BANACH FUNCTION SPACES

AND SPECTRAL MEASURES

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ABSTRACT

The fundamental link between prespectral measures and Banach function spaces is to be found in a theorem of T.A.Gillespie which relates cyclic spaces isomorphically to certain Banach function spaces. We obtain here an extension of this result to the wider class of precyclic spaces.

We then consider the properties of weak sequential completeness and reflexivity in Banach function spaces: necessary and sufficient conditions are obtained which in turn, via the afore-mentioned isomorphisms, both extend and simplify analogously formulated existing results for cyclic spaces.

Finally the concept of a homomorphism between pairs of Banach function spaces is examined.The class of such mappings is determined and a complete description obtained in the form of a (unique) disjoint sum of two mappings, one of which is always an isomorphism and the other of which is arbitrary in a certain sense, or null.It is shown moreover that the isomorphic component itself is composed of two other isomorphisms in a manner analogous to the geometrical composition of a rotation and a dilatation.

Ne la cherchez plus puisqu'elle est partie
Il l'a appelée et elle a dit oui,
Ne la cherchez plus car elle a suivi
Celui qui un jour lui a souri...

Mais au loin dans le vent, écoutez cette voix
Chanter un printemps d'amour et de joie...

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PREFACE

The theory of normed vector lattices of functions was initiated in the thirties by G.Köthe, then pursued by several others, most notably by Toeplitz. These function lattices became known as normed Köthe spaces, or also, when their norm is complete, as Banach function spaces.

The theory was properly established and standardised by A.C. Zaanen and W.A.J. Luxemburg who produced a series of very detailed papers entitled "Notes on Banach function spaces" in Proc. Acad. Sci. Amsterdam and Indagationes Mathematicae. This extensive material was later distilled into the succinct and attractive theory rendered in [Z], § 63-73.

The fundamental link between Banach function spaces and prespectral measures was made by T.A. Gillespie in 1978 when a Representation Theorem for cyclic spaces was obtained in terms of certain Banach function spaces. This theorem, as one might expect, will be exploited quite considerably in this thesis. Indeed after Chapter I which summarises the background notions pertinent to the following chapters, we begin, in Chapter II, by extending the theorem to yield a representation for the wider class of precyclic spaces.

The impulse for Chapter III came from two papers by L. Tzafriri ([T_1], [T_2]) where necessary and sufficient conditions for weak sequential completeness and for reflexivity of cyclic spaces are discussed. The important results are proved again here differently and more concretely by invoking the representation theorems and working within function spaces. In the process they are extended as well as simplified.

In Chapter IV which pertains essentially only to Banach function space theory, the object was to explore the concept of a homomorphic relation between Banach function spaces. The literature appears to have a gap in this area, beyond the mere application of Riesz homomorphism theory to this subclass of Riesz spaces. Riesz spaces in general are not endowed with a norm and it was found that the function norm, by its monotonicity, forces any pair of Riesz (i.e. lattice-) homomorphic Banach function spaces to be homomorphic in a given sense in the norm topology. Moreover it turns out that every Riesz homomorphism of one Banach function space onto another arises as the restriction to the former space of a unique surjective Riesz homomorphism between the respective parent spaces of measurable functions. We show that this always consists of two uniquely defined disjoint components, one of which is injective (and consequently an isomorphism on its domain) and continuous in the order topology, whilst the other is nowhere injective and everywhere discontinuous in this topology, unless it vanishes. It is also shown that the isomorphic component has a strongly geometric character in that it is composed of two isomorphisms in a manner analogous to the Euclidean composition of a rotation and a dilatation.

CHAPTER I. INTRODUCTION AND PRELIMINARY RESULTS.

In this chapter, we present a brief account, in two independent sections, of the elementary properties of prespectral measures and of Banach function spaces, as far as is relevant to the rest of this thesis. A comprehensive account can be found for § 1 in [DS_2] and for § 2 in [Z].

We begin by giving some notation, for the most part standard.

If X is a Banach space, X^* will denote its Banach dual space and B(X) will denote the set of bounded linear operators on X. When $x \in X$ and $\varphi \in X^*$, $\langle x, \varphi \rangle$ will denote the value of the functional φ at the point x.

All spaces will be over $\mathbb{C}$ unless otherwise specified.

The symbol $\|\cdot\|$ will always denote the usual norm on any given space, e.g. if $\varphi \in X^*$, $\|\varphi\|$ will mean the dual norm of φ, when no confusion arises from this convention.

If (Ω, Σ, μ) is a measure space, $L^1(\mu)$ and $L^\infty(\mu)$ will denote respectively the usual spaces of (equivalence classes of) μ-integrable and μ-essentially bounded functions defined on Ω.

The linear span of elements $x_1, x_2, \ldots$ will be denoted by lin $\{x_1, x_2, \ldots\}$ and their closed linear span by $\overline{\text{lin}}\ \{x_1, x_2, \ldots\}$.

The symbols $\vee$, $\wedge$ will denote lattice supremum and infimum respectively.

§ 1. Prespectral and spectral measures.

Let X be a Banach space.

1.1 Definitions. A Boolean algebra of projections $\mathcal{B}$ on X is a commutative subset of B(X) such that

(i) $P^2 = P \ (P \in \mathcal{B})$,

(ii) $O \in \mathcal{B}$,

(iii) if $P \in \mathcal{B}$ then $I - P \in \mathcal{B}$ (where I is the identity operator on X),

(iv) if $P, Q \in \mathcal{B}$ then $P \vee Q = P + Q - PQ \in \mathcal{B}$ and $P \wedge Q = PQ \in \mathcal{B}$.

A Boolean algebra of projections $\mathcal{B}$ is called bounded if there is a real constant K with $\|P\| \le K \ (P \in \mathcal{B})$.

The Boolean algebra $\mathcal{B}$ is said to be abstractly σ-complete if each sequence in $\mathcal{B}$ has a greatest lower bound and a least upper bound in $\mathcal{B}$; $\mathcal{B}$ is said to be σ-complete if it is abstractly σ-complete and if for every sequence $\{P_n\}$ in $\mathcal{B}$,

$$(\bigvee_n P_n)X = \overline{\text{lin}}\ \{P_nX : n = 1,2,\ldots\} ,$$

$$(\bigwedge_n P_n)X = \bigcap_n \{P_nX : n = 1,2,\ldots\} .$$

1.2 LEMMA ([DS$_2$],XVII.3.3). If a Boolean algebra of projections is abstractly σ-complete, then it is bounded.

1.3 LEMMA ([DS$_2$],XVII.3.11). The restriction of a σ-complete Boolean algebra of projections to an invariant subspace is σ-complete.

A subset Γ of X* is called total whenever $y \in X$ and

$\langle y, \varphi \rangle = 0$ for all $\varphi \in \Gamma$ together imply that $y = 0$.

1.4 Definition. A prespectral measure of class (Σ, Γ) with values in $B(X)$ is a mapping $E(\cdot)$ from some σ-algebra Σ of subsets of an arbitrary set Ω into a Boolean algebra of projections on X, satisfying the following conditions for all $\delta, \delta_1, \delta_2 \in \Sigma$:

(i) $E(\delta_1) + E(\delta_2) = E(\delta_1 \cup \delta_2) + E(\delta_1)E(\delta_2)$;

(ii) $E(\delta_1)E(\delta_2) = E(\delta_1 \cap \delta_2)$;

(iii) $E(\Omega) = I$;

(iv) $\|E(\delta)\| \leq K$ for some constant $K > 0$;

(v) if $\Gamma = \{\varphi \in X^* : \langle E(\cdot)x, \varphi\rangle$ is a countably additive complex measure on Σ for every $x \in X\}$, then Γ is a total linear subspace of X^*.

A spectral measure in $B(X)$ is a prespectral measure of class (Σ, X^*). It can be shown that a prespectral measure in $B(X)$ is spectral if and only if it is strongly countably additive.

1.5 LEMMA ([DS_2],XVII.3.10). Let $\mathcal{B}$ be a Boolean algebra of projections on X. Then $\mathcal{B}$ is σ-complete if and only if $\mathcal{B}$ is the range of a spectral measure defined on a σ-field of subsets of a compact space.

§ 2. Banach function spaces.

Let Σ be a σ-algebra of subsets of a non-empty set Ω and let μ be a σ-finite measure defined on (Ω, Σ). Let M_μ (resp. M_μ^+) denote the set of all complex-valued (resp.

non-negative extended-real-valued) measurable functions defined on Ω.

Note: a function in M_μ (resp. M_μ^+) will be called measurable if the inverse image of every Borel subset of $\mathbb{C}$ (resp. $[0, \infty]$) belongs to Σ. Although this differs slightly from the definition of measurability in [Z], which is our principal source for the theory of Banach function spaces, the difference is of little consequence and the theory developed in [Z] applies here entirely.

Elements of M_μ (resp. M_μ^+) which agree μ-a.e. are identified and we shall not normally distinguish between a function f and the equivalence class of functions that are equal to f a.e. This means, in particular, that the support of f is defined only up to a μ-null set. Thus, in fact, any two elements of Σ whose symmetric difference is a μ-null set could essentially be identified and we shall not require to make a formal distinction between the σ-algebra Σ and its measure algebra Σ/N, where N denotes the collection of μ-null sets. (See for instance Theorem 11.5.)

2.1 Notation and terminology.

(i) The characteristic function of a set σ will be denoted by χ_σ.

(ii) If $\sigma \in \Sigma$ and L is any subset of M_μ, then $\chi_\sigma L$ will denote $\{f\chi_\sigma : f \in L\}$.

(iii) If $f \in M_\mu$, supp $f = \{x \in \Omega : f(x) \neq 0\}$ and if $f \in M_\mu^+$, $\{f \leq n\} = \{x \in \Omega : f(x) \leq n\}$, etc. (defined up to a μ-null set).

(iv) If $A_n \in \Sigma$ $(n = 1,2,\ldots)$, we shall write $A_n \uparrow A$ to mean that $A_n \subseteq A$ for each n and $\chi_{A_n} \uparrow \chi_A$ μ-a.e. as $n \to \infty$. It is easily shown (by a measure-theoretic argument) that whenever we have $A_n \uparrow A$ and $A'_n \uparrow A$ (A_n, $A'_n \in \Sigma$; $n = 1,2,\ldots$) then

$$A_n \cap A'_n \uparrow A .$$

(v) When a simple measurable function is expressed in the form $\sum_{i=1}^{n} \alpha_i \chi_{\delta_i}$, this will always mean that the scalars α_i are distinct and the sets δ_i are pairwise disjoint ($i = 1,\ldots,n$).

(vi) Two functions are called mutually <u>disjoint</u> whenever their supports intersect only in a null set; if L is a subset of M_μ, the function $f \in M_\mu$ is called disjoint to L if $fg = 0$ a.e. for every $g \in L$.

When referring to functions in M_μ, "a.e." will always mean μ-a.e. unless otherwise indicated.

The remainder of this section provides an outline of the theory of Banach function spaces. Proofs and further detail may be found in [Z], §§ 63-73.

2.2 Function norms.

A <u>function norm</u> on (Ω, Σ, μ) is a mapping $\rho : M_\mu^+ \to [0, \infty]$ such that for all $f, g \in M_\mu^+$ and all $\alpha \in [0, \infty[$,

(i) $\rho(f + g) \leq \rho(f) + \rho(g)$;

(ii) $\rho(\alpha f) = \alpha\rho(f)$;

(iii) $\rho(f) \leq \rho(g)$ whenever $f \leq g$ a.e.;

(iv) $\rho(f) = 0$ if and only if $f = 0$ a.e.

Setting $\rho(f) = \rho(|f|)$ for $f \in M_\mu$, we define the <u>normed Köthe space</u>

$$L_\rho = \{f \in M_\mu : \rho(f) < \infty\} .$$

When L_ρ is norm complete, it is known as the <u>Banach function space</u> derived from ρ.

2.3 Properties of function norms.

(i) ρ is called <u>saturated</u> if, whenever $\sigma \in \Sigma$ and $\mu(\sigma) > 0$, there exists $\sigma' \subseteq \sigma$ with $0 < \rho(\chi_{\sigma'}) < \infty$.

We shall always, for convenience, make the assumption that ρ is a saturated norm: this assumption is equivalent to simply deleting from Ω a maximal ρ-purely infinite set, i.e. a maximal set δ such that for every $f \in L_\rho$, $f\chi_\delta = 0$ a.e. In this situation we can, and shall frequently, invoke the <u>Exhaustion Theorem</u> ([Z], Theorem 67.3): this measure-theoretic result has important consequences for our work, most notably the following:

> <u>if ρ is a saturated function norm based on (Ω, Σ, μ) and if $\sigma \in \Sigma$, then there exists in Σ a sequence</u> $\sigma_n \uparrow \sigma$ <u>with</u> $\rho(\chi_{\sigma_n}) < \infty$ <u>$(n = 1,2,\ldots)$</u>.

(ii) The function norm ρ has the <u>Riesz-Fischer property</u> if, whenever $f_i \in M_\mu^+$ $(i = 1,2,\ldots)$ and $\sum_{i=1}^{\infty} \rho(f_i) < \infty$, then $\rho(\sum_{i=1}^{\infty} f_i) < \infty$.

(iii) ρ has the <u>weak Fatou property</u> if it follows from $0 \leq u_1 \leq u_2 \leq \ldots \uparrow u$ a.e., with each $u_n \in M_\mu^+$ and $\sup_n \rho(u_n) < \infty$, that $\rho(u) < \infty$.

(iv) ρ has the <u>Fatou property</u> if it follows from $0 \leq u_1 \leq u_2 \leq \ldots \uparrow u$ a.e., with each $u_n \in M_\mu^+$, that $\rho(u) = \sup_n \rho(u_n)$.

The Riesz-Fischer property is equivalent to completeness of the normed space L_ρ ([Z], 64.2), and properties (ii) - (iv) are listed in increasing order of strength, i.e. (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) ([Z], 65.1).

2.4 The Associate Norms.

For each $f \in M_\mu^+$, define

$$\rho'(f) = \sup \{\int fg\, d\mu : g \in M_\mu^+,\ \rho(g) \le 1\} \quad .$$

It is easily checked that ρ' satisfies conditions (i) - (iii) of 2.2, so is a function seminorm. If ρ is saturated, then ρ' also satisfies condition (iv), and is then called the associate norm of ρ. With some more work it can also be shown that ρ' is saturated ([Z], § 69). Instead of $L_{\rho'}$ we will write L'_ρ and call this function space the associate space of L_ρ.

We can define the second and n^{th} associates of ρ by $\rho'' = (\rho')'$ and, inductively, $\rho^{(n)} = (\rho^{(n-1)})'$ for $n \ge 3$. The following fundamental results should be noted:

(i) ρ' always has the Fatou property ([Z], 68.1);

(ii) $\rho'' \le \rho$, with equality if and only if ρ has the Fatou property; if ρ has the weak Fatou property, then ρ and ρ'' are equivalent norms, so L_ρ and L''_ρ then contain the same elements ([Z], 68.2, 71.3);

(iii) From (i) and (ii) clearly $\rho^{(n+2)} = \rho^{(n)}$ $(n \ge 1)$;

(iv) If ρ is a saturated function norm, then for all $u, v \in M_\mu^+$,

$$\int uv\, d\mu \le \rho(u)\rho'(v) \qquad \text{(Hölder inequality)}$$

([Z], 68.5). In particular, if $u \in L_\rho$ and $v \in L'_\rho$, then $uv \in L^1(\mu)$.

2.5 The order ideal L^a_ρ.

An order ideal of M_μ is a linear subspace I such that, if $f \in I$ and $g \in M_\mu$ with $|g| \le |f|$ a.e., then $g \in I$.

Every ideal I has a <u>carrier set</u> $C \in \Sigma$ which is defined to be the complement in Ω of a maximal set $C' \in \Sigma$ such that $f\chi_{C'} = 0$ a.e. for every $f \in I$.

An element f of L_ρ is said to be of <u>absolutely continuous norm</u> if, whenever $f_i \in L_\rho$ $(i = 1,2,\ldots)$ and $|f| \geq f_1 \geq f_2 \geq \ldots \downarrow 0$ pointwise a.e. on Ω, then it follows that $\rho(f_i) \to 0$, as $i \to \infty$. Let L_ρ^a denote the set of all functions of absolutely continuous norm in L_ρ.

(i) L_ρ^a is a norm-closed ideal of L_ρ, and we shall denote by Ω_a the carrier set of L_ρ^a.

(ii) If ρ is an absolutely continuous norm, i.e. if $L_\rho = L_\rho^a$, and if ρ has the weak Fatou property, then ρ has the Fatou property ([Z], 73.α).

(iii) L_ρ^a is an <u>order-dense</u> ideal of $\chi_{\Omega_a} L_\rho$, i.e. whenever $0 \leq f = f\chi_{\Omega_a} \in L_\rho$, then there exists a sequence $\{f_n\}$ in L_ρ^a with $0 \leq f_n \uparrow f$ a.e.

<u>Proof</u>. Let $0 \leq f = f\chi_{\Omega_a} \in L_\rho$. By the Exhaustion Theorem ([Z], 67.3) and the definition of the carrier of L_ρ^a, we can find a sequence of sets $\Omega_n' \uparrow \Omega_a$ with $\chi_{\Omega_n'} \in L_\rho^a$ $(n = 1,2,\ldots)$. Let $\Omega_n'' = \{f \leq n\}$ and define $\Omega_n = \Omega_n' \cap \Omega_n''$ $(n = 1,2,\ldots)$. Since $f < \infty$ a.e., $\Omega_n'' \uparrow \Omega$ and so (from 2.1(iv)), $\Omega_n \uparrow \Omega_a$. Hence $0 \leq f\chi_{\Omega_n} \uparrow f$ a.e., and for each n,

$$f\chi_{\Omega_n} \leq n\,\chi_{\Omega_n} \leq n\,\chi_{\Omega_n'} \in L_\rho^a \quad .$$

Taking $f_n = f\chi_{\Omega_n}$ $(n = 1,2,\ldots)$ we have a suitable sequence.

Note (iii) implies, in particular, that for all $f \in M_\mu^+$,

$$\rho'(f\chi_{\Omega_a}) = \sup \{\textstyle\int fg \, d\mu : 0 \leq g \in L_\rho^a, \rho(g) \leq 1\} \; .$$

2.6 The dual space of L_ρ.

The dual space L_ρ^* is partially ordered by defining that $G_1 \leq G_2$ whenever $\langle f, G_1\rangle \leq \langle f, G_2\rangle$ for every $f \in L_\rho^+$ $(G_1, G_2 \in L_\rho^*)$.

The non-negative linear functionals on L_ρ are precisely those elements G of L_ρ^* satisfying $G \geq \Theta$, where Θ denotes the null functional. Let $\Theta \leq G \in L_\rho^*$: then G is said to be

(i) an order continuous linear functional if, whenever $f_i \in L_\rho$ $(i = 1,2,\dots)$ and $f_1 \geq f_2 \geq \dots \downarrow 0$ a.e., we have

$$\langle f_n, G\rangle \to 0 ;$$

(ii) a singular linear functional if, whenever $G \geq G_1 \geq \Theta$ in L_ρ^* and G_1 is order continuous, then $G_1 = \Theta$.

THEOREM (see [Z], § 48). Any $G \in L_\rho^*$ has a Standard (Jordan) Decomposition as

$$G = G_1 - G_2 + i\,(G_3 - G_4)$$

where each $G_i \geq \Theta$ $(i = 1,\dots,4)$, and where this is the most efficient decomposition in the sense that if we also have

$$G = G_5 - G_6 + i\,(G_7 - G_8)$$

with $G_i \geq \Theta$ $(i = 5,\dots,8)$, then $G_5 - G_1 = G_6 - G_2$ is non-negative and $G_7 - G_3 = G_8 - G_4$ is non-negative.

Let $G \in L_\rho^*$: we call G order continuous (resp. singular) if each of the non-negative components of G in its Standard Decomposition is order continuous (resp. singular).

In [Z], § 70 it is shown that the mutually exclusive properties (i), (ii) for linear functionals define disjoint linear subspaces of L^*_ρ, and moreover that every element G of L^*_ρ has a unique decomposition as

$$G = G_c + G_s$$

where G_c is an order continuous, and G_s a singular linear functional. It is shown also that an element G of L^*_ρ is order-continuous if and only if there exists a function $g \in L'_\rho$ such that for every $f \in L_\rho$,

$$\langle f, G \rangle = \int fg \, d\mu \ ,$$

and in this case the dual norm $\|G\|$ of G is precisely $\rho'(g)$. It is therefore standard practice to identify the subspace of order continuous linear functionals with L'_ρ, and, letting $L^*_{\rho,s}$ denote the space of singular linear functionals, we write

$$L^*_\rho = L'_\rho \oplus L^*_{\rho,s} \ .$$

Moreover, L'_ρ is a total linear subspace of L^*_ρ.

Standard Decomposition of functions.

Note finally that every element f of M_μ has a unique decomposition as $f_1 - f_2 + i(f_3 - f_4)$ where for $j = 1,\ldots,4$, f_j is an element of M^+_μ, f_1 and f_2 are disjoint, and f_3 and f_4 are disjoint. This fact enables us to obtain results more easily by proving them first for non-negative valued functions and extending them by the Standard Decomposition above, to all of M_μ. This extension process is usually trivial, and when occurring, will therefore not be made explicit.

CHAPTER II. REPRESENTATION OF PRECYCLIC SPACES.

In [G_1], T.A. Gillespie gave a representation theorem for cyclic spaces, in terms of the ideal of functions of absolutely continuous norm in a Banach function space. In the present chapter we shall obtain a slight generalisation of this result, and pursue one or two points arising, particularly in relation to Banach function spaces themselves. We begin by summarising Gillespie's representation.

Definition. Let X be a Banach space and $\mathcal{B}$ be a σ-complete Boolean algebra of projections on X. Then X is called cyclic with respect to $\mathcal{B}$ if there exists an element x_o of X such that

$$X = \overline{\text{lin}}\ \{Px_o : P \in \mathcal{B}\} \quad . \tag{1}$$

Such an element x_o is called a cyclic vector for X.

REPRESENTATION THEOREM ([G_1],§ 3). Let $\mathcal{B}$ be a σ-complete Boolean algebra of projections on a Banach space X and let $x_o \in X$ be such that

$$X = \overline{\text{lin}}\ \{Px_o : P \in \mathcal{B}\} \quad .$$

Then there exist a finite measure μ (*) defined on a σ-algebra Σ of subsets of a compact Hausdorff space Ω, a saturated function norm ρ based on (Ω, Σ, μ), and a linear isomorphism U from the ideal L^a_ρ of L_ρ onto X such that

(i) ρ has the Fatou property;

(ii) the constant function 1 belongs to L^a_ρ and to L'_ρ;

(iii) $U1 = x_o$;

(iv) $U(\varphi f) = T_\varphi Uf$ for $f \in L^a_\rho$ and $\varphi \in L^\infty(\mu)$;

(*) and a spectral measure E with range $\mathcal{B}$

(v) $\|Uf\| \leq \rho(f) \leq 4K\|Uf\|$ for $f \in L_\rho^a$,

where $T_\varphi = \int \varphi(\lambda)E(d\lambda)$ is well-defined in the norm topology of $B(X)$ ([DS_2],p. 1929), and K is a uniform bound on the norms $\|E(\sigma)\|$ $(\sigma \in \Sigma)$ (Lemma 1.2).

Via this isomorphism, the Banach space X inherits a natural lattice ordering given by

$$x_1 \leq x_2 \qquad \text{if} \quad U^{-1}x_1 \leq U^{-1}x_2 \quad \mu\text{-a.e.}$$

The σ-completeness of $\mathcal{B}$ ensures that for each $x \in X$ a Bade functional can be found ([DS_2],XVII.3.12), that is to say an element x^* of X^* such that

(i) $\langle Px, x^*\rangle \geq 0$ $(P \in \mathcal{B})$, and

(ii) $\langle Px, x^*\rangle = 0$ only if $Px = 0$.

The measure μ of the theorem is defined by taking a Bade functional x_o^* corresponding to x_o and letting

$$\mu(\sigma) = \langle E(\sigma)x_o, x_o^*\rangle \qquad (\sigma \in \Sigma) .$$

Since $\mathcal{B}$ is σ-complete we also have (Lemma 1.5) that $\mathcal{B}$ is the range of a strongly countably additive spectral measure $E(\cdot)$ on X. We shall presently be considering Banach spaces X of a form analogous to (1), where $\mathcal{B}$ is replaced by the range of a prespectral measure on X.

§ 3. Prespectral measures in L_ρ and L_ρ^a.

We begin with two lemmas of a general character which will play an important part in later chapters also.

3.1 LEMMA. $(L_\rho^a)^* = \chi_{\Omega_a}L_\rho'$ (cf. [Z], 72.5, 6).

Proof. Recall that ([Z], § 70, § 73)

$$L^*_\rho = L'_\rho \oplus L^*_{\rho,s} \tag{2}$$

and $$L^*_{\rho,s} \subseteq (L^a_\rho)^\perp . \tag{3}$$

Let $0 \leq g \in L'_\rho$ and write

$$g_a = g\chi_{\Omega_a} \quad , \quad g_b = g - g_a .$$

For any $h \in L^a_\rho$, $h = h\chi_{\Omega_a}$, so

$$\langle h, g_b\rangle = \int hg_b \, d\mu = 0 \, ,$$

i.e. $g_b \in (L^a_\rho)^\perp$. Hence

$$\chi_{\Omega\setminus\Omega_a} L'_\rho \subseteq (L^a_\rho)^\perp . \tag{4}$$

Suppose that $g_a \in (L^a_\rho)^\perp$. Then for any $\sigma \in \Sigma$ with $\chi_\sigma \in L^a_\rho$,

$$0 = \langle\chi_\sigma, g_a\rangle = \int_\sigma g_a \, d\mu .$$

Since $g_a = g\chi_{\Omega_a} \geq 0$ a.e., this means that $g_a\chi_\sigma = 0$ a.e. By the Exhaustion Theorem ([Z], 67.3) we can find sets $\sigma_n \uparrow \Omega_a$ with each $\chi_{\sigma_n} \in L^a_\rho$; it follows easily that $g_a = 0$ a.e. Hence $\chi_{\Omega_a} L'_\rho$ is disjoint from $(L^a_\rho)^\perp$. Putting this fact together with (3) and (4), compare then with (2) which may be rewritten as

$$L^*_\rho = \chi_{\Omega_a} L'_\rho \oplus \chi_{\Omega\setminus\Omega_a} L'_\rho \oplus L^*_{\rho,s} .$$

The result then follows.

3.2 LEMMA. Let ρ be a saturated function norm with the Riesz-Fischer property. Then

$$L^a_\rho = \chi_{\Omega_a} L^a_{\rho''} .$$

Proof. We begin by showing that L^a_ρ is a closed subspace of L''_ρ. Recall that $\rho = \rho''$ on L^a_ρ ([Z], 72.3). If $f \in L^a_\rho$ and $f \geq f_1 \geq f_2 \geq \ldots \downarrow 0$ a.e., then $\rho''(f_n) = \rho(f_n) \to 0$ as $n \to \infty$. Hence $f \in L^a_{\rho''}$ and

$$L^a_\rho \subseteq L^a_{\rho''} \quad . \tag{5}$$

Suppose that $f_n \in L^a_\rho$ $(n = 1,2,\ldots)$ and $f \in L''_\rho$ satisfy $\rho''(f-f_n) \to 0$ as $n \to \infty$. The sequence $\{f_n\}$ is ρ''-Cauchy and hence ρ-Cauchy so, since L^a_ρ is a closed ideal of L_ρ, there exists $g \in L^a_\rho$ with $\rho(g-f_n) \to 0$.

However then,

$$\rho''(g-f_n) = \rho(g-f_n) \to 0 \ .$$

So $g = f$ a.e. and $f \in L^a_\rho$ as required.

Let Ω_C denote the carrier of $L^a_{\rho''}$. From (5), $\Omega_a \subseteq \Omega_C$ and $L^a_\rho \subseteq \chi_{\Omega_a} L^a_{\rho''}$. Hence, viewing L^a_ρ as a subspace of $L^a_{\rho''}$,

$$(\chi_{\Omega_a} L^a_{\rho''})^\perp \subseteq (L^a_\rho)^\perp \quad .$$

We shall show the converse inclusion also holds. We have that $L^*_{\rho''} = L_\rho''' \oplus L^*_{\rho'',s} = L_\rho' \oplus L^*_{\rho'',s}$, and viewing L^a_ρ as a subspace of L''_ρ,

$$(L^a_\rho)^\perp \supseteq (L^a_{\rho''})^\perp \supseteq L^*_{\rho'',s} \quad . \tag{6}$$

Since

$$L^*_{\rho''} = L_\rho''' \oplus L^*_{\rho'',s} = L_\rho' \oplus L^*_{\rho'',s} = \chi_{\Omega_a} L'_\rho \oplus \chi_{\Omega\setminus\Omega_a} L'_\rho \oplus L^*_{\rho'',s} \ ,$$

it follows from (6) and from Lemma 3.1 that

$$(L^a_\rho)^\perp = \chi_{\Omega\setminus\Omega_a} L'_\rho \oplus L^*_{\rho'',s} \quad .$$

If $g = g\chi_{\Omega\setminus\Omega_a} \in L'_\rho$, then trivially $\langle h, g\rangle = 0$ whenever $h = h\chi_{\Omega_a} \in L^a_{\rho''}$. Now suppose $0 \leq g = g\chi_{\Omega_a} \in L'_\rho \cap (\chi_{\Omega_a} L^a_{\rho''})^\perp$. Then for any $\sigma \in \Sigma_a = \{\delta \cap \Omega_a : \delta \in \Sigma\}$ with $\chi_\sigma \in L^a_{\rho''}$,

$$0 = \langle \chi_\sigma, g \rangle = \int_\sigma g \, d\mu \quad ,$$

so $g\chi_\sigma = 0$ a.e. By the Exhaustion Theorem ([Z], 67.3), we can find a sequence $\sigma_n \uparrow \Omega_a$ with $\chi_{\sigma_n} \in L^a_{\rho''}$ for each n. It now follows that $g\chi_{\Omega_a} = 0$ a.e. and thus

$$(\chi_{\Omega_a} L^a_{\rho''})^\perp \cap L'_\rho = \chi_{\Omega \setminus \Omega_a} L'_\rho = (L^a_\rho)^\perp \cap L'_\rho \quad .$$

So finally, since L^a_ρ and $\chi_{\Omega_a} L^a_{\rho''}$ are both closed ideals of L''_ρ,

$$L^a_\rho = (L^a_\rho)^{\perp\perp} = (\chi_{\Omega_a} L^a_{\rho''})^{\perp\perp} = \chi_{\Omega_a} L^a_{\rho''} \quad .$$

<u>3.3 COROLLARY</u>. $L^a_\rho = L^a_{\rho''}$ if and only if $\Omega_a = \Omega_c$, where Ω_c is the carrier of $L^a_{\rho''}$.

We now give a further three lemmas which are crucial to the present chapter.

<u>3.4 LEMMA</u>. Let ρ be a saturated function norm based on (Ω, Σ, μ). Then the set of multiplication operators $M_{\chi_\sigma} : f \mapsto f\chi_\sigma$ $(\sigma \in \Sigma,\ f \in L_\rho)$ constitutes a bounded Boolean algebra of projections on L_ρ and the mapping $E(\cdot) : \sigma \mapsto M_{\chi_\sigma}$ on Σ is a prespectral measure of class L'_ρ. Moreover each element of L_ρ has a corresponding Bade functional in L'_ρ. This prespectral measure $E(\cdot)$ is spectral (i.e. strongly countably additive) if and only if ρ is an absolutely continuous norm.

<u>Proof</u>. It is routine to check that the operators $\{M_{\chi_\sigma} : \sigma \in \Sigma\}$ form a bounded Boolean algebra of projections each of norm at most one; whenever $f \in L_\rho$ and $g \in L'_\rho$, then $fg \in L^1(\mu)$ so the mapping

$$\sigma \mapsto \langle M_{\chi_\sigma} f, g \rangle = \int_\sigma fg \, d\mu \qquad (\sigma \in \Sigma)$$

is a countably additive complex measure on Σ. Let $f \in L_\rho$ and suppose that $\langle f, g\rangle = 0$ for all $g \in L'_\rho$. We show that $\langle |f|, g\rangle = 0$ for all $g \in L'_\rho$. Let θ be a unimodular function satisfying $f\theta = |f|$. Then for any $g \in L'_\rho$, $g\theta \in L'_\rho$ also, and

$$0 = \langle f, g\theta\rangle = \int fg\theta \, d\mu = \int |f|g \, d\mu = \langle |f|, g\rangle .$$

Therefore there is no loss in assuming that $f \in L^+_\rho$. Since ρ' is a saturated norm, the Exhaustion Theorem ([Z], 67.3) allows us to find a sequence $\Omega_n \uparrow \Omega$ with $\chi_{\Omega_n} \in L'_\rho$ for each n. By hypothesis,

$$\int_{\Omega_n} f \, d\mu = \langle f, \chi_{\Omega_n}\rangle = 0$$

for each n. Hence $f\chi_{\Omega_n} = 0$ a.e. for each n, and it follows that $f = 0$ a.e. on Ω. Hence L'_ρ is a total subspace of L^*_ρ, and thus $E(\cdot)$ is a prespectral measure on L_ρ, of class (Σ, L'_ρ).

Now choose an element h of L'_ρ with $h > 0$ a.e. For any fixed $f \in L_\rho$, let $\gamma = \operatorname{supp} f$ and let θ be a function in M_μ with $|\theta| \leq 1$ a.e. and $\operatorname{supp} \theta = \gamma$, satisfying $f\theta = |f|$ a.e. For any $\sigma \in \Sigma$,

$$\langle \chi_\sigma f, h\theta \rangle = \int_\sigma fh\theta \, d\mu = \int_\sigma |f|h \, d\mu \geq 0 \quad ,$$

and $\langle \chi_\sigma f, h\theta\rangle = 0$ only if $|f|h\chi_\sigma = 0$ μ-a.e. Since $h > 0$ a.e. this occurs only when $|f|\chi_\sigma = 0$ a.e., i.e. when $f\chi_\sigma = 0$ a.e. Therefore $h\theta$ is a Bade functional for f, and is in L'_ρ since $|h\theta| \leq h$ a.e.

Suppose finally that $E(\cdot)$ is strongly countably additive. Let $f \in L_\rho$ and $\{\sigma_n\}$ be a sequence in Σ such that $\sigma_n \downarrow \emptyset$. Then we have

$$\rho(f\chi_{\sigma_n}) = \rho(E(\sigma_n)f) \to 0 .$$

But this shows that f is of absolutely continuous norm (by [Z], 72.1).
Hence $L_\rho = L_\rho^a$.

Conversely if $L_\rho = L_\rho^a$, then $L_\rho' = L_\rho^*$, so the measure $E(\cdot)$ is of class (Σ, L_ρ^*) and hence is strongly countably additive (Def. 1.4).

3.5 COROLLARY. If L_ρ is weakly sequentially complete, then $L_\rho = L_\rho^a$.

Proof. Assume L_ρ is weakly sequentially complete. Then by [G3], Theorem 1 and Corollary, every prespectral measure on L_ρ is spectral. In particular, the measure $\sigma \mapsto M_{\chi_\sigma}$ is spectral. From Lemma 3.4 therefore, $L_\rho = L_\rho^a$.

In [G1] it was shown that whenever $1 \in L_\rho^a$, then L^∞ is dense in L_ρ^a. The next lemma generalises this fact.

3.6 LEMMA. Let ρ be a saturated function norm with the Riesz-Fischer property, and let $f \in L_\rho^a$ have support Ω_a. Then, denoting by J_f the principal ideal generated by f,

$$\bar{J}_f = L_\rho^a \quad .$$

More generally, for any $g \in L_\rho^a$ with supp $g = \gamma \in \Sigma$,

$$\bar{J}_g = \chi_\gamma L_\rho^a \quad .$$

Proof. L_ρ^a is a closed order ideal of L_ρ, so clearly since $f \in L_\rho^a$, the closed principal ideal $\bar{J}_f$ generated by f is contained in L_ρ^a.

For the converse inclusion, suppose $0 \leq h \in L_\rho^a$. Since $J_f = J_{|f|}$, we shall assume that $f \geq 0$ a.e. Let $E_n = \{h \leq n\}$ $(n = 1,2,\ldots)$. By the Exhaustion Theorem ([Z], 67.3), we may choose a sequence $\Omega_n \uparrow \Omega_a$ with

$\chi_{\Omega_n} \in L^a_\rho$ for each n. Let $F_n = E_n \cap \Omega_n$ $(n = 1,2,\ldots)$. Since $h < \infty$ a.e., $E_n \uparrow \Omega$ so $F_n \uparrow \Omega_a$ and $h\chi_{F_n} \uparrow h$ a.e. Since $h \in L^a_\rho$ and $h \geq h - h\chi_{F_1} \geq h - h\chi_{F_2} \geq \ldots \downarrow 0$ a.e., we have that $\rho(h - h\chi_{F_n}) \to 0$; it is therefore sufficient to prove that for each n, $h\chi_{F_n} \in \overline{J}_f$.

Let n be fixed and write $h_n = h\chi_{F_n}$; letting $G_m = \{f \geq \frac{1}{m}\}$ $(m = 1,2,\ldots)$, then $\chi_{G_m} \leq mf\chi_{G_m} \leq mf$, so $\chi_{G_m} \in J_f$ $(m = 1,2,\ldots)$. Since $G_m \uparrow \Omega_a$, $F_n \cap G_m \uparrow F_n$ as $m \to \infty$ and so $h_n\chi_{G_m} \uparrow h_n$ a.e. Since $h_n \in L^a_\rho$, $\rho(h_n - h_n\chi_{G_m}) \to 0$ as $m \to \infty$. Now for each m,

$$h_n\chi_{G_m} \leq h\chi_{E_n \cap G_m} \leq n\chi_{G_m} \in J_f \quad ;$$

hence $h_n \in \overline{J}_f$ and the first assertion of the lemma follows.

It is clear that the more general statement also follows in a similar manner.

Note that by Lemma 1.5, Lemma 3.4 tells us that whenever ρ is an absolutely continuous norm, then L_ρ is a cyclic space. It is easy to see that more generally, the ideal L^a_ρ of L_ρ is cyclic with respect to the restricted projections $M_{\chi_\sigma}\big|_{L^a_\rho}$ $(\sigma \in \Sigma)$, by virtue of its invariance under each M_{χ_σ}, and by Lemma 3.1. Recall however that the ideal of functions of absolutely continuous norm occurring in Gillespie's representation displayed several special properties. The following lemma will derive directly from any ρ which is complete and saturated, another function norm τ displaying properties (i) - (v) of the Representation Theorem, together with an appropriate linear isomorphism between L^a_ρ and L^a_τ.

<u>3.7 LEMMA</u>. Let ρ be a saturated function norm with the Riesz-Fischer property, based on the σ-finite measure space (Ω, Σ, μ). Let $0 \leq f_o \in L^a_\rho$ with supp $f_o = \Omega_a$.

Then L_ρ^a is a cyclic space with respect to the Boolean algebra of projections $\{M_{\chi_\sigma}|_{L_\rho^a} : \sigma \in \Sigma\}$ and f_o is a cyclic vector.

Proof. From Lemma 3.4 and Lemma 3.1, the projections $\{M_{\chi_\sigma}|_{L_\rho^a} : \sigma \in \Sigma\}$ form a σ-complete Boolean algebra. From Lemma 3.6, $L_\rho^a = \overline{J}_{f_o}$. Define a measure ν on $\Sigma_a = \{\sigma \cap \Omega_a : \sigma \in \Sigma\}$ by

$$\nu(\sigma) = \int_\sigma f_o \varphi_o \, d\mu \quad ,$$

where φ_o can be taken to be any fixed non-negative element of L_ρ' whose support is Ω_a; ν and $\mu|_{\Sigma a}$ are then equivalent measures. Now define a function norm τ on M_ν by

$$\tau(g) = \rho''(gf_o) \qquad (g \in M_\nu) \quad ,$$

and a mapping $U_1 : M_\mu \to M_\nu$ by

$$U_1 f = ff_o^{-1}\chi_{\Omega a} \qquad (f \in M_\mu) \quad .$$

Then for each $f \in M_\mu$, $\tau(U_1 f) = \rho''(f\chi_{\Omega a})$ and

$$L_\tau = \{ff_o^{-1} : f = f\chi_{\Omega_a} \in L_\rho''\} = U_1(\chi_{\Omega a} L_\rho'') \quad .$$

It is clear that τ is a saturated norm. Now suppose we have $0 \le u_1 \le u_2 \le \ldots \uparrow u$ a.e., with each $u_n \in L_\tau$. Then, since ρ'' has the Fatou property,

$$\tau(u) = \rho''(uf_o) = \sup_n \rho''(u_n f_o) = \sup_n \tau(u_n) \quad .$$

Hence τ has the Fatou property.

For any $0 \le f \in M_\nu$,

$$\begin{aligned} \tau'(f) &= \sup \{\textstyle\int fg \, d\nu : \tau(g) \le 1, \ g \ge 0 \text{ a.e.}\} \\ &= \sup \{\textstyle\int fgf_o\varphi_o \, d\mu : \rho''(gf_o) \le 1, \ g \ge 0 \text{ a.e.}\} \end{aligned}$$

$$= \sup \{\int fg\varphi_o \, d\mu : \rho''(g) \leq 1, \ g \geq 0 \text{ a.e.}\}$$

$$= \rho''' (f\varphi_o)$$

$$= \rho' (f\varphi_o) \ .$$

In particular, $\tau'(1) = \rho'(\varphi_o) < \infty$ (where clearly, 1 denotes χ_{Ω_a}).

Let $0 \leq g \in L_\tau^a$ and suppose we have $gf_o \geq g_1 \geq g_2 \geq \ldots \downarrow 0$ μ-a.e. Then $g \geq g_1 f_o^{-1} \geq g_2 f_o^{-1} \geq \ldots \downarrow 0$ μ-a.e. on Ω_a, therefore also ν-a.e., and we have $\tau(g_i f_o^{-1}) \to 0$ as $i \to \infty$, i.e. $\rho''(g_i) \to 0$ as $i \to \infty$. Hence $gf_o \in L_{\rho''}^a$. Since $gf_o = gf_o\chi_{\Omega_a}$, then from Lemma 3.2, $gf_o \in L_\rho^a$.

An equally easy argument in the converse direction allows us to conclude that $g \in L_\tau^a$ if and only if $gf_o \in L_\rho^a$, or equivalently that

$$f \in L_\rho^a \quad \text{if and only if} \quad U_1 f \in L_\tau^a \ .$$

In particular $U_1 1 = f_o$, so $1 \in L_\tau^a$ and thus the norm τ displays all the desired properties.

Since Ω_a is the carrier of L_ρ^a, it is evident that $U = U_1 \big|_{L_\rho^a}$ is one-one. Hence $U : L_\rho^a \to L_\tau^a$ is a bijective linear isomorphism. For each $f \in L_\rho^a$, $\tau(Uf) = \rho''(f) = \rho(f)$, so U is bicontinuous. Thus U is the required isomorphism of L_ρ^a, and finally,

$$L_\rho^a = \overline{\text{lin}} \ \{\chi_\sigma f_o : \sigma \in \Sigma\} \ .$$

Lemmas 3.6 and 3.7 show that the principal order ideal generated in L_ρ by any function of absolutely continuous norm is a cyclic subspace of L_ρ. In fact a broader result is true as we shall see in Prop. 3.9. But first we need the definition of a precyclic space.

<u>3.8 Definition</u>. Let X be a Banach space and $E(\cdot)$ be a prespectral measure on X, of class (Σ, Γ), where Σ is a σ-algebra of sets, and $\Gamma = \{x^* \in X^* : \langle E(\cdot)x, x^*\rangle$ is a countably additive complex measure on Σ for every $x \in X\}$. If there exists an element x_o of X such that

$$X = \overline{\text{lin}}\ \{E(\sigma)x_o : \sigma \in \Sigma\} ,$$

then X is called <u>precyclic</u> with respect to the range of $E(\cdot)$, and x_o is called a <u>cyclic vector</u> for X.

<u>Notation</u>. Given a prespectral measure $E(\cdot)$ on a Banach space X as above, we shall denote by $M(x)$ the precyclic subspace $\overline{\text{lin}}\ \{E(\sigma)x : \sigma \in \Sigma\}$ of X, generated by the element x of X.

<u>3.9 PROPOSITION</u>. Let ρ be a complete saturated function norm based on (Ω, Σ, μ) and let $g \in L_\rho$. Then the closed principal order ideal $\overline{J}_g$ is a precyclic subspace of $L\rho$ with respect to the prespectral measure $E(\cdot) : \sigma \to M\chi_\sigma$ $(\sigma \in \Sigma)$, and g is a cylic vector, i.e.

$$\overline{J}_g = \overline{\text{lin}}\ \{\chi_\sigma g : \sigma \in \Sigma\} .$$

<u>Proof</u>. For each $\sigma \in \Sigma$, $|\chi_\sigma g| \leq |g|$ therefore $\chi_\sigma g \in J_g$; since an ideal is a linear subspace, $\text{lin}\ \{\chi_\sigma g : \sigma \in \Sigma\} \subseteq J_g$ and hence $M(g) \subseteq \overline{J}_g$. For the reverse inclusion, note firstly that whenever σ and δ are elements of Σ, then

$$M\chi_\delta(\chi_\sigma g) = \chi_{\sigma\cap\delta}g \in M(g) ,$$

so $M(g)$ is closed under multiplication by characteristic functions and therefore also under multiplication by simple functions. Now let $\varphi \geq 0$ be a bounded measurable function: there is a sequence $\{s_n\}$ of simple functions such that $0 \leq s_n \uparrow \varphi$ a.e. uniformly as $n \to \infty$, and we may therefore assume (or take a suitable subsequence to ensure) that for each n,

$$|s_n - \varphi| \leq 2^{-n} \quad \text{a.e.}$$

For any $f \in M(g)$, we have $s_n f \in M(g)$ from above, and

$$\rho(\varphi f - s_n f) = \rho(|\varphi - s_n| f) \leq 2^{-n} \rho(f) \to 0$$

as $n \to \infty$. Since $M(g)$ is norm-closed, $\varphi f \in M(g)$ and so $M(g)$ is closed under multiplication by any bounded measurable function. Since $g \in M(g)$, it now follows that the order ideal J_g is contained in $M(g)$ and hence that $\overline{J}_g \subseteq M(g)$.

§ 4. A representation theorem for precyclic spaces.

Throughout this section, we let $E(\cdot)$ be a prespectral measure of class (Σ, Γ) on a Banach space X, and we suppose that there is an element e of X such that

$$X = \overline{\text{lin}}\, \{E(\sigma)e : \sigma \in \Sigma\} \quad .$$

Further, we assume that a Bade functional e* may be found for e, in Γ. We introduce definitions for a measure space μ on Σ and a function norm ρ on M_μ, similar to those of [G1], viz.

$$\mu(\sigma) = \langle E(\sigma)e, e^* \rangle \qquad (\sigma \in \Sigma)$$

and

$$\rho(f) = \sup\, \{\|T_\varphi e\| : |\varphi| \leq |f| \text{ a.e.}, \varphi \in L^\infty(\mu)\} \ (f \in M_\mu),$$

recalling that $T_\varphi = \int_\Omega \varphi(\lambda) E(d\lambda)$ is well-defined in the uniform topology of $B(X)$, for each $\varphi \in L^\infty$, and satisfies

$\|T_\varphi\| \leq 4K \|\varphi\|_\infty$, where K is a uniform bound for $\{\|E(\sigma)\| : \sigma \in \Sigma\}$ ([DS$_2$], p. 1929).

That ρ is then a saturated function norm follows exactly as in [G$_1$]. We include at this stage a purely technical lemma, and derive a corollary thereof which will enable us to show completeness of the norm ρ. The setting of this lemma is an arbitrary Riesz space (or vector lattice): since the corollary, and its application in Prop. 4.3 use none but the most elementary lattice theory and the fact that L_ρ is a Riesz space, it is not pertinent to say any more about Riesz spaces here. A fuller introduction to their properties will be given at the beginning of Chapter IV, or may be found in [LZ$_2$] and [F].

4.1 LEMMA ([F], 14 Jb). Let E be a Riesz space. If $\{x_i : i = 1,\ldots,n\}$ is a finite sequence in E^+, and $y \in E$ satisfies $|y| \leq \sum_{i=1}^{n} x_i$, then there exists a finite sequence $\{y_i : i = 1,\ldots,n\}$ in E such that $y = \sum_{i=1}^{n} y_i$ and $|y_i| \leq x_i$ for each i.

Remark. From the proof of the lemma, it also follows that the sequence obtained satisfies $|y_i| \leq |y|$ for each i.

4.2 COROLLARY. Suppose we have $x, x_i \in E^+$ $(i = 1,2,\ldots)$ satisfying $x = \sum_{i=1}^{\infty} x_i$, and $y \in E$ with $|y| \leq x$. Then there exists a sequence $\{y_i\}$ in E such that $|y_i| \leq x_i \wedge |y|$ $(i = 1,2,\ldots)$, and for each n, $y_n' = y - \sum_{i=1}^{n} y_i$ satisfies $|y_n'| \leq \sum_{i=n+1}^{\infty} x_i$.

Proof. For $m = 1,2,\ldots$, define $x_m' = \sum_{r=m+1}^{\infty} x_r$ so that $x = \sum_{r=1}^{m} x_r + x_m'$. We apply the lemma and remark iteratively with $n = 2$ at each step.

Step 1. Since $|y| \leq x = x_1 + x_1'$, there exist elements y_1, y_1' of E with $|y_1| \leq x_1 \wedge |y|$, $|y_1'| \leq x_1' \wedge |y|$ and $y = y_1 + y_1'$.

Step n+1 ($n \geq 1$). We have $|y_n'| \leq x_n' = x_{n+1} + x_{n+1}'$; so there exist y_{n+1}, y_{n+1}' in E with

$|y_{n+1}| \leq x_{n+1} \wedge |y_n'| \leq x_{n+1} \wedge |y|$,

$|y_{n+1}'| \leq x_{n+1}' \wedge |y_n'| \leq x_{n+1}' \wedge |y|$ and $y_n' = y_{n+1} + y_{n+1}'$.

The sequences $\{y_i\}$, $\{y_i'\}$ thus defined satisfy the required conditions.

4.3 PROPOSITION. ρ has the Riesz-Fischer property.

Proof. Suppose $f_i \in M_\mu^+$ ($i = 1,2,\ldots$) and $\sum \rho(f_i) < \infty$. Let $f = \sum_{i=1}^{\infty} f_i \in M_\mu$: we show that $f \in L_\rho$.

Let $\varphi \in L^\infty$ with $|\varphi| \leq f$ a.e. Applying Cor. 4.2, there are sequences $\{\varphi_i\}_{i\in\mathbb{N}}$, $\{\psi_i\}_{i\in\mathbb{N}}$ such that for any $N \in \mathbb{N}$,

$$\varphi = \sum_{i=1}^{N} \varphi_i + \psi_N$$

where $|\varphi_i| \leq \inf(f_i, |\varphi|)$ ($i = 1,\ldots,N$), and

$|\psi_N| \leq \sum_{i=N+1}^{\infty} f_i$. We claim now that $\psi_N \to 0$ a.e. as $N \to \infty$. Indeed, letting $\tau_\infty = \{f = \infty\}$ and $g_N = \sum_{i=1}^{N} f_i$ ($N = 1,2,\ldots$), then $g_N \uparrow \infty$ pointwise a.e. on τ_∞. Since $\mu(\Omega) = \langle e, e^*\rangle < \infty$, a variant of Egoroff's Theorem applies and yields an increasing sequence of measurable sets τ_k contained in τ_∞, with $\mu(\tau_\infty \setminus \bigcup_k \tau_k) = 0$ and such that $g_N \uparrow \infty$ uniformly on τ_k for each k. Since e^* is a Bade functional, it follows that $E(\tau_\infty \setminus \bigcup_k \tau_k)e = 0$.

Suppose that $E(\tau_\infty)e \neq 0$. Then $\mu(\tau_\infty) = \mu(\bigcup_k \tau_k) \neq 0$. Let $\tau_1' = \tau_1$ and $\tau_r' = \tau_r \setminus \tau_{r-1}$ $(r \geq 2)$; then the sets τ_r' are pairwise disjoint and $\bigcup_r \tau_r' = \bigcup_r \tau_r$, so

$$0 < \mu(\bigcup_r \tau_r) = \mu(\bigcup_r \tau_r') = \sum_r \mu(\tau_r') .$$

Hence for some $k \in \mathbb{N}$, $\mu(\tau_k') > 0$, which implies that $E(\tau_k')e \neq 0$, and therefore that $E(\tau_k)e \neq 0$. Since $g_N \uparrow \infty$ uniformly on τ_k, there is a subsequence $\{n_m\}$ of $\mathbb{N}$ such that for each m,

$$g_{n_m}\chi_{\tau_k} \geq m\chi_{\tau_k} \quad \text{a.e.}$$

Therefore

$\rho(g_{n_m}) \geq \rho(g_{n_m}\chi_{\tau_k}) \geq m\ \rho(\chi_{\tau_k}) \geq m\|T_{\chi_{\tau_k}}e\| = m\|E(\tau_k)e\|$

$(m = 1,2,\ldots)$, and hence $\rho(g_{n_m}) \to \infty$ as $m \to \infty$. However,

$$\rho(g_{n_m}) = \rho(\sum_{i=1}^{n_m} f_i) \leq \sum_{i=1}^{n_m} \rho(f_i) \leq \sum_{i=1}^{\infty} \rho(f_i) < \infty .$$

From this contradiction, we must have that $E(\tau_\infty)e = 0$. Thus, $\mu(\tau_\infty) = 0$ so $\sum f_i$ is convergent μ-a.e. and $|\sum_{N+1}^{\infty} f_i| \to 0$ a.e. as $N \to \infty$. Since $|\psi_N| \leq \sum_{N+1}^{\infty} f_i$, it follows that $\psi_N \to 0$ a.e. as was claimed above, and we may thus write

$$\varphi = \sum_{i=1}^{\infty} \varphi_i \quad \text{a.e.}$$

For $m, n \in \mathbb{N}$, $\sum_{r=m}^{n} \|T_{\varphi_r}e\| \leq \sum_{r=m}^{n} \rho(f_r)$, therefore by the initial hypothesis, $\sum_{n=1}^{\infty} T_{\varphi_n}e$ converges in X, to an element z, say. We now show that $z = T_\varphi e$. Let $x^* \in \Gamma$: then

$$\begin{aligned}\langle z, x^*\rangle &= \langle \sum_{n=1}^{\infty} T_{\varphi_n}e, x^*\rangle \\ &= \sum_{n=1}^{\infty} \langle T_{\varphi_n}e, x^*\rangle\end{aligned}$$

$$= \sum_{n=1}^{\infty} \int \varphi_n(\lambda) \langle E(d\lambda)e, x^*\rangle . \tag{7}$$

If we can now show that (7) is equal to $\int \sum \varphi_n(\lambda) \langle E(d\lambda)e, x^*\rangle$, then it follows immediately, by the totality of Γ, that

$$z = \int \sum \varphi_n(\lambda)E(d\lambda)e = \int \varphi(\lambda)E(d\lambda)e = T_\varphi e .$$

To obtain this equality, we shall resort to a dominated convergence argument, and to this end we define

$$\nu(\tau) = \langle E(\tau)e, x^*\rangle \qquad (\tau \in \Sigma) .$$

There exist unimodular functions θ, θ_n $(n = 1,2,\ldots)$ satisfying

$$\theta d\nu = d|\nu| \quad \text{and} \quad \varphi_n\theta_n = |\varphi_n| \qquad (n = 1,2,\ldots) .$$

For each $N \in \mathbb{N}$, we then have

$$\begin{aligned}
\sum_{n=1}^{N} \int |\varphi_n| d|\nu| &= \sum_{n=1}^{N} \int \varphi_n\theta_n\theta \, d\nu \\
&= \sum_{n=1}^{N} \langle T_{\theta_n\theta} T_{\varphi_n}e, x^*\rangle \\
&\leq \sum_{n=1}^{N} \| T_{\theta_n\theta} \| \, \| T_{\varphi_n} e \| \, \| x^* \| \\
&\leq 4K \, \| x^* \| \sum_{n=1}^{N} \rho(f_n) \\
&\leq 4K \, \| x^* \| \sum_{n=1}^{\infty} \rho(f_n) \\
&< \infty .
\end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \int |\varphi_n| d|\nu| < \infty .$$

Since $|\nu|$ is a positive measure, the Monotone Convergence Theorem is applicable and yields

$$\int \sum_{n=1}^{\infty} |\varphi_n| d|\nu| = \sum_{n=1}^{\infty} \int |\varphi_n| d|\nu| < \infty \quad . \tag{8}$$

Putting $\xi_n = \sum_{m=1}^{n} \varphi_m$, we have for each n,

$$|\xi_n \theta^{-1}| = |\xi_n| = |\sum_{m=1}^{n} \varphi_m| \leq \sum_{m=1}^{n} |\varphi_m| \leq \Phi$$

say, where $\Phi = \sum_{n=1}^{\infty} |\varphi_n| \in L^1(\nu)$, from (8). So for each n, $\xi_n \theta^{-1} \in L^1(\nu)$ and, applying Lebesgue's Dominated Convergence Theorem, we have

$$\int \lim_n \xi_n \theta^{-1} d|\nu| = \lim_n \int \xi_n \theta^{-1} d|\nu| \ ,$$

$$\text{i.e. } \int \lim_n \sum_{m=1}^{n} \varphi_m \theta^{-1} d|\nu| = \lim_n \int \sum_{m=1}^{n} \varphi_m \theta^{-1} d|\nu|$$

$$= \lim_n \sum_{m=1}^{n} \int \varphi_m \theta^{-1} d|\nu| \ .$$

$$\text{i.e. } \int \sum_{n=1}^{\infty} \varphi_n d\nu = \sum_{n=1}^{\infty} \int \varphi_n d\nu$$

as required, and thus $z = T_\varphi e$. Hence finally

$$\| T_\varphi e \| = \| \sum_i T_{\varphi_i} e \| \leq \sum_i \| T_{\varphi_i} e \| \leq \sum_i \rho(f_i) \ ,$$

so by the definition of ρ,

$$\rho(f) \leq \sum \rho(f_i) < \infty \ ,$$

and thus ρ has the Riesz-Fischer property.

4.4 THEOREM. There is a bicontinuous bijection U of the closed principal order ideal $\overline{J}_1$ generated by the constant function $1 \in L_\rho$, onto X, satisfying $U1 = e$

and

$$U(\chi_\sigma f) = E(\sigma)Uf \qquad (f \in \bar{J}_1,\ \sigma \in \Sigma) .$$

Proof. Since the range of $E(\cdot)$ is bounded in norm by K, $\rho(1) \leq 4K\|e\|$, so 1 is indeed an element of L_ρ. If the function φ is bounded a.e., then for some constant $c \geq 0$, $|\varphi| \leq c1$ a.e., so $\varphi \in J_1$; conversely, since it is clear that every element of J_1 is bounded a.e., we have that $\bar{J}_1$ is precisely the ρ-closure of $L^\infty(\Omega, \mu)$. Define $U_1 : J_1 \to X$ by

$$U_1 f = T_f e \qquad (f \in J_1) .$$

Since $U_1(\chi_\sigma) = E(\sigma)e$ $(\sigma \in \Sigma)$, then

$$U_1(J_1) \supseteq \text{lin}\,\{E(\sigma)e : \sigma \in \Sigma\} . \tag{9}$$

Clearly U_1 is linear, and since for any $\psi \in J_1$ we have

$$\|U_1\psi\| = \|T_\psi e\| \leq \sup\,\{\|T_{\psi'}e\| : \psi' \in J_1,\ |\psi'| \leq |\psi|\} = \rho(\psi) ,$$

U_1 is ρ-continuous and therefore extends continuously to a mapping $U : \bar{J}_1 \to X$. For any bounded measurable functions s, t with $|t| \leq |s|$ a.e., there is a measurable function θ with $|\theta| \leq 1$ a.e. and $t = \theta s$: we have

$$\|T_t e\| = \|T_\theta T_s e\| \leq \|T_\theta\|\ \|T_s e\| \leq 4K\|\theta\|_\infty \|T_s e\| \leq 4K\|T_s e\| .$$

Hence $\rho(s) \leq 4K\|T_s e\| \leq 4K\|U_1 s\| \leq 4K\ \rho(s)$ and it follows that whenever $g \in \bar{J}_1$,

$$\rho(g) \leq 4K\|Ug\| \leq 4K\ \rho(g) ,$$

and thus that U is bicontinuous. Since $\bar{J}_1$ is closed, by the Riesz-Fischer property it is complete. From (9),

$$U(\bar{J}_1) \supseteq \text{lin}\,\{E(\sigma)e : \sigma \in \Sigma\} ;$$

since U is bicontinuous $U(\bar{J}_1)$ is complete and therefore closed, so

$$U(\bar{J}_1) \supseteq \overline{\mathrm{lin}}\ \{E(\sigma)e : \sigma \in \Sigma\} = X .$$

Thus $U(\bar{J}_1) = X$ and so U is a linear isomorphism which satisfies $U1 = e$; finally, since the relation

$$U(\chi_\sigma f) = E(\sigma)Uf \qquad (\sigma \in \Sigma)$$

holds for $f \in J_1$, it holds also, by continuity, for $f \in \bar{J}_1$.

In order to construct the norm ρ at the beginning of this section, we assumed that e, the cyclic vector for X, had a Bade functional in Γ. From [G4], Theorem 6, this assumption is legitimate provided only that the prespectral measure $E(\cdot)$ is countably decomposable at e, i.e. provided that whenever Σ' is a subset of Σ whose elements are pairwise disjoint, then $E(\sigma)e \neq 0$ for only countably many $\sigma \in \Sigma'$.

Conversely, in the situation where the cyclic vector e is known to have a Bade functional, then by the same theorem we may conclude that the prespectral measure $E(\cdot)$ is countably decomposable at e and, moreover, that a Bade functional can be found in Γ, as required for the isomorphism theorem. Since $X = \overline{\mathrm{lin}}\ \{E(\sigma)e : \sigma \in \Sigma\}$, it follows also that $E(\cdot)$ is then countably decomposable at each $x \in X$ and so that each $x \in X$ has a corresponding Bade functional in Γ. For the precyclic subspace $\bar{J}_1$ of L_ρ, these Bade functionals were easily found, in Lemma 3.4.

The ideals $\bar{J}_1$ occurring in both representation theorems (noting that $L_\rho^a = \bar{J}_1$ in the case where $L_\rho = L_\rho^a$ and $1 \in L_\rho^a$) are examples of Banach lattices with topological order unit.

4.5 Definitions.

(1) Let E be a vector lattice. A norm $\|\cdot\|$ on E is called a lattice norm if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ $(x, y \in E)$. If E is complete with respect to the lattice norm $\|\cdot\|$, the pair $(E, \|\cdot\|)$ is called a Banach lattice.

(2) An element $u \geq 0$ of the Banach lattice $(E, \|\cdot\|)$ is a topological order unit for E if the closure of the principal order ideal E_u generated by u is E. An alternative characterisation of a topological order unit $u \in E^+$ is the property that for every $x \in E^+$,

$$\|x - x \wedge nu\| \to 0$$

as $n \to \infty$.

If u satisfies the weaker condition that for every x in E^+

$$x = \bigvee_n (x \wedge nu) \quad ,$$

then u is called a weak order unit for E.

Note that in the function space L_ρ, any element f_o which is strictly positive a.e. is a weak order unit. If the norm ρ is absolutely continuous, then by Lemma 3.6, f_o is also a topological order unit. Note however that the order continuity of ρ at f_o is not the crucial factor. Indeed, in general, if h is any element of L_ρ^+ whose support contains Ω_a, then the closed principal order ideal $\bar{J}_h$ must contain L_ρ^a. (For any $0 \leq f \in L_\rho^a$, $f \wedge nh \uparrow f$ a.e. so $\rho(f - f \wedge nh) \to 0$; $f \wedge nh \in J_h$ for each n, hence $f \in \bar{J}_h$.)

We have seen that every cyclic or precyclic space X may be endowed with a Banach lattice structure by means of a linear isomorphism with the principal order ideal generated by the constant function 1; this isomorphism matches the function 1 with the cyclic vector for X, and hence, in the vector lattice X, the cyclic vector is a topological order unit.

However although the ideal L_ρ^a, and indeed every closed principal order ideal of L_ρ, has a topological order unit, L_ρ itself need not have one at all, even when the norm ρ has strong continuity properties such as the Fatou property. The following example illustrates this.

4.6 EXAMPLE. Consider the space of Gould, described in [LZ_1]. Here the norm ρ is based on an infinite, σ-finite atomfree measure μ and defined by

$$\rho(f) = \inf \{\|f_1\|_1 + \|f_2\|_\infty : f = f_1 + f_2,$$
$$f_1 \in L^1(\mu),\ f_2 \in L^\infty(\mu)\} .$$

Thus $L_\rho = L^1 + L^\infty$. The associate norm ρ' is given by

$$\rho'(f) = \sup \{\|f\|_1, \|f\|_\infty\} . \tag{10}$$

Thus $L_\rho' = L^1 \cap L^\infty$. For a fuller account, see [LZ_1]. Let X denote L_ρ' and suppose that $f_o \in X$ is a topological order unit for X. Then $\overline{J}_{f_o} = X$ and for each $0 \leq g \in X$,

$$\rho'(g - g \wedge nf_o) \to 0$$

as $n \to \infty$. Now f_o must be strictly positive a.e. for otherwise, if δ is any non-null measurable subset of $\Omega \setminus \operatorname{supp} f_o$, and g is any element of L_ρ^+ with supp g = δ, then for every $n \in \mathbb{N}$,

$$g \wedge nf_o = 0 \quad \text{a.e.} ,$$

so that $\rho'(g - g \wedge nf_o) = \rho'(g) \not\to 0$ as $n \to \infty$. Define sets $\sigma_n' = \{\frac{1}{n} < f_o \leq \frac{1}{n-1}\}$ $(n \geq 2)$ in Σ. Then,

$$\|f_o\chi_{\sigma_n'}\|_\infty \leq \frac{1}{n-1} \to 0$$

as $n \to \infty$. If there existed a positive integer n_o such that for all $n \geq n_o$, $\mu(\sigma_n') = 0$, then

$$0 = \mu(\bigcup_{n \geq n_o} \sigma_n') = \mu(\{f_o \leq \frac{1}{n_o-1}\}) \; ,$$

and $f_o \geq \frac{1}{n_o-1}$ on Ω. But since $f_o \in L^1(\mu)$ this cannot occur. So we may delete from the sequence $\{\sigma_n'\}$ any elements which are μ-null and be left with an infinite sequence of sets $\{\sigma_n\}$, each of positive measure, and satisfying $\|f_o\chi_{\sigma_n}\|_\infty \to 0$ as $n \to \infty$.

Let $X_o = \text{lin}\,\{f_o\chi_\sigma \; : \; \sigma \in \Sigma\}$. Since bounded functions can be approximated uniformly by simple functions,

$$\overline{X}_o \supseteq \{\varphi f_o \; : \; \varphi \in L^\infty\} = J_{f_o} \; .$$

Since $\overline{X}_o$ is closed, $\overline{X}_o \supseteq \overline{J}_{f_o} = X$. Hence,

$$\overline{X}_o = X \quad . \tag{11}$$

Now for any $g \in X_o$, we have $g = \sum_{i=1}^{m} \alpha_i f_o \chi_{\delta_i}$, for some $m \in \mathbb{N}$, where the $\alpha_i \in \mathbb{C}$ are distinct, and the sets $\delta_i \in \Sigma$ are pairwise disjoint. Then,

$$\|g\chi_{\sigma_n}\|_\infty \leq \max_{1 \leq i \leq m} |\alpha_i| \|f_o\chi_{\sigma_n \cap \bigcup_i \delta_i}\| \leq \max_{1 \leq i \leq m} |\alpha_i| \|f_o\chi_{\sigma_n}\|_\infty \to 0 \tag{12}$$

as $n \to \infty$. For any $g \in \overline{X}_o$, there is a sequence $\{g_k\}$ in X_o with $\rho'(g - g_k) \to 0$, so that from (10), each of $\|g - g_k\|_1$ and $\|g - g_k\|_\infty$ tends to zero as $k \to \infty$. Hence, given $\varepsilon > 0$,

$$\begin{aligned} \|g\chi_{\sigma_n}\|_\infty &\leq \|(g - g_k)\chi_{\sigma_n}\|_\infty + \|g_k\chi_{\sigma_n}\|_\infty \\ &\leq \frac{\varepsilon}{2} + \|g_k\chi_{\sigma_n}\|_\infty \qquad \text{(for } k \geq k_1\text{, say)} \\ &\leq \varepsilon \end{aligned}$$

for sufficiently large n, by (12) (applied to g_k). Thus $\|g\chi_{\sigma_n}\|_\infty \to 0$ as $n \to \infty$.

However, consider now the function

$$h = \sum_{n=1}^{\infty} \chi_{\tau_n} \quad ,$$

where for each n, $\tau_n \subseteq \sigma_n$ and $0<\mu(\tau_n) < 2^{-n}$. Then $\|h\|_\infty = 1$, $\|h\|_1 < \sum_n 2^{-n} = 1$, so $h \in X$. But $\|h\chi_{\sigma_n}\|_\infty = 1 \not\to 0$ as $n \to \infty$. Hence $h \notin \overline{X}_o$. From this contradiction of (11) if follows that X can have no topological order unit.

CHAPTER III. WEAK SEQUENTIAL COMPLETENESS IN L_ρ.

The main result and the objective of this chapter is stated at the very beginning as Theorem 5.1 but will only be arrived at in three stages, starting from a similar but less general result. This initial result (Theorem 5.4) arises from a reinterpretation of a theorem stated for cyclic spaces by L. Tzafriri [T_2], in the light of T.A. Gillespie's Representation Theorem [G_1] given in Chapter II. This theorem represents any cyclic space over a σ-complete Boolean algebra of projections as the ideal L_ρ^a of a Banach function space L_ρ, in which the norm ρ has the Fatou property.

§ 5. Conditions for weak sequential completeness in L_ρ and L_ρ^a.

5.1 THEOREM. Let ρ be a saturated function norm based on (Ω, Σ, μ) and possessing the Riesz-Fischer property. The following statements are equivalent.

(a) L_ρ is weakly sequentially complete.

(b) L_ρ^a is weakly sequentially complete, and $\Omega_a = \Omega$.

(c) L_ρ^a contains no subspace isomorphic to c_o, and $\Omega_a = \Omega$.

(d) L_ρ contains no subspace isomorphic to c_o.

(e) $L_\rho = L_\rho^a$ and ρ has the (weak) Fatou property.

(f) L_ρ'' contains no subspace isomorphic to l_∞, and $\Omega_a = \Omega$.

It will be useful to be aware at an early stage of certain simple facts which we therefore record at this point.

5.2 PROPOSITION. A subset B of M_μ (resp. L_ρ) is a band of M_μ (L_ρ) if and only if there is a measurable subset γ of Ω such that

$$B = \{f\chi_\gamma : f \in M_\mu \ (L_\rho)\} \quad ,$$

and then γ is the carrier of B.

Proof. Clearly any subset B of the given form is an order closed solid linear subspace of M_μ.

Conversely, if B is a band of M_μ, then B has a carrier set γ and $B \subseteq \chi_\gamma M_\mu$. Let $0 \leq f = f\chi_\gamma \in M_\mu$. By the definition of carrier sets, we may choose a sequence $\gamma_n' \uparrow \gamma$ in Σ with $\chi_{\gamma_n'} \in B$ for each n. If we let $\gamma_n'' = \{f \leq n\}$, then $\gamma_n'' \uparrow \Omega$: hence writing $\gamma_n = \gamma_n' \cap \gamma_n''$, we have $\gamma_n \uparrow \gamma$ and $f\chi_{\gamma_n} \leq n\chi_{\gamma_n'}$, so $f\chi_{\gamma_n} \in B$ for each n.

Since B is order closed and $f = \sup_n f\chi_{\gamma_n}$, then $f \in B$ and thus $\chi_\gamma M_\mu \subseteq B$, as required.

The same proof holds replacing M_μ throughout by L_ρ. In this chapter only the result for L_ρ will be required, but that for M_μ will become relevant in Chapter IV.

We shall make use several times of the following result due to T.A. Gillespie ([G3], Theorem 1).

5.3 THEOREM. Let X be a complex Banach space with dual space X*. The following statements are equivalent:

(i) X does not contain any subspace isomorphic to l_∞;

(ii) every prespectral measure on X of arbitrary class (Σ, Γ), where Σ is a σ-algebra of sets and Γ is a total subset of X^*, is strongly countably additive.

5.4 THEOREM. Let ρ be a saturated function norm with the Fatou property, such that the constant function 1 is an element of both L^a_ρ and L'_ρ. The following statements are equivalent.

(a) L^a_ρ is weakly sequentially complete.

(b) L^a_ρ contains no subspace isomorphic to c_o.

(c) L^a_ρ contains no complemented subspace isomorphic to c_o.

(d) L^a_ρ is a complemented subspace of L^{**}_ρ.

(e) $L^a_\rho = L_\rho$.

(f) L_ρ contains no subspace isomorphic to l_∞.

Proof.

(a) ⇒ (b). Trivial, since c_o is not weakly sequentially complete (w.s.c.) and any closed subspace of a w.s.c. space must also be w.s.c..

(b) ⇒ (c). A fortiori.

(c) ⇒ (e). We shall suppose that (e) does not hold and show that L^a_ρ must then admit a bounded projection onto an isomorphic copy of c_o, thus precluding (c).

So suppose we have $0 \leq g \in L_\rho \setminus L^a_\rho$ and choose a sequence $\{g_n\}$ of simple functions with $0 \leq g_n \uparrow g$ a.e. Since $1 \in L^a_\rho$, $L^\infty \subseteq L^a_\rho$ so for each n, $g_n \in L^a_\rho$. Since ρ has the Fatou property, $\rho(g) = \sup_n \rho(g_n)$; but since L^a_ρ is a norm-closed ideal of L_ρ, we cannot have $\rho(g - g_n) \to 0$. Define $\eta_m = \{m-1 \leq g < m\}$ for $m = 1,2,\ldots$. Since $g \in L_\rho$, g is finite-valued a.e. so $\Omega = \bigcup_m \eta_m$ and, if

$$\varphi_j = \sum_{k=1}^{j} k\, \chi_{\eta_k} \qquad (j = 1,2,\ldots) ,$$

then $0 \leq \varphi_1 \leq \varphi_2 \leq \ldots$ and $\varphi_j \in L_\rho^a$ for each j. Also,

$$\rho(\varphi_j) \leq \rho\left(\sum_{k=1}^{j} (k-1)\chi_{\eta_k} \right) + \rho\left(\sum_{k=1}^{j} \chi_{\eta_k} \right)$$

$$\leq \rho\left(g\, \chi_{\bigcup_{k=1}^{j} \eta_k}\right) + \rho\left(\chi_{\bigcup_{k=1}^{j} \eta_k}\right)$$

$$\leq \rho(g) + \rho(1) .$$

Let $\varphi = \sup_j \varphi_j$. Then φ takes the constant value k on each η_k, hence $\varphi\chi_{\eta_k} \geq g\chi_{\eta_k}$ $(k = 1,2,\ldots)$ and so $\varphi \geq g$ a.e. which implies that $\varphi \notin L_\rho^a$. Thus the functions φ and φ_n $(n = 1,2,\ldots)$ satisfy $0 \leq \varphi_n \uparrow \varphi$ a.e. with each $\varphi_n \in L_\rho^a$ but $\varphi \notin L_\rho^a$, and also

$$\rho(\varphi) = \sup_j \rho(\varphi_j) \leq \rho(g) + \rho(1)$$

whilst

$$\rho(\varphi - \varphi_j) \not\to 0 \tag{1}$$

as $j \to \infty$. Hence the sequence $\{\varphi_j\}$ cannot converge in L_ρ, for if it had a limit $\overline{\varphi}$ say, in L_ρ, then $\{\varphi - \varphi_j\}$ would converge to $\varphi - \overline{\varphi}$, and so, from the proof of [Z] Theorem 64.2, some subsequence $\{\varphi - \varphi_{n_j}\}_{j\in\mathbb{N}}$ would converge pointwise a.e. to $\varphi - \overline{\varphi}$; since every subsequence of $\{\varphi - \varphi_j\}$ converges pointwise to zero a.e., this would mean that $\varphi = \overline{\varphi}$ a.e., contradicting (1). It follows that for some $\varepsilon > 0$, we can find subsequences $\{j_n\}$ and $\{k_n\}$ of $\mathbb{N}$ with

$$k_{n+1} \geq j_n > k_n$$

and

$$\rho(\varphi_{j_n} - \varphi_{k_n}) \geq \varepsilon \qquad (n = 1,2,\ldots) \; .$$

Setting $\psi_n = \varphi_{j_n} - \varphi_{k_n}$, observe that the functions ψ_n have mutually disjoint supports - namely $\bigcup_{k=k_n+1}^{j_n} \eta_k$ - which we shall denote by δ_n, and that

$$\varepsilon \leq \rho(\psi_n) \leq 2(\rho(g) + \rho(1)) \qquad (n = 1,2,\ldots) \; .$$

Let $\alpha = \{\alpha_n\}$ be any bounded sequence: then,

$$\rho(\sum_{n=1}^{N} \alpha_n\psi_n) = \rho(\sum_{n=1}^{N} |\alpha_n| \; \psi_n) \quad \text{since the } \psi_n\text{'s are disjoint}$$

$$\leq \sup_n |\alpha_n| \; \rho(\sum_{n=1}^{N} \psi_n)$$

$$= \|\alpha\|_\infty \; \rho(\sum_{n=1}^{N} (\varphi_{j_n} - \varphi_{k_n}))$$

$$\leq \|\alpha\|_\infty \; \rho(\varphi_{j_N} - \varphi_{k_1})$$

$$\leq \|\alpha\|_\infty \; \rho(\varphi_{j_N})$$

$$\leq \|\alpha\|_\infty \; (\rho(g) + \rho(1)) \; .$$

So by the Fatou property, it follows that $\sum_n \alpha_n\psi_n \in L_\rho$ and

$$\rho(\sum_n \alpha_n\psi_n) \leq \|\alpha\|_\infty \; (\rho(g) + \rho(1)) \; . \tag{2}$$

On the other hand,

$$\rho(\alpha_m\psi_m) = \rho(\chi_{\delta_m} \cdot \sum_n \alpha_n\psi_n) \leq \rho(\sum_n \alpha_n\psi_n)$$

and

$$\rho(\alpha_m\psi_m) = |\alpha_m| \ \rho(\psi_m) \geq |\alpha_m|\varepsilon \quad (m = 1,2,\ldots) \ .$$

So

$$\rho(\sum_n \alpha_n\psi_n) \geq \varepsilon\|\alpha\|_\infty. \tag{3}$$

Define $f_\alpha = \sum_n \alpha_n\psi_n$: then the mapping $\alpha \mapsto f_\alpha$ ($\alpha \in l_\infty$) is positive and linear, it is one-one since the ψ_n's are disjoint, and bicontinuous by (2) and (3); hence its range which we shall denote by $\hat{l}_\infty$ is a subspace of L_ρ isomorphic to l_∞.

Now choose for each n a function $0 \leq u_n \in L'_\rho$ with $\rho'(u_n) \leq 1$ and supp $u_n \subseteq \delta_n$, satisfying

$$\int \psi_n u_n \, d\mu \geq \frac{\varepsilon}{2} \ .$$

For $0 \leq f \in L_\rho$, define

$$Pf = \sum_{s=1}^{\infty} \frac{\int fu_s \, d\mu}{\int \psi_s u_s \, d\mu} \psi_s \ . \tag{4}$$

This defines Pf as an element of $\hat{l}_\infty$, because for each s,

$$\left| \frac{\int fu_s \, d\mu}{\int \psi_s u_s \, d\mu} \right| \leq \frac{\rho(f)\rho'(u_s)}{\frac{\varepsilon}{2}} \leq \frac{2}{\varepsilon} \rho(f) \ ;$$

so the sequence of coefficients of ψ_s in (4) is bounded. For all r and s,

$$\psi_r u_s = \delta_{rs}\psi_r u_s \quad , \tag{5}$$

so letting $c_r(f)$ denote the coefficient of ψ_r in (4),

$$P^2f = P(\sum_{r=1}^{\infty} c_r(f)\psi_r)$$

$$= \sum_{s=1}^{\infty} \frac{\int (\sum c_r(f)\psi_r)u_s \, d\mu}{\int \psi_s u_s d\mu} \psi_s$$

$$= \sum_{s=1}^{\infty} \varsigma_s(f)\psi_s$$

$$= Pf \ .$$

Hence $P^2 = P$. It also follows from (5) that for each $\alpha \in l_\infty$, $Pf_\alpha = f_\alpha$, and P is a bounded linear projection of L_ρ onto $\hat{l}_\infty$. Now if $f \in L_\rho^a$, then, since we have $\bigcup_{k=1}^{s} \eta_k \uparrow \Omega$ as $s \to \infty$,

$$\rho(f\chi_{\Omega \setminus \bigcup_{k=1}^{s} \eta_k}) \to 0$$

and for each s, $\delta_s \subseteq \bigcup_{k=k_s+1}^{\infty} \eta_k = \Omega \setminus \bigcup_{k=1}^{k_s} \eta_k$, so

$\rho(f\chi_{\delta_s}) \to 0$. Hence,

$$\left| \frac{\int f u_s \, d\mu}{\int \psi_s u_s \, d\mu} \right| \leq \frac{2}{\varepsilon} \rho'(u_s)\rho(f\chi_{\delta_s}) \to 0$$

as $s \to \infty$. Thus the sequence of coefficients of ψ_s in Pf is an element of c_o, and so Pf is in $\hat{c}_o$, the isomorphic copy of c_o imbedded in $\hat{l}_\infty$.

It now remains to be shown that $\hat{c}_o \subseteq L_\rho^a$. Let $\alpha \in c_o$, suppose $f_\alpha \geq f_1 \geq f_2 \geq \ldots \downarrow 0$ a.e. and let $\varepsilon > 0$. Since $f_\alpha = \sum \alpha_n \psi_n \in L_\rho$, then for some $N \in \mathbb{N}$, $\rho(\sum_{n \geq N} \alpha_n \psi_n) < \frac{\varepsilon}{2}$. Writing $\delta = \bigcup_{n \geq N} \delta_n = \bigcup_{n \geq N} \operatorname{supp} \psi_n$, then $\rho(f_i \chi_\delta) \leq \rho(f_\alpha \chi_\delta) < \frac{\varepsilon}{2}$ for each i. Now since $\psi_i \in L_\rho^a$ $(i = 1,2,\ldots)$, $f_\alpha \chi_{\Omega \setminus \delta} = \sum_{n=1}^{N-1} \alpha_n \psi_n \in L_\rho^a$. Therefore since $f_\alpha \chi_{\Omega \setminus \delta} \geq f_i \chi_{\Omega \setminus \delta} \downarrow 0$ a.e., it follows that for sufficiently large j, $\rho(f_j \chi_{\Omega \setminus \delta}) < \frac{\varepsilon}{2}$, and then

$$\rho(f_j) \leq \rho(f_j \chi_\delta) + \rho(f_j \chi_{\Omega \setminus \delta}) < \varepsilon \ .$$

This shows that $f_\alpha \in L^a_\rho$ as required.

(e) ⇒ (a). Let $\{f_n\}$ be a weakly Cauchy sequence in L_ρ, where $L_\rho = L^a_\rho$. Let K be a uniform bound on the norms $\rho(f_n)$. Since $1 \in L'_\rho$, $L_\rho \subseteq L^1(\mu)$ (algebraically); by assumption, $\{\int hf_n d\mu\}$ is a Cauchy sequence in $\mathbb{C}$ for each $h \in L'_\rho = L^*_\rho$, so in particular, letting $h \in L^\infty$ shows that $\{f_n\}$ is weakly Cauchy in L^1. Let $f \in L^1$ be the weak L^1 limit of $\{f_n\}$ so that whenever $h \in L^\infty$,

$$\lim_n \int hf_n \, d\mu = \int hf \, d\mu \; . \tag{6}$$

Now let $g \in L'_\rho$: then $g < \infty$ a.e. For any $\sigma \in \Sigma$ such that g is bounded on σ,

$$\begin{aligned} \left| \int_\sigma fg \, d\mu \right| &= \lim_n \left| \int f_n g\chi_\sigma \, d\mu \right| \\ &\le \limsup \rho(f_n)\rho'(g\chi_\sigma) \\ &\le K\rho'(g) \; . \end{aligned} \tag{7}$$

Choose a sequence $\{\sigma_n\}$ in Σ with $\sigma_n \uparrow \Omega$ and with g bounded on each σ_n, and let θ be a unimodular function satisfying $fg\theta = |fg|$. By (7),

$$\int_{\sigma_n} fg\theta \, d\mu \le K \, \rho'(g\theta) = K \, \rho'(g) \; .$$

Hence, applying the Monotone Convergence Theorem,

$$\int fg\theta \, d\mu = \lim_n \int_{\sigma_n} fg\theta \, d\mu \le K \, \rho'(g) \; ,$$

i.e. $fg \in L^1(\mu)$ for each $g \in L'_\rho$, and

$$\rho(f) = \rho''(f) = \sup_{\rho'(g) \le 1} \left| \int fg \, d\mu \right| \le K \quad ;$$

thus $f \in L_\rho$.

We show finally that for any $g \in L'_\rho$

$$\lim_n \int f_n g \, d\mu = \int fg \, d\mu \ . \tag{8}$$

Let $g \in L'_\rho$. For $\delta \in \Sigma$, define

$$\nu_n(\delta) = \int_\delta (f_n - f) g \, d\mu \ .$$

Since $f_n - f \in L_\rho$ and $g \in L'_\rho$, $(f_n-f)g \in L^1(\mu)$, so ν_n is a countably additive measure on Σ. Since $\lim_n \int_\delta f_n g \, d\mu$ exists, $\lim_n \nu_n(\delta)$ exists for each $\delta \in \Sigma$. Hence from Nikodym's theorem ([DS$_1$],III.7.4), the countable additivity of $\nu_n(\cdot)$ is uniform in n. Fix $\varepsilon > 0$. Let $\delta_m = \{|g| \leq m\}$. Then $\delta_m \uparrow \Omega$ so we can find an integer m_o such that

$$|\nu_n(\Omega \setminus \delta_m)| < \frac{\varepsilon}{2} \quad \text{for } m \geq m_o, \text{ and all } n = 1,2,\ldots;$$

so,

$$\limsup |\nu_n(\Omega \setminus \delta_m)| \leq \frac{\varepsilon}{2} \quad \text{for } m \geq m_o \ .$$

Hence

$$\begin{aligned} \left|\int (f_n-f)g \, d\mu\right| &= \left|\int (f_n-f)g\chi_{\delta_m} \, d\mu + \nu_n(\Omega \setminus \delta_m)\right| \\ &\leq \left|\int (f_n-f)g\chi_{\delta_m} \, d\mu\right| + \frac{\varepsilon}{2} \qquad (m \geq m_o) \\ &\leq \varepsilon \end{aligned}$$

by (6), if n is sufficiently large. Thus (8) follows, and hence L_ρ is w.s.c. as required.

<u>(e) ⇒ (d)</u>. Elementary, for since the Fatou property implies $\rho = \rho''$, then if $L_\rho = L^a_\rho$,

$$(L^a_\rho)^{**} = (L'_\rho)^* = L''_\rho \oplus L^*_{\rho',s} = L_\rho \oplus L^*_{\rho',s} = L^a_\rho \oplus L^*_{\rho',s} \ .$$

<u>(d) ⇒ (e)</u>. Let Q be the restriction to L_ρ of a bounded projection mapping L^{**}_ρ onto L^a_ρ, and suppose $L_\rho \neq L^a_\rho$.

Obtain, as in the proof of (c) $\Rightarrow$ (e) a linear subspace $\hat{l}_\infty$ of L_ρ which is isomorphic to l_∞ and is the range of a projection P on L_ρ. Then P fixes $\hat{l}_\infty$ and Q fixes L_ρ^a, and since $\hat{c}_o$ is contained in each of these subspaces, $\hat{c}_o$ is fixed by each of P and Q and thus also by PQ. Hence PQ is a projection of L_ρ into $P(L_\rho^a) = \hat{c}_o$ and

$$\hat{c}_o = PQ\,\hat{c}_o \subseteq PQ\,\hat{l}_\infty \subseteq P(L_\rho^a) = \hat{c}_o \quad .$$

So PQ $\hat{l}_\infty = \hat{c}_o$. However it is well-known that c_o is not complemented in l_∞ (see [P]), and from this contradiction it follows that L_ρ^a must equal L_ρ.

We now have that (a), (b), (c), (d), (e) are all equivalent. Assume that one and hence each of these conditions holds, and suppose (f) does not. By (e), this means that L_ρ^a contains an isomorphic copy of l_∞ and hence also an isomorphic copy of c_o, contradicting (b). So <u>(a) - (e) $\Rightarrow$ (f)</u>.

<u>(f) $\Rightarrow$ (e)</u>. If $L_\rho \neq L_\rho^a$, then from the proof of (c) $\Rightarrow$ (e), L_ρ must contain a subspace isomorphic to l_∞; but this contradicts (f).

<u>Note</u>. In the proof of (c) $\Rightarrow$ (e), the subspace $\hat{l}_\infty$ is by construction also lattice isomorphic to l_∞, as is therefore $\hat{c}_o$ to c_o. Hence in each of statements (b), (c), (d), (f) of the theorem, the word "subspace" may be equivalently replaced by "closed sublattice".

Now suppose that ρ is a saturated function norm based on (Ω, Σ, μ) but endowed only with the Riesz-Fischer property. Let $f_o \in L_\rho^a$ be positive-valued a.e. on Ω_a so that $L_\rho^a = \bar{J}_{f_o}$ (Lemma 3.6). Choosing any function $\varphi_o \in L_\rho'$ with φ_o positive a.e. on Ω_a, define a measure ν on

$\Sigma_a = \{\sigma \cap \Omega_a : \sigma \in \Sigma\}$ by

$$\nu(\sigma) = \int_\sigma f_o \varphi_o \, d\mu \qquad (\sigma \in \Sigma_a) ;$$

then μ and ν are equivalent in Σ_a. Now define a mapping $U : \chi_{\Omega_a} M_\mu \to M_\nu$, by

$$Uf = f f_o^{-1} \qquad (f = f\chi_{\Omega_a} \in M_\mu) .$$

U is clearly linear, one-one, onto and increasing. Define a norm τ on M_ν by

$$\tau(h) = \rho''(hf_o) \qquad (h \in M_\nu) ,$$

i.e. $\quad \tau(Uf) = \rho''(f) \qquad (f = f\chi_{\Omega_a} \in M_\mu)$.

That this indeed defines a Banach function norm is easily verified, and taking τ to be based on $(\Omega_a, \Sigma_a, \nu)$, then τ is saturated; moreover, τ has the Fatou property since ρ'' does, and $\tau'(h) = \rho'(h\varphi_o)$ for each $h \in M_\nu$, so in particular $1 \in L'_\tau$ (here 1 denotes χ_{Ω_a} clearly). Since $\rho''(f) \leq \rho(f)$ for every f, $\| U|_{L^a_\rho} \| \leq 1$; however if $f \in L^a_\rho$, $\rho''(f) = \rho(f)$, so with $U_a = U|_{L^a_\rho}$ we have

$$\tau(U_a f) = \rho(f) \qquad (f \in L^a_\rho) . \tag{9}$$

It is a routine exercise to check that $h \in L^a_\tau$ if and only if $U^{-1}h = f_o h \in L^a_{\rho''}$.

To conclude recall that by Lemma 3.2, $L^a_\rho = \chi_{\Omega_a} L^a_{\rho''}$ and then, since for every $h \in L_\tau$, $h = h\chi_{\Omega_a}$, it is clear that $h \in L^a_\tau$ if and only if $hf_o \in L^a_\rho$. Hence and from (9) it follows finally that U_a is a norm-preserving isomorphism between L^a_ρ and L^a_τ. Consequently, since τ satisfies the conditions of Theorem 5.4, statements (b), (c) and (f) for L_τ are immediately equivalent to weak sequential completeness of L^a_ρ, even with the present relaxed conditions on ρ. Before completing the theorem, we give another lemma.

<u>5.5 LEMMA</u>. Let ρ be a saturated function norm with the Riesz-Fischer property. Then, if L_ρ^a is weakly sequentially complete, L_ρ^a is order-closed, i.e. $L_\rho^a = \chi_{\Omega_a} L_\rho$.

<u>Proof</u>. Let $f = f\chi_{\Omega_a} \in L_\rho^+$ and suppose that L_ρ^a is w.s.c. Define sets $\delta_p = \{f \leq p\} \cap \Omega_a$ $(p = 1,2,\ldots)$; then $\delta_p \uparrow \Omega_a$ as $p \to \infty$. Now choose a sequence of sets $\tau_p \uparrow \Omega_a$ with $\tau_p \subseteq \delta_p$ and $\chi_{\tau_p} \in L_\rho^a$ for each p. Writing $f_p = f\chi_{\tau_p}$ we have $f_p \uparrow f$ a.e. and

$$f_p \leq p\chi_{\tau_p} \in L_\rho^a \quad .$$

Recall that $(L_\rho^a)^* = \chi_{\Omega_a} L_\rho'$ (Lemma 3.1). Let $0 \leq g \in L_\rho'$: since g is an integral linear functional on L_ρ, it follows that

$$\langle f - f_p, g\rangle \to 0$$

as $p \to \infty$. So,

$$\langle f_p - f_q, g\rangle \to 0$$

as $p, q \to \infty$, i.e. $\{f_p\}$ is weakly Cauchy, and hence weakly convergent in L_ρ^a. Let $f_o \in L_\rho^a$ be its weak limit, so that for every $g \in L_\rho'$,

$$\langle f_o, g\rangle = \lim_p \langle f_p, g\rangle = \langle f, g\rangle \quad .$$

Since L_ρ' is a total subset of L_ρ^*, this shows $f_o = f$ a.e. Hence $f \in L_\rho^a$ and $\chi_{\Omega_a} L_\rho \subseteq L_\rho^a$. The converse inclusion is immediate and hence $\chi_{\Omega_a} L_\rho = L_\rho^a$.

<u>5.6 Remark</u>. The converse of this lemma need not hold. Consider, as an example, the case where Ω is $\mathbb{N}$ with the discrete measure μ, and let

$$\rho(\{\alpha_n\}) = \begin{cases} \sup |\alpha_n| & , \alpha_n \to 0 , \\ \infty & , \text{otherwise} . \end{cases}$$

Here $L_\rho = c_o = L_\rho^a$, but L_ρ is not w.s.c.

However it is easily shown that with the additional hypothesis of the weak Fatou property, the converse of Lemma 5.5 does hold.

<u>5.7 THEOREM</u>. Let ρ be a saturated function norm based on (Ω, Σ, μ) and having the Riesz-Fischer property. The following statements are equivalent.

(a) L_ρ^a is w.s.c.

(b) L_ρ^a contains no subspace isomorphic to c_o.

(c) L_ρ^a contains no complemented subspace isomorphic to c_o.

(d) L_ρ^a is complemented in $(L_\rho^a)^{**}$.

(e) $L_\rho^a = \chi_{\Omega_a} L_\rho$ and the conclusion of the Fatou property holds for increasing sequences in L_ρ^a.

(f) $\chi_{\Omega_a} L_\rho''$ contains no subspace isomorphic to l_∞.

[<u>Note</u>. We can express (e) alternatively as follows. Let λ be the function norm based on $(\Omega_a, \Sigma_a, \mu_a)$, where μ_a denotes $\mu|_{\Sigma_a}$, defined by

$$\lambda(f) = \begin{cases} \rho(f) & , \text{ if } f \in L_\rho^a \\ \infty & , \text{ otherwise} , \end{cases}$$

so that λ is an absolutely continuous saturated norm. Then

(e)' λ has the Fatou property.

The equivalence of (e) and (e)' is a routine exercise.]

Proof. As remarked earlier, that (b), (c) and (f) are each equivalent to (a) follows from the isomorphism U_a of L_ρ^a and L_τ^a described on p. 44, and from the application of Theorem 5.4 to L_τ^a.

Suppose now that (d) holds. Since $L_\rho^a \subseteq \chi_{\Omega_a} L_\rho \subseteq \chi_{\Omega_a} L_\rho''$ and $(L_\rho^a)^{**} = (\chi_{\Omega_a} L_\rho')^* \supseteq \chi_{\Omega_a} L_\rho''$, it follows that there is a bounded projection P of $\chi_{\Omega_a} L_\rho''$ onto L_ρ^a. Let $Q = UPU^{-1}$: then $Q^2 = Q$ and Q is thus a bounded projection of L_τ onto L_τ^a. Thus L_τ^a satisfies (d) of Theorem 5.4. If conversely, there is a bounded projection Q of L_τ onto L_τ^a, put $P = U^{-1}QU$. Then similarly P is a bounded projection of $\chi_{\Omega_a} L_\rho''$ onto L_ρ^a. Hence by the equivalence of (a) and (d) in Theorem 5.4, (a) and (d) of the present theorem are equivalent.

Finally we show (a) and (e) are equivalent. Assume that L_ρ^a is w.s.c. By Lemma 5.5, $L_\rho^a = \chi_{\Omega_a} L_\rho$. Now suppose that $0 \leq v_1 \leq v_2 \leq \ldots \uparrow v$ a.e. with $v_i \in L_\rho^a$ for each i. If $\rho(v_i) \to \infty$, then $\rho(v) = \sup \rho(v_i)$ trivially. If $\rho(v_i) \uparrow K < \infty$, then $\rho''(v) = K$. So,

$$\tau(Uv) = \tau(vf_o^{-1}) = \rho''(v) = K < \infty .$$

But by isomorphism with L_ρ^a, L_τ^a is w.s.c. So $L_\tau = L_\tau^a$. Hence $v \in L_{\rho''}^a$. Since $v = \sup v_i$, $v = v\chi_{\Omega_a}$ and so by Lemma 3.2, $v \in L_\rho^a$. Hence $\rho(v-v_i) \to 0$ and $\lim \rho(v_i) = \rho(v)$ as required.

Now assume conversely that (e) holds. We shall show that $L_\tau = L_\tau^a$. Let $0 \leq f \in L_\tau$. Since $1 \in L_\tau^a$, we can choose a sequence $\{f_n\}$ in L_τ^a with $0 \leq f_n \uparrow f$ a.e. For each n, $U^{-1}f_n = f_n f_o \in L_\rho^a$ and $0 \leq f_n f_o \uparrow ff_o$ a.e. By hypothesis therefore,

$$\rho(ff_o) = \sup \rho(f_n f_o) = \sup \rho''(f_n f_o) = \sup \tau(f_n) = \tau(f) < \infty,$$

i.e. $ff_o \in L_\rho$; since $\operatorname{supp} ff_o \subseteq \Omega_a$, then $ff_o \in L_\rho^a$ and

$f \in L_\tau^a$ as required. It follows that L_τ^a is w.s.c. and hence that L_ρ^a is w.s.c.

We are now in a position to prove the main theorem.

Proof of Theorem 5.1.

(a) ⇒ (b). It follows from [DS_1],XVII.3.8 that every prespectral measure on a weakly complete space is in fact spectral, i.e. strongly countably additive. Hence if L_ρ is w.s.c., the measure $E(\cdot) : \sigma \to M_{\chi_\sigma}$ is spectral and so from Lemma 3.4, $L_\rho = L_\rho^a$ and (b) follows.

Note. In (e) the word "weak" may be equivalently read or omitted, because while usually, the Fatou property is stronger than the weak Fatou property, in the case of absolutely continuous norms, the two properties are equivalent (see 2.5 or [Z], 73α) i.e. if ρ is absolutely continuous and has the weak Fatou property, then ρ has the Fatou property.

(b) ⇒ (e) ⇒ (a). Both implications follow from the equivalence in Theorem 5.7 of conditions (a) and (e).

(a) ⇒ (d). Clear, since c_o is not w.s.c.

(d) ⇒ (c). If L_ρ contains no isomorphic copy of c_o, nor can it contain an isomorphic copy of l_∞ and so by Theorem 5.3 the prespectral measure $E(\cdot) : \sigma \to M_{\chi_\sigma}$ is spectral, so as before $L_\rho = L_\rho^a$. Thus L_ρ^a contains no isomorphic copy of c_o and $\Omega_a = \Omega$ as required.

(c) ⇒ (b), (e) ⇒ (f) ⇒ (c). Immediate from Theorem 5.7.

Notes. Consider the alternative statements:

(c)' L_ρ^a contains no complemented subspace isomorphic to c_o and $\Omega_a = \Omega$;

(d)' L_ρ contains no complemented subspace isomorphic to

c_o and $\Omega_a = \Omega$;

(f)' L_ρ contains no subspace isomorphic to l_∞, and $\Omega_a = \Omega$.

1. From Theorem 5.7, (c) and (c)' of Theorem 5.1 are equivalent.

2. (d)' is genuinely weaker than (d). For if we take ρ to be the norm $\|\cdot\|_\infty$ on the space of all complex sequences, so that $L_\rho = l_\infty$, then L_ρ^a is c_o so we have $\Omega_a = \Omega$. Since every complemented infinite-dimensional subspace of l_∞ is isomorphic to l_∞ itself ([LT1],2.a.7), L_ρ cannot contain a complemented subspace isomorphic to c_o. Thus (d)' holds; however $L_\rho \neq L_\rho^a$ and L_ρ is not w.s.c.

3. (f)' is genuinely weaker than (f). For taking again the space of complex sequences with, this time,

$$\rho(\{\alpha_n\}) = \begin{cases} \|\alpha\|_\infty , & \text{if } \alpha_n \to 0 , \\ \infty , & \text{otherwise} , \end{cases}$$

then $L_\rho = L_\rho^a$, so $\Omega = \Omega_a$ and L_ρ contains no copy of l_∞. However c_o is not w.s.c.

§ 6. Applications of Theorem 5.1.

A.C. Zaanen has given necessary and sufficient conditions for a Banach function space L_ρ to be reflexive, namely, that ρ and ρ' should both be absolutely continuous norms and that ρ should have the weak Fatou property ([Z], § 73). Applying the results of the present chapter, we can give an alternative characterisation of reflexivity in terms of the geometry of the function space.

<u>6.1 THEOREM</u>. Let ρ be a saturated function norm with the Riesz-Fischer property. Then L_ρ is reflexive if and only if no subspace of L_ρ is isomorphic to either c_o or l_1.

<u>Proof</u>. The necessity of the condition is obvious, since neither c_o nor l_1 is reflexive.

For the sufficiency, assume that L_ρ contains no isomorphic copy of either c_o or l_1. The method for this part of the proof borrows from that of [T_1], Lemma 4. By Theorem 5.1 ((d) $\Leftrightarrow$ (e)), $L_\rho = L_\rho^a$ and ρ has the Fatou property, so $L_\rho = L_\rho''$.

It remains to be shown that $L_\rho' = L_{\rho'}^a$: for then,

$$L_\rho^{**} = (L_\rho^a)^{**} = (L_\rho')^* = (L_{\rho'}^a)^* = L_\rho'' = L_\rho \quad ,$$

and the required conclusion follows.

We consider initially the case where μ is a finite measure, and suppose that $L_\rho' \neq L_{\rho'}^a$. Then we can find functions $h_n \in L_\rho'$ $(n = 1,2,\ldots)$ with $h_1 \geq h_2 \geq \ldots \downarrow 0$ a.e. while $\inf \rho'(h_n) = \varepsilon > 0$.

By Egoroff's theorem, we can now find in Σ a sequence $\Omega_i' \downarrow \emptyset$ such that $h_n \downarrow 0$ a.e. uniformly on $\Omega \setminus \Omega_i'$ for each i. By the saturation of ρ' and the Exhaustion Theorem ([Z], 67.3), we can also choose sets $\Omega_j'' \downarrow \emptyset$ with $\chi_{\Omega\setminus\Omega_j''} \in L_\rho'$ for each j. Let $\Omega_i = \Omega_i' \cup \Omega_i''$ $(i = 1,2,\ldots)$. Then $\Omega \setminus \Omega_i = \Omega \setminus \Omega_i' \cap \Omega \setminus \Omega_i'' \uparrow \Omega$; since $\chi_{\Omega\setminus\Omega_i} \in L_\rho'$ and $h_n \downarrow 0$ a.e. uniformly on $\Omega \setminus \Omega_i$, there is a subsequence $\{n_i\}$ of $\mathbb{N}$ such that $h_{n_i} < \dfrac{\varepsilon}{2\rho'(\chi_{\Omega\setminus\Omega_i})+1}$ a.e. on $\Omega \setminus \Omega_i$ $(i = 1,2,\ldots)$. Therefore for each i,

$$\rho'(h_{n_i}\chi_{\Omega\setminus\Omega_i}) \leqslant \frac{\varepsilon}{2}$$

and

$$\rho'(h_1\chi_{\Omega_i}) \geq \rho'(h_{n_i}\chi_{\Omega_i}) \geq \rho'(h_{n_i}) - \rho'(h_{n_i}\chi_{\Omega\setminus\Omega_i}) \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2} .$$

By the definition of ρ', therefore, there exist functions $g_i \in L_\rho^+$ with $\rho(g_i) \leq 1$ and

$$\int g_i h_1 \chi_{\Omega_i} \, d\mu \geq \frac{\varepsilon}{4} \qquad (i = 1,2,\ldots) .$$

However in general, the functions $g_i\chi_{\Omega_i}$ are not pairwise disjoint so we now choose a subsequence $\{i_k\}$ of $\mathbb{N}$ such that the functions

$$\varphi_k = g_{i_k}\chi_{\Omega_{i_k}\setminus\Omega_{i_{k+1}}} ,$$

clearly pairwise disjoint, satisfy

$$\int h_1\varphi_k \, d\mu \geq \frac{\varepsilon}{8} \qquad (k = 1,2,\ldots) .$$

Now let $\alpha = \{\alpha_i\} \in l_1$. For any $N \in \mathbb{N}$,

$$\begin{aligned} \rho\Big(\sum_{k=1}^{N} \alpha_k\varphi_k\Big) &\leq \sum_{k=1}^{N} |\alpha_k| \; \rho(\varphi_k) \\ &\leq \sum_{k=1}^{N} |\alpha_k| \; \rho(g_{i_k}) \\ &\leq \sum_{k=1}^{N} |\alpha_k| \\ &\leq \sum_{k} |\alpha_k| . \end{aligned}$$

Since ρ has the Fatou property,

$$\rho\Big(\sum_k \alpha_k\varphi_k\Big) = \sup_N \rho\Big(\sum_{k=1}^{N} \alpha_k\varphi_k\Big) \leq \sum_k |\alpha_k| < \infty .$$

It follows that

$$\rho'(h_1)\sum_k |\alpha_k| \geq \rho'(h_1)\rho\Big(\sum_k \alpha_k\varphi_k\Big)$$

$$= \rho'(h_1)\rho(|\sum_k \alpha_k\varphi_k|)$$

$$= \rho'(h_1)\rho(\sum_k |\alpha_k|\varphi_k) \tag{10}$$

$$\geq \int h_1 \ (\sum_k |\alpha_k| \ \varphi_k)d\mu$$

$$= \sum_k |\alpha_k| \int h_1\varphi_k \ d\mu \tag{11}$$

$$\geq \frac{\varepsilon}{8} \ (\sum_k |\alpha_k|) \ ,$$

where each of (10) and (11) follows immediately from the preceding line by the disjointness of the φ_k's. Hence,

$$\frac{\varepsilon}{8\rho'(h_1)} \|\alpha\|_1 \leq \rho(\sum \alpha_k\varphi_k) \leq \|\alpha\|_1 \quad . \tag{12}$$

The mapping $\alpha \mapsto \sum \alpha_k\varphi_k \in L_\rho$ is thus a linear bijection of l_1 into L_ρ, which by (12) is bicontinuous, and hence $\overline{\text{lin}}\ \{\varphi_k : k \in \mathbb{N}\} = \{\sum_k \alpha_k\varphi_k : \{\alpha_k\} \in l_1\}$ is a linear subspace of L_ρ isomorphic to l_1. Since this contradicts the hypothesis, we must in fact have $L_\rho' = L_{\rho'}^a$.

In general, μ is a σ-finite measure: in this case, choose a function $\xi_o \in L^1(\mu)$, with $\xi_o > 0$ a.e. (let $\xi_o = \xi_1\xi_2$ where $\xi_1 \in L_\rho$, $\xi_2 \in L_\rho'$ and $\xi_i > 0$ a.e. $(i = 1,2)$). Define

$$\mu_1(\sigma) = \int_\sigma \xi_o \ d\mu \qquad (\sigma \in \Sigma) \ .$$

Then μ_1 is a finite measure on Σ and is equivalent to μ so $M_{\mu_1} = M_\mu$. Define a norm ρ_1 based on (Ω, Σ, μ_1) by

$$\rho_1(f) = \rho(f) \qquad (f \in M_{\mu_1}) \ .$$

Then ρ_1 is complete and saturated and

$$\rho_1'(f) = \sup \{|\int hf \, d\mu_1| : \rho_1(h) \leq 1\}$$

$$= \sup\{|\int hf\xi_o \, d\mu| : \rho_1(h) \leq 1\}$$

$$= \rho'(f\xi_o)$$

for each $f \in M_{\mu_1}$. Suppose we have $0 \leq g \in L_\rho' \setminus L_{\rho'}^a$. Then there is a sequence $\{g_n\}$ in L_ρ' with $g \geq g_n \downarrow 0$ a.e. on Ω, while $\rho'(g_n) \geq \delta > 0$ for every n. Then $\rho_1'(g\xi_o^{-1}) = \rho'(g)$ so $g\xi_o^{-1} \in L_{\rho_1}'$, and $g\xi_o^{-1} \geq g_1\xi_o^{-1} \geq \ldots \downarrow 0$ a.e. but for each n, $\rho_1'(g_n\xi_o^{-1}) = \rho'(g_n) \geq \delta$. Hence $g\xi_o^{-1} \notin L_{\rho_1'}^a$.

From the first case, we deduce that L_{ρ_1} contains a subspace isomorphic to l_1. But since $L_{\rho_1} = L_\rho$, this contradicts the hypothesis and hence in fact $L_\rho' = L_{\rho'}^a$.

The following result also is based on a theorem for cyclic spaces, given by Tzafriri ([T2], Theorem 10).

<u>6.2 THEOREM</u>. Let ρ be a saturated function norm with the Riesz-Fischer property. Then L_ρ^a is isomorphic to the dual of some cyclic space $Z = \overline{\text{lin}}\,\{Ez_o : E \in \mathcal{B}\}$ where $z_o \in Z$ and $\mathcal{B}$ is a σ-complete Boolean algebra of projections on Z whose adjoints correspond to the multiplication operators $M_{\chi_\sigma} : f \mapsto f\chi_\sigma$ ($\sigma \in \Sigma$, $f \in L_\rho^a$), if and only if

(a) $L_\rho^a = \chi_{\Omega_a} L_\rho$, and

(b) $\Omega_a \subseteq \Omega_b$, the carrier of $L_{\rho'}^a$.

Proof.

Case 1. Assume that ρ has the Fatou property (so $\rho = \rho''$) and that $\Omega_a = \Omega$.

Suppose that (a) and (b) both hold. Then $\Omega_b = \Omega_a = \Omega$ and

$$(L^a_{\rho'})^* = \chi_{\Omega_b} L''_\rho = L''_\rho = L_\rho = L^a_\rho$$

and, since by 3.4, 3.6 and 3.9 $L^a_{\rho'}$ is a cyclic space with respect to the Boolean algebra of multiplications $M\chi_\sigma : f \mapsto f\chi_\sigma$ $(\sigma \in \Sigma,\ f \in L^a_{\rho'})$, the result follows.

Conversely suppose that $L^a_\rho \simeq Z^*$ for Z as in the statement of the theorem. Identify L^a_ρ with Z^* and for each $\sigma \in \Sigma$, denote by E_σ the element of $\mathcal{B}$ whose adjoint is $M\chi_\sigma$. If we imbed Z canonically in $Z^{**} = (L^a_\rho)^* = \chi_{\Omega_a} L'_\rho = L'_\rho$, then the projections in $\mathcal{B}$ correspond to the multiplications $M\chi_\sigma$ $(\sigma \in \Sigma)$ on L'_ρ, and if we let $g \in L'_\rho$ be the function corresponding to the cyclic vector z_o, then Z becomes the ρ'-closed principal ideal $\bar{J}_g$ in L'_ρ (Prop. 3.9). We may assume $g \geq 0$ a.e. since $J_g = J_{|g|}$. We show now that

(i) supp $g = \Omega$, and

(ii) $g \in L^a_{\rho'}$.

If $\sigma \subseteq \Omega \setminus$ supp g, then for any $f \in L^a_\rho$,

$$\langle g, \chi_\sigma f\rangle = \int_\sigma gf\, d\mu = 0 .$$

Hence $\langle g', \chi_\sigma f\rangle = 0$ for every $g' \in Z$ and $f \in L^a_\rho$; therefore $\chi_\sigma L^a_\rho \subseteq Z^\perp$ and $Z^* \subseteq \chi_{\Omega\setminus\sigma} L^a_\rho$. So σ must be a null set and supp $g = \Omega$.

Now let $\{\sigma_n\}$ be a decreasing sequence of sets in Σ, whose intersection is a null set. For each n,
$g\chi_{\sigma_n} = E^{**}_{\sigma_n} g = E_{\sigma_n} z_o$, and $\inf_n M\chi_{\sigma_n} = 0$, so $\inf_n E_{\sigma_n} = 0$.

Since $\mathcal{B}$ is σ-complete, $\inf \|E_{\sigma_n} z_o\| = 0$ and hence, $\rho'(\chi_{\sigma_n} g) \downarrow 0$, which shows that $g \in L^a_{\rho'}$. Since $\Omega_b \supseteq$ supp g, it follows that $\Omega_b = \Omega$.

Finally, from a result of Bessaga and Pelczýnski ([BP], Cor.4), L_ρ^a, being a Banach dual space, can contain no complemented subspace isomorphic to c_o. Hence by Theorem 5.7, $L_\rho = L_\rho^a$ and the theorem now follows.

Case 2. Now consider ρ a saturated norm having the Riesz-Fischer property. We shall apply the isomorphism U of L_ρ^a onto L_τ^a, described following Theorem 5.4, since the norm τ will satisfy the conditions of Case 1.

Assume that (a) and (b) both hold for L_ρ^a. Then by the isomorphism U, $L_\tau^a = L_\tau$; since $\tau'(f) = \rho'(f\varphi_o)$ for $f = f\chi_{\Omega_a}$, it is easily seen that

$$L_{\tau'}^a = \{f\varphi_o^{-1} : f = f\chi_{\Omega_a} \in L_{\rho'}^a\} = \varphi_o^{-1}\chi_{\Omega_a}L_{\rho'}^a = \varphi_o^{-1}L_{\rho'}^a ,$$

so that the carrier of $L_{\tau'}^a$ is supp $\varphi_o^{-1} \cap \Omega_b = \Omega_a \cap \Omega_b = \Omega_a$, from (b). By Case 1 therefore, L_τ^a is isomorphic to the dual of a cyclic space Z and hence so also is L_ρ^a.

Conversely, if L_ρ^a is the dual of a cyclic space Z as in the statement of the theorem, then also $L_\tau^a \simeq Z^*$, so $L_\tau^a = L_\tau$ and the carrier of $L_{\tau'}^a$ is Ω_a. Consequently $L_\rho^a = \chi_{\Omega_a}L_\rho$ and Ω_a = supp $\varphi_o^{-1} \cap \Omega_b = \Omega_a \cap \Omega_b$; hence $\Omega_a \subseteq \Omega_b$ as required.

We shall return to this theorem later in the light of the results of Chap. IV.

6.3 PROPOSITION. Let ρ be a saturated function norm with the Riesz-Fischer property. Then L_ρ is isomorphic to the dual of a cyclic space $Z = \overline{\text{lin}}\,\{Ez_o : E \in \mathcal{B}\}$, where $z_o \in Z$ and $\mathcal{B}$ is a σ-complete Boolean algebra of projections on Z whose adjoints correspond to the multiplication operators $M\chi_\sigma : f \mapsto f\chi_\sigma$ ($\sigma \in \Sigma$, $f \in L_\rho$), if and only if

(a) ρ has the weak Fatou property, and

(b) $\Omega_b = \Omega$.

Proof. Suppose $L_\rho \simeq Z^*$ as in the statement of the theorem. Identifying Z^{**} with $L_\rho^* = L_\rho' \oplus L_{\rho,s}^*$, the canonical image $\hat{Z}$ of Z in Z^{**} consists of order continuous linear functionals on L_ρ, therefore is contained in L_ρ' and is of the form $\overline{\text{lin}}\ \{M_{\chi_\sigma}^* g : \sigma \in \Sigma\}$ for some $g \in L_\rho'$. Since $\mathcal{B}$ is σ-complete, the restriction of $\mathcal{B}^{**} = \{E^{**} : E \in \mathcal{B}\} = \{M_{\chi_\sigma}^* : \sigma \in \Sigma\}$ to $\hat{Z}$, forms a σ-complete Boolean algebra of projections, and so by the Representation Theorem and Lemmas 3.4 and 1.5, the norm of $\hat{Z}$ is absolutely continuous, i.e. $\hat{Z}$ is an ideal of $L_{\rho'}^a$. By Prop. 3.9 and Lemma 3.6, $\hat{Z} = \overline{J}_g = \chi_{\text{supp } g} L_{\rho'}^a$. However it is easily shown, just as in Theorem 6.2, that supp $g = \Omega$. Hence $\overline{J}_g = L_{\rho'}^a$ and clearly $\Omega_b = \Omega$. Thus by Lemma 3.1,

$$L_\rho = (L_{\rho'}^a)^* = \chi_{\Omega_b} L_\rho'' = L_\rho'' \ .$$

Hence ρ and ρ'' are equivalent norms, and so ρ has the Fatou property.

Conversely if condition (a) of the theorem holds, then $L_\rho = L_\rho''$. If (b) also holds, then we can find $f \in L_{\rho'}^a$ with $f > 0$ a.e. By Lemmas 3.9 and 3.6, $\overline{J}_f = L_{\rho'}^a = \overline{\text{lin}}\ \{\chi_\sigma f : \sigma \in \Sigma\}$ and

$$\overline{J}_f^* = (L_{\rho'}^a)^* = \chi_{\Omega_b} L_\rho'' = L_\rho'' = L_\rho \ .$$

Taking Z to be $L_{\rho'}^a$, the theorem now follows.

§ 7. Appendix.

The original theorems stated by L. Tzafriri, to which we referred at the beginning of the chapter, concerned cyclic Banach spaces, namely, spaces of the form

$$X = \overline{\mathrm{lin}}\ \{Px_o : P \in \mathcal{B}\} ,$$

where $x_o \in X$, and $\mathcal{B}$ is a σ-complete Boolean algebra of projections on X. These theorems, in [T_1] and [T_2], gave conditions for weak sequential completeness and for reflexivity of cyclic spaces, as follows, similar in form to those of our present theorems 6.1 and 5.1, 5.4, 5.7. The notation of 7.2(c) and 7.3 is defined in [T_1] and [T_2] respectively.

1. THEOREM ([T_1], Theorem 5). The cyclic space $X = \overline{\mathrm{lin}}\ \{Px_o : P \in \mathcal{B}\}$ is reflexive if and only if no subspace of it is isomorphic to either c_o or l_1.

2. THEOREM ([T_2], Theorem 4). Let $X = \overline{\mathrm{lin}}\ \{Px_o : P \in \mathcal{B}\}$ be a cyclic space. Then the following conditions are equivalent:

(a) X is weakly sequentially complete;

(b) No subspace of X is isomorphic to c_o;

(c) For any sequence of Borel functions $0 \leq f_1(\omega) \leq \ldots \leq f_n(\omega) \leq \ldots \quad (\omega \in \Omega)$ with $\sup_n \| S(f_n)x_o \| < \infty$, we have $x_o \in \mathcal{D}(S(\sup_n f_n))$, i.e. $S(\sup_n f_n)x_o \in X$;

(d) No complemented subspace of X is isomorphic to c_o;

(e) X is complemented in X^{**}.

By [G_1], Theorem 3.4 , every cyclic space is linearly isomorphic to the ideal L_ρ^a of a Banach function space L_ρ whose norm ρ has the Fatou property. The results of this chapter on weak sequential completeness and reflexivity therefore apply, modulo isomorphism, to cyclic spaces in particular, and they yield Tzafriri's results. However the present proofs are considerably easier to handle, and since more concrete, are perhaps more transparent than the proofs given in $[T_1]$ and $[T_2]$.

P. Meyer-Nieberg has also formulated some similar results, this time for Banach lattices, obtained by different methods again ([M], Theorems 13 and 16).

Since every Banach function space and every ideal thereof is a Banach lattice, these results are of wider application than ours; however their statements are slightly weaker.

In $[T_2]$, Tzafriri also gave the following theorem, which simplifies very considerably when cyclic spaces are reinterpreted as Banach function spaces, and gives rise to our Theorem 6.2.

3. THEOREM ($[T_2]$, Theorem 10). A cyclic space $X = \overline{\text{lin}}\,\{Px_o : P \in \mathcal{B}\}$ is isomorphic to the conjugate of a cyclic space Z if and only if it is weakly sequentially complete and at least one of the following conditions is satisfied:

(a) $P(s(\Gamma)) = I$;

(b) there exists a strictly positive functional $x_1^* \in \Gamma$ such that $x^{**}x_1^* = \sup\,\{x_1^*x : 0 \leq x \leq x^{**},\ x \in X\}$ for $0 \leq x^{**} \in X^{**}$;

(c) the closure of Γ in the $\sigma(X^*, X)$ topology contains x_o^*.

CHAPTER IV. HOMOMORPHISMS OF BANACH FUNCTION SPACES.

Our intention in this chapter is to develop and study the notion of a homomorphic relation between Banach function spaces. For any space of functions M, M^r (respectively M^+) will denote the subspace of real-valued (respectively non-negative valued) functions in M. As usual, we do not distinguish betwen a function f and the equivalence class of functions that are equal to f a.e. Note, in the case where M is M_μ, that $(M_\mu)^+$ is a strictly smaller class than M_μ^+ as defined earlier.

§ 8. Preliminaries and definitions.

We follow the notation of [LZ_2].

8.1 Definition. A Riesz space, or vector lattice, is a partially ordered real linear space $(L, +, \cdot, \leq)$ such that $(L, \leq)$ is a lattice.

The complexification of L is the space of elements of the form $x + iy$ $(x, y \in L)$, often denoted as the direct sum $L \oplus iL$. However, in [MW], Mittelmeyer and Wolff have axiomatised the notion of absolute value in a vector space and hence established the definition of a complex Riesz space. Moreover, whenever L is a complex Riesz space and L^r denotes the real vector lattice generated by the positive cone L^+ of L (the cone being determined by the absolute value defined on L), then L is precisely the standard complexification of L^r. The basic concepts of real Riesz spaces carry over easily to the complex setting, e.g.

an ideal of L is the complexification of an ideal of L^r, and so on. For a fuller discussion we refer the reader to [S] and we shall always use the term "Riesz space" meaning "complex Riesz space". Familiarity with the elementary properties of vector lattices is assumed;

these may be found in [LZ_2] and [F]. However the following fact merits explicit mention.

<u>8.2 LEMMA</u> ([F], 14D).. If L is a vector lattice, then L is an infinitely distributive lattice, i.e. if A is a subset of L such that $\sup A$ exists in L, then for every $y \in L$,

$$y \wedge \sup A = \sup \{y \wedge x : x \in A\} \quad ;$$

similarly, if $\inf A$ exists in L, then for every $y \in L$,

$$y \vee \inf A = \inf \{y \vee x : x \in A\} \; .$$

<u>8.3 Linear maps between Riesz spaces</u>.

(a) A linear map T is <u>increasing</u> (or <u>positive</u>) if $Tx \geq 0$ whenever $x \geq 0$.

(b) A <u>Riesz homomorphism</u> is a linear map which is also a lattice homomorphism. If T is a linear map of L_1 to L_2, the following statements are equivalent ([F], 14E(b)):

(i) T is a Riesz homomorphism;

(ii) $(Tx)^+ = Tx^+$ for all $x \in L_1^r$;

(iii) $|Tx| = T|x|$ for all $x \in L_1^r$;

(iv) $Tx \wedge Ty = 0$ whenever $x \wedge y = 0$.

It is easily checked that if T is a Riesz homomorphism of L_1 to L_2, then T is increasing and (using (ii) and (iv)) whenever $x \in L_1^r$ then $Tx \in L_2^r$.

(c) A <u>Riesz isomorphism</u> is a bijective Riesz homomorphism.

(d) An increasing linear map T is <u>order continuous</u> if whenever A is a non-empty directed subset of L^+, then $A \downarrow 0$ implies $T(A) \downarrow 0$, or equivalently, $A \uparrow x$ implies $T(A) \uparrow Tx$.

(e) An increasing linear map T is <u>sequentially order</u>

continuous if the condition of (d) holds with the directed set A replaced by any monotone sequence.

8.4 Riesz subspaces.

(a) A subset A of a vector lattice L is called solid if it follows, whenever $x \in L$ and $|x| \leq a$ for some $a \in A$, that $x \in A$.

(b) An ideal of L is a solid linear subspace of L, and is thus always a Riesz subspace.

(c) A band of L is an order closed ideal, i.e. an ideal M, say, such that, if $A \subset M$ and sup A exists in L, then $\sup A \in M$.

(d) A Riesz subspace M of L is order dense if for each $x \in L^+$,

$$x = \sup \{y : 0 \leq y \leq x,\ y \in M\} .$$

The range of a Riesz homomorphism is a Riesz subspace of the codomain; the kernel of a Riesz homomorphism is an ideal of the domain ([F], 14F).

8.5 Quotient spaces and homomorphic images.

If L is a Riesz space and I is an ideal of L, then L/I is a Riesz space with respect to a partial order $\leq$ defined as follows:

given $f, g \in L$, we say $[f] \leq [g]$ whenever there exist $f_1 \in [f]$ and $g_1 \in [g]$ with $f_1 \leq g_1$.

The canonical mapping of L onto L/I is a Riesz homomorphism with kernel I; conversely, any Riesz homomorphic image T(L) of L is Riesz isomorphic to L/ker T ([LZ$_2$], 18.7,18.9).

8.6 Dedekind completeness.

(a) The Riesz space L is called Dedekind complete (resp. Dedekind σ-complete) if every non-empty (resp. at most countable non-empty) subset of L which is bounded from above has a supremum.

(b) L is called <u>super Dedekind complete</u> if L is Dedekind complete and every non-empty subset possessing a supremum contains a countable subset with the same supremum.

Note that (i) if L is a Dedekind complete Riesz space and M is an ideal of L, then M is also Dedekind complete;

(ii) in a super Dedekind complete space, sequentially order continuous mappings are order continuous.

<u>8.7 The Riesz spaces M_μ and L_ρ.</u>

Let ρ be a saturated function norm based on (Ω, Σ, μ). When endowed with the natural (pointwise) ordering, whereby for f and g in M_μ^r, $f \leq g$ if and only if $f(x) \leq g(x)$ for almost every $x \in \Omega$, M_μ is a Riesz space with real part M_μ^r and positive cone M_μ^+.

The following facts are fundamental to many results concerning Banach function spaces, and will normally be used without explicit reference.

<u>8.7 (i) THEOREM.</u> M_μ is a super Dedekind complete Riesz space ([LZ_2], 23.3(iv)).

<u>8.7 (ii) LEMMA.</u> L_ρ is an order dense ideal of M_μ.

<u>Proof</u>. Since whenever $0 \leq f \leq g$ a.e. we have $\rho(f) \leq \rho(g)$, clearly L_ρ is a solid subspace, and hence a Riesz subspace, of M_μ. Since ρ is saturated, then by the Exhaustion Theorem ([Z], 67.3), there exists a sequence $\Omega_n \uparrow \Omega$ in Σ with $\rho(\chi_{\Omega_n}) < \infty$ for each n. Let $0 \leq f \in M_\mu$ and let $\sigma_n = \{f \leq n\}$ $(n = 1,2,\ldots)$; then $\sigma_n \uparrow \Omega$ so if we let $\delta_n = \sigma_n \cap \Omega_n$, we have $\delta_n \uparrow \Omega$ (2.1(iv)), and for each n,

$$f\chi_{\delta_n} \leq n\chi_{\delta_n} \leq n\chi_{\Omega_n} \in L_\rho \quad .$$

So $\{f\chi_{\delta_n}\}$ is a sequence in L_ρ with supremum f. Hence,

$$f = \sup \{f\chi_{\delta_n}\} \leq \sup \{g : 0 \leq g \leq f, g \in L_\rho\} \leq f .$$

Note that the existence of the second supremum here is ensured by the Dedekind completeness of L_ρ (which follows from 8.6 (i)). So f equals this supremum a.e. and the result follows.

<u>8.7 (iii) COROLLARY</u>. If $f \in M_\mu^+$, we can always find a sequence $\{f_n\}$ in L_ρ with $0 \leq f_n \uparrow f$ a.e.

<u>§ 9. Homomorphisms between Banach function spaces</u>.

Throughout the present and the following sections, ρ and τ will be saturated function norms with the Riesz-Fischer property, based on the σ-finite measure spaces (Ω, Σ, μ) and $(\in, \Lambda, \nu)$ respectively.

It is clear that since L_ρ is a Riesz subspace of M_μ, the restriction to L_ρ of any Riesz homomorphism on M_μ is also a Riesz homomorphism.

Now let Π be any Riesz homomorphism of L_ρ onto L_τ. Let the null ideal ker Π have carrier set $A \subseteq \Omega$; denote by q the canonical quotient map of L_ρ to $L_\rho/\ker \Pi$ and by α the induced isomorphism of $L_\rho/\ker \Pi$ to L_τ. Thus the following diagram commutes:

$$\begin{tikzcd} L_\rho \arrow[r, "\Pi"] \arrow[dr, "q"'] & L_\tau \\ & L_\rho/\ker \Pi \arrow[u, "\alpha"'] \end{tikzcd}$$

It follows from [F], 25D that the completeness of L_ρ and monotonicity of the norm τ are sufficient to ensure that Π is continuous, and hence that α is continuous with respect to the usual quotient norm, denoted by $\hat{\rho}$ and given by

$$\hat{\rho}([f]) = \inf \{\rho(f') : f - f' \in \ker \Pi\} .$$

Thus, since α is both one-one and onto, it must, by the Closed Graph Theorem, be bicontinuous. Consequently, if $f \in L_\rho$ and $g \in L_\tau$ we have

(i) $\tau(\Pi f) \leq k_1 \hat{\rho}([f])$,

(ii) $\hat{\rho}(\alpha^{-1} g) \leq k_2 \tau(g)$,

where $k_1 = \|\alpha\|$, $k_2 = \|\alpha^{-1}\|$. From (i) and the definition of $\hat{\rho}$, clearly $\|\Pi\| = k_1$ also, and from (ii), for an arbitrary $\varepsilon > 0$ and for each $g \in L_\tau$ we can find an element h in the coset $\alpha^{-1} g$ satisfying $\rho(h) \leq (k_2 + \varepsilon)\tau(g)$. Hence, choosing any positive constant $c_1 < k_2^{-1}$ and letting $c_2 = k_1$, there exists for every $g \in L_\tau$, some $h \in L_\rho$ satisfying $\Pi h = g$ and

$$c_1 \rho(h) \leq \tau(g) \leq c_2 \rho(h) . \tag{1}$$

<u>9.1 Definition</u>. The Banach function space L_τ shall be called a <u>homomorphic image</u> of the Banach function space L_ρ if there exists a Riesz homomorphism of L_ρ onto L_τ.

<u>9.2 Remark</u>. We shall consider only the case where the Riesz homomorphism Π, say, is surjective, since otherwise the image of an ideal of L_ρ, and indeed the range $\Pi(L_\rho)$ itself, need not be ideals of L_τ nor, therefore, Banach function spaces, under any monotone renorming. For example,

take $L_\rho = L_\tau$ to be the space l_∞ of all bounded complex sequences with the usual norm $\|\cdot\|_\infty$ and the pointwise ordering of elements; for $\alpha = \{\alpha_n\} \in L_\rho$, define

$$\Pi\alpha = (\alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_3, \ldots) ;$$

clearly Π is linear, increasing and bounded, and commutes with the lattice operations $\vee$, $\wedge$: hence Π is a Riesz homomorphism on L_ρ, but is certainly not onto; now if $0 \neq \alpha \in L_\rho$, let

$$\beta = (\alpha_1, \tfrac{1}{2}\alpha_1, \alpha_2, \tfrac{1}{2}\alpha_2, \alpha_3, \ldots) ;$$

then $|\beta| \leq |\Pi\alpha|$, but $\beta \notin \Pi(L_\rho)$ and the range of Π is therefore not an order ideal.

Thus, throughout the chapter, the homomorphism $\Pi : L_\rho \to L_\tau$ will be assumed surjective. Moreover, A will always denote the carrier of ker Π, and c_1, c_2 will be the constants of the inequalities (1).

9.3 LEMMA. Let $\{v_n\}$ be a monotone sequence of elements of L_τ^+. Then there exists a monotone sequence $\{u_n\}$ in L_ρ^+, which is increasing or decreasing according as $\{v_n\}$ is increasing or decreasing, and such that for each n,

$$\Pi u_n = v_n \ .$$

Furthermore, in the case where $\{v_n\}$ decreases, each u_n may be chosen to satisfy additionally

$$c_1 \rho(u_n) \leq \tau(v_n) \leq c_2 \rho(u_n) \ .$$

Proof. Suppose the sequence $\{v_n\}$ increases. Since Π is onto, we can find $u_n' \in L_\rho$ with $\Pi u_n' = v_n$ for each n. Since

$$\Pi(|u_n'|) = |\Pi u_n'| = |v_n| = v_n \ ,$$

we can replace, if necessary, each u_n' by $|u_n'|$ and ensure that the sequence lies in L_ρ^+. Now write $u_1 = u_1'$ and $u_n = u_n' \vee u_{n-1}$ $(n \geq 2)$. Then $u_n \geq u_{n-1}$ a.e.; also,

$$\rho(u_n) \leq \rho(u_n' + u_{n-1}) \leq \rho(u_n') + \rho(u_{n-1}) \leq \ldots \leq \sum_{r=1}^{n} \rho(u_r') < \infty \ ,$$

and

$$\Pi u_n = \Pi u_n' \vee \Pi u_{n-1} = \ldots = \sup_{1 \leq r \leq n} \Pi u_r' = \sup_{1 \leq r \leq n} v_r = v_n \ .$$

This inductively defines the required sequence.

A similar procedure applies in the case where $\{v_n\}$ decreases, by putting $u_1 = u_1'$ and $u_n = u_n' \wedge u_{n-1}$ $(n \geq 2)$. But furthermore, we could choose the original sequence $\{u_n'\}$ to satisfy

$$c_1\rho(u_n') \le \tau(v_n)$$

as in (1). This condition is preserved if we replace u_n' by $|u_n'|$, since $\rho(f) = \rho(|f|)$ for every $f \in M$; finally, since for each n, $u_n \le u_n'$, we have

$$c_1\rho(u_n) \le c_1\rho(u_n') \le \tau(v_n) \ .$$

The inequality $\tau(v_n) \le c_2\rho(u_n)$ is immediate.

<u>9.4 LEMMA</u>. If σ, $\sigma_1 \in \Sigma$ with $\chi_\sigma \in L_\rho$ and $\sigma_1 \subseteq \sigma$, then

$$\Pi\chi_{\sigma_1} = (\Pi\chi_\sigma)\chi_{\text{supp } \Pi\chi_{\sigma_1}} \ .$$

<u>Proof</u>. Let $\sigma_2 = \sigma \setminus \sigma_1$, and $\delta_i = \text{supp } \Pi\chi_{\sigma_i}$ $(i = 1,2)$; let $f = \Pi\chi_\sigma$ and $\delta = \text{supp } f$. Since $\sigma_i \subseteq \sigma$, clearly $\chi_{\sigma_i} \in L_\rho$ and $\Pi\chi_{\sigma_i} \le \Pi\chi_\sigma$, so $\delta_i \subseteq \delta$ $(i = 1,2)$. In fact $\Pi\chi_\sigma = \Pi\chi_{\sigma_1} + \Pi\chi_{\sigma_2}$ so $\delta = \delta_1 \cup \delta_2$; since σ_1 and σ_2 are disjoint, $\Pi\chi_{\sigma_1} \wedge \Pi\chi_{\sigma_2} = 0$, so δ_1 and δ_2 are disjoint. It follows that $\Pi\chi_{\sigma_i} = f\chi_{\delta_i}$ $(i = 1,2)$.

<u>9.5 COROLLARY</u>. In the case where $1 \in L_\rho$, then letting $\phi_o = \Pi 1$, we have for every $\sigma \in \Sigma$,

$$\Pi\chi_\sigma = \phi_o\chi_\delta$$

where $\delta = \text{supp } \Pi\chi_\sigma$.

We conclude this section with two simple but important properties of Π.

<u>9.6 LEMMA</u>. If $u \in L_\rho^a$, then $\Pi u \in L_\tau^a$.

<u>Proof</u>. We may assume $u \ge 0$ a.e. Write $v = \Pi u$ and suppose $v \ge v_1 \ge v_2 \ge \dots \downarrow 0$ a.e. on Ω. Applying Lemma 9.3, we can find a decreasing sequence $\{u_n\}$ in L_ρ^+, majorised by u and satisfying $\Pi u_n = v_n$ for each n.

Define $\bar{u} = \inf u_n \ge 0$. Since $\bar{u} \le u_n$ a.e., $\Pi\bar{u} \le \Pi u_n$ $(n = 1,2,\dots)$ so $\Pi\bar{u} \le \inf \Pi u_n = \inf v_n = 0$, i.e. $\bar{u} \in \ker \Pi$.

Now $u - \bar{u} \geq u_n - \bar{u} \geq \ldots \downarrow 0$ a.e., and $u - \bar{u} \leq u \in L_\rho^a$, so $\rho(u_n - \bar{u}) \to 0$ as $n \to \infty$. Hence

$$\tau(v_n) = \tau(\Pi u_n - \Pi\bar{u}) \leq c_2\rho(u_n - \bar{u}) \to 0$$

as $n \to \infty$, showing that $v \in L_\tau^a$.

<u>9.7 PROPOSITION</u>. Let $\Pi_a = \Pi|_{L_\rho^a}$. Then Π_a is order continuous.

<u>Proof</u>. Recalling 8.6 (ii), it is sufficient to show that Π_a is sequentially order continuous. Let $f \in L_\rho^a$, and suppose $0 \leq f_1 \leq f_2 \leq \ldots \uparrow f$ a.e. Then $\rho(f - f_n) \to 0$, and hence $\tau(\Pi_a f - \Pi_a f_n) \to 0$ as $n \to \infty$. The sequence $\{\Pi_a f_n\}$ is increasing and bounded above by $\Pi_a f$, so $\sup_n \Pi_a f_n \leq \Pi_a f$ and by Lemma 9.6, this implies that $\sup_n \Pi_a f_n \in L_\tau^a$. Hence, since $\Pi_a f_1 \leq \Pi_a f_2 \leq \ldots \uparrow \sup_n \Pi_a f_n$ a.e., $\tau(\sup_n \Pi_a f_n - \Pi_a f_m) \to 0$ as $m \to \infty$. Thus $\Pi_a f = \sup_n \Pi_a f_n$ a.e., as required.

In particular, it follows from Prop. 9.7, that if ρ is an absolutely continuous norm, every Riesz homomorphism of L_ρ onto L_τ is order continuous. Hence this is true, for example, when L_ρ is reflexive or when L_ρ is weakly sequentially complete, since both these conditions imply $L_\rho = L_\rho^a$ (Theorems 5.1 and 6.1).

<u>§ 10. Order continuity</u>.

It is appropriate to record here some elementary facts about Riesz homomorphisms.

<u>10.1 PROPOSITION</u>. If L and M are Riesz spaces and T is a Riesz homomorphism of L onto M, then

(i) whenever I is an ideal of L, T(I) is an ideal of M;

(ii) if I_1 and I_2 are disjoint ideals of L, then $T(I_1)$ and $T(I_2)$ are disjoint ideals of M;

(iii) T is order continuous if and only if ker T is a band.

Proof. The proofs of (i) and (ii) are an easy exercise; for (iii) see [LZ_2],18.13.

Recalling from Prop. 5.2 that the bands of L_ρ are precisely the subspaces of the form $\chi_C L_\rho$ for some set $C \in \Sigma$ which is then the carrier of that band, we can apply (iii) directly to $\Pi : L_\rho \to L_\tau$:

10.2 COROLLARY. Π is order continuous if and only if $\ker \Pi = \chi_A L_\rho$.

Before the main theorem of this section, we require a further lemma.

10.3 LEMMA. Let the Riesz homomorphism Π of L_ρ onto L_τ be order continuous. Let $h \in L_\tau$: then there is a unique element f of L_ρ disjoint to $\ker \Pi$ and satisfying $\Pi f = h$.

Proof. We first assume $h \geq 0$ a.e. Since Π is onto, $h = \Pi f_1$ for some $f_1 \in L_\rho$. Let $f = f_1 \chi_{\Omega \setminus A}$.
By Cor. 10.2, $f_1 - f \in \ker \Pi$. Since $\operatorname{supp} f \subseteq \Omega \setminus A$, f is disjoint to $\ker \Pi$, and $\Pi f = \Pi f_1 - \Pi(f_1 \chi_A) = \Pi f_1 = h$.
Suppose $g \in L_\rho$ is also disjoint to $\ker \Pi$ and satisfies $\Pi g = h$. Since $\Pi g^- = (\Pi g)^- = h^- = 0$ a.e., i.e. $g^- \in \ker \Pi$, we must have $g^- = 0$ a.e.; hence $g \in L_\rho^+$, and

$$\Pi(f \wedge g) = \Pi f \wedge \Pi g = h = \Pi f\,.$$

So $f - (f \wedge g) \in \ker \Pi$; but since $0 \leq f - (f \wedge g) \leq f$ a.e. and f is disjoint to $\ker \Pi$, this means $f = f \wedge g$, i.e. $f \leq g$. Similarly $\Pi(f \wedge g) = \Pi g$ and it follows that $g \leq f$ a.e. Hence $g = f$ a.e.; so $f \geq 0$ a.e. and is unique in satisfying the required condition.

In general, $h = h_1 - h_2 + ih_3 - ih_4$ where $h_i \geq 0$ a.e. ($i = 1,\dots,4$) and $h_1 h_2 = 0 = h_3 h_4$ a.e. For each i, there is a unique element f_i of L_ρ disjoint to $\ker \Pi$ and satisfying $\Pi f_i = h_i$. Define $f = f_1 - f_2 + if_3 - if_4$: then supp $f = \bigcup_i$ supp f_i so f is disjoint to ker Π, and $\Pi f = h$. Suppose $g \in L_\rho$ is also disjoint to ker Π and $\Pi g = h$. If $g = g_1 - g_2 + ig_3 - ig_4$ in its standard decomposition, it is easily deduced from 8.3 (b) that for each i, $\Pi g_i = h_i$; since supp $g_i \subseteq$ supp g, each g_i is disjoint to ker Π, and so, by uniqueness, $g_i = f_i$ a.e. Hence $g = f$ a.e.

For f and h related as in the lemma, we shall write

$$f = \Pi_c^{-1} h$$

(for reasons to become apparent) and call f the <u>fundamental inverse of h under Π</u>. It follows without difficulty from Lemma 10.3 that Π_c^{-1} is an increasing linear map of L_τ into L_ρ. Furthermore, observe that for any $g \in L_\rho$ with $\Pi g = h$, $g\chi_{\Omega \setminus A} = f$ a.e., so $g\chi_A$ and f are mutually disjoint and $g - f = (g - f)\chi_A$; therefore

$$\rho(g) = \rho(|g|) = \rho(|f + (g-f)|) = \rho(f + |g-f|) \geq \rho(f) .$$

Hence,

$$\rho(\Pi_c^{-1} h) = \inf\{\rho(g) : \Pi g = h\} . \tag{2}$$

In particular, $f = \Pi_c^{-1} h$ satisfies the left-hand inequality in (1), i.e. for $h \in L_\tau$,

$$c_1 \rho(\Pi_c^{-1} h) \leq \tau(h) . \tag{3}$$

<u>10.4 THEOREM</u>. Let Π be a Riesz homomorphism of L_ρ onto L_τ. There is a decomposition of Π in the form

$$\Pi = \Pi_c \oplus \Pi_n$$

where

(a) Π_c is one-one and order continuous,

(b) Π_n is order discontinuous at every point where it does not vanish,

(c) the domains of Π_c and Π_n are mutually complementary bands of L_ρ and the ranges of Π_c and Π_n are mutually complementary bands of L_τ.

<u>Proof</u>. Let A be, as usual, the carrier of the null ideal of Π and let A' be $\Omega \setminus A$; let Π_c and Π_n denote the restrictions of Π to $\chi_{A'} L_\rho$ and $\chi_A L_\rho$ respectively.

(a) That Π_c is one-one is clear. Now suppose $f_n = f_n \chi_{A'} \downarrow 0$ a.e. in L_ρ and let $g_n = \Pi f_n = \Pi_c f_n \in L_\tau$ $(n = 1,2,\ldots)$. Then the sequence $\{g_n\}$ is decreasing a.e. Let $g_o \in L_\tau$ be its infimum. Then $g_o \geq 0$ a.e. and for some $h \in L_\rho^+$, $\Pi h = g_o$. For each $n \in \mathbb{N}$,

$$\Pi(h \wedge f_n) = \Pi h \wedge \Pi f_n = g_o \wedge g_n = g_o = \Pi h \ .$$

So

$$h - (h \wedge f_n) \in \ker \Pi \ , \tag{4}$$

and therefore $\mathrm{supp}(h - (h \wedge f_n)) \subseteq A$. Thus $(h - (h \wedge f_n))\chi_{A'} = 0$ a.e., i.e. $h\chi_{A'} = (h \wedge f_n)\chi_{A'} \leq f_n$. Hence $h\chi_{A'} \leq \inf f_n = 0$ a.e., so $h = h\chi_A$ and

$$h \wedge f_n = h\chi_A \wedge f_n \chi_{A'} = 0 \quad \text{a.e.}$$

Thus from (4), $h \in \ker \Pi$ and $g_o = \Pi h = 0$ a.e., as required.

(b) Let $0 \leq h = h\chi_A \in L_\rho \setminus \ker \Pi$. Put $X_m = \{h \leq m\}$ for $m = 1,2,\ldots,$ and choose a sequence $Y_m \uparrow A$ in Σ with $\chi_{Y_m} \in \ker \Pi$ for each m. Since $X_m \uparrow \Omega$, we have $X_m \cap Y_m \uparrow A$ (by 2.1(iv)) so that $h\chi_{X_m \cap Y_m} \uparrow h$ a.e. For each m however,

$$h\chi_{X_m \cap Y_m} \leq m\chi_{Y_m} \in \ker \Pi \ ,$$

so $\sup \Pi_n(h\chi_{X_m \cap Y_m}) = 0$ and by assumption is not equal to $\Pi_n h$. This shows that Π_n is not order continuous at h.

(c) Since $\chi_A L_\rho$ and $\chi_{A'} L_\rho$ are disjoint ideals of L_ρ, the first assertion is immediate. Their images $\Pi(\chi_A L_\rho)$ and $\Pi(\chi_{A'} L_\rho)$ are disjoint ideals of L_τ by Prop. 10.1 (ii), and hence have disjoint carriers B and B' respectively, say, in Λ. Thus

$$L_\tau = \Pi(L_\rho) = \Pi(\chi_A L_\rho \oplus \chi_{A'} L_\rho) = \Pi_n(\chi_A L_\rho) \oplus \Pi_c(\chi_{A'} L_\rho) ,$$

and it follows that $\mathcal{E} = B \cup B'$ and that (c) holds. Hence we can express Π as the direct sum $\Pi_c \oplus \Pi_n$.

<u>10.5 COROLLARY</u>. If $f \in L_\rho^a$, $\Pi f = \Pi(f\chi_{A'})$, i.e. $f\chi_A \in \ker \Pi$.

<u>Proof</u>. Let $0 \le f \in L_\rho^a$ and choose any sequence $0 \le f_m \uparrow f\chi_A$ a.e. Since $f\chi_A$ and f_m are in L_ρ^a $(m = 1,2,\ldots)$ we have from Prop. 9.7 that

$$\Pi(f\chi_A) = \Pi_a(f\chi_A) = \lim_m \Pi_a f_m = \lim_m \Pi f_m ,$$

$$\text{i.e.} \quad \Pi_n f\chi_A = \lim_m \Pi_n f_m .$$

Hence Π_n is order continuous at $f\chi_A$, so in fact, from Theorem 10.4, $\Pi(f\chi_A) = 0$ a.e.

<u>10.6 COROLLARY</u>. Π is order continuous if and only if $\Pi|_{\chi_A L_\rho} = 0$.

<u>10.7 COROLLARY</u>. If Π is an isomorphism of L_ρ onto L_τ, then Π is necessarily order continuous.

<u>Proof</u>. If Π is an isomorphism, Π is one-one so $\ker \Pi = \{0\}$. Hence $A = \emptyset$ and $\Pi = \Pi_c$.

<u>10.8 Remark</u>. From the theorem, $\Pi_c = \Pi|_{\chi_{A'} L_\rho}$ is a Riesz isomorphism of $\chi_{A'} L_\rho$ onto $\chi_{B'} L_\tau$. In particular, this means that the mapping Π_c^{-1}, which we had encountered already in Lemma 10.3 for the case when Π was order continuous (and therefore $B' = \mathcal{E}$), is a Riesz isomorphism of $\chi_{B'} L_\tau$ onto $\chi_{A'} L_\rho$.

<u>10.9 EXAMPLES</u>. A simple example of an order continuous homomorphism is the mapping $M_\varphi : f \to f\varphi$ for some fixed measurable function $\varphi \ge 0$. If the domain here is L_ρ based

on (Ω, Σ, μ), then the range $L_\tau = \{f\varphi : f \in L_\rho\}$(*), while based on the same measurable space Ω, need not however be based on the same measure algebra Σ.

If φ is essentially bounded on Ω, then M_φ maps L_ρ into itself; if also, $\varphi \geq \varepsilon > 0$ a.e., then M_φ maps L_ρ one-one onto itself (since for each $f \in L_\rho$, $f\varphi \in L_\rho$ and $f\varphi^{-1} \in L_\rho$) and hence M_φ is a Riesz isomorphism on L_ρ.

We now give an example of a homomorphism which is not order continuous, to illustrate that Π_n need not always be trivial.

<u>10.10 EXAMPLE</u>. Let $\Omega = \mathbb{N}$, let Σ be the σ-algebra of all subsets of $\mathbb{N}$, and μ the discrete measure. Let $\rho = \tau = \|\cdot\|_\infty$, both based on (Ω, Σ, μ). Then L_ρ $(= L_\tau) = l_\infty$, $L_\rho^a = c_o$ and ρ has the Fatou property. Now l_∞ is linearly isomorphic to $C(\beta\mathbb{N})$ ($\beta\mathbb{N}$ being the Stone-Čech compactification of the positive integers), so let $\hat{x}$ denote the canonical image in $C(\beta\mathbb{N})$ of $x = \{x_i\} \in l_\infty$, and choose an element φ of $\beta\mathbb{N} \setminus \mathbb{N}$ lying in the compactification of the even integers. Define

$$\Pi x = (\hat{x}(\varphi), x_1, x_3, x_5, \ldots) \quad (x \in l_\infty) .$$

(i) Clearly, <u>Π is linear</u>.

(ii) $\|\Pi x\|_\infty \leq \max\{|\hat{x}(\varphi)|, \|x\|_\infty\} \leq \|x\|_\infty \max\{|\varphi|, 1\}$,

so $\Pi x \in l_\infty$ and <u>Π is bounded</u>.

(iii) Since $\mathbb{N}$ is dense in $\beta\mathbb{N}$ and $\hat{x}$ is continuous, then whenever $x \geq 0$, $\hat{x}(\varphi) \geq 0$, and so <u>Π is increasing</u>.

(iv) <u>Π is a lattice homomorphism</u>. It is sufficient to show that $|x|\hat{\ }(\varphi) = |\hat{x}(\varphi)|$ $(x \in l_\infty)$, so let $\{n_\alpha\}$ be a net of even integers such that $n_\alpha \to \varphi$ in $\beta\mathbb{N}$; for each α,

$$|\hat{x}(n_\alpha)| = |x_{n_\alpha}| = |x|\hat{\ }(n_\alpha) ,$$

(*) where τ may be obtained from ρ using φ, in the obvious way, so the continuity of $\hat{x}$ gives the result.

(v) Π is onto. Let $y \in l_\infty$. Define x by

$$x_{2n} = y_1 \quad (n \geq 1) \quad \text{and} \quad x_{2n-3} = y_n \quad (n \geq 2) \ :$$

then $x \in l_\infty$ and $\|x\|_\infty = \|y\|_\infty$. For any $k \in \mathbb{N}$, $\hat{x}(2k) = x_{2k} = y_1$; hence, by continuity, $\hat{x}(\varphi) = y_1$. It now follows easily that $\Pi x = y$.

(vi) Π is not order continuous. Consider the sequence of elements $\{x^n\}$ of l_∞, where $x^n = (\frac{1}{2}, \frac{2}{3}, \ldots, 1 - \frac{1}{n+1}, 0, 0, \ldots)$. For each n, $\hat{x}^n(\varphi) = 0$. Let $x \in l_\infty$ be defined by $x_m = 1 - \frac{1}{m+1}$ $(m = 1,2,\ldots)$; then $x^n \uparrow x$ as $n \to \infty$. However

$$\lim x_{2m} = \lim \hat{x}(2m) = 1 \ ,$$

so $\hat{x}(\varphi) = 1 \neq \hat{x}^n(\varphi)$. Thus $\Pi x^n \not\uparrow \Pi x$.

It follows by Prop. 10.1 (iii) that ker Π is not a band. Indeed, consider the carrier of ker Π: if $x \in \ker \Pi$, then $x_{2n-1} = 0$ $(n = 1,2,\ldots)$, so $A \subseteq \mathbb{N}_{even}$; on the other hand, if k is any even integer, and $x \in l_\infty$ is defined by

$$x_n = \delta_{nk} \qquad (n \in \mathbb{N}) \ ,$$

then $\Pi x = 0$ so $\{k\} = \operatorname{supp} x \subseteq A$, and $A \supseteq \mathbb{N}_{even}$. Thus $A = \mathbb{N}_{even}$. However, if $\alpha \in l_\infty$ is given by

$$\alpha_{2n} = 1 \ , \ \alpha_{2n-1} = 0 \qquad (n = 1,2,\ldots) \ ,$$

then $\operatorname{supp} \alpha = A$, but $\hat{\alpha}(\varphi) \neq 0$; so $\alpha \notin \ker \Pi$ and thus $\ker \Pi \neq \chi_A L_\rho$.

Observe that if α has support contained in $A = \mathbb{N}_{even}$, then $(\Pi\alpha)_n = 0$ for $n \geq 2$, and whenever β has support contained in $\Omega \setminus A = \mathbb{N}_{odd}$, then $\hat{\beta}(\varphi) = 0$ so $\operatorname{supp} \Pi\beta \subset \mathbb{N} \setminus \{1\}$. It follows easily that the sets B and B', defined in the proof of Theorem 10.4 (c), are respectively $\{1\}$ and $\mathbb{N} \setminus \{1\}$. Thus $\chi_B L_\tau \simeq \mathbb{C}$ and $\chi_{B'} L_\tau \simeq l_\infty$, and we can illustrate the decomposition theorem, for this example, as follows. We write l_∞^e and l_∞^o to denote the bounded complex sequences with supports in the even and odd integers respectively.

$$\begin{array}{ccccccc} L_\rho & = & l_\infty & = & l_\infty^o & \oplus & l_\infty^e \\ \Pi\big\downarrow & & & & \big\downarrow\Pi_c & & \big\downarrow\Pi_n \\ L_\tau & = & l_\infty & \simeq & l_\infty & \oplus & \mathbb{C} \end{array}$$

where $\Pi_c : (y_1, 0, y_3, 0, y_5, 0, \ldots) \mapsto (y_1, y_3, y_5, \ldots)$, $\Pi_n : (0, y_2, 0, y_4, 0, y_6, \ldots) \mapsto \varphi(0, y_2, 0, y_4, \ldots)$.

§ 11. Isomorphisms between Banach function spaces.

We have seen in Theorem 10.4 that any Riesz homomorphism Π of one Banach function space L_ρ say, onto another, has a component acting isomorphically on a band of L_ρ. Π will itself be order continuous only if the other component, acting on the orthogonal complement of that band, vanishes on its domain.

This observation motivates the closer inspection of isomorphisms: indeed, it turns out that every such, from L_ρ to L_τ say, arises uniquely from a Riesz isomorphism between the Riesz spaces M_μ and M_ν which is specified completely by a pair (θ, φ_o) where θ is a measure algebra isomorphism between Σ and Λ, and φ_o is a fixed strictly positive Λ-measurable function.

Throughout this section, we assume therefore, that Π is one-one, i.e. we let Π be a Riesz isomorphism of L_ρ onto L_τ.

For convenience, define

$$\theta\sigma = \operatorname{supp} \Pi\chi_\sigma \quad (\sigma \in \Sigma,\ \chi_\sigma \in L_\rho) . \tag{5}$$

In fact the order continuity of Π will enable us to extend this definition consistently to all of Σ. We begin with a simple but far-reaching observation.

<u>11.1 PROPOSITION</u>. Let $\sigma \in \Sigma$ with $\chi_\sigma \in L_\rho$, and suppose $\sigma = \operatorname{supp} f$ for some $f \in L_\rho$. Then

$$\operatorname{supp} \Pi f = \theta\sigma .$$

<u>Proof</u>. We may assume that $f \geq 0$ a.e. Let $\sigma_n = \{\frac{1}{n} \leq f \leq n\}$. Then $\sigma_n \uparrow \sigma$ so $f\chi_{\sigma_n} \uparrow f$ a.e. Since $\frac{1}{n}\chi_{\sigma_n} \leq f\chi_{\sigma_n} \leq n\chi_{\sigma_n}$, we have

$$\frac{1}{n}\Pi\chi_{\sigma_n} \leq \Pi(f\chi_{\sigma_n}) \leq n\Pi\chi_{\sigma_n}$$

and hence $\operatorname{supp} \Pi(f\chi_{\sigma_n}) = \theta\sigma_n$ for each $n \in \mathbb{N}$.

By order continuity, $\Pi\chi_{\sigma_n} \uparrow \Pi\chi_\sigma$ and so considering the supports it follows that $\theta\sigma_n \uparrow \theta\sigma$; hence, since $\Pi(f\chi_{\sigma_n}) \uparrow \Pi f$,

$$\operatorname{supp} \Pi f = \bigcup_n \operatorname{supp} \Pi(f\chi_{\sigma_n}) = \bigcup_n \theta\sigma_n = \theta\sigma .$$

<u>11.2 Definition</u>. Let $\Omega_n \uparrow \Omega$ in Σ with $\chi_{\Omega_n} \in L_\rho$ for each n. For any $\sigma \in \Sigma$, $\sigma = \bigcup_n \sigma \cap \Omega_n$ and $\chi_{\sigma\cap\Omega_n} \in L_\rho$ $(n = 1,2,\ldots)$. Define

$$\theta\sigma = \bigcup_n \theta(\sigma \cap \Omega_n) \tag{6}$$

where the r.h.s. is obtained from (5).

Because Π is order continuous, (5) and (6) are clearly consistent in the case where $\chi_\sigma \in L_\rho$. In general, the r.h.s. of (6) is an element of Σ and if we choose any $u \in L_\rho^+$ with $\operatorname{supp} u = \sigma$, we have, from Prop. 11.1, that for each $n \in \mathbb{N}$, $\operatorname{supp} \Pi(u\chi_{\sigma\cap\Omega_n}) = \theta(\sigma \cap \Omega_n)$. Hence,

$$\begin{aligned}\theta\sigma &= \bigcup_n \operatorname{supp} \Pi(u\chi_{\sigma\cap\Omega_n}) \\ &= \operatorname{supp} \sup_n \Pi(u\chi_{\sigma\cap\Omega_n}) \\ &= \operatorname{supp} \Pi u .\end{aligned}$$

This shows that θ is well-defined, independently of the sequence $\{\Omega_n\}$ and moreover, that the restrictions of Prop. 11.1 can be weakened:

11.1' PROPOSITION. For any $\sigma \in \Sigma$ and any $f \in L_\rho$ with $\operatorname{supp} f = \sigma$,

$$\operatorname{supp} \Pi f = \theta\sigma \quad .$$

11.3 PROPOSITION. θ is continuous with respect to set containment ($\subseteq$) in Σ.

Proof. Let $\sigma_n \uparrow \sigma$ ($\sigma, \sigma_n \in \Sigma$, $n = 1,2,\ldots$). Choose $f \in L_\rho^+$ such that $\operatorname{supp} f = \sigma$. Then $f\chi_{\sigma_n} \uparrow f$ a.e., and by the order continuity of Π, $\Pi(f\chi_{\sigma_n}) \uparrow \Pi f$ a.e. Hence, by Prop. 11.1',

$$\theta\sigma = \operatorname{supp} \Pi f = \bigcup_n \operatorname{supp} \Pi(f\chi_{\sigma_n}) = \bigcup_n \theta\sigma_n \; .$$

11.4 PROPOSITION. $\theta\Omega = \mathcal{E}$.

Proof. Suppose the contrary, so that $\nu(\mathcal{E} \setminus \theta\Omega) > 0$. By the Exhaustion Theorem ([Z], 67.3), we can find in Λ a sequence $\{\delta_n\}$ increasing to $\mathcal{E} \setminus \theta\Omega$ with $\chi_{\delta_n} \in L_\tau$ for each n. Fix some $m \in \mathbb{N}$ such that $\nu(\delta_m) > 0$ and let δ, χ denote δ_m, χ_{δ_m} respectively. Let $u = \Pi^{-1}\chi$ and $\sigma = \operatorname{supp} u$. From Prop. 11.1', $\theta\sigma = \operatorname{supp} \Pi u = \delta$. Now choose a sequence $\Omega_n \uparrow \Omega$ with each χ_{Ω_n} in L_ρ; by Def. 11.2, $\theta\Omega = \bigcup_n \theta\Omega_n$ and for each n, $\chi_{\sigma\cap\Omega_n} \in L_\rho$; however since $\sigma \cap \Omega_n \subseteq \Omega_n$, $\theta(\sigma \cap \Omega_n) \subseteq \theta\Omega_n$ by (5). Hence,

$$\theta\sigma = \bigcup_n \theta(\sigma \cap \Omega_n) \subseteq \bigcup_n \theta\Omega_n = \theta\Omega \quad .$$

Since $\theta\sigma = \delta$, this is a contradiction.

Note. An entirely analogous proof shows that for each $\sigma \in \Sigma$, $\theta(\Omega \setminus \sigma) = \mathcal{E} \setminus \theta\sigma$, but this will also follow from the next theorem.

11.5 THEOREM. θ is a measure algebra isomorphism of Σ onto Λ.

<u>Proof</u>. If $\sigma \in \Sigma$ and $\theta\sigma = \emptyset$, then for any $\sigma' \subseteq \sigma$ such that $\chi_{\sigma'} \in L_\rho$ we have supp $\Pi\chi_{\sigma'} = \theta\sigma' = \emptyset$, so $\chi_{\sigma'} \in \ker \Pi$. Since Π is an isomorphism, $\chi_{\sigma'} = 0$ a.e. It follows easily that $\chi_\sigma = 0$ a.e. Hence θ is one-one, and it remains to show that

(a) θ commutes with the algebraic operations $\cup$, $\cap$, and

(b) θ is onto.

<u>(a)</u>. Let $\sigma, \gamma \in \Sigma$. Choose $f, g \in L_\rho^+$ such that supp $f = \sigma$, and supp $g = \gamma$. Then supp $\sigma \cup \gamma$ = supp $f \vee g$ and $\sigma \cap \gamma$ = supp $f \wedge g$, so by Prop. 11.1',

$$\theta(\sigma\cup\gamma) = \operatorname{supp} \Pi(f\vee g) = \operatorname{supp}(\Pi f \vee \Pi g) = \operatorname{supp} \Pi f \cup \operatorname{supp} \Pi g = \theta\sigma \cup \theta\gamma,$$
$$\theta(\sigma\cap\gamma) = \operatorname{supp} \Pi(f\wedge g) = \operatorname{supp}(\Pi f \wedge \Pi g) = \operatorname{supp} \Pi f \cap \operatorname{supp} \Pi g = \theta\sigma \cap \theta\gamma.$$

<u>(b)</u>. Let $\delta \in \Lambda$ and choose a sequence $\delta_n \uparrow \delta$ with $\chi_{\delta_n} \in L_\tau$, and let $w_n = \Pi^{-1}\chi_{\delta_n}$ $(n = 1,2,\ldots)$. For each n,

$$\Pi(w_n \wedge w_{n+1}) = \Pi w_n \wedge \Pi w_{n+1} = \chi_{\delta_n} \wedge \chi_{\delta_{n+1}} = \chi_{\delta_n} = \Pi w_n .$$

Hence $w_n \wedge w_{n+1} = w_n$, i.e. $w_n \leq w_{n+1}$ $(n = 1,2,\ldots)$ so the sequence $\{w_n\}$ is increasing a.e. Let σ_n = supp w_n and $\sigma = \bigcup_n \sigma_n$. Define

$$\sigma_n' = \{w_n \geq \tfrac{1}{n}\} \quad (n = 1,2,\ldots) .$$

We shall show that (i) $\bigcup_n \sigma_n' = \sigma$, and (ii) $\theta\sigma_n' \uparrow \delta$.

(i). Let $x \in \sigma \setminus \bigcup \sigma_n'$ be such that $w_n(x)$ increases as $n \to \infty$. This condition excludes only a μ-null subset of $\sigma \setminus \bigcup_n \sigma_n'$ from consideration. Note that in this instance, for σ and for each σ_n' we have to choose a particular (fixed) representative of the μ-equivalence class of sets normally denoted by each of these symbols.

Since $x \in \sigma$, there is a positive integer k for which $w_k(x) \neq 0$. Since $x \notin \sigma_n'$, $w_n(x) < \frac{1}{n}$ for each n. Hence for each $n > k$, $w_k(x) \leq w_{k+1}(x) \leq \ldots \leq w_n(x) < \frac{1}{n}$. This implies that $w_k(x) = 0$, contradicting the choice of k. Thus, $\sigma = \bigcup_n \sigma_n'$.

(ii). For each n, $\Pi(w_n\chi_{\sigma_n'}) \leq \Pi w_n$. Hence by Prop. 11.1',

$$\theta\sigma_n' = \text{supp}\ \Pi(w_n\chi_{\sigma_n'}) \subseteq \text{supp}\ \Pi w_n = \delta_n\ .$$

So $\bigcup_n \theta\sigma_n' \subseteq \bigcup_n \delta_n = \delta$.

Suppose that this containment is strict, and choose a non-null subset $\delta' \in \Lambda$ of $\delta \setminus \bigcup_n \theta\sigma_n'$ satisfying $\chi_{\delta'} \in L_\tau$. For $r = 1,2,\ldots$, let $\eta_r = \delta' \cap \delta_r$. The sequence $\{\eta_r\}$ is clearly increasing in Λ, and by Lemma 9.4 applied to Π^{-1},

$$\Pi^{-1}\chi_{\eta_r} = w_r\chi_{\gamma_r}\ ,$$

where $\gamma_r = \text{supp}\ \Pi^{-1}\chi_{\eta_r} \subseteq \text{supp}\ \Pi^{-1}\chi_{\delta_r} = \sigma_r$ $(r = 1,2,\ldots)$, and the sequence $\{\gamma_r\}$ is increasing in Σ.

However δ' is disjoint from each $\theta\sigma_n'$ $(n = 1,2,\ldots)$, therefore so is each η_r $(r = 1,2,\ldots)$. In particular, each η_r is disjoint from $\theta\sigma_r'$, and hence,

$$\chi_{\eta_r} \wedge \Pi(w_r\chi_{\sigma_r'}) = 0 \qquad (r = 1,2,\ldots)\ .$$

It follows that

$$0 = \Pi^{-1}\chi_{\eta_r} \wedge \Pi^{-1}\Pi(w_r\chi_{\sigma_r'}) = w_r\chi_{\gamma_r} \wedge w_r\chi_{\sigma_r'} \quad ;$$

hence $\gamma_r \subseteq \sigma_r \setminus \sigma_r'$ for each r.

Now $\sigma_r \uparrow \sigma$ and $\sigma_r' \uparrow \sigma$ so $\chi_{\sigma_r \setminus \sigma_r'} \to 0$, and therefore $\chi_{\gamma_r} \to 0$ as $r \to \infty$. But $\{\chi_{\gamma_r}\}$ is an increasing sequence. So in fact, we must have $\chi_{\gamma_r} = 0$ on Ω and hence $\chi_{\eta_r} = 0$ on $\in$ $(r = 1,2,\ldots)$. It follows that $\delta = \bigcup_n \theta\sigma_n'$ as required.

Hence finally, from (i) and (ii) together with Prop. 11.3, we obtain

$$\theta\sigma = \theta(\bigcup_n \sigma_n') = \bigcup_n \theta\sigma_n' = \delta\ .$$

<u>11.6 LEMMA</u>. If $u \in L_\rho$ and $\sigma \in \Sigma$, then $\Pi(u\chi_\sigma) = \Pi u\ \chi_{\theta\sigma}$.

<u>Proof</u>. First let u be a characteristic function χ_γ, say,

where $\gamma \in \Sigma$. Then,

$$\Pi(u\chi_\sigma) = \Pi\chi_{\gamma\cap\sigma}$$

$$= (\Pi\chi_{\sigma\cap\gamma})\chi_{\theta(\sigma\cap\gamma)} \qquad \text{(from def. of } \theta)$$

$$= \Pi\chi_\gamma \ \chi_{\theta(\sigma\cap\gamma)} \qquad \text{(from 9.4)}$$

$$= \Pi\chi_\gamma \ \chi_{\theta\sigma} \qquad \text{(from def. of } \theta)$$

$$= \Pi u \ \chi_{\theta\sigma} \ .$$

It follows immediately that the lemma holds also when u is a simple measurable function. Now let $0 \leq u \in L_\rho$ and choose a sequence $0 \leq u_n \uparrow u$ a.e., with each u_n simple and measurable. Then $0 \leq u_n\chi_\sigma \uparrow u\chi_\sigma$ a.e. and so by the order continuity of Π,

$$\Pi(u\chi_\sigma) = \sup_n \Pi(u_n\chi_\sigma) = \sup_n \Pi u_n\chi_{\theta\sigma} = \Pi u\chi_{\theta\sigma}$$

as required.

<u>11.7 EXAMPLE</u>. Taking Π as in Example 10.10, then $\Pi_\epsilon = \Pi|_{l^o_\infty}$ is an isomorphism and the corresponding measure algebra isomorphism of the subsets of $\mathbb{N}_{odd}$ onto those of $\mathbb{N}\setminus\{1\}$ is given by

$$\theta(\{2n-3\}) = \{n\} \qquad (n = 2,3,\ldots) \ .$$

The measure algebra isomorphism θ induces a natural mapping Π_1, say, between the sets of measurable functions on Ω and ϵ, given by

$$\Pi_1\chi_\sigma = \chi_{\theta\sigma} \qquad (\sigma \in \Sigma) \ . \tag{7}$$

This can be extended immediately to simple functions by linearity, but before extending the domain to all of M_μ, we require the following lemma.

11.8 LEMMA. Let $\{u_n\}$ be a sequence of simple measurable functions such that $u_n \to 0$ a.e. on Ω. Then $\Pi_1 u_n \to 0$ a.e. on $\mathcal{E}$.

Proof. First assume that each u_n is real-valued a.e. Then, as $0 \leq |u_n| \to 0$ a.e. and $u_n^+ \leq |u_n|$, we have $0 \leq u_n^+ \to 0$ a.e. Hence

$$\liminf u_n^+ = 0 = \limsup u_n^+ \quad . \tag{8}$$

The sequence $\{\sup_{n \geq k} u_n^+\}_{k \in \mathbb{N}}$ is decreasing pointwise a.e. and from (8) has infimum zero a.e. Choose a sequence $\Omega_m \uparrow \Omega$ in Σ with $\chi_{\Omega_m} \in L_\rho$ and $\mu(\Omega_m) < \infty$ for each m. Fix $\varepsilon > 0$: we can then apply Egoroff's theorem ([DS$_1$],III.6.12) to infer, for each m, the existence of a set $\Omega_m' \subseteq \Omega_m$ with $\mu(\Omega_m \setminus \Omega_m') < m^{-1}\varepsilon$, such that

$$\sup_{n \geq k} u_n^+ \chi_{\Omega_m'} \downarrow 0 \quad \text{(uniformly as } k \to \infty) \quad .$$

Let $\Omega_m'' = \bigcup_{r=1}^{m} \Omega_r'$. For $1 \leq r \leq m$, $\Omega_r' \subseteq \Omega_r \subseteq \Omega_m$ so $\Omega_m'' \subseteq \Omega_m$, and the sequence $\{\Omega_m''\}$ is increasing; so $\mu(\Omega_m \setminus \Omega_m'') \leq \mu(\Omega_m \setminus \Omega_m') < m^{-1}\varepsilon$ and

$$\sup_{n \geq k} u_n^+ \chi_{\Omega_m''} \downarrow 0$$

uniformly as $k \to \infty$, since the convergence is uniform on Ω_r' for $1 \leq r \leq m$. Fix $m \in \mathbb{N}$ and extract a subsequence $\{n_j\}$ of $\mathbb{N}$ such that for each j

$$\sup_{n \geq n_j} u_n^+ \chi_{\Omega_m''} \leq 2^{-j} \chi_{\Omega_m''} \quad . \tag{9}$$

Then, the l.h.s. of (9) is in L_ρ and

$$\sup_{n\geq n_j} \Pi(u_n^+ \chi_{\Omega''_m}) = \Pi(\sup_{n\geq n_j} u_n^+ \chi_{\Omega''_m}) \leq 2^{-j} \Pi\chi_{\Omega''_m} . \tag{10}$$

By lemma 9.4, if $\sigma_1 \subseteq \sigma$ and $\chi_\sigma \in L_\rho$, then

$$\Pi\chi_{\sigma_1} = \Pi\chi_\sigma\chi_{\theta\sigma_1} = \Pi\chi_\sigma\Pi_1\chi_{\sigma_1} ,$$

and it follows that if s is a simple measurable function with supp $s \subseteq \sigma$, then

$$\Pi s = \Pi\chi_\sigma\Pi_1 s . \tag{11}$$

For each n, let $\sigma_n = \Omega''_m \cap \operatorname{supp} u_n^+$ and let $\psi = \Pi\chi_{\Omega''_m}$. Then $\theta\sigma_n \subseteq \theta\Omega''_m$ and, by lemma 9.4, $\Pi\chi_{\sigma_n} = \psi\chi_{\theta\sigma_n}$ for each n. So, from (11)

$$\Pi(u_n^+ \chi_{\Omega''_m}) = \Pi\chi_{\sigma_n}\Pi_1 u_n^+ = \psi\chi_{\theta\sigma_n}\Pi_1 u_n^+$$

and hence from (10),

$$\sup_{n\geq n_j} (\psi\chi_{\theta\sigma_n}\Pi_1 u_n^+) \leq 2^{-j}\psi ;$$

therefore, since $\psi \geq 0$ a.e. and $\theta\sigma_n \subseteq \theta\Omega''_m$,

$$\sup_{n\geq n_j} (\Pi_1 u_n^+ \chi_{\theta\sigma_n}) \leq 2^{-j} \chi_{\theta\Omega''_m} ,$$

and

$$\lim_j \sup_{n\geq n_j} (\Pi_1 u_n^+ \chi_{\theta\sigma_n}) = 0 \quad \text{a.e.}$$

Now for each large enough $k \in \mathbb{N}$, there is a $j \in \mathbb{N}$ such that $n_j \leq k < n_{j+1}$ and since the sequence $\{\sup_{n\geq k} \Pi_1 u_n^+\}$ is decreasing a.e.,

$$\sup_{n\geq k} \Pi_1 u_n^+ \chi_{\theta\Omega''_m} \leq \sup_{n\geq n_j} \Pi_1 u_n^+ \chi_{\theta\Omega''_m} \leq 2^{-j} \chi_{\theta\Omega''_m} .$$

Thus $\limsup \Pi_1 u_n^+ \chi_{\theta\Omega''_m} = 0$ a.e. for every $m \in \mathbb{N}$.

Now since $\Omega_n \uparrow \Omega$ and $\mu(\Omega_n \setminus \Omega_n'') < n^{-1}\varepsilon$, clearly $\Omega_n'' \uparrow \Omega$ and so by the continuity of θ (Prop. 11.3), $\theta\Omega_n'' \uparrow \mathcal{E}$. Hence in fact,

$$\limsup \Pi_1 u_n^+ = 0 \quad \text{a.e.}$$

on $\mathcal{E}$, and finally it follows that

$$\lim \Pi_1 u_n^+ = 0 \quad \text{a.e.}$$

Likewise, $u_n^- \leq |u_n|$, so $0 \leq u_n^- \to 0$ a.e. and similarly we obtain

$$\lim \Pi_1 u_n^- = 0 \quad \text{a.e.} \;,$$

so therefore

$$\lim_n \Pi_1 u_n = \lim_n \Pi_1 u_n^+ - \lim_n \Pi_1 u_n^- = 0 \quad \text{a.e.}$$

In general $u_n = v_n + iw_n$ where v_n, w_n are real-valued; if $u_n \to 0$ a.e., then $v_n \to 0$ and $w_n \to 0$ a.e. Hence from the preceding part, $\Pi_1 v_n \to 0$ and $\Pi_1 w_n \to 0$ a.e. so finally,

$$\Pi_1 u_n = \Pi_1 v_n + i\Pi_1 w_n \to 0 \quad \text{a.e.}$$

Now let $0 \leq f \in M_\mu$. Choose a sequence of non-negative functions s_n, each simple and measurable, satisfying $0 \leq s_n \uparrow f$ a.e. Define

$$\Pi_1 f = \sup_n \Pi_1 s_n \quad . \tag{12}$$

This definition of $\Pi_1 f$ is independent of the particular choice of sequence $\{s_n\}$. Indeed if $\{t_n\}$ is another sequence of simple measurable functions with $0 \leq t_n \uparrow f$ a.e., then for each n, $s_n - t_n$ is also simple and $s_n - t_n \to 0$ a.e. From the lemma therefore,

$$\Pi_1 s_n - \Pi_1 t_n = \Pi_1(s_n - t_n) \to 0 \quad \text{a.e.} \ ,$$

i.e. $\lim_n \Pi_1 t_n = \lim_n \Pi_1 s_n$.

We now show that $\Pi_1 f \in M_\nu$. For each $p \in \mathbb{N}$, let

$$A_p = \{f > p\} = \bigcup_n \{s_n > p\} \in \Sigma .$$

Since $f \in M_\mu$, $f < \infty$ a.e. so $A_p \downarrow \emptyset$. By the continuity of θ (Prop. 11.3), $\theta A_p \downarrow \emptyset$ in Λ. Now

$$\begin{aligned}
\theta A_p &= \theta \bigcup_n \{s_n > p\} \\
&= \bigcup_n \theta\{s_n > p\} && \text{(by continuity of } \theta\text{)} \\
&= \bigcup_n \{\Pi_1 s_n > p\} && \text{(from (7))} \\
&= \{\Pi_1 f > p\} && \text{(from (12))} \ .
\end{aligned}$$

Hence $\{\Pi_1 f = \infty\} = \bigcap_p \{\Pi_1 f > p\} = \bigcap_p \theta A_p = \emptyset$. Thus $\Pi_1 f < \infty$ as required.

It follows from (7) that for every simple function $s \in L_\rho$, $\operatorname{supp} \Pi_1 s = \operatorname{supp} \Pi s$. If $f \in L_\rho^+$ and we choose a sequence of simple functions f_n with $0 \leq f_n \uparrow f$ a.e. then,

$$\begin{aligned}
\operatorname{supp} \Pi_1 f = \operatorname{supp} \sup_n \Pi_1 f_n &= \bigcup_n \operatorname{supp} \Pi_1 f_n \\
&= \bigcup_n \operatorname{supp} \Pi f_n = \operatorname{supp} \Pi f \ ,
\end{aligned}$$

by the order continuity of Π. Hence if f and g in L_ρ are disjoint, then $\Pi_1 f$ and $\Pi_1 g$ are disjoint. If f and g in M_μ^+ are disjoint, choose simple functions f_n, g_n $(n = 1,2,\ldots)$ with $0 \leq f_n \uparrow f$ a.e. and $0 \leq g_n \uparrow g$ a.e.; then for any $m, n \in \mathbb{N}$

$$f_m \wedge g_n \leq f \wedge g = 0 \quad \text{a.e.} ,$$

therefore

$$\Pi_1 f_m \wedge \Pi_1 g_n = 0 \quad \text{a.e.}$$

Since supp $\Pi_1 f = \bigcup_m$ supp $\Pi_1 f_m$, $\Pi_1 f$ is certainly disjoint from each $\Pi_1 g_n$ $(n = 1,2,\ldots)$ and so since supp $\Pi_1 g = \bigcup_n$ supp $\Pi_1 g_n$, $\Pi_1 f$ is disjoint from $\Pi_1 g$.

Hence since Π_1 is clearly linear, it follows from 8.3 (b) that Π_1 is a lattice homomorphism of M_μ to M_ν.

<u>11.9 PROPOSITION</u>. Π_1 is bijective and order continous.

<u>Proof</u>. (i) <u>Π_1 is one-one</u>. Suppose $0 \leq u \in \ker \Pi_1$. If $\{u_n\}$ is any sequence of simple functions with $0 \leq u_n \uparrow u$ a.e., $\Pi_1 u \geq \sup_n \Pi_1 u_n$ and so $u_n \in \ker \Pi_1$ for each n. Letting $\sigma_n =$ supp u_n, and α_n be the infimum of u_n on σ_n or zero if σ_n is null $(n = 1,2,\ldots)$,

$$0 \leq \Pi_1 \ (\alpha_n \chi_{\sigma n}) \leq \Pi_1 u_n = 0 \quad ;$$

hence, either $\alpha_n = 0$ or $\chi_{\theta\sigma_n} = \Pi_1 \chi_{\sigma_n} = 0$ a.e.; since θ is an isomorphism, we have in either case that σ_n is Σ-null $(n = 1,2,\ldots)$ and since supp $u = \bigcup_n \sigma_n$, it follows that $u = 0$ a.e.

(ii) <u>Π_1 is onto</u>. Let $v \in M_\nu^+$ and $\{v_n\}$ be a sequence of simple functions with $0 \leq v_n \uparrow v$ a.e. Since θ is bijective, it follows from (7) that for each $\delta \in \Lambda$, $\chi_\delta = \Pi_1 \chi_\sigma$ for some $\sigma \in \Sigma$. By linearity, each simple ν-measurable function is also in the range of Π_1, so for each n, we can find a simple μ-measurable function u_n with $\Pi_1 u_n = v_n$; since Π_1 is one-one, clearly $0 \leq u_1 \leq u_2 \leq \ldots \leq u_n \leq \ldots$. It is sufficient to prove that $u = \sup_n u_n < \infty$ a.e., for then $u \in M_\mu^+$ and

$$\Pi_1 u = \sup_n \Pi_1 u_n = \sup_n v_n = v \ .$$

Therefore suppose $\delta \in \Sigma$ and $\delta \subseteq \{u = \infty\}$. By shrinking δ if necessary, we may assume that $u_n \uparrow \infty$ uniformly on δ; furthermore, by extracting an appropriate subsequence of $\{u_n\}$ we may also assume that

$$u_n \chi_\delta \geq n \chi_\delta \qquad (n = 1,2,\ldots) \ .$$

However we then have

$$v_n \chi_{\theta\delta} = \Pi_1 (u_n \chi_\delta) \geq n \Pi_1 \chi_\delta = n \chi_{\theta\delta} \ ,$$

and hence $v\chi_{\theta\delta} \geq n\chi_{\theta\delta}$ for every n, i.e. $\theta\delta \subseteq \{v = \infty\}$. It follows that $\nu(\theta\delta) = 0$ and hence that $\mu(\delta) = 0$. So $u < \infty$ a.e. as required.

(iii) Suppose that Π_1 is not order continuous on M_μ. Then there exists a sequence $\{f_n\}$ in M_μ with $f_n \downarrow 0$ a.e. and a function $0 \neq g_o \in M_\nu$ such that

$$\Pi_1 f_n \downarrow g_o \quad \text{a.e.} \ .$$

Let $\delta = \operatorname{supp} g_o \in \Lambda$ and let δ_1 be some subset of δ on which g_o is bounded away from zero, so that

$$g_o \chi_{\delta_1} \geq \varepsilon \chi_{\delta_1} \quad \text{a.e.} \ ,$$

say. Since Π_1 is a lattice isomorphism, so is Π_1^{-1}; hence for each $n \in \mathbb{N}$,

$$\Pi_1^{-1}(\Pi_1 f_n \chi_{\delta_1}) \geq \Pi_1^{-1}(g_o \chi_{\delta_1}) \geq \varepsilon \Pi_1^{-1} \chi_{\delta_1}$$

i.e. $$f_n \chi_{\theta^{-1}\delta_1} \geq (\Pi_1^{-1} g_o)\chi_{\theta^{-1}\delta_1} \geq \varepsilon \chi_{\theta^{-1}\delta_1} \ .$$

But since $f_n \downarrow 0$ a.e., it follows that $\mu(\theta^{-1}\delta_1) = 0$ and therefore that $\nu(\delta_1) = 0$. Hence in fact

$$\inf_n \Pi_1 f_n = 0 \quad \text{a.e.}$$

and Π_1 is order continuous.

11.10 The order continuous extension of Π.

As the mapping $\Pi_1 : M_\mu \to M_\nu$ is in essence a lifting into M_μ of the measure algebra isomorphism θ, it is becoming evident that the Riesz homomorphism Π has a strongly geometric character. This is best illustrated in the case where $1 \in L_\rho$ since then, with $\varphi_o = \Pi 1$, we obtain from equation (11) that

$$\Pi s = \varphi_o \Pi_1 s$$

whenever s is a simple function in L_ρ and since we can approximate any $h \in L_\rho^+$ by an increasing sequence of simple functions each in L_ρ^+, it follows by the order continuity of Π and Π_1 that

$$\Pi h = \varphi_o \Pi_1 h$$

whenever $h \in L_\rho$. Since pointwise multiplication by a non-negative measurable function is order continuous, the mapping

$$\Pi_e : f \mapsto \varphi_o \Pi_1 f \qquad (f \in M_\mu)$$

is an order continuous extension of Π. Moreover since L_ρ is order dense in M_μ, such an extension is necessarily unique.

With this in mind, we pass to the more general case where no assumption is made about the norm of 1. Let $\{\Omega_n\}$ be a sequence of mutually disjoint elements of Σ, whose union is Ω, such that $\chi_{\Omega n} \in L_\rho$ for each n. If $\epsilon_n = \theta\Omega_n$, then by Theorem 11.5, $\bigcup_n \epsilon_n = \epsilon$. Let $\varphi_i = \Pi\chi_{\Omega i}$: by Prop. 11.1', supp $\varphi_i = \mathcal{E}_i$. Let $\varphi_o = \sup_i \varphi_i$: then the set $\{\varphi_o = \infty\}$ is at most a countable union of ν-null sets, hence is also null, so $\varphi_o \in M_\nu$ and

$$\text{supp } \varphi_o = \bigcup_n \text{supp } \varphi_n = \bigcup_n \mathcal{E}_n = \mathcal{E} \ ;$$

thus $\varphi_o > 0$ a.e. So now for $f \in M_\mu$, define

$$\Pi_e f = \varphi_o \Pi_1 f \ . \tag{13}$$

Since both Π_1 and the multiplication M_{φ_o} are order continuous Riesz isomorphisms on M_μ, so is Π_e. Suppose $\sigma \in \Sigma$ with $\chi_\sigma \in L_\rho$: if $\sigma \subseteq \Omega_i$ for some i, then $\theta\sigma \subseteq \mathcal{E}_i$ and

$$\Pi_e \chi_\sigma = \varphi_o \Pi_1 \chi_\sigma = \varphi_o \chi_{\theta\sigma} = \varphi_i \chi_{\theta\sigma} = \Pi \chi_\sigma \quad ;$$

in general,

$$\begin{aligned} \Pi_e \chi_\sigma &= \Pi_e \left(\sum_{i=1}^{\infty} \chi_{\sigma \cap \Omega_i} \right) \\ &= \varphi_o \Pi_1 \left(\sum_{i=1}^{\infty} \chi_{\sigma \cap \Omega_i} \right) \\ &= \varphi_o \sum_i \chi_{\theta(\sigma \cap \Omega_i)} && \text{(by order continuity of } \Pi_1) \\ &= \sum_i \varphi_i \chi_{\theta(\sigma \cap \Omega_i)} && \text{(by disjointness of } \theta(\sigma \cap \Omega_i)) \\ &= \sum_i \Pi \chi_{\sigma \cap \Omega_i} \\ &= \Pi \chi_\sigma \quad . \end{aligned}$$

By linearity we have that $\Pi_e s = \Pi s$ for simple functions $s \in L_\rho$. If $f \in L_\rho^+$ and the simple functions s_n increase to f a.e., then

$$\Pi_e f = \sup_n \Pi_e f_n = \sup_n \Pi f_n = \Pi f \ ,$$

using, in turn, the order continuity of Π and Π_e. It follows that Π_e as defined in (13), does indeed extend Π and by the order density of L_ρ in M_μ it does so uniquely. Note, finally, that $\varphi_o = \Pi_e 1$ and that Prop. 11.1' generalises further:

<u>11.11 LEMMA</u>. If $\sigma \in \Sigma$ and $u \in M_\mu$ with supp $u = \sigma$, then supp $\Pi_e u = \theta\sigma$.

The inverse map $\Pi^{-1} : L_\tau \to L_\rho$ is also a Riesz isomorphism and hence has a unique order continuous extension $(\Pi^{-1})_e : M_\nu \to M_\mu$. In fact,

<u>11.12 PROPOSITION</u>. $(\Pi^{-1})_e = \Pi_e^{-1}$.

<u>Proof</u>. If $g \in L_\tau$, then $\Pi_e^{-1} g = f$ only if $g = \Pi_e f = \Pi f$. Hence $\Pi_e^{-1}|_{L_\tau} = \Pi^{-1}$, i.e. Π_e^{-1} is an extension of Π^{-1}. Suppose $g_n \in M_\nu$ $(n = 1,2,\ldots)$ and $g_n \downarrow 0$ a.e. Let $\Pi_e^{-1} g_n = f_n \in M_\mu$ and $f_o = \inf\limits_n f_n$. Then

$\Pi_e f_o = \inf\limits_n \Pi_e f_n = \inf\limits_n g_n = 0.$

By uniqueness therefore, Π_e^{-1} is precisely $(\Pi^{-1})_e$.

Not surprisingly, the measure algebra isomorphism underlying Π^{-1} and Π_e^{-1} is precisely θ^{-1}.

<u>11.13 LEMMA</u>. If $\sigma \in \Sigma$ and $\theta\sigma = \delta \in \mathcal{E}$, then supp $\Pi_e^{-1} \chi_\delta = \sigma$.

<u>Proof</u>. Let $f \in M_\mu$ with supp $f = \sigma$; let $g = \Pi_e f$ so that supp $g = \delta$. Then by lemma 11.11,

$$\text{supp } \Pi_e^{-1} \chi_\delta = \text{supp } \Pi_e^{-1} g = \text{supp } f = \sigma = \theta^{-1}\delta .$$

If we denote by $(\Pi^{-1})_1$, the Riesz isomorphism of M_ν onto M_μ, derived from θ^{-1} as was Π_1 from θ, then the preceding lemma may be equivalently restated as follows.

<u>11.13' LEMMA</u>. $(\Pi^{-1})_1 = (\Pi_1)^{-1}$.

<u>Proof</u>. Let $\delta = \theta\sigma \in \Lambda$. Then, by 11.13, $(\Pi^{-1})_1\chi_\delta = \chi_{\theta^{-1}\delta}$, so $\Pi_1(\Pi^{-1})_1\chi_\delta = \Pi_1\chi_\sigma = \chi_\delta$ and the result then follows easily.

Define $\psi_o = \Pi_e^{-1}1$ (here 1 denotes χ_ϵ). Then for all $g \in M_\nu$,

$$\Pi_e^{-1}g = \psi_o\Pi_1^{-1}g \quad .$$

Our final observations in this section relate φ_o and ψ_o, and yield an alternative description of Π and Π_e to the form given by (13).

<u>11.14 PROPOSITION</u>.

(a) $\Pi_1\psi_o = \varphi_o^{-1}$ ν-a.e. ; $\Pi_1^{-1}\varphi_o = \psi_o^{-1}$ μ-a.e.

(b) If $f \in M_\mu$ then

$$\Pi_e f = \varphi_o\Pi_1 f = \Pi_1(f\psi_o^{-1}) \quad .$$

<u>Proof</u>.

(a) Let $\sigma \in \Sigma$ be arbitrary and let $\theta\sigma = \delta$. Then,

$$\chi_\sigma = \Pi_e^{-1}\Pi_e\chi_\sigma = \Pi_e^{-1}(\varphi_o\chi_\delta) = (\Pi_e^{-1}\varphi_o)\chi_{\theta^{-1}\delta} = (\psi_o\Pi_1^{-1}\varphi_o)\chi_\sigma \quad .$$

Now let $\delta \in \Lambda$ be arbitrary and let $\theta^{-1}\delta = \gamma$. Then,

$$\chi_\delta = \Pi_e\Pi_e^{-1}\chi_\delta = \Pi_e(\psi_o\chi_\gamma) = (\Pi_e\psi_o)\chi_{\theta\gamma} = (\varphi_o\Pi_1\psi_o)\chi_\delta \quad .$$

The result follows easily. Note that at the third step of each sequence of equalities, we use the obvious extension of Lemma 9.4.

(b) Using Lemma 11.13',

$$\Pi_e f = \Pi_e(\psi_o(f\psi_o^{-1})) = \Pi_e(\psi_o \Pi_1^{-1}\Pi_1(f\psi_o^{-1}))$$

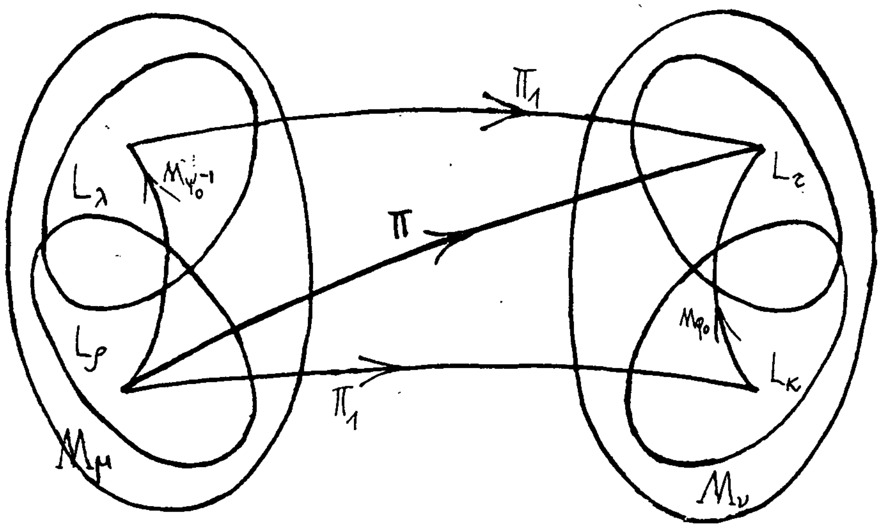
$$= \Pi_e \Pi_e^{-1}(\Pi_1(f\psi_o^{-1})) = \Pi_1(f\psi_o^{-1}) \ .$$

Thus, from Prop. 11.14 (b), Π_e can be composed, either as a multiplication (M_{φ_o}) on M_ν following a measure algebraic transformation (Π_1) or as a multiplication ($M_{\psi_o^{-1}}$) on M_μ followed by the (same) measure algebraic transformation. The net and composite actions of Π_e on L_ρ are depicted in the following commutative diagram, where we define the Banach function norms λ and κ on M_μ and M_ν respectively by

$$\lambda(f) = \rho(\psi_o f) \qquad (f \in M_\mu) \ ;$$

$$\kappa(g) = \rho(\Pi_1^{-1} g) \qquad (g \in M_\nu) \ ,$$

so that $L_\lambda = \{\psi_o^{-1}h : h \in L_\rho\}$ and $L_\kappa = \{\Pi_1 h : h \in L_\rho\}$.



$\Pi = \Pi_e|_{L_\rho}$; $\Pi_e = M_{\varphi_o} \circ \Pi_1 = \Pi_1 \circ M_{\psi_o^{-1}}$.

11.15 Remark. In the case where Ω and $\mathcal{E}$ are separable complete metric spaces and Σ and Λ are the σ-algebras of their respective Borel subsets, then in fact the measure algebra θ is derived from a pointwise isomorphism (non-uniquely of course) of the sets Ω, $\mathcal{E}$. This fact depends on a theorem given by P. Billingsley in [B].

11.16 Definition. Let Π be a Riesz isomorphism of L_ρ onto L_τ and Π_e its extension to M_μ. We call Π, Π_e unitary if they are isometric with respect to the norm $\|\cdot\|_\infty$, where we admit $+\infty$ as a possible value for the ess sup norm by defining

$$\|f\|_\infty = \infty \qquad (f \in M_\mu \setminus L^\infty(\mu)) ,$$

and similarly in M_ν.

Note that if Π is unitary, $\Pi(L_\rho \cap L^\infty(\mu)) = L_\tau \cap L^\infty(\nu)$.

11.17 LEMMA. Let Π_e be a Riesz isomorphism of M_μ onto M_ν. The following statements are equivalent:

(a) Π_e is unitary;

(b) Π_e maps characteristic functions to characteristic functions.

Proof. We keep the notation Π_e, θ as earlier.

(a) $\Rightarrow$ (b). Let $\sigma \in \Sigma$. By hypothesis, $0 \leq |\Pi_e\chi_\sigma| \leq 1$ a.e. Let $\delta = \theta\sigma = \operatorname{supp} \Pi_e\chi_\sigma$ and suppose there is a non-null subset δ_1 of δ with

$$(\Pi_e\chi_\sigma)\chi_{\delta_1} \leq \alpha\chi_{\delta_1} \tag{14}$$

a.e., for some $0 < \alpha < 1$. Since Π_e is onto,

$(\Pi_e\chi_\sigma)\chi_{\delta_1} = \Pi_e f$ for some $f \in M_\mu^+$. Let $\sigma_1 = \operatorname{supp} f$. By Lemma 11.11, $\theta\sigma_1 = \operatorname{supp} \Pi_e f = \delta_1 \subseteq \delta$. Hence, since θ is a measure algebra isomorphism

$$\sigma_1 = \theta^{-1}\delta_1 \subseteq \theta^{-1}\delta = \sigma .$$

By Lemma 9.4

$$\Pi_e\chi_{\sigma_1} = (\Pi_e\chi_\sigma)\chi_{\theta\sigma_1} = (\Pi_e\chi_\sigma)\chi_{\delta_1} = \Pi_e f .$$

Since Π_e is one-one, $f = \chi_{\sigma_1}$ a.e., so by hypothesis $\|\Pi_e f\| = \|f\|_\infty = 1$. But this contradicts (14). So in fact $\Pi_e\chi_\sigma \geq 1$ a.e. on its support and hence finally we have $\Pi_e\chi_\sigma = 1\cdot\chi_{\theta\sigma} = \chi_\delta$ a.e.

<u>(b) $\Rightarrow$ (a)</u>. Let $f \in M_\mu^+ \cap L^\infty(\mu)$ and let $\sigma = \operatorname{supp} f$. By hypothesis, $\Pi_e\chi_\sigma = \chi_{\theta\sigma}$. Let $a = \|f\|_\infty$; then $0 \leq f \leq a\chi_\sigma$ a.e., so $0 \leq \Pi_e f \leq a\Pi\chi_\sigma = a\chi_{\theta\sigma}$ a.e. Hence

$$\|\Pi_e f\|_\infty \leq a \quad .$$

Suppose that $\|\Pi_e f\|_\infty = b \lneq a$. Let $\gamma = \{\frac{a+b}{2} \leq f \leq a\}$. By definition of the norm $\|\cdot\|_\infty$, γ is a non-null set and we have

$$\Pi_e(f\chi_\gamma) \geq \frac{a+b}{2}\,\Pi_e\chi_\gamma \gneq b\Pi_e\chi_\gamma = b\chi_{\theta\gamma} .$$

But $\Pi_e(f\chi_\gamma) \leq \Pi_e f \leq b$ a.e. From this contradiction it follows that

$$\|\Pi_e f\|_\infty = a = \|f\|_\infty$$

as required.

In the case where $\|f\|_\infty = \infty$, a similar type of argument, considering the sets $\gamma_M = \{M \leq f \leq M+1\}$ $(M \in \mathbb{N})$ shows that $\|\Pi_e f\|_\infty = \infty$. Thus the result follows.

From the remarks following Prop. 11.14, we saw that the most general Riesz isomorphism of M_μ onto M_ν is a composition of a unitary isomorphism and a pointwise multiplication.

Note 1. Π_e is purely unitary if and only if $\Pi_e 1 = 1$ a.e.

Proof. If $\Pi_e 1 = 1$ a.e., then, by Lemma 9.4, for each $\sigma \in \Sigma$,

$$\Pi_e \chi_\sigma = (\Pi_e 1)\chi_{\theta\sigma} = \chi_{\theta\sigma} .$$

So by Lemma 11.17, Π_e is unitary.

Conversely if Π_e is unitary, then $\Pi_e 1 = \chi_{\mathcal{E}'}$ for some $\mathcal{E}' \subseteq \mathcal{E}$. Let $\delta \subseteq \mathcal{E} \setminus \mathcal{E}'$ be Λ-measurable. By the characterisation given in Lemma 11.17, Π_e^{-1} must also be unitary. Hence $\chi_\delta = \Pi_e \chi_\sigma$ for some $\sigma \in \Sigma$. However, since $\chi_\sigma \leq 1$ a.e., $\chi_\delta = \Pi_e \chi_\sigma \leq \Pi_e 1 = \chi_{\mathcal{E}'}$ a.e., so $\delta \subseteq \mathcal{E}'$. Thus δ a null set and $\Pi_e 1 = \chi_{\mathcal{E}}$.

Note 2. Π_e is a pure multiplication (in the case where $(\Omega, \Sigma) = (\mathcal{E}, \Lambda)$) if and only if

$$\operatorname{supp} \Pi_e f = \operatorname{supp} f \qquad (f \in M_\mu) .$$

Proof. If $\operatorname{supp} \Pi_e f = \operatorname{supp} f$ for every $f \in M_\mu$, then for each $\sigma \in \Sigma$, $\theta\sigma = \sigma$ and

$$\Pi_e \chi_\sigma = (\Pi_e 1)\chi_{\theta\sigma} = \varphi_o \chi_\sigma .$$

By linearity, $\Pi_e s = \varphi_o s$ for every simple function s, and by order continuity, $\Pi_e f = \varphi_o f$ for every $f \in M_\mu$.

Conversely suppose that $\Pi_e f = \varphi_o f$ $(f \in M_\mu)$ for some $\varphi_o \in M_\mu$. Since Π_e is a Riesz isomorphism we must have $\varphi_o > 0$ a.e. and then clearly $\operatorname{supp} \Pi_e f = \operatorname{supp} f$.

§ 12. The Associated Homomorphism.

Let Π be once again a Riesz homomorphism of L_ρ onto L_τ, with kernel carried on $A \in \Sigma$. It is an elementary exercise in lattice theory to show that the adjoint of any surjective lattice homomorphism of a lattice L is also a lattice homomorphism, when L* is endowed with the usual algebraic dual lattice structure (namely, that whereby F is in the positive cone of L* if $\langle f, F\rangle \geq 0$ for every $f \in L^+$). Moreover this adjoint is always an order continuous mapping.

When we identify with the Banach function space L'_τ, that subspace of L^*_τ which consists of the order continuous linear functionals on L_τ, the restriction of Π^* to this subspace is then a Riesz homomorphism on L'_τ. The question then poses itself naturally of what conditions enable us to apply to this homomorphism some of the results developed so far. The first two lemmas of this section give an answer to this question.

<u>12.1 LEMMA</u>. If $H \in L^*_\tau$ and $\Pi^*H \in L'_\rho$, then $H \in L'_\tau$, i.e.

$$\Pi^*(L^*_\tau) \cap L'_\rho \subseteq \Pi^*(L'_\tau) \quad .$$

<u>Proof</u>. Suppose $0 \leq f = \Pi^*H \in L'_\rho$ and the sequence $\{v_n\}$ in L_τ satisfies $v_1 \geq v_2 \geq \ldots \downarrow 0$ a.e. By Lemma 9.3, we can find a decreasing sequence $\{u_n\}$ in L^+_ρ with $\Pi u_n = v_n$ $(n = 1,2,\ldots)$. We may assume without loss that $\inf_n u_n = 0$ a.e. (for if $0 \leq u_o \leq u_n$ $(n = 1,2,\ldots)$, then $\Pi u_o \leq \Pi u_n$ $(n = 1,2,\ldots)$; hence $\Pi u_o \leq \inf_n \Pi u_n = 0$, i.e. $u_o \in \ker \Pi$; so replace each u_n if necessary by $u_n - \inf_r u_r$). Then, since $f \in L'_\rho$, f is an order continuous linear functional on L_ρ, so

$$\langle v_n, H\rangle = \langle \Pi u_n, H\rangle = \langle u_n, \Pi^* H\rangle = \langle u_n, f\rangle \to 0$$

showing that $H \in L'_\tau$.

<u>12.2 LEMMA</u>. $\Pi^*(L'_\tau) \subseteq L'_\rho$ if and only if Π is order continuous.

<u>Proof</u>. Suppose first that Π is order continuous and let $0 \leq v \in L'_\tau$. Let the sequence $\{f_n\}$ in L_ρ satisfy $f_1 \geq f_2 \geq \ldots \downarrow 0$ a.e. Then $\{\Pi f_n\}$ is decreasing in L_τ and $\inf_n \Pi f_n = \Pi(\inf_n f_n) = 0$, so, since v is an order continuous linear functional,

$$\inf_n \langle f_n, \Pi^* v\rangle = \inf_n \langle \Pi f_n, v\rangle = 0 ,$$

and hence $\Pi^* v \in L'_\rho$.

Suppose now that Π is not order continuous: then we can find $h \in L_\rho$ and a sequence $\{h_n\}$ satisfying

$$0 \leq h_n \uparrow h \quad \text{a.e.} \quad \text{but} \quad \Pi h \gneq \sup_n \Pi h_n .$$

So $\delta = \operatorname{supp}(\Pi h - \sup_n \Pi h_n)$ is non-null, and choosing some $0 \leq v \in L'_\tau$ whose support is non-trivial and contained in δ, we have,

$$\langle \Pi h - \sup_n \Pi h_n, v\rangle = \int_\delta (\Pi h - \sup_n \Pi h_n) v \, d\mu > 0 .$$

So $\langle h, \Pi^* v\rangle = \langle \Pi h, v\rangle \gneq \sup_n \langle \Pi h_n, v\rangle = \sup_n \langle h_n, \Pi^* v\rangle$. Hence $\Pi^* v$ is not order continuous, i.e. $\Pi^* v \notin L'_\rho$.

It follows by these two lemmas that if $\Pi : L_\rho \to L_\tau$ is order continuous, then $\Pi^* : L^*_\tau \to L^*_\rho$ satisfies

$$\Pi^*(L'_\tau) = L'_\rho \cap \Pi^*(L^*_\tau) ,$$

and so $\tilde{\Pi} = \Pi^*|_{L'_\tau}$ is an order continuous Riesz homomorphism of L'_τ into L'_ρ. We can make this more precise:

12.3 LEMMA. If $\Pi : L_\rho \to L_\tau$ is order continuous, then $\tilde{\Pi}(L'_\tau) = \chi_{\Omega\setminus A}L'_\rho$.

Proof. Let $0 \leq v \in L'_\tau$: from Lemma 12.2, $\tilde{\Pi}v \in L'_\rho$. For any $f \in L_\rho$, $\langle f, (\tilde{\Pi}v)\chi_A\rangle = \langle f\chi_A, \Pi^*v\rangle = \langle \Pi(f\chi_A), v\rangle = 0$, by Cor. 10.2. Hence $(\tilde{\Pi}v)\chi_A = 0$ a.e., so $\tilde{\Pi}(L'_\tau) \subseteq \chi_{\Omega\setminus A}L'_\rho$.
Conversely, let $0 \leq g = g\chi_{\Omega\setminus A} \in L'_\rho$. Define $H \in L^*_\tau$ by

$$\langle v, H\rangle = \int (\Pi_c^{-1}v)g\, d\mu \qquad (v \in L_\tau) ,$$

where $\Pi_c^{-1}v$ is the fundamental inverse of v introduced in Lemma 10.3, i.e. $\Pi_c^{-1}v = w\chi_{\Omega\setminus A}$ for any w such that $\Pi w = v$. Since the mapping Π_c^{-1} is positive, linear and bounded (by (3)), H is a bounded positive linear functional and

$$\|H\| \leq \|\Pi_c^{-1}\| \; \rho'(g) \leq c_1^{-1}\rho'(g) .$$

Furthermore, if $v_1 \geq v_2 \geq \ldots \downarrow 0$ $(v_i \in L_\tau;\ i = 1,2,\ldots)$, then the sequence $\{\Pi_c^{-1}v_i\}$ also decreases to zero a.e. Hence, since $g \in L'_\rho$,

$$\langle v_i, H\rangle = \int (\Pi_c^{-1}v_i)g\, d\mu \to 0$$

as $i \to \infty$. So H is order continuous and may therefore be identified with some element $h \in L'_\rho$ such that $\rho'(h) \leq c_1^{-1}\rho'(g)$. Finally, for any $u \in L_\rho$,

$$\langle u, \tilde{\Pi}h\rangle = \langle \Pi u, H\rangle = \langle \Pi_c^{-1}\Pi u, g\rangle = \langle u\chi_{\Omega\setminus A}, g\rangle = \langle u, g\rangle ,$$

since $g = g\chi_{\Omega\setminus A}$. Hence $g = \tilde{\Pi}h$ and so $\tilde{\Pi}$ maps L'_τ onto $\chi_{\Omega\setminus A}L'_\rho$.

<u>12.4 PROPOSITION</u>. For any $g \in L'_\tau$,

$$c_1\tau'(g) \leq \rho'(\tilde{\Pi}g) \leq c_2\tau'(g) \quad ,$$

where c_1, c_2 are the constants of (1).

<u>Proof</u>. For any $G \in L^*_\tau$,

$$\begin{aligned} \|\Pi^*G\| &= \sup\{|\langle f, \Pi^*G\rangle| : \rho(f) \leq 1\} \\ &= \sup\{|\langle \Pi f, G\rangle| : \rho(f) \leq 1\} \\ &\leq \sup\{\tau(\Pi f)\,\|G\| : \rho(f) \leq 1\} \\ &\leq c_2\,\|G\| \quad , \end{aligned}$$

where $\|\cdot\|$ is used to denote the Banach dual norm both in L^*_ρ and in L^*_τ. So when G is some $g \in L'_\tau$, $\rho'(\tilde{\Pi}g) = \|\Pi^*G\| \leq c_2\|G\| = c_2\tau'(g)$. Also, if $0 \leq g \in L'_\tau$, then there exists a sequence $\{f_n\}$ in L_τ such that $\tau(f_n) \leq 1$ and

$$\langle f_n, g\rangle = \int f_n g \, d\nu \uparrow \tau'(g)$$

as $n \to \infty$. Let $h_n = \Pi_c^{-1} f_n$, so that $h_n = h_n\chi_{\Omega\setminus A}$ $(n = 1,2,\ldots)$. Then from (2), $\rho(h_n) = \inf\{\rho(h) : \Pi h = f_n\}$; so $c_1\rho(h_n) \leq \tau(f_n) \leq 1$, i.e. $\rho(h_n) \leq c_1^{-1}$ for each n, and

$$\langle h_n, \tilde{\Pi}g\rangle = \langle \Pi h_n, g\rangle = \langle f_n, g\rangle \uparrow \tau'(g) \quad .$$

Hence, $\sup\{\langle h, \tilde{\Pi}g\rangle : \rho(h) \leq 1\} \geq c_1\tau'(g)$, i.e. as required,

$$\rho'(\tilde{\Pi}g) \geq c_1\tau'(g) \quad .$$

Since Π is onto, clearly Π^* is one-one, so $\tilde{\Pi}$ is a Riesz isomorphism of L'_τ to $\chi_{\Omega\setminus A}L'_\rho$. Now the latter is a Banach function space in its own right: if we redefine $\mu(A)$ to be zero, and ρ' to be based on the accordingly modified measure space, then the restricted ρ' is a saturated norm.

The results of § 11 can now be applied to deduce that $\tilde{\Pi}$ has a unique order continuous extension $\tilde{\Pi}_e$, which is a Riesz isomorphism of M_ν onto $\chi_{\Omega\setminus A}M_\mu$. Let $\tilde{\theta}$ denote the underlying measure algebra isomorphism of Λ onto $\Sigma_{\Omega\setminus A} = \{\sigma \setminus A : \sigma \in \Sigma\}$, and let $\xi_o = \tilde{\Pi}_e 1$ (here 1 denotes $\chi_{\mathcal{E}}$): then supp $\xi_o = \Omega\setminus A$ and for each $\delta \in \Lambda$,

$$\tilde{\Pi}_e \chi_\delta = \xi_o \chi_{\tilde{\theta}\delta} \ .$$

An order continuous mapping $\tilde{\Pi}_1$ of M_ν onto $\chi_{\Omega\setminus A}M_\mu$ arises naturally from $\tilde{\theta}$, as did Π_1 from θ in § 11, and we can write

$$\tilde{\Pi}_e g = \xi_o \tilde{\Pi}_1 g \qquad (g \in M_\nu) \ .$$

Our final result describes, albeit implicitly, the structural connection between Π and $\tilde{\Pi}$. Take Π_1 and θ to be derived (as in §11) from the isomorphic component $\Pi_c = \Pi|_{\chi_{\Omega-A}L_\rho}$ of Π.

<u>12.5 THEOREM</u>. Let $0 \le f \in M_\mu$, and let $\sigma, \gamma \in \Sigma$ and $\delta \in \Lambda$ satisfy $\theta\sigma = \delta$ and $\tilde{\theta}\delta = \gamma$. Then,

$$\int_\sigma \xi_o f \, d\mu = \int_\delta \varphi_o \Pi_1 f \, d\nu = \int_\gamma \xi_o \tilde{\Pi}_1(\Pi_1 f) d\mu \ . \tag{15}$$

In particular, with $f = 1$ a.e., we obtain

$$\int_\sigma \xi_o \, d\mu = \int_\delta \varphi_o \, d\nu = \int_\gamma \xi_o \, d\mu \ . \tag{16}$$

<u>Proof</u>. We prove (16) for simplicity, but (15) follows very similarly. Suppose first that $\chi_\sigma \in L_\rho$. Choose a sequence $\mathcal{E}_n \uparrow \mathcal{E}$ with $\chi_{\mathcal{E}_n} \in L'_\tau$ for each n. Then $\tilde{\Pi}\chi_{\mathcal{E}_n} \in L'_\rho$, so $(\tilde{\Pi}\chi_{\mathcal{E}_n})\chi_\sigma \in L^1(\mu)$ for $n = 1,2,\ldots$, and by the Monotone Convergence Theorem (MCT),

$$\int_\sigma \tilde{\Pi}\chi_{\epsilon_n}\, d\mu \uparrow \int_\sigma \sup_n \tilde{\Pi}\chi_{\epsilon_n}\, d\mu = \int_\sigma \tilde{\Pi}_e 1\, d\mu \ . \tag{17}$$

Hence,

$$\begin{aligned}
\int_\sigma \xi_o\, d\mu &= \sup_n \int_\sigma \tilde{\Pi}\chi_{\epsilon_n}\, d\mu && \text{(from (17))} \\
&= \sup_n \langle \chi_\sigma, \tilde{\Pi}\chi_{\epsilon_n} \rangle \\
&= \sup_n \langle \Pi\chi_\sigma, \chi_{\epsilon_n} \rangle \\
&= \sup_n \int_{\epsilon_n} \varphi_o \chi_{\theta\sigma}\, d\nu \\
&= \int_{\theta\sigma} \varphi_o\, d\nu \ . && \text{(by MCT.)}
\end{aligned}$$

More generally, for $\sigma \in \Sigma$, we can find a sequence $\sigma_n \uparrow \sigma$ with $\chi_{\sigma_n} \in L_\rho$ for each n, and so, by two further applications of MCT,

$$\begin{aligned}
\int_\sigma \xi_o d\mu = \sup_n \int_{\sigma_n} \xi_o d\mu &= \sup_n \int_{\theta\sigma_n} \varphi_o d\nu \\
&= \int_{\bigcup_n \theta\sigma_n} \varphi_o d\nu = \int_{\theta\sigma} \varphi_o d\nu \ .
\end{aligned}$$

The second equality in (16) is obtained analogously.

<u>Note</u>. The equations (15) will hold for any $f \in M_\mu^r$ provided that at least one of $\int_\sigma \xi_o f^+ d\mu$ and $\int_\sigma \xi_o f^- d\mu$ is finite, and indeed they hold for any $f \in M_\mu$, provided always that the integrals in (15) do exist.

§ 13. Applications.

13.1 EXAMPLE. Consider the case where ρ and τ are both based on the same measure space (Ω, Σ, μ) and where Π is a Riesz isomorphism of L_ρ onto L_τ such that the underlying measure algebra isomorphism θ fixes Σ, i.e. $\theta\sigma = \sigma$ for every $\sigma \in \Sigma$. Then the action of Π and of its extension Π_e on M_μ is a pure multiplication, i.e.

$$\Pi_e f = \varphi_o \Pi_1 f = \varphi_o f \qquad (f \in M_\mu) .$$

As usual we let $\varphi_o = \Pi_e 1$, $\psi_o = \Pi_e^{-1} 1$ and it follows from Prop. 11.14 (a), since $\Pi_1 f = f$ for every $f \in M_\mu$, that $\varphi_o = \psi_o^{-1}$ a.e.

Denote by η the associated isomorphism $\tilde{\Pi}$ of L'_τ onto L'_ρ, and let $\xi_o = \eta_e 1$. We shall show that

(a) $\xi_o = \varphi_o$ a.e., and

(b) for each $\sigma \in \Sigma$, $\tilde{\theta}\sigma = \sigma$ so that for each $f \in M_\mu$,

$$\eta_e f = \xi_o f = \varphi_o f = \Pi_e f ,$$

i.e. $\eta_e = \Pi_e$.

Proof.

(a) Let $\sigma \in \Sigma$ with $\chi_\sigma \in L'_\tau$ and let $f \in L_\rho^+$. Then $\Pi(f\chi_\sigma) = (\Pi f)\chi_{\theta\sigma} = \varphi_o f \chi_\sigma$ and

$$\int_\sigma \varphi_o f \, d\mu \leq \tau(\varphi_o f)\tau'(\chi_\sigma) < \infty .$$

Let $\Omega_n \uparrow \Omega$ in Σ with $\chi_{\Omega n} \in L'_\tau$ $(n = 1,2,\ldots)$. Then

$$\begin{aligned} \int_\sigma \varphi_o f \, d\mu &= \int_\Omega \Pi(f\chi_\sigma) d\mu \\ &= \sup_n \int_{\Omega n} \Pi(f\chi_\sigma) d\mu \qquad \text{(by MCT)} \\ &= \sup_n \langle \Pi(f\chi_\sigma), \chi_{\Omega n} \rangle \end{aligned}$$

$$= \sup_n \langle f\chi_\sigma, n\chi_{\Omega_n} \rangle$$

$$= \sup_n \int_\sigma (n\chi_{\Omega_n}) f \, d\mu$$

$$= \int_\sigma (\eta_e 1) f \, d\mu \qquad \text{(by MCT and def.of } \eta_e\text{)}$$

$$= \int_\sigma \xi_o f \, d\mu \, .$$

Since τ' is a saturated norm, it follows that $\varphi_o = \xi_o$ a.e. on Ω.

(b) Now let $\sigma \in \Sigma$ with $\chi_\sigma \in L_\rho \cap L'_\rho \cap L'_\tau$ so that, in particular, $\mu(\sigma) \leq \rho(\chi_\sigma)\rho'(\chi_\sigma) < \infty$, and let $\tilde{\theta}\sigma = \gamma$. By the Exhaustion Theorem ([Z], 67.3), choose a sequence $\Omega_n \uparrow \Omega$ in Σ such that $\chi_{\Omega_n} \in L'_\tau$ and $\varphi_o^{-1}\chi_{\Omega_n} \in L_\rho$ for each n; then

$$\mu(\sigma) = \int_\Omega \varphi_o\varphi_o^{-1}\chi_\sigma^2 \, d\mu$$

$$= \int_\Omega \Pi_e(\varphi_o^{-1}\chi_\sigma)\chi_\sigma \, d\mu$$

$$= \int \sup_n \Pi(\varphi_o^{-1}\chi_{\sigma\cap\Omega_n})\chi_\sigma \, d\mu \qquad \text{(by def. of } \Pi_e\text{)}$$

$$= \sup_n \int \Pi(\varphi_o^{-1}\chi_{\sigma\cap\Omega_n})\chi_\sigma \, d\mu \qquad \text{(by MCT)}$$

$$= \sup_n \int \varphi_o^{-1}\chi_{\sigma\cap\Omega_n}(\eta\chi_\sigma) d\mu$$

$$= \sup_n \int \varphi_o^{-1}\chi_{\sigma\cap\Omega_n} \xi_o\chi_\gamma \, d\mu$$

$$= \int_{\sigma\cap\gamma} 1 \, d\mu \qquad \text{(from (a))}$$

$$= \mu(\sigma \cap \gamma) \, .$$

Hence $\sigma \subseteq \gamma$. On the other hand,

$$\mu(\gamma) = \int \varphi_o^{-1}\varphi_o\chi_\gamma \, d\mu$$

$$= \int \varphi_o^{-1}\eta_e(\chi_\sigma) d\mu$$

$$= \int \varphi_o^{-1} \sup_n \eta(\chi_{\sigma\cap\Omega_n}) d\mu$$

$$= \sup_n \int \varphi_o^{-1} \eta(\chi_{\sigma\cap\Omega_n}) d\mu \qquad \text{(by MCT)}$$

$$= \sup_n \sup_m \int \varphi_o^{-1} \chi_{\Omega_m} \eta(\chi_{\sigma\cap\Omega_n}) d\mu$$

$$= \sup_{n,m} \int_{\sigma\cap\Omega_n} \Pi(\varphi_o^{-1} \chi_{\Omega_m}) d\mu$$

$$= \sup_{n,m} \int_{\sigma\cap\Omega_n} \chi_{\Omega_m} d\mu$$

$$= \sup_{n,m} \mu(\sigma \cap \Omega_n \cap \Omega_m)$$

$$= \mu(\sigma) .$$

Hence $\sigma = \gamma$, and so $\tilde{\theta}\sigma = \sigma$ whenever $\chi_\sigma \in L_\rho \cap L'_\rho \cap L'_\tau$; since each of ρ, ρ', τ' is a saturated norm, it follows that $\tilde{\theta}\sigma = \sigma$ for every $\sigma \in \Sigma$. Hence

$$\eta f = \varphi_o f = \Pi f \qquad \text{for} \quad f \in L_\rho \cap L'_\tau$$

and $$\eta_e f = \varphi_o f = \Pi_e f \qquad \text{for all} \quad f \in M_\mu .$$

We now return briefly to Theorem 6.2 as promised.

13.2. Let $E(\cdot)$, $F(\cdot)$ be spectral measures on the Banach spaces X, Z respectively, such that for some elements x_o of X and z_o of Z,

$$X = \overline{\text{lin}} \{E(\sigma)x_o : \sigma \in \Sigma\}$$

and $$Z = \overline{\text{lin}} \{F(\delta)z_o : \delta \in \Lambda\} ,$$

where $E(\cdot)$, $F(\cdot)$ are of class (Σ, X^*), (Λ, Y^*) respectively. From the Representation Theorem (Chapter II) we know there exist saturated function norms ρ and τ, based on finite measure spaces (Ω, Σ, μ) and $(\mathcal{E}, \Lambda, \nu)$ say, where

$$1_\Omega \in L^a_\rho \cap L'_\rho \quad \text{and} \quad 1_{\mathcal{E}} \in L^a_\tau \cap L'_\tau ,$$

and such that X and Z may be identified isomorphically with the ideals L^a_ρ and L^a_τ respectively. From this identification, each of X and Z inherits a Riesz space structure.

Suppose X is Riesz isomorphic to Z*.

Define $\rho_1(f) = \begin{cases} \rho(f) & \text{if } f \in L^a_\rho , \\ \infty & \text{otherwise} . \end{cases}$

Then ρ_1 is also a saturated norm based on (Ω, Σ, μ), $L_{\rho_1} = L^a_{\rho_1} = L^a_\rho$ and $1 \in L_{\rho_1} \cap L'_{\rho_1}$. (Note that since the carrier of L^a_ρ is Ω — i.e. L^a_ρ is order dense in L_ρ — then $\rho'_1 = \rho'$ (see 2.4)). Since $1 \in L^a_\tau$, $(L^a_\tau)^* = L'_\tau$ and hence our hypothesis is equivalent to the hypothesis: L_{ρ_1} is Riesz isomorphic to L'_τ. Letting $\Pi : L_{\rho_1} \to L'_\tau$ denote this isomorphism, then from the results of § 11 there exist a positive function $\varphi_o \in L'_\tau$ and a measure algebra isomorphism θ of Σ onto Λ such that

$$\Pi 1_\Omega = \varphi_o$$

and

$$\Pi \chi_\sigma = \varphi_o \chi_{\theta\sigma} \qquad (\sigma \in \Sigma) .$$

Since ρ_1 is an absolutely continuous norm, it follows from Lemma 9.6 that $L'_\tau = L^a_{\tau'}$, and since τ' has the Fatou property, then by Theorem 5.1, L'_τ is weakly sequentially complete. By the isomorphism, L_{ρ_1} is also weakly sequentially complete and therefore, again by Theorem 5.1, ρ_1 has the Fatou property. This implies that whenever $u \in L_\rho$, $u_n \in L_\rho^a$ $(n = 1,2,\ldots)$ and $0 \leq u_n \uparrow u$ a.e., then

$$\rho_1(u) = \sup_n \rho_1(u_n) = \sup_n \rho(u_n) \leq \rho(u) < \infty ,$$

i.e. $u \in L_{\rho_1} = L_\rho^a$; hence L_ρ^a is order closed. So in fact, since its carrier is Ω, $L_\rho^a = L_\rho$ and $\rho_1 = \rho$. Thus, ρ has the Fatou property. Now observe that letting $\simeq$ denote Riesz isomorphism,

$$L_\rho = L_\rho^a \simeq Z^* \simeq (L_\tau^a)^* = L_\tau' = L_{\tau'}^a \quad ;$$

so $L_\rho' = (L_\rho^a)^* \simeq (L_{\tau'}^a)^* = L_\tau'' = L_\tau$. Since Π is an isomorphism of L_ρ onto L_τ', then $\tilde{\Pi} = \Pi^*|_{L_\tau''}$ is an isomorphism of $L_\tau'' = L_\tau$ onto L_ρ', and $\tilde{\Pi}|_{L_\tau^a}$ is an isomorphism of L_τ^a onto $L_{\rho'}^a$. Since L_τ^a is order dense in L_τ, it follows that $L_{\rho'}^a$ is order dense in L_ρ', i.e. that the carrier of $L_{\rho'}^a$ is Ω.

It is an easy exercise to verify that the hypothesis of Riesz isomorphism between X and Z^* is equivalent to the hypothesis of Theorem 6.2, viz. that provided we identify X with L_ρ^a as above from the outset, then the adjoints of projections $F(\delta)$ $(\delta \in \Lambda)$ on Z should correspond, under the given isomorphism of X and Z^*, to the natural multiplication operators on L_ρ^a. Hence in 13.2 above, we have an alternative proof of the forward implication of Theorem 6.2.

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