sigma}(x,y) = \int_I e^{i\left(x\frac{P_1(t)}{p_{1,j_1}} + y\frac{P_2(t)}{p_{2,j_2}}\right)} dt \lesssim \frac{1}{\rho(x,y)}.$$

REMARKS. 1. This proposition fails for $j_1 = 1 < j_2$. For this case though we have been able to prove a similar proposition using Euclidean balls. This is done in Section 3.2.2.

2. Proposition 3.2.1 fails more substantially in higher dimensions.

PROOF. The statement is equivalent to proving that if, on $I = [B, D]$,

$$P_i(t) \sim t^{j_i}, \quad i = 1, 2,$$

then

$$\int_I e^{i(xP_1(t)+yP_2(t))} dt \lesssim \frac{1}{\rho(x,y)}. \tag{3.12}$$

We recall from Lemma 2.3.1 that in this case we have

$$A_1 t^{j_1-1} \leq |P_1'(t)| \leq A_1' t^{j_1-1},$$

$$B_1 t^{j_2-1} \leq |P_2'(t)| \leq B_1' t^{j_2-1},$$

where A_1, A_1', B_1, B_1' are constants only depending on the degree of P_1, P_2. Let us assume w.l.o.g. that $j_2 > j_1$ and consider first the range where $|y|^{1/j_2} \geq |x|^{1/j_1}$. We split the integral in (3.12) at $C(A_1'/B_1)^{1/(j_2-j_1)}|y|^{-1/j_2}$ for $C > 1$ to obtain

$$\begin{aligned} &\int_I e^{i(xP_1(t)+yP_2(t))} dt \\ =& \int_B^{C\left(\frac{A_1'}{B_1}\right)^{\frac{1}{j_2-j_1}}|y|^{-1/j_2}} e^{i(xP_1(t)+yP_2(t))} dt + \int_{C\left(\frac{A_1'}{B_1}\right)^{\frac{1}{j_2-j_1}}|y|^{-1/j_2}}^D e^{i(xP_1(t)+yP_2(t))} dt \\ =:& \ \mathrm{I} + \mathrm{II}. \end{aligned}$$

Integral I is clearly bounded above by $C(A_1'/B_1)^{1/(j_2-j_1)}|y|^{-1/j_2} \lesssim \rho(x,y)^{-1}$. For II we want to use van der Corput's lemma (see e.g. [S2]) to bound the integral from above. In order to use van der Corput's lemma, we need to bound some derivative of the phase function $xP_1(t) + yP_2(t)$ from below. Using Lemma 2.3.1, we have for the first derivative (we can use simple calculus to verify that the expression $|y|B_1 t^{j_2-1} - |x|A_1' t^{j_1-1}$ is monotonic and increasing in $(C(A_1'/B_1)^{\frac{1}{j_2-j_1}}|y|^{-1/j_2}, D))$,

$$\begin{aligned}
|xP_1'(t) + yP_2'(t)| &\geq |y|B_1 t^{j_2-1} - |x|A_1' t^{j_1-1} \\
&\geq |y|B_1 \left(C \left(\frac{A_1'}{B_1} \right)^{\frac{1}{j_2-j_1}} |y|^{-\frac{1}{j_2}} \right)^{j_2-1} \\
&\quad -|x|A_1' \left(C \left(\frac{A_1'}{B_1} \right)^{\frac{1}{j_2-j_1}} |y|^{-\frac{1}{j_2}} \right)^{j_1-1} \\
&\geq C^{j_2-1} A_1'^{\frac{j_2-1}{j_2-j_1}} B_1^{\frac{1-j_1}{j_2-j_1}} |y|^{\frac{1}{j_2}} - C^{j_1-1} A_1'^{\frac{j_2-1}{j_2-j_1}} B_1^{\frac{1-j_1}{j_2-j_1}} |y|^{\frac{1}{j_2}} \\
&\sim |y|^{\frac{1}{j_2}}.
\end{aligned}$$

Hence, II $\lesssim |y|^{-\frac{1}{j_2}} \lesssim \rho(x,y)^{-1}$. This analysis takes care of the range $|y|^{1/j_2} \geq |x|^{1/j_1}$. We now turn to the oposite range $|y|^{1/j_2} \leq |x|^{1/j_1}$. For this range we split the integral in (3.12) as follows:

$$\begin{aligned}
&\int_I e^{i(xP_1(t)+yP_2(t))} dt \\
= &\int_B^{\delta\left(\frac{A_1}{B_1'}\right)^{\frac{1}{j_2-j_1}}|x|^{-\frac{1}{j_1}}} e^{i(xP_1(t)+yP_2(t))} dt + \int_{\delta\left(\frac{A_1}{B_1'}\right)^{\frac{1}{j_2-j_1}}|x|^{-\frac{1}{j_1}}}^{\delta\left(\frac{A_1}{B_1'}\frac{|x|}{|y|}\right)^{\frac{1}{j_2-j_1}}} e^{i(xP_1(t)+yP_2(t))} dt \\
&+ \int_{\delta\left(\frac{A_1|x|}{B_1'|y|}\right)^{\frac{1}{j_2-j_1}}}^{C\left(\frac{A_1'|x|}{B_1|y|}\right)^{\frac{1}{j_2-j_1}}} e^{i(xP_1(t)+yP_2(t))} dt + \int_{C\left(\frac{A_1'|x|}{B_1|y|}\right)^{\frac{1}{j_2-j_1}}}^{D} e^{i(xP_1(t)+yP_2(t))} dt \\
=: &\ \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},
\end{aligned}$$

where $\delta < 1$ and $C > 1$. Integral I is clearly bounded above by $\delta(A_1/B_1')^{1/(j_2-j_1)}$ $|x|^{-1/j_1} \lesssim \rho(x,y)^{-1}$. For integral II we use van der Corput's lemma. For the phase function we have

$$|xP_1'(t) + yP_2'(t)| \geq |x|A_1 t^{j_1-1} - |y|B_1' t^{j_2-1}.$$

The right side of this inequality is minimised at the two endpoints of II. At $\delta(A_1|x|/B_1'|y|)^{1/(j_2-j_1)}$ we have

$$\begin{aligned}
|x|A_1 t^{j_1-1} - |y|B_1' t^{j_2-1} &= |x|A_1 \delta^{j_1-1} \left(\frac{A_1|x|}{B_1'|y|} \right)^{\frac{j_1-1}{j_2-j_1}} - |y|B_1' \delta^{j_2-1} \left(\frac{A_1|x|}{B_1'|y|} \right)^{\frac{j_2-1}{j_2-j_1}} \\
&= (\delta^{j_1-1} - \delta^{j_2-1}) \left(\frac{(A_1|x|)^{j_2-1}}{(B_1'|y|)^{j_1-1}} \right)^{\frac{1}{j_2-j_1}} \\
&\gtrsim |x|^{\frac{1}{j_1}},
\end{aligned}$$

since we are in the range $|y|^{1/j_2} \le |x|^{1/j_1}$. At the other endpoint, $\delta(A_1/B_1')^{1/(j_2-j_1)}$ $|x|^{-1/j_1}$, we have

$$\begin{aligned} |x|A_1 t^{j_1-1} - |y|B_1' t^{j_2-1} &= |x|A_1\delta^{j_1-1}\left(\frac{A_1}{B_1'}\right)^{\frac{j_1-1}{j_2-j_1}}|x|^{-\frac{j_1-1}{j_1}} \\ &\quad - |y|B_1'\delta^{j_2-1}\left(\frac{A_1}{B_1'}\right)^{\frac{j_2-1}{j_2-j_1}}|x|^{-\frac{j_2-1}{j_1}} \\ &\ge (\delta^{j_1-1}-\delta^{j_2-1})\left(\frac{A_1^{j_2-1}}{B_1'^{j_1-1}}\right)^{\frac{1}{j_2-j_1}}|x|^{\frac{1}{j_1}}. \end{aligned}$$

So no matter which endpoint the phase function is minimised, we obtain the same bound from below and hence from van der Corput's lemma we see that II $\lesssim \rho(x,y)^{-1}$. Let us now consider integral IV. Using simple calculus we see that the expression $|y|B_1 t^{j_2-1} - |x|A_1' t^{j_1-1}$ is minimised at the endpoint $C(A_1'|x|/B_1|y|)^{1/(j_2-j_1)}$. Hence

$$\begin{aligned} |xP_1'(t)+yP_2'(t)| &\ge |y|B_1' t^{j_2-1} - |x|A_1 t^{j_1-1} \\ &\ge |y|B_1 C^{j_2-1}\left(\frac{A_1'|x|}{B_1|y|}\right)^{\frac{j_2-1}{j_2-j_1}} - |x|A_1' C^{j_1-1}\left(\frac{A_1'|x|}{B_1|y|}\right)^{\frac{j_1-1}{j_2-j_1}} \\ &= C^{j_2-1}(A_1'|x|)^{\frac{j_2-1}{j_2-j_1}}(B_1|y|)^{\frac{1-j_1}{j_2-j_1}} \\ &\quad - C^{j_1-1}(A_1'|x|)^{\frac{j_2-1}{j_2-j_1}}(B_1|y|)^{\frac{1-j_1}{j_2-j_1}} \\ &= \left(\frac{(A_1'|x|)^{j_2-1}}{(B_1|y|)^{j_1-1}}\right)^{\frac{1}{j_2-j_1}}(C^{j_2-1}-C^{j_1-1}) \\ &\gtrsim |x|^{\frac{1}{j_1}}, \end{aligned}$$

where in the last inequality we have used the fact that we are in the range $|x|^{1/j_1} \ge |y|^{1/j_2}$. Hence we can bound IV from above by $\rho(x,y)^{-1}$. So we are now left with estimating III in the range $|x|^{1/j_1} \ge |y|^{1/j_2}$. We define $\psi(t) = xP_1(t) + yP_2(t)$. Using matrix notation we write

$$\begin{pmatrix} \psi'(t) \\ \psi''(t) \end{pmatrix} = \begin{pmatrix} P_1'(t) & P_2'(t) \\ P_1''(t) & P_2''(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

This implies

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{P_1'(t)P_2''(t) - P_2'(t)P_1''(t)}\begin{pmatrix} P_2''(t) & -P_2'(t) \\ -P_1''(t) & P_1'(t) \end{pmatrix}\begin{pmatrix} \psi'(t) \\ \psi''(t) \end{pmatrix}.$$

By taking norms on both sides we have

$$|x|+|y| \le \frac{1}{|P_1'(t)P_2''(t) - P_2'(t)P_1''(t)|}\left\|\begin{pmatrix} P_2''(t) & -P_2'(t) \\ -P_1''(t) & P_1'(t) \end{pmatrix}\right\|(|\psi'(t)|+|\psi''(t)|).$$

At this point we use Proposition 2.7.1 to obtain

$$|\psi'(t)|+|\psi''(t)| \gtrsim (|x|+|y|)t^{-j_1-j_2+1}\frac{1}{\max(|P_1'(t)|,|P_2''(t)|,|P_2'(t)|,|P_1''(t)|)}.$$

If $t<1$ we have from Lemma 2.3.3 that $\max(|P_1'(t)|,|P_2''(t)|,|P_2'(t)|,|P_1''(t)|) \lesssim t^{j_1-2}$. Hence

$$\begin{aligned}|\psi'(t)|+|\psi''(t)| &\gtrsim (|x|+|y|)t^{-2j_1-j_2+3}\\ &\gtrsim |x|\left(\frac{|x|}{|y|}\right)^{\frac{-2j_1-j_2+3}{j_2-j_1}}\\ &\geq |x||x|^{\left(\frac{j_1-j_2}{j_1}\right)\left(\frac{-2j_1-j_2+3}{j_2-j_1}\right)}\\ &= |x|^{\frac{3j_1+j_2-3}{j_1}}.\end{aligned}$$

Now split the interval of integration into a union of O(1) intervals on which either

$$|\psi'(t)| \gtrsim |x|^{\frac{3j_1+j_2-3}{j_1}} \quad \text{or} \quad |\psi''(t)| \gtrsim |x|^{\frac{3j_1+j_2-3}{j_1}}$$

holds. This can be done since ψ is a polynomial and therefore we can split the interval of integration into a union of O(1) intervals on which either $|\psi'(t)| \geq |\psi''(t)|$ or $|\psi'(t)| \leq |\psi''(t)|$. In the first case we can then see that $|\psi'(t)| \gtrsim |x|^{1/j_1}$ and in the second case we see that $|\psi''(t)| \gtrsim |x|^{2/j_1}$, so in both cases, using van der Corput's lemma we obtain the desirable bound III $\lesssim |x|^{-1/j_1}$. If $t>1$ we have from Lemma 2.3.3 that $\max(|P_1'(t)|,|P_2''(t)|,|P_2'(t)|,|P_1''(t)|) \lesssim t^{j_2-1}$. Then

$$\begin{aligned}|\psi'(t)|+|\psi''(t)| &\gtrsim (|x|+|y|)t^{-j_1-2j_2+2}\\ &\gtrsim |x|\left(\frac{|x|}{|y|}\right)^{\frac{-j_1-2j_2+2}{j_2-j_1}}\\ &\geq |x||x|^{\left(\frac{j_1-j_2}{j_1}\right)\left(\frac{-j_1-2j_2+2}{j_2-j_1}\right)}\\ &= |x|^{\frac{2j_1+2j_2-2}{j_1}}.\end{aligned}$$

Like before, we split the interval of integration into a union of O(1) intervals on which either

$$|\psi'(t)| \gtrsim |x|^{\frac{2j_1+2j_2-2}{j_1}} \quad \text{or} \quad |\psi''(t)| \gtrsim |x|^{\frac{2j_1+2j_2-2}{j_1}}.$$

The first inequality implies $|\psi'(t)| \gtrsim |x|^{1/j_1}$ and the second inequality implies $|\psi''(t)| \gtrsim |x|^{2/j_1}$, so in either case, using van der Corput's lemma we obtain the desirable bound III $\lesssim |x|^{-1/j_1}$. This proves that

$$\int_I e^{i(xP_1(t)+yP_2(t))}dt \lesssim \frac{1}{\rho(x,y)}$$

in the range $|x|^{1/j_1} \geq |y|^{1/j_2}$, concluding the proof of Proposition 3.2.1.

Proposition 3.2.2 *With $2 \leq j_1 \neq j_2$ and σ, ρ, defined as in (3.10) and (3.11), if we let*

$$B_\theta(x,y) = \{(u,v) : \rho((u,v)-(x,y)) < \theta\},$$

then $\sigma(B_\theta(x,y)) \lesssim \theta$.

PROOF. With $|\cdot|$ denoting Lebesgue measure, we have

$$\sigma(B_\theta(x,y)) = |I \cap \{t : |x - \bar{P}_1(t)|^{\frac{1}{j_1}} + |y - \bar{P}_2(t)|^{\frac{1}{j_2}} \leq \theta\}|,$$

where $\bar{P}_1(t) = P_1(t)/p_{1,j_1}$ and $\bar{P}_2(t) = P_2(t)/p_{2,j_2}$. Hence

$$\begin{aligned} \sigma(B_\theta(x,y)) &\leq |I \cap \{t : |x - \bar{P}_1(t)|^{\frac{1}{j_1}} \leq \theta\}| \\ &\leq |I \cap \{t : ||x| - |\bar{P}_1(t)|| \leq \theta^{j_1}\}| \\ &= |I \cap \{t : |x| - \theta^{j_1} \leq |\bar{P}_1(t)| \leq |x| + \theta^{j_1}\}| \\ &\leq |\{t : |x| - \theta^{j_1} \leq |\bar{P}_1(t)| \leq |x| + \theta^{j_1}\}|, \end{aligned}$$

since by Lemma 2.3.1, $|\bar{P}_1(t)|$ is increasing on I. Without loss of generality, we may assume that $\bar{P}_1(t)$ is positive and split the analysis in two cases. For the first case we assume $|x| \geq 2\theta^{j_1}$. We then have

$$|\{t : |x| - \theta^{j_1} \leq \bar{P}_1 \leq |x| + \theta^{j_1}\}| = \beta - \alpha,$$

where $\bar{P}_1(\alpha) = |x| - \theta^{j_1}$ and $\bar{P}_1(\beta) = |x| + \theta^{j_1}$. Then

$$\int_\alpha^\beta \bar{P}_1'(t)dt = \bar{P}_1(\beta) - \bar{P}_1(\alpha) = 2\theta^{j_1}.$$

Also

$$\int_\alpha^\beta \bar{P}_1'(t)dt \gtrsim \int_\alpha^\beta t^{j_1-1}dt \geq \alpha^{j_1-1}(\beta - \alpha).$$

From Lemma 2.3.1 we have that $\alpha^{j_1} \gtrsim \bar{P}_1(\alpha) = |x| - \theta^{j_1}$, which implies $\alpha \gtrsim (|x| - \theta^{j_1})^{1/j_1}$. Putting this together we obtain

$$\theta^{j_1} \gtrsim (|x| - \theta^{j_1})^{\frac{j_1-1}{j_1}}(\beta - \alpha) \geq \theta^{j_1\frac{j_1-1}{j_1}}(\beta - \alpha).$$

Hence $(\beta - \alpha) \lesssim \theta$. For the second case we have $|x| < 2\theta^{j_1}$. Thus

$$\begin{aligned} |\{t : |x| - \theta^{j_1} \leq |\bar{P}_1(t)| \leq |x| + \theta^{j_1}\}| &\leq |\{t : |\bar{P}_1(t)| \leq |x| + \theta^{j_1}\}| \\ &\leq |\{t : t^{j_1} \lesssim |x| + \theta^{j_1}\}| \\ &\leq |\{t : t^{j_1} \lesssim 3\theta^{j_1}\}| \\ &\lesssim \theta, \end{aligned}$$

completing the proof of Proposition 3.2.2.

We now come back to the proof of Theorem 3.1.1. Let us define $\varphi \in C_c^\infty$ such that $\operatorname{supp}\varphi \subseteq B_1(0,1)$ and $\varphi \equiv 1$ near 0. We split our measure σ on the Fourier transform side, in the following way: $\sigma = \sigma_1 + \sigma_2$ where

$$\widehat{\sigma}_1(x,y) = \widehat{\sigma}(x,y)\varphi(\theta^{j_1}x, \theta^{j_2}y)$$

and

$$\widehat{\sigma}_2(x,y) = \widehat{\sigma}(x,y)(1 - \varphi(\theta^{j_1}x, \theta^{j_2}y)),$$

where θ is a parameter to be chosen later. Then

$$\begin{aligned}
&\int_I |\widehat{a}(P_1(t), P_2(t))|^2 dt \\
= &\int_I \left(\int_{\mathbb{R}^2} a(u,v)e^{-i(uP_1(t)+vP_2(t))}dudv\right)\left(\int_{\mathbb{R}^2} \overline{a(x,y)}e^{i(xP_1(t)+yP_2(t))}dxdy\right) dt \\
= &\int_{\mathbb{R}^2} a(u,v)\left(\int_{\mathbb{R}^2} \overline{a(x,y)}\int_I e^{-i((u-x)P_1(t)+(v-y)P_2(t))}dtdxdy\right) dudv \\
= &\int_{\mathbb{R}^2} a(u,v)\left(\bar{a} * \widehat{d\sigma}(p_{1,j_1}\cdot, p_{2,j_2}\cdot)\right)(u,v)dudv \\
= &\int_{\mathbb{R}^2} a(u,v)\left(\bar{a} * \widehat{d\sigma_1}(p_{1,j_1}\cdot, p_{2,j_2}\cdot)\right)(u,v)dudv \\
&+ \int_{\mathbb{R}^2} a(u,v)\left(\bar{a} * \widehat{d\sigma_2}(p_{1,j_1}\cdot, p_{2,j_2}\cdot)\right)(u,v)dudv \\
\leq\ &||a||_2||\bar{a} * \widehat{d\sigma_1}(p_{1,j_1}\cdot, p_{2,j_2}\cdot)||_2 + ||a||_1||\bar{a} * \widehat{d\sigma_2}(p_{1,j_1}\cdot, p_{2,j_2}\cdot)||_\infty \\
\leq\ &||a||_2^2\frac{1}{|p_{1,j_1}p_{2,j_2}|}||d\sigma_1(p_{1,j_1}^{-1}\cdot, p_{2,j_2}^{-1}\cdot)||_\infty + ||a||_1^2||\widehat{d\sigma_2}(p_{1,j_1}\cdot, p_{2,j_2}\cdot)||_\infty \\
= &||a||_2^2\frac{1}{|p_{1,j_1}p_{2,j_2}|}||d\sigma_1||_\infty + ||a||_1^2||\widehat{d\sigma_2}(p_{1,j_1}\cdot, p_{2,j_2}\cdot)||_\infty. \qquad (3.13)
\end{aligned}$$

We look at the second term of the last expression and recall that $||a||_1 = 1$ and

$$|\widehat{d\sigma_2}(p_{1,j_1}x, p_{2,j_2}y)| = |\widehat{\sigma}(p_{1,j_1}x, p_{2,j_2}y)||1 - \varphi(p_{1,j_1}\theta^{j_1}x, p_{2,j_2}\theta^{j_2}y)|.$$

We note that the second term of the above product is nonzero when

$$\rho(p_{1,j_1}\theta^{j_1}x, p_{2,j_2}\theta^{j_2}y) = \theta\rho(p_{1,j_1}x, p_{2,j_2}y) \gtrsim 1.$$

That is the above product is nonzero only for those (x,y)'s that satisfy

$$\frac{1}{\rho(p_{1,j_1}x, p_{2,j_2}y)} \lesssim \theta.$$

Using Proposition 3.2.1 we then have that

$$\begin{aligned}
|\widehat{d\sigma_2}(p_{1,j_1}x, p_{2,j_2}y)| &= |\widehat{\sigma}(p_{1,j_1}x, p_{2,j_2}y)||1 - \varphi(p_{1,j_1}\theta^{j_1}x, p_{2,j_2}\theta^{j_2}y)| \\
&\lesssim \frac{1}{\rho(p_{1,j_1}x, p_{2,j_2}y)} \\
&\lesssim \theta.
\end{aligned}$$

For the first term in (3.13) we need an estimate on $\sigma_1(x,y)$. We have that $\sigma_1(x,y) = \sigma * \check{\varphi}_\theta(x,y)$, where $\varphi_\theta(u,v) = \varphi(\theta^{j_1}u, \theta^{j_2}v)$. So

$$\begin{aligned}
\sigma_1(x,y) &= \frac{1}{\theta^{j_1+j_2}} \int_{\mathbb{R}^2} \check{\varphi}\left(\frac{x-u}{\theta^{j_1}}, \frac{y-v}{\theta^{j_2}}\right) d\sigma(u,v) \\
&= \frac{1}{\theta^{j_1+j_2}} \left[\int_{\rho((u,v)-(x,y))\leq\theta} \check{\varphi}\left(\frac{x-u}{\theta^{j_1}}, \frac{y-v}{\theta^{j_2}}\right) d\sigma(u,v)\right. \\
&\quad \left. + \sum_{n\geq 1} \int_{2^{n-1}\theta\leq\rho((u,v)-(x,y))\leq 2^n\theta} \check{\varphi}\left(\frac{x-u}{\theta^{j_1}}, \frac{y-v}{\theta^{j_2}}\right) d\sigma(u,v)\right].
\end{aligned}$$

Now since $\check{\varphi} \in \mathcal{S}$, we have $\rho(x,y)^N |\check{\varphi}(x,y)| \leq C_N$. Therefore

$$\begin{aligned}
\sigma_1(x,y) &\lesssim \frac{1}{\theta^{j_1+j_2}} \left[\sigma(B_\theta(x,y))\right. \\
&\quad \left. + \sum_{n\geq 1} \sigma(B_{2^n\theta}(x,y)) \int_{2^{n-1}\theta\leq\rho((u,v)-(x,y))\leq 2^n\theta} \frac{C_N}{\rho\left(\frac{x-u}{\theta^{j_1}}, \frac{y-v}{\theta^{j_2}}\right)^N} dudv\right] \\
&\lesssim \frac{1}{\theta^{j_1+j_2}} \left[\theta + \sum_{n\geq 1} 2^n\theta \int_{2^{n-1}\theta\leq\rho((u,v)-(x,y))\leq 2^n\theta} \frac{C_N\theta^N}{\rho(x-u, y-v)^N} dudv\right] \\
&\lesssim \frac{1}{\theta^{j_1+j_2}} \left(\theta + \theta \sum_{n\geq 1} 2^n 2^{(1-n)N}\right) \\
&\lesssim \frac{\theta}{\theta^{j_1+j_2}},
\end{aligned}$$

for N large enough. Hence putting the estimates for the two terms in (3.13) together we obtain

$$\int_I |\widehat{a}(P_1(t), P_2(t))|^2 dt \lesssim \theta + \frac{\theta}{|Q|\theta^{j_1+j_2}|p_{1,j_1}p_{2,j_2}|}.$$

Thus choosing $\theta = (|p_{1,j_1}p_{2,j_2}||Q|)^{-1/(j_1+j_2)}$ we complete the proof of Theorem 3.1.1 in the case $2 \leq j_1 \neq j_2$.

3.2.2 The case $1 = j_1 < j_2$

To prove Theorem 3.1.1, for $1 = j_1 < j_2 =: j$, we first note that by performing a change of vaiables and using Lemma 2.3.1, it is enough to prove a rescaled analogue of it, that is with $I = [B, 1]$. As before, we define the measure

$$\sigma(\phi) = \int_B^1 \phi\left(\frac{P_1(t)}{p_{1,1}}, \frac{P_2(t)}{p_{2,j}}\right) dt \tag{3.14}$$

and we denote by $\rho(x,y)$ the 'Euclidean' metric

$$\rho(x,y) = |x| + |y|. \tag{3.15}$$

We will need the following two propositions which are similar to Propositions 3.2.1 and 3.2.2.

Proposition 3.2.3 *With* $1 < j$ *and* σ, ρ, *defined as in (3.14) and (3.15), we have*

$$\widehat{\sigma}(x,y) = \int_B^1 e^{-i\left(x\frac{P_1(t)}{p_{1,1}}+y\frac{P_2(t)}{p_{2,j}}\right)}dt \lesssim \frac{1}{\rho(x,y)^{\frac{1}{j}}}.$$

PROOF. It is enough to show that if $P_1(t) \sim t$ and $P_2(t) \sim t^j$ on $(B,1)$, then

$$\int_B^1 e^{-i(xP_1(t)+yP_2(t))}dt \lesssim \frac{1}{(|x|+|y|)^{\frac{1}{j}}}. \tag{3.16}$$

We recall from Lemma 2.3.1 that we have

$$A_1 \leq |P_1'(t)| \leq A_1',$$

$$B_1 t^{j-1} \leq |P_2'(t)| \leq B_1' t^{j-1},$$

where A_1, A_1', B_1, B_1' are constants only depending on the degrees of P_1, P_2. We first consider the range $A_1|x|/B_1' \geq 2|y|$. In this range,

$$\begin{aligned} |xP_1'(t)+yP_2'(t)| &\geq A_1|x| - B_1'|y|t^{j-1} \\ &\gtrsim |x|. \end{aligned}$$

So by van der Corput's lemma,

$$\int_B^1 e^{-i(xP_1(t)+yP_2(t))}dt \lesssim \frac{1}{|x|} \leq \frac{1}{|x|^{\frac{1}{j}}} \lesssim \frac{1}{(|x|+|y|)^{\frac{1}{j}}},$$

if $|x| \geq 1$. In case $|x| < 1$, we use the trivial estimate $|\int_B^1 e^{-i(xP_1(t)+yP_2(t))}dt| \leq 1$ which implies the desired estimate since $|x|, |y| \lesssim 1$. For the range $|y| \geq |x|^j$, we split the integral in (3.16), for suficiently large C, in the following way:

$$\begin{aligned} \int_B^1 e^{-i(xP_1(t)+yP_2(t))}dt &= \int_B^{C|y|^{-\frac{1}{j}}} e^{-i(xP_1(t)+yP_2(t))}dt + \int_{C|y|^{-\frac{1}{j}}}^1 e^{-i(xP_1(t)+yP_2(t))}dt \\ &= \mathrm{I}+\mathrm{II}. \end{aligned}$$

Integral I is clearly bounded above by $|y|^{-1/j}$. For II we have

$$\begin{aligned} |xP_1'(t)+yP_2'(t)| &\geq B_1|y|t^{j-1} - A_1'|x| \\ &\gtrsim |y|^{\frac{1}{j}}, \end{aligned}$$

which by van der Corput's lemma implies that II $\lesssim |y|^{-1/j}$. Thus we have shown that

$$\int_B^1 e^{-i(xP_1(t)+yP_2(t))}dt \lesssim (|x|+|y|)^{-1/j},$$

for $|y| \geq |x|^j$, if $|y| \geq 1$. Note that if $|y| < 1$ we can use the trivial estimate $|\int_B^1 e^{-i(xP_1(t)+yP_2(t))}dt| \leq 1$. We are now left with the range $|x| \lesssim |y| \lesssim |x|^k$ (note that this forces $x, y \gtrsim 1$). Here we split the integral in (3.16) as follows:

$$\begin{aligned}
&\int_B^1 e^{-i(xP_1(t)+yP_2(t))}dt \\
= &\int_B^{\delta\left(\frac{A_1|x|}{B_1'|y|}\right)^{\frac{1}{j-1}}} e^{-i(xP_1(t)+yP_2(t))}dt + \int_{\delta\left(\frac{A_1|x|}{B_1'|y|}\right)^{\frac{1}{j-1}}}^{C\left(\frac{A_1'|x|}{B_1|y|}\right)^{\frac{1}{j-1}}} e^{-i(xP_1(t)+yP_2(t))}dt \\
&+ \int_{C\left(\frac{A_1'|x|}{B_1|y|}\right)^{\frac{1}{j-1}}}^1 e^{-i(xP_1(t)+yP_2(t))}dt \\
= &\ \mathrm{I} + \mathrm{II} + \mathrm{III},
\end{aligned}$$

for sufficiently small δ and sufficiently large C. For I we have

$$|xP_1'(t) + yP_2'(t)| \geq A_1|x| - B_1'|y|t^{j-1} \gtrsim |x| \gtrsim |y|^{\frac{1}{j}},$$

implying, by van der Corput's lemma, that $\mathrm{I} \lesssim |y|^{-1/j} \lesssim \rho(x,y)^{-1/j}$. For III we have

$$|xP_1'(t) + yP_2'(t)| \geq B_1|y|^{j-1} - A_1'|x| \gtrsim |y|t^{j-1} \gtrsim |y|\frac{|x|}{|y|} = |x| \gtrsim |y|^{\frac{1}{j}},$$

implying, by van der Corput's lemma, that $\mathrm{III} \lesssim |y|^{-1/j} \lesssim \rho(x,y)^{-1/j}$. To estimate II we first write $\psi(t) = xP_1(t) + yP_2(t)$, so

$$\begin{pmatrix} \psi'(t) \\ \psi''(t) \end{pmatrix} = \begin{pmatrix} P_1'(t) & P_2'(t) \\ P_1''(t) & P_2''(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

This implies that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{P_1'(t)P_2''(t) - P_1'(t)P_2''(t)} \begin{pmatrix} P_2''(t) & -P_2'(t) \\ -P_1''(t) & P_2'(t) \end{pmatrix} \begin{pmatrix} \psi'(t) \\ \psi''(t) \end{pmatrix}$$

and consequently that

$$|x| + |y| \leq \frac{1}{|P_1'(t)P_2''(t) - P_1'(t)P_2''(t)|} \left\| \begin{pmatrix} P_2''(t) & -P_2'(t) \\ -P_1''(t) & P_2'(t) \end{pmatrix} \right\| (|\psi'(t)| + |\psi''(t)|).$$

Now using the upper bound for $P_1''(t)$ from Lemma 2.3.3 and the fact that $t < 1$, we have

$$\left\| \begin{pmatrix} P_2''(t) & -P_2'(t) \\ -P_1''(t) & P_2'(t) \end{pmatrix} \right\| \sim \max(|P_1'(t)|, |P_1''(t)|, |P_2'(t)|, |P_2''(t)|) \lesssim 1.$$

Thus, using Proposition 2.7.1 we have

$$\begin{aligned}
|x| + |y| &\lesssim \frac{1}{|P_1'(t)P_2''(t) - P_1'(t)P_2''(t)|}(|\psi'(t)| + |\psi''(t)|)) \\
&\lesssim t^j(|\psi'(t)| + |\psi''(t)|),
\end{aligned}$$

implying that

$$|\psi'(t)| + |\psi''(t)| \gtrsim t^{-j}(|x| + |y|) \gtrsim t^{-j}x.$$

We now split the interval of integration into a union of O(1) intervals on which either $\psi'(t) \gtrsim t^{-j}x$ or $\psi''(t) \gtrsim t^{-j}x$ holds. But,

$$t^{-j}x \gtrsim \left(\frac{x}{y}\right)^{-\frac{j}{j-1}} x = x^{1-\frac{j}{j-1}} y^{\frac{j}{j-1}} = x^{-\frac{1}{j-1}} y^{\frac{j}{j-1}} \gtrsim y^{-\frac{1}{j(j-1)}} y^{\frac{j}{j-1}} = y^{\frac{j^2-1}{j(j-1)}} = y^{\frac{j+1}{j}}.$$

So in the case that $\psi'(t) \gtrsim t^{-j}x$, we have that

$$\psi'(t) \gtrsim t^{-j}x \gtrsim y^{\frac{j+1}{j}} \gtrsim y^{\frac{1}{j}},$$

giving the required estimate for II, using van der Corput's lemma, and in the case that $\psi''(t) \gtrsim t^{-j}x$, we have

$$\psi''(t) \gtrsim t^{-j}x \gtrsim y^{\frac{j+1}{j}} \gtrsim y^{\frac{2}{j}},$$

again giving the required estimate for II using van der Corput's lemma. This completes the proof of Proposition 3.2.3.

Proposition 3.2.4 *With $1 < j$ and σ, ρ, defined as in (3.14) and (3.15), if we let*

$$B_\theta(x, y) = \{(u, v) : \rho((u, v) - (x, y)) < \theta\},$$

then $\sigma(B_\theta(x, y)) \lesssim \theta$.

PROOF. With $|\cdot|$ denoting Lebesgue measure, we have

$$\sigma(B_\theta(x, y)) = |I \cap \{t : |x - \bar{P}_1(t)| + |y - \bar{P}_2(t)| \leq \theta\}|,$$

where $\bar{P}_1(t) = P_1(t)/p_{1,1}$ and $\bar{P}_2(t) = P_2(t)/p_{2,j}$. Hence

$$\begin{aligned}
\sigma(B_\theta(x, y)) &\leq |I \cap \{t : |x - \bar{P}_1(t)| \leq \theta\}| \\
&\leq |I \cap \{t : ||x| - |\bar{P}_1(t)|| \leq \theta\}| \\
&= |I \cap \{t : |x| - \theta \leq |\bar{P}_1(t)| \leq |x| + \theta\}| \\
&\leq |\{t : |x| - \theta \leq |\bar{P}_1(t)| \leq |x| + \theta\}|,
\end{aligned}$$

since by Lemma 2.3.1, $|\bar{P}_1(t)|$ is increasing on I. Without loss of generality, we may assume that $\bar{P}_1(t)$ is positive and split the analysis in two cases. For the first case we have $|x| \geq 2\theta$. We have

$$|\{t : |x| - \theta \leq \bar{P}_1 \leq |x| + \theta\}| = \beta - \alpha,$$

where $\bar{P}_1(\alpha) = |x| - \theta$ and $\bar{P}_1(\beta) = |x| + \theta$. Then

$$\int_\alpha^\beta \bar{P}_1'(t)dt = \bar{P}_1(\beta) - \bar{P}_1(\alpha) = 2\theta.$$

Also, from Lemma 2.3.1,

$$\int_\alpha^\beta \bar{P}_1'(t)dt \gtrsim \int_\alpha^\beta \gtrsim \beta - \alpha,$$

implying that $\beta - \alpha \lesssim \theta$. For the second case we have $|x| < 2\theta$, so

$$\begin{aligned} |\{t : |x| - \theta \le \bar{P}(t) \le |x| + \theta\}| &\le |\{t : \bar{P}(t) \le |x| + \theta\}| \\ &\le |\{t : t \lesssim |x| + \theta\}| \\ &\le |\{t : t \lesssim 3\theta\}| \\ &\lesssim \theta, \end{aligned}$$

completing the proof of Proposition 3.2.4.

We now come back to the proof of Theorem 3.1.1. Let us define $\varphi \in C_c^\infty$ such that $\operatorname{supp} \varphi \subseteq B_1(0,0)$ and $\varphi \equiv 1$ near 0. We split our measure σ on the Fourier transform side, in the following way: $\sigma = \sigma_1 + \sigma_2$ where

$$\widehat{\sigma}_1(x,y) = \widehat{\sigma}(x,y)\varphi(\theta x, \theta y)$$

and

$$\widehat{\sigma}_2(x,y) = \widehat{\sigma}(x,y)(1 - \varphi(\theta x, \theta y)),$$

where θ is a parameter to be chosen later. Then

$$\begin{aligned} &\int_I |\widehat{a}(P_1(t), P_2(t))|^2 dt \\ = &\int_I \left(\int_{\mathbb{R}^2} a(u,v) e^{-i(uP_1(t)+vP_2(t))} dudv\right) \left(\int_{\mathbb{R}^2} \overline{a(x,y)} e^{i(xP_1(t)+yP_2(t))} dxdy\right) dt \\ = &\int_{\mathbb{R}^2} a(u,v) \left(\int_{\mathbb{R}^2} \overline{a(x,y)} \int_I e^{-i((u-x)P_1(t)+(v-y)P_2(t))} dtdxdy\right) dudv \\ = &\int_{\mathbb{R}^2} a(u,v) \left(\bar{a} * \widehat{d\sigma}(p_{1,1}\cdot, p_{2,j}\cdot)\right)(u,v) dudv \\ = &\int_{\mathbb{R}^2} a(u,v) \left(\bar{a} * \widehat{d\sigma_1}(p_{1,1}\cdot, p_{2,j}\cdot)\right)(u,v) dudv \\ &+ \int_{\mathbb{R}^2} a(u,v) \left(\bar{a} * \widehat{d\sigma_2}(p_{1,1}\cdot, p_{2,j}\cdot)\right)(u,v) dudv \\ \le &\|a\|_2 \|\bar{a} * \widehat{d\sigma_1}(p_{1,1}\cdot, p_{2,j}\cdot)\|_2 + \|a\|_1 \|\bar{a} * \widehat{d\sigma_2}(p_{1,1}\cdot, p_{2,j}\cdot)\|_\infty \\ \le &\|a\|_2^2 \frac{1}{|p_{1,1}p_{2,j}|} \|d\sigma_1(p_{1,1}^{-1}\cdot, p_{2,j}^{-1}\cdot)\|_\infty + \|a\|_1^2 \|\widehat{d\sigma_2}(p_{1,1}\cdot, p_{2,j}\cdot)\|_\infty \\ = &\|a\|_2^2 \frac{1}{|p_{1,1}p_{2,j}|} \|d\sigma_1\|_\infty + \|a\|_1^2 \|\widehat{d\sigma_2}(p_{1,1}\cdot, p_{2,j}\cdot)\|_\infty. \end{aligned} \tag{3.17}$$

We look at the second term of the last expression and recall that $||a||_1 = 1$ and

$$|\widehat{d\sigma_2}(p_{1,j_1}x, p_{2,j_2}y)| = |\widehat{\sigma}(p_{1,1}x, p_{2,j}y)||1 - \varphi(p_{1,1}\theta x, p_{2,j}\theta y)|.$$

We note that the second term of the above product is nonzero when

$$\rho(p_{1,1}\theta x, p_{2,j}\theta y) = \theta\rho(p_{1,1}x, p_{2,j}y) \gtrsim 1.$$

That is the above product is nonzero only for those (x, y)'s that satisfy

$$\frac{1}{\rho(p_{1,1}x, p_{2,j}y)} \lesssim \theta.$$

Using Proposition 3.2.3 we then have that

$$\begin{aligned} |\widehat{d\sigma_2}(p_{1,1}x, p_{2,j}y)| &= |\widehat{\sigma}(p_{1,1}x, p_{2,j}y)||1 - \varphi(p_{1,1}\theta x, p_{2,j}\theta y)| \\ &\lesssim \frac{1}{\rho(p_{1,1}x, p_{2,j}y)^{\frac{1}{j}}} \\ &\lesssim \theta^{\frac{1}{j}}. \end{aligned}$$

For the first term in (3.17) we need an estimate on $\sigma_1(x, y)$. We have that $\sigma_1(x, y) = \sigma * \check{\varphi}_\theta(x, y)$, where $\varphi_\theta(u, v) = \varphi(\theta u, \theta v)$. So

$$\begin{aligned} \sigma_1(x, y) &= \frac{1}{\theta^2}\int_{\mathbb{R}^2} \check{\varphi}\left(\frac{x-u}{\theta}, \frac{y-v}{\theta}\right) d\sigma(u, v) \\ &= \frac{1}{\theta^2}\left[\int_{\rho((u,v)-(x,y))\leq\theta} \check{\varphi}\left(\frac{x-u}{\theta}, \frac{y-v}{\theta}\right) d\sigma(u, v)\right. \\ &\quad \left. + \sum_{n\geq 1}\int_{2^{n-1}\theta\leq\rho((u,v)-(x,y))\leq 2^n\theta} \check{\varphi}\left(\frac{x-u}{\theta}, \frac{y-v}{\theta}\right) d\sigma(u, v)\right]. \end{aligned}$$

Now since $\check{\varphi} \in \mathcal{S}$, we have $\rho(x, y)^N|\check{\varphi}(x, y)| \leq C_N$. Therefore

$$\begin{aligned} \sigma_1(x, y) &\lesssim \frac{1}{\theta^2}\left[\sigma(B_\theta(x, y))\right. \\ &\quad \left. + \sum_{n\geq 1}\sigma(B_{2^n\theta}(x, y))\int_{2^{n-1}\theta\leq\rho((u,v)-(x,y))\leq 2^n\theta} \frac{C_N}{\rho\left(\frac{x-u}{\theta}, \frac{y-v}{\theta}\right)^N} dudv\right] \\ &\lesssim \frac{1}{\theta^2}\left[\theta + \sum_{n\geq 1} 2^n\theta \int_{2^{n-1}\theta\leq\rho((u,v)-(x,y))\leq 2^n\theta} \frac{C_N\theta^N}{\rho(x-u, y-v)^N} dudv\right] \\ &\lesssim \frac{1}{\theta^2}\left(\theta + \theta\sum_{n\geq 1} 2^n 2^{(1-n)N}\right) \\ &\lesssim \frac{1}{\theta}, \end{aligned}$$

for N large enough. Hence putting the estimates for the two terms in (3.17) together we obtain

$$\int_I |\widehat{a}(P_1(t), P_2(t))|^2 dt \lesssim \theta^{\frac{1}{j}} + \frac{1}{|Q|\theta|p_{1,1}p_{2,j}|}.$$

Thus choosing $\theta = (|p_{1,1}p_{2,j}||Q|)^{-j/(1+j)}$ we complete the proof of Theorem 3.1.1 in the case $1 = j_1 < j_2 = j$.

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