A STUDY OF LOGICAL PARADOXES

by

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CONTENTS

INTRODUCTION

By a paradox we understand a seemingly true statement or set of statements which lead by valid deduction to contradictory statements. Logical paradoxes - paradoxes which involve logical concepts - are in fact as old as the history of logic. The Liar paradox, for instance, goes back to Epimenides (6th century B.C.?). In the late 19th century a new impetus was given to the investigation of logical paradoxes by the discovery of new logico-mathematical paradoxes such as those of Russell and Burali-Forti. This came about in the course of attempts to give mathematics a rigorous axiomatic foundation.

Sometimes a distinction is maintained between a paradox and an antinomy. In a paradox, it is said, semantical notions are involved and a certain "oddity", "strangeness", or what may be called "paradoxical situation", resides in its construction. The resolution of a paradox is therefore not simply a matter of removing contradiction, but also requires clarifying and removing the "oddity". On the other hand, an antinomy is said to consist in the derivation of a contradiction in an axiomatic system and its resolution lies in revising the system so as to avoid the contradiction. In discussing paradoxes and antinomies, we shall not be strictly bound by this usage of these terms: we use "paradox" and "antinomy" interchangeably. Indeed, from our point of view, even antinomies in an axiomatic system ultimately need semantic clarification and thus removal of paradoxical situations.

Chapter I

RUSSELL ON PARADOXES

A. Russell's vicious circle principle

§1. Formulation of the principle

Russell agreed with Henri Poincaré that logical paradoxes result from a 'vicious circle', but dissented from Poincaré's view that this 'vicious circle' is the outcome of taking infinity as actual or completed. Poincaré was one of the initiators of intuitionism and was essentially concerned with Cantorian antinomies, i.e. antinomies which arise within the context of Cantor's set-theory. Russell was interested in other paradoxes as well and wanted to attack them with some uniform principle presumably because of their structural similarity. Russell concluded from his analysis of paradoxes that:

> they all result from a certain kind of vicious circle. The vicious circle in question arises from supposing that a collection of objects may contain members which can only be defined by means of the collection as a whole.*

Russell's phraseology here is unhappy, for in employing the vocabulary of set-theory, namely "class" and "class-membership", he gives the impression

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* A. N. Whitehead and B. Russell, Principia Mathematica, 2nd edn., p.37. (Hereafter abbreviated PM).

that what he says has application to set-theory only. Russell's view about 'vicious circle' may be better appreciated by his 'vicious circle principle' in which he intended 'to avoid illegitimate totalities', namely:

> Whatever involves all of a collection must not be one of the collection

or

> If provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total.*

According to Russell if we transgress the 'vicious circle principle', we have then an 'illegitimate totality', and thus we commit a 'vicious circle fallacy', which leads to paradoxical consequences.

Russell's formulation of the vicious circle principle is not free from vagueness, but by considering his illustrations we get a clear idea of what he wants to say. Russell would like to say that once we have an aggregate or a statement or an idea, say x, then nothing, say y, defined or explained in terms of x, can come under or be comprehended in or be a member of x. This elucidation of the vicious circle principle becomes quite evident if we keep in view Russell's examples of the violation of this principle. According to Russell, thus, any reflexive statement or any self-inclusive one would transgress the principle and hence involve a vicious circle fallacy, because its application would extend to itself or come under itself. Russell himself explicitly mentions as the purpose of his principle to exclude self-reference

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* Ibid.

or reflexiveness in any statement or class. For this reason a statement about all statements must be 'meaningless' because it is self-inclusive. Also the law of excluded middle taken as a proposition of the form, "All propositions are true or false", would involve a vicious circle fallacy.*

§2. Basis of the principle

It seems that Russell asserted his 'vicious circle principle' on pragmatic and empirical grounds. He nowhere proves it by purely rational or logical arguments. It is a merit of Poincaré's vicious circle principle that it is based on reasoning and is not grounded simply on empirical generalisation. As Russell seems to base his principle on inductive generalisation, by citing various examples as evidence, it is quite sufficient to refute Russell's contention by citing counter-examples where his principle does not hold - i.e. where reflexive statements make good sense and may even be true.

Let us first take Russell's own example of an imaginary sceptic** 'who asserts that he knows nothing'. Russell argues, in case this assertion be taken as self-inclusive, it commits the vicious circle fallacy and thus leads to paradoxical consequences. This is so, according to Russell, because the man is refuted by the implication of his own statement, namely, that he knows that he knows nothing. This implication would then be similar to the wise saying of Socrates, namely, "I know my ignorance". But, in fact, knowing this much is knowing too much! A thorough-going sceptic does not mean

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* Cf. Ibid. p.38

** Ibid.

to assert this important implication. He would not mean to put any limitation on the application of his assertion, 'I know nothing'. A consistent sceptic would like to assert that he does not know whether or not he has any (unconscious) knowledge of anything. There may be some other arguments to raise against an all-round sceptic, but his statements are quite consistent and sensible. Hence Russell is mistaken in thinking that the sceptic's assertion leads to nonsensical results, because his assertion does not imply that he knows that he knows nothing if it is taken to be self-referring.

It is not only that the sceptic's assertion is quite consistent, but there are many statements, indeed true statements, which are reflexive, self-referring or self-inclusive, but which according to Russell's principle must be discarded as meaningless and absurd. For instance: "All sentences are compounded of words" is self-referring and necessarily true; "I always tell lies" with the assumption that I make several other significant statements is a self-referring, meaningful and necessarily false assertion; "Whatever sentence is written on this page is in English" is self-inclusive and not only meaningful but also empirically true; "This sentence consists of six words" is self-inclusive and empirically true, whereas "This sentence is in French" is self-inclusive, empirical and a false assertion. On similar lines the following sentence may or may not be meant to be self-inclusive: "I have been told by our teacher that I may let you know that there will not be any lecture next week." It seems that if we accept Russell's view of vicious circle many obviously meaningful self-referring or self-inclusive statements would be rendered meaningless. Imagine a limited company which has all limited companies as its members (shareholders). Now, this limited company of all limited companies is self-inclusive because it is itself a limited company. Hence according to Russell's vicious circle principle the notion of such a limited company shall be meaningless and inconceivable. The principle is: "Whatever involves all of a collection must not be one of the collection." But contrary to such a principle we may actually create a limited company which includes itself such that its share-holders are all and only those limited companies which are share-holders of themselves. It seems that according to Russell's ligislation we cannot even speak significantly of any property (or perhaps even of a word) as being self-predicable (or autological) because then that property (or predicate designated by the word) can be applied to itself and that involves self-reference, which Russell would like to exclude from any statement in accordance with his own principle. But in fact we may reasonably say that abstract is abstract or concept is concept or the word 'polysyllabic' is polysyllabic or 'noun' is noun, etc.

§3. Extension of Russell's principle

We might be accused of misinterpreting Russell's vicious circle principle and it might be argued that Russell did not mean to extend the notion of vicious circle to such lengths; or that we can make the principle valid by minor modifications. Thus it may be remarked that the principle does not extend to the case of saying "'polysyllabic' is polysyllabic" or "'black' is black" because the principle extends only where the meaning of a phrase is involved and does not refer to the written or spoken aspects of words. But we have provided ample examples of self-reference where only meanings are involved, like an idea (or concept) is an idea (or concept), or the example given in the previous subsection about the information given by the teacher. Or take the following assertion: "What I am saying is meaningful". It is self-referring and also meaningful. It cannot be regarded as meaningless, for then it would contradict itself and entail the falsity of the assertion and this falsity would, in turn, imply the meaningfulness of the assertion. Hence this self-referring expression must be regarded as meaningful.

Again it might be suggested that Russell's principle is in fact a combination of several principles and it is only, e.g. in the case of definition that this principle should be regarded as properly applicable. In other cases where there is just self-description or self-reference, it should not be taken seriously. Against this suggestion, however, it must be said that in the first place, this was not the intention of Russell as evidenced in his explanation and samples of vicious circle. Secondly, we have given above diverse examples to refute various sorts of formulation of the principle. And unless we have been given a definite and unambiguous formulation, it is not illuminating to comment at random on various possible formulations of the vicious circle principle. Russell himself has given us no clue as to restrictions imposed on the use of the principle and hence we are justified in giving above an "extended" interpretation of his principle.

In the case of Poincaré's view of vicious circle we have a criterion for deciding where to find a vicious circle, namely, when we treat infinity as completed. But Russell has provided us with no such criterion. Moreover he has not offered any rational justification for his principle except one based on inductive or pragmative grounds. Hence we have to rely on his examples of vicious circle for the interpretation of his principle.

It is amply clear, therefore, from the above illustrations that Russell is mistaken in believing that the so-called vicious circle fallacy (as leading to absurdity or contradiction) will always ensue if we transgress the vicious circle principle. Even if Russell were to modify his statement and say that all paradoxes originate in a 'vicious circle' and not that all 'vicious circle' statements, that is, self-inclusive, reflexive, or self-referring expressions, lead to paradoxical consequences, he could still be easily refuted. There are several paradoxes in which no question of 'vicious circle' arises i.e. which do not seem to involve any self-inclusive statements. For instance, the paradox of the Unpredictable Examination and the paradox of the sophist Protagoras and his pupil*. Nevertheless, even if Russell were to restrict himself to well-known paradoxes which have been favourites of philosophers or which he himself discusses, and so restricts the applicability or relevancy of his principle to such paradoxes, he does not seem to be successful in his account, as we shall now show.

§4. Russell's logic and his principle

Before examining the actual examples of paradoxes which Russell considers to confirm his principle, let us first familiarise ourselves with his logical apparatus. As a proposition would be meaningless if it suffered from vicious circle, his logical symbols to be well-formed must not violate the vicious circle principle. Accordingly, Russell says that in a given function

> the values of a function cannot contain terms only definable in terms of the function. Now given a function $\phi\hat{x}$, the values for the function are all propositions of the form ϕx. It follows that there must be no propositions, of the form ϕx, in which x has a value which involves $\phi\hat{x}$.**

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* See the Appendix, pp. 181.

** PM, p.40.

Russell argues that this cannot be the case, otherwise the values of the function would not all be determinate until the function were determinate, but we know the function not to be determinate unless its values are previously determinate. Hence, we can thereby clearly see that it would be then a case of vicious circle. Therefore Russell concludes that there must be no such thing as the value for $\phi\hat{x}$ with the argument $\phi\hat{x}$ or with any other argument which involves $\phi\hat{x}$. The symbol "$\phi(\phi\hat{x})$" must not express a proposition, as "ϕa" does if ϕa is a value for $\phi\hat{x}$. The symbol "$\phi(\phi\hat{x})$" does not express anything; it is not significant. On this point Russell remarks:

> Thus given any function $\phi\hat{x}$, there are arguments with which the function has no value ..... We will call the arguments with which $\phi\hat{x}$ has a value "possible values of x". We will say that $\phi\hat{x}$ is "significant with the argument x" when $\phi\hat{x}$ has a value with the argument x.*

By such legislation Russell has been able to avoid the occurrence of 'vicious circle' in his symbolic apparatus.**

B. <u>On the paradoxes discussed by Russell</u>

We have just quoted Russell to see how vicious circle principle affects his symbolism. Keeping in view the above symbolic structure let us consider the paradoxes enumerated by Russell as evidence for the claim that the paradoxes are the result of a vicious circle.

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* <u>Ibid</u>.

** In fact, in Russell's logical symbolism two rules corresponding to his simple theory of types and his ramified theory of types are required to avoid the vicious circle fallacy. More will be said on a later occasion. See ch.II, pp. 52-3.

Russell says that in all these paradoxes:

> there is a common characteristic which we may describe as self-reference or reflexiveness ...... In each contradiction something is said about all cases of some kind, and from what is said a new case seems to be generated, which both is and is not of the same kind as the cases of which all were concerned in what was said. But this is the characteristic of illegitimate totalities, as defined them in stating the vicious circle principle. Hence all our contradictions are illustrations of vicious circle fallacies. It only remains to show, therefore, that the illegitimate totalities involved are excluded by the hierarchy of types .....*

That is, using the symbolic expressions as delineated and directed by the above sub-section we should avoid vicious circle fallacies in terms of his theory of types, which will be discussed at a later time.

Our first objection to the above statement is that even if we grant in all the illustrations we come across a 'vicious circle', it is by no means certain that the 'vicious circle' alone would only be a necessary but also sufficient basis for the contradiction (paradox) and hence solely by the removal of 'vicious circle' shall we be able to uproot the contradiction and thus remove the paradox in a plausible and reasonable way. A vicious circle in the following illustrations is not a sufficient condition, but may be one of the necessary conditions for the paradox to arise. Let us now turn to his illustrations in sequence.

§1. The Liar paradox

Russell first takes up the following version of the Liar paradox: "I am lying." Taken as a statement which must be either true or false,

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* Ibid., pp. 61-62.

this statement leads to contradiction. In order to avoid the contradiction, Russell would say, we must avoid the self-inclusiveness of the statement, that is, we must restrict its application so that it cannot be taken as self-inclusive. Thus, this statement is to be interpreted in accordance with Russell's theory of types. The statement "I am lying" according to Russell means: "There is a proposition which I am affirming and which is false". That is: "I assert *p* and *p* is false". If this proposition is supposed to apply to itself, it involves then self-reference and hence commits the vicious circle fallacy. In order to avoid this unpleasant situation Russell asserts that the word "false" is ambiguous and that in order to make it unambiguous, precise and univocal we must pinpoint the order of falsehood. This order of falsehood is determined by specifying the order of the proposition to which falsehood is ascribed. There are propositions about individual existent things; again there may be propositions about such previous propositions in turn and so on and so forth. To quote Russell:

> We saw also that, if *p* is a proposition of the *n*th order, a proposition in which *p* occurs as an apparent variable is not of the *n*th order, but of a higher order. Hence the kind of truth or falsehood which can belong to the statement "there is a proposition *p* which I am affirming and which has falsehood of the *n*th order" is truth or falsehood of a higher order than the *n*th. Hence the statement of Epimenides does not fall within its scope and therefore no contradiction emerges.*

Thus, according to Russell the statement "I am lying" taken meaningfully consists of a combination of an infinite number of statements in the form: "I am asserting a false proposition of the first order", "I am asserting a false proposition of the second order" and so on. Now,

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* Ibid. p.62.

according to Russell's analysis, as no proposition of the first order is asserted, the statement "I am asserting a false proposition of the first order", is to be regarded false: as according to Russell this statement consists of two statements (1) I assert *p* (2) *p* is false, of which (1) is false and so the conjunction is false. Hence we get a true statement of the second order: "I am asserting a false proposition of the first order" is false. Hence the statement "I am making a false statement of the second order" is true. Again this is the only statement of the third order. Hence the statement "I am making a false statement of the third order" is false. Thus Russell concludes:

> Thus we see that the statement "I am making a false statement of order $2n+1$" is false, while the statement "I am making a false statement of order $2n$" is true. But in this state of things there is no contradiction.

Both Russell's analysis and his solution of the Liar paradox are objectionable. If the contradiction in the Liar paradox arises solely from a vicious circle, then why does it not arise in the following parallel statement "I am speaking the truth"? This shows that the cause of contradiction (paradox) is not solely due to the vicious circle, hence that Russell has failed to specify the sufficient cause of the contradiction. Nor does Russell's solution in terms of a hierarchy of "true" and "false" seem convincing. His claim that the words "true" and "false" are ambiguous and have different meanings (in fact an infinite number of meanings according to the order of the proposition) does not reflect the real meanings or the logical behaviour of the words "true" and "false". When I say "All statements on this page are true", this statement applies to itself, i.e., it is self-inclusive, and it may well be a true statement. The truth of this statement no doubt banks upon the truthfulness of the rest of the statements on this page. That is, in order to verify this statement, we must verify the rest of the statements on this page. Also the question of truthfulness in this arises only if there are other statements on the page so that the question of the truthfulness of these statements could be settled without further reference to the question of truthfulness of some other statements. This shows that the method of verification of a statement which concerns all other statements on this page is different from that of other statements on the page. But it does not follow from this that every value ϕx has "first truth" i.e., the truth of the first order, and the sort of truth appropriate to $(x).\phi x$ is "second truth" i.e., the truth of the second order different from that of the first order, as Russell wants us to believe. Because of the confusion between the method of verification and the question of truth (i.e. whether we can sensibly raise the question of whether the statement is true or false) with the notion of truth, Russell seems to have been misled into thinking that "I am lying" paraphrased as "I am asserting a false proposition of the first order" is false. In fact, according to his analysis, we are led to the unhappy consequence that the statements "I am lying" and "I am speaking the truth" should have the same truth-value, namely falsehood, because in both cases no proposition of the first order is being asserted. He should have argued rather that since there is no proposition (i.e. a statement which is true or false) of the first order, the question of truth and thus the question of the verification of the statement "I am asserting a false proposition of the first order" - i.e. the statement "I am lying"-does not arise: insofar as such a question is necessarily linked with the truth of the statement, we cannot talk of this sentence being true or false. It nowhere follows from Russell's argument that the notion of truth is ambiguous and that there is an hierarchy of different meanings of the word "true" or the word "false".

§2. Russell's paradox

In the case of the paradox of the class of all classes which are not members of themselves, now called Russell's paradox, the contradiction results from posing the question whether this class is a member of itself or not. According to Russell:

> a proposition about a class is always to be reduced to a statement about a function which defines the class, i.e. about a function which is satisfied by the members of the class and by no other arguments. Thus a class is an object derived from a function and presupposing a function, just as, for example, $(x).\phi x$ presupposes the function $\phi\hat{x}$. Hence a class cannot, by the vicious circle principle, significantly be the argument to its defining function, that is to say, if we denote by "$\hat{z}(\phi z)$" the class defined by $\phi\hat{z}$, the symbol "$\phi\{\hat{z}(\phi z)\}$" must be meaningless ..... *

Hence, according to Russell, no class can be a member of itself: class-self-membership is rather meaningless, and so to say that such-and-such class is a member of or is not a member of itself is meaningless. Russell's argument seems to be highly implausible. He tries to show by inductive generalisation that all paradoxes are the result of a vicious circle; he makes the generalisation that where there is a vicious circle, it leads to paradoxes, meaninglessness, and absurdity. Russell gives no other reason for rejecting the notion of self-membership of any set except that it involves a vicious circle. This is a very unconvincing analysis of the situation. In fact, he could have reasonably argued that the rejection of self-membership of the set follows directly from the very concept of set and set-membership - a point to which we shall return at a later stage. However the meaninglessness of the concept of self-membership for sets does not follow. That is, "$\sim(x \in x)$" holds good by virtue of the concept of set-membership but not the meaninglessness

* Ibid. p.62-3.

of "$x \in x$". Russell fails to provide any rational ground for the belief that a vicious circle necessarily leads either to contradiction or to meaninglessness.

§3. Class and Relation paradoxes

Russell was able to derive the contradiction by using both the notions: "x is a member of x" and "x is not a member of x". His layout of the paradox is similar to that of the Barber paradox and the Relation paradox (discussed below). But it does not follow from the fact that if these paradoxes have some structural similarity, then they are structurally identical.* Our objection to Russell's analysis of the Class, i.e., Russell paradox, is that he does not bring out the essential structure involved in the paradox. Apart from the structural features which it shares with the Barber paradox and the Relation paradox, the more important and essential structure involved in the Class paradox is similar to that of, say, the Class of all classes (discussed in Chapter V). Again, if we follow Russell's description of the Class paradox, that it results from a 'vicious circle' then there should occur a contradiction even if we remould the paradox so as to concern the class of all classes which are members of themselves. But, in fact, no contradiction emerges according to his analysis. That is, Russell cannot show us any contradiction if in our argument we bring in "$\{x : x \in x\}$", although "$x \in x$" is clearly a case of a vicious circle. Hence Russell cannot show the meaninglessness of "$x \in x$" in that it results from contradiction solely derived from his analysis of the paradox in question.

Russell's presentation of the Class paradox is similar to the paradox

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* The notion of structure of paradox is discussed in the next chapter.

of 'the relation (T) which holds between the relations R and S whenever R does not have the relation R to S'*. Let the relation T be R, then "R has the relation R to S" is equivalent to "R does not have the relation R to S", leading thus to contradiction. As it was in the case of classes, a class cannot by the vicious circle principle significantly be the argument to its defining function, that is to say, if we denote by "$\hat{z}(\phi z)$" the class defined by $\phi\hat{z}$, the symbol "$\phi\{\hat{z}(\phi z)\}$" must be meaningless. Similarly, if we let the relation R be defined, say by the function $\phi(x,y)$, (i.e. R holds between x and y whenever $\phi(x,y)$ is true, but not otherwise). Then to interpret "R has the relation R to S" we shall have to suppose that R and S can significantly be the arguments to ϕ. But this would require that ϕ should be able to take as an argument an object which is defined in terms of ϕ, and this no function can do because of the vicious circle principle. Hence "R has the relation R to S" is meaningless, and thus Russell says, the contradiction vanishes. Our objection is the same as mentioned above, that if we confine ourselves in our argument only to the notion "R has the relation R to S" and do not bring in the notion of "R does not have the relation R to S" no contradiction arises, although in the notion of "R has the relation R to S" the vicious circle principle is exemplified. Hence, violation of the vicious circle principle alone cannot be deemed responsible for the emergence of contradiction. Russell, in fact, would agree that in most cases the conclusions of the arguments which involve vicious circle fallacies will not be self-contradictory, but he states:

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* Cf. Ibid. pp.60, 63.

> wherever we have an illegitimate totality, a little ingenuity will enable us to construct a vicious circle fallacy leading to a contradiction, which disappears as soon as the typically ambiguous words are rendered typically definite, i.e. are determined as belonging to this or that type.*

This is a very elusive reply and Russell provides no example of his ingenuity in cases where we use only the notion of self-membership of classes, or the notion of relation R having the relation R to S such that they would lead to contradictions. Possibly what Russell calls a 'little ingenuity' would discover some factor other than 'vicious circle' which becomes mainly responsible for contradictions. In the paradox under discussion the notion of "R does not have the relation R to S" enters in just as in the case of the class paradox the notion of "not a member of itself" creeps in to bring about the contradiction. We may interpret Russell's concealed argument as follows: As the notion "R has the relation R to S" commits the vicious circle fallacy, it is meaningless. Hence the opposite notion "R does not have the relation R to S" is also meaningless (or ill-formed). As the argument involving the notion "R has the relation R to S" and its opposite leads to contradiction as shown in the above paradox, we need to expose and eliminate the root-cause of the evil, namely, vicious circle. That is, in order to remove the paradox, we should make the notions significant and this is only possible if we avoid the vicious circle. Russell ignores to discuss the structure or skeleton of the argument which occurs while creating a paradoxical argument with the help of notions like "R has and does not have the relation R to S". But this structure is of great importance, and hence by ignoring this, Russell has failed to present an adequate analysis of the

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* Ibid. p.64. Russell uses many expressions like 'illegitimate totality', 'systematic ambiguity', 'vicious circle fallacy', 'insignificant', meaningless' etc. etc. without specifying their discriminating characteristics. As it stands, the expression 'illegitimate totality' may be taken as equivalent to 'systematic ambiguity' or even to 'vicious circle fallacy', although these expressions seem to denote different concepts which are nevertheless connected with one another.

paradox. Russell vaguely saw some similarity in structure between the class paradox and the relation paradox, although he proffered an inaccurate account of it.

As already mentioned, Russell rejected Poincaré's contention that the contradictions arise because of introducing the notion of actual or completed infinity.* Russell saw that we encounter contradictions even if there is a finite number of things and no question of infinity could arise. But he mistakenly assumed that there could only be one common root-cause behind all the paradoxes. It might be that in certain cases the trouble-spot may be treating infinity as completed, although in some such cases there might be an additional factor in common with other paradoxes which concern only a finite number of objects. It seems Russell was blinded by his eagerness to dig out a uniform formula to mend all the paradoxes. But there may be several ailments needing diverse treatments, each treatment or medicine best-suited to cure some particular sort of ailment, and no panacea to cure all the paradoxes. The urge to attain uniformity, harmony and oneness is deeply-seated; it is as old as man. The search for elixir and philosopher's stone, in order to cure all diseases and to convert baser metals into gold, and Russell's vicious circle principle are examples of this natural urge.

§4. Berry's paradox

Next, we come to Berry's paradox of 'the least integer not nameable in fewer than nineteen syllables', the paradox of 'the least indefinable ordinal', and the Richard paradox. It is difficult to see how the vicious

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* See H. Poincaré, 'The Logic of Infinity' and 'Mathematics and Logic' transl. in <u>Mathematics and Science: Last Essays</u>; <u>Science and Method</u>, Part II.

circle fallacy is responsible for the rise of such paradoxes. Russell tries to show how these paradoxes embody the vicious circle fallacy. He points out that the Berry paradox results from thinking that 'the least integer not nameable in fewer than nineteen syllables' is itself a name. First we come to know that there is in fact a least integer which is not nameable in fewer than nineteen syllables. This we know because the number of syllables in English names of finite integers grow larger, and must generally increase indefinitely. Hence the names of some integers must consist of at least nineteen syllables and among them there must be a least. Hence this particular number is both nameable and not nameable in less than nineteen syllables. It is nameable in fewer than nineteen syllables, because we regard 'the least integer not nameable in fewer than nineteen syllables' as itself a name, which consists of less than nineteen syllables. Here we are confronted with a contradiction. But this contradiction arises only in confusing two different methods of naming integers. According to one method of naming, which is our ordinary way of counting numbers, the order and number of definite syllables is essential and all-important. According to this procedure, the discussed number would be that number chosen by selecting the naming expression which would just occur when the counting expression just exceeds eighteen syllables i.e. nineteen or more than nineteen syllables. This method actually determines both the name (numeral) as well as the number because by convention we usually understand such names as numbers. Once we have the number by the above method we may designate this number by the expression "the least integer not nameable in fewer than nineteen syllables". Obviously this expression would be really equivalent to "the least integer not nameable in fewer than nineteen syllables in the English language as used in ordinary counting". In case the above equivalence does not hold and the "nineteen syllables" in the expression "the least integer not nameable in fewer than nineteen syllables" should mean "nineteen syllables in the expression formed by any meaningful English expression", then we do not have any clue as to what number we are talking about. The simple designation "the least integer not nameable in fewer than nineteen syllables" does not apply to any particular number unless we already know some particular method by which the significant English expressions are to be arranged. The manner of classification of expressions is very important. It may be that according to one classification one number is named or described in more than nineteen syllables and according to some other classification the same number is named or described in fewer than nineteen syllables. Russell himself remarks that the number mentioned in the expression 'the least integer not nameable in fewer than nineteen syllables' in fact is 111777. Russell obviously arrived at the number 111777 by considering expressions which are arranged in accordance with the normal way of counting numbers in English. Thus the number 111777 would be expressed by "One hun/dred and e/le/ven thou/sand se/ven hun/dred and se/ven/ty se/ven", which of course is constituted of just nineteen syllables. But the same number (111777) can be expressed by some other English expression having less than nineteen syllables e.g., we may simply omit two "ands" in the above 19-syllabic expression, or we may express the same number by the expression "Three thou/sand and twen/ty one mul/ti/plied by thir/ty se/ven". But if there is some precise, clear-cut, unambiguous and well-determined rule for writing series of numbers, there will not appear any contradiction as we have above. From the foregoing discussion it is clear that the expression "the least integer not nameable in fewer than nineteen syllables" cannot be said to have full significance unless we have already some unambiguous rule for framing names for numbers by combining definite syllables. According to Russell the word "nameable" in the Berry expression refers to the totality of names and hence the expression cannot stand by itself without infringing the 'vicious circle principle'.* In accordance with the theory of types we have thus a hierarchy of functions. We, therefore, have to distinguish names of different orders. Elementary names will be true 'proper names' without involving any description. First-order names involve description by means of first-order function - that is, if $\phi!\hat{x}$ is a first-order function, then the term satisfying this function will be a first-order name, although according to Russell, there may not be any corres-poinding object to be named. Second-order names involve similarly the second order function. Also such names may involve reference to the totality of first-order names. So the phrase "nameable" remains systematically ambiguous unless we mention the corresponding order, and the name in which the unambiguous phrase "nameable by names of order n" occurs is necessarily of higher order than the nth. Hence no paradox arises. Accordingly Russell would like the 19-syllable expression to be paraphrased as "the least integer not nameable in fewer than nineteen syllables of order n", and since this expression (considered as a name) would belong to order $n+1$, there is no contradiction if it has less than nineteen syllables. So two or more names of different orders may denote the same number without involving any contradiction. The question arises how to make a hierarchy of names. Take the following expressions: (a) "ten multiplied by two" (b) "four multiplied by five" (c) "ten added to ten" (d) "twenty" etc. These expressions represent the same number but have a different number of syllables. Do these expressions as names belong to the same order or not? If they do not belong to the same order, then we should not treat them on the same level as we actually do in arithmetic. Again, according to Russell's ramified theory of types, irrational numbers should be regarded of higher order than rational numbers; for the former are to be defined as classes of

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* Cf. PM, p.63-4.

rational numbers. Hence it follows that the name of a number in terms of irrational numbers would be of different order from that of in terms of rational numbers. But we do not maintain this distinction in arithmetic*. The fact is that in the Berry paradox we are not concerned with the systematic ambiguity of the concept "name", and hence there is no need of a hierarchy of names. The real problem is simply to remove the ambiguity. Once it is clarified, the paradoxical tangle is resolved. There is no question of systematic ambiguity corresponding to different types of naming. Again the contradiction does not arise because of any vicious circle involved in Berry's puzzle; it arises because of ambiguity and vagueness. Russell would himself agree that if we write the expression "the least integer which is not nameable in less than nineteen syllables", then the contradiction would cease, for the new expression makes up nineteen syllables.

Whether the expression is self-inclusive or not is another matter. Once we formulate an unambiguous method for the generation of names, it is then an empirical question whether or not the said formula involves the expression used for the formation of the formula. That is, self-inclusiveness of the formula is incidental. Consider e.g. the expression: "The least integer which is nameable (using expressions based on our ordinary counting system) only by syllables of the words used on this page such that each definite and distinguished syllable is used only once".

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* To avert the unpleasant situation which results from the ramified theory of types, Russell introduces the axiom of reducibility, which asserts that functions of different order may be equivalent. This correction, however, does not affect our argument here.

Now this expression is meant to include itself. Our objective is to avoid ambiguity and it is not always or necessarily the case that this objective is achieved by applying the vicious circle principle. Again Russell's account presents an unfair account of the Berry expression "the least integer not nameable in fewer than nineteen syllables" by calling it a name, because this expression is indeterminate and cannot attain the status of name unless we already know a specific method of arranging names. Hence, to say of it that it is a second-order name seems unfortunate.

§5. Other paradoxes treated by Russell

Similar remarks can be applied to the paradox of 'the least transfinite ordinal' and Richard's paradox discussed in Chapter IV. Russell states the Burali-Forti paradox in the following words:

> It can be shown that every well-ordered series has an ordinal number, that the series of ordinals up to and including any given ordinal exceeds the given ordinal by one, and (on certain very natural assumptions) that the series of all ordinals (in order of magnitude) is well-ordered. It follows that the series of all ordinals has an ordinal number, Ω say. But in that case the series of all ordinals including Ω has an ordinal number $\Omega+1$, which must be greater than Ω. Hence Ω is not the ordinal number of all ordinals.*

To put the matter crudely, the paradox arises from two premisses:

(a) the series of all ordinals up to and including any given ordinal exceeds the given ordinal; and

(b) the series of all ordinals has an ordinal number.

The first assertion implies that given any ordinal there is a greater

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* Ibid. p.60-1.

ordinal. The second assertion implies that there is the greatest ordinal which contradicts the first implication. To avoid the paradoxical situation, Russell employs his hierarchy of types that is in this case hierarchy of ordinals. He says:

> .... a series is a relation, and an ordinal number is a class of series .... Hence a series of ordinal numbers is a relation between classes of relations, and is of higher type than any of the series which are members of the ordinal numbers in question. Burali-Forti's "ordinal number of all ordinals" must be the ordinal number of all ordinals of a given type, and must therefore be of higher type than any of these ordinals. Hence it is not one of these ordinals, and there is no contradiction in its being greater than any of them.*

Our objections to this resolution are the same as we levied against his theory of types in general. We return to this paradox in Chapter V.

C. Meaninglessness and type-theory

§1. Vicious circle and meaninglessness

Russell's basis for asserting that vicious circle leads to meaninglessness may be: (a) that the vicious circle leads to contradiction; or (b) that common-sense suggests so. It seems clear that the fact that certain statements involving a vicious circle lead to contradiction, in no way implies that they are meaningless. Even the notion of "square-circle" is meaningful, though it is self-contradictory. Even in an axiomatic system the combination of symbols $p.\sim p$ is regarded as a well-formed though self-contradictory formula. As far as the other Consideration is concerned, of course, common-sense does perceive many instances where self-inclusive sentences lead to meaninglessness. Taking Russell's own example, if the propositional function '$\phi\hat{x}$' means '$\hat{x}$ is a man', then (also in accordance with the vicious circle principle) '$\phi(\phi\hat{x})$' becomes meaningless, and this can be directly seen when we consider '$\phi\hat{x}$ is a man'.** As Russell

*Ibid. p.63.
**Cf. Ibid. p.41.

remarks: 'we cannot legitimately deny "the function '$\hat{x}$ is a man' is a man", because this is nonsense'. From direct inspection and reflection, Russell is led to think that the things which we say about individuals, we cannot significantly say about the properties (functions) of individuals; and again the things which we sensibly say about the properties (functions) of individuals, we cannot say about the properties (functions) of the properties (functions) of individuals, etc. We significantly say of persons that they are honest or dishonest, but to say that certain characteristics of man are honest or dishonest would be non-sensical, because it is only about persons that we can sensibly say that they are honest or dishonest.

Russell is quite aware of the distinction between a false statement and a meaningless sentence (although he does not make any distinction between a meaningful sentence which is neither true nor false and one which can be said to be true or false). Now, Russell appeals solely to commonsense to support the claim that self-inclusive functions are meaningless. It is easy to refute him, as we have already done earlier, by actually citing instances where we can not only make significant statements by means of self-inclusive sentences, but also we can reasonably say whether they could be true or false. In the above example we can reasonably say: "All Greek men's opinions are honest" and hence the word "honest" is not only used to apply for men but also for their opinions. In fact, commonsense would rather suggest that it is quite meaningful to ask whether a set can or cannot be a member of itself; in fact it is necessarily false (contradictory) that a set is a member of itself and it is necessarily true that it is not a member of itself, as will be explained in Chapter V.

rding to Russell, the sentence "this proposition is false" when considered exively will be meaningless. A little reflection will show that it cannot nything but meaningful. For suppose a proposition p be meaningless. the statement saying that p is true or p is false is false and hence ingful. But in this special case "This proposition is false", "p is e" is nothing else than p itself and hence p is meaningful contradicting original hypothesis. There seems to be no reason why meaninglessness d follow from the vicious circle. Our judgement that such-and-such a e or sentence is meaningless or meaningful depends upon the context of situation*. In certain contexts the vicious circle leads to meaningless- in others it does not. In certain other contexts where there is no ion of vicious circle the sentences may still be regarded as meaningless. nstance the assertion "My watch is sweet and delicious", or "An elephant square root of two" are meaningless and the reason is that the watch lephant are not the sort of things which could be sweet and a square of two, respectively. No fixed and definite criterion can be given to nguish meaningless phrases or sentences. Again it is not certain that e assertion that certain relation or property is meaningless, its con- ctory must also be meaningless. If "$x \in x$" is meaningless, it does not w that "$x \notin x$" is also meaningless. As Russell bases his theory of meaning- ess on the vicious circle principle, he should have shown independently in the case of "$x \notin x$" i.e. "x is not a member of x", the vicious circle iple is violated in order to show that "$x \notin x$" is meaningless. It seems he assumes another principle, namely that if any sentence is meaningless,

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is point is discussed in Chapter III, pp.94-8.

then its negation is also meaningless, although he does not state it explicitly. In certain cases this principle may hold but it all depends on the contextual situation*.

§2. Systematic ambiguity: type theory and meaninglessness

In certain cases certain propositions appear to be meaningfully self-inclusive. Such a proposition according to Russell is in fact ambiguous, that is, it combines many statements having different types - for example the proposition "$(p).p$ is false" the proposition asserting that all propositions are false. Such a proposition is infected with systematic ambiguity. The above proposition in fact says that propositions of a certain type are false but the proposition making such an assertion would be of a higher type. So there is an ambiguity in the terms "true" and "false" and they can be applied unambiguously only to propositions of a certain order (type) and not to all propositions of whatever order. Hence, in fact these terms have an infinite number of meanings corresponding to an infinite number of orders of propositions. We must not confuse the various types of meanings; otherwise we shall get into systematic ambiguity, and hence contradiction and meaninglessness. Therefore, according to Russell's type theory, it would seem that the phrases "x is a member of y" ($x \in y$) or "x is not a member of y" ($x \notin y$) would be meaningless unless x is of type n and y of type $n+1$.

Discussing this point, L. Goddard** considers the problem whether

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* This point is discussed in Chapter III, pp. 94-8.

** See L. Goddard, 'Sense and Nonsense', Mind 1964, pp.309-31.

we can reasonably say that the class-membership is transitive, intransitive, reflexive, irreflexive or not. Class-membership ($\in$) cannot be transitive, for it is then $((x_n \in y_{n+1} \cdot y_{n+1} \in z_{n+2}) \supset x_n \in z_{n+2})$ - (subscripts denoting the types) should hold, but then the deduction of nonsense would follow from sense, for in the conclusion $x \in z$, the class z is not of immediate higher type than that of its member and hence the ambiguity of types occurs. For the same reason it cannot be intransitive (i.e., $((x \in y . y \in z) \supset x \notin z))$), nor can it be non-transitive. The same comments hold for the relation of symmetry and reflexivity. The relation "next to" is symmetric as it satisfies "$xRy \supset yRx$ the relation "father of" is assymmetric: it satisfies always "$xRy \supset y\not{R}x$". The relation non-symmetric is neither symmetric nor assymmetric. But in case the relation R is $\in$, x and y cannot be of ascending consecutive types. Again $\in$ cannot be reflexive, since "$x \in x$" is meaningless for all x, nor irreflexive for "$x \notin x$" too is meaningless, nor non-reflexive since "$\sim(x)(x \in x) . \sim(x)(x \notin x)$" is likewise meaningless. Thus Goddard comments:

> Thus whereas every two-termed relation has (at least) three properties: one drawn from the transitivity-trio, one from the symmetry-trio, and one from the reflexivity-trio, $\in$ has none of these. And this faces us with a choice between the theory of types and otherwise reputable laws of logic. For it is easy to establish as laws theorems which express the fact that every relation has one property in each trichotomy. Thus, for example, that every two-termed relation is either reflexive, irreflexive or non-reflexive is expressed by,
> (a) $(x)(xRx) \vee (x)\sim(xRx) \vee (\sim(x)(xRx) . \sim(x)(x\not{R}x))$
> which is an immediate consequence of the sentential law "$p \vee \sim p$".*

Goddard gives another example to show that the acceptance of type-theory does lead to far more serious consequences, namely, rejection of the accepted laws of logic which Russell himself admits. We present the example

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Ibid., p.312.

as follows:

If x is a member of the Pan-Hellenic League then x is a state.

If x is a state then x is not an individual.

Hence, if x is a member of the Pan-Hellenic League then x is not an individual.

That is, if x is an individual then x is not a member of the Pan-Hellenic League.

Hence, if Pericles is an individual then Pericles is not a member of the Pan-Hellenic League.

But, Pericles is an individual; therefore,
Pericles is not a member of the Pan-Hellenic League.

The conclusion when expressed in accordance with Russell's theory of types is meaningless. But the argument is validly drawn from true premisses. So either we must reject the theory or the validity. Goddard shows that the premisses can be meaningfully stated according to the theory of types and thus he concludes:

> The argument thus faces us with a dilemma: a choice between the theory of types and the rest of logic. For if we accept the truth of the conclusion or deny the truth of the premisses, we have to reject the theory of types; but if we try to save the theory by denying the validity of the argument, we have to reject otherwise cherished laws such as, '$(p \supset q \,.\, q \supset r) \supset p \supset r$' and '$(p \supset q) \equiv (\sim q \supset \sim p)$'; or the rule of substitution ('Pericles' for 'x'); or the rule of detachment. And of course we cannot reject any one of these in isolation. The rejection of any one would entail a major logical breakdown.
>
> The dilemma, then, is not simply a choice between the theory of types and (a)-laws, it is a choice between the theory of types and the rest of logic.*

It is clear from Goddard's criticism that Russell's theory of meaninglessness as it emerges from Russell's theory of types cannot be accepted without first limiting the range of application of his theory of types. We shall soon take up this point again.

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* Ibid., p.315.

Apart from the above criticism, the very idea of systematic ambiguity suggests that there should be a common notion of which we could talk of its systematic ambiguity. The systematic ambiguity of the word "false" is quite different from the ambiguity of the word "bat". Russell nowhere explains why in certain cases there is a question of systematic ambiguity and why for instance we use the same word "false" in all the types of falsehood, but in certain other cases the vicious circle leads to meaninglessness.

Again, there is a problem linked with the above point as to how to formulate the type-theory in a Russellian type-theoretical language. Pap discusses this point*. He asserts the impossibility of formulating statements about types within a type-theoretical language; to express, for instance, the theorem that distinct types do not overlap. Pap expresses this in the language of quantification:

> for any class A and B, if A is a type and B is a type and A is distinct from B, then there is no x such that x is a member of both A and B.*

Clearly there is a reference to all classes, which is meaningless according to Russellian type-theory. Again if we express semantically by saying that if "P" and "Q" are different type-predicates, then it is meaningless to say "$Px.\ Qx$". But how to express that the same entity, say a, cannot occur as "$Pa\ .\ Qa$" without using an unrestricted variable ("entity"). Hence Pap concludes:

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* A. Pap, 'Types and Meaninglessness', Mind 1960, p.44.

> it cannot be expressed in a logically perfect language of the Russellian kind. The old difficulty of how to formulate Russellian type theory without violating it in the very act of formulation has reappeared.*

Also there is a problem of how to arrange entities in a hierarchy of types and also what is meant by saying that such-and-such things are of the same logical type. Colours and shapes are taken by logicians as entities of the same type, being attributes of the first level (i.e., of individuals), yet colours can be said to be bright while it would be regarded as non-sensical to say that a triangle is bright.

It is clear from the above discussion that Russell's whole machinery to eliminate unwanted and meaningless expressions, namely, that of vicious circle, systematic ambiguity and type-theory, abounds with difficulties and obscurities. Perhaps Russell's aim could be interpreted as a search for an ideal language which would satisfy the requirements of type-theory. And then the requirements of type-theory would amount to the requirements for well-formed formula in this system or language. But in that case type-theory and the vicious circle principle should not be taken seriously outside the limited range of the system in question. In particular, also, the language in which type-theory is described should not be subject to such restrictions. Just as we regard symbols or expressions such as $p \supset \vee q$ as meaningless and ill-formed in the propositional calculus, likewise the expressions involving vicious circle and systematic ambiguity would be deemed meaningless and ill-formed in the system under scrutiny. In that case,

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* Ibid.

Russell should show the need and utility for such a system and why such requirements to prevent meaninglessness are necessary and relevant for its construction and what are the limitations for the application of this system. But Russell nowhere suggests that his vicious circle principle of his type-theory should have restricted application. Moreover he did not even lay down strict rules for the formation (or prohibition) of such well-formed (or respectively ill-formed) formulae. That is, he did not provide us with clear-cut, strict and unambiguous criteria for meaningless phrases. Gödel criticizes Russell for his lack of precision, though perhaps for different reasons. He says:

> It is to be regretted that his first comprehensive and thorough going presentation of a mathematical logic and the derivation of mathematics from it is so greatly lacking in formal precision in the foundations (contained in *I-*2I of Principia) that it presents in this respect a considerable step backwards as compared with Frege. What is missing, above all, is a precise statement of the syntax of the formalism. Syntactical considerations are omitted even in cases where they are necessary for the cogency of the proofs, in particular in connection with "incomplete symbols".......*

D. Russell's philosophy of mathematics

§1. Russell and Poincaré

Russell seems to be chiefly interested in obviating the paradoxes rather than of proper analysing and studying of the concepts involved in them. He thinks that a vicious circle is responsible for all the above paradoxes and hence if it is avoided, his purpose of averting the paradoxes is achieved. With this aim in view he was led to create his theory of types. His main argument by empirical justification, that

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* K. Gödel, 'Russell's Mathematical Logic', in The Philosophy of Bertrand Russell, p.126.

vicious circle is the only source or cause of paradox, we have already criticised. His exposé remains unconvincing and undemonstrated because he provides no rationale, no logical or purely rational argument to support his general thesis that the vicious circle leads to paradoxical results (and to meaningless or systematic ambiguity). On the other hand Poincaré provides a reason why the vicious circle leads to paradoxical consequences. Confining himself essentially to the Cantorian antinomies where the notion of completed infinity is involved, he points out that the vicious circle results whenever we use in our argument the notion of completed infinity.* While discussing the Richard paradox he says:

> Now we have defined N by a finite number of words, it is true, but only with the help of the notion of the aggregate E, and that is the reason why N does not form a part of E.
>
> In the above example chosen by M. Richard, the conclusion is presented with completed evidence, and the evidence becomes more apparent on a reference to the actual text of the letter. But the same explanation serves for the other antinomies, as may be easily verified.
>
> Thus <u>the definitions that must be regarded as non-predicative are those which contain a vicious circle.</u>**

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* In the notion of "potential infinity", say of integers, we are not given all the integers at once and we can thus always add a further unit (i.e., one), and so here we have the possibility of going on ad infinitum. But in the notion of "completed infinity", we may entertain all the integers given altogether all at once. Hence we have infinity as completed totality. Some philosophers (e.g., Aristotle) not only simply reject the notion of completed infinity but regard it as absurd or even self-contradictory.

** H. Poincaré, <u>Science and Method</u>, p.190.

Poincaré now explains the genesis of non-predicative definitions; that is, the definitions involving vicious circle, in these words:

> It is the belief in the existence of actual infinity that has given birth to these non-predicative definitions. I must explain myself. In these definitions we find the word all, as we saw in the examples quoted above. The word all, has a very precise meaning when it is a question of a finite number of objects; but for it still to have a precise meaning when the number of objects is infinite, it is necessary that there should exist an actual infinity. Otherwise all these objects cannot be conceived as existing prior to their definitions, and then, if the definition of a notion N depends on all the objects A, it may be tainted with the vicious circle, if among the objects A there is one that cannot be defined without bringing in the notion N itself.*

Poincaré explains why vicious circle or non-predicative definitions lead to contradictions. He draws a distinction** between predicative and non-predicative classification. The former classification remains unchanged by the introduction of new elements. But, in the latter classification, the introduction of new elements necessitates constant modification. Poincaré elucidates his idea of classification by an example of arranging integers or points in space. Suppose we legislate to arrange them in alphabetical order, etc., and to choose the first of them. Poincaré remarks that such a classification would not be predicative, since by the introduction of new integers the sentences which were unmeaning previously, would acquire new meaning and some of these sentences should come first according to our alphabetical arrangement. Hence a new classification emerges, and so on and so forth. Non-predicative

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*Ibid., p.194-5.

** Cf. H. Poincaré, 'The Logic of Infinity', op. cit., pp.47-8.

classifications or definitions may occur both in finite and infinite domains, e.g., the Berry paradox and the Richard paradox, respectively. But in the latter case the fallacy is not so easily detectable. Poincaré says:

> The antinomies to which certain logicians have been led arise from the fact that they have been unable to avoid certain vicious circles. This happened when they considered finite collections, but this happened much more often when they laid claim to treating of infinite collections. In the first case, they could have easily avoided the trap .... Very different are those generated by the notion of infinity; it often happens that the logicians fall into it without doing it on purpose ......*

So Poincaré provides a sound rationale as to why a vicious circle in certain cases, such as where we bring in the notion of completed infinity, results in antinomies. This is so because it leads us to a impredicative classification. He rejects the notion of completed infinity and affirms:

> There is no actual infinity. The Cantorian forgot this, and so fell into contradiction.**

He passes a sarcastic remark against logistics by pointing out:

> Logistics is not barren, it engenders antinomies.***

In short, Russell has failed to give any rational ground for holding the vicious circle principle but Poincaré gives a sound reason why a vicious circle in definition leads to antinomies.

§2. Russell's logicism

Russell was eager to show that mathematics is just a part of logic,

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* Ibid. p.63.

** H. Poincaré, Science and Method, p.195.

*** Ibid. p.194.

and hence that paradoxes in mathematics are just paradoxes in logic. Accordingly, as he maintained the Cantorian view of mathematics, he thought that it was wrong to suppose that the source of contradiction in the Cantorian paradoxes was a mistaken view of infinity, i.e., the notion of actual infinity. Further he observed some structural similarities between Cantorian paradoxes and other paradoxes, like that of the Liar, where no question of infinity was involved. Therefore, Russell concluded that the Cantorian paradoxes have nothing to do with infinity and all paradoxes are the result of some defect in our formulation of logic. Russell thus points out:

> The paradoxes of symbolic logic concern various sorts of objects: propositions, classes, cardinal and ordinal numbers, etc. All various sorts of objects, as we shall show, represent illegitimate totalities, and are therefore capable of giving rise to vicious circle fallacies ...... *

Why do all these objects concern logic? Russell's reason for saying so stems from his theory, according to which the statements concerning classes and relations can be reduced to statements concerning propositional functions. Russell points out that the paradoxes which involve propositions are indirectly relevant to mathematics, while those that more nearly concern the mathematicians are all concerned with propositional functions. Because of his own belief that mathematics is ultimately reducible to logic, also observing that Cantorian paradoxes suffer from self-inclusiveness and thus from 'vicious circle', Russell came to the conclusion that logic in general must be purged of the 'vicious circle fallacy'. Any class, any relation, any propositional function, any

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* PM, p.38.

proposition must not be self-inclusive or reflexive. Hence, Russell's theory of types was the result of his general effort to reduce mathematics to logic. But in his effort to do so, he was led to present a distorted account of paradoxes, including the Cantorian paradoxes. If we apply the 'vicious circle principle' only to the Cantorian paradoxes, which pertain to the notion of completed infinity, our approach would be based on rational grounds (as Poincaré tried to show), and thus our account for such paradoxes might be well justified. But in doing so, we would distinguish paradoxes into: (a) those to which 'vicious circle principle' necessarily applies i.e., where the notion of 'completed infinity' is involved, or, as we shall argue later, where the notion of set is involved; and (b) those paradoxes to which the vicious circle principle does not necessarily apply, that is, for instance where simply the general concept of proposition or propositional function is involved. By doing so we shall weaken the claim of Russell that mathematics is a part of logic.

Now, if the 'vicious circle principle' is to be regarded as a metalogical principle, it must be applicable to all cases and not only to the Cantorian or set paradoxes. In logic we would like to have minimum axioms and rules. From these axioms we derive theorems by applying rules of transformation. Thus, we create a logical system. Now if the 'vicious circle principle' is a metalogical principle and is supposed to be applicable to every proposition (or propositional variable) and every propositional function, it must be applicable in all cases and not merely to some propositional functions and not to others. As this principle is to be extended to the whole of logic and not merely confined to some specific, somehow limited system, it should be deemed to have universal validity. As it fails to apply in all cases, it cannot be a metalogical rule as Russell enunciated and extended to every particular instance of propositional function. But as this principle is essentially applied to mathematical paradoxes, it is specially relevant in the theory of sets which is supposed to serve as a framework for the mathematical system. This would illustrate the fact, granted Russell's suppositions, that the theory of sets falls outside the ambit of logic. Another way Russell's program may be stated is that logic should deal with propositions and propositional functions; but insofar as other entities like class, relation, number, ordinal, cardinal and so forth are reducible to propositional functions, these entities also come into our logical system. It becomes sometimesquite essential to speak of all such entities together, e.g. all the cardinal numbers or even the set of all x such that $x=x$. But to refer to all of such entities would transgress the vicious circle principle in general. To circumvent this difficulty, Russell introduces the axiom of reducibility. He states:

> Hence many kinds of general statements become possible which would otherwise involve vicious-circle paradoxes. These general statements are none of them such as lead to contradictions, and many of them such as it is very hard to suppose illegitimate. The fact that they are rendered possible by the axiom of reducibility, and that they would otherwise be excluded by the vicious-circle principle, is to be regarded as an argument in favour of the axiom of reducibility.*

Hence, Russell himself demolishes the generality of his vicious circle principle. But he does not tell us explicitly what would then be the remaining principle when some types of vicious circle no longer come under the vicious circle principle. It seems that Russell is only making *ad hoc* arrangements to suit mathematical needs and ends, and there appears to be no question of intuitive or logical rules being involved in it. We may on similar lines agree to bring in some other *ad hoc* rule to cover those

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* *Ibid.* p.76.

cases where neither the general principle of vicious circle nor the axiom of reducibility would apply. But it would further lead to a loosening of the generality of rules in our logical system and symbolism. To construct a logical system we are supposed, or rather obliged, to employ intuitive or logical rules and should not rest content with *ad hoc* arrangements. Hence, Russell has failed to supply a logical or intuitive principle or a criterion as a general remedy for the removal of paradoxical situations in the paradoxes, or for that matter, to construct a logical system.

8

Chapter II

CLASSIFICATION OF PARADOXES

A. Distinction between syntactic and semantic paradoxes

§1. Three such distinctions

Are there common features in all or in some of the paradoxes? According to Russell, all paradoxes suffer from the 'vicious circle' fallacy. But as we tried to show in the first chapter, this view is mistaken. Indeed there seems to be no characteristic which may be deemed common to all the paradoxes except that we regard them as leading to contradiction. Attempts have been made to classify paradoxes. Sometimes a distinction is maintained between "syntactic" and "semantic" paradoxes. Logicians appear to have offered various accounts of this distinction.* We make the following three specifications of this distinction to clarify the matter:

(a) In one sense, as the very meanings of the terms "syntactic" and "semantic" suggest, the contradiction in a syntactic paradox results from

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* For example see:

(1) Hao Wang, A Survey of Mathematical Logic, p.387.
(2) A.A. Fraenkel and Y. Bar-Hillel, Foundations of Set Theory, pp.5, 12.
(3) S.C. Kleene, Introduction to Metamathematics, p.45
(4) D. Hilbert & W. Ackermann, Principles of Mathematical Logic, p.151.
(5) F.P. Ramsey, The Foundation of Mathematics, pp. 20, 24-5.

the manipulation of symbols of certain symbolic systems only; that is, the symbols occurring in syntactic paradoxes are completely uninterpreted and have been assigned no meanings except through syntactical "grammatical" rules which govern their usage; while in semantic paradoxes we assign meanings to the symbols used in expressing the paradoxical statements.

(b) In another sense, this distinction between syntactic and semantic paradoxes may be identified with Ramsey's distinction between logical and epistemological paradoxes; namely, that in logical paradoxes only logical or mathematical concepts are involved but epistemological paradoxes essentially involve epistemological concepts, like meaning, truth, etc.

(c) Again in another sense, the syntactic paradoxes may be regarded as those which occur in an axiomatic system and the semantic paradoxes as those which do not arise within any axiomatic system.

The above distinctions are our own. Usually the above three senses of the distinction between semantic and syntactic paradoxes are not clearly distinguished but are confused with one another. Let us now consider in turn these preliminary three ways of drawing the distinction of paradoxes into semantic and syntactic.

§2. Are any paradoxes purely "syntactic"?

In the first sense the division of paradoxes is trivial. We are not interested in just creating a syntactic system and showing its consistency. The distinction between a wholly syntactic system - in which we are supposed to deal only with uninterpreted symbols, arbitrary definitions and rules - and the meta-language - which concerns the interpretation of the system - does not mean that we first create an uninterpreted symbolic system and then try to interpret it. Both processes go together, and the distinction may be drawn to appreciate and facilitate the solving of some other problems connected with the system e.g. the phenomenon of isomorphism. But some writers seem to express the extreme position of making a distinction in an uninterpreted, i.e. completely syntactic system. R.M. Martin, for example, says:

> Strictly speaking we should distinguish between a formalised logistic system (or calculus) and a formalised language-system (or interpreted language) as follows: The former is determined by grammatical rules or definitions which refer exclusively to symbols and expressions, regarded in abstraction from any specific interpretation. A language-system, on the other hand, is a logistic system with a fixed, determinate interpretation given to certain of its expressions.*

From our standpoint such an extreme position is untenable. Firstly it is extremely difficult to construct a system of any worth without keeping in view some interpretation of it. In general, we have to define what is inconsistency or contradiction in the system. We may approach the problem of consistency in a purely syntactic way and say that a system is consistent when we have certain well-formed formulae of the system which are not derivable in the system, or that the system is consistent if we cannot derive both p and $\sim p$ in the system. But the above notion of Post-consistency is not enough for a complicated system. As we know, for predicate calculus, we need a semantic notion of consistency as well. It shows that we need semantic clarification and explanation of the concept "consistency". Secondly, we are here concerned with logico-mathematical systems and not with any other axiomatic system. In the logical system we already have some semantic understanding about the symbols used in the system. Hence, the distinction between paradoxes in the first sense has to be rejected.

* R.M. Martin, Truth and Denotation, p.2.

A weaker form of distinction may be proposed by starting to build a mathematico-logical system with some intuitively clear axioms, rules and definitions. We express these basic rules and axioms in symbolic form. By manipulation of these symbols in accordance with transformation rules we may arrive at contradictory symbolic formulae. In order to rectify the contradictory situation in the system we may no longer bother about the interpretations or meanings of the terms involved in the system and so may just try to change the axioms or rules by only observing how to avoid the deduction of contradictory formulae. One may go so far as to assert that this is the only reasonable or correct approach to constructing an axiomatic system. In that case it may be said that any contradiction occurring in such a system may be referred to as a syntactic paradox (as a reminder that we have only to look to the symbols, i.e. the syntactic aspect of the system, for reconsideration and solution), and the other sorts of paradoxes, like the Liar paradox, where we have to look to the meanings of the terms involved, may be called semantic paradoxes.

Even this revised formulation of the first distinction between paradoxes is misleading. Firstly, the so-called syntactic paradoxes are not purely syntactic because even they can be traced back to the interpreted original axioms, rules and concepts such as set and set-membership. Secondly, the syntactic procedure to avoid contradiction in the system is not plausible. We need semantic clarification of concepts for rational understanding of the situation. It should be specially necessary in the case of a logico-mathematical system, i.e., the system incapsulating mathematical and logical theorems; namely, the propositional, predicate calculus, and set theory. Because we usually hold that the logical and mathematical theorems (i.e. of mathematics which may properly be regarded true) have special truth and they are not just deductions which follow from our arbitrary definitions and axioms - the definitions, axioms and rules being chosen by our determination without reference to the notion of truth. Again, just as in other fields of enquiry we should like to be clear about the conceptual scheme involved, so we would like to be clear about the conceptual scheme embodied in our axiomatic system of logico-mathematics. Just as in our common notions of free-will and determinism or in our concept of God as all-powerful, all-just, all-merciful, etc. we may find contradictions and try to remove them and thus clarify our notions about them, so it is no less necessary that we should clarify the conceptual foundations of logic and mathematics. Hence in order to remove the contradiction in the system we should look to the nature of the contradiction arrived at, that is to say, we should approach the contradiction semantically, determining the meaning behind the symbolic expressions which lead to contradiction. In this way we come to know the real nature of paradox. As the contradictory formulae encasing the paradox are traced back to axioms and rules of the system, ultimately we rectify the basic axioms and rules of the system. That is to say, we in fact amend and clarify our concepts about the main principles of the system. We may thus come to know some general principle for avoiding the paradoxes because we now come to know some real paradoxical malady. We may thus profitably apply this principle to rectify our basic axioms or rules and thereby create a sound and flawless system.

It may be interesting to note that Russell's 'vicious circle principle' for avoiding self-inclusiveness is a semantic principle. As Russell himself pointed out, he arrived at this principle by studying the nature of paradoxes.

Hence, in order to avoid the paradoxes, he said to avoid a 'vicious circle' in symbolic expressions in the system. Hence, Russell's approach to resolve the paradoxes is essentially semantic.

B. Ramsey's distinction

§1. Its basis

From above, it follows that all paradoxes are properly called "semantic" according to our first interpretation of the distinction between semantic and syntactic paradoxes, because even in considering formal systems semantic elements are involved. Ramsey, in his article "The Foundations of Mathematics" (1925), makes a distinction between paradoxes concerned with logical or mathematical systems and other sorts of paradoxes which involve linguistic or epistemological concepts.* This distinction corresponds to our second interpretation of the distinction between syntactic and semantic paradoxes. Hao Wang says:

> In 1926, Ramsey introduced a distinction between logical (mathematical) and semantic (epistemological) paradoxes. ...., while the semantic ones involve notions of truth designation, expressibility, and the like. The distinction is not entirely sharp.**

Paradoxes of group A, on Ramsey's account, involve only logical or mathematical terms such as class and number, and thus show that there must be something wrong with our logic or mathematics. Paradoxes of group B, on the other hand, admit some reference to thought, language or symbolism, which according to Ramsey are not formal but empirical terms. They betray faults embodied in

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* F. P. Ramsey, The Foundations of Mathematics, pp.20-21

** Hao Wang, A Survey of Mathematical Logic, p.387.

our ideas about thought and language. To group A belong paradoxes like Russell's class paradox and the relation paradox and the Burali-Forti paradox; and to group B belong paradoxes like the Liar paradox, Richard's paradox, Berry's paradox, Grelling's paradox, etc. The paradoxes of group B 'are not purely logical and cannot be stated in logical terms alone' and thus Ramsey concludes that these paradoxes are not relevant to mathematics or logic.

Ramsey says that Russell confused the two groups of contradictions, although the distinction is implied in his distinction between the theory of types and the theory of orders (i.e. ramified theory of types), respectively. Thus Ramsey has it:

> The contradictions of group A are removed by pointing out that a propositional function cannot significantly take itself as an argument, and by dividing functions and classes into a hierarchy of types according to their possible arguments ...
>
> The first part of the theory, then, distinguishes types of propositional functions by their arguments; thus there are functions of individuals, functions of functions of individuals, and so on. The second part designed to meet the second group of contradictions requires further distinctions between the different functions which take the same arguments, for instance between the different functions of individuals.*

Let us examine still more closely the differentiating marks between the simple theory of types and the ramified theory of types. In the simple theory of types the type of a function is determined by its arguments. If the arguments comprise individuals of type n, then the type of the function would be $n + 1$; if the arguments have the type n then their functions would have the type $n + 1$. So according to the simple theory of types, the type of a

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* F. P. Ramsey, The Foundations of Mathematics, pp.24-25.

function is determined by its arguments. According to the ramified theory of types we further subdivide each type (above the level of individuals), though having the same type of arguments, into different orders. The propositional functions of type I (i.e., functions of individuals), in which there is either no quantification or quantification only on individual variables, are called first order functions. For instance, $F(x)$ or $(x)F(x)$ where x is an individual and F a certain property of the individuals. In case we have quantification over the functions of type I we have the second-order functions of individuals. We give Russell's own examples:

$$'(\phi).f!(\phi!\hat{z},x)',\quad '(\exists\phi).f(\phi\hat{z},x)',\quad '(\phi,\psi).f!(\phi.\hat{z},\psi!\hat{z},x)' \; *$$

We can create higher order functions of type I by quantifying the second-order function of individuals. An nth-order function of type I thus has quantifiers on functions of order n-1. The conception of Russell's theory of orders will be further clarified by considering the heterological paradox of Grelling below in Section C.

§2. Our assessment

It is difficult to understand Ramsey's equating the two distinctions of paradoxes; the one based on the contrast between logical and epistemological paradoxes, the other based on the contrast between the solutions offered (or simply the expressibility of the paradoxes) by two theories of types. Let us consider an example which appears to go counter to Ramsey's thesis that the epistemological paradoxes are only expressible and resoluble in terms of Russell's ramified theory of types.

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* P.M., p.53.

Consider the following version of the Liar paradox:*

'Every proposition asserted by A during the interval c is false'

Let us call this proposition (or sentence) P. We are further given:

'"P" is the only proposition asserted by A during the interval c'

We now construct the paradox symbolically as follows:

(1) $P = (p)[\phi(p) \supset \sim p]$ ..... (i.e., Every proposition (p) asserted by A during the interval c (ϕ) is false)

(2) $\phi(P) . \{(p)[\phi(p) \supset (p = P)]\}$ (i.e., "P" is asserted by A during the interval c and "P" is the only proposition he asserted during the interval c)

In order to obtain the contradiction we proceed as follows:

(3) $P = P$ .... Theorem of propositional calculus
(4) $P \supset (p)[\phi(p) \supset \sim p]$ .... Substitution: $P / (p)[\phi(p) \supset \sim p]$
(5) $P \supset [\phi(P) \supset \sim P]$ .... Instantiation: $\dfrac{A \supset (x)B(x)}{A \supset B(t)}$
(6) $\phi(P) \supset (P \supset \sim P)$ .... Theorem of propositional calculus
(7) $P \supset \sim P$ .... $\phi(P)$ being given

Therefore if P is true then P is false. Let us now prove "$\sim P \supset P$".

(8) $\sim P \supset \sim P$ .... Theorem of propositional calculus
(9) $\sim P \supset \sim\{(p)[\phi(p) \supset \sim p)\}$ Substitution: $P / (p)[\phi(p) \supset \sim p]$
(10) $\sim P \supset (\exists p)[\phi(p) . p]$ .... from (9)
(11) $(p)\{[\phi(p) . p] \supset [(p = P) . p]$ From (2) and the theorem: $(x)[A(x) \supset B(x)] \supset (x)[(Ax . Q) \supset (Bx . Q)]$
(12) $(\exists p)[\phi(p) . p] \supset (\exists p)[(p = P) . p]$ From (11)
(13) $\sim P \supset (\exists p)[(p = P) . p]$ From (12) & (10)
(14) $[(p = P) . p] \supset P$
(15) $(\exists p)[(p = P) . p] \supset P$ From (14)
(16) $\sim P \supset P$ .... From (13) & (15)

Hence, we derive both "$P \supset \sim P$" and "$\sim P \supset P$" (i.e., $P \equiv \sim P$) and that is a contradiction. In order to obvert the contradiction we may, following the Russell simple theory of types, say that proposition P belongs to a higher type than the variable p because the function ϕ is involved in P, and thus

* Cf. D. Hilbert & W. Ackermann, Principles of Mathematical Logic, p.145.

the notion of falsity which attaches to propositions of the type of p cannot be unambiguously attached to P. (Other solutions would be, to deny that P is a proposition and hence the question of truthfulness does not arise; or to claim that P is a sort of proposition to which law of excluded middle does not apply, so that we cannot regard P as an instance for the variable p.)

From the above formulation of the Liar paradox, it is quite obvious that there is no need to quantify over any function, that is to say we do not have to refer to the orders of the function or bring in the ramified theory of types. A similar symbolisation after the manner of the simple theory of types is sufficient in the case of the Barber paradox, where a barber asserts that he shaves all and only those who do not shave themselves. This may be symbolised as:

(1) $(x)[S(B,x) \equiv \sim S(x,x)]$ i.e., the barber B shaves every man x (or has the relation of shaving S to every man x) who does not shave himself. A contradiction arises by applying universal instantiation: $S(B, B) \equiv \sim S(B, B)$. There is an assumption in the premiss (1) that (1a) $(\exists x)(x = B)$. In this case, as in the version of the Liar paradox given above, there is no need to quantify over the relational function S, hence there is no need for the ramified theory of types. It follows from the above illustration that Ramsey's interpretation of Russell's simple theory of types and ramified theory of types as correlated with the logical and epistemological paradoxes is not well-founded.

Let us now examine Ramsey's distinction between logical or mathematical paradoxes and epistemological paradoxes as such; i.e., the class of paradoxes which involve logical or mathematical terms only, as opposed to the class of paradoxes which involve other terms as well. We may notice that the Barber paradox, when symbolised, involves only logical concepts, e.g. relation (S), negation (~), individual variable (x), etc. The structure of the paradox as expressed in symbolism has nothing to do specifically with any semantic or epistemological concepts. In the case of the Liar paradox we symbolise it with logical concepts like property ϕ, negation ~, propositional variable p. Now if we place any particular semantic property or relation in "ϕ" or "S", the paradoxical structures remain. It follows from Ramsey's description of logico-mathematical paradoxes, as those which embody only logical or mathematical terms, that these "structural paradoxes" should be counted as logico-mathematical paradoxes; but as soon as we make a particular interpretation of "ϕ", "S", "x" we get epistemological paradoxes. If this is the distinction between paradoxes, it is obviously a trivial one. Moreover, it is hard to maintain a distinction between logico-mathematical and epistemological concepts. Ramsey has given us no clue how to maintain this distinction. The essential feature of logico-mathematical concepts appears to lie in their generality. If we regard "modal logic" as part of logic, we would include concepts like possibility, impossibility, etc., as part of logical vocabulary. Again, the general concepts involved in conditionals, counter-factuals, causal statements, etc., may come in purview of logic. As much as we map our field of reasoning, we are widening the horizon of logic and hence therewith the terms involved in the widened scope of logic.

Although Ramsey does not elaborate his position, it seems that he did not intend to make the distinction described above. Ramsey seems to have combined our second and third interpretation of the distinction of semantic and syntactic paradoxes. He would claim that logico-mathematical paradoxes are those which occur in the construction of logico-mathematical systems, i.e., they are syntactic paradoxes in our third sense. These contradictions occur in logico-mathematical systems and not in any axiomatic system like Euclidian or non-Euclidian system. By logico-mathematical system we should understand the propositional and predicate logic and set-theoretical system for the foundation of pure mathematics. In this sense the Liar paradox is not logical, because we may affirm that we do not get the premiss, for instance $\{(p)[\phi(p) \supset (p = P)]\}$, of the Liar paradox in a logical system. Hilbert and Ackermann seem to hold this distinction.* Their contention is that, since the premisses of paradoxes like that of Liar cannot occur in our axiomatic system, they do not form universally valid formulae; they are to be regarded as semantical paradoxes as distinguished from logical paradoxes.

C. Heterological paradox

Ramsey discusses his point of view at some length in the case of the heterological paradox. This paradox was not discussed by Russell and it seems an unhappy selection on the part of Ramsey for discussing Russell's theory of types. We will discuss the heterological paradox in some detail, in order to assess both Russell's and Ramsey's solutions. This is a paradox about words. There are some words which designate properties which are exemplified by the words themselves. These words may be called autological. The characteristics for which these words stand can truly be applied to the words themselves. There are other words which specify properties which are not exemplified by the words themselves. These words therefore cannot be validly

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* D. Hilbert & W. Ackermann, Principles of Mathematical Logic, p.151.

applied to themselves. They may be called heterological. For instance, "English" is English, but "German" is not German; "noun" is a noun, but "adjective" is not an adjective; "polysyllabic" is polysyllabic, but "monosyllabic" is not monosyllabic; "short" is short, but "long" is not long. As "English", "noun", "short", "polysyllabic" are self-applicable or self-predicable, in the sense that they instantiate their designated characteristics, they may be called autological. But "German", "adjective", "monosyllabic", "long" are not self-applicable in the sense that the specified properties are not attributable to themselves, hence they may be regarded as heterological. Now, the question may be raised whether "heterological" is heterological or not. The word "heterological" indicates that common property of heterologicality shared by all and only those words which name properties which the words themselves do not possess or exemplify. Now, suppose the word "heterological" is heterological, that is, it has the property of being heterological; thus this word "heterological" does not have the property which it designates. But this is a contradiction because by supposition it possesses the property of being heterological to that which it designates. Now, suppose "heterological" is autological, then the word "heterological" does not have the property which it designates. If so, then it must be heterological. Hence, a contradiction follows from the fact that we must assume that "heterological" is either heterological or autological.

Let us express the paradox after the manner of Ramsey*. When we say the word "noun" means noun, we mean that the word "noun" designates the property of being a noun. In Russell's terminology, to assert that a word designates

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* Ramsey did not give a full symbolic formulation of the heterological paradox. I.M. Copi in his book Symbolic Logic, pp.33-34, has given a symbolic formulation of the heterological paradox patterned after Ramsey's.

a certain property is to say that the word designates or means a certain propositional function. So we symbolise '"x" designates propositional function $\phi\hat{x}$' by '"x"R($\phi\hat{z}$)'. To express the heterological paradox, we have to suppose that there is in fact a single property designated by the word (i.e., the word is univocal). Furthermore that word does or does not have the designated property (i.e., the law of excluded middle holds good here). We thus symbolise '"x" is heterological' as: $(\exists\phi)[$"x"$R(\phi\hat{z}) \cdot \sim\phi$"$x$"$]$. Granted the supposition of univocality of the word, it can easily be seen that this expression is equivalent to $(\phi)[$"x"$R(\phi\hat{z}) \cdot \phi$"$x$"$]$, or simply to $[$"x"$R(\phi\hat{z}) \cdot \sim\phi$"$x$"$]$. We now symbolise '"heterological" is heterological' as:

$\underline{het}$("$\underline{het}$") ≡ $(\exists\phi)[$"$\underline{het}$"$R(\phi\hat{z}) \cdot \sim\phi($"$\underline{het}$"$)]$

⊃ "$\underline{het}$"$R(\phi\hat{z}) \cdot \sim\phi($"$\underline{het}$"$)$ .... because "het"$R(\phi\hat{z})$ is equal to "$\underline{het}$"R($\underline{het}$ $\hat{z}$) and is given as univocal, hence the expression above is equivalent to $(\phi)[$"$\underline{het}$"$R(\phi\hat{z}) \cdot \sim\phi($"$\underline{het}$"$)]$

⊃ $\sim\underline{het}$("$\underline{het}$")

Hence the assumption that "heterological" is heterological leads to contradictory results. Let us now assume "heterological" is not heterological, then expressing it in symbols we have:

$\sim\underline{het}$("$\underline{het}$") ≡ $\sim\{(\exists\phi)[$"$\underline{het}$"$R(\phi\hat{z}) \cdot \sim\phi($"$\underline{het}$"$)]\}$

⊃ $(\phi)[$"$\underline{het}$"$R(\phi\hat{z}) \supset \phi($"$\underline{het}$"$)]$

⊃ ["$\underline{het}$"R($\underline{het}$ $\hat{z}$) ⊃ ϕ("$\underline{het}$")]

$\underline{het}$("$\underline{het}$") .... because "$\underline{het}$"R($\underline{het}$ $\hat{z}$) already known to be true. (Modus Ponens)

Hence whether we assume "heterological" is heterological or is not heterological we are led to contradictory consequences that "heterological" is not heterological and is heterological, respectively.

Ramsey rightly points out that according to Russell the contradiction is removed by saying that the function $\underline{het}$ ($\hat{z}$) is of higher order than any of the functions $(\phi\hat{z})$, and hence it cannot be an instance of $(\phi\hat{z})$. If we distinguish the hierarchy of orders by applying subscripts, that is to say, if we avoid systematic ambiguity in this way, then we derive no contradiction. For example, in saying '"adjective" is heterological$_1$' and '"heterological" is heterological$_2$' the functions $\underline{het}_1(\hat{z})$ and $\underline{het}_2(\hat{z})$ are different and do not mean the same thing. What the difference of meanings between $\underline{het}_1$ and $\underline{het}_2$ may be, however, it is hard to imagine; this is the sort of general criticism we have already levied against the notion of systematic ambiguity. Anyhow, Russell's theory of orders avoids the present contradiction. It can be easily seen that a 'vicious circle' as defined by Russell is involved in the Grelling paradox. To simplify, let us define any word being not heterological (i.e. autological) as:

$\underline{aut}$ ("x") $\equiv (\phi)[$"x"$R(\phi\hat{z}) \supset \phi($"$x$"$)]$ .... That is, a word "x" is autological if and only if "x" possesses whatever property it designates. The symbolism above clearly shows that the function of autologicality $\underline{aut}(\hat{z})$ involves all the functions "$\phi\hat{z}$". And in the case of '"autological" is autological' the function $\underline{aut}(\hat{z})$ itself becomes one of such functions (i.e., an instance of $(\phi\hat{z})$). Hence the vicious circle fallacy is committed. As the definition of this function requires that we quantify over functions (or properties) it involves the ramified theory of types. This example shows how a vicious circle is involved even in the ramified theory of types. And because of the vicious circle the function "$\underline{aut}(\hat{z})$" becomes meaningless as an instance of the functional variable "$\phi\hat{z}$", so its negation "$\underline{het}(\hat{z})$" (i.e., $\sim\underline{aut}(\hat{z})$) becomes meaningless as we discussed in the last chapter section C. Hence, according to Russell's ramified theory of types we cannot put "$\underline{het}\ \hat{z}$" as an instance of "$\phi\hat{z}$" and thus the contradiction is avoided.

Ramsey's solution also involves a hierarchy of types. He eliminates the ramified theory of types and therewith the principle of reducibility. Ramsey would interpret a formula like '$(\phi)\phi a$' as the logical product of the propositions ϕa, of which it is itself one. He says:

> To take a particularly simple case, '$(\phi)\phi a$' is the logical product of the propositions ϕa, of which it is itself one; but this is no more remarkable and no more vicious than the fact that '$p \cdot q$' is the logical product of the set 'p', 'q', '$p \cdot q$', of which it is itself a member. The only difference is that, owing to our inability to write propositions of infinite length, which is logically a mere accident, '$\phi \cdot \phi a$' cannot, like '$p \cdot q$', be elementarily expressed, but must be expressed, as the logical product of a set of which it is also a member.*

Hence, according to Ramsey, quantification over functions does not necessitate the hierarchy of orders and therewith the principle of reducibility. But Ramsey's solution is similar to that of Russell's. According to Ramsey, the sense of the verb "means" in '"heterological" means heterological' is different from its sense in '"adjective" means adjective'. Therefore '"<u>het</u>"R(<u>het</u> $\hat{z}$)' cannot be an instance of '"x"R($\phi\hat{z}$)'. He says:

> the contradiction is simply due to an ambiguity in the word 'meaning' and has no relevance to mathematics whatsoever.**

It is because of the epistemological or semantic nature of the paradox that such a situation arises. He says that in the case of '"<u>het</u>"R(<u>het</u> $\hat{z}$)' the R is complicated. To illustrate this point he gives the following illustration. Consider the case of '$a\ s\ b$' where 'a', 'b', 's' mean in the simplest way the separate objects a, b and s. If we define now 'ϕx' as '$a\ s\ x$', then 'ϕ' is substituted for '$a\ s$' and does not mean a single object; it has meaning in a

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* F. P. Ramsey, <u>The Foundations of Mathematics</u>, p.41.

** <u>Ibid</u>., p.43.

more complicated way by virtue of the 3-termed relation to both 'a' and 's'. In the same way, R in '"het"R($\underline{\text{het}}\ \hat{z}$)' is complicated and has a different meaning from its meaning in '"x"R($\phi\hat{z}$)'. According to Ramsey's interpretation, therefore, we can substitute '$\underline{\text{het}}\ \hat{z}$' for '$\phi\hat{z}$'; but due to the epistemological nature (of this property), the meaning of 'R' alters. Hence, Ramsey was led to create a hierarchy of different meanings of 'R', i.e., of semantic expressions like 'means' or 'designates'.

Ramsey's solution does not appear to be satisfactory. We rather think that the meaning of 'means' in '"heterological" means heterological' is the same as it is in the expressions '"adjective" means adjective' or '"noun" means noun'. If the meaning of 'means' differs from one expression to another, we can legitimately raise the question why the same word 'means' has been used in various expressions, or in other words, to ask what is the meaning of 'means' in general. The fact is that in order for a word to be heterological (or autological) - i.e., in order for a word to have the characteristic of heterologicality (or autologicality) - it must designate some property other than that of heterologicality (or autologicality. In order to judge whether a certain word is autological, we have to determine whether this word designates some property (other than heterologicality or autologicality) and further whether or not it has the designated property. And this is quite different from saying that there is a systematic ambiguity of the word 'means'. The property of heterologicality (or autologicality) is a <u>dependent property</u>; i.e., its existence is logically dependent on that of some other property. The truth-value of an assertion of heterologicality of some word depends on there being some other property just as the truth-value of the sentence, "I am lying", depends on there being some other proposition. The property of being admirable is a dependent property; it may depend on the property of being honest or of being beautiful, etc. So, whether something is admirable depends on its having or lacking a property such as honesty or beauty. Similarly, the property of autologicality depends on there being some other property designated and owned by some word. The property of heterologicality depends on some other property designated but not owned by certain words. If there is no designated property already present, we cannot sensibly ask whether or not the words are heterological. There is no need to make a hierarchy of meanings of "autological" or "heterological" or "means". We have only to learn the logical behaviour of the properties of heterologicality or autologicality, that is to say, the requirement for asserting that certain words are or are not heterological; these words must have some other designated property about which we could sensibly raise the question of their being or not being heterological.

D. <u>The structure and solution of paradoxes</u>

§1. Structure

We may note that no contradiction ensues if we ask whether "autological" is autological or is not. As we remarked earlier, it is the structure or pattern which is responsible for the emergence of contradiction in a paradox. There is some pattern or structure of the argument involved in a paradox. We try to clarify the notion of structure which leads to contradiction in three stages: first, by considering some paradoxes in symbolic formulae; secondly, by examining Thomson's theorem; and thirdly, by graphic representation of the Barber structure.

The pattern in the heterological paradox may be symbolised in the simplest form as:

$(\phi)\{["x"R(\phi) \supset \sim\phi("x")] \equiv \underline{het}("x")\}$ .... i.e., to say that any word ("x") designates (R) a particular property (ϕ) and that that word does not possess the designated property ($\sim\phi("x")$) is equivalent to saying that that word possesses the property of heterologicality ($\underline{het}("x")$). (Instead of using Russellian cumbersome notation of propositional function ($\phi(\hat{z})$) we shall simply write the property (ϕ).) In order to get the contradiction, we take the word "heterological" for "x", and then obviously follows the designated property "heterological" as an instance for "ϕ". Thus, by universal instantiation we get the following:

$["\underline{het}"R(\underline{het}) \supset \sim\underline{het}("\underline{het}")] \equiv \underline{het}("\underline{het}")$. With the justifiable assumption of '$"\underline{het}"R(\underline{het})$' the above formula is self-contradictory, being equivalent to: $\sim\underline{het}("\underline{het}") \equiv \underline{het}("\underline{het}")$. If we have the formula "$P \equiv Q$", we may conjoin any true or valid formula to P or Q without any change in the truth-value of "$P \equiv Q$".

In this particular paradox, the structure of the paradoxical argument has become a bit more complicated and involved because of the additional factor of words designating their meanings. In order to appreciate the basic or primitive structure behind it, let us consider the paradox of Impredicability, which has apparent similarity to the Grelling paradox.* The Impredicability paradox deals with properties only and does not concern words. Individuals have properties and properties themselves may have further properties. For example, the property of being beautiful has the property of being admirable.

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* Cf. B. Russell, The Principles of Mathematics, pp.96-97, 102.

Now, there are some properties which may be predicated of themselves. For instance, we may say that the property of being abstract possesses the property of being abstract (i.e., abstract is abstract); but the property of being concrete does not seem to have the property of being concrete. A self-predicable property may be called Predicable. A property which is not self-predicable may be called Impredicable. Now, the property of being impredicable is a property of all and only those properties which are not self-predicable. Let us raise the question whether the property of being impredicable is predicable or not. If it is predicable, then it must be impredicable; and if it is impredicable, then being self-predicable it is predicable. The paradoxical structure may be symbolised as: $(x)(\mathrm{I\ R}\ x \equiv \sim(x\ \mathrm{R}\ x))$; by universal instantiation we get: $(\mathrm{I\ R\ I} \equiv \sim(\mathrm{I\ R\ I}))$, which is a self-contradictory formula.

The structure or the symbolic formula exhibited by the above paradox is self-contradictory and under our usual interpretation of I, R and x, it would lead to contradiction by valid deductions. Let us interpret x as individual variable for persons, I as Barber, and R being the relation of shaving; then the symbolic formula "$(x)(\mathrm{I\ R}\ x \equiv \sim(x\ \mathrm{R}\ x))$" would mean that the Barber shaves all but only those persons who do not shave themselves. By this interpretation we get the Barber paradox; the paradox of a barber who claimed that he shaves only and all those persons who do not shave themselves. Now, if the barber does shave himself, he must not shave himself in case his assertion is taken to be true. If the barber does not shave himself, he must shave himself. This sufficiently illustrates that the contradiction occurs because of the peculiar paradoxical structure. If we negate the whole expression of the structure, it will yield a universally valid formula as expected, and hence a theorem of logic; the formula being then, $\sim(x)(\mathrm{I}\ \mathrm{R}\ x \equiv \sim(x\ \mathrm{R}\ x))$, i.e., $(\exists x)\{(\mathrm{I}\ \mathrm{R}\ x\ .\ x\ \mathrm{R}\ x) \vee [\sim(\mathrm{I}\ \mathrm{R}\ x)\ .\ \sim(x\ \mathrm{R}\ x)]\}$, which is valid even in the domain of one object. In this basic self-contradictory formula of paradoxical structure we may insert some additional symbolic expressions which may not affect the truth-value of this formula, and these added formulae would also yield contradiction as we noticed in the case of the Grelling paradox.

J. F. Thomson brings out the structure common to the paradoxes like that of heterological, Russell, and of the Barber. This common structure we label the "Barber structure", as it is exhibited in the Barber paradox in a most simplified form. By considering the relations in sets he states the following theorem:

> Let *S* be any set and *R* ~~xx~~ any relation defined at least on *S*. Then no element of *S* has *R* to all and only those *S*-elements which do not *R* to themselves.*

It follows from this theorem that there cannot be a man (barber) who can shave all and only those men who do not shave themselves. But the Barber paradox asserts this, thus showing that the structure of the Barber paradox is self-contradictory. Similarly there cannot be a word (e.g., "heterological") which applies to all and only those words which do not apply to themselves. That is, the structure of the heterological paradox is likewise against the theorem and hence leads to contradiction. Similarly, there cannot be a class whose members are all and only those classes which are not members of themselves.

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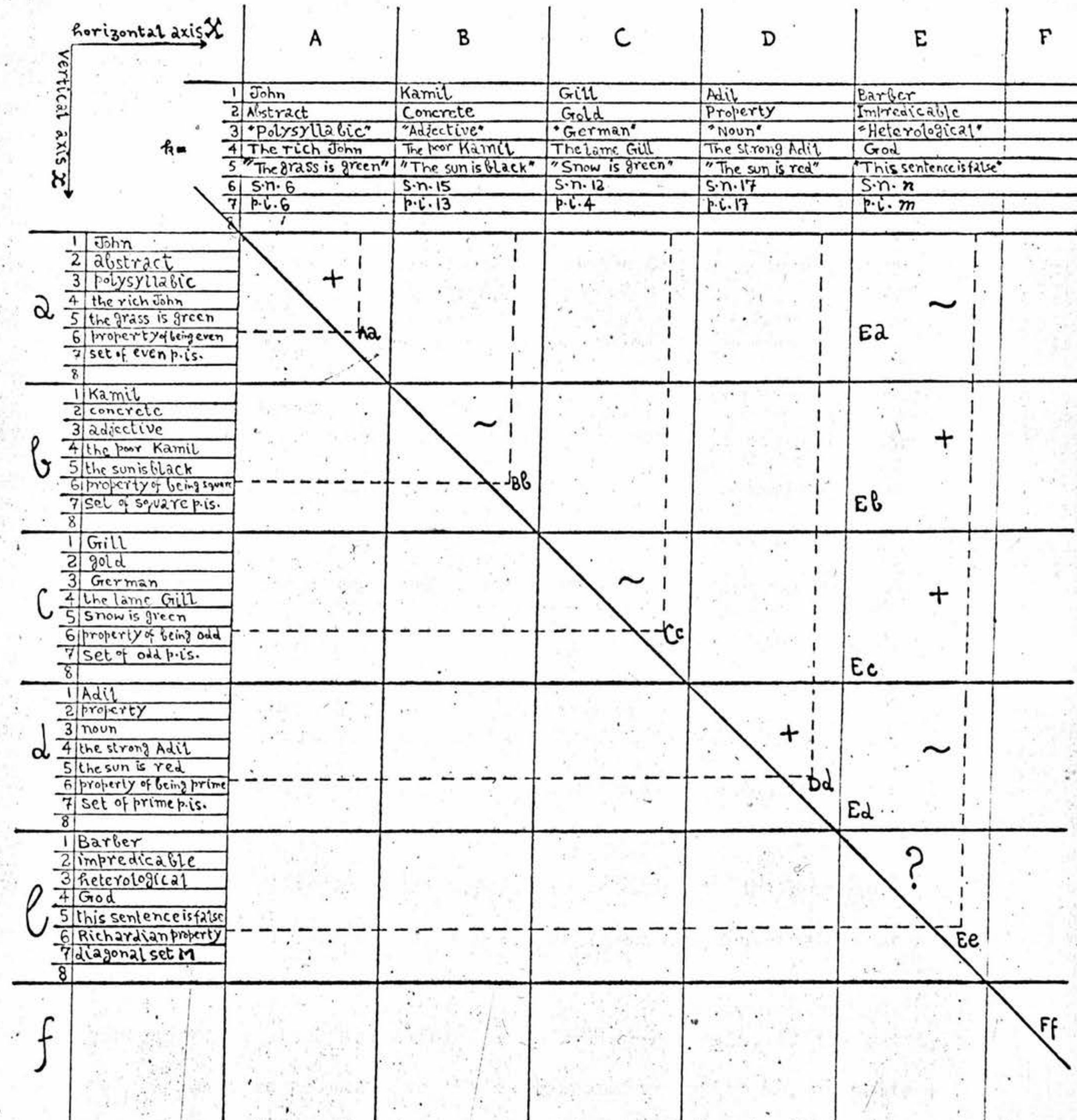
* J. F. Thomson, 'On some paradoxes', in Analytical Philosophy I, ed. R. J. Butler, p.104.

In Russell's paradox such a class is defined, so we are bound to meet a contradiction. The application of the theorem can be extended to more complicated cases like that of Richardian and the Cantor general argument (to be discussed later) where the relation involved is not so simple. In the Richardian paradox we have the set of serial numbers correlated with properties and the relation on the set being the possession of the correlated property by the serial numbers. According to the theorem above, there cannot be a serial number *n* correlated with the property (e.g., of being Richardian) which is owned by all and only those serial numbers which do not have the correlated properties. In the Richardian paradox we suppose such a serial number *n* and hence there follows the contradiction. In the Cantor general argument, we have a class of positive integers correlated with sets and the relation on the class being the membership of positive integers with their correlated sets. Now, there cannot be a positive integer *m* correlated with the set M whose members are all and only those positive integers which are not members of their correlated sets. Supposition of such a positive integer *m* in the Cantor general argument leads to contradiction. We can make our relation in a certain paradox still more complicated and involved, and it may even then be difficult to trace its connection with the theorem above.

§2. Diagrammatic representation of the Barber structure

We draw the diagram to illustrate the Barber structure involved in the main paradoxes discussed in the present thesis. The numbers 1-7 correspond to different paradoxes: the Barber, Impredicable, heterological, God who helps all and only those who do not help themselves and then raising the question whether or not God helps himself, we are led to contradiction (- this paradox


horizontal axis X
vertical axis x
A
B
C
D
E
F
k =
1 John | Kamil | Gill | Adil | Barber
2 Abstract | Concrete | Gold | Property | Impredicable
3 "Polysyllabic" | "Adjective" | "German" | "Noun" | "Heterological"
4 The rich John | The poor Kamil | The lame Gill | The strong Adil | God
5 "The grass is green" | "The sun is black" | "Snow is green" | "The sun is red" | "This sentence is false"
6 S.n. 6 | S.n. 15 | S.n. 12 | S.n. 17 | S.n. n
7 p.i. 6 | p.i. 13 | p.i. 4 | p.i. 17 | p.i. m
8
a
1 John
2 abstract
3 polysyllabic
4 the rich John
5 the grass is green
6 property of being even
7 set of even p.is.
8
b
1 Kamil
2 concrete
3 adjective
4 the poor Kamil
5 the sun is black
6 property of being square
7 set of square p.is.
8
c
1 Gill
2 gold
3 German
4 the lame Gill
5 Snow is green
6 property of being odd
7 set of odd p.is.
8
d
1 Adil
2 property
3 noun
4 the strong Adil
5 the sun is red
6 property of being prime
7 set of prime p.is.
8
e
1 Barber
2 impredicable
3 heterological
4 God
5 this sentence is false
6 Richardian property
7 diagonal set M
8
f
+
Aa
Ea
~
Bb
+
Eb
~
Cc
+
Ec
+
Dd
~
Ed
?
Ee
Ff

is mentioned to illustrate that we can construct as many paradoxes as we like by maintaining the Barber structure in the argument), Liar, Richardian and the Cantor general diagonal argument to prove the higher cardinality of the power-set (here: set of all sub-sets of positive integers) of any set (here: set of positive integers), respectively. The Liar paradox is fully discussed in the next chapter, the Richardian paradox and the Cantor general diagonal argument are dealt with in Chapter IV.

An entry in a horizontal axis, Xk, stands in a certain relation to the corresponding entry in the vertical axis xk, or it does not stand in this relation. E.g. John (= A1) does or does not stand in the relation of shaving to John (= a1); "Snow is green" (= C5) does or does not stand in the relation of truly asserting to the fact that snow is green (= c5); and the positive integer (p.i.) 17 (= D7) does or does not stand in the relation of being a member of the set to which the p.i.17 is correlated in an enumeration of all subsets of the set of positive integers (= d7). Entities Ek in the horizontal axis E, like Barber, Impredicable, serial number (s.n.) *n* etc., bear certain simple or complex relations to the corresponding entities Xk. It is to be observed that if any entity Xk bears a positive relation to the corresponding entity xk, then Ek would bear negative relation to Xk and conversely. Accordingly, if a certain entity, say A1, bears a positive relation to a1, we write + in the corresponding block Aa; otherwise we write ~. Again, if the entity, say E1, bears a positive relation to the corresponding entity A1, we write + in the corresponding block Ea; otherwise we write ~. For example, if John shaves himself and so we write + in the diagonal block Aa, then the Barber does not shave John and so we write ~ in block Ea of vertical column E. Hence if we write + in the diagonal block, then we have to write ~ in the corresponding block of vertical column E. The paradoxical situation arises when we raise the question as to what should be written in the block Ee, + or ~.

To show how the diagram works, we pick out one instance from each paradox. Suppose Gill does not shave himself and hence the Barber shaves Gill; accordingly we write ~ in the diagonal block Cc and + in the block Ec. Again, the property of being concrete does not have the property of being concrete and hence it has the property of being impredicable; accordingly we write ~ in the block Bb and + in Eb. Again, "noun" is a noun and hence it is not heterological. Therefore the word "heterological" cannot be applied to it; accordingly we write + in the diagnoal block Dd, but ~ in the block Ed. Again, the lame Gill cannot help himself; therefore God helps him. Hence we write ~ in the diagonal block Cc, but + in the block Ec. Again, the proposition, "The sun is black", falsely asserts the state of affairs that the sun is black; but the proposition, "This sentence is false that the sun is black", does truly assert the state of affairs that this sentence is false that the sun is black. Hence we write ~ in the diagonal block Bb, but + in the block Eb. Again, in the Richardian paradox, suppose the s.n. 12 is correlated with the property of being odd, then as the serial number 12 does not possess the property of being odd, we write ~ in the diagonal block Cc; but as this s.n. does not possess the correlated property, it owns the Richardian property supposedly correlated with the s.n. n and hence we write + in the block Ec. Again, in the Cantor general diagonal argument, suppose the set of prime numbers of the power set of positive integers is correlated with the integer

17 as it is shown to be the case in 7 in the diagram, then it means that the p.i. 17 being a prime is a member of the set of prime numbers, and hence we write + in the diagonal block Dd. But the p.i. 17 being the member of the correlated set, it cannot be a member of the diagonal set M supposedly correlated with the p.i. m, hence we write ~ in the block Ed. The contradiction becomes apparent when we think of putting the mark + or ~ in the block Ee. The marks on the diagonal blocks run opposite to the marks on the blocks of column E, but the block Ee is common both to the diagonal and the vertical column E. Hence we write both + and ~ in the block Ee, and this is a contradiction.

Thomson also draws diagrams to explain the paradoxical nature of the heterological paradox. But our diagram has the advantage of being comprehensive so as to include other paradoxes of the Barber type and also it shows clearly how the contradiction is bound to occur in the block Ee. Our graphic representation also shows the evolution of the complication in the Barber structure. In the Barber and Impredicable paradox the relation involved is quite simple. In the heterological paradox, the relation gets involved; we say, e.g., the word "polysyllabic" is applied to the word "polysyllabic" because the word "polysyllabic" is polysyllabic, since the word has the property designated by it. The element of designation comes in and makes the relation more involved. In the case of the Richardian paradox, the relation of serial number having certain property becomes more complicated because it is conjoined with the relation of correlation. A similar situation arises in the case of the Cantor general argument where the relation of membership of positive integers to the set is conjoined with the relation of correlation with the set. We can make our relation in a paradox still more complicated, though the basic structure which is exhibited by the Barber paradox remains intact. There may be paradoxical structure other than that of the Barber but interlaced with the latter, thus making the structure still more complex.

§3. Nature of the solution of paradox

In solving the paradoxes we have not only to look at the structure which leads us into contradiction. We have also to examine the logical behaviour of the terms involved in the paradox. In solving the paradoxes we should keep the following three things distinct even though they may be closely linked with another and may be fully understood in context with the others: (a) particular concepts involved in the paradox (b) the paradoxical structure, and (c) solution of the paradox. Identical paradoxical structure does not necessarily imply identical solutions. For an adequate resolution of a paradox, we have to consider the logical behaviour of the concepts involved and thus rectify our beliefs concerning these concepts. In our examples of various paradoxes above, the basic paradoxical structure is the same but there are various ways to avoid or change the paradoxical structure. In the Barber paradox, we cannot sensibly deny that the barber is a person or the relation of shaving to oneself or to others. And it also seems that there is no essential difference between the relation of shaving as it exists with oneself or with somebody else. The relation of shaving seems quite unambiguous and we need no recourse to the hierarchy of this relation. Here both Russell's and Ramsey's solutions are unsound; their answers to such paradoxes fail to achieve satisfactory solutions. But we may reasonably argue that the extension (or application) of the relation of shaving by a barber such that the barber shaves all and only those who do not shave themselves, is objectionable and responsible for contradiction. In the extension we must either deny that this relation of shaving applies to <u>all</u> persons who do not shave themselves or that this relation applies to <u>only</u> those persons who do not shave themselves. The terms involved in the Barber paradox are quite clear and need no clarification. But in the case of the Impredicable paradox the situation is different and we may reasonably deny that we can by universal instantiation put the property of impredicability I in place of the variable x (any property) because the sense in which impredicability or predicability is a property is different from that of other properties like coloured, red, abstract, etc. We may by our own definition even deny the status of property to impredicability or predicability. The same remarks apply to the property of heterologicality and of autologicality. This is so because in order that any property should have the characteristic of impredicability or of predicability, it must be other than the property of impredicability or of predicability. For as we argued above, the property of impredicability or of predicability is a <u>dependent property</u>, like the property of heterologicality. Just as we cannot call a thing admirable or good unless it has some other characteristic like beauty or honesty, we cannot characterise a property as impredicable unless it has some other property. There seems to be no ambiguity in the relation of predication or in the characteristic of being impredicable or predicable, as Russell and Ramsey would suggest. In the heterological paradox we have the word "heterological" which means heterological but we cannot conjoin this fact with the legitimacy of raising the question whether "heterological" is heterological or not, because as said earlier, in order that any word be heterological or autological it must designate some property other than that of heterologicality or autologicality.

It is clear from the above examples that even if their paradoxical structure is the same, the solutions may be diverse. Of course, any proposed solution must eliminate the paradoxical structure. But essentially any solution of the paradox must also clarify and bring to light the logical behaviour of the basic notions involved in the paradox. Russell seems to have committed the fallacy of presuming that if all paradoxes involve the vicious circle fallacy, their solutions must be similar. He overlooked the peculiarities of different paradoxes (or different sorts of paradoxes); - peculiarities which may accompany the paradoxical structure or the vicious circle fallacy. Russell says:

> In all of them (i.e. paradoxes), the appearance of contradiction is produced by the presence of some word which has systematic ambiguity of type, such as truth, falsehood, function, property, class, relation, cardinal, ordinal, name, definition. Any such word, if its typical ambiguity is overlooked,.... will thus give rise to vicious circle fallacies.*

And thus he says that once ambiguity is removed by his theory of types, the vicious circle is removed and thus paradoxes are eliminated. His whole attention was directed to eliminate the vicious circle instead of studying the logical behaviour of concepts involved in the paradoxes.

§4. Classification of paradoxes

As already mentioned there seems to be no common characteristic shared by all paradoxes. We may classify paradoxes according to the sort of solutions they offer or according to their common paradoxical structure. There seems to be no particular advantage in making a strict classification. One paradox

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* P.M., p.64. Bracketed expression is our own.

insofar as it involves a particular sort of concept, may have different formulations and hence may involve different paradoxical structures. Even the notion of paradoxical structure is not absolute, because in complicated cases it may be difficult to decide whether to put a certain formulation of paradox under this or that paradoxical structure. We may appreciate such a situation in complicated cases like that of Richardian and the Cantor general diagonal argument, as shown in our graphic representation of the Barber structure. Again, one may find similarities amongst any things, such as between stones and men. But one should select similarities which serve some useful purpose. There is some convenience of arranging and tackling paradoxes according to their common paradoxical structure. Along these lines, we shall try to arrange various paradoxes primarily in accord with their common paradoxical structure. We call all those paradoxes "Barber paradoxes" which are structurally similar to the Barber paradox. To the Barber paradoxes belong, for example, all the paradoxes mentioned in our previous diagram. The moral we drew from that discussion is that it is logically impossible for there to exist a man (barber) who shaves every man that does not shave himself. Taking any entity instead of man and any relation instead of shaving, but keeping otherwise the same structure, we get antinomical results as in the case of the specific Barber paradox. The Barber structure or pattern makes us realise that it is logically impossible that in any specific class of things, whether this class or group be of persons, ideas, relations, or of any sort of entities, there could not be one member of this class which bears some specific relation to all and only those members of the group or class in question which do not have the said relation to themselves. In the next chapter we chall discuss the Liar paradox as a further example of a "Barber" paradox.

Finally, in Chapter V, we shall discuss paradoxes which occur in logico-mathematical systems. It is profitable and convenient to group together such paradoxes as that of Russell, Burali-Forti, and of Cantor because for each of them we can trace their origin to defective axioms. Two or more paradoxes occurring in an axiomatic system will certainly help us in understanding the nature of all sorts of paradoxes occurring in the system. We may, for the sake of better understanding, make a distinction between semantic and syntactic paradoxes, meaning by the latter those paradoxes which occur in logico-mathematical systems and by the former those which do not occur in such systems. Even if we separate "syntactic paradoxes" as outlined above, it would still be useful to keep in view other paradoxes which have an identical or similar paradoxical structure or involve some common conceptual error.

Chapter III

THE LIAR PARADOX

A. Introduction

§1. The Barber structure of the Liar paradox

The Liar paradox may be formulated in several ways. One version of it has already been discussed in the last chapter. The Barber structure which underlies most of the common versions of the Liar paradox becomes apparent when we keep in view that a true sentence is that which truly asserts the state of affairs asserted by the sentence and similarly a false sentence is that which falsely asserts the state of affairs asserted by the sentence. That is to say, suppose sentence x asserts state of affairs y, then x is true if and only if x asserts y and y obtains or holds; and sentence x is false only if y does not obtain or hold. The entity in the Liar paradox which corresponds to the man in the Barber paradox is a proposition, statement or sentence (we make no distinction of these terms unless specifically mentioned), and the relation analogous to the relation of shaving and not shaving is that of truly or falsely asserting the state of affairs asserted by the sentence. We choose one statement among all the statements in question, of which we say that it truly asserts the states of affairs asserted by all and only those statements which do not truly assert the states of affairs asserted by themselves; and then, by raising the question whether that statement truly asserts the state of affairs asserted by itself, we are led to contradiction. To elucidate the comparison between the Liar paradox and the Barber paradox, let us take the following two sentences: (a) "This statement is false", and (b) "This page is green". Let us suppose that sentence (a) says something about the statement made by sentence (b). Then sentence (a) says that it is false that this page is green. In short, sentence (a) says that it is not the case that this page is green; i.e. that this page is not green. And this is true. Hence sentence (a) truly asserts the state of affairs asserted by sentence (b). Whereas sentence (b) taken by itself does not truly assert the state of affairs asserted by itself, because sentence (b) states or asserts that this page is green, which it is not. Now, suppose there are only two sentences, namely (a) and (b), and we arise the question whether sentence (a), taken by itself and without reference to (b), truly asserts the state of affairs asserted by itself. In short, we ask whether sentence (a) is itself true or not. By raising this question we arrive at the Barber structure. This is so, because we ask whether or not sentence (a) truly asserts the state of affairs asserted by itself, which truly asserts the states of affairs asserted by all and only those sentences which do not truly assert the states of affairs asserted by themselves. More explicitly but concisely:

<u>The Barber paradox</u>: A man b shaves all and only those men who do not shave themselves. $(x)(Sbx \equiv \sim Sxx)$, where Sxy symbolises "x shaves y". Does the man b shave himself or not?

<u>The Liar paradox</u>: A sentence (i.e. the Liar sentence) truly asserts the states of affairs asserted by all and only those sentences which do not truly assert the states of affairs asserted by themselves. $(x)(Tsx \equiv \sim Txx)$, where Txy symbolises "x truly asserts the state of affairs asserted by y". Does sentence s truly assert the state of affairs asserted by itself?

§2. A version of the Liar paradox

Let us consider the following version of the Liar paradox:

Whatever is written in this rectangle is not true

The sentence written in the rectangle concerns the truth-value of all sentences written in the rectangle. But this very sentence, namely "whatever is written in this rectangle is not true", is itself written in the rectangle. Hence, if we take this sentence to be true, then by its own very assertion, it is not true. And likewise if taken to be false then it is true. The same pattern is exemplified by another version of the paradox, namely: "I am lying".

It might be argued that in these two versions of the Liar paradox, the paradoxical situation or contradiction may be averted by including some other statements in the rectangle in question in the one case, and in the other by the person uttering some other statements beside "I am lying". But this is not a correct assessment of the situation. Even if there were other sentences in the rectangle or uttered by the person, the paradox would remain unaltered, provided we could maintain the Barber structure involved in the Liar paradox. If we write in the above rectangle the following sentences: (a) "A square is a three-sided figure", (b) "A polygon is a four-sided figure", (c) "This page is green", the paradoxical result would ensue by raising the

(1) Whatever is written in this rectangle is not true
(a) A square is a 3-sided figure
(b) A polygon is a 4-sided figure
(c) This page is green

question whether sentence (1), namely "whatever is written in this rectangle is not true" is true or not. This is so because it fulfils the requirements for the Barber structure. The sentence (1) in the new rectangle says of all sentences in the rectangle that they are not true, that is they do not truly assert the states of affairs which they assert. Sentences (a), (b), (c) are in fact false (not true). Hence it is clear that sentence (1) truly asserts the states of affairs asserted by sentences (a), (b), (c) although they themselves do not truly assert the states of affairs asserted by themselves. It is the very nature of the sentence (1) that it can truly assert the states of affairs of only those sentences in the rectangle which do not truly assert the states of affairs asserted by themselves. The sentence (1), being itself in the rectangle, the requirements for the Barber structure are satisfied and thus the question whether sentence (1) truly asserts the state of affairs asserted by itself, in other words whether sentence (1) is true or not, leads to a contradiction. If in this rectangle in question we add the sentence (d) "Snow is white" or "An equilateral triangle is equiangular", even then the paradox would ensue. But the paradox may not seem to ensue just because for psychological reasons we may tend to cling to the reference provided by (d) and the trick of self-reference of the Liar statement does not enter as such. (This point also relates to the problem of shifting reference in the Liar paradox, which will be discussed at a later stage.) For then the sentence (1) does exclude the application to itself as such and so the requirement (b) described below for the genesis of the Barber structure is lacking. The requirements for the genesis of the Barber structure are (a) that there should be one entity (here: the Liar statement) which claims to bear a certain relation to all entities which do not bear this relation to themselves, and (b) that this unique entity belongs to the group of entities under discussion.

§3. Resolution and structure of the paradox

It is clear that the above versions of the Liar paradox have basically the Barber structure. But the main point of the Liar paradox concerns the concepts of truth and falsity. We are not simply interested in averting the contradiction involved in the Liar paradox, i.e., in just removing the Barber structure. Our primary object is to clarify the logic or the proper usage of the concepts involved in the paradox and the logical status of sentences like "I am telling a lie". Even if by some clever device we manage to eliminate the Barber structure, such a resolution of the paradox may not illuminate and clarify the notions of "true", "false" and the proper usage of sentences like "I am telling a lie", and hence should not be regarded as an adequate resolution of the paradox. It is possible that we may arrange to prevent the Barber structure from occurring in our usage of words like "true" or "false" but contradiction may still arise through the emergence of some other paradoxical structure. The Liar paradox is not necessarily connected with the Barber structure. The Liar paradox in general concerns the notion of truth and falsity insofar as the incorrect and illegitimate usages of "true" and "false" lead to contradictory results. And the resolution of the Liar paradox lies in removing the incorrect usage of these words and thus clarifying their proper usage. The incorrect or inappropriate usage of these terms may lead to a paradoxical structure, and this paradoxical structure may be the Barber structure or some other. The reason why we have treated the Liar paradox as one of the Barber paradoxes is that the usual versions of the Liar paradox fit the Barber structure, and we seem to be able to appreciate the Liar paradox in a more fruitful way by examining especially its connexion with the Barber structure.

Of course we may present other versions of the Liar paradox which do not involve the Barber structure. For example, if on one side of a blackboard is written: "whatever is written on the other side is true" and nothing else, but on the other side of the blackboard is written: "whatever is written on the other side is false" and nothing else, then raising the question of the truth or falsity of either of the sentences leads to contradiction. Or, to take a still simpler example, consider the following sentences:

The next sentence is true. The previous sentence is false.

Here we do not have the Barber structure. We simply meet a straightforward contradiction because what is asserted by one sentence is denied by the other. We may in fact combine the above two sentences in one:

This sentence is (both) true and false

which, as one can easily see, violates the principle of contradiction; namely, a sentence cannot be both true and false together. As formulated in this way, it is just a circuitous way of violating the principle of contradiction. One may easily make the situation more complicated, so that it becomes all the more difficult to notice the violation of the principle of contradiction at one glance. For example:

(The second sentence in the bracketed space is true. The third sentence in the bracketed space is false. The first sentence in the bracketed space is true.)

By raising the question of truth or falsity about any sentence in the above bracketed space, we are led to contradiction. The proper resolution of this contradiction should make us realise where or why the principle of contradiction has been broken.

From the above discussion it is clear what is needed for a proper resolution of the paradox, i.e., one which does not simply avert the paradoxical structure. An adequate resolution of the paradox must clarify the concepts involved in the paradox, and should tell us how they should be properly used. A solution of the Liar paradox should throw light on any version of the Liar paradox and should enable us to understand the flaws involved in the Liar paradox and to appreciate the logical behaviour of terms essentially involved in it.

3. <u>Tarski's resolution of the paradox</u>

1. Importance of the Liar paradox

Tarski discusses the Liar paradox and comes to the conclusion that atural languages, say English or French, are essentially inconsistent.* The nherent inconsistency of natural language is concealed by the fact that it as no specific structure. The Liar paradox and other 'semantic' paradoxes re just reminders that we have inconsistency in our language. So, Tarski's onclusion is that because of this inherent inconsistency of our language, e come across such 'semantic' paradoxes. Once we reform our language and emove the inconsistency in it, the paradoxes naturally disappear. If what arski says is true, his resolution of the Liar and other 'semantic' paradoxes ould satisfy the test for a proper resolution of paradox; that is, it would larify the concepts involved in the paradoxes. Indeed, should Tarski's olution prove acceptable, it would amount to a revolutionary advance in emantics. Thus he impresses upon us the importance of the Liar paradox in hese words:

Tarski, 'The Semantic Conception of Truth', (<u>Philosophy and Phenomenological Research</u>, 1944); repr. in <u>Readings in Philosophical Analysis</u>, ed. H. Feigl & W. Sellars, pp.52-84.

> And just as class-theoretical antinomies, and in particular Russell's antinomy (of the class of all classes that are not members of themselves), were the starting point for the successful attempts at a consistent formalization of logic and mathematics, so the antinomy of the Liar and other semantic antinomies give rise to the construction of theoretical semantics.*

lence, for Tarski the proper resolution of the Liar paradox would lead to the removal of imprecision and inconsistency in our common language and thus the onstruction of a self-consistent language for scientific purposes.

Tarski's comparison of the Liar paradox with Russell's paradox is nfortunate. He overlooks the fact that in logic and mathematics a few basic otions are involved which can be rendered precise in a deductive system. But n the case of ordinary language we have many notions, call them "semantic" or therwise, which extend to various paradoxes. Neither Tarski nor any other emanticist has shown that these concepts can be systematized in a form like hat of a geometrical system, and thus has not exhibited a closely inter-onnected and interwoven system. Showing some connections between a few emantic concepts together is a different sort of thing from the close con-ection of concepts involved in a formal system. And unless this is shown, he proper resolution of the Liar paradox may remain restricted to the pecific Liar paradox or to a few other semantic paradoxes, and may not throw ny light on the notions involved in all the 'semantic' paradoxes. To say hat the solution of Russell's paradox has bearing upon or resolves other aradoxes occurring in a logical or mathematical system has a sound rational asis because such a solution involves the rectification and clarification f basic concepts and axioms responsible for the construction of the whole f a deductive system. But there seems to be no sound rationale or justifi-ation for the assertion that the solution of the Liar paradox has bearing

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Ibid., p.59.

upon or would resolve other 'semantic' paradoxes. Tarski offers only an indirect argument, through his theory of language-levels. But Tarski's theory may be adequate to resolve the Liar paradox but not to resolve other 'semantic' paradoxes. So Tarski must provide some reason for the assertion that resolving the Liar paradox would also result in the resolution of other paradoxes. His argument as such is circular and has a metaphysical flavour: he argues for the inconsistency of ordinary language through the inconsistency of the Liar paradox and then he brings forth his comprehensive view of the inconsistency of ordinary language to resolve the Liar and other 'semantic' paradoxes. He does not offer any independent proof for the inconsistency of ordinary language. Let us now turn to his actual presentation and solution of the Liar paradox.

2. Tarski's presentation and resolution of the paradox

Tarski considers the following version of the Liar paradox. Given the sentence

The sentence written on page 78, line 16 is not true,

hich we abbreviate by the symbol S, according to Tarski's semantic theory of ruth (i.e., a sentence is true if it asserts an existing state of affairs), e assert:

1) 'S' is true if and only if, the sentence written on page 78 line 16 is not true.

ut, keeping in view the meaning of the symbol S, we establish empirically, s Tarski says, the following fact in accordance with the theory of identity Leibniz's law):

2) 'S' is identical with the sentence written on page 78 line 16.

o we arrive at the contradictory conclusion

3) 'S' is true if and only if 'S' is not true.

After presenting the paradox, Tarski embarks on an analysis of the assumptions which lead to the antinomy. He says that we have assumed that the language, in which the above antinomy is constructed obeys the ordinary laws of logic and moreover that (in this language) we can formulate and assert an empirical premiss such as (2) in the above argument. Tarski thinks that the second assumption is not necessary since some paradoxes may be constructed without any empirical premiss but that the first assumption is true and must be accepted. There is also a third assumption underlying the above construction of antinomy, namely - as expressed in Tarski's words:

> We have implicitly assumed that the language in which the antinomy is constructed contains, in addition to its expressions, also the names of these expressions, as well as semantic terms such as the term "true" referring to sentences of this language; we have also assumed that all sentences which determine the adequate usage of this term can be asserted in the language. A language with these properties will be called "semantically closed".*

It is true that Tarski expresses the Liar paradox in a 'semantically closed' language as defined above. Tarski blames the 'semantically closed' language for the paradox. Hence, according to Tarski, if we decide not to use any language which is semantically closed, we get no paradox. Tarski then goes on to say:

> Since we have agreed not to employ semantically closed languages, we have to use two different languages in discussing the problem of the definition of truth and, more generally, any problems in the field of semantics. The first of these languages is the language which is "talked about" and which is the subject-matter of the whole discussion; the definition of truth which we are seeking applies to the sentences of this language. The second is the language in which we "talk about" the first language, and in terms of which we wish, in particular, to construct the definition of truth for the first language. We shall refer to the first language as "the object language" and to the second as "the meta-language".**

* Ibid., p.59 ** Ibid., p.60

Tarski reminds us that the terms "object language" and "metalanguage" have only relative meaning. If we want to apply the notion of truth to the sentences belonging to the metalanguage of the original object-language, we have to create a new metalanguage to talk about truth in this metalanguage, which now may be regarded as an object-language. And in order to define or talk about the truth of statements expressed in this new metalanguage, we require yet another metalanguage of a higher level. So Tarski introduces the idea of an unending hierarchy of languages.

In fact Tarski's solution of the paradox is analogous to Russell's theory of types. As Russell arrived at his theory of types in order to avoid a 'vicious circle', so Tarski arrives at his hierarchy of languages in order to avoid 'semantically closed' language. Again, as Russell discovers the principle of the vicious circle by generalizing the fact that all paradoxes suffer from self-inclusiveness (reflexivity) and from the general observation of propositions involving the vicious circle, so Tarski arrives at the idea of metalanguages simply by examining the conditions of the Liar paradox and concluding that the paradox arises because it is expressed in a 'semantically closed' language. With regard to Tarski's solution, we can raise general objections which may be raised against any solution in terms of a levels of language theory. We have already mentioned these objections and more will be said later.

The concept of levels of truth seems very odd. That is, it is odd to suppose there is no concept of truth in general but there can be truth of the first order, truth of the second order and so on ad infinitum. We employ the word "true" without taking it to stand for an infinite series of levels of

ruth, for we believe that there is a general concept of truth as such. But
t may be said that the proponents of the levels of language theory have found
his belief to be false. Sometimes truth is stranger than fiction. Simplicity
s a virtue, but contradiction is worse than complication. The proponents
ssert that the construction of levels of language is the only proper way to
vert the contradiction in semantic paradoxes and so Occam's Razor fails to
ork here. But the theory of levels of language remains implausible unless
ts proponents can give us a clear-cut indication as to how to construct a
istinctly-demarcated series of levels, so as to overcome the following
jection. The general idea of levels of language is that, given any language,
y L1, a language about certain existing objects, then the language in which
talk about the language L1 would be of higher level, say L2; and the language
which we talk about language L2 would be of still higher level, say L3, and
on. And we must not confuse the levels or orders of languages with each
her. That is, if we are talking about sentences of language L1 in terms of
, then we cannot talk of the sentences of language L2 in terms of L1, and
so we must not talk about sentences in one language in the language of the
me level. We have already argued in the first chapter that in many cases
lf-reference is quite legitimate and consistent, e.g. "This sentence is in
glish". Admission of such self-referential expressions goes against the
vels of language theory. To regard such expressions as illegitimate in any
nsistent language seems remote from the truth and we find no plausible
ound for such conviction. Tarski may retort that in such expressions as
ose quoted above no semantical concepts are involved and hence self-reference
harmless. But in the first place, neither Tarski nor any other semanticist
s offered us any criterion to decide whether or not a certain concept is to

be regarded as a semantical concept. Although the semanticists do give some examples of semantic concepts like "truth", "designation", "satisfaction", they do not provide any general criterion to distinguish them from non-semantic concepts. In the second place, Tarski does not seem to restrict himself to semantic concepts only, and he does seem to make a wider claim that we should not talk about any language in the language itself, whatsoever concepts are involved. In the third place, in the specific case of the concept "truth", which every semanticist has acknowledged to be a semantical concept, the following statement-form is valid but according to the levels of language theory it should be dismissed as illegitimate:

> If x says that whatever y says is false and y says that something which x says is true, then something which x says is false and something which y says is true.

Obviously, the levels of language theory should not regard such a statement-form as legitimate. If two statements attach truth-value to each other, then there is no reason for regarding one as of higher level than the other. In our daily discourse we often employ sentences attributing truth-value to each other. In the case of the Liar paradox we have already presented some versions of it where two sentences attribute truth-value to each other - for instance

The next sentence is true. The previous sentence is false.

If there is no objection based on the levels of language theory against the valid statement-form presented above, then why should these versions of the Liar paradox, or the self-reference of the Liar paradox in general, be attacked on that ground?

The other point which Tarski wants to emphasize is that we must not employ names of expressions in which the antinomy is contructed in the very e language (- names or designating expressions being semantic concepts), ause then we would be employing a "semantically closed" language. But s point is nullified by the fact that we can present the Liar paradox in h a way that the problem of the name and designation belonging to the same guage may not arise. W. Rozeboom formulates the Liar paradox in this way making a distinction between sentence-token and statement.* His formulation the paradox is as follows. Let us call S the expression inside the Figure 1.

In Figure 1 there is a sentence-token which conveys a false statement

Figure 1.

us assume that it designates a statement, call it Σ. So we have:

S is a sentence-token and, moreover, the only sentence-token in Fig. 1.

If Σ is a statement, S conveys Σ.

S conveys no statement other than Σ.

Σ is a statement.

Σ is true only if it is the case that in Figure 1 there is a sentence-token which conveys a false statement.

Σ is false only if it is not the case that in Figure 1 there is a sentence-token which conveys a false statement.

No statement is both false and true.

All statements are either true or false.

, suppose (H1): Σ is true. Then from (H1), (2), (4), it follows S conveys rue statement. From (H1), (5), (1), it follows that S conveys a false tement; hence from (3), Σ is false. Hence by (H1), (1) - (5), (7) we are to contradiction. Now suppose (H2): Σ is false. Then (H2), (2), (4) ly that S conveys a false statement. From (H2), (1), (6), it follows that

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/. Rozeboom, 'Is Epimenides still lying?', <u>Analysis</u> 1958 , pp.105-113.

loes not convey a false statement. (1), (2), (4), (6) entail that Σ is not
lse. But since (1)-(7) imply Σ is a statement which is neither true nor
lse, (1)-(8) are logically inconsistent.

It follows from the above discussion that the levels of language theory
ter the lines of Tarski is not justified. Thus we have to look for some
ner solution of the Liar paradox.

Significance of the Liar paradox

Meaninglessness and the Liar statement

We mentioned in our first chapter that according to Russell self-
ferring expressions are to be regarded as meaningless or insignificant.
the usual versions of the Liar paradox are straightforwardly self-
erential, expressions in these versions may be regarded as meaningless.
enko has attempted to show that Russell's solution - that the expression of
Liar paradox is meaningless - itself leads to contradiction.* Let us
sider the following version of the Liar paradox which is discussed by Toms:**

No true sentence is written within the rectangle of Figure 2.

Figure 2.

if the sentence or statement within Figure 2 asserts no statement to be
e or false, it follows that no true statement is written within Figure 2.
s, the sentence "No true sentence is written within Figure 2." is true, and
refore meaningful, and so contradicts Russell's hypothesis. Symbolically
ressed:

A. P. Ushenko, Problems of Logic, pp.78-81. Another argument to refute
Russell on this point has already been dealt with in the first chapter p.25.
. Toms, Being, Negation and Logic, pp.11-13.

Let fx mean "x is written in the rectangle of Figure 2."

Tx mean "x belongs to the class of true sentences."

Mx mean "x belongs to the class of meaningful sentences."

s be a variable for which the name of a sentence may be substituted.

a name the sentence "No true sentence is written within the rectangle of Figure 2."

e have : (1) fa - (i.e., a is written in Figure 2.)

(2) $(s)(fs \supset s=a)$ - (i.e., only a is written in Figure 2.)

o, Figure 2 is:

$$\boxed{(s)(fs \supset \sim Ts)}$$

Figure 2.

A) Suppose Ta, then:

(3) $(s)(fs \supset \sim Ts)$ - i.e. assertion of the sentence supposed true.

(4) $fa \supset \sim Ta$ - from (3) for value (2) of s.

(5) $\sim Ta$ - (1), (4).

o, this alternative leads to contradiction.

B) Suppose $\sim Ta$, then:

(6) $\sim(s)(fs \supset \sim Ts)$ - i.e., assertion of the sentence supposed false.

(7) $(\exists s)(fs \,.\, Ts)$ - from (6) by logical transformation

(8) $fy \,.\, Ty$ - by EI

(9) $fy \supset y=a$ - by UI from (2)

(10) $y=a$ - by MP and simplification from (8), (9)

(11) $fa \,.\, Ta$ - by replacement and principle of extentionality: $((\phi x \,.\, x=y) \supset \phi y)$.

(12) Ta - from (11) by simplification.

o, this alternative also leads to contradiction.

C) Suppose $\sim Ma$, then Ushenko and Toms argue:

(13) $\sim Ta$, and then by argument (B) we get

(14) Ta, and hence

(15) Ma.

Section (C) above needs explanation. The sentence, "No true sentence written within the rectangle of Figure 2.", is as true as the sentence, "No lse sentence is written within the rectangle of Figure 2."; if there is no sentence written within the Figure 2, or if the sentence written in Figure 2 cannot be regarded as true or false, i.e., it cannot be regarded as conveying a proper statement, - in that case it is not right to conclude merely that no true statement is written within the rectangle of Figure 2, or that no false statement is written within the rectangle of Figure 2: one should draw both these conclusions. Hence Ushenko and Toms' argument of part (C) is misleading. Symbolisation does not necessarily bring out the clarity and validity in our thinking; sometimes it rather leads us astray from correct and valid arguments. Symbolisation follows rigid patterns to which we may doggedly try to fit arguments which do not follow those patterns. From "$\sim Ma$" we get the "combined" assertion, namely: "No true sentence is written within the rectangle of Figure 2 and no false sentence is written within the rectangle of Figure 2.". This combined assertion or sentence cannot be treated as the combination of two sentences in truth-functional logic. So the symbolisation of truth-functional logic cannot hold here. In truth-functional logic a combined assertion, say "$A \cdot B$", can be broken up into two sentences "A" and "B" which can be treated separately and independently from each other. But this cannot be done with the above combined assertion while remaining within the framework of truth-functional logic. This is so because by the application of law of excluded middle we are soon led to contradiction. Hence, unless we construct another system in which the law of excluded middle does not apply, we cannot express correctly the logic behind the present combined entailment. So the argument in part (C) is not a valid one, although the conclusion that "$(s)(fs \supset \sim Ts)$" is meaningful can be justified by other considerations. We may reasonably argue that since $\sim Ta$ is a part of the meaningful "combined" sentence in an important sense, hence "$(s)(fs \supset \sim Ts)$" may also be reasonably regarded as eaningful. We shall further consider the problem of meaningfulness at the nd of this chapter.

2. Dichotomy of either-or attacked

By regarding the sentence "No true sentence is written within the ectangle of Figure 2." as meaningful but neither true nor false, we have in act shown that there is a third possibility; that is, we can reasonably say f a sentence or statement that it is neither true nor false. In certain ases we can deny both these alternatives. In all the formulations of the aradoxes considered above, the law of excluded middle (of traditional logic) s been assumed to hold. Tarski explicitly admitted that traditional laws logic underlie the formulation of the Liar paradox. He could easily have oided constructing his cumbersome hierarchy of languages by attacking the aditional law of excluded middle. It is not the law of contradiction but e law of excluded middle that we regard as questionable. We can say both o true sentence is written on the blackboard" and "No false sentence is itten on the blackboard" in case the blackboard is either clean or there is imperative sentence, etc. written on it. If on the blackboard the words e written "The blackboard is black (or white)", then we can determine the uth-value of the sentence "No true sentence is written on the blackboard". t it has a truth-value because its truth-value has become dependent on the uth-value of the sentence "The blackboard is black (or white)". If it ands on its own, we cannot say whether it is true or not true. We have ready touched on this point, especially in chapter II. Now, let us explore re closely the logical status of expressions contained in the Liar paradox ke "This sentence is false" or "I am lying".

• The logical behaviour of "true" and "false"

As said above, we can avoid contradiction in the Liar paradox if we gard the sentence involved as neither true nor false, though meaningful. ere are several sorts of sentences which cannot be regarded as true or lse. For instance, imperative sentences are of such a nature. But the ntences involved in the Liar paradox are neutral (i.e., neither true nor lse) for some other reason than e.g. imperative sentences. Sentences like am lying", "This sentence is false" taken by themselves are incomplete its of thought because they lack reference. We raise the question "What e am I telling?", "Which sentence is false?" or "Why is this sentence se?" There is no possible or actual state of affairs about which judgnts are passed in the above sentences when taken by themselves. We must ow what we are referring to, or what statement is to be assessed as true false before we can say that it is true or false. R.C. Skinner has ably gued that to describe a statement (sentence or assertion) as true or false to say something about the statement.* He says:

> what is properly describable as being true or false is always a sentence (or statement), or group of words that could stand alone as a sentence (or could be used to make a statement), <u>which does not itself contain the words "true" or "false"</u>;

argues that if A says "It is false that the conservatives are out of our at present", and B replies "It is true", then B would be taken as erting that the conservatives are out of favour at present. This is so, ause in fact, the sentence to which B refers is: "The conservatives are of favour at present", - a sentence not containing the word "false" -

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R.C. Skinner, 'The paradox of the Liar', <u>Mind</u> 1959, p.326.

nd not "It is false that the conservatives are out of favour at present". It is true that it is false that ....." means "It is false that ....", and It is false that it is false that ....." means "It is true that .....". ltimately in order to say that a certain sentence is true or false, there ust be some reference, some possible state of affairs asserted by the sentence - nd if this reference is missing, there is nothing to be assessed as true or alse. The single sentence written on a blackboard "The sentence written on his blackboard is in English" though self-referring is true; the single entence written on a blackboard "The sentence written on this blackboard is n French" is self-referring but false. But to ask whether the single entence on a blackboard "The sentence written on this blackboard is false" true or false (as in one of the versions of the Liar paradox) is to misuse e words "true" or "false"; that is, it is to ignore the proper usage of the rds. In the first two cases "true" and "false" refer to the question ether or not the sentence written on the blackboard is in English and French spectively. Here we find a reference, so we can reasonably raise the estion whether it is the case or it is not the case; that is, whether a rtain state of affairs exists or not - the state of affairs being whether e sentence written on the blackboard is in English (or in French). But in e last case we do not find any reference to any state of affairs of which could decide whether or not it exists. When I say "This sentence is false", must refer to something expressing a certain state of affairs. When imenides the Cretan utters only the sentence "Nothing true is ever asserted a Cretan", the sentence lacks reference to something that is the case or not the case. There may be other sentences uttered by Epimenides himself by some other Cretan to which this sentence is supposed to apply, but this nnot change the oddity unless the sentences concerned express states of fairs and so can be assessed as true or false. Another Cretan might say: hatever Epimenides says is true"; but even then we are led to contradiction. seems obviously odd to ask whether the sentence "This sentence is false" true or false if there is no sentence to be judged true or false, just as is odd to tell a man to go and shut the door when there is no door to shut open. Telling somebody to shut the door implies that there exists a door; herwise it seems absurd to tell him. Similarly, to say that a certain ntence is true or false implies that there is a sentence which can be asonably said to be true or false; i.e., the sentence in question must press a certain state of affairs. If this precondition is not satisfied, oddity is apparent and it becomes nonsensical to ask whether or not the ntence "This sentence is true" is true.

Skinner elaborates his point of view, which we fully endorse, on the esis of contradiction in the Liar paradox by citing the following example. sider whether the sentence "'Socrates is still alive' is false" is true false. He says we can play a trick with this sentence by remarking that it is true then it is false and vice versa; but if we look for its refer- e and keep our eyes on it, no paradox arises. We may paraphrase the above tence by: "Is it in fact true that Socrates is still alive?". We must w what we are referring to. Do the predicates "true" and "false" apply "Socrates is still alive" or to "'Socrates is still alive' is false"? nner ably argues that a paradox arises because the argument switches from to another. The first part of the paradox runs: If <u>it</u> is false (assumption) n, since this is what <u>it</u> is declared to be, <u>it</u> is true. The first and ond "it" in the argument refer to "Socrates is still alive", but the third t" refers to "'Socrates is still alive' is false". Similarly with the cond part of the argument. Skinner argues that a similar sort of malady fects the statement of the Liar paradox. The situation here becomes more mplicated because the expression "The sentence written on this blackboard" not a sentence. But if we regard it as having a sentence or statement as ference, a similar situation arises. It is the shifting of reference which ads to contradiction and it occurs owing to our oversight in not clinging one fixed reference. If we accept the convention that "It is true that it false" means "It is false", and "It is false that it is false" means "It s true", but keep the same reference then we get no contradiction. For The sentence written on this blackboard is false' is true" just means "The ntence written on this blackboard is false" and "'The sentence written on is blackboard is false' is false" means "The sentence written on this ackboard is true". Skinner further argues that if we do not accept the ove convention and do not change the reference, but clearly fix our atten- on on the expression taken into account, then also no paradox arises. ppose that the sentence written on this blackboard is true, - then the nclusion "'The sentence written on this blackboard is false' is false" s follow, but it is not paradoxical because what is asserted to be false the conclusion is "The sentence written on this blackboard is false". nce, truth or falsity has not been assigned to the same reference and hance contradiction arises. In the example considered above, namely "'Socrates still alive' is false", if we fix the reference and take care not to shift reference, no contradiction ensues. To say that this sentence is false s not lead to the conclusion that this (whole) sentence, namely "'Socrates still alive' is false" is true. Only by confusing the two references, namely "Socrates is still alive" and "'Socrates is still alive' is false", do we get the contradiction. Similarly with the case "This is a false sentence". The reference shifts here from "This" to "This is a false sentence". If we suppose the above sentence to be false then, for reasons given previously, we cannot say "This is a false sentence" is true. As in statements of the Liar paradox there is in fact no reference: we imperceptibly shift the reference, e.g., from "This" to "This is a false sentence", and fall into a trap. The absence of the real reference or mention of state of affairs in the assertions judged to be true or false prevents us from detecting the change of reference.

It is clear that the proper resolution of the Liar paradox should safeguard against this change of reference and also it should provide a definite and legitimate reference for the Liar assertion. Once we have ultimately a fixed and definite reference, which may or may not contain the words "true" or "false" (e.g., in the case "All sentences on this page are true" the reference or statement describing a certain state of affairs contains the word "true"), we can use the rules governing the correct usages of the words "true" or "false" - these rules being e.g. that it is false that it is true means it is false and that it is false that it is false means it is true.

. <u>Misleading aspects of the Liar paradox</u>

1. Grammar and the Liar paradox

The conclusion from the above discussion is that we must have some fixed reference in order to speak of truth or falsity. For proper usage of the terms "true" and "false" there should be determinate reference to a certain state of ffairs. Compare the following two sentences: (a) "I am writing" (b) "I am lying". Their grammatical structure seems to be the same, but the logical behaviour of the words "writing" and "lying" is different. "Writing" is purely a factual or descriptive word, but "lying" is essentially an assessment of the assertions of certain man: it is used to pass a certain judgment. The words "true" and "false" are likewise "assessments" used in passing judgment upon whether such-and-such is the case or not. And hence to use them significantly there must be something to assess or to be judged true or false. This something to be judged as true or false is an assertion expressing some reference or state of affairs. When I utter the sentence "I am uttering a sentence" then it is true that I am uttering a sentence. The sentence "I am uttering a sentence" asserts a state of affairs which actually exists. But in the case of "This sentence is false" or "I am lying", the relevant reference is lacking and unless this vacuum is filled by asserting a certain state of affairs, we cannot say of these sentences whether they are true or false. In the first case, there is an answer to the question "What is it that I am uttering?"; but in the second case, there is no answer to the question "What state of affairs is being falsely asserted?", or "What lie is it that I am telling?".

The adverse encroachment of grammar on thought is as old as the origin of thought. Many wrong philosophical arguments and speculations have their roots in grammar. Two or more sentences may have similar grammatical structure but widely different logical forms. "This man has four pounds" and "This chair has four legs" have similar grammatical structures which may beguile us into thinking that their logical behaviour is of the same kind. But in fact, the sort of relation which pounds have to the man is of a quite different nature from the relation which legs bear to the chair. The distinction of substance (substratum) and property in philosophical literature seems to derive from the misleading effect of grammar. The sentence "The sentence written on the blackboard is an English sentence" and the sentence "The sentence written on the blackboard is false" have grammatical similarities, but it is misleading to assume that their logical behaviour is also the same. The problem involved in the Liar paradox is to get at the logical behaviour of the words "true" and "false" and to get rid of mistakes created or suggested by grammar. Our account of the Liar paradox in terms of the Barber structure assists us in bringing out the defect with which the paradox is infected. To recapitulate, we argued that we have a certain statement (call it the Liar statement) which truly asserts the state of affairs asserted by all and only those statements which do not truly assert the states of affairs asserted by themselves. Now, we cannot raise the question whether or not the Liar statement truly asserts the state of affairs asserted by itself. The question is illegitimate because no state of affairs is asserted by the Liar statement itself. This also brings out our point that we can legitimately raise the question of the truth or falsity of the Liar statement (e.g., "This sentence is false") only when it refers to some other statement which ultimately describes a certain state of affairs.

§2. The problem of reference and significance

The problem of reference in the Liar paradox can perhaps be better appreciated if we compare it with other sentences like "The present king of France is bald". This sentence, as pointed out by Strawson, cannot reasonably be regarded as either true or false.* For there exists no present king of

* See P.F. Strawson, Introduction to Logical Theory, pp.184-192.

rance, and hence the question of the sentence's truth or falsity does not rise. But there are stronger grounds for denying that the expression of he Liar paradox, for instance, "I am lying", can be regarded as true or alse. The sentence "The present king of France is bald" expresses a certain ossible state of affairs, even though there is no present king of France - he state of affairs that someone referred to by "the present king of France" s bald. But in the case of "I am lying" there is no expression of any state f affairs. In order to judge any sentence to be true or false, the first and oremost requirement is that the sentence must express a certain state of ffairs. This condition is necessary though not sufficient. The second equirement for such a sentence to be judged true or false is that if it ontains referring expressions then there must exist a subject about which omething is asserted by the sentence. In the case of "The present king of rance is bald" there exists no person corresponding to the present king of rance about whom it is said that he is bald. The question of truthfulness rises only when granted the supposition that there exists one and only one resent king of France. Both the sentences above are significant and intel-.gible. But when we raise the question of their being true or false, con-.dering these sentences as they are and without reference to other relevant entences, the Liar sentence becomes insignificant. It is then devoid of roper significance because it lacks the proper background or expressions to scribe any state of affairs. As regards the second sentence, if we know ere is no present king of France, we cannot then significantly raise the estion of its being true or false. We understand the expressions in certain ntexts and the expressions gain significance and relevance with the provision relevant contexts. Unless these contextual requirements are satisfied, e whole expression may become insignificant though we may understand the dividual words involved in it.

The significance of a sentence depends on the context. Russell ably ings home this point in connection with his theory of descriptions, although did not pursue our line of argument. He says:

> The central point of the theory of descriptions was that a phrase may contribute to the meaning of a sentence without having any meaning at all in isolation. Of this, in the case of descriptions there is precise proof: If 'the author of Waverley' meant anything other than 'Scott', 'Scott is the author of Waverley' would be false, which it is not. If 'the author of Waverley' meant 'Scott', 'Scott is the author of Waverley' would be a tautology, which it is not. Therefore, 'the author of Waverley' means neither 'Scott' nor anything else - i.e. 'the author of Waverley' means nothing, Q.E.D.*

above quotation is given just to illustrate how the phrase "the author of verley" attains proper significance only in the context of the whole sentence. ay utter any sort of noise or scribble down any sort of dots, say "☓"; they nify nothing. But when I say that they are examples of such-and-such a sort noise or dot, they attain significance. Similarly, to take the extreme mples, "The law of excluded middle is green" or "The square root of an phant is a cow" would be regarded as absurd, but in certain contexts, for mple where the speaker is ignorant about what is the law of excluded middle, may have significance and then we may tell him that he is mistaken about law of excluded middle so that his utterance cannot be regarded as true or n significant when viewed in a different context. A similar situation may se with the assertion that prime numbers are heavier than natural numbers. tences sometimes gain significance when viewed in a metaphorical or poetic text. With an appropriate background or context the sentences gain or lose

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3. Russell, My Philosophical Development, p.85.

aningfulness or significance. If a speaker is aware that x has no children, en the speaker's assertion that x's children are asleep is absurd. But if e speaker does not know whether x has or does not have children, then we can asonably say that his assertion is false.

There can be no hard and fast rules for determining whether a sentence meaningful. Significance depends on contextual matters, and as the number possible contexts is uncountable and extensible we cannot make strict rules significance. In our everyday language we use many words to express the aninglessness of expressions - like "absurd", "insignificant", "nonsensical", enseless", etc. They exhibit different shades, different nuances or degrees meaninglessness with respect to our contextual requirements. It is also portant to note that the logic behind all contextual situations (and hence our language) cannot be formalised. A formal system has its own limitations. is delimited with rigid, inflexible rules which do not apply to every tuation. As contextual situations are numerous and can spring up unexpec- ily, the logical or deductive rules applying to them are equally innumerable l indeterminate. Our common natural language is able to express newly ergent situations. It follows that there cannot be a formal system to vern all the deductive rules of ordinary natural language. Of course, we for our own limited purposes demarcate sentences having a certain degree significance. We may concentrate our attention only on those sentences out which we could significantly talk of their truth or falsity - the realm ere true and false are contradictory and hence where the law of excluded ddle does apply. This limitation, in fact, we have imposed upon ourselves le discussing the Liar paradox. We may consider sentences where true and se are not contradictory and the concept of "false" extends far wider than that of truth - where the negation of negation (i.e., falsity of falsity) does not imply the truth. It is quite useful and interesting to investigate for these purposes the problem of negation. In certain other cases a sentence can be significant without implying that it has any truth-value (ie., truth or falsity as we generally understand these terms), like some interrogative, imperative, and even indicative sentences. Modal logic and deontic logic are attempts to understand the logic behind concepts and sentences to which we do not ordinarily apply the terms "true" or "false".

In some cases phrases signify like demonstrative pronouns. "It is false", "It is true", "It is condemnable", "It is commendable", "It is provable", "It is unprovable", etc. are intelligible sentences. But their significance lies in that they prepare us for something whose truth-value or provability, etc. is in question. But if we try to understand the above phrases in another sense, where they do not act demonstratively and refer solely to themselves, they would appear absurd, insignificant and devoid of content. They are significant when they are appreciated and understood in their proper role and function.

We have not explained what sort of sentence is supposed to be assessed as true or false; i.e., what sort of reference is lacking in the Liar expressions like "This sentence is false". The sentence to be assessed must express either that certain thing, say that a round golden table exists or does not exist or that a certain law, relation, connection, or claim holds good or not. Even a hypothetical or conditional statement can be judged true or false. It may assert that a certain causal nexus holds good or not. There is a limit to our ability to explain a thing and perhaps we cannot do more than use the above phrases to explain our point. To explain further we might be simply using circuitous and unnecessary phrases.

Concluding remarks

Our resolution of the paradox by first providing a proper reference and then raising the question of truth or falsity is adequate to remove the Barber structure and hence the contradiction involved in it. This is so because the Liar sentence, which attributes a truth-value to itself, is devoid of a proper reference and unless a proper reference is provided we cannot raise the question of its truth or falsity. Hence we cannot legitimately ask whether or not the Liar sentence is true. Our solution not only removes the contradiction but also explains and clarifies the notions of truth and falsity involved in the Liar paradox. Hence our solution satisfies the requirements for the proper resolution of a paradox. It may be objected that in the version cited earlier there was a reference, but still we got the contradiction. The example was that sentence (1) in the rectangle leads to

> (1) Whatever is written in this rectangle is not true
> (a) A square is a 3-sided figure
> (b) A polygon is a 4-sided figure
> (c) This page is green

contradiction if one is asked whether it is true or false, although ample reference is provided by the sentences (a), (b), (c). But the contradiction arises here because the conditions of the Barber structure are satisfied. If we write sentence (1) outside the rectangle and thus do not regard this sentence as falling into the domain of sentences inside the rectangle, we get no contradiction. Then we can reasonably say that sentence (1) is true because all the sentences inside the rectangle, namely (a), (b), (c), are, in fact, false. The sentence (1) has indirect reference not provided by (1) itself. If there had been another sentence, call it (1a) instead of (1), which has direct reference, that is reference provided by (1a) itself, i.e. the states of affairs asserted by (1a) itself, even then there would have arisen no contradiction because in that case we could not have the Barber structure. Because then we could not have the sentence (1a) which would truly assert the state of affairs provided by all the statements (i.e. in the rectangle) which do not truly assert the states of affairs provided by themselves. This is so, because <u>then</u> (1a) would have no direct reference. It may be noted that the Barber structure as such has nothing to do with the lack of reference as such. It just happens that in the Liar paradox, if we provide the reference, we eliminate the Barber structure as well, for the reason just mentioned in the case of (1a).

In conclusion, we cannot speak of statements of the Liar paradox as true or false because they lack reference, that is they do not express or assert any states of affairs, and as soon as we provide the reference for the Liar statement, the paradox disappears. Considered independently from external reference we may simply regard the Liar statement as neither true nor false, but as a sentence which can be used to assess as true or false any statement having proper reference, i.e. assertion of certain states of affairs. We have provided an adequate solution because it not only helps us to eliminate the self-contradictory Barber structure but also helps us to understand the logical behaviour of the words "true" and "false" and the Liar statement, and thereby leads us to a better understanding and appreciation of the concepts "true", "false", "reference", etc. It teaches us that in order to apply significantly the words "true" and "false", there must be some reference - a sentence describing a certain state of affairs, and that we should be careful not to shift our reference in the argument.

Chapter IV

DIAGONAL PARADOXES

In this chapter we shall discuss the Richard paradox along with ntor's diagonal argument. As the main theme of this chapter is to aborate the Richard paradox, which in its formal structure follows e Cantor diagonal, we have called the chapter "Diagonal Paradoxes". shall also discuss Cantor's general argument, Cantor's argument nested intervals to prove the indenumerability of the real numbers d the "Richardian" paradox (our name given to a paradox in order to stinguish it from the Richard paradox). For they have some essential atures in common with the Cantor diagonal and the Richard paradox, though Cantor's general argument and the Richardian paradox manifest e Barber structure and not the diagonal structure.

Introduction to Richard's paradox

. Richard's paradox and its importance

For the sake of clear exposition we present the paradox in chard's own words:

I am going to define a certain set of (ensemble) numbers which I shall call the set *E*, by means of the following considerations:

Let us write all the arrangements of the twenty-six letters of the French alphabet (taken) two by two, arranging these arrangements in alphabetical order; then all the arrangements three by three, ranged in alphabetical order; then those four by four, etc. These arrangements may contain the same letter repeated several times; they are arrangements with repetition.

Whatever whole number *p* may be, every arrangement of the twenty-six letters *p* by *p* will be found in this table, and as everything that can be written with a finite number of words is an arrangement of letters, everything that can be written will be found in the table of which we have just shown the manner of construction.

As numbers are defined by means of words, and the latter by means of letters, some of these arrangements will be definitions of numbers. Let us cancel from our arrangements all those which are not definitions of numbers.

Let u_1 be the first number defined by an arrangement, u_2 the second, u_3 the third, etc.

There have thus been arranged in a determinate order all the numbers defined by means of a finite number of words.

Therefore: all the numbers that can be defined by means of a finite number of words form a denumerable set.

This now is where the contradiction lies. We can form a number which does not belong to this set.

'Let *p* be the *n*-th decimal of the *n*-th number of the set *E*; let us form a number having zero for its integral part, *p*+1 for its *n*-th decimal if *p* is equal neither to eight nor to nine, and otherwise unity.'

This number *N* does not belong to the set *E*. If it was the *n*-th number of the set *E*, its *n*th figure would be the *n*th decimal figure of that number, which it is not.

I call *G* the group of letters in inverted commas.

The number *N* is defined by the words of the group *G*, *i.e.* by a finite number of words; it ought therefore to belong to the set *E*. But we have seen that it does not belong. That is the contradiction.*

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* J. Richard, 'Les Principes des Mathématiques et Le Problème des ensembles', Revue générale des Sciences, (1905), p.541 (as translated by Ivo Thomas in I.M. Bochénski, A History of Formal Logic, p.390).

After presenting the above paradox, Richard goes on to discuss its resolution. He says:

> Let us show that this contradiction is only apparent. Let us return to our arrangements. The group of letters *G* is one of these arrangements. It will exist in my table but at the place which it occupies, it has no meaning. There it is a question of the set *E* and this set is not yet defined. Therefore I ought to cancel it. The group *G* has meaning only if the set *E* is totally defined and this can come about only by an infinite number of words. Hence there is no contradiction.

It seems clear from above that Richard seems to resolve the contradiction by denying the existence and constructibility of the 'diagonal' number; i.e., the number which was proposed to be constructed by the instructions embodied in the group of letters *G*. If we accept Richard's solution we have to reject Cantor's diagonal proof of the indenumerability of the set of real numbers.* In fact, before presenting the actual paradox, Richard claims that contradictions arise from the supposition of an indenumerable set:

> It is not necessary to go to the theory of ordinal numbers in order to find such contradictions. Here is one which presents itself in the study of continuum.

This important aspect of the Richard paradox may be summed up by saying that it concerns the definability of real numbers. We suppose, or think that we have some reason to believe, that the set of real numbers is indenumerable. But the real numbers are definable in words or sentences which constitute a denumerable set. Hence we conclude that the set of all real numbers is denumerable, thus contradicting

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* Subsequently Richard regarded the 'diagonal' number as valid. We shall discuss this point at a later stage.

the previous assumption about the indenumerability of real numbers. This is the general argument which may be put forward. For his particular or definite argument, he exploited Cantor's diagonal argument. He formulated the paradox following Cantor's diagonal argument and tried to disentangle the riddle by rejecting the diagonal derivative real number. Let us therefore turn to Cantor's diagonal argument in order to elucidate Richard's presentation and resolution of the paradox.

§2. Cantor's diagonal and the Richard paradox

Cantor's diagonal to prove the indenumerability of real numbers may be stated as follows: Let us consider the real numbers between 0 and 1 (excluding 0, including 1) taking each number expressed as an infinite non-terminating decimal. For instance, a real number 0.5 (i.e. 1/2) may be expressed as 0.49999..... Let us start enumerating these real numbers according to some enumerating method. Let the first few numbers run as follows:

0.214322...............

0.134264...............

0.215432...............

0.321653...............

0...........

Now, according to our supposition of the denumerability of the set of all the real numbers, every number in the interval $0 < x \leq 1$ must have a place at some stage in this list of numbers. But we can construct a number which cannot occur in this list. We construct the new real number by inserting in the first decimal place a digit 3 (or in fact any digit different from 2 - the digit 2 occurring at the first decimal place of the first number of the list in our example), in the second decimal place 2 (or again in fact any digit different from 3 - the digit 3 occurring at the second decimal place of the second number of the list), in the third decimal place again 3 (or in fact any digit - different from 5 - the digit 5 occurring at the third decimal place of the third number of the list in the above example), in the *n*th decimal place again 3 unless there is a digit 3 at the *n*th decimal place of the *n*th number in the list and in that case 2 (or in fact any digit - different from the digit occurring at the *n*th decimal place of the *n*th number in the list). Now, this newly-constructed diagonal-derivative number cannot occur in our proposed enumeration or list because it must differ from every number in the list. The list was arranged under the assumption that we have a rule or rules whereby to enumerate all the real numbers between $0<x\leqslant 1$. But this so-called diagonal derivative number, though lying between 0 and 1, cannot occur in the list. Thus we conclude that the proposed rules were insufficient or incapable of enumerating all the real numbers in question because the diagonal derivative number cannot occur in the list. We may then start with some different rules so as to include this newly-constructed diagonal derivative real number in the list. But once again we may construct another diagonal derivative number which cannot occur in the new list. Hence no procedure can serve to enumerate all the real numbers in the interval $0<x\leqslant 1$, and so we have demonstrated the indenumerability of the set of all real numbers between 0 and 1.

By this method we get a diagonal derivative number which cannot occur in the supposed denumerable list. If our denumerable list contains all the rational numbers, though expressed in decimals, then our diagonal derivative must be an irrational number. As the diagonal number purports to be a real number, no denumerable list can be claimed to exhaust all the real numbers.

Now, here comes Richard's onslaught! Consider a list of all possible arrangements formed of letters from the French alphabet. This alphabet is made up of 26 letters and we arrange them in a certain alphabetical order. Sentences and words consist of letters and hence must occur among these arrangements. Now every real number can be expressed in words: let every such number be expressed or defined by some sentence. And hence all verbal definitions of real numbers are included in our list of arrangements of French letters. Hence the set of all verbal definitions of all real numbers is denumerable. Thus the set of all real numbers is denumerable. This contradicts the conclusion of Cantor's diagonal argument. Before examining the underlying assumption about the denumerability of the set of all possible arrangements of the French letters, which is central to Richard's argument, we shall first look at Richard's paradox more closely. That is, we shall assess and examine the validity of the diagonal argument and see whether we can actually define a definite diagonal derivative number or not.

B. Validity of Cantor's diagonal argument

In general there are two main arguments against the Cantor diagonal, namely: (a) that the diagonal number is not a proper number; and (b) that even if we regard it as a proper number, its nature must be regarded as different from that of the listed numbers and thus the argument as such does not work. We do not discuss these two arguments completely at one place because they involve for their elucidation and explanation other arguments and considerations.

The first objection (a) can equally be levelled against Cantor's other argument by nested intervals, discussed in subsection 4 below. The second objection (b) also holds good for Cantor's general argument, discussed in the next section, to prove the Cantor theorem. These objections therefore will also be discussed along with the respective arguments. In the present section, the second objection is discussed in the next sub-section and the first objection in sub-sections 2, 3 and 4.

§1. Assessment of the argument

The diagonal argument has an air of plausibility but it is in fact unsatisfactory. The given diagonal rule to produce a diagonal number can define a definite number - a definite number in the proper sense - only under certain conditions. The diagonal procedure obviously defines a definite number in the case where the list consists of a finite number of members (i.e. decimal numbers). The rule, call it a "diagonal rule", justifiably applicable if the list is finite, but the applicability of is rule to an infinite list is open to question. In one sense there erges a diagonal number even if the list is infinite; namely, in the nse that we can tell what digit there would be at the nth decimal place r any n. We just have to look at the nth digit of the nth decimal number the list and determine the required digit. This feature may be regarded one of the essential requirements for determining any decimal number. We st have some definite method or formula to find out what digit is to placed at the nth decimal place. We may construct a decimal number throwing a diè; if at the nth throw we get the digit k, we put the git k at the nth decimal place of the designed number. But in this cimal number the digit at the nth decimal place depends upon chance-ctors or factors not completely comprehensible by us. What is required r a definite number is that the digits at any nth decimal place be termined by some rule, as e.g. are the digits at the nth decimal ace of say π or 1/3. Our diagonal number does satisfy this particular ndition because we have a list determined by definite rules of enumera-on to determine the digits at the nth decimal place for any n of the agonal number. But we shall show in the next subsection that this aracteristic, though necessary, is not sufficient in order to have a per number.

For the sake of argument we may agree for the time being that we e the diagonal number. We can show, however, that the number charac-rized and determined by the diagonal procedure does not demonstrate what tor wanted to demonstrate. We would like to argue that the diagonal ber must be regarded as a different sort of number from the numbers the list. The diagonal number has some special features which are ferent from those of the real numbers in the initially planned list. existence of a diagonal number is dependent upon the existence of a tain infinite list; it cannot be constructed independent of this list.* the case where the given list is finite, the diagonal derivative can constructed independent of the list which initially generated it. the Cantor diagonal number is inextricably bound up with an infinite t. The infinite list is responsible for the emergence and existence the diagonal number - no list, no number. That is, we cannot describe determine the decimal digits except through the infinite list. The damental specification of this connection between the Cantor diagonal ber and the infinite list is that the digit at any nth decimal place the diagonal number must be different from the digit at the nth decimal ce of the nth number of the list. This specification is the defining perty of the Cantor diagonal and not just an accidental feature adjoined a definite and independent number. Hence Cantor's diagonal number cannot placed in the designed list. But from these facts it also follows that the very nature of such a number it can only be constructed in terms of the numbers in the list. The numbers in the list are to be in the st place "independent" numbers; that is, they do not depend for their stence upon any list of numbers, and their decimal digits are determined fixed by rules independent of the infinite list. We must start to nerate in the list only such numbers, for we cannot put dependent numbers

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Ve have already talked about dependence and independence in connection with the heterological paradox in Chapter II.

(e.g. the diagonal number which depends for its very construction upon the infinite list) in the list unless we first have a list of independent numbers. To specify a dependent number, like the diagonal number, we have to resort to a certain infinite list. So, we start to enumerate in the list only independent numbers. That is, those which do not depend for their existence upon any infinite list. Later on we are asked to include in the list the diagonal number which is of a different nature and status.

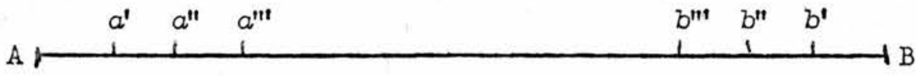
The situation may be compared with the following example. If we list numbers by a formula which permits only rational square numbers to occur in the list, it is unjustifiable to complain that there are no rational prime numbers in the list. So, when we create a diagonal number which is bound to be different in nature from the listed numbers, it is illegitimate to insist that it should occur in the list. This by no means follows from the argument that the list of all real numbers is indenumerable because the real numbers which we had initially in view were of a different kind from that of Cantor's diagonal numbers. If we insist that genuine real numbers be independent numbers and not dependent numbers, the answer is very simple: the diagonal number is not a real number and hence its non-occurrence in the list by no means proves the indenumerability of the real numbers. Suppose now that diagonal (or dependent) numbers are also real numbers; then we must formulate our question unambiguously. We must say (a) whether independent real numbers are denumerable; (b) whether the diagonal numbers (constructed from the list of independent real numbers, i.e. of type (a)) are denumerable; (c) whether the diagonal numbers (constructed out of the diagonal numbers of type (b)) are denumerable, and so on. Ultimately the question arises whether the independent numbers are denumerable or not. We may say:

ive us an infinite list of independent numbers and we shall let you know he rule whereby to enumerate all the diagonal numbers constructed out of hat list. It is only through not making the distinction between dependent nd independent numbers that Cantor's diagonal argument can carry any eight. Further discussion of this point is resumed in Section C on the eneral diagonal method.

2. The nature of real numbers

The question naturally emerges from our distinction between the diagonal umber and real numbers, as to what is the nature of a real number and what ort of definition is needed for it. A real number may be regarded as a non-erminating decimal such that, as mentioned earlier, there is a certain rule hereby to determine the digit at the nth decimal place. There is another equirement which we explain by the following analogy. Let the points on straight line AB (A C ———— B) represent all real numbers, say be-ween 0 and 1. To every point there is then a definite corresponding real umber. Suppose the point at C represents the real number 1/3. Accordingly, is number may be expressed by an infinite decimal 0.33333...... Now, $\dot{3}$ (or 0.333...) is another way of saying 3/10 + 3/100 + 3/1000 + 3/10000 + ... Similarly, if we go on adding indefinitely we cannot get any number ich is less than or greater than 1/3. We say that the series 3/10 + /100 + 3/1000 + ... converges to a limit 1/3. We may also express e series as the sequence 3/10, 33/100, 333/1000, ... and so on, and y that this ascending sequence has a limit 1/3. Since there is an infinite mber of points (or real numbers) between any two points (or real numbers) locate any definite point (and so, real number) it is sufficient to indicate that such-and-such a series or sequence has a limit - the limit being the point (or number). The symbols 1/2 and 0.5 and 0.49999...... represent the same number - i.e., that which corresponds to the point which lies in the middle of line AB. Given the above series, we cannot have any lesser or greater number than the limit. For if we choose the greater, then our series cannot reach the point (number). In other words there is a lesser number or nearer point which the series cannot exceed. This indicates that our chosen point or number cannot be represented by the series. Whereas, if we choose the lesser, then sooner or later by developing further the series or sequence, we shall reach a number greater than the chosen one, indicating again that our chosen number cannot be represented by the series. This shows that in order to have a definite point (number), we must have a convergent series (or sequence) with a limit. For instance, a series 3/10 + 3/100 + 3/1000 + ...(i.e. $0.\dot{3}$) is a convergent series, because however many terms we add in this series, the total number will never grow beyond a certain fixed number or point. Here we may say roughly that this series cannot grow beyond say 1/2. At the same time, this series has a limit in the sense that this series cannot represent a number lesser or greater than 1/3: it cannot grow beyond 1/3 and it tends towards reaching 1/3.

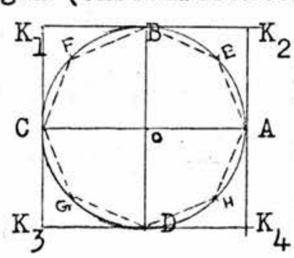
But it is important to note that it may also happen that we have a bounded series which may not converge to any limit. This situation may arise in the following case. Let there be a straight line AB consisting of an infinite number of points (representing real numbers). We may choose any two points (numbers), say $a'b'$, lying inside the points A and B, And again we choose two points (numbers) $a''b''$ inside $a'b'$. We go on repeating this process. We thereby create an infinite number of nested intervals, each including the earlier ones. Obviously the sequences a', a'', a''', ... and b', b'', b''' ..... are bounded, for evidently the sequence



a', a'', a''', ... cannot go beyond say B (taking the extreme point, or in fact any determinate b^n), and the sequence b', b'', b''', ... cannot go beyond say A or any determinate a^n, say a'''. It is also obvious that if they converge to a limit then they both can have only one limit, for if they have two limiting points, then we could always create a shorter interval $a^k b^k$, which would lie between these two points. But it is also conceivable that they (both sequences) may not have any limiting point. As there is always an infinite distance in the sense of an infinite number of points between any $a^{\overbrace{1....1}^{k}}$ and $b^{\overbrace{1...1}^{k}}$, it may happen that our consideration of a sequence at a certain stage would give us the impression that it is converging to the limiting point c. But at a later stage, it may give the impression that it converges to another point say c', and at some further stage to c" and so on. This situation may arise where there is no definite and fixed rule by which we choose the sequences a', a'', a''', ..., and b', b'', b''', ... But it would of course be impossible where we have such a rule, i.e., where we know the rate at which the sequences seem to converge. In the case of a sequence like 3/10, 3/100, 3/1000, ... we have a certain rule of convergence, and so we know the rate at which the number is increasing so as to approach a definite point or number. We know that in this number

(0.$\dot{3}$) a digit 3 must necessarily occur at every decimal place: it betrays the rate of convergence, or boundedness too. This rule is indissolubly linked with the number concerned. What rational number is to occur say at the nth member of the convergent sequence is determined by a rule which is also applicable to ascertain the numbers which occur earlier in the sequence. The rule in question for getting a particular nth member of the sequence is applied with the aid of previous members of the sequence. A limiting convergent series 1/2 + 1/4 + 1/8 + 1/16 + ... represents the number 1. To obtain the nth member of the series we have simply to multiply by 2 the denumerator of the n-1th member of the series. But if such a rule of boundedness is lacking we have no guarantee that a bounded convergent series or sequence has a limit.

For further elucidation we resort to the following example. Take as an example, the number π, which equals half the circumference of a circle of a radius 1. How do we determine such a length? Let us draw a figure. We have a circle ABCD. CA and BD are diagonals at right angles meeting at the centre O. Obviously the length (circumference) of the circle lies between the total length of the sides of the bigger square $K_1K_2K_3K_4$ and the total length of the sides of the smaller square ABCD. The total length of the sides of the square ABCD is obviously less than the circumference of the circle. Let us divide the arcs (AB, BC, CD, DA) at E, F, G, H and make an 8-sided polygon. The total length of the sides of this polygon is nearer to the length of the circle than the total length of the sides of the



quare ABCD. Again, let us divide the smaller arcs (AE, EB, BE,......) nd construct a 16-sided polygon. The total length of the sides of this olygon is even still nearer to the circumference of the circle than is hat of an 8-sided polygon. We go on constructing polygons of more and ore sides (of 32, 64,... etc.) and we notice the total length of the sides ets closer and closer, though it never becomes equal to the circumference f the circle. The total length of the sides of any such polygon cannot xceed the circumference of the circle; hence the sequence consisting of he total lengths of the sides of the successive polygons forms a convergent sequence. But this sequence must also have a limit and that limit is ne circumference of the circle because we can always make a polygon of a reater number of sides approaching though never exceeding the circumference f the circle. Of course we may have a rule how this convergent sequence rows, for we can easily observe the sides of the polygon growing in a egular fashion, namely 4, 8, 16, 32, 64, ... showing thereby that we ave a limit which approaches at most to 2π. But even if we have increased ne number of sides arbitrarily, sometimes trebling, sometimes adding say 00 sides to the previous polygon, we shall have a limit because we have .ready a limit of the circumference of a circle. Suppose we have a line 3 of one unit length consisting of points representing all real numbers etween 0 and 1. If we want to create a convergent series representing ne point B (i.e. number 1), we may simply divide the line AB at the middle oint c (thus representing A c c' c" B umber 1/2), again dividing cB at c' and again sub-dividing at c" and so on. is series (1/2 + 1/4 + 1/8 + 1/16 + ...) must converge to the point

B, thus representing the number 1. In this case we already have a limiting point at B, and thus any infinite convergent series which cannot limit any point before B is sufficient to determine the number. In short, the bounded series or sequence denotes a definite number only when it indicates that it has a certain definite limit.

§3. Cantor's diagonal number

Now, reverting to the Cantor diagonal number, we notice that it is associated with a bounded sequence. But a simple bounded sequence may be generated by just throwing a die and putting the digit that appears on the die at a decimal place corresponding to the number of the throw. In fact any decimal number is a bounded or convergent sequence of rational numbers. For example the decimal number $0.\dot{3}$ is the convergent sequence of 3/10, 33/100, 333/1000,..... But it may happen that an infinite bounded sequence or series has no limit. And in this respect we find little difference between the diagonal number and the number generated by throwing a die. We find a definite diagonal number only if we have a limit. That is, Cantor should have proved that by his diagonal method we always get a limit. We shall construct an example in order to emphasize our point.

We know that the set of all rational numbers is denumerable. We show that a rational number of the same nature as the diagonal one may be generated which cannot occur in the list of rational numbers. Let us enumerate the positive rational numbers between 0 and 1. Let us construct or define a number which lies, say between 1/5 and 1/1. We start constructing the new number by the following instructions or rules:

	A		B
1	1/1		1/2
2	1/2	$\frac{+1}{+2+1}$	2/5
3	1/3	$\frac{+2}{+5+3}$	3/11
4	2/3	$\frac{+3}{+11+3+2}$	5/19
5	1/4		6/29
6	1/5		9/41
7	3/4		12/59

(a) Change the denominator of the first number of our list so as to make it the next prime number. In our example it would turn out to be 1/2. Obviously this resultant number 1/2 would be different from the one upon which it is constructed, i.e., here from 1/1.

(b) Add the numerator and the denominator of the resultant number and the next number of the list A. In our example it would be 1+1/2+2 = 2/4. Then put the next prime number as denominator. Here it becomes 2/5.

(c) See whether the resultant number is less than 1/5; if it is, then add to the numerator some number to make it greater than 1/5. Ensure also that each resultant number is less than the previous one. If it is not, then by selecting the next prime number as denominator and adding or subtracting some digits from numerator, make it so. In our example we have done this, making the resultant numbers 9/41 and 12/59. We have shown the resultant numbers in list B.

So, we have an infinite bounded sequence of resultant rational numbers, just as in Cantor's diagonal we have a bounded sequence of, in the example cited, 3/10, 32/100, 323/100, ... But in fact we have a series of resultant numbers and not just one definite number; just as in the diagonal procedure we have a series of real numbers (in fact rationals) differing from the numbers occurring previously in the proposed list, and not one definite real number. One purpose of this illustration is to make ourselves immune from any sort of deception with regard to the decimal notation. We may in fact translate our example into decimal notation. In order to show that there is a definite number, we have to prove that by such procedure the bounded series does tend to a limit. Unless we have a limit, how can we possibly assert the existence of a definite number? We may take a decimal whose initial few decimal-places are determined by the following digits: 0.2387....... We may create nested intervals from it as follows: The first interval lies between 1/10 and 3/10, the second between 22/100 and 24/100, the third between 237/1000 and 239/1000 and so on. But this sequence does not give any clue as to the limit. In the case of our example of the number π the sequence had a limit. In fact in that example we already had a limit and we were only trying to express this limit by the lengths of the sides of successive polygons. But in the present case, though the intervals get closer and closer we do not know whether we have a definite limit. Because there are an infinite number of numbers in any interval it can be any one of the infinite numbers represented by the bounded sequence, and hence we have no definite number. Exactly the same is true in the case of the diagonal number.

§4. Cantor's proof by nested intervals

There is another proof of the indenumerability of real numbers, due also to Cantor, which involves nested intervals and commits the error mentioned above.* The proof is as follows: Suppose we have an infinite series I of real numbers given according to a certain rule. Cantor tries to prove that in every interval of the series there is a number which cannot occur in the series. We choose an arbitrary interval $[\alpha,\beta]$ from the series I and let $\alpha<\beta$. We then choose another interval $[\alpha',\beta']$, such that it lies within the previous interval $[\alpha,\beta]$. The numbers α',β' being chosen as the first x and y to occur in I, such that $\alpha\leqslant x$ and $y<\beta$. Likewise we create other intervals $[\alpha'',\beta'']$, $[\alpha''',\beta''']$, ..., occurring within the previous intervals, where $\alpha^{\overbrace{|\cdots|}^{n}}$ and $\beta^{\overbrace{|\cdots|}^{n}}$ are similarly the first x and y to occur in I such that $\alpha^{\overbrace{|\cdots|}^{n-1}}\leqslant x$ and $y<\beta^{\overbrace{|\cdots|}^{n-1}}$. Now either the number of intervals is infinite or finite. If finite, then we can obviously choose a number which lies within the last interval but does not occur in the list, otherwise it would make a further interval contrary to our supposition. If the number of intervals is infinite then Cantor says:

> ..... the numbers α,α',α'' ...... have a certain limit $\overset{\infty}{\alpha}$, because they are always increasing in value without becoming infinite; the same holds for the numbers β,β',β'', ..., because they are constantly decreasing, let their limit be $\overset{\infty}{\beta}$.*

Now either the two limits are the same, i.e. $\overset{\infty}{\alpha} = \overset{\infty}{\beta}$; or they are different, and so they cannot occur in the list I. For the limit is

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* G. Cantor 'Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen' in Journal für die reine und angewandte Mathematik, vol. 77, pp.260-1 (or in Gesammelte Abhandlungen, edited by E. Zermelo, pp.115-8).

supposedly the last component of the sequence and yet cannot be one of the components of the sequence.

Our objection is that the infinite sequence $\alpha, \alpha', \alpha''$, ... and β, β', β'', ... may not converge to any limit unless we are given a definite rule of convergence, and hence the derivation of any definite number from the above sequences is illegitimate. It is evident in the above proof that it is assumed that every bounded monotone increasing or decreasing sequence has a limit. No proof is provided: it is just assumed. We tried to show in the last subsection that it might not always be the case. If we take it as a hypothesis for the foundations of mathematics, then we have to forego the claim to base mathematics on logic or on intuition, because this assumption is not intuitively obvious, neither is it derivable from any logical or intuitive principles. On the other hand our analysis given above runs counter to such an assumption because, as we remarked, we cannot locate any definite point on a line simply by a bounded sequence or series: we need a certain definite rule to measure the rate of boundedness or convergence to locate a definite number or point.

C. The diagonal fallacy

§1. General diagonal method

Let us examine now the general diagonal procedure, which is also supposed to offer us a proof for the indenumerability of real numbers. The general diagonal procedure is applied when we have a set or sets of ertain things, say of sets or of numbers, and we construct a new thing - new set or number as the case may be - which cannot occur in the original et, or cannot be correlated with a member of another set with which it as assumed that it could be correlated. We exhibit this method by onsidering Cantor's assertion that the set $\mathcal{U}S$ of all subsets of set S as a higher cardinality than its set. This assertion is called by ne name Cantor's theorem. To prove the indenumerability of real numbers, t is sufficient to prove that the set $\mathcal{U}S$ of all sets of positive integers s not denumerable. Hence from the Cantor theorem, indenumerability of ne set of all real numbers follows.

The theorem is proved in the following way. Suppose S and $\mathcal{U}S$ re equinumerous. That is, we suppose there exists a scheme for mapping embers of $\mathcal{U}S$ with the members of S in one-one correlation. Now as the embers of $\mathcal{U}S$ are all subsets of S, we can raise the question whether r not any members of S is a member of its correlate - its correlate being ne correlated member of $\mathcal{U}S$. Let us construct a set, say M, that would ontain only and all those members of S which are not members of their orrelates. Now this set M should be correlated with some particular ember, say m, of S, because of our supposition of correlation between and $\mathcal{U}S$. Now, as m is supposed to be correlated with M, we raise the uestion whether m is a member of M or not. If it is a member of M, nen according to the definition of M, m cannot be correlated with M, nich in fact it is by our supposition. If on the other hand, m is not

. member of M, then according to the definition of M, *m* should be a member
f M because *m* is not a member of its correlate. So we arrive at a contra-
iction. Hence our supposition that S and $\mathcal{V}S$ are equinumerous is false.
f we try to correlate $\mathcal{V}S$ with S by some other scheme, we can always like-
ise construct a set M, and the equivalence in cardinality between $\mathcal{V}S$ and S
s bound to break down. From this the conclusion follows that, since the
ardinality of $\mathcal{V}S$ cannot be less than S and is proved non-equal to S,
herefore the cardinality of $\mathcal{V}S$ must be greater than S.

Let us examine the above general diagonal argument. We shall try
o demonstrate that it embodies an illegitimate inference. The nature
f the constructed set M is quite different in kind from the members of
S, initially laid down for correlation. The members of $\mathcal{V}S$ to be cor-
elated with the members of S can be defined independently of any member
f S, that is independently of any consideration of correlation between
S and S. So, once we get a member of $\mathcal{V}S$, it could be described or
efined independently of any consideration of S. But the constructed
et M cannot be described or defined on its own: it depends upon those
embers of S, which are not members of their own correlates - for in
rder to determine M we have to determine how the members of S are
orrelated. In case S is a finite set, the situation is very simple.
hen we can always determine M and can then express it independently as
ny other member of $\mathcal{V}S$. Suppose $S=\{1,2\}$ and let the correlation with
he set $\mathcal{V}S$ be: 1 —— $\{1\}$, 2 —— $\{1,2\}$; we have then the null set,
hich is a subset of S but which does not contain any member of S, which

can be a member of their correlate, and we can easily conclude that $\mathcal{U}S$ must have greater cardinality than S. But if S is infinite, we can determine M as independently describable only if at some stage of our correlation we get a certain rule by which we can then tell which members of S are the members of M without resorting to further examination of the correlation. If this is not obtained then our raising the question whether or not *m* can be correlated with M is illegitimate.*

The impossibility of a correlation follows simply from the process of defining the set M. To simplify this point we resort to the following analogy: Let us suppose we have two entities a and A, a being a particular positive integer and A being a subset of the set of all positive integers. We further suppose, as is obvious in this particular case, that it is justified to raise the question whether a is a member of A or not. Now, given any two entities we can obviously correlate them one to one. Now, if we define the set A such that its correlation with a becomes impossible, the legitimate conclusion would be that such a definition of A is itself illegitimate. Let us suppose that the defining characteristic of the subset A is that it contains only and all those positive integers which are not members of their correlated sets. If we now correlate one to one a to A, and raise the question whether or not a is a member of A, we meet a contradiction. This shows that something has gone wrong with our

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* Actually we are repeating the same sort of argument as advanced when dealing with the Cantor diagonal number earlier.

definition of the set A, because it seems quite legitimate to correlate a to A and to raise the question whether or not a is a member of A. The definition of the set A reminds us of the description of the barber in the Barber paradox. We then said that there cannot be a barber who could shave all and only those men who do not shave themselves. Once we grant the barber to be a man, our definition of the barber is self-contradictory and so illegitimate for defining a barber. As no self-contradictory thing can exist, the barber described cannot exist. A similar situation appears in the case of the set A and the set M above. We tried to define the set A (and also the set M and in fact the diagonal number too) so that its correlation (and in the case of Cantor's diagonal argument, the occurrence of the diagonal number in the enumerating list) becomes impossible. If we insist on the legitimacy of the set (and the diagonal numbers) we should also add that they must differ from other correlated sets (and numbers) just as the above barber must be regarded a different person from every man, if the definition of the barber is to be accepted. The moral we draw from this argument is that in order to have a correlation between any two things or sets, their definitions should be independent of the correlation. So the definition of the set M should not make it impossible that it could be correlated with any member of S. Hence Cantor's argument to prove the higher cardinality of the set of all subsets of S than S is not valid - the set M cannot be regarded as a member of $\mathcal{U}$S as other subsets of S would be, for the set M is not the sort of set, as others are, of which we could sensibly raise the question of correlation with any member of S. Our argument is in fact the same as it was in the case of Cantor's diagonal number. So, the set M should not be regarded as a properly defined set because of its dependent character. But even if we regard it as a set, the question of its correlation with a member of the set S is illegitimate, because by our very definition we make it impossible. We may call this way of defining an entity a <u>diagonal fallacy</u>. It occurs in defining Cantor's diagonal number and the Cantor diagonal set M, and in the Barber paradox as well.

§2. The Richardian paradox

Our point may perhaps be better appreciated if one considers the following paradox. This paradox is given by Nagel and Newman*, who inaccurately call it the Richard paradox. For the sake of clarity we shall call it the Richardian paradox - this name suggesting that there is some characteristic common to it and the Richard paradox.

Let us start by defining the numerical properties of cardinal numbers, e.g. a prime number may be defined as "not divisible by any integer other than 1 and itself". Each of these definitions contains a finite number of words. Let us place the definitions in serial order, using some definite rule like the rule that definitions containing fewer words should come first and definitions containing the same number of words are arranged in alphabetical order. On this basis a unique serial number will correspond to each definition. It may in certain cases happen that a serial number possesses the very property designated by the definition with which the integer is correlated. For example, if the defining expression of "prime number" happens

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E. Nagel & J. R. Newman, <u>Gödel's Proof</u>, pp.60-2.

to be correlated with the serial number 17, then 17 itself has the designated property. If the defining expression "product of some integer by itself"

Serial numbers	Properties of numbers
1	
2	
15	Property of being square
17	Property of being prime
n	Property of being Richardian

were to be correlated with serial number 15, 15 does not have the property designated. We shall say, then, 15 has the property of being Richardian, while 17 does not have the property of being Richardian. Now, let us consider the property of being Richardian. The defining expression for the property of being Richardian describes a certain sort of numerical property of integers. We take the expression itself as one of the series of definitions proposed. Suppose this expression is correlated with the serial number n. Now the paradoxical situation emerges when the question is raised whether or not n is Richardian. For n is Richardian if and only if n does not have the designated property, that is, if and only if it is not Richardian. And if it is not Richardian, then it must have the designated property, i.e., it must be Richardian. Hence, a contradiction results.

The heterological (or the Barber) structure of the Richardian paradox discussed above - that is, the structural pattern exhibited by the heterological paradox - has been mentioned in chapter II. First we have the statement: "The serial number n is correlated with the property of being a serial number which does not have the property correlated with it", and then we raise the question whether or not n has its correlated property. The resolution of the paradox is now clear. We can sensibly ask whether n has the correlated Richardian property only if n has some correlated property other than the Richardian property. The semantical extrication of the paradoxical question "Does N have the correlated property or not?" is the same as in the heterological question "Is 'heterological' heterological or not?", or "Is Impredicable impredicable or not?". We observed while discussing the heterological paradox that in order for a certain word to be heterological or not, it must have some property other than that of being heterological. The property of being heterological is in fact a property of certain other properties of the word and indeed it is dependent on other such properties, and hence unless there is some other such property by virtue of which the word can be regarded as heterological, we cannot sensibly talk of the word as heterological or not. In the same way the Richardian property is dependent on some other properties and unless we know that other property it is not legitimate to ask whether or not n is Richardian.

The general diagonal method also exemplifies the heterological or Barber pattern. It has the same structure as the Richardian paradox. Instead of a list of serial numbers, we have the members of the set S

f positive integers and instead of a list of properties of numbers we
ave the members of the power set $\mathcal{U}S$ of the set of positive integers.
nstead of the constructed Richardian property and its correlated serial
umber n, we have the constructed set M and its correlated member m
f S. In both cases an attempt to correlate n with the Richardian property
nd m with M leads to contradiction. Not only do these paradoxes have
he same structure, they seem to suffer from the same malady. The set
and the Richardian property are both constructed by an already planned
orrelation - the fact of correlation constitutes the core of the defini-
ion of the set M and the Richardian property. The Richardian property
eems to be different from other properties insofar as it depends on the
orrelation; similarly the set M is different from other sets since it
epends on the correlation. If it is illegitimate to raise the question
f correlation in one case, why not in the other case? If we turn, in parti-
ular, to Cantor's diagonal to prove the indenumerability of real numbers,
e notice that although in structure it seems to be different from the
eneral diagonal argument or from the heterological, it suffers also from
he same defect. The construction of a diagonal number is similar in
ature to the construction of the set M. The diagonal number depends on
very real decimal in the infinite list for its very existence (insofar
s it must differ from each of them by at least one digit), and then it
s asked whether it occurs in the list or not. It is as illegitimate to
sk whether or not the diagonal number occurs in the list as it is to
sk for a correlation of the set M. In both cases we did not follow
he original plan (in the sense specified in our discussion of Cantor's

diagonal number), but instead of producing an independent set or number, we constructed a dependent number or set. We made them by their very definition unable to occur in the list in one case and to be correlated in the other case. Hence the Richardian property, the diagonal number and the set M are all illegitimately constructed concepts in their respective contexts. If we are to say that *n* possesses the Richardian property or not, that the diagonal number occurs in the list or not, that the set M is correlated with *m* or not, we should first establish their independent status, i.e. they must be shown to be describable independent of their correlated serial number, of the infinite list of numbers, and of the correlated members of S, respectively. If this is a valid objection, namely that by definition we render impossible what we want to establish, then the Richardian paradox, Barber paradox and Cantor's diagonal procedure all suffer from the defect of the general diagonal method, i.e. from the diagonal fallacy. Indeed the structure of the argument in each case may be dissimilar in some ways, and each paradox or argument may have some special features peculiar to itself. For example, Cantor's diagonal argument does not exhibit the heterological or the Barber pattern, as explained diagrammatically earlier, and the notion of convergence, boundedness and limit is peculiarly relevant to it. None the less, despite these differences, these paradoxes involve the same basic fallacy.

Hence, we come to the conclusion that the Cantor diagonal method, Cantor's proof by nested intervals and the general diagonal method fail to demonstrated the indenumerability of the set of real numbers. Of course t does not follow that the real numbers are denumerable. We now ttempt to show the indenumerability of real numbers, thus showing the nadequacy of any attempt to resolve the Richard paradox by denying he legitimacy of the diagonal number or the indenumerability of real umbers in general.

A proof of the indenumerability of real numbers

The real numbers may be divided into two groups, namely algebraic mbers and non-algebraic (transcendental) numbers. For the definition algebraic numbers only algebraic symbols and operations are needed. gebraic operations are addition, subtraction, multiplication, division, tracting roots and raising to powers. In order to express an algebraic mber we need not use an infinite process, i.e. the notion of limit. , an algebraic number may be expressed as a positive or negative rational mber or as the root of an integer. Algebraic operations on algebraic mbers can yield only algebraic numbers in turn. Hence, algebraic mbers form a closed system, that is, the manipulation of algebraic numrs by algebraic operations always results in algebraic numbers, which n be studied without resorting to the notion of limit or some such finite process. Every algebraic number can be represented as the lution of an algebraic equation $a_0x^n + a_1x^{n-1} + \dots\dots\dots a_n = 0$, ere $a_0 \dots\dots\dots a_n$ are rational coefficients.

Now, from the very description of algebraic numbers, it follows at they can be co-ordinated with natural numbers or integers. Both sitive and negative integers are, crudely speaking, only twice as "numerous" as natural numbers: we have only to put negative and positive signs before numerals representing natural numbers to represent integers. Hence, if we can enumerate natural numbers we can enumerate integers too. In fact, it can easily be shown that the union of two or more denumerably infinite sets has the cardinality of a denumerable set. Again, any rational number construed as a fraction whose denominator is greater than 1, may be correlated with an integer equivalent to the product of numerator and denominator of the rational number concerned. So all the rational numbers having denominators greater than 1 can be correlated with integers. The class of rational numbers having 1 as denominator is obviously equinumerous with (or even may be regarded as the same as) the class of integers. Now, insofar as addition of any denumerably infinite number of integers to the infinite class of integers makes no difference to the cardinality of the class of integers, the rational numbers have the same cardinality as that of integers. Similarly, any algebraic number expressed in the form of a root $\sqrt[n]{x}$, when multiplied by itself n times, gives rise to an integer. Other algebraic numbers involving complicated symbols of roots and square can be correlated likewise with integers, hence every algebraic number can be correlated with an integer. So, we have been able to show that the class of algebraic numbers has the same cardinality as that of a denumerably infinite class, say of natural numbers, though we have not provided any systematic rule for enumerating the algebraic numbers. The denumerability of the set of algebraic numbers results from the fact that each algebraic number can be derived within a closed system: we get the natural numbers out of a few axioms (Peano's axioms, or by recursive function theory) and with a finite number of operations performed on natural numbers we get the algebraic numbers. We have showed only the possibility that algebraic numbers can be enumerated. However, Cantor showed how to enumerate the algebraic numbers.

It is about transcendental numbers that we are told that they constitute an indenumerably infinite set. What we require, however, is a proof to show that by the very nature of transcendental numbers, it is impossible for them to form a denumerable set. Transcendental numbers are negatively defined as non-algebraic numbers. It is obvious, however, that the cardinality of the set of transcendental numbers cannot be less than that of the set of algebraic numbers, and in a vulgar and unscientific sense they must be "more numerous" than algebraic numbers - just as rational numbers or integers may be said unscientifically to be "more numerous" than square integers. For once we have one transcendental number, say π or e, its multiplication, division, addition, subtraction, etc. with any algebraic number would always yield a transcendental number; otherwise it could not be a transcendental number. A transcendental number when multiplied or divided by, subtracted from or added to another transcendental number may yield an algebraic or non-algebraic number. Now the problem of denumerability of any given kind of numbers, rational or irrational, real or prime etc., is whether it is possible to give an effective method for enumerating them in a list, such that any number of the kind under examination may be shown to occur sooner or later somewhere in the enumerating list. If, for instance, the transcendental (or real) numbers form a denumerable set and we have some rule according to which we start enumerating them, then any transcendental (or real) number, say π, should occur in the list at some stage. If it is impossible for there to be such a finite number of methods for enumerating the given numbers, then we conclude that the set of those numbers is not denumerable, i.e. that its cardinality is greater than that of a denumerably infinite set.

Let us examine the question whether it is possible to give such rules for enumerating real or transcendental numbers. If a set is denumerable, its members can be effectively enumerated; and if the members of the set can be enumerated, it is denumerable. Also a set is not denumerable, if its members cannot be enumerated and vice versa.

In order to have a clear view of the situation, let us consider the position in the case of algebraic numbers. As we said earlier, we may construct the natural numbers by Peano's finite number of axioms. Again, by a finite number of effective operations we construct the series of all algebraic numbers. (For the sake of simplicity we shall avoid any reference to complex numbers.) No infinite process is needed. In short, algebraic numbers can be generated from a few axioms and operations which constitute a formal system. The completeness of the algebraic number system (i.e. its capacity to generate every algebraic number) may be achieved through these instructions. We have primitive recursive notions like zero, successor, variable and we recursively define operations like +, ~, ÷, $\sqrt{}$ by number theoretic functions with the help of initial functions like the zero function ($Zx = 0$), identity function ($Ix = x$) and successor function ($Sx = x+1$). We can generate the natural numbers by the rule that every variable has a successor. We can construct other complicated functions for ÷, √, etc. from the earlier ones. The class of all formulae in such a system has to be denumerable just as the class of formulae, say in the propositional calculus. We have an effective means of deriving formulae from the finite number of primitive notions, definitions, axioms or functions.

But in the case of real numbers (i.e. including transcendental numbers) we do not have such a generating system. Indeed our proof for the indenumerability of the set of real numbers lies in showing that we cannot have such a system. We have amply discussed the nature of a real number. To define a real number is to specify the rate of convergence in a series or sequence. We may think of each such number as a non-terminating decimal with a plan or pattern which specifies the digit that occupies the nth decimal place. Thus such-and-such a sort of number examplifies such-and-such a definite pattern at its decimal places. We can show that any rational number is necessarily represented only by repeating decimals and vice versa. For example, 1/3 as $0.\dot{3}$, 1/7 as $0.\overline{14285\dot{7}}$ - i.e. the whole series 142857 is infinitely repeatable. Mathematicians have not yet succeeded in mapping out a differentiating pattern in the non-terminating decimal system corresponding to irrational algebraic numbers as they have for the case of rational numbers. For our purpose, it is enough to remark that such a mapping of the class of irrational algebraic numbers expressed in terms of of only algebraic operations and expressed in decimal terms must exist, though the pattern or patterns of decimal digits on decimal places would certainly be much more complicated than that of rational numbers.

Let us take the transcendental number e. It may be defined by: 1 + 1/1! + 1/2! + 1/3! + ... This expresses a limiting series which tends towards the limit e. In fact every real number, whether algebraic or non-algebraic, can be expressed in a limiting sequence or series, e.g. 1 may be represented by the sequence 9/10, 99/100, 999/1000, ... (=0.9°). There must exist ultimate patterns or types in decimal digits corresponding to certain types of numbers. Rational numbers as stated have a certain type or pattern in decimal digits: the pattern is exhibited by the very nature of rational numbers. A definite number, say $\frac{1}{3}$ has a definite unique pattern exhibited in decimal digits, here recurrence of digit 3. But it has also a common or generic pattern which is shared by all rational numbers as described earlier. Similarly irrational algebraic numbers must have corresponding patterns. In like manner the numbers which arise out of operating with a transcendental number, say π with algebraic numbers must have certain patterns. Let us ask whether it is possible to have only a finite number of ultimate types of limiting series or sequences which express all the real numbers. Our proof for the indenumerability of real numbers lies in showing that the number of such ultimate types cannot be finite. Hence there cannot be an effective enumeration of all the real numbers; hence they are indenumerable.

By the term "types of limiting series or sequences" we mean the common features or patterns according to which the limiting series or sequences can be grouped. This idea of grouping further clarifies the notion of the general pattern exhibited by the digits on the decimal places. For example, the group of rational numbers are mapped by repeating decimals as already mentioned. For the sake of precision, let us talk about all

eal numbers between 1 and 2. By a few types of limiting series (and con-
equently of patterns exhibited by the digits in our non-terminating
ecimal system), say A_1, we can start enumerating the rational numbers.
ith some other types, say type B_1 to B_m we start enumerating irrational
lgebraic numbers. By some other types, say C_1 to C_k, we start enumer-
ting the transcendental numbers which result from performing operations
ike +, ~, ÷, √ with algebraic numbers by, say π. But the types C_1 to
$_k$ will not exhaust all the transcendental numbers. There will still be
eal transcendental numbers which cannot occur in our enumerating lists,
imply because they belong to different patterns. This results from the
ollowing considerations: We may have other patterns which express the
esult of performing algebraic operations with another transcendental
umber, say e, with algebraic numbers. But still there will be real
ranscendental numbers which cannot occur in the lists for the above
eason, because the possible types or patterns in which the symbols from
to 9 can be arranged are infinite in number. For instance in one kind
arrangement we have the rule that at every 125th decimal place we
ve the digit 5, then in the other pattern we stipulate that at every
6th place there shall be a digit 5, or even two or more 5's, the other
nditions remaining the same. In short, we can change the pattern by
bitrary legislation. These possible types of patterns cannot be enumer-
ed. For if they could be enumerated it would mean that we could subsume
ese possible patterns under still more general patterns or under a
ngle pattern. The possibility of such a suggestion is ruled out for
e reason that we can arbitrarily change even the most general pattern
an arbitrary way and there is no rule how this alteration of pattern is

effected. We can generate a real number by formulating a rule as to how different digits should occur in a decimal expansion, e.g. we generate a real number by the rule that in the proposed real number only two digits, namely 1 and 2, occur, such that first the digits 1 and 2 occur only once in their natural order (i.e., 0.12.....), then these digits further occur two times each in the same order (thus becoming 0.121122...) and then three times each, and so on. The possibility of such rules or patterns for forming real numbers is unlimited and cannot be subsumed under any general principle, rule or most general patterns. We can always alter any fixed pattern for generating real numbers at will.

We thus reach the crucial conclusion that there can only be a non-enumerably infinite number of patterns or arrangements of digits from 0 to 9. Hence we cannot have a finite number of principles, axioms or rules for defining the transcendental numbers such that by applying these rules we arrive at any transcendental number at some stage. About algebraic numbers we know how they can be generated by a few operations and primitive symbols, and hence a finite number of types exhaust all sorts of algebraic numbers. This is so because algebraic numbers are characterized by a few common algebraic operations and hence they must correspondingly exhibit certain common patterns in decimal digits. The same cannot be said of transcendental numbers. For this reason we cannot enumerate them like algebraic or rational numbers. Hence the set of all transcendental numbers is indenumerable. Denumerability in the case of infinite sets implies that we can have some finite number of principles according to which the numbers in question can be arranged so that we can construct or enumerate any given number sooner or later. Now this is impossible in the case of non-algebraic numbers. A finite number of principles to generate numbers implies that we have a finite number of general types or patterns in our decimal system. And if we have an infinite number of general patterns to describe numbers, this implies correspondingly that we cannot have a finite number of principles for generating numbers. In order to have an infinite number of general patterns, we need an infinite number of principles to enumerate all the numbers. So, no matter how many principles or ways of enumerating real numbers we use, there will always be an infinite number of real (transcendental) numbers which cannot occur in the enumerated lists, simply because they can have no place in the given principles or methods of enumerations: they would belong to different patterns in the decimal system. It is just as if we started enumerating natural numbers divisible by two; then odd numbers would naturally be excluded from the list or enumeration.

C. Resolution of the Richard paradox

1. The Richard paradox and the indenumerability of real numbers

We have been able to show that the set of all real numbers is indenumerable. Richard himself at a later stage changed his mind and admitted the meaningfulness of the diagonal.* Hence his solution, described earlier on page 104 can no longer hold. Now, as the set of all real numbers is indenumerable, there cannot be any finite number of rules to enumerate them such that sooner or later any arbitrarily chosen number occurs in the list. But there is one list in which any such number could occur. That list is the one in which numbers are defined or described in

* J. Richard, 'Considérations sur la Logique et les Ensembles', Revue des Métaphysiques et Morales, 1920, pp.356-69.

finite number of words, which are arranged in a certain lexicographical rder as Richard describes in presenting the paradox. Now, as every umber can be expressed in a finite number of words, it must occur in he list. For Richard there would be a clear contradiction because he as already assumed that such a list is denumerable. Therefore he oncludes in his presentation of the paradox:

> .... all the numbers that can be defined by means of finite number of words form a denumerable set.

et us investigate this supposition. This takes us back again to he question: What is a denumerable set?

If the number of things under review is finite, obviously they onstitute a denumerable set*: we can count them so that none is left ver. If the set of things under consideration is infinite, the problem how to count them, such that every member of the set has a place in he enumeration. The set of valid formulae, or even of well-formed ormulae in the propositional calculus is an example of a denumerably nfinite set. Here we have a few basic signs, and rules are given for ombining these signs in a certain order. Even the ill-formed formulae onstitute a denumerable set because we can always make rigid rules of eir various ways of combining in the denumerably infinite number of mbols. In fact, given any denumerably infinite number of symbols, the t of finite formulae arising out of the combinations of the initially ven symbols is always denumerable. We give a simple proof of this sertion. We know that the set of natural numbers is denumerable. We

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Sometimes the word "denumerable" is used only for infinite sets. We need not make here any such distinction between "countable" and "denumerable" sets.

have a finite number of original symbols 0, 1, ..., 9. Any finite formula originating by the combination of these symbols in fact represents a number, hence the infinite set of these formulae is denumerable. In the duodecimal system we have twelve original symbols. In another system we may choose some other number of initial symbols to construct natural numbers. In a natural language we have also a finite number of initial symbols and therefore the set of sentences (or in fact any combinations of letters making finite formulae) must be denumerable.

But a natural language is not simply a combination of symbols. In the case of symbols representing numbers, they are unambiguous and have a definite and fixed place in our thought; similarly in the case of an axiomatic system. But the symbols in a natural language (say, words here) do not stand for definite, unambiguous and immutable objects. Not only are they ambiguously used but also the symbols or combination of symbols attain a different meaning or significance depending on the particular contexts. We cannot count the contexts or the situations on which the meaning or significance of the symbols depends. The contexts or situations cannot be enumerated: they emerge and no formulae can predict their evolution or emergence. Our language serves to deal with all occasions, contexts and situations; hence the symbols have to be flexible to attain new meanings. This means that although the set of phrases of a natural language, considered only as combinations of symbols, is denumerable, considered with respect to their significance it is indenumerable.

Let us return to the Richard paradox. From the above discussion we see that there cannot be a finite number of formulae to enumerate each and every phrase of a natural language as intelligible units. Now, insofar as each member of the indenumerable set of real numbers is expressible or definable correspondingly by members of the indenumerable set of a given language, there is no contradiction - both being indenumerable sets. The trouble arises if we suppose that the set of phrases of the language is only denumerably infinite. Of course if the set of all sentences were denumerable, the set of all real numbers would also have to be denumerable.

In this connection there may arise some confusion about the notion of denumerable set. Suppose we have a diagonal number which does not occur in our enumeration, then we can always make another enumeration in which this particular number occurs along with others. Once numbers are known, i.e. once they are properly defined or we come to know some definite method for defining them, then they always form a denumerable set. No definite, defined or known number can be said to be indenumerable: it is the set consisting of numbers which is properly called indenumerable. A set is thus indenumerable when we cannot by any enumeration exhaust all the members; that is, sooner or later every member must appear in the enumerating list.

§2. Concluding remarks

While discussing the Richard paradox, J. Tucker remarks:

> The source of the confusion is two-fold. It arises in part from the usual failure to distinguish between the diagonal and heterological patterns, in part from the superstition that there has to be a one-one correlation between the things talked about and the words used in talking about them. There are accordingly two stages of clarification. The Richard as a diagonal has to be distinguished from the heterological contradictions. Then the grammatical confusion peculiar to the Richard has to be cleared up.*

have already sufficiently discussed in what respects the diagonal d the heterological patterns are similar and in what respects dis-milar, and hence we need not comment on this point. On Tucker's cond remark we would like to quote him:

> The grammar of the Richard is the grammar of 'talking about' and the trouble peculiar to the Richard is peculiar to 'talking about'.**

then indicates the nature of this trouble in the following words:

> How, it is asked (at various levels of sophistication) we can talk about the indenumerable if the number of words in any possible language is merely denumerable? How, in the case of one particular way of talking about, namely defining, can we define each indenumerable number if the supply of definitions is only denumerable while the things to be defined exceed that supply?***

have underlined "each indenumerable number" in quoting Tucker above. is difficult to understand what Tucker means by these words. Are ey transcendental numbers? His formulation of the Richard paradox likewise difficult to understand. He states:

John Tucker, 'Constructivity and Grammar' P.A.S., 1962-3, p.55.

Ibid. p.54.

Ibid. p.57.

The Richard consists of two 'conflicting' statements:

(i) Only a denumerable number of numbers can be finitely defined;

(ii) Indenumerable numbers can be finitely defined by means of the diagonal.*

We have already pointed out that the so-called diagonal number can occur in an enumeration other than the already planned one. So this diagonal number is only indenumerable relative to the previous enumeration but denumerable according to a new enumeration. We pointed out that in fact no known particular number can be properly called indenumerable; it is only a set which may be properly called indenumerable.

In a way Cantor seems to commit this error too. He tries to prove that a particular number, i.e., the diagonal derivative number, cannot occur in the denumerable list. What he should have tried to show was that a particular set or class of numbers cannot occur in the list, and not one particular number. Perhaps Tucker wants to emphasise that there is no need of an infinite number of words to represent or define the diagonal number as Richard in his first presentation of the paradox suggested. Tucker is eager to emphasise the incorrectness of the assumption that 'talking about' involves a 1-1 correspondence between the things talked about and the words used in talking about them. Even so, this does not seem to resolve the paradox. It soon appears that Tucker wants to criticise those who think it impossible to talk properly about an indenumerable set like the set of all real numbers because the supply of definitions (in this formal system) can only be denumerable. We quote Tucker once again:

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* _Ibid._ p.54.

> Dixon assumed that 'talking about' involves a one-one correlation between the things talked about and the words used in talking about them; and Church makes no advance on Dixon. Dixon took the argument as a sign of what would now be called incompleteness: since the set of finite definitions is denumerable, whereas the real numbers are indenumerable there must be real numbers about which we cannot talk adequately and unambiguously. Church argues that since there is no one-one correlation between any possible formal system (denumerable) and number theory (indenumerable) we cannot adequately talk about number theory by means of formal systems.*

He further comments on Church:

> .... He has not noticed that a simple extension of his argument demolishes it. For the denumerability which he attributes only to formal systems applies, if it applies to them, to all possible languages, formal or informal. But number theory must be in some language and this language too must be denumerable. It follows that there is no difference in size between the informal language in which number theory is expressed and any formal system and, on Church's own premiss, no such thing as the Richard paradox. If it were true that the number of words in any possible language is denumerably infinite, and if it were also true that we could talk about indenumerable sets adequately and unambiguously only if we had an indenumerable number of words available, then there would be no Richard paradox because it could not be adequately and unambiguously stated. But both premisses are false.*

Tucker seems to be confused about the nature of formal systems and natural language. It is not the words as such that are important, but their arrangements in phrases as intelligible units. We have already remarked that the set of all phrases as intelligible units in natural language is not denumerable. In a strict formal system like propositional or functional calculus, where every symbol has a fixed signifi-

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* J. Tucker, 'Constructivity and grammar' pp.57-8.
See A. C. Dixon, 'On well-ordered Aggregates', Proceedings of the London Mathematical Society, 1907, pp. 18-20.
See A. Church, 'The Richard Paradox', American Mathematical Monthly (1934), pp.356-61.

cance and where we can deal with symbols irrespective of their interpretation, the formulae are bound to be denumerable. Hence there is a marked difference between a formal system and a natural language. It follows that in a strict formal system we cannot talk about every real number because in such a system we can only produce a denumerable set of formulae. And this denumerable set must exclude some real numbers. But as natural languages can be used in forming phrases constituting an indenumerable set, we can talk in these languages about each real number in sentences (definitions). Hence, Dixon is right in asserting that 'In any other aggregate whose cardinal number exceeds N_0 it is similarly impossible to give a finite description of each individual number'. This follows from his assertion that 'by defining, describing or specifying an object is meant stating such properties of it as distinguish it from all other objects of mental activity'. Obviously there must be a one-one correlation between numbers and their descriptions, and therefore not all real numbers can be individually defined. But the inability to define each individual member of the set of all real numbers does not mean we cannot talk about the indenumerable set taken as a whole. Tucker seems to be unclear about this distinction, too.

In conclusion we may remark summarily once again that the Richard paradox arises from the incorrect supposition that the set of all sentences (definitions) in natural language is denumerable. It makes us aware of the important distinction between a natural language and a rigid formal system.

Chapter V

SET-THEORETICAL PARADOXES

In this chapter we discuss the antinomies which we come across in attempting to construct a consistent axiomatic set theory. We have intentionally labelled these antinomies "set-theoretical" paradoxes and not "logical" paradoxes in order to avoid any suggestion as to the status of set theory - that is, whether or not axiomatic set theory should be counted as logic.

We now outline three such paradoxes: (1) Russell's paradox of the class of all non-self-membered classes; (2) Cantor's paradox of the class of all cardinal numbers; and (3) the Burali-Forti paradox of the class of all ordinal numbers. We have already touched upon these paradoxes in the first chapter. We assume certain basic axioms in our set theory from which we construct the above-mentioned classes, which in turn lead to inconsistent derived formulae. We start with the Russell paradox.

A. The Russell and related paradoxes

Cantor, the pioneer of set theory, in his earlier developments of set theory did not explicitly lay down his basic axioms in order to serve as a basis for deriving theorems. But it seems clear that he required the following three axioms:*

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* See P. Suppes, Axiomatic set theory, p.5.

i) <u>The axiom of extensionality</u>: Two sets are identical if they have the same members: $(x)(x \in A \equiv x \in B) \supset A = B$

i) <u>The axiom of abstraction</u>: Given any property or description ϕ, there exists a set whose members are just those entities having that property: $(\exists y)(x)(x \in y \equiv \phi\ (x))$; and

i) <u>The axiom of choice</u>.

Russell's paradox

Russell's antinomy is derived by uncritically applying the axiom of
traction. The axiom states that for every property or description there
corresponding set. But let us consider a description of the class of
classes which are not members of themselves. We observe that this des-
otion results in a contradiction. We may briefly state the antinomy
ormally as follows:

(I) A class Y contains as members just those classes which are not
ers of themselves.

(II) Suppose Y is a member of itself; but by definition Y contains only
e classes which are not members of themselves - hence Y is not a member
tself.

(III) Suppose now Y is not a member of itself; but by definition Y
ains all those classes which are not members of themselves - hence Y is
mber of itself.

(IV) Hence Y can neither be or not be a member of itself. But by our
osition Y must either be or not be a member of itself. These statements
contradictory - $(P \vee Q)$ is contradictory to $(\sim P \,.\, \sim Q)$. Or we may express
contradiction by remarking that as Y is a member of itself $(Y \in Y)$ and Y
ot a member of itself $(Y \notin Y)$ are contradictory, either alternative lead-
to its contradictory, i.e., $((P \supset \sim P) \,.\, (\sim P \supset P)) \equiv (P \,.\, \sim P)$.

The emergence of this antinomy is not a trivial matter. When Russell
formed Frege of the antinomy in a letter, Frege was deeply shocked, for it
s as though the entire foundation of his edifice which he took such pains
construct was shaken.* In an axiomatic system of set theory the antinomy
be derived as follows:-

(I) $(\exists y)(X)(X \in Y \equiv \phi(X))$ .... Axiom of Abstraction

To obtain Russell's antinomy we take $\phi(X)$ to assert that X is not a member of itself, i.e. $X \notin X$. We have then an instance of the above axiom-schema of abstraction:

(II) $(\epsilon Y)(X)(X \in Y \equiv X \in X)$

(III) $Y \in Y \equiv Y \in Y$ .... by instantiation in (II). (III) is logically equivalent to $(Y \in Y \,.\, Y \notin Y)$ .... contradiction.

In order to avoid the antinomy there are several alternatives. One is
uphold the axiom of abstraction but deny that there could be such a class
Y on the ground that the description of this class is meaningless. For
defining characteristic of membership in Y .... $X \notin X$ .... is meaningless
so cannot be an instance in the axiom of abstraction, and hence cannot
lace $\phi(X)$. Russell takes up this line of argument. Accepting the basic
umption (axiom-schema of abstraction), he declared that there are meaning-
s phrases which nevertheless appear to be meaningful. He offered a touch-
ne in the vicious circle principle whereby meaningless phrases like "member
itself" or "not a member of itself" are excluded. We have already sufficient-
discussed this principle. We pointed out that the principle is not semanti-
ly unimpeachable and that there are several cases where the so-called

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ee: Frege, Grundgesetze der Arithmetik, vol.ii, Appendix, pp.253-265 (transl.
n Philosophical Writings of Gottlob Frege, eds. Geach & Black, pp.234-244).

"vicious circle" is quite harmless. It may of course be offered as a pragmatic justification or as an <u>ad hoc</u> arrangement to avoid certain paradoxes. But this in no way proves the logical or intuitive certainty of the principle. As far as Russell's solution in terms of his theory of types is concerned it does avoid the paradox because we cannot then meaningfully use $X \in X$ and $X \notin X$. But his theory of types as a basis for set theory is not only semantically objectionable but also technically cumbersome, in addition to needing some extra-logical assumptions. We have already on several occasions criticized the theory of types. W. V. Quine also has aptly criticized the theory of types in the following words:

> But the theory of types has unnatural and inconvenient consequences. Because the theory allows a class to have members only of uniform type, the universal class V gives way to an infinite series of quasi-universal classes, one for each type. The negation $\sim x$ ceases to comprise all non-members of x, and comes to comprise only those non-members of x, which are next lower in type than x. Even the null class Λ gives way to an infinite series of null classes. The Boolean class algebra no longer applies to classes in general, but is reproduced rather within each type. The same is true of the calculus of relations. Even arithmetic, when introduced by definitions on the basis of logic, proves to be subject to the same reduplication. Thus the members cease to be unique; a new 0 appears for each type, likewise a new 1, and so on, just as in the case of V and Λ. Not only are all these cleavages and reduplication intuitively repugnant, but they call continually for more or less technical manoeuvres by way of restoring severed connections.*

It is to be noted that the vicious circle principle does not necessarily lead to type theory. It is quite possible and consistent to uphold the principle as far as set theory is concerned, not accepting a type-theory, but adopting some other methods to avoid the vicious circle. Let us now attempt to discover the real ground for the genesis of this paradox.

* W. V. Quine, 'New foundations for mathematical logic', <u>American Mathematical Monthly</u>, Feb. 1937, p.78, vol.XLIV. Reprinted in <u>From a Logical Point of View</u>, pp.91-2.

§2. The Curry paradox and others

The structure of the Russell paradox as such is obviously that of the Barber paradox. Our aim is not merely to legislate so as to prevent the occurrence of such a structure in our axiomatic system, but to determine the real cause behind the paradoxical pattern of the Russell paradox. It may be that we shall be able to adopt measures so as to prevent Russell's paradox, but that other paradoxes having different structures arise because of the same underlying fault. For example, no direct derivation of the Russell paradox is possible in Quine's system ML, but J. B. Rosser showed that this system embodies the Burali-Forti paradox.** It may be that the real cause ehind the two paradoxes is the same. An axiomatic set theory constitutes n inter-related whole and in order to find the real cause of any set-heoretical antinomy we have to find some fault in its basic axioms. And n order to discover the faults in the basic axioms and rectify them, it is seful to study various sorts of antinomies arising in the system. With this im in mind, let us consider a few more antinomies originating out of the xiom of abstraction. This will also throw light on the concepts of set and et-membership.

As already remarked, the Russell paradox emerges by instantiation in he axiom of abstraction. It may be surmised that one way of avoiding this ontradiction would be to construct a set theory based on many-valued logic nd not on classical or two-valued logic. In this way the meaning of negation ould be changed and thus a set may both be a member of itself and not be a ember of itself. But the Curry paradox may be cited against such a

* See P. Suppes, Axiomatic Set theory, p.9.

suggestion, for this paradox arises even when we do not employ any negation sign. This paradox may be symbolically expressed as follows:

(1) $(\exists Y)(X)(X \in Y \equiv \phi(X))$ ...... Axiom of abstraction

(2) $(\exists Y)(X)(X \in Y \equiv (X \in X \supset (P \,.\, \sim P))$ ...... $\phi(X)$ interpreted as $(X \in X \supset (P \,.\, \sim P))$ in (1).

(3) $Y \in Y \equiv (Y \in Y \supset (P \,.\, \sim P))$ ...... by instantiation of (2)

(4) $Y \in Y \supset (P \,.\, \sim P)$ ...... from (3) because $[(P \supset (P \supset Q)) \supset (P \supset Q)]$ holds in any finitely-valued logic.

(5) $Y \in Y$ ...... by (3) & (4), Modus Ponens.

(6) $P \,.\, \sim P$ .... by (4) & (5), Modus Ponens.

hence any statement, e.g. $(P \,.\, \sim P)$ above, may be deduced.

The above discussion brings out the fact that the assumption that there is a set corresponding to every predicate is untenable. Zermelo took this line. The problem remains, however, as to which sort of predicate gives rise to sets. For Russell, those predicates which do not give rise to sets may be regarded simply as meaningless. Zermelo's approach is positive as compared with Russell's. He tries to choose "good" predicates, i.e. those which serve genuinely to describe sets. But Russell tries to eliminate "bad" predicates, i.e. those which do not describe sets. Let us examine a few more predicates which purport to describe sets through the axiom of abstraction, thereby to determine what sorts of predicates lead to paradoxical results. The paradoxes arising from the following predicates may be regarded as belonging to the same family as the Russell paradox. Let $\phi(X)$ be interpreted as $(Z) \sim (Z \in X \,.\, X \in Z)$ or $(Z)(\underline{W}) \sim (Z \in X \,.\, X \in N \,.\, \underline{W} \in Z)$ or $(Z)(X = Z \supset X \not\in Z)$ or $(Z)(\underline{W})((X = Z \,.\, Z = \underline{W}) \supset X \not\in \underline{W})$ etc. In each case the axiom of abstraction leads to contradictory results. What is at the root of these contradictions?

B. Description and existence of sets and the paradoxes

§1. Set-membership and the existence of sets

We shall argue that the above predicates run counter to the very nature of set and set-membership. The identification of a set requires solely the determination of its members. The axiom of extensionality states that two sets are in fact identical if they have the same members; hence we may have two or more different definitions of the same set. Two or more predicates may seem to characterize different sets, but in fact these sets may contain the same members, just as "evening star" and "morning star" denote the same star. Now, the problem is what sort of definition is permissible to identify the members of a set. To identify the members of a group is, in a way, to define a set. The point to be noticed is that in trying to define a set, we already presuppose the possibility of its "existence". Hence the defining expression of a set should satisfy conditions for existence, among which is that the defining characteristics are not self-contradictory. For example, by defining a figure as a square-circle we rule out even the possibility of its existence. There simply cannot be such a figure. Existence implies possibility and impossibility rules out even the possibility of existence. A figure being a square-circle is impossible because it is impossible for a square figure to be circular and vice versa, so that a square-circle cannot exist. Similarly we cannot define a set by a contradictory description. Construction of a set is tantamount to its existence, and the theory of sets involves the construction of sets. Hence the defining description of a set within axiomatic set theory must not involve a contradiction. For instance, the null set may be defined as a set having no members or as a set which is included in every other set; but not as a set whose members are those objects which are not equal to themselves, i.e. $\{z: z \neq z\}$. Just as we cannot assert the ideal existence of, say, a geometrical line defined as straight-crooked, similarly we cannot assert the existence of an empty set by giving a contradictory condition of its membership. This fact is concealed in the case of the null set because it is suggested that, as there cannot be an object z which would satisfy $z \neq z$, it follows that the set in question is an empty set. But one may reasonably argue that insofar as we define the empty set as a set having only those members which satisfy the required condition ($z \neq z$) and that sort of object cannot exist, it follows that there is no such set and hence no such thing as the null set.

§2. Self-membership

The antinomies may emerge by overlooking the above point in defining a set; that is, the contradictory definition of a set may lead to contradictory results, as we noticed in the case of the Russell set of all non-self-membered sets. To show that this definition is in fact contradictory, let us consider another example: the set of all sets. This set, being a set, must be a member of itself. We can show, on the other hand, that this self-membership is a contradiction in terms. By the axiom of extensionality given above two sets are the same if they have the same members. This implies that if two sets have different members or a different number of members (a special case of having different members), they are different sets; although under the above assertion of extensionality this does not strictly follow. Suppose two sets be the same but with different members, and in particular some set a member of itself. But this supposition conflicts with the very concept of set. A set has been defined solely in terms of its membership, as we intuitively nderstand the meaning of "set" or "class". Cantor defined a set (Menge) as any collection into a whole M of definite and separate objects m of our ntuition or thought, which are called the "elements" of M'.* Patrick Suppes efines a set as 'something which has members or is the empty set', i.e.: is a set $=_{df} [(\exists x)(x \in y) \vee y = 0]$.**The essential characteristic of a set, erefore, concerns the notion of membership. This is the characteristic of e concept in general. No other characteristic is given on the basis of ich we could distinguish one set from another. If this has been assumed e sole defining characteristic of a set, why should it not be regarded as e sole differentiating characteristic of sets? That is by virtue of this aracteristic we differentiate not only sets from non-sets, but also one set om another. If some other characteristic is regarded as a differentiating ature, this would imply that membership cannot be regarded as the sole cessary and sufficient condition of sets. As a set has always been treated defined solely in terms of its members, it follows not only that two sets e the same if they have the same members, but also conversely. Accordingly, me logicians, e.g. P.R. Halmos, have given as the axiom of extensionality wo sets are equal if and only if they have the same elements."*** So, if o sets have different elements or a different number of elements, then they e necessarily different sets. From this it immediately follows that a set nnot be a member of itself. For as soon as we suppose a set to be a member itself, then it no longer remains the same set; a new set emerges which ll have one new different member, namely the set in question. Consider a

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G. Cantor, Contributions to the Founding of the Theory of Transfinite Numbers, translated by Philip E.B. Jourdain, p.85.

P. Suppes, Axiomatic Set Theory, p.19.

* Paul R. Halmos, Naive Set Theory, p.2. See also, P. Bernays, Axiomatic Set Theory, pp.8-9, 51-53.

set A. A cannot be conceived of as a member of itself, for what would be conceived would be not A, but rather the set $A \cup \{A\}$, which is different from A. Suppose a set consists of two members x and y, i.e. $\{x, y\}$. If we try to make this set self-membered, a new set appears, namely $\{x, y, \{x, y\}\}$. If the set in question is finite, the effort to make it self-membered will lead to the addition of another member to the members of the original set. As in the case given, the new set would have an extra member and hence be a different set. If the set in question is infinite, the new set will not have more members, but it would have one member (i.e. the original set itself) which would be different from the members of the original set. Let us take the set of natural numbers, i.e. $\{1,2,3,\ldots\ldots\}$, then the new set becomes $\{\{1,2,3,\ldots\},$ $1,2,3,\ldots\ldots\}$. Hence there cannot be a self-membered set.

It does not follow that there cannot be anything which is self-membered, or that there cannot be something to which the axiom of extensionality does not apply. The axiom of extensionality, for instance, does not hold in the case of attributes. For a self-membered entity, we can cite the example of a limited company. A limited company is composed of share-holders and the share-holders may be regarded as members of the company. Further, a company may be a share-holder (member) of another company or even of itself. The point to be noticed is that a company will remain the same even if some of the share-holders cease to be members of it or some new share-holders join it. This shows that unlike sets, the full nature of a company cannot be determined solely in terms of membership or extension. One may construct a theory in which an entity can be a member of itself, but that would not be a theory of sets. In the theory of sets, self-membership is a contradiction in terms. Hence it is no wonder that where we have a set which is immediately or ltimately self-membered we arrive at contradictory results. An obvious case uld be the class of all classes. This class, being a class, must by definition a member of itself. But then the resultant class would have one new member amely, the class in question) and thus a new class would emerge, and so on. follows that there cannot possibly exist a class of all classes. The Russell radox may be regarded as a special case of the above fallacy. As we know at no class can be self-membered, it follows that the class of all non-self-mbered classes falls into the same category as the class of all classes.

. Formulae to avoid self-contradictory descriptions of sets

Similarly, we may arrive at a paradoxical situation if we posit a set ose members are all sets having more than three members. Clearly there are re than three such sets. Hence the posited set has more than three members; is it should be, by definition, a member of itself. But this is what a set nnot be. Thus we have given a self-contradictory description. Now, the oblem arises as to how to avoid such defining expressions which immediately in a round-about way lead to self-membership. We may take as an axiom: $\notin A$. But there are other expressions like "$A \notin B \cdot B \notin A$" or "$A \notin B \cdot B \notin C$ $C \notin A$" and so on, which are self-contradictory for the same reason. To prevent h cycles, we may introduce rules to prevent the construction of such cycles, assert that these cycles cannot occur in the system. That is, formulae such "$A \notin A$" or "$\sim(A \in B \cdot B \in A)$" must be taken as valid. Von Neumann introduced axiom of Fundierung for this purpose. The simplified version due to Zermelo, axiom of regularity, is given here:

$$A \neq 0 \supset (\exists X)[X \in A \cdot (y)(y \in X \supset y \notin A)].$$

s says that for any non-empty set A, there is a member X of A such that the ntersection of A and X is empty. This axiom can be easily justified intuit-vely in accordance with the conception of set outlined above. Think of any on-empty set A, and choose one of its members, say X. Now if X does not appen to have any member, then the above axiom necessarily holds. If X has member, say y_1, which is also a member of A, thus contradicting $y_1 \notin A$, hen instead of choosing X as our member of A, we select y_1 - for the axiom ontains existential quantifier. The same process may then be repeated with he set y_1. If it has a member, say y_2, which is also a member of A, we hoose y_2 the member of A instead of y_1 and so on, till we reach a point here the intersection of the set A and at least one of its members is empty, hus satisfying the axiom. Cyclic formulae like "$A \in B \, . \, B \in A$" go against he axiom.*

Returning to the main problem, viz. what sort of description is legiti-ate for defining a set, the one thing that has become evident is that this efinition or description must not run contrary to the axiom of regularity. t there may be descriptions like "set of all sets exceeding three members" hich are deceptive and we may only come to know their illegitimacy when we eet a contradiction. To meet this problem Zermelo invented the axiom-schema separation (Aussonderungs Axiom), given below, by modifying the axiom-schema abstraction. We shall show how the paradoxes are avoided by this amendment. e Russell paradox is avoided as follows:

(1) $(\exists Y)(X)(X \in Y \equiv (X \in Z \, . \, \phi(X)))$ .... Axiom of Separation

(2) $(\exists Y)(X)(X \in Y \equiv (X \in Z \, . \, \sim(X \in X)))$ ... $\sim(X \in X)$ as an instance of $\phi(X)$ in (1).

(3) $Y \in Y \equiv (Y \in Z \, . \, Y \notin Y)$ .... taking $X = Y$.

t the result (3) is not self-contradictory because both sides could be false.

* Cf. P. Suppes, Axiomatic Set Theory, pp.53-55.

do not get a contradiction because in this axiom we have to be given the t Z; only then are we permitted to assert the existence of the sub-set Y. the very name of the axiom suggests, it is used to separate off from a ven set only those elements which satisfy a certain predicate, thus forming set consisting of just those elements. For instance, given the set of all man beings, we may construct another set, the set of all human females, by parating off all the members of the given set who possess the property of mininity. The acceptance of this axiom leads to the following interesting nclusion. Let us replace $\phi(X)$ by $X \notin X$, thus arriving at (2) as above. We n show that $Y \notin Z$ must hold. Suppose it does not, then we have $Y \in Z$. Now have either $Y \in Y$ or $Y \notin Y$. If $Y \in Y$, then with our assumption of $Y \in Z$, get the contradiction: $(Y \in Y \equiv (Y \in Z \,.\, Y \notin Y))$, i.e. (True ≡ (True . False)). $Y \notin Y$, even then we have the contradiction, i.e. (False ≡ (True . True)). nce we must have $Y \notin Z$. (Thus also showing that in our result (3) above both des of ≡ must be false, thus avoiding Russell's paradox.) It follows that ere can exist a set Y that does not belong to Z. As the set Z is any bitrary set, it follows that there cannot be a set which includes every set. fact, the admission of the universal class would lead from the axiom of paration to the axiom of abstraction and therefore to antinomies. Hence, cording to Zermelo's system, there cannot be such a set as the set of all ts.

The above discussion leads to the question as to what sort of condition, property or predicate is to be represented by $\phi(X)$. Not every property can , for otherwise we could posit a set which is a member of the given set. rmelo insisted that this condition must be 'definite'. In plain words, this ndition consists in predicates having significance for the elements of the ɡiven set, where a "significant" predicate is a predicate P such that if $x \in \underline{M}$, then either x has the property P or it does not. E.g., the predicates "prime" and "square" are significant, relative to the set of natural numbers. Thus Zermelo's axiom may be informally stated as follows: If $\underline{M}$ is a given set and P is a property having significance relative to $\underline{M}$, then there exists a subset of $\underline{M}$ having those and only those elements that have the property P. Fraenkel replaced the notion of "significant property" by determined formulae and thus removed the vagueness.*

It is clear that the step from the axiom-schema of abstraction to that of separation is immense. By the principle of abstraction we could have "over-big" sets like the class of all classes - as big as we like. But according to the principle of separation we cannot construct a set bigger than that of a given set - the given set being already well-established by other axioms. We cannot obtain, for instance, the set of all sets; rather, its impossibility may be demonstrated. In brief, in the Zermelo-Fraenkel set theory we construct sets out of given sets by certain axioms like that of sum (*Vereinigung*), pairing and sub-set formation. We cannot have the set of all sets having more than three members because we cannot just arbitrarily define a set. A definition of a set must be governed by the restrictions imposed by given sets and the formulae to determine the "significant property" in the axiom-schema of separation. We can, in accordance with the axiom of separation, construct a set of all sets having more than three members only by separating or choosing such sets among the members (sets) of a certain already given set. Moreover, this constructed set cannot be the given set. Otherwise it would become a

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See G.T. Kneebone, *Mathematical Logic*, pp.288-290: W. & M. Kneale, *The Development of Logic*, p.682.

self-membered set which cannot be constructed according to the axiom of separation. For we have shown above that $Y \notin Z$ must hold true.

C. The Cantor paradox and the Burali-Forti paradox

We quoted Cantor's definition of set to illustrate the view that the differentiating and defining characteristics of a set lie simply in its membership. Our conclusion was that on this meaning of "set", a set could not be a member of itself. Further reflection on Cantor's definition of "set" shows, however, that if we strictly follow it, we are led to contradiction. That is, the definition itself seems to embody a contradiction. He defined a set as 'any comprehension into a whole (Zusammenfassung zu einem Ganzen) M of definite and separate objects *m* of our intuition or our thought'. We can avoid any reference to "mind" here and concentrate our attention on "sets", as consisting of objects which are definite and separate. If the objects are finite in number, we can count them and we can individually specify them simply by proper names. But if the number of objects is infinite, then we have to resort to a descriptive or definitional method of specifying them. In this way we define the set of all natural numbers. As every set is definite and separate, we feel justified in talking of the set of all sets, but this leads to contradiction, as observed earlier. Insofar as numbers may be identified with sets, a similar situation arises in the case of the set of all cardinal numbers and the set of all ordinal numbers. For we are led to talk of the greatest cardinal and the greatest ordinal, whereas according to Cantor, there cannot be such greatest numbers. Thus we come to the Cantor and Burali-Forti paradoxes.

§1. The Cantor paradox

Cantor defines the cardinal number of a set M as follows:

> .... the general concept which, by means of our active faculty of thought, arises from the aggregate M when we make abstraction of the nature of its various elements *m* and of the order in which they are given.
>
> We denote the result of this double act of abstraction, the cardinal number or power of M, by $\overline{\overline{M}}$.*

If, for instance, we have two sets, namely: (A) {Napoleon, Caesar} and (B) {1, 2}, and we think of these sets irrespective of the nature of the members and their order, then we are thinking about the cardinal numbers or powers of these sets. According to Cantor, each member then becomes a 'unit', so that the cardinal number $\overline{\overline{M}}$ is a definite set composed of units.' Thus Cantor says:

> Since every single element *m*, if we abstract from its nature, becomes a "unit", the cardinal number $\overline{\overline{M}}$ is a definite aggregate composed of units, and this number has existence in our mind as an intellectual image or projection of the given aggregate M.**

It is quite clear that by this double abstraction the above sets (A) and (B) become one and the same set; they both have two units and so they have the same cardinal number. It is clear that if two sets are equivalent in the sense of equinumerous, i.e. if it is possible to find a law whereby they are put in relation to one another, so that to every element of one there corresponds one and only one element of the other - then the sets have the same number of units and hence have the same cardinal number. So we have: if $M \backsim N$, then $\overline{\overline{M}} = \overline{\overline{N}}$, where $M \backsim N$ symbolises " M and N are equinumerous". This means that to every set there is a corresponding cardinal number and every cardinal number is, furthermore, itself a set. Two different sets may have

* G. Cantor, Contributions to the Founding of the Theory of Transfinite Numbers, translated by P.E.B. Jourdain, p.86.

** *Ibid.*, p.86.

the same cardinal number. One set has a higher cardinal number than another if it has a proper subset equivalent to the other. But the reverse is not true. In addition to finite cardinal numbers, Cantor talked of transfinite cardinal numbers, A set, consisting of all finite cardinals, has a transfinite cardinal number; it cannot have a finite cardinal number since then it would be a member of itself and could not be the set of all finite cardinal numbers. Now, according to Cantor there are an infinite number of transfinite cardinal numbers, because by Cantor's theorem the set of all sub-sets of any given set has a greater cardinality than that of the given set. Thus we have the assertion that there is no greatest cardinal number. But let us consider, on the other hand, the set of all cardinal numbers. This set, being a set, has a cardinal number. In fact, it should be the greatest cardinal, thus contradicting the previous assertion.

§2. The Burali-Forti paradox

This is also called the paradox of the greatest ordinal. An ordinal number in Cantor's theory is constructed as follows:* A set is ordered if the arrangement of its elements in the set is an essential feature of the set considered. A set is simply ordered by the binary relation $\dashv$, if for any two distinct elements x and y of the set either $x \dashv y$ or $y \dashv x$ holds, $\dashv$ being non-reflexive but transitive. We shall call this the relation of precedence. Two simply ordered sets M and N may be called <u>similar</u> ($M \simeq N$) if there exists a one-one correlation between them which preserves their relation of precedence. From the concept of similarity we have the concept

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<u>Ibid</u>., pp.110-118, 137-159. See also G.T. Kneebone, <u>Mathematical Logic and the Foundation of Mathematics</u>, pp.160-161.

of ordinal-type. If two sets are similar they have the same ordinal-type, otherwise not. Thus any simply ordered set M has a property, the ordinal-type $\overline{\overline{M}}$. If every non-empty sub-set of the simply ordered set M has a first member with respect to the relation of precedence (i.e. the set is <u>well-ordered</u> by ⊣), then the ordinal-type of this set is called an <u>ordinal number</u>. The set $^*\omega = \{\ldots\ldots 3,2,1\}$ is not well-ordered but the set $\omega = \{1,2,3,\ldots\ldots\}$ is well-ordered. The set $\{1,3,5,\ldots\ldots 6,4,2\}$ is not well-ordered though it has a first element, because one of its sub-sets, the set of even numbers, has no first element. It is obvious that every sub-set of a well-ordered set is well-ordered. If we obtain from a well-ordered set M a subset A which constitutes all the elements preceding a definite element *m* of M, this sub-set may be called a segment of M - e.g. the segment determined by the first element of M is the null set. Any two well-ordered sets are either similar to each other or one of them is similar to a segment of the other. Hence well-ordered sets can always be compared with respect to their cardinal numbers. A well-ordered set cannot be similar to any of its segments. It is because of this characteristic of comparability that ordinal-types of well-ordered sets are called ordinal numbers. For instance, the sets $N = \{3,4,5,\ldots\ldots\}$ and $N' = \{4,5,6,\ldots\ldots 1\}$ have ordinal numbers: the ordinal number of N is less than N' because N is similar to the segment of N' determined by the element 1.

We may now succinctly express the Burali-Forti paradox by two conflicting statements:

(1) The series of all ordinals up to and including any given ordinal exceeds the given ordinal by one. For instance, suppose we have an ordinal number (well-ordered set), say 3: $\{0,1,2\}$. The series of all ordinals up to and including 3 would be: $\{0,1,2,3\} = 4$, thus giving rise to a higher ordinal than the given one.

(2) The series of all ordinals has an ordinal number, i.e. the set of all ordinals is well-ordered.

(2) implies that there is a greatest ordinal number but (1) implies that here cannot be a greatest ordinal because a still higher ordinal emerges out f the given one. Hence the contradiction arises.

3. General remarks

Both paradoxes can be constructed in certain systems of set theory. heir exact formulation depends upon what axioms we have in our set theory nd how we define cardinal and ordinal numbers in terms of sets. The above ormulation of the Burali-Forti paradox, for instance, assumes the well-rdering theorem that every set can be well-ordered, which is in fact equi-alent to the axiom of choice. If we reject this proposition, then we may say hat the set of all ordinal numbers is not itself well-ordered. We can define ardinal and ordinal numbers in terms of sets in several ways. The purpose of set theory is fulfilled if it exhibits in sets the structural patterns and nterrelations characteristic of numbers. The paradoxes therefore can be xpressed in terms of sets. We may, for example, express the Burali-Forti aradox in terms of Von Neumann's theory without reference to the axiom of hoice. We present below this paradox, following the definition of ordinal mber due to R.M. Robinson:*

Definition of transitivity: $T(\alpha) \leftrightarrow (Y)(Z)((Y \in Z \,.\, Z \in \quad) \supset Y \in \alpha)$

Definition of connexivity: $C(\alpha) \leftrightarrow (Y)(Z)((Y \in \alpha \,.\, Z \in \alpha) \supset (Y \in Z \vee Z \in Y \vee Z = Y))$

Definition of foundation: $F(\alpha) \leftrightarrow (Y)[(Y \supset \alpha \,.\, Y \neq \Lambda) \supset (\exists Z)((Z \in Y \,.\, Z \cap Y) = \Lambda)]$

Definition of ordinal: $O(\alpha) \leftrightarrow (T(\alpha) \,.\, C(\alpha) \,.\, F(\alpha))$

See, R.M. Robinson, 'The Theory of Classes, A modification of Von Neumann's System', Journal of Symbolic Logic, 1937, pp.35-36. F.v. Kutschera, Die Antinomien der Logik, pp.40-41.

So a set is an ordinal number if and only if it satisfies the predicates of transitivity, connexivity and foundation. For example, the set $\{\wedge, \{\wedge\}, \{\wedge, \{\wedge\}\}\}$ clearly satisfies all these conditions. Consider now the set of all ordinals. We can easily show that this set should satisfy all three conditions and hence itself must be regarded as an ordinal number. Hence this set must be a member of itself. But the definition of foundation goes against the self-membership of a set. Thus the ordinal number in question satisfies the requirement of foundation, but being self-membered also goes against it. Hence the contradiction.

. Resolution of the paradoxes

1. Zermelo-Fraenkel's and von Neumann's approaches

As the paradoxes are considered in a context of particular definitions, their avoidance is also dependent on these definitions. The common feature of these formulations is that ultimately we reach "biggest" sets, like the set of all cardinals or ordinals, which appear to satisfy the conditions of cardinality or ordinality. These sets either ultimately have to be self-membered - as we have shown in the case of the Burali-Forti paradox, following Robinson's definition of ordinal - or go against Cantor's theorem. Once we exclude such "biggest" sets, we may consistently hold that there is no largest cardinal, ordinal or set of all sets. Zermelo took this line of approach. We cannot construct such "biggest" sets in Zermelo-Fraenkel's axiomatic set theory. The existence of self-membered sets and hence the self-membership of those "biggest" sets is denied by the axiom of regularity. The sets are constructed by axioms like that of Elementarmengen, Potenzmenge or Vereinigung. We cannot talk in this set theory about the set of all cardinals or ordinals as there is an unending hierarchy of sets having ever-increasing cardinality or ordinality.

That we cannot talk in this set theory about such "biggest" sets may seem an unsuitable restriction on mathematical predicates and entities. We may wish to talk about properties such as self-identity which belong to all entities. Or we may likewise wish to talk about each and every cardinal or ordinal. Thus we may wish to define a set which contains all such entities. Von Neumann's approach is to provide for these "biggest" sets like the set of all ordinals or cardinals. He regarded the limitations imposed by the Zermelo-Fraenkel set theory as excessively severe. Accordingly, he divided classes into _sets_ and _proper classes_. All sets are classes as well, but not _vice versa_. Those classes which are not sets are called proper classes. The distinguishing mark of proper classes is that they are not members of any class. So, von Neumann's sets are the same as Zermelo's, and hence his set theory incorporates Zermelo's set theory. But his theory of proper classes is an additional feature. To draw an analogy, we may say that every event has a cause and so construct a series of connected causes; but then we may decide to posit an ultimate event, which is not caused by any other event. Similarly, we may go on constructing classes, or ordinals and cardinals in terms of classes, and then decide that there are ultimate or proper classes which are not members of other classes. These proper classes are typified by the class of all sets or class of all ordinals or cardinals. As these proper classes cannot be members of any class, no self-membership is involved and hence no contradiction results, as long as we keep in view the definition of cardinality and ordinality given above.

Now the following difficulty arises: at what stage of our construction should we say that we have now reached proper classes? For instance, when we reach the ordinal number $\omega^{\omega^{\omega^{\omega^{\omega}}}}$, would the next class only be a proper class? To say simply that we have a proper class of all sets or of all cardinals is just an ad hoc stipulation made to avoid contradiction. Also it seems that the very concept of class as such implies that it can always be a member of another class. Hence, we can always go on forming new classes ad infinitum. It seems artificial to put an end to this infinite process of forming new and different classes.

§2. Intuitionistic approach

So far we have discussed solutions for the antinomies of axiomatic set theory (1) in terms of Russell's theory of types, through applying the vicious circle principle to eliminate characteristics which do not make sense; (2) in terms of Zermelo-Fraenkel's limitation concerning the existence of sets; and (3) in terms of von Neumann's distinction between proper classes and sets. But there is another approach to the problem and that is to dissolve it, saying that there is in reality no problem and no paradoxes insofar as the existence of these is simply the result of our own misguided attempts to construct an axiomatic set theory. This is the intuitionistic approach. For intuitionists, the problem of set-theoretical antinomies does not exist; in their theory we do not have to deal with the paradoxes since the very question of paradox does not arise. Indeed, they take the paradoxes as a sign that the logicist's and the formalist's approaches to set theory are incorrect. According to an intuitionist, we have an intuition of 'units' in succession and thus we have natural numbers. Kronecker, a precursor of the intuitionists, remarked:

"The natural numbers were made by God, all else is the work of man".* The intuitionists thus try to build the edifice of mathematics by construction from natural numbers. Indeed, for them mathematical existence lies in constructibility. "Construction" means the production of something else by starting from simple objects in whose nature we have insight (i.e. natural numbers) through a finite number of steps. For example, the well-ordering theorem is not acceptable to the intuitionists because it asserts the existence of a "well-ordering" without really showing how to construct it. For the same reason, intuitionists do not wholly accept the law of excluded middle. They accept this law only in a limited way, namely where it is "intuitively" clear, as in the case of a finite set. For instance, if S were a finite set of rational numbers, they would accept that S either contains a prime number or it does not. But if S were infinite they would not accept that the falsity of the falsity of an assertion about S implies the truth of the assertion. For example, we do not know yet whether the sequence 1234567 occurs in the decimal expansion of π. If we show that rejection of the hypothesis that this sequence occurs in π leads to absurdity, this is no proof that π contains this sequence. Verifiability is truth. For this reason the intuitionists do not accept Cantor's Theorem. Moreover, because of the concept of constructibility, the intuitionists admit only denumerably infinite sets. On their view, as we cannot construct ascending series of transfinite ordinals and cardinals, we do not come across any paradoxical situation.

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* Cf. R. L. Wilder, Introduction to the Foundation of Mathematics, pp.192-195.

§3. Our suggestion

The mathematics which intuitionism offers is no doubt soundly based, but it puts excessively stringent limitations on our mathematical methods. Cantor's theory of transfinite numbers is a bold attempt to construct a mathematical edifice using only laws of logic and without resorting to introspective construction. We argued in the last chapter that Canto's theory of transfinite numbers, involving the construction of an infinite number of transfinite numbers, cannot, by his proof, be rationally constructed out of and linked with the rational or real number system. Hence, unless some other proof is provided, the Cantorian transfinite edifice tends to obscure the distinction between <u>universal</u> and <u>particular</u> mathematics, which we shall explain now.

Our common-sense belief is that there is one and only one proper mathematics, just as there is one and only one logic (i.e. a system of rules of deduction or entailment). From this common-sense point of view, just as the laws of logic are universal and immutable, so are the theorems of proper mathematics. We endorse the common-sense opinion that a mathematical proposition, say "2 + 2 = 4" holds true in any mathematical or numerical universe just as an entailment rule or a valid logical formula, say "$p \supset p$", holds in any argumentative universe. We may describe a mathematical system which has such validity as universal mathematics. Hence, the theorems of universal mathematics must be presupposed as valid in order to construct any mathematical or numerical system. The theorems of universal mathematics do not tell us how things in the universe actually or possibly <u>are</u>. Although "2 + 2" is valid, i.e. holds universally, we may develop a different sort of mathematical or numerical system in which "2 + 2 = 1" is derivable; and although a x b = b x a holds universally in the domain of real numbers, we may nevertheless develop a non-commutative algebra which holds true for the behaviour of certain actual objects, say in the domain of quantum physics. For the construction of a mathematical system applicable to an actual or imaginary realm we have to regard the theorems of universal mathematics as valid. We shall moreover describe such a mathematical system as an instance of particular mathematics. An instance of particular mathematics lacks the validity which universal mathematics possesses. Just as particular rules concerning the physico-mental universe (say Newton's system, many-valued logic or Hegelian logic) presuppose the rules of entailment as such, similarly the instances of particular mathematics do not affect the validity of universal mathematics. The theorems of particular mathematics are derivable within their particular systems, and which, if regarded as universally true, may even contradict each other's theorems or theorems of universal mathematics. We may assert after Heraclitus that the whole universe is in flux and hence make a rule $p \cdot \sim p$. But even in arguing for or with this principle in rational arguments we have to presuppose the validity of entailment rules or laws of logic where $\sim(p \cdot \sim p)$ holds. Laws of logic are not laws about how things actually or possible behave. Likewise theorems of universal mathematics are not formulae expressing how things in the universe actually (or possibly) behave. The laws of logic are laws of valid reasoning which we employ whatever arguments we are discussing. They are presupposed and implicitly or explicitly exploited in any argument in everyday discourse or in discussing laws or theorems about physical or mental phenomena. Similarly, whatever instance of particular mathematics we construct, it must presuppose and exploit universal mathematics. For example, let us construct a numerical system in which the multiplication of matrices is given as follows:

$$Q \times P = \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} \times \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} c_1a_1 + c_2b_1 & c_1a_2 + c_2b_2 \\ d_1a_1 + d_2b_1 & d_1a_2 + d_2b_2 \end{vmatrix}$$

It can be easily seen that this multiplication is non-commutative. But in defining such a multiplication and thereby creating such a numerical system we have to use commutative algebra when multiplying say c_1a_1 etc. (unless c_1c_2 etc. stand for other complicated concepts involving commutative algebra). Non-commutative algebra is simply a particular numerical system with matrices. Consistency is required both in systems of particular mathematics and in the system of universal mathematics. But the construction of any possible particular mathematics presupposes universal mathematics. And it is in this fact that the validity of universal mathematics lies. In our sense of the term, particular mathematics may or may not be applicable to any particular objects in the universe and existence amounts simply to freedom from contradiction within the particular system. There can be several sorts of particular mathematics. But we hold that there is one and only one universal mathematics to which validity is attached and which cannot be otherwise without foregoing this validity. The objective criterion for this validity of universal mathematics lies in the fact that any numerical or mathematical system must presuppose for its construction the validity of universal mathematics. The subjective criterion for the validity of universal mathematics is provided by our own subjective intuition of realizing that the theorems of universal mathematics are immutably, universally and necessarily true - as, for example, that "2 + 2 = 4" always holds.

Now, what are instances of universal mathematics? Number theory may be cited as an instance of universal mathematics. Again construction of a system of rational or algebraic numbers is to be included in universal mathematics. Now, how do we achieve the validity of universal mathematics? The intuitionists answer that universal mathematics is to be founded on our intuition of 'units' in succession and 'introspective constructions'.* The logicists would say that insofar as mathematics is derivable from logic, it has the validity of logical principles. But it seems clear that mathematics cannot be derived from logic, however sophisticated the logical manipulations we perform. We cannot show that the axiom of infinity, for example, is derivable from logic as we commonly understand the term "logic". On the other hand, if we accept the intuitionistic thesis, we cannot have the full real number system. Nor would we be permitted to employ the law of excluded middle where it seems intuitively legitimate. For example in the case of the number π we seem to be justified in our assertion that its decimal expansion either contains the series 1234567 or it does not contain it.

Our suggestion would be to build up mathematics step-by-step as the intuitionists suggest, but furthermore to widen the concept of "constructibility" and to eliminate the "subjectivity" involved in it. Let us quote Heyting to illustrate the intuitionistic view of mathematical construction:

> (Brouwer's programme) consisted in the investigation of mental mathematical construction as such, without reference to questions regarding the nature of the constructed objects, such as whether these objects exist independently of our knowledge of them. That this point of view leads immediately to the rejection of the principle of excluded middle, I can best demonstrate by an example.**

Then he gives as an example the definition of an integer <u>l</u>:'<u>l</u> is the greatest prime such that <u>l</u>-2 is also a prime, or <u>l</u>=1 if such a number does not exist'.

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* See L.E.J. Brouwer, 'Intuitionism and Formalism' in <u>Philosophy of Mathematics</u> edited by P. Benacerraf and H. Putnam.

** A. Heyting, <u>Intuitionism</u>, pp.1-2. Words in brackets are our own.

Heyting argues that this definition must be rejected. The intuitionists 'consider an integer to be well-defined only if a method for calculating it is given'. It may be argued that we may prove at some later stage that such an integer does exist and that our knowledge is not yet advanced enough to say for certain. So in a conceivable case l may be defined though we cannot actually calculate the number. In reply Heyting says simply:

> In the study of mental mathematical constructions "to exist" must be synonymous with "to be constructed."*

Hence if we cannot construct the integer l, we cannot say whether l=1 or not. Hence the law of excluded middle does not apply. It is clear that the intuitionists identify the concept of "true" (in mathematics) with that of "verifiable". If we cannot show the manner of verification or falsification of a mathematical theorem, we cannot talk of its truth or falsity. We may reasonably reject this thesis and assert that "true" is a wider concept than verifiable and cannot be identified with it. Why can only our 'introspective constructions' lead to the truth of validity of mathematics? Our extension of the view of constructibility would obviously permit the law of excluded middle as for instance in the case of the assertion that π either has or has not the sequence 1234567 in its decimal expansion. No doubt the intuitionists' method of 'introspective construction' provides us with methods to prove and construct mathematical theorems. But why should we limit ourselves to this method? If we attach objective truth or validity to logical theorems, why should we not do the same with mathematical truth? Thus the intuitionists may be accused of having a myopic view of reality.

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* Ibid.

Furthermore our insight or intuitive understanding guides us to a decision as to which methods are to be used as we continue to develop mathematics. In our effort to build up mathematics we are aided by the notation and the various concepts used in mathematics. They do help us in our understanding and appreciation of mathematical lay-out and structure, and they shed light on the reasoning involved in mathematics. The change from Roman numerals to Arabic ones was a great step in the development of mathematics. The work performed by algebraic numbers cannot be achieved by rational numbers alone. The introduction of the concept of limit marked a step further forward in the formation and expansion of mathematics. The things which we would like to express in terms of limit cannot be expressed in terms of definite rational numbers alone. Similarly, the introduction of the concept of set is another step in the development of mathematics. At each stage of development we meet several sorts of difficulties and we have to use methods of reasoning which we were not accustomed to before. We simply cannot legistlate beforehand that only such-and-such types of arguments are permitted in mathematics. At each stage of development we have to assess and examine the validity of the arguments used. These considerations can be summed up in two conclusions:

(1) The whole of mathematics cannot be moulded and shaped in a strictly formalized system. Godel's incompleteness theorem supports this thesis.

(2) We cannot legislate beforehand that certain types of arguments are the only ones to be used in mathematics. This assertion runs counter to the formalists' programme as well as against the intuitionists who insist that mathematical existence is only possible through 'constructions'.

We noted in the last chapter that Cantor's argument, demonstrating that the cardinality of a given set is less than that of the set of all sub-sets of the set, is not sound. Were Cantor's argument valid, we would be justified in constructing a series of an infinite number of transfinite cardinals.

Our point of view is to eliminate the predominant note of subjectivity in the thesis of the intuitionists. Intuitionists assert that we have an intuition of 'units' in succession. From this we derive natural numbers and thence by 'construction' we arrive at rational and algebraic numbers. In short, mathematics is valid only so long as it is constructed from this basic intuition. One may say: we accept that we have this intuition, but why should the mathematical system based on this intuition have a superior status? It used to be said that we have 'intuition' of Euclidean space, but how can Euclidean geometry be regarded as superior to non-Euclidean geometries? The construction of a non-Euclidean geometry in no way logically presupposes the theorems of Euclidean geometry. Again our perception of Euclidean space cannot necessarily be regarded as a universal feature of the universe, as our investigation into the universe has led us to believe. On similar lines one may argue that mathematics based on the intuitionist's thesis of intuition of 'units' in succession and 'construction' may not be valid. It is no more than a reflection of the intuitive powers of the human being. Another being, say a superman, may have a different intuition of "units"; he may, for example, always apprehend two units and two units as three units in succession. Our reply to this criticism and fantastic claim is that this sort of criticism may be levelled against logic or entailment rules in general as well. Our position against intuitionists is that we claim the objective validity of mathematics. Our mark of validity for entailment or logical rules like p implies p is that if our attempts to reason are out of harmony with logical rules, then we soon find ourselves in confusion. We cannot argue unless we recognise implicitly or explicitly the validity of entailment rules. Similarly, a mark of validity for universal mathematics lies in the fact that we cannot construct any mathematical or numerical system without the assumption that the former is valid. From our point of view, our intuition of 'units' in succession and the intuitionists' method of 'construction' are simply means to discover universal mathematics. We may even furthermore assert that we have an intuition of "duration", a "mental continuum" and further that we have an innate intuition or capacity to break any duration or continuum into parts. The analysis of such intuitions may be expressed in the real number system. Again, one may assert that we derive natural numbers from our external observation of distinct objects and the real number system is an effort to express in a number system an objectively seen line. All these intuitions, insights or observations are no more than means of discovering and constructing universal mathematics and they help each other in the development and mapping out of the structure of universal mathematics. And our assertion that there is a universal mathematics is based on the fact that it has validity which particular mathematical systems cannot help but acknowledge and exploit. Our basis of universal mathematics seems to involve circularity: it rests on our intuitions, clear perceptions and certain insights which are corrected and justified by the objective criterion of validity and vice versa. But the same thing holds true for logical truths, i.e. for entailment rules. Logical principles are intuitively given and justified by their consistency and validity and vice versa.

It follows from the above discussion that we should not have such a narrow view of construction as that of intuitionists, but rather accept any argument or method which seems to our rationality sound. As suggested these rational arguments cannot be specified once and for all, but are discovered as we proceed with the construction of universal mathematics.

We are justified in positing only two transfinite cardinals, namely that associated with a denumerably infinite set (e.g. the set of all natural numbers) and that associated with an indenumerably infinite set (e.g. the set of all real numbers), as discussed in the last chapter. We are not justified in constructing a series of ascending transfinite cardinals without any rationally sound proof. Hence we are not presented with the paradox of the greatest cardinal or ordinal. Our view is to maintain strictly the distinction between universal and particular mathematics which we formulated earlier. Hence to our mind, the status of the Cantor edifice of transfinite numbers in the system, insofar as it admits more than two transfinite cardinals, is that of particular mathematics, unless it is demonstrated to be otherwise. That is, theorems relating to Cantor's system, which imply that we have an infinite number of transfinite numbers, do not have universal validity like the theorems of finite cardinal arithmetic, e.g. $2 + 2 = 4$. If this system serves some useful purpose as particular mathematics, there is all the more reason to develop it. And for this overgrowth on universal mathematics we need to have consistency as a criterion for the development of transfinite set theory. One proof that axiomatic set theory, involving infinite number of transfinite cardinals, should not be regarded as a part of universal mathematics, is that we notice that different sorts of axiomatic set theories, as for example that of Zermelo-Fraenkel or that of von Neumann, have been constructed, and that the theorems from these even contradict each other when given the status of universal mathematics. As these set-theories are logically independent of each other, they do not show the sort of validity which universal mathematics is supposed to possess.

It may be asserted that there exists no ultimate distinction between universal and particular mathematics; that a sheer conceptual approach in building a mathematical or numerical system may not presuppose the validity of universal mathematics; that all the mathematical systems are just conceptual games in which the respective systems of rules of each are essentially logically independent of each other. But we have no proof for such an assertion and it runs counter to our intuitive belief and the arguments which we have been pursuing. We see the impossibility of creating a particular mathematical system without presupposing the validity of universal mathematics. Unless we have strong reason to repudiate this distinction we should uphold it. Further research may throw helpful light on this point. In case there is ultimately no such distinction, then there are in fact no such paradoxes to be resolved, in the sense that their emergence does not lie in our rationality and that they emerge because of our own resolution to construct such a system. In short, then, there is no philosophical problem about these paradoxes because philosophical speculation in general lies in the elucidation and construction of rational thought.

Appendix

The paradox of the Unpredictable Examination

The paradox may be stated as follows: A teacher announces in class that there will be an unexpected examination on some day during the next week; i.e. either on Monday, or on Tuesday, or on Wednesday, or on Thursday, or on Friday. By an unexpected examination he means that the students will not be able to know on which day there will be an examination before the day of the examination. One student objects by saying that there cannot be such an examination and that the teacher has contradicted himself. He argues, if the examination is held on Friday, the students will come to know the day of the examination on Thursday night because Friday will be the only possible day left for the examination. Further, such an examination cannot be held on Thursday because the students will come to know on Wednesday night that as the examination cannot be held on Friday, the only day left for the examination is Thursday. For similar reasons the examination cannot be held on any of the other days. But the teacher retorts that if he holds the examination, say on Tuesday, how could the students know it beforehand? So, such an examination can be held. Hence the contradiction emerges that such an examination can be and cannot be held. It arises from the fact that we permit the students to suppose that they will be able to make more than one (here it would be five) prediction in the course of the week; but the teacher's conclusion that the examination can be held, follows from forbidding more than one prediction. Hence the two contradictory conclusions do not arise from the same premisses. The paradox is resolved, once the nature and number of predictions (the nature of prediction being interpreting prediction of logical deduction) permitted is made unambiguous and definite.

The paradox of Protagoras and his pupil

It is stated that Protagoras tutored his pupil on the agreement that the pupil would pay the fees in a lump sum only when he first won a case at court. The pupil abandoned the idea of practising law after completing his studies. Protagoras got worried and impatient about his fees. At last he sued his pupil in court and argued that if he (Protagoras) won the case, then by the decision of the court his pupil would have to pay the fees; but if he lost the case, then by the agreement his pupil would have to pay the fees. So whether he lost or won, he would get the payment. The pupil, in his defence, argued that if he lost the case, then by the agreement he would not have to pay the fees; and if he won, then by the decision of the court he did not have to pay. The judge got confused and adjourned the case. Perhaps the solution lies in simply dismissing the case and thus no question of losing or winning the case on either side would arise, that is just maintaining the status quo.

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