# ON PERIOD DOUBLING BIFURCATIONS

# AND

# ON COMPACT ANALYTIC SEMIGROUPS

by

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Dedicated

To

My Mother and Father

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# DECLARATION

This thesis has been composed by myself and has not been submitted for any other degree or professional qualification. The work is my own, except where otherwise indicated.

## ABSTRACT

The contents of this thesis may be divided into two main branches; the first one deals with a detailed analysis of period doubling bifurcations of the Henon map and a short account of this bifurcation of the Duffing equation, and the second one consists of the study of the relationship of compact analytic semigroups (CAS) with the hermitian approximation property (HAP) and with the metric approximation property (MAP) on a separable Banach space.

Section One of Chapter One provides some numerical devices to evaluate periodic points and bifurcation values (and their limiting values) of the Henon map, $H_{M,B}(x,y) = (1 - Mx^2 + y, Bx)$. Section Two deals with the smoothness of the curve $M_\infty = M_\infty(B)$, $B \in (-\infty,\infty)$ and suggests that this curve is only $C^\alpha$ smooth, $\alpha = \log_2\tilde{\delta}$, $\tilde{\delta} = 8.721097200 \ldots$ near $B = \pm 1$ and $C^\infty$ smooth otherwise. In Section Three we develop some geometric insight which enables us to explore the domain of attraction of a periodic orbit, and the existence of a homoclinic point and a Horseshoe for higher values of $B$ and smaller values of $M$.

In Chapter Two, we describe some computational methods for finding periodic points and period doubling bifurcation values of the Duffing Equation. The numerical methods developed and the results obtained in Chapter One and Chapter Two suggest very wide scope of conducting analogous study in other ordinary differential equations.

Chapter Three establishes mainly three facts, viz. (1) a separable Banach space having HAP possesses a CAS, $a^t$, $t \in H$ satisfying

the relations $(T_1)\,(a^t X)^- = X$ and $(T_2)\ \| a^t \| \leq 1$ for every $t$ in $H$, (2) the converse of (1) is false in general and (3) a separable Banach space having MAP also has a CAS, $a^t$, $t \in H$ satisfying the relations $(T_1)\ (a^t X)^- = X$ for all $t$ in $H$ and $(T_3)\ \| a^t \| \leq 1$ for every $t$ in $\mathbb{R}^+$. The converse of (3) is unknown at this moment, and we close this chapter by citing it as an open problem.

# CONTENTS

# CHAPTER ONE

# PERIOD DOUBLING BIFURCATIONS OF THE HENON MAP

## 1.0 Introduction

This chapter is primarily concerned with the answers of four questions connected with the Henon map. Henon in [34] has introduced and studied a remarkable map, $H_{M,B}(x,y) = (1 - Mx^2 + y, Bx)$, (where $M,B$ are parameters), which is a canonical form for the most general quadratic maps of the plane into itself, having constant Jacobians, (here equal to $-B$ everywhere). Feigenbaum in [24] and [26] has studied numerically a number of one-parameter families, such as $M \to 1 - Mx^2$ ($x \in [-1,1]$ and $M \in [0,2]$) and discovered that period doubling bifurcations occur in such systems. After his exciting discovery, many authors (refer to [12], [18], [29] and [56]) have extended his theory to higher dimensional systems. Amongst them, Derrida et al. in [18] has computed with good accuracy period doubling bifurcations in the Henon map with the parameter value $B = 0.3$, and Collet et al. in [12] has discussed in great detail this bifurcation theory for one parameter families of analytic maps from $\mathbb{C}^n$ to $\mathbb{C}^n$, whose restriction to $\mathbb{R}^n$ is real. Moreover Hitzl in [36], and Hitzl and Zele in [37] have extensively studied this theory for the Henon map and displayed for a few cases the boundary curves in the $M - B$ parameter plane, of the occurrence of period doubling bifurcations. Now a natural question arises, "If the period doubling bifurcation values of a parameter in a higher dimensional system having at least two parameters occur, can we find a suitable numerical method to calculate them for all values of the other parameter(s) lying in a certain domain?" In Section One, an affirmative answer to this question in the case of the Henon map is discussed and very effective numerical methods are devised to find a periodic point and a bifurcation value for every value of $B$ in $(-\infty, \infty)$. We are also hopeful that our method would work to some extent with the other higher dynamical systems (see Remark 1.1.6). Again Quispel in [51] and [52] has shown analytically the existence of bifurcation values for all values of $B$ and derived an approximate analytical expression of $M_n(B)$. However his formula does not give very accurate results and our method gives much more accurate results than those given by his formula.

Next, if $M_\infty(B)$ is the limit of the bifurcation values $M_n(B)$ for each $B$, then $M_\infty$ is a function of $B$, that is, $M_\infty = M_\infty(B)$. Obviously this curve is continuous. Now the second question is, "How smooth is this curve?" Section Two deals with the study of this curve, and our numerical explorations here suggest that this curve is $C^\alpha$, $\alpha = \log_2 \tilde{\delta}$, $\tilde{\delta} = 8.721097200 \ldots$, near $B = \pm 1$ and $C^\infty$ otherwise.

Since to find a periodic point depends extremely sensitively on the initial conditions, we can raise the third question as, "How big is the domain of attraction of a periodic orbit?" In Section Three some graphical pictures are presented to illustrate this question. The Henon map has two fixed points whose coordinates are given by

$x = \frac{(B-1) \pm \sqrt{(1-B)^2 + 4M}}{2M}$, $y = Bx$. When both of these two points become unstable, (one is always unstable), for some specific parameter values, then the pictorial behaviour of the stable manifold at one of these two points indicates that the domain of attraction of a periodic orbit lies in a region whose boundary is mostly formed by this manifold.

Section Three is also devoted to some discussion of the existence of homoclinic points and of a Smale Horseshoe. Marotto [49] has shown analytically the existence of a transversal homoclinic point for small values of $B$ and some appropriate values of $M$, say if $M > 1.55$. Curry in his paper [14] has also suggested the existence of such a point for $B = 0.3$ and $M = 1.4$. Later, Misiurewicz and Szewc in [50] have proved rigorously that there does exist a transversal homoclinic point for $B = 0.3$ and $M = 1.4$. Now the fourth question can be put as, "Does there exist a transversal homoclinic point for a higher value of $B$ and a smaller value of $M$?" This question has also an affirmative answer. We present this fact for $B = 0.8$ (and for $B = 0.35$ briefly) and $M = 0.9$. Eventually we show the existence of a Horseshoe for these parameter values.

Now before laying out the plan of our main study, we want to state some definitions and elementary results which are needed for our study.

<u>Definition 1.0.1, Diffeomorphisms</u>. Let $A$ and $B$ be two subsets of $\mathbb{R}^n$. A $C^k$-differmorphism $f: A \to B$ is a mapping $f$ which is one-to-one, onto and has the property that both $f$ and $f^{-1}$ are k-times differentiable.

<u>Definition 1.0.2, Flows.</u> Let us consider a system of differential equation as

$$\frac{d\underline{x}}{dt} \overset{\text{def}}{=} \dot{\underline{x}} = f(\underline{x}) , \tag{1}$$

where $\underline{x} = \underline{x}(t) \in \mathbb{R}^n$ is a vector valued function of an independent variable (usually time) and $f: U \to \mathbb{R}^n$ is a smooth function defined on some subset $U \subseteq \mathbb{R}^n$. The vector field $f$ is said to generate a flow $\phi_t: U \to \mathbb{R}^n$, where $\phi_t(\underline{x}) = \phi(\underline{x},t)$ is a smooth function defined for all $\underline{x}$ in $U$ and $t$ in some interval $I = (a,b) \subseteq \mathbb{R}$, $(0 \in I)$, and $\phi$ satisfies (1) in the sense that

$$\frac{d}{dt}(\phi(\underline{x},t))\Big|_{t=\tau} = f(\phi(\underline{x},\tau))$$

for all $\underline{x} \in U$ and $\tau \in I$. Moreover $\phi_t$ satisfies the group properties (i) $\phi_o = id$ and (ii) $\phi_{t+s} = \phi_t \circ \phi_s$. If an initial condition $\underline{x}(0) = \underline{x}_o \in U$ is given, then we seek a solution $\phi(\underline{x}_o,t)$ such that $\phi(\underline{x}_o,0) = \underline{x}_o$. In this case, $\phi(\underline{x}_o, \cdot): I \to \mathbb{R}^n$ defines a solution curve, trajectory, or orbit of the differential equation (1) based at $\underline{x}_o$.

<u>Definition 1.0.3, Stable And Unstable Periodic Points.</u> Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. A point $\underline{x}$ in $\mathbb{R}^n$ is called a fixed point of $f$ if $f(\underline{x}) = \underline{x}$. A fixed point $\underline{x}$ is said to be stable if for every neighbourhood $U$ of $\underline{x}$, there exists a neighbourhood $V$ of $\underline{x}$ whose images $f^k(V)$ lie in $U$ for all positive integers $k$. Otherwise $\underline{x}$ is known as unstable. A periodic orbit of $f$ is a finite sequence of distinct points each

of which is the image of the previous one, and whose first point is the image of the last. Its period $m$ is the number of points in the sequence, which are called periodic points of period $m$. Fixed points can be included under this definition as periodic points of period one. A periodic orbit of period $m$ (or $m$-cycle) is said to be stable or unstable according as each of its points is stable or unstable when considered as a fixed point of $f^m$. By continuity, they are all stable or unstable together, so it is sufficient to examine the stability of one of them.

Lemma 1.0.4. A sufficient condition for a periodic point $\underline{x}$ of period $k$ for a diffeomorphism $f$ to be stable is that the eigenvalues of the derivative $Df^k(\underline{x})$ are less than one in absolute value.

The proof is simple and so, omitted.

Definition 1.0.5, Hyperbolic Periodic Points. A periodic point $\underline{x}$ of period $k$ for a diffeomorphism $f$ is called hyperbolic if the derivative $Df^k(\underline{x})$ at $\underline{x}$ has no eigenvalues of absolute value 1. It is easy to check that if $\underline{x}$ is a hyperbolic periodic point of $f$, then it is also a hyperbolic periodic point of $f^{-1}$ and vice-versa.

Definition 1.0.6, Bifurcations. The systems of physical interest typically have parameters which appear in the defining systems of equations. As these parameters are varied, changes may occur in the qualitative structure of the solutions for certain parameter values. These changes are called bifurcations and the parameter values are called bifurcation values (or bifurcation points).

In the case of a diffeomorphism $f$, period doubling bifurcations (or flip bifurcations or subharmonic bifurcations) occur when one of the eigenvalues of the derivative $Df^k(\underline{x})$ equals -1.

Definition 1.0.7, The Domain of Attraction for A Periodic Point. Let $\underline{x}$ be an attracting periodic point of period $k$ for a diffeomorphism $f$. Then the set, $D = \{y \in \mathbb{R}^n \mid (f^k)^m(\underline{y}) \to \underline{x} \text{ as } m \to \infty\}$, is the domain of attraction for $\underline{x}$ .

Definition 1.0.8, Stable and Unstable Manifolds. We define the stable manifold of a hyperbolic periodic point $\underline{x}$ having period $k$ as the set of points $\underline{y}$ for which $(f^k)^m(\underline{y}) \to \underline{x}$ as $m \to \infty$. Also the unstable manifold of a hyperbolic periodic point $\underline{x}$ is defined as the set of all points $\underline{z}$ for which $(f^k)^m(\underline{z}) \to \underline{x}$ as $m \to -\infty$. The stable and unstable manifolds of a periodic point $\underline{x}$ are denoted by $W^s(\underline{x})$ and $W^u(\underline{x})$ respectively.

Definition 1.0.9, Homoclinic and Heteroclinic Points. Let $\underline{x}$ be a hyperbolic periodic point for a diffeomorphism $f$. Then a point $p \in W^s(\underline{x}) \cap W^u(\underline{x}) - \{\underline{x}\}$ is called a homoclinic point. If the intersection of $W^s(\underline{x})$ and $W^u(\underline{x})$ at $P$ is transverse, the homoclinic point is called transverse. Similarly, if the stable manifold $W^s(\underline{x})$ at a hyperbolic periodic point $\underline{x}$ intersects the unstable manifold $W^u(\underline{y})$ at another hyperbolic periodic point $\underline{y}$ transversally at some point $q$, say, this point $q$ is then said to be transversally heteroclinic.

Definition 1.0.10, $C^\alpha$ ($\alpha$ is a positive real number) Smooth Curve; Holder's Continuity.

A mapping $f: \mathbb{R} \to \mathbb{R}$ is said to be $C^\alpha$-smooth if $f$ is n-times differentiable and $f^{(n)}$ is Holder's continuous with the exponent $\mu$ in the sense that $|f^{(n)}(x) - f^{(n)}(y)| \leqslant C|x-y|^\mu$, where $C$ is a positive constant, $n$ is the largest non-negative integer less than $\alpha$ and $\mu = \alpha - n$.

Definition 1.0.11, Poincaré Maps. Let $\gamma$ be a periodic orbit of some flow $\phi_t$ in $\mathbb{R}^n$ arising from a non-linear vector field $f(\underline{x})$. A cross-section to $\gamma$ at $\underline{x}$ is a submanifold $\Sigma \subset \mathbb{R}^n$ which has codimension 1 and intersects $\gamma$ transversally at $\underline{x}$. Let $U \subseteq \Sigma$ be some neighbourhood of $\underline{x}$. Then the first return or the Poincaré map $P: U \to \Sigma$ is defined for a point $\underline{y} \in U$ by setting $P(\underline{y})$ to be the point $\phi_\tau(\underline{y})$ where $\tau$ is chosen to be the smallest positive number such that $\phi_\tau(\underline{y}) \in \Sigma$. Clearly $\underline{x}$ is a fixed point for the map $P$. The criterion for hyperbolicity is that its derivative $DP(\underline{x})$ has no eigenvalue of absolute value one.

Definition 1.0.12, Hyperbolic Sets. Let $\Lambda$ be a closed invariant set for a differmorphism $f$ defined on $\mathbb{R}^n$. $\Lambda$ is called hyperbolic for $f$ if there is a continuous invariant direct sum decomposition $T_\Lambda \mathbb{R}^n = E^u_\Lambda \oplus E^s_\Lambda$ with the property that there are constants $c > 0$, $0 < \lambda < 1$ such that:

(i) if $v \in E^u_{\underline{x}}$, $\underline{x} \in \Lambda$, then $|Df^{-n}(\underline{x})v| \leqslant c\lambda^n|v|$ ;

(ii) if $v \in E^s_{\underline{x}}$, $\underline{x} \in \Lambda$, then $|Df^n(\underline{x})v| \leqslant c\lambda^n|v|$ ,

where $T_\Lambda \mathbb{R}^n$ consists of all the tangent vectors to $\mathbb{R}^n$ at all points of $\Lambda$, and for each $\underline{x} \in \Lambda$, $T_{\underline{x}}\mathbb{R}^n$ is the tangent space at $\underline{x}$ and $T_{\underline{x}}\mathbb{R}^n = E^u_{\underline{x}} \oplus E^s_{\underline{x}}$ is a direct sum splitting of this vector space into subspaces of dimensions $n_u$ and $n_s$ $(n_u + n_s = n)$. Moreover, here

$$E^u_{\underline{x}} = \text{Span}\ \{u^1, u^2, \ldots, u^{n_u}\}$$

and

$$E^s_{\underline{x}} = \text{Span}\ \{v^1, v^2, \ldots, v^{n_s}\}$$

, where $u^1, u^2, \ldots, u^{n_u}$ are the $n_u$ eigenvectors whose eigenvalues are greater than one in absolute value and $v^1, v^2, \ldots, v^{n_s}$ are $n_s$ eigenvectors whose eigenvalues are less than one in modulus.

<u>Definition 1.0.13, (Smale) Horseshoes.</u> Let $\Lambda$ be a hyperbolic set for a diffeomorphism, $f: \mathbb{R}^n \to \mathbb{R}^n$. Let $\Sigma_2 = \{1,2\}^{\mathbb{Z}}$ be the set of bi-infinite sequences of 1's and 2's with the product (compact-open) topology, and write elements of $\Sigma_2$ by $\underline{x}$, where $\underline{x}(i) = 1$ or $2$ and $i \in \mathbb{Z}$. Define the shift map $\sigma: \Sigma_2 \to \Sigma_2$ by $\sigma(\underline{x})(i) = \underline{x}(i+1)$; that is, $\sigma$ shifts a sequence $\underline{x}$ one step to the left. If there exists a homeomorphism $h: \Lambda \to \Sigma_2$ such that $hf = \sigma h$ or $hfh^{-1} = \sigma$, then the set $\Lambda$ is called a Horseshoe and $f$ is called a Horseshoe map.

## 1.1 <u>Section One</u>: <u>Numerical Methods and Evaluations</u>

Our chief aim in this section is to demonstrate some numerical algorithms in order to find periodic points and to obtain the limiting value $M_\infty$ of the sequence of period doubling bifurcation values of $M$ for every $B$ lying in $(-\infty,\infty)$.

### 1.1.1 Feigenbaum Theory For Period Doubling Bifurcations Of The Henon Map

Before embarking upon our exposition of the numerical methods and results, we wish to broach the beautiful phenomenon of the sequence of period doubling bifurcations appearing with the Henon map for $|B| \leq 1$. Some intuitive pictures of these phenomena are depicted later. In this context, we also wish to point out that the stability theory is intimately connected with the Jacobian matrix of the map, and that the trace of the Jacobian matrix is the sum of its eigenvalues and the product of the eigenvalues equals the Jacobian determinant. For a particular value of $B$ in the closed interval $[-1,1]$, the Henon map $H$ depends on the real parameter $M$, and so a fixed point $\underline{x}_o$ ( or a periodic point $\underline{x}_o$) of this map depends on the parameter value $M$, that is, $\underline{x}_o = \underline{x}_o(M)$. Now, first consider the open interval $I_1 = (-\frac{1}{4}(1 - B)^2, \frac{3}{4}(1 - B)^2)$. The fixed point $\underline{x}_o$ remains stable for all values of $M$ lying in this interval and a stable periodic trajectory of period one appears around it, (for more technical details, refer to [37]). This means, the two eigenvalues of the Jacobian matrix $J = \begin{pmatrix} -2Mx & 1 \\ B & 0 \end{pmatrix}$ at $\underline{x}_o$ remain less than one in modulus, and as a result all the neighbouring points (that is, points in the domain of attraction) are attracted towards $\underline{x}_o(M)$, $M$ lying in $I_1$. Again some negative values of $B$ for which $M$ lies in the region sandwiched between the boundary curves $M = -B \pm (1 - B)\sqrt{-B}$ yield complex eigenvalues for the Jacobian $J_1$. This region is exhibited in Fig. 1 by the striped lines between the curves $M_o$ and $M_1$. The significance of complex eigenvalues is that the successive iterations of the map spiral into the stable fixed point, and that of real

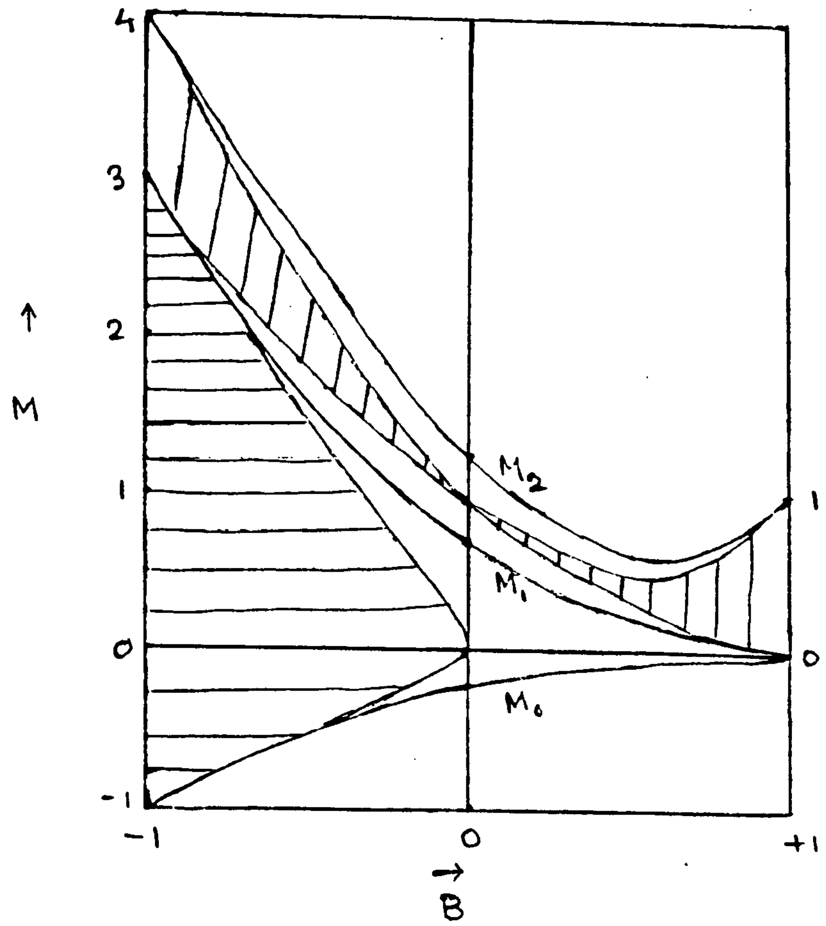


Fig.1: Striped regions indicate the existence of complex eigenvalues for periods 1 and 2 .

eigenvalues is that consecutive iterations approach the stable fixed point along the direction of the eigenvector corresponding to the higher eigenvalue in modulus. If we now begin to increase the value of M, then it happens that one of the eigenvalues starts decreasing through -1 and the other remains less than one in modulus, because their product is always equal to (-B). When M equals $\frac{3}{4}(1-B)^2$, one of the eigenvalues becomes -1 and then $\underline{x}_o$ loses its stability,

$M_1 = \frac{3}{4}(1 - B)^2$ emerging as the first bifurcation value of M. Again if we keep increasing the value M, the point $\underline{x}_o(M)$ becomes unstable and there arises around it two points, say, $\underline{x}_{21}(M)$ and $\underline{x}_{22}(M)$ forming a stable periodic trajectory of period 2. All the neighbouring points except the stable manifold of $\underline{x}_o(M)$ are attracted towards these two points and this phenomenon continues for all M lying in the open interval $I_2 = (\frac{3}{4}(1-B)^2, \frac{1}{4}(1+B)^2 + (1-B)^2)$. Since the period as emerged becomes double, the previous eigenvalue which was -1 becomes +1 and as we keep increasing M, one of the eigenvalues starts decreasing from +1 to -1. The values of M in $I_2$ for which we obtain the inequality $[2(1-B)^2 - 2M + B]^2 - B^2 < 0$ (see the relation (19) in [37]) give complex eigenvalues for the Jacobian $J_2$. The shaded portion in Fig. 1 between the curves $M_1$ and $M_2$ shows this region. Since the trace is always real, when eigenvalues are complex they are conjugate to each other moving along the circle of radius $\sqrt{B_e}$, where $B_e = B^{2^n}$ is the effective Jacobian, in the opposite directions as shown in Fig. 2.

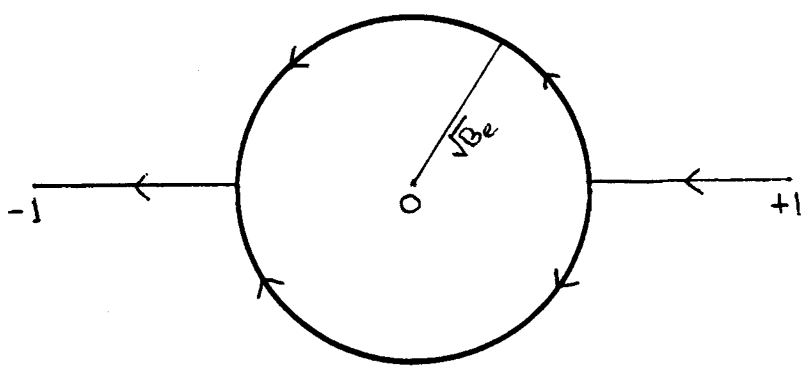


Fig. 2.

When we reach $M = \frac{1}{4}(1+B)^2 + (1-B)^2$, we find that one of the eigenvalues of the Jacobian of $H^2$ (because of the Chain rule of differentiation, it does not matter at which periodic point one evaluates the eigenvalues) becomes $-1$, indicating the loss of stability of the periodic trajectory of period two. Thus, the second bifurcation takes place at this value $M_2$ of M. We can then repeat the same arguments, and find that the periodic trajectory of period 2 becomes unstable and a periodic trajectory of period 4 appears in its neighbourhood. This phenomenon continues up to a particular value of M, say $M_3(B)$, at which the periodic trajectory of period 4 loses its stability in such a way that one of the eigenvalues at any of its periodic points becomes $-1$, and thus it gives the third bifurcation at $M_3(B)$. For any period doubling trajectory, when eigenvalues run from $+1$ to $-1$ they show the similar behaviour as represented by Fig. 2.

Increasing the value M further and further, and repeating the same arguments we obtain a sequence $\{M_n(B)\}$ (see Figs. 3 and 4) as bifurcation values for the parameter M such that at $M = M_n(B)$ a periodic trajectory of period $2^n$ arises and all periodic trajectories of period $2^m (m < n)$ remain unstable. The sequence $\{M_n(B)\}$ behaves in a universal manner such that $M_\infty(B) - M_n(B) \sim C\delta^{-n}$, where C is a constant and $\delta$ is the Feigenbaum Universal constant. Since the Henon map has constant Jacobian $-B$, $|B| < 1$ gives the dissipative case, that is, contraction of area and in this case $\delta$ equals 4.6692016091029 ... . For $|B| = 1$ we have the conservative case, i.e. the preservation of area and in this case $\delta$ equals 8.721097200 ... . Furthermore, the Feigenbaum theory says that the Henon map H at $M = M_\infty(B)$ has an invariant set F of Cantor type encompassed by infinitely many unstable periodic orbits of period $2^n$

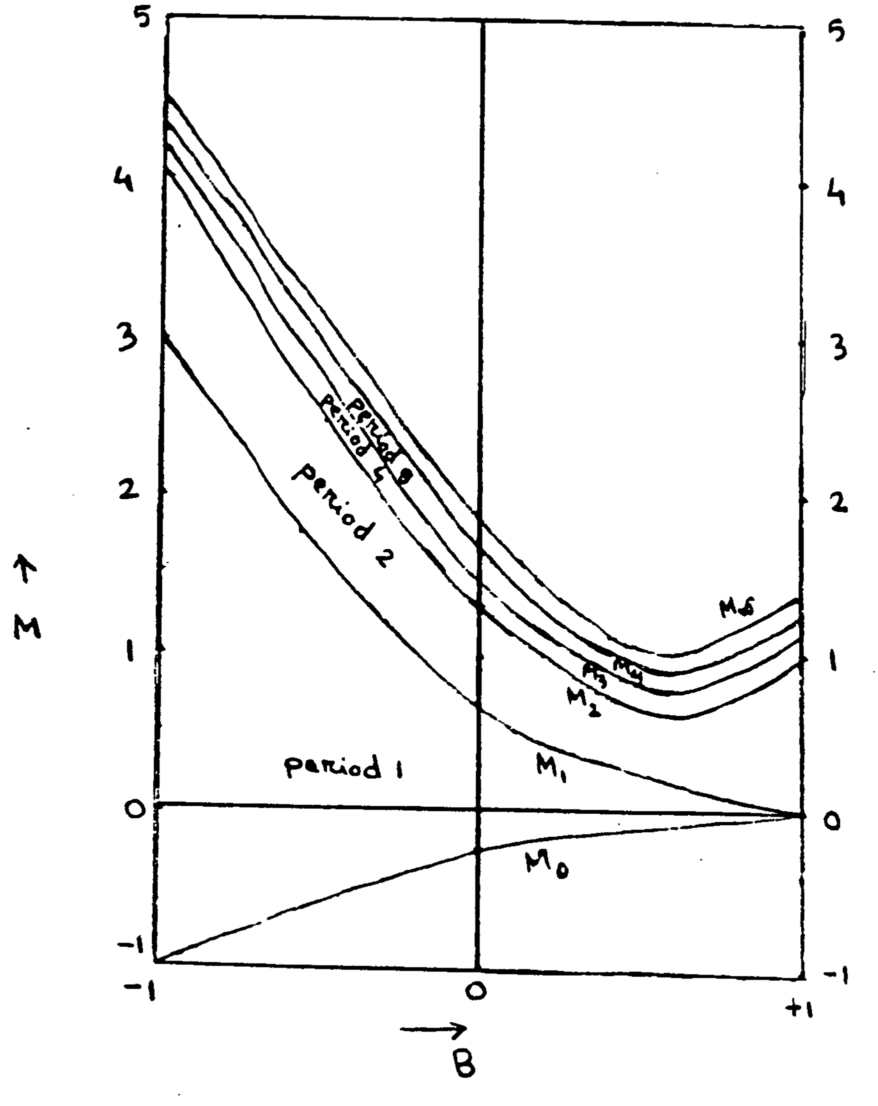


Fig.3:A Putative picture of the regions of different periods .

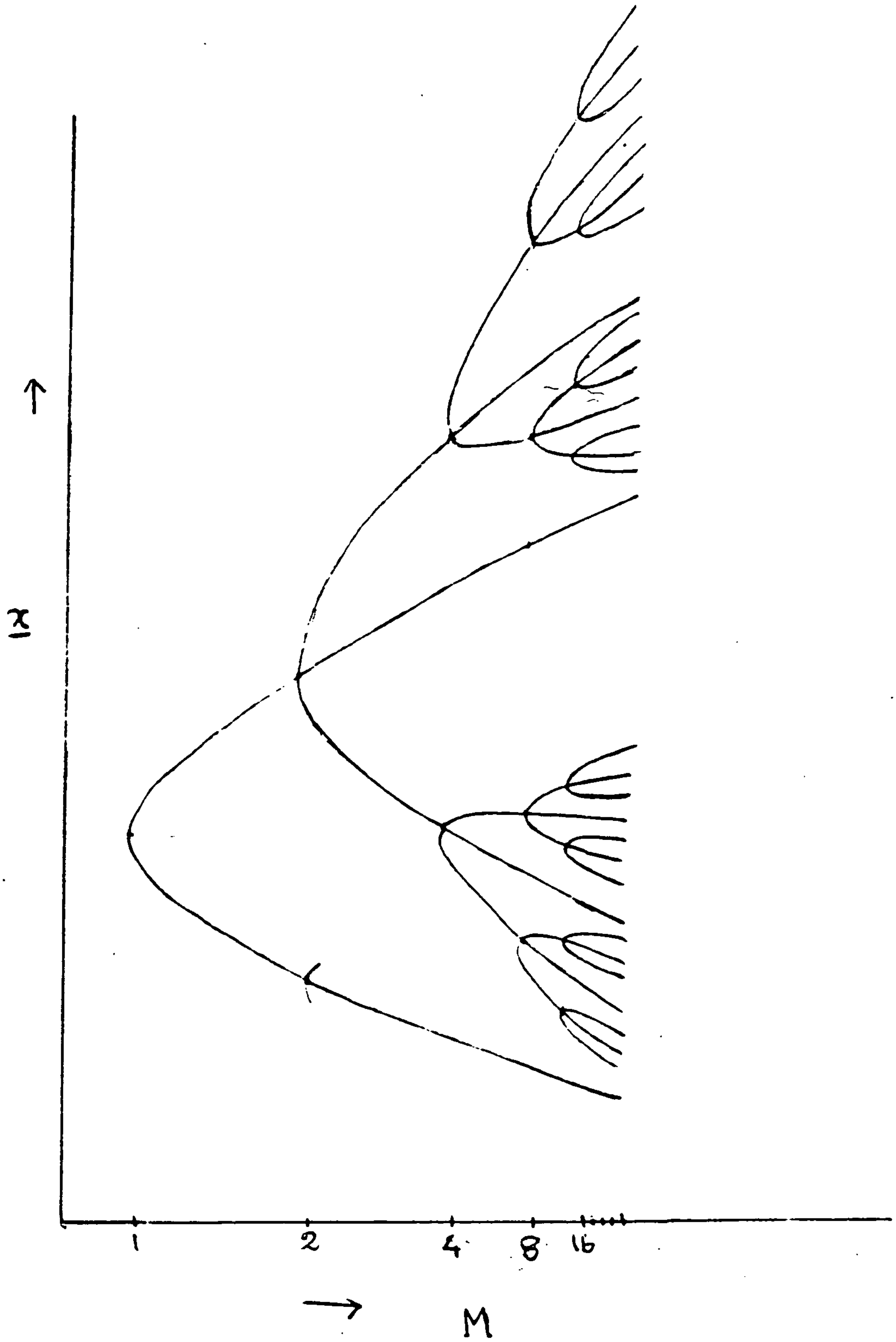


Fig.4:A typical sequence of period doubling bifurcations .

, $(n = 0,1,2, \ldots)$, and that all the neighbouring points except those belonging to these unstable orbits and their stable manifolds are attracted to $F$ under the iterations of $H_{M_\infty,B}$.

The case when $|B| > 1$.

When $|B| > 1$, the Jacobian of the map $H^k$, $k = 2^n$, $n = 0,1,2,3, \ldots$ is always greater than one and so we have expansion of area for this map. Consequently a trajectory of any period $k$ is always unstable. However we can obtain bifurcations for the repelling orbit of period $k$ because of the following reasons.

We have

$$H_{M,B}(x,y) = (1 - Mx^2 + y, Bx) .$$

Again,
$$H^{-1}_{M,B}(x,y) = (B^{-1}y, -1 + x + B^{-2}My^2) .$$

Now, consider a homeomorphism $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by
$T(x,y) = (-y, -x)$ .

Then

$$\begin{aligned} & T^{-1} \circ H^{-1}_{M,B} \circ T(x,y) \\ &= T^{-1} \circ H^{-1}_{M,B}(-y, -x) \\ &= T^{-1}(-B^{-1}x, -1-y + B^{-2}Mx^2) \\ &= (1+y - B^{-2}Mx^2, B^{-1}x) \\ &= H_{B^{-2}M,\, B^{-1}} . \end{aligned}$$

Therefore, $H_{M,B}^{-1}$ is topologically conjugate to $H_{B^{-2}M,B^{-1}}$. Since the topological conjugacy preserves all the dynamical structure of $H_{M,B}$ and of $H_{M,B}^{-1}$, the above conjugacy implies that the B-values with absolute value greater than one do not exhibit new behaviour and that whenever $H_{M,B}$ has a stable orbit of period $k$, then $H_{M,B}^{-1}$ or $H_{B^{-2}M,B^{-1}}$ has a repelling orbit of the same period. In addition, this conjugacy gives the relation $M_k(B^{-1}) = B^{-2}M_k(B)$. Hence whenever the bifurcation values for all B with $|B| < 1$ are available, this relation gives the bifurcation values for all B with $|B| > 1$. Therefore, it establishes the remarkable result that the bifurcation values for the Henon map, of any period $k$, can be obtained for each value of B lying in $(-\infty, \infty)$.

We now wish to describe some suitable numerical methods with the help of which two important ingredients, namely a periodic point $\underline{x}$ of the map for a period $k$ and the bifurcation values of the parameter M for different $k$ can be obtained.

### 1.1.2 Numerical Methods For Finding A Periodic Point

To find a periodic point of the Henon map H (for simplicity, we write H, instead of $H_{M,B}$) for the period $k$, we can apply the following three numerical methods.

(i) the Newton recurrence formula,

$$\underline{x}_{n+1} = \underline{x}_n - Df(\underline{x}_n)^{-1} f(\underline{x}_n), \quad \text{where } n = 0,1,2, \ldots ,$$

and $(Df)(\underline{x})$ is the Jacobian of the map f at the vector $\underline{x}$. (We see later that this map f is equal to $H^k - I$ in our case.)

(ii) the averaging iteration method.

(iii) the direct iteration method.

(i) The Newton formula actually gives the zero(es) of a map, and to apply this numerical tool in the Henon map one needs a number of recurrence formulae which are given below.

Let the initial point be $(x_o, y_o)$. Then

$$H(x_o,y_o) = (1 - Mx_o^2 + y_o,\ Bx_o) = (x_1,\ y_1),$$

where $x_1 = 1 - Mx_o^2 + y_o$ and $y_1 = Bx_o$.

$$H^2(x_o,y_o) = H(x_1,\ y_1) = (x_2,\ y_2),$$ where $x_2 = 1 - Mx_1^2 + y_1$

and $y_2 = Bx_1$.

Proceeding in this manner, the following recurrence formulae for the Henon map can be established.

$$x_n = 1 - Mx_{n-1}^2 + y_{n-1} \quad \text{and} \quad y_n = Bx_{n-1}, \quad \text{where} \quad n = 1,2,3,4, \ldots .$$

Since the Jacobian of $H^n$, (n time iterations of the Henon map), is the product of the Jacobian of each iteration of the map, we proceed as follows to describe our recurrence mechanism for the Jacobian matrix.

The Jacobian $J_1$ for the transformation $H(x_o,y_o) = (1 - Mx_o^2 + y_o,\ Bx_o)$ is

$$J_1 = \begin{pmatrix} -2Mx_o & 1 \\ B & 0 \end{pmatrix} = \begin{pmatrix} D_1 & E_1 \\ N_1 & T_1 \end{pmatrix} = \begin{pmatrix} L_1 & E_1 \\ N_1 & T_1 \end{pmatrix},$$

where, $L_1 = D_1 = -2Mx_o$, $E_1 = 1$, $N_1 = B$ and $T_1 = 0$. Next the Jacobian $J_2$ for the transformation $H^2(x_o,y_o) = (x_2,y_2)$, where

$x_2$ and $y_2$ are as mentioned above, is the product of the Jacobians for the transformations $H(x_1,y_1) = (1 - Mx_1^2 + y_1, Bx_1)$ and $H(x_o,y_o) = (1 - Mx_o^2 + y_o, Bx_o)$. So we obtain

$$J_2 = \begin{pmatrix} -2Mx_1 & 1 \\ B & 0 \end{pmatrix} \cdot \begin{pmatrix} L_1 & E_1 \\ N_1 & T_1 \end{pmatrix} = \begin{pmatrix} L_2 & E_2 \\ N_2 & T_2 \end{pmatrix}$$

, where $D_2 = -2Mx_1$, $L_2 = D_2L_1+N_1$, $E_2 = D_2E_1+T_1$, $N_2 = BL_1$ and $T_2 = BE_1$.

Continuing the process in this way, we have the Jacobian for $H^m$ as

$$J_m = \begin{pmatrix} L_m & E_m \\ N_m & T_m \end{pmatrix}$$ with a set of recursive formulae as

$D_m = -2Mx_{m-1}$, $L_m = D_mL_{m-1} + N_{m-1}$, $E_m = D_mE_{m-1} + T_{m-1}$, $N_m = BL_{m-1}$, and $T_m = B\ E_{m-1}$, $m = 2,3,4, \ldots$ (in addition to the particular initial values $L_1 = D_1 = -2Mx_o$, $N_1 = B$, $E_1 = 1$ and $T_1 = 0$). Since a fixed point of the map $H$ is a zero of the map $H'(x,y) = H(x,y) - (x,y)$, the Jacobian of $H'^k$ is given by

$$J_k - I = \begin{pmatrix} L_k-1 & E_k \\ N_k & T_k-1 \end{pmatrix},$$ where $L_k, E_k, N_k, T_k$ are as mentioned above. So its inverse is $(J_k-I)^{-1} = \frac{1}{\Delta}\begin{pmatrix} T_k-1 & -E_k \\ -N_k & L_k-1 \end{pmatrix}$

where $\Delta = (T_k-1)(L_k-1) - E_kN_k$, the Jacobian determinant. Therefore, Newton's method gives the following recurrence formulae in order to yield a periodic point of $H^k$.

$$x_{n+1} = x_n - ((T_k-1)(\bar{x}_n-x_n) - E_k(\bar{y}_n-y_n))/\Delta$$

and $y_{n+1} = y_n - ((-N_k)(\bar{x}_n - x_n) + (L_k - 1)(\bar{y}_n - y_n))/\Delta$

, where $H^k \underline{x}_n = (\bar{x}_n, \bar{y}_n)$, and with the initial point $(x_o, y_o)$ and the initial conditions of L, D, N, E, T are as said before. For practical purposes, this method is very useful and requires generally at most 20 iterations to yield a periodic point of the given map. However, it involves a number of complicated recurrence formulae for the Jacobian of the map.

(ii) We now wish to describe some averaging iteration methods on the Henon map in order to obtain a periodic point of period k. Suppose an initial value $\underline{x}_o = (x_o, y_o)$ is given. By this method, after first iteration we take the average value $\underline{x}_1^1 = \frac{1}{2}(\underline{x}_o + H^k \underline{x}_o)$ or $\underline{x}_1^1 = \frac{1}{2}(H^k \underline{x}_o + H^{2k} \underline{x}_o)$ as the initial value for the second iteration, instead of $\underline{x}_1$. We repeat this process by applying the averaging formula $\underline{x}_{n+1} = \frac{1}{2}(\underline{x}_n + H^k \underline{x}_n)$ or $\underline{x}_{n+1} = \frac{1}{2}(H^k \underline{x}_n + H^{2k} \underline{x}_n)$ and find that this process gives the fast convergence of the values of $\underline{x}$, owing to the following reasons.

Suppose our averaging formula is $\underline{x}_{n+1} = \frac{1}{2}(\underline{x}_n + H^k \underline{x}_n)$. Assume that $\underline{x}$ is a fixed point of $H^k$ and that $\underline{x}_n = \underline{x} + e_n$ for some vector $e_n$. Let U and V be the basis eigenvectors for the Jacobian operator $J_k$ at $\underline{x}$ with the corresponding eigenvalues $\sigma$ and $\mu$ such that for some scalars $\alpha_n$ and $\beta_n$, we have $e_n = \alpha_n U + \beta_n V$. Then

$$e_{n+1} = \tfrac{1}{2}(e_n + J_k e_n) + O(\|e_n\|^2)$$

$$= \tfrac{1}{2}(\alpha_n U + \beta_n V) + \tfrac{1}{2}(\alpha_n \sigma U + \beta_n \mu V) + O(\|e_n\|^2)$$

$$= \left(\frac{1+\sigma}{2}\right) \alpha_n U + \left(\frac{1+\mu}{2}\right) \beta_n V + O(\|e_n\|^2) .$$

If $e_{n+1} = \alpha_{n+1}U + \beta_{n+1}V$, then we have $\alpha_{n+1} = (\frac{1+\sigma}{2})\ \alpha_n$ + some small error term and $\beta_{n+1} = (\frac{1+\mu}{2})\ \beta_n$ + some small error term. At a stable periodic point, the absolute values of $\sigma$ and $\mu$ are less than 1 and their product is always equal to $(-B)^k$. It is noted that we are concerned with period doubling bifurcations when one of the eigenvalues becomes -1. As M approaches to a bifurcation value, one of the eigenvalues, say $\sigma$ approaches -1 and $\mu$ tends to $-(-B)^k$. Then $\frac{1+\sigma}{2} \sim 0$ and $\frac{1+\mu}{2} \sim \frac{1}{2}$ and so each iteration of the map reduces the error at least by half of the previous error. Consequently, after a reasonably small number of iterations these two scalars tend to zero. This leads $e_{n+1}$ to zero approximately and, therefore, gives a fast convergence of $\underline{x}_{n+1}$ to $\underline{x}$.

However the second factor $\frac{1+\mu}{2}$ in the first averaging method is not very small and so we apply the second averaging method. Consider the second averaging method $\underline{x}_{n+1} = \frac{1}{2}(H^k\underline{x}_n + H^{2k}\underline{x}_n)$, and then analogous assumptions yield $\alpha_{n+1} = \frac{1}{2}\sigma(1+\sigma)\alpha_n$ and $\beta_{n+1} = \frac{1}{2}\mu(1+\mu)\beta_n$. In this case, when M approaches a bifurcation value, $\frac{1}{2}\sigma(1+\sigma) \sim 0$ and $\frac{1}{2}\mu(1+\mu) \sim 0$. Here the scalar $\frac{1}{2}\mu(1+\mu)$ is smaller than the term $\frac{1}{2}(1+\mu)$ obtained in the first case, and hence the same sort of argument leads us to conclude that $\underline{x}_{n+1}$ converges to $\underline{x}$ much faster than it does in the previous case. This averaging method is one of the most suitable and effective methods for finding a periodic point near a bifurcation value. Table 2 shows that with the judicious choices of initial values of M and $\underline{x}$, this method gives a periodic point even after one iteration.

(iii) The direct iteration method means to achieve a periodic point just by iterating directly the map itself. Recalling our assumptions made in the first averaging method, and using the direct iteration method, we obtain

$$e_{n+1} = \alpha_n \sigma U + \beta_n \mu V + O(\|e_n\|^2) .$$

So, $\alpha_{n+1} = \alpha_n \sigma$ + some small error term and

$\beta_{n+1} = \beta_n \mu$ + some small error term.

These relations do not indicate any significant reduction of the error term, and so the direct iteration method requires a large number of iterations to yield a periodic point. Therefore, this method gives very slow convergence near the bifurcation values, and as such this method is much too time consuming and tedious.

### 1.1.3 Numerical Methods For Finding Bifurcation Values

For our purposes, we use two numerical methods, namely, (i) Trial and Error Method and (ii) Secant Method.

(i) The Trial and Error method can be applied as follows:

First of all, we recall our recurrence relations for the Jacobian matrix of the map $H^k$ described in Newton's method and then the eigenvalue theory gives the relation $L_k + T_k = -1 - (-B)^k$ at the bifurcation value. Again the Feigenbaum theory says that

$$M_{n+2} \sim M_{n+1} + \frac{M_{n+1} - M_n}{\delta} , \qquad (*)$$

where $n = 1,2,3, \ldots$ and $\delta$ is the Feigenbaum Universal constant as stated in 1.1.1.

In the case of the Henon map, the first two bifurcation values $M_1$ and $M_2$ can be evaluated by their explicit formulae, viz., $M_1 = \frac{3}{4}(1-B)^2$ and $M_2 = \frac{1}{4}(1+B)^2 + (1-B)^2$. Furthermore, it is easy to find the periodic points for these $M_1$ and $M_2$ for any value of B. So we start to apply this method in order to find bifurcation values from $M_3$ onwards. After fixing B and obtaining a periodic point, say $\underline{x}_2$, at $M_2$, we make a judicious choice of an initial value $M_3'$ by using (*) for the bifurcation value $M_3$ of period 4. Because a periodic point at $M_2$ may not be an attracting point for a periodic point at $M_3'$ (or at $M_3$), our next primary task is to obtain a stable periodic point for $M_3'$. We also recall that the modulus of the sum of two eigenvalues at a stable periodic point is less than two. So, in order to find a stable periodic point for $M_3'$, we consider a closed region around $\underline{x}_2$ bounded by a simple Jordan curve, say a square of length 0.2, and then search for a point inside the square in such a way that the absolute value of $L_4+T_4$ remains less than 2. It is always possible to have such a point, because there exists a stable periodic point for $M_3'$ near $\underline{x}_2$. After obtaining a stable periodic point for $M_3'$, we adopt it as an initial point for a periodic point at $M_3$, and then go on steadily increasing the value of $M_3'$ with close observation of the value of $L_4 + T_4$. Ultimately, that value of $M_3'$ for which the value of $L_4+T_4$ equals $-1 - (-B)^4$ appears to be the third bifurcation value $M_3$. In order to find out the next higher bifurcation value of $M_4$ of period 8, we employ the same mechanism to have an initial value $M_4'$ for $M_4$. This time we slightly reduce the side length of the square around a stable periodic point $\underline{x}_3$ at $M_3$, because the larger the period k, the nearer the periodic points for different bifurcation values. Proceeding in the same way as for $M_3$, we can obtain $M_4$. We can continue the process to obtain further higher bifurcation values, as many as we want. Although this method is cumbersome in the sense that it takes a lot of time, and that to get a stable periodic point for an initial value of M is not so immediate, it is very useful when other methods fail.

(ii) To discuss the Secant method, we notice that if we put $I = L_k + T_k + 1 + (-B)^k$, then I turns out to be a function of the parameter M. The bifurcation value of M of the period k occurs when I(M) equals zero. This means, in order to find a bifurcation value of period k, one needs the zero of the function I(M), which is given by the Secant method, $M_{n+1} = M_n - \frac{I(M_n)(M_n - M_{n-1})}{I(M_n) - I(M_{n-1})}$, applied on the function I(M). This method depends very sensitively on the initial conditions. If an initial value is very far from an actual bifurcation value, this method fails to give the convergence of the values of M in general. With the right choice of an initial value of M, it is found that at most 20 iterations are sufficient to give a bifurcation value. The results in different Tables furnished later justify this statement. We also find that although some methods are complicated, all these methods give almost the same accuracy of the results.

### 1.1.4 The Numerical Tools Employed In Our Results

We are now in a position to elucidate our final version of the numerical mechanism which gives the results listed in different Tables later. We first describe how the Secant method is made suitable to work for yielding a bifurcation value. Feigenbaum theory also tells us the following result

$$M_\infty \sim M_{n+1} + \frac{(M_{n+1} - M_n)}{(\delta - 1)} \quad . \tag{**}$$

First of all, our substantial discovery is the achievement of some common attracting points for applying the averaging methods, namely, (-0.8, 0.7), (0.6, 0.275), (0.55, 0.275) and (0.4953, -0.04758); the first one acts as a common attracting point for all values of B in $0 \leq B \leq 1$, the second for all B in $-0.85 \leq B \leq 0$, the third for all B in $-0.95 \leq B < -0.85$ and the fourth for all B in $-1 \leq B < -0.95$. In our study, exact (true) values mean they are correct up to 12 or 18 decimal places unless stated to the contrary. The results are calculated up to 18 decimal places in order to study the smoothness of the curve $M_\infty = M_\infty(B)$ in Section Two and up to 12 decimal places otherwise.

The first two bifurcation values $M_1$ and $M_2$ can be evaluated by their explicit formulae. Then using the relation (*), an approximate value $M_3'$ of $M_3$ is obtained. Since the Secant method needs two initial values, we use $M_3'$ and a slightly larger value, say $M_3' + 10^{-4}$ as the two initial values to apply this method and ultimately obtain $M_3$. In like manner, the same procedure is employed to obtain the successive bifurcation values $M_4$, $M_5$ ... etc. to our requirement. However to do so, a great difficulty arises in using (*) with the true value of $\delta$ (here true values of $\delta$ means either $\delta = 8.721$ or $\delta = 4.669$ correct up to 3 decimal places, depending upon the values of B). For a value of B in the vicinity of 1 and -1, the approximate values of M given by (*) with exact value of $\delta = 4.669$ are very far from their respective true values. For instance, for $B = -0.9$, ($M_1 = 2.7075$, $M_2 = 3.6125$), the first four approximate values are given as tabulated below.

In such a situation, the Secant method may not converge, and so one needs to give a delicately chosen number in place of $\delta$ while

TABLE 1

| Using $\delta = 4.669$, the approximate values of $M_3$, $M_4$, $M_5$ and $M_6$ | Approximate values of $M_3$ $M_4$, $M_5$ and $M_6$ by our method | True values of $M_3$, $M_4$, $M_5$ and $M_6$ |
|---|---|---|
| 3.806323286241 | 3.726923685819 | 3.722670082410 |
| 3.746265140331 | 3.736081626121 | 3.735914971523 |
| 3.738751621172 | 3.737507301162 | 3.737661378613 |
| 3.738035405532 | 3.737891651415 | 3.737949035969 |

using (*). At least it should be noticed that the approximate value of $M_3$ should not be higher than the true value of $M_4$, and similarly for other cases as well. So, what we have done is that for $0 \leq B \leq 1$, $\delta$ is replaced by the number $\delta' = \delta + 3.B$ in order to obtain the initial value $M_3'$ for $M_3$. After obtaining $M_1$, $M_2$, $M_3$, the number $\frac{M_n - M_{n+1}}{M_{n+1} - M_{n+2}}$, (n = 1, 2, ...), is put in the formula (*) to replace $\delta$ in order to achieve the initial values for the higher successive bifurcation values. Again for $-1 < B < 0$, $\delta$ is replaced by the number $\delta'' = \delta - 3.6B$ in order to get an approximate value $M_3'$ for $M_3$, and afterwards the same technique is applied for higher subsequent bifurcation values. For this purpose one could also replace $\delta$ by the value $\delta(B_e)$ given by the formula stated in page 3924 in [51]. However, our method gives much more suitable approximation than that given by $\delta(B_e)$. Moreover, things look far worse when $B = -1$. Since at this value B, the domain of attraction for a periodic point becomes much smaller and smaller with the larger value of k, an initial value $M'_{n+1}$ for the bifurcation value $M_{n+1}$ should be chosen in such a way that it becomes compatible to the Secant method as well as to the averaging methods so that a periodic point of $M_n$ can be used as an initial point for a periodic point of $M'_{n+1}$. Our computer computation shows that if $\delta$ is replaced by 13.5, then the formula (*) gives a good approximation for all the bifurcation values.

We next discuss how the two averaging methods are made compatible to work for achieving a periodic point. In this case, we choose an appropriate point out of the four common attracting points mentioned before, as an initial point for a periodic point at $M_3'$. Then eventually one of the averaging methods yields a periodic point for $M_3$ and this periodic point is used as an initial point for a periodic point at $M_4'$. The reason for doing this is that a periodic point at $M_3$ may not be an attracting point for a periodic point at $M_4$. So $M_4'$ should be chosen in such a way that a periodic point at $M_3$ becomes an attracting point for a periodic point at $M_4'$, and besides $M_4'$ becomes a suitable initial value in order to apply the Secant method for evaluating $M_4$. The process can be continued in a similar fashion.

As mentioned before, the second averaging method gives faster convergence than the first one does. But since in the neighbourhood of $B = \pm 1$, both eigenvalues approximately equal -1 at a bifurcation value, the second averaging method is not suitable for small values of k. Our computational test shows that for any period k, this method works for all values of B lying in $-0.7 \lesssim B \lesssim 0.95$. For $0.95 < B < 1$, the first averaging method is used up to $k = 2^8$ and the second one is used for other values of k. For $-1 < B < -0.7$ the first method is used up to $k = 2^9$ and otherwise the second one. For $B = \pm 1$ the first method is more useful than the second one , (also see 1.1.9).

The effect of this numerical machinery is significantly strong. This drastically reduces the number of iterations to give the convergence of the Secant method as well as of the averaging methods. Tables 2, 3 and 4 are furnished to show the effect of this scheme for $B = -0.7$. In this scheme, the Secant method requires at most 10 iterations to give the next higher bifurcation value from its predecessor, and the number of iterations in order to yield a periodic point by the averaging methods is at most 50; sometimes even one iteration is sufficient for this purpose. In addition, the values of $M_\infty$ given by (**) with exact value of $\delta$ converge very fast. For instance, $M_\infty$ converges (correct up to 12 decimal places) after 7 steps at $B = -1$, after 9 steps at $B = 1$ and after at most 12 steps for other values of $B$.

Remark 1.1.5 To get faster convergence of $M_\infty$, one could use Aitken's extrapolation method, namely,

$$M'_{\infty,n} = M_{\infty,n} - \frac{(M_{\infty,n} - M_{\infty,n-1})^2}{(M_{\infty,n} - 2M_{\infty,n-1} + M_{\infty,n-2})} .$$

However, near $B = \pm 1$ this method does not help much. Moreover, by using the formula $\delta \sim \frac{M_n - M_{n+1}}{M_{n+1} - M_{n+2}}$ or by applying Aitken's extrapolation formula, $\delta$ can also be evaluated.

Remark 1.1.6 This numerical scheme suggests the following conjecture:

'If $F: \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism having some parameters such that period-doubling bifurcations occur with $F$, then in order to evaluate the periodic points and the bifurcation values, the same numerical machinery including the averaging methods and the Secant method described in 1.1.4, can be applied with the judicious choice of the initial values of the periodic points and of the bifurcation values.' For example, we apply this machinery to the map F described in connection with Duffing's equation in Chapter 2.

### 1.1.7 Illustration Of The Results In Tables 2, 3 and 4

To show more clearly how effective and accurate our numerical methods are, some results obtained for $B = -0.7$ are furnished in Tables 2, 3 and 4. For this case, a Computer program with some special comments is also displayed (see Program 1). All calculations in Table 2 are executed in longreal precision (double precision) on a Amdahl 470 V/7 Computer and those in Table 3 in longlongreal precision (quadruple precision).

In Table 2, firstly the period $k = 4$ is shown, and then 1st, 2nd, 3rd and 4th columns show respectively the number of iterations required to yield a periodic point, the successive values of $M$ starting from an initial value to the bifurcation value, $x$ and $y$ coordinates of a periodic point. To be more precise, for $B = -0.7$ and for the period $k = 4$, the initial value of $M$ is 3.016127640524, and for this value of $M$, the initial values -0.8 and 0.7 of $x$ and $y$ take 13 iterations to go to the periodic point (0.571286672010, 0.056446636624).

The Secant method needs 5 iterations to yield the bifurcation value 3.012363618984 with the periodic point (0.571762110574, 0.055530484303). After completing these steps, the Computer program gives approximate $\delta$ value and $M_\infty$ value. Next, the period $k = 8$ starts, and exactly the same procedure is continued to yield the next higher bifurcation value 3.027939554995 with the periodic point (0.576644536483, 0.054577594261). The process is continued and it is found that at $k = 1024$, the value of $M_\infty$ starts convergence. These results are exhibited up to $k = 4096$. For Table-3, we use the same Computer program just by changing the word 'longreal' with the word 'longlongreal'.

PROGRAM 1

```
!COMMENT:THE FOLLOWING PROGRAM GIVES THE SUCCESSIVE %C
BIFURCATION VALUES(M) STARTING FROM M(3) FOR B=-0.7

%BEGIN
%INTEGER K,GG,G,W,H,R,RR,WW
%LONGREAL  Z,MMMM,O,ZZ,X,I,II,Y,M,B,A,D,C,E,J,S,T,U, %C
JJJ,XX,YY,MM,MMM,AA,DD,CC,EE,JJ,SS,TT,UU,LL,NN,VV,QQ,PP, %C
OO,L,N,V,Q,P
SELECTOUTPUT(1)
B=-0.7;OO=4.6692016091029

!COMMENT:X,Y AND DELTA VALUES ARE INITIALISED AS FOLLOWS

X=.6;Y=.275;O=OO-3.6*B;XX=X;YY=Y

!COMMENT:THE FIRST TWO BIFURECATION VALUES ARE GIVEN %C
BY J AND JJ.

J=((1-B)**2)*3/4
JJ=((1+B)**2)/4+((1-B)**2)

!COMMENT:THE IMMEDIATELY FOLLOWING CYCLE GIVES %C
THE CONSECUTIVE BIFURCATION VALUES(M).

%CYCLE RR=2,1,12
K=2\\RR
SPACES(20);PRINTSTRING("K=");PRINT(K,1,1);NEWLINE

!COMMENT:BY USING (*), TWO INITIAL VALUES M AND MM  %C
ARE CALCULATED TO START THE SECANT METHOD.  %C
HERE OUR SECANT METHOD IS %C
MMM=MM-(II*(MM-M))/(II-I),WHERE I=I(M) AND II=I(MM).

M=JJ+(JJ-J)/O
MM=M+(10**(-RR-2))

!COMMENT:THE IMMEDIATELY FOLLOWING CYCLE YIELDS A- %C
PERIODIC POINT FOR M WITH THE AID OF THE SECOND  %C
AVERAGING METHOD AND EVENTUALLY ESTIMATES I.

%CYCLE GG=1,1,100
A=X;C=Y
%CYCLE W=1,1,K
Z=1-M*X*X+Y
Y=B*X
X=Z
%REPEAT
%CYCLE WW=1,1,2*K
ZZ=1-M*XX*XX+YY
YY=B*XX
XX=ZZ
```

```
%REPEAT
X=(X+XX)/2;Y=(Y+YY)/2
XX=X;YY=Y

!COMMENT:THE FOLLOWING CONVERGENCE CONDITION IS PUT %C
TO GET A PERIODIC POINT  CORRECT UPTO 12 DECIMAL  %C
PLACES AND SIMILAR CONDITION IS PUT LATER WHENEVER %C
NECCESSARY.

%EXITIF ((X-A)**2)+((Y-C)**2)<10**(-24)
%REPEAT
PRINT(GG,3,1);PRINT(M,3,12);PRINT(X,3,12);PRINT(Y,3,12)
NEWLINE

!COMMENT:THE IMMEDIATELY FOLLOWING CYCLE CALCULATES   %C
THE JACOBIAN OF THE APPROPRIATE MAP WITH INITIAL %C
CONDITIONS STATED BELOW.

T=0;S=0;U=B;E=1;V=X;L=-2*M*X;Q=Y
%CYCLE H=1,1,K-1
P=B*V
V=1-M*V*V+Q
D=-2*M*V
Q=P
T=B*E;N=B*L
L=D*L+U;E=D*E+S
S=T;U=N
%REPEAT
I=L+T+1+((-B)**K)

!COMMENT:NOW THE SECANT METHOD STARTS WITH THE FOLLOWING  %C
FIRST CYCLE AND THEN EXACTLY SAME STEPS DESCRIBED ABOVE  %C
IN CASE OF YIELDING I ARE REPEATED TO OBTAIN II.

%CYCLE G=1,1,20
%CYCLE GG=1,1,100
A=X;C=Y
%CYCLE W=1,1,K
Z=1-MM*X*X+Y
Y=B*X
X=Z
%REPEAT
%CYCLE WW=1,1,2*K
ZZ=1-MM*XX*XX+YY
YY=B*XX
XX=ZZ
%REPEAT
X=(X+XX)/2;Y=(Y+YY)/2
XX=X;YY=Y
%EXITIF ((X-A)**2)+((Y-C)**2)<10**(-24)
%REPEAT
PRINT(GG,3,1);PRINT(MM,3,12);PRINT(X,3,12);PRINT(Y,3,12)
NEWLINE
T=0;S=0;U=B;E=1
```

```
V=X;L=-2*MM*X;Q=Y
%CYCLE H=1,1,K-1
P=B*V
V=1-MM*V*V+Q
D=-2*MM*V
Q=P
T=B*E;N=B*L
L=D*L+U;E=D*E+S
S=T;U=N
%REPEAT
II=L+T+1+((-B)**K)
%EXITIF MOD(II-I)=0
MMM=MM-II*(MM-M)/(II-I)
I=II;M=MM;MM=MMM
%EXITIF  MOD(MM-M)<10**(-12)
%REPEAT

!COMMENT:THE APPROXIMATE DELTA VALUE IS GIVEN BY MMMM.

MMMM=(J-JJ)/(JJ-MMM)
O=MMMM
J=JJ;JJ=MMM

!COMMENT:THE APPROXIMATE LIMITING VALUE  OF M  IS GIVEN BY JJJ.

JJJ=JJ+(JJ-J)/(OO-1)
SPACES(11);PRINT(MMMM,3,12);PRINT(JJJ,3,12);NEWLINE
%REPEAT
%ENDOFPROGRAM
```

# TABLE 2

(B=-0.7;IN LONGREAL PRECISION)

| | | | |
|---|---|---|---|
| | K=4.0 | | |
| 13.0 | 3.016127640524 | 0.571286672010 | 0.056446636624 |
| 8.0 | 3.016227640524 | 0.571274085712 | 0.056470704182 |
| 10.0 | 3.012369256614 | 0.571761395983 | 0.055531871684 |
| 7.0 | 3.012363627439 | 0.571762109502 | 0.055530486384 |
| 5.0 | 3.012363618984 | 0.571762110574 | 0.055530484303 |
| | 7.460174261466 | 3.039580335213 | |
| | K= 8.0 | | |
| 10.0 | 3.025749850389 | 0.576344855997 | 0.054528780926 |
| 9.0 | 3.025759850389 | 0.576346299720 | 0.054528968626 |
| 7.0 | 3.027956662844 | 0.576646757138 | 0.054578032110 |
| 6.0 | 3.027939422048 | 0.576644519219 | 0.054577590862 |
| 4.0 | 3.027939554987 | 0.576644536481 | 0.054577594261 |
| 2.0 | 3.027939554995 | 0.576644536483 | 0.054577594261 |
| | 6.411404034447 | 3.032184602739 | |
| | K= 16.0 | | |
| 11.0 | 3.030368966077 | 0.574882852109 | 0.056352126455 |
| 8.0 | 3.030369966077 | 0.574882458442 | 0.056352572313 |
| 6.0 | 3.031017210803 | 0.574640289762 | 0.056630188720 |
| 4.0 | 3.031007551890 | 0.574643738756 | 0.056626189209 |
| 3.0 | 3.031007692487 | 0.574643688519 | 0.056626247455 |
| 2.0 | 3.031007692518 | 0.574643688508 | 0.056626247467 |
| | 5.076674658352 | 3.031843879201 | |
| | K= 32.0 | | |
| 8.0 | 3.031612052208 | 0.575330636100 | 0.056147344497 |
| 6.0 | 3.031612152208 | 0.575330692393 | 0.056147311298 |
| 5.0 | 3.031664436179 | 0.575359499401 | 0.056130453810 |
| 3.0 | 3.031664116374 | 0.575359326836 | 0.056130554009 |
| 2.0 | 3.031664118310 | 0.575359327880 | 0.056130553402 |
| | 4.674005136592 | 3.031843019843 | |
| | K= 64.0 | | |
| 6.0 | 3.031804560131 | 0.575057111971 | 0.056393646128 |
| 3.0 | 3.031804570131 | 0.575057100681 | 0.056393656529 |
| 3.0 | 3.031804835274 | 0.575056801470 | 0.056393932195 |
| 2.0 | 3.031804835234 | 0.575056801515 | 0.056393932153 |
| | 4.664867416155 | 3.031843186063 | |
| | K= 128.0 | | |
| 6.0 | 3.031835000491 | 0.575174103135 | 0.056302640902 |
| 3.0 | 3.031835001491 | 0.575174105116 | 0.056302639420 |
| 3.0 | 3.031834989243 | 0.575174080851 | 0.056302657570 |
| | 4.666607485830 | 3.031843207382 | |
| | K= 256.0 | | |
| 6.0 | 3.031841450898 | 0.575126437373 | 0.056342116383 |
| 3.0 | 3.031841450998 | 0.575126436991 | 0.056342116705 |
| 3.0 | 3.031841448117 | 0.575126447988 | 0.056342107426 |
| | 4.668616592658 | 3.031843208411 | |
| | K= 512.0 | | |
| 6.0 | 3.031842831584 | 0.575108702383 | 0.056356667159 |
| 2.0 | 3.031842831594 | 0.575108702321 | 0.056356667211 |
| 2.0 | 3.031842831450 | 0.575108703216 | 0.056356666467 |

4.669068644347 3.031843208462

K= 1024.0

| | | | |
|---|---|---|---|
| 4.0 | 3.031843127726 | 0.575111161929 | 0.056354710844 |
| 2.0 | 3.031843127727 | 0.575111161932 | 0.056354710842 |
| 2.0 | 3.031843127719 | 0.575111161912 | 0.056354710857 |

4.669173765120 3.031843208464

K= 2048.0

| | | | |
|---|---|---|---|
| 5.0 | 3.031843191171 | 0.575112252843 | 0.056353834562 |
| 1.0 | 3.031843191171 | 0.575112252844 | 0.056353834561 |

4.669197491804 3.031843208464

K= 4096.0

| | | | |
|---|---|---|---|
| 6.0 | 3.031843204760 | 0.575112492847 | 0.056353641728 |
| 1.0 | 3.031843204760 | 0.575112492847 | 0.056353641728 |

4.669195373917 3.031843208464

---

TABLE 3

(B=-0.7;IN LONGLONGREAL PRECISION)

K= 4.0

| | | | |
|---|---|---|---|
| 13.0 | 3.016127640524 | 0.571286672010 | 0.056446636624 |
| 8.0 | 3.016227640524 | 0.571274085712 | 0.056470704182 |
| 10.0 | 3.012369256614 | 0.571761395983 | 0.055531871684 |
| 7.0 | 3.012363627439 | 0.571762109502 | 0.055530486384 |
| 5.0 | 3.012363618984 | 0.571762110574 | 0.055530484303 |
| | 7.460174261466 | 3.039580335213 | |
| | K= 8.0 | | |
| 10.0 | 3.025749850389 | 0.576344855997 | 0.054528780926 |
| 9.0 | 3.025759850389 | 0.576346299720 | 0.054528968626 |
| 7.0 | 3.027956662844 | 0.576646757138 | 0.054578032110 |
| 6.0 | 3.027939422048 | 0.576644519219 | 0.054577590862 |
| 4.0 | 3.027939554987 | 0.576644536481 | 0.054577594261 |
| 2.0 | 3.027939554995 | 0.576644536483 | 0.054577594261 |
| | 6.411404034448 | 3.032184602739 | |
| | K= 16.0 | | |
| 11.0 | 3.030368966077 | 0.574882852109 | 0.056352126455 |
| 8.0 | 3.030369966077 | 0.574882458442 | 0.056352572313 |
| 6.0 | 3.031017210804 | 0.574640289761 | 0.056630188720 |
| 4.0 | 3.031007551890 | 0.574643738756 | 0.056626189209 |
| 3.0 | 3.031007692487 | 0.574643688519 | 0.056626247455 |
| 2.0 | 3.031007692518 | 0.574643688508 | 0.056626247467 |
| | 5.076674658348 | 3.031843879201 | |
| | K= 32.0 | | |
| 8.0 | 3.031612052208 | 0.575330636100 | 0.056147344497 |
| 6.0 | 3.031612152208 | 0.575330692393 | 0.056147311298 |
| 5.0 | 3.031664436179 | 0.575359499400 | 0.056130453811 |
| 3.0 | 3.031664116374 | 0.575359326836 | 0.056130554009 |
| 2.0 | 3.031664118310 | 0.575359327880 | 0.056130553402 |
| | 4.674005136602 | 3.031843019843 | |
| | K= 64.0 | | |
| 6.0 | 3.031804560131 | 0.575057111971 | 0.056393646128 |
| 3.0 | 3.031804570131 | 0.575057100681 | 0.056393656529 |
| 3.0 | 3.031804835274 | 0.575056801470 | 0.056393932195 |
| 2.0 | 3.031804835234 | 0.575056801515 | 0.056393932153 |
| | 4.664867416163 | 3.031843186063 | |
| | K= 128.0 | | |
| 6.0 | 3.031835000491 | 0.575174103135 | 0.056302640902 |
| 3.0 | 3.031835001491 | 0.575174105116 | 0.056302639420 |
| 3.0 | 3.031834989243 | 0.575174080851 | 0.056302657570 |
| | 4.666607486204 | 3.031843207382 | |
| | K= 256.0 | | |
| 6.0 | 3.031841450898 | 0.575126437373 | 0.056342116383 |
| 3.0 | 3.031841450998 | 0.575126436991 | 0.056342116705 |
| 3.0 | 3.031841448117 | 0.575126447988 | 0.056342107426 |
| | 4.668616589653 | 3.031843208411 | |
| | K= 512.0 | | |
| 6.0 | 3.031842831584 | 0.575108702383 | 0.056356667159 |
| 2.0 | 3.031842831594 | 0.575108702321 | 0.056356667211 |
| 2.0 | 3.031842831450 | 0.575108703216 | 0.056356666467 |

| | | | |
|---|---|---|---|
| | 4.669068665659 | 3.031843208462 | |
| | K= 1024.0 | | |
| 4.0 | 3.031843127726 | 0.575111161929 | 0.056354710844 |
| 2.0 | 3.031843127727 | 0.575111161932 | 0.056354710842 |
| 2.0 | 3.031843127719 | 0.575111161912 | 0.056354710857 |
| | 4.669173734149 | 3.031843208464 | |
| | K= 2048.0 | | |
| 5.0 | 3.031843191171 | 0.575112252843 | 0.056353834562 |
| 1.0 | 3.031843191171 | 0.575112252844 | 0.056353834561 |
| | 4.669195548191 | 3.031843208464 | |
| | K= 4096.0 | | |
| 6.0 | 3.031843204760 | 0.575112492848 | 0.056353641728 |
| 1.0 | 3.031843204760 | 0.575112492848 | 0.056353641728 |
| | 4.669200321759 | 3.031843208464 | |

TABLE 4

(B=-0.7;IN LONGLONGREAL PRECISION WITH THE CHANGE OF CONVERGENCE CONDITION )

| | | | |
|---|---|---|---|
| | K= 4.0 | | |
| 16.0 | 3.016127640524 | 0.571286672010 | 0.056446636624 |
| 11.0 | 3.016227640524 | 0.571274085712 | 0.056470704182 |
| 13.0 | 3.012369256619 | 0.571761395982 | 0.055531871686 |
| 10.0 | 3.012363627439 | 0.571762109502 | 0.055530486384 |
| 8.0 | 3.012363618984 | 0.571762110574 | 0.055530484303 |
| | 7.460174261462 | 3.039580335213 | |
| | K= 8.0 | | |
| 14.0 | 3.025749850389 | 0.576344855997 | 0.054528780926 |
| 12.0 | 3.025759850389 | 0.576346299720 | 0.054528968626 |
| 9.0 | 3.027956662779 | 0.576646757129 | 0.054578032108 |
| 8.0 | 3.027939422049 | 0.576644519219 | 0.054577590862 |
| 6.0 | 3.027939554987 | 0.576644536481 | 0.054577594261 |
| 4.0 | 3.027939554995 | 0.576644536483 | 0.054577594261 |
| | 6.411404034475 | 3.032184602739 | |
| | K= 16.0 | | |
| 14.0 | 3.030368966077 | 0.574882852109 | 0.056352126455 |
| 12.0 | 3.030369966077 | 0.574882458442 | 0.056352572313 |
| 7.0 | 3.031017210746 | 0.574640289782 | 0.056630188697 |
| 5.0 | 3.031007551891 | 0.574643738756 | 0.056626189209 |
| 4.0 | 3.031007692487 | 0.574643688519 | 0.056626247455 |
| 3.0 | 3.031007692518 | 0.574643688508 | 0.056626247467 |
| | 5.076674658337 | 3.031843879201 | |
| | K= 32.0 | | |
| 10.0 | 3.031612052208 | 0.575330636100 | 0.056147344497 |
| 9.0 | 3.031612152208 | 0.575330692393 | 0.056147311298 |
| 6.0 | 3.031664436169 | 0.575359499395 | 0.056130453814 |
| 3.0 | 3.031664116374 | 0.575359326836 | 0.056130554009 |
| 3.0 | 3.031664118310 | 0.575359327880 | 0.056130553402 |
| | 4.674005136595 | 3.031843019843 | |
| | K= 64.0 | | |
| 7.0 | 3.031804560131 | 0.575057111971 | 0.056393646128 |
| 4.0 | 3.031804570131 | 0.575057100681 | 0.056393656529 |
| 4.0 | 3.031804835274 | 0.575056801470 | 0.056393932195 |
| 2.0 | 3.031804835234 | 0.575056801515 | 0.056393932153 |
| | 4.664867416162 | 3.031843186063 | |
| | K= 128.0 | | |
| 7.0 | 3.031835000491 | 0.575174103135 | 0.056302640902 |
| 4.0 | 3.031835001491 | 0.575174105116 | 0.056302639420 |
| 3.0 | 3.031834989243 | 0.575174080851 | 0.056302657570 |
| | 4.666607486204 | 3.031843207382 | |
| | K= 256.0 | | |
| 7.0 | 3.031841450898 | 0.575126437373 | 0.056342116383 |
| 3.0 | 3.031841450998 | 0.575126436991 | 0.056342116705 |
| 3.0 | 3.031841448117 | 0.575126447988 | 0.056342107426 |
| | 4.668616589653 | 3.031843208411 | |
| | K= 512.0 | | |
| 7.0 | 3.031842831584 | 0.575108702383 | 0.056356667159 |
| 3.0 | 3.031842831594 | 0.575108702321 | 0.056356667211 |

| | | | |
|---|---|---|---|
| 3.0 | 3.031842831450 | 0.575108703216 | 0.056356666467 |
| | 4.669068665659 | 3.031843208462 | |
| | K= 1024.0 | | |
| 5.0 | 3.031843127726 | 0.575111161929 | 0.056354710844 |
| 2.0 | 3.031843127727 | 0.575111161932 | 0.056354710842 |
| 2.0 | 3.031843127719 | 0.575111161912 | 0.056354710857 |
| | 4.669173734149 | 3.031843208464 | |
| | K= 2048.0 | | |
| 6.0 | 3.031843191171 | 0.575112252843 | 0.056353834562 |
| 2.0 | 3.031843191171 | 0.575112252844 | 0.056353834561 |
| | 4.669195548216 | 3.031843208464 | |
| | K= 4096.0 | | |
| 7.0 | 3.031843204760 | 0.575112492848 | 0.056353641728 |
| 2.0 | 3.031843204760 | 0.575112492848 | 0.056353641728 |
| | 4.669200321619 | 3.031843208464 | |

In order to verify whether the convergence condition for periodic points affects the bifurcation values and hence the $M_\infty$ values, the results in Table 4 were performed in longlongreal precision, but with a new convergence condition as $(X - A)^2 + (Y - C)^2 < 10^{-30}$. However with this new condition we can not get correct figures in longreal precision due to the reasons stated in the next article for Table 5.

Remark 1.1.8. The number of iterations taken by the Secant method and by the averaging methods can be further reduced by choosing more accurately the initial values of M and the periodic points.

## 1.1.9. Illustration Of The Results In Tables 5, 6 and 7

Tables 5, 6 and 7 are shown with an idea of how the methods work in the vicinity of $B = 1$, (similar tables can be provided for a value of B near -1). As said earlier, the second averaging method is not very suitable for small values of k in general and not even suitable for large values of k when the results are calculated in longreal precision, (see also 1.1.10). The results in Tables 5 and 6, prepared for $B = 0.999$, are evaluated respectively in longreal and longlongreal precisions. We divide the Computer program into two parts, each part being similar to the program used for $B = -0.7$; in the first part, the first averaging method is applied up to $k = 256$ and in the second part, the second averaging method is used for the next higher values of k. Unfortunately, some of the results in Table 5 after the period $k = 256$ are not correct.
It is seen that when the second averaging method began for the

TABLE 5

| | | | |
|---|---|---|---|
| (B=0.999;IN LONGREAL PRECISION) | | | |
| | K= 4.0 | | |
| 22.0 | 1.129313567747 | -0.564839447767 | 0.565391829323 |
| 11.0 | 1.129323567747 | -0.564822175406 | 0.565374576881 |
| 7.0 | 1.132834905416 | -0.558804738916 | 0.559364076948 |
| 4.0 | 1.132842036703 | -0.558792613178 | 0.559351965182 |
| 3.0 | 1.132842051204 | -0.558792588522 | 0.559351940555 |
| | 7.464095335765 | 1.169318869722 | |
| | K= 8.0 | | |
| 21.0 | 1.150773333587 | -0.625844116377 | 0.626289235787 |
| 11.0 | 1.150774333587 | -0.625845684691 | 0.626290800914 |
| 6.0 | 1.150195028122 | -0.624926741994 | 0.625373719805 |
| 4.0 | 1.150194334007 | -0.624925628215 | 0.625372608276 |
| 3.0 | 1.150194333172 | -0.624925626875 | 0.625372606938 |
| | 7.713152739634 | 1.154923504206 | |
| | K= 16.0 | | |
| 37.0 | 1.152444033464 | -0.602759111488 | 0.603243027255 |
| 21.0 | 1.152444133464 | -0.602758581564 | 0.603242498172 |
| 8.0 | 1.152201031739 | -0.604078379610 | 0.604560195131 |
| 4.0 | 1.152199853055 | -0.604084941530 | 0.604566746577 |
| 3.0 | 1.152199847288 | -0.604084973634 | 0.604566778630 |
| | 8.652286128912 | 1.152746427806 | |
| | K= 32.0 | | |
| 15.0 | 1.152431637371 | -0.608774883108 | 0.609249151972 |
| 4.0 | 1.152431647371 | -0.608774982643 | 0.609249251344 |
| 5.0 | 1.152430303334 | -0.608761584471 | 0.609235875139 |
| 3.0 | 1.152430303008 | -0.608761581210 | 0.609235871884 |
| | 8.702383797429 | 1.152493111145 | |
| | K= 64.0 | | |
| 11.0 | 1.152456784915 | -0.607539135611 | 0.608015474009 |
| 5.0 | 1.152456785915 | -0.607539111852 | 0.608015450289 |
| 6.0 | 1.152456737717 | -0.607540257445 | 0.608016593974 |
| 3.0 | 1.152456737713 | -0.607540257543 | 0.608016594071 |
| | 8.717922627410 | 1.152463942198 | |
| | K= 128.0 | | |
| 11.0 | 1.152459769938 | -0.607837758300 | 0.608313610500 |
| 5.0 | 1.152459770038 | -0.607837763249 | 0.608313615441 |
| 6.0 | 1.152459771870 | -0.607837853851 | 0.608313705896 |
| | 8.712374484679 | 1.152460598795 | |
| | K= 256.0 | | |
| 13.0 | 1.152460120128 | -0.607762983574 | 0.608238959736 |
| 4.0 | 1.152460120138 | -0.607762982479 | 0.608238958644 |
| 13.0 | 1.152460121134 | -0.607762873492 | 0.608238849837 |
| 13.0 | 1.152460121130 | -0.607762873933 | 0.608238850277 |
| | 8.687375542262 | 1.152460216317 | |
| | K= 512.0 | | |
| 100.0 | 1.152460161333 | -0.607781507640 | 0.608257453418 |
| 21.0 | 1.152460161533 | -0.607781554594 | 0.608257500296 |
| 11.0 | 1.152460161794 | -0.607781615615 | 0.608257561216 |
| | 8.588973395533 | 1.152460172876 | |
| | K= 1024.0 | | |

| | | | |
|---|---|---|---|
| 18.0 | 1.152460166528 | -0.607786263428 | 0.608262201371 |
| 49.0 | 1.152460166548 | -0.607786273219 | 0.608262211146 |
| 5.0 | 1.152460166734 | -0.607786363245 | 0.608262301024 |
| | 8.231638693359 | 1.152460168080 | |
| | K= 2048.0 | | |
| 100.0 | 1.152460167334 | -0.607786341689 | 0.608262279515 |
| 59.0 | 1.152460167336 | -0.607786341876 | 0.608262279702 |
| 16.0 | 1.152460167417 | -0.607786350139 | 0.608262287952 |
| | 7.216865539175 | 1.152460167605 | |
| | K= 4096.0 | | |
| 53.0 | 1.152460167513 | -0.607786483586 | 0.608262421177 |
| 100.0 | 1.152460167513 | -0.607786483815 | 0.608262421406 |
| 100.0 | 1.152460167530 | -0.607786498324 | 0.608262435891 |
| 50.0 | 1.152460167540 | -0.607786506009 | 0.608262443563 |
| | 5.643759977738 | 1.152460167572 | |
| | K= 8192.0 | | |
| 100.0 | 1.152460167561 | -0.607786474654 | 0.608262412261 |
| 9.0 | 1.152460167561 | -0.607786474496 | 0.608262412103 |
| | 5.518503925075 | 1.152460167567 | |

# TABLE 6

(B=0.999;IN LONGLONGREAL PRECISION)

| | | | |
|---|---|---|---|
| | K= 4.0 | | |
| 22.0 | 1.129313567747 | -0.564839447767 | 0.565391829323 |
| 11.0 | 1.129323567747 | -0.564822175406 | 0.565374576881 |
| 7.0 | 1.132834905416 | -0.558804738916 | 0.559364076948 |
| 4.0 | 1.132842036703 | -0.558792613178 | 0.559351965182 |
| 3.0 | 1.132842051204 | -0.558792588522 | 0.559351940555 |
| | 7.464095335765 | 1.169318869722 | |
| | K= 8.0 | | |
| 21.0 | 1.150773333587 | -0.625844116377 | 0.626289235787 |
| 11.0 | 1.150774333587 | -0.625845684691 | 0.626290800914 |
| 6.0 | 1.150195028123 | -0.624926741995 | 0.625373719806 |
| 4.0 | 1.150194334007 | -0.624925628215 | 0.625372608276 |
| 3.0 | 1.150194333172 | -0.624925626875 | 0.625372606938 |
| | 7.713152739634 | 1.154923504206 | |
| | K= 16.0 | | |
| 37.0 | 1.152444033464 | -0.602759111488 | 0.603243027255 |
| 21.0 | 1.152444133464 | -0.602758581564 | 0.603242498172 |
| 8.0 | 1.152201031736 | -0.604078379625 | 0.604560195145 |
| 4.0 | 1.152199853055 | -0.604084941530 | 0.604566746577 |
| 3.0 | 1.152199847288 | -0.604084973634 | 0.604566778630 |
| | 8.652286128914 | 1.152746427806 | |
| | K= 32.0 | | |
| 15.0 | 1.152431637371 | -0.608774883108 | 0.609249151972 |
| 4.0 | 1.152431647371 | -0.608774982643 | 0.609249251344 |
| 5.0 | 1.152430303334 | -0.608761584471 | 0.609235875140 |
| 3.0 | 1.152430303008 | -0.608761581210 | 0.609235871884 |
| | 8.702383797480 | 1.152493111145 | |
| | K= 64.0 | | |
| 11.0 | 1.152456784915 | -0.607539135611 | 0.608015474009 |
| 5.0 | 1.152456785915 | -0.607539111852 | 0.608015450289 |
| 6.0 | 1.152456737717 | -0.607540257457 | 0.608016593986 |
| 3.0 | 1.152456737713 | -0.607540257543 | 0.608016594072 |
| | 8.717922626134 | 1.152463942198 | |
| | K= 128.0 | | |
| 11.0 | 1.152459769938 | -0.607837758300 | 0.608313610500 |
| 5.0 | 1.152459770038 | -0.607837763249 | 0.608313615441 |
| 6.0 | 1.152459771869 | -0.607837853845 | 0.608313705889 |
| | 8.712374522829 | 1.152460598795 | |
| | K= 256.0 | | |
| 13.0 | 1.152460120128 | -0.607762983575 | 0.608238959738 |
| 5.0 | 1.152460120138 | -0.607762982481 | 0.608238958645 |
| 7.0 | 1.152460121130 | -0.607762873921 | 0.608238850265 |
| | 8.687375847293 | 1.152460216317 | |
| | K= 512.0 | | |
| 14.0 | 1.152460161333 | -0.607781507643 | 0.608257453421 |
| 7.0 | 1.152460161533 | -0.607781554590 | 0.608257500291 |
| 7.0 | 1.152460161794 | -0.607781615634 | 0.608257561235 |
| | 8.588964083775 | 1.152460172876 | |
| | K= 1024.0 | | |
| 11.0 | 1.152460166528 | -0.607786263436 | 0.608262201379 |

| | | | |
|---|---|---|---|
| 6.0 | 1.152460166548 | -0.607786273231 | 0.608262211158 |
| 6.0 | 1.152460166734 | -0.607786363235 | 0.608262301014 |
| | 8.231846004257 | 1.152460168080 | |
| | K= 2048.0 | | |
| 7.0 | 1.152460167334 | -0.607786341669 | 0.608262279495 |
| 5.0 | 1.152460167336 | -0.607786341863 | 0.608262279689 |
| 5.0 | 1.152460167419 | -0.607786350277 | 0.608262288090 |
| | 7.215956034426 | 1.152460167605 | |
| | K= 4096.0 | | |
| 7.0 | 1.152460167513 | -0.607786483616 | 0.608262421207 |
| 5.0 | 1.152460167513 | -0.607786483790 | 0.608262421381 |
| 5.0 | 1.152460167540 | -0.607786506076 | 0.608262443630 |
| | 5.642834312037 | 1.152460167572 | |
| | K= 8192.0 | | |
| 6.0 | 1.152460167561 | -0.607786474611 | 0.608262412218 |
| 4.0 | 1.152460167561 | -0.607786474600 | 0.608262412207 |
| 5.0 | 1.152460167565 | -0.607786472420 | 0.608262410030 |
| | 4.764167392389 | 1.152460167572 | |

TABLE 7

(B=0.999;IN LONGLONGREAL PRECISION WITH THE CHANGE OF CONVERGENCE CONDITION )

| | | | |
|---|---|---|---|
| | K= 4.0 | | |
| 26.0 | 1.129313567747 | -0.564839447767 | 0.565391829323 |
| 14.0 | 1.129323567747 | -0.564822175406 | 0.565374576881 |
| 8.0 | 1.132834905412 | -0.558804738922 | 0.559364076954 |
| 5.0 | 1.132842036703 | -0.558792613178 | 0.559351965182 |
| 4.0 | 1.132842051204 | -0.558792588522 | 0.559351940555 |
| | 7.464095335765 | 1.169318869722 | |
| | K= 8.0 | | |
| 26.0 | 1.150773333587 | -0.625844116377 | 0.626289235787 |
| 15.0 | 1.150774333587 | -0.625845684691 | 0.626290800914 |
| 8.0 | 1.150195028129 | -0.624926742004 | 0.625373719815 |
| 5.0 | 1.150194334007 | -0.624925628215 | 0.625372608276 |
| 4.0 | 1.150194333172 | -0.624925626875 | 0.625372606938 |
| | 7.713152739634 | 1.154923504206 | |
| | K= 16.0 | | |
| 47.0 | 1.152444033464 | -0.602759111488 | 0.603243027255 |
| 30.0 | 1.152444133464 | -0.602758581564 | 0.603242498172 |
| 10.0 | 1.152201031797 | -0.604078379283 | 0.604560194804 |
| 6.0 | 1.152199853055 | -0.604084941528 | 0.604566746575 |
| 5.0 | 1.152199847288 | -0.604084973634 | 0.604566778630 |
| | 8.652286128914 | 1.152746427806 | |
| | K= 32.0 | | |
| 18.0 | 1.152431637371 | -0.608774883108 | 0.609249151972 |
| 8.0 | 1.152431647371 | -0.608774982647 | 0.609249251348 |
| 7.0 | 1.152430303335 | -0.608761584476 | 0.609235875145 |
| 5.0 | 1.152430303008 | -0.608761581210 | 0.609235871884 |
| | 8.702383797480 | 1.152493111145 | |
| | K= 64.0 | | |
| 14.0 | 1.152456784915 | -0.607539135611 | 0.608015474009 |
| 7.0 | 1.152456785915 | -0.607539111852 | 0.608015450289 |
| 8.0 | 1.152456737717 | -0.607540257457 | 0.608016593986 |
| 5.0 | 1.152456737713 | -0.607540257543 | 0.608016594072 |
| | 8.717922626137 | 1.152463942198 | |
| | K= 128.0 | | |
| 14.0 | 1.152459769938 | -0.607837758300 | 0.608313610500 |
| 7.0 | 1.152459770038 | -0.607837763249 | 0.608313615441 |
| 8.0 | 1.152459771869 | -0.607837853845 | 0.608313705889 |
| | 8.712374522857 | 1.152460598795 | |
| | K= 256.0 | | |
| 16.0 | 1.152460120128 | -0.607762983576 | 0.608238959738 |
| 8.0 | 1.152460120138 | -0.607762982481 | 0.608238958645 |
| 10.0 | 1.152460121130 | -0.607762873921 | 0.608238850266 |
| | 8.687375848205 | 1.152460216317 | |
| | K= 512.0 | | |
| 18.0 | 1.152460161333 | -0.607781507643 | 0.608257453421 |
| 10.0 | 1.152460161533 | -0.607781554590 | 0.608257500291 |
| 11.0 | 1.152460161794 | -0.607781615634 | 0.608257561235 |
| | 8.588964064977 | 1.152460172876 | |
| | K= 1024.0 | | |

| | | | |
|---|---|---|---|
| 14.0 | 1.152460166528 | -0.607786263436 | 0.608262201379 |
| 9.0 | 1.152460166548 | -0.607786273231 | 0.608262211158 |
| 10.0 | 1.152460166734 | -0.607786363238 | 0.608262301016 |
| | 8.231845142297 | 1.152460168080 | |
| | K= 2048.0 | | |
| 11.0 | 1.152460167334 | -0.607786341669 | 0.608262279495 |
| 8.0 | 1.152460167336 | -0.607786341863 | 0.608262279689 |
| 8.0 | 1.152460167419 | -0.607786350279 | 0.608262288092 |
| | 7.215964453454 | 1.152460167605 | |
| | K= 4096.0 | | |
| 10.0 | 1.152460167513 | -0.607786483616 | 0.608262421207 |
| 8.0 | 1.152460167513 | -0.607786483790 | 0.608262421381 |
| 6.0 | 1.152460167540 | -0.607786506082 | 0.608262443636 |
| | 5.642825647998 | 1.152460167572 | |
| | K= 8192.0 | | |
| 9.0 | 1.152460167561 | -0.607786474612 | 0.608262412218 |
| 7.0 | 1.152460167561 | -0.607786474600 | 0.608262412207 |
| 6.0 | 1.152460167565 | -0.607786472426 | 0.608262410037 |
| | 4.764123947302 | 1.152460167572 | |

period $k = 512$, it failed to yield a periodic point correctly at $M = 1.152460161333$ even after 100 iterations. It does not mean that the second averaging method needs more than 100 iterations to do so correctly. It is rather found that $x$ and $y$ values fluctuated around the actual periodic point without converging to it because of the rounding errors. Since the results started going wrong from the period $k = 512$, the program afterwards started giving wrong bifurcation values and periodic points. Consequently the $M_\infty$ value is not correct. However this difficulty can be averted by applying the first averaging method throughout and by choosing more accurately the initial values of $M$, or by calculating the results in longlongreal precision.

On the other hand, Table 6 shows that the second averaging method took 14 iterations to yield the periodic point $(-0.607781507643,\ 0.608257453421)$ at $M = 1.152460161333$ for the period $k = 512$. The different figures in this table can be described like Table 2. It is seen that the numbers of iterations taken by the Secant method and by the averaging methods are atmost 5 and 22 respectively. Furthermore, the $M_\infty$-values start convergence from $k = 4096$ and the convergence of $\delta$-values is slower than that done for $B = -0.7$. The results in Table 7 were executed in longlongreal precision, but with a different convergence condition for periodic points, namely, $(X - A)^2 + (Y - C)^2 < 10^{-30}$.

### 1.1.10 Error Estimates

In our Computer programs we impose two stopping conditions for convergence in order to save Computer time, one for the periodic points, viz., exitif $(x - A)^2 + (y - C)^2 < 10^{-24}$, where A and C are respectively last but one values of x and y, and another one for M-values, viz., exitif Mod(MM - M) < $10^{-12}$, where M and MM are the last two consecutive values of M. (These two conditions are applied if the results are calculated up to 12 decimal places, and similarly the negative power of 10 should be rightly chosen according to the number of decimal places one needs.) As such it seems that there may be some truncation errors in calculations. Moreover, our Computer can retain values up to 16 and 36 significant digits if calculations are performed in longreal and longlongreal precisions respectively. Therefore, the results may involve some rounding errors. But it is quite illuminating that the results, except $\delta$-values in Tables 2 and 3, have good agreement in both precisions. The different figures, except $\delta$-values in Table 4, have also good agreement with those in Tables 2 and 3. The $\delta$-values, which are not our prime concern, are slightly affected by these errors. The $\delta$-values calculated in longreal and longlongreal precisions appear to be correct up to 3 and 5 decimal places respectively, and so these errors are not very significant.

For the values of B near 1 and -1, if we evaluate the results up to $\ell$ ($\ell \leqslant 11$) decimal places, then our numerical methods can be applied without any difficulty in both precisions. Also in that case all the results have good agreement in both precisions. In the case of a higher number of decimal places, say 12 or more, the bifurcation values and the periodic points may differ in the last one or two decimal places. As an example, we can cite the case for the period k = 128 in Table 5. Here the bifurcation value, and x and y values differ from those tabulated in Table 6 by $10^{-12}$, $6 \times 10^{-12}$ and $7 \times 10^{-12}$

respectively, which are very small indeed. Similarly, the bifurcation values and $M_\infty$ value in Table 7 are the same as those in Table 6. This indicates that the increase of the negative power of 10 in the convergence condition for periodic points does not affect the $M_\infty$-values. Moreover, as mentioned in 1.1.9, if we want to calculate the results up to m $(m \geqslant 12)$ decimal places in longreal precision, then our numerical methods seem to be not very suitable. Hence in this case, we evaluate the results in longlongreal precision and so the results are very accurate. In addition, it is very interesting to note that whenever our method works, the $M_\infty$ values are the same in either precision, indicating that the above-mentioned errors do not have any significant effect on them.

In a nutshell, we can conclude from the results in different Tables that the $M_\infty$ values for all values of B are very accurate as they are the same in either precision, and that the bifurcation values and the periodic points are also very accurate for small values of B and they may be slightly different near $B = 1$ and $-1$, having very small errors.

## 1.2 Section Two. The Smoothness Of The Curve $M_\infty = M_\infty(B)$, $B \in (-\infty,\infty)$

This section consists of a study of the smoothness of the curve $M_\infty = M_\infty(B)$, $B \in (-\infty,\infty)$. Our theoretical discussions,(see 1.2.2), motivate the following conjecture.

Conjecture: The curve $M_\infty = M_\infty(B)$ is $C^\alpha$ smooth in the sense of Holder continuity near $B = \pm 1$, where $\alpha = \log_2 \tilde{\delta}$, $\tilde{\delta}$† the area

† In general the symbol ∿ is used to mean 'approximately equal to', but here it is used with δ to specify the Feigenbaum constant for the conservative case.

preserving Feigenbaum constant being equal to 8.721097200 ..., but not for $\alpha > \log_2 \tilde{\delta}$, and $C^\infty$ smooth otherwise.

To study this conjecture, the theory of finite differences is applied and suggests that the conjecture is true. The results in different Tables provided for this section are calculated in longlong-real precision and so they are very accurate. The same sort of Computer program as used for B =-0.7 is also employed for these results with an appropriate averaging method. To show distinctly the finite difference values up to the fifth order, the $M_\infty$ values are calculated up to 18 decimal places, but the finite difference values are kept up to certain decimal places in order to save space.

### 1.2.1 The Curve $M_\infty = M_\infty(B)$ In General

The explicit formulae, $M_1 = \frac{3}{4}(1-B)^2$ and $M_2 = \frac{1}{4}(1+B)^2 + (1-B)^2$, say that their graphs are parabolas. From the graphs of $M_n = M_n(B)$, $n = 3,4,5,...$, drawn in the computer, it seems that $M_n = M_n(B)$ can be approximately represented by a parabolic equation of the form

$$M_n = C + L(B - A)^2$$

, where A is the value of B at which $M_n$ attains its minimum that lies between 0.6 and 0.61, (observed from the graphs and their numerical values), C is the bifurcation value of M at A, L is a suitably chosen constant depending on $M_n$. For example, the curve $M_\infty = M_\infty(B)$ can be approximately represented by a parabolic equation as $M_\infty = C + L(B - A)^2$, where A = 0.608, C = 0.934099116562 and L = 1.238392802522. Of course, the errors are noticeably high and lie between $-22 \times 10^{-3}$ and $29 \times 10^{-3}$.

TABLE 8

| | | | |
|---|---|---|---|
| 1.00 | 1.153612188859 | 0.49 | 0.952775397350 |
| 0.99 | 1.142235043825 | 0.48 | 0.956036908783 |
| 0.98 | 1.131175273874 | 0.47 | 0.959556532975 |
| 0.97 | 1.120431922315 | 0.46 | 0.963333393776 |
| 0.96 | 1.110003961624 | 0.45 | 0.967366621325 |
| 0.95 | 1.099890315473 | 0.44 | 0.971655351119 |
| 0.94 | 1.090089868786 | 0.43 | 0.976198723186 |
| 0.93 | 1.080601473674 | 0.42 | 0.980995881366 |
| 0.92 | 1.071423953941 | 0.41 | 0.986045972704 |
| 0.91 | 1.062556108371 | 0.40 | 0.991348146972 |
| 0.90 | 1.053996713145 | 0.39 | 0.996901556319 |
| 0.89 | 1.045744523968 | 0.38 | 1.002705355050 |
| 0.88 | 1.037798278173 | 0.37 | 1.008758699547 |
| 0.87 | 1.030156696809 | 0.36 | 1.015060748319 |
| 0.86 | 1.022818486639 | 0.35 | 1.021610662187 |
| 0.85 | 1.015782342011 | 0.34 | 1.028407604605 |
| 0.84 | 1.009046946650 | 0.33 | 1.035450742106 |
| 0.83 | 1.002610975408 | 0.32 | 1.042739244866 |
| 0.82 | 0.996473096057 | 0.31 | 1.050272287397 |
| 0.81 | 0.990631971127 | 0.30 | 1.058049049341 |
| 0.80 | 0.985086259830 | 0.29 | 1.066068716369 |
| 0.79 | 0.979834620037 | 0.28 | 1.074330481185 |
| 0.78 | 0.974875710285 | 0.27 | 1.082833544606 |
| 0.77 | 0.970208191792 | 0.26 | 1.091577116726 |
| 0.76 | 0.965830730436 | 0.25 | 1.100560418156 |
| 0.75 | 0.961741998683 | 0.24 | 1.109782681312 |
| 0.74 | 0.957940677429 | 0.23 | 1.119243151769 |
| 0.73 | 0.954425457749 | 0.22 | 1.128941089650 |
| 0.72 | 0.951195042539 | 0.21 | 1.138875771047 |
| 0.71 | 0.948248148035 | 0.20 | 1.149046489472 |
| 0.70 | 0.945583505204 | 0.19 | 1.159452557322 |
| 0.69 | 0.943199861011 | 0.18 | 1.170093307344 |
| 0.68 | 0.941095979538 | 0.17 | 1.180968094106 |
| 0.67 | 0.939270642962 | 0.16 | 1.192076295456 |
| 0.66 | 0.937722652377 | 0.15 | 1.203417313956 |
| 0.65 | 0.936450828465 | 0.14 | 1.214990578295 |
| 0.64 | 0.935454011997 | 0.13 | 1.226795544671 |
| 0.63 | 0.934731064175 | 0.12 | 1.238831698121 |
| 0.62 | 0.934280866801 | 0.11 | 1.251098553816 |
| 0.61 | 0.934102322273 | 0.10 | 1.263595658292 |
| 0.60 | 0.934194353426 | 0.09 | 1.276322590625 |
| 0.59 | 0.934555903200 | 0.08 | 1.289278963543 |
| 0.58 | 0.935185934156 | 0.07 | 1.302464424461 |
| 0.57 | 0.936083427854 | 0.06 | 1.315878656443 |
| 0.56 | 0.937247384091 | 0.05 | 1.329521379090 |
| 0.55 | 0.938676820029 | 0.04 | 1.343392349336 |
| 0.54 | 0.940370769213 | 0.03 | 1.357491362161 |
| 0.53 | 0.942328280521 | 0.02 | 1.371818251218 |
| 0.52 | 0.944548417041 | 0.01 | 1.386372889362 |
| 0.51 | 0.947030254908 | 0.00 | 1.401155189092 |
| 0.50 | 0.949772882119 | | |

Continued

| | |
|---|---|
| -0.01 | 1.416165102889 |
| -0.02 | 1.431402623466 |
| -0.03 | 1.446867783919 |
| -0.04 | 1.462560657778 |
| -0.05 | 1.478481358968 |
| -0.06 | 1.494630041667 |
| -0.07 | 1.511006900075 |
| -0.08 | 1.527612168092 |
| -0.09 | 1.544446118902 |
| -0.10 | 1.561509064468 |
| -0.11 | 1.578801354950 |
| -0.12 | 1.596323378036 |
| -0.13 | 1.614075558198 |
| -0.14 | 1.632058355871 |
| -0.15 | 1.650272266573 |
| -0.16 | 1.668717819950 |
| -0.17 | 1.687395578773 |
| -0.18 | 1.706306137872 |
| -0.19 | 1.725450123033 |
| -0.20 | 1.744828189845 |
| -0.21 | 1.764441022517 |
| -0.22 | 1.784289332661 |
| -0.23 | 1.804373858062 |
| -0.24 | 1.824695361416 |
| -0.25 | 1.845254629075 |
| -0.26 | 1.866052469775 |
| -0.27 | 1.887089713372 |
| -0.28 | 1.908367209590 |
| -0.29 | 1.929885826776 |
| -0.30 | 1.951646450680 |
| -0.31 | 1.973649983264 |
| -0.32 | 1.995897341532 |
| -0.33 | 2.018389456408 |
| -0.34 | 2.041127271644 |
| -0.35 | 2.064111742779 |
| -0.36 | 2.087343836140 |
| -0.37 | 2.110824527894 |
| -0.38 | 2.134554803157 |
| -0.39 | 2.158535655152 |
| -0.40 | 2.182768084421 |
| -0.41 | 2.207253098098 |
| -0.42 | 2.231991709228 |
| -0.43 | 2.256984936149 |
| -0.44 | 2.282233801921 |
| -0.45 | 2.307739333803 |
| -0.46 | 2.333502562787 |
| -0.47 | 2.359524523159 |
| -0.48 | 2.385806252119 |
| -0.49 | 2.412348789427 |
| -0.50 | 2.439153177078 |
| -0.51 | 2.466220459011 |
| -0.52 | 2.493551680840 |
| -0.53 | 2.521147889597 |
| -0.54 | 2.549010133493 |
| -0.55 | 2.577139461691 |
| -0.56 | 2.605536924076 |
| -0.57 | 2.634203571035 |
| -0.58 | 2.663140453233 |
| -0.59 | 2.692348621390 |
| -0.60 | 2.721829126050 |
| -0.61 | 2.751583017349 |
| -0.62 | 2.781611344783 |
| -0.63 | 2.811915156971 |
| -0.64 | 2.842495501418 |
| -0.65 | 2.873353424282 |
| -0.66 | 2.904489970151 |
| -0.67 | 2.935906181819 |
| -0.68 | 2.967603100085 |
| -0.69 | 2.999581763555 |
| -0.70 | 3.031843208464 |
| -0.71 | 3.064388468511 |
| -0.72 | 3.097218574704 |
| -0.73 | 3.130334555215 |
| -0.74 | 3.163737435238 |
| -0.75 | 3.197428236840 |
| -0.76 | 3.231407978806 |
| -0.77 | 3.265677676459 |
| -0.78 | 3.300238341455 |
| -0.79 | 3.335090981547 |
| -0.80 | 3.370236600313 |
| -0.81 | 3.405676196852 |
| -0.82 | 3.441410765462 |
| -0.83 | 3.477441295293 |
| -0.84 | 3.513768770009 |
| -0.85 | 3.550394167445 |
| -0.86 | 3.587318459279 |
| -0.87 | 3.624542610686 |
| -0.88 | 3.662067579940 |
| -0.89 | 3.699894317923 |
| -0.90 | 3.738023767478 |
| -0.91 | 3.776456862582 |
| -0.92 | 3.815194527361 |
| -0.93 | 3.854237675003 |
| -0.94 | 3.893587206579 |
| -0.95 | 3.933244009537 |
| -0.96 | 3.973208955444 |
| -0.97 | 4.013482896672 |
| -0.98 | 4.054066661898 |
| -0.99 | 4.094961048250 |
| -1.00 | 4.136166803904 |

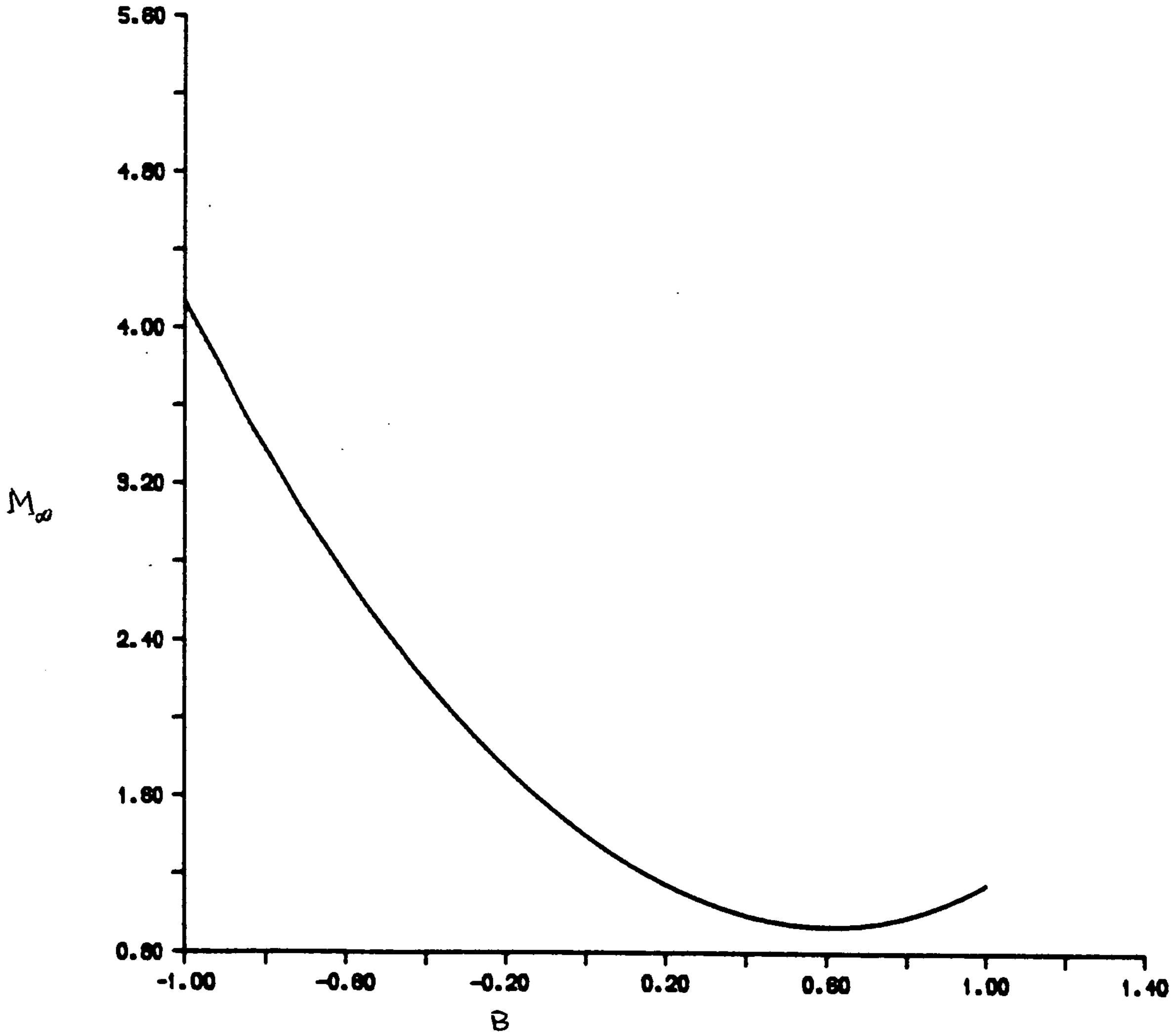


Fig.5:The graph of M(infinity)
between B=-1 AND B=1 .
Scales:B-axis,1cm=0.2
$M_\infty$-axis,1cm=0.4

Table 8 is provided for the values of $M_\infty$ for the whole range of B-values from +1 to −1 with the difference 0.01 and the graph of $M_\infty$ for these values is shown in Fig. 5. However the computational exploration suggests that this curve is not as smooth as a parabola is near $B = \pm 1$.

### 1.2.2 Some Theoretical Discussions

We present below some theoretical discussions which motivate the above-mentioned conjecture.

Let T be the period doubling operator with suitable coordinate adjustments on the space S of all smooth mappings from the plane into itself. Also let $\phi$ and $\psi$ be the area preserving and the dissipative fixed points of T respectively. Then the Jacobian $DT(\phi)$ of the operator T at $\phi$ has two eigenvalues greater than one, namely $\tilde{\delta}$ and 2 (see [10]).

Then by the manifold theorem, there exist at $\phi$ an unstable manifold $W^u(\phi)$ of dimension 2 spanned by the eigenvectors corresponding to the eigenvalues $\tilde{\delta}$ and 2, and a stable manifold $W^s(\phi)$ of codimension 2. Since the space is infinite dimensional, this stable manifold is also infinite dimensional. Moreover, these manifolds are invariant under T.

Let $\lambda_1 (= \tilde{\delta})$, $\lambda_2$ (= 2), $\lambda_3$, ... be the eigenvalues of $DT(\phi)$. Linearize T, if possible, near $\phi$ and choose coordinates $x_i$, $i = 1,2, \ldots$, in such a way that the following hold:

(i) $Tx_i = \lambda_i x_i$

(ii) $Tx = \Sigma \alpha_i \lambda_i x_i$, for $x = \Sigma \alpha_i x_i$ with some scalars $\alpha_i$'s.

(iii) $W^s(\psi)^- = \{x: x = \Sigma\alpha_i x_i$ and $x_1 = f(x_2, x_3, \ldots)$ for some suitable map $f\}$

Now $\quad x_1 = f(x_2, x_3, \ldots\ldots)$ .

So, $\lambda_1^{-r} x_1 = f(2^{-r} x_2, \lambda_3^{-r} x_3, \ldots)$, for all $r > 0$.

This gives

$$\tilde{\delta}^{-r} f(x_2, x_3, \ldots) = f(2^{-r} x_2, \lambda_3^{-r} x_3, \ldots).$$

Taking m-times partial derivative with respect to $x_2$, we obtain

$$\tilde{\delta}^{-r} \frac{\partial^m f}{\partial x_2^m} (x_2, x_3, \ldots) = 2^{-rm} \frac{\partial^m f}{\partial x_2^m} (2^{-r} x_2, \lambda_3^{-r} x_3, \ldots) .$$

This implies

$$\frac{\partial^m f}{\partial x_2^m} (2^{-r} x_2, \lambda_3^{-r} x_3, \ldots) = \left(\frac{2^m}{\tilde{\delta}}\right)^r \frac{\partial^m f}{\partial x_2^m} (x_2, x_3, \ldots) \qquad \text{(I)}$$

We now see that at the point $A = (1\ (=x_2), 0(=x_3), \ldots)$ the right hand side of (I) is bounded for all sufficiently large values of $r$ only if $\frac{2^m}{\tilde{\delta}} \leq 1$, i.e., $m \leq \log_2 \tilde{\delta}$. Hence in order that $\frac{\partial^m f}{\partial x_2^m}$ is continuous, $m$ must be less than or equal to $\log_2 \tilde{\delta}$. Since $m$ is an integer, the greatest value of $m$ is 3. Moreover, if $f \in C^\alpha$, $\alpha = 3+\mu$, $\mu > 0$, then

$$\frac{\partial^3 f}{\partial x_2^3} (2^{-r}, 0, 0 \ldots) = \left(\frac{8}{\tilde{\delta}}\right)^r \frac{\partial^3 f}{\partial x_2^3} (1, 0, \ldots) . \qquad \text{(II)}$$

Since $\frac{\partial^3 f(0)}{\partial x_2^3} = 0$ from (II) we obtain

$$\left| \frac{\partial^3 f}{\partial x_2^3} (2^{-r}, 0, \ldots) - \frac{\partial^3 f}{\partial x_2^3} (0, 0, \ldots) \right|$$

$$= \left(\frac{8}{\tilde{\delta}}\right)^r \left| \frac{\partial^3 f}{\partial x_2^3} (1, 0, \ldots) \right|$$

$$\leq C(2^{-r})^{\mu}, \quad \text{where} \quad C = \left(\frac{8}{\tilde{\delta}} \cdot 2^{\mu}\right)^{r} \left|\frac{\partial^3 f}{\partial x_2^3}(1,0,\ldots)\right| .$$

Here $\bar{C}$ remains bounded for all values of $r$ if $\mu \leq \log_2\tilde{\delta} - 3$. This suggests that $W^s(\psi)^-$ is not a smooth submanifold of $S$.

Next, the class $\{H_{M,B}$: $M$ and $B$ are reals$\}$ forms a two-dimensional manifold in the function space $S$ and

$H_{M_\infty(B),B} \in W^s(\psi)^- \cap W^s(\phi)$ if $|B| = 1$.

The arguments cited above lead us to imagine that the curve $M_\infty = M_\infty(B)$ is only $C^\alpha$ smooth, where $\alpha = \log_2\tilde{\delta}$ near $B = \pm 1$.

On the other hand, the stable manifold $W^s(\psi)$ is a smooth submanifold of $S$ (see [12]). Again $H_{M_\infty(B),B}$ belongs to $W^s(\psi)$ for all $B$ with $|B| < 1$. Therefore, the curve

$$\{(M_\infty,B) : |B| < 1 \text{ and } H_{M_\infty,B} \in W^s(\psi)\} \text{ is } C^\infty\text{-smooth.}$$

The relation $M_\infty(B^{-1}) = B^{-2} M_\infty(B)$ implies that this curve is $C^\infty$ for $|B| > 1$ as well.

Nevertheless, our arguments above do not guarantee any conclusion rigorously, but shed light on the conjecture stated above. So our next goal is to discuss the theory of finite differences through which we want to study the curve $M_\infty = M_\infty(B)$.

### 1.2.3. The Theory Of Finite Differences And the Curve $M_\infty = M_\infty(B)$, $B \in (-\infty,\infty)$

The finite difference operator $\Delta$ is defined for a real valued function $y = f(x)$ in the interval $[a,b]$ by

$\Delta f(x_o) = f(x_o+h) - f(x_o)$ , where $x_o$ is a particular value of $x$ in $[a,b]$ and $h$, a step-length. For higher differences the following symbolisms are used

$$\Delta(\Delta f(x)) = \Delta^2 f(x), \ldots, \Delta(\Delta^r f(x)) = \Delta^{r+1} f(x).$$

If $D$ is the differential operator, then the theory of finite differences gives the relation $\Delta = e^{hD} - 1$. From this we have

$$\Delta^i = h^i D^i \left(1 + \frac{hD}{2!} + \ldots\right)^i ,$$

where $i = 1,2,3, \ldots$ .

Therefore, for small $h$ we have an approximate relation as $\Delta^i \sim h^i D^i$.

More precisely, we have

$$|\Delta_h^i f(x_o)| \sim |h|^i |D^i f(x_o)| , \quad i = 1,2,3,\ldots, \qquad \text{(I)}$$

where the lower suffix $h$ in $\Delta$ is written to emphasize the step-length. If the step-length $h$ is increased to $2h$, then the same sort of relation can be achieved as

$$|\Delta_{2h}^i f(x_o)| \sim |2h|^i |D^i f(x_o)| . \qquad \text{(II)}$$

The relations (I) and (II) jointly imply that

$$\frac{|\Delta_h^i f(x_o)|}{|\Delta_{2h}^i f(x_o)|} \sim 2^{-i} , \quad i = 1, 2, \ldots .$$

Replacing $f(x_o)$ by $M_\infty(B_o)$, we obtain

$$\frac{|\Delta_h^i M_\infty(B_o)|}{|\Delta_{2h}^i M_\infty(B_o)|} \sim 2^{-i} . \tag{III}$$

From this we can say that if the function $M_\infty$ is i-times differentiable, then a relation of the form (III) can be obtained.

The Case When $|B| < 1$.

It can be shown numerically that when $|B| < 1$, the result (III) is true for all i. To support this assertion we cite the case when B = 0.6. Around B = 0.6, 41 values starting from 0.580 to 0.620 are considered and the $M_\infty$-values at these B are calculated up to 18 decimal places in longlongreal precision. To justify the claim that these $M_\infty$-values are correct up to 18 decimal places, we list in Table 9 the $M_\infty$-values for different periods k at B = 0.60 and B = 0.999. It is also found that the increase of negative power of 10 in the convergence condition for the periodic points does not affect these values. The B-values, their corresponding $M_\infty$-values, and the successive finite differences up to fifth order with the step-length 0.001 and with some special comments are listed in Tables 10 and 11. Also their successive finite differences with the step-length 0.002 are shown in Table 12.

If the ratios, $\dfrac{|\Delta_{0.001}^i M_\infty(B_o)|}{|\Delta_{0.002}^i M_\infty(B_o)|}$,

$i = 1,2,3,4,5$ and $B_o = 0.580, \ldots, 0.618,$

are calculated, it is found that they are approximately equal to $2^{-i}$.

To allude specifically, suppose $B_o = 0.580$.

Then
$$\frac{|\Delta_{0.001}\ M_\infty(0.580)|}{|\Delta_{0.002}\ M_\infty(0.580)|}$$

$$= \frac{0.750555 \times 10^{-4}}{0.147436 \times 10^{-3}}$$ , substituting the values from Tables 10 and 12.

$$\sim 0.509074$$

$$\sim 2^{-1} .$$

Similarly,

$$\frac{|\Delta^2_{0.001}\ M_\infty(0.580)|}{|\Delta^2_{0.002}\ M_\infty(0.580)|} \sim 0.249905 \sim 2^{-2} ,$$

$$\frac{|\Delta^3_{0.001}\ M_\infty(0.580)|}{|\Delta^3_{0.002}\ M_\infty(0.580)|} \sim 0.124659 \sim 2^{-3} ,$$

$$\frac{|\Delta^4_{0.001}\ M_\infty(0.580)|}{|\Delta^4_{0.002}\ M_\infty(0.580)|} \sim 0.061973 \sim 2^{-4} ,$$

and
$$\frac{|\Delta^5_{0.001}\ M_\infty(0.580)|}{|\Delta^5_{0.002}\ M_\infty(0.580)|} \sim 0.032529 \sim 2^{-5} .$$

Analogous results can be found for other values of $B_o$. If further higher finite differences are evaluated, the ratios with step-lengths 0.001 and 0.002 show similar characterizations. This leads

TABLE 9

!COMMENTS:IN THE FOLLOWING TABLE ,1ST,2ND AND %C
3RD COLUMNS REPRESENT RESPECTIVELY THE PERIODS %C
(K-VALUES),THE M(INFINITY)-VALUES FOR B=0.6 %C
AND THE M(INFINITY)-VALUES FOR B=0.999 .

| | | |
|---|---|---|
| 4.0 | 0. 937051895951745786 | 1.169318869722449742 |
| 8.0 | 0. 934261315005137582 | 1.154923504206398399 |
| 16.0 | 0. 934188780301218141 | 1.152746427806330732 |
| 32.0 | 0. 934194020477330677 | 1.152493111144941790 |
| 64.0 | 0. 934194330850821958 | 1.152463942197569717 |
| 128.0 | 0. 934194352446653112 | 1.152460598794910413 |
| 256.0 | 0. 934194353378878220 | 1.152460216316855307 |
| 512.0 | 0. 934194353424139706 | 1.152460172876164771 |
| 1024.0 | 0. 934194353426158145 | 1.152460168079801284 |
| 2048.0 | 0. 934194353426252336 | 1.152460167604650976 |
| 4096.0 | 0. 934194353426256614 | 1.152460167572461434 |
| 8192.0 | 0. 934194353426256812 | 1.152460167571796193 |
| 16384.0 | 0. 934194353426256821 | 1.152460167571804477 |
| 32768.0 | 0. 934194353426256821 | 1.152460167571805621 |
| 65536.0 | 0. 934194353426256821 | 1.152460167571805706 |
| 131072.0 | 0. 934194353426256821 | 1.152460167571805710 |
| 262144.0 | 0. 934194353426256821 | 1.152460167571805710 |

TABLE 10

A(1)=10000
!COMMENT:IN THE FOLLOWING TABLE,THE FIRST,THE SECOND AND THE THIRD COLUMNS GIVE RESPECTIVELY THE VALUES OF B,M(INFINITY) AND FIRST FINITE DIFFERENCES FOR STEP-LENGTH 0.001.

| | | |
|---|---|---|
| 0.580 | 0.935185934155621779 | |
| 0.581 | 0.935110878577981541 | -0.750555/A(1) |
| 0.582 | 0.935038498622734149 | -0.723799/A(1) |
| 0.583 | 0.934968795301744995 | -0.697033/A(1) |
| 0.584 | 0.934901769628721421 | -0.670256/A(1) |
| 0.585 | 0.934837422619220645 | -0.643470/A(1) |
| 0.586 | 0.934775755290657563 | -0.616673/A(1) |
| 0.587 | 0.934716768662312423 | -0.589866/A(1) |
| 0.588 | 0.934660463755338382 | -0.563049/A(1) |
| 0.589 | 0.934606841592768929 | -0.536221/A(1) |
| 0.590 | 0.934555903199525183 | -0.509383/A(1) |
| 0.591 | 0.934507649602423062 | -0.482535/A(1) |
| 0.592 | 0.934462081830180320 | -0.455677/A(1) |
| 0.593 | 0.934419200913423453 | -0.428809/A(1) |
| 0.594 | 0.934379007884694469 | -0.401930/A(1) |
| 0.595 | 0.934341503778457526 | -0.375041/A(1) |
| 0.596 | 0.934306689631105432 | -0.348141/A(1) |
| 0.597 | 0.934274566480966006 | -0.321231/A(1) |
| 0.598 | 0.934245135368308302 | -0.294311/A(1) |
| 0.599 | 0.934218397335348696 | -0.267380/A(1) |
| 0.600 | 0.934194353426256821 | -0.240439/A(1) |
| 0.601 | 0.934173004687161374 | -0.213487/A(1) |
| 0.602 | 0.934154352166155765 | -0.186525/A(1) |
| 0.603 | 0.934138396913303633 | -0.159552/A(1) |
| 0.604 | 0.934125139980644207 | -0.132569/A(1) |
| 0.605 | 0.934114582422197523 | -0.105575/A(1) |
| 0.606 | 0.934106725293969498 | -0.078571/A(1) |
| 0.607 | 0.934101569653956844 | -0.051556/A(1) |
| 0.608 | 0.934099116562151841 | -0.024530/A(1) |
| 0.609 | 0.934099367080546951 | 0.002505/A(1) |
| 0.610 | 0.934102322273139291 | 0.029551/A(1) |
| 0.611 | 0.934107983205934939 | 0.056609/A(1) |
| 0.612 | 0.934116350946953092 | 0.083677/A(1) |
| 0.613 | 0.934127426566230076 | 0.110756/A(1) |
| 0.614 | 0.934141211135823185 | 0.137845/A(1) |
| 0.615 | 0.934157705729814377 | 0.164945/A(1) |
| 0.616 | 0.934176911424313804 | 0.192056/A(1) |
| 0.617 | 0.934198829297463186 | 0.219178/A(1) |
| 0.618 | 0.934223460429439028 | 0.246311/A(1) |
| 0.619 | 0.934250805902455677 | 0.273454/A(1) |
| 0.620 | 0.934280866800768209 | 0.300608/A(1) |

TABLE 11

A(2)=100000;A(3)=100000000;A(4)=100000000000
A(5)=100000000000000
!COMMENTS:DIVIDE THE FOLLOWING 4 COLUMNS RESPECTIVELY BY A(2),A(3),A(4) AND A(5).THEN THESE COLUMNS SHOW 2ND,3RD,4TH AND 5TH FINITE DIFFERENCES RESPECTIVELY WITH THE STEP-LENTH 0.001.

| | | | |
|---|---|---|---|
| 0.267562 | | | |
| 0.267663 | 0.101187 | | |
| 0.267765 | 0.101371 | 0.184195 | |
| 0.267866 | 0.101556 | 0.184988 | 0.7926 |
| 0.267968 | 0.101741 | 0.185768 | 0.7802 |
| 0.268070 | 0.101928 | 0.186535 | 0.7674 |
| 0.268172 | 0.102115 | 0.187291 | 0.7557 |
| 0.268274 | 0.102303 | 0.188033 | 0.7423 |
| 0.268377 | 0.102492 | 0.188763 | 0.7298 |
| 0.268480 | 0.102682 | 0.189480 | 0.7169 |
| 0.268582 | 0.102872 | 0.190184 | 0.7037 |
| 0.268686 | 0.103063 | 0.190874 | 0.6906 |
| 0.268789 | 0.103254 | 0.191551 | 0.6770 |
| 0.268892 | 0.103446 | 0.192215 | 0.6638 |
| 0.268996 | 0.103639 | 0.192865 | 0.6500 |
| 0.269100 | 0.103833 | 0.193501 | 0.6361 |
| 0.269204 | 0.104027 | 0.194124 | 0.6224 |
| 0.269308 | 0.104222 | 0.194732 | 0.6087 |
| 0.269412 | 0.104417 | 0.195326 | 0.5935 |
| 0.269517 | 0.104613 | 0.195906 | 0.5807 |
| 0.269622 | 0.104809 | 0.196471 | 0.5649 |
| 0.269727 | 0.105006 | 0.197023 | 0.5516 |
| 0.269832 | 0.105204 | 0.197559 | 0.5361 |
| 0.269937 | 0.105402 | 0.198081 | 0.5217 |
| 0.270043 | 0.105601 | 0.198588 | 0.5074 |
| 0.270149 | 0.105800 | 0.199080 | 0.4914 |
| 0.270255 | 0.105999 | 0.199557 | 0.4773 |
| 0.270361 | 0.106199 | 0.200018 | 0.4614 |
| 0.270467 | 0.106400 | 0.200466 | 0.4473 |
| 0.270574 | 0.106601 | 0.200896 | 0.4306 |
| 0.270681 | 0.106802 | 0.201312 | 0.4158 |
| 0.270788 | 0.107004 | 0.201713 | 0.4010 |
| 0.270895 | 0.107206 | 0.202097 | 0.3839 |
| 0.271002 | 0.107408 | 0.202466 | 0.3696 |
| 0.271110 | 0.107611 | 0.202819 | 0.3530 |
| 0.271218 | 0.107814 | 0.203157 | 0.3374 |
| 0.271326 | 0.108018 | 0.203479 | 0.3217 |
| 0.271434 | 0.108221 | 0.203784 | 0.3057 |
| 0.271543 | 0.108426 | 0.204073 | 0.2887 |

TABLE 12

A(6)=1000;A(7)=10000;A(8)=100000000;
A(9)=10000000000;A(10)=1000000000000
!COMMENTS:DIVIDE THE FOLLOWING 5 COLUMNS RESPECTIVELY BY A(6),A(7),A(8),A(9) AND A(10).THEN THESE COLUMNS GIVE 1ST,2ND,3RD,4TH AND 5TH FINITE DIFFERENCES RESPECTIVELY WITH THE STEP-LENGTH 0.002.

| | | | | |
|---|---|---|---|---|
| -0.147436 | | | | |
| -0.142083 | | | | |
| -0.136729 | 0.107065 | | | |
| -0.131373 | 0.107106 | | | |
| -0.126014 | 0.107147 | 0.811707 | | |
| -0.120654 | 0.107187 | 0.813190 | | |
| -0.115292 | 0.107228 | 0.814680 | 0.297218 | |
| -0.109927 | 0.107269 | 0.816175 | 0.298447 | |
| -0.104561 | 0.107310 | 0.817676 | 0.299655 | 0.243659 |
| -0.099192 | 0.107351 | 0.819183 | 0.300843 | 0.239631 |
| -0.093821 | 0.107392 | 0.820696 | 0.302010 | 0.235551 |
| -0.088449 | 0.107433 | 0.822215 | 0.303157 | 0.231433 |
| -0.083074 | 0.107474 | 0.823739 | 0.304283 | 0.227273 |
| -0.077697 | 0.107516 | 0.825269 | 0.305388 | 0.223061 |
| -0.072318 | 0.107557 | 0.826804 | 0.306471 | 0.218804 |
| -0.066937 | 0.107598 | 0.828344 | 0.307533 | 0.214508 |
| -0.061554 | 0.107640 | 0.829890 | 0.308573 | 0.210165 |
| -0.056169 | 0.107682 | 0.831440 | 0.309591 | 0.205777 |
| -0.050782 | 0.107723 | 0.832995 | 0.310586 | 0.201358 |
| -0.045393 | 0.107765 | 0.834556 | 0.311560 | 0.196897 |
| -0.040001 | 0.107807 | 0.836120 | 0.312510 | 0.192385 |
| -0.034608 | 0.107849 | 0.837690 | 0.313438 | 0.187840 |
| -0.029212 | 0.107891 | 0.839264 | 0.314343 | 0.183263 |
| -0.023814 | 0.107933 | 0.840842 | 0.315225 | 0.178642 |
| -0.018415 | 0.107975 | 0.842425 | 0.316083 | 0.173981 |
| -0.013013 | 0.108017 | 0.844011 | 0.316918 | 0.169293 |
| -0.007609 | 0.108060 | 0.845602 | 0.317729 | 0.164574 |
| -0.002203 | 0.108102 | 0.847197 | 0.318516 | 0.159805 |
| 0.003206 | 0.108144 | 0.848795 | 0.319279 | 0.155000 |
| 0.008616 | 0.108187 | 0.850397 | 0.320017 | 0.150185 |
| 0.014029 | 0.108230 | 0.852002 | 0.320732 | 0.145337 |
| 0.019443 | 0.108272 | 0.853611 | 0.321422 | 0.140433 |
| 0.024860 | 0.108315 | 0.855223 | 0.322087 | 0.135508 |
| 0.030279 | 0.108358 | 0.856838 | 0.322727 | 0.130574 |
| 0.035700 | 0.108401 | 0.858456 | 0.323343 | 0.125596 |
| 0.041124 | 0.108444 | 0.860078 | 0.323933 | 0.120582 |
| 0.046549 | 0.108487 | 0.861701 | 0.324498 | 0.115552 |
| 0.051977 | 0.108530 | 0.863328 | 0.325038 | 0.110501 |
| 0.057406 | 0.108574 | 0.864957 | 0.325553 | 0.105428 |

us to conclude that the curve $M_\infty = M_\infty(B)$ for $-1 < B < 1$ is $C^\infty$-smooth.

The Case When $|B| = 1$.

In the neighbourhood of $B = 1$ and $-1$, it can be shown numerically that the relation (III) is true up to $i = 3$ and that for $i > 3$ the ratio is very far from the value $2^{-i}$. To verify this fact, we consider just as above 41 B-values around $B = 1$, namely, $(0.999)^Y$, $Y = 20, 19, \ldots, -20$. Then the finite differences are applied to $M_\infty(e^u)$ as a function of $u$ with step-length $H = -\log(0.999)$, where $u = \log B$. The reason for doing this is that if we evaluate the values $M_\infty(e^u)$ for $B = (0.999)^Y$, $Y = 20, 19, \ldots, 0$, then the other values $M_\infty(e^u)$ for $B = (0.999)^Y$, $Y = -1, -2, \ldots, -20$ can be evaluated just by using the formula $M_\infty(B^{-1}) = B^{-2}M_\infty(B)$. These B-values, the corresponding $M_\infty$ values and the successive finite difference values up to 5th order are listed in Tables 13 and 14. Further, the finite difference values up to 5th order with step-length 2H are shown in Table 15. If we observe the values of these finite differences in these Tables, it is seen that there is a symmetry in values up to third differences, but fourth and fifth difference values show irregular behaviour. This indicates reasonably that the relation (III) is not satisfied when $i > 3$. To be more precise, let $B_o = 0.999$.

Then
$$\frac{|\Delta_H M_\infty(B_o)|}{|\Delta_{2H} M_\infty(B_o)|} = \frac{0.114769 \times 10^{-2}}{0.229971 \times 10^{-2}}$$

$$\sim 0.499059 \sim 2^{-1},$$

$$\frac{|\Delta_H^2 M_\infty(B_o)|}{|\Delta_{2H}^2 M_\infty(B_o)|} = \frac{0.432862 \times 10^{-5}}{0.173582 \times 10^{-4}}$$

$$\sim 0.249371 \sim 2^{-2}$$

$$\frac{|\Delta_H^3 M_\infty(B_o)|}{|\Delta_{2H}^3 M_\infty(B_o)|} = \frac{0.111370 \times 10^{-7}}{0.821066 \times 10^{-7}}$$

$$\sim 0.135640 \sim 2^{-3}$$

$$\frac{|\Delta_H^4 M_\infty(B_o)|}{|\Delta_{2H}^4 M_\infty(B_o)|} = \frac{0.845435 \times 10^{-9}}{0.210925 \times 10^{-8}}$$

$$\sim 0.400822 \not\sim 2^{-4}$$

and

$$\frac{|\Delta_H^5 M_\infty(B_o)|}{|\Delta_{2H}^5 M_\infty(B_o)|} = \frac{0.584835 \times 10^{-9}}{0.202068 \times 10^{-8}}$$

$$\sim 0.289424 \not\sim 2^{-5} \quad .$$

We can carry out similar calculations for higher finite differences and can find that the relation (III) is highly unsatisfied. So we can draw the conclusion that the curve $M_\infty = M_\infty(B)$ near $B = 1$ is $C^3$ only.

In Table 16, 21 values of $B$ around $B = -1$, the corresponding $M_\infty$-values, and the finite differences up to 5th order with step-length $-H$ are shown, while in Table 17, the finite differences with step-length -2H are listed. It is evident from these Tables that there is a regularity in the finite difference values up to 3rd order, but that there is no such regularity in the case of 4th and 5th order finite differences. If we calculate the values

TABLE 13

!COMMENTS:HERE N(1)=100,X=0.999 AND THE EXPRESSION
X**Y MEANS THAT THE EXPONENT OF X IS Y,(Y=20,19,..,-20). %C
THE THREE COLUMNS IN THE FOLLOWING TABLE SHOW %C
RESPECTIVELY THE VALUES OF B,M(INFINITY) AND %C
THE FIRST FINITE DIFFERENCES OF THE FUCTION %C
M(INFINITY) WITH STEP-LENGTH EQUAL TO H=-LOG(.999) .

| | | |
|---|---|---|
| X**20 | 1.131381219307043391 | |
| X**19 | 1.132452941176066519 | 0.107172/N(1) |
| X**18 | 1.133528787734328562 | 0.107585/N(1) |
| X**17 | 1.134608770161625452 | 0.107998/N(1) |
| X**16 | 1.135692899659255966 | 0.108413/N(1) |
| X**15 | 1.136781187449718716 | 0.108829/N(1) |
| X**14 | 1.137873644776346336 | 0.109246/N(1) |
| X**13 | 1.138970282902865887 | 0.109664/N(1) |
| X**12 | 1.140071113112880811 | 0.110083/N(1) |
| X**11 | 1.141076146709278335 | 0.110503/N(1) |
| X**10 | 1.142285395013571740 | 0.110925/N(1) |
| X**9 | 1.143398869365178994 | 0.111347/N(1) |
| X**8 | 1.144516581120604523 | 0.111771/N(1) |
| X**7 | 1.145638541652425755 | 0.112196/N(1) |
| X**6 | 1.146764762347921667 | 0.112622/N(1) |
| X**5 | 1.147895254607197664 | 0.113049/N(1) |
| X**4 | 1.149030029840788276 | 0.113478/N(1) |
| X**3 | 1.150169099466611584 | 0.113907/N(1) |
| X**2 | 1.151312474905250401 | 0.114338/N(1) |
| X**1 | 1.152460167571805711 | 0.114769/N(1) |
| X**0 | 1.153612188858759193 | 0.115202/N(1) |
| X**-1 | 1.154768549903061932 | 0.115636/N(1) |
| X**-2 | 1.155929260996230460 | 0.116071/N(1) |
| X**-3 | 1.157094332169181304 | 0.116507/N(1) |
| X**-4 | 1.158263773424284349 | 0.116944/N(1) |
| X**-5 | 1.159437594752278002 | 0.117382/N(1) |
| X**-6 | 1.160615806140290558 | 0.117821/N(1) |
| X**-7 | 1.161798417576483278 | 0.118261/N(1) |
| X**-8 | 1.162985439053049948 | 0.118702/N(1) |
| X**-9 | 1.164176880568593056 | 0.119144/N(1) |
| X**-10 | 1.165372752129996270 | 0.119587/N(1) |
| X**-11 | 1.166573063753804824 | 0.120031/N(1) |
| X**-12 | 1.167777825467256020 | 0.120476/N(1) |
| X**-13 | 1.168987047309121836 | 0.120922/N(1) |
| X**-14 | 1.170200739330462119 | 0.121369/N(1) |
| X**-15 | 1.171418911595321235 | 0.121817/N(1) |
| X**-16 | 1.172641574181365812 | 0.122266/N(1) |
| X**-17 | 1.173868737180452922 | 0.122716/N(1) |
| X**-18 | 1.175100410699123642 | 0.123167/N(1) |
| X**-19 | 1.176336604859025752 | 0.123619/N(1) |
| X**-20 | 1.177577329797275960 | 0.124072/N(1) |

TABLE 14

N(2)=100000;N(3)=10000000
N(4)=1000000000 ;N(5)=N(4).
!COMMENTS:DIVIDE THE FOLLOWING 4 COLUMNS
BY N(2),N(3),N(4) AND N(5) RESPECTIVELY,
AND THEN THESE COLUMNS GIVE RESPECTIVELY
2ND,3RD,4TH AND 5TH FINITE DIFFERENCES
WITH THE STEP-LENGTH H

| | | | |
|---|---|---|---|
| 0.412469 | | | |
| 0.413587 | 0.111798 | | |
| 0.414707 | 0.112013 | 0.021503 | |
| 0.415829 | 0.112225 | 0.021200 | -0.000303 |
| 0.416954 | 0.112433 | 0.020834 | -0.000366 |
| 0.418080 | 0.112637 | 0.020394 | -0.000440 |
| 0.419208 | 0.112836 | 0.019876 | -0.000518 |
| 0.420339 | 0.113029 | 0.019284 | -0.000593 |
| 0.421471 | 0.113215 | 0.018626 | -0.000658 |
| 0.422605 | 0.113394 | 0.017905 | -0.000721 |
| 0.423740 | 0.113565 | 0.017086 | -0.000818 |
| 0.424878 | 0.113726 | 0.016073 | -0.001013 |
| 0.426016 | 0.113873 | 0.014702 | -0.001371 |
| 0.427156 | 0.114001 | 0.012826 | -0.001875 |
| 0.428297 | 0.114105 | 0.010429 | -0.002397 |
| 0.429439 | 0.114179 | 0.007384 | -0.003046 |
| 0.430581 | 0.114206 | 0.002665 | -0.004719 |
| 0.431723 | 0.114151 | -0.005482 | -0.008147 |
| 0.432862 | 0.113925 | -0.022619 | -0.017137 |
| 0.433976 | 0.111370 | -0.255531 | -0.232911 |
| 0.435005 | 0.102915 | -0.845435 | -0.589904 |
| 0.436008 | 0.100309 | -0.260600 | 0.584835 |
| 0.437008 | 0.100024 | -0.028547 | 0.232053 |
| 0.438007 | 0.099907 | -0.011631 | 0.016915 |
| 0.439006 | 0.099871 | -0.003610 | 0.008021 |
| 0.440005 | 0.099882 | 0.001033 | 0.004643 |
| 0.441004 | 0.099922 | 0.004033 | 0.003000 |
| 0.442004 | 0.099986 | 0.006409 | 0.002376 |
| 0.443005 | 0.100069 | 0.008281 | 0.001872 |
| 0.444006 | 0.100165 | 0.009662 | 0.001380 |
| 0.445009 | 0.100272 | 0.010692 | 0.001031 |
| 0.446013 | 0.100388 | 0.011535 | 0.000843 |
| 0.447018 | 0.100511 | 0.012288 | 0.000753 |
| 0.448024 | 0.100640 | 0.012985 | 0.000697 |
| 0.449032 | 0.100777 | 0.013622 | 0.000638 |
| 0.450041 | 0.100919 | 0.014190 | 0.000568 |
| 0.451052 | 0.101065 | 0.014684 | 0.000494 |
| 0.452064 | 0.101216 | 0.015107 | 0.000423 |
| 0.453078 | 0.101371 | 0.015469 | 0.000362 |

(TABLE 15)

N(6)=10000;N(7)=100000000
!COMMENTS:THE FOLLOWING COLUMNS SHOW 1ST,
2ND,3RD,4TH AND 5TH FINITE DIFFERENCES
WITH THE STEP-LENGTH EQUAL TO 2H
,IF THEY ARE RESPECTIVELY DIVIDED
BY N(1),N(6),N(3),N(7) AND N(7).

| | | | | |
|---|---|---|---|---|
| 0.214757 | | | | |
| 0.215583 | | | | |
| 0.216411 | 0.165435 | | | |
| 0.217242 | 0.165883 | | | |
| 0.218075 | 0.166332 | 0.896945 | | |
| 0.218910 | 0.166782 | 0.898625 | | |
| 0.219747 | 0.167232 | 0.900273 | 0.033276 | |
| 0.220586 | 0.167684 | 0.901882 | 0.032569 | |
| 0.221428 | 0.168136 | 0.903447 | 0.031743 | -0.001533 |
| 0.222272 | 0.168588 | 0.904962 | 0.030801 | -0.001768 |
| 0.223119 | 0.169042 | 0.906422 | 0.029747 | -0.001996 |
| 0.223967 | 0.169496 | 0.907819 | 0.028563 | -0.002238 |
| 0.224818 | 0.169951 | 0.909139 | 0.027176 | -0.002572 |
| 0.225671 | 0.170407 | 0.910362 | 0.025432 | -0.003131 |
| 0.226527 | 0.170863 | 0.911453 | 0.023132 | -0.004043 |
| 0.227384 | 0.171319 | 0.912371 | 0.020094 | -0.005338 |
| 0.228245 | 0.171776 | 0.913060 | 0.016078 | -0.007054 |
| 0.229107 | 0.172232 | 0.913411 | 0.010402 | -0.009692 |
| 0.229971 | 0.172689 | 0.913175 | 0.001141 | -0.014938 |
| 0.230838 | 0.173142 | 0.909803 | -0.036086 | -0.046488 |
| 0.231707 | 0.173582 | 0.892949 | -0.202254 | -0.203394 |
| 0.232578 | 0.173999 | 0.857088 | -0.527148 | -0.491062 |
| 0.233451 | 0.174403 | 0.821066 | -0.718830 | -0.516576 |
| 0.234326 | 0.174803 | 0.803821 | -0.532669 | -0.005521 |
| 0.235203 | 0.175203 | 0.799974 | -0.210925 | 0.507905 |
| 0.236082 | 0.175602 | 0.799241 | -0.045798 | 0.486870 |
| 0.236963 | 0.176002 | 0.799088 | -0.008857 | 0.202068 |
| 0.237846 | 0.176402 | 0.799268 | 0.000266 | 0.046065 |
| 0.238731 | 0.176802 | 0.799674 | 0.005863 | 0.014720 |
| 0.239618 | 0.177202 | 0.800252 | 0.009840 | 0.009574 |
| 0.240507 | 0.177603 | 0.800961 | 0.012869 | 0.007006 |
| 0.241398 | 0.178004 | 0.801770 | 0.015181 | 0.005340 |
| 0.242291 | 0.178405 | 0.802656 | 0.016951 | 0.004081 |
| 0.243186 | 0.178807 | 0.803608 | 0.018377 | 0.003197 |
| 0.244083 | 0.179210 | 0.804618 | 0.019612 | 0.002661 |
| 0.244983 | 0.179613 | 0.805680 | 0.020727 | 0.002350 |
| 0.245884 | 0.180017 | 0.806792 | 0.021741 | 0.002129 |
| 0.246787 | 0.180421 | 0.807945 | 0.022646 | 0.001919 |
| 0.247692 | 0.180826 | 0.809135 | 0.023438 | 0.001698 |

TABLE 16

!COMMENTS: HERE U=0.999 AND THE EXPRESSION
U ** V MEANS THE EXPONENT OF U IS V,
(V=10,9,....,-10).
THE FOLLOWING 3 COLUMNS GIVE THE
VALUES OF B,M(INFINITY) AND THE 1ST FINITE
DIFFERENCES RESPECTIVELY WITH THE STEP-LENGTH
EQUAL TO -H.

| | | |
|---|---|---|
| -U ** 10 | 4.095145284431820540 | |
| -U ** 9 | 4.099215149832903502 | 0.406987/N(1) |
| -U ** 8 | 4.103292152688709156 | 0.407700/N(1) |
| -U ** 7 | 4.107376306969043645 | 0.408415/N(1) |
| -U ** 6 | 4.111467626663355739 | 0.409132/N(1) |
| -U ** 5 | 4.115566125779267603 | 0.409850/N(1) |
| -U ** 4 | 4.119671818340691261 | 0.410569/N(1) |
| -U ** 3 | 4.123784718385430683 | 0.411290/N(1) |
| -U ** 2 | 4.127904839961456277 | 0.412012/N(1) |
| -U ** 1 | 4.132032197120459739 | 0.412736/N(1) |
| -U ** 0 | 4.136166803904275415 | 0.413461/N(1) |
| -U ** -1 | 4.140308674160105790 | 0.414187/N(1) |
| -U ** -2 | 4.144457821072506695 | 0.414915/N(1) |
| -U ** -3 | 4.148614257627411568 | 0.415644/N(1) |
| -U ** -4 | 4.152777996796256821 | 0.416374/N(1) |
| -U ** -5 | 4.156949051549415599 | 0.417105/N(1) |
| -U ** -6 | 4.161127434862575070 | 0.417838/N(1) |
| -U ** -7 | 4.165313159720433280 | 0.418572/N(1) |
| -U ** -8 | 4.169506239119091949 | 0.419308/N(1) |
| -U ** -9 | 4.173706686067954631 | 0.420045/N(1) |
| -U ** -10 | 4.177914513591225160 | 0.420783/N(1) |

!COMMENTS:DIVIDE THE FOLLOWING 4 COLUMNS BY
N(2),N(3),N(4) AND N(5) RESPECTIVELY AND THEN
THEY REPRESENT RESPECTIVELY 2ND,3RD,4TH AND 5TH
FINITE DEFFERENCES WITH THE STEP-LENGTH
EQUAL TO -H.

| | | | |
|---|---|---|---|
| 0.713745 | | | |
| 0.715142 | 0.139698 | | |
| 0.716541 | 0.139894 | 0.019643 | |
| 0.717942 | 0.140076 | 0.018173 | |
| 0.719345 | 0.140239 | 0.016290 | -0.001884 |
| 0.720748 | 0.140378 | 0.013892 | -0.002398 |
| 0.722153 | 0.140480 | 0.010166 | -0.003726 |
| 0.723558 | 0.140517 | 0.003721 | -0.006445 |
| 0.724962 | 0.140418 | -0.009857 | -0.013579 |
| 0.726347 | 0.138472 | -0.194632 | -0.184775 |
| 0.727666 | 0.131846 | -0.662647 | -0.468015 |
| 0.728964 | 0.129859 | -0.198622 | 0.464024 |
| 0.730261 | 0.129714 | -0.014497 | 0.184125 |
| 0.731558 | 0.129704 | -0.001063 | 0.013434 |
| 0.732856 | 0.129757 | 0.005314 | 0.006377 |
| 0.734154 | 0.129847 | 0.009011 | 0.003697 |
| 0.735454 | 0.129961 | 0.011404 | 0.002393 |
| 0.736755 | 0.130094 | 0.013302 | 0.001898 |
| 0.738057 | 0.130242 | 0.014800 | 0.001498 |

TABLE 17

N(8)=1000000
!COMMENTS:IF WE DIVIDE THE FOLLOWING FIVE COLUMNS RESPECTIVELY BY N(1),N(6),N(8),N(7) AND N(7),THEN THEY GIVE 1ST,2ND,3RD,4TH AND 5TH FINITE DIFFERENCES WITH THE STEP-LENGTH EQUAL TO -2H.

| | | | | |
|---|---|---|---|---|
| 0.814687 | | | | |
| 0.816116 | | | | |
| 0.817547 | 0.286057 | | | |
| 0.818982 | 0.286617 | | | |
| 0.820419 | 0.287177 | 0.111985 | | |
| 0.821859 | 0.287738 | 0.112122 | | |
| 0.823302 | 0.288299 | 0.112241 | 0.025581 | |
| 0.824748 | 0.288861 | 0.112333 | 0.021107 | |
| 0.826196 | 0.289423 | 0.112379 | 0.013788 | -0.011793 |
| 0.827648 | 0.289983 | 0.112176 | -0.015718 | -0.036825 |
| 0.829102 | 0.290532 | 0.110903 | -0.147527 | -0.161315 |
| 0.830558 | 0.291064 | 0.108123 | -0.405271 | -0.389553 |
| 0.832018 | 0.291586 | 0.105330 | -0.557325 | -0.409798 |
| 0.833479 | 0.292105 | 0.104027 | -0.409600 | -0.004330 |
| 0.834944 | 0.292623 | 0.103787 | -0.154306 | 0.403019 |
| 0.836411 | 0.293142 | 0.103794 | -0.023272 | 0.386328 |
| 0.837880 | 0.293662 | 0.103848 | 0.006058 | 0.160364 |
| 0.839353 | 0.294182 | 0.103927 | 0.013317 | 0.036590 |
| 0.840827 | 0.294702 | 0.104025 | 0.017779 | 0.011721 |

for further higher order finite differences, they show the same sort of behaviour as they do for 4th and 5th order. So, we can draw the same conclusion that the curve $M_\infty = M_\infty(B)$ is $C^3$ near $B = -1$.

Remark 1.2.4 This study suggests the following conjecture.

"The behaviour of the curve $M_\infty = M_\infty(B)$ is universal, that means, if $T: \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism having two parameters P and R, like M and B in the Henon map, such that for each value of R in a nicely chosen interval in the real line, period doubling bifurcations occur with the parameter P, then the curve $P_\infty = P_\infty(R)$ is $C^\alpha$, $\alpha = \log_2 \delta$, near that value of R which corresponds to $B = 1$ or $B = -1$ in the Henon map and $C^\infty$ for the other values of R in the chosen interval."

To have a clearer picture of this conjecture, we can explain it with the map F defined in Chapter 2. As mentioned there, this map has constant Jacobian $e^{-2\pi R}$ and therefore, here $R = 0$ corresponds to $B = 1$ in the Henon map. If we consider an interval $[0,L]$, L being a finite positive real number, then the curve $P_\infty = P_\infty(R)$ in this interval will possess the similar virtue as the curve $M_\infty = M_\infty(B)$ in the interval $(0,1]$ has.

## 1.3 Section Three: The Domain Of Attraction Of A Periodic Orbit, Homoclinic Points

The principal purpose in this section is to set forth some ideas behind the concept of the domain of attraction of a stable periodic orbit. Our computational tests suggest that the stable manifold at a particular unstable periodic point forms a part of the boundary of a region in which the domain of attraction of a stable periodic orbit lies. It is also hinted that there exist a transversal homoclinic point and a Horseshoe for higher values of B, say $B = 0.8$.

To present this theory in a comprehensive manner, we consider the parameter values as $B = 0.8$ and $M = 0.9$. At this value of B, the second and the third bifurcation values of M are 0.85 and 0.964285570069 respectively. So, for these B and M there exists a stable trajectory of period 4 comprising of the following periodic points.

P (-1.149409918717, 1.080458237198)

Q (-0.634709659927, 0.713143513662)

R (0.891429392077, -0.919527934974)

and S (1.350572796497, -0.507767727942).

Again for these B and M, there are two unstable periodic points of period 2 as

T (-0.925264339232, 0.917989249163)

and U (1.147486561454, -0.740211471386) .

Furthermore, another two unstable fixed points given by the equations

$$x = \frac{(B-1) \pm \sqrt{(1-B)^2 + 4M}}{2M}, \quad y = Bx, \quad \text{are}$$

V (0.948821334908, 0.759057067926)

and W (-1.171043557130, -0.936834845704).

The topological structure of the domains of attraction says that the domains of attraction of the periodic points P, Q, R, S are respectively separated by the stable manifolds at the unstable points T, V and U (see Fig. 6).

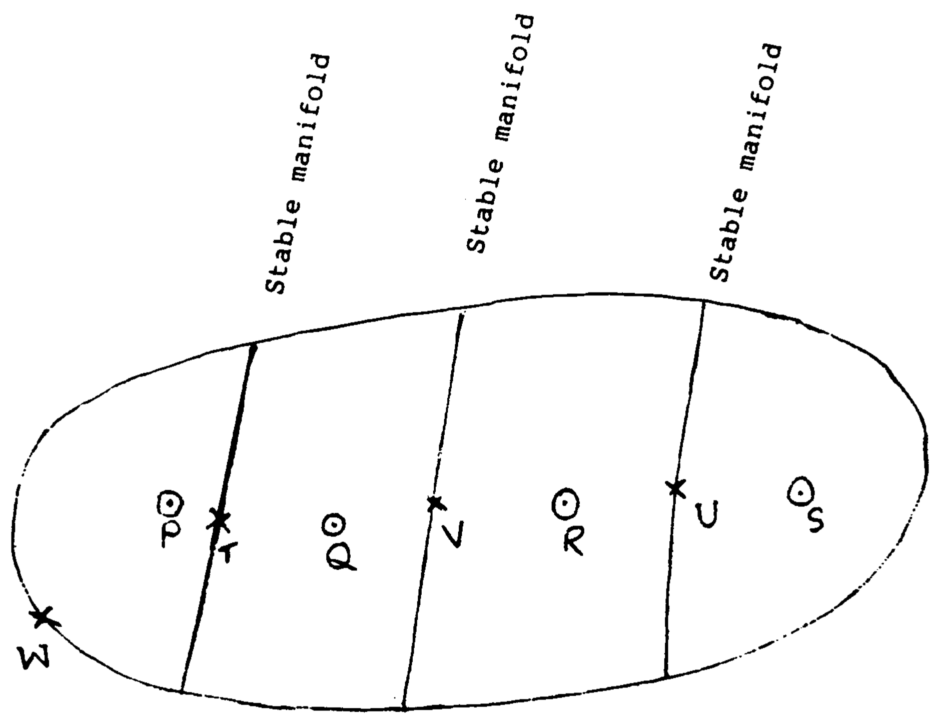


Fig.6:Topological structure of the domains of attraction.
The symbols O-for stable periodic points and X-for unstable periodic points .

But in reality, these manifolds show very wild behaviour indicating that the domains of attraction have complicated boundaries. Now, to exhibit stable and unstable manifolds at these unstable periodic points some numerical techniques are employed as described below:

First of all, the Stable Manifold Theorem (Th.1.4.2, p. 18 in [33]) states that if there is a hyperbolic point of period $k$, $A(\underline{x}_o)$, such that the eigenvalues of the Jacobian of the transformation $H^k$, ($k$ is the appropriate period), at this point are $\lambda$ and $\mu$ with $|\lambda| < 1$ and $|\mu| > 1$, then there exist stable and unstable manifolds $W^s$, $W^u$ at $A(\underline{x}_o)$ such that they are tangent to the eigenspaces $E^\lambda$, $E^\mu$ generated by the eigenvectors corresponding to the eigenvalues $\lambda$ and $\mu$ respectively. Let us consider eigenvectors $\underline{U}$ and $\underline{V}$ at the respective eigenvalues $\lambda$ and $\mu$ . Then any point $\underline{x} = (x,y)$ in the plane can be represented by $\underline{x} = \alpha\underline{U} + \beta\underline{V}$, with $\alpha$, $\beta$ some scalars.

Then, $$H^k(\underline{x_o}+\underline{x}) = H^k\,(\underline{x_o} + \alpha\underline{U} + \beta\underline{V}) \sim (\underline{x_o} + \lambda\alpha\underline{U} + \beta\mu\underline{V}) .$$

Hence, in coordinate wise, we have

$$H^k(x_o+x,\ y_o+y) \sim (x_o + \lambda x,\ y_o + \mu y) .$$

For convenience, considering $(x_o,y_o)$ as the origin, this gives

$$H^k(x,y) \sim (\lambda x,\ \mu y) \qquad \text{and so}$$

$$H^{-k}(x,y) \sim (\lambda^{-1}x,\ \mu^{-1}y) .$$

Besides, $$d(H^{-k}\underline{x},W^s) \sim \mu^{-1}\, d(\underline{x},\ W^s) \tag{I}$$

(d denotes the distance from a point to the manifold).

Let $\underline{x}$ be a point on the eigenspace $E^\lambda$ and in a suitable neighbourhood of the origin (origin being considered as a hyperbolic periodic point). In this case $d(\underline{x},\ W^s)$ is small initially and so after a certain suitable number of iterations the right-hand side of (I) becomes very small. As a result, the point $\underline{x}$ is mapped by $H^{-k}$ to a point which lies on a curve close to the stable manifold.

In the same manner, we can deduce that

$$d(H^k\underline{x},\ W^u) \sim \lambda d(\underline{x},\ W^u) . \tag{II}$$

From (II), we can conclude similarly that if $\underline{x}$ is a point in a suitable neighbourhood of the origin and lies on the eigenspace $E^{\mu}$, then the point $\underline{x}$ is brought under some suitable number of iterations of the map $H^k$ to a curve close to the unstable manifold. Bearing this idea in mind, we proceed to obtain our desired manifold pictures. Again it is noted that in order to obtain these manifolds at a point, we mention a certain number of iterations which is the best possible we can give in a specific situation considered. Further, the calculations are executed in double precision. The stable manifolds are shown in Fig. 8, whereas the unstable manifolds are shown in Fig. 10.

### 1.3.1 Stable and Unstable Manifolds:

We now wish to illustrate our procedures of how to obtain the stable and the unstable manifolds at the unstable periodic points T, U, V and W.

#### At the point T

Consider first the unstable point T which is a periodic point of period 2. The Computer program used to obtain the stable manifold at T is furnished in Program 2. The eigenvalues of the Jacobian of $H^2$ at this point are $\lambda = -0.465687332337$ and $\mu = -1.374312667662$. So by the stable manifold theorem, there exist stable and unstable manifolds at T. Since the eigenspaces $E^{\lambda}$, $E^{\mu}$ generated by the eigenvectors corresponding to the eigenvalues $\lambda$ and $\mu$ are 1-dimensional, they are straight lines and parallel to the respective equations:

$$y = -1.052693358347x \quad \text{(III)}$$

and $$y = -0.612782452271x \quad . \quad \text{(IV)}$$

Consider now the line that passes through the point T and parallel to (III). Next, consider a closed interval on this line, centred at T, having the length 0.01 units, and then pick up 1000 equally spaced points from this interval.

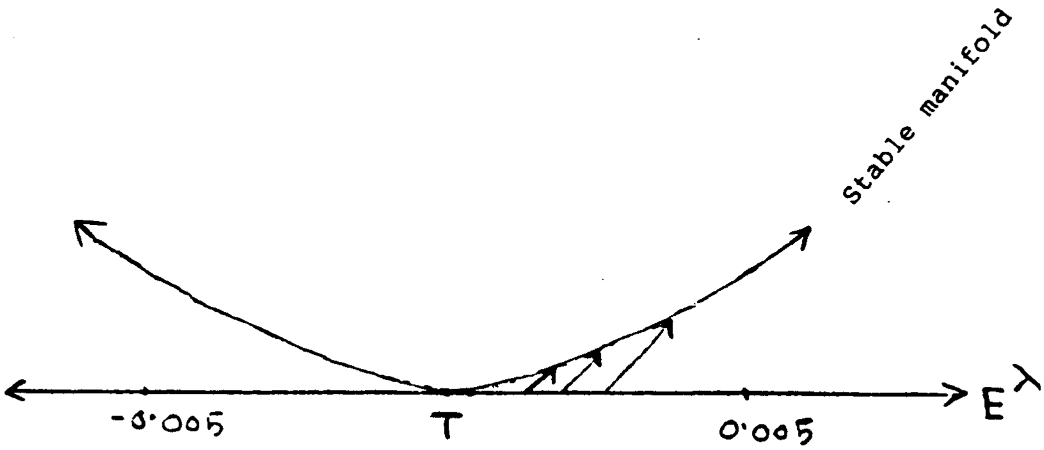


Fig.7:A rough picture of how our method works for the stable manifold at T.A real picture is shown in fig 8.

It is found that the inverse map $H^{-2}$ at each of these 1000 points can be iterated up to 10 times without overflowing to a large value, although at some points more iterations are possible. Therefore, the points obtained by iterating ten times the inverse map $H^{-2}$ at these points are plotted to obtain the stable manifold at T. This graph is marked as $S_1$ in Fig. 8. We draw only a part of the stable manifold by retaining the absolute values of x and y less than or equal to 8 so that we can draw the other manifolds in the same plane with the same scales. The same restrictions on x and y values are imposed for all manifold graphs shown in Figs. 8 and 10 except for the stable manifold at W, where we retain their absolute values less than or equal to 15, so as to get a better picture of that manifold.

Next, consider the line that also passes through T, but parallel to (IV). To obtain the unstable manifold we use the same Computer program as shown in Program 2 with the only exceptions that the cycle K is executed with the direct Henon map and that the cycle J goes up to 20 iterations. Just like the stable manifold, we consider a segment of the line $E^{\mu}$, centred at T, containing 1000 equally spaced points and having the total length 0.01 units. It is observed in this case that even the large number of iterations of the map $H^2$ does not give real overflow and that the points obtained by these iterations approach normally towards the periodic points P and Q. The number of iterations of the map $H^2$ we give is 20 and the unstable manifold is marked by $U_1$ in Fig. 10.

We would like to point out that similar Computer programs with slight appropriate changes are used for the manifolds at U, V and W.

At the point U

Since U is the image of T under the map H, the images of the stable manifold $S_1$ and the unstable manifold $U_1$ under H give respectively the stable manifold $S_2$ and the unstable manifold $U_2$ at the point U.

At the point V

This point exists as an unstable fixed point from the direct

Henon map H. The eigenvalues of the Jacobian of H at this point are $\lambda = 0.382673979702$ and $\mu = -2.090552382536$. The eigenspaces $E^{\lambda}$ and $E^{\mu}$ are given by the equations

$y = 2.090552382536x + \ell_1$ and $y = -0.382673979702x + m_1$

respectively, where $\ell_1$ and $m_1$ are some constants.

Now, in order to obtain the stable manifold at V, on the line $E^{\lambda}$ we consider a segment of length 0.003 units, containing 300 points, towards the right of point V, and another segment of length 0.003 units, containing 1200 points towards the left of V. These two segments are continuous in the sense that they form an interval on the line $E^{\lambda}$ of total length 0.006 units. Iterating the inverse map $H^{-1}$ 8 times at the points lying on the right-side segment of length 0.003 and 12 times at the points lying on the left-side segment of length 0.003, and then plotting the resulting points, we achieve the stable manifold at V. It is marked by $S_3$ in Fig. 8.

In the case of the unstable manifold at V, we consider a segment of the line $E^{\mu}$, centred at V, having length 0.006, and then pick up 600 equally spaced points from this segment. We give 12 iterations of the map H at these points. The points obtained under this scheme are plotted in order to obtain the unstable manifold at V,(see mark $U_3$ in Fig. 10).

<u>At the point W</u>

This point also exists as an unstable fixed point from the direct Henon map. The eigenvalues of the Jacobian of H at this point are $\lambda = -0.328373295916$ and $\mu = 2.436251698750$. The eigenspaces $E^{\lambda}$ and $E^{\mu}$ are generated by the lines

$y = -2.436251698750\ x + \ell_2$ and $y = 0.328373295916\ x + m_2$ respectively, where $\ell_2$ and $m_2$ are some constants. This time we consider a segment of the line $E^\lambda$ centred at W, having the length 0.006 units and then pick up 600 equally spaced points. Iterating this segment nine times by the inverse map $H^{-1}$, the stable manifold at W is obtained. It is marked by $S_4$ in Fig. 8. In this Fig. 8, the dotted marks Z and ZZ are put in order to indicate that this manifold does not take turns at these points; in other words the curve goes far beyond these points and then takes turns,(see also Fig. 9).

To achieve the unstable manifold at W, we consider on the line $E^\mu$ a segment of length 0.001, containing 100 points, towards the right of the point W and another segment of length 0.003 units, containing 300 points towards the left of W . All these points are equally spaced and these two segments are continuous in the sense that they form an interval on the line $E^\lambda$ of total length 0.004 units. Iterating the direct map H twelve times at the points lying on the right-side segment of length 0.001 units and eight times at the points lying on the left-side segment of length 0.003 units, and then plotting the resulting points, we obtain the unstable manifold at W. (See mark $U_4$ in Fig. 10).

Some important remarks are now in order.

<u>Remark 1.3.2</u> The stable manifold at the unstable point W is quite interesting. For, it is apparent from its graph (Fig. 9) that all the stable and the unstable points (other than W which is on the curve) are in a region whose boundary is mostly covered by this manifold. Further, it is also evident from Fig. 8 that

the stable manifolds at T, U and V , (which separate the domains of attraction of the periodic points P, Q, R and S) , lie inside this region. So it is important to note that this stable manifold forms mostly the boundary of a region which contains the domain of attraction of the stable periodic orbit containing the periodic points P, Q, R and S. However, the violent winding of this manifold implies that the domain of attraction has a complicated boundary.

Remark 1.3.3 (See Fig. 11). The unstable manifold $U_3$ at V intersects the stable manifold $S_3$ at H other than V. Moreover, it is clear from Fig. 11 that this intersection is transversal. This shows the existence of a transversal homoclinic point H for the parameter values $B = 0.8$ and $M = 0.9$ in the Henon map. Similarly, we can show by considering a suitable interval on the appropriate eigen-space that the unstable manifold at W intersects transversally the stable manifold $S_4$ at a point other than W, indicating also the existence of transversal homoclinic points.

Remark 1.3.4 Remark 1.3.3 says that there exists a transversal homoclinic point for stable and unstable manifolds at V and W. Therefore, by the Smale-Birkhoff Homoclinic Theorem, (see Th. 5.3.5, p. 252 in [33]) the Henon map has a Horseshoe for the parameter values $B = 0.8$ and $M = 0.9$. This extends the results of Devaney and Nitecki, who have proved in [19] that the Henon map has a Horseshoe for $M > (5 + 2\sqrt{5})(1 + |B|^2)/4$ and $B \neq 0$.

<u>Remark 1.3.5</u> We have carried out similar analysis for $B = 0.35$ and $M = 0.9$, as described in this section for $B = 0.8$ and $M = 0.9$. For $B = 0.35$ and $M = 0.9$, there is a stable periodic orbit of period 4, consisting of the following stable periodic points.

A (-0.625970810088, 0.454391626438)

B (-0.311530007613, 0.385607640902)

C (1.101736116864, -0.219089783531)

and

D (1.298261789824, -0.109035502664) .

Again, there is an unstable periodic trajectory of period two containing two unstable points, E (-0.487362463850, 0.423354640125) and F (1.209584686072, -0.170576862347). In addition, there are two unstable periodic points having period 1 as follows

G (0.753120617785, 0.263592216225)

and

H (-1.475342840007, -0.516369994003) ,

The stable manifolds at E, F, G and H, and the unstable manifold at H are shown in Fig. 12. Here also it is found that the unstable manifold at H intersects transversally the stable manifold at another point I. Though the points H and I are seen to coincide in Fig. 12, they can not coincide on theoretical grounds and that they are distinct points can be seen by choosing some suitable scales. By the same sort of arguments as made in Remark 1.3.4, we can conclude that there is a Horseshoe for the Henon map at the parameter values $B = 0.35$ and $M = 0.9$.

Another important finding from Fig. 13 is that all the stable and the unstable periodic points (except H) are contained in a region bounded mostly by the stable manifold at H. Also it can be shown that the stable manifolds at E, F and G lie in this region. So we can conclude that the stable manifold at H forms mostly the boundary of a region in which the domain of attraction of the stable periodic orbit containing the points A, B, C and D lies.

On balance, we wish to conjecture that for all values of B and M for which there is a stable periodic orbit of period k ($k = 2^N$, $N = 1,2,\ldots$), the stable manifold at an unstable fixed point, (which is given by the direct Henon map at the particular parameter values of B and M), forms mostly the boundary of a region in which the domain of attraction of a stable periodic orbit lies.

PROGRAM 2

```
!COMMENTS:THE FOLLOWING PROGRAM IS %C
USED TO FIND OUT SOME POINTS ON THE    %C
STABLE MANIFOLD AT THE UNSTABLE %C
POINT T=(AA,AAA),WHERE AA=-0.925264339232  %C
AND AAA=0.917989249163.  %C
C IS THE TANGENT OF THE SLOPE OF %C
THE EIGEN-SPACE.

%BEGIN
%INTEGER I,K,J,N
%LONGREAL X,R,XX,YY,ZZ,Y,M,B,A,C,AA,AAA
SELECTOUTPUT(1)
READ(N);READ(A)
B=.8;M=.9
C=-1.052693358347;AA=-.925264339232;AAA=.917989249163
!COMMENTS:THE IMMEDIATELY FOLLOWING CYCLE %C
STARTS TO GIVE POINTS ON THE EIGEN- %C
SPACE. HERE N=500 AND A=0.005 .


%CYCLE I=0,1,N
R=A*I/N
X=AA+R*(1/SQRT(1+C*C))
Y=AAA+R*(C/SQRT(1+C*C))
XX=X;YY=Y
!COMMENTS:THE IMMEDIATELY FOLLOWING %C
CYCLE STARTS TO GIVE POINTS ON THE %C
STABLE MANIFOLD.

%CYCLE J=1,1,10
%CYCLE K=1,1,2
ZZ=(B**(-1))*YY
YY=-1+XX+M*(B**(-2))*YY*YY
XX=ZZ
%IF XX*XX+YY*YY>10000 %THEN->NN
%REPEAT
%REPEAT
PRINT(XX,3,12);PRINT(YY,3,12);NEWLINE
NN:%REPEAT
%ENDOFPROGRAM
```

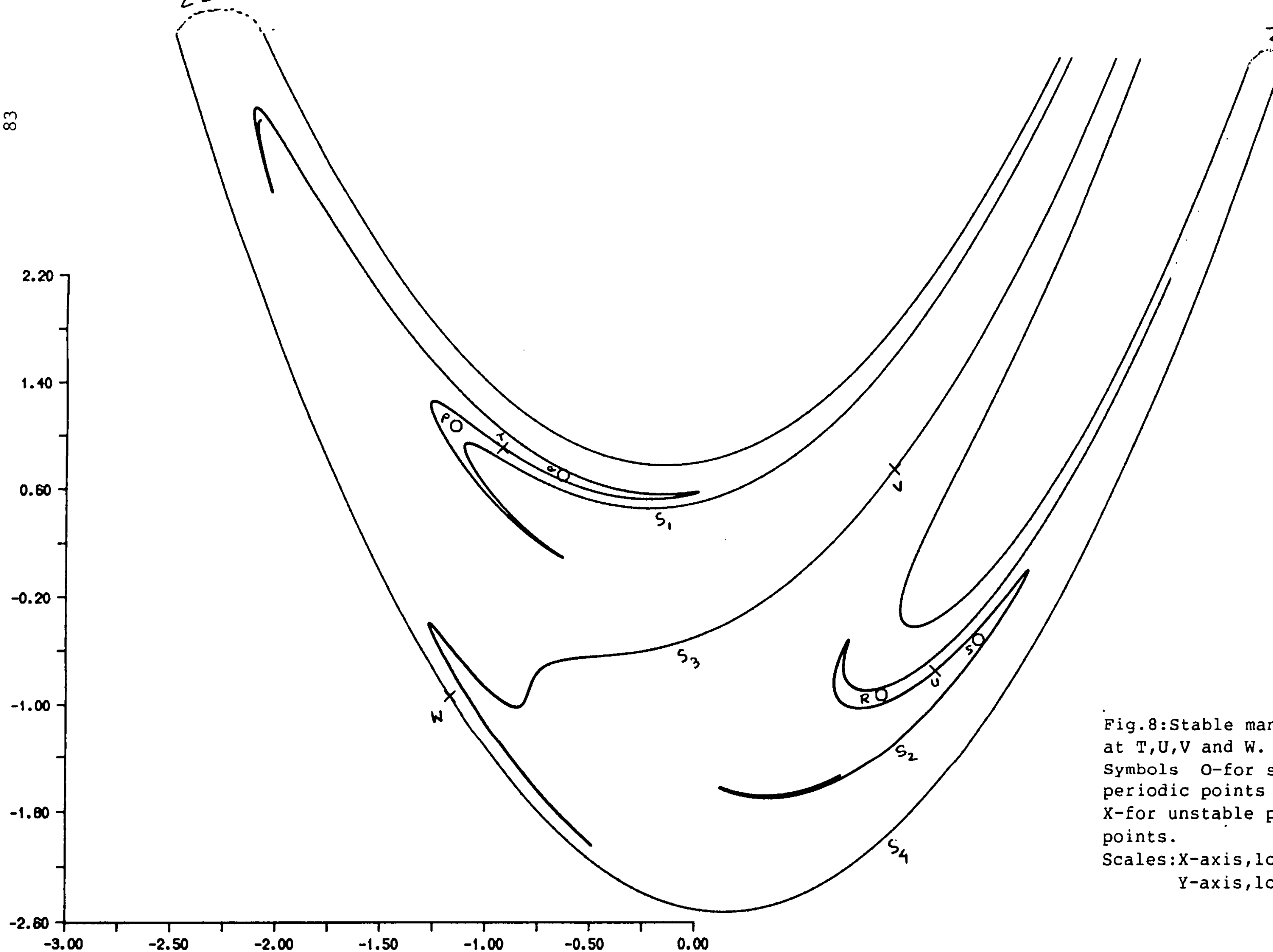


Fig.8:Stable manifolds at T,U,V and W. Symbols O-for stable periodic points and X-for unstable periodic points.
Scales:X-axis,1cm=0.25
Y-axis,1cm=0.4

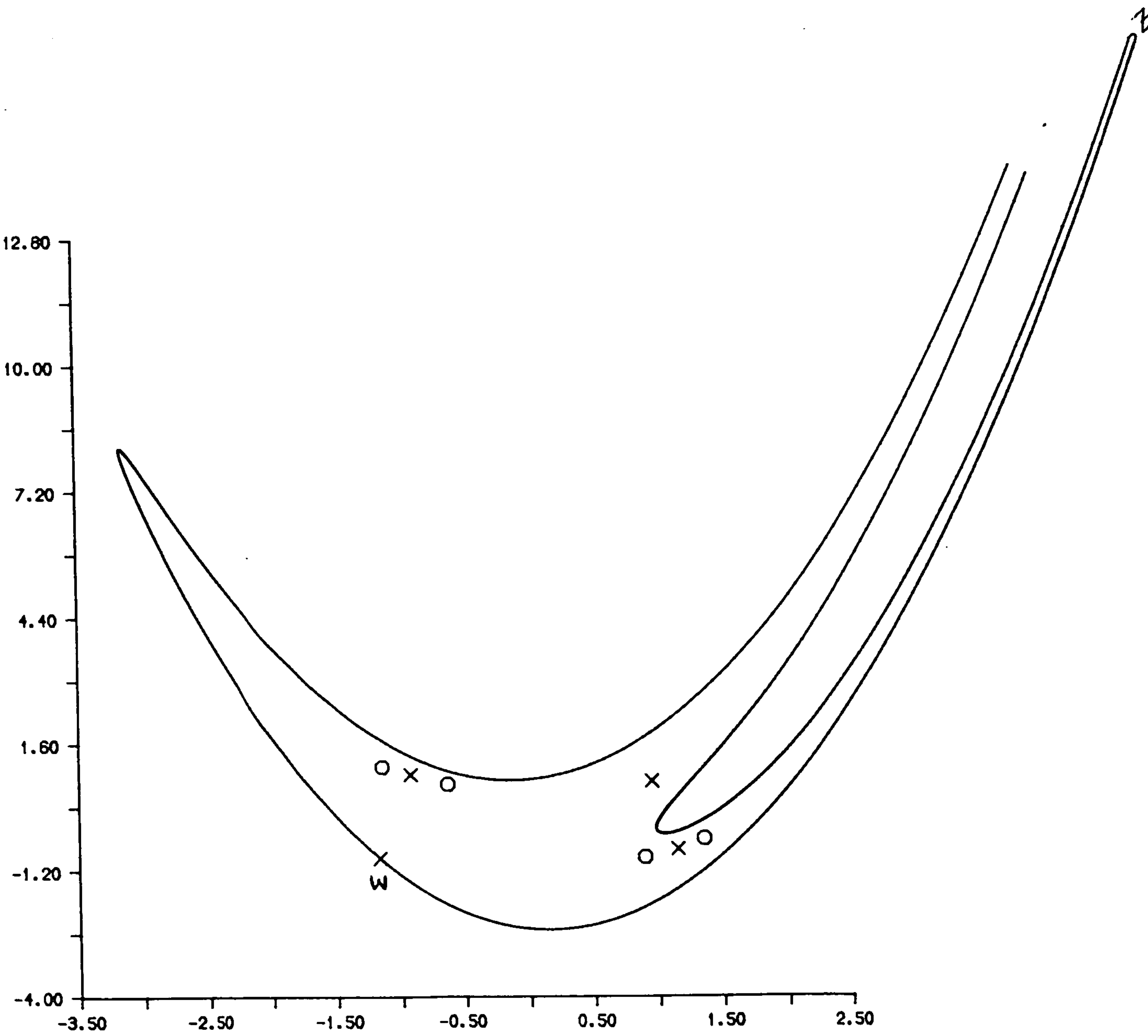


Fig.9:Stable manifold at W.
All stable and unstable points
are surrounded by this manifold.
Scales:X-axis,1cm=0.5
Y-axis,1cm=1.4

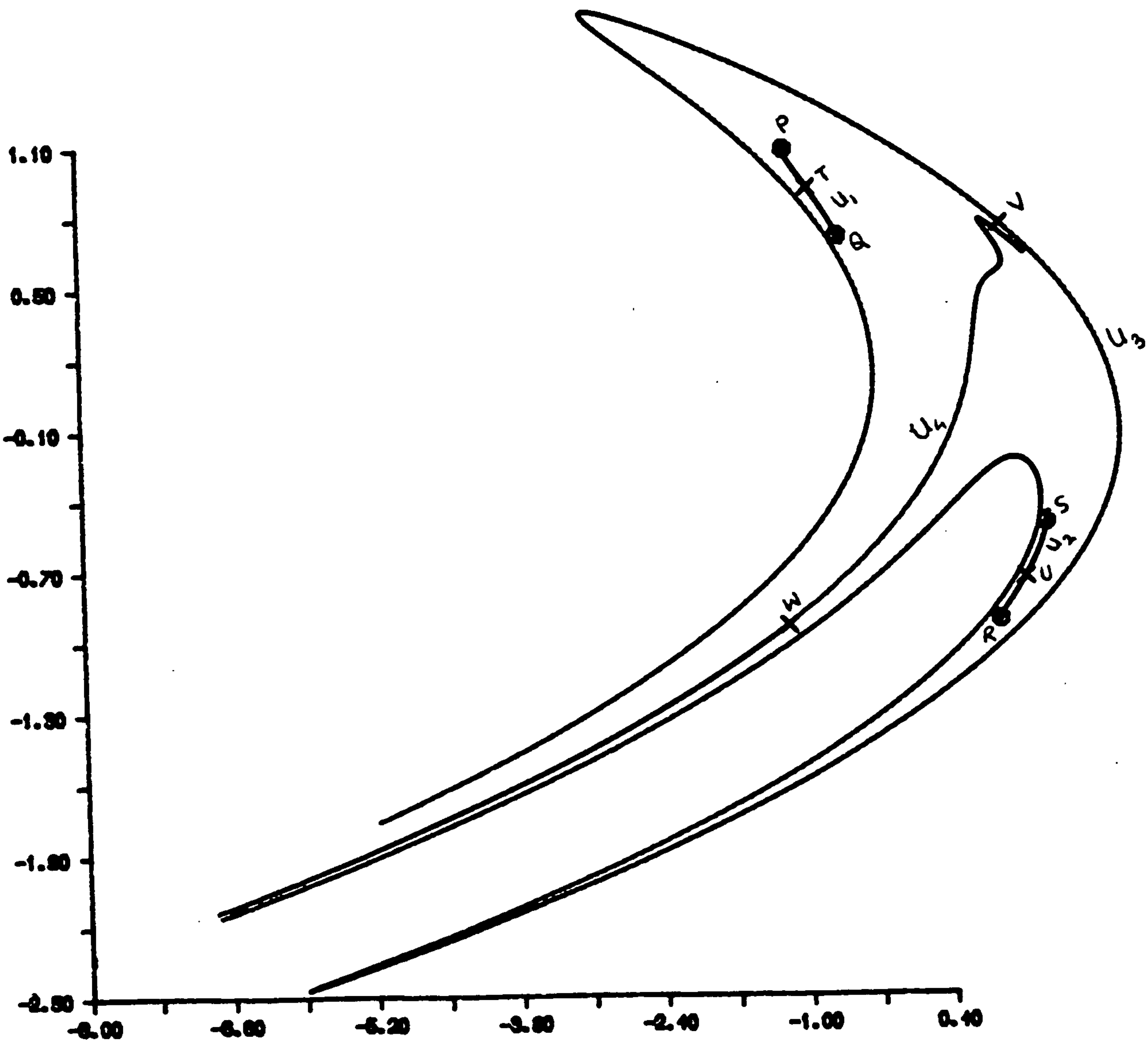


Fig.10:Unstable manifolds
at T,U,V and W.
Scales:X-axis,1cm=0.7
Y-axis,1cm=0.3

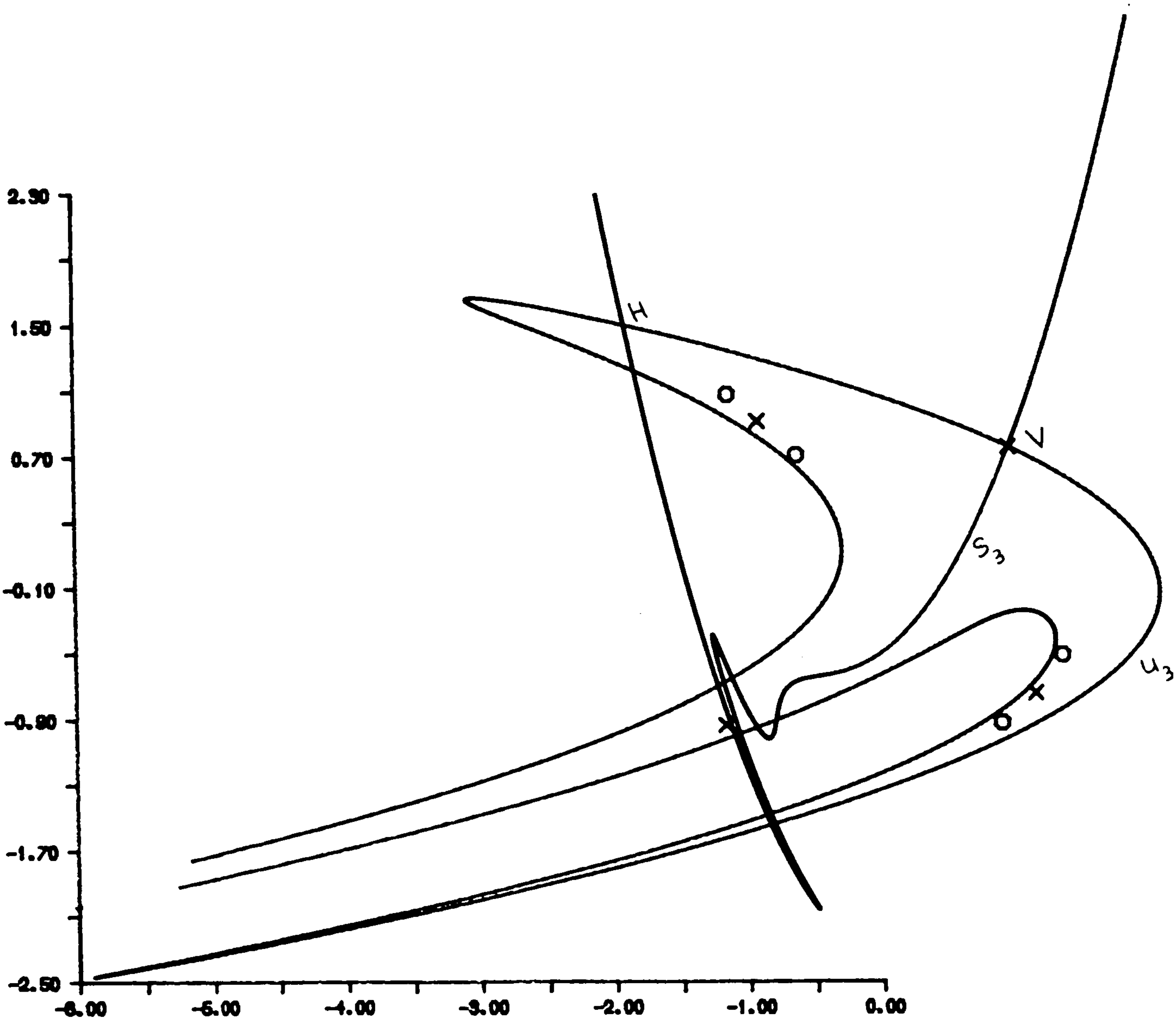


Fig.11:Stable and unstable manifolds at V show the existence of a transversal homoclinic point H.
Scales:X-axis,1cm=0.5
Y-axis,1cm=0.4

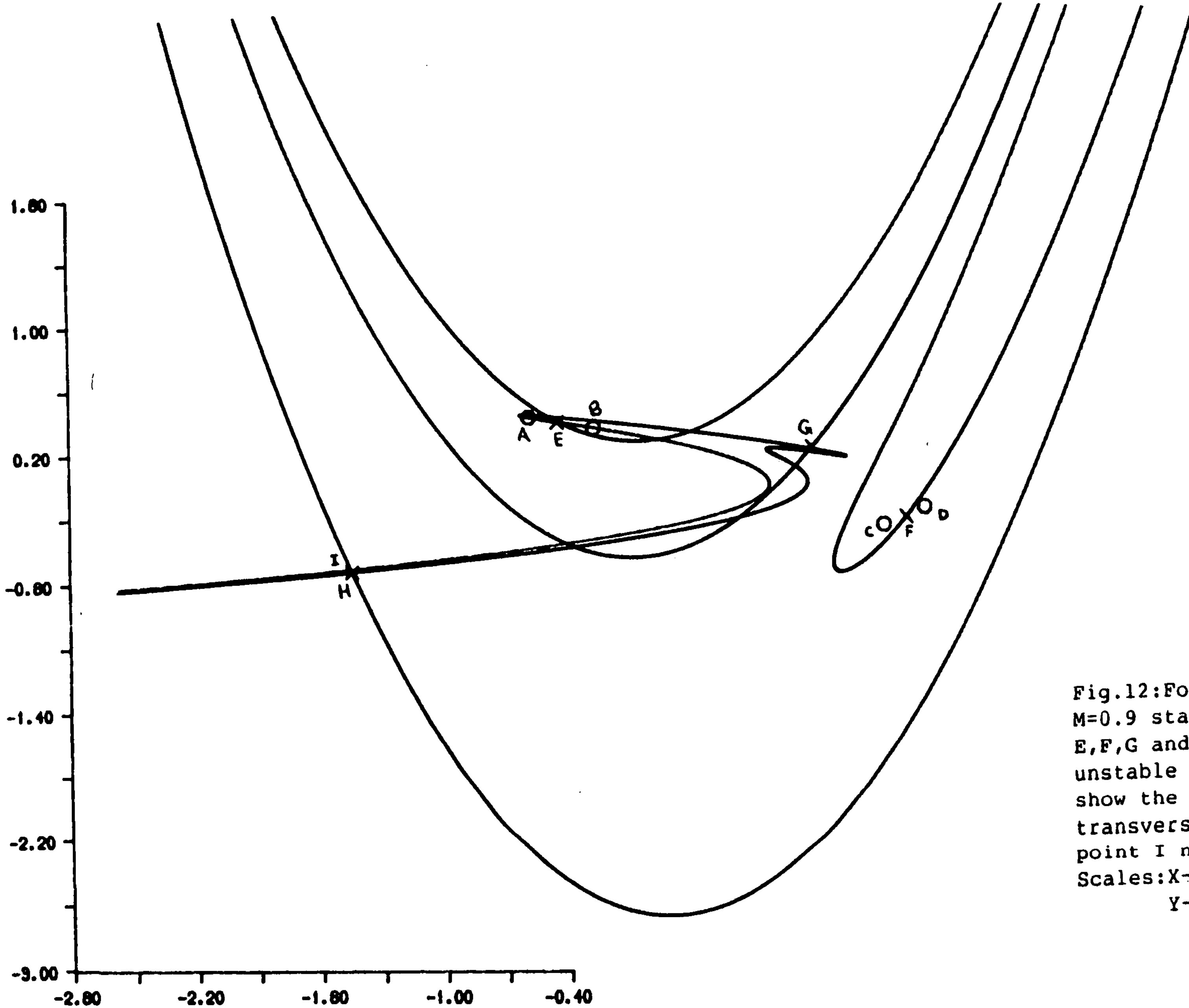


Fig.12:For B=0.35 and M=0.9 stable manifolds at E,F,G and H.Stable and unstable manifolds at H show the existence of a transversal homoclinic point I near H.
Scales:X-axis,1cm=0.3
Y-axis,1cm=0.4

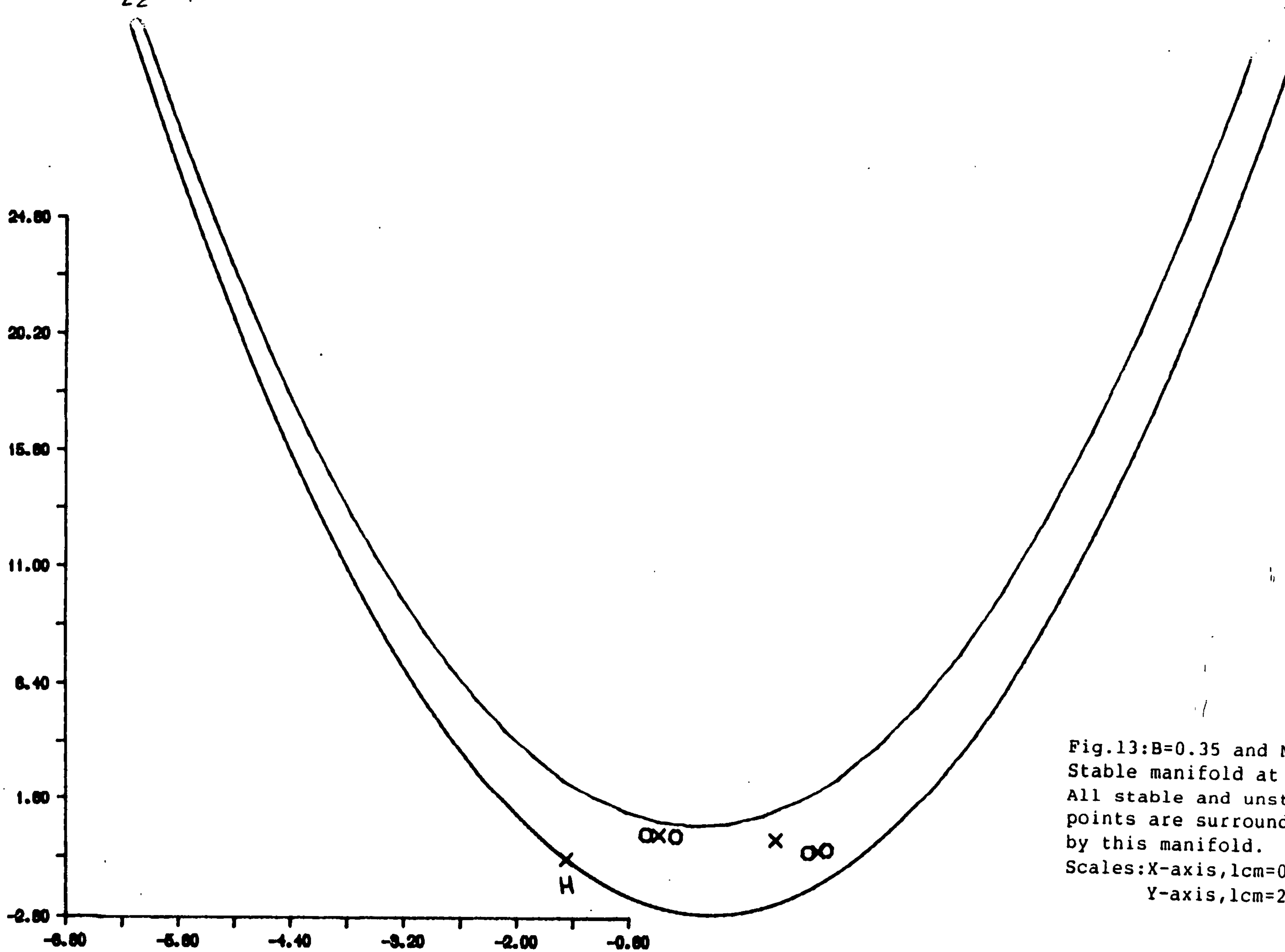


Fig.13:B=0.35 and M=0.9
Stable manifold at H.
All stable and unstable
points are surrounded
by this manifold.
Scales:X-axis,1cm=0.6
Y-axis,1cm=2.3

# CHAPTER TWO

# PERIOD DOUBLING BIFURCATIONS OF THE DUFFING EQUATION

## 2.0 Introduction

Feigenbaum's fascinating sequence of period doubling bifurcations can also be observed in some ordinary differential equations, such as Duffing's and Van-der Pol's equations. Although various bifurcations of the forced Duffing's and Van-der Pol's equations have been studied by many authors (see [7], [39], [40] and [41]), so far as we are concerned a systematic theory of period doubling bifurcations in these equations is yet to be made. So the bifurcations of these equations are worth examining in great detail.

In this short chapter, we describe our attempts of obtaining period doubling bifurcation values of the Duffing equation by using partly some of the numerical apparatus developed in Chapter One and partly some other numerical tools introduced here. Kubiĉek and Holodniok in [43] have provided some algorithms for the determination of such bifurcation points in ordinary differential equations. However our numerical methods are considerably different from those.

## 2.1 Runge-Kutta Fourth Order Method

One of the most generally used methods to solve a system of first order differential equations is the Runge-Kutta fourth order method. We wish to state this method for a system in the plane which is useful for our purpose.

Suppose we are given the following system with an initial value $(x_o, y_o)$ at time $t_o$.

$$\dot{x} = f(x,y,t)$$

$$\dot{y} = g(x,y,t) .$$

Then this method gives the following recursive formulae with a step-length $h$.

$$x_{n+1} = x_n + \frac{1}{6}(K_{1x} + 2K_{2x} + 2K_{3x} + K_{4x})$$

$$y_{n+1} = y_n + \frac{1}{6}(K_{1y} + 2K_{2y} + 2K_{3y} + K_{4y}) ,$$

where, $t_n = t_o + nh$,

$$K_{1x} = hf(x_n, y_n, t_n),$$

$$K_{1y} = hg(x_n, y_n, t_n),$$

$$K_{2x} = hf(x_n + \tfrac{1}{2}K_{1x}, y_n + \tfrac{1}{2}K_{1y}, t_n + \tfrac{1}{2}h),$$

$$K_{2y} = hg(x_n + \tfrac{1}{2}K_{1x}, y_n + \tfrac{1}{2}K_{1y}, t_n + \tfrac{1}{2}h).$$

$$K_{3x} = hf(x_n + \tfrac{1}{2}K_{2x}, y_n + \tfrac{1}{2}K_{2y}, t_n + \tfrac{1}{2}h),$$

$$K_{3y} = hg(x_n + \tfrac{1}{2}K_{2x}, y_n + \tfrac{1}{2}K_{2y}, t_n + \tfrac{1}{2}h),$$

$$K_{4x} = hf(x_n + K_{3x}, y_n + K_{3y}, t_n + h),$$

$$K_{4y} = hg(x_n + K_{3x}, y_n + K_{3y}, t_n + h) .$$

For detailed theory of this method, refer to [44].

Again, this Runge-Kutta method involves mainly truncation errors. Although there are some bounds for this error (see page 125-26, in [44]), it is difficult to derive an explicit formula for this error and so in practice one keeps track of the errors by repeating the computation with $^h/2$ instead of $h$ and comparing the results.

## 2.2 The Map To Be Considered With The Duffing Equation

We wish to outline period doubling bifurcations with the following case of the Duffing equation having $2\pi$-periodic forcing term.

$$\ddot{x} + R\,\dot{x} - x + x^3 = P\cos t, \qquad P \geq 0 . \qquad \text{(I)}$$

Our first intention is to show that for all values of $R$ and $t$ (time variable) in the real line, equation (I) has a finite solution.

First, let $t \geq 0$.

Multiplying (I) by $\dot{x}$, we obtain

$$\dot{x}\,\ddot{x} + R\,\dot{x}^2 - x\,\dot{x} + x^3\,\dot{x} = P\,\dot{x}\cos t$$

Then, $$\frac{d}{dt}\left(\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4\right) = -R\,\dot{x}^2 + P\,\dot{x}\cos t$$

$$\leq -R\,\dot{x}^2 + P\,|\dot{x}|$$

$$\leq -R\dot{x}^2 + \frac{P}{2}(1 + |\dot{x}|^2)$$

$$\leq R'\dot{x}^2 + \frac{P}{2},$$ $R'$ can be chosen to be positive.

This gives,

$$\tfrac{1}{2}\dot{x}^2 \leq c + \int_0^t (R'\dot{x}^2 + \frac{P}{2})dt,$$ where $c$ is a constant.

Let $g(t) = \tfrac{1}{2}\dot{x}^2$

and $f(t) = c + \int_0^t (2R' f(t) + \frac{P}{2})dt$ .

Then, $g(t) \leq c + \int_0^t (2R' g(t) + \frac{P}{2})dt.$

Set, $h(t) = g(t) - f(t).$

So $h(t) \leq 2R' \int_0^t h(t)dt$ and $h(0) \leq 0.$

This implies $h(t) \leq 0$, for all $t \geq 0$.

Consequently, $\tfrac{1}{2}\dot{x}^2 \leq f(t)$, (II)

where $f(t) = Ae^{2R't} - \frac{P}{4R'}$ with $A - \frac{P}{4R'} = c.$

The inequality (II) implies that $x(t)$ and $\dot{x}(t)$ are finite for all $t \geq 0$.

Next, let $t < 0$ and $t = -s.$

Setting $x' = \frac{dx}{ds}$ and $x'' = \frac{d^2x}{ds^2}$, we get from (I)

$$x'' - Rx' - x + x^3 = P \cos s \quad . \tag{III}$$

The equation (III) is of the form (I), and so with the same sort of arguments given above we can show that $x(s)$ and $x'(s)$ remain finite for $s > 0$, that is to say, $x(t)$ and $\dot{x}(t)$ are finite for $t < 0$.

Now, take an initial point $(x(0), y(0))$ and define a map F by

$$F(x(0), y(0)) = (x(2\pi), y(2\pi)),$$

where $x(t)$ is the solution of (I) and $y(t) = \dot{x}(t)$.

Then, by the $2\pi$-periodicity of the above equation, the following hold.

$$F^2(x(0), y(0)) = (x(4\pi), y(4\pi))$$

$$F^3(x(0), y(0)) = (x(6\pi), y(6\pi))$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$F^n(x(0), y(0)) = (x(2n\pi), y(2n\pi)) \ ,$$

where $F^n$ is the n-times functional composition of the map F. Keeping R fixed we see that F depends on P, i.e. $F = F(P)$. Again by the Floquet theory (see Ch. 8, in [42]), this map has the constant Jacobian $e^{-2\pi R}$. If a comparison between the Duffing equation and the Henon map is made, it is noticed that $R = 0$ and $R = \infty$ correspond to $B = 1$ and $B = 0$ respectively in the

Henon map. Since period doubling bifurcations occur in the Henon map for each B in [0,1], it seems that the map F has these bifurcations for every R in $[0,\infty)$. Byatt-Smith in his paper [7] has fixed R = 0.25 and shown some specific values of P where this bifurcation occurs. Furthermore, like the Henon map near B = 1, the case near R = 0 becomes more complicated to handle than that with the larger value of R.

## 2.3 The Jacobian Matrix For The Transformation F

Let the solution x(t) of (I) be perturbed to x(t) + a(t) with a(t) small. Putting this perturbed value in (I), we have
$\ddot{x} + \ddot{a} + R(\dot{x} + \dot{a}) - (x + a) + (x + a)^3 = P\cos t.$

This gives

$$\ddot{a} + R\dot{a} - a + 3x^2 a = 0 \qquad \text{(IV)}$$

(retaining only first order terms in a).
Now (IV) is a linear differential equation which can be written as a system of first order equations by means of

$$\left.\begin{aligned} \dot{a} &= b \\ \dot{b} &= -Rb + a - 3x^2 a \end{aligned}\right\} \qquad \text{(V)}$$

If (a(0), b(0)) is an initial value of (a(t), b(t)) and if we define a map D by

$$D(a(0),\, b(0)) = (a(2\pi),\, b(2\pi)),$$

then there exists a $2 \times 2$ matrix, $J = \begin{pmatrix} Q(1) & Q(3) \\ Q(2) & Q(4) \end{pmatrix}$

such that

$$\begin{pmatrix} a(2\pi) \\ b(2\pi) \end{pmatrix} = D \begin{pmatrix} a(0) \\ b(0) \end{pmatrix} = \begin{pmatrix} Q(1) & Q(3) \\ Q(2) & Q(4) \end{pmatrix} \cdot \begin{pmatrix} a(0) \\ b(0) \end{pmatrix}$$

Now by the Floquet theory (see also p. 25 in [33]), the matrix $J$ is the Jacobian of the transformation $F$ at the point $(x(0), y(0))$. Besides, the initial values $a(0) = 1$, $b(0) = 0$ result $a(2\pi) = Q(1)$, $b(2\pi) = Q(2)$, and the initial values $a(0) = 0$, $b(0) = 1$ give $a(2\pi) = Q(3)$, $b(2\pi) = Q(4)$. So with these initial values, the solutions of the system (V) yield the values of the Jacobian elements, $Q(i)$, $i = 1,2,3,4$.

## 2.4 The Computational Scheme To Evaluate Bifurcation Values

We now wish to outline the computational scheme which gives the bifurcation values for the map F. In our study we put $R = 0.25$ so that $F$ is a function of $P$ alone. We want to recall the Jacobian theory which says that the sum of the eigenvalues of the Jacobian of the map F is equal to its trace and the product of the eigenvalues is $e^{-2\pi R}$. A bifurcation value occurs when one of the eigenvalues equals $-1$ and so the trace $Q(1) + Q(4) = -1 - e^{-2\pi R}$. If we put $I = Q(1) + Q(4) + 1 + e^{-2\pi R}$, then $I$ turns out to be a function of $P$. In order to obtain the bifurcation values of $P$, we need to find the zeroes of I and this can be achieved by the Secant method. To give an initial value of P, we do not use the relation $P_{n+2} \sim P_{n+1} + \frac{P_{n+1} - P_n}{\delta}$, because it is not easy here to find a suitable $\delta$-value. Consequently, slightly complicated tricks are employed for our purpose.

Step One: We first use the Computer Program 3. Initially we put some arbitrary initial values of P, x and y, keeping an eye that the Runge-Kutta method gives the convergence of x and y. Then by the Trial and Error method, we keep increasing the value of P in such a way that the value of I approaches to zero. For each value of P, the Runge-Kutta method yields a periodic point, say, $(x_1, y_1)$, and so this periodic point is used as initial values of x and y for the next chosen higher value of P. At the same time, to check whether the program runs properly we notice the value of the Jacobian determinant which should approximately equal $e^{-2\pi R}$ and which is given by Z(9) in the program. The process is continued and gives a rough estimate of the first bifurcation value as 2.65485, (evaluated up to five decimal places), with a periodic point (0.99799968, 2.36515406). Since our intention is to obtain this value up to 12 decimal places, we could have continued this method to do so, but the method is too time-consuming and tedious. So, estimating approximately the bifurcation value up to 5 decimal places by this method, we then apply the Secant method to obtain up to 12 decimal places. Of course the result(s) may not be correct up to 12 decimal places because of truncation error, (see 2.7). It is also noted that the Secant method essentially needs two suitable initial values of P and this goal is accomplished by the Trial and Error method.

Step Two: Secondly, we use the computer Program 4. Here $P = 2.65485$ and $PP = 2.654855$ are taken as two initial values of P, and $x_1 = 0.99799968$ and $y_1 = 2.36515406$ are taken as initial values of x and y. In this program, N is the number of divisions of the period $2\pi$ and equals 250. So the step-length is $H = \frac{2\pi}{N}$. To achieve a periodic point, the first averaging method,(described in Chapter One), is used, and the same sort of convergence condition, viz., $(X - AA)^2 + (Y - AAA)^2 < 10^{-24}$ is imposed. The results given by the Secant method for the period $2\pi$ are listed in Tables 18 and 19. Table 18 shows that the first bifurcation value is 2.654850042763 with a periodic point $A_1 = (0.998012083515, 2.365155496697)$.

Step Three: Next, the period $2\pi$ is increased to $4\pi$, but the step-length $H = \frac{4\pi}{N}$ is kept fixed by putting $N = 500$. In order to apply the Trial and Error method for evaluating some approximate second bifurcation value, a slightly larger value of the first bifurcation value is considered as an initial value of P and $A_1$ is taken as an initial point for (x,y). This process gives 2.76459 as an approximate second bifurcation value with a periodic point ($x_2 = 1.29150051$, $y_2 = 2.49939170$). Then in Program 4, necessary alterations of the values of N and H are made, two initial values 2.76459 and 2.764591 are put for P and $(x_2,y_2)$ is used as an initial value for (x,y). Ultimately this procedure yields 2.764589999694 as the second bifurcation value with a periodic point

$$A_2 = (1.291489423615, \ 2.499385725247).$$

Step Four: We next repeat the Step Three for the periods $8\pi$, $16\pi$, $32\pi$ and $64\pi$ with the necessary alterations of the values of N and H, and of initial values of P, PP, x and y. For all periods we keep the step-length H fixed by choosing N rightly in order to have higher accuracy in values. Then the Trial and Error method computes the following approximate values:

| Periods | P values | x values | y values |
|---|---|---|---|
| $8\pi$ | P = 2.79344<br>PP = 2.793441 | 1.41105997 | 2.51319511 |
| $16\pi$ | P = 2.79948<br>PP = 2.799481 | 1.39558224 | 2.51845406 |
| $32\pi$ | P = 2.80099<br>PP = 2.800991 | 1.40583830 | 2.51809560 |
| $64\pi$ | P = 2.801208<br>PP = 2.8012081 | 1.40271141 | 2.51859976 |

Having considered the above-mentioned values as initial values with an appropriate period, the Secant method determines 3rd, 4th, 5th and 6th bifurcation values respectively, as given below:

| Bifurcation values | Periodic Points | |
|---|---|---|
| | x value | y value |
| 2.793440001689 | 1.411059957611 | 2.513195117261 |
| 2.799480005153 | 1.395582258661 | 2.518454044130 |
| 2.800989993105 | 1.405838449499 | 2.518095605683 |
| 2.801207999882 | 1.402658349000 | 2.518604862011 |

Remark 2.5: Following the same computational mechanism, we can evaluate further higher bifurcation values. However to keep H fixed, N should be made considerably larger with the higher periods. As a result the Computer program takes a very long time to produce the required results.

Remark 2.6: An approximate bifurcation value evaluated up to 5 or 6 decimal places by the Trial and Error method may not be suitable values in order to apply the Secant method to yield further higher bifurcation values. In other words, we may need to obtain more than six decimal places in a bifurcation value by applying initially the Trial and Error method.

2.7 Error Analysis. The results in Table 18 were given by Program 4 for the period $2\pi$ and executed in longreal precision.

Replacing the convergence condition for $x$ and $y$ by

$(x - AA)^2 + (y - AAA)^2 < 10^{-30}$, we find that except for the number of iterations, the bifurcation value and the values of the periodic point remain unchanged. This implies that the effect of rounding errors is not noticeable. However truncation error is very significant here. The results in Table 19 were calculated with the half of the step-length used for Table 18. It is found that the truncation error involved with the first bifurcation value is approximately equal to $91 \times 10^{-11}$. This error can be similarly estimated for other bifurcation values. This suggests that our results are correct up to 8 decimal places.

Problem 2.8: One can carry out analogous study on Duffing's and Van der Pol's equations, as we have made on the Henon map.

PROGRAM 3

```
%BEGIN
%INTEGER K,N,L,S,SS,T
%LONGREAL PPP,R,AA,AAA,II,I,PP,H,B,C,D,E,F,G,J,X,Y,P
%LONGREALARRAY Z(1:9),A(1:9),Q(1:4)
READ(N);READ(K);READ(P);READ(X);READ(Y)
R=0.25;Q(1)=1;Q(2)=0;Q(3)=0;Q(4)=1
H=(2*PI*K)/N
!COMMENTS:IN THE FOLLOWING TWO CYCLES %C
S & L, THE RUNGE-KUTTA FOURTH ORDER METHOD %C
IS APPLIED TO ACHIEVE THE CONVERGENCE OF X %C
AND Y.SINCE WE ARE CONCERNED TO YIELD BIFUR- %C
CATION POINTS,FIRST AVERAGING METHOD DESCRI- %C
BED IN CHAPTER ONE IS USED FOR THIS PURPOSE.

%CYCLE S=1,1,100
AA=X;AAA=Y
%CYCLE L=1,1,N
A(9)=H*Y;B=H*(-R*Y+X-(X**3)+P*COS(((S-1)*N+(L-1))*H))
C=H*(Y+B/2)
D=H*(-R*(Y+B/2)+(X+A(9)/2)-((X+A(9)/2)**3)  %C
+P*COS(((S-1)*N+(L-1))*H+H/2))
E=H*(Y+D/2)
F=H*(-R*(Y+D/2)+(X+C/2)-((X+C/2)**3) %C
+P*COS(((S-1)*N+(L-1))*H+H/2))
G=H*(Y+F)
J=H*(-R*(Y+F)+(X+E)-((X+E)**3)+P*COS(((S-1)*N+(L-1)) %C
*H+H))
X=X+(A(9)+2*C+2*E+G)/6;Y=Y+(B+2*D+2*F+J)/6
%REPEAT
%EXITIF (((X-AA)**2)+((Y-AAA)**2))<10**(-12)
X=(X+AA)/2;Y=(Y+AAA)/2
%REPEAT
!COMMENTS:THE IMMEDIATELY FOLLOWING  %C
CYCLE IS EMPLOYED TO EVALUATE THE FUCTION %C
I FOR A PARTICULAR VALUE OF P.

%CYCLE SS=1,1,N
A(9)=H*Y;B=H*(-R*Y+X-(X**3)+P*COS((SS-1)*H))
C=H*(Y+B/2)
D=H*(-R*(Y+B/2)+(X+A(9)/2)-((X+A(9)/2)**3)  %C
+P*COS(((SS-1)*H+H/2)))
E=H*(Y+D/2)
F=H*(-R*(Y+D/2)+(X+C/2)-((X+C/2)**3) %C
+P*COS(((SS-1)*H+H/2)))
G=H*(Y+F)
J=H*(-R*(Y+F)+(X+E)-((X+E)**3)+P*COS(((SS-1) %C
*H+H)))
A(1)=H*Q(2);A(2)=H*(-R*Q(2)+Q(1)-3*X*X*Q(1))
A(3)=H*(Q(2)+A(2)/2)
A(4)=H*(-R*(Q(2)+A(2)/2)+(Q(1)+A(1)/2)-3*X*X*(Q(1)+A(1)/2))
```

```
A(5)=H*(Q(2)+A(4)/2)
A(6)=H*(-R*(Q(2)+A(4)/2)+(Q(1)+A(3)/2)-3*X*X*(Q(1)+A(3)/2))
A(7)=H*(Q(2)+A(6))
A(8)=H*(-R*(Q(2)+A(6))+(Q(1)+A(5))-3*X*X*(Q(1)+A(5)))
Z(1)=H*Q(4);Z(2)=H*(-R*Q(4)+Q(3)-3*X*X*Q(3))
Z(3)=H*(Q(4)+Z(2)/2)
Z(4)=H*(-R*(Q(4)+Z(2)/2)+(Q(3)+Z(1)/2) %C
-3*X*X*(Q(3)+Z(1)/2))
Z(5)=H*(Q(4)+Z(4)/2)
Z(6)=H*(-R*(Q(4)+Z(4)/2)+(Q(3)+Z(3)/2)-3*X*X*(Q(3)+Z(3)/2))
Z(7)=H*(Q(4)+Z(6))
Z(8)=H*(-R*(Q(4)+Z(6))+(Q(3)+Z(5))-3*X*X*(Q(3)+Z(5)))
Q(1)=Q(1)+(A(1)+2*A(3)+2*A(5)+A(7))/6
Q(2)=Q(2)+(A(2)+2*A(4)+2*A(6)+A(8))/6
Q(3)=Q(3)+(Z(1)+2*Z(3)+2*Z(5)+Z(7))/6
Q(4)=Q(4)+(Z(2)+2*Z(4)+2*Z(6)+Z(8))/6
X=X+(A(9)+2*C+2*E+G)/6;Y=Y+(B+2*D+2*F+J)/6
%REPEAT
I=Q(1)+Q(4)+1+EXP(-2*PI*.25*K)
Z(9)=Q(1)*Q(4)-Q(2)*Q(3)
PRINT(I,3,6);PRINT(X,3,8);PRINT(Y,3,8);PRINT(Z(9),3,8)
NEWLINE
%ENDOFPROGRAM
```

PROGRAM 4

```
%BEGIN
%INTEGER K,N,L,S,SS,T
%LONGREAL PPP,R,AA,AAA,II,I,PP,H,B,C,D,E,F,G, %C
X,Y,P,J
%LONGREALARRAY Z(1:9),A(1:9),Q(1:4)
READ(N);READ(K);READ(P);READ(PP);READ(X);READ(Y)
R=0.25;Q(1)=1;Q(2)=0;Q(3)=0;Q(4)=1
H=(2*PI*K)/N
!COMMENTS:THE PART OF THE PROGRAM 3  %C
STARTING FROM THE FIRST COMMENT  TO THE  %C
EQUATION WITH I IS PUT HERE. THIS EVALUATES I %C
WITH P.

PRINT(S,3,1);PRINT(P,3,12);PRINT(X,3,12) %C
;PRINT(Y,3,12);NEWLINE
!COMMENTS:THE SECANT METHOD STARTS WITH THE %C
FOLLOWING CYCLE T.

%CYCLE T=1,1,20
!COMMENTS:AGAIN THE PART OF THE PROGRAM 3  %C
STARTING WITH THE FIRST COMMENT TO THE     %C
EQUATION I IS PUT HERE .P IS REPLACED BY   %C
PP AND I BY II.THE VALUE OF II %C
GIVES THE VALUE OF I AT PP.

%EXITIF MOD(II-I)=0
PPP=PP-II*(PP-P)/(II-I)
I=II;P=PP;PP=PPP
%EXITIF MOD(PP-P)<10**(-12)
PRINT(S,3,1);PRINT(PPP,3,12);PRINT(X,3,12) %C
;PRINT(Y,3,12);NEWLINE
%REPEAT
%ENDOFPROGRAM
```

TABLE 18

!COMMENTS:IN THIS TABLE FIRST,SECOND,THIRD AND %C
FOURTH COLUMNS GIVE RESPECTIVELY THE NUMBER OF %C
ITERATIONS REQUIRED BY THE RUNGE-KUTTA METHOD %C
,THE BIFURCATION VALUES (P),X-AND Y VALUES %C
CORRESPONDING TO DIFFERENT P. HERE STEP-LENGTH %C
H=2*PI/250 .

| | | | |
|---|---|---|---|
| 19.0 | 2.654850000000 | 0.998012136785 | 2.365155502863 |
| 18.0 | 2.654849999902 | 0.998005908057 | 2.365154781881 |
| 18.0 | 2.654849514909 | 0.998012136907 | 2.365155502877 |
| 15.0 | 2.654850047915 | 0.998012741087 | 2.365155572810 |
| 16.0 | 2.654850103230 | 0.998012077095 | 2.365155495954 |
| 13.0 | 2.654850042169 | 0.998012008187 | 2.365155487978 |
| 13.0 | 2.654850035810 | 0.998012084253 | 2.365155496783 |
| 11.0 | 2.654850042831 | 0.998012092175 | 2.365155497700 |
| 11.0 | 2.654850043563 | 0.998012083428 | 2.365155496687 |
| 8.0 | 2.654850042755 | 0.998012082516 | 2.365155496582 |
| 9.0 | 2.654850042671 | 0.998012083523 | 2.365155496698 |
| 6.0 | 2.654850042764 | 0.998012083628 | 2.365155496710 |
| 6.0 | 2.654850042774 | 0.998012083512 | 2.365155496697 |
| 4.0 | 2.654850042763 | 0.998012083500 | 2.365155496695 |
| 4.0 | 2.654850042762 | 0.998012083513 | 2.365155496697 |
| 1.0 | 2.654850042763 | 0.998012083515 | 2.365155496697 |

TABLE 19

!COMMENTS:FOR THE FOLLOWING TABLE ,THE SAME PROGRAM %C
WHICH GIVES THE RESULTS IN TABLE 18 IS USED %C
JUST BY REPLACING THE STEP-LENGTH H WITH %C
H/2=2*PI/500 .

| | | | |
|---|---|---|---|
| 19.0 | 2.654850000000 | 0.998009555254 | 2.365154116044 |
| 18.0 | 2.654850003738 | 0.998003326527 | 2.365153395046 |
| 18.0 | 2.654849537398 | 0.998009550597 | 2.365154115505 |
| 15.0 | 2.654850048058 | 0.998010131540 | 2.365154182749 |
| 15.0 | 2.654850097767 | 0.998009495385 | 2.365154109114 |
| 13.0 | 2.654850043226 | 0.998009433461 | 2.365154101946 |
| 13.0 | 2.654850038039 | 0.998009501406 | 2.365154109811 |
| 11.0 | 2.654850043718 | 0.998009507867 | 2.365154110558 |
| 11.0 | 2.654850044244 | 0.998009500793 | 2.365154109740 |
| 8.0 | 2.654850043669 | 0.998009500138 | 2.365154109664 |
| 8.0 | 2.654850043618 | 0.998009500853 | 2.365154109747 |
| 6.0 | 2.654850043674 | 0.998009500918 | 2.365154109754 |
| 6.0 | 2.654850043679 | 0.998009500848 | 2.365154109746 |
| 3.0 | 2.654850043673 | 0.998009500841 | 2.365154109745 |

# CHAPTER THREE

## COMPACT ANALYTIC SEMIGROUPS (CAS)

### 3.0 Introduction

The chief aim in this chapter is to study the following two problems.

Problem 1. Let $X$ be a separable Banach space. Is there a compact analytic semigroup $t \to a^t: H \to CL(X)$ such that $(T_1)\, (a^t X)^- = X$ and $(T_2)\, \|a^t\| \leq 1$ for all $t$ in $H$ if and only if $(*)$ $X$ has the hermitian approximation property?

Problem 2. Is there a compact analytic semigroup $t \to a^t: H \to CL(X)$ such that $(T_1)\, (a^t X)^- = X$ for every $t$ in $H$ and $(T_3)\, \|a^t\| \leq 1$ for all $t$ in $\mathbb{R}^+$ if and only if $(**)$ $X$ has the metric approximation property?

$(T_1)$, $(T_2)$ and $(T_3)$ represent the same conditions as mentioned above throughout this chapter, and play an important role in our study in the sense that they restrict the construction of a compact analytic semigroup $a^t$, $t \in H$, on a given separable Banach space. For instance, we can not construct such a semigroup with the properties $(T_1)$ and $(T_2)$ on $C[0,1]$, (see Theorem 3.3.7.) Sinclair [54] and many other authors have studied analytic semigroups with a bounded approximate identity; and we study compact analytic semigroups with a bounded compact left approximate identity. In fact, in this chapter we establish a road to a compact analytic semigroup via (*) and (**), and show that the reverse road in the case of (*) is not possible in general. Although the reverse road in the case of (**) is possible in some particular case (see Theorem 3.3.8), whether or not it is possible in general is kept as an open problem (see Problem 3.3.9).

In Section One, we discuss the problem 1 on the space $C_o$ so that it can give a clear picture of problem 1,(problem 2 can also be discussed similarly on $C_o$). Although we are not directly concerned to study a compact analytic semigroup on this space, it is very interesting to see some enlightening results briefly. In particular, the theorem 3.1.4 is significant and shows the special importance of the conditions $(T_1)$ and $(T_2)$.

Section Two is mainly devoted to the study of problem 1. We give an affirmative answer of the 'only if part' of the problem completely (Theorem 3.2.6) and show with a counter example (3.2.7) that 'the if part' is false in general. Eventually, it is shown that the scalar multiples of the identity operator are the only hermitians in the disc algebra A.

In Section Three we present problem 2, showing that its 'only if part' is true with the condition $(T_3)$, (Theorem 3.3.5), but not true with the condition $(T_2)$, (Theorem 3.3.7); and that 'the if part' is also true with the condition $(T_3)$, provided CL(X) is the norm closure of the finite rank operators on X,(Theorem 3.3.8).

Now some basic definitions are provided in order to carry out our main study.

Definition 3.0.1. Let $X$ be a unital Banach algebra with dual space $X^*$. The numerical range of an element $x$ in $X$ is defined by

$$V(x) = \{f(x): f \in X^*, \|f\| = f(1) = 1\} .$$

An element $x \in X$ is called hermitian if one of the following equivalent conditions is satisfied.

(a) the numerical range of $x$, $V(x)$, is contained in the real line.

(b) $\lim_{\alpha \to 0} \frac{1}{\alpha}\{\|1+i\alpha x\| - 1\} = 0$

(c) $\|\exp(i\alpha x)\| = 1 \quad (\alpha \in \mathbb{R})$ .

Definition 3.0.2. A one-parameter semigroup on a complex Banach space $X$ is a family $a^t$, $t \in \mathbb{R}^+$, of bounded linear operators $a^t: X \to X$, satisfying the following relations.

(i) $a^o = 1$

(ii) $a^s a^t = a^{s+t}$, for all $s, t \in \mathbb{R}^+$

(iii) $\lim_{t \downarrow 0} a^t x = x$, for every $x$ in $X$.

If the parameter $t$ ranges over the whole real line $\mathbb{R}$, we then call it a one-parameter group. The (infinitesimal) generator $Z$ of a one-parameter semigroup $a^t$ is defined by

$$Zx = \lim_{t \downarrow 0} t^{-1}(a^t x - x).$$

The domain, $Dom(Z)$, of $Z$ is the set of $x$ for which the limit exists.

Definition 3.0.3. Let $X$ be a Banach space and let $S_\alpha = \{z \in \mathbb{C} : \operatorname{Re} z > 0 \text{ and } |\operatorname{Arg}(z)| < \alpha\}$ be a sector in $\mathbb{C}$, where $\alpha$ lies in $(0, \pi/2]$. An analytic semigroup $a^t$ on $X$ is a family of bounded linear operators, $a^t: X \to X$, defined for $t \in S_\alpha$, where $\alpha$ is fixed and satisfying the following conditions:

(i) $a^t a^s = a^{t+s}$, for all $t, s \in S_\alpha$

(ii) $a^t$ is an analytic function of $t \in S_\alpha$

(iii) If $x \in X$ and $\varepsilon > 0$ then $\lim_{t \to 0} a^t x = x$, provided $t$ remains within $S_{\alpha-\varepsilon}$.

We define the generator $Z$ of $a^t$ by

$$Zx = \lim_{t \downarrow 0} t^{-1}(a^t x - x),$$

where $t > 0$ and Dom $(Z)$ is the set of $x$ for which the limit exists. If all $a^t$ are compact operators, then we call it a compact analytic semigroup.

Definition 3.0.4. A Banach space $X$ is said to have the hermitian approximation property if for each compact subset $F$ of $X$ and each $\varepsilon > 0$ there is a compact hermitian operator $R$ on $X$ such that

(i) $\|Rx - x\| < \varepsilon$ for all $x \in F$ and (ii) $\|R\| \leq 1$.

Definition 3.0.5. A Banach space $X$ is said to have the metric approximation property if for each compact subset $F$ of $X$ and each $\varepsilon > 0$ there is a finite rank operator $T$ on $X$ such that

(i) $\|Tx - x\| < \varepsilon$ for all $x \in F$ and (ii) $\|T\| \leq 1$.

Approximate Identities. 3.0.6. Let A be a Banach algebra over $\mathbb{C}$. We assume that A does not have an identity. Let $\Lambda$ be a directed set. A net $\{e_\lambda\}_{\lambda\in\Lambda}$ in A is called a left (res. right, two-sided) approximate identity in A, if for all $x \in A$,

$$\lim_{\lambda\in\Lambda} e_\lambda x = x,$$

(resp. $\lim_{\lambda\in\Lambda} xe_\lambda = x$, $\lim_{\lambda\in\Lambda} e_\lambda x = x = \lim_{\lambda\in\Lambda} xe_\lambda$).

It is said to be bounded if there is a constant K such that $\|e_\lambda\| \leq K$ for all $\lambda \in \Lambda$; in this case, we define the bound of $\{e_\lambda\}_{\lambda\in\Lambda}$ by $\sup\{\|e_\lambda\| : \lambda \in \Lambda\}$, and the norm by $\|\{e_\lambda\}\| = \limsup_\lambda \|e_\lambda\|$.

Let X be a left Banach A module. Then a bounded approximate identity in A for X is a bounded net $\{e_\lambda : \lambda \in \Lambda\}$ in A such that $\lim_{\lambda\in\Lambda} e_\lambda x = x$ for all x in X. We can define similarly an approximate identity for a right or two sided Banach-A module. However we are mainly interested in a left approximate identity.

If all $e_\lambda$, $\lambda \in \Lambda$ are compact and hermitian bounded operators, then we call it a bounded compact hermitian approximate identity for X.

Definition 3.0.7. Let $\Omega$ be a connected open set in $\mathbb{C}$ and $H(\Omega)$ be the collection of all analytic functions on $\Omega$. Suppose $F \subset H(\Omega)$. We call F a normal family if every sequence of members of F contains a subsequence which converges uniformly on compact subsets of $\Omega$. The limit function is not required to belong to F .

## 3.1. Section One: CAS on The Space $C_o$

In this section we focus our attention on the construction of a compact analytic semigroup and of a compact hermitian approximate identity on the space $C_o$. It is a folklore fact that every Hilbert space has hermitian approximation property, and so does $C_o$. However the space $C_o$ has its own compact analytic semigroups and approximate identity (Theorem 3.1.1) which are, we think, worth-looking at. We show in Theorem 3.1.4 that a compact analytic semigroup on $C_o$ with the conditions $(T_1)$ and $(T_2)$ consists of multipliers only.

Theorem 3.1.1. There exists a compact analytic semigroup $t \to a^t : H \to CL(C_o)$ such that $(T_1)(a^t C_o)^- = C_o$ and $(T_2)\ \|a^t\| \leq 1$ for every $t \in H$. Furthermore, $C_o$ has a compact hermitian approximate identity.

Proof. We define an analytic semigroup $a^t$, $t \in H$ as $a^t$ = Multiplication by $(1, e^{-t}, e^{-2t}, \ldots, e^{-nt}, \ldots)$. It can be checked without difficulty that $a^t$ is an analytic semigroup on $C_o$. We now show that $a^t$ is a compact operator on $C_o$.

Let $a_n{}^t = (1, e^{-t}, \ldots, e^{-nt}, 0, \ldots)$. Then $a_n{}^t$ being a finite rank operator, is compact. Now $\|a_n{}^t - a^t\| \to 0$ as $n \to \infty$. Since the limit of a compact operator sequence is compact, $a^t$ is compact.

Next,
$$\|a^t\| = \sup_n \{|e^{-nt}|\} \leq 1 .$$

Also it is straight-forward to show that $(a^t C_o)^- = C_o$. This completes the arguments for the first part of the theorem.

To prove the second part we construct an approximate identity as follows.

Let $P_n : C_o \to C_o$

$$(x_1, x_2, \ldots x_n, \ldots) \to (x_1, x_n, \ldots, x_n, 0, 0 \ldots) .$$

Then clearly, (i) $\| P_n \| \leq 1$

(ii) $\| P_n x - x \|_\infty \to 0$ as $n \to \infty$ for all $x \in C_o$.

(iii) $P_n$ is compact.

We finally wish to prove that $P_n$ is hermitian.

Let $x \in C_o$ with $\| x \| \leq 1$.

Now, $\| x + i\alpha P_n(x) \|$, $(\alpha \in \mathbb{R})$

$$= \| ((1+i\alpha)x_1, (1+i\alpha)x_2, \ldots , (1+i\alpha)x_n, x_{n+1}, \ldots) \|$$

$$= \text{either } \sup_k \{ \| x_{k+1} \| : k = n, n+1, \ldots \}$$

$$\text{or } \max_{1 \leq k \leq n} \{ |(1+i\alpha)x_k| \}$$

$\leq 1$, in the first case,

and in the second case, say, it is equal to

$$\| (1+i\alpha)x_\ell \|$$

$$\leq |1+i\alpha| = 1 + \tfrac{1}{2}\alpha^2 + \ldots .$$

Therefore, in either case

$$\lim_{\alpha\to 0+}\left(\frac{\|x + i\alpha P_n x\| - 1}{\alpha}\right) \leq 0$$

Thus, for every $x \in C_o$ with $\|x\| \leq 1$, we obtain

$$\lim_{\alpha\to 0+}\left(\frac{\|x + i\alpha P_n x\| - 1}{\alpha}\right) \leq 0$$

So, $$\lim_{\alpha\to 0+}\frac{\|1 + i\alpha P_n\| - 1}{\alpha} \leq 0 \qquad \text{(iv)}$$

Next, consider the element $x_o = (\underbrace{1,1, \ldots, 1}_{\text{up to n times}},0,0, \ldots, 0 \ldots)$

Then, $x_o \in C_o$ and $\|x_o\| = 1$.

Now $$\lim_{\alpha\to 0+}\frac{\|x_o + i\alpha P_n x_o\| - 1}{\alpha}$$

$$= \lim_{\alpha\to 0+}\frac{|1 + i\alpha| - 1}{\alpha}$$

$$= 0$$

This gives $$\lim_{\alpha\to 0+}\frac{\|1 + i\alpha P_n\| - 1}{\alpha} \geq 0 \qquad \text{(v)}$$

The inequalities (iv) and (v) combinedly imply that

$$\lim_{\alpha \to 0+} \frac{\| 1 + i\alpha P_n \| - 1}{\alpha} = 0 .$$

Hence by definition 3.0.1, $P_n$ is hermitian (vi)

The results (i), (ii), (iii) and (vi) combinedly assert that $\{P_n\}$ is a left compact hermitian approximate identity.

The results now follow.

Q.E.D.

Before discussing the main result in Theorem 3.1.4, it is enlightening to look at the following lemmas.

Lemma 3.1.2. If $a^t$, $t \in H$, is an analytic semigroup on a separable Banach space $X$ satisfying the conditions $(T_1)$ and $(T_2)$, then there exists a one parameter group $S^r$, $r \in \mathbb{R}$ on $X$ such that $S^r x = \lim_{s \downarrow 0} a^{s+ir} x$ exists for all $r \in \mathbb{R}$.

It is not hard to prove the lemma and so the proof is omitted. Also refer to problem 2.36 in page 63 in [16] .

Lemma 3.1.3. Let $\ell_1$ be the usual sequence space. If the operators $T: \ell_1 \to \ell_1$ and $T^*: \ell_\infty \to \ell_\infty$ are conjugate to each other, then $T$ is a multiplier if and only if $T^*$ is a multiplier.

The proof is straightforward and so, omitted.

We now move on to the main result on $C_o$. In addition to the above two lemmas, we need more technical apparatus, namely, the

Poisson integral on the right half complex plane, for proving the following theorem.

Theorem 3.1.4. If there is a compact analytic semigroup $a^t$, $t \in H$, on $C_o$ satisfying the conditions $(T_1)$ and $(T_2)$, then the generator Z of $a^t$ is of the form $(\mu_1, \mu_2, \ldots)$ with $\mu_i \leq 0$, $i = 1,2, \ldots$ and $a^t$ is a multiplier so that

$$a^t x = (e^{\mu_1 t} x_1, e^{\mu_2 t} x_2, \ldots), \quad \text{where } x = (x_1, x_2, \ldots)$$

in $C_o$.

Proof. For each $r \in \mathbb{R}$, let $S^r = \lim_{s \downarrow 0} a^{s+ir}$. By lemma 3.1.2, the right hand limit exists and $S^r$ is a one-parameter group on $C_o$ such that $\| S^r \| = 1$. Therefore, $S^r$, $r \in \mathbb{R}$, is an isometry on $C_o$. If $x \in C_o$ and $x^* \in C_o^* = \ell_1$, then the conjugate operator $S^{r*}$ defined by $(S^{r*} x^*)x = x^*(S^r x)$ is readily seen to be an onto isometric mapping on $C_o^*$.

Let $\sigma_n = (0,0,\ldots, \underbrace{1}_{\text{nth place}}, 0\ldots)$ .

Again, for each $x \in C_o$,

$(S^{r*} \sigma_n)x = \sigma_n(S^r x) \to \sigma_n x$ as $r \to 0$ by group properties.

Hence this gives

$$S^{r*} \sigma_n \to \sigma_n \quad \text{as} \quad r \to 0.$$

As a result, given a positive integer N we can choose $r_o \in \mathbb{R}$ such that

$$\| S^{r*}\sigma_n - \sigma_n \| < 1 \quad \text{for} \quad n \leq N, \quad 0 < r < r_o . \tag{I}$$

Since an isometry on $\ell_1$ sends an extreme point of the unit ball of $\ell_1$ to an extreme point, and since an extreme point of the unit ball of $\ell_1$ is of the form $(0,0, \ldots e^{i\theta},0,\ldots)$, $\theta \in \mathbb{R}$, we obtain

$$S^{r*}\sigma_n = e^{i\theta_n(r)} \sigma_{\tau(n)}, \text{ where } \tau \text{ is a map from } \mathbb{N}$$

onto $\mathbb{N}$. Now (I) gives

$$\| e^{i\theta_n(r)} \sigma_{\tau(n)} - \sigma_n \| < 1, \text{for } n \leq N, \quad 0 < r < r_o . \tag{II}$$

We claim that (II) gives $\tau(n) = n$ for all $n \leq N$. Suppose if possible, $\tau(n_o) \neq n_o$ for some $n_o \leq N$. Now consider the element

$$\sigma_{n_o} = (0, \ldots, \underbrace{1}_{n_o\text{th place}}, 0, \ldots ) .$$

Then

$$\| \sigma_{n_o} \| = 1, \quad \text{and} \quad (\sigma_{\tau(n_o)})_{n_o} = 0 . \tag{III}$$

Applying $\ell_1$ norm in (II), we get

$$\sum_{k=1}^{\infty} | e^{i\theta_n(r)} (\sigma_{\tau(n)})_k - (\sigma_n)_k | < 1 . \tag{IV}$$

But putting $k = n_o$, we have

$$\left| e^{i\theta_{n_o}(r)} (\sigma_{\tau(n_o)})_{n_o} - (\sigma_{n_o})_{n_o} \right| = 1 ,$$

by using (III) .

This offends the inequality (IV) and hence, a contradiction. So we must have $\tau(n) = n$ for all $n \leqslant N$. Hence $S^{r*}\sigma_n = e^{i\theta_n(r)} \sigma_n$,

for each $n \leqslant N$. If $k$ is any integer, then by group-law $S^{kr*} = (S^{r*})^k$ . This yields
$S^{kr*}\sigma_n = e^{ik\theta_n(r)} \sigma_n$ .

Hence, for every $r \in \mathbb{R}$

$$S^{r*}(x) = (e^{i\theta_1(r)} x_1, \; e^{i\theta_2(r)} x_2, \ldots) ,$$

where $x = (x_i) \in \ell_1$ .

Since $S^r$ is a group, the map $r \to e^{i\theta_n(r)}$ is a character on the additive group $\mathbb{R}$, and therefore by Theorem 35C in [47] there exists a real $\mu_n$ such that $\theta_n(r) = \mu_n r$. Consequently we obtain

$$S^{r*}(x) = (e^{i\mu_1 r} x_1, \; e^{i\mu_2 r} x_2, \ldots\ldots) .$$

So by lemma 3.1.3

$$a^{ir}y = S^r y = (e^{i\mu_1 r} y_1, \; e^{i\mu_2 r} y_2, \ldots\ldots) \quad \text{where} \quad y = (y_i) \in C_o .$$

Also, an analytic function is uniquely determined by its boundary values by the Poisson Integral on the right-half plane. Our semigroup $a^t$ is analytic in $H$ and continuous in $H^-$, and so, by the Poisson Integral on the right half plane

$$(a^{s+ir}(\sigma_n))_k = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(a^{iy}(\sigma_n))\,\delta_{kn}s}{s^2 + (y-r)^2}\,dy \qquad (s > 0)$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{i\mu_n y}\,s\,\delta_{kn}}{s^2 + (y-r)^2}\,dy$$

$$= e^{(s+ir)\mu_n}\,\delta_{kn} \quad .$$

This yields $a^{s+ir}(x) = (e^{(s+ir)\mu_n} x_n)$ .

Thus the assertion follows.

Q.E.D.

<u>Remark 3.1.5</u>. If we drop the condition $\| a^t \| \leq 1$ for each $t$ in $H$, then the above theorem may not be true. For simplicity, if we consider the space $\mathbb{C}^2$ and the analytic semigroup

$a^t = e^{t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$ then it can be checked without difficulty that $a^t$ is not a multiplier.

## 3.2. Section Two: CAS And Hermitian Approximation Property

This section mainly presents the proofs of the facts that a separable Banach space having the hermitian approximation property possesses a compact analytic semigroup with the properties $(T_1)$ and $(T_2)$ (Theorem 3.2.6) and that its converse is not true in general (Counter Example 3.2.7). The somewhat contrived construction of an analytic semigroup in Theorem 3.2.6 is based on the paper [54], where the author has shown many interesting properties of an analytic semigroup. Dixon in his paper [20] has studied the existence of various approximate identities in the algebra CL(X) of all compact operators on a Banach space X. However our approximate identity in this algebra needs essentially to be hermitian. For the converse problem (3.2.7) the references [38] and [53] are important ones.

We begin with some lemmas.

Lemma 3.2.1. Let X be a separable Banach space. Then the following two conditions are equivalent.

(i) For each compact subset F of X and each $\varepsilon > 0$, there is a compact hermitian operator R with $\| R \| \leq 1$ such that $\| Rx - x\| < \varepsilon$ for every x in F.

(ii) There exists a sequence $\{h_j\}$ with $\| h_j \| \leq 1$ of compact hermitian operators on X such that $\| h_j x - x\| \to 0$ as $j \to \infty$ for every x in X.

Proof. Since $X$ is a separable Banach space, it has a countable dense subset, say, $S = \{x_1, x_2, \ldots, x_j \ldots\}$.

We first prove that (i) implies (ii).

Let $F_j$ be a compact subset containing $x_1, x_2, \ldots, x_j$. Then, by the given hypothesis, there exists a compact hermitian operator $h_j$ with $\|h_j\| \leq 1$ such that $\|h_j y - y\| < \frac{1}{j}$ for all $y$ in $F_j$...(I). Let $x \in X$ and $\varepsilon > 0$. Then there is an $i$ such that $\|x_i - x\| < \varepsilon$ (II).

Now, for $j \geq i$

$$\|h_j x - x\| \leq \|h_j x - h_j x_i\| + \|h_j x_i - x_i\| + \|x_i - x\|$$

$$\leq \|h_j\| \|x - x_i\| + \|h_j x_i - x_i\| + \|x_i - x\|$$

$$< 2\varepsilon + \frac{1}{j} .$$

Since $x$ and $\varepsilon$ were arbitrary, $\|h_j x - x\| \to 0$ as $j \to \infty$ for every $x$ in $X$.

The next attempt is to show that (ii) implies (i). Let $\varepsilon > 0$ and a compact subset $F$ of $X$ be given. Now, for each $x$ in $F$, there exists a positive integer $N_x$ such that the open ball $B_{x,\varepsilon/4}$ centred on $x$ and with radius $\varepsilon/4$, contains $h_j x$ for all $j \geq N_x$.

Now, for $y \in B_{x,\varepsilon/4}$ and $j \geq N_x$, we obtain

$$\| h_j y - x \| \leq \| h_j y - h_j x \| + \| h_j x - x \| < \varepsilon/2$$

and

$$\| h_j y - y \| \leq \| h_j y - x \| + \| x - y \| < \varepsilon/2 + \varepsilon/2 = \varepsilon . \qquad \text{(II)}$$

Again all these open balls $B_{x,\varepsilon/4}$, $x \in F$ form an open cover for $F$. Since $F$ is compact there is a finite number of balls, say, $B_{x_{11}}, \varepsilon/4$, $B_{x_{22}}, \varepsilon/4$, ... $B_{x_{kk}}, \varepsilon/4$, ($x_{11}$, $x_{22}$, ..., $x_{kk}$ are in F) such that $F$ is contained in their union. Again by virtue of (II), $\| h_j y - y \| < \varepsilon$ for all $y$ in $B_{x_{ii}}, \varepsilon/2$ and for all $j \geq N_{x_{ii}}$.

Let $N = \max \{N_{x_{11}}, N_{x_{22}}, \ldots, N_{x_{kk}}\}$

Then for any $y$ in $F$, we obtain $\| h_N y - y \| < \varepsilon$. Putting $h_N = R$, we get our desired result.

Q.E.D.

Remark 3.2.2. Following the definition 3.0.6 the sequence $\{h_j\}$ can be treated as a left approximate identity for $X$ bounded by 1.

Remark 3.2.3. The proof of the second part of the above lemma shows that for each $x$ in any compact set $F$, $h_j x \to x$ uniformly on $F$.

The following lemma gives a key result for satisfying the hypothesis of the lemma 3.2.5. The mechanism of proving it goes through the standard compactness arguments.

Lemma 3.2.4. Let $X$ be a complex Banach space having the hermitian approximation property. Then for each compact operator $T$ on $X$, each $\eta > 0$ and each finite subset $x_1, x_2, \ldots, x_m$ in $X$, there exists a compact hermitian operator $U$ with $\|U\| \leq 1$ such that $\|(U-1)x_k\| + \|(U-1)T\| < \eta$ for $k = 1,2, \ldots, m$.

<u>Proof</u>. Because $X$ has the hermitian approximation property, lemma 3.2.1 together with remark 3.2.2 gives a left approximate identity $\{h_j\}$ consisting of compact-hermitian operators on $X$ and bounded by 1. Now for each $x_k$, $(k = 1,2,\ldots, m)$, $h_j x_k \to x_k$ as $j \to \infty$. So, given $\eta > 0$, there exists a positive integer $N_k$ such that $\|h_j x_k - x_k\| < \eta/2$ for all $j \geq N_k$ (I)

Next, let $X_1$ be the unit ball of $X$. Since $T$ is compact, $(TX_1)^-$ is a compact subset of $X$.

Now for each $Tx$ in $(TX_1)^-$,
$h_j Tx \to Tx$ uniformly on $(TX_1)^-$, (see Remark 3.2.3). So, there exists a positive integer $N$ such that

$$\|(h_N - 1)Tx\| < \eta/2 \quad \text{for all } x \text{ in } X_1. \qquad \text{(II)}$$

Let $N' = \max\{N_1, N_2, \ldots, N_m, N\}$.

Then setting $U = h_{N'}$ in (I) and (II), we have

$$\|(U - 1)x_k\| + \|(U - 1)T\| < \eta, \quad \text{for } k = 1,2, \ldots, m,$$

which is what was wanted.

Q.E.D.

The proof of the following lemma is omitted, because it follows almost exactly the line of proving the lemma 5(b) in [54]. The main difference is that the treatment there is in terms of two-sided bounded approximate identity rather than only left approximate identity (which is in our case).

Lemma 3.2.5. Let $X$ be a separable Banach space, $T \in CL(X)$, $x \in X$, $\varepsilon > 0$ and $K$ be a bounded subset of $\mathbb{C}$. Then there exists $\eta > 0$ such that $\| \exp t(T + (U-1)) \cdot x - \exp t\, T \cdot x \| < \varepsilon$ for all $t \in K$ and all $U \in CL(X)$ with $\| U \| \leq 1$ and $\| (U-1)x \| + \| (U-1)T \| < \eta$.

We have now achieved all the principal ingredients in order to prove the following theorem 3.2.6. It is apparent that some of the results proved in this theorem may follow from the paper [54]. However, in order to have a consistent notation and a completely self-contained exposition, we give almost independent proof by repeating a little analysis in that paper.

Theorem 3.2.6. Let $X$ be a separable Banach space with the hermitian approximation property. Then there exists a compact analytic semigroup $t \to a^t : H \to CL(X)$ such that $(T_1)\ (a^t X)^- = X$ and $(T_2)\ \| a^t \| \leq 1$ for all $t \in H$.

Proof. Since $X$ has the hermitian approximation property, by lemma 3.2.1 there is a sequence $\{h_j\}$ of compact hermitian operators on $X$ such that $\| h_j \| \leq 1$ for all $j \in \mathbb{N}$ and $\| h_j x - x \| \to 0$ as $j \to \infty$ for every $x$ in $X$. For each $n \in \mathbb{N}$, let $\Delta(n) = \{t \in \mathbb{C} : |t| \leq n\}$ and $CL(X)_1 = CL(X) + \mathbb{C}.1$. The separability of $X$ gives a countable dense subset

$S = \{x_1, x_2, \ldots, x_k, \ldots\}$.

Now our first aim is to select a subsequence $\{U_n\}$ from $\{h_j\}$ and to construct a sequence $\{b_n^t\}$, $t \in \mathbb{C}$, in $CL(X)_1$ such that for all positive integers $n$, the following hold:

(I) $\|U_n\| \leq 1$.

(II) $b_o = 1$, $b_n^t = \exp t(\sum_{j=1}^{n} (U_j - 1))$, for each $t \in \mathbb{C}$.

(III) $\|b_{n-1}^t x_k - b_n^t x_k\| < 2^{-n}$, for each $x_k \in S$,
$k = 1, 2, \ldots, n$
and for each $t \in \Delta(n)$.

Our arguments are organized as follows.

We first pick up an element $U_1$ from the sequence $\{h_j\}$ in such a way that the lemma 3.2.5 with $T = 0$ and $\varepsilon = 2^{-1}$ is satisfied. This choice $U_1$ is possible, because $\|U_1 x_k - x_k\|$, $(k = 1, 2, \ldots, n)$ can be chosen as small as we require by using Lemma 3.2.4. Let $b_1^t = \exp t(U_1 - 1)$. Owing to lemma 3.2.5, $b_1^t$ satisfies condition (III), and hence $U_1$ and $b_1^t$ satisfy (I), (II) and (III). Thus the case $n=1$ is handled. In order to apply the induction hypothesis, assume now $n > 1$ and that we obtain $U_1, U_2, \ldots, U_{n-1}$ and $b_1^t, b_2^t, \ldots, b_{n-1}^t$ satisfying (I), (II) and (III). Next, putting $T = \sum_{j=1}^{n-1} U_j$, $\varepsilon = 2^{-n}$ and $K = \Delta(n)$ in lemma 3.2.5, we can choose $U_n$ from the sequence $\{h_j\}$ to satisfy the conditions (I), (II) and (III). The hypotheses of lemma 3.2.5 may be satisfied by $U_n$ because of lemma 3.2.4. This completes the inductive choice of the sequence $\{U_n\}$.

Our next objective is to construct $a^t$ from $b_n^{\,t}$ satisfying the following relations.

(IV) $\|a^t\| \leq 1$ for all $t$ in $H$.

(V) $t \to a^t : H \to CL(X)$ is analytic.

(VI) $a^t$ is compact for every $t$ in $H$.

(VII) $(a^t X)^- = X$, for every $t$ in $H$.

(VIII) $a^t x \to x$ as $t \to 0+$, for every $x$ in $X$.

(IX) $a^{t+s} = a^t a^s$, $t, s \in H$.

Now, for $t \in \mathbb{R}^+$, $\|b_n^{\,t}\| = \|\exp t \sum_{j=1}^{n} (U_j - 1)\|$

$\leq \exp(-nt)\exp nt$

$= 1$, for every positive integer $n$.

Again, let $t = \alpha + i\beta$, $\alpha$, $\beta \in \mathbb{R}$ and $\alpha > 0$.

Then, $\|b_n^{\,t}\| = \|\exp(\alpha+i\beta) \sum_{j=1}^{n} (U_j - 1)\|$

$\leq \|\exp \alpha \sum_{j=1}^{n} (U_j - 1)\| \cdot \|\exp(i\beta \sum_{j=1}^{n} (U_j - 1)\|$

$\leq 1$, since $\sum_{j=1}^{n} (U_j - 1)$ is hermitian and

and $\|\exp \alpha \sum_{1}^{n} U_j\| \leq e^{\alpha n}$.

Hence, $\|b_n^{\,t}\| \leq 1$ for all $t \in H$ and for $n \geq 1$. (X)

Let $\varepsilon > 0$ and $y \in X$. Now choose $n \geq k$ such that $2^{-n} < \varepsilon/3$ and $\| y - x_n \| < \varepsilon/3$ for some $x_n \in S$. (XI)

Therefore, for each $t \in \Delta(n) \cap H = D(n)$

$$\| b_{n-1}^t y - b_n^t y \|$$

$$\leq \| b_{n-1}^t y - b_{n-1}^t x_n \| + \| b_{n-1}^t x_n - b_n^t x_n \| + \| b_n^t x_n - b_n^t y \|$$

$$< \varepsilon, \text{ by applying (X) and (XI).}$$

Thus, $\| b_{n-1}^t y - b_n^t y \| < \varepsilon$ for all $t \in D(m)$ and all $n \geq m$. So, the sequence $b_n^t y$ is Cauchy in $X$ uniformly in $t$ in $D(m)$ for each positive integer $m$. Hence, $\lim_{n\to\infty} b_n^t y$ exists for all $t \in H$ and we denote this limit by $a^t y$. Now the sequence $\{b_n^t y\}$ converges to $a^t y$ uniformly on the compact set $D(m)$ for each $m$. Moreover each function $t \to b_n^t y$: $H \to X$ is analytic, and so $t \to a^t y$: $H \to X$ is analytic. We are thus led to the conclusion that for each $y \in X$, $t \to a^t y$ is analytic and therefore, that $t \to a^t$: $H \to CL(X)$ is analytic. This gives (V), and now (IV) follows immediately from (X).

Next we want to show that $a^t$ is compact for every $t$ in $H$. Now for every $t$ in $H$,

$$b_n^t = \exp t \sum_{j=1}^{n} (U_j - 1)$$

$$= \exp(-nt) \exp t \sum_{j=1}^{n} U_j$$

$$= \exp(-nt)\cdot 1 + \exp(-nt) \sum_{i=1}^{\infty} \frac{(tM)^i}{i!} \quad ,$$

$$\text{where} \quad M = \sum_{j=1}^{n} U_j .$$

Since every $U_j$ is compact, $M = \sum_{j=1}^{n} U_j$ is compact. Moreover, since $M$ is compact, $\lambda M$ and $M^i$ are compact for any scalar $\lambda$ and any integer $i$. Again the limit of a sequence of compact operators is compact. Hence

$$\exp(-nt) \sum_{i=1}^{\infty} \frac{(tM)^i}{i!} \quad \text{is compact.}$$

Now,
$$a^t = \lim_{n\to\infty} b_n^{\,t}$$

$$= \lim_{n\to\infty} \exp(-nt)\cdot 1 + \lim_{n\to\infty} \exp(-nt) \sum_{i=1}^{\infty} \frac{(tM)^i}{i!}$$

$$= \lim_{n\to\infty} \exp(-nt) \sum_{i=1}^{\infty} \frac{(tM)^i}{i!}$$

So, $a^t$ is compact.

Again, given an $x_i \in S$, we can obtain (as we obtained the inequality (III))

$$\| b_{n-1}^{-t} x_i - b_n^{-t} x_i \| < 2^{-n} \quad \text{for each} \quad t \in D(m),$$

$m$ is a positive integer.

Hence the sequence $\{b_n^{-t} x_i\}$ is Cauchy in $X$ uniformly in $t$ in $D(m)$ for each positive integer $m$. Thus $\lim_{n\to\infty} b_n^{-t} x_i$ exists for each $t \in H$.

We can write $x_i = b_n^t \cdot (b_n^{-t} x_i)$.

This gives, $x_i = \lim_n b_n^t \cdot (b_n^{-t} x_i)$

$$= a^t \cdot x^t, \text{ where } \lim_{n\to\infty} b_n^{-t} x_i = x^t \text{ is in } X.$$

The assertion (VII) $(a^t X)^- = X$ follows now.

Next, fix $t \in H$. Given $x \in X$, $\varepsilon > 0$, there exists $y$ in $X$ such that $\| x - a^t y \| < \varepsilon/3$. For $s \in H$, $\| a^s(x - a^t y) \| < \varepsilon/3$. This implies $\| a^s x - a^{s+t} x \| < \varepsilon/3$. Since $t \to a^t$ is analytic for $t \in H$, $a^{s+t} y \to a^t y$ as $s \to 0$. So, there is a $\delta > 0$ such that $|s| < \delta$ implies $\| a^{s+t} y - a^t y \| < \varepsilon/3$. As a result, we have
$|s| < \delta$ implies $\| x - a^s x \| \leq \| x - a^t y \| + \| a^t y - a^{s+t} y \| + \| a^{s+t} y - a^s x \|$

$$< \varepsilon .$$

This gives, $a_s x \to x$ as $s \to 0$.

The last result (IX) follows readily from the fact that
$b_n^{t+s} = b_n^t \, b_n^s$ .

The results (IV) through (IX) give the existence of a compact analytic semigroup $a^t$ $(t \in H)$ satisfying the conditions $(T_1)$ and $(T_2)$.

Q.E.D.

We now shift our attention to the converse of theorem 3.2.6 which is stated below.

"Let $X$ be a separable Banach space. If there is a compact analytic semigroup $t \to a^t : H \to CL(X)$ satisfying the conditions $(T_1)$ and $(T_2)$, does $X$ have the hermitian approximation property?"

We give a negative answer with a counter example.

Counter Example 3.2.7

We recall some notations and definitions. The set $D = \{z \in \mathbb{C}: |z| < 1\}$ is the open unit disc in the complex plane $\mathbb{C}$ and $D^-$ its closure. A is the collection of functions which are continuous on $D^-$ and analytic on D. Then A is a Banach space under the sup norm, $\|f\|_\infty = \sup_{|z| \leq 1} |f(z)|$, $f \in A$. In fact, A is a uniformly closed linear algebra of continuous complex-valued functions on the closed disc $D^-$. Moreover, the space $C(D^-)$ of all complex-valued continuous functions on $D^-$ is separable. A being a subspace of $C(D^-)$ is separable.

We now define a map $a^t: A \to A$, parametrized by $t \in H$ as follows: $(a^t f)(z) = f(e^{-t}z)$, $t \in H$, $f \in A$, $z \in D^-$. Clearly, for every $t \in H$, $a^t$ is a bounded linear operator on A. In order to prove that $a^t$ is compact, we proceed as follows.

Let $\{f_n\}$ with $\|f_n\| \leq 1$, $\forall n$, be a sequence in A. We shall show that $\{a^t f_n\}$ has a convergent subsequence. Because $\{f_n\}$ is uniformly bounded on each compact subset of D, it is a normal family, (refer to Theorem 14.6 in [53]). Therefore, it has a convergent subsequence, say $\{f_{n_k}\}$, which converges uniformly on compact subsets of D. Since $a^t f_{n_k}$ maps $D^-$ onto the compact subset K, where $K = \{\zeta: |\zeta| \leq e^{-\mathrm{Re}\,t}\}$, $\{a^t f_{n_k}\}$ converges uniformly on $D^-$. Hence the compactness of $a^t$ is a consequence of Montel's theorem. The other properties of an analytic semigroup are easily verified. Thus, $a^t$, $t \in H$, is a compact analytic semigroup on $A$.

We next prove that it is contraction, that is,

$$\| a^t \| \leq 1 \quad \text{for each} \quad t \in H.$$

Now, for $f \in A$ with $\| f \| \leq 1$,

$$\| a^t f \| = \sup_{|z| \leq 1} \{ \| a^t f(z) \| \}$$

$$= \sup_{|z| \leq 1} \{ | f(e^{-t} z) | \}$$

$$\leq 1 .$$

Our next objective is to show that the closure of the range of $a^t$ is the whole space $A$, that is, $(a^t A)^- = A$, for every $t$ in $H$. For this purpose, we first show that $A$ is the uniform closure of the polynomials $P(z) = \sum_{k=0}^{n} \beta_k z^k$, $\beta_k \in \mathbb{C}$, $z \in D^-$. Let $f \in A$. Then $f$ has a convergent power series representation as $f(z) = \sum_{n=1}^{\infty} \alpha_n z^n$, $z \in D$. Let $f_r(z) = f(rz)$, $0 < r < 1$.

Since $f$ is uniformly continuous, $f_r$ tends to $f$ as $r \to 1$.

Also $f_r(z) = \Sigma \, \alpha_n r^n z^n$ converges uniformly on $D^-$ and so $f_r$ is in the closure of all polynomials $P(z)$. Hence $f$ is in the uniform closure of all polynomials $P(z)$.

Next, for $z \in D$, let $P_n(z)$ be a polynomial,

$$P_n(z) = \alpha_o + \alpha_1 z + \dots + \alpha_n z^n \qquad (\alpha_i\text{'s are scalars}).$$

Let us define a function $g_n$ by

$$g_n(z) = z^n , \quad n = 0,1,2, \ldots .$$

Then $g_n \in A$, $\forall n$. Now consider the function $f$ defined by

$$f = \sum_{k=0}^{n} \alpha_k e^{kt} g_k .$$

Because $A$ is a unital Banach algebra, $f$ is in $A$.

Hence, $(a^t f)(z) = f(e^{-t}z)$

$$= \sum_{k=0}^{n} \alpha_k e^{kt} g_k(e^{-t}z)$$

$$= \sum_{k=0}^{n} \alpha_k e^{kt} e^{-kt} z^k$$

$$= \sum_{k=0}^{n} \alpha_k z^k = P_n(z) .$$

Thus $a^t A$ contains all the polynomials $P(z)$. Since polynomials are dense in $A$, $(a^t A)^- = A$, as desired. Ultimately we have proved that there exists an analytic semigroup $a^t$, $(t \in H)$ of compact operators on $A$ such that $(T_1)$ $(a^t A)^- = A$ and $(T_2)$ $\| a^t \| \leq 1$ for every $t$ in $H$.

Our final task is to establish the fact that every hermitian operator $S$ on $A$ is a scalar multiple of the identity operator $I$, that is, $S = \gamma I$, $\gamma$ is a scalar. Fix $t \in \mathbb{R}$ sufficiently small such that

$$\| e^{itS} - I \| < \tfrac{1}{2} \tag{I}$$

Now an isometry $T$ of the space $A$ onto $A$ is of the form $(Tf)(z) = (\alpha f)(\tau(z))$, where $f$ is in $A$, $z$ in $D^-$, $\alpha$ is a complex constant of modulus 1 and $\tau$ is a conformal map of the unit disc onto itself, (refer to [38], p. 147). Since $S$ is hermitian, $e^{itS}$ is an isometry and so $e^{itS}f(z) = \alpha_t f(\tau z)$, $\forall f \;\varepsilon^{\dagger} A$, $\forall\, z \,\varepsilon\, D^-$ and $\alpha_t$ is a complex constant of modulus 1.

The inequality (I) now yields

$$|\alpha_t f(\tau(z)) - f(z)| < \tfrac{1}{2} \| f \| , \quad \forall\, f \in A \text{ and } \forall\, z \in D^- . \text{ (II)}$$

We claim that the above inequality holds only when

$$\tau(z) = z \qquad \forall\, z \in D^- .$$

Suppose, if possible, $\tau(z_o) \neq z_o$ for some $z_o$ on the boundary of $D$. Let $z_o = e^{i\theta}$. Since $\tau$ maps boundary to boundary, $\tau(z_o)$ is on the boundary. Let $\tau(z_o) = e^{i\phi}$, $\phi \neq \theta$. Then

$$\left| \frac{1 + e^{-i\theta}\, \tau(z_o)}{2} \right| < 1 .$$

Choose $n \in \mathbb{N}$ sufficiently large such that

$$\left| \frac{1 + e^{-i\theta}\tau(z_o)}{2} \right|^n < \tfrac{1}{2} .$$

For this $n$, let us define a function $g$ on $D^-$ by

$$g(z) = \left( \frac{1 + e^{-i\theta}\, z}{2} \right)^n .$$

Now obviously, (III) $g(e^{i\theta}) = 1$, (IV) $\| g \| = 1$ and (V) $g \in A$.

† The symbol $\varepsilon$ is sometimes used for the phrase 'belongs to'.

Again, $\quad |\alpha_t \, g(\tau(z_o)) - g(z_o)|$

$$= \left| \alpha_t \left( \frac{1 + e^{-i\theta} \, \tau(z_o)}{2} \right)^n - 1 \right|$$

$$\geq \left| |\alpha_t| \, \left| \frac{1 + e^{-i\theta} \, \tau(z_o)}{2} \right|^n - 1 \right|$$

$$> \left| \tfrac{1}{2} - 1 \right| = \tfrac{1}{2} \| g \| .$$

This contradicts the inequality (II).

Therefore, $\tau(z) = z$ for all $z$ on the boundary of the unit disc. Hence, by the Maximum Modulus Theorem, $\tau(z) = z, \ \forall z \in D^-$. Thus, we obtain

$$e^{itS} f(z) = \alpha_t f(z), \ \forall f \text{ in } A, \ \forall z \in D^- .$$

Consequently, there exists $\delta > 0$ such that $|t| < \delta$ implies $\| e^{itS} - I \| < \frac{1}{2}$. This in turn gives $e^{itS} = \alpha_t I$ for some $\alpha_t \in \mathbb{C}$ .

Again, $\quad S = \lim_{t \to 0} \dfrac{e^{itS} - 1}{it}$

$$= \lim_{t \to 0} \left( \frac{\alpha_t - 1}{it} \right) I$$

$$= LI, \quad L \text{ is a scalar.}$$

Therefore, every hermitian operator on $A$ is a scalar multiple of the identity operator. But the identity operator is not a compact operator on A. So, A does not have the hermitian approximation property.

Q.E.D.

## 3.3 Section Three. CAS And Metric Approximation Property

In Section Two (3.2), we dealt with the hermitian approximation property and now we wish to investigate analogous results when the space X has the metric approximation property. In fact, most of the results in this section are the harvest of the fruits of our preceding work. However, some of the technical difficulties here are that if the property 'hermitian' is eliminated from the theorem 3.2.6, the elements $b_n^t$ constructed in that theorem may not satisfy the condition $\| b_n^t \| \leq 1$ for all t in H, and that lemma 3.3.3. which plays a vital role in constructing the family $c_n^t$ in Theorem 3.3.5, may not be true in general unless we have a commutative bounded left approximate identity. The first handicap is averted by restricting the condition $\| b_n^t \| \leq 1$ for only non-negative reals t, and to surmount the second one we first prove in Theorem 3.3.5 the existence of a one-parameter semigroup $a^t$, $t \in \mathbb{R}^+$ and then in the light of this semigroup, we construct a commutative bounded left approximate identity.

A fairly concrete example in Theorem 3.3.7 shows that the main result in Theorem 3.3.5 may not be true with the substantial condition $(T_2)$. The book [21] is a standard reference for this investigation. In order to study the converse problem, one technical difficulty is that the space of compact operators on a separable

Banach space X may not be the norm closure of the finite rank operators, although this result is true in most of our familiar separable Banach spaces. Enflo in [23] has given a counter example of this. But with this hypothesis, the converse can be established (Theorem 3.3.8). Nevertheless, there may be some sophisticated strategy for solving this converse problem, and so we keep it as an open problem (3.3.9). Since all Hilbert spaces have the metric approximation property, the converse is true for all Hilbert spaces.

Lemma 3.3.1. Let X be a separable Banach space. Then the following two conditions are equivalent:

(i) For each compact subset F of X and each $\varepsilon > 0$ there is a bounded operator T on X with finite rank such that $\| Tx - x \| < \varepsilon$ for every x in F and $\| T \| \leq 1$.

(ii) There exists a sequence $\{h_j\}$, with $\| h_j \| \leq 1$, of finite rank bounded operators on X such that $\| h_j x - x \| \to 0$ as $j \to \infty$ for every x in X.

Proof: Replace the word 'hermitian' by the words 'finite rank' in lemma 3.2.1 and then the proof of this lemma follows closely the lines of the proof of that lemma.

Remark 3.3.2. From the second part of the proof of lemma 3.3.1, it follows that $h_j \to I$ uniformly on compact sets.

The following lemma is very useful, particularly to satisfy the hypotheses of the lemma 3.3.4. We notice that this lemma needs a commutative left approximate identity.

Lemma 3.3.3. Let $X$ be a separable Banach space such that it has a commutative left approximate identity $\{h_j\}$ bounded by one and consisting of compact operators on $X$. If $T$ is in the closed subalgebra of $CL(X)$ generated by $\{h_j\}$, then for each $\eta > 0$ there exists $U$ in $\{h_j\}$ such that $\|(U-1)T\| + \|T(U-1)\| < \eta$.

Proof: Because $h_i h_j = h_j h_i$, $\forall\ i,j$, $T$ commutes with each member of $\{h_j\}$. So, $T(U-1) = (U-1)T$ for every $U$ in $\{h_j\}$. Hence we need to show that given $\eta > 0$ there is $U$ in $\{h_j\}$ such that $\|(U-1)T\| < \eta/2$.

Let $X_1$ be the unit ball of $X$. Since $T$ is compact, $(TX_1)^-$ is a compact subset of $X$. So, appealing to remark 3.3.2, we obtain $h_n \to I$ uniformly on $(TX_1)^-$. This tells us that there exists a positive integer $N$, independent of any $x$ in $X_1$, such that $\|(h_N - 1)Tx\| < \eta/2$ for all $x$ in $X_1$. Putting $U = h_N$, we get $\|(U-1)T\| < \eta/2$. Hence the result now follows as desired.

Q.E.D.

Lemma 3.3.4. Let $X$ be a separable Banach space, let $CL(X)_1$ be the Banach algebra obtained by adjoining an identity to $CL(X)$, let $K$ be a bounded subset of the complex plane and let $\varepsilon > 0$. If $S = T + \mu.1$ with $T \in CL(X)$ and $\mu \in \mathbb{C}$, then there exists $\eta > 0$ such that

$$\|\exp t(S + (U-1)) - \exp t\, S\|$$

$$\leqslant (\varepsilon + \exp 2|t| - 1)\exp \operatorname{Re}(t\mu)$$

for all $t \in K$ and all $U \in CL(X)$ with $\|U\| \leq 1$ and $\|(U-1)T\| + \|T(U-1)\| < \eta$.

Proof. See lemma 5(a) in [54].

We now turn to the main result in this section. We have already aired the general philosophy behind similar development in former sections when X has the hermitian approximation property. The guiding idea is to find a family of elements $c_n^t$ in $CL(X)_1$ which ultimately leads to the wanted conclusions.

Theorem 3.3.5. Let X be a separable Banach space having the metric approximation property. Then there exists an analytic semi-group $t \to a^t : H \to CL(X)$ such that $(T_1)\ (a^tX)^- = X$ for all $t \in H$ and $(T_3)\ \|a^t\| \leq 1$ for every $t$ in $\mathbb{R}^+$.

Proof: By the given hypothesis, X has the metric approximation property, and so by lemma 3.3.1 we can have a sequence $\{h_j\}$ consisting of finite rank bounded operators on X such that $\|h_j\| \leq 1$, $\forall\, j \in \mathbb{N}$ and $\|h_jx - x\| \to 0$ as $j \to \infty$, $\forall\, x \in X$. We can now regard the sequence $\{h_j\}$ as a countable bounded left approximate identity for X. We adopt here the same sort of notations as used in Theorem 3.2.6. If we replace the word 'hermitian' by the words 'finite rank' in lemmas 3.2.4 and 3.2.5, these results are still true. So applying these lemmas and retracing the same sort of arguments as we did in theorem 3.2.6, we can select a subsequence $\{U_n\}$ from $\{h_j\}$ in $CL(X)$ and a sequence $\{b_n^t\}$ in $CL(X)_1$ such that for all positive integers n, the following hold:

(i) $\|U_n\| \leq 1$.

(ii) $b_o^t = 1$, $b_n^t = \exp t(\sum_{j=1}^{n} (U_j - 1))$ for all $t$ in $\mathbb{C}$.

(iii) $\| b_{n-1}^t x_k - b_n^t x_k \| < 2^{-n}$ for each $x_k$, $k = 1,2, \ldots, n$ ($x_k \in S$) and for each $t \in \Delta(n)$.

From this construction of $b_n^t$, it can be proved easily that $\|b_n^t\| \leq 1$ only for $t \in \mathbb{R}^+$. Now, given $\varepsilon > 0$ and $y \in X$, it can be seen without difficulty that $\| b_{n-1}^t y - b_n^t y \| < \varepsilon$ for all $t$ in $\mathbb{R}^+$. Therefore, the sequence $b_n^t y$ is Cauchy in X uniformly in $t$ in $\mathbb{R}^+$. Hence $\lim_{n\to\infty} b_n^t y$ exists for all $t \in \mathbb{R}^+$ and let this limit be $a^t y$. Employing the similar techniques as applied in Theorem 3.2.6, we can show that:

(iv) $t \to a^t : \mathbb{R}^+ \to CL(X)$ is analytic.

(v) $\|a^t\| \leq 1$, $t \in \mathbb{R}^+$.

(vi) $a^t a^s = a^{t+s}$, $t, s \in \mathbb{R}^+$, and

(vii) $a^t x \to x$, $\forall\, x \in X$ as $t \to o+$ .

This gives $a^t$, $t \in \mathbb{R}^+$ is a one-parameter contraction semigroup.

We recall that our principal goal is to show the existence of a compact analytic semigroup on X where the parameter ranges over H. Next, let $f_\ell = a^{1/\ell}$, $\ell \in \mathbb{N}$, then clearly $\{f_\ell\}$ is a commutative left approximate identity for X consisting of compact operators and bounded by one. Now we want to choose a subsequence $\{V_n\}$ from $\{f_\ell\}$ in CL(X) and to construct a sequence $\{c_n^t\}$ in $CL(X)_1$ such that for all positive integers n, the following hold:

(viii) $\| V_n \| \leq 1.$

(ix) $c_o^t = 1, \quad c_n^t = \exp t(\sum_{k=1}^{n} (V_k - 1))$ for all $t \in \mathbb{C}$ .

(x) $\| c_{n-1}^{-t} x_k - c_n^{-t} x_k \| < 2^{-n}$ for each $x_k \in S$, $k = 1,2,\ldots, n$ and for each $t \in \Delta(n)$.

(xi) $\| c_{n-1}^{t} - c_n^{t} \| \leq 2^{-n} + \{\exp 2|t| - 1\} \exp - (n-1)\mathrm{Re}\, t,$ for all $t \in \Delta(n)$.

We choose an element $V_1$ from the sequence $\{f_\ell\}$ with $S = T = 0$ and $\varepsilon = 2^{-1}$ in lemmas 3.2.5 and 3.3.4. This choice is possible by invoking the lemmas 3.2.4 and 3.3.3. Let $c_o^t = 1$ and $c_1^t = \exp t(V_1 - 1)$. Then $V_1$ and $c_1^t$ satisfy the relations from (viii) through (xi). We now apply induction hypothesis to obtain the sequences $\{V_n\}$ and $\{c_n^t\}$ . Suppose we obtain $V_1, V_2, \ldots, V_n$ and $c_1^t, \ldots, c_n^t$ satisfying our required relations. Then by putting $S = -n + T$, $T = \sum_{k=1}^{n} V_k$, $\varepsilon = 2^{-n-1} \exp(-n(n+1))$ and $K = \Delta(n+1)$ in lemmas 3.2.5 and 3.3.4 we can choose a $V_{n+1}$ from the sequence $\{f_\ell\}$ satisfying our desired relations (viii) to (xi). This choice of $V_{n+1}$ is possible because of lemmas 3.2.4 and 3.3.3. Thus we obtain a sequence $\{V_n\}$ as wanted. Next, consider the compact set,

$$A_\delta = \{t \in \mathbb{C} : \mathrm{Re}\, t \geq \delta^{-1}, \ |t| \leq \delta \ \text{ and } \ \delta > 0\}.$$

Then obviously, $H = \bigcup_{\delta > 0} A_\delta$. The inequality (xi) yields that

$$\| c_{n-1}^t - c_n^t \| \leq 2^{-n} + \exp (2\delta - (n-1)\delta^{-1}), \quad \text{for all } t \in A_\delta .$$

This inequality shows that $c_n^t$ is Cauchy uniformly for all $t$ in each compact subset $A_\delta$, $\delta > 0$. Consequently $\lim_n c_n^t$ exists in $CL(X)_1$ for all $t$ in $H$. We denote this limit by $d^t$. Now each function $t \to c_n^t : H \to CL(X)_1$ is analytic and the sequence $c_n^t$ converges uniformly to $d^t$ for all $t$ in each compact set $A_\delta$, $\delta > 0$. Therefore $t \to d^t : H \to CL(X)$ is analytic (xii).

Again for each $t \in \mathbb{R}^+$

$$\| c_n^t \| = \| \exp t( \sum_{k=1}^{n} (V_k - 1)) \|$$

$$\leq \exp(-nt) \exp(nt)$$

$$= 1 \text{, for every positive integer } n.$$

This concludes that $\| d^t \| \leq 1$ for every $t \in \mathbb{R}^+$ (xiii).

The result (xiv) $d^t d^s = d^{t+s}$ is obvious from the construction of $c_n^t$. Moreover, using the inequality (x) and with the aid of the analogous mechanism as led up the relations (VII) and (VIII) in Theorem 3.2.6 we can prove the following relations

(xv) $(d^t X)^- = X, \quad \forall\, t \in H$

and (xvi) $d^t x \to x, \ \forall\, x \in X$ as $t \to 0+$.

Moreover, clearly $d^t$ is compact for every $t$ in $H$. (xvii)

Finally the relations (xii) to (xvii) ensure the existence of a compact analytic semigroup $d^t$, $t \in H$ satisfying the conditions $(T_1)$ and $(T_3)$.

Q.E.D.

We now show that a separable Banach space $X$ having the metric approximation property may not have a compact analytic semigroup satisfying the conditions $(T_1)$ and $(T_2)$. We illustrate below that the space $C[0,1]$ can not have such a semigroup. Before discussing this result we need the following lemma.

<u>Lemma 3.3.6</u> Consider the space $C[0,1]$ with the uniform norm. Let $f$ be a non-zero element in $C[0,1]$. Then the multiplier $L_f$ defined by means of $L_f g = fg$, $\forall\, g \in C[0,1)$ can not be compact on $C[0,1]$.

This lemma can be proved easily, and therefore, the proof is omitted.

<u>Theorem 3.3.7</u>. The space $C[0,1]$ with the uniform norm can not have a compact analytic semigroup $a^t$, $t \in H$, satisfying the relations $(T_1)(a^t X)^- = X$ and $(T_2)$ $\|a^t\| \leq 1$ for all $t$ in $H$.

<u>Proof</u>. We know that $C[0,1]$ is a separable Banach space with its uniform norm. Also $C[0,1]$ has the metric approximation property,(refer to §30 in [21]). We now assume that, if possible, this space has a compact analytic semigroup $a^t$, $t \in H$ satisfying $(T_1)$ and $(T_2)$. Then by Theorem 3.1.4 in Section One, there exists an isometry $S^r$ on this space. So, by the Banach-Stone Theorem, (Theorem 8, p. 442, in [22]), there is a homeomorphism $\tau$ from $[0,1]$ onto $[0,1]$ and a function $\alpha$ in $C[0,1]$ with $|\alpha(x)| = 1$, $x \in [0,1]$ such that

$$(S^r f)(x) = \alpha(x)\, f(\tau(x)), \quad \forall\, x \in [0,1] \text{ and } \forall\, f \in C[0,1].$$

By the similar arguments given in the counter example 3.2.7 in Section Two and constructing a suitable function $g$ in $C[0,1]$, we can prove that $\tau(x) = x$ for all $x \in [0,1]$. Thus $S^r$ is a multiplier and so $S^r$ can not be compact. We thus reach a contradiction and hence the result follows.

Q.E.D.

Theorem 3.3.8. Let $X$ be a separable Banach space such that the set of compact operators is the norm closure of the finite rank operators. If there is a compact analytic semigroup $t \to a^t : H \to CL(X)$ such that $(T_1)\ (a^t X)^- = X$ for all $t \in H$ and $(T_3)\ \| a^t \| \leq 1,\ \forall\ t \in \mathbb{R}^+$, then $X$ has the metric approximation property.

Proof: By the given hypothesis, we have

$$\| a^t \| \leq 1, \quad \forall\ t \in \mathbb{R}^+ \quad \text{and} \quad a^t x \to x, \quad \forall\ x \in X, \quad \text{as} \quad t \to 0+ .$$

Let $T_n = a^{1/n}$, $n \in \mathbb{N}$. Then $T_n$ is compact, $T_n x \to x$ as $n \to \infty$ for every $x$ in $X$ and $\| T_n \| \leq 1$, $\forall\ n \in \mathbb{N}$.

Choose a finite rank operator $S_n$ such that

$$\| S_n \| \leq 1 \quad \text{and} \quad \| S_n - T_n \| < \frac{1}{n} .$$

Now, for each $x$ in $X$,

$$\begin{aligned} \| S_n x - x \| &\leq \| S_n x - T_n x \| + \| T_n x - x \| \\ &\leq \frac{1}{n} \| x \| + \| T_n x - x \| \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Consequently, we obtain a sequence $\{S_n\}$ consisting of finite rank operators such that $\| S_n \| \leq 1$, $\forall$ n and $S_n x \to x$, $\forall x \varepsilon X$ as $n \to \infty$. Hence, by lemma 3.3.1 we obtain the required result.

Q.E.D.

We close this chapter by citing the following open problem.

Problem 3.3.9. Let X be a separable Banach space. If there exists a compact analytic semigroup $t \to a^t : H \to CL(X)$ satisfying the relations $(T_1)$ and $(T_3)$, does X have the metric approximation property?

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# LIST OF SYMBOLS

We give below a list of some of the symbols used, and what they mean.

| | |
|---|---|
| $\mathbb{N}$ | The set of natural numbers. |
| $\mathbb{Z}$ | The set of integers. |
| $\mathbb{R}$ | The real line. |
| $\mathbb{C}$ | The complex plane. |
| $\mathbb{R}^n$ $(\mathbb{C}^n)$ | The set of all n-tuples $(x_1, x_2, \ldots, x_n)$, $x_i$'s in $\mathbb{R}$ ($x_i$'s in $\mathbb{C}$). |
| H | The open right half, $\{t \in \mathbb{C} : \text{Re } t > 0\}$, of the complex plane $\mathbb{C}$. |
| $S^-$ | The closure of a set S. |
| BL(X) | The set of all bounded linear operators on the space X. |
| CL(X) | The set of all compact operators on the space X. |
| $c_o$ | The separable Banach space consisting of all convergent sequences with the limit 0. |
| C[0,1] | The separable Banach space of bounded continuous real functions on [0,1]. |
| $\sim$ | Approximately equal to. |
| $f^n$ | n times functional composition of a map f . |
| $f^{(n)}$ | n times derivative of a map f . |