ALLIED SUBSETS OF TOPOLOGICAL GROUPS AND LINEAR SPACES

By

GRAHAM J.O. JAMESON

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Faculty of Science

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CONTENTS

NOTATION

Notational devices

The results in section n are numbered n.1, n.2, ... , with no attempt to classify them into theorems, propositions, lemmas and corollaries. The end of a proof is marked by ‡.

Index of definitions and symbols

INTRODUCTION

Let X be a topological linear space. The condition for a cone in X to be normal and the condition for the sum of two complete linear subspaces of X to be complete are both special cases of a relation between pairs of subsets of X which is the object of study of this thesis. All that is required for the definition is the structure of a topological group. Let A, B be subsets of a topological group X. We say that A is ALLIED to B, and write A al B , if, given a neighbourhood M of the identity e, there exists a neighbourhood N of e such that

$$a \in A,\ b \in B,\ ab \in N \Rightarrow a, b \in M.$$

The thesis is intended primarily as an attempt to examine the implications of this definition for their own sake. However, some of the most interesting results are applications, in the sense that allied sets appear in the proof but not in the statement. The deepest result of this sort is a kind of open mapping theorem (5.6).

The basic theory is dealt with in section 1. Some equivalent forms of the definition are given; the one using nets (1.5) is particularly useful. 1.13 is basic to all applications involving compactness. An example is given to show that, in a non-commutative group, the relation need not be symmetric.

In section 2, we study the extra theory peculiar to linear spaces. For stars, alliedness is determined within any neighbourhood of 0 (2.1). 2.2 and 2.3 show the relation between allied sets and boundedness. Numerical characterisations are available in normed linear spaces, and further ones in inner product spaces.

If A, B are allied subgroups of a commutative topological group, and (a_n+b_n) is a Cauchy net $(a_n \in A,\ b_n \in B)$, then the nets (a_n) and (b_n) are Cauchy. Hence if A is complete, then A+B is complete (or closed) if B is. In section 3, we show that this and further statements can be formulated also in non-commutative groups. Restricted converses are obtained in section 4, where it is shown that alliedness of subgroups is equivalent to the continuity of certain homomorphisms.

Section 5 is concerned with locally compact subsets of topological linear spaces. The results here are mainly applications, in the sense defined above.

Section 6 is concerned with the question of whether sets which are allied with respect to one topology are allied with respect to another one. It is shown that two different topologies for the same group will certainly give rise to some pairs which are allied with respect to one but not the other. Some positive results are then obtained for certain kinds of subsets of topological linear spaces.

Sections 7,8 and 9 form the part of the thesis that is most obviously related to the theory of partially ordered groups and linear spaces. A cone or semigroup is self-allied if and only if it is normal. This is a concept of fundamental importance in the theory of partially ordered linear spaces, and our definition enables us to generalise results about normal cones to statements about allied pairs of sets. Many of the results of section 7 are of this sort, although some of them are "generalisations" of theorems which do not appear to have been explicitly stated before. 7.7 and 7.11 are the most important.

Sections 8 ("Open decomposition") and 9 ("Applications to lattices") are not primarily devoted to allied sets, but show their relevance to the topics considered.

In section 10, we show how, in a commutative group, allied families of sets can be defined, and how some of the properties of allied pairs can be extended to allied families.

Appendices 1 and 2 deal with related topics which could not be introduced elsewhere without disturbing the continuity. Appendix 1 is relevant to section 5, and appendix 2 to section 6.

Considerable prominence is given to counter-examples throughout the work. The author feels that, in places, these are more elegant than the positive results !

Most of the material is being published in (7) and (8). Appendix 3 specifies which parts are not.

I am greatly indebted to my research supervisor, Prof. F.F. Bonsall, who has been a constant source of inspiration and guidance during the period in which this work was done. I am also grateful to Dr. G. Brown for drawing my attention to (21).

1. BASIC THEORY

Let X be a topological group (X may or may not be a topological linear space). The following notation will be used consistently. The identity in X will be denoted by e, or, when X is known to be commutative, by 0. The family of all neighbourhoods of the identity will be denoted by $\eta(X)$. By a LOCAL BASE we mean a base of neighbourhoods of e. It is well-known that the closed, symmetric neighbourhoods of e form a local base (a subset A of X is SYMMETRIC if $A^{-1} = A$). Our basic definition, already given in the introduction, is the following: if A, B are subsets of X, then A is said to be ALLIED to B if, given $M \in \eta(X)$, there exists $N \in \eta(X)$ such that

$$a \in A,\ b \in B,\ ab \in N \Rightarrow a, b \in M.$$

This statement will be denoted by A al B , and the contrary statement by A nal B .

It is clearly sufficient for A al B if the condition above holds for all M in some local base. Further, if it holds, then we may take N to be closed, symmetric, or contained in M, or with any combination of these properties.

Our first result shows that an apparently weaker condition is equivalent:

<u>1.1.</u> A al B if and only if, given $M \in \eta(X)$, there exists $N \in \eta(X)$ such that

$$a \in A,\ b \in B,\ ab \in N \Rightarrow a \text{ or } b \in M.$$

<u>Proof</u>. Suppose the condition holds and $M \in \eta(X)$ is

given. Take symmetric $N \in \mathcal{N}(X)$ such that $N^2 \subseteq M$. There exists $P \in \mathcal{N}(X)$ such that $P \subseteq N$ and

$$a \in A,\ b \in B,\ ab \in P \ \Rightarrow\ a \text{ or } b \in N.$$

Suppose that $a \in A$, $b \in B$, and $ab \in P$. Then $a \in N$ or $b \in N$. If $a \in N$, then $b = a^{-1}(ab) \in NP \subseteq M$. If $b \in N$, then $a = (ab)b^{-1} \in PN \subseteq M$. In either case, both a and b are in M, giving the result. ‡

In particular, it is sufficient for A al B if, given $M \in \mathcal{N}(X)$, there exists $N \in \mathcal{N}(X)$ such that

$$a \in A,\ b \in B,\ ab \in N \ \Rightarrow\ a \in M.$$

This fact will be used repeatedly. As an immediate corollary, we have:

<u>1.2.</u> A nal B if and only if there exists $M \in \mathcal{N}(X)$ such that, for each $N \in \mathcal{N}(X)$, there exist $a_N \in A \sim M$, $b_N \in B$ with $a_N b_N \in N$. ‡

We note some trivial consequences of the definition. If A al B and $A' \subseteq A$, $B' \subseteq B$, then A' al B'. If A al B, then $(A \cup \{e\})$ al $(B \cup \{e\})$. If $e \notin \overline{AB}$ (i.e. the closure of AB), then A al B , by vacuous implication. By 1.1, A al $\{e\}$ and $\{e\}$ al A for all A. A nal A^{-1} unless $A \subseteq \overline{\{e\}}$.

Two simple examples follow. Examples of a less trivial nature will be given when the requisite theory has been developed.

(i) In the additive group R of real numbers with the usual topology, let A denote the closed interval $[0,1]$, and B the open interval $(-2,-1)$. Given $\varepsilon > 0$, put $a = 1$,

$b = -1-\varepsilon$. Then $|a+b| = \varepsilon$, so A nal B .

(ii) Let X be the additive group R^2 with the usual topology, and let

$$A = \{(\xi,\eta) : \xi > 0\},$$
$$B = \{(\xi,\eta) : \xi > \delta\}$$

for some $\delta > 0$. All elements of $A+B$ are at a distance at least δ from 0, so A al B . However, A nal A since, for any $\varepsilon > 0$, $(\varepsilon,1)$ and $(\varepsilon,-1)$ are in A, and have sum $(2\varepsilon,0)$.

Further equivalent forms of the definition are given by the next theorem.

<u>1.3.</u> Each of the following statements is equivalent to A al B :

(i) given $M \in \mathfrak{N}(X)$, there exists $N \in \mathfrak{N}(X)$ such that
$$(NA)\cap(NB^{-1}) \subseteq M ,$$

(ii) given $M \in \mathfrak{N}(X)$, there exists $N \in \mathfrak{N}(X)$ such that
$$A \cap(NB^{-1}) \subseteq M ,$$

(iii) given $M \in \mathfrak{N}(X)$, there exists $N \in \mathfrak{N}(X)$ such that
$$\text{for } a \in A\sim M, \quad (Na)\cap B^{-1} = \emptyset .$$

<u>Proof</u>. Suppose that A al B and $M \in \mathfrak{N}(X)$ is given. Take $M_1 \in \mathfrak{N}(X)$ such that $M_1^2 \subseteq M$. There exists $N_1 \in \mathfrak{N}(X)$ such that $N_1 \subseteq M_1$ and

$$a \in A,\ b \in B,\ ab \in N_1 \implies a \in M_1 .$$

Take symmetric $N \in \mathfrak{N}(X)$ such that $N \subseteq M_1$ and $N^2 \subseteq N_1$. Suppose that $x = n_1a = n_2b^{-1}$, where $a \in A$, $b \in B$ and $n_i \in N$ $(i = 1,2)$. Then $ab = n_1^{-1}n_2 \in N_1$, so $a \in M_1$, and

$x \in NM_1 \subseteq M$. Hence (i) holds.

(i) implies (ii) a priori. If (ii) holds, with N chosen symmetric, and $na \in B^{-1}$ (where $n \in N$, $a \in A$), then $a \in A \cap (NB^{-1})$, so $a \in M$. Hence (iii) holds.

Suppose that (iii) holds, with N chosen symmetric, and that $a \in A$, $b \in B$ and $ab = n \in N$. Then $n^{-1}a \in B^{-1}$, so $a \in M$. Hence A al B. ‡

An even stronger form of the definition follows with ease:

1.4. A al B if and only if, given $M \in \eta(X)$, there exists $N \in \eta(X)$ such that $\overline{NA} \cap \overline{NB^{-1}} \subseteq M$.

Proof. The condition is clearly sufficient. Suppose that A al B. By 1.3(i), there exists $P \in \eta(X)$ such that $(PA) \cap (PB^{-1}) \subseteq M$. Take $N \in \eta(X)$ such that $N^2 \subseteq P$. Then $\overline{NA} \subseteq PA$, $\overline{NB^{-1}} \subseteq PB^{-1}$, so $\overline{NA} \cap \overline{NB^{-1}} \subseteq M$. ‡

Loosely speaking, A al B means that, away from e, A is remote from B^{-1}. If we put $ab^{-1} \in N$ instead of $ab \in N$ in the defining condition, we would ensure that A was remote from B away from e. This definition might seem more natural - in fact, the author used it at first - but it turns out to be less convenient to work with.

Characterisation by nets

Considerable use is made of directed nets in the sequel. There will normally be no need to mention the underlying directed set, and we shall often use the notation

(x_n) for a net, instead of writing $\{x_n : n \in D\}$.

Let (x_n) be a net in AB. Then each x_n can be expressed (not necessarily uniquely) in the form $a_n b_n$, where $a_n \in A$, $b_n \in B$. Having chosen such an expression for each n, (a_n) and (b_n) are nets in A and B respectively. We can specify both the net (x_n) and the choice of a_n, b_n by speaking of the net $(a_n b_n)$, and this policy will be adopted henceforth.

Alliedness is characterised in terms of nets as follows:

<u>1.5</u>. (i) If A al B and $(a_n b_n)$ is a net convergent to e (where $a_n \in A$, $b_n \in B$), then the nets (a_n) and (b_n) converge to e.

(ii) If for each net $(a_n b_n)$ convergent to e (with $a_n \in A$, $b_n \in B$), e is a cluster point of (a_n), then A al B. In a metrisable group, it is sufficient if this condition holds for sequences.

<u>Proof</u>. (i) Given $M \in \mathcal{N}(X)$, there exists $N \in \mathcal{N}(X)$ such that

$$a \in A,\ b \in B,\ ab \in N \implies a, b \in M .$$

There exists n_o such that for $n \geq n_o$, $a_n b_n \in N$. For such n, a_n and b_n are in M, and the result follows.

(ii) If A nal B, then there exists $M \in \mathcal{N}(X)$ such that, for each $N \in \mathcal{N}(X)$, there exist $a_N \in A \sim M$, $b_N \in B$ with $a_N b_N \in N$. Let $\mathcal{B}$ be a local base. Then $\{a_N b_N : N \in \mathcal{B}\}$ is a net convergent to e, while e is not a cluster point of (a_N). In a metrisable group, we can take $\mathcal{B}$ countable, thus obtaining a sequence. ‡

Semigroups, orderings and filters

Let A be a subset of a commutative topological group X. By 1.3(i), A al A if and only if, given $M \in \mathfrak{N}(X)$, there exists $N \in \mathfrak{N}(X)$ such that $(N+A) \cap (N-A) \subseteq M$. If A is a semigroup containing 0, then an associated partial ordering $\leqslant$ is defined by

$$x \leqslant y \iff y-x \in A .$$

In terms of this notation, A al A if and only if, given $M \in \mathfrak{N}(X)$, there exists $N \in \mathfrak{N}(X)$ such that if $y, z \in N$ and $y \leqslant x \leqslant z$, then $x \in M$. This is the well-known definition of a NORMAL semigroup or associated ordering (though the concept has been applied mostly to cones in linear spaces). However, we shall speak of SELF-ALLIED sets rather than normal ones, to avoid confusion with normal subgroups in the algebraic sense.

Returning to the case of a general topological group, if a pair of subsets A, B is given, write

$$[E] = (EA) \cap (EB^{-1})$$

for any subset E. 1.3 says that A al B if and only if, given $M \in \mathfrak{N}(X)$, there exists $N \in \mathfrak{N}(X)$ such that $[N] \subseteq M$. We notice that if $\mathcal{F}$ is a filter base, then so is $[\mathcal{F}]$, since $[F_1 \cap F_2] \subseteq [F_1] \cap [F_2]$.

If A induces the ordering $\leqslant$ as above, and $B = A$, then $[[E]] = [E]$, and $[E]$ is the order-convex cover of E. A is self-allied if and only if there is a local base consisting of neighbourhoods N such that $[N] = N$, in other words if and only if X is locally order-convex.

The next theorem shows how alliedness can be characterised in terms of filters. Nets, however, turn out to be of more interest than filters in the sequel.

1.6. Each of the following statements is equivalent to A al B :

(i) if a filter base $\mathcal{F}$ converges to x, then $[\mathcal{F}] \to x$,

(ii) if a filter base $\mathcal{F}$ converges to e, then $[\mathcal{F}] \to e$.

Proof. Suppose that A al B and $\mathcal{F} \to x$. Given $M \in \mathcal{N}(X)$, there exists $N \in \mathcal{N}(X)$ such that $[N] \subseteq M$. There is a member F of $\mathcal{F}$ contained in xN. Then $(x^{-1}FA) \cap (x^{-1}FB^{-1})$ is contained in M, so $[F] \subseteq xM$. Hence $[\mathcal{F}] \to x$.

(i) implies (ii) a priori. Suppose that (ii) holds. $\mathcal{N}(X)$ is a filter convergent to e, so, by (ii), given $M \in \mathcal{N}(X)$, there exists $N \in \mathcal{N}(X)$ such that $[N] \subseteq M$. Hence A al B , by 1.3. ‡

Some elementary properties

1.7. If A al B and A al C , then A al $(B \cup C)$.

Proof. Given $M \in \mathcal{N}(X)$, there exist $N, P \in \mathcal{N}(X)$ such that

$$a \in A,\ b \in B,\ ab \in N \Rightarrow a \in M ,$$

$$a \in A,\ c \in C,\ ac \in P \Rightarrow a \in M .$$

If $a \in A$, $x \in B \cup C$, and $ax \in N \cap P$, then $a \in M$. ‡

1.8. If A al B, then $A \cap B^{-1} \subseteq \overline{\{e\}}$, $A^{-1} \cap B \subseteq \overline{\{e\}}$. In particular, if X is Hausdorff, then $A \cap B^{-1}$ is $\{e\}$ or $\emptyset$.

Proof. Suppose $x \in A \cap B^{-1}$. Take $M \in \mathcal{N}(X)$. We show that $x \in M$, from which the result follows. There exists $N \in \mathfrak{n}(X)$ such that

$$a \in A,\ b \in B,\ ab \in N \ \Rightarrow\ a \in M .$$

Now $x \in A$, $x^{-1} \in B$ and $xx^{-1} = e \in N$. Hence $x \in M$, as required. ‡

<u>1.9.</u> If A al B, then $\bar{A}$ al $\bar{B}$.

<u>Proof.</u> By 1.3, given $M \in \mathfrak{n}(X)$, there exists $N \in \mathfrak{n}(X)$ such that $(NA) \cap (NB^{-1}) \subseteq M$. Take $P \in \mathfrak{n}(X)$ such that $P^2 \subseteq N$. Then $P\bar{A} \subseteq NA$, $P\bar{B}^{-1} \subseteq NB^{-1}$, so, again by 1.3, $\bar{A}$ al $\bar{B}$. ‡

Combining the last two results, we have:

<u>1.10.</u> If A al B, then $\bar{A} \cap \bar{B}^{-1} \subseteq \overline{\{e\}}$. ‡

<u>1.11.</u> If A al (BC) and B al C, then (AB) al C.

<u>Proof.</u> Given $M \in \mathfrak{n}(X)$, there exists $N \in \mathfrak{n}(X)$ such that

$$b \in B,\ c \in C,\ bc \in N \ \Rightarrow\ c \in M .$$

There exists $P \in \mathfrak{n}(X)$ such that if $a \in A$, $b \in B$, $c \in C$ and $a(bc) \in P$, then $bc \in N$. Thus $(ab)c \in P \Rightarrow c \in M$. ‡

<u>1.12.</u> Let X be commutative. If A, B are self-allied semigroups, and A al B, then $A+B$ is self-allied.

<u>Proof.</u> Suppose that

$$(a_n+b_n)+(a_n'+b_n') \rightarrow 0 ,$$

this being a net with $a_n, a_n' \in A$ and $b_n, b_n' \in B$. Since $a_n+a_n' \in A$, $b_n+b_n' \in B$, and A al B, we have $a_n+a_n' \rightarrow 0$, by 1.5(i). Since A al A, $a_n \rightarrow 0$. Similarly, $b_n \rightarrow 0$. Hence $a_n+b_n \rightarrow 0$, and $A+B$ is self-allied, by 1.5(ii). ‡

Symmetry of the relation

B al A means: given $M \in \mathcal{N}(X)$, there exists $N \in \mathcal{N}(X)$ such that

$$a \in A,\ b \in B,\ ba \in N \Rightarrow a, b \in M .$$

Taking symmetric neighbourhoods, it is clear that B al A is equivalent to A^{-1} al B^{-1}. Thus one condition which is sufficient for B al A to be equivalent to A al B is that A and B should be symmetric.

Another condition which is obviously sufficient is

$$a \in A,\ b \in B \Rightarrow ab = ba ,$$

a statement which will be denoted by A comm B .

We now give an example of non-symmetry. The set

$$\{(\xi_1, \xi_2) : \xi_1 > 0 \text{ and } \xi_2 \text{ real}\}$$

is a group under the operation

$$(\xi_1, \xi_2)(\eta_1, \eta_2) = (\xi_1\eta_1 , \xi_2 + \xi_1\eta_2) .$$

This is, in fact, the multiplication obtained by regarding the elements as matrices of the form

$$\begin{pmatrix} \xi_1 & \xi_2 \\ 0 & 1 \end{pmatrix}.$$

The identity is (1,0), and the inverse of (ξ_1, ξ_2) is $(\xi_1^{-1}, -\xi_1^{-1}\xi_2)$. From the continuity of addition, multiplication and inversion of real numbers, it is clear that the usual topology for R^2 makes this a topological group.

Let

$$A = \{(\alpha_1, \alpha_2) : \alpha_1 \text{ and } \alpha_2 > 0\} ,$$

$$B = \{(\beta, 1) : \beta > 0\} .$$

$a_n = (n^{-1}, n^{-2}) \in A$, $b_n = (n, 1) \in B$. $a_n b_n = (1, n^{-1}+n^{-2})$, so $a_n b_n \rightarrow (1,0)$, while $a_n \not\rightarrow (1,0)$. Thus, by 1.5, A nal B.

On the other hand,

$$(\beta,1)(\alpha_1,\alpha_2) = (\beta\alpha_1 \, , \, 1+\beta\alpha_2) \, ,$$

and the distance of all such elements from (1,0) is greater than 1. Hence B al A , by vacuous implication.

<u>One set compact</u>

The next result is basic to much of our later work (especially section 5). We give two different proofs, one using 1.3 and the other using directed nets.

<u>1.13.</u> If A is compact, and $A \cap \bar{B}^{-1} \subseteq \overline{\{e\}}$, then A al B and B al A .

<u>Proof 1.</u> Take open $M \in \mathcal{N}(X)$. If $a \in A \sim M$, then $a \notin \bar{B}^{-1}$, so there exists $N(a) \in \mathcal{N}(X)$ such that

$$N(a)a \cap B^{-1} = \emptyset \, .$$

Take symmetric $P(a) \in \mathcal{N}(X)$ such that $P(a)^2 \subseteq N(a)$. Since $A \sim M$ is compact, there exists a finite set of points $a_1, \ldots, a_n$ such that $A \sim M \subseteq \bigcup_{i=1}^{n} (P_i a_i)$, where $P_i = P(a_i)$. Let $P = \bigcap_{i=1}^{n} P_i$. If $a \in A \sim M$, then $a \in P_i a_i$ for some i, and $Pa \subseteq N(a_i)a_i$, so does not meet B^{-1}. Therefore A al B , by 1.3(iii). ‡

<u>Proof 2.</u> Let $(a_n b_n)$ be a net convergent to e $(a_n \in A,$ $b_n \in B)$. Since A is compact, (a_n) has a cluster point a_o in A. Then a_o is also a cluster point of the net (b_n^{-1}), and so $a_o \in A \cap \bar{B}^{-1}$, giving $a_o \in \overline{\{e\}}$. It follows that e is a

cluster point of (a_n). Hence A al B, by 1.5.

Since A^{-1} is compact and $A^{-1} \cap \bar{B} \subseteq \overline{\{e\}}$, we have also A^{-1} al B^{-1}, i.e. B al A. ‡

We saw on p. 5 that, in the additive group R with the usual topology, $[0,1]$ is not allied to $(-2,-1)$. This is sufficient to show that the condition $A \cap \bar{B}^{-1} \subseteq \overline{\{e\}}$ cannot be replaced by the weaker condition $A \cap B^{-1} \subseteq \overline{\{e\}}$ in 1.13.

Homomorphic images

We shall say that a pair of subsets A, B of a topological group is PSEUDO-DISJOINT if $A \cap B^{-1} \subseteq \{e\}$. Alliedness is clearly a topological embellishment of this relation. If T is a homomorphism with kernel K, and the pairs A, K and TA, TB are pseudo-disjoint, then so is the pair A, B. An analogous result holds for allied sets:

1.14. Let X, Y be topological groups, and let T be a continuous homomorphism $X \to Y$ wth kernel K. Suppose that (TA) al (TB), A al K, and that, given $Q \in \mathfrak{n}(X)$, there exists $R \in \mathfrak{n}(Y)$ such that $(TQ) \cap (TA) = R \cap (TA)$. Then A al B.

Proof. Suppose that A nal B. Then there exists $M \in \mathfrak{n}(X)$ such that, given $N \in \mathfrak{n}(X)$, there exist $a \in A \sim M$, $b \in B$ with $ab \in N$.

Since A al K, there exists $Q \in \mathfrak{n}(X)$ such that

$$a \in A,\ k \in K,\ ak \in Q \Rightarrow a \in M .$$

There exists $R \in \mathfrak{n}(Y)$ such that $(TQ) \cap (TA) = R \cap (TA)$.

Take $P \in \mathfrak{n}(Y)$. There exists $N \in \mathfrak{n}(X)$ such that $TN \subseteq P$. Take $a \in A \sim M$ and $b \in B$ such that $ab \in N$. Since $a \notin M$, we have $ak \notin Q$ for $k \in K$. Hence $Ta \notin TQ$, so $Ta \notin R$. But $(Ta)(Tb) \in P$. Thus (TA) nal (TB). ‡

The last condition in the theorem is satisfied, in particular, if $e \in A$ and the restriction of T to A is open at e.

If T is a homomorphism with kernel K, and the pairs A, B and AB, K are pseudo-disjoint, then so is the pair TA, TB. Again we have a corresponding result for allied sets:

<u>1.15</u>. Let X, Y be topological groups, and let T be a continuous homomorphism $X \to Y$ with kernel K. Suppose that A al B, (AB) al K, and that, given $Q \in \mathfrak{n}(X)$, there exists $R \in \mathfrak{n}(Y)$ such that $(TQ) \cap (TAB) = R \cap (TAB)$. Then (TA) al (TB).

<u>Proof</u>. Suppose that (TA) nal (TB). Then there exists $M \in \mathfrak{n}(Y)$ such that, given $R \in \mathfrak{n}(Y)$, there exist $a \in A$, $b \in B$ with $Ta \notin M$ and $T(ab) \in R$.

Take $N \in \mathfrak{n}(X)$. Since (AB) al K, there exists $Q \in \mathfrak{n}(X)$ such that

$$a \in A,\ b \in B,\ k \in K,\ abk \in Q \implies ab \in N .$$

There exists $R \in \mathfrak{n}(Y)$ such that $(TQ) \cap (TAB) = R \cap (TAB)$. Take $a \in A$, $b \in B$ such that $Ta \notin M$ and $T(ab) \in R$. Then $T(ab) \in TQ$, so there exists $k \in K$ such that $abk \in Q$. Thus we have $a \notin T^{-1}M$, while $ab \in N$. Hence A nal B. ‡

Since the projection onto a quotient group is an open, continuous homomorphism, we have:

1.16. Let K be a normal subgroup of X, and P the projection of X onto X/K. If A al B and (AB) al K , then (PA) al (PB) . ǂ

An example given at the end of section 2 shows that this result can fail if AB is not allied to K, even if it is still pseudo-disjoint to it.

2. ALLIED SETS IN LINEAR SPACES

Basic property of stars

A subset A of a real or complex linear space will be said to be A STAR, POSITIVE HOMOGENEOUS or HOMOGENEOUS if $\lambda A \subseteq A$ holds, respectively, for $0 \leqslant \lambda \leqslant 1$, for all $\lambda \geqslant 0$, or for all scalars λ. If A is convex and $a \in A$, then $A-a$ is a star.

By a WEDGE we mean a positive homogeneous set A such that $a, b \in A \Rightarrow a+b \in A$. If, in addition, $A \cap (-A) = \{0\}$, we call A a CONE.

Throughout this section, X will be a topological linear space. We notice that a positive homogeneous subset A of X is closed if and only if $A \cap M$ is closed for some $M \in \mathfrak{n}(X)$. The basic theorem relating alliedness to scalar multiplication is:

2.1. Let A, B be stars in a topological linear space. If there is a neighbourhood M of 0 such that $(A \cap M)$ al $(B \cap M)$, then A al B.

Proof. Suppose that A nal B. Let $M \in \mathfrak{n}(X)$ be given. Take circled $N \in \mathfrak{n}(X)$ such that $N+N \subseteq M$. There exists $P \in \mathfrak{n}(X)$ such that $P \subseteq \frac{1}{2} N$ and such that, for each circled $Q \in \mathfrak{n}(X)$, there exist $a \in A \sim P$, $b \in B$ with $a+b \in Q \cap N$.

There exists $\lambda \in (0, 1]$ such that $\lambda a \in N \sim P$. Then $\lambda(a+b) \in N$, so $\lambda b \in N+N \subseteq M$. Thus we have $\lambda a \in (A \cap M) \sim P$ and $\lambda b \in B \cap M$, while $\lambda a + \lambda b \in Q$. Hence $(A \cap M)$ nal $(B \cap M)$. ‡

Obviously, the behaviour near 0 gives no information if A or B is not a star.

Boundedness

2.2. (i) If A, B are allied stars, and $(a_n + b_n)$ is a bounded net $(a_n \in A,\ b_n \in B)$, then the nets (a_n) and (b_n) are bounded.

(ii) If X is a metrisable topological linear space, and A, B are non-allied, positive homogeneous subsets, then there is a sequence $(a_n + b_n)$ convergent to 0 $(a_n \in A,\ b_n \in B)$ with (a_n) and (b_n) unbounded.

Proof. (i) Given $M \in \mathcal{N}(X)$, there exists $N \in \mathcal{N}(X)$ such that

$$a \in A,\ b \in B,\ a+b \in N \ \Rightarrow\ a, b \in M .$$

There exists $\lambda \in (0,1]$ such that $\lambda(a_n + b_n) \in N$ for all n. Then $\lambda a_n, \lambda b_n \in M$ for all n. Hence (a_n) and (b_n) are bounded.

(ii) Take a countable local base $\{M_n : n = 1, 2 \ldots\}$. Since A nal B, there exists $M \in \mathcal{N}(X)$ such that, for each n, there exist $a_n' \in A \sim M$, $b_n' \in B$ with $a_n' + b_n' \in n^{-1} M_n$. Let $a_n = n a_n'$, $b_n = n b_n'$. Then $a_n + b_n \in M_n$, so $a_n + b_n \to 0$, while (a_n) is unbounded, since $a_n \notin nM$. ‡

We note that a convergent sequence is bounded, while a convergent net need not be.

As in section 1, for fixed A and B, we write

$$[E] = (E+A) \cap (E-B).$$

2.3. If A,B are allied stars, and E is bounded, then so is $[E]$.

Proof. Let $M \in \mathcal{N}(X)$ be given. By 1.3, there exists $N \in \mathcal{N}(X)$ such that $[N] \subseteq M$. There exists $\lambda \in (0,1]$ such that $\lambda E \subseteq N$. Then $\lambda [E] \subseteq M$. ‡

In the language of partially ordered linear spaces, the special case of 2.3 where A is a wedge and $B = A$ states that, if the positive cone is self-allied, then the order-convex cover of a bounded set is bounded, a result which seems to have been first stated by Schaefer in (19) (p. 216). In particular, order-intervals are bounded.

Corresponding results for total boundedness and compactness are not to be expected, except in very special circumstances (cf. 7.5). Consider the space m of all bounded real sequences, with the usual norm and ordering. It is shown at the end of the section that the positive cone is self-allied (ex. (ii), p. 27). Denote by e_n the sequence having 1 in place n and 0 elsewhere, and by e the sequence having every term equal to 1 (this notation will be used consistently in examples below; no confusion will arise with the use of e to denote the identity of a group). The order-interval $\{x : 0 \leqslant x \leqslant e\}$ contains each e_n, so is not totally bounded.

If A,B are allied stars, then 2.3 shows that $(A-x)\cap(y-B)$ is bounded for any x,y. If x,y are interior points of A,B respectively, then this set is a neighbourhood of 0. Hence we have:

2.4. If X is a topological linear space, and there exist convex allied subsets A, B that contain 0 and have interior points, then the topology of X is pseudo-normable.

Proof. A, B are stars, and $(A-x)\cap(y-B)$ is a bounded, convex neighbourhood of 0 (where x, y are interior points of A, B respectively). ‡

Applied to a partially ordered linear space, 2.4 states that if the positive cone is self-allied and has an interior point, then the space must be pseudo-normable. This result, in its simplicity, does not seem to have been stated previously, though it is clear that a pseudo-norm inducing the topology is the "order-unit pseudo-norm" associated with the interior point.

Proper values and vectors

For spaces of proper vectors, we have the following simple and elegant result:

2.5. Let X be a topological linear space, and T a continuous linear mapping $X \to X$. If $\lambda \neq \mu$, then

$$\{x : Tx = \lambda x\} \ \text{al} \ \{y : Ty = \mu y\}.$$

Proof. If $(x_n + y_n)$ is a net convergent to 0, where $Tx_n = \lambda x_n$, $Ty_n = \mu y_n$, then

$$T(x_n + y_n) = \lambda x_n + \mu y_n \to 0 .$$

Hence $x_n \to 0$, and the result follows, by 1.5. ‡

For proper values, we have the following generalisation of a result of Krein and Rutman ((12), lemma 4.2). If Λ is a subset of a topological linear space X, then a is said to be an INTERNAL point of Λ if, given $x \in X$, there exists $\delta > 0$ such that, for $|\lambda| \leq \delta$, $a+\lambda x \in \Lambda$.

2.6. Let Λ be a self-allied, positive homogeneous subset of a Hausdorff topological linear space X, and let T be a linear mapping $X \to X$ such that $T\Lambda \subseteq \Lambda$. Suppose that, for some internal point a of Λ, we have $Ta = \rho a$. If λ is any proper value of T, then $|\lambda| \leq |\rho|$.

Proof. Suppose that $Tx = \lambda x$ for some x, λ. There exists $\delta > 0$ such that $a \pm \delta x \in \Lambda$. Applying the mapping T^n, we have $\rho^n a \pm \delta\lambda^n x \in \Lambda$. Let

$$b_n = |\lambda|^{-n}(\rho^n a + \delta\lambda^n x) ,$$

$$c_n = |\lambda|^{-n}(\rho^n a - \delta\lambda^n x) .$$

Then $b_n, c_n \in \Lambda$, and $b_n + c_n = 2\left(\frac{\rho}{|\lambda|}\right)^n a$. If $|\lambda| > |\rho|$, then $b_n + c_n \to 0$, so $b_n \to 0$, since Λ is self-allied. This implies that $x = 0$, and so that λ is not a proper value of T. ‡

A simple automatic continuity theorem

Continuity of T was not required in 2.6. However, slightly stronger conditions ensure that T is continuous, as the next theorem shows. If TM is bounded for some $M \in \mathcal{N}(X)$, then T is, of course, continuous.

<u>2.7.</u> Let X,Y be topological linear spaces, and A a positive homogeneous subset of X with non-empty interior. If T is a linear mapping $X \rightarrow Y$ such that TA is self-allied, then T is continuous.

<u>Proof.</u> There exist $e \in A$ and symmetric $M \in \mathcal{N}(X)$ such that $e+M \subseteq A$. Write $E = (A-e) \cap (e-A)$. Then $M \subseteq E$, and, by 2.3, TE is bounded. ‡

We mention that the same conclusion will hold if Y is locally convex and TA is self-allied with respect to the weak topology for Y, since weak boundedness in Y implies boundedness. However, the question of alliedness with respect to different topologies for the same space is left over to section 6.

<u>A generalisation of monotonic sequences</u>

It was shown by Bonsall in (2) that if a partial ordering in a locally convex space is given by a normal cone, then a monotonic, weakly convergent sequence is convergent. A simpler proof was given by Weston in (21). Our next result shows how the normal cone can be replaced by a pair of allied sets.

<u>2.8.</u> Suppose that A, B are allied subsets of a locally convex space, B being convex. Let (a_n) be a net in A with the property that

$$m \leqslant n \Rightarrow a_m - a_n \in B .$$

If 0 is in the weak closure of $\{a_n\}$, then $a_n \rightarrow 0$.

Proof. Given $M \in \mathcal{N}(X)$, there exists $N \in \mathcal{N}(X)$ such that

$$a \in A,\ b \in B,\ a+b \in N \ \Rightarrow\ a \in M .$$

0 is in the weak closure of the convex cover of $\{a_n\}$, which is the same as the closure of this set in the given topology. Hence there exist a finite set of indices n_i and corresponding $\lambda_i \in (0,1]$ such that $\sum_i \lambda_i = 1$ and $\sum_i \lambda_i a_{n_i} \in N$. If $n \geq n_i$ for all i, then

$$a_n + \sum_i \lambda_i (a_{n_i} - a_n) = \sum_i \lambda_i a_{n_i} \in N .$$

But $\sum_i \lambda_i (a_{n_i} - a_n) \in B$, so $a_n \in M$. Hence $a_n \to 0$. ‡

Normed linear spaces

Most of the examples given later concern subsets of normed linear spaces, and use one of the formulations given by the next two results. The set $\{x : \|x\| = 1\}$ will be denoted by S.

2.9. If A, B are non-zero, positive homogeneous subsets of a normed linear space, then each of the following statements is equivalent to A al B :

(i) $\phi(A,B) > 0$, where

$$\phi(A,B) = \inf \{\|a+b\| : a \in A \cap S,\ b \in B\} ;$$

(ii) there exists $\delta > 0$ such that

$$a \in A,\ b \in B \ \Rightarrow\ \|a+b\| \geq \delta \|a\| .$$

Proof. If A al B, then there exists $\delta > 0$ such that

$$a \in A,\ b \in B,\ \|a+b\| < \delta \ \Rightarrow\ \|a\| < 1 .$$

If $a \in A \cap S$ and $b \in B$, then $\|a+b\| \geq \delta$. Hence $\phi(A,B) \geq \delta$.

If $\phi(A,B) \geq \delta$ and $a \in A \sim \{0\}$, $b \in B$, then $\|c\| \geq \delta$, where $c = (a+b)/\|a\|$. Hence $\|a+b\| \geq \delta \|a\|$, and (ii) holds.

If (ii) holds, then A al B, since, given $\varepsilon > 0$, we have

$$a \in A,\ b \in B,\ \|a+b\| \leq \varepsilon\delta \Rightarrow \|a\| \leq \varepsilon . \ \ddagger$$

$\phi(B,A)$ need not be equal to $\phi(A,B)$. To show this, consider again the space m of bounded real sequences. With e_n, e defined as on p. 19, let A, B be the linear subspaces generated by e_1, e respectively. For any λ, μ, we have $\|\lambda e_1 + \mu e\| \geq |\mu|$, so $\phi(B,A) = 1$. But $\|e_1 - \frac{1}{2}e\| = \frac{1}{2}$, so $\phi(A,B) \leq \frac{1}{2}$ (in fact, $\phi(A,B) = \frac{1}{2}$, by the next theorem). A symmetrical function is introduced by the definition

$$\psi(A,B) = \inf \{\|a+b\| : a \in A \cap S,\ b \in B \cap S\}.$$

In terms of this, we have:

<u>2.10.</u> If A, B are non-zero, positive homogeneous subsets of a normed linear space, then

$$\phi(A,B) \leq \psi(A,B) \leq 2\,\phi(A,B),$$

so A al B if and only if $\psi(A,B) > 0$.

<u>Proof</u>. Clearly, $\phi(A,B) \leq \psi(A,B)$.

Take $a \in A \cap S$ and $b \in B \sim \{0\}$. Write $b' = b/\|b\|$. We show that $\|a+b'\| \leq 2\|a+b\|$, from which the result follows. Now $a+b' = (a+b) + (b'-b)$, and $b'-b = b(\|b\|^{-1}-1)$, so

$$\begin{aligned} \|b'-b\| &= |1 - \|b\|| \\ &= |\|a\| - \|b\|| \\ &\leq \|a+b\|. \ \ddagger \end{aligned}$$

Other properties of ϕ and ψ are easily established. For instance, $\psi(\bar{A},\bar{B}) = \psi(A,B)$, and if $\lambda = \phi(A,B)$, $\mu = \phi(B,A)$, then $|\lambda-\mu| \leq \lambda\mu$. It is not our intention here to give a full treatment of such results, since they do not belong to the theory of allied sets. ψ is related to a very natural metric on the set of closed subspaces which is studied in (5).

The converse of 2.3 holds in normed spaces. As always, $[E]$ denotes $(E+A)\cap(E-B)$.

2.11. Let A, B be non-zero, positive homogeneous subsets of a normed linear space, and let T be the unit ball. Then A al B if and only if $[T]$ is bounded.

Proof. If A al B, then $[T]$ is bounded, by 2.3. Conversely, suppose that $\|x\| \leq K$ for $x \in [T]$. Take $a \in A$, $b \in B$ with $\|a+b\| \leq 1$. Then $a \in [T]$, since $a = 0+a = (a+b)-b$. Hence $\|a\| \leq K$, and A al B, by 2.9. ‡

Inner product spaces

Further numerical results apply in inner product spaces:

2.12. Let A, B be non-zero, homogeneous subsets of an inner product space. Let

$$\gamma = \sup \{(a,b) : a \in A\cap S,\ b \in B\cap S\}.$$

Then:

(i) $\phi(A,B) = \phi(B,A) = \phi$, where $\phi^2 + \gamma^2 = 1$.

(ii) $\psi^2 = 2(1-\gamma)$, so $\phi \leq \psi \leq \sqrt{2}\,\phi$.

(iii) A al B $\iff \gamma < 1$;

A $\perp$ B $\iff \phi = 1 \iff \psi = \sqrt{2}$.

Proof. Take $a \in A \cap S$, $b \in B \cap S$. Write $(a,b) = \mu$. Then

$$(a+\lambda b\ ,\ a+\lambda b) = 1 + \lambda\bar{\mu} + \bar{\lambda}\mu + \lambda\bar{\lambda} \qquad (1)$$
$$= 1 + (\lambda+\mu)(\bar{\lambda}+\bar{\mu}) - \mu\bar{\mu}\ .$$

This is least when $\lambda = -\mu$, giving then

$$\|a+\lambda b\|^2 = 1 - |\mu|^2\ .$$

It follows that $\phi(A,B)^2 = 1-\gamma^2$. Similarly, $\phi(B,A)^2 = 1-\gamma^2$.

If $|\lambda| = 1$, then, by (1),

$$\|a+\lambda b\|^2 \geq 2 - 2|\mu|.$$

Putting $\lambda = -\mu/|\mu|$ (if $\mu \neq 0$), we obtain equality. Equality also holds if $\mu = 0$. Hence $\psi^2 = 2(1-\gamma)$. Since $0 \leq \gamma \leq 1$, we have $1-\gamma \leq 1-\gamma^2 = \phi^2$, and $\psi \leq \sqrt{2}\,\phi$.

(iii) follows, since A al B if and only if $\phi > 0$, while $A \perp B$ if and only if $\gamma = 0$. ‡

The "angle" between A and B can be defined by: $\cos\theta = \gamma$, $\sin\theta = \phi$. See (5).

Examples

In giving counter-examples, we shall frequently make use of spaces of sequences. The notations m, e_n were introduced on p. 19. In accordance with usual practice, c_o will denote the subspace of m consisting of sequences convergent to 0, and l_p (for any $p \geq 1$) will denote the space of real sequences $x = (\xi_n)$ such that $\sum_n |\xi_n|^p$ is convergent, with the norm defined by $\|x\|^p = \sum_n |\xi_n|^p$.

We shall call (ξ_n) a "finite sequence" if it has only a finite number of non-zero terms, and use the notation $(\xi_1, \ldots, \xi_n)$ to indicate that $\xi_r = 0$ for $r > n$.

We now give some examples for the sake of illustration.

(i) In m or l_p (any $p \geq 1$), let A be the set of sequences (α_n) having $\alpha_{2n} = 0$ for all n, and B the set of sequences (β_n) having $\beta_{2n-1} = 0$ for all n. A and B are closed subspaces, and A al B, since, for $a \in A$, $b \in B$, we have $\|a+b\| \geq \|a\|$.

(ii) In m or l_p (any $p \geq 1$), let A be the set of non-negative sequences, i.e. sequences having every term non-negative. A is self-allied, since if $a_1, a_2 \in A$, then $\|a_1+a_2\| \geq \|a_1\|$. This, of course, is the positive cone for the usual ordering of these spaces.

(iii) In l_p (any $p \geq 1$), let A be the set of sequences (α_n) having $\alpha_{2n} = 0$ for all n, and let B be the set of sequences (β_n) having $\beta_{2n} = 2^{-n}\beta_{2n-1}$ for all n. A and B are closed subspaces with intersection $\{0\}$.

Now $e_{2n-1} \in A$, and $f_{2n-1} = e_{2n-1} + 2^{-n}e_{2n} \in B$. $\|e_{2n-1}\| = 1$, while $\|f_{2n-1} - e_{2n-1}\| = 2^{-n}$. Hence A nal B, by 2.9.

We show that A+B is not closed, though this follows from a later theorem (4.7). Now

$$(0, 2^{-1}, \ldots, 0, 2^{-n}) \in A+B,$$

so

$$(0, 2^{-1}, \ldots, 0, 2^{-n}, 0, 2^{-n-1}, \ldots) \in \overline{A+B}.$$

If this is equal to

$$(\alpha_1, 0, \ldots, \alpha_{2n-1}, 0, \ldots)$$
$$+ (\beta_1, 2^{-1}\beta_1, \ldots, \beta_{2n-1}, 2^{-n}\beta_{2n-1}, \ldots),$$

then $\beta_{2n-1} = 1$ for each n, giving a sequence (β_n) which is not in l_p.

(iv) In l_1, let K be the subspace consisting of sequences (β_n) such that

$$\beta_1 = 0,$$
$$\beta_{2n+1} = \beta_{2n} \quad (n \geq 1).$$

Let A be the set of sequences (α_n) such that

$$\alpha_{2n-1} = \alpha_{2n} \geq 0$$

for all n. Then it is clear that A al A and $A \cap K = \{0\}$. We show that (PA) nal (PA), where P is the projection onto the quotient space l_1/K (cf. 1.16). Let x_n (respectively y_n) have components $4i-1$ and $4i$ (respectively, $4i-3$ and $4i-2$) equal to 1 for $i = 1, 2, \ldots, n$, and all other components 0. Then $x_n, y_n \in A$, and $\|x_n\| = \|y_n\| = 2n$. If $b = (\beta_n) \in K$, then

$$\|x_n + b\| > \sum_{i=1}^{2n} \left(|\beta_{2i}| + |1-\beta_{2i}|\right) \geq 2n.$$

Hence $\|Px_n\| = 2n$.

Let b_n have components $2, 3, \ldots, 4n+1$ equal to 1, and all others 0. Then $b_n \in K$, and $\|x_n + y_n - b_n\| = 2$. Hence $\|P(x_n+y_n)\| \leq 2$ (in fact, equality holds), and (PA) nal (PA), as stated.

3. CAUCHY NETS AND COMPLETENESS

The results of the next two sections are presented in their maximum generality. At times this results in rather complicated statements. It will be noticed, however, that if the results are specialised to commutative topological groups, they become a lot simpler.

We start by summarising certain definitions and facts connected with the right, left and two-sided uniformities of a topological group. The distinction, of course, disappears in the commutative case.

A net (x_n) in a topological group is R-CAUCHY (respectively, L-CAUCHY) if, given $M \in \mathfrak{N}(X)$, there exists n_o such that for $m,n \geqslant n_o$, $x_m x_n^{-1} \in M$ (respectively, $x_m^{-1} x_n \in M$). (x_n) is R-Cauchy if and only if (x_n^{-1}) is L-Cauchy. A subset A is R-complete if and only if A^{-1} is L-complete.

A net is U-CAUCHY (U denoting the two-sided uniformity) if and only if it is both R-Cauchy and L-Cauchy. A set which is complete with respect to either R or L is also U-complete.

In most of this section, we shall require A comm B, i.e. $a \in A$, $b \in B$ => $ab = ba$. This implies $\bar{A}$ comm $\bar{B}$, A^{-1} comm B^{-1}, and also A^{-1} comm B, since

$$ba^{-1} = a^{-1}(ab)a^{-1} = a^{-1}(ba)a^{-1} = a^{-1}b \ .$$

It also implies that A al B is equivalent to B al A, and that if (a_n) and (b_n) are R-Cauchy nets $(a_n \in A,\ b_n \in B)$, then $(a_n b_n)$ is R-Cauchy (and similarly for L-Cauchy nets).

We now prove the basic theorem of this section.

3.1. Let A, B be subsets of a topological group such that A comm B and (AA^{-1}) al (BB^{-1}). Then:

(i) If $(a_n b_n)$ is an R-Cauchy net $(a_n \in A,\ b_n \in B)$, then (a_n) and (b_n) are R-Cauchy.

(ii) If A and B are R-complete, then so is AB.

(iii) If A is R-complete and B is closed, then AB is closed.

Similar results hold for L if $(A^{-1}A)$ al $(B^{-1}B)$.

Proof. (i) Given $M \in \mathfrak{n}(X)$, there exists $N \in \mathfrak{n}(X)$ such that if $x \in AA^{-1}$, $y \in BB^{-1}$ and $xy \in N$, then $x, y \in M$. There exists n_o such that, for $m, n \geqslant n_o$,

$$(a_m b_m)(a_n b_n)^{-1} \in N.$$

Now

$$\begin{aligned}(a_m b_m)(a_n b_n)^{-1} &= a_m b_m b_n^{-1} a_n^{-1} \\ &= (a_m a_n^{-1})(b_m b_n^{-1}).\end{aligned}$$

Hence, for $m, n \geqslant n_o$, $a_m a_n^{-1}$ and $b_m b_n^{-1}$ are in M.

(ii) Suppose that A, B are R-complete, and that (x_n) is an R-Cauchy net in AB. For each n, there exist $a_n \in A$, $b_n \in B$ (not necessarily unique) such that $x_n = a_n b_n$. By (i), (a_n) and (b_n) are R-Cauchy, so have limits $a \in A$ and $b \in B$. Then $x_n \to ab$.

(iii) Take $x \in \overline{AB}$. There exist $a_n \in A$, $b_n \in B$ such that $(a_n b_n)$ is a net convergent to x. By (i), (a_n) is R-Cauchy, so has a limit $a \in A$. Then

$$b_n = a_n^{-1}(a_n b_n) \to a^{-1}x ,$$

and this is in B, since B is closed. Hence $x \in AB$. ‡

(ii) remains true if we substitute "sequentially complete" for "complete".

Note that $AB = BA$, because of the condition A comm B. If A and B are subgroups, this shows that AB is also a subgroup, and the condition (AA^{-1}) al (BB^{-1}) reduces to A al B. Even if A and B are not subgroups, (i) will hold under the assumption A al B if (a_n) and (b_n) are "monotonic" sequences in the sense that $a_m a_n^{-1} \in A$ and $b_m b_n^{-1} \in B$ for $m \leqslant n$.

We now give some corollaries of 3.1.

<u>3.2.</u> If X is Hausdorff and R-complete, and A, B are subsets such that A comm B and (AA^{-1}) al (BB^{-1}), then $\overline{AB} = \overline{A}\,\overline{B}$.

<u>Proof.</u> $\overline{AA^{-1}}$ al $\overline{BB^{-1}}$, by 1.9, so $(\overline{A}\,\overline{A}^{-1})$ al $(\overline{B}\,\overline{B}^{-1})$. Also, $\overline{A}$ comm $\overline{B}$. Thus, by 3.1(ii), $\overline{A}\,\overline{B}$ is R-complete, and therefore closed. ‡

<u>3.3.</u> If A is compact, B is R-complete, A comm B and $(AA^{-1}) \cap \overline{BB^{-1}} \subseteq \overline{\{e\}}$, then AB is R-complete.

<u>Proof.</u> AA^{-1} is compact, so (AA^{-1}) al (BB^{-1}) , by 1.13. ‡

We notice that there is no mention of alliedness in the statement of 3.3. It is, in fact, the first application of our theory. Further such results appear later, especially in section 5.

Combining the results for the right and left uniformities, we obtain:

3.4. Suppose that A comm B, (AA^{-1}) al (BB^{-1}) and $(A^{-1}A)$ al $(B^{-1}B)$. Then:

(i) If $(a_n b_n)$ is a Cauchy net with respect to R, L or U $(a_n \in A,\ b_n \in B)$, then so are (a_n) and (b_n).

(ii) If A and B are complete with respect to R, L or U, then so is AB.

(iii) If A is U-complete and B is closed, then AB is closed.

Proof. (i) says nothing new. (ii) (for U) and (iii) follow exactly as in the proof of 3.1. ‡

Some results in the converse direction

Motivated by 1.5(ii), we now look for results in the converse direction to 3.1(i), deducing alliedness from the hypothesis that if $(a_n b_n)$ is a Cauchy net (with $a_n \in A$, $b_n \in B$), then so are (a_n) and (b_n). It is clear that (as in 3.1) some additional conditions are required: take $a \notin \overline{\{e\}}$, and let $A = \{a\}$, $B = \{a^{-1}\}$. Then A nal B, but every net in A or B is Cauchy. We find that, in the presence of additional conditions, the hypothesis above can be weakened to the assumption that if $(a_n b_n)$ converges to e, then (a_n) and (b_n) are Cauchy.

The simplest additional condition which works is $e \in A \cap B$, giving the following theorem (the method of proof is perhaps of more interest than the result itself):

3.5. If A, B are subsets of a topological group that contain the identity, and for every net $(a_n b_n)$ with $a_n \in A$, $b_n \in B$ which converges to e, (a_n) is R-Cauchy or L-Cauchy, then A al B.

Proof. If X has the indiscrete topology, then A al B trivially. We suppose the contrary, so that there are neighbourhoods of e which are properly contained in X.

Suppose that A nal B . Then there exists $M \in \mathcal{N}(X)$ such that, for each $N \in \mathcal{N}(X)$, there exist $a_N \in A \sim M$, $b_N \in B$ with $a_N b_N \in N$. Let

$$D = \{(x, N) : x \in X,\ N \in \mathcal{N}(X)\} .$$

If we define $(x, N) \leqslant (y, P)$ if and only if $N \supseteq P$, then $\leqslant$ directs D. We define nets in A and B over the directed set D by putting:

$$a_{(x,N)} = a_N , \quad b_{(x,N)} = b_N \quad \text{if} \quad x \in N,$$

$$a_{(x,N)} = b_{(x,N)} = e \quad \text{if} \quad x \notin N .$$

Then $a_{(x,N)} b_{(x,N)} \in N$ in all cases, so the net $\{a_p b_p : p \in D\}$ converges to e. However, given $(x, N) \in D$, there exist elements p, q of D following (x, N) such that $a_p \notin M$, $a_q = e$. Hence $\{a_p : p \in D\}$ is neither R-Cauchy nor L-Cauchy. $\ddagger$

Of course, 1.5(i) shows that, under the conditions of the theorem, (a_n) and (b_n) converge to e. It is easily seen that the condition $e \in A \cap B$ can be weakened to $e \in \overline{A} \cap \overline{B}$. A different set of additional conditions is given by:

<u>3.6.</u> Suppose that $\overline{A} \cap \overline{B}^{-1} \subseteq \overline{\{e\}}$, $\overline{A}$ is V-complete for some uniformity V inducing the topology, and that for every net $(a_n b_n)$ convergent to e (with $a_n \in A$, $b_n \in B$), (a_n) is V-Cauchy. Then A al B .

If the group is metrisable, it is sufficient if this is true for sequences.

<u>Proof</u>. Suppose that A nal B . Let $\mathcal{B}$ be a local base, countable in the metrisable case. There exists open $M \in \mathcal{N}(X)$ such that, for each $N \in \mathcal{B}$, there exist $a_N \in A \sim M$, $b_N \in B$ with $a_N b_N \in N$. $\{a_N b_N : N \in \mathcal{B}\}$ is a net convergent to e, so (a_N) is V-Cauchy, and converges to a point a of $\overline{A}$. Since M is open, $a \notin M$. Now

$$b_N^{-1} = (b_N^{-1} a_N^{-1}) a_N \rightarrow ea = a ,$$

so $a \in \overline{B}^{-1}$. This implies that $a \in \overline{\{e\}}$, which is a contradiction. ‡

It is sufficient in 3.6 if A is V-complete for some uniformity V inducing a topology not smaller than the given one. Applying 3.6 to the "natural" uniformities R, L, we obtain:

<u>3.7.</u> Suppose that $\overline{A} \cap \overline{B}^{-1} \subseteq \overline{\{e\}}$, $\overline{A}$ is R-complete,

and that for every net (a_nb_n) convergent to e (with $a_n \in A$, $b_n \in B$), (b_n) is I-Cauchy. Then A al B . Sequences are sufficient in the metrisable case.

<u>Proof</u>. The result follows by 3.6 if we show that $a_nb_n \to e$ implies that (a_n) is R-Cauchy. Given $M \in \mathcal{N}(X)$, take symmetric $N \in \mathcal{N}(X)$ such that $N^3 \subseteq M$. There exists n_o such that:

$$n \geqslant n_o \quad \Rightarrow \quad a_nb_n \in N \ ,$$

$$m,n \geqslant n_o \quad \Rightarrow \quad b_m^{-1}b_n \in N \ .$$

Then, for $m,n \geqslant n_o$:

$$a_ma_n^{-1} = (a_mb_m)(b_m^{-1}b_n)(b_n^{-1}a_n^{-1}) \in N^3 \subseteq M \ . \ \ddagger$$

Restricted converses to 3.1(ii), deducing alliedness from completeness of AB, are obtained in the next section. We finish this section with two counter-examples.

<u>Closed allied subspaces with a non-closed sum</u>

Let F be the space of all finite real sequences, with norm defined by $\|(\xi_n)\| = \sup |\xi_n|$. Let A be the subspace consisting of sequences (α_n) having $\alpha_{2n} = \frac{1}{2}\,\alpha_{2n-1}$ for all n, and B the subspace consisting of sequences (β_n) having $\beta_1 = 0$ and $\beta_{2n+1} = \frac{1}{2}\,\beta_{2n}$ for all n. A and B are closed, even with respect to the topology of pointwise convergence.

Suppose that $a = (\alpha_n) \in A$, $b = (\beta_n) \in B$, $\|a\| = 1$ and $\|a-b\| < \frac{1}{2}$. We show that $\|b\| > 1$; it then follows by 2.10 that A al B . Now $|\alpha_r| = 1$ for some r, and $r > 1$, since $\beta_1 = 0$. Then $|\alpha_r - \frac{1}{2}\beta_{r-1}| < \frac{1}{2}$, so

$$|\beta_{r-1}| \geq 2|\alpha_r| - |2\alpha_r - \beta_{r-1}| > 2-1 = 1 ,$$

and $\|b\| > 1$, as stated.

Now

$$(1, 2^{-1}, 2^{-2}, \ldots, 2^{-2n+1}) \in A ,$$

$$(0, 2^{-1}, 2^{-2}, \ldots, 2^{-2n+1}, 2^{-2n}) \in B ,$$

so

$$(1, 0, \ldots, 0, -2^{-2n}) \in A+B ,$$

and $(1) \in \overline{A+B}$. By considering the position of the last non-zero term, we see that $(1) \notin A+B$.

This shows that completeness is an essential ingredient of 3.1.

Complete allied wedges with a non-closed sum

3.1 shows that the sum of complete allied subgroups of a commutative topological group is complete. We give an example where this fails for wedges in a Banach space. We can even do so while allowing one of the wedges to be a subspace.

In l_1, let A be the set of sequences (α_n) for which

$$\alpha_{2n} = \alpha_{2n+1} \geq 0 \quad (n \geq 1)$$

and

$$\alpha_1 = \sum_{n=2}^{\infty} \alpha_n .$$

A is a closed wedge, and for $a \in A$, $\|a\| = 2\alpha_1$.

Let B be the closed subspace consisting of sequences (β_n) for which $\beta_{2n-1} = \beta_{2n}$ for all n.

Take $a \in A$, $b \in B$. If $|\alpha_1 - \beta_1| < \frac{1}{4}\alpha_1$, then $\beta_1 > \frac{3}{4}\alpha_1$. But $\alpha_2 \leqslant \frac{1}{2}\alpha_1$, so $\beta_1 - \alpha_2 > \frac{1}{4}\alpha_1$. Hence

$$\|a-b\| \geqslant \frac{1}{4}\alpha_1 = \frac{1}{8}\|a\|,$$

so A al B .

Let

$$a_n = (1, \tfrac{1}{2n}, \ldots, \tfrac{1}{2n}), \qquad \|a_n\| = 2,$$

$$b_n = (\tfrac{1}{2n}, \ldots, \tfrac{1}{2n}), \qquad \|b_n\| = 1.$$

(By closing the brackets, we imply that all further terms are zero; the given norms determine the number of non-zero terms.) Then $a_n \in A$, $b_n \in B$, and

$$a_n - b_n = (1-\tfrac{1}{2n}, 0, \ldots, 0, \tfrac{1}{2n})$$

$$\to (1) \quad \text{as} \quad n \to \infty.$$

Hence $(1) \in \overline{A+B}$. But if $(1) = a-b$, where

$$a = (\alpha_1, \alpha_2, \alpha_2, \ldots, \alpha_{2n}, \alpha_{2n}, \ldots),$$

$$b = (\beta_1, \beta_1, \beta_3, \beta_3, \ldots, \beta_{2n-1}, \beta_{2n-1}, \ldots),$$

then

$$\beta_1 = \alpha_2 = \beta_3 = \alpha_4 = \ldots,$$

so, since a and b are in l_1, $\beta_n = 0$ for all n, and $\alpha_n = 0$ for $n \geqslant 2$, while $\alpha_1 = 1$. Thus $a \notin A$.

It is instructive to consider A-A, the subspace generated by A. A-A is, in fact, the set of sequences $z = (\zeta_n)$

for which

$$\zeta_{2n} = \zeta_{2n+1} \quad (n \geqslant 1)$$

and

$$\zeta_1 = \sum_{n=2}^{\infty} \zeta_n .$$

For, given such z, let

$$\alpha_1 = \sum_{n=2}^{\infty} \zeta_n^+ , \quad \alpha_2 = \sum_{n=2}^{\infty} \zeta_n^- ,$$

$$a_1 = (\alpha_1, \zeta_2^+, \zeta_2^+, \ldots, \zeta_{2n}^+, \zeta_{2n}^+, \ldots) ,$$

$$a_2 = (\alpha_2, \zeta_2^-, \zeta_2^-, \ldots, \zeta_{2n}^-, \zeta_{2n}^-, \ldots) .$$

Then $a_1, a_2 \in A$ and $a_1 - a_2 = z$.

Hence A-A is closed, and $(A-A) \cap B = \{0\}$. It is clear from 3.1 that (A-A) nal B , and it is easy to verify this directly. The argument above shows that $(1) \notin (A-A)+B$. We notice that the sequences (a_n) and (b_n) are bounded (cf. ex. (iii), p. 27).

In section 5 we shall see that positive results about the sum of two wedges can be obtained when one of them is locally compact.

4. SUBGROUPS AND SUBSPACES

In this section, we show that alliedness of subgroups is equivalent to the continuity of certain homomorphisms, and use the closed graph theorems to deduce partial converses to 3.1. As in the last section, the statements appropriate to the commutative case are a good deal simpler.

Projection onto A

Let A, B be subsets of a topological group X. The expression ab ($a \in A$, $b \in B$) for elements of AB is unique if and only if $(A^{-1}A) \cap (BB^{-1}) = \{e\}$. The natural projection π of AB onto A is then defined by $\pi(ab) = a$. If, also, $e \in A \cap B$, then A al B is equivalent to the continuity of π at e with respect to the relative topologies, since this occurs if and only if, given $M \in \mathcal{N}(X)$, there exists $N \in \mathcal{N}(X)$ such that

$$a \in A,\ b \in B,\ ab \in N \implies a \in M .$$

If A, B are subgroups, then the expression ab for elements of AB is unique if and only if $A \cap B = \{e\}$, i.e. if and only if A and B are pseudo-disjoint (see p. 14). If A comm B or $B \triangleleft X$ (i.e. B is a normal subgroup), then AB is a subgroup and π is a homomorphism, for in either case, given $a_i \in A$, $b_i \in B$ $(i = 1, 2)$, there exists $c \in B$ such that $b_1 a_2 = a_2 c$, and then

$$(a_1 b_1)(a_2 b_2) = (a_1 a_2)(c b_2) .$$

Hence we have the following theorem:

4.1. Suppose that X is a topological group, and that A, B are pseudo-disjoint subgroups such that A comm B or $B \triangleleft X$. Then A al B if and only if π is continuous.

If A al B and θ is a continuous homomorphism defined on A, then a continuous homomorphism θ_1 is defined on AB by $\theta_1(ab) = \theta(a)$. ‡

Isomorphism with the direct product

Let $A \times B$ denote the topological product of A and B. $A \times B$ is mapped onto AB by ϕ, where $\phi(a,b) = ab$. ϕ is continuous, by the definition of a topological group. ϕ is one-to-one if and only if $(A^{-1}A) \cap (BB^{-1}) = \{e\}$.

If A and B are subgroups, then ϕ is a homomorphism if and only if A comm B , the implication "only if" following from the identity

$$(e,b)(a,e) = (a,b) .$$

A sufficient condition for this is that A, B are pseudo-disjoint and both normal in X.

With this notation, we have:

4.2. If $(A^{-1}A) \cap (BB^{-1}) = \{e\}$ and $e \in A \cap B$, then A al B if and only if ϕ^{-1} is continuous at e.

If A, B are pseudo-disjoint subgroups, and A comm B, then A al B if and only if ϕ is a topological isomorphism.

Proof. ϕ^{-1} is continuous at e if and only if, given $M \in \mathfrak{N}(X)$, there exists $N \in \mathfrak{N}(X)$ such that

$$a \in A,\ b \in B,\ ab \in N \implies a, b \in M . \quad ‡$$

Quotient spaces

If A, B are subgroups of X, and A comm B, then AB is a subgroup, and $A \triangleleft AB$, since, for $a \in A$, $b \in B$:

$$A(ab) = Ab = bA = (ba)A .$$

A continuous homomorphism χ of B onto the quotient group AB/A is then defined by $\chi b = Ab$. If $A \cap B = \{e\}$, then χ is one-to-one. With this notation, we have:

4.3. Let X be a topological group, and let A, B be pseudo-disjoint subgroups such that A comm B. Then A al B if and only if χ^{-1} is continuous (so that AB/A is topologically isomorphic to B).

Proof. (i) Suppose that A al B. By 1.3, given $M \in \eta(X)$, there exists $N \in \eta(X)$ such that $(AN) \cap B \subseteq M$. $\{An : n \in N\}$ is a neighbourhood of the identity in AB/A. If $\chi b = Ab = An$ for some $n \in N$, then $b \in AN$, so $b \in M$. Hence χ^{-1} is continuous at the identity (and therefore continuous everywhere).

(ii) Suppose that A nal B. Let Q be a neighbourhood of the identity in AB/A. Then $\cup Q$ is a neighbourhood of e in AB. There exist $M \in \eta(X)$, $a \in A$ and $b \in B \sim M$ such that $ab \in \cup Q$. Then $\chi b = Ab \in Q$. Hence χ^{-1} is not continuous. ‡

Application of closed graph theorems

Following Day (4), we will say that a mapping is cg if its graph is a closed set. We show that, under certain conditions, the projection π of AB onto A is cg, and use the well-known results about cg mappings to deduce sufficient conditions for alliedness.

4.4. Let X be a topological group, and let A, B be closed, pseudo-disjoint subgroups such that AB is closed. Then the projection of AB onto A is cg.

Proof. Suppose that $(a_n b_n)$ is a net convergent to ab, where $a_n, a \in A$ and $b_n, b \in B$. Suppose, further, that $a_n \to a_o \in A$. We must show that $a_o = a$. Now

$$b_n = a_n^{-1}(a_n b_n) \to a_o^{-1}ab = b_o \quad \text{(say)}.$$

$b_o \in B$, since B is closed. Thus $a_o^{-1}a = b_o b^{-1} = e$, since this is an element of $A \cap B$. Hence $a_o = a$, as required. ‡

We mention in passing that if AB is closed, and $\overline{A} \cap \overline{B} = \{e\}$, then A and B must be closed.

The simplest "closed graph theorem" states that if S, T are topological spaces, T being locally compact and regular, then a nearly continuous, cg mapping $S \to T$ is continuous (the proof is easy).

If S is a non-meagre topological group, and T is a Lindelöf one, then every homomorphism $S \to T$ is nearly continuous (see, e.g., (9), p. 213).

A locally compact topological group is non-meagre, and if it is also connected, then it is σ-compact (and so Lindelöf), since if M is a compact neighbourhood of e, then $\bigcup_{n=1}^{\infty} M^n = X$.

Various sets of sufficient conditions for A al B can be put together from these facts. A relatively simple one is:

4.5. Suppose that X is a locally compact topological group, and that A, B are closed, pseudo-disjoint subgroups such that:

(i) AB is closed,

(ii) A comm B or $B \triangleleft X$,

(iii) A is connected.

Then A al B. ‡

The usual form of the closed graph theorem for topological groups states that if X, Y are topological groups, Y being complete and metrisable, then a nearly continuous, cg homomorphism $X \to Y$ is continuous (see (9), p. 213). This yields:

4.6. Let X be a topological group, and suppose that A, B are closed, pseudo-disjoint subgroups such that:

(i) AB is closed and non-meagre in itself,

(ii) A comm B or $B \triangleleft X$,

(iii) A is complete, metrisable and Lindelöf.

Then A al B. ‡

If X is complete and metrisable, then conditions (i) and (iii) reduce to AB being closed and A being Lindelöf.

The situation is simpler in topological linear spaces: if X is non-meagre, and Y is complete and metrisable, then a cg linear mapping $X \to Y$ is continuous (see (10), p. 97). Combining this with 3.1, we obtain:

4.7. If X is a complete, metrisable topological linear space, and A, B are closed, pseudo-disjoint subspaces, then A+B is closed if and only if A al B . ‡

As a single application of this result, we mention:

4.8. If X is a complete, metrisable topological linear space, and A, B, C are closed subspaces such that $A \cap C = \{0\}$, $B \subset C$, and A+C is closed, then so is A+B.

Proof. A al C , so A al B . ‡

We can elaborate on our example of closed allied subspaces with a non-closed sum (pp. 35-36) to show that we cannot dispense with the completeness condition here. With A, B as before, let C be the set of sequences (β_n) having $\beta_{2n+1} = \frac{1}{2}\beta_{2n}$ for all n, and β_1 arbitrary. Consideration of the position of the last non-zero term shows that $A \cap C = \{0\}$, and it is easily seen that $e_n \in A+C$ for all n, so that A+C = F . It is also easily seen that A nal C (cf. 4.7).

Closed allied subspaces A, B such that A+B = X are called TOPOLOGICAL COMPLEMENTS. Some results on the existence of these are given in (11), section 31.

We finish the section by giving an example of a wedge A and a subspace B in a Banach space such that $A \cap B = \{0\}$, A+B is closed, and A nal B . In l_1, let A be the set of sequences (α_n) having $\alpha_n \geq \alpha_{n+1} \geq 0$ for all n. Let B be the set of sequences (β_n) with $\beta_1 = 0$. A, B are closed, and $A \cap B = \{0\}$. Let

$$a_n = (1, \ldots, 1), \quad \|a_n\| = n,$$

$$b_n = (0, 1, \ldots, 1), \quad \|b_n\| = n-1.$$

$\|a_n - b_n\| = 1$, so A nal B .

A+B is the (closed) set consisting of all sequences (ξ_n) having $\xi_1 \geq 0$. For, given such a sequence, we have

$$(\xi_1) \in A,$$

$$(0, \xi_2, \xi_3, \ldots) \in B,$$

while it is clear that all elements of A+B satisfy this condition.

In the light of 3.1, it would be interesting to have such an example subject to the stronger condition $(A-A) \cap (B-B) = \{0\}$.

5. LOCALLY COMPACT SUBSETS OF TOPOLOGICAL LINEAR SPACES

Let X be a topological linear space. A positive homogeneous subset A is locally compact if and only if there exists $M \in \mathcal{N}(X)$ such that $A \cap M$ is compact. For if this holds, and $a \in A$, then a scalar λ exists such that a is an interior point of λM. Then $A \cap (\lambda M) = \lambda(A \cap M)$, and this set is a compact A-neighbourhood of a.

If A is positive homogeneous and locally compact, and X is Hausdorff, then A is closed. If X is metrisable, then A is complete.

We shall denote by pos A the wedge generated by A, i.e. the set of all linear combinations of elements of A with positive coefficients.

A Hausdorff topological linear space is locally compact if and only if it is finite-dimensional, and it then has the Euclidean topology. In fact, it is sufficient if there is a totally bounded neighbourhood of 0 (see, e.g. (10), 7.8). Thus, in a Hausdorff space, the wedge generated by a finite set is locally compact. However, a locally compact wedge may contain an infinite linearly independent set: an example is the set of all sequences (ξ_n) in l_1 having $0 \leq \xi_n \leq \frac{1}{2} \xi_{n-1}$ for all n.

As an immediate deduction from two of our basic results, we have:

5.1. If A, B are stars such that $A \cap (-\bar{B}) \subseteq \overline{\{0\}}$, and A is locally compact, then A al B .

Proof. There exists $M \in \mathcal{N}(X)$ such that $A \cap M$ is compact. By 1.13, $(A \cap M)$ al $(B \cap M)$. Therefore, by 2.1, A al B . ‡

Clearly, the existence of a compact A-neighbourhood of 0 is sufficient here (since A is only a star, this does not necessarily imply that A is locally compact).

The remaining results of this section are applications of our theory, in the sense that allied sets appear in the proof but not in the statement.

5.2. Let X be a Hausdorff topological linear space, A a finite-dimensional subspace, and B any subspace. Then:

(i) if B is complete, so is A+B ;

(ii) if B is closed, so is A+B.

Proof. $A+B = A_1+B$, where A_1 is a subspace of A such that $A_1 \cap B = \{0\}$. A_1 is locally compact, so, by 5.1, A_1 al B . The results follow, by 3.1. ‡

(ii) is well-known, but the literature does not seem to contain an explicit statement of (i).

It is difficult to obtain corresponding results for more general sets using 3.1, since this requires the condition (A-A) al (B-B) . However, we can make use of the fact that if A is compact, then A+B is compact (or closed) if B is.

5.3. Suppose that A, B are positive homogeneous sets such that $A \cap (-\overline{B}) \subseteq \overline{\{0\}}$, and that A is locally compact. Then:

(i) if B is locally compact, so is A+B ;

(ii) if B is closed, so is A+B.

Proof. There exists $M \in \mathfrak{n}(X)$ such that $A \cap M$ is compact and $B \cap M$ is compact (case (i)) or closed (case (ii)). Then $A \cap M + B \cap M$ is compact (case (i)), or closed (case (ii)). We show that this is a neighbourhood of 0 in A+B, from which the results follow. Now A al B , by 5.1, so there exists $N \in \mathfrak{n}(X)$ such that

$$(A+B) \cap N \subseteq (A \cap M + B \cap M) ,$$

giving the required result. ‡

A simple example shows that we cannot dispense with the disjointness condition in 5.3. Consider the space s of all real sequences, with the topology of pointwise convergence. Let B be the set of non-negative sequences in s, and e the sequence having every term equal to 1. We show that pos e − B is not closed. Let

$$x_n = ne - (n-1, n-2, \ldots, 1)$$
$$= (1, 2, \ldots, n, n, \ldots).$$

Then

$$x_n \rightarrow (1, 2, \ldots, n, n+1, \ldots) ,$$

which is not in pos e − B , since it is not bounded above.

Putting A = pos x in 5.3(ii), we obtain theorem 1(c) of (20). Our proof, unlike that in (20), does not use nets at any stage. What we can say about nets is the following:

5.4. Suppose that A, B are stars such that $A \cap (-\overline{B}) \subseteq \overline{\{0\}}$, and that A is locally compact. Let $(a_n + b_n)$ be a net which is either bounded or Cauchy ($a_n \in A$, $b_n \in B$). Then (a_n) has a convergent subnet.

Proof. Take $M \in \mathcal{N}(X)$ such that $A \cap M$ is compact. Now A al B, by 5.1, so $N \in \mathcal{N}(X)$ exists such that

$$a \in A,\ b \in B,\ a+b \in N \implies a \in M.$$

(i) If (a_n+b_n) is bounded, then there exists $\lambda \in (0,1]$ such that $\lambda(a_n+b_n) \in N$ for all n. Then $\lambda a_n \in A \cap M$ for all n, so (λa_n) has a convergent subnet.

(ii) If (a_n+b_n) is Cauchy, take circled $P \in \mathcal{N}(X)$ such that $P+P \subseteq N$. There exists p such that for $n \geq p$, $(a_n+b_n)-(a_p+b_p) \in P$. For some $\lambda \in (0,1]$, $\lambda(a_p+b_p) \in P$. Then for $n \geq p$, $\lambda(a_n+b_n) \in P+P \subseteq N$, so $\lambda a_n \in A \cap M$. The result follows. ‡

We can deduce the following variant of 5.3, in which B is allowed to be a star instead of being positive homogeneous:

5.5. Suppose that A is locally compact and positive homogeneous, B is a star, and $A \cap (-\bar{B}) \subseteq \{0\}$. Then:

(i) if B is complete, so is A+B ;

(ii) if B is closed, so is A+B.

Proof. (i) Let (a_n+b_n) be a Cauchy net ($a_n \in A$, $b_n \in B$). By 5.4, (a_n) has a subnet (a_{n_i}) convergent to $a \in A$. (b_{n_i}) is Cauchy, so has a limit $b \in B$. Then $a_{n_i} + b_{n_i} \to a+b$, so $a_n+b_n \to a+b$.

(ii) Suppose that $a_n+b_n \to x$ ($a_n \in A$, $b_n \in B$). (a_n)

has a subnet (a_{n_i}) convergent to $a \in A$. Then $b_{n_i} \to x-a$, and this is in B, since B is closed. Hence $x \in A+B$. ‡

The conditions of 5.4 do not ensure that the expressions (a_n+b_n) are unique. One might suppose that by altering these expressions, one could cause the whole net (a_n) to converge, but the following example shows that this is not the case.

Consider again the space s of all real sequences with the topology of pointwise convergence. Let A be the set of all non-negative sequences, and let $B = \text{pos } b$, where $\beta_n = (-1)^{n+1}$, $b = (\beta_n)$. Let

$$a_{2n-1} = (1,0, \dots ,1,0) \quad (n \text{ 1's}),$$

$$a_{2n} = (0,1, \dots ,0,1) \quad (n \text{ 1's}).$$

Let $x_{2n-1} = a_{2n-1}$, $x_{2n} = a_{2n} + b$. Then $x_n \to x$, where

$$x = (1,0, \dots ,1,0, \dots).$$

If $a_n = a + \lambda b$, then the components of a alternate eventually between λ and $-\lambda$, so this is only possible with $a \in A$ if $\lambda = 0$. It follows that, in this case, the expressions for x_n as elements of A+B are unique, so that we cannot alter the sequence (a_n) to make it converge.

Slightly more can, in fact, be said about nets of the form $(a_n + \lambda_n b)$, without using allied sets. For this, see appendix 1.

An open mapping theorem

Let A be a locally compact, positive homogeneous subset of a normed linear space, and let T be a continuous linear mapping such that if $a \in A \sim \{0\}$, then $Ta \neq 0$. Then $\{ \|Ta\| : a \in A \text{ and } \|a\| = 1 \}$ is compact and does not contain 0, so has a positive infimum, say δ. Hence

$$a \in A, \quad \|a\| \geqslant 1 \quad \Rightarrow \quad \|Ta\| \geqslant \delta ,$$

or, to put it another way, $\{ Ta : a \in A \text{ and } \|a\| < 1 \}$ is a neighbourhood of 0 in TA.

The generalisation of this result to all topological linear spaces is perhaps the most significant application of our theory of allied sets. The mapping T need not be linear, but only positive homogeneous, i.e. such that $T(\lambda x) = \lambda(Tx)$ for $\lambda \geqslant 0$ and $x \in X$. The set of elements which are mapped onto 0 will still be called the kernel of T. The theorem is:

5.6. Let X,Y be topological linear spaces, and let T be a continuous, positive homogeneous mapping $X \rightarrow Y$ with kernel K. Let A be a star in X such that 0 has a compact A-neighbourhood and $A \cap K \subseteq \overline{\{0\}}$. Then, for every A-neighbourhood N' of 0, TN' is a (TA)-neighbourhood of 0.

Proof. It is sufficient to prove the result for compact A-neighbourhoods of 0, since these form a base of A-neighbourhoods. Take $N \in \mathfrak{n}(X)$ such that $N' = A \cap N$ is compact. K is closed, since T is continuous, so A al (-K), by 5.1, and $Q \in \mathfrak{n}(X)$ exists such that

$$a \in A,\ k \in K,\ a-k \in Q \quad \Rightarrow \quad a \in N \,. \qquad (1)$$

Suppose that TN' is not a (TA)-neighbourhood of 0, so that, given circled $P \in \mathcal{N}(Y)$, there exists $y_P \in (P \cap TA) \sim TN'$. Take $x_P \in A \sim N'$ such that $Tx_P = y_P$. Then $x_P \notin N$, and there exists $\mu_P \in (0,1]$ such that

$$\mu_P x_P = x'_P \in (2N) \sim N \,.$$

Since A is a star, $x'_P \in 2(A \cap N) = 2N'$, and since P is circled, $Tx'_P = \mu_P y_P \in P$.

Let $\mathcal{B}$ be a base of circled neighbourhoods of 0 in Y. Then $\{x'_P : P \in \mathcal{B}\}$ is a net, and $Tx'_P \to 0$. Since $2N'$ is compact, (x'_P) has a convergent subnet. We denote this subnet by (z_n), and its limit by z. Then $Tz = \lim (Tz_n) = 0$, so $z \in K$.

If $x'_P \in K+Q$, then, by (1), $x'_P \in N$, contrary to hypothesis. Hence, for all $P \in \mathcal{B}$, $x'_P \notin z+Q$. This contradicts the fact that (z_n) converges to z, and the result follows. ‡

Various corollaries follow with ease. We start with:

<u>5.7</u>. Let X, Y be topological linear spaces, and let T be a continuous, positive homogeneous mapping $X \to Y$ with kernel K. Let A be a locally compact, positive homogeneous subset of X such that $A \cap K \subseteq \overline{\{0\}}$. Then TA is locally compact.

<u>Proof</u>. TA is positive homogeneous, and there is a compact (TA)-neighbourhood of 0, by 5.6. ‡

Next we have a generalisation of a result which is used repeatedly in (3) ((3), lemma 2):

5.8. Let X be a topological algebra, and let A be a positive homogeneous, locally compact subset of X. If x is an element such that $a \in A \sim \{0\} \Rightarrow xa \neq 0$, then xA is locally compact. ‡

Applying 5.7 to the identity map, we obtain:

5.9. A locally compact, positive homogeneous subset of a topological linear space is locally compact with respect to a smaller linear topology. ‡

5.6 says that T is an open mapping on A at 0. We finish by giving sufficient conditions for it to be open at a general point of a convex set. Naturally, linearity is needed for this.

5.10. Let X,Y be topological linear spaces, and T a continuous linear mapping $X \to Y$. Let A be a convex subset of X, and suppose that $a \in A$ is such that a has a compact A-neighbourhood, and that if $x \in A$ and $Tx = Ta$, then $x = a$. Then, for every A-neighbourhood N' of a, TN' is a (TA)-neighbourhood of Ta.

Proof. N'-a is a neighbourhood of 0 in the star A-a. If $x \in A$ and $T(x-a) = 0$, then $Tx = Ta$, so $x-a = 0$. Therefore, by 5.6, $T(N'-a)$ is a neighbourhood of 0 in $T(A-a)$, i.e. there exists $Q \in \mathfrak{n}(Y)$ such that $T(N'-a) = Q \cap T(A-a)$. Then $TN' = (Q+Ta) \cap TA$, giving the result. ‡

6. ALLIED SETS WITH RESPECT TO DIFFERENT TOPOLOGIES

This section is concerned with the question of whether sets which are allied with respect to one topology are allied with respect to another one. Firstly, we show that two different topologies for the same group will certainly give rise to some pairs which are allied with respect to one but not the other. In fact, it is only necessary to consider pairs of sets of which one is a singleton. Then we return to linear spaces and obtain some positive results for subspaces and positive homogeneous subsets.

We shall write A al B (τ) to denote the statement that A al B with respect to the topology τ.

Determination of topologies by allied pairs

We start with a very simple lemma:

6.1. Let X be a topological group with identity e.

(i) If $x^{-1} \notin \overline{A}$, then $\{x\}$ al A, A al $\{x\}$.

(ii) If $x \notin \overline{\{e\}}$ and $\{x\}$ al A, then $x^{-1} \notin \overline{A}$.

Proof. (i) There exists $M \in \mathfrak{N}(X)$ such that $(x^{-1}M) \cap A = \emptyset$, or $(xA) \cap M = \emptyset$. Hence, by vacuous implication, $\{x\}$ al A. Similarly, A al $\{x\}$.

(ii) If $\{x\}$ al A, then, by 1.10, $\{x^{-1}\} \cap \overline{A} \subseteq \overline{\{e\}}$. ‡

This enables us to prove the result stated in the introduction to the section:

6.2. Two group topologies for X giving the same allied pairs are the same.

Proof. Let τ_1 and τ_2 be the two topologies, and let $\overline{}^{i}$ denote closure with respect to τ_i $(i = 1,2)$.

If $x \in \overline{\{e\}}^{1} \sim \overline{\{e\}}^{2}$, then $\{x\}$ al $\{x^{-1}\}$ (τ_1), but $\{x\}$ nal $\{x^{-1}\}$ (τ_2), by 1.10. Hence $\overline{\{e\}}^{1} = \overline{\{e\}}^{2}$, and both sets can be denoted by $\overline{\{e\}}$. The result certainly holds if $\overline{\{e\}} = X$, so we suppose that $\overline{\{e\}} \subset X$.

Take any $A \subseteq X$. We show that $\overline{A}^{1} = \overline{A}^{2}$.

If $x \notin \overline{\{e\}}$, then, by 6.1,

$$
\begin{aligned}
x^{-1} \notin \overline{A}^{1} \quad &\Leftrightarrow \quad \{x\} \text{ al } A \quad (\tau_1)\\
&\Leftrightarrow \quad \{x\} \text{ al } A \quad (\tau_2)\\
&\Leftrightarrow \quad x^{-1} \notin \overline{A}^{2}.
\end{aligned}
$$

Now suppose that $x \in \overline{\{e\}}$. Take $y \notin \overline{\{e\}}$. Then $xy \notin \overline{\{e\}}$, so $xy \in \overline{Ay}^{1} \Leftrightarrow xy \in \overline{Ay}^{2}$, by the result just proved. But $x \in \overline{A}^{i} \Leftrightarrow xy \in \overline{Ay}^{i}$ $(i = 1,2)$, since τ_i is a group topology. ‡

It is natural to ask whether a group topology can be defined by specifying the allied pairs and defining:

$$x \in \overline{\{e\}} \quad \Leftrightarrow \quad \{x\} \text{ al } \{x^{-1}\} ;$$

$$\text{if } \ x \notin \overline{\{e\}}, \text{ then } \ x \in \overline{A} \quad \Leftrightarrow \quad \{x^{-1}\} \text{ nal } A ;$$

$$\text{if } \ x \in \overline{\{e\}}, \text{ then } \ x \in \overline{A} \quad \Leftrightarrow \quad xy \in \overline{Ay} \ \text{ for some } \ y \notin \overline{\{e\}}.$$

The correspondence between some of the axioms is immediate. For instance, the condition

$$(A \cup B) \text{ al } C \iff A \text{ al } C \text{ and } B \text{ al } C$$

(cf. 1.7) implies $\overline{A \cup B} = \overline{A} \cup \overline{B}$. But it does not seem to be easy to give a set of axioms to be satisfied by the relation A al B which ensures that $\overline{\overline{A}} = \overline{A}$ and that multiplication is continuous.

Subspaces of complete topological linear spaces

The results of sections 3 and 4 give at once:

6.3. Suppose that σ and τ are complete, Hausdorff topologies for a linear space X such that σ is metrisable and every τ-closed linear subspace is σ-closed (which will occur, in particular, if $\tau < \sigma$). If A, B are subspaces, and A al B (τ), then A al B (σ).

Proof. By 1.9, $\overline{A}$ al $\overline{B}$ (τ), the closures being taken with respect to τ. Hence, by 3.1, $\overline{A} + \overline{B}$ is τ-complete, so τ-closed. Hence $\overline{A}, \overline{B}$ and $\overline{A} + \overline{B}$ are all σ-closed, so A al B (σ), by 4.7. ‡

It is well-known that if τ is complete, $\tau < \sigma$, and σ has a local base consisting of τ-closed sets, then σ is complete. However, 6.3 is of limited application, and we proceed to deduce results of greater interest by different methods.

Dual topologies

Let (X,Y) be a dual pair of linear spaces, not necessarily separated. We shall often regard the elements of Y as linear functionals on X, and denote the scalar $\langle x,y \rangle$ by $y(x)$. The weak and strong topologies induced by Y on X and its subsets will be denoted by $w(Y)$ and $s(Y)$ respectively.

If X is a topological linear space, then X^* will denote the space of all continuous linear functionals on X (with respect to the given topology), and $w(X^*)$ will be called the corresponding weak topology.

We start by examining the meaning of alliedness with respect to a weak topology:

6.4. Let (X,Y) be a dual pair, and let A,B be subsets of X. Then A al B with respect to $w(Y)$ if and only if, given $y \in Y$, there exists a $w(Y)$-neighbourhood N of 0 such that

$$a \in A,\ b \in B,\ a+b \in N \ \Rightarrow\ |y(a)| \leqslant 1 .$$

Proof. The condition is clearly necessary. Suppose that it holds, and that $y_1, \ldots ,y_n \in Y$ are given. Then there exist $w(Y)$-neighbourhoods N_i of 0 $(i = 1, \ldots ,n)$ such that

$$a \in A,\ b \in B,\ a+b \in N_i \ \Rightarrow\ |y_i(a)| \leqslant 1 .$$

If $a \in A$, $b \in B$, and $a+b \in \bigcap N_i$, then $|y_i(a)| \leqslant 1$ for each i. It follows that A al B with respect to $w(Y)$. ‡

Now we have:

6.5. Let (X,τ) be a Hausdorff, locally convex space. If subspaces A, B are allied with respect to τ, then they are allied with respect to $w(X^*)$.

Proof. Since τ is Hausdorff, $A\cap B = \{0\}$. Hence elements of A+B can be expressed uniquely in the form a+b ($a \in A$, $b \in B$), and, given $y \in X^*$, we can define a linear functional y' on A+B by: $y'(a+b) = y(a)$. By 4.1, y' is τ-continuous on A+B, so has an extension $z \in X^*$. If $a \in A$, $b \in B$ and $|z(a+b)| \leqslant 1$, then $|y(a)| \leqslant 1$. Thus A al B with respect to $w(X^*)$, by 6.4. ‡

Later, we shall see how to extend this result to wedges (see 7.13). Results in the converse direction can be obtained for more general sets still. Recall that if τ is a locally convex topology and w the corresponding weak topology, then a set is w-bounded if and only if it is τ-bounded, so that we may speak simply of "bounded" sets (see, e.g., (10), 17.5).

6.6. Let (X,τ) be a locally convex, metrisable space, and let w denote the associated weak topology. If A, B are positive homogeneous subsets such that A al B (w), then A al B (τ).

Proof. If A nal B (τ), then, by 2.2(ii), there is a bounded sequence (a_n+b_n) with $a_n \in A$, $b_n \in B$, and (a_n) unbounded. Hence, by 2.2(i), A nal B (w). ‡

Next we consider the topologies w(X) and s(X) for X^*. If X is a normed space, then s(X) is the norm topology for

X^*. By the uniform boundedness theorem ((10), 18.5), if X is locally convex and sequentially complete, then subsets of X^* are w(X)-bounded if and only if they are s(X)-bounded. Thus, by the same argument as in 6.6, we have:

6.7. Suppose that X is locally convex and sequentially complete, and that the topology s(X) for X^* is metrisable. If A, B are positive homogeneous subsets of X^* such that A al B (w(X)), then A al B (s(X)).

In particular, if X is a Banach space, and A, B are positive homogeneous subsets of X^* such that A al B (w(X)), then A al B with respect to the norm topology. ∤

Lastly, we make use of the fact that if X, Y are Hausdorff, locally convex spaces, and a linear mapping $X \to Y$ is continuous with respect to the weak topologies w_X, w_Y associated with the given topologies, then it is continuous with respect to the Mackey topologies m_X, m_Y (see (11), 21,4(6)). Also, the Mackey topology for a subspace of X is not smaller than the relative topology induced by m_X (no direct proof of this entirely elementary result seems to exist in the literature, so one is given in appendix 2). Recall that X is a MACKEY SPACE if it is Hausdorff, locally convex, and its topology coincides with the associated Mackey topology (in particular, all metrisable, locally convex spaces and all barrelled spaces are Mackey spaces). Using these facts, we have:

6.8. Suppose that (X, τ) is a Mackey space, and that A, B are weakly allied subspaces such that A+B = X. Then A al B (τ).

Proof. $A \cap B = \{0\}$, so the projection $A+B \to A$ is defined and continuous with respect to w_X and w_A. Therefore it is continuous with respect to $m_X = \tau_X$, and m_A, which is not smaller than τ_A. Hence it is continuous with respect to τ_X and τ_A, so by 4.1, A al B (τ) . ‡

It will be noticed that, in the case when X is bornological, 6.8 follows easily from the equivalence of weak boundedness and boundedness. However, it does not seem to be easy to replace A and B by more general sets in 6.8, even when X is bornological.

Examples

(i) Let $\| \ \|$ be the usual norm of l_1. Another norm on l_1 is the usual norm p of m, defined by $p(x) = \sup |\xi_n|$, where $x = (\xi_n)$.

Let A be the set of decreasing, non-negative sequences in l_1, and let B be the set of sequences having $\beta_1 = 0$. Let

$$a_n = (1,1, \ldots ,1), \quad \|a_n\| = n,$$

$$b_n = (0,1, \ldots ,1), \quad \|b_n\| = n-1.$$

Then $a_n \in A$, $b_n \in B$ and $\|a_n - b_n\| = 1$. Hence A nal B with respect to $\| \ \|$.

For any $a = (\alpha_n) \in A$, $p(a) = \alpha_1$. If $b \in B$, then the first component of a+b is α_1, so $p(a+b) \geq p(a)$. Hence A al B with respect to p.

(ii) Let A be the subspace of m consisting of sequences (α_n) having $\alpha_{2n} = 0$ for all n, and B the subspace consisting of sequences (β_n) having $\beta_{2n} = 2^{-n} \beta_{2n-1}$ for all n (cf. section 2, ex. (iii), p. 27). Clearly, A nal B with respect to the norm topology.

Let τ be the topology (for m) of pointwise convergence. We note in passing that this is a metrisable topology, and that it is w(F), where F is the space of all finite real sequences. A basic τ-neighbourhood of 0 is the set M of all (ξ_n) having $|\xi_i| \leq \varepsilon$ for $i \leq 2k$ (for some ε,k). Suppose that $(\alpha_n) \in A$, $(\beta_n) \in B$, and $|\beta_i - \alpha_i| \leq 2^{-k}\varepsilon$ for $i \leq 2k$. The even components show that $|\beta_i| \leq \varepsilon$ for $i = 1, 3, \ldots, 2k-1$. Hence $(\beta_n) \in M$. It follows that A al B (τ).

Note that A and B are τ-closed (though not τ-complete) subsets of m, while A+B is not even closed in the norm topology. Thus we have another example of closed allied subspaces with a non-closed sum (cf. p. 35-36).

We have found subspaces which are allied with respect to τ but not with respect to $\| \|$. Conversely, let K be a subspace which is $\| \|$-closed but not τ-closed (e.g. c_0). Take an element x which is in the τ-closure of K, but not in K. Let L be the one-dimensional subspace spanned by x. Then K,L are allied with respect to $\| \|$, by 5.1, but not with respect to τ.

7. DECOMPOSITION THEOREMS

Let (X,Y) be a real dual pair of linear spaces. For $A \subseteq X$, write

$$A^+ = \{y \in Y : y(a) \geqslant 0 \quad \text{for all } a \in A\},$$

$$A^o = \{y \in Y : |y(A)| \leqslant 1 \quad \text{for all } a \in A\}.$$

These are $w(X)$-closed subsets of Y. A^+ is a wedge, and A^o a star. If A is symmetric, then A^o is the set of elements of Y which are not greater than 1 on A, and A^+ is the set of elements of Y which vanish on A. For a subset B of Y, B^+ and B^o are defined similarly as subsets of X.

We continue to denote by pos A the wedge generated by A. A^{++} is the closure with respect to $w(Y)$ of pos A.

The motivation for this section derives from the fact that if $A-B = X$, then $A^+ \cap (-B^+) = \{0\}$, since an element of this would be non-negative on X. This is the necessary disjointness condition for A^+ al B^+, and indeed we have:

<u>7.1</u>. If (X,Y) is a real dual pair, and $A-B = X$, then A^+ al B^+ with respect to $w(X)$.

<u>Proof</u>. Take $x \in X$. There exist $a, a' \in A$ and $b, b' \in B$ such that $x = a-b$, $-x = a'-b'$. Suppose that $f \in A^+$, $g \in B^+$ and $f+g \leqslant 1$ at each of a, a', b, b'. Then

$$g(a) \leqslant f(a) + g(a) \leqslant 1,$$

so

$$g(x) = g(a) - g(b) \leqslant g(a) \leqslant 1.$$

Similarly, $g(-x) \leqslant 1$, so $|g(x)| \leqslant 1$. The result follows, by 6.4. ‡

From 7.1 and 6.7 we have immediately:

7.2. If X is a real Banach space, and $A-B = X$, then A^+ al B^+ with respect to the norm topology of X^*. ‡

The special case of this appropriate to partially ordered linear spaces, viz. when A is a wedge and $B = A$, was obtained by Bonsall ((1), lemma 2), and also by Namioka ((14), 8.11), under certain extra hypotheses.

If A and B are symmetric, then we may read $A^{\perp}$ for A^+ (and similarly for B) in 7.1 and 7.2, obtaining results which strengthen certain statements by Köthe in (11) (20,5).

7.3. If (X,Y) is a real dual pair, and A,B are subsets of X such that $A^+-B^+ = Y$, then A al B with respect to w(Y).

Proof. A^{++} al B^{++}, by 7.1. ‡

7.4. If (X,τ) is a locally convex, metrisable space, and A,B are positive homogeneous subsets of X such that $A^+-B^+ = X^*$, then A al B with respect to τ.

Proof. By 7.3, A al B with respect to $w(X^*)$. The result follows, by 6.6. ‡

As in sections 1 and 2, for fixed A,B, we write

$$[E] = (E+A) \cap (E-B) .$$

We recall 2.3 and the example following it: if A,B are allied stars, and E is bounded, then so is [E], but there is no corresponding result for compactness. Our next theorem gives one situation in which compactness of E implies that of $[E]$. If X is barrelled, and a subset K of X^* is w(X)-bounded, then K is w(X)-compact (the closure being taken with respect to w(X)), since K^o is a neighbourhood of 0, so that K^{oo} is w(X)-compact.

<u>7.5</u>. Suppose that X is barrelled and that $A-B = X$. For $E \subseteq X^*$, write $[E] = (E+A^+) \cap (E-B^+)$. Then, for X^* with the topology w(X):

(i) if E is bounded, then $\overline{[E]}$ is compact;

(ii) if E is compact, then so is $[E]$.

<u>Proof</u>. By 7.1, A^+ al B^+ with respect to w(X). Therefore, by 2.3, if E is w(X)-bounded, then so is $[E]$, so that $\overline{[E]}$ is w(X)-compact.

If E is w(X)-compact, then $[E]$ is w(X)-closed, so (ii) follows. ‡

If $B = A$ and $\leqslant$ is the ordering induced on X^* by the wedge A^+, then 7.5 shows that order-intervals in X^* are w(X)-compact. In the case when $\leqslant$ is a lattice ordering, this implies that X^* is order-complete, i.e. that a subset of X^* which is bounded above has a least upper bound.

<u>$\mathcal{F}$-decomposition</u>

Given a family $\mathcal{F}$ of subsets of a commutative topological group X, we say that the pair of subsets (A,B) gives an $\mathcal{F}$-DECOMPOSITION of X if, given $F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ such that $F \subseteq A\cap G - B\cap G$. If (A,A) gives an $\mathcal{F}$-decomposition of X, we will simply say that A does so. (Cf. the concept "strict $\mathcal{F}$-cone" introduced in (19), p. 217).

$\mathcal{F}$ will usually be chosen with its members symmetric, so that we will have, equivalently, $F \subseteq B\cap G - A\cap G$. Provided that $\bigcup \mathcal{F} = X$, it is clear that $A-B = X$.

Let X,Y be topological linear spaces, and let Z be the space of all continuous linear mappings from X to Y (or a subspace). Let $\mathcal{F}$ be a family of bounded, symmetric subsets of X such that the union of any two members is contained in a third. Then a local base for the topology $\tau(\mathcal{F})$ (for Z) of uniform convergence on $\mathcal{F}$ is the family of sets of the form $\{T : TF \subseteq M\}$, where $F \in \mathcal{F}$ and $M \in \mathfrak{N}(Y)$. If X and Y are normed spaces, and $\mathcal{F}$ is the set of spheres in X with centre 0, then $\tau(\mathcal{F})$ is the norm topology for Z.

Under these circumstances, we have the following theorem:

<u>7.6</u>. If A gives an $\mathcal{F}$-decomposition of X, and K,L are allied subsets of Y, then

$$\{T : TA \subseteq K\} \text{ al } \{U : UA \subseteq L\}$$

with respect to $\tau(\mathcal{F})$.

<u>Proof</u>. Take $F \in \mathcal{F}$ and $M \in \mathfrak{N}(Y)$. There exists $M_1 \in \mathfrak{N}(Y)$ such that $M_1-M_1 \subseteq M$. Since K al L , there exists $N \in \mathfrak{N}(Y)$ such that

$$k \in K,\ l \in L,\ k+l \in N \Rightarrow k \in M_1 .$$

Also, there exists $G \in \mathcal{F}$ such that $F \subseteq A\cap G - A\cap G$.

Suppose that $TA \subseteq K$, $UA \subseteq L$ and $(T+U)G \subseteq N$. If $x \in A\cap G$, then $Tx \in K$, $Ux \in L$ and $Tx + Ux \in N$, so $Tx \in M_1$. Hence $TF \subseteq M_1-M_1 \subseteq M$, and the theorem is proved. ‡

We notice that it is sufficient if, instead of an

$\mathcal{F}$-decomposition of X, we have the condition that, given $F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ such that $F \subseteq \overline{H}$, where $H = A \cap G - A \cap G$. To see this, take M closed in the proof. Cf. Schaefer's definition of "$\mathcal{F}$-cone" ((19), p. 217).

7.6, in the case when K = L, is given in (19) (p. 226), under certain extra hypotheses. A second generalisation of Schaefer's result is obtained by keeping K = L but replacing A by two sets in X:

7.7. If (A,B) gives an $\mathcal{F}$-decomposition of X, and K is a self-allied semigroup in Y, then

$$\{T : TA \subseteq K\} \text{ al } \{U : UB \subseteq K\}$$

with respect to $\tau(\mathcal{F})$.

Proof. Take $F \in \mathcal{F}$ and $M \in \mathcal{N}(Y)$. There exists $G \in \mathcal{F}$ such that $F \subseteq A \cap G - B \cap G$. Take symmetric $M_1 \in \mathcal{N}(Y)$ such that $M_1 + M_1 + M_1 \subseteq M$. There exists $N_1 \in \mathcal{N}(Y)$ such that $N_1 \subseteq M_1$ and such that, if $k_i \in K$ $(i = 1,2,3,4)$ and $k_1+k_2+k_3+k_4 \in N_1$, then $k_i \in M_1$ for each i (this is the stage at which we need the fact that K is a semigroup). Take $N \in \mathcal{N}(Y)$ such that $N+N \subseteq N_1$.

Suppose that $TA \subseteq K$, $UB \subseteq K$ and $(T+U)G \subseteq N$. We show that $TF \subseteq M$. Take $f \in F$. Since F is symmetric, there exist $a, a' \in A \cap G$ and $b, b' \in B \cap G$ such that $f = a-b$, $-f = a'-b'$. Then $a+a' = b+b'$, so

$$Ta + Ta' + Ub + Ub' = (T+U)(a+a') \in N+N \subseteq N_1 ,$$

and $Ta, Ta', Ub, Ub' \in M_1$. Now $Tb + Ub \in N \subseteq M_1$, so

$$Tf = Ta - Tb = Ta + Ub - (Tb + Ub) \in M_1+M_1+M_1 \subseteq M ,$$

as required. ‡

In the special case when $Y = R$, Z becomes a subspace of X^*, and $\tau(\mathcal{F})$ becomes the polar topology induced on Z by $\mathcal{F}$, having local base $\mathcal{F}^o = \{F^o : F \in \mathcal{F}\}$. The strong topology $s(X)$ and the Mackey topology $m(X)$ are obtained by letting $\mathcal{F}$ be the set of symmetric subsets of X which are (i) bounded or (ii) compact with respect to $w(Z)$. Putting $K = R^+$ in 7.7, we obtain:

<u>7.8</u>. If (X,Y) is a real dual pair, $\mathcal{F}$ is a family of symmetric, $w(Y)$-bounded subsets of X, and (A,B) gives an $\mathcal{F}$-decomposition of X, then A^+ al B^+ with respect to $\tau(\mathcal{F})$.‡

7.1 is a special case of this result - a fact which illustrates the considerable generality of 7.7.

$\mathcal{N}^o$-decompositions of the real dual space X^* are of particular interest, where $\mathcal{N} = \mathcal{N}(X)$. We repeat the definition: (C,D) gives an $\mathcal{N}^o$-decomposition of X^* if and only if, given $M \in \mathcal{N}(X)$, there exists $N \in \mathcal{N}(X)$ such that $M^o \subseteq C \cap N^o - D \cap N^o$. An $\mathcal{N}^o$-decomposition of X^* is the same as a $\mathcal{B}^o$-decomposition, where $\mathcal{B}$ is any local base in X. If (X,τ) is a locally convex space, then $\tau = \tau(\mathcal{N}^o)$, so 7.8 gives:

<u>7.9</u>. Let X be a real, locally convex space, and write $\mathcal{N} = \mathcal{N}(X)$. If (C,D) gives an $\mathcal{N}^o$-decomposition of X^*, then C^+ al D^+ in X. ‡

In particular, if $A,B \subseteq X$ and (A^+,B^+) gives an $\mathcal{N}^o$-decomposition of X^*, then A al B .

We notice that if X is a normed space, then (C,D)

gives an η^{o}-decomposition of X^{*} if and only if there exists $K > 0$ such that every $f \in X^{*}$ is expressible as g-h, where $g \in C$, $h \in D$ and $\|g\|, \|h\| \leq K \|f\|$.

We now turn our attention to results in the converse direction, i.e. deducing that some pair of sets gives a decomposition. The main result of Bonsall (2) is that if A is a self-allied cone in a locally convex space X, then A^{+} gives an η^{o}-decomposition of X^{*}. Schaefer obtained this result independently ((16), 1.3). With a proof substantially similar to Bonsall's, we show that this can be generalised to two wedges A, B; (the resulting theorem is the converse of the remark after 7.9, in the case when A and B are wedges). We use the following lemma, proved by Bonsall ((2), theorem 1):

<u>7.10.</u> Let X be a real linear space, p a sublinear functional on X, and B a wedge in X. Suppose that q is a functional defined on B and superlinear there (i.e. -q is sublinear), and that $q(b) \leq p(b)$ for $b \in B$. Then there is a linear functional f on X such that

$$f(x) \leq p(x) \quad (x \in X),$$
$$f(b) \geq q(b) \quad (b \in B). \quad \ddagger$$

Our theorem is:

<u>7.11.</u> Let X be a real, locally convex space, and write $\eta = \eta(X)$. If A, B are allied wedges in X, then (A^{+}, B^{+}) gives an η^{o}-decomposition of X^{*}.

<u>Proof</u>. Let $M \in \eta(X)$ be given. There exists convex,

symmetric $N \in \mathcal{N}(X)$ such that

$$a \in A,\ b \in B,\ a+b \in N \Rightarrow a, b \in M. \qquad (1)$$

Let p_N be the Minkowski functional of N. For all $x \in X$, define

$$p(x) = \inf \{p_N(y) : y \in x+A\}.$$

$p(x) \geq 0$, and for $\lambda \geq 0$, $p(\lambda x) = \lambda p(x)$. If $x \in -A$, then $0 \in x+A$, so $p(x) = 0$. If $y_i \in x_i+A$ and $p_N(y_i) \leq p(x_i) + \varepsilon$ $(i = 1,2)$, then $y_1+y_2 \in x_1+x_2+A$, and

$$p_N(y_1+y_2) \leq p(x_1) + p(x_2) + 2\varepsilon .$$

Hence $p(x_1+x_2) \leq p(x_1) + p(x_2)$, and p is sublinear.

Take $f \in M^o$. Suppose that $y \in (x-A) \cap B$, and take $\delta > p(x)$. $x \in y+A$, so $x+A \subseteq y+A$, and $p(y) \leq p(x)$. Hence $p(y/\delta) \leq p(x/\delta) < 1$, so there exists $z \in y/\delta + A$ such that $p_N(z) < 1$, i.e. $z \in N$. $y/\delta \in B$, so by (1), $y/\delta \in M$, and $|f(y)| \leq \delta$. Hence $f(y) \leq p(x)$.

Thus, for $b \in B$, we may define

$$q(b) = \sup \{f(y) : y \in (b-A) \cap B\},$$

and $q(b) \leq p(b)$. Putting $y = b$, we see that $q(b) \geq f(b)$. If $y_i \in (b_i-A) \cap B$ and $f(y_i) > q(b_i) - \varepsilon$ $(i = 1,2)$, then $y_1+y_2 \in (b_1+b_2-A) \cap B$, and

$$f(y_1+y_2) > q(b_1) + q(b_2) - 2\varepsilon .$$

Hence q is superlinear on B.

By 7.10, there is a linear functional f_1 on X such that

$$f_1(x) \leq p(x) \quad (x \in X),$$
$$f_1(b) \geq q(b) \quad (b \in B).$$

If $x \in -A$, then $f_1(x) \leq p(x) = 0$. Thus $f_1 \in A^+$. Also,

$f_1 \leqslant p \leqslant p_N$; so $f_1 \in N^o$. Putting $f_2 = f_1 - f$, we have

$$f_2(b) \geqslant q(b) - f(b) \geqslant 0 \qquad (b \in B).$$

Finally, let $P = \frac{1}{2}(M \cap N)$. $|f|$ and $|f_1| \leqslant \frac{1}{2}$ on P, so $|f_2| \leqslant 1$ on P. Hence

$$M^o \subseteq (A^+ \cap P^o) - (B^+ \cap P^o) \text{ . } \ddagger$$

Combining 7.1 and 7.11, we have:

<u>7.12.</u> Let (X,Y) be a real dual pair, and let A,B be $w(Y)$-closed wedges in X. Then the following statements are equivalent:

(i) $A-B = X$;

(ii) A^+ al B^+ with respect to $w(X)$.

<u>Proof</u>. (i) implies (ii), by 7.1. If (ii) holds, then $A^{++} - B^{++} = X$, by 7.11. (i) follows, since $A^{++} = A$, $B^{++} = B$. ‡

Under the same conditions, it is elementary that $A-B$ is $w(Y)$-dense in X if and only if $A^+ \cap (-B^+) = \{0\}$.

7.11 enables us to improve upon 6.5. Note that if X is a complex topological linear space, and X_R^*, X_C^* denote respectively the spaces of real and complex continuous linear functionals on X, then the topologies $w(X_R^*)$ and $w(X_C^*)$ are the same. Thus, in all cases, $w(X_R^*)$ is the same topology as $w(X^*)$.

<u>7.13.</u> If (X,τ) is a locally convex space, and A,B are τ-allied wedges in X, then A al B with respect to $w(X^*)$.

<u>Proof</u>. By 7.11, $A^+ - B^+ = X_R^*$. The result follows, by 7.3. ‡

Another easy consequence of 7.11 is:

7.14. Let (X,τ) be a real, locally convex metrisable space, and write $\mathfrak{N} = \mathfrak{N}(X)$. If C, D are $w(X)$-closed wedges in X^* such that $C-D = X^*$, then (C, D) gives an $\mathfrak{N}^o$-decomposition of X^*.

Proof. By 7.1 and 6.6, C^+ al D^+ (τ). Since $C^{++} = C$ and $D^{++} = D$, the result follows, by 7.11. ‡

Denoting by $\Delta(E)$ the convex, circled cover of a set E, we have:

7.15. Let X be a locally convex space, and $\mathcal{F}$ the family of finite subsets of X. If A, B are closed wedges such that $A-B = X$, then (A, B) gives a $\Delta(\mathcal{F})$-decomposition of X.

Proof. Consider the dual pair (X, X^*_R). By 7.1, A^+ al B^+ with respect to $w(X)$. $\{F^o : F \in \mathcal{F}\}$ is a local base for $w(X)$, and $F^{oo} = \Delta(F)$. The result follows, by 7.11. ‡

Let X be a barrelled space. Write $\mathfrak{N} = \mathfrak{N}(X)$, and let $\mathcal{B}$ be the set of $w(X)$-bounded subsets of X^* (which, incidentally, coincides with the set of $s(X)$-bounded subsets). If $E \in \mathcal{B}$, then $E^o \in \mathfrak{N}$, so $E \subseteq E^{oo} \in \mathfrak{N}^o$. Conversely, if $N \in \mathfrak{N}$, then N^o is $w(X)$-compact, so certainly $N^o \in \mathcal{B}$. It follows that the $\mathfrak{N}^o$-decompositions of X^* are precisely the $\mathcal{B}$-decompositions. Hence we have:

7.16. If X is a barrelled space, and $\mathcal{B}$ denotes the set of $w(X)$-bounded subsets of X^*, then wedges A, B in X are allied if and only if (A^+, B^+) gives a $\mathcal{B}$-decomposition of X^*. ‡

8. OPEN DECOMPOSITION

Let X be a commutative topological group. We say that the pair of subsets (A,B) gives an OPEN DECOMPOSITION of X if $A-B = X$ and, for each neighbourhood M of 0, the set $A\cap M - B\cap M$ is also a neighbourhood of 0. If (A,A) gives an open decomposition of X, we shall simply say that A does so.

This section is devoted to the subject of open decomposition in its own right, but we shall see that it is closely connected with allied sets. We start with some very elementary results (8.1 - 8.6). X, throughout, denotes a commutative topological group.

<u>8.1.</u> If $A-B = X$ and A al (−B), then (A,B) gives an open decomposition of X.

<u>Proof.</u> Take $M \in \mathcal{N}(X)$. There exists $N \in \mathcal{N}(X)$ such that

$$a \in A,\ b \in B,\ a-b \in N \Rightarrow a,b \in M .$$

Since $A-B = X$, it follows that $A\cap M - B\cap M \supseteq N$. ‡

<u>8.2.</u> If $(A-A) \cap (B-B) = \{0\}$, and (A,B) gives an open decomposition of X, then A al (−B).

<u>Proof.</u> Given $M \in \mathcal{N}(X)$, let $N = A\cap M - B\cap M$. Then $N \in \mathcal{N}(X)$, and if $a-b \in N$ (where $a \in A$, $b \in B$), then $a,b \in M$, since expressions of the form $a-b$ ($a \in A$, $b \in B$) are unique. ‡

Combining these two results, we have, for subgroups:

<u>8.3.</u> If A,B are subgroups such that $A\cap B = \{0\}$ and $A+B = X$, then (A,B) gives an open decomposition of X if and only if A al B. ‡

8.4. If there exist locally compact subsets A, B which contain 0 and give an open decomposition of X, then X is locally compact.

Proof. There exists $M \in \mathcal{N}(X)$ such that $A \cap M$ and $B \cap M$ are compact. Then $A \cap M - B \cap M$ is a compact neighbourhood of 0. ‡

Note that if (A, B) gives an open decomposition of X, then $\overline{A}$ and $\overline{B}$ contain 0.

8.5. Let X, Y be commutative topological groups, and suppose that A, B contain 0 and give an open decomposition of X. If T is a homomorphism $X \to Y$ which is continuous on A and B at 0, then T is continuous on X.

Proof. Given $P \in \mathcal{N}(Y)$, take $Q \in \mathcal{N}(Y)$ such that $Q - Q \subseteq P$. There exists $M \in \mathcal{N}(X)$ such that $T(A \cap M) \subseteq Q$ and $T(B \cap M) \subseteq Q$. $(A \cap M - B \cap M) \in \mathcal{N}(X)$, and $T(A \cap M - B \cap M) \subseteq P$. ‡

The following characterisation by nets and sequences should be compared with 1.5. Equivalence is only obtained in the metrisable case here.

8.6. Suppose that $A - B = X$, and consider the statements:

(i) If (x_n) is a net convergent to 0, then there exist, for each n, $a_n \in A$ and $b_n \in B$ such that $x_n = a_n - b_n$ and the nets $(a_n), (b_n)$ converge to 0.

(i)(s) Statement (i), with "net" replaced by "sequence".

(ii) (A, B) gives an open decomposition of X.

Then (i) implies (ii), and if X is metrisable, then (i)(s) is equivalent to (ii).

<u>Proof</u>. Suppose that (A,B) does not give an open decomposition of X, so that, for some $M \in \mathcal{N}(X)$, $A \cap M - B \cap M \notin \mathcal{N}(X)$. Write $A \cap M - B \cap M = L$. Let $\mathcal{B}$ be a local base, countable in the metrisable case. For each $N \in \mathcal{B}$, there exists $x_N \in N \sim L$. $\{x_N : N \in \mathcal{B}\}$ is a net convergent to 0 (a sequence in the metrisable case). If $x_N = a_N - b_N$ (where $a_N \in A$, $b_N \in B$), then one of a_N, b_N is not in M. Thus neither of the nets $(a_N), (b_N)$ converges to 0, for if one did, then both would, and both would eventually stay inside M.

Now suppose that X is metrisable, and that (A,B) gives an open decomposition of X. There is a countable, contracting local base $\{M_n : n = 1, 2, ..\}$. Let (x_n) be a sequence convergent to 0. For each positive integer i, there exists $n_i > n_{i-1}$ such that

$$n \geq n_i \implies x_n \in (A \cap M_i - B \cap M_i) .$$

For $n_i \leq n < n_{i+1}$, choose $a_n \in A \cap M_i$, $b_n \in B \cap M_i$ such that $x_n = a_n - b_n$. Then the sequences $(a_n), (b_n)$ converge to 0. $\ddagger$

The case where X is metrisable and A = B is essentially given by Nachbin ((13), proposition 15, p. 87).

<u>Linear spaces</u>

First we notice that if X is a topological linear space, and A,B are positive homogeneous subsets such that, for some $M \in \mathcal{N}(X)$, $(A \cap M - B \cap M) \in \mathcal{N}(X)$, then $A-B = X$,

Open decompositions of linear spaces are of interest in two quite different contexts:

(i) Open decomposition by pseudo-disjoint subspaces A,B. Of course, we can equally well write $A+B = X$ in this case. By 8.3, open decomposition occurs if and only if A al B .

(ii) Open decomposition by a wedge A which induces a partial ordering of X. Nachbin actually incorporates this condition in his definition of a "locally convex directed space" ((13), p. 86). One elementary consequence is the continuity of any linear functional g satisfying $0 \leqslant g \leqslant f$, where f is continuous.

A simple result which applies in context (ii) is:

<u>8.7</u>. If X is a topological linear space, and A is a positive homogeneous subset with non-empty interior, then A gives an open decomposition of X.

<u>Proof</u>. Take open $M \in \mathcal{N}(X)$, and let $G = (\text{int } A) \cap M$. Then G is open and non-empty, since A is positive homogeneous. Hence $G-G \in \mathcal{N}(X)$, and the result follows. ‡

A question which arises naturally is that of open decomposition with respect to different topologies. 8.3 enables us to answer this in context (i), using 6.6 and 7.13:

8.8. Let (X,τ) be a locally convex space, and let w be the associated weak topology. Then:

(i) If (A,B) gives an open decomposition of X with respect to τ, then it does with respect to w.

(ii) If τ is metrisable, and (A,B) gives an open decomposition of X with respect to w, then it does with respect to τ. ‡

Positive homogeneous subsets A,B of a normed linear space X give an open decomposition if and only if $K > 0$ exists such that every $x \in X$ is expressible in the form a-b, where $a \in A$, $b \in B$ and $\|a\|, \|b\| \leq K \|x\|$. Thus an open decomposition of a normed space by positive homogeneous subsets is the same as a bounded decomposition (i.e. a $\mathcal{B}$-decomposition, where $\mathcal{B}$ is the family of bounded sets). We now attempt to discover how much of this remains true in more general spaces:

8.9. Let X be a topological linear space. If A,B are stars such that $A-B = X$ and A al $(-B)$, then (A,B) gives a bounded decomposition of X.

Proof. For each $x \in X$, there exist $a_x \in A$, $b_x \in B$ such that $x = a_x - b_x$. Take a bounded set E, and write $A_E = \{a_x : x \in E\}$, $B_E = \{b_x : x \in E\}$. Then $E \subseteq A_E - B_E$. We show that A_E and B_E are bounded. Take $M \in \mathcal{N}(X)$. There exists circled $N \in \mathcal{N}(X)$ such that

$$a \in A,\ b \in B,\ a-b \in N \Rightarrow a,b \in M.$$

There exists $\lambda \in (0,1]$ such that $\lambda E \subseteq N$. If $x \in E$, then

$\lambda x = \lambda a_x - \lambda b_x \in N$, so $\lambda a_x, \lambda b_x \in M$. Hence $\lambda A_E \subseteq M$, $\lambda B_E \subseteq M$, and the theorem is proved. ‡

From this and 8.2, we have:

8.10. Let A,B be pseudo-disjoint subspaces of a topological linear space X. If (A,B) gives an open decomposition of X, then it gives a bounded decomposition of X. ‡

Now we show that converse results apply in bornological spaces, but it is necessary to make a distinction between the real and complex cases. Every complex topological linear space is, of course, a real one. However, two different concepts of "circled set" are available: complex-circled and (a weaker condition) real-circled. Let $\mathfrak{A}$ denote the family of convex subsets M of X such that, given a bounded set E, there exists $\lambda > 0$ such that $\lambda E \subseteq M$. As a complex space, X is bornological if every complex-circled set in $\mathfrak{A}$ is a neighbourhood of 0. As a real space, X is bornological if (a stronger condition) every real-circled set in $\mathfrak{A}$ is a neighbourhood of 0. In fact, in this case, every set in $\mathfrak{A}$ is a neighbourhood of 0, since if $M \in \mathfrak{A}$, then $M \cap (-M)$ is a real-circled set in $\mathfrak{A}$. Curiously, this rather basic point seems to have been ignored in all the literature. Our converse result is:

8.11. (i) If X is a real bornological space, and A,B are wedges which give a bounded decomposition of X, then (A,B) gives an open decomposition of X.

(ii) If X is a complex bornological space, and A,B are linear subspaces which give a bounded decomposition of X, then (A,B) gives an open decomposition of X.

Proof. Take convex $M \in \mathcal{N}(X)$. Let $N = A\cap M - B\cap M$. Then N is convex. In case (ii), it is also circled. Take a bounded set E. There exists a bounded set F such that $E \subseteq A\cap F - B\cap F$. For some $\lambda > 0$, $\lambda F \subseteq M$. Then $\lambda E \subseteq N$. Since X is bornological, it follows that $N \in \mathcal{N}(X)$. ‡

The class of bornological spaces is fairly wide; in particular, it includes all locally convex, metrisable spaces.

Bounded decompositions are , of course, the same with respect to a locally convex topology and its associated weak topology. The concept has no meaning in topological groups.

Metrisable spaces and groups

The next theorem is our deepest result on open decomposition. It generalises results of Nachbin ((13), theorem 11, p. 92) and Namioka ((14), 5.3, p. 23).

8.12. Let X be a complete, metrisable topological linear space. Suppose that A,B are closed wedges such that, given $x \in X$, there exist bounded sequences (a_n) in A, (b_n) in B such that $a_n - b_n \to x$. Then (A,B) gives an open decomposition of X.

Proof. The topology is given by an invariant metric d. We show first that, given $M \in \mathcal{N}(X)$, there exists $\delta > 0$ such that if $d(x,0) \leqslant \delta$, then there exist sequences (y_n)

in $A\cap M$, (z_n) in $B\cap M$ such that $y_n - z_n \to x$.

Take circled $N \in \mathcal{N}(X)$ such that $N+N \subseteq M$. Define K to be the set of elements x for which there exist sequences $(a_n),(a_n')$ in $A\cap N$ and $(b_n),(b_n')$ in $B\cap N$ such that

$$a_n - b_n \to x , \qquad a_n' - b_n' \to -x .$$

Given $x \in X$, there exist bounded sequences $(a_n),(a_n')$ in A and $(b_n),(b_n')$ in B such that $a_n - b_n \to x$, $a_n' - b_n' \to -x$. There exists $\lambda > 0$ such that $\lambda a_n, \lambda a_n', \lambda b_n, \lambda b_n' \in N$ for all n. Then $\lambda x \in K$. Hence $X = \bigcup_{n=1}^{\infty} (nK)$.

Suppose that $x \in \overline{K}$. There exists $x_n \in K$ such that $d(x, x_n) \leqslant 2^{-n}$. There exist $a_n, a_n' \in A\cap N$ and $b_n, b_n' \in B\cap N$ such that $d(x_n, a_n - b_n) \leqslant 2^{-n}$ and $d(-x_n, a_n' - b_n') \leqslant 2^{-n}$. Then $a_n - b_n \to x$, $a_n' - b_n' \to -x$, so $x \in K$. Hence K is closed.

Therefore, by Baire's theorem, there exists $x_o \in X$ and $\delta > 0$ such that $d(x, x_o) \leqslant \delta \Rightarrow x \in K$. Take x such that $d(x, 0) \leqslant \delta$. There exist $a_n, a_n' \in A\cap N$ and $b_n, b_n' \in B\cap N$ such that

$$a_n - b_n \to x_o + x ,$$

$$a_n' - b_n' \to -x_o .$$

Then

$$(a_n + a_n') - (b_n + b_n') \to x .$$

$(a_n + a_n') \in A\cap M$, $(b_n + b_n') \in B\cap M$, so the assertion above is proved.

Write $M_n = \{x : d(x, 0) \leqslant 2^{-n}\}$. There exists $\delta_n > 0$

such that if $d(x,0) \leqslant \delta_n$, then there exist sequences (y_r) in $A \cap M_n$, (z_r) in $B \cap M_n$ such that $y_r - z_r \to x$. Clearly, $\delta_n \leqslant 2^{-n+1}$.

The proof is completed by showing that, for each k,

$$\{x : d(x,0) \leqslant \delta_{k+1}\} \subseteq (A \cap M_k - B \cap M_k) .$$

Take x with $d(x,0) \leqslant \delta_{k+1}$. There exist $a_{k+1} \in A \cap M_{k+1}$, $b_{k+1} \in B \cap M_{k+1}$ such that $d(x_{k+1},0) \leqslant \delta_{k+2}$, where $x_{k+1} = x - (a_{k+1} - b_{k+1})$. Having obtained x_{r-1} with $d(x_{r-1},0) \leqslant \delta_r$, choose $a_r \in A \cap M_r$, $b_r \in B \cap M_r$ such that $d(x_r,0) \leqslant \delta_{r+1}$, where $x_r = x_{r-1} - (a_r - b_r)$.

Since A and B are complete,

$$\sum_{k+1}^{\infty} a_r = a \in A \cap M_k , \qquad \sum_{k+1}^{\infty} b_r = b \in B \cap M_k .$$

Now

$$\sum_{k+1}^{l} (a_r - b_r) = x - x_l ,$$

and $x_l \to 0$ as $l \to \infty$. Thus $x = a-b$, and the proof is complete. ‡

This gives us another automatic continuity theorem (cf. 2.7):

<u>8.13</u>. Let X,Y be topological linear spaces, X being complete and metrisable. Suppose that A,B are closed wedges in X such that $A-B = X$ (or such that the condition of 8.12 is satisfied), and that T is a linear mapping $X \to Y$ such that TA and TB are self-allied. Then T is continuous.

Proof. Let d be an invariant metric giving the topology of X, and let $M_n = \{x : d(x,0) \leqslant 2^{-n}\}$. By 8.12, $n^{-1}(A \cap M_n - B \cap M_n)$ is a neighbourhood of 0.

If T is not continuous, then there exist $N \in \mathcal{N}(Y)$ and $a_n \in A \cap M_n$, $b_n \in B \cap M_n$ such that $n^{-1}T(a_n - b_n) \notin N$.

Since X is complete, $\sum_{n=1}^{\infty} a_n$ is convergent, say to a. $a \in A$, since A is closed. Further, $a = a_n + a_n'$, where $a_n' \in A$. Now

$$n^{-1}(Ta_n + Ta_n') = n^{-1}Ta \rightarrow 0 ,$$

so $n^{-1}Ta_n \rightarrow 0$, since TA is self-allied. Similarly, $n^{-1}Tb_n \rightarrow 0$. This is a contradiction, and the result follows. ‡

This applies, in particular, to "positive" linear mappings into a space with an order given by a self-allied cone (cf. (13), theorem 12, p. 95). Also, putting $Y = R$, we have:

8.14. If X is a complete, metrisable topological linear space, and A, B are closed wedges such that $A - B = X$, then every linear functional in $A^+ \cap B^+$ is continuous. ‡

A simple example shows that we cannot dispense with completeness here. Consider the space F of finite real sequences, with the usual ordering and the topology given by the supremum norm. Let $f(x) = \sum \xi_n$, where $x = (\xi_n)$. Then f is a discontinuous positive linear functional.

The following partial converse to 8.12 should be compared with 8.4:

8.15. Suppose that X is a metrisable, commutative topological group, and that complete sub-semigroups A, B exist which give an open decomposition of X. Then X is complete.

Proof. Let d be an invariant metric giving the topology, and let $M_n = \{x : d(x,0) \leqslant 2^{-n}\}$. $P_n = A \cap M_n - B \cap M_n$ is a neighbourhood of 0. Suppose that (x_n) is a sequence such that $x_n - x_{n-1} \in P_n$ for $n \geqslant 2$ (any Cauchy sequence has a subsequence with this property). For $n \geqslant 2$, there exist $a_n \in A \cap M_n$, $b_n \in B \cap M_n$ such that $x_n - x_{n-1} = a_n - b_n$. Also, there exist $a_1 \in A$, $b_1 \in B$ such that $x_1 = a_1 - b_1$.

Then $x_n = \sum_{r=1}^{n} (a_r - b_r)$. For $l > k > n$, $\sum_{r=k}^{l} a_r \in M_n$, so, since A is a complete semigroup, $\sum_{1}^{\infty} a_n = a \in A$. Similarly, $\sum_{1}^{\infty} b_n = b \in B$. Thus $x_n \rightarrow a-b$, and X is complete. ‡

9. APPLICATIONS TO LATTICES

In this section we show how alliedness and open decomposition play an essential part in the theory of commutative topological groups and topological linear spaces with a lattice ordering. No attempt is made to replace the positive cone by two sets.

Let X be a commutative lattice group, i.e. a commutative group with a lattice ordering such that $x \leqslant y$ implies $x+z \leqslant y+z$ for all z. Write $x^+ = x \vee 0$, $x^- = (-x) \vee 0$, $|x| = x \vee (-x)$. Then $|x| = x^+ + x^- = x^+ \vee x^-$, and $|y| \leqslant x$ if and only if $-x \leqslant y \leqslant x$. The following inequalities hold:

$$|x \vee y| \leqslant |x| \vee |y| , \quad |x \wedge y| \leqslant |x| \vee |y| ,$$
$$|x+y| \leqslant |x| + |y| .$$

If X is a real linear space, we also have $|\lambda x| = |\lambda|\,|x|$. (For these and other elementary properties, see (10),(19)).

In accordance with the notation used in earlier sections, we denote by $[E]$ the order-convex cover of E. We recall that the positive cone is self-allied if and only if X is locally order-convex (see p. 9). A subset A of X is said to be SOLID if $a \in A$ and $|x| \leqslant |a|$ implies that $x \in A$.

If X has a topology, it is natural to consider continuity of the lattice operations. It is clear that continuity of the mapping $x \to x^+$ at 0 implies that the positive cone P gives an open decomposition of X, and that, if X is Hausdorff, continuity of $x \to x^+$ for all x implies that P is closed. (We notice that if P is closed, then X must be Hausdorff, since if $x \neq 0$, then $x \notin P$ or $x \notin -P$). The basic result connecting the lattice operations with the topology is the following. It is essentially a combination of results published in (13)(p. 89), (14) (p. 40) and (19) (p. 234), but no straightforward proof of the equivalence of all five statements seems to have appeared yet, so we give one:

9.1. Let X be a commutative lattice group, and let P denote the set of non-negative elements. Then the following conditions are equivalent:

(i) the mapping $(x,y) \to x \vee y$ is uniformly continuous on $X \times X$;

(ii) the mapping $x \to x^+$ is uniformly continuous on X;

(iii) P is self-allied and gives an open decomposition of X;

(iv) P is self-allied and $x \to x^+$ is continuous at 0;

(v) X is locally solid.

Proof. (i) => (ii). A priori.

(ii) => (iii). Open decomposition follows from continuity of $x \to x^+$, as mentioned above. To show that P is self-allied, take $M \in \mathcal{N}(X)$. Then there exists $N \in \mathcal{N}(X)$ such that $x-y \in N$ implies $x^+ - y^+ \in M$. If $a,b \in P$ and $a+b \in N$, then $a = a^+ - (-b)^+ \in M$.

(iii) => (iv). Since P is self-allied, the order-convex neighbourhoods of 0 form a local base. Take order-convex $M \in \mathcal{N}(X)$. Let $N = P \cap M - P \cap M$. Then $N \in \mathcal{N}(X)$. Take

$x \in N$, so that $x = u_1 - u_2$, where $u_1, u_2 \in P \cap M$. Then $0 \leqslant x^+ \leqslant u_1$, so $x^+ \in M$.

(iv) => (v). Take order-convex $M \in \mathcal{N}(X)$. The mapping $x \to |x|$ is continuous at 0, so there exists $N \in \mathcal{N}(X)$ such that $x \in N$ implies $\pm|x| \in M$. If $x \in N$ and $|y| \leqslant |x|$, then $y \in M$. Hence the set

$$\{ y : \exists\, x \in N \text{ such that } |y| \leqslant |x| \}$$

is a solid neighbourhood of 0 contained in M.

(v) => (i). Take solid $M \in \mathcal{N}(X)$. There exists solid $N \in \mathcal{N}(X)$ such that $N+N \subseteq M$. Take $v_1, v_2 \in N$ and $x, y \in X$. Let

$$z = (x+v_1) \vee (y+v_2) - x \vee y .$$

Then

$$v_1 \wedge v_2 \leqslant z \leqslant v_1 \vee v_2 ,$$

so $|z| \leqslant |v_1| + |v_2|$, and $z \in M$. ‡

If X is a locally convex topological linear space that satisfies the conditions of 9.1, it is a straightforward matter to show that the solid, convex neighbourhoods of 0 form a local base. Taking the Minkowski functionals of such neighbourhoods, it follows that the topology can be given by seminorms p satisfying $|x| \leqslant |y| \Rightarrow p(x) \leqslant p(y)$.

We shall see presently that $x \to x^+$ can be continuous everywhere without being uniformly continuous (contradicting a statement by Schaefer in (19), p. 234).

Let X be a commutative lattice group with a topology. We say that X satisfies CONDITION (M) if there is a local base consisting of sublattices. If M is a sublattice, then

$M \cap (-M)$ is a symmetric sublattice, so the condition implies that there is a local base consisting of symmetric sublattices. It is clear that condition (M) implies that $x \to x^+$ is continuous at 0. However, it does not imply that this mapping is continuous at other points, or even (in a Hausdorff space) that the positive cone is closed, as is shown by the lexicographic ordering of R^2 (since the ordering here is total, every subset is a sublattice).

A treatment of spaces satisfying condition (M) is given in (6) and (8), and no attempt is made to reproduce it here, since it does not belong to the theory of allied sets. However, we notice that if condition (M) is satisfied and the positive cone is self-allied, then 9.1 shows that the space is locally solid. For topological linear spaces, we have the rather remarkable fact that these conditions imply local convexity, as the following shows:

9.2. Let X be a linear lattice with a topology. If condition (M) is satisfied and the positive cone is self-allied, then there is a local base consisting of solid, convex sublattices.

Proof. Take order-convex $M \in \mathcal{N}(X)$. M contains a symmetric sublattice $N \in \mathcal{N}(X)$. If $v \in N$, then $|v| = v \vee (-v) \in N$. We show that $x \in [N]$ if and only if there exists $v \in N$ such that $|x| \leq |v|$. The condition is clearly sufficient, by the above. Conversely, if $x \in [N]$, then there exist $v_1, v_2 \in N$ such that $-v_1 \leq x \leq v_2$. Then $\pm x \leq v_1 \vee v_2$, so $|x| \leq v_1 \vee v_2 \in N$, as required.

It follows that $[N]$ is solid. Clearly, $[N] \subseteq M$.

Take $x, y \in [N]$. Then there exist $u, v \in N$ such that $|x| \leqslant u$, $|y| \leqslant v$. Let $w = u \vee v$. Then $w \in N$, and $|x \vee y| \leqslant |x| \vee |y| \leqslant w$, so $x \vee y \in [N]$. Similarly, $x \wedge y \in [N]$. Hence $[N]$ is a sublattice. It remains to show that it is convex. Take $\lambda \in (0,1)$. Then

$$|\lambda x + (1-\lambda)y| \leqslant \lambda|x| + (1-\lambda)|y|$$
$$\leqslant \lambda w + (1-\lambda)w = w ,$$

so $\lambda x + (1-\lambda)y \in [N]$, as required. ‡

It follows that the topology can be given by seminorms p satisfying:

$$|x| \leqslant |y| \Rightarrow p(x) \leqslant p(y) ,$$
$$p(x \vee y) = p(x) \vee p(y) \quad \text{for } x, y \geqslant 0 .$$

Conversely, it is clear that if the topology can be given by such seminorms, then the conditions of 9.2 are satisfied.

9.2 applies to commutative lattice groups if the word "convex" is omitted.

Alliedness also arises in another context. Elements x,y of a commutative lattice group are said to be DISJOINT if $|x| \wedge |y| = 0$. Let $A^{\perp}$ denote the set of elements which are disjoint to each member of A. $A^{\perp}$ is a subgroup, closed if X is locally solid, and if X is order-complete, then $A^{\perp} + A^{\perp\perp} = X$ ((19), pp. 210, 235). The following holds:

<u>9.3.</u> If X is a locally solid commutative lattice group, and A is any subset of X, then A al $A^{\perp}$.

<u>Proof</u>. Take solid $M \in \mathcal{N}(X)$, and suppose that $x \in A$,

$y \in A^{\perp}$, and $x+y \in M$. Then $|x+y| \in M$. But $|x+y| = |x|+|y|$ ((19), cor. 1, p. 208). Hence $x \in M$. ‡

The partial-sum cone

In either m or c_0, let P denote the set of sequences having all partial sums non-negative. This is a closed cone, and (as a subset of m) it is the dual of the cone of decreasing positive sequences in l_1. It provides examples of several situations of interest in the theory of partially ordered linear spaces.

Firstly, P nal P with respect to the norm topology. Given n, two elements of P are:

$$x_n = (1, \ldots ,1,-n),$$
$$y_n = (0, \ldots ,0,n),$$

the terms -n and n occurring in place n+1. $\|x_n\| = \|y_n\| = n$, while $\|x_n+y_n\| = 1$.

However, P al P with respect to the topology of pointwise convergence (cf. the examples in section 6). To show this, suppose that $(\xi_i),(\eta_i) \in P$ and $|\xi_i+\eta_i| \leq \varepsilon$ for $i \leq n$. Then, for $k \leq n$, $\sum_{i=1}^{k} (\xi_i+\eta_i) \leq k\varepsilon$, so $0 \leq \sum_{i=1}^{k} \xi_i \leq k\varepsilon$, $0 \leq \sum_{i=1}^{k} \eta_i \leq k\varepsilon$. It follows that $|\xi_k|, |\eta_k| \leq k\varepsilon$ for $k \leq n$.

We notice that the sequence (e_n) is monotonic with respect to P, and converges to 0 with respect to the weak topology for m, but not with respect to the norm topology (cf. 2.8).

We show that P gives a lattice ordering of m and c_o which is such that $\|x \vee y\| \leqslant \|x\| \vee \|y\|$ (so that condition (M) is satisfied). Given $x = (\xi_n)$, $y = (\eta_n)$, let

$X_n = \xi_1 + \ldots + \xi_n$, $Y_n = \eta_1 + \ldots + \eta_n$, and

$$\zeta_n = (X_n \vee Y_n) - (X_{n-1} \vee Y_{n-1}) \qquad (n \geqslant 2),$$

$$\zeta_1 = X_1 \vee Y_1 .$$

Write $z = (\zeta_n)$. It is easily seen that $|\zeta_n| \leqslant |\xi_n| \vee |\eta_n|$ for all n, so that $\|z\| \leqslant \|x\| \vee \|y\|$. Also, $\zeta_1 + \ldots + \zeta_n = X_n \vee Y_n$, so that, with respect to the ordering given by P, $z = x \vee y$.

In particular, the mapping $x \to x^+$ is continuous at 0 (we consider only the norm topologies from now on). In m, it is easy to verify that it is discontinuous at the point (1,-1,1,-1, ...). By contrast, in c_o it is continuous for all x (though 9.1 shows that it is not uniformly continuous). To show this, take $x = (\xi_n) \in c_o$ and $\varepsilon > 0$. There exists an integer N such that $|\xi_r| \leqslant \varepsilon$ for r > N. Take $y = (\eta_n) \in c_o$ such that $\|y-x\| \leqslant \varepsilon/N$. Now $x^+ = (\alpha_n)$, $y^+ = (\beta_n)$, where

$$\alpha_r = X_r^+ - X_{r-1}^+ \quad (r \geqslant 2), \quad \alpha_1 = X_1^+ ,$$

$$\beta_r = Y_r^+ - Y_{r-1}^+ \quad (r \geqslant 2), \quad \beta_1 = Y_1^+ ,$$

and X_r, Y_r are as above. We use the fact that $|\lambda^+ - \mu^+| \leqslant |\lambda - \mu|$ for real λ, μ. For $r \leqslant N$,

$$|Y_r - X_r| = \left|\sum_{s=1}^{r} (\eta_s - \xi_s)\right| \leq r\varepsilon/N \leq \varepsilon ,$$

so $|Y_r^+ - X_r^+| \leq \varepsilon$, and $|\beta_r - \alpha_r| \leq 2\varepsilon$. For $r > N$, we have $|\alpha_r| \leq |\xi_r| \leq \varepsilon$, $|\beta_r| \leq |\eta_r| \leq \varepsilon + \|y-x\| \leq 2\varepsilon$, so $|\beta_r - \alpha_r| \leq 3\varepsilon$. Hence $\|y^+ - x^+\| \leq 3\varepsilon$.

Roughly speaking, the continuity of $x \to x^+$ in c_o is due to the fact that each element is close to a finite-dimensional subspace, but the continuity is not uniform because the dimension of the subspace required depends on x.

P is too large to be self-allied, and large enough to give an open decomposition of c_o. By contrast, the cone of decreasing positive sequences is small enough to be self-allied, and too small to give an open decomposition (in fact, it only generates a dense subspace).

Finally, we show that there is a continuous linear functional which is unbounded on an order interval (this shows (16), 3.4 to be false; cf. 2.3). Let z_n denote the sequence (ξ_j), where

$$\xi_{rn} = 1 \quad (r = 1, 2, \ldots),$$

$$\xi_j = 0 \quad \text{for other } j.$$

By considering the first non-zero term in a linear combination, we see that the sequence (z_n) is linearly independent. Define a linear functional f on the subspace spanned by the z_n by putting $f(z_n) = 1$ for each n. Then f is bounded, since its value at $(\lambda_1 z_1 + \ldots + \lambda_n z_n)$ is equal to term n! of this sequence. Hence f has a bounded extension to m. Now $0 \leq z_1 - nz_n \leq z_1$, and $f(z_1 - nz_n) = 1-n$.

10. ALLIED FAMILIES

Let X be a commutative topological group, and write $E' = E \cup \{0\}$ for any subset E. A finite family $\{A_1, \dots, A_n\}$ of subsets of X is said to be ALLIED if, given $M \in \mathcal{N}(X)$, there exists $N \in \mathcal{N}(X)$ such that if $a_i \in A_i'$ $(i = 1, \dots, n)$ and $a_1 + \dots + a_n \in N$, then $a_i \in M$ for each i.

By taking A_i' instead of A_i, we ensure that every subfamily of an allied family is allied. The family $\{A, B\}$ is allied if and only if A al B (which is equivalent to A' al B').

We say that an infinite family is allied if every finite subfamily is.

The next theorem gives some equivalent formulations. In particular, we notice that alliedness of a finite family is equivalent to a finite number of statements of the form A al B.

<u>10.1</u>. Each of the following statements is equivalent to $\{A_1, \dots, A_n\}$ being allied:

(i) A_i al $\left(\sum_{j \neq i} A_j'\right)$ for each i ;

(ii) A_i al $\left(\sum_{j < i} A_j'\right)$ for $i = 2, \dots, n$;

(iii) if (x^r) is a net convergent to 0, and for each r, $x^r = a_1^r + \dots + a_n^r$ (where $a_i^r \in A_i'$), then $a_i^r \to 0$ for each i.

<u>Proof</u>. If $\{A_i\}$ is allied, it is clear that (i) holds. Conversely, if (i) holds and $M \in \mathcal{N}(X)$ is given, there exists, for each i, $N_i \in \mathcal{N}(X)$ such that if $a_j \in A'_j$ $(j = 1, \ldots, n)$ and $a_1 + \ldots + a_n \in N_i$, then $a_i \in M$. If $a_1 + \ldots + a_n \in \bigcap N_i$, then $a_i \in M$ for all i. Hence $\{A_i\}$ is allied.

The proof is completed by showing (i) => (ii) => (iii) => (i). (i) implies (ii) a priori. (ii) implies (iii), by repeated applications of 1.5(i), and (iii) implies (i), by 1.5(ii). ‡

Some easy deductions follow.

<u>10.2</u>. If $\{A_i\}$ is allied, then so is $\{\overline{A}_i\}$.

<u>Proof</u>. It is sufficient to consider finite families. Let $B_i = \sum_{j \neq i} A'_j$. Then A_i al B_i for each i, so $\overline{A}_i$ al $\overline{B}_i$, by 1.9. But $\overline{B}_i \quad \sum_{j \neq i} \overline{A'_j}$, and the result follows. ‡

<u>10.3</u>. If $\{A_i\}$ is a finite allied family, and A al $(\sum A'_i)$, then $\{A_i\} \cup \{A\}$ is allied.

<u>Proof</u>. Immediate, by 10.1(ii). ‡

<u>10.4</u>. If $\{A_i\}$ is a finite allied family, A is compact, and $(-A) \cap (\overline{\sum A'_i}) \subseteq \{0\}$, then $\{A_i\} \cup \{A\}$ is allied.

<u>Proof</u>. By 1.13, A al $(\sum A'_i)$. ‡

The proof of the following requires slightly more effort:

<u>10.5</u>. If $\{A_i\}$ is a family of stars in a topological linear space, and for some $M \in \mathcal{N}(X)$, the family $\{A_i \cap M\}$ is allied, then so is $\{A_i\}$.

<u>Proof</u>. By 2.1, each pair of members of the family is allied. Suppose that all subfamilies with n-1 members are allied, and take a subfamily $\{A_1, \ldots, A_n\}$ with n members. Note that stars contain 0, so that $A_i' = A_i$. For each i, we have

$$(A_i \cap M) \text{ al } \left(\sum_{j \neq i} (A_j \cap M)\right).$$

Since $\{A_j : j \neq i, 1 \leq j \leq n\}$ is allied, there exists $N \in \mathcal{N}(X)$ such that $N \subseteq M$ and

$$\left(\sum_{j \neq i} A_j\right) \cap N \subseteq \sum_{j \neq i} (A_j \cap M).$$

Hence

$$(A_i \cap N) \text{ al } \left[\left(\sum_{j \neq i} A_j\right) \cap N\right],$$

so A_i al $\left(\sum_{j \neq i} A_j\right)$, by 2.1. Thus $\{A_i\}$ is allied, by 10.1. ‡

Other generalisations of earlier results are apparent on inspection. We mention a few of them.

<u>10.6</u>. If $\{A_1, \ldots, A_n\}$ is an allied family of stars in a topological linear space, and (x^r) is a bounded net, where $x^r = a_1^r + \ldots + a_n^r$ $(a_i^r \in A_i)$, then, for each i, the net (a_i^r) is bounded. The converse holds for positive homogeneous subsets of a metrisable space. ‡

<u>10.7</u>. A family $\{A_1, \ldots, A_n\}$ of positive homogeneous subsets of a normed linear space is allied if and only if there exists $\delta > 0$ such that if $a_i \in A_i$ for each i, then

$$\|a_1 + \ldots + a_n\| \geq \delta \|a_i\|$$

for each i. ‡

<u>10.8</u>. Suppose that $\{A_i - A_i : i = 1, \ldots, n\}$ is allied.

(i) If (x^r) is a Cauchy net, where $x^r = a_1^r + \ldots + a_n^r$ $(a_i^r \in A_i)$, then, for each i, the net (a_i^r) is Cauchy.

(ii) If each A_i is complete, then so is $\sum A_i$.

(iii) If A_i is complete for $i \neq j$, and A_j is closed, then $\sum A_i$ is closed. ‡

To see that we cannot expect corresponding results for an infinite allied family, it is sufficient to consider the one-dimensional subspaces of m spanned by e_n $(n = 1, 2, \ldots)$.

A natural example of an infinite allied family is given by the next theorem. It is an extension of 2.5.

<u>10.9</u>. If X is a topological linear space, and T is a continuous linear mapping $X \to X$, then the sets $\{x : Tx = \lambda x\}$ (for all scalars λ) form an allied family.

<u>Proof</u>. By 2.5, each pair of members of the family is allied. Suppose that all subfamilies with n-1 members are allied. Take distinct $\lambda_1, \ldots, \lambda_n$, and let (x^r) be a net convergent to 0, where $x^r = x_1^r + \ldots + x_n^r$, and $Tx_i^r = \lambda_i x_i^r$. Then

$$Tx^r = \lambda_1 x_1^r + \dots + \lambda_n x_n^r \to 0 ,$$

so

$$(\lambda_1 - \lambda_n)x_1^r + \dots + (\lambda_{n-1} - \lambda_n)x_{n-1}^r \to 0 .$$

Thus, by the induction hypothesis, $x_i^r \to 0$ for $1 \leq i \leq n-1$. Hence also $x_n^r \to 0$, and the result follows. ‡

Thus, if X is complete, the sum of a finite number of such subspaces is closed.

Families may, of course, be pairwise allied but not allied: any three distinct one-dimensional subspaces of R^2 form such a family. Pairwise allied families seem to be of no particular interest.

It follows from the form of the definition that every allied family is contained in a maximal one. The same is true for allied families of subspaces of a topological linear space X, and it is clear that if A_i is a maximal allied family of subspaces of X, then $\overline{\sum A_i} = X$, for otherwise the family could be extended by adding a one-dimensional subspace.

APPENDIX 1. NETS IN $A + \text{pos } b$.

We shall need the following result, which is an immediate application of the fact that every topological linear space has a completion (a direct proof is easy to give):

<u>A1.1</u>. If $\{x_n : n \in D\}$ is a Cauchy net in a topological linear space, and $\{\lambda_n : n \in D\}$ is a net convergent to 0 in the underlying field, then $\lambda_n x_n \to 0$. ‡

Using this, we have:

<u>A1.2</u>. Suppose that A is a star and $-b \notin \bar{A}$. Then:

(i) If $(a_n+\lambda_n b)$ is a Cauchy net $(a_n \in A,\ \lambda_n \geqslant 0)$, then there is a subnet $(a_m+\lambda_m b)$ such that the nets $(a_m), (\lambda_m)$ are Cauchy. If the original net was a sequence, then there is a subsequence with this property.

(ii) If A is complete, or sequentially complete, then so is $A + \text{pos } b$.

(iii) If A is closed, then so is $A + \text{pos } b$.

<u>Proof</u>. (i) Suppose that there is a cofinal subset F such that the net $\{\lambda_r : r \in F\}$ tends to ∞. We may suppose that each $\lambda_r > 1$. Then $\lambda_r^{-1} \to 0$, so, by A1.1,

$$\lambda_r^{-1}(a_r+\lambda_r b) = b + \lambda_r^{-1} a_r \to 0 .$$

This implies that $-b = \lim (\lambda_r^{-1} a_r) \in \bar{A}$, contrary to hypothesis.

Hence there exists $K > 0$ such that $\{n : \lambda_n \leqslant K\}$ is residual, giving a net in the compact set $[0,K]$. This has a convergent subnet (a subsequence if the original net was a sequence), and the result follows.

(ii) Take a Cauchy net $(a_n+\lambda_n b)$. By (i), there is a subnet $(a_m+\lambda_m b)$ such that the nets (a_m) and (λ_m) converge, say to $a \in A$ and $\lambda \geqslant 0$. Then the original net converges to $a+\lambda b$, since it is Cauchy. The usual variant of this argument proves (iii). ‡

This method is essentially due to Simons (see (20), theorem 1). A1.2 should be compared with 5.4 and 5.5; it is not a special case of these results, because $-b \notin \overline{A}$ does not imply that $(\text{pos } b) \cap (-\overline{A}) \subseteq \overline{\{0\}}$. Note that the examples following 5.3 and 5.5 both involve nets of the form $(a_n+\lambda_n b)$.

Lastly, we consider the case where A is positive homogeneous and $b \notin A-A$, so that expressions of the form $a+\lambda b$ are unique. By 3.1 and 5.1, we know that if $b \notin \overline{A-A}$ and $(a_n+\lambda_n b)$ is a Cauchy net, then (a_n) and (λ_n) are Cauchy. Without requiring $b \notin \overline{A-A}$, we have:

<u>A1.3</u>. Suppose that A is a closed, positive homogeneous subset of a Hausdorff topological linear space, and that $b \notin A-A$. If $(a_n+\lambda_n b)$ is a convergent net $(a_n \in A,\ \lambda_n \geqslant 0)$, then the nets (a_n) and (λ_n) are convergent.

<u>Proof</u>. By A1.2, there is a subnet $\{a_m+\lambda_m b : m \in E\}$ such that $a_m \to a \in A$ and $\lambda_m \to \lambda \geqslant 0$. Then $a+\lambda b$ is

the limit of the original net (unique, since the space is Hausdorff).

If $\lambda_n \not\to \lambda$, then there exist $\varepsilon > 0$ and a cofinal set F such that $|\lambda_r - \lambda| \geq \varepsilon$ for $r \in F$. Applying A1.2 to the net $\{a_r + \lambda_r b : r \in F\}$, we see that $\{\lambda_r : r \in F\}$ has a subnet (λ_s) convergent to μ (say), where $\mu \neq \lambda$, while $a_s \to a' \in A$. Hence we have $a+\lambda b = a'+\mu b$, so that $b \in A-A$, contrary to hypothesis. ‡

If A is not closed, the result fails. For instance, let $X = m$, and let A be the set of finite, non-negative sequences. Let $b = (\beta_n)$, where $\beta_n = n^{-1}$. Putting

$$a_{2n-1} = \left(1, \tfrac{1}{2}, \ldots, \tfrac{1}{2n-1}\right), \qquad \lambda_{2n-1} = 0,$$

$$a_{2n} = 0, \quad \lambda_{2n} = 1,$$

we obtain an example of this situation.

APPENDIX 2. THE MACKEY TOPOLOGY OF A SUBSPACE.

Let X be a locally convex space, and A a subspace of X. Let X^*, A^* denote the spaces of continuous linear functionals on X, A respectively. For each $f \in X^*$, let f' denote the restriction of f to A. Then $f \to f'$ is a linear mapping on X^* into A^* (in fact, onto A^*, snnce X is locally convex). Furthermore, it is continuous with respect to the topologies $w(X), w(A)$. To show this, take $a_1, \ldots, a_n \in A$, and let

$$N = \{\phi \in A^* : |\phi(a_i)| \leq 1 \text{ for each } i\}.$$

Then N is a basic w(A)-neighbourhood of 0 in A^*. If $f \in X^*$ and $|f(a_i)| \leq 1$ for each i, then $f' \in N$.

It follows that if K is a w(X)-compact subset of X^*, then $K' = \{f' : f \in K\}$ is a w(A)-compact subset of A^*. Thus we obtain the result used in 6.8:

<u>A2.1</u>. Let X be a locally convex space, and A a subspace of X. Then the Mackey topology $m(A^*)$ for A is not smaller than the topology induced on A by the Mackey topology $m(X^*)$ for X.

<u>Proof</u>. A basic neighbourhood of 0 in the induced topology is $A \cap K^o$, where K is a convex, circled, w(X)-compact subset of X^*, and the polar is taken in X. With the notation used above, K' is a convex, circled, w(A)-compact subset of A^*, and $A \cap K^o$ is the polar in A of K'. Thus $A \cap K^o$ is a neighbourhood of 0 with respect to the topology $m(A^*)$ for A. ‡

APPENDIX 3. PUBLICATION OF RESULTS.

Almost all of section 9 is included in (8). Most of the remaining material is included in (7). The following are the main exceptions:

1.7; 1.11; 1.14; 1.15; 1.16.

2.6; 2.8; 2.11; example (iv), p. 28.

3.4; 3.5; 3.6; 3.7.

Examples following 4.8.

Example following 5.5.

6.4; 6.5.

7.12.

8.4; 8.6; 8.8; 8.9; 8.10; 8.11; 8.15.

Appendix 1.

Appendix 2.

The thesis includes the whole of (7), but only a small proportion of (8). The style of (7) is consistently briefer than that of the thesis, and a number of proofs omitted in (7) are given in full here. Note that, while a similar system of numbering theorems is used in (7), the same result often appears with different numbers there and in the thesis.

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