Game Semantics for Probabilistic Modal μ-Calculi

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Abstract

The *probabilistic* (or *quantitative*) modal μ-calculus is a fixed-point logic designed for expressing properties of probabilistic labeled transition systems (PLTS's). Two semantics have been studied for this logic, both assigning to every process state a value in the interval $[0, 1]$ representing the probability that the property expressed by the formula holds at the state. One semantics is *denotational* and the other is a *game semantics*, specified in terms of two-player stochastic games. The two semantics have been proved to coincide on all *finite* PLTS's. A first contribution of the thesis is to extend this coincidence result to *arbitrary* PLTS's.

A shortcoming of the probabilistic μ-calculus is the lack of expressiveness required to encode other important temporal logics for PLTS's such as *Probabilistic Computation Tree Logic* (PCTL). To address this limitation, we extend the logic with a new pair of operators: *independent product* and *coproduct*, and we show that the resulting logic can encode the *qualitative fragment* of PCTL. Moreover, a further extension of the logic, with the operation of *truncated sum* and its dual, is expressive enough to encode full PCTL.

A major contribution of the thesis is the definition of appropriate game semantics for these extended probabilistic μ-calculi. This relies on the definition of a new class of games, called *tree games*, which generalize standard 2-player stochastic games. In tree games, a play can be split into concurrent subplays which continue their evolution independently. Surprisingly, this simple device supports the encoding of the whole class of imperfect-information games known as *Blackwell* games. Moreover, interesting open problems in game theory, such as *qualitative determinacy* for 2-player stochastic parity games, can be reformulated as determinacy problems for suitable classes of tree games. Our main technical result about tree games is a proof of determinacy for 2-player stochastic *meta-parity* games, which is the class of tree games that we use to give game semantics to the extended probabilistic μ-calculi. In order to cope with measure-theoretic technicalities, the proof is carried out in ZFC set theory extended with Martin's Axiom at the first uncountable cardinal ($\mathrm{MA}_{\aleph_1}$).

The final result of the thesis shows that the game semantics of the extended logics coincides with the denotational semantics, for arbitrary PLTS's. However, in contrast to the earlier coincidence result, which is proved in ZFC, the proof of coincidence for the extended calculi is once again carried out in $\mathrm{ZFC} + \mathrm{MA}_{\aleph_1}$.

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Declaration

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

(*Matteo Mio*)

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Chapter 1

Introduction

By the end of the 70's, defining the *semantics* of programming languages in terms of *transition systems*, i.e., directed graphs whose nodes represent program states and transitions represent possible evolutions from states to states, was already a common and well established technique. This way of interpreting the meaning of a program is now well known under the name of *operational semantics* [96]. In the last three decades the methods of operational semantics have been extensively studied and many concepts and techniques have been developed, most notably by deep insights of Gordon Plotkin [95] and Robin Milner [79]. As of today, operational semantics is one of the most adopted tools for giving formal semantics to programming languages and, more generally, concurrent systems exhibiting computational features such as nondeterminism, stochasticity, timed-transition steps, *etcetera*.

One of the most important gains one gets from giving a formal semantics to programs, or more generally to computing systems, is the possibility of being able to *formally express* interesting properties of systems, and of *formally verifying* if a certain property is fulfilled by a given system. For example, one might want to express the fact that "the program P always terminates its execution, whenever it is fed with input(s) satisfying a given property F", or similarly, "the system c never reaches bad configurations, no matter how its execution is driven by inputs received from an unpredictable controller", where C is the software governing the execution of a nuclear plant, and *bad configurations* occur when the temperature of the reactor goes beyond a safety threshold. The formal language used to describe such properties is given by the set of formulas of a *program logic*.

A program logic is specified by a grammar defining the set of its formulas, i.e., the language of expressible properties, together with a *semantics* which provides them with a mathematical meaning. Once we understand the meaning of a system P as a transition system with set S of states by means of an appropriate operational semantics, a natural choice for describing the semantics of a program-logic formula F is as a map $[\![F]\!]: S \rightarrow \{0,1\}$ specifying which program states $s \in S$ satisfy the property expressed by F. We would then say that the program P satisfies the property F at the state s, whenever $[\![F]\!](s) = 1$.

Since the introduction, due to Saul Kripke (see, e.g., [64]), of *possible-world semantics*, *modal logics* have been recognised as an important tool for expressing properties of directed graphs. Although relatively inexpresive in themselves, modal logics provided the basis for the subsequent development of so-called *temporal (program) logics*, a line of research pioneered by the seminal works of Arthur Prior [98] and subsequently developed by Amir Pnueli [97], E. Allen Emerson [34], Moshe Y. Vardi [109] and Dexter Kozen [62] among many others. The idea is to enrich modal logics, and in particular the logic **K** [25], which are capable of expressing (local) properties of the transition relations of directed graphs, with further *temporal operators* allowing the expression of properties of interest for program analysis. A typical example is given by the binary (existentially quantified) *until* operator $\mathcal{U}$. Roughly speaking, the formula $\mathcal{U}(F, G)$ holds at a state s_0 of a directed graph if there is a *path* $s_0, s_1, s_2, \ldots s_n$ in the graph starting at s_0, reaching at some point a state s_n such that G holds at s_n and F holds at all previous visited states s_i, for $i < n$. Temporal logics of this kind, enriching modal logics with operators capable of expressing properties of *infinite sequences of transitions* in graph structures, include *Computation Tree Logic* CTL [23] and some of its extensions such as CTL* [33] and ECTL [109], which are currently among the most well-known and widely applied temporal program logics.

In 1983, Dexter Kozen introduced in [62] a new temporal logic, today well known under the name of *modal μ-calculus* (Lμ). The logic Lμ is obtained by enriching the base modal system **K** with greatest and least fixed-point operators allowing the specification of recursive properties defined by (co)inductive definitions. This logic has been subsequently widely studied [18]. Its semantics, which can be formulated in a (mathematically) straightforward way, can benefit from the reasoning methods coming from (lattice) fixed point theory [2]. At the same time the logic Lμ is very expressive, as it can encode most of the temporal program logics appeared in the literature, including CTL [18] and CTL* [26]. A precise expressivity result was obtained by D. Janin and I. Walukiewicz in [58]: Lμ is the fragment of second order monadic logic (interpreted over tree structures) consisting of those formulas which can not distinguish between *bisimilar* models. This is an extremely satisfactory property: monadic second order logic is a very expressive theory, well understood and admitting important decidability results [99]; moreover, bisimilarity is a central concept in Robin Milner's theory of concurrent systems (see, e.g., [79]), often taken as a satisfactory notion of behavioral equivalence.

The modal μ-calculus is a *low level* logic, in the sense that interesting properties require the formulation of non trivial composite formulas. This is both a benefit and a shortcoming. The syntax and semantics of Lμ are simple and minimal. This allows fruitful connections to be established with other natural mathematical concepts coming from, e.g., game theory and automata theory. However, it is often difficult to correctly translate an intended property into the low-level formalism. At the same time, it is hard to grasp the meaning of a Lμ formula, expecially if it contains several nested occurrences of fixed-point operators. In [32], E. A. Emerson and C. S. Jutla introduced an alternative semantics for Lμ, often referred to as *game semantics*, which partially addresses this problem. The game semantics follows the tradition initiated by Jaakko Hintikka and his 2-player game semantics for first order predicate (classical) logic [55]: to each model M and formula F of a given logic L, a *game* $\mathcal{G}^F_M$ played by two players (named Player 1 and Player 2 for simplicity) is constructed. The formula F is satisfied (under the game semantics) by the model M, if Player 1 has a winning strategy in $\mathcal{G}^F_M$, and it is not satisfied otherwise. Here the words *players*, *game*, *strategy* and *winning strategy* have a formal meaning in the context of *game theory*, an increasingly important branch of mathematics dating back to the seminal work of John von Neumann [89, 112], which analyzes competitive dynamics involving rational agents called *players*. The class of games used to give game semantics to Lμ is known under the name of *parity games*.

A fundamental result in the theory of the modal μ-calculus is that the standard (denotational) semantics and the game semantics, given in terms of parity games, coincide on all models [32, 106]: for every transitions system $\mathcal{L}$, the formula F holds at s under the denotational semantics if and only if Player 1 has a winning strategy in the associated parity game $\mathcal{G}^F_{\mathcal{L}}$. This allows one to take the most useful viewpoint when reasoning about the logic Lμ: the denotational semantics offers powerful reasoning techniques coming from fixed-point theory, while the game semantics is much more concrete and allows one to understand the meaning of a formula by means of the interactions occurring in the parity game between the two players, which generally reflect more transparently the properties one wants to express with Lμ formulas. In particular it is often useful to consider Player 1 as the *controller* and Player 2 as an adversarial *environment* which tries to cause some undesirable behavior. Although having a radically different and complementary semantics for a logic is already an achievement, the game semantics for Lμ has proved to be very valuable, inspiring deep theoretical results (see, e.g., [114] and [17], for two of the most celebrated theoretical results on Lμ) as well as model checking algorithms [4, 24].

Directed-graph structures are sufficient for modeling concurrent and non-deterministic programs and computational systems but, of course, they can not be used to represent other important aspects of computations, such as *probabilistic behaviors*, *timed transitions* and other *quantitative* aspects one might need to express. To address this limitation, since the late 80's, a lot of research has focused on the identification of appropriate structures for expressing these quantitative aspects (see, e.g., [67], [50] and [9]), and in particular for modeling probabilistic behaviors. One of the most successful such models is today known under several names: Segala systems [6], concurrent Markov chains [50], probabilistic automata [101] or simply *probabilistic transition systems* (PTS). Today PTS's are the mathematical structures, generalizing standard transition systems, most often used for providing an operational semantics to probabilistic and non-deterministic languages [53, 66, 6]. A PTS is given by a set of states S, and a *transition relation* E which relates states s with probability distributions d over S. The intended interpretation is that the system, at some state s, can evolve by non-deterministically choosing one of the *accessible* distributions d, i.e., such that $(s, d) \in E$, and then continuing its execution from the state s' with probability $d(s')$. PTS's can be visualized, using graphs labeled with probabilities in a natural way. For example the PTS having set of states $S = \{p, q\}$ and accessibility relation $E = \{(p, d_1), (p, d_2)\}$, with $d_1(q) = d_1(p) = \frac{1}{2}$ and $d_2(q) = 1$, can be depicted as in Figure 1.1. This combination of non-deterministic choices immediately followed by probabilistic ones, allows the modeling of concurrency, non-determinism and probabilistic behaviors in a natural way.

Once PTS's are fixed as a model of non-deterministic and probabilistic compu-

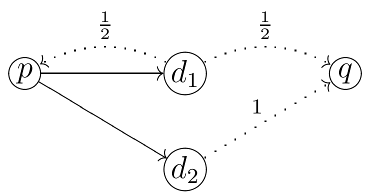


Figure 1.1: Example of PTS

tation, and flexible techniques for defining the operational semantics of languages in terms of PTS's are available (see, e.g., [6]), one naturally wants to define logics for expressing interesting properties of such systems. For example one might want to express the fact that "the program P, no matter how its input is chosen, always terminates its execution with at least probability $\frac{1}{2}$" or "with probability 1, the system C never reaches bad configurations, no matter how its execution is driven by external inputs". In particular, note how in the second specification, the safety requirement is expressed in terms of a condition which is required to hold *almost surely*, but not necessarily *surely*, thus allowing negligible (i.e., having probability 0) bad behaviors.

Already by the mid 90's, the research community had developed the first logics for expressing properties of PTS's. These are extensions of standard (i.e., designed for transition systems) temporal logics, primarily CTL and CTL*, obtained by adding to the syntax of the logics *threshold operators* which allow the specification of the desired quantitative properties. For instance, $\mathbb{P}_{>\frac{1}{2}}F$ holds at a state v if for every possible execution of the system, guided by the non-deterministic and the probabilistic choices, the probability of observing a computation satisfying F is greater than $\frac{1}{2}$. This way of extending standard temporal logics to express properties of PTS's led to the definition of *probabilistic CTL* (PCTL) and *probabilistic CTL** (PCTL*), among others [50, 51, 9, 5]. The semantics of a PCTL formula F is, by analogy with the semantics of CTL formulas, a map $[\![F]\!] : S \rightarrow \{0, 1\}$ specifying which states of the PTS satisfy the property expressed by F. The logics PCTL and PCTL* are currently among the most well understood and practically used probabilistic temporal logics for expressing properties of PTS's [65].

However, a problem with the approach described above, which we shall refer to as the *threshold-based* or *boolean* approach to probabilistic temporal logics, is that it apparently does not help one define a satisfactory probabilistic extension of the modal μ-calculus. The most natural way to attempt to get such an extension, adopting the boolean approach, is to start with a probabilistic version of the base modal logic **K** obtained by replacing the modalities $\Diamond$ and $\Box$ with decorated (with probabilities) counterparts $\Diamond_{\geq\lambda}$ and $\Box_{\geq\lambda}$, with $\lambda \in [0,1]$, and then enrich the logic by adding fixed point operators. The intended interpretation is that the formula $\Diamond_{\geq\lambda}F$ holds at a state s if there exists some accessible probability distribution d such that d assigns probability greater or equal than λ to states satisfying the subformula F. This approach, however, leads to a logic which can not express many properties of interest. The point is that one is often interested in expressing conditions over the probabilities associated with *long-term* events, while the above described logic can only express *local* constraints which are not sufficient, when composed by means of (co)inductive definitions, to capture the desired global behaviors. Even though PCTL and PCTL* are sufficient for most practical purposes, it seems quite important, and not only from a purely theoretical point of view, to find an appropriate probabilistic variant of the modal μ-calculus. One hopes to be able to transfer to the probabilistic setting the elegant mathematical theory of Lμ and, in particular, the flexibility offered by the possibility of formulating arbitrarily (co)inductive definitions, which allow the formulation in Lμ of some interesting properties not expressible in CTL and CTL*.

One important step towards a satisfactory probabilistic version of the modal μ-calculus was provided by the insights of M. Huth and M. Kwiatkowska [56] and, independently, by those of C. Morgan and A. McIver [84]. The main idea is to move the probabilistic nature of the logic, from the syntax (i.e., the probabilistic thresholds) to the semantics. Indeed, the formulas of the logics considered in [56, 84] are interpreted, in a PTS with set S of states, as maps of type $[\![F]\!] : S \to [0,1]$, assigning to program states a *value* in the real interval $[0,1]$. The intended interpretation is that $[\![F]\!](s)$ represents the probability of the *property* expressed by F to hold at the state s. We shall call this approach to probabilistic temporal logics the $[0,1]$-*valued* or *quantitative approach*. The probabilistic variant of the logic Lμ, proposed in [84] following this approach, in known under the name *quantitative modal μ-calculus*. Since the adjective "quantitative" has been adopted in the literature for other non-probabilistic logics (see, e.g., [35]), in this thesis we refer to the logic of [84] as *probabilistic modal μ-calculus* (pLμ).

A central point in the definition of the logic pLμ is the choice for the interpretations of the conjunction and disjunction operators. Indeed there are several possible binary operators $f : [0,1]^2 \to [0,1]$ which, when restricted to the two element set $\{0,1\}$, act as ordinary boolean conjunction, and similarly for disjunction. As a matter of fact, in [56], the authors study three such operators, namely $\sqcap$ (i.e., the min operation), $\cdot$ (standard multiplication on reals) and $\ominus$ (defined as $x \ominus y = \max\{0, x+y-1\}$), arguing in favor of $\ominus$ on the basis of considerations[1] about "mathematical convenience". On the other hand, the logic pLμ as defined in [84], and subsequently also in [29], takes the function $\sqcap$ as interpretation for conjunctions. This situation raises an immediate question: if $[\![F]\!](s)$ is supposed to represent the probability of the property expressed by F holding at s, what, precisely, is the *property* expressed by F, since it clearly changes if we modify the semantical interpretation of conjunctions? Indeed, if in boolean-based logics, such as PCTL, the property associated with a formula F can be always considered to be well-specified as the *set* of states $\{s \mid [\![F]\!](s) = 1\}$, in quantitative logics such as pLμ some additional explanation seems to be required. The situation is even worse when one think about enriching the logic pLμ with additional operators. For example, if we define[2] $x \star y = \log_7\big(1 + \frac{(7^x-1)\cdot(7^y-1)}{7-1}\big)$ and $[\![F \star G]\!](p) = [\![F]\!](p) \star [\![G]\!](p)$, what is the property associated with $F \star G$ of which $[\![F \star G]\!](s)$ is supposed to be the corresponding probability at the state s? What we can conclude from this discussion is that the quantitative approach to probabilistic temporal logics requires, in order to be a meaningful tool for expressing properties of PTS's, a description for the meaning of formulas going beyond the mere numerical function corresponding to the denotational semantics.

In order to address this conceptual issue, C. Morgan and A. McIver introduced, in [78], a game semantics for the logic pLμ. As for Lμ, the games are played by two players named Player 1 and Player 2 by following rules similar to those for Lμ-calculus games, but unlike in Lμ games, a third agent named *Nature* takes part in the game. Nature is a purely probabilistic agent whose role is to make choices in accordance with certain prescribed probabilities, and this allows the incorporation of the stochastic choices associated with the PTS. Since a probabilistic agent is taking part in the game, rather than restricting attention

[1] Roughly speaking, their observation is based on the fact that, given a measurable space X, a probability measure μ on X and two (unknown) measurable sets A and B, $x \oplus y$ is the best possible upper-bound for the measure $\mu(A \cup B)$ when the available information is just $\mu(A) = x$ and $\mu(B) = y$.

[2] The function $\star$ is known as *Frank T-norm* with parameter $p = 7$. When restricted to the set $\{0,1\}$ it acts as ordinary boolean conjunction, for every $p \in (1, \infty)$.

just to winning strategies, one more generally considers strategies that guarantee that the probability of Player 1 winning is above a given threshold. Given a PTS $\mathcal{L}$ with set S of states, the pLμ *game semantics* of a formula F at a state s, denoted by $(\!| F |\!)(s)$, is then the best (limit) probability of Player 1 winning the game $\mathcal{G}^F_{\mathcal{L}}$ starting from a specific game state associated with s. The class of games used for giving game semantics to pLμ is known in game thoery as that of *two player stochastic parity games*, or $2\frac{1}{2}$*-player parity games* for short. Fundamental theoretical results about $2\frac{1}{2}$-player (parity) games, critically exploited in the precise mathematical definition of the game semantics for pLμ, were developed only at the end of the 90's by remarkable achievements in game theory of, most notably, Donald A. Martin [74]. One of the most important aspects of the game semantics of pLμ is that it is a straightforward generalization of that of Lμ, where the actions of Nature are used to model the probabilistic choices corresponding to the probability distributions occurring in the PTS. Therefore the game semantics offers a clear interpretation for the *properties* associated to the formulas, explained in terms of the interactions between the controller (Player 1) and a hostile environment (Player 2) in the context of the stochastic choices occurring in the PTS (Nature).

A main result of [78] is that, as one would hope, the game and denotational semantics of pLμ coincide on all *finite models*. One of the contributions of the present thesis is to generalize this result: as we shall see (Theorem 3.2.14), the game and denotational semantics for pLμ coincide on all models, whether finite or infinite. Thus, as for Lμ, one can take the preferred viewpoint when reasoning about pLμ formulas.

The logic pLμ is, however, not completely satisfactory because it is apparently not expressive enough to encode other important probabilistic temporal logics such as PCTL and PCTL* and, as observed earlier, one of the crucial features of Lμ is its ability to encode most other temporal logics for transition systems. This is due to the fact that the logic pLμ, and its $[0,1]$-valued interpretation, can not describe *sets* of states (which constitute the interpretations of PCTL formulas) and is instead limited to quantitative specifications. An important achievement of the present thesis is to show that this is not an intrinsic limit of the $[0,1]$-valued approach to probabilistic temporal logics, but rather a lack of expressivity in the logic pLμ which can be overcome by moving to stronger logics. We shall consider primarily two logics built on top of pLμ: the *probabilistic modal μ-*

calculus with independent product (pL$\mu^{\odot}$) and the *probabilistic modal μ-calculus with independent product and truncated sum* (pL$\mu^{\odot}_{\oplus}$).

The logic pL$\mu^{\odot}$ is obtained from pLμ by adding to the syntax of the logic the new binary *product* operator, denoted by the $\cdot$ symbol, and its associated dual operation, called *coproduct* and denoted by the $\odot$ symbol, whose denotational interpretations are given as follows: $[\![F \cdot G]\!](s) = [\![F]\!](s) \cdot [\![G]\!](s)$ and $[\![F \odot G]\!](s) = [\![F]\!](s) + [\![G]\!](s) - [\![F]\!](s) \cdot [\![G]\!](s)$, where the symbols $\cdot$ and $+$ denotes standard multiplication and sum on reals. As mentioned earlier, these operators have been already investigated in [56] as *alternative* interpretations for the pLμ connectives $\{\wedge, \vee\}$. Here, we take the small but apparently novel step of considering the (co)product operations $\{\odot, \cdot\}$ in combination with the lattice operations $\{\sqcup, \sqcap\}$, which we continue calling $\vee$ and $\wedge$. The logic pL$\mu^{\odot}$ is very rich and can, for instance, express the meaning of *qualitative* modalities *à la* PCTL such as $\mathbb{P}_{>0}F$ and $\mathbb{P}_{=1}F$ which, in turn, allow the encoding of the *qualitative* fragment of PCTL into pL$\mu^{\odot}$ (Theorem 7.2.16).

The logic pL$\mu^{\odot}_{\oplus}$ is a further extension of pLμ obtained by adding to the syntax of pL$\mu^{\odot}$ the pair of dual operators $\oplus$ and $\ominus$. Once again, these operations were considered as *alternatives* to $\{\vee, \wedge\}$ in [56], and we take the small but novel step of considering them in combination with the other operations already discussed. The denotational semantics of pL$\mu^{\odot}_{\oplus}$ is, again, straightforwardly specified by extending the denotational semantics of pL$\mu^{\odot}$ by the following definitions: $[\![F \oplus G]\!] = \min\{1, [\![F]\!] + [\![G]\!]\}$ and $[\![F \ominus G]\!] = \{0, [\![F]\!] + [\![G]\!] - 1\}$. This extension is of concrete interest because it is possible to encode *quantitative* threshold modalities *à la* PCTL such as $\mathbb{P}_{>\frac{1}{2}}F$ and $\mathbb{P}_{\geq\frac{2}{3}}F$ which, in turn, allow the encoding of *full* PCTL into in pL$\mu^{\odot}_{\oplus}$ (Theorem 7.2.16).

While the denotational semantics is straightforwardly defined this way, our major goal is to define appropriate game semantics for pL$\mu^{\odot}$ and pL$\mu^{\odot}_{\oplus}$. As discussed earlier, we consider this as a fundamental task required in order to understand the kinds of properties corresponding to formulas. The game-semantics we build is based on the intuition that $[\![F \cdot G]\!](s)$ might be interpreted as the probability of the property expressed by F and the property expressed by G both holding at the state s when verified *independently*. The idea of independent verification of two properties is captured in the games used to give game semantics to pL$\mu^{\odot}$ as follows: the formula $F \cdot G$ is interpreted in the game dynamics as generating two *concurrent* sub-games, continuing their executions following the game interpretation of F and G respectively. These two sub-games are played *independently*, in the sense that a player acting on the sub-game associated with F has no information whatsoever about the choices happening in the other sub-game, and *viceversa*. Player 1 is declared to have won the game associated with $F \cdot G$ if they manage to win *both* generated "concurrent and independent" sub-games. The dual operator $\odot$ is interpreted in a similar way: in the game, the formula $F \odot G$ generates two concurrent and independent sub-games, but unlike in $F \cdot G$ configurations, Player 1 is declared to have won the $F \odot G$ game if they manage to win in *at least one* of the two generated sub-games.

In the pL$\mu^{\odot}$ games outlined above, a *play* is not just a sequence of game states as in most logical games (including Hintikka games for first order logic, Lμ-games and pLμ-games) but rather a tree structure, which we refer to as a *branching play*, where the nodes x having more than one child, which we refer to as *branching states*, correspond to game states of the form $F \cdot G$ and $F \odot G$ on which the game splits into two concurrent sub-games, represented in the tree by the children of x.

The simple and intuitive game-interpretation for the new connectives $\cdot$ and $\odot$ of pL$\mu^{\odot}$ formulas provides, building on top of the game-interpretation of the other connectives as in pLμ games, a satisfactory and straightforward game-interpretation for the fixed-point-free fragment of the logic pL$\mu^{\odot}$: a play in a pL$\mu^{\odot}$ game associated with a formula without fixed points, can always be seen as a *finite* branching play, i.e., a finite tree of degree at most 2, and the winning condition described above constitute a precise specification for the set of branching plays which are considered to be winning for Player 1. However, when the full logic pL$\mu^{\odot}$ is considered, defining precisely the set of branching plays winning for Player 1 becomes a surprisingly technical undertaking. Indeed, pL$\mu^{\odot}$ games associated with formulas defined by (co)inductive definitions may generate infinite branching plays containing infinitely many interleaved occurrences of product and coproduct operations, so that the simple explanation given above for the winning condition at such nodes does not suffice. To account for this, branching plays are themselves considered as ordinary 2-player (parity) games with coproduct nodes as Player-1 nodes, and product nodes as Player-2 nodes. Player 1's goal in the *outer* pL$\mu^{\odot}$ game is to produce a branching play for which, when itself considered as a game, the *inner game*, they have a winning strategy.

In order to formalize these ideas, we introduce a general notion of 2-*player* *stochastic tree game*, which generalizes standard 2-player stochastic games by introducing a new type of game state, the *branching states*, at which the game is split into concurrent and independent sub-games. As briefly discussed earlier, plays in $2\frac{1}{2}$-player tree games are tree structures called *branching plays*, modeling in a natural way concurrent executions. One of the contributions of this thesis is the study of some interesting properties of tree games. We show how the simple form of imperfect information formalized in $2\frac{1}{2}$-player tree games, namely the fact a player can not observe the execution of the game in other concurrent sub-games, can be used to faithfully model *Blackwell games* (Theorem 4.2.18), an important class of games of *imperfect information* [12, 110, 74]. We also show how some problems in stochastic games, such as the open problem of *qualitative determinacy* of 2-stochastic player parity games [19, 20], can be formulated as appropriate determinacy problems for *non-stochastic* $2\frac{1}{2}$-player tree games (Theorem 4.4.7).

Although $2\frac{1}{2}$-player tree games are interesting in their own right, their introduction is mainly motivated by wanting to provide a mathematically precise description of the pL$\mu^{\odot}$ games described informally above. To this end, we identify the class of 2-player stochastic *meta-games*, consisting of those $2\frac{1}{2}$-player tree games whose set of winning branching plays is described using *inner games*: branching plays are themselves interpreted as ordinary 2-player games (referred to as *inner games*), and a branching play is declared to be winning for Player 1 in the meta-game (referred to as the *outer game*) if Player 1 has a winning strategy in the inner game. In particular, we identify the class of $2\frac{1}{2}$-player meta-*parity* games, given by those meta-games whose inner-games are ordinary 2-player parity games. This is the class which we use for providing a precise game semantics for the logic pL$\mu^{\odot}$.

The main technical achievement of the thesis is a proof of *determinacy* for $2\frac{1}{2}$-player meta-parity games (Theorem 6.4.2), a property of fundamental interest. Roughly speaking, determinacy asserts that a player does not gain any advantage if, when choosing a strategy for playing in a $2\frac{1}{2}$-player meta-parity game, they are informed in advance about the choice made by the adversary. Our proof is novel in the sense that it can not be obtained from (or at least does not seem straightforwardly reducible to) standard results in game theory, such as the determinacy for classes of $2\frac{1}{2}$-player (Gale–Stewart or Blackwell) games, because we work with the novel class of $2\frac{1}{2}$-player tree games. The general technique we adopt is however, one that is well established [35, 100]. We reduce the determinacy of a $2\frac{1}{2}$-player meta-parity game of some given *complexity* to the determinacy of a $2\frac{1}{2}$-player meta-parity game of lower complexity, and we directly prove that all $2\frac{1}{2}$-player meta-parity games of *minimal* complexity are determined. Thus our proof is by induction on the complexity of $2\frac{1}{2}$-player meta-parity games. One of the crucial steps in our proof, is a transfinite (up to the first uncountable ordinal ω_1) inductive characterization of the winning sets of $2\frac{1}{2}$-player meta-parity games, which allows us to reason inductively on the sets otherwise defined, somewhat declaratively, by means of *inner games* as discussed earlier. A key observation about the sets of winning branching plays is that they are quite complicated from the point of view of descriptive set theory. In general, the winning set of a $2\frac{1}{2}$-player meta-parity game is a $\mathbf{\Delta}^1_2$ set, thus neither Borel nor even (co)analytic. Sets of this kind are quite difficult to work with, especially in the context of measure-theory. For instance, it is consistent with ZFC[3] that there exists a non-Lebesgue-measurable $\mathbf{\Delta}^1_2$-set. For this reason our proof is carried out in ZFC extended with an extra axiom, Martin's Axiom at the first uncountable cardinal ($\mathrm{MA}_{\aleph_1}$) [75], which is known to be equiconsistent with ZFC. Therefore our determinacy result is at least consistent, i.e., it can not be disproved, within ZFC. We leave open the question of whether the result is provable in ZFC alone.

Interestingly, $2\frac{1}{2}$-player meta-parity games are expressive enough to provide a game semantics for the extended logic $\mathrm{pL}\mu^{\odot}_{\oplus}$. The meaning of the $\mathrm{pL}\mu^{\odot}_{\oplus}$ operators $\{\oplus, \ominus\}$ is captured in the game semantics by means of certain infinitary protocols involving operations of product and coproduct.

A central result of the thesis is the equivalence of the denotational and the game semantics, given in terms of $2\frac{1}{2}$-player meta-parity games, for the logic $\mathrm{pL}\mu^{\odot}_{\oplus}$, and thus also for its fragment $\mathrm{pL}\mu^{\odot}$ (Theorem 7.1.10). The proof is obtained by application of the techniques and results developed in the proof of determinacy for $2\frac{1}{2}$-player meta-parity games. Thus the equivalence between the denotational and game semantics for $\mathrm{pL}\mu^{\oplus}_{\odot}$ is formally valid in $\mathrm{ZFC} + \mathrm{MA}_{\aleph_1}$ set theory.

The straightforward game interpretation of the new operations of product $(\cdot)$ and coproduct $(\odot)$ as generating two concurrent and independent games, and the description of the winning objective which requires Player 1 to win in *both* (respectively *at least one*) generated sub-games at product (respectively coproduct)

[3] *Zermelo-Fraenkel* set theory with the *Axiom of Choice*: the first-order set theory that forms the most widely accepted foundation of mathematics.

configurations, provides a satisfactory *interactive* or *operational* description for the meaning of $\text{pL}\mu^{\odot}$ formulas. As for other logics, having two complementary semantics allows one to pick the preferred viewpoint when trying to prove the desired property. The game semantics for $\text{pL}\mu^{\odot}$ also offers a straightforward interpretation for the meaning of the qualitative threshold modalities $\{\mathbb{P}_{>0}, \mathbb{P}_{=1}\}$. The formula $\mathbb{P}_{>0}F$ is interpreted as generating countably many instances of the game associated with F, and Player 1 is required to win in *at least one* generated sub-game. Similarly, the the formula $\mathbb{P}_{=1}F$ is interpreted in the game semantics as generating countably many instances of the game associated with F, and Player 1 is required to win in *all* generated sub-games. We shall make full use of the two complementary semantics for discussing the expressive power of the logic $\text{pL}\mu^{\odot}$ by means of concrete examples. Some of our examples are used to prove interesting properties of the logic. For instance, we show that there are $\text{pL}\mu^{\odot}$ formulas which can be satisfied, with probability one, only by PTS's having an infinite state space (Proposition 7.2.5). Thus the logic $\text{pL}\mu^{\odot}$ does not satisfy the the so called *finite model property* [106].

The game interpretation of the two $\text{pL}\mu^{\odot}_{\oplus}$ connectives $\oplus$ and $\ominus$ is, however, not as transparent and straightforward as the one given for the connectives $\cdot$ and $\odot$. Indeed, as anticipated above, the games for $\text{pL}\mu^{\odot}_{\oplus}$ capture the operational meaning of the new connective $\oplus$ and $\ominus$ as specific infinitary protocols involving infinitely many operations of product and coproduct, and thus do not necessarily provide a clean and intuitive interpretation for the new connectives. Nevertheless we suggest that our results about $\text{pL}\mu^{\odot}_{\oplus}$ are interesting, at the very least because of the following observations. The game semantics for $\text{pL}\mu^{\odot}_{\oplus}$ is defined in terms of $2\frac{1}{2}$-player meta-parity games, and thus it serves as an expressivity result for this novel class of games introduced in the present thesis. Secondly, it is certainly interesting to know that $\text{pL}\mu^{\odot}_{\oplus}$, and its fragment PCTL, can be given an *operational* interpretation in terms of the primitives available in stochastic tree games, albeit not as transparent as one might hope for.

We conclude this section by recalling, from the second paragraph of the present introduction, that one of the advantages coming from a formal semantics is the possibility of being able to formally express interesting properties of systems, and of formally verifying if a certain property is fulfilled by a given system. In this thesis, we identify expressive logics for formulating interesting properties of probabilistic transition systems which, as discussed earlier, are the mathematical structures used to give operational semantics for concurrent probabilistic programs and systems. We entirely focus on the task of providing appropriate semantics for these logics (pLμ, pL$\mu^{\odot}$ and pL$\mu^{\odot}_{\oplus}$) and we succeed in finding interesting game semantics agreeing with the denotational semantics. On the other hand, we completely ignore the problem of *verification*. Providing verification methods for probabilistic concurrent systems using probabilistic μ-calculi is an interesting area for further research. It seems likely that the semantical foundations laid in this thesis will be of help towards this endeavor.

1.1 Synopsis

The rest of the thesis is organized as follows.

In Chapter 2 we provide the required mathematical background. We cover the relevant notions and results from order theory, topology, measure theory, set theory and game theory.

In Chapter 3 we provide the necessary background on program logics. We first consider temporal logics for *Labeled Transition Systems* (LTS's), and in particular the logic CTL and the modal μ-calculus (Lμ). We then turn our attention to temporal logics for expressing properties of *Probabilistic LTS's* (PLTS's). We introduce, delving into the historical and conceptual details, the logics PCTL and pLμ. We finally start on the novel content of the thesis by introducing pL$\mu^{\odot}_{\oplus}$, the most expressive logic considered in this thesis. We motivate our interest for pL$\mu^{\odot}_{\oplus}$ by considering the expressive power of some of its fragments, including the logic pL$\mu^{\odot}$. We conclude the chapter by discussing, informally, the intuitions which will lead us toward the definition of appropriate game semantics.

In Chapter 4 we define the class of two player stochastic tree games and study some of its properties. We prove that every Blackwell game can be faithfully encoded as a finite 2-player tree game and that $2\frac{1}{2}$-player tree games can be modeled by 2-player (non-stochastic) tree games. The latter fact will be used to discuss an interesting open problem in the literature: the *qualitative determinacy* of standard 2-player stochastic games.

In Chapter 5 we identify a class of $2\frac{1}{2}$-player tree games, called $2\frac{1}{2}$*-player meta games*, whose winning sets are specified by means of *inner games*. Two player stochastic meta-*parity* games, the class of games we use for giving game semantics to the logic pL$\mu^{\odot}_{\oplus}$ and its fragments, have inner games specified as standard 2-player parity games.

In Chapter 6 we prove our main technical result: a proof, valid in ZFC + $MA_{\aleph_1}$ set theory, of determinacy for $2\frac{1}{2}$-player meta-parity games. The proof, as discussed earlier, is by induction on the complexity, i.e., on the number of priorities used in $2\frac{1}{2}$-player meta-parity games.

In Chapter 7 we give a game semantics for the extended probabilistic μ-calculi identified in Chapter 3. The game and denotational semantics are proved to coincide on all models. We consider several examples of formulas expressing useful properties of PLTS's, and we show how the logic $pL\mu^{\odot}_{\oplus}$, and its fragment $pL\mu^{\odot}$, can encode the logic PCTL, and its qualitative fragment, respectively.

Lastly, Chapter 8 contains our final conclusions and gives suggestions for future work.

Chapter 2

Technical Background

This chapter provides an introduction to the relevant mathematical notions used in this thesis. We start, in Section 2.1, by discussing some basic concepts of lattice theory, topology, measure theory and set theory. In Section 2.2 we focus on properties of the real unit interval $[0, 1]$ and define a few operations on it which will be used to give semantics to the probabilistic logics considered in our work. In Section 2.3 we discuss, in detail, the relevant concepts and results from game theory.

2.1 Classical Mathematical Background

In this section we discuss some basic notions from lattice theory, topology and measure theory, which are going to be used in the present thesis. Given the size and maturity of all these mathematical fields, we limit ourselves to a very concise, and necessarily limited, presentation which can not be considered, by any means, as self contained. The mathematics hereafter discussed is valid in ZFC, *Zermelo-Fraenkel Set Theory with the Axiom of Choice* [59], which is currently one of the most common axiomatic systems for the foundations of mathematics. At the end of this section we shall discuss the extension of ZFC, denoted by $\text{ZFC} + \text{MA}_{\aleph_1}$, obtained by validating an instance of the so-called *Martin's Axiom*. Some of the consequences of $\text{ZFC} + \text{MA}_{\aleph_1}$, relevant for our work, will be discussed.

2.1.1 Basic mathematical notation

In this subsection we summarize the mathematical notation and a few conventions to which we adhere in this thesis.

We write $\mathbb{N}$ or ω for the set of natural numbers, $\mathbb{R}$ for the sets of reals, $[0,1]\subseteq\mathbb{R}$ for the unit interval. Given sets X and Y, we denote with $|X|$ the cardinality of X (often taken as the least ordinal admitting a bijection with X) and with Y^X the set of functions from X to Y. We denote with 2 the two element set $\{\emptyset,\{\emptyset\}\}$ (or more generally any set with two elements) and with 2^X, or sometimes with $\mathcal{P}(X)$, the collection of subsets of X, i.e., the set $\{Y \mid Y\subseteq X\}$, thus identifying subsets of X with their characteristic functions. We often think of the space X^Y as the set of Y-indexed sequences $\{x_y\}_{y\in Y}$ of elements in X. We write $f\colon X\to Y$ to specify that $f\in Y^X$. Given a subset Y of X, we denote with $\overline{Y}$ its complement, i.e., the set $\overline{Y}=X\setminus Y$. We denote with $X\times Y$ the cartesian product of X and Y, and with $\prod_i X_i$ the I-indexed cartesian product of the sets X_i, $i\in I$. Given a set $E\subseteq X\times Y$, we denote with $E(x)\subseteq Y$ the set $\{y \mid (x,y)\in E\}$. We use the greek letters α, β and γ to range over the well-ordered class of ordinals. We shall in particular consider the least infinite ordinal ω and the least uncountable ordinal ω_1, which is the limit of all countable ordinals. We use the letter κ to range over cardinal numbers and the letter $\aleph$ to identify cardinals via their *Aleph number*. In particular we shall consider the following cardinals: $\aleph_0$ (the cardinality of ω), $\aleph_1$ (the cardinality of ω_1), and $2^{\aleph_0}$ the cardinality of the continuum $\mathbb{R}$.

Definition 2.1.1. Given a set X, we say that a function $d: X \to [0,1]$ is a *(discrete) probability distribution* on X, if $d(x)\geq 0$ for all $x\in X$ and $\sum_{x\in X} d(x)= 1$. We denote with *supp*(d) the necessarily countable set $\{x \mid d(x) > 0\}$. We denote with $\mathcal{D}(X)$ the set of all (discrete) probability distributions on X.

2.1.2 Lattice Theory

In this subsection we follow the excellent introduction of [2, §1], to which we refer for a much deeper introduction to the topic.

Definition 2.1.2 (Poset)**.** A *partially ordered set*, or just a *poset*, is a pair $(X,\sqsubseteq)$, where X is a *set* and $\sqsubseteq$ is a relation on X which is reflexive, antisymmetric and transitive. The relation $\sqsubseteq$ is called a *partial order* on the set X.

Definition 2.1.3 (Pointwise ordering)**.** Let $(X,\sqsubseteq)$ be a poset and I an index set. The relation $\preceq$ on X^I defined as

$$\{x_i\}_{i\in I} \preceq \{y_i\}_{i\in I} \text{ iff } \forall i\in I.\ x_i\sqsubseteq y_i$$

is called the *pointwise order* obtained by lifting $\sqsubseteq$. It is simple to see that $(X^I, \preceq)$ is a poset.

Definition 2.1.4. Given two posets $(X, \sqsubseteq)$ and $(Y, \preceq)$, a function $f : X \to Y$ is *monotone* if for all $x, x' \in X$, $f(x) \preceq f(x')$ whenever $x \sqsubseteq x'$. The function f is called an *order isomorphism* if it is bijective and the inverse map $f^{-1} : Y \to X$ is monotone.

Definition 2.1.5. An *ω-chain* or an *increasing sequence* in a poset $\langle X, \sqsubseteq\rangle$ is a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \sqsubseteq x_{n+1}$ for every $n \in \mathbb{N}$.

Definition 2.1.6. Given a poset $\langle X, \sqsubseteq\rangle$, an *antichain* in X is a set $A \subseteq X$ such that for all $a, b \in A$, if $a \sqsubseteq b$ then $a = b$.

Definition 2.1.7. A poset $(X, \sqsubseteq)$ is *well-founded* if every (strictly) $\sqsubseteq$-descending chain is finite.

Definition 2.1.8. Given a poset $(X, \sqsubseteq)$, we say that a subset $A \subseteq X$ is *up-closed* if and only if for all $a \in A$ and $b \in X$ with $a \sqsubseteq b$, it holds that $b \in A$. Similarly, we say that A is down-closed if and only if $a \in A$ and $b \in X$ with $b \sqsubseteq a$, it holds that $b \in A$. Given an element $a \in X$, we denote with $a\uparrow$ the up-closed set $\{b \in X \mid a \sqsubseteq b\}$. Similarly we write $a\downarrow$ for the the down-closed set $\{b \in X \mid a \sqsubseteq b\}$. Given a set $A \subseteq X$ we denote with $A\uparrow$ the up-closed set $\bigcup\{a\uparrow \mid a \in A\}$ which we refer to as the *up-closure of* A. Similarly, the *down-closure of* A is defined as the down-closed set $\bigcup\{a\downarrow \mid a \in A\}$.

Definition 2.1.9. Given a poset $(X, \sqsubseteq)$ and a subset $A \subseteq X$, we say that $b \in X$ is an *upper-bound* for A if $a \sqsubseteq b$ holds for all $a \in A$. An upper-bound b of A is called the *least upper-bound of* A and denoted by $\bigsqcup A$, if for any other upper-bound b' of A the inequality $b \sqsubseteq b'$ holds. Similarly we say that $b \in X$ is a *lower-bound* for A if $a \sqsupseteq b$ holds for all $a \in A$, and that b is the *greatest lower-bound of* A, denoted by $\bigsqcap A$, if for any other lower-bound b' of A, $b \sqsupseteq b'$ holds.

We are now ready to define the notion of *lattice*.

Definition 2.1.10 (Lattice)**.** A poset $(X, \sqsubseteq)$ is a *lattice* if for every $x, y \in X$, the two element set $\{x, y\}$ has a least upper-bound and a greatest lower-bound, simply denoted by $x \sqcup y$ and $x \sqcap y$ respectively. Furthermore, the poset $(X, \sqsubseteq)$ is a *complete lattice* if every $A \in X$ has a least upper bound ($\bigsqcup A$) and greatest lower bound ($\bigsqcap X$). In a complete lattice $(X, \sqsubseteq)$, we denote with $\top$ and $\bot$ the greatest and least elements $\bigsqcup X = \bigsqcap \emptyset$ and $\bigsqcap X = \bigsqcup \emptyset$ respectively. A lattice $(X, \sqsubseteq)$ is *bounded* if there exists two elements $\top, \bot \in X$ such that for all $x \in X$, $\bot \sqsubseteq x \sqsubseteq \top$. Note that every complete lattice is bounded.

Proposition 2.1.11. *Let $(X, \sqsubseteq)$ be a (complete) lattice. Then, for every index set I, the poset $(X^I, \preceq)$, where $\preceq$ is the pointwise order, is a (complete) lattice and the following equalities hold:*

$$(f \sqcup g)(x) = f(x) \sqcup g(x) \qquad (f \sqcap g)(x) = f(x) \sqcap g(x)$$
$$\left(\bigsqcup F\right)(x) = \bigsqcup\{f(x) \mid f \in F\} \qquad \left(\bigsqcap F\right)(x) = \bigsqcap\{f(x) \mid f \in F\}$$

for every $f, g \in X^I$ and $F \subseteq X^I$.

An important concept in lattice theory is the so-called *principle of symmetry* [2] which we now discuss.

Definition 2.1.12. Given a poset $(X, \sqsubseteq)$, we defined its *dual* posed $(X, \sqsubseteq^*)$ as follows: $x \sqsubseteq^* y$ if and only if $y \sqsubseteq x$. It is immediate to verify that $\sqsubseteq^*$ is indeed a partial order on X.

Observe that, if the least upper-bound of a set $A \subseteq X$ exists in $(X, \sqsubseteq)$, then it equals the greater lower-bound of the same set in $(X, \sqsubseteq^*)$ and viceversa. Hence $(X, \sqsubseteq)$ is a (complete) lattice if and only if $(X, \sqsubseteq^*)$ is a (complete) lattice, and we have

$$x \sqcup y = x \sqcap^* y \qquad x \sqcap y = x \sqcup^* y \qquad \bigsqcup A = \bigsqcap^* A \qquad \bigsqcap A = \bigsqcup^* A$$

An important result in lattice theory is the fixed-point theorem discovered by B. Knaster and A. Tarski [61, 108].

Theorem 2.1.13 (Knaster-Tarski)**.** *Let $(X, \sqsubseteq)$ be a complete lattice and $f : X \to X$ a monotone function. Let us denote with Fix(f) the set of fixed points of f defined as Fix$(f) = \{x \mid x = f(x)\}$. The set Fix$(f)$ equipped with the order $\sqsubseteq$ (restricted to Fix(f)) is a complete lattice. There is thus a least fixed point* $\mathrm{lfp}(f)$ *and a greatest fixed point* $\mathrm{gfp}(f)$*. Moreover the following equalities hold:*

1. $\mathrm{lfp}(f) = \bigsqcap\{x \mid f(x) \sqsubseteq x\}$,
2. $\mathrm{gfp}(f) = \bigsqcup\{x \mid x \sqsubseteq f(x)\}$,

3. $\mathrm{lfp}(f) = \bigsqcup\{f_\alpha \mid \alpha$ *an ordinal*$\}$,

4. $\mathrm{gfp}(f) = \bigsqcap\{f^\alpha \mid \alpha$ *an ordinal*$\}$,

where $f_0 = \bot$, $f_{\beta+1} = f(f_\beta)$ *and* $f_\lambda = \bigsqcup\{f_\alpha \mid \alpha < \lambda\}$ *for every limit ordinal* λ, *and similarly,* $f^0 = \top$, $f^{\beta+1} = f(f^\beta)$ *and* $f^\lambda = \bigsqcap\{f^\alpha \mid \alpha < \lambda\}$ *for every limit ordinal* λ.

Definition 2.1.14. Let $(X, \sqsubseteq)$ be a complete lattice. We say that $f : X \to X$ is *ω-continuous* if $f(\bigsqcup\{x_n\}_{n\in\mathbb{N}}) = \bigsqcup\{f(x_n)\}_{n\in\mathbb{N}}$, for every $\sqsubseteq$-increasing sequence $\{x_n\}_{n\in\mathbb{N}}$. Similarly we say that f is *ω-cocontinuous* if $f(\bigsqcap\{x_n\}_{n\in\mathbb{N}}) = \bigsqcap\{f(x_n)\}_{n\in\mathbb{N}}$, for every $\sqsubseteq$-decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$.

It is simple to verify that if a function f is ω-continuous or ω-cocontinuous then it is also monotone. For ω-continuous and ω-cocontinuous functions, the statement of Knaster-Tarski theorem can be strengthened as follows:

Theorem 2.1.15. *Let* $(X, \sqsubseteq)$ *be a complete lattice and* $f : X \to X$ *a monotone function. Then the following assertions hold:*

1. *if* f *is* ω*-continuous, then* $\mathrm{lfp}(f) = \bigsqcup\{f_n \mid n < \omega\}$, *and*

2. *if* f *is* ω*-cocontinuous, then* $\mathrm{gfp}(f) = \bigsqcap\{f^n \mid n < \omega\}$,

where $f_0 = \bot$, $f^0 = \top$, $f_{n+1} = f(f_n)$ *and* $f^{n+1} = f(f^n)$ *for all* $n \in \mathbb{N}$.

Proof. See, e.g., Theorem 1.2.14 in [2]. □

An important operation definable in many interesting lattices, and in particular in those considered in this thesis, is that of *negation* or *dual operation*.

Definition 2.1.16 (Negation)**.** Let $(X, \sqsubseteq)$ be a lattice. A function $\eta : X \to X$ is called a *negation on* X, if it satisfies the following properties:

1. for every $x \in X$, $\eta(\eta(x)) = x$,

2. for every $x, y \in X$, $x \sqsubseteq y$ implies $\eta(y) \sqsubseteq \eta(x)$.

Definition 2.1.17. A lattice $(X, \sqsubseteq)$ is *distributive* if it satisfies the two equivalent conditions:

1. For all $x, y, z \in X$, $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$,

2. For all $x, y, z \in X$, $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$.

Definition 2.1.18 (De Morgan and Boolean algebras)**.** A bounded (complete) lattice $(X, \sqsubseteq, \top, \bot, \neg)$, equipped with a negation $\neg : X \to X$, is a (complete) *De Morgan algebra* if it is distributive. It is a (complete) *Boolean algebra* if it also satisfies the equalities $x \sqcup \neg x = \top$ and $x \sqcap \neg x = \bot$ for all $x \in X$.

In complete De Morgan algebras there is an important form of duality relating least and greatest fixed points of monotone operators.

Theorem 2.1.19. *Let $(X, \sqsubseteq, \top, \bot, \neg)$ be a complete De Morgan algebra. For every monotone function $f : X \to X$ we define its* dual*, denoted by $\overline{f} : X \to X$, as follows: $\overline{f}(x) = \neg(f \neg(x))$. Then the following assertions hold:*

1. *$\overline{f}$ if monotone,*
2. *$\mathrm{lfp}(\overline{f}) = \neg\big(\mathrm{gfp}(f)\big)$, and*
3. *$\mathrm{gfp}(\overline{f}) = \neg\big(\mathrm{lfp}(f)\big)$.*

Proof. See, e.g., Proposition 1.2.25 of [2]. □

2.1.3 Topology and Polish spaces

In this section we present the main definitions and concepts from topology which are going to be used in this thesis. We shall be mainly interested in Polish spaces, which bridge the world of topology and *classical descriptive set theory*. We follow closely the presentation of [60], including many definitions and facts *verbatim*, to which we refer for an extensive treatment of the subject.

Definition 2.1.20. A *topological space* is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ a collection of subsets of X such that $\emptyset, X \in \mathcal{T}$, and $\mathcal{T}$ is closed under arbitrary unions and finite intersections. The collection $\mathcal{T}$ is called a *topology* on X and its members are called *open sets*. If $\mathcal{T}$ is the entire collection of subsets of X, then $\mathcal{T}$ is called the *discrete topology* on X. The complement of an open set is a *closed set*. A subset of X which is both open and closed is a *clopen set*: $\emptyset$ and X are always clopen sets. Given a point $x \in X$, an *open neighborhood* of x is an open set containing x. A topology is *Hausdorff* if distinct points in X have disjoint open neighborhoods. A subset $D \subseteq X$ is *dense* in the topology $\mathcal{T}$, if $D \cap U \neq \emptyset$, for all $U \in \mathcal{T}$. A topological space $(X, \mathcal{T})$ admitting a countable dense set is called *separable*. A *subspace* of $(X, \mathcal{T})$ consists of a subset $Y \subseteq X$ with the *relative* or *subspace topology* $\mathcal{T}|Y = \{Y \cap U \mid U \in \mathcal{T}\}$.

Definition 2.1.21. Given topological spaces X and Y, a map $f\colon X\to Y$ is:

1. *continuous* if the inverse image of each open set is open,
2. *open* (*closed*), or *preserving the open (closed) sets*, if the image of each open set is open (closed),
3. a *homeomorphism* if it is a continuous bijection such that f^{-1} is continuous.

We say that X and Y are homeomorphic if there exists a homeomorphism $f:X\to Y$.

Definition 2.1.22. Given a subspace $(Y,\mathcal{S})$ of $(X,\mathcal{T})$ we say that a continuous function $f\colon X\to Y$ is a *retraction* if, for every $y\in Y$, $f(y)=y$.

Definition 2.1.23. A *basis* $\mathcal{B}$ for a topology $\mathcal{T}$ on X, is a collection $\mathcal{B}\subseteq\mathcal{T}$ with the property that every open set is the union of elements of $\mathcal{B}$ (the empty union gives $\emptyset$). A *subbasis* for $\mathcal{T}$ is a collection $\mathcal{S}\in\mathcal{T}$ such that the set of finite intersections of sets in $\mathcal{S}$ is a basis for $\mathcal{T}$. For any family $\mathcal{S}$ of subsets of a set X, there is a smallest topology $\mathcal{T}$ containing $\mathcal{S}$, called the *topology generated by* $\mathcal{S}$, of which $\mathcal{S}$ is easily shown to be a sub-basis. A topological space is *second countable* if it has a countable basis, and it is 0-*dimensional* if it is Hausdorff and has a basis consisting of clopen sets.

Definition 2.1.24. The product $\prod_{i\in I}X_i$ of a I-indexed family of topological spaces $\{X_i\}_{i\in I}$ is the topological space consisting of the cartesian product of the sets X_i with the topology generated by the following basis consisting of the sets of the form $\prod_{i\in I}U_i$ where U_i is open in X_i for all $i\in I$ and $U_i=X_i$ for all but finitely many $i\in I$. The projection function $\pi_i:\prod_{i\in I}X_i\to X_i$ is continuous and open, for every $i\in I$.

The sum $\bigvee_{i\in I}X_i$ of an I-indexed family of topological spaces $\{X_i\}_{i\in I}$ is the topological space consisting of the disjoint union $\bigcup X_i$ of the sets X_i (which we can always assume to be disjoint by considering homeomorphic copies) with the topology defined as follows: $U\subseteq\bigcup X_i$ is open iff $U\cap X_i$ is open in X_i, for every $i\in I$.

Definition 2.1.25. A topological space $(X,\mathcal{T})$ is *compact* if every open cover of X has a finite subcover, i.e., if $\{U_i\}_{i\in I}$ is a family of open sets and $X=\bigcup_{i\in I}U_i$, then there is a finite $J\subseteq I$ such that $X=\bigcup_{j\in J}U_j$. A subset $A\in X$ is compact in $(X,\mathcal{T})$ if it is compact in the relative topology.

The following proposition states important properties of compact sets and spaces.

Proposition 2.1.26. *The following assertions hold:*

- *Compact subsets of Hausdorff spaces are closed.*
- *The union of finitely many compact sets is compact. Finite sets are compact.*
- *The continuous image of a compact set is compact.*
- *(Tychonoff's Theorem) Any product of compact spaces is compact.*
- *The sum of finitely many compact spaces is compact.*
- *Any compact Hausdorff space X is normal: given disjoint compact sets $A, B \subseteq X$, there exists disjoint open sets $U \supseteq A$ and $V \supseteq B$.*

Proof. For a proof of points 1-5 see, e.g., Proposition 4.1 in [60]. See, e.g., Theorem 32.3 in [85] for a proof of the last assertion. □

Definition 2.1.27. A *metric space* is a pair (X, d), with X a set and d a function of type $d : X^2 \to [0, +\infty)$ satisfying:

i. $d(x, y) = 0$ if and only if $x = y$,

ii. $d(x, y) = d(y, x)$,

iii. $d(x, y) \leq d(x, z) + d(z, y)$,

for every $x, y, z \in X$. The function d is called a *metric* on X. The *open ball* with center at $x \in X$ and radius $r \in \mathbb{R}$ is defined by $B(x, r) = \{y \in X \mid d(x, y) < r\}$. These balls form a basis for a topology on X called the *topology of the metric space*, which is clearly Hausdorff. A topological space $(X, \mathcal{T})$ is metrizable if there is a metric d on X so that $\mathcal{T}$ is the topology of (X, d). In this case we say that d is *compatible* with $\mathcal{T}$.

Definition 2.1.28. Let (X, d) a metric space. A *Cauchy sequence* is a sequence $\{x_n\}_{n \in \mathbb{N}}$ of elements of X such that $\lim_{m,n} d(x_n, x_m) = 0$. We call (X, d) *complete* if every Cauchy sequence has a limit in X, i.e., an element $y \in X$ such that for every $\epsilon > 0$ there exists some $n \in \mathbb{N}$ such that $d(x_m, y) < \epsilon$ for all $m > n$.

Definition 2.1.29. A topological space $(X, \mathcal{T})$ is *(completely) metrizable* if it admits a compatible (complete) metric $d : X^2 \to [0, +\infty)$. If X is metrizable, then X is separable if and only if X is second countable, so we use these terms interchangeably in this case.

Definition 2.1.30. A topological space $(X, \mathcal{T})$ is *Polish* if it is completely metrizable and second countable.

Example 2.1.31. The following are important examples of Polish spaces:

1. $\mathbb{R}$, the set of reals numbers with its standard topology generated by the open intervals with rational endpoints.
2. $[0, 1]$, the unit interval with its standard (subspace) topology.
3. $\{0, 1\}$, the two-point set endowed with the discrete topology,
4. $\mathbb{N}$, the set of natural numbers endowed with the discrete topology.
5. The *Cantor space* $\{0, 1\}^{\mathbb{N}}$, the space of infinite sequences of elements in $\{0, 1\}$, i.e., functions of type $\mathbb{N} \to \{0, 1\}$, endowed with the product topology.
6. The *Baire space* $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$, the space of infinite sequences of natural numbers, endowed with the product topology.

See, e.g., [60, §3], for the description of the relevant complete metrics.

The following proposition provides useful facts about Polish spaces, and makes it simpler to recognize the topological spaces that are Polish.

Proposition 2.1.32. *The following assertions hold:*

1. *A closed subspace of a Polish space is Polish.*
2. *Every countable set endowed with the discrete topology is Polish.*
3. *The product and the sum of a sequence of Polish spaces are Polish.*
4. *Every Polish space is homeomorphic to a closed subspace of* $\mathbb{R}^{\mathbb{N}}$.

Proof. See, e.g., Proposition 3.3 and Theorem 4.17 in [60]. □

Note how the fact that $[0,1]$, $\mathbb{R}$, $\{0,1\}$, $\mathbb{N}$, $\{0,1\}^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ are Polish follows immediately from the previous proposition. A useful technique for proving that a topological space is Polish X is to find a homeomorphism with a closed subset of a known Polish space Y.

Definition 2.1.33. A *graph*, is a pair $G=(V,E)$, where V is a set of *vertices* or *nodes* and $E \subseteq X \times X$ is the *edge* or *accessiblity relation*. The graph G is *countable* if V is a countable set. A *finite path* in G is a non-empty finite sequence $\{v_0,\dots,v_n\}$, such that $(v_m,v_{m+1}) \in E$, for all $0 \leq m < n$. The set of finite paths in G is denoted by $\mathcal{P}_G^{<\omega}$. Given a finite path $\vec{v}=\{v_i\}_{0\leq i\leq n}$ in G, we denote with *first*$(\vec{v})$ and *last*$(\vec{v})$ the states v_0 and v_n, which we refer to as the *first* and *last* states of $\vec{v}$, and we denote with $|\vec{v}|$ the length of the path $\vec{v}$, i.e., the number $n+1$. We say that a finite path $\vec{v}\in\mathcal{P}_G^{<\omega}$ is *terminated* if $E(last(\vec{v}))=\emptyset$. We denote with $\mathcal{P}_G^t$ the set of terminated paths in G. An *infinite path* in G is an infinite sequence $\{v_n\}_{n\in\mathbb{N}}$ such that $(v_n,v_{n+1}) \in E$ for all $n \in \mathbb{N}$. The set of infinite paths in G is denoted with $\mathcal{P}_G^{\omega}$. The function *first* is defined on $\mathcal{P}_G^{\omega}$ in the expected way. We denote with $\mathcal{P}_G$ the set $\mathcal{P}_G^t \cup \mathcal{P}_G^{<\omega}$ which we refer to as the set of *completed paths* in G. We say that a finite path $\{v_0,\dots,v_n\}$ is a *prefix* of a completed path $\{t_i\}_{i\in I}$ if $n \in I$ and for all $0 \leq j \leq n$, the equality $v_j = t_j$ holds. Given two finite paths $\vec{v} = \{v_i\}_{0\leq i\leq m}$ and $\vec{t} = \{t_i\}_{0\leq i\leq n}$ in G, with $t_0 \in E(v_m)$, we denote with $\vec{v}.\vec{t}$ the finite path $\{h_0,\dots h_{n+m+1}\}$, where $h_i = v_i$, for all $0 \leq i \leq m$, and $h_i = t_{i-(m+1)}$, for all $m < i \leq m+n+1$. We call the operation associated with the *dot* symbol, the *concatenation operator* on finite paths. The concatenation of a finite path with an infinite one is defined similarly. We write $\vec{v}\lhd\vec{t}$ if $\vec{v}$ is a prefix of $\vec{t}$. Observe that $\lhd$ is a partial order on $\mathcal{P}_G^{<\omega}$. Given two finite paths $\vec{v}=\{v_i\}_{0\leq i\leq m}$ and $\vec{t}=\{t_i\}_{0\leq i\leq n}$ in G, with $last(\vec{v}) = first(\vec{t})$, we denote with $merge(\vec{v},\vec{t})$ the path $\{u_i\}_{0\leq i\leq n+m}$ where $u_i = s_i$, for all $0 \leq i \leq m$, and and $u_i = t_{i-m}$, for all $m \leq i \leq m+n$. In other words $merge(\vec{v},\vec{t})$ is the path obtained by concatenating $\vec{v}$ with $\vec{t}$ merging the last state of $\vec{v}$ with the first state of $\vec{t}$. The operation of merging a finite path with an infinite one is defined similarly. Given a finite path $\vec{v}$ and a set of finite or infinite paths $X = \{\vec{t_k}\}_{k\in K}$, such that $first(\vec{t_k}) = last(\vec{v})$, for every $k \in K$, we denote with $merge(\vec{v},X)$ the set $\{merge(\vec{v},\vec{t_k})\}_{k\in K}$.

We endow the set $\mathcal{P}_G$ with the topology generated by the basis consisting of the sets $O_{\vec{v}}=\{\vec{t}\in\mathcal{P}_G \mid \vec{v}\lhd\vec{t}\}$ of all completed paths having $\vec{v}$ as prefix, for every finite path $\vec{v}$ in G.

Proposition 2.1.34. *Let G be a countable graph. The space $\mathcal{P}_G$ of completed paths in a graph G is Polish and 0-dimensional.*

Proof. To illustrate the technique, we prove the result by finding a homeomorphism with a closed subset F of $\mathcal{N}$. Let us consider a bijection b between V and the set of positive natural numbers $\mathbb{N} \setminus \{0\}$. Let us define the function $g : \mathcal{P}_G \to \mathcal{N}$ as follows:

- $g(\{v_0, \ldots, v_n\}) = \{b(v_0), \ldots, b(v_n), n_k, \ldots\}_{k>n}$ where $n_k = 0$ for all $k > n$,
- $g\big(\{v_n\}_{n\in\mathbb{N}}\big) = \{b(v_n)\}_{n\in\mathbb{N}}$.

Clearly g is injective, hence $f = g|_F$ is a bijective function of type $\mathcal{P}_G \to F$, for $F = g(\mathcal{P}_G)$. Note that a sequence $\{n_i\}_{i\in\mathbb{N}} \in \mathcal{N}$ is not in F if either:

1. there is a prefix $\{n_0, \ldots, n_k\}$ of $\{n_i\}_{i\in\mathbb{N}}$ such that $\big(b(n_{k-1}), b(n_k)\big) \notin E$, or
2. there is a prefix $\{n_0, \ldots, n_k, n_{k+1}\}$, where $n_{k+1} = 0$ is the first occurrence of a 0 in the prefix, and the state $b(n_k)$ has some outgoing edge, i.e., $\{v \mid \big(b(n_k), v\big) \in E\} \neq \emptyset$.

Thus, $\mathcal{N} \setminus F$ is the union of the open sets $O_{\vec{n}}$, for $\vec{n}$ of the two kinds discussed above. It follows that F is closed as desired. The fact that f and f^{-1} are continuous is easily proved. To show that $\mathcal{P}_G$ is a 0-dimensional space we just need to show the basic sets $O_{\vec{v}}$ are clopen. Note that

$$\mathcal{P}_G \setminus O_{\vec{v}} = \bigcup_{\vec{t}} O_{\vec{t}}$$

where $\vec{t}$ ranges over the set of finite paths in G which are prefix-incompatible with v, i.e., such that $\vec{t}$ is not a prefix of $\vec{v}$ and $\vec{v}$ is not a prefix of $\vec{v}$. In other words, the set of completed paths not contained in $O_{\vec{v}}$ is given by those paths that at some finite point deviate from the initial prefix $\vec{v}$. It then follows that $O_{\vec{v}}$ is closed as desired. □

In the rest of the thesis, we will just mention that a given topological space is Polish or 0-dimensional without providing a formal proof. Without exception, such a proof can always be obtained following the lines of the one discussed above.

We now define the notion of *trees* in a graph.

Definition 2.1.35 (Trees in G)**.** A *tree* in a graph $G = (V, E)$ is a non-empty set T of finite paths, such that

1. T is down-closed under $\lhd$: if $\vec{v}\in T$ and $\vec{v}\lhd\vec{t}$, i.e, if $\vec{t}$ is a prefix of $\vec{v}$, then $\vec{t}\in T$.

2. T has a root: there exists exactly one finite path $\{s\}$ of length one in T. The state s, denoted by $root(T)$, is called *the root* of the tree T.

We denote with $\mathcal{T}_G$, or just $\mathcal{T}$ if G is clear from the context, the set of trees in G. We consider the nodes $\vec{v}$ of T as labeled by the *last* function. For $\vec{s},\vec{t}\in T$, we say that $\vec{t}$ is a child of $\vec{s}$ in T if $\vec{t}=\vec{s}.\{last(\vec{t})\}$, i.e., if $\vec{t}$ is obtained by concatenating the finite path $\vec{s}$ with just one state.

We now discuss a special kind of trees over a graph.

Definition 2.1.36 (Uniquely and fully branching nodes of a tree)**.** Given a graph G and a tree T in G, a node $\vec{v}\in T$ is said to be *uniquely branching* in T if either $E(last(\vec{v}))=\emptyset$ (in which case $\vec{v}$ does not have children in T) or $\vec{v}$ has a unique child in T. Similarly, $\vec{v}$ is *fully branching* in T if, for every $t\in E(last(\vec{v}))$, it holds that $\vec{v}.\{t\}\in T$.

Definition 2.1.37. Given a graph $G=(V,E)$ and a partition (V_1,V_2) of V, we define the *trees uniquely branching in V_1 and fully branching in V_2* as the set, denoted by $\mathcal{T}_G^{(V_1,V_2)}$, of trees T in G such that for every node $\vec{v}\in T$ the following conditions hold:

1. If $last(\vec{s})\in V_1$ then $\vec{s}$ branches uniquely in T.

2. If $last(\vec{s})\in V_2$ then $\vec{s}$ branches fully in T.

Definition 2.1.38 (Topology on $\mathcal{T}_G^{(V_1,V_2)}$)**.** Given a finite tree $F\in\mathcal{T}_G$ in G, we denote with O_F the set of trees in $\mathcal{T}_G^{(V_1,V_2)}$ such that $F\subseteq T$. The topology on $\mathcal{T}_G^{(V_1,V_2)}$ is generated by the the sets O_F, for any finite tree $F\in\mathcal{T}_G$.

Proposition 2.1.39. *Let $G=(V,E)$ be a countable graph and (V_1,V_2) a partition of V. The topology on $\mathcal{T}_G^{(V_1,V_2)}$ is Polish and 0-dimensional. In particular, the sets O_F are clopen, for any finite tree $F\in\mathcal{T}_G$. Moreover the sets of the form $\{O_{\vec{v}\downarrow}\mid\vec{v}\in\mathcal{P}_G^{<\omega}\}$, with $\vec{v}\downarrow=\{\vec{t}\mid\vec{t}\lhd\vec{v}\}$, form a sub-basis for $\mathcal{T}_G^{(V_1,V_2)}$.*

Proof. The proof goes by showing that $\mathcal{T}_G^{(V_1,V_2)}$ is homeomorphic to a closed subset of $\mathbb{N}^{\mathbb{N}}$, through an encoding of finite trees as natural numbers. □

Observation 2.1.40. Fix a graph G and a tree $T \in \mathcal{T}_G$. Then T can be seen as a graph, and indeed as a tree in the sense of Graph Theory, (T, E_T) by defining $(\vec{v}, \vec{t}) \in E_T$ if and only if $\vec{t}$ is a child of $\vec{v}$ in T. We can then talk about finite, terminated, infinite and completed paths in T. Note that, if $T \in \mathcal{T}_G^{(V_1,V_2)}$, given a completed path $\{\vec{v}_i\}_{i\in I}$ in (T, E_T), the sequence $\{last(\vec{v}_i)\}_{i\in I}$ is a completed path in G. Similarly for finite, terminated and infinite paths. We often say that T contains the completed path $\{v_i\}_{i\in I} \in \mathcal{P}_G$ if $\{v_i\}_{i\in I}$ can be obtained by the pointwise application of the function *last* to a path $\{\vec{t}_i\}_{i\in I}$ in T starting from the root of T.

Another example of a Polish space is given by the set of *trees over a set*.

Definition 2.1.41. Given a countable set Σ, a *tree* T over Σ is a set $T \subseteq \Sigma^{<\mathbb{N}}$ of finite sequences of elements in Σ which is closed under taking prefixes, i.e., if $\vec{s} \in T$ and $\vec{t}$ is a prefix-sequence of $\vec{s}$ then $\vec{t} \in T$. In particular $\emptyset \in T$ if T is non-empty. We denote with $\mathcal{T}(\Sigma)$ the set of trees over Σ. We endow $\mathcal{T}(\Sigma)$ with the topology generated by the sets of the form $\{T \mid \vec{s} \in T\}$ and $\{T \mid \vec{s} \notin T\}$, for every $\vec{s} \in \Sigma^{<\mathbb{N}}$.

Proposition 2.1.42. *For every countable set Σ the space $\mathcal{T}(\Sigma)$ is Polish and 0-dimensional.*

Definition 2.1.43. Given a tree $T \in \mathcal{T}(\Sigma)$ we say that T is *well-founded* if it does not contain infinite branches. Equivalently, T is well founded when the poset $(T, \lhd^*)$ is well founded (see Definition 2.1.7), where $\lhd^*$ is the reversed prefix relation over sequences in T. We say that a tree $T \in \mathcal{T}$ is *not terminating*[1] if for every $\vec{s} \in T$ there is some $\vec{t} \in T$ such that $\vec{s} \lhd \vec{t}$ and $\vec{s} \neq \vec{t}$.

Descriptive Set Theory is the study of *definable sets* in Polish spaces [60]. In this theory, sets are classified in hierarchies, according to the complexity of their definitions. The two most studied such hierarchies are the *Borel hierarchy* and the *Projective hierarchy* which we now recall.

Definition 2.1.44. Let X be a set. A *σ-algebra* on X is a collection $\mathcal{F}$ of subsets of X containing the empty set ($\emptyset$) and closed under complements and countable unions (so also under countable intersections). Given $\mathcal{E} \subseteq 2^X$, there is a smallest σ-algebra containing $\mathcal{E}$, called the σ-algebra *generated* by $\mathcal{E}$ and denoted by $\sigma(\mathcal{E})$.

[1]The adjective *pruned* is adopted in [60].

A *measurable space* is a pair $(X,\mathcal{F})$ where X is a set and $\mathcal{F}$ is a σ-algebra on X. Given a family $(X_i,\mathcal{S}_i)$ of measurable spaces, the *product measurable space* $(\prod_i X_i,\prod_i \mathcal{S}_i)$, where $\prod_i X_i$ is the cartesian product, is generated by the sets of the form $\prod_i A_i$ where $A_i\in\mathcal{S}_i$ for all i, and $A_i=X_i$ for all but finitely many i.

Definition 2.1.45. Let $(X,\mathcal{F})$, $(Y,\mathcal{H})$ be measurable spaces. A map $f:X\to Y$ is called *measurable* if $f^{-1}(H)\in\mathcal{F}$ for every $H\in\mathcal{H}$. If $\mathcal{H}=\sigma(\mathcal{E})$ it is enough to require this for $H\in E$. A *measurable isomorphism* between X and Y is a bijection such that both f and f^{-1} are measurable. Given a set X, a σ-algebra $(Y,\mathcal{S})$ and a collection $\mathcal{F}$ of functions from X to Y, the σ-algebra on X *generated by* $\mathcal{F}$ is the smallest σ-algebra such that every function $f\in\mathcal{F}$ is measurable.

Definition 2.1.46. Let $(X,\mathcal{T})$ be a topological space. The collection of *Borel sets* of X is the σ-algebra $\mathcal{B}(X)=\sigma(\mathcal{T})$ generated by the open sets of X. We call $\big(X,\mathcal{B}(X)\big)$ the Borel space of X. Note that, whenever X is second countable, $\mathcal{B}(X)=\sigma(\mathcal{S})$ for every subbasis $\mathcal{S}$ for the topology on X. Given topological spaces $(X,\mathcal{T})$ and $(X,\mathcal{U})$, we say that $f:Y\to X$ is *Borel measurable* if it is measurable with respect to $(X,\mathcal{B}(X))$ and $(Y,\mathcal{B}(Y))$. Clearly every continuous function is Borel measurable. A function $f:X\to Y$ is a *Borel isomorphism* if it is a bijection with both f and f^{-1} Borel measurable.

Theorem 2.1.47 (Lebesgue, Hausdorff)**.** *Let X be a Polish space. The class of Borel measurable functions $f:X\to\mathbb{R}$ is the smallest class of functions from X into $\mathbb{R}$ which contains all the continuous functions and is closed under taking pointwise limits of sequences of functions (i.e., if $f_n:X\to[0,1]$ are in the class, and $f(x)=\lim_n f_n(x)$, then f is in the class too).*

Proof. See, e.g., Theorem 11.6 in [60]. □

Definition 2.1.48 (The Borel Hierarchy)**.** Let $(X,\mathcal{T})$ be a topological space. We define the classes $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$, for $1\leq\alpha<\omega_1$ a countable ordinal, as follows:

- $\mathbf{\Sigma}^0_1=\{U \mid U \textit{ is open}\}$,
- $\mathbf{\Pi}^0_\alpha=\neg\mathbf{\Sigma}^0_\alpha=\{A \mid (X\setminus A)\in\mathbf{\Sigma}^0_\alpha\}$,
- $\mathbf{\Sigma}^0_\alpha=\big\{\bigcup_n A_n \mid A_n\in\mathbf{\Pi}^0_{\beta_n}\ ,\ \beta_n<\alpha\ ,\ n\in\mathbb{N}\big\}$, for $\alpha>1$.

We also define the classes $\mathbf{\Delta}^0_\alpha$ as $\mathbf{\Sigma}^0_\alpha\cap\mathbf{\Pi}^0_\alpha$.

Proposition 2.1.49. *Let $(X, \mathcal{T})$ be a topological space. The following assertions hold:*

1. $\mathbf{\Delta}^0_1$ *and* $\mathbf{\Pi}^0_1$ *are the collections of clopen and closed sets in* X *respectively,*
2. $\mathbf{\Sigma}^0_\alpha$ *is closed under countable unions and finite intersections,*
3. $\mathbf{\Pi}^0_\alpha$ *is closed under countable intersections and finite unions,*
4. $\mathbf{\Delta}^0_\alpha$ *is closed under finite unions, finite intersections and complements.*

If in addition X *is metrizable, then following assertions hold too:*

5. $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Delta}^0_{\alpha+1}$,
6. $\mathcal{B}(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Delta}^0_\alpha$.

From the last proposition we have that the Borel sets are organized in a hierarchy of at most ω_1 levels.

Not all interesting sets in Polish spaces are Borel. In particular, continuous images of Borel sets are, in general, not Borel. We now introduce the so-called projective hierarchy which extends the Borel hierarchy by considering new kinds of definable sets.

Definition 2.1.50. Let X be a Polish space and let $A \subseteq X$. The set A is *analytic* if one of the following equivalent condition hold:

- there is a Polish space Y and a continuous map $f : Y \to X$ with $f(Y) = A$,
- there is a Polish space Y and Borel $B \subseteq X \times Y$ with $A = \{x \mid \exists y.(x, y) \in B\}$,
- there is a closed set $F \subseteq \mathcal{N}$ with $A = \{x \mid \exists y.(x, y) \in F\}$.

We denote with $\mathbf{\Sigma}^1_1$ the collection of all analytic sets.

Theorem 2.1.51. *Let* X *be a Polish space. The following assertions hold:*

1. *if* X *is uncountable, then* $\mathcal{B}(X) \subsetneq \mathbf{\Sigma}^1_1$,
2. *for every sequence* $\{A_n\}_{n\in\mathbb{N}}$ *of analytic sets,* $\bigcup_n A_n \in \mathbf{\Sigma}^1_1$ *and* $\bigcap_n A_n \in \mathbf{\Sigma}^1_1$,
3. *if* Y *is a Polish space and* $f : X \to Y$ *is Borel measurable, then* $f(A)$ *and* $f^{-1}(B)$ *are analytic, for all* $A \subseteq X$ *and* $B \subseteq Y$ *analytic.*

Proof. See, e.g., Theorem 14.2 and Proposition 14.4 in [60]. □

Definition 2.1.52. Let X be a Polish space and let $A \subseteq X$. The set A is *co-analytic* if one of the following equivalent conditions hold:

- $X \setminus A$ is analytic,
- there is a Polish space Y and Borel $B \subseteq X \times Y$ with $A = \{x \mid \forall y.(x,y) \in B\}$,
- there is an open set $G \subseteq \mathcal{N}$ with $A = \{x \mid \forall y.(x,y) \in F\}$.

We denote with $\mathbf{\Pi}^1_1$ the collection of all coanalytic sets. The *bi-analytic sets* are those that are both analytic and co-analytic. Their class is denoted by $\mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$.

The following theorem, due to Mikhail Y. Suslin, states that the bi-analytic sets are precisely the Borel sets.

Theorem 2.1.53. *Let X be a Polish space. Then $\mathcal{B}(X) = \mathbf{\Delta}^1_1$.*

Proof. See, e.g., Theorem 14.11 in [60]. □

We are now ready to introduce the Projective hierarchy.

Definition 2.1.54. For each $n \geq 1$ we define the *projective* classes $\mathbf{\Sigma}^1_n(X)$, $\mathbf{\Pi}^1_n(X)$, $\mathbf{\Delta}^1_n(X)$ of sets a Polish space X as follows: we have already defined the $\mathbf{\Sigma}^1_1$ (analytic), $\mathbf{\Pi}^1_n$ (co-analytic) and $\mathbf{\Delta}^1_n$ (bi-analytic) sets. Then we let, in general,

$$\begin{array}{lcl} \mathbf{\Sigma}^1_{n+1} & = & \Big\{ A \subseteq X \mid A = \{x \mid \exists y.(x,y) \in B\} \textit{ for some } B \in \mathbf{\Pi}^1_n(X \times \mathcal{N}) \Big\} \\ \mathbf{\Pi}^1_{n+1} & = & \{ A \subseteq X \mid (X \setminus A) \in \mathbf{\Sigma}^1_{n+1}(X) \} \\ \mathbf{\Delta}^1_n & = & \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n \end{array}$$

Clearly $\mathbf{\Sigma}^1_n \subseteq \mathbf{\Sigma}^1_{n+1}$. It then follows easily that $\mathbf{\Sigma}^1_n \cup \mathbf{\Pi}^1_n \subseteq \mathbf{\Sigma}^1_{n+1}$. The sets in the class $\mathbb{P}$, defined as

$$\mathbb{P} = \bigcup_n \mathbf{\Sigma}^1_n = \bigcup_n \mathbf{\Pi}^1_n = \bigcup_n \mathbf{\Delta}^1_n.$$

are called the *projective sets.*

Thus the class of projective sets is defined, starting from the Borel sets $\mathbf{\Delta}^1_1$ by iterating the operations of projection and complementation.

Proposition 2.1.55. *The following assertions hold:*

1. *The classes* $\boldsymbol{\Sigma}^1_n$ *are closed under continuous pre-images, countable intersections and unions, and continuous images. Moreover the collection* $\boldsymbol{\Sigma}^1_{n+1}$ *can be, alternatively, characterized as follows:*

$$\boldsymbol{\Sigma}^1_{n+1} = \{f(A) \mid A \in \boldsymbol{\Pi}^1_n(Z)\ ,\ f : Z \to X \text{ continuous, } X, Y \text{ Polish }\}$$

2. *The classes* $\boldsymbol{\Pi}^1_n$ *are closed under continuous pre-images, countable intersections and unions, and co-projections (i.e., universal quantification over Polish spaces).*

3. *The classes* $\boldsymbol{\Delta}^1_n$ *are closed under continuous pre-images, countable intersections and unions, and complements (i.e., they form a* σ*-algebra).*

4. *For every uncountable Polish space,* $\boldsymbol{\Delta}^1_n \subsetneq \boldsymbol{\Sigma}^1_n \subsetneq \boldsymbol{\Delta}^1_{n+1}$*. In particular* $\sigma(\boldsymbol{\Sigma}^1_n) \subsetneq \boldsymbol{\Delta}^1_{n+1}$*.*

The organization of definable sets in Polish spaces into hierarchies, such as the Borel and Projective hierarchies, provides a tool for comparing the complexity of sets. Another important notion for measuring the relative complexity of sets in Polish spaces is given by the notion of *Wadge reducibility*, which we now introduce.

Definition 2.1.56. Let X, Y be topological spaces. We say that a set $A \subseteq X$ is *Wadge reducible* to $B \subseteq Y$, written as $A \leq_W B$, if there is a continuous map $f : X \to Y$ such that $f^{-1}(B) = A$, i.e., $x \in A$ if and only if $f(x) \in B$. The $\leq_W$ relation is a preorder, i.e., it is transitive and reflexive. Given any collection Γ of subsets of the Baire space $\mathbb{N}^{\mathbb{N}}$, we say that A is Γ-*hard* if $B \leq_W A$ for any $B \in \Gamma$. Moreover if $A \in \Gamma$, we say that A is Γ-*complete*.

If $A \leq_W B$, then A can be considered *simpler* than B. Moreover, if A is Γ-complete, for some class Γ of sets in the Baire space, then A is as complicated as a set in Γ can be. One method for showing that a given set A is not in some class Γ, is to choose carefully some Γ-hard set B, with $B \notin \Gamma$, and prove that $B \leq_W A$.

Theorem 2.1.57. *The following assertions hold:*

1. *The set* $A \subseteq \mathcal{T}(\mathbb{N})$ *of well-founded trees over* $\mathbb{N}$ *is* $\boldsymbol{\Pi}^1_1$*-complete.*

2. *The set* $B \subseteq \mathcal{T}(\{0,1\})$ *of non-terminating trees over* $\{0,1\}$ *not containing paths with infinitely many* 1*'s, is* $\boldsymbol{\Pi}^1_1$*-complete.*

where the notion of (well-founded and non-terminating) tree over a set is specified as in Definition 2.1.41.

Proof. See, e.g., [60, §33.A]. □

We conclude this section by introducing the notion of *partially ordered topological space*, also known as Nachbin's spaces from the name of the mathematician Leopoldo Nachbin[88].

Definition 2.1.58 ([1])**.** A *partially ordered topological space*, or just *pospace*, $(X, \mathcal{T}, \leq)$ is a triple where X is a set, $\mathcal{T}$ is a topology on X and $\leq$ is a partial order on X such that $\leq \subseteq X \times X$ is closed in the product topology. Equivalently, for every $x \not\leq y$, there are open sets U and V such that $x \in U$, $y \in V$, and for all $x' \in U$ and $y' \in Y$, $x \not\leq y$. It follows that every pospace is Hausdorff.

Example 2.1.59. An example of pospace is $(\{0,1\}^{\mathbb{N}}, \mathcal{T}, \sqsubseteq)$ where $(\{0,1\}^{\mathbb{N}}, \mathcal{T})$ is the Cantor space with its standard topology and $\sqsubseteq$ is the pointwise order of $\mathbb{N}$-indexed sequences of elements in $\{0,1\}$ induced by the trivial order $0 \sqsubseteq 1$ on $\{0,1\}$. Clearly the relation $\sqsubseteq$ is closed.

The following is an important property of partially ordered topological spaces.

Theorem 2.1.60 ([88])**.** *In a pospace $(X, \mathcal{T}, \sqsubseteq)$, the upper-closure set $K\uparrow = \{y \sqsupseteq x \mid x \in K\}$ and the down-closure set $K\downarrow = \{\ y \sqsubseteq x \mid x \in K\}$ of a compact set $K \subseteq X$ are compact sets.*

Lemma 2.1.61 (Order Normality [88])**.** *In a compact pospace $(X, \mathcal{T}, \sqsubseteq)$, let A and B be disjoint compact sets, where A is upper-closed and B is a down-closed. Then there exist disjoint open sets $U \supseteq A$ and $V \supseteq B$ where again U is upper-closed and V is down-closed.*

Proof. By normality of compact Hausdorff spaces (see Proposition 2.1.26), there exist disjoint open sets $U' \supseteq A$ and $V' \supseteq B$. Set $U = X \setminus \big((X \setminus U')\downarrow\big)$ and $V = X \setminus \big((X \setminus V')\uparrow\big)$. Clearly U and V are upper-closed and down-closed respectively, and they are open by Theorem 2.1.60. □

We follow closely the presentation of [60], including many definitions and facts *verbatim*, to which we refer for an extensive treatment of the subject.

2.1.4 Measure Theory

The material contained in this section closely follow the presentations of [60] and [107], including many definitions and facts *verbatim*.

Definition 2.1.62. Let $(X, \mathcal{S})$ be a measurable space. A *measure* μ on $(X, \mathcal{S})$ is a map $\mu\colon X \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and $\mu\big(\bigcup_n A_n\big) = \sum_n \big(\mu(A_n)\big)$, for any pairwise disjoint countable family $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}$. A measure μ on $(X, \mathcal{S})$ is called σ-finite if $X = \bigcup_n X_n$, with $X_n \in \mathcal{S}$, and $\mu(X_n) < \infty$, *finite* if $\mu(X) < \infty$ and a *probability measure* if $\mu(X) = 1$. A set $A \subseteq X$ is called *μ-null* if there is $B \in \mathcal{S}$ with $A \subseteq B$ and $\mu(B) = 0$. The collection of μ-null sets, which is denoted by NULL_μ is clearly a σ-ideal on X, i.e., it contains the emptyset $\emptyset$, it is downward closed with respect to the $\subseteq$-relation and is closed under countable unions. The σ-algebra generated by $\mathcal{S} \cup \mathrm{NULL}_\mu$ is called the σ-algebra of *μ-measurable sets*, and it is denote by MEAS_μ. A μ-measurable set is always of the form $A \cup N$, for $A \in \mathcal{S}$ and $N \in \mathrm{NULL}_\mu$. The measure μ is extended to a measure $\overline{\mu}$ on MEAS_μ, called its *completion*, by $\overline{\mu}(A \cup N) = \mu(A)$.

Definition 2.1.63. An *outer measure* of a set X is a map $\mu^* : 2^X \to [0, \infty]$ such that $\mu^*(\emptyset) = 0$, $A \subseteq B$ implies $\mu^*(A) \leq \mu^*(B)$ and $\mu^*\big(\bigcup_n A_n\big) \leq \sum_n \big(\mu^*(A_n)\big)$. A set $A \subseteq X$ is *μ^*-measurable* if for every $E \subseteq X$, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$. The μ^*-measurable sets form a σ-algebra, denoted by MEAS_{μ^*}, and μ^* restricted to MEAS_{μ^*} is a measure on $(X, \mathrm{MEAS}_{\mu^*})$. Every measure μ on $(X, \mathcal{S})$ gives rise to an outer measure μ^* defined as follows: $\mu^*(A) = \bigsqcap\{\mu(B) \mid B \in \mathcal{S}\ ,\ B \supseteq A\}$. If μ is σ-finite, then $\mathrm{MEAS}_{\mu^*} = \mathrm{MEAS}_\mu$ and the completion of μ and μ^* agree on MEAS_μ.

Definition 2.1.64. For measurable spaces $(X, \mathcal{R})$, $(Y, \mathcal{S})$ and measure μ on $(X, \mathcal{R})$, we say that a map $f : X \to Y$ is *μ-measurable* if the inverse image of a measurable set in Y is μ-measurable. Given a μ-measurable bounded function $f : X \to [0, 1]$, we write $\int_X f \,\mathrm{d}\mu$ for the its Lebesgue integral.

Definition 2.1.65. Let $(X, \mathcal{T})$ be a Polish space. A probability *(Borel) measure* on X is a probability measure μ on $\big(X, \mathcal{B}(X)\big)$. We denote with $\mathcal{M}_1(X)$ the collection of all probability measures on X. The set $\mathcal{M}_1(X)$ is endowed with the smallest topology such that every function $\lambda\mu.\big(\int_X f \,\mathrm{d}\mu\big)$, of type $\mathcal{M}_1(X) \to [0, 1]$, is continuous for $f : X \to [0, 1]$ continuous. This is called the *weak topology* on $\mathcal{M}_1(X)$ [94]. We define the map $\delta : X \to \mathcal{P}(X)$ as $\delta(x)(A) = 1$ if $x \in A$ and 0 otherwise, for all $A\in\mathcal{B}(X)$. We often denote with δ_x the probability measure $\delta(x)$, for $x\in X$, and call it the *Dirac measure with mass at x*. If $(Y,\mathcal{S})$ is a measurable space and $f:X\to Y$ is Borel measurable, we define the *image measure* f_μ as $f_\mu(B)=\mu\big(f^{-1}(B)\big)$, for all $B\in\mathcal{S}$. Given a Polish space Y and a Borel function $f:X\to Y$, we denote with $\mathcal{M}_1(f)$ the function of type $\mathcal{M}_1(X)\to\mathcal{M}_1(Y)$ defined as $\mathcal{M}_1\big(f\big)(\mu)=f_\mu$.

Theorem 2.1.66. *Let $(X,\mathcal{T})$ be a Polish space. The following assertions hold:*

1. *$\mathcal{M}_1(X)$ is Polish.*

2. *If X is compact so is $\mathcal{M}_1(X)$.*

3. *Every $\mu\in\mathcal{M}_1(X)$ is* regular *and* inner regular *(or* tight*), i.e.,*

$$\begin{aligned} \textit{regularity:} \quad & \mu(A) = \bigsqcup\{\mu(F)\mid F\subseteq A\ ,\ F\ \textit{closed}\} \\ & = \bigsqcap\{\mu(U)\mid U\subseteq A\ ,\ U\ \textit{open}\} \\ \textit{inner regularity:} \quad & \mu(A) = \bigsqcup\{\mu(K)\mid K\subseteq A\ ,\ K\ \textit{compact}\}. \end{aligned}$$

   for every μ-measurable set A.

4. *The sets $U_{O,\lambda}=\{\mu\mid\mu(O)>\lambda\}$, for $O\in\mathcal{T}$ and $\lambda\in[0,1]$ rational, form a subbasis for $\mathcal{M}_1(X)$.*

5. *If X is 0-dimensional and $\mathcal{B}$ is a countable basis of clopen sets such that, for every $A,B\in\mathcal{B}$, the set $A\setminus B$ is expressible as a disjoint union of clopen sets in $\mathcal{B}$, then the sets $U_{B,\lambda}=\{\mu\mid\mu(B)>\lambda\}$, with $B\in\mathcal{B}$ and $\lambda\in[0,1]$ a rational number, form a subbasis for the weak topology on $\mathcal{M}_1(X)$.*

6. *The Borel σ-algebra $\mathcal{B}\big(\mathcal{M}_1(X)\big)$ is generated by the maps of the form $\lambda\mu.\mu(A)$, for $A\in\mathcal{B}(X)$ and also generated by the maps of the form $\lambda\mu.\big(\int_X f\,\mathrm{d}\mu\big)$, for $f:X\to[0,1]$ Borel measurable.*

7. *The function $\delta:X\to\mathcal{M}_1(X)$ is continuous.*

8. *For every Polish space Y and continuous (Borel measurable) map $f:X\to Y$ the function $\mathcal{M}_1(f):\mathcal{M}_1(X)\to\mathcal{M}_1(Y)$ is continuous (Borel measurable).*

9. *the map $\mathtt{m}:\mathcal{M}_1\big(\mathcal{M}_1(X)\big)\to\mathcal{M}_1(X)$, specified on every Borel $A\in\mathcal{B}(X)$ as*

$$\mathtt{m}\big(\mu'\big)(A)=\int_{\mathcal{M}_1(X)}\rho_A\,\mathrm{d}\mu'$$

with $\rho_A(\mu)=\mu(A)$ for all $\mu\in\mathcal{M}_1(X)$, is well defined (i.e., for every Borel set A, ρ_A is μ'-measurable for every $\mu'\in\mathcal{M}_1\big(\mathcal{M}_1(X)\big)$) and continuous.

10. *The triple $\big(\mathcal{M}_1,\delta,\mathtt{m}\big)$ is a monad on the category **Pol** of Polish spaces with continuous maps. For Polish spaces X and Y, we shall consider the* bind *operation associated with the monad, i.e., the function* $\mathtt{b}$ *of type* $\mathtt{b}\colon\Big(\mathcal{M}_1(X)\times\big(X\to\mathcal{M}_1(Y)\big)\Big)\to\mathcal{M}_1(Y)$ *defined as follows:*

$$\mathtt{b}(\mu,f)=\mathtt{m}\big(\mathcal{M}_1(f)(\mu)\big).$$

Proof. See, e.g., Theorem 17.22 in [60] for a proof of point 1; Theorem 17.23 in [60] for a proof of point 2; Theorem 17.10 and Theorem 17.11 in [60] for a proof of point 3; Lemma A.1.7 and Lemma A.1.8 in the Appendix for a proof of points 4 and 5; Theorem 17.24 in [60] for a proof of point 6. The results of points 7 and 8 follow from Theorem 17.26 in [60]: in particular, see Exercise 17.27, 17.28. For a proof of point 9 and 10 we refer to Theorem 1 in [44]. □

In many circumstances a probability measure μ on a measurable space $(X,\mathcal{S})$ is uniquely determined by the probability assignments on a restricted collection of elements in $\mathcal{S}$. This is due to the following important theorem.

Definition 2.1.67. For a set X, a *boolean algebra of sets* on X is a collection $\mathcal{A}$ of subsets of X closed under complements and *finite* intersections such that $\emptyset\in\mathcal{A}$. A function $\mu\colon\mathcal{A}\to[0,\infty]$ is a *pre-measure* on $(X,\mathcal{A})$ if:

1. $\mu(\emptyset)=0$,
2. (Pre-countable additivity) if $A_1,A_2,\dots$ are disjoint sets in $\mathcal{A}$ and $\bigcup_n A_n\in\mathcal{A}$, then $\mu\big(\bigcup_n A_n\big)=\sum_n \mu(A_n)$.

A pre-measure μ on $(X,\mathcal{A})$ is *σ-finite* if the following condition hold:

1. X can be covered by countably a countable collection $\{A_n\}$ of sets in $\mathcal{A}$, and $\mu(A_n)<\infty$ for all $n\in\mathbb{N}$.

Lastly, a pre-measure μ on $(X,\mathcal{A})$ is a *probability pre-measure* if $\mu(X)=1$.

Theorem 2.1.68 (Carathéodory's Extension Theorem)**.** *Let $(X,\mathcal{A})$ be a boolean algebra on X and μ a pre-measure on $(X,\mathcal{A})$. Then there exists a measure ν on the measurable space $(X,\sigma(\mathcal{A}))$, where $\sigma(A)$ is the σ-algebra generated by $\mathcal{A}$,* *extending μ, i.e., such that $\nu(A) = \mu(A)$ for all $A \in \mathcal{A}$. Moreover if μ is a probability (σ-finite) pre-measure then ν is unique and is a probability (σ-finite) pre-measure.*

Proof. See, e.g., Theorem 1.7.3 in [107]. □

Carathéodory's extension theorem is very useful when defining (Borel) probability measures on spaces $(X, \mathcal{T})$ having a boolean algebra $\mathcal{A} \in 2^X$ generating either the topology $\mathcal{T}$ (and therefore also the Borel σ-algebra), or just the Borel σ-algebra $\mathcal{B}(X)$. An important example of the first kind is given by 0-dimensional Polish spaces such as the Cantor space $\{0,1\}^{\mathbb{N}}$ or the Baire space $\mathbb{N}^{\mathbb{N}}$ having a basis of clopen sets constituting a boolean algebra. An example of the second kind is given by the space of reals $\mathbb{R}$, or the unit interval $[0,1]$, where the boolean algebra generating the Borel sets is given by finite unions of semiopen intervals, i.e., sets of the form $(a, b]$ and $[b, a)$. On such spaces, in order to uniquely specify a (Borel) probability measure μ it is enough to provide a probability pre-measure ν on $\mathcal{A}$. The probability measure μ can then be extended to its completion $\overline{\mu}$ on all μ-measurable sets.

The following definition makes use of Carathéodory's extension theorem.

Definition 2.1.69 (Product measure)**.** Given a pair $(X_1, \mathcal{S}_1)$ and $(X_2, \mathcal{S}_1)$ of measurable spaces and probability measures μ_1 and μ_2 on X_1 and X_2 respectively, we define the *product (probability) measure* $\mu_1 \times \mu_2$ on the product space $X_1 \times X_2$ defined as:

$$\mu(B) = \textstyle\sum_n \mu_1(A_1^n) \cdot \mu_2(A_2^n)$$

for every finite union $B = \bigcup_n B_n$ of disjoint sets of the form $B_n = A_1^n \times A_2^n$, with $A_1^n \in \mathcal{S}_1$ and $A_2^n \in \mathcal{S}_2$. This kind of sets are closed under complement and intersection, i.e., they form a boolean algebra of sets on $X_1 \times X_2$ which generates the product σ-algebra on $X_1 \times X_2$. By Carathéodory's extension theorem, there is a unique probability measure μ on $X_1 \times X_2$ compatible with this specification. The definition extends as expected to the product measure $\prod_{0 \leq k \leq n} \mu_k$ of any finite collection $\{\mu_k\}_{0 \leq k \leq n}$ of probability measures on the collection $\{(X_k, \mathcal{S}_k)\}_{0 \leq k \leq n}$ of measurable spaces.

The following theorem will be useful for specifying probability measures on product spaces.

Theorem 2.1.70 (Kolmogorov's Extension Theorem). *For a given index-set I, let $(X_i, \mathcal{T}_i)$ be a Polish space and μ_i a probability measure on $\mathcal{B}(X_i)$, for every $i \in I$. There exists a unique probability measure on the product space $\prod_i X_i$, denoted by $\prod_i \mu_i$, specified as $\big(\prod_i \mu_i\big)(A) = \mu_k(A)$ on every set A of the form $B \times \prod_{j \in J \setminus k} X_j$, with $B \in \mathcal{B}(X_k)$, for some $k \in J$.*

Proof. See, e.g., Theorem 2.4.3 in [107, §2.4]. □

We now turn our attention to the notions of *universally measurable* sets and functions.

Definition 2.1.71. Given a Polish space $(X, \mathcal{T})$, we denote with $\mathrm{UM}(X)$ the σ-algebra consisting of those sets that are μ-measurable for every (Borel) probability measure μ on X, i.e., the σ-algebra defined as follows:

$$\mathrm{UM}(X) = \bigcap_{\mu \in \mathcal{M}_1(X)} \mathrm{MEAS}_\mu.$$

A set $A \in \mathrm{UM}(X)$ is called a *universally measurable subset of X*. This is a good definition as σ-algebras are closed under arbitrary intersections. Given a measurable space $(Y, \mathcal{S})$ and a function $f : X \to Y$ we say that f is *universally measurable* if $f^{-1}(A) \in \mathrm{UM}(X)$ for all $A \in \mathcal{S}$.

Clearly every Borel set is universally measurable. Moreover, as stated in the following theorem every (co)analytic set is universally measurable.

Theorem 2.1.72. *For each Polish space $(X, \mathcal{T})$ the following assertions hold:*

1. *$\sigma\big(\mathbf{\Sigma}^1_1(X)\big) \subseteq \mathrm{UM}(X) \subseteq 2^X$ and, if X is uncountable, the inclusions are strict.*
2. *If $(X, \mathcal{T}) = (\prod_i X_i, \prod_i \mathcal{T}_i)$, and $\{E_i\}_{i \in I}$, with $E \subseteq X_i$, is a collection of universally measurable sets (i.e., $E_i \in \mathrm{UM}(X_i)$), then $\prod_i E_i \in \mathrm{UM}(X)$,*

Proof. For a proof of the strict inequality $\sigma\big(\mathbf{\Sigma}^1_1(X)\big) \subsetneq \mathrm{UM}(X)$ see, e.g., Proposition B.9 in [8]. The inequality $\mathrm{UM}(X) \subsetneq 2^X$ follows from the existence of a set $A \subseteq \mathbb{R}$ which is not μ-measurable, where μ is the standard Lebesgue measure on $\mathbb{R}$, see e.g. [59]. For a proof of the second assertion see, e.g., [39, 434X(e)]. □

As we shall see in the next section, it is not possible, in ZFC alone, to show that universally measurable sets extend any further up the projective hierarchy. Indeed already the inclusion $\mathbf{\Delta}^1_2(X) \subseteq \mathrm{UM}(X)$ is not decidable in ZFC set theory.

The concepts of universally measurable set and universally measurable function are very useful. Indeed, given a universally measurable set $A \subseteq X$, the probability $\mu(X)(A)$ and the integral $\int_X f \,\mathrm{d}\mu$ are well defined for every $\mu \in \mathcal{M}_1(X)$ and universally measurable function $f : X \to [0,1]$. Moreover universally measurable functions satisfy several important closure properties which we now discuss.

Theorem 2.1.73. *Let X and Y be Polish spaces, and $f : X \to Y$ a universally measurable map, i.e., such that the inverse image of Borel sets is universally measurable. For every universally measurable $W \subseteq Y$, the set $f^{-1}(W)$ is universally measurable.*

Proof. See, e.g., Corollary 7.44.1 in [8]. □

As an immediate consequence of the previous theorem we have the following fundamental property of universally measurable maps.

Corollary 2.1.74. *Given Polish spaces X, Y and Z and universally measurable maps $f : X \to Y$ and $g : Y \to Z$ the composition $f \circ g : X \to Z$ is universally measurable.*

Note how, in contrast, the composition $f \circ g$ of two Lebesgue (but not universally) measurable functions $f, g : \mathbb{R} \to \mathbb{R}$ need not be Lebesgue measurable. We shall also make use of the following closure properties.

Theorem 2.1.75. *Let X be a Polish space and $f : X \to [0,1]$ a universally measurable (respectively Borel measurable) function. Then the map $\tilde{f} : \mathcal{M}_1(X) \to [0,1]$, defined as $\tilde{f}(\mu) = \int_X f \,\mathrm{d}\mu$ is universally (respectively Borel) measurable.*

Proof. By definition of the weak topology on $\mathcal{M}_1(X)$, the map $\tilde{f}$ is continuous whenever f is continuous. The desired result for Borel measurable functions then immediately follows from the fact that, for every Polish space Y, the set of Borel measurable functions of type $Y \to [0,1]$ is the smallest one containing the continuous functions and closed under taking bounded pointwise limits of sequences (see Theorem 2.1.47). For a proof of the analogous result for universally measurable functions see, e.g., Proposition 7.46 in [8]. □

Lemma 2.1.76. *Let X, Y and Z be Polish spaces, and $f : X \times Y \to Z$ be a universally measurable function, where $X \times Y$ is endowed with the product topology. The function $f_x : Y \to Z$, defined as $f_x(y) = f(x,y)$, is universally measurable for every $x \in X$.*

Proof. The result follows by application of Lemma 2.1.73. ☐

Lemma 2.1.77. *Let $(X_n, \mathcal{T}_n)$ be a countable collection of Polish spaces and $f_n : X_n \to Y_n$ a universally measurable (respectively Borel or continuous) function, for every $n \in \mathbb{N}$. Then the map $f : \prod_n X_n \to \prod_n Y_n$, defined as $f(\{x_n\}_{n\in\mathbb{N}}) = \{f_n(x_n)\}_{n\in\mathbb{N}}$, is universally measurable (respectively Borel or continuous).*

Proof. We just need to show that $f^{-1}(U)$ is universally measurable for every set $U \subseteq \prod_n Y_n$ of the form $U = \prod_n U_n$, with $U_n \in \mathcal{T}_n$ and $U_n = Y_n$ for all but finitely many n. The sets U of this shape form a sub-basis for the product topology on $\prod_n Y_n$. The set $f^{-1}(U)$ is of the form $V = \prod_n V_n$ with $V_n = f^{-1}(U_n)$. Since f is universally measurable by hypothesis, V_n is universally measurable for every $n \in \mathbb{N}$. The proof is concluded by application of Theorem 2.1.72(3). The analogous result for f continuous or Borel measurable is proved in the same way. ☐

We conclude this section by stating two important theorems in measure theory: the *monotone convergence theorem* and the *Fubini theorem*.

Theorem 2.1.78 (Monotone Convergence Theorem)**.** *Let $(X, \mathcal{S})$ be a measurable space and $\{f_n\}_{n\in\mathbb{N}}$ a sequence of $\mathcal{S}$-measurable functions $f : X \to [0,1]$ such that $f_n \sqsubseteq f_{n+1}$, where $\sqsubseteq$ is the pointwise order, and $f = \bigsqcup_n f_n$ the associated least upper bound in the complete lattice of functions $\big((X \to [0,1]), \sqsubseteq\big)$. Then the following assertions hold:*

1. *the function f is $\mathcal{S}$-measurable, and*
2. $\displaystyle\int_X f \,\mathrm{d}\mu = \bigsqcup_n \Big(\int_X f_n \,\mathrm{d}\mu\Big)$,

for every probability measure μ on $(X, \mathcal{S})$.

Theorem 2.1.79 (Fubini's Theorem)**.** *Let $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ be measurable spaces, and $f : X \times Y \to [0,1]$ a measurable map, where $X \times Y$ is endowed with the product σ-algebra. If the functions $f_x : Y \to [0,1]$ and $f_y : X \to [0,1]$, defined as $f_x(y) = f(x,y) = f_y(x)$ for all $x \in X$ and $y \in Y$, are $\mathcal{R}$-measurable and $\mathcal{S}$-measurable respectively, the following assertions hold:*

1. *the maps $x \mapsto \Big(\int_Y f_x \,\mathrm{d}\mu_2\Big)$ and $y \mapsto \Big(\int_X f_y \,\mathrm{d}\mu_1\Big)$ are $\mathcal{R}$-measurable and $\mathcal{S}$-measurable respectively,*
2. $\displaystyle\int_{X\times Y} f \,\mathrm{d}(\mu_1 \times \mu_2) = \int_{x\in X} \Big(\int_Y f_x \,\mathrm{d}\mu_2\Big) \mathrm{d}\mu_1 = \int_{y\in Y} \Big(\int_X f_y \,\mathrm{d}\mu_1\Big) \mathrm{d}\mu_2$

for every probability measure μ_1 on $(X, \mathcal{R})$ and μ_2 on $(Y, \mathcal{S})$.

Both theorems can be further generalized to deal with σ-finite measures and $\mathbb{R}$-valued functions [107].

2.1.5 Set Theory

As we briefly mentioned in the previous sections, there are mathematical statements, such as "$\mathbf{\Delta}^1_2 \subseteq \mathrm{UM}(X)$ for all uncountable Polish X", which neither admit a proof nor a disproof in ZFC.

Definition 2.1.80. Let us denote with $\mathbf{\Delta}^1_2$-UM the assertion: "$\mathbf{\Delta}^1_2 \subseteq \mathrm{UM}(X)$ for all uncountable Polish X".

Theorem 2.1.81. *If* ZFC *is consistent, the following assertions hold:*

1. ZFC $\nvdash \mathbf{\Delta}^1_2$-UM,
2. ZFC $\nvdash \neg\big(\mathbf{\Delta}^1_2\text{-UM}\big)$.

Proof. The result of the first assertion is due to Kurt Gödel, see, e.g., Corollary 25.28 of [59]. A proof of the second assertion can be found in, e.g., [75]. □

These kinds of statements are called *undecidable* or *independent* of ZFC. One of the best known such statements is the so-called *Continuum Hypothesis* (CH), originally formulated by Georg Cantor in 1877, which can be formulated as follows: there is no set whose cardinality is strictly between that of the integers and that of the real numbers, or more concisely in ZFC [59], as $\aleph_1 = 2^{\aleph_0}$. The Continuum Hypothesis quickly became one of the most challenging problems in modern mathematics. It was chosen as the first of the famous 23 problems formulated by David Hilbert at the beginning of the 20th century [69]. The following theorem is due to Kurt Gödel (1940) and Paul Cohen (1963):

Theorem 2.1.82. *If* ZFC *is consistent, the following assertions hold:*

1. *(Gödel)* ZFC $\nvdash \neg$CH,
2. *(Cohen)* ZFC $\nvdash$ CH.

Proof. We refer to [59] for detailed proofs of both theorems. □

Since both CH and its negation ¬CH are not provable in ZFC, several philosophical positions are legitimate: simply dispose of the problem, search for new convincing axioms to be added to ZFC in order to decide CH or even debate on the *truth* of CH or ¬CH by means of some *intuitive*, yet informal, arguments (see e.g., [37] for Freiling's arguments against CH). We refer to [69] for an overview on these positions.

In the field of *set theory*, a lot of attention is dedicated to the investigation of the consequences induced by extensions of ZFC with other axioms, such as CH. This process, which shed light on the deep implications of the axioms, provides a lot of useful tools which can sometimes be used to simplify proofs. An important example is the following. Suppose we want to prove a complicated theorem T, and that we realize that T follows simply by one of the consequences of CH; then we can conclude that ZFC + CH $\vdash T$. We would then say that T *consistently holds in* ZFC. Note that this implies that $\neg T$ is *not* provable in ZFC. Of course it also leaves the question of wether or not T can be proved in ZFC alone, which could be a much harder question to settle.

As we shall see later in this thesis, the proof of one of our main theorems is a *consistent* result, in the sense specified above, and in particular formalized in ZFC extended with the set-theoretic axiom $\mathrm{MA}_{\aleph_1}$, introduced by D. A. Martin and R. M. Solovay in 1970 [75], which we now discuss.

We start with a few preliminary order theoretic definitions.

Definition 2.1.83. Let $(X, \leq)$ be a partially ordered set. We say that $x_1, x_2 \in X$ are *compatible* if there is some $y \in X$ with $x_1 \geq y$ and $x_2 \geq y$; x_1 and x_2 are *incompatible* otherwise. A nonempty subset $D \subseteq X$ is *dense down-closed* if

1. $D = D\downarrow$, i.e., if $x \in D$, $y \in X$ and $y \leq x$ then $y \in D$, and

2. for every $x \in X$ there is some $y \in D$ such that $x \geq y$

or, equivalently, if D is a dense and open set in the topology on X generated by the basic sets $\{y \mid y \leq x\}$, for $x \in X$. A subset $F \subseteq X$ is a *filter* on X if:

1. $F = F\uparrow$, i.e., if $x \in F$, $y \in X$ and $y \geq x$ then $y \in F$, and

2. for every $x, y \in F$ there exists an element $z \in F$ such that $x \geq z$ and $y \geq z$, i.e., all elements of F are compatible.

A filter F on X is a $\mathcal{D}$*-generic filter*, for some collection $\mathcal{D}$ of dense down-closed subsets of X, if $F \cap D \neq \emptyset$, for all $D \in \mathcal{D}$. A *forcing antichain*[2] in X is a set $A \subseteq X$ such that for all $x, y \in A$, x and y are incompatible. We say that $(X, \leq)$ satisfies the *countable chain condition* (c.c.c.) if every forcing antichain in X is countable.

We are now ready to define the assertion MA_κ, for any infinite cardinal κ.

Definition 2.1.84. For any infinite cardinal κ, we denote with MA_κ the following assertion: "If $(X, \leq)$ is a partially ordered set satisfying the c.c.c. and $\mathcal{D}$ is a collection of cardinality $\leq \kappa$ of dense down-closed subsets of X, then there is a $\mathcal{D}$-generic filter on X". We refer to MA_κ as *Martin's Axiom at* κ. We also define the assertion MA as "for every infinite cardinal κ, if $\kappa < 2^{\aleph_0}$ then MA_κ holds", which we refer to as *Martin's Axiom.*

The following theorem provides important information about the consequences of $\mathrm{MA}_\aleph$.

Theorem 2.1.85. *The following assertions hold:*

1. $\mathrm{ZFC} \vdash \mathrm{MA}_{\aleph_0}$*,*
2. $\mathrm{ZFC} \vdash \mathrm{MA}_\aleph$ *implies* $\aleph < 2^{\aleph_0}$*,*
3. $\mathrm{ZFC} \vdash \neg\big(\mathrm{MA}_{2^{\aleph_0}}\big)$*,*
4. $\mathrm{ZFC} \vdash \mathrm{CH}$ *implies* MA*,*
5. $\mathrm{ZFC} \vdash \mathrm{MA}_{\aleph_1}$ *implies* $\neg\mathrm{CH}$.

Moreover $\mathrm{ZFC} + \mathrm{MA}_{\aleph_1}$ *is consistent relative to* ZFC*, i.e., if* ZFC *is consistent then* $\mathrm{ZFC} + \mathrm{MA}_{\aleph_1}$ *is consistent.*

Proof. For a proof of the relative consistency $\mathrm{ZFC} + \mathrm{MA}_{\aleph_1}$ and of Assertion 1, which is known as Rasiowa-Sikorski Lemma, see e.g., [59]. For a proof of Assertion 2 see, e.g., Theorem 1 in the original paper by Martin and Solovay [75]. The last three assertions are immediate consequences of the first two. □

It is clear, from the results of Theorem 2.1.85, than Martin's Axiom asserts properties of cardinals strictly between $\aleph_0$ and the continuum $2^{\aleph_0}$. When the

[2] The adjective *forcing* is use to avoid confusion with the standard notion of antichain in posets as specified in Definition 2.1.6. Forcing antichains are also known as *strong* antichains.

Continuum Hypothesis holds, there are no such cardinals, and MA holds by the first point of Theorem 2.1.85. Therefore Martin's Axiom becomes interesting when $\neg$CH is assumed. In this setting, MA can informally be considered to say that all infinite cardinals less than the cardinality of the continuum, behave roughly like $\aleph_0$ [38]. The main consequence of $\neg\text{CH}+\text{MA}$, or even of $\text{MA}_{\aleph_1}$, that will be used in this thesis has to do with measure theory and in particular on closure properties, related to the limit ordinal ω_1, of the σ-algebra $\text{UM}(X)$ on a Polish space X.

Definition 2.1.86. Let $(X, \mathcal{T})$ be a Polish space, and $\mu \in \mathcal{M}_1(X)$ a probability measure on X. We say that the σ-algebra MEAS_μ is ω_1-*complete* if the following condition holds:

- For every ω_1-indexed collection $\{A_\beta\}_{\beta<\omega_1}$ of μ-measurable sets $A_\beta \in \text{MEAS}_\mu$, $\bigcup\limits_{\beta<\omega_1} A_\beta \in \text{MEAS}_\mu$.

Similarly, we say that the σ-ideal NULL_μ is ω_1-*additive* if the following condition holds:

- For every ω_1-indexed collection $\{A_\beta\}_{\beta<\omega_1}$ of μ-null sets $A_\beta \in \text{NULL}_\mu$, $\bigcup\limits_{\beta<\omega_1} A_\beta \in \text{NULL}_\mu$.

Lastly, if MEAS_μ is ω_1-complete, we say that μ is ω_1-*continuous* if the following condition holds:

- For every ω_1-indexed increasing chain $\{A_\beta\}_{\beta<\omega_1}$ of sets $A_\beta \in \text{MEAS}_\mu$, $\mu\Big(\bigcup\limits_{\beta<\omega_1} A_\beta\Big) = \bigsqcup\limits_{\beta<\omega_1} \Big(\mu(\bigcup\limits_{\alpha<\beta} A_\alpha)\Big)$.

The notion ω_1-*continuity* is often formulated as ω_1-*additivity*, which asserts that for every ω_1-indexed collection $\{A_\beta\}_{\beta<\omega_1}$ of disjoint sets in MEAS_μ, the equality $\mu(\bigcup\limits_{\beta<\omega_1} A_\beta) = \sum\limits_{\beta<\omega_1} \mu(A_\beta)$ holds.

It is clear that if CH holds then NULL_μ is, in general, not ω_1-additive: take for example the unit $[0,1]$ interval, having cardinality $2^{\aleph_0}$, with the standard Lebesgue measure μ; then $\mu(\{r\})=0$, for all $r \in [0,1]$, but $\mu([0,1])=1$. Similarly, the σ-algebra MEAS_μ is not ω_1-complete if CH holds.

Theorem 2.1.87 ($\text{MA}_{\aleph_1}$). *Let $(X, \mathcal{T})$ be a Polish space. For every probability measure $\mu \in \mathcal{M}_1(X)$, the σ-ideal NULL_μ is ω_1-additive.*

Proof. See Theorem A.2.2 in the Appendix. □

We now list three useful consequences of Theorem 2.1.87.

Proposition 2.1.88 ($\mathrm{MA}_{\aleph_1}$). *The following assertions hold for every Polish space $(X, \mathcal{T})$:*

1. *for every $\mu \in \mathcal{M}_1(X)$, the σ-algebra MEAS_μ is ω_1-complete and the probability measure μ is ω_1-continuous,*
2. *$\mathbf{\Sigma}^1_2$-UM holds, i.e., every $A \in \mathbf{\Sigma}^1_2$ is universally measurable and hence, in particular, every $A \in \mathbf{\Delta}^1_2$ is universally measurable.*
3. *the σ-algebra $\mathrm{UM}(X)$ is ω_1-complete.*

Proof. We first prove that the σ-algebra MEAS_μ is ω_1-complete. Let $\{A_\alpha\}_{\alpha<\omega_1}$ a collection of μ-measurable sets, i.e., $A_\alpha \in \mathrm{MEAS}_\mu$. Recall that $A_\alpha = B_\alpha \cup N_\alpha$, with B_α Borel and $N_\alpha \in \mathrm{NULL}_\alpha$. Since $\bigcup_{\alpha<\omega_1} A_\alpha = \bigcup_{\alpha<\omega_1} B_\alpha \cup \bigcup_{\alpha<\omega_1} N_\alpha$, by Theorem 2.1.87 we just need to show that $\bigcup_{\alpha<\omega_1} B_\alpha$ is μ-measurable. Without loss of generality we can assume that the sets $\{B_\alpha\}_{\alpha<\omega_1}$ are pairwise disjoint. To do so, define $B'_{\alpha+1} = B_{\alpha+1} \setminus B'_\alpha$ for every successor ordinal $\alpha + 1$, and $B'_\alpha = B_\alpha \setminus \bigcup_{\beta<\alpha} B'_\beta$, for every limit ordinal α. Clearly every B'_α is Borel, since every $\alpha < \omega_1$ is countable and $\bigcup B_{\alpha<\omega} = \bigcup B'_{\alpha<\omega}$. Now observe that only countably many B'_α, say $\{B'_{\alpha_n}\}_{n\in\mathbb{N}}$ can have measure greater than 0. The desired result then follows trivially from Theorem 2.1.87.

The fact that μ is ω_1-continuous is proved with a similar technique. For every collection $\{A_\alpha\}_{\alpha<\omega_1}$ of disjoint μ-measurable sets, only countably many of them, say $\{A_{\alpha_n}\}_{n\in\mathbb{N}}$ can have measure greater than 0. Again, the desired result the trivially follows from Theorem 2.1.87.

The fact that $\mathbf{\Delta}^1_2$-UM holds, follows from the following theorem of Sierpiński: every $\mathbf{\Sigma}^1_2$ set A can be expressed as the ω_1-union of a collection $\{A_\alpha\}_{\alpha<\omega_1}$ of Borel sets. See, e.g., Theorem 25.19 in [59].

Lastly, the ω_1-completeness of $\mathrm{UM}(X)$ follows immediately from the fact that MEAS_μ is ω_1-complete for every $\mu \in \mathcal{M}_1(X)$. □

The assertions of Proposition 2.1.88 are the only consequences of $\mathrm{MA}_{\aleph_1}$ we use in the thesis. As a mater of fact, rather than considering Martin's Axiom at $\aleph_1$, these consequences follow from the weaker assumption that NULL_μ is ω_1-additive, for every Borel probability measure μ on a Polish space. This is assumption, often denoted by *add*(NULL) $\geq \aleph_2$, is studied in the theory of *cardinal characteristics of the Continuum* (see, e.g., [15]).

2.2 The lattice $[0,1]$

In this section we consider in some detail the closed set $[0,1]$ of reals, with its standard order, and introduce a few operations on it which will be used extensively in this thesis. As a starting point, we state the following proposition:

Proposition 2.2.1. *The map* $f : [0,1] \to \mathcal{M}_1(\{0,1\})$ *defined as* $f(\lambda) = \mu_\lambda$ *is a homeomorphism, where* $\{0,1\}$ *is endowed with the product topology and* μ_λ *is the probability measure assigning probability* λ *to the set* $\{1\}$*, and* $1-\lambda$ *to the set* $\{0\}$.

Therefore we can look at the real numbers in $[0,1]$ as probability measures over the two element set $\{0,1\}$. An important property of the closed set $[0,1]$ with its standard order, and the associated operations of meet and join, is the following:

Proposition 2.2.2. *The algebraic structure* $([0,1], \sqcup, \sqcap, 0, 1)$ *is a distributive complete lattice.*

The first operation we define on the lattice $[0,1]$ is the involutive map $x \mapsto 1-x$, which we often refer to as the *negation* operator on $[0,1]$ and simply denote with $(1-)$. It is immediate to verify that $(1-)$ is a negation (see Definition 2.1.16) on the bounded distributive lattice $\big([0,1], \sqcup, \sqcap, 0, 1\big)$. Therefore the algebraic structure $\big([0,1], \sqcup, \sqcap, (1-), 0, 1\big)$ is a De Morgan algebra (see Definition 2.1.18).

Beside the operations of meet, join and negation on $[0,1]$ we shall make frequent use of other operators which, when thinking at $[0,1]$ as a set of quantitative truth values, have some logical character. As we will discuss at the end of this section, the mathematical field *Fuzzy logic* studies appropriate generalizations of the boolean logical connectives to the real interval $[0,1]$.

Definition 2.2.3. We define the operations $\odot$, $\oplus$, $\ominus$ and $+_\lambda$, for $\lambda \in (0,1)$, all of type $([0,1] \times [0,1]) \to [0,1]$, as follows:

$$
\begin{array}{llll}
x \odot y & \stackrel{\text{def}}{=} & x + y - (x \cdot y) & \textit{binary coproduct} \text{ on } [0,1] \\
x \oplus y & \stackrel{\text{def}}{=} & \min\{1, x+y\} & \textit{truncated sum} \text{ on } [0,1] \\
x \ominus y & \stackrel{\text{def}}{=} & \max\{0, x+y-1\} & \textit{truncated co-sum} \text{ on } [0,1] \\
x +_\lambda y & \stackrel{\text{def}}{=} & (\lambda \cdot x) + \big((1-\lambda) \cdot y\big) & \textit{binary weighted sum} \text{ on } [0,1]
\end{array}
$$

where $\cdot$, $+$ and $-$ are the standard operations of *product* (on $[0,1]$), sum and subtraction (on $\mathbb{R}$). The operations of $\cdot$, $\odot$, $\oplus$ and $\ominus$ are clearly commutative and associative, and thus extend trivially to n-ary operations of type $[0,1]^n \to [0,1]$. In particular we denote with $\prod$ and $\coprod$ the operations of product and coproduct having, in general, more than two arguments. The operations of product and coproduct extend uniquely to infinitary operations of type $[0,1]^I \to [0,1]$, for any index set I, as follows:

$$
\prod_{i\in I} x_i \stackrel{\text{def}}{=} \bigsqcap_{J\subseteq_{fin} I} \prod_{j\in J} x_j \qquad \text{and} \qquad \coprod_{i\in I} x_i \stackrel{\text{def}}{=} \bigsqcup_{J\subseteq_{fin} I} \coprod_{j\in J} x_j
$$

where J ranges over the finite subsets of I, and $\prod_{j\in J} x_j$ and $\coprod_{j\in J} x_j$ denote the n-ary, for $n = |J|$, (co)products of the tuple $\{x_j\}_{j\in J}$. We shall often make use of an infinitary operation of convex combination as well. We write $\sum_{i\in I} \lambda_i \cdot x_i$, for I at most countable, $\lambda_i \in [0,1]$, $\sum_{i\in I} \lambda_i = 1$ and tuple $\{x_i\}_{i\in I} \in [0,1]^I$, to denote the weighted sum of the tuple $\{x_i\}_{i\in I}$ with the weights $\{\lambda_i\}_{i\in I}$, defined as the notation suggests.

The following is an important property of the family of operators introduced above.

Proposition 2.2.4. *The operators $\cdot$, $\odot$, $\oplus$, $\ominus$ and $+_\lambda$, for $\lambda \in (0,1)$, are monotone and continuous.*

Moreover the operators satisfy several De Morgan dualities as we now show

Lemma 2.2.5. *The following assertions hold:*

I) The operations $\cdot$ and $\odot$ are De Morgan duals:

$$
1 - (x \odot y) = (1-x) \cdot (1-y) \quad \textit{and} \quad 1 - (x \cdot y) = (1-x) \odot (1-y).
$$

II) The operations $\oplus$ and $\ominus$ are De Morgan duals:

$$
1 - (x \oplus y) = (1-x) \ominus (1-y) \quad \textit{and} \quad 1 - (x \ominus y) = (1-x) \oplus (1-y)
$$

III) The operation $+_\lambda$ *is self dual, for every* $\lambda \in (0,1)$*:*

$$1-(x+_\lambda y) = (1-x)+_\lambda (1-y).$$

Proof. All assertions are trivial. For illustration we just prove the third one.

$$\begin{aligned}(1-x)+_\lambda(1-y) &= \lambda\cdot(1-x)+(1-\lambda)\cdot(1-y)\\ &= \lambda-(\lambda\cdot x)+1-\lambda-y+(\lambda\cdot y)\\ &= 1-(\lambda\cdot x)-\big((1-\lambda)\cdot y\big)\\ &= 1-(x+_\lambda y).\end{aligned}$$

□

The same dualities hold for the infinitary operators $\prod_i$, $\coprod_i$ and $\big(\sum_i \lambda_i \cdot \_\big)$.

Lemma 2.2.6. *The following equalities hold:*

IV) $1-\big(\prod_{i\in I} x_i\big) = \coprod_{i\in I}(1-x_i)$,

V) $1-\big(\coprod_{i\in I} x_i\big) = \prod_{i\in I}(1-x_i)$,

VI) $1-\big(\sum_{i\in\mathbb{N}} \lambda_i x_i\big) = \sum_{i\in\mathbb{N}} \lambda_i\cdot(1-x_i)$.

Proof. The third assertion is trivial recalling that $\sum_i \lambda_i = 1$ by assumption. The first two follow from the definition of the infinitary operations as the limit of their finitary counterparts, and application of Lemma 2.2.5. □

We now discuss a useful property of the (countably) infinitary operations of product and coproduct.

Lemma 2.2.7. *Let* $\{x_i\}_{i\in\mathbb{N}}$ *be a sequence of real numbers* $x_i \in [0,1]$*, and let* $\epsilon \in (0,1]$*. Then the following inequality holds:*

$$\prod_{i\in\mathbb{N}}(x_i+\frac{\epsilon}{2^{2^i+1}}) \leq \big(\prod_{i\in\mathbb{N}} x_i\big)+\epsilon$$

Proof. See Appendix A.3.1 □

Lemma 2.2.8. *Let* $\{x_i\}_{i\in\mathbb{N}}$ *be a sequence of real numbers* $x_i \in [0,1]$*, and let* $\epsilon \in (0,1]$*. Then the following inequality holds:*

$$\prod_{i\in\mathbb{N}}(x_i-\frac{\epsilon}{2^{2^i+1}}) \geq \big(\prod_{i\in\mathbb{N}} x_i\big)-\epsilon$$

Proof. See Appendix A.3.2 □

The approximations of products described in lemmas 2.2.7 and 2.2.8 are going to be used so often in the rest of the thesis that we find helpful to introduce a special notation for the number 2^{2^i+1}.

Definition 2.2.9 (Function #)**.** We define the map $\# : \mathbb{N} \to \mathbb{N}$ as: $\#(n) = 2^{2^n+1}$.

This allows us to re-state Lemma 2.2.7 and Lemma 2.2.8 as follows:

Lemma 2.2.10. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers $x_i \in [0,1]$, and let $\epsilon \in (0,1]$. Then the following inequalities hold:*

$$\prod_{i\in\mathbb{N}}(x_i + \frac{\epsilon}{\#(i)}) \leq \Big(\prod_{i\in\mathbb{N}} x_i\Big) + \epsilon \quad \text{and} \quad \prod_{i\in\mathbb{N}}(x_i - \frac{\epsilon}{\#(i)}) \geq \Big(\prod_{i\in\mathbb{N}} x_i\Big) - \epsilon.$$

As a corollary we have the following dual result concerning the approximation of co-products.

Lemma 2.2.11. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers $x_i \in [0,1]$, and let $\epsilon \in (0,1]$. Then the following inequalities hold:*

$$\coprod_{i\in\mathbb{N}}(x_i + \frac{\epsilon}{\#(i)}) \leq \big(\coprod_{i\in\mathbb{N}} x_i\big) + \epsilon \quad \text{and} \quad \coprod_{i\in\mathbb{N}}(x_i - \frac{\epsilon}{\#(i)}) \geq \big(\coprod_{i\in\mathbb{N}} x_i\big) - \epsilon.$$

Proof. See Appendix A.3.3 □

We conclude with a brief discussion about the relationship between the material presented in this section and the mathematical field known as *fuzzy logic* [49]. Fuzzy logic is a form of many-valued logic suited for reasoning about approximate informations, rather than fixed and exact. In contrast with traditional logic and its bivalent interpretation, formulas in fuzzy logic are interpreted in a range of values which is often assumed to be the real interval $[0,1]$: the least and top elements, 0 and 1, play the role of *falsity* and *truth* respectively, while the values in $(0,1)$ are used to represent some in-between level of truth. Thus the study of the $[0,1]$ interval and the operations defined of it, is of great importance in fuzzy logic. Fuzzy logic have been studied extensively using algebraic tools, in the style of algebraic logic [49]. The simplest example is the logic associated with the algebra $\big([0,1], \sqcup, \sqcap, (-1), 0, 1\big)$, where $\sqcup$ and $\sqcap$ play the role of disjunction and conjunction respectively and the operation $(1-)$ is used as negation: the resulting theory of equality coincides with the so-called *Lukasiewicz three-valued logic* [41].

Other operations on $[0,1]$ can be used as $[0,1]$-counterparts of standard conjunction and disjunction. For example, also the pairs $\cdot/\odot$ and $\ominus/\oplus$ are reasonable generalization of standard conjunction and disjunction known under the name of *product* and *Lukasiewicz* t-(co)norms (see [49] for an extensive treatment of the subject). The operation of $+_\lambda$ is called a *mean* operator in [42], and the logic associated with the algebraic structure $\big([0,1], \sqcup, \sqcap, (-1), 0, 1, +_{\frac{1}{2}}\big)$ a *Frank system.* In, e.g., [41] and [113], several logics associated to algebraic structures combining the operators considered in this thesis are studied: $\big([0,1], \sqcup, \sqcap, (-1), 0, 1, \cdot, \odot\big)$, $\big([0,1], \sqcup, \sqcap, (-1), 0, 1, \ominus, \oplus\big)$, $\big([0,1], \sqcup, \sqcap, (-1), 0, 1, \ominus, \oplus, +_{\frac{1}{2}}\big)$, *etcetera.*

2.3 Game Theory

2.3.1 Gale–Stewart Games

In this section we introduce the class of Gale–Stewart games, named after David Gale and Frank M. Stewart who introduced them in [40]. We refer to [60, §20] and [111] for complementary introductions to the topic.

Gale–Stewart games are infinite duration games of perfect information played by two players, named Player 1 and Player 2. Given an arbitrary non empty set X (henceforth endowed with the discrete topology) a Gale–Stewart game is played as follows: Player 1 starts by choosing an element x_0 from X, then Player 2, aware of what Player 1 previously played, chooses an element x_1 from X, then it is again Player 1's turn to pick an element x_2 from X basing their decision on the list of previously played moves, i.e., (x_0, x_1), and so on. In general at each stage $2n$ of the game, with $n \in \mathbb{N}$, Player 1 chooses an element x_{2n} from X basing their choice on the list $\{x_i\}_{i<2n}$ of previously played moves, and at each stage $2n+1$ of the game, with $n \in \mathbb{N}$, Player 2 chooses an element x_{2n+1} from X basing their choice on the list $\{x_i\}_{i\leq 2n}$ of previously played moves. A *play* of the game, which can be depicted as follows,

Player 1:	x_0		x_2		x_4		$\dots$	x_{2n}		$\dots$
Player 2:		x_1		x_3		x_5	$\dots$		x_{2n+1}	$\dots$

is therefore as an infinite sequence, denoted by $\vec{x} \in X^\omega$ of elements in X. We endow X^ω with the product topology. A Gale–Stewart game is specified by the set X of playable moves, and by a subset $A \subseteq X^\omega$ of plays called the *winning set.* The outcome $\vec{x}$ of a play is said to be winning for Player 1 if $\vec{x} \in A$, and winning for Player 2 otherwise. We denote with $\mathsf{GS}(X, A)$ the Gale–Stewart game having X as set of playable moves and A as winning set for Player 1.

In order to formally specify how the two players interact in a Gale–Stewart game, we need to define the notion of deterministic strategy.

Definition 2.3.1 (Deterministic strategy in a Gale–Stewart game)**.** A *deterministic strategy* for Player 1 in the Gale–Stewart game $\mathsf{GS}(X, A)$ is a function $\sigma_1^{\mathsf{GS}} : X^{<\omega} \to X$. Similarly a deterministic strategy for Player 2 in the Gale–Stewart game $\mathsf{B}(X, A)$ is a function $\sigma_2^{\mathsf{GS}} : X^{<\omega} \to X$. A pair of deterministic strategies $(\sigma_1^{\mathsf{GS}}, \sigma_2^{\mathsf{GS}})$ is called a *Gale–Stewart deterministic strategy profile.* We denote with Σ^{GS} the sets of deterministic strategies for Player 1 (and Player 2) in the game $\mathsf{GS}(X, A)$.

Definition 2.3.2 (Topology of Σ^{GS})**.** For every history $h \in X^{<\omega}$ and and element $x \in X$, let us denote with $O_{h \mapsto x}$ the set of deterministic strategies σ^{GS} for Player 1 (and Player 2) such that $\sigma^{\mathsf{GS}}(h) = x$. We fix the topology on Σ^{GS}, generated by the basis for the open sets given by the sets $O_{h \mapsto x}$, for every pair (h, x) as defined above. If X is countable, this is a 0-dimensional Polish space. The space $\Sigma^{\mathsf{GS}} \times \Sigma^{\mathsf{GS}}$ of Gale–Stewart deterministic strategy profiles is endowed with the product topology.

A Gale–Stewart deterministic strategy profile $(\sigma_1^{\mathsf{GS}}, \sigma_2^{\mathsf{GS}})$ induces a unique play $\{x_n\}_{n\in\omega}$ in X^ω specified as follows $x_{2n} = \sigma_1^{\mathsf{GS}}(\{x_m\}_{m<2n})$, and $x_{2n+1} = \sigma_2^{\mathsf{GS}}(\{x_m\}_{m\leq 2n})$, for every $n \in \mathbb{N}$. We denote with $\langle \_ , \_ \rangle^{\mathsf{GS}} : \Sigma^{\mathsf{GS}} \times \Sigma^{\mathsf{GS}} \to X^\omega$ the function which maps a deterministic strategy profile to its induced play[3]. It is clear that $\langle \_ , \_ \rangle^{\mathsf{GS}}$ is continuous. We denote with $\langle \sigma_1^{\mathsf{GS}} , \sigma_2^{\mathsf{GS}} \rangle^{\mathsf{GS}}$ the play $\big(\langle \_ , \_ \rangle^{\mathsf{GS}}\big)(\sigma_1^{\mathsf{GS}}, \sigma_2^{\mathsf{GS}})$.

We are now ready to define the concepts of lower and upper values of a Gale–Stewart game under deterministic strategies.

Definition 2.3.3. Let $\mathsf{GS}(X, A)$ be a Gale–Stewart game. We define the *lower value* and the *upper value* of $\mathsf{GS}(X, A)$ under deterministic strategies, denoted as $\mathsf{VAL}_\downarrow\big(\mathsf{GS}(X, A)\big)$ and $\mathsf{VAL}_\uparrow\big(\mathsf{GS}(X, A)\big)$ respectively, as follows:

$$\mathsf{VAL}_\downarrow\big(\mathsf{GS}(X, A)\big) = \begin{cases} 1 & \text{if } \exists \sigma_1^{\mathsf{GS}} \forall \sigma_2^{\mathsf{GS}}. \big(\langle \sigma_1^{\mathsf{GS}}, \sigma_2^{\mathsf{GS}} \rangle^{\mathsf{GS}} \in A\big) \\ 0 & \text{otherwise} \end{cases}$$

[3]Note that, although our notion of strategy is quite convenient for its simplicity, it models some redundant information: Player 1's choices at odd positions and Player 2's choices at even position are never considered by the map $\langle \_ , \_ \rangle^{\mathsf{GS}}$.

$$\mathsf{VAL}_{\uparrow}\big(\mathsf{GS}(X,A)\big) = \begin{cases} 0 & \text{if } \exists \sigma_2^{\mathsf{GS}} \forall \sigma_1^{\mathsf{GS}}.\big(\langle \sigma_1^{\mathsf{GS}}, \sigma_2^{\mathsf{GS}} \rangle^{\mathsf{GS}} \not\in A\big) \\ 1 & \text{otherwise} \end{cases}$$

We say that Player 1 has a winning deterministic strategy in $\mathsf{GS}(X,A)$ if $\mathsf{VAL}_{\downarrow}\big(\mathsf{GS}(X,A)\big) = 1$. Similarly we say that Player 2 has a winning strategy in $\mathsf{GS}(X,A)$ if $\mathsf{VAL}_{\uparrow}\big(\mathsf{GS}(X,A)\big) = 0$. It is clear that if Player 1 has winning strategy, then Player 2 does not, and vice versa, i.e., $\mathsf{VAL}_{\downarrow}\big(\mathsf{GS}(X,A)\big) \leq \mathsf{VAL}_{\uparrow}\big(\mathsf{GS}(X,A)\big)$ trivially holds.

Definition 2.3.4 (Determinacy)**.** We say that the Gale–Stewart game $\mathsf{GS}(X,A)$ is *determined under deterministic strategies*, or just *determined*, if one of the two players has a winning deterministic strategy, i.e., if $\mathsf{VAL}_{\downarrow}\big(\mathsf{GS}(X,A)\big) = \mathsf{VAL}_{\uparrow}\big(\mathsf{GS}(X,A)\big)$, in which case we just write $\mathsf{VAL}\big(\mathsf{GS}(X,A)\big)$ to denote its unique value.

One might also be interested in considering the dynamics of the game $\mathsf{GS}(X,A)$ when the two players can use randomized procedures to make their choices. One of the simplest, yet powerful, scenarios consists in allowing the players to choose randomly, at the very beginning of the game, which deterministic strategy to use in the rest of the game. This leads to the notion, standard[4] in Game theory, of *mixed strategy* which we now formalize.

Definition 2.3.5 (Mixed strategy in a Gale–Stewart game)**.** A *mixed strategy* for Player 1 in the Gale–Stewart game $\mathsf{GS}(X,A)$ is a probability measure $\eta_1^{\mathsf{GS}} \in \mathcal{M}_1(\Sigma^{\mathsf{GS}})$ over Σ^{GS}. Similarly for Player 2. A pair of mixed strategies $(\eta_1^{\mathsf{GS}}, \eta_2^{\mathsf{GS}})$, one for each player, is called a *Gale–Stewart mixed strategy profile.*

A Gale–Stewart mixed strategy profile $(\eta_1^{\mathsf{GS}}, \eta_2^{\mathsf{GS}})$ naturally induces a probability measure $\eta_1^{\mathsf{GS}} \times \eta_2^{\mathsf{GS}}$ over deterministic strategy profiles, and therefrom a probability measure $\mathbb{P}^{\mathsf{GS}}_{\eta_1^{\mathsf{GS}},\eta_2^{\mathsf{GS}}} = \mathcal{M}_1(\langle \_ , \_ \rangle^{\mathsf{GS}})(\eta_1^{\mathsf{GS}} \times \eta_2^{\mathsf{GS}})$ over X^ω via the continuous function $\langle \_ , \_ \rangle^{\mathsf{GS}}$, i.e., the unique probability measure specified by the assignment $\mathbb{P}^{\mathsf{GS}}_{\eta_1^{\mathsf{GS}},\eta_2^{\mathsf{GS}}}(S) = \big(\eta_1^{\mathsf{GS}} \times \eta_2^{\mathsf{GS}}\big)(\langle \_ , \_ \rangle^{\mathsf{GS}^{-1}}(S))$, on every Borel set $S \subseteq X^\omega$.

When considering mixed strategy profiles, the notions of lower and upper values of the game under deterministic mixed strategies are replaced by the similar notions of upper and lower values of the game under mixed strategies.

[4]Mixed strategies are sometimes defined as functions from finite histories to (discrete) probability distributions over X. It is possible to show that this definition coincide with ours. See e.g. [74, 111].

Definition 2.3.6. Given a Gale–Stewart game $\mathsf{GS}(X,A)$, where A is universally measurable, we define its *lower* and *upper values* under mixed strategies, denoted by $\mathsf{MVAL}_{\downarrow}\big(\mathsf{GS}(X,A)\big)$ and $\mathsf{MVAL}_{\uparrow}\big(\mathsf{GS}(X,A)\big)$, as

$$\mathsf{MVAL}_{\downarrow}\big(\mathsf{GS}(X,A)\big)=\bigsqcup_{\eta_1^{\mathsf{GS}}}\bigsqcap_{\eta_2^{\mathsf{GS}}}\mathbb{P}^{\mathsf{GS}}_{\eta_1^{\mathsf{GS}},\eta_2^{\mathsf{GS}}}(A)\ \text{, and}$$

$$\mathsf{MVAL}_{\uparrow}\big(\mathsf{GS}(X,A)\big)=\bigsqcap_{\eta_2^{\mathsf{GS}}}\bigsqcup_{\eta_1^{\mathsf{GS}}}\mathbb{P}^{\mathsf{GS}}_{\eta_1^{\mathsf{GS}},\eta_2^{\mathsf{GS}}}(A)$$

respectively. Note that this is a good definition since A is assumed to be universally measurable, hence $\mathbb{P}^{\mathsf{GS}}_{\eta_1^{\mathsf{GS}},\eta_2^{\mathsf{GS}}}$-measurable.

When working with Gale–Stewart mixed strategies, we will always assume implicitly that the winning set A is universally measurable[5]. $\mathsf{MVAL}_{\downarrow}\big(\mathsf{GS}(X,A)\big)$ represents the limit probability of Player 1 winning the game when they choose their mixed strategy η_1^{GS} first, and then Player 2 chooses their mixed strategy η_2^{GS}, possibly making their choice based on the strategy η_1^{GS} previously chosen by Player 1. Similarly $\mathsf{MVAL}_{\uparrow}\big(\mathsf{GS}(X,A)\big)$ represents the limit probability of Player 1 winning the game when Player 2 chooses their mixed strategy η_2^{GS} first, and then Player 1 chooses their mixed strategy η_1^{GS}, possibly making their choice based on the strategy η_2^{GS} previously chosen by Player 2. The inequality $\mathsf{MVAL}_{\downarrow}\big(\mathsf{GS}(X,A)\big)\leq\mathsf{MVAL}_{\uparrow}\big(\mathsf{GS}(X,A)\big)$ trivially holds.

Definition 2.3.7. We say that the Gale–Stewart game $\mathsf{GS}(X,A)$ is *determined under mixed strategies* if and only if $\mathsf{MVAL}_{\downarrow}\big(\mathsf{GS}(X,A)\big)=\mathsf{MVAL}_{\uparrow}\big(\mathsf{GS}(X,A)\big)$. If $\mathsf{GS}(X,A)$ is determined under mixed strategies, then we just write $\mathsf{MVAL}\big(\mathsf{GS}(X,A)\big)$ do denote its unique value.

The following lemma trivially holds:

Lemma 2.3.8. *If $\mathsf{GS}(X,A)$ is determined under deterministic strategies, then it is also determined under mixed strategies.*

Proof. If Player 1 has a winning strategy σ_1^{GS} in $\mathsf{GS}(X,A)$, the η_1^{GS} defined as the probability measure $\delta(\sigma_1^{\mathsf{GS}})$ with unit mass at σ_1^{GS} is such that $\bigsqcap_{\eta_2^{\mathsf{GS}}}\mathbb{P}^{\mathsf{GS}}_{\eta_1^{\mathsf{GS}},\eta_2^{\mathsf{GS}}}(A)=1$, hence $\mathsf{MVAL}\big(\mathsf{GS}(A,X)\big)=1$. Similarly if Player 2 has a winning deterministic strategy, then $\mathsf{MVAL}\big(\mathsf{GS}(A,X)\big)=0$. □

[5] Actually, following [111, §5] and [74], one could give meaningful definitions for arbitrary sets A by replacing $\mathbb{P}^{\mathsf{GS}}_{\eta_1^{\mathsf{GS}},\eta_2^{\mathsf{GS}}}$, in the definitions of $\mathsf{MVAL}_{\downarrow}$ and $\mathsf{MVAL}_{\uparrow}$, with its inner and outer measures respectively. This generalization goes beyond the purpose of this introductory chapter.

A natural question is then if the converse of Lemma 2.3.8 holds. The answer is, rather surprisingly, negative. In [74, §3] a counter-example, credited to Greg Hjorth, is discussed.

Proposition 2.3.9. *There exists a universally measurable set* $A \subseteq \mathbb{N}^\omega$*, such that* $\mathsf{GS}(\mathbb{N}, A)$ *is determined under mixed strategies, with* $\mathsf{MVAL}\big(\mathsf{GS}(\mathbb{N}, A)\big) = 0$*, but is not determined under deterministic strategies.*

We sketch the main lines of the construction (which uses the Axiom of Choice in an essential way) and provide the necessary references, which are implicitly assumed in [74] but not necessarily well-known, in Theorem A.4.1. The result of Proposition 2.3.9 shows that mixed strategies are stronger, in the sense of allowing strictly more rational behaviors, than deterministic strategies. As an immediate reaction to the result of Lemma 2.3.9, one might ask if it possible to have a Gale–Stewart game $\mathsf{GS}(X, A)$ determined under mixed strategies with $\mathsf{MVAL}(\mathsf{GS}(X, A)) \notin \{0, 1\}$. A definitive answer, for the special case $X = \mathbb{N}$, has been provided by Martin and Vervoort (see [74] and [111, 5.33 in §5]).

Theorem 2.3.10 (0-1 Law for mixed strategies)**.** *For every Gale–Stewart game* $\mathsf{GS}(\mathbb{N}, A)$ *(with A universally measurable) if* $\mathsf{GS}(\mathbb{N}, A)$ *is determined under mixed strategies, then* $\mathsf{MVAL}\big(\mathsf{GS}(\mathbb{N}, A)\big) \in \{0, 1\}$.

We are not aware of any generalization of Theorem 2.3.10 to arbitrary sets X with $|X| > \aleph_0$. We discussed mixed strategies for Gale–Stewart games because the concept of random behavior which is captured by such strategies, will be used in several sections to come. However Gale–Stewart games have been deeply investigated mostly in the context of deterministic strategies, on which we now focus.

David Gale and Frank M. Stewart themselves proved in their seminal paper [40] the following theorem:

Theorem 2.3.11. *For every set* X*, with* $|X| \geq 2$*, there exists a set* $A \subset X^\omega$ *such that* $\mathsf{GS}(X, A)$ *is not determined under deterministic strategies.*

This negative result makes essential use of the Axiom of Choice, and actually implies it [72]. On the other hand, in the same paper, the following positive result is proven:

Theorem 2.3.12. *For every set* X*, if* $A \subseteq X^\omega$ *is a closed or open set, then* $\mathsf{GS}(X, A)$ *is determined under deterministic strategies.*

Also this result makes an essential use of the Axiom of Choice, which is needed to deal with arbitrary[6] sets X, and indeed implies it [72]. These negative and positive results showed that, even though it is a fact of life[7] that not all Gale–Stewart games are determined, still there exist interesting classes of winning sets which guarantee determinacy. The quest for finding larger classes of determined winning sets had begun. In [115] Philip Wolfe showed that every Gale–Stewart game $\mathsf{GS}(X, A)$, with A a $\mathbf{\Sigma}^0_2$ set, is determined. Only after 10 years, Morton Davis improved this result by proving that every Gale–Stewart game $\mathsf{GS}(X, A)$, with A a $\mathbf{\Sigma}^0_3$ set, is determined, and shortly after Jeff B. Paris [92] improved the result to $\mathbf{\Sigma}^0_4$ winning sets. In the meanwhile, Mycielski and Swierczkowski proved in [87] that it is not provable[8] in ZFC that every $\mathsf{GS}(X, A)$ with $A \in \mathbf{\Sigma}^1_1$ set, i.e., with A analytical, is determined, even with $|X| = 2$. On the other hand Donald Martin proved in [71] that, under reasonable set-theoretic assumptions[9], every game $\mathsf{GS}(X, A)$ with $A \in \mathbf{\Sigma}^1_1$ set is determined. Hence we have the following theorem:

Theorem 2.3.13. *If* ZFC *is consistent, the following assertions hold:*

1. $\mathrm{ZFC} \nvdash$ *Every* $\mathsf{GS}(X, A)$ *is determined,*

2. $\mathrm{ZFC} + \exists(\text{Measurable cardinal}) \vdash$ *Every* $\mathsf{GS}(X, A)$ *is determined,*

where A *ranges over* $\mathbf{\Sigma}^1_1(X)$.

As a matter of fact, it was later proved by Martin and Steel [76], that the following stronger theorem holds:

Theorem 2.3.14. *If* ZFC *is consistent and infinitely many Woodin cardinals exist, then every Gale–Stewart game* $\mathsf{GS}(X, A)$*, with* A *a projective set, is determined.*

Only in 1975, Donald Martin proved [72] what is widely considered[10] the best result possible in ZFC.

[6]Indeed if $|X| \leq \aleph_0$ the result is provable in ZF alone [73].

[7]Assuming, of course, that ZFC captures *true* aspects of *life*.

[8]They prove the result working in a model of ZFC + V=L. V=L is known to be consistent with ZFC, see e.g. [59].

[9]Assuming the consistency of ZFC+ "there exists a measurable cardinal" [59].

[10]This sentiment is justified in light of Theorem 2.3.13, which however does not exclude the possibility of proving determinacy for interesting point-classes including all Borel sets but not all analytic sets.

Theorem 2.3.15 (Borel Determinacy of Gale–Stewart games)**.** *For every set X and Borel $A \subseteq X^\omega$, the game $\mathsf{GS}(X, A)$ is determined.*

According to [72, page 368], the proof of determinacy for $A \in \boldsymbol{\Sigma}^0_n$, with $n \in \mathbb{N}$, only relies on the existence of a well-ordering of X. Hence the Axiom of Choice is needed in general, but if $X = \mathbb{N}$ the proof goes through in ZF alone. This is not the case if $A \in \boldsymbol{\Sigma}^0_\alpha$, for some infinite countable ordinal α. The Axiom of Countable Choice is indeed needed [73], for all X with $|X| \leq \aleph_0$, to prove the result.

The result of Theorem 2.3.15, henceforth just referred as *Borel Determinacy*, constituted a tremendous achievement in Game Theory which required more than 20 years of intensive research, during which theoretical Game Theory became an important sub-field of Descriptive Set Theory. For instance much study was devoted to the analysis of extensions of ZFC obtained by axioms expressed in terms of determinacy of interesting point-classes. We now list a few of these axioms. For a comprehensive reference to the topic see [59].

Definition 2.3.16 (Axiom of $\boldsymbol{\Sigma}^1_1$-Determinacy)**.** The game $\mathsf{GS}(\mathbb{N}, A)$ is determined for every $A \in \boldsymbol{\Sigma}^1_1$.

More generally we have for each projective point-class $\boldsymbol{\Gamma}^1_n$, with $\Gamma \in \{\boldsymbol{\Sigma}, \boldsymbol{\Pi}, \boldsymbol{\Delta}\}$, the following axiom:

Definition 2.3.17 (Axiom of $\boldsymbol{\Gamma}^1_n$-Determinacy)**.** The game $\mathsf{GS}(\mathbb{N}, A)$ is determined for every $A \in \boldsymbol{\Gamma}^1_n$.

Lastly, the following axiom asserts that every projective set is determined.

Definition 2.3.18 (Axiom of Projective Determinacy - PD)**.** The game $\mathsf{GS}(\mathbb{N}, A)$ is determined for every $A \in \boldsymbol{\Sigma}^1_n$, with $n \in \mathbb{N}$.

All these axioms, which as a result of theorems 2.3.13 and 2.3.14 follow from appropriate large cardinal hypothesis, are believed to be true by several set-theorists[11] in name of the so-called principle of *definable determinacy*, which asserts that every "definable" subset of $\mathbb{N}^\omega$ is determined[12].

We refer to [59] and [60] for detailed analysis of the consequences of these axioms. Here we just list those that are relevant to our work.

[11] Of course, this must be understood as a personal feeling about the *real* nature of the universe of sets.

[12] Alexander Kechris [60, §26.B] expresses his position by claiming an overwhelming evidence in favor of this principle, originally proposed in [86].

Theorem 2.3.19. *The following assertions hold for every Polish space X:*

1. ZFC $+$ $\mathbf{\Sigma}^1_n$-Determinacy $\vdash$ *Every $\mathbf{\Sigma}^1_{n+1}$ set in X is universally measurable.*

2. *Let $f : X \to [0,1]$ be a function whose graph $G_f \subseteq X \times [0,1]$, defined as $G_f = \{(x,y) \mid f(x){=}y\}$, is a $\mathbf{\Pi}^1_n$ set. Then the following assertion holds:* ZFC $+$ $\mathbf{\Sigma}^1_n$-Determinacy $\vdash$ *f is universally measurable.*

Proof. See [60, 36.20 in §36.E] for a proof of point 1. For a proof of point 2 it is enough to observe that $f^{-1}(O) {=} \{x \mid \exists \lambda. \big(\lambda {\in} O \ \wedge (x,\lambda) {\in} G_f\big)\}$ is a $\mathbf{\Sigma}^1_{n+1}$ set, for every open set $O {\subseteq} [0,1]$. Then the result follows from point 1. □

2.3.2 Blackwell Games

In this section we introduce the class of Blackwell games, named after David Blackwell who introduced them in [10]. We refer to Marco Vervoort's PhD thesis [111] for a comprehensive introduction to the topic.

Blackwell games are infinite duration games of *imperfect information* played by two players, named Player 1 and Player 2. Given two non-empty *finite* sets X and Y (henceforth endowed with the discrete topologies) of moves available to Player 1 and Player 2 respectively, a Blackwell game is played as follows: at each stage $n \in \mathbb{N}$ of the game, Player 1 chooses an element x_n from X and, independently and simultaneously[13], Player 2 chooses an element y_n in Y. After these concurrent choices are made the two players synchronize again becoming aware of the choice made by the opponet, and the same protocol is repeated forever. A play of the game, which can be depicted as follows,

$$\begin{array}{lccccc} \text{Player 1:} & x_0 & x_1 & \dots & x_n & \dots \\ \text{Player 2:} & y_0 & y_1 & \dots & y_n & \dots \end{array}$$

is therefore as an infinite sequence, denoted by $\vec{x}{\times}\vec{y} {\in} (X {\times} Y)^\omega$ of pairs of elements of X^ω and Y^ω.

A Blackwell game is specified by a payoff function, which is defined[14] to be a function $\phi : (X \times Y)^\omega \to [0,1]$. In what follows we endow the set $(X \times Y)^\omega$ with the product topology. Note that, unlikely the Gale–Stewart games considered in

[13] It is this lack of knowledge about the adversary's concurrent choice that makes such a game a game of imperfect information. Blackwell himself describes this class as "games of slight imperfect information" in [12].

[14] The definition can be extended to arbitrary $\mathbb{R}$-valued *bounded* functions.

Section 2.3.1, Blackwell games deal with payoff functions which express quantitative rewards. This is clearly a generalization, as every set $A \subseteq (X \times Y)^\omega$ can be represented by its characteristic function $\chi_A : (X \times Y)^\omega \to \{0,1\}$. If the outcome of a play is $\vec{x} \times \vec{y}$, Player 1 receives the reward $\phi(\vec{x} \times \vec{y})$. The objective of Player 1 is to maximize their reward, while the dual objective of Player 2 it to minimize it. We denote with $\mathsf{B}(X, Y, \phi)$ the Blackwell game having sets of actions X and Y for Player 1 and Player 2 respectively, and payoff function ϕ.

In order to formally specify how the two players interact in a Blakwell game, we need to define the notion of deterministic strategy in a Blackwell game.

Definition 2.3.20 (Deterministic strategy in a Blackwell game)**.**
A *deterministic strategy* for Player 1 in the Blackwell game $\mathsf{B}(X, Y, \phi)$ is a function $\sigma_1^{\mathsf{B}} : (X \times Y)^{<\omega} \to X$. Similarly a deterministic strategy for Player 2 in the Blackwell game $\mathsf{B}(X, Y, \phi)$ is a function $\sigma_2^{\mathsf{B}} : (X \times Y)^{<\omega} \to Y$. A pair of deterministic strategies $(\sigma_1^{\mathsf{B}}, \sigma_2^{\mathsf{B}})$ is called a *Blackwell deterministic strategy profile.* We denote with Σ_1^{B} and Σ_2^{B} the sets of deterministic strategies for Player 1 and Player 2 respectively in the game $\mathsf{B}(X, Y, \phi)$.

We now endow the sets Σ_1^{B} and Σ_2^{B} with Polish topologies.

Definition 2.3.21 (Topologies of Σ_1^{B} and Σ_2^{B})**.** For every history $h \in (X \times Y)^{<\omega}$ and and element $x \in X$, let us denote with $O_{h \mapsto x}$ the set of deterministic strategies σ_1^{B} for Player 1 such that $\sigma^{\mathsf{B}}(h) = x$. We fix the topology on Σ_1^{B}, generated by the basis for the open sets given by the sets $O_{h \mapsto x}$, for every pair (h, x) as defined above. This is a 0-dimensional Polish space where all basic open sets are clopen. The set Σ_2^{B} is endowed with the 0-dimensional Polish topology defined in the similar way. The space $\Sigma^{\mathsf{B}} \times \Sigma^{\mathsf{B}}$ of Blackwell deterministic strategy profiles is endowed with the product topology.

Definition 2.3.22. A Blackwell deterministic strategy profile $\langle \sigma_1^{\mathsf{B}}, \sigma_2^{\mathsf{B}} \rangle$ induces a unique play $\{(x_n, y_n)\}_{n \in \omega} \in (X \times Y)^\omega$ specified as follows:

$$(x_n, y_n) = \big(\sigma_1^{\mathsf{B}}(\{x_m, y_m\}_{m<n}), \sigma_2^{\mathsf{B}}(\{x_m, y_m\}_{m<n})\big).$$

for every $n \in \mathbb{N}$. We denote with $\langle \_ , \_ \rangle^{\mathsf{B}} : (\Sigma_1^{\mathsf{B}} \times \Sigma_2^{\mathsf{B}}) \to (X \times Y)^\omega$ the function that maps a Blackwell strategy profile to its induced play. It is clear that $\langle \_ , \_ \rangle^{\mathsf{B}}$ is continuous. We denote with $\langle \sigma_1^{\mathsf{B}} , \sigma_2^{\mathsf{B}} \rangle^{\mathsf{B}}$ the play $\big(\langle \_ , \_ \rangle^{\mathsf{B}}\big)(\sigma_1^{\mathsf{B}}, \sigma_2^{\mathsf{B}})$.

Since the payoff of a Blackwell game is, in general, not the characteristic function of a set, we adapt the notion of lower and upper values of the game under deterministic strategies given in Definition 2.3.3 for Gale–Stewart games, as follows:

Definition 2.3.23. Let $\mathsf{B}(X, Y, \phi)$ be a Blackwell game. We define the *lower* and the *upper values* under deterministic strategies of $\mathsf{B}(X, Y, \phi)$, denoted as $\mathsf{VAL}_{\downarrow}(\mathsf{B}(X, Y, \phi)$ and $\mathsf{VAL}_{\uparrow}(\mathsf{B}(X, Y, \phi)$ respectively, as follows:

$$\mathsf{VAL}_{\downarrow}(\mathsf{B}(X, Y, \phi) = \bigsqcup_{\sigma_1^{\mathsf{B}}} \bigsqcap_{\sigma_2^{\mathsf{B}}} \phi\big(\langle \sigma_1^{\mathsf{B}}, \sigma_2^{\mathsf{B}} \rangle^{\mathsf{B}}\big),$$

$$\mathsf{VAL}_{\uparrow}(\mathsf{B}(X, Y, \phi) = \bigsqcap_{\sigma_2^{\mathsf{B}}} \bigsqcup_{\sigma_1^{\mathsf{B}}} \phi\big(\langle \sigma_1^{\mathsf{B}}, \sigma_2^{\mathsf{B}} \rangle^{\mathsf{B}}\big).$$

Note that when ϕ is the characteristic function χ_A of some set A, then the above definition is equivalent to the one adopted for Gale–Stewart games in Definition2.3.3. $\mathsf{VAL}_{\downarrow}\big(\mathsf{B}(X, Y, \phi)\big)$ represents the limit reward assigned to Player 1 when Player 1 chooses its deterministic strategy σ_1^{B} first, and then Player 2 chooses its deterministic strategy σ_2^{B}, possibly making their choice based on the strategy σ_1^{B} previously chosen by Player 1. Similarly $\mathsf{VAL}_{\uparrow}\big(\mathsf{B}(X, Y, \phi)\big)$ represents the limit reward assigned to Player 1 when Player 2 chooses its deterministic strategy σ_2^{B} first, and then Player 1 chooses its deterministic strategy σ_1^{B}, possibly making their choice based on the strategy σ_2^{B} previously chosen by Player 2. The inequality $\mathsf{VAL}_{\downarrow}\big(\mathsf{B}(X, Y, \phi)\big) \leq \mathsf{VAL}_{\uparrow}\big(\mathsf{B}(X, Y, \phi)\big)$ trivially holds.

It is natural, given the above description for the meaning of the lower and upper values of a Blackwell game under deterministic strategies, to introduce the notion of ϵ-optimal deterministic strategy.

Definition 2.3.24. A deterministic strategy σ_1^{B} for Player 1 in a Blackwell game $\mathsf{B}(X, Y, \phi)$ is called *ϵ-optimal*, for $\epsilon \in [0, 1]$, if and only if the following inequality holds: $\bigsqcap_{\sigma_2^{\mathsf{B}}} \phi\big(\langle \sigma_1^{\mathsf{B}}, \sigma_2^{\mathsf{GS}} \rangle^{\mathsf{B}}\big) \geq \mathsf{VAL}_{\downarrow}\big(\mathsf{B}(X, Y, \phi)\big) - \epsilon$. Similarly, a deterministic strategy σ_2^{B} for Player 2 in a Blackwell game $\mathsf{B}(X, Y, \phi)$ is called *ϵ-optimal*, for $\epsilon \in [0, 1]$, if and only if the following inequality holds: $\bigsqcup_{\sigma_1^{\mathsf{B}}} \phi\big(\langle \sigma_1^{\mathsf{B}}, \sigma_2^{\mathsf{B}} \rangle^{\mathsf{B}}\big) \leq \mathsf{VAL}_{\downarrow}\big(\mathsf{B}(X, Y, \phi)\big) + \epsilon$. A 0-optimal deterministic strategy, for either Player 1 or Player 2, is simply called *optimal*.

Clearly ϵ-optimal deterministic strategies for both players always exist for every $\epsilon > 0$, but not necessarily so for $\epsilon = 0$. We are now ready to introduce the notion of Blackwell determinacy under deterministic strategies.

Definition 2.3.25. We say that the Blackwell game $\mathsf{B}(X, Y, \phi)$ is *determined under deterministic strategies*, if $\mathsf{VAL}_{\downarrow}\big(\mathsf{B}(X, Y, \phi)\big) = \mathsf{VAL}_{\uparrow}\big(\mathsf{B}(X, Y, \phi)\big)$.

Unlike Gale–Stewart games, even very simple Blackwell games are not determined under deterministic strategies. This is because the concurrent choice of moves, that Blackwell games capture, allows to model games such as the well known “rock-scissor-paper” game, which are clearly not determined under deterministic strategies. We refer to [111] for a detailed exposition of this kind of example. Blackwell games become more interesting when the two players are allowed to make their choices randomly, according with some probabilistic method they may adopt. Probabilistic behaviors are formalized, as done for Gale–Stewart games, by the notion of mixed strategies which we now introduce.

Definition 2.3.26 (Mixed strategy in a Blackwell game)**.** A *mixed strategy* for Player 1 in the Blackwell game $\mathsf{B}(X, Y, \phi)$ is a probability measure $\eta_1^{\mathsf{B}} \in \mathcal{M}_1(\Sigma_1^{\mathsf{B}})$ over Σ_1^{B}. Simlarly, a mixed strategy for Player 2 in the Blackwell game $\mathsf{B}(X, Y, \phi)$ is a probability measure $\eta_2^{\mathsf{B}} \in \mathcal{M}_1(\Sigma_2^{\mathsf{B}})$ over Σ_2^{B}. A pair of mixed strategies $(\eta_1^{\mathsf{B}}, \eta_2^{\mathsf{B}})$ is called a Blackwell mixed strategy profile. The space $\mathcal{M}_1(\Sigma_1^{\mathsf{B}}) \times \mathcal{M}_1(\Sigma_2^{\mathsf{B}})$ of Blackwell strategy profiles is endowed with the product topology.

Definition 2.3.27. Every Blackwell mixed strategy profile $(\eta_1^{\mathsf{B}}, \eta_2^{\mathsf{B}})$ naturally determines a probability measure $\mathbb{P}_{\eta_1^{\mathsf{B}}, \eta_2^{\mathsf{B}}} \in \mathcal{M}_1\big((X \times Y)^{\omega}\big)$ over plays, defined as $\mathcal{M}_1\big(\langle \_ , \_ \rangle^{\mathsf{B}}\big)(\eta_1^{\mathsf{B}}, \eta_2^{\mathsf{B}})$, or equivalently as the unique probability measure assigning probability $\mathbb{P}_{\eta_1^{\mathsf{B}}, \eta_2^{\mathsf{B}}}(S) = \eta_1^{\mathsf{B}} \times \eta_2^{\mathsf{B}}\big(\langle \_, \_ \rangle^{\mathsf{B}^{-1}}(S)\big)$, to every Borel set $S \subseteq (X \times Y)^{\omega}$. More concretely, $\mathbb{P}_{\eta_1^{\mathsf{B}}, \eta_2^{\mathsf{B}}}$ is the unique probability measure induced by the probability assignment on basic open subsets of $(X \times Y)^{\omega}$, specified as follows:

$$\mathbb{P}_{\eta_1^{\mathsf{B}}, \eta_2^{\mathsf{B}}}(O_h) = \eta_1^{\mathsf{B}}\Big(\bigcap_{i=0}^{n} O_{h|_i \mapsto x_i}\Big) \cdot \eta_2^{\mathsf{B}}\Big(\bigcap_{i=0}^{n} O_{h|_i \mapsto y_i}\Big)$$

where $O_h \subseteq (X \times Y)^{\omega}$ denotes the basic opens set of plays in $\mathsf{B}(X, Y, \phi)$ having the finite history $h = \big((x_0, y_0), ..., (x_n, y_n)\big)$ as prefix, and $h|_i$, for $0 \leq i \leq n$, denotes the prefix of h of length i, so that $h|_0 = \epsilon$ and $h|_n = \big((x_0, y_0), ..., (x_{n-1}, y_{n-1})\big)$. One can calculate $\mathbb{P}_{\eta_1^{\mathsf{B}}, \eta_2^{\mathsf{B}}}(O_h)$ with the following formula:

$$\mathbb{P}_{\eta_1^{\mathsf{B}}, \eta_2^{\mathsf{B}}}(O_h) = \prod_{i=0}^{n} \eta_1^{\mathsf{B}}(O_{h|_i \mapsto x_i} | U_{\{h|_j \mapsto x_j\}_{j<i}}) \cdot \eta_2^{\mathsf{B}}(O_{h|_i \mapsto x_i} | U_{\{h|_j \mapsto x_j\}_{j<i}})$$

where $U_{\{h|_j \mapsto x_j\}_{j<i}}$ denote the open set of strategies $\big(\bigcap_{j=0}^{i-1} O_{h|_i \mapsto x_j}\big)$ and the conditional probability $\mu(O|U)$ is defined as $\frac{\mu(O \cap U)}{\mu(U)}$.

Intuitively, $\mathbb{P}_{\eta_1^{\mathsf{B}},\eta_2^{\mathsf{B}}}(S)$, for some $\mathbb{P}_{\eta_1^{\mathsf{B}},\eta_2^{\mathsf{B}}}$-measurable set $S\subseteq(X\times Y)^\omega$, is the probability of producing a play $\vec{x}\times\vec{y}$ in the set S, when the two players play according with the mixed strategies η_1^{B} and η_2^{B} respectively.

As for Gale–Stewart games, when considering Blackwell mixed strategy profiles, the notions of lower and upper values of the game under deterministic strategies are replaced by the notions of upper and lower values of the game under mixed strategies, which we now introduce.

Definition 2.3.28. Given a Blackwell game $\mathsf{B}(X,Y,\phi)$, where the payoff function ϕ is universally measurable, we define its *lower* and *upper values* under mixed strategies as the values denoted by $\mathsf{MVAL}_{\downarrow}\big(\mathsf{B}(X,Y,\phi)\big)$ and $\mathsf{MVAL}_{\uparrow}\big(\mathsf{B}(X,Y,\phi)\big)$ respectively, specified as follows:

$$\mathsf{MVAL}_{\downarrow}\big(\mathsf{B}(X,Y,\phi)\big)=\bigsqcup_{\eta_1^{\mathsf{B}}}\bigsqcap_{\eta_2^{\mathsf{B}}}\Big(\int_{(X\times Y)^\omega}\phi\ \mathrm{d}\mathbb{P}_{\eta_1^{\mathsf{B}},\eta_2^{\mathsf{B}}}\Big),$$

$$\mathsf{MVAL}_{\uparrow}\big(\mathsf{B}(X,Y,\phi)\big)=\bigsqcap_{\eta_2^{\mathsf{B}}}\bigsqcup_{\eta_1^{\mathsf{B}}}\Big(\int_{(X\times Y)^\omega}\phi\ \mathrm{d}\mathbb{P}_{\eta_1^{\mathsf{B}},\eta_2^{\mathsf{B}}}\Big).$$

Note that the integrals are well defined since, by definition, ϕ is a universally measurable function, hence $\mathbb{P}_{\eta_1^{\mathsf{B}},\eta_2^{\mathsf{B}}}$-measurable. Also observe that if ϕ is the characteristic function χ_A, of some universally measurable set A, then this definition is equivalent to the one adopted for Gale–Stewart games in Definition 2.3.6.

When working with Blackwell mixed strategies, i.e., most of the time, we will always assume implicitly that the function ϕ is universally measurable[15]. $\mathsf{MVAL}_{\downarrow}\big(\mathsf{B}(X,Y,\phi)\big)$ represents the limit expected reward assigned to Player 1 when Player 1 chooses their mixed strategy η_1^{B} first, and then Player 2 chooses their mixed strategy η_2^{B}, possibly making their choice based on the strategy η_1 previously chosen by Player 1. Similarly $\mathsf{MVAL}_{\uparrow}\big(\mathsf{B}(X,Y,\phi)\big)$ represents the limit expected reward assigned to Player 1 when Player 2 chooses their mixed strategy η_2^{B} first, and then Player 1 chooses their mixed strategy η_1^{B}, possibly making their choice based on the strategy η_2 previously chosen by Player 2. The inequality $\mathsf{MVAL}_{\downarrow}\big(\mathsf{B}(X,Y,\phi)\big)\leq\mathsf{MVAL}_{\uparrow}\big(\mathsf{B}(X,Y,\phi)\big)$ trivially holds.

We now introduce the notion of ϵ-optimal mixed strategy, as the natural modification of the notion of ϵ-optimal deterministic strategy of Definition 2.3.24.

[15] As already pointed out in Definition 2.3.6, following [111, §5] and [74], one could give meaningful definitions for arbitrary functions ϕ. This generalization goes beyond the purpose of this introductory section.

Definition 2.3.29. A mixed strategy η_1^{B} for Player 1 in a Blackwell game $\mathsf{B}(X, Y, \phi)$ is called *ϵ-optimal*, for $\epsilon \in [0, 1]$ if and only if the following inequality holds:

$$\prod_{\eta_2^{\mathsf{B}}} \Big(\int_{(X \times Y)^\omega} \phi \, \mathrm{d}\mathbb{P}_{\eta_1^{\mathsf{B}}, \eta_2^{\mathsf{B}}} \Big) \geq \mathsf{MVAL}_{\downarrow}\big(\mathsf{B}(X, Y, \phi)\big) - \epsilon.$$

Similarly, a mixed strategy η_2^{B} for Player 2 in a Blackwell game $\mathsf{B}(X, Y, \phi)$ is called *ϵ-optimal*, for $\epsilon \in [0, 1]$ if and only if the following inequality holds:

$$\bigsqcup_{\eta_1^{\mathsf{B}}} \Big(\int_{(X \times Y)^\omega} \phi \, \mathrm{d}\mathbb{P}_{\eta_1^{\mathsf{B}}, \eta_2^{\mathsf{B}}} \Big) \leq \mathsf{MVAL}_{\downarrow}\big(\mathsf{B}(X, Y, \phi)\big) + \epsilon.$$

A 0-optimal mixed strategy, for either Player 1 or Player 2, is simply called *optimal*.

Clearly ϵ-optimal mixed strategies for both players always exist for every $\epsilon > 0$, but not necessarily so for $\epsilon = 0$. We are now ready to introduce the notion of Blackwell determinacy under mixed strategies.

Definition 2.3.30. We say that the Blackwell game $\mathsf{B}(X, Y, \phi)$ (with ϕ assumed to be universally measurable) is *determined under mixed strategies*, or just determined[16], if and only if the equality $\mathsf{MVAL}_{\downarrow}\big(\mathsf{B}(X, Y, \phi)\big) = \mathsf{MVAL}_{\uparrow}\big(\mathsf{B}(X, Y, \phi)\big)$ holds. If $\mathsf{B}(X, Y, \phi)$ is determined, we denote with $\mathsf{MVAL}\big(\mathsf{B}(X, Y, \phi)\big)$ its value under mixed strategies.

David Blackwell himself proved in 1969 that if ϕ is the characteristic function of a $\mathbf{\Pi}_2^0$ subset of $(X \times Y)^\omega$, then $\mathsf{B}(X, Y, \phi)$ is determined [10, 11]. Much later, in 1996, Marco Vervoort proved that if ϕ is the characteristic function of a $\mathbf{\Sigma}_3^0$ subset of $(X \times Y)^\omega$, then $\mathsf{B}(X, Y, \phi)$ is determined [110]. Finally in 1998, Donald A. Martin proved the following theorem[17]:

Theorem 2.3.31 (Martin [74])**.** *For every Borel measurable payoff function ϕ, the game $\mathsf{B}(X, Y, \phi)$ is determined.*

This result had a tremendous impact on the development of (especially applied) game theory in the last decade, as we are going to discuss in later sections.

[16]Note that in the context of Blackwell game we use the word *determined* as a shorthand for *determined under mixed strategies*. This contrasts with the convention adopted in the context of Gale–Stewart games, and highlights the shift of interest towards mixed strategies when working with Blackwell games.

[17]Note that the restriction to finite sets of moves X and Y in Blackwell games in necessary if one is interested in determinacy results. Indeed it is simple to define a Blackwell game $\mathsf{B}(X, Y, \phi)$, with $X = Y = \mathbb{N}$, which is not determined under mixed strategies.

Martin's paper [74] is so rich in content that its Theorem 2.3.31 is just one of the important contributions. One remarkable technical result of [74] is the following theorem, which improves[18] an earlier result of Marco Vervoort [110, Theorem 4.5.7].

Theorem 2.3.32 (Approximation property)**.** *For every Borel set $A \subseteq (X \times Y)^\omega$, the following equalities hold:*

$$\mathsf{MVAL}\big(\mathsf{B}(X,Y,\chi_A)\big) = \bigsqcap_{O \supseteq A} \Big\{\mathsf{MVAL}\big(\mathsf{B}(X,Y,\chi_O)\big)\Big\} = \bigsqcup_{C \subseteq A} \Big\{\mathsf{MVAL}\big(\mathsf{B}(X,Y,\chi_C)\big)\Big\}$$

where O and C ranges over open and closed subsets of $(X \times Y)^\omega$, and χ_S denotes the characteristic function of S, for $S \subseteq (X \times Y)^\omega$.

Proof. See Theorem 5 in [74]. □

A strengthening of Theorem 2.3.32 allowing the approximation of Borel measurable payoffs with simpler payoffs[19] is also discussed in [74].

Another important aspect of Martin's work is that Theorem 2.3.31 was derived as a consequence of Theorem 2.3.15, i.e., Borel determinacy for Gale–Stewart games. More generally, as discussed in [74, §3], the following theorem holds:

Theorem 2.3.33. *For every Blackwell game $\mathsf{B}(X,Y,\phi)$ whose payoff function ϕ is $\mathbf{\Sigma}^1_n$-measurable*[20]*, the following assertion holds:*

$$\text{ZFC} + \mathbf{\Sigma}^1_n\text{-Determinacy} \vdash \text{“}\, \mathsf{B}(X,Y,\phi) \textit{ is determined } \text{”}.$$

Note that, by Theorem 2.3.19, under ZFC + $\mathbf{\Sigma}^1_n$-Determinacy the function ϕ is universally measurable, thus $\mathsf{B}(X,Y,\phi)$ is well defined.

We now introduce a useful generalization of Blackwell games, discussed in [74] and concurrently developed by Ashok P. Maitra and William D. Sudderth in [70], which allows one to model stochastic games.

A *stochastic* Blackwell game $\mathsf{B}(X,Y,Z,\phi,\eta^{\mathsf{B}}_N)$ is similar to an ordinary Blackwell game except that at each step of the game, after Player 1 and Player 2 have concurrently played their moves in X and Y respectively, another player named Nature chooses an element z in the *finite*[21] set Z. Thus a play in $\mathsf{B}(X,Y,Z,\phi,\eta^{\mathsf{B}}_N)$ can be depicted as follows:

[18]The result of [110] is valid only for χ_A with $A \in \mathbf{\Sigma}^0_3$.

[19]These *simple* payoffs are technically called `lim-sup` and `lim-inf` payoffs functions. See, e.g., [22] for a survey.

[20]By this we mean that $\phi^{-1}(S) \in \mathbf{\Sigma}^1_n$, for every Borel $S \subseteq [0,1]$.

[21]As remarked in [74], one can work with countable sets Z without altering, in any significant way, the theory. However for the sake of uniformity we consider finite sets Z.

Player 1:	x_0		x_1		$\dots$	x_n	$\dots$
Player 2:	y_0		y_1		$\dots$	y_n	$\dots$
Nature:		z_0		z_1	$\dots$	z_n	$\dots$

The deterministic strategies available to Player 1 and Player 2 become then of type $(X \times Y \times Z)^{<\omega} \to X$ and $(X \times Y \times Z)^{<\omega} \to Y$ respectively, while the possible deterministic behaviors of Nature are captured by strategies of type $\big((X \times Y \times Z)^{<\omega} \times (X \times Y)\big) \to Z$. The behavior sustained by Nature in the stochastic Blackwell game $\mathsf{B}(X, Y, Z, \phi, \eta_N^{\mathsf{B}})$ is fixed in advance by the mixed strategy η_N^{B}, which is defined as expected, i.e., as a probability measure over deterministic strategies for Nature. The strategy η_N^{B} is known in advance by Player 1 and Player 2, who can then make their rational choices in accordance with this information. The payoff function ϕ in $\mathsf{B}(X, Y, Z, \phi, \eta_N^{\mathsf{B}})$ is a universally measurable function of type $(X \times Y \times Z)^{\omega} \to [0, 1]$. The goal of Player 1 in the stochastic Blackwell game $\mathsf{B}(X, Y, Z, \phi, \eta_N^{\mathsf{B}})$ is to maximize the payoff function, while the dual goal of Player 2 is to minimize it, as in an ordinary Blackwell game.

Remarkably, Martin's proof of Blackwell Borel determinacy under mixed strategies goes through in the stochastic generalization without significant changes.

Theorem 2.3.34. *Let $\mathsf{B}(X, Y, Z, \phi, \eta_N^{\mathsf{B}})$ be a stochastic Blackwell game where the payoff function ϕ is Borel-measurable. Then the following assertion holds: $\mathsf{MVAL}_{\downarrow}\big(\mathsf{B}(X, Y, Z, \phi, \eta_N^{\mathsf{B}})\big) = \mathsf{MVAL}_{\uparrow}\big(\mathsf{B}(X, Y, Z, \phi, \eta_N^{\mathsf{B}})\big)$, where $\mathsf{MVAL}_{\downarrow}$ and $\mathsf{MVAL}_{\uparrow}$ are defined as expected, by adapting Definition 2.3.28 to the new setting.*

More generally, the following theorem holds.

Theorem 2.3.35. *For every stochastic Blackwell game $\mathsf{B}(X, Y, Z, \phi, \eta_N^{\mathsf{B}})$ whose payoff function ϕ is $\boldsymbol{\Sigma}^1_n$-measurable, the following assertion holds:*

$$\text{ZFC} + \boldsymbol{\Sigma}^1_n\text{-Determinacy} \vdash \mathsf{MVAL}_{\downarrow}\big(\mathsf{B}(X, Y, Z, \phi, \eta_N^{\mathsf{B}})\big) = \mathsf{MVAL}_{\uparrow}\big(\mathsf{B}(X, Y, Z, \phi, \eta_N^{\mathsf{B}})\big).$$

Moreover the equivalent of the Theorem 2.3.32 for approximating stochastic Blackwell games holds as well.

We conclude this section by mentioning some recent developments in the theory of Blackwell games. The fact that Borel-determinacy for Blackwell games follows from Borel-determinacy for Gale–Stewart games, and more generally the result of Theorem 2.3.33, are slightly surprising (see, e.g., [57]) as Blackwell games intuitively look more complicated than Gale–Stewart games[22]. A natural question is then if Blackwell determinacy-based axioms (defined in analogy with those formulated in terms of Gale–Stewart games, see e.g. [111, §5], [57] and [74, §3]) imply the corresponding Gale–Stewart determinacy-based axioms such as those discussed at the end of Section 2.3.1. Donald A. Martin conjectured in [74] that this should be the case, and suggested possible directions to approach the problem. However several questions are still open after more than a decade [57, 68].

2.3.3 Generalized Gale–Stewart games

As we saw in Section 2.3.2, Blackwell games $\mathsf{B}(X, Y, \phi)$ deal with real valued payoff functions $\phi\colon (X \times Y)^\omega \to [0,1]$ rather than just winning sets, i.e., characteristic functions χ_A of subsets of $(X \times Y)^\omega$. It is then natural to consider Gale–Stewart games $\mathsf{GS}(X, \phi)$ having real-valued payoff functions as well. As observed in [74], a Gale–Stewart game $\mathsf{GS}(X, \phi)$, having a finite set X of moves, can be seen as a particular kind of Blackwell game $\mathsf{B}(X, X, \phi')$ where the payoff function ϕ' ignores Player 2 moves at even positions, and Player 1 moves at odd positions[23]. More formally the Blackwell game $\mathsf{B}(X, X, \phi')$ which corresponds to the Gale–Stewart game $\mathsf{GS}(X, \phi)$ has function $\phi'\colon (X \times X)^\omega \to [0,1]$ defined as:

$$\phi'\big(\{(x^1_n, x^2_n)\}_{n\in\mathbb{N}}\big) = \phi\big(\{y_n\}_{n\in\mathbb{N}}\big)$$

where $y_{2n} = x^1_{2n}$ and $y_{2n+1} = x^2_{2n+1}$, for every $n \in \mathbb{N}$. As a matter of fact, the restriction on X being a finite set can be relaxed. Indeed one can encode every Gale–Stewart game $\mathsf{GS}(X, \phi)$, with $|X| \leq \aleph_0$ as a Blackwell game $\mathsf{B}(Y, Y, \phi')$, even with $|Y| = 2$. Intuitively it is possible to mimic countable choices by infinite repetitions of binary choices, and slightly more formally this can be captured by an appropriate Borel isomorphism between X^ω and $(Y \times Y)^\omega$. We refer to [74] and [111] for further details.

Gale–Stewart games with real-valued payoff functions can be generalized to stochastic Gale–Stewart games with real-valued payoffs in a similar way. A

[22] We refer to Gale–Stewart games $\mathsf{GS}(X, A)$, with $|X| \leq \aleph_0$, which are indeed those considered in most determinacy-based axioms.

[23] One could also see the Blackwell game $\mathsf{B}(X, X, \phi')$ as a Gale–Stewart game $\mathsf{GS}(X, \phi')$ in which the set of strategies available to Player 2 is restricted to those which *ignore* the last move played by Player 1. Arguably, one could say that a game is of *imperfect information* whenever it is not encodable as a standard Gale–Stewart game without restricting the set of strategies available to the players. Thus Blackwell games are games of imperfect information in this sense.

stochastic Gale–Stewart game $\mathsf{B}(X, \phi, \eta_N^{\mathsf{GS}})$, with $|X| \leq \aleph_0$, is similar to an ordinary Gale–Stewart game except that after Player 2's move, Nature (rather than directly Player 1) makes a move, and then the process is repeated, by allowing Player 1 to make another move and so on, for ever. Thus a play in $\mathsf{GS}(X, \phi, \eta_N^{\mathsf{GS}})$ can be depicted as follows:

Player 1:	x_0			x_3			$\dots$	x_{3n}			$\dots$
Player 2:		x_1			x_4		$\dots$		x_{3n+1}		$\dots$
Nature:			x_2			x_5	$\dots$			x_{3n+2}	$\dots$

The definition of deterministic strategies for Player 1 and Player 2 stay unchanged, i.e., they are maps $X^\omega \to X$, and the set of deterministic strategies for Nature, ranged over by σ_N^{GS}, is defined in the same way. The map $\langle \_ , \_ \rangle^{\mathsf{GS}}$ mapping strategy profiles to plays in X^ω is extended to $\langle \_ , \_ , \_ \rangle^{\mathsf{GS}}$ in the obvious way in order to cope with Nature's deterministic strategies. As for stochastic Blackwell games, the behavior of Nature in the stochastic Gale–Stewart game $\mathsf{GS}(X, \phi, \eta_N^{\mathsf{GS}})$ is fixed in advance by the mixed strategy η_N^{GS}, which is defined as a probability measure over deterministic strategies for Nature. The strategy η_N^{GS} which is going to be used by Nature in the game $\mathsf{GS}(X, \phi, \eta_N^{\mathsf{GS}})$ is known in advance by Player 1 and Player 2, who can then make their rational choices in accordance with this information. Lastly, the payoff function ϕ is a universally measurable function of type $X^\omega \to [0, 1]$.

Given a pair of deterministic strategies $(\sigma_1^{\mathsf{GS}}, \sigma_2^{\mathsf{GS}})$ in $\mathsf{GS}(X, \phi, \eta_N^{\mathsf{GS}})$ for Player 1 and Player 2 respectively, a unique probability measure $\mathbb{P}_{\sigma_1^{\mathsf{GS}}, \sigma_2^{\mathsf{GS}}}$ over X^ω is induced as follows.

Definition 2.3.36. We define $\mathbb{P}_{\sigma_1^{\mathsf{GS}}, \sigma_2^{\mathsf{GS}}}$ as $\mathcal{M}_1\big(\langle \sigma_1^{\mathsf{GS}}, \sigma_2^{\mathsf{GS}}, \_ \rangle^{\mathsf{GS}}\big)(\eta_N^{\mathsf{GS}})$, or equivalently as the probability measure uniquely specified by the following assignment

$$\mathbb{P}_{\sigma_1^{\mathsf{GS}}, \sigma_2^{\mathsf{GS}}}(S) = \eta_N^{\mathsf{GS}}\Big(\{\sigma_N^{\mathsf{GS}} \mid \langle \sigma_1^{\mathsf{GS}}, \sigma_2^{\mathsf{GS}}, \sigma_N^{\mathsf{GS}} \rangle^{\mathsf{GS}} \in S\}\Big)$$

on all Borel sets $S \subseteq \mathbb{N}^\omega$.

The upper and lower values under deterministic strategies of the game $\mathsf{GS}(X, \phi, \eta_N^{\mathsf{GS}})$ are generalized, in the expected way, as follows:

Definition 2.3.37. We define the lower and upper values $\mathsf{VAL}_\downarrow\big(\mathsf{GS}(X, \phi, \eta_N^{\mathsf{GS}})\big)$ and $\mathsf{VAL}_\uparrow\big(\mathsf{GS}(X, \phi, \eta_N^{\mathsf{GS}})\big)$ of $\mathsf{GS}(X, \phi, \eta_N^{\mathsf{GS}})$ under deterministic strategies as follows:

$$\mathsf{VAL}_{\downarrow}\big(\mathsf{GS}(X,\phi,\eta_N^{\mathsf{GS}})\big) = \bigsqcup_{\eta_1^{\mathsf{GS}}} \bigsqcap_{\eta_2^{\mathsf{GS}}} \Big(\int_{X^\omega} \phi \, \mathrm{d}\mathbb{P}_{\sigma_1^{\mathsf{GS}},\sigma_2^{\mathsf{GS}}} \Big),$$

$$\mathsf{VAL}_{\uparrow}\big(\mathsf{GS}(X,\phi,\eta_N^{\mathsf{GS}})\big) = \bigsqcap_{\eta_2^{\mathsf{GS}}} \bigsqcup_{\eta_1^{\mathsf{GS}}} \Big(\int_{X^\omega} \phi \, \mathrm{d}\mathbb{P}_{\sigma_1^{\mathsf{GS}},\sigma_2^{\mathsf{GS}}} \Big).$$

What makes stochastic Gale–Stewart games (with countable set of moves) with real valued payoff functions interesting, and not just a natural fragment of Blackwell games, is that they are determined under *deterministic* strategies whenever the payoff function ϕ is Borel-measurable, as the following theorem, again due to Donald Martin [74], states.

Theorem 2.3.38. *Every stochastic Gale–Stewart game* $\mathsf{GS}(\mathbb{N},\phi,\eta_N^{\mathsf{GS}})$*, with* ϕ *Borel-measurable, is determined under deterministic strategies, i.e., the following equality holds:* $\mathsf{VAL}_{\downarrow}\big(\mathsf{GS}(X,\phi,\eta_N^{\mathsf{GS}})\big) = \mathsf{VAL}_{\uparrow}\big(\mathsf{GS}(X,\phi,\eta_N^{\mathsf{GS}})\big)$.

More generally the following theorem holds [74]:

Theorem 2.3.39. *For every stochastic Gale–Stewart game* $\mathsf{GS}(\mathbb{N},\phi,\eta_N^{\mathsf{GS}})$ *whose payoff function* ϕ *is* $\mathbf{\Sigma}^1_n$*-measurable, the following assertion holds:*

$$\text{ZFC} + \mathbf{\Sigma}^1_n\text{-Determinacy} \vdash \mathsf{VAL}_{\downarrow}\big(\mathsf{GS}(X,\phi,\eta_N^{\mathsf{GS}})\big) = \mathsf{VAL}_{\uparrow}\big(\mathsf{GS}(X,\phi,\eta_N^{\mathsf{GS}})\big).$$

where the set-theoretic axiom of $\mathbf{\Sigma}^1_n$-Determinacy *is formulated as in Definition 2.3.17, i.e., in terms of standard Gale–Stewart games with winning sets.*

Moreover the equivalent of the Theorem 2.3.32 for approximating stochastic Blackwell games holds as well.

We refer to [74] for detailed proofs of these theorems. However we find it useful to explain why these theorems hold for stochastic Gale–Stewart games with real-valued payoff while they do not hold for general Blackwell games, which are indeed determined only under mixed strategies. The technical tool adopted in Martin's proof for proving determinacy (under mixed strategies) of Blackwell games consists in constructing an infinite sequence of *one-step* games, choosing *optimal* strategies for each of them, and combining these locally optimal strategies to produce an ϵ-optimal strategy in the original Blackwell game. In the case general Blackwell games, the one-step games which are considered are *concurrent one-step zero-sum* games. These games are known to be determined under mixed strategies and to admit optimal mixed strategies by Von Neumann's celebrated Minimax theorem [89]. That's why the ϵ-optimal strategy for the Blackwell game, which is built from this collection of local mixed strategies, is mixed. Similarly, in stochastic Blackwell games, the local games are *two-steps games*, where the first step is played concurrently by Player 1 and Player 2, and the third step is probabilistic, i.e., made by Nature. Again, these games are known to be determined under mixed strategies and to admit optimal mixed strategies by Von Neumann's theorem. When considering stochastic Gale–Stewart games $\mathsf{GS}(\mathbb{N}, \phi, \eta_N^{\mathsf{GS}})$ (seen as Blackwell games $\mathsf{B}(X, X, X, \phi', \eta_N^{\mathsf{B}})$, with $|X| = 2$, as discussed before), the collection of local games consists of *three-step turn-based* games, where the third step is probabilistic, i.e., made by Nature. These games are known to be determined under deterministic strategies and to admit optimal deterministic strategies. Hence, the ϵ-optimal strategy for the stochastic Gale–Stewart game, which is built from this collection of local deterministic strategies, is deterministic as desired. We remark that even if every stochastic Gale–Stewart game $\mathsf{GS}(\mathbb{N}, \phi, \eta_N^{\mathsf{GS}})$ is determined under deterministic strategies, there may not exist an optimal strategy, i.e., a 0-optimal strategy in the sense of Definition 2.3.24, for either player.

We conclude this section by remarking, once again, how deep the results of Donald A. Martin in [74] are. Indeed they provide solid mathematical foundations for two players games with real-valued payoffs, both for the turn based model (i.e., Gale–Stewart games) and for the concurrent model (i.e., Blackwell games), and for their stochastic generalizations. We remark that this theory only copes with Blackwell games having finite sets of moves available to the players, and Gale–Stewart games with at most countably many moves.

2.3.4 Two player games on graphs

As discussed in Section 2.3.3, Blackwell and Gale–Stewart games provide a mathematical foundation for two player games of infinite duration having real-valued payoff functions, when the players play their moves concurrently (Blackwell games) or when they alternate their choices (Gale–Stewart games), and for their stochastic generalizations.

However, it is often quite clumsy to model games directly in terms of abstract Blackwell games or Gale–Stewart games. Indeed it is often much more convenient to work with *games with rules*. Suppose for example that we want to model the game of Chess as a Gale–Stewart game $\mathsf{GS}(X, \phi)$, where X is the set of all possible moves[24], such as "move the piece which is currently in the `A4` square to the `A8` square", and the payoff function ϕ assigns 1 and 0 to all plays which at some point reach a checkmate position winning for white and black respectively, and $\frac{1}{2}$ to all other plays, thus modeling a draw. To model the game properly we would need to prevent both players making illegal moves, such as trying to move a piece from `A4` to `A8` when `A4` is empty, or trying to move an opposite-color piece, *etcetera*. This can be done by specifying the function ϕ to assign rewards 1 and 0 to every play on which Black or White made the first illegal move, respectively. However, even if mathematically clean, this way of specifying the payoff functions is quite indirect. Furthermore, once a checkmate position for either player is reached, there is little point in carrying on playing an infinity of other moves because the reward assigned by ϕ is already fixed.

For these reasons it is often convenient to play the game on a *tree* or *graph* structure rather than just allowing the two players to pick their choices from an initially given sets of available moves. If the current game-position is at some node in the game-graph, then the player who ought to move (in a turn based game, say) has to choose among one of the successor nodes of the current node from which the rest of the game will continue. If no successor states are available, then the game just ends. As a matter of fact this way of describing the dynamics of the game is not only convenient for modeling purposes, but also for establishing theoretical results. For example all the results of Donald A. Martin in [74], are proven by working with Blackwell games played on trees. Note that games played on graph structures can always be converted into games on trees, just by unraveling the graph into a tree in the obvious way. Thus there is no significant mathematical difference in the tree and graph models. Graphs, however, are often useful to describe in a more succinct and informative way the corresponding tree. As an extreme example, the tree corresponding to a finite graph is, in general, infinite.

We now introduce the graph-based formulation of Gale–Stewart games and their real-valued payoffs and stochastic generalizations. We refer to [21] for an exposition of the graph-based formulation of Blackwell games. Before embarking upon the technical definitions, we remark once again that two player games played on graphs or trees are just convenient ways to describe the desired Gale–Stewart or Blackwell game, and all the theoretical results of sections 2.3.1, 2.3.2 and 2.3.3 apply to their graph-based formulations.

[24] $|X| < 64^2$.

Definition 2.3.40. A *two player stochastic game arena*, or just a $2\frac{1}{2}$-player arena, is a tuple $\mathcal{A}=\langle(S,E),(S_1,S_2,S_N),\pi\rangle$ where S is a *countable* set of game-states and $E\subseteq S\times S$ is called the transition relation of $\mathcal{A}$. The sets (S_1,S_2,S_N) form a partition of S, and $\pi: S_N \rightarrow \mathcal{D}(S)$ is a map assigning to each state in S_N a probability distribution over S. For each $s\in S$, we denote with $E(s)$ the set $\{t \mid (s,t)\in E\}$, and we refer to it as the set of successor states of s in $\mathcal{A}$. The states $s\in S$ such that $E(s)=\emptyset$ are called *terminal states* of $\mathcal{A}$. As a technical requirement, we impose that $E(s)=supp(\pi(s))$, for every $s\in S_N$. Thus, $E(s)\neq\emptyset$ for all $s\in S_N$. The states in S_1 and S_2 are called Player 1's states and Player 2's states respectively, and the states in S_N are called *probabilistic* states. We say that $\mathcal{A}$ is a 2*-player arena* if $S_N=\emptyset$.

Definition 2.3.41 (Paths in $\mathcal{A}$)**.** The sets of *finite*, *terminated*, *infinite* and *completed paths* in $\mathcal{A}=\langle(S,E),(S_1,S_2,S_N,\pi)\rangle$, denoted by $\mathcal{P}^{<\omega}_{\mathcal{A}}$, $\mathcal{P}^{t}_{\mathcal{A}}$, $\mathcal{P}^{\omega}_{\mathcal{A}}$ and $\mathcal{P}_{\mathcal{A}}$ respectively, are defined as the corresponding sets of paths in the graph (S,E) (see Definition 2.1.33). We denote with $\mathcal{P}^{<\omega}_{1}$ and $\mathcal{P}^{<\omega}_{2}$ the sets of finite paths $\vec{s}$ such that $last(\vec{s})$ is in S_1 and S_2 respectively. We often just write $\mathcal{P}$ for $\mathcal{P}_{\mathcal{A}}$, and similarly for the other sets defined on $\mathcal{A}$, if the arena $\mathcal{A}$ is clear from the context. The set $\mathcal{P}$ of completed paths in $\mathcal{A}$ is endowed with a 0-dimensional Polish topology specified as in Definition 2.1.33.

Definition 2.3.42 (Two player stochastic game)**.** A two player stochastic game, or just a $2\frac{1}{2}$-player game, is a pair $\mathcal{G}=\langle\mathcal{A},\Phi\rangle$, where $\mathcal{A}$ is a $2\frac{1}{2}$-player game arena and the payoff function Φ is a universally measurable function of type $\mathcal{P}_{\mathcal{A}}\rightarrow[0,1]$. The game $\mathcal{G}$ is called a 2*-player game* if $\mathcal{A}$ is a 2-player arena.

A $2\frac{1}{2}$-player game played on the arena $\mathcal{A}$, starts at a given state $s_0\in S$. If the game's current state is some state $s\in S_1$, i.e., under the control of Player 1, then Player 1 has to choose a successor state in the set $E(s)$ from which the game will proceed. If $E(s)=\emptyset$, then Player 1 gets stuck, and the game terminates immediately[25]. Similarly if $s\in S_2$, then Player 2 has to choose a successor state in $E(s)$ and if $E(s)$ is empty the game terminated. If the game's current state is some probabilistic state $s\in S_N$, then Nature moves to the successor state t with probability $\pi(s)(t)$. Note that, since $E(s)\neq\emptyset$, the game can never terminate at a probabilistic state. The outcome of a play of the three players in a $2\frac{1}{2}$-player game

[25] It is often useful to work with 2-player games where the player who gets stuck loses, but this situation is not assumed, nor required, in general.

is a completed path $\vec{s} \in \mathcal{P}_{\mathcal{A}}$ in $\mathcal{A}$, i.e., either a finite path ending in a terminal state, or an infinite path. If $\vec{s}$ is the outcome of the game, Player 1 receives the payoff $\Phi(\vec{s})$. The objective of Player 1 in the $2\frac{1}{2}$-player game $\mathcal{G}$ is to maximize the payoff function Φ. The dual objective of Player 2 is to minimize it.

To specify formally how Player 1 and Player 2 interact in the 2-player game $\mathcal{G}$ played on the arena $\mathcal{A}$, we define the notion of deterministic strategies.

Definition 2.3.43 (Deterministic strategies)**.** A *deterministic strategy* σ_1 for Player 1 in $\mathcal{A}$ is defined as a function $\sigma_1 : \mathcal{P}_1^{<\omega} \rightarrow S \cup \{\bullet\}$ such that $\sigma_1(\vec{s}) \in E(last(\vec{s}))$ if $E(last(\vec{s})) \neq \emptyset$ and $\sigma_1(\vec{s}) = \bullet$ otherwise. Similarly a deterministic strategy σ_2 for Player 2 is defined as a function $\sigma_2 : \mathcal{P}_2^{<\omega} \rightarrow S \cup \{\bullet\}$. A pair $\langle \sigma_1, \sigma_2 \rangle$ of deterministic strategies, one for each player, is called a *deterministic strategy profile* and determines the behavior of both players. We denote with Σ_1 and Σ_2 the set of deterministic strategies available to Player 1 and Player 2 respectively.

Definition 2.3.44 (Topologies on Σ_1 and Σ_2)**.** Let us denote with $O_{\vec{s} \mapsto s}$, for $\vec{s} \in \mathcal{P}_1^{<\omega}$ and $s \in S$, the set of all strategies σ_1 for Player 1 such that $\sigma_1(\vec{s}) = s$. We fix the topology on Σ_1, where the countable basis for the open sets is given by the clopen sets $O_{\vec{s} \mapsto s}$, for every pair $(\vec{s}, s)$ as defined above. This is a 0-dimensional Polish space. The topology on Σ_2 is defined in a similar way.

A particular kind of deterministic strategy is given by the so-called *positional* or *memoryless* strategies which, intuitively, base their choices on a particular path $\vec{s}$ only considering the last state $last(\vec{s})$ of the path $\vec{s}$.

Definition 2.3.45. Let $\sigma_1 : \mathcal{P}_1^{<\omega} \rightarrow S \cup \{\bullet\}$ be a deterministic strategy for Player 1 in the arena $\mathcal{A}$. We say that σ_1 is a *positional strategy* if $\sigma_1(\vec{s}) = f\big(last(\vec{s})\big)$ for some function $f : S_1 \rightarrow S \cup \{\bullet\}$. A positional strategy for Player 2 is defined in the analogous way.

It is going to be useful to consider a structure, which we call *Markov play*[26], that formalizes the notion of outcome of the game *up-to* the behavior of Nature.

Definition 2.3.46 (Markov play)**.** A *Markov play* M in a $2\frac{1}{2}$-player game arena $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N), \pi \rangle$ is a tree in (S, E) (see Definition 2.1.35) uniquely branching in $S_1 \cup S_2$ and fully branching in S_N, in the sense of Definition 2.1.36.

[26] We use this terminology because a Markov play is a *Markov chain* defining a probability measure over plays. Many authors refer to them simply as Markov chains (see, e.g., see e.g. [21, 117]).

The nodes of a Markov play M with more than one child are all labeled with a state $s \in S_N$ and their children represent the possible choices of Nature at that state. Note that if $\mathcal{A}$ is a 2-player arena, the a Markov play M can be considered as a completed path in $\mathcal{P}_{\mathcal{A}}$.

Markov plays are useful structures because it is possible to extract from them the probability that Nature, with its probabilistic choices, will produce an outcome contained in some given set of completed paths. This is formally captured by the following definition.

Definition 2.3.47 (Probability measure $\mathbb{P}(M)$)**.** Every Markov play M determines a probability assignment $\mathbb{P}_M(O_{\vec{s}})$ to every basic clopen set $O_{\vec{s}} \subseteq \mathcal{P}$, for $\vec{s} = \{s_i\}_{0 \leq i \leq n}$ a finite path, defined as follows:

$$\mathbb{P}_M(O_{\vec{s}}) \stackrel{\text{def}}{=} \prod \big\{ \pi(s_i)(s_{i+1}) \mid i < n \text{ and } s_i \in S_N \big\}$$

More informally, $\mathbb{P}_M(O_{\vec{s}})$ is the multiplication of all probabilities labeling, via the map π, the edges connecting the probabilistic states in $\vec{s}$ with their successors. The assignment $\mathbb{P}_M$ on basic clopen sets extends to a unique complete probability measure $\mathbb{P}_M \in \mathcal{M}_1(\mathcal{P})$.

The value $\mathbb{P}_M(X)$, for some $\mathbb{P}_M$-measurable set $X \subseteq \mathcal{P}$, models the probability that the outcome of the game, when M is the result of a play in the $2\frac{1}{2}$-player game up-to the behavior of Nature, is a completed path in X.

As discussed earlier, a Markov play M represents the result of a play of the two players up to the behavior of Nature. This is made precise by the following definition:

Definition 2.3.48. Given an initial state $s_0 \in S$ and a strategy profile (σ_1, σ_2) a unique Markov play, denoted by $M^{s_0}_{\sigma_1,\sigma_2}$, is determined:

1. the root of M is s_0,
2. for every $\vec{s} \in M^{s_0}_{\sigma_1,\sigma_2}$, if $last(\vec{s}) = s$ with $s \in S_1$ not a terminal state, then the unique child of $\vec{s}$ in $M^{s_0}_{\sigma_1,\sigma_2}$ is $\vec{s}.\{\sigma_1(\vec{s})\}$,
3. for every $\vec{s} \in M^{s_0}_{\sigma_1,\sigma_2}$, if $last(\vec{s}) = s$ with $s \in S_2$ not a terminal state, then the unique child of $\vec{s}$ in $M^{s_0}_{\sigma_1,\sigma_2}$ is $\vec{s}.\{\sigma_2(\vec{s})\}$.

The probability $\mathbb{P}_{M^{s_0}_{\sigma_1,\sigma_2}}(X)$, for some $(\mathbb{P}_{M^{s_0}_{\sigma_1,\sigma_2}})$-measurable set X of completed paths, formally captures the probability that the outcome of the $2\frac{1}{2}$-player game played on the arena $\mathcal{A}$ and starting at s_0, is in X, given that Player 1 and Player 2 follow the deterministic strategies σ_1 and σ_2 respectively.

Definition 2.3.49. Let $\langle \mathcal{A}, \Phi \rangle$ be a $2\frac{1}{2}$-player game, and let $M^{s_0}_{\sigma_1,\sigma_2}$ be the Markov play induced by the strategy profile (σ_1, σ_2) when the game starts at s_0. The *expected payoff* associated with $M^{s_0}_{\sigma_1,\sigma_2}$, denoted by $E(M^{s_0}_{\sigma_1,\sigma_2})$, is defined as follows:

$$E(M^{s_0}_{\sigma_1,\sigma_2}) = \int_{\mathcal{P}} \Phi \, \mathrm{d}\mathbb{P}_{\mathrm{M}^{s_0}_{\sigma_1,\sigma_2}}.$$

This is a good definition because Φ is assumed to be universally measurable.

Definition 2.3.50. Given a $2\frac{1}{2}$-player game $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$, we define its *lower* and *upper values* under deterministic strategies when the game starts at $s \in S$, denoted by $\mathsf{VAL}^s_\downarrow(\mathcal{G})$ and $\mathsf{VAL}^s_\uparrow(\mathcal{G})$ respectively, as follows:

$$\mathsf{VAL}^s_\downarrow(\mathcal{G}) = \bigsqcup_{\sigma_1} \bigsqcap_{\sigma_2} E\big(M^s_{\sigma_1,\sigma_2}\big),$$

$$\mathsf{VAL}^s_\uparrow(\mathcal{G}) = \bigsqcap_{\sigma_2} \bigsqcup_{\sigma_1} E\big(M^s_{\sigma_1,\sigma_2}\big).$$

We can now state the following theorems.

Theorem 2.3.51. *For any $2\frac{1}{2}$-player game $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$, with Φ a Borel measurable payoff function, and for every state $s \in S$, the following equality holds:*

$$\mathsf{VAL}^s_\downarrow(\mathcal{G}) = \mathsf{VAL}^s_\uparrow(\mathcal{G}).$$

Theorem 2.3.52. *For every $2\frac{1}{2}$-player game $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$ whose payoff function Φ is $\mathbf{\Sigma}^1_n$-measurable, the following assertion holds:*

$$\text{ZFC} + \mathbf{\Sigma}^1_n\text{-Determinacy} \vdash \mathsf{VAL}^s_\downarrow(\mathcal{G}) = \mathsf{VAL}^s_\uparrow(\mathcal{G}).$$

Both theorems follows from the the corresponding theorems for stochastic Gale–Stewart with real valued payoff functions of Section 2.3.3. Is it indeed simple enough to convert a $2\frac{1}{2}$-player game $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$ into an equivalent stochastic Gale–Stewart game $\mathsf{GS}(S, \phi, \eta_N)$.

2.3.4.1 Two player parity games

Two player (stochastic) games on graphs have found plenty of applications in computer science in the last decades. One important example arises when the vertices and edges of the game-graph represent the states and transitions of a reactive system, and the two players represent controllable versus uncontrollable decisions during the execution of the system. A reactive system satisfies a certain specification if Player 1 has a winning strategy in the corresponding two player, possibly stochastic, game.

An important class of two player (stochastic) games, which is sufficient to model many interesting specifications (see, e.g., [21]), is given by those games $\mathcal{G} = \langle \mathcal{A}, \chi_C \rangle$, whose payoff function χ_C is the characteristic function of a *parity set* C of paths, which we now formally define.

Definition 2.3.53 (Parity assignment)**.** Let $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N), \pi \rangle$ be a $2\frac{1}{2}$-player arena. A *parity assignment*, or a *priority assignment*, for a $\mathcal{A}$ is a function $\mathrm{Pr} : S \to \mathbb{N}$, such that the set $\mathrm{Pr}(S) = \{n \mid \exists s \in S. \mathrm{Pr}(s) = n\}$ is finite. In other words Pr assigns to each state $s \in S$ a natural number, also referred to as a *priority*, taken from a finite pool of options $\{n_0, \ldots, n_k\} = \mathrm{Pr}(S)$. We denote with $\max(\mathrm{Pr})$, $\min(\mathrm{Pr})$ and $|\mathrm{Pr}|$ the natural numbers $\max\{n_0, \ldots, n_k\}$, $\min\{n_0, \ldots, n_k\}$ and $|\{n_0, \ldots, n_k\}|$ respectively.

Definition 2.3.54. Let $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N), \pi \rangle$ be a $2\frac{1}{2}$-player arena and Pr a parity assignment for it. Let $\mathcal{W}_{\mathrm{Pr}} \subseteq \mathcal{P}$ be the set of all completed paths $\vec{s} \in \mathcal{P}_{\mathcal{A}}$ such that:

1. $\vec{s}$ is a finite terminated path, i.e., $\vec{s} \in \mathcal{P}^t$, and the priority assigned to the last state of $\vec{s}$ is odd, i.e., $\mathrm{Pr}\big(last(\vec{s})\big) \equiv 1 \pmod 2$, or
2. $\vec{s}$ is infinite, i.e., $\vec{s} \in \mathcal{P}^\omega$ with $\vec{s} = \{s_i\}_{i \in \mathbb{N}}$, and the greatest priority assigned to infinitely many states s_i in $\vec{s}$ is even, i.e., $\big(\limsup_{i \in \mathbb{N}} \mathrm{Pr}(s_i)\big) \equiv 0 \pmod 2$.

The set $\mathcal{W}_{\mathrm{Pr}}$ is called the *parity set* induced by the parity assignment Pr. A subset $X \subseteq \mathcal{P}$ is a *parity set* if $X = \mathcal{W}_{\mathrm{Pr}}$ for some parity assignment Pr for $\mathcal{A}$.

We are now ready to define the class of two player (stochastic) parity games.

Definition 2.3.55. A $2\frac{1}{2}$-player game $\langle \mathcal{A}, \chi_C \rangle$, where $\chi_C : \mathcal{P}_{\mathcal{A}} \to \{0, 1\}$ is the characteristic function of a set $C \subseteq \mathcal{P}_{\mathcal{A}}$, is called a $2\frac{1}{2}$-player *parity game* if $C = \mathcal{W}_{\mathrm{Pr}}$ for some parity assignment Pr for $\mathcal{A}$.

Lemma 2.3.56. *Let $\langle \mathcal{A}, \mathcal{W}_{\mathrm{Pr}} \rangle$ be a $2\frac{1}{2}$-player parity game, where $\mathcal{W}_{\mathrm{Pr}}$ is the winning set induced by the parity assignment* Pr *for $\mathcal{A}$. Then the set $\mathcal{W}_{\mathrm{Pr}}$ is a $\mathbf{\Delta}^0_3$ set, hence a Borel set.*

Proof. See e.g. [21]. □

In particular, parity winning sets are closed under complementation.

Proposition 2.3.57. *Let* $\mathrm{Pr} : S \to \mathbb{N}$ *be a parity assignment on some* $2\frac{1}{2}$*-player game arena* $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N), \pi \rangle$ *and let* $\neg \mathrm{Pr}$ *be the parity assignment defined as* $\neg \mathrm{Pr}(s) = \mathrm{Pr}(s) + 1$*. Then the following assertions hold:*

1. $\min(\neg \mathrm{Pr}) = \min(\mathrm{Pr}) + 1$,
2. $\max(\neg \mathrm{Pr}) = \max(\mathrm{Pr}) + 1$,
3. $|\,\mathrm{Pr}\,| = |\neg \mathrm{Pr}\,|$,
4. $\mathcal{W}_{\neg \mathrm{Pr}} = \overline{\mathcal{W}_{\mathrm{Pr}}}$.

As a corollary of Theorem 2.3.51 we have that every $2\frac{1}{2}$-player (stochastic) parity game is determined under deterministic strategies. Moreover the following stronger theorem holds.

Theorem 2.3.58 (Positional Determinacy)**.** *If* $\mathcal{G}$ *is a (possibly infinite) 2-player (non-stochastic) parity game or a finite* $2\frac{1}{2}$*-player parity game, then Player 1 and Player 2 have optimal positional strategies (see Definition 2.3.45).*

Proof. For a proof of positional determinacy for (possibly infinite) 2-player parity games see, e.g., [116]. For a proof of positional determinacy for finite $2\frac{1}{2}$-player parity games see, e.g., [117]. □

Two player (stochastic) parity games received in the past twenty years a lot of attention for their theoretical as well as practical interest [21, 106, 116, 117]. Moreover, as we shall see in the next section, they provide a game-theoretical semantics for the modal μ-calculus $\mathrm{L}\mu$. Another important class of winning sets is given by the so-called *prefix-independent* winning sets, also known as *tail* winning sets [43, 47].

Definition 2.3.59 (Prefix independent set)**.** Let $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N), \pi \rangle$ be a $2\frac{1}{2}$-player game arena. A set $X \subseteq \mathcal{P}_{\mathcal{A}}$ is a *prefix independent set* if the following property holds: if $\vec{s} = \vec{t}.\vec{r}$ is a completed path in $\mathcal{P}_{\mathcal{A}}$ then, $\vec{r} \in X$ if and only if $\vec{s} \in X$.

Lemma 2.3.60. *If* $X \subseteq \mathcal{P}_{\mathcal{A}}$ *is a prefix independent set, then also* $\overline{X}$ *is prefix independent.*

It is immediate to verify that every parity set $\mathcal{W}_{\mathrm{Pr}}$ is a prefix independent set.

Proposition 2.3.61. *Let $\mathcal{A}$ be a $2\frac{1}{2}$-player game arena and* Pr *a parity assignment for it. The winning set $\mathcal{W}_{\mathrm{Pr}}$ is prefix independent.*

Prefix independent sets include many other interesting classes of winning sets. For instance natural generalizations of the notion of parity set, considering parity assignments having infinite (rather than finite) range, lead to interesting prefix independent sets (see e.g. [47]). We remark that prefix independent sets can be complicated objects. See, e.g., [13] for an example of prefix independent set which is not universally measurable.

Chapter 3

Program Logics

This chapter is organized as follows. In Section 3.1 we provide the necessary background on temporal logics from Labeled Transition Systems (LTS's), and in particular we consider Computation Tree Logic (CTL) and the modal μ-calculus (Lμ). In Section 3.2 we turn our attention to temporal logics from expressing properties of Probabilistic Labeled Transition Systems (PLTS's). We consider Probabilistic CTL (PCTL) and the probabilistic modal μ-calculus (pLμ). We define the game and denotational semantics of pLμ as specified in [78], and discuss a few examples of interesting properties expressed by pLμ formulas. In Section 3.3 we introduce the logic pL$\mu^{\odot}_{\oplus}$, the strongest probabilistic logic considered in this thesis, obtained by extending pLμ with additional connectives. We motivate our interest for this rich logic by discussing the expressive power of some of its fragments in terms of the ability to encode useful operators, such as the qualitative and quantitative modalities. We conclude Section 3.3 by discussing, informally, the intuitions which will lead us towards the definition of appropriate *game semantics* for each of the proposed μ-calculi. Chapters 4, 5, 6 and 7 are devoted to the formalization and mathematical development of these ideas.

3.1 Temporal logics

The term *temporal logic* has been broadly used, during the 20th century, to cover all approaches to the representation of temporal information within a logical framework. Of particular interest for our discussion is the so-called *modal* approach to temporal logics which dates back to the seminal work of Arthur Prior [98]. In this approach standard classical (propositional) logic is extended with *modalities* which express temporal statements. After the seminal work of Amir Pnueli [97], the modal style of temporal logic has found extensive application in the area of Computer Science concerned with the specification and verification of programs, especially concurrent programs in which the computation is performed by two or more components working in parallel. Programs are represented by Kripke-structures (or transition systems) which describe the global state on which they are executed and the way they change. Temporal (modal) logics can then express properties of programs such as: "in the next state ϕ holds", "in every reachable state, ϕ holds", "there is a reachable state on which ϕ holds", *etcetera*.

During the 70–80's the theory of process-calculi was being developed, most notably by Robin Milner [79]. An essential component of this line of work was the use of labeled Kripke structures, also known as *labeled transition systems* or LTS for short, as models for concurrent programs. The labels, which decorate the transitions of a LTS, describe the type of such transitions, and can be used to model several aspects of reactive systems, such as interactions occurring on named-channels, message-passing, actions, *etcetera*. In [52], Robin Milner and Matthew Hennessy introduced a primitive modal logic in which the modalities directly refer to the labels. This logic is today very well known as the Hennessy-Milner (HML) logic. One of the key aspect of HML is that its induced logical equivalence coincides[1] with the important notion of behavioral equivalence given by *bisimilarity* [79]. Thus HML can express enough properties to distinguish processes that are not behavioral equivalent, and at the same time no HML formula can distinguish between bisimilar processes. Even though, thanks to this correspondence, the logic HML is theoretically very interesting, it is not possible to express, within this minimal modal logic, interesting properties such as liveness and safety of programs.

In 1983, Dexter Kozen introduced in [62] a logic combining the simple modalities of HML together with fixed-point operators, to provide a form of recursion. This logic, known as the modal μ-calculus (Lμ), has since then been widely studied. The modal μ-calculus has a simple syntax and an easily given denotational semantics, and yet it has a great expressive power. Indeed, most of nowadays popular temporal logics can be seen as proper fragments of Lμ. This is under-

[1]The coincidence holds when finite-branching LTS's are considered. Alternatively, without restricting the class of transition systems, the equivalence holds consider an infinitary version of the logic HML.

lined by the fact that every formula of monadic second order logic over LTS which does not distinguish between bisimilar models is equivalent to a Lμ formula [58]. An important result in the theory of the modal μ-calculus was obtained in [32] by E. A. Emerson and C. S. Jutla. They introduced an alternative semantics for Lμ based on 2-player parity games (see Definition 2.3.55). Providing semantics to logics in terms of logical games dates back at least to the seminal work of Jaakko Hintikka [55] and his 2-player *game semantics* for first order classical logic. Game semantics for logics, especially temporal logics for program verification (of which Lμ is an important example), proved to be very useful. Not only do they provide a different way of thinking about the meaning of formulas, but they suggest methods for proving interesting results. An important example is the celebrated theorem of Igor Walukiewicz [114] which establishes the completeness of a deductive system for deriving valid Lμ-formulas using game-based methods. Game semantics also turned out to be very helpful in designing algorithms for Model Checking, i.e., for verifying automatically if a given (finitely presentable) model satisfies a given formula.

The rest of this section is organized as follows: we first introduce the notion of labeled transition system (LTS) and the syntax and denotational semantics of the modal μ-calculus (Lμ); then we discuss the game semantics for Lμ of Emerson and Jutla; lastly we discuss another important temporal logic, *computation tree logic* (CTL), and we briefly discuss how CTL can be seen as a proper fragment of Lμ. This introduction, given the immense literature covering the topic of temporal logics for verification and the modal μ-calculus, is necessarily very limited. We suggest [106] and [18] for two extensive introductions to these topics.

3.1.1 Labeled Transition Systems and Modal μ-calculus

We start by defining the notion of labeled transition system (LTS), which is a mathematical structure often used to model concurrent programs and systems [79, 95].

Definition 3.1.1. A *labeled transition systems*, or a LTS for short, is a pair $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$, where P is a set of *process states*, L is a set of *labels*, and the relation $\xrightarrow{a} \subseteq P \times P$ is called the *a-transition relation*, for every $a \in L$. We write $p \xrightarrow{a} q$ for $(p, q) \in \xrightarrow{a}$. For $p, q \in P$, we say that q is a a-successor of p if $p \xrightarrow{a} q$. We write $p \not\xrightarrow{a}$ if the set of a-successors of p is empty. A labeled transition system $\mathcal{L}$ is called *finite-branching* if for every $a \in L$ and $p \in P$, the set of a-successors of p is a finite set.

The intended interpretation of a LTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a \in L}\rangle$ is the following: the process states $p \in P$ represent the possible configurations of the system; at a process state p, the system can react to an a-action, for $a \in L$, by changing its state to a process q with $p \xrightarrow{a} q$. In case p has several a-successors, the choice of which one is reached as a consequence of the a-action is *non-deterministic*, i.e., not predictable. Furthermore, if $p \not\xrightarrow{a}$, the system gets stuck, or does not respond, to the a-action. Often the metaphor given by the idea of reacting to an a-action, is replaced with similar ones, such as: performing an a-action, synchronizing through the a-channel, *etcetera*.

We now introduce the syntax of the modal μ-calculus Lμ.

Definition 3.1.2 (Syntax of Lμ)**.** Given a countable set $\mathcal{V}$ of variables, the syntax of the modal μ-calculus formulas is generated by the following context-free grammar:

$$F, G ::= X \mid F \vee G \mid F \wedge G \mid \langle a \rangle G \mid [a]\, G \mid \mu X.F \mid \nu X.F$$

where X ranges over the set $\mathcal{V}$ of variables, and a over a fixed set L of labels. The variable X is bound in $\mu X.F$ and in $\nu X.F$ by the fixed point operators μ and ν. Given an Lμ formula we denote with *free*(F) and *bound*(F) the sets of free and bound variables in F defined as usual. We say that an Lμ formula F is *closed* if *free*$(F) = \emptyset$. Adopting standard terminology, we say that two formulas are α-equivalent, if they are identical up-to renaming of the bound variables.

It is going to be useful to formally define the set of subformulas of a given Lμ formula.

Definition 3.1.3. The set *Sub*(F) of *subformulas* of a Lμ formula F is defined by case analysis on F as follow:

$$\begin{array}{lcl}
Sub(X) & = & \{X\} \\
Sub(F \vee G) & = & \{F \vee G\} \cup Sub(F) \cup Sub(G) \\
Sub(F \wedge G) & = & \{F \wedge G\} \cup Sub(F) \cup Sub(G) \\
Sub(\langle a \rangle F) & = & \{\langle a \rangle F\} \cup Sub(F) \\
Sub([a]\, F) & = & \{[a]\, F\} \cup Sub(F) \\
Sub(\mu X.F) & = & \{\mu X.F\} \cup Sub(F) \\
Sub(\nu X.F) & = & \{\nu X.F\} \cup Sub(F)
\end{array}$$

A useful syntactical convention consists in restricting our attention to *normal* Lμ formulas [106].

Definition 3.1.4 ([106])**.** A Lμ formula F is *normal* provided that:

1. If $\sigma_1 X_1.G_1$ and $\sigma_2 X_2.G_2$ are distinct sub-formulas of F, for $\sigma_1, \sigma_2 \in \{\mu, \nu\}$, then $X_1 \neq X_2$, and
2. no occurrence of a free variable X is also used in a binder μX or νX in F.

Clearly every Lμ formula can be converted into a α-equivalent normal formula by renaming the bound variables. Working with normal formulas, beside improving readability, allows us to give the following definition in a simple and direct way.

Definition 3.1.5 (Variable Subsumption [106])**.** Given a normal Lμ formula F, we say that the variable X subsumes Y in F if:

1. $\sigma_1 X.G,\ \sigma_2 Y.H \in \mathit{Sub}(F)$ for $\sigma_1, \sigma_2 \in \{\mu, \nu\}$, i.e., both variables appears bound in F, and
2. $\sigma_2 Y.H \in \mathit{Sub}(\sigma_1 X.G)$, i.e., the sub-formula bounded by $\sigma_2 Y$ appears in the scope of the $\sigma_1 X$ binder.

The following proposition follows immediately [106].

Proposition 3.1.6. *The subsumption relation is a partial order, i.e.,*

1. *X subsumes X,*
2. *if X subsumes Z and Z subsumes Y, then X subsumes Y,*
3. *if X subsumes Y and $X \neq Y$, then Y does not subsumes X.*

Modal μ-calculus formulas are interpreted over a LTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a \in L}\rangle$ and their semantics is a predicate specifying which process states satisfy the given formula, i.e., a map in the function space $\{0,1\}^P$ (simply denoted simply as 2^P) assigning to each process state a boolean value in the complete lattice $0 \sqsubseteq 1$. The space 2^P forms a complete lattice lifting the order on $\{0,1\}$ to 2^P pointwise.

The *denotational semantics* is specified compositionally by induction on the structure of Lμ formulas. In order to do this, we first define the notion of interpretation of the variables.

Definition 3.1.7. Given a LTS $\mathcal{L}=\langle P,\{\xrightarrow{a}\}_{a\in L}\rangle$, an *interpretation* of the Lμ variables $\mathcal{V}$ in $\mathcal{L}$ is a map $\rho:\mathcal{V}\to 2^P$. Given an interpretation $\rho\in(\mathcal{V}\to 2^P)$ and a predicate $f\in 2^P$, we denote with $\rho[f/X]$, for $X\in\mathcal{V}$, the interpretation defined as follows:

$$\rho[f/X](Y) = \begin{cases} \rho(Y) & \text{if } X\neq Y \\ f & \text{if } X = Y \end{cases}$$

We are now ready to define the denotational semantics of the modal μ-calculus.

Definition 3.1.8. Given a LTS $\mathcal{L}=\langle P,\{\xrightarrow{a}\}_{a\in L}\rangle$, an interpretation $\rho\in\mathcal{V}\to 2^P$ and a Lμ formula F, we define the *denotational semantics* of F over $\mathcal{L}$ as the map $[\![F]\!]^{\mathcal{L}}_{\rho}\in 2^P$, defined by case analysis on F as follows:

$$\begin{array}{lcl}
[\![X]\!]^{\mathcal{L}}_{\rho}(p) & = & \rho(X)(p) \\
[\![G\vee H]\!]^{\mathcal{L}}_{\rho}(p) & = & [\![G]\!]^{\mathcal{L}}_{\rho}(p)\sqcup[\![H]\!]^{\mathcal{L}}_{\rho}(p) \\
[\![G\wedge H]\!]^{\mathcal{L}}_{\rho}(p) & = & [\![G]\!]^{\mathcal{L}}_{\rho}(p)\sqcap[\![H]\!]^{\mathcal{L}}_{\rho}(p) \\
[\![\langle a\rangle G]\!]^{\mathcal{L}}_{\rho}(p) & = & \bigsqcup_{p\xrightarrow{a}q}[\![G]\!]^{\mathcal{L}}_{\rho}(q) \\
[\![[a]\,G]\!]^{\mathcal{L}}_{\rho}(p) & = & \bigsqcap_{p\xrightarrow{a}q}[\![G]\!]^{\mathcal{L}}_{\rho}(q) \\
[\![\mu X.G]\!]^{\mathcal{L}}_{\rho}(p) & = & \mathrm{lfp}\big(\lambda f\in 2^P.[\![G]\!]_{\rho[f/X]}\big)(p) \\
[\![\nu X.G]\!]^{\mathcal{L}}_{\rho}(p) & = & \mathrm{gfp}\big(\lambda f\in 2^P.[\![G]\!]_{\rho[f/X]}\big)(p)
\end{array}$$

Note that this is a good definition since the space 2^P is a complete lattice and every $L\mu$ connective is interpreted as a monotone operator. Hence least and greatest fixed points exist by the Knaster-Tarski theorem. We say that a process state $p\in P$ is satisfied by the formula F under the interpretation ρ if $[\![F]\!]^{\mathcal{L}}_{\rho}(p)=1$. We often omit the superscript $\mathcal{L}$ in $[\![F]\!]^{\mathcal{L}}_{\rho}$ if the LTS $\mathcal{L}$ is clear from the context.

The denotational semantics defined above is often presented in a slightly more readable way, namely looking at 2^P as the (isomorphic) set of all subsets of P ordered by inclusion. Thus one can replace the lattice-operations of meet and join with the more familiar set-theoretic intersection and union. We opted for the predicate-inclined presentation of the denotational semantics of Lμ for uniformity with the way we will discuss the denotational semantics of probabilistic modal μ-calculi in later sections.

We presented the syntax of Lμ in *positive-form*, i.e., without including a *negation* operator explicitly, in order to simplify the definition of the denotational semantics. However a negation operator can be defined by induction of the syntax of Lμ in a straightforward way, exploiting the dualities between the operators $\langle a \rangle$, $\vee$, $(\mu X.)$ and $[a]$, $\wedge$, $(\nu X.)$.

Definition 3.1.9. Given a closed Lμ formula F, we define its *dual formula* $\overline{F}$ by induction on the structure of F as follows:

$$\overline{\overline{X}} = X$$
$$\overline{F \vee G} = \overline{F} \wedge \overline{G} \qquad \overline{F \wedge G} = \overline{F} \vee \overline{G}$$
$$\overline{\langle a \rangle F} = [a]\,\overline{F} \qquad \overline{[a]\,F} = \langle a \rangle \overline{F}$$
$$\overline{\mu X.F} = \nu X.\overline{F[\overline{X}/X]} \qquad \overline{\nu X.F} = \mu X.\overline{F[\overline{X}/X]},$$

where $F[\overline{X}/X]$ denotes the Lμ formula F where all occurrences of the (free) variable X are replaced by $\overline{X}$.

Proposition 3.1.10. *Given a LTS $\mathcal{L}$, an interpretation ρ and a closed Lμ-formula F, the following assertion holds: $[\![F]\!]^{\mathcal{L}}_{\rho}(p) = 1$ if and only if $[\![\overline{F}]\!]^{\mathcal{L}}_{\rho}(p) = 0$, for every process state p in $\mathcal{L}$.*

Proof. See e.g. [106, §4.6]. □

3.1.2 Game semantics for Lμ

In this subsection we discuss the *game semantics* of the modal μ-calculus, which is given in terms of 2-player parity games, introduced by Emerson and Jutla in [32]. The main idea is that in order to specify the semantics of a Lμ formula over an LTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a \in L} \rangle$ under an interpretation ρ, one adopts game-theoretic rather than lattice and fixed-point methods: the formula F holds at the process state p under the interpretation ρ if Player 1 has a winning strategy in a 2-player parity game $\mathcal{G}(F, \rho)$ which is constructed in a canonical way from $\mathcal{L}$, ρ and F. Otherwise, by the determinacy of 2-player parity games (see Section 2.3.4.1), Player 2 has a winning strategy and the formula F does not hold at p. The logical game $\mathcal{G}(F, \rho)$ has the set of pairs $\langle p, G \rangle \in P \times Sub(F)$ as game states and is played as follows:

- If the current game state is $\langle p, X \rangle$, with $X \in free(F)$ then the game ends in favor of Player 1 if $\rho(X)(p) = 1$, and in favor of Player 2 otherwise.
- If the current game state is $\langle p, G \vee H \rangle$, then Player 1 can move either to the state $\langle p, G \rangle$ or to $\langle p, H \rangle$.

- If the current game state is $\langle p, G \wedge H \rangle$, then Player 2 can move either to the state $\langle p, G \rangle$ or to $\langle p, H \rangle$.

- If the current game state is $\langle p, \langle a \rangle G \rangle$, then Player 1 can move to any state $\langle q, G \rangle$, with $p \xrightarrow{a} q$; if $p \not\xrightarrow{a}$, then Player 1 gets stuck and the game ends in favor of Player 2.

- If the current game state is $\langle p, [a]\, G \rangle$, then Player 2 can move to any state $\langle q, G \rangle$, with $p \xrightarrow{a} q$; if $p \not\xrightarrow{a}$, then Player 2 gets stuck and the game ends in favor of Player 1.

- If the current game state is $\langle p, \mu X.G \rangle$ or $\langle p, \nu X.G \rangle$, then the game automatically progresses to the state $\langle p, G \rangle$.

- If the current game state is $\langle p, X \rangle$, with $X \in \mathit{bound}(F)$, then the game automatically progresses to the state $\langle p, G \rangle$, where $\sigma X.G$ is the (unique since F is normal) sub-formula binding X in F, for $\sigma \in \{\mu, \nu\}$.

A play of the game $\mathcal{G}(F, \rho)$ can either end in a finite time in favor of Player 1 or Player 2 (if the play reaches a state $\langle p, X \rangle$, with $X \in \mathit{free}(F)$, or a state $\langle p, \langle a \rangle G \rangle$ or $\langle p, [a]\, G \rangle$, with $p \not\xrightarrow{a}$) or can last an infinite number of steps. In the latter case it is easily verified that the infinite sequence $\{\langle p_n, G_n \rangle\}_{n \in \mathbb{N}}$ of visited game states is such that for infinitely many $m \in \mathbb{N}$, it holds that $G_m \in \mathit{bound}(F)$. Moreover it is easy to prove the following lemma [106, page 139]:

Lemma 3.1.11. *If $\{\langle p_n, G_n \rangle\}_{n \in \mathbb{N}}$ is an infinite length play in $\mathcal{G}(F, \rho)$ then there is a unique variable $X \in \mathit{bound}(F)$ such that:*

1. *occurs infinitely often, that is for infinitely many $m \in \mathbb{N}$, $X = G_m$, and*

2. *if Y also occurs infinitely often, then X subsumes Y in F.*

The variable X is called the *dominant variable* of the play $\{\langle p_n, G_n \rangle\}_{n \in \mathbb{N}}$.

An infinite play $\{\langle p_n, G_n \rangle\}_{n \in \mathbb{N}}$ in $\mathcal{G}(F, \rho)$ is won by Player 1 if the dominant variable X is bound in F by a greatest fixed point operator ($\nu X.$), and is won by Player 2 otherwise, i.e., if X is bound in F by a least fixed point operator ($\mu X.$).

The above described winning criterion for the game $\mathcal{G}(F, \rho)$ can be formalized as the parity winning set (see Definition 2.3.54) induced by any parity assignment $\mathrm{Pr}\colon \big(P \times \mathit{Sub}(F)\big) \to \mathbb{N}$ satisfying the following specification:

1. If X is a free variable in F then every game state s of the form $\langle p, X\rangle$ is terminal. We then define $\Pr(s)=\rho(X)(p)$. By Definition 2.3.54, this implies that Player 1 wins (loses) when the game reaches states of the form $\langle p, X\rangle$ with $\rho(X)(p)=1$ ($\rho(X)(p)=0$).

2. If s is a terminal game state of the form $\langle p, \langle a\rangle G\rangle$ then $\Pr(s)=0$ and if s of the form $\langle p, [a]\, G\rangle$ then $\Pr(s)=1$. This formalize the idea that the player who gets stuck choosing a transition loses.

3. Let $\alpha : \mathit{bound}(F) \rightarrow \mathbb{N}$ an assignment of natural numbers to the variables bound in F such that: $\alpha(X)$ is odd if X is bound in F by a least fixed point operator ($\mu X.$), $\alpha(X)$ is even if X is bound in F by a greatest fixed point operator ($\nu X.$) and $\alpha(X) > \alpha(Y)$ if X subsumes Y in F. We then specify the priority assigned to game states of the form $\langle p, X\rangle$, with $X \in \mathit{bound}(F)$, as follows: $\Pr(\langle p, X\rangle)=\alpha(X)$.

4. $Pr(s)=0$, for all other game states $s \in \big(P \times Sub(F)\big)$.

It is simple to verify that the parity winning set formally defined as above coincides with the winning criterion of the $\mathcal{G}(F,\rho)$, discussed earlier. Thus $\mathcal{G}(F,\rho)$ is a 2-player parity game.

We are now ready to formally define the game semantics of a Lμ formula.

Definition 3.1.12. Given a LTS $\mathcal{L}=\langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$, an interpretation $\rho \in \mathcal{V} \rightarrow 2^P$ and a Lμ formula F, we define the *game semantics* of F over $\mathcal{L}$ as the map $(\!|\, F \,|\!)^{\mathcal{L}}_{\rho} \in 2^P$, defined as follows:

$$(\!|\, F \,|\!)^{\mathcal{L}}_{\rho}(p) = \mathsf{VAL}\big(\mathcal{G}(F,\rho)\big)$$

i.e., $(\!|\, F \,|\!)^{\mathcal{L}}_{\rho}(p) = 1$ if and only if Player 1 has a winning strategy in the 2-player parity game $\mathcal{G}(F,\rho)$.

The following fundamental theorem, which establishes the equivalence of the denotational and game semantics of Lμ, was proved in [32]. We also point to [106] for a detailed exposition.

Theorem 3.1.13. *Given a LTS $\mathcal{L}=\langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$, an interpretation $\rho \in \mathcal{V} \rightarrow 2^P$ and a Lμ formula F, the following equality holds: $[\![F]\!]^{\mathcal{L}}_{\rho} = (\!|\, F \,|\!)^{\mathcal{L}}_{\rho}$.*

The result of Theorem 3.1.13 allows us to switch freely between denotational and game semantics, thus adopting the most suitable viewpoint depending on the context. The denotational semantics is often more suitable, for example, when one want to reason algebraically about the logic Lμ. On the other hand, the use of game semantics is often considered convenient for grasping the meaning of a Lμ formula F by trying to understand it in terms of the dynamics of the game $\mathcal{G}(F, \rho)$. Indeed the game semantics offers an *operational* (or dynamic) interpretation for the meaning of Lμ formulas, which can be summarized as follows:

- The formula X holds at p if and only if $\rho(X)(p) = 1$.
- The formula $G \vee H$ holds at p if and only if either G holds at p *or* H holds at q: this because in the game $\mathcal{G}(G \vee H, \rho)$ Player 1 can choose to move either to $\langle p, G\rangle$ or to $\langle p, H\rangle$, from which the rest of the game continues as in $\mathcal{G}(G, \rho)$ or $\mathcal{G}(H, \rho)$ respectively.
- Similarly, the formula $G \wedge H$ holds at p if and only if either G holds at p *and* H holds at q: this because in the game $\mathcal{G}(G \wedge H, \rho)$ Player 2 can choose to move either to $\langle p, G\rangle$ or to $\langle p, H\rangle$, from which the rest of the game continues as in $\mathcal{G}(G, \rho)$ or $\mathcal{G}(H, \rho)$ respectively. Thus Player 1 can win the game $\mathcal{G}(G \wedge H, \rho)$ if and only if they can win in both sub-games.
- With a similar argument, the formula $\langle a\rangle G$ holds at p if and only if G holds at q, for *some* process state q such that $p \stackrel{a}{\longrightarrow} q$. In particular if $p \stackrel{a}{\not\longrightarrow}$ then $\langle a\rangle G$ does not hold at p.
- Similarly, the formula $[a]\, G$ holds at p if and only if G holds at q, for *all* process state q such that $p \stackrel{a}{\longrightarrow} q$. Thus if $p \stackrel{a}{\not\longrightarrow}$ then $[a]\, G$ holds at p.
- The formula $\mu X.G$ holds at p, if G holds at p by *unfolding* the recursive definition of $G(X)$ finitely many times. Indeed if in the game $\mathcal{G}(\mu X.G)$, states of the form $\langle q, X\rangle$, for $q \in P$, are visited infinitely often, then Player 1 loses because $\mathrm{Pr}(\langle q, X\rangle)$ is odd by definition and X is the outermost variable.
- The formula $\nu X.G$ holds at p, if G holds at p by unfolding the recursive definition of $G(X)$ possibly infinitely many times. Indeed if in the game $\mathcal{G}(\mu X.G)$, states of the form $\langle q, X\rangle$, for $q \in P$, are visited infinitely often,

the Player 1 wins because $\Pr(\langle q, X\rangle)$ is even by definition and X is the outermost variable.

In particular the last two points, which provides an *operational* interpretation for the meaning of the fixed point constructors, are often useful when tyring to understand the meaning of complicated formulas with several nested fixed point operators, and provide some formal support to the well known slogan: "ν means looping and μ means finite looping"[18].

3.1.3 Computation Tree Logic

Computation Tree Logic (CTL), introduced by Clarke, Emerson and Sistla in [23], is one of the most studied temporal logics for the specification of properties of programs [18, 106], and provides the basis for several variants and enrichments, such as CTL*[33] and ECTL[109]. In this section, following the presentation of [106], we formally introduce CTL and we discuss its encoding into the modal μ-calculus.

A central notion, which is at the basis of the logic CTL, is that of *run*, also known as *trace*, *execution* or simply *path*, in a label transition system which is usually assumed[2] to have just one label denoted by the $\bullet$ symbol.

Definition 3.1.14. Let $\mathcal{L} = \langle P, \{\xrightarrow{\bullet}\}\rangle$ be a labeled transition system with just one label. A run $\vec{r}$ in $\mathcal{L}$ is *completed path* in $\mathcal{L}$ (see Definition 2.1.33), i.e., a finite or infinite sequence of transitions

$$p_0 \xrightarrow{\bullet} p_1 \xrightarrow{\bullet} p_2 \xrightarrow{\bullet} p_3 \ldots$$

with $p_n \in P$ for $n \in \mathbb{N}$, of maximal length. This means that, if a run has finite length then its final process p_n is unable to perform a transition (i.e., $p \not\xrightarrow{\bullet}$), because otherwise, the sequence would be extended. We denote with $Run(p)$, for $p \in P$, the set of runs in $\mathcal{L}$ having p as first state.

The notion of *run* in $\mathcal{L}$ captures the idea of a possibly infinite computation, seen as a sequence of actions (described by the unique label $\bullet$) which induce changes in the states (the process states) of the system. The syntax of CTL is specified by a two sorted grammar: *state-formulas* describe properties of process

[2]The logic CTL, although originally defined on Kripke structures[23] (i.e., LTS's with just one label), can be defined on general labeled transition systems as well, see e.g. [90].

states (and are interpreted, as for $L\mu$-formulas, as predicates over process states) and *path-formulas* describe properties of runs. The combined use of these two kind of expressions allow the formulation of numerous properties of systems.

Definition 3.1.15. The syntax of CTL is defined by the following context-free grammars:

$$\begin{array}{lll} \text{state-formulas:} & F, G & ::= X \mid \mathrm{tt} \mid F \vee G \mid \neg F \mid \exists\phi \\ \text{path-formulas:} & \phi, \psi & ::= \circ F \mid F\ \mathcal{U}\ G \mid \mathcal{A}\ F, \end{array}$$

where X range over a countable set $\mathcal{V}$ of variables.

The state-formula variables $\mathcal{V}$ are often ranged over by the letters P, Q in the literature (see, e.g., [23, 33]), as they are interpreted as atomic predicates over states. We opted for the letters X, Y for uniformity with the syntax of Lμ formulas. The semantics of state-formulas is given as a predicate over states, i.e., as maps in 2^P, where the boolean connectives are interpreted as usual, and the new formula $\exists\phi$ holds at a state p if there exists a run $\vec{r} \in Run(p)$ such that $\vec{r}$ satisfies the path-formula ϕ. A run $\vec{r} \in Run(p)$ satisfies $\circ F$, if $\vec{r}$ has at least length two and its second state satisfies F; the run $\vec{r}$ satisfies $F\ \mathcal{U}\ G$ if at some state $p_n \in \vec{r}$ the formula G holds, and all previous states $p_0, \ldots, p_{n-1}$ satisfy F; lastly, $\vec{r}$ satisfies $\mathcal{A}F$ if all of its process states (of which there are finitely many if $\vec{r}$ is finite) satisfy F. The path operators $\circ$, $\mathcal{U}$ and $\mathcal{A}$ are respectively called *next*, *until* and *always* (sometimes denoted by the letter G rather than A).

Definition 3.1.16 (Semantics of CTL)**.** Given an LTS $\mathcal{L} = \langle P, \xrightarrow{\bullet} \rangle$ with one label, an interpretation $\rho \in \mathcal{V} \to 2^P$ of the variables into $\mathcal{L}$ and a CTL formula F, we define the semantics $\|F\|^{\mathcal{L}}_{\rho} \in 2^P$ of F by induction on the structure of F as follows:

$$\begin{array}{lcl} \|X\|^{\mathcal{L}}_{\rho}(p) & = & \rho(X)(p) \\ \|\mathrm{tt}\|^{\mathcal{L}}_{\rho}(p) & = & 1 \\ \|G \vee H\|^{\mathcal{L}}_{\rho}(p) & = & \|G\|^{\mathcal{L}}_{\rho}(p) \sqcup \|H\|^{\mathcal{L}}_{\rho}(p) \\ \|\neg G\|^{\mathcal{L}}_{\rho}(p) & = & \begin{cases} 1 & \text{if } \|G\|^{\mathcal{L}}_{\rho}(p) = 0 \\ 0 & \text{if } \|G\|^{\mathcal{L}}_{\rho}(p) = 1 \end{cases} \\ \|\exists\phi\|^{\mathcal{L}}_{\rho}(p) & = & \displaystyle\bigsqcup_{\vec{r} \in Run(p)} \|\phi\|^{\mathcal{L}}_{\rho}(\vec{r}) \end{array}$$

where $\|\phi\|^{\mathcal{L}}_{\rho}(\vec{r})$ is defined by case analysis on ϕ as follows:

$$\|\circ F\|^{\mathcal{L}}_{\rho}(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r}{=}p_0 \xrightarrow{\bullet} p_1 \ldots, \text{ and } \|F\|^{\mathcal{L}}_{\rho}(p_1) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\|F\ \mathcal{U}\ G\|^{\mathcal{L}}_{\rho}(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r}{=}p_0 \xrightarrow{\bullet} p_1 \xrightarrow{\bullet} p_2 \ldots \text{ and } \exists i \geq 0 \text{ such that:} \\ & \forall j < i.\ \|F\|^{\mathcal{L}}_{\rho}(p_j){=}1, \text{ and } \|G\|^{\mathcal{L}}_{\rho}(p_i){=}1 \\ 0 & \text{otherwise} \end{cases}$$

$$\|\mathcal{A}F\|^{\mathcal{L}}_{\rho}(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r}{=}p_0 \xrightarrow{\bullet} p_1 \xrightarrow{\bullet} p_2 \ldots \text{ and } \forall p_i {\in} \vec{r}.\ \|F\|^{\mathcal{L}}_{\rho}(p_i){=}1 \\ 0 & \text{otherwise} \end{cases}$$

As anticipated earlier, we will now discuss how to encode CTL formulas as Lμ formulas. To do so it is convenient to assume that for each variable X in the countable set $\mathcal{V}$, there is a *dual variable* $\overline{X}$, such that for every interpretation $\rho \in \mathcal{V} \to 2^P$ into some LTS $\mathcal{L}$, $\rho(X)(p) = 1$ if and only if $\rho(\overline{X})(p) = 0$, for every process state $p \in P$. It is then natural to extend the negation operator (see Definition 3.1.9) to open Lμ formulas by defining $\overline{(X)} {=} \overline{X}$. It is clear that with this convention, the property expressed by Proposition 3.1.10 holds for arbitrary open formulas.

Definition 3.1.17. We define the encoding $\mathbb{E}$ from CTL formulas to Lμ formulas, by induction on the structure of the CTL formula F as follows:

1. $\mathbb{E}(X) = X$,
2. $\mathbb{E}(\mathrm{tt}) = \nu X.X$,
3. $\mathbb{E}(F \vee G) = \mathbb{E}(F) \vee \mathbb{E}(G)$,
4. $\mathbb{E}(\neg F) = \overline{\mathbb{E}(F)}$,
5. $\mathbb{E}(\exists(\circ F)) = \langle \bullet \rangle \mathbb{E}(F)$,
6. $\mathbb{E}(\exists(F\ \mathcal{U}\ G)) = \mu X.\Big(\mathbb{E}(G) \vee \big(\mathbb{E}(F) \wedge \langle \bullet \rangle X\big)\Big)$,
7. $\mathbb{E}(\exists(\mathcal{A}F)) = \nu X.\Big(\mathbb{E}(F) \wedge \big(\langle a \rangle X \vee [a]\,(\mu Z.Z)\big)\Big)$.

The soundness of the encoding is formalized by the following standard result:

Proposition 3.1.18. *Given a LTS* $\mathcal{L} {=} \langle P, \xrightarrow{\bullet} \rangle$ *with one label, an interpretation* $\rho {\in} \mathcal{V} {\to} 2^P$ *of the variables into* $\mathcal{L}$*, and a CTL formula* F*, the following equality holds:* $\|F\|^{\mathcal{L}}_{\rho} = [\![\mathbb{E}(F)]\!]^{\mathcal{L}}_{\rho}$.

We refer to, e.g., [18] for a detailed explanation of the encoding. We mention that Proposition 3.1.18 can be proved in a straightforward way by using the game semantics of Lμ: it is easy to translate, in both directions, between the properties of paths described by CTL path formulas and winning strategies in the games associated with Lμ formulas.

3.2 Probabilistic temporal logics

3.2.1 Probabilistic Labeled Transition Systems

Labeled transition systems allow the description of systems in a qualitative way, i.e., not considering quantitative details such as: time, probabilistic behavior, quantities such as energy-level or temperature, *etcetera*. However these quantitative aspects of reactive systems are often too important to be ignored. For example it is often desirable to consider systems that must respond in a fixed amout of time or, similarly, that must respond, with a high probability, in a small amount of time. Moreover some problems arising in concurrent systems (such as, e.g, the *dining philosophers* synchronization problem [77]), require probabilistic procedures for a correct solution, which can not be modeled within ordinary labeled transition systems.

Since the beginning of the 90's, a lot of research has focused on the identification of appropriate generalizations of the notion of LTS, with the goal of modeling quantitative aspects of concurrent computations. In particular, the problem of modeling probabilistic behaviors has received a lot of interest. Models of purely probabilistic system evolving during time, such as *stochastic processes* and *Markov chains*, have been intensively studied in probability theory since the 50's. However models of concurrent probabilistic systems must also crucially be able to represent the non-deterministic behaviors induced, for instance, by the unpredictable process-interleaving managed by a scheduler. One of the first models designed to describe probabilistic and non-deterministic aspects of computations was introduced by K. G. Larsen and A. Skou [67], and is today known as the *reactive* model [45], also known as *Labeled Markov Processes* in case that the state space is not discrete [91]. Alternative models, such as *generative* and *stratified* models, have been subsequently introduced (see, e.g., [45]). In 1995, R. Segala introduced in his PhD thesis [101] a new model of concurrent probabilistic computation. The model, with its slight variations, has been named in the literature in several ways: Segala systems [6], probabilistic automata [101], concurrent Markov chains [50] or just probabilistic labeled transition systems (PLTS). In what follows we will always refer with PLTS to this class of models. Since its introduction, PLTS's have been successfully adopted as models for formal languages describing concurrent probabilistic systems, such as the *probabilistic π-calculus* [53], *probabilistic mobile ambients* [66] and the class of PGSOS languages of [6]. We now formally define the class of probabilistic labeled transition systems.

Definition 3.2.1. A *probabilistic labeled transition system*, or a PLTS for short, is a pair $\mathcal{L}=\langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$, where P is a set of *process states*, L is a set of *labels*, and the relation $\xrightarrow{a} \subseteq P \times \mathcal{D}(P)$, where $\mathcal{D}(P)$ is the set of (discrete) probability distributions over P, is called the *a-transition relation*, for every $a\in L$. We write $p \xrightarrow{a} d$ for $(p, d) \in \xrightarrow{a}$. For $p\in P$ and $d\in\mathcal{D}(P)$, we say that d is a a-successor distribution of p if $p \xrightarrow{a} d$. We write $p \not\xrightarrow{a}$ if the set of a-successor distributions of p is empty. A PTLS $\mathcal{L}$ is called *finite-branching* if for every $a\in L$ and $p\in P$, the set of a-successor distributions of p is a finite set. Similarly we say that a PLTS is *countably branching* if for every $a \in L$ and $p\in P$, the set of a-successor distributions of p is countable. Lastly, a PLTS is *finite* if it is finite-branching and its set of states P is finite.

The intended interpretation of a PLTS $\mathcal{L}=\langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$ is the following: the process states $p\in P$ represent the possible configurations of the system; at a process state p, the system can react to an a-action, for $a\in L$, by changing its state to a process q according to some nondeterministically chosen probability distribution $d\in\mathcal{D}(P)$ such that $p \xrightarrow{a} d$. Thus the kind of non-deterministic probabilistic computation modeled by a PLTS consists in a sequence of non-deterministic choices (induced by a-actions, for $a \in L$) each immediately followed by a corresponding probabilistic choice (induced by the non-deterministically chosen probability distribution d).

It is clear, under this interpretation, that an LTS can always be seen as a particular kind of PLTS where for each process state p and each a-successor distribution d of p, for $a\in L$, d is a Dirac distribution over P, i.e., the probabilistic choice associated with d is actually deterministic. More formally, in what follows, we will adopt the following convention:

Convention 3.2.2. We automatically refer to a LTS $\mathcal{L} = \langle P, \{\xrightarrow{a}_1\}_{a\in L}\rangle$ as the corresponding PLTS $\mathcal{L} = \langle P, \{\xrightarrow{a}_2\}_{a\in L}\rangle$ defined as follows: $p \xrightarrow{a}_1 q$ if and only if $p \xrightarrow{a}_2 \delta_q$, where δ_q is the unique probability distribution with $supp(\delta_q) = \{q\}$, for $q \in P$. Thus PLTS's constitute a generalization of standard LTS's.

An important consequence of modeling, within a PLTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$, the probabilistic choices as probability distributions $d \in \mathcal{D}$ is that we necessarily restrict such choices to countable ones. More general choices could be modeled by working instead with an appropriate generalization of the notion of PLTS, where the process state space is a (possibly uncountable) measurable-set, and the a-successors elements are probability measures rather than discrete probability distributions. Such generalizations, however, introduce significant complications by involving in a crucial way notions from measure theory. We refer to [91] for a detailed exposition of a generalization in this direction of the reactive model of [67].

In this thesis we restrict to the sort of countable probabilistic behaviors which can be modeled with PLTS's. Moreover, for uniformity and technical convenience, we restrict our attention to countably branching PLTS's, i.e., those having at most countable non-deterministic choices, having a countable state-space. This class of models is arguably sufficient for modeling most of the kind of systems arising in Computer Science. Indeed program-states are generally thought as consisting of a finite piece of program-code and a finite portion of memory or data. However further research, following the lines of [91], towards generalizations of the notion of PLTS modeling uncountable choices, is very likely to be at least useful in order to model system arising, for instance, from biology and physics.

Assumption 3.2.3. In the rest of the thesis, a PLTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$ is assumed to have a countable set P of process states, a countable set L of labels and to be countably branching.[3] We denote with $\mathcal{D}(\mathcal{L})$ the countable set $\bigcup_{p\in P}\bigcup_{a\in L}\{d \mid p \xrightarrow{a} d\}$ to which we refer to as the *set of probability distributions in* $\mathcal{L}$.

[3]Note that there exist PLTS's with a countable set of process-states which are not countably branching. This is due to the fact that, when $|P| \geq 2$, $\mathcal{D}(P)$ is uncountable.

3.2.2 Probabilistic CTL

Temporal logics for specification of properties of concurrent programs formalized as standard LTS's, such as CTL or Lμ, had become, by the end of the '80s, one of the most successful formal methods for the verification of software, with important real-world applications [4, 24], as well as a flourishing research community working on the field. Therefore, with the goal of extending these techniques, the research community started investigating logics for expressing properties of probabilistic concurrent programs formalized as PLTS's. One of the first such logics, introduced by H. Hansson in [50] and by A. Bianco and L. De Alfaro in [9], is a probabilistic generalization of CTL and is today well known as Probabilistic CTL, or PCTL for short. The logic PCTL, together with its extensions (e.g., PCTL* [9]), still constitutes nowadays one of the most popular and well understood logics for probabilistic concurrent programs, with several real-world applications (see, e.g., the project PRISM [65]).

Following the tradition of CTL, the logic PCTL is interpreted over PLTS's having only one label, i.e., with $L=\{\bullet\}$.

Observation 3.2.4. It is possible to look at a PLTS $\mathcal{L}=\langle P, \xrightarrow{\bullet}\rangle$ as a countable graph $G(\mathcal{L}) = \big(P \cup \mathcal{D}(\mathcal{L}), E\big)$ by defining $(p,d) \in E$ if and only if $p \xrightarrow{a} d$ and $(d,p) \in E$ if and only if $p \in supp(d)$, where the edge connecting d with p can be thought as labeled with the the probability $d(p)$.

As for CTL (see Section 3.1.3), a key concept at the basis of PCTL is that of *run* in a PLTS. The following definition of run in a PLTS is very similar to the corresponding notion for standard LTS's.

Definition 3.2.5. Let $\mathcal{L} = \langle P, \{\xrightarrow{\bullet}\}\rangle$ be a PLTS with just one label. A run $\vec{r}$ in $\mathcal{L}$ is *completed path* (see Definition 2.1.33) in $\mathcal{L}$ (seen as the corresponding graph), i.e., a finite or infinite sequence of transitions

$$p_0 \xrightarrow{\bullet} d_1 \rightsquigarrow p_1 \xrightarrow{\bullet} d_2 \rightsquigarrow p_2 \xrightarrow{\bullet} d_3 \rightsquigarrow p_3 \dots$$

with $p_n \xrightarrow{\bullet} d_{n+1}$ and $p_{n+1} \in supp(d_{n+1})$, for $n \in \mathbb{N}$, of *maximal* length. We denote with $Run(p)$, for $p \in P$, the set of runs in $\mathcal{L}$ having p as first state, and with Run the set of all runs in $\mathcal{L}$. The set Run is endowed with a Polish 0-dimensional topology defined as in 2.1.33.

In analogy with the definition for LTS's, the notion of *run* in a PLTS $\mathcal{L}$ captures the idea of a possibly infinite computation, seen as a sequence of actions (described by the unique label $\bullet$), leading to probabilistic choices (described by the chosen probability distributions) which lead to possible (in the sense of having probability of being reached greater than zero) successor states.

However, when working with PLTS's, another useful concept is necessary to interpret the probabilistic behavior induced by the random choices following each action-step. This leads to the notion of *Markov run* (also frequently called *Markov chain*) in a PLTS. A Markov run in a PLTS $\mathcal{L}$ can be understood as an ordinary run *up-to* the result of the stochastic choices.

Definition 3.2.6 (Markov run)**.** Let $\mathcal{L} = \langle P, \{\xrightarrow{\bullet}\}\rangle$ be a PLTS with just one label. A Markov run in $\mathcal{L}$ is a is a tree in $G(\mathcal{L}) = \big(P \cup \mathcal{D}(\mathcal{L}), E_{\mathcal{L}}\big)$ (see Definition 2.1.35) uniquely branching in P and fully branching in $\mathcal{D}(\mathcal{L})$, in the sense of Definition 2.1.36.

Every Markov run induces a probability measure over runs.

Definition 3.2.7 (Probability measure $\mathbb{P}(M)$)**.** Every Markov run M determines a probability assignment $\mathbb{P}_M(O_{\vec{p}})$ to every basic clopen set $O_{\vec{p}} \subseteq \mathit{Run}$, for $\vec{p} = p_0 \xrightarrow{\bullet} d_1 \rightsquigarrow p_1 \xrightarrow{\bullet} d_2 \rightsquigarrow p_2 \xrightarrow{\bullet} \ldots p_n$ a finite sequence of transitions, defined as follows:

$$\mathbb{P}_M(O_{\vec{p}}) \stackrel{\text{def}}{=} \prod \big\{ d_i(p_{i+1}) \mid i < n \big\}$$

More informally, $\mathbb{P}_M(O_{\vec{p}})$ is the multiplication of all probabilities associated with the random choices necessary for the sequence to have place in the PLTS $\mathcal{L}$. The assignment $\mathbb{P}_M$ on basic clopen sets extends to a unique complete probability measure $\mathbb{P}_M \in \mathcal{M}_1(\mathit{Run})$.

The value $\mathbb{P}_M(X)$, for some $\mathbb{P}_M$-measurable set $X \subseteq \mathit{Run}$, models the probability that the outcome of the stochastic execution of the PLTS $\mathcal{L}$, formalized as the Markov run M, is a run in X.

We are now ready to introduce Probabilistic Computation Tree Logic (PCTL). If CTL can be thought as a logic specifying properties of process states (of a LTS) in terms of the set of runs, or computations, starting from them, PCTL might be thought as a logic specifying properties of process states (of a PLTS) in terms of the set runs, like in CTL, *and* in terms of the the Markov runs, or probabilistic computations.

As done for CTL, we specify the syntax of PCTL by a two sorted grammar: *state-formulas* describe properties of process states (and are interpreted, as in standard CTL, as predicates over process states) and *path-formulas* describe properties of runs. The novelty introduced in PCTL consists in two new threshold operators $\mathbb{P}_{\geq\lambda}$ and $\mathbb{P}_{>\lambda}$ for state formulas, with $\lambda \in [0,1]$.

Definition 3.2.8. The syntax of the PCTL logic is defined by the following context-free grammars:

$$\begin{array}{lll} \text{state-formulas:} & F, G & ::= X \mid \text{tt} \mid F \vee G \mid \neg F \mid \exists\psi \mid \mathbb{P}_{\geq\lambda}\psi \mid \mathbb{P}_{>\lambda}\psi \\ \text{path-formulas:} & \psi & ::= \circ F \mid F \ \mathcal{U} \ G \mid \mathcal{A} \ F, \end{array}$$

where X range over the set $\mathcal{V}$ of variables, and $\lambda \in [0,1]$. We will refer to the *qualitative* PCTL logic, or PCTL$^{\{0,1\}}$ for short, as the fragment of PCTL where the allowed threshold modalities in state formulas are $\mathbb{P}_{>0}$ and $\mathbb{P}_{=1}$.

As for CTL, the state-formula variables $\mathcal{V}$ are interpreted as atomic predicates over states. The semantics of state-formulas is given as a predicate over process states, where the boolean connectives are interpreted as usual and the formula $\exists\phi$, as in CTL, existentially quantifies over runs satisfying the path formula ϕ. The important novelty is given by the state formulas $\mathbb{P}_{\geq\lambda}\psi$ and $\mathbb{P}_{>\lambda}\psi$. The PCTL formula $\mathbb{P}_{\geq\lambda}\phi$ holds at a process state p if there exists a Markov play $M \in MRUN(p)$ assigning probability greater than $\lambda - \epsilon$ to the set of runs satisfying the path formula ϕ, for every $\epsilon > 0$. Similarly for $\mathbb{P}_{\geq\lambda}\phi$. Thus the formulas $\mathbb{P}_{\geq\lambda}\psi$ and $\mathbb{P}_{>\lambda}\psi$ are satisfied at a state p if, under an angelic interpretation of the non-determinism modeled in a PLTS $\mathcal{L}$, the (limit) probability associated with runs satisfying ϕ is greater (or equal) than a given threshold. The meaning of the path formulas operators *next*, *until* and *always* can be understood as in CTL.

We are now ready to formally define the semantics of PCTL.

Definition 3.2.9 (Semantics of PCTL [9])**.** Given a PLTS $\mathcal{L} = \langle P, \xrightarrow{\bullet} \rangle$ with one label, an interpretation $\rho \in \mathcal{V} \to 2^P$ of the variables and a PCTL formula F, we define the semantics $\|F\|^{\mathcal{L}}_{\rho} \in 2^P$ of F by induction on the structure of F as follows:

$$
\begin{array}{lcl}
\|X\|_{\rho}^{\mathcal{L}}(p) & = & \rho(X)(p) \\
\|\mathrm{tt}\|_{\rho}^{\mathcal{L}}(p) & = & 1 \\
\|G \vee H\|_{\rho}^{\mathcal{L}}(p) & = & \|G\|_{\rho}^{\mathcal{L}}(p) \sqcup \|H\|_{\rho}^{\mathcal{L}}(p) \\
\|\neg G\|_{\rho}^{\mathcal{L}}(p) & = & \begin{cases} 1 & \text{if } \|G\|_{\rho}^{\mathcal{L}}(p) = 0 \\ 0 & \text{if } \|G\|_{\rho}^{\mathcal{L}}(p) = 1 \end{cases} \\
\|\exists\phi\|_{\rho}^{\mathcal{L}}(p) & = & \bigsqcup_{\vec{r} \in Run(p)} \|\phi\|_{\rho}^{\mathcal{L}}(\vec{r}) \\
\|\mathbb{P}_{\geq\lambda}\phi\|_{\rho}^{\mathcal{L}}(p) & = & \begin{cases} 1 & \text{if } \bigsqcup \Big\{ \mathbb{P}_M\big(\{\vec{r} \mid \|\phi\|_{\rho}^{\mathcal{L}}(\vec{r}) = 1\}\big) \mid M \in MRun(p) \Big\} \geq \lambda \\ 0 & \text{otherwise} \end{cases} \\
\|\mathbb{P}_{>\lambda}\phi\|_{\rho}^{\mathcal{L}}(p) & = & \begin{cases} 1 & \text{if } \bigsqcup \Big\{ \mathbb{P}_M\big(\{\vec{r} \mid \|\phi\|_{\rho}^{\mathcal{L}}(\vec{r}) = 1\}\big) \mid M \in MRun(p) \Big\} > \lambda \\ 0 & \text{otherwise} \end{cases}
\end{array}
$$

where the predicate over runs $\|\psi\|_{\rho}^{\mathcal{L}}(\vec{r})$ is defined by case analysis on ψ, like in CTL, as follows:

$$
\begin{array}{lcl}
\|\circ F\|_{\rho}^{\mathcal{L}}(\vec{r}) & = & \begin{cases} 1 & \text{if } \vec{r} = p_0 \xrightarrow{\bullet} d_0 \rightsquigarrow p_1 \ldots, \text{ and } \|F\|_{\rho}^{\mathcal{L}}(p_1) = 1 \\ 0 & \text{otherwise} \end{cases} \\
\|F \,\mathcal{U}\, G\|_{\rho}^{\mathcal{L}}(\vec{r}) & = & \begin{cases} 1 & \text{if } \vec{r} = p_0 \xrightarrow{\bullet} d_0 \rightsquigarrow p_1 \ldots \text{ and } \exists i \geq 0 \text{ such that:} \\ & \quad \forall j < i.\, \|F\|_{\rho}^{\mathcal{L}}(p_j) = 1, \text{ and } \|G\|_{\rho}^{\mathcal{L}}(p_i) = 1 \\ 0 & \text{otherwise} \end{cases} \\
\|\mathcal{A}F\|_{\rho}^{\mathcal{L}}(\vec{r}) & = & \begin{cases} 1 & \text{if } \vec{r} = p_0 \xrightarrow{\bullet} d_0 \rightsquigarrow p_1 \ldots \text{ and } \forall p_i \in \vec{r}.\, \|F\|_{\rho}^{\mathcal{L}}(p_i) = 1 \\ 0 & \text{otherwise} \end{cases}
\end{array}
$$

It should be noted that the semantic interpretation of the state formulas $\mathbb{P}_{\geq\lambda}\phi$ and $\mathbb{P}_{>\lambda}\phi$ is well defined if and only if the set $\{\vec{r} \mid \|\psi\|_{\rho}^{\mathcal{L}}(\vec{r}) = 1\}$ is $\mathbb{P}_M$-measurable for every path-formula ψ and Markov run M. It is easy to verify that any such set is in fact Borel, hence universally measurable. We refer to, e.g., [50, 9] for a detailed proof.

3.2.3 Probabilistic modal μ-calculus (pLμ)

Due to the success of the modal μ-calculus Lμ as a temporal logic for specifying properties of LTS's, researchers quickly started searching for probabilistic generalizations of this logic to express properties of PLTS's. The power of Lμ comes from the combination of the very simple Hennessy-Milner (HM) modal logic and the fixed-point constructors. Thus, researchers focused immediately on the problem of identifying a probabilistic modal logic having some of the key properties of Hennessy-Milner logic. One such logic was immediately identified for the reactive model of Larsen and Skou [67] and subsequently extended to the alternating model [31], a slight variation of the notion of PLTS[4]. The formulas are generated by the following syntax: $F ::= \mathrm{tt} \mid F \wedge F \mid \neg F \mid \langle a \rangle_{\geq\lambda} F \mid \langle a \rangle_{>\lambda} F$. Thus the only difference with the HM-logic is that diamonds formulas are decorated with an inequality with a threshold value $\lambda \in [0,1]$. As for PCTL, the meaning of a formula, provided a PLTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a \in L} \rangle$, is given as a predicate over states, i.e., as a map in 2^P. The interpretation of the boolean connectives is the usual one and the formula $\langle a \rangle_{>\lambda} F$ holds at p if and only if there exists an a-successor distribution d, i.e., such that $p \xrightarrow{a} d$, assigning probability greater than λ to the set of process states satisfying F; similarly for $\langle a \rangle_{\geq\lambda} F$. This logic shares some important properties of Hennessy-Milner logic, and in particular its induced logical equivalence coincide with a notion of bisimulation for the alternating model [31]. However adding fixed-point operators to this modal logic does not yield a logic expressive enough to specify properties of interest such as those defined by PCTL formulas: the main issue is concerned with the fact that expressing probability constraints *locally* (in the modalities) does not allow the expression of interesting *global* properties, concerning probabilistic computations rather than single transition steps.

Thus a satisfactory probabilistic logic, based on a theory of fixed points, had to be found by a different path. An important idea was proposed in 1997 by M. Huth and M. Kwiatkowska [56] (even if conceived for the reactive model of Larsen and Skou) and independently by C. Morgan and A. McIver [84]: they introduced logics whose formulas are not interpreted as predicates (i.e., maps from process states to the two element set $\{0,1\}$), like in PCTL, but as maps from process states to the real-unit interval $[0,1]$. The intended interpretation is that the property expressed by a formula holds with some probability at a given state, and that probability constitutes the semantics of the formula at p. Thus there is a significant shift in the way formulas must be understood: if in predicate-based probabilistic logics, such as PCTL, formulas directly express properties involving probability constraints, and therefore just hold or not hold at a given process state, in $[0,1]$-valued predicate logics, formulas describe properties which

[4]Although the difference between the alternating model of [31] and PLTS's is apparently minimal, and indeed one model can faithfully mimic the other [102], the problem of identifying a satisfactory minimal HM-style modal logic characterizing bisimilarity [101] for PLTS's is still an active area of research. See, e.g., [93] and [54] for recent developments.

have some probability of being satisfied at a given state, and that probability constitutes their meaning. The *quantitative modal μ-calculus* of [84] and [56] is a probabilistic variant of Lμ based of these ideas. Since the adjective "quantitative" has been adopted in the literature for other non-probabilistic logics (see, e.g., [35]), in this thesis we refer to the logic of [84] as *probabilistic modal μ-calculus* (pLμ).

Definition 3.2.10. Given a PLTS $\langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$ we denote with $[0,1]^P$ the space of maps from process states to the real interval $[0,1]$. In the following we refer[5] to elements in $[0,1]^P$ as $[0,1]$-*predicates* or *quantitative predicates.* The space $[0,1]^P$ forms a complete lattice by extending the linear order on $[0,1]$ to $[0,1]^P$ pointwise. Note that the space 2^P is a subspace of $[0,1]^P$.

We are now ready to introduce the *probabilistic modal μ-calculus*, as in [84]. Its syntax is the *same* as the one of Lμ (see Definition 3.1.2). The semantics of pLμ formulas is defined in a similar way to that of Lμ, using the lattice operations $\sqcup$ and $\sqcap$ defined on the new truth-set $[0,1]$.

Definition 3.2.11. Given a PLTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$, a $[0,1]$-interpretation $\rho \in \mathcal{V} \to [0,1]^P$ of the variables and a pLμ formula F, we define the *denotational semantics* of F over $\mathcal{L}$ as the map $[\![F]\!]^{\mathcal{L}}_{\rho} \in [0,1]^P$, defined by case induction on F as follows:

$$
\begin{array}{lcl}
[\![X]\!]^{\mathcal{L}}_{\rho}(p) & = & \rho(X)(p) \\
[\![G \vee H]\!]^{\mathcal{L}}_{\rho}(p) & = & [\![G]\!]^{\mathcal{L}}_{\rho}(p) \sqcup [\![H]\!]^{\mathcal{L}}_{\rho}(p) \\
[\![G \wedge H]\!]^{\mathcal{L}}_{\rho}(p) & = & [\![G]\!]^{\mathcal{L}}_{\rho}(p) \sqcap [\![H]\!]^{\mathcal{L}}_{\rho}(p) \\
[\![\langle a\rangle G]\!]^{\mathcal{L}}_{\rho}(p) & = & \bigsqcup_{p \xrightarrow{a} d} \Big(\sum_{q \in supp(d)} d(q) \cdot [\![G]\!]^{\mathcal{L}}_{\rho}(q)\Big) \\
[\![[a]\, G]\!]^{\mathcal{L}}_{\rho}(p) & = & \bigsqcap_{p \xrightarrow{a} d} \Big(\sum_{q \in supp(d)} d(q) \cdot [\![G]\!]^{\mathcal{L}}_{\rho}(q)\Big) \\
[\![\mu X.G]\!]^{\mathcal{L}}_{\rho}(p) & = & \mathrm{lfp}\big(\lambda f \in [0,1]^P. [\![G]\!]_{\rho[f/X]}\big)(p) \\
[\![\nu X.G]\!]^{\mathcal{L}}_{\rho}(p) & = & \mathrm{gfp}\big(\lambda f \in [0,1]^P. [\![G]\!]_{\rho[f/X]}\big)(p)
\end{array}
$$

Note that this is a good definition since the space $[0,1]^P$ is a complete lattice and every constructor is interpreted as a monotone operator, hence least and greatest fixed points exist by the Knaster-Tarski theorem. We say that a process state $p \in P$, under the interpretation ρ, *satisfies the formula F with probability λ*, if

[5] Elements in $[0,1]^P$ are also known as *fuzzy-sets* or *fuzzy predicates* [49].

$[\![F]\!]^{\mathcal{L}}_{\rho}(p) = \lambda$. We often omit the superscript $\mathcal{L}$ in $[\![F]\!]^{\mathcal{L}}_{\rho}$ if the PLTS $\mathcal{L}$ is clear from the context.

The main novelty in the definition of the semantics of pLμ resides in the interpretation of the modalities $\langle a \rangle$ and $[a]$, for $a \in L$. The definitions resemble the corresponding ones for Lμ but, crucially, in PLTS's transitions lead to probability distributions over processes, rather than processes. The most natural way to interpret the meaning of a formula G at a probability distribution d is to consider the *expected probability* of the formula G holding at a process q, associated by the random choice over processes induced with d, and this is formalized by the weighted sums in the definition above.

As an immediate observation, note that for every LTS $\mathcal{L}$, process state p, standard interpretation of the variables $\rho \in \mathcal{V} \to 2^P$ and formula F, the standard Lμ-semantics of F at p under ρ and the pLμ semantics of F, interpreted in the PLTS $\mathcal{L}'$ corresponding to $\mathcal{L}$ as discussed earlier (see convention 3.2.2), coincide at p. In other words when considering non-probabilistic PLTS's (i.e., LTS's) and $\{0,1\}$-valued interpretations of the variables, the logic pLμ collapses to the standard modal μ-calculus Lμ. Therefore pLμ should be considered as a conservative $[0,1]$-generalization of Lμ, pretty much as the concept of PLTS generalize that of LTS.

Being a conservative generalization of Lμ is certainly a good property for the logic pLμ. However there are plenty of other possible definitions for a $[0,1]$-valued interpretation of pLμ formulas. For instance one could modify the definitions for the connectives $\wedge$ and $\vee$ given above as follows:

$$\begin{aligned} [\![G \vee H]\!]^{\mathcal{L}}_{\rho}(p) &= [\![G]\!]^{\mathcal{L}}_{\rho}(p) \odot [\![H]\!]^{\mathcal{L}}_{\rho}(p) \\ [\![G \wedge H]\!]^{\mathcal{L}}_{\rho}(p) &= [\![G]\!]^{\mathcal{L}}_{\rho}(p) \cdot [\![H]\!]^{\mathcal{L}}_{\rho}(p) \end{aligned}$$

or as follows:

$$\begin{aligned} [\![G \vee H]\!]^{\mathcal{L}}_{\rho}(p) &= [\![G]\!]^{\mathcal{L}}_{\rho}(p) \oplus [\![H]\!]^{\mathcal{L}}_{\rho}(p) \\ [\![G \wedge H]\!]^{\mathcal{L}}_{\rho}(p) &= [\![G]\!]^{\mathcal{L}}_{\rho}(p) \ominus [\![H]\!]^{\mathcal{L}}_{\rho}(p) \end{aligned}$$

where the pairs of binary (De Morgan dual) operators $\cdot$ with $\odot$, and $\oplus$ with $\ominus$, have been defined in Section 2.2. Since all these operations are monotone the existence of the fixed points is guaranteed by the Knaster-Tarski theorem, thus these alternative definitions for the semantics of pLμ formulas are well-specified. Moreover, when restricted to the space 2^P, the operators $\cdot$ and $\ominus$ collapse to the meet operation $\sqcap$, and $\odot$ and $\oplus$ to the join operation $\sqcup$. Therefore both these alternative versions for the semantics of pLμ formulas constitute conservative generalizations of Lμ. More generally, there are numerous possible monotone operators, defined on the real interval $[0, 1]$, which might be considered as *quantitative*-generalizations of the classical conjunction and disjunction operators on the two element boolean lattice $\{0, 1\}$. Such generalizations are well known in field of *fuzzy-logic* (see, e.g., [49]), and there is no single operator which can be considered the *right* generalization for all purposes. As a matter of fact, in the paper of M. Huth and M. Kwiatkowska, the authors consider all the three pairs of operators cited above, and suggest that the most mathematically convenient (see Footnote 1 in Chapter 1) one for a probabilistic temporal logic might be the pair $\oplus$ and $\ominus$.

Justifying the definition of a probabilistic temporal logic, such as the probabilistic modal μ-calculus, just in terms of "mathematical conveniency" is somewhat unsatisfactory. A probabilistic temporal logic should be able to express properties of interest, and its $[0, 1]$-semantics should reflect a clear probabilistic interpretation corresponding to the chance of the property specified by the formula holding at a given state. Fortunately, as we shall see in the next subsection, the logic pLμ, and its semantics specified as in Definition 3.2.11, have a very strong justification provided by a game-theoretic interpretation of the meaning of formulas in terms of $2\frac{1}{2}$-player games. However the logic pLμ, as far as expressivity is concerned, is not completely satisfactory. For example, it is not possible to encode PCTL formulas in pLμ, and this is quite a severe limitation since these are the kind of properties mostly used nowadays (for example in the model checker PRISM [65]).

The $[0, 1]$-quantitative approach to (temporal) logics is not, by any means, restricted to probabilistic logics and conceived for expressing properties of PLTS's. For example, another Lμ-like logic for expressing properties of quantitative-LTS's, labeled transition systems whose transitions are decorated with real values representing the cost (for instance in terms of energy, or requested time) associated with that transition, has been defined in [35] and named *quantitative μ-calculus*. Again formulas are interpreted as $[0, 1]$-predicates[6], but unlike for the probabilistic logic discussed in this section, the real value represents, roughly speaking, the long term cost of a computation satisfying the property expressed by the formula at a give state. Recent results show how the (non probabilistic) quanti-

[6]Or equivalently as $\big(\mathbb{R} \cup \{-\infty, +\infty\}\big)$-valued predicates.

tative approach to temporal logics might find interesting application in describing properties of hybrid systems [36]. Furthermore, $[0, 1]$-quantitative analogous of the notion of bisimulation, capturing *how much* two process states are similar by means of a (pseudo-)metric have been defined [28, 91].

The growing interest in quantitative logics unavoidably leads to some confusion induced, for example, by the use of the common adjective *quantitative* referring to very different interpretations of the formulas. For this reasons we adopted the adjective *probabilistic*, rather than the original *quantitative*, to refer to the logic pLμ.

We conclude this section by pointing to another promising direction followed, for instance, in [93], [54] and [30], for the development of a satisfactory probabilistic temporal logics based on a theory of fixed points. The key idea is to define the semantics of a formula, given a PLTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}\rangle$, not as a $[0, 1]$-predicate over processes but as an ordinary predicate over probability distributions over processes: a map in the function space $\mathcal{D}(P) \to \{0, 1\}$. The meaning of a formula at a single process state p can be recovered by looking at the meaning of a formula at the Dirac probability distribution δ_p with mass at p. Therefore this approach, like the $[0, 1]$-quantitative interpretation, can be considered as a conservative generalization of the standard predicate-based approach. In accordance with [54], one of the main advantages of this approach lies in the fact that the resulting logics (e.g., the fixed point logic proposed in [30]) characterize the standard notion of bisimilarity for PLTS, in the sense that two processes are bisimilar if and only if the satisfy the same formulas. However, a possible drawback is that the properties of probability distributions expressed by formulas can be hard to understand as concrete properties of process states. For example the formula $\bigoplus\big(\lambda : F_1, 1-\lambda : F_2\big)$ is satisfied by a probability distribution d if there exist two probability distributions d_1 and d_2 such that d_i satisfies F_i, for $i \in \{1, 2\}$, and d is the weighted combination of d_1 and d_2. The two distributions d_1 and d_2 do not need to appear in the PLTS, in the sense that they might be not reachable by any process state p in the PLTS. Therefore, when this kind of formulas are used in recursive specifications using fixed-point operators, it might become difficult to grasp the meaning of a formula as a property of process states where, as discussed above, p can be thought as satisfying F if the Dirac distribution δ_p satisfies F.

Thus, after more than two decades of research, at least three possible approaches for the identification of a satisfactory probabilistic temporal logic based on a theory of fixed points emerged: the standard predicate based interpretation (à la PCTL) of formulas, the $[0,1]$-quantitative interpretation (à la pLμ) and the interpretation based on predicates over process distributions just mentioned.

3.2.4 Game semantics for pLμ

As briefly discussed in the previous section, the specification of the denotational semantics for pLμ is a very delicate matter. Logics whose semantics are given as ordinary predicates over process states are, provided a mathematically sound definition, always meaningful because the interpretation of a formula is the set of all processes satisfying the formula which, as a consequence, describe some *property* of PLTS's. Logics whose semantics is given as a $[0,1]$-predicate are, on the other hand, not necessarily meaningful. What might it mean for a process state p to assign F the denotational value $\frac{\sqrt{2}}{2}$, say, if the formula F does not specify any property, and thus the value $\frac{\sqrt{2}}{2}$ is not naturally assigned a probabilistic interpretation? At the very basis of the $[0,1]$-predicate approach is the assumption that formulas express *properties* of PLTS's and their semantics corresponds, somehow, to the probability of that property being satisfied at a given process state. Thus, when specifying the $[0,1]$-semantics of a logic like pLμ, providing a clear description of the intended properties specified by formulas is of great conceptual importance.

We saw in Section 3.1.2, how 2-player parity games offer a *dynamic* (or *operational*) interpretation for the meaning of standard $L\mu$ formulas: a Lμ formula F holds at a state p under an interpretation ρ if and only if Player 1 has a winning strategy in the 2-player parity game $\mathcal{G}(F,\rho)$. The game-theoretic machinery captures the idea of a hostile environment (which makes moves as Player 2) and of a program-controller (Player 1) which tries to satisfy the desired property (modeled by the winning condition of the game).

In 2003, C. Morgan and A. McIver, introduced a *game semantics* for the logic pLμ in terms of $2\frac{1}{2}$-player parity games [78]. The game is played by Player 1 and Player 2 (as in Lμ games) together with Nature, the third agent modeling probabilistic choices. The novelty in pLμ games is that, on configurations $\langle p, \langle a\rangle G\rangle$ and $\langle p, [a]\, G\rangle$, Player 1 and Player 2 respectively can choose a transition $p \xrightarrow{a} d$ in the PLTS and move the game to $\langle d, G\rangle$. If no a-transition is available, the player who gets stuck loses. In the configuration $\langle d, G\rangle$, d is a probability distribution (this is the key difference between pLμ and Lμ games) and the configuration $\langle d, G\rangle$ belongs to Nature, who moves on to the next configuration $\langle q, G\rangle$ with probability $d(q)$.

The rules of the pLμ game $\mathcal{G}(F,\rho)$ associated with a pLμ formula F and a $[0,1]$-interpretation for the variables into a PLTS $\mathcal{L}$, can be specified by extending those for Lμ games, as follows:

- at a configuration $\langle p, X\rangle$, with X free in F, the game ends in favor of Player 1 and Player 2 with probability $\rho(X)(p)$ and $1-\rho(X)(p)$, respectively.

- at $\langle p, \langle a\rangle G\rangle$, Player 1 chooses to move to one of the configurations in $\{\langle d, F\rangle \mid p \xrightarrow{a} d\rangle\}$; if $p \not\xrightarrow{a}$, Player 1 gets stuck and loses.

- at $\langle p, [a]\, G\rangle$, Player 2 chooses to move to one of the configurations in $\{\langle d, F\rangle \mid p \xrightarrow{a} d\rangle\}$; if $p \not\xrightarrow{a}$, Player 2 gets stuck and loses.

- at $\langle d, G\rangle$, Nature moves to the state $\langle q, G\rangle$ with probability $d(q)$.

The parity winning set in pLμ games is defined, as for Lμ-games, as the set of paths of congurations along which the outermost xed point variable X unfolded innitely often is bound by a greatest fixed point.

Thus, the game semantics for pLμ is a straightforward generalization of that for Lμ, where the actions of Nature are used to model the probabilistic choices corresponding to the probability distributions reachable by transitions in PLTS's. Therefore the game semantics offers a clear interpretation for the *properties* associated to the formulas, explained in terms of the interactions between the controller (Player 1) and an hostile environment (Player 2) in the context of the stochastic choices modeled by PLTS's (Nature).

We are now ready to formally define the game semantics of a pLμ formula.

Definition 3.2.12. Given a PLTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$, a $[0,1]$-interpretation $\rho \in \mathcal{V} \to [0,1]^P$ and a formula F, we define the *game semantics* of F over $\mathcal{L}$ as the map $(\!| F |\!)^{\mathcal{L}}_{\rho} \in [0,1]^P$, defined as follows:

$$(\!| F |\!)^{\mathcal{L}}_{\rho}(p) = \mathsf{VAL}\big(\mathcal{G}(F,\rho)\big)$$

where $\mathsf{VAL}\big(\mathcal{G}(F,\rho)\big)$ (see Definition 2.3.50) is the (limit) probability of Player 1 winning the $2\frac{1}{2}$-player parity game $\mathcal{G}(F,\rho)$.

The following fundamental theorem, which establishes the equivalence of the denotational and game semantics of pLμ on *finite models*, was proved by C. Morgan and A. McIver in [78].

Theorem 3.2.13 ([78])**.** *Given a finite-branching PLTS* $\mathcal{L}=\langle P,\{\xrightarrow{a}\}_{a\in L}\rangle$ *with a finite set of states* P*, an interpretation* $\rho\in\mathcal{V}\to[0,1]^P$ *and a pL*μ *formula* F*, the following equality holds:* $[\![F]\!]^{\mathcal{L}}_{\rho}=(\!|\,F\,|\!)^{\mathcal{L}}_{\rho}$.

Theorem 3.2.13 provides a strong justification for the denotational semantics of Definition 3.2.11, because it coincides with a game-theoretic interpretation which is arguably the right, straightforward generalization of the game interpretation of Lμ. Thus one can understand the property expressed by a pLμ formula as for Lμ, and its meaning as the *limit probability* of a program-controller of satisfying the property against a hostile environment in the context of stochastic choices.

However, Theorem 3.2.13, from [78], only applies to finite state finite-branching PLTS's. One of the results of this thesis consists in proving the generalization of Theorem 3.2.13 to arbitrary PLTS's. We now state the general theorem[7] whose proof will be discussed in Chapter 7 (Theorem 7.1.10).

Theorem 3.2.14. *Given an arbitrary PLTS* $\mathcal{L}=\langle P,\{\xrightarrow{a}\}_{a\in L}\rangle$*, an interpretation* $\rho\in\mathcal{V}\to[0,1]^P$ *and a pL*μ *formula* F*, the following equality holds:* $[\![F]\!]^{\mathcal{L}}_{\rho}=(\!|\,F\,|\!)^{\mathcal{L}}_{\rho}$.

3.2.5 Examples of pLμ formulas

In this section we consider a few examples of interesting pLμ formulas, and discuss their meaning using both the denotational and the game semantics, thus taking advantage of the result of Theorem 3.2.14.

Consider the following pLμ formulas:

1. $F_1 \stackrel{\text{def}}{=} \langle a\rangle\langle a\rangle\underline{1}$
2. $F_2 \stackrel{\text{def}}{=} \nu X.\langle a\rangle X$
3. $F_3 \stackrel{\text{def}}{=} \mu X.\big(F_2 \vee \langle b\rangle X\big)$
4. $F_4 \stackrel{\text{def}}{=} \nu X.\langle a\rangle\underline{1} \wedge [a]\, X$

[7]The result was announced at the *7th Workshop on Fixed Points in Computer Science (FICS)* [81]. The proof will appear in [82].

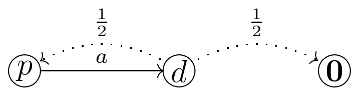


Figure 3.1: Example of PLTS

5. $F_5 \stackrel{\text{def}}{=} \langle a\rangle\langle a\rangle \underline{1} \wedge [a]\,[a]\,\underline{0}$.

where we denote with $\underline{1}$ and $\underline{0}$ the formulas $\nu X.X$ and $\mu X.X$ respectively.

The properties they express can be explained as in the ordinary μ-calculus Lμ using the game semantics for pLμ. As discussed earlier, we adopt the metaphor of a controller (Player 1) who tries to satisfy a logical specification against an hostile environment (Player 2) in the context of the probabilistic choices (Nature) modeled in PLTS's.

The formula F_1 expresses the possibility of the controller performing two consecutive a-steps. Thus in the PLTS of Figure 3.1, at the state p the formula F_1 is satisfied with probability $\frac{1}{2}$, because after the first a-step, which is always possible, the probability of reaching a state where no a-action is available, namely the state $\mathbf{0}$, is $\frac{1}{2}$.

Similarly, the formula F_2 expresses the possibility of performing an infinite sequence of a-actions. In the PLTS of Figure 3.1 the probability of satisfying this property is clearly 0, as the computation will reach the bad state $\mathbf{0}$ at some finite stage almost surely.

The third formula F_3 expresses the possibility of reaching some state, by means of a sequence of b-actions, and from there produce an infinite sequence of a-actions. Thus the formula F_3 at the state p in some PLTS is interpreted as the limit probability of producing a computation of type b^*a^ω. Note how the intended interpretation is significantly different from the following similar, but meaningless, one: the limit probability of reaching, by means of b-actions, a state where F_2 *holds*. Indeed, as opposed to PCTL-like logics, in pLμ we do not have any notion of a formula holding at a given state. The right way to understand the meaning of pLμ formulas is as the limit probability of the controller (i.e., Player 1) satisfying the property against an adversary environment (i.e., Player 2).

For example the meaning of the formula F_4, at a given state p of some PLTS, is the limit probability of the controller being able to perform an a-action after any possible sequence of a-actions chosen by an adversary environment. As it is easy to see in the context of the PLTS of Figure 3.1, $[\![F]\!](p)=0$, because the adversary environment can just keep making a-actions until the state $\mathbf{0}$ is reached, and this will happen with probability 1, and then ask the controller to satisfy their task, i.e., making an a-action in a state where this is clearly not possible.

Our last example, expressed as formula F_5, is useful to expose an important point: the logic pLμ is not boolean. Indeed the formula F_5, or equivalently $F_1 \wedge \overline{F_1}$, where F_1 denotes the negation of the closed formula F_1, is not trivially false (i.e., with value 0) at avery state of a PLTS. The interpretation of formula F_5 at the state p of the PLTS of figure 3.1, for instance, is $\frac{1}{2}$. Indeed the probability of satisfying $\langle a\rangle\langle a\rangle\underline{1}$ is, as discussed in the first example, $\frac{1}{2}$. But also the probability of satisfying $[a]\,[a]\,\underline{0}$, i.e., the probability associated to the possible failure of the environment in producing two consecutive a-steps, is $\frac{1}{2}$. More generally, when the value of $\langle a\rangle\langle a\rangle\underline{1}$ is λ, the value of $[a]\,[a]\,\underline{0}$ is always $1-\lambda$, since the two formulas are dual.

Thus when working with the logic pLμ, one should understand the connectives $\vee$ and $\wedge$ as expressing the properties *take the best of the two* and *take the worst of the two* respectively, forgetting about the (boolean) algebraic properties often associated with these operators.

3.3 Extensions of the probabilistic modal μ-calculus

In this Section we define several *probabilistic modal μ-calculi*, each extending the base $[0,1]$-valued logic pLμ introduced in Section 3.2.3, for expressing properties of PLTS's. The presentation is based on the *denotational semantics* of these logics. Thus the meaning of a formula F, interpreted on a PLTS $\mathcal{L}=\langle P, \langle \xrightarrow{a} \rangle_{a\in L}\rangle$, is a $[0,1]$-valued predicate (see Definition 3.2.10) $[\![F]\!]^{\mathcal{L}}_{\rho} : P \to [0,1]$ assigning to each process state $p \in P$ a real value in $[0,1]$, with the intended interpretation that $[\![F]\!](p)$ represents the probability of the formula F holding at the state p, as discussed in Section 3.2.3. In Section 3.3.3 we shall provide, informally, the intuitions which will lead us towards the definition of appropriate *game semantics* for each of the proposed μ-calculi.

3.3.1 Formal definitions

We start by defining the most expressive logic considered in this thesis, which we call the *probabilistic modal μ-calculus with independent product, truncated sum and convex combinations*, or just $pL\mu^{\odot}_{\oplus}$ for short.

Definition 3.3.1 (Syntax of $pL\mu^{\odot}_{\oplus}$)**.** The syntax of $pL\mu^{\odot}_{\oplus}$ formulas F is given by the following context free grammar:

$$
\begin{array}{rcl}
F, G & ::= & X \;\mid\; F \vee G \;\mid\; F \wedge G \;\mid\; \langle a \rangle G \;\mid\; [a]\, G \;\mid\; \mu X.F \;\mid\; \nu X.F \\
 & & F +_{\lambda} G \mid F \odot G \mid F \cdot G \mid F \oplus G \mid F \ominus G
\end{array}
$$

where X ranges over the set $\mathcal{V}$ of variables, a over a fixed set L of labels and λ is a real number in the open interval $(0, 1)$.

Thus the logic $pL\mu^{\odot}_{\oplus}$ is obtained by extending the logic $pL\mu$ with the following new connectives: $\odot$ (*coproduct*), $\cdot$ (*product*), $\oplus$ (*truncated sum*), $\ominus$ (*truncated co-sum*) and with the family of *convex combination* operators $\{+_{\lambda}\}_{\lambda \in (0,1)}$.

The notion of subformula of a $pL\mu^{\odot}_{\oplus}$ formula is obtained extending Definition 3.1.3 to the new connectives as follows:

$$
\begin{array}{rcl}
Sub(F +_{\lambda} G) & = & \{F +_{\lambda} G\} \cup Sub(F) \cup Sub(G) \\
Sub(F \cdot G) & = & \{F \cdot G\} \cup Sub(F) \cup Sub(G) \\
Sub(F \odot G) & = & \{F \odot G\} \cup Sub(F) \cup Sub(G) \\
Sub(F \ominus G) & = & \{F \ominus G\} \cup Sub(F) \cup Sub(G) \\
Sub(F \oplus G) & = & \{F \oplus G\} \cup Sub(F) \cup Sub(G)
\end{array}
$$

We also restrict our attention, as done for $L\mu$, to *normal* $pL\mu^{\odot}_{\oplus}$ formulas (see Definition 3.1.4), which are defined as expected. The *subsumption* relation on bound variables of $pL\mu^{\odot}_{\oplus}$ formulas is defined as in Definition 3.1.5.

Definition 3.3.2. Given a PLTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a \in L}\rangle$, a $[0,1]$-interpretation $\rho \in \mathcal{V} \to [0,1]^P$ of the variables and a $pL\mu^{\odot}_{\oplus}$ formula F, we define the *denotational semantics* of F over $\mathcal{L}$ as the map $[\![F]\!]^{\mathcal{L}}_{\rho} \in [0,1]^P$, by extending Definition 3.2.11 as follows:

$$
\begin{array}{rcl}
[\![F +_{\lambda} G]\!]^{\mathcal{L}}_{\rho}(p) & = & [\![F]\!]^{\mathcal{L}}_{\rho}(p) +_{\lambda} [\![G]\!]^{\mathcal{L}}_{\rho}(p) \\
[\![F \odot G]\!]^{\mathcal{L}}_{\rho}(p) & = & [\![F]\!]^{\mathcal{L}}_{\rho}(p) \odot [\![G]\!]^{\mathcal{L}}_{\rho}(p) \\
[\![F \cdot G]\!]^{\mathcal{L}}_{\rho}(p) & = & [\![F]\!]^{\mathcal{L}}_{\rho}(p) \cdot [\![G]\!]^{\mathcal{L}}_{\rho}(p) \\
[\![F \oplus G]\!]^{\mathcal{L}}_{\rho}(p) & = & [\![F]\!]^{\mathcal{L}}_{\rho}(p) \oplus [\![G]\!]^{\mathcal{L}}_{\rho}(p) \\
[\![F \ominus G]\!]^{\mathcal{L}}_{\rho}(p) & = & [\![F]\!]^{\mathcal{L}}_{\rho}(p) \ominus [\![G]\!]^{\mathcal{L}}_{\rho}(p)
\end{array}
$$

where the interpretation for the new connectives is given by the binary operations of type $[0,1]^2 \to [0,1]$ defined in Section 2.2. Note that this is a good definition since the space $[0,1]^P$ is a complete lattice and every constructor (see Proposition 2.2.4) is interpreted as a monotone operator, hence least and greatest fixed points exist by the Knaster-Tarski theorem.

Adopting the terminology already introduced for pLμ, we say that a process-state $p \in P$, under the interpretation ρ, *satisfies the formula F with probability* λ, if $[\![F]\!]^{\mathcal{L}}_{\rho}(p) = \lambda$.

As discussed in Section 3.2.3, the operations $\{\cdot, \odot\}$ and $\{\ominus, \oplus\}$ were considered in [56] as alternative interpretations for the pLμ connectives $\{\wedge, \vee\}$, thus not as operations to consider in combination as in pL$\mu^{\odot}_{\oplus}$.

We presented the syntax of pL$\mu^{\odot}_{\oplus}$ in *positive-form*, i.e. without including a *negation* operator explicitly, in order to simplify the definition of the denotational semantics. As discussed in Section 3.1.1 in the context of the logic Lμ, a negation operator for pL$\mu^{\odot}_{\oplus}$ can be defined by induction on the syntax of *closed* formulas in a straightforward way, exploiting the dualities between the operators $\{\langle a\rangle, \vee, (\mu X.), \odot, \oplus\}$ and $\{[a], \wedge, (\nu X.), \cdot, \ominus\}$ and the *self-duality* of $+_\lambda$ (see Lemma 2.2.5).

Definition 3.3.3. Given a closed pL$\mu^{\odot}_{\oplus}$ formula F, we define its *dual formula* $\overline{F}$ by induction on the structure of F as follows:

$$\overline{\overline{X}} = X$$
$$\overline{F \vee G} = \overline{F} \wedge \overline{G} \qquad \overline{F \wedge G} = \overline{F} \vee \overline{G}$$
$$\overline{\langle a\rangle F} = [a]\,\overline{F} \qquad \overline{[a]\,F} = \langle a\rangle \overline{F}$$
$$\overline{\mu X.F} = \nu X.\overline{F[\overline{X}/X]} \qquad \overline{\nu X.F} = \mu X.\overline{F[\overline{X}/X]}$$
$$\overline{F +_\lambda G} = \overline{F} +_\lambda \overline{G}$$
$$\overline{F \odot G} = \overline{F} \cdot \overline{G} \qquad \overline{F \cdot G} = \overline{F} \odot \overline{G}$$
$$\overline{F \oplus G} = \overline{F} \ominus \overline{G} \qquad \overline{F \ominus G} = \overline{F} \oplus \overline{G}$$

where $F[\overline{X}/X]$ denotes the pL$\mu^{\odot}_{\oplus}$ formula F where all occurrences of the (free) variable X are replaced by $\overline{X}$.

Proposition 3.3.4. *Given a PLTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$, a $[0,1]$-interpretation $\rho \in \mathcal{V} \to [0,1]^P$ of the variables and a closed pL$\mu^{\odot}_{\oplus}$ formula F, the following assertion holds:* $[\![F]\!]^{\mathcal{L}}_{\rho}(p) = 1 - [\![\overline{F}]\!]^{\mathcal{L}}_{\rho}$.

Proof. The desired result follows immediately by De Morgan dualities of $\mathrm{pL}\mu^{\odot}_{\oplus}$ operators. In particular the duality of $\langle a \rangle$ with $[a]$ follows from the self-duality of the (countably indexed) weighted sum operator (see Proposition 2.2.5) and from the duality of $\sqcup$ (join) with $\sqcap$ (meet). $\square$

It is going to be convenient to define the following $\mathrm{pL}\mu^{\odot}_{\oplus}$ formulas:

Definition 3.3.5. We denote with $\underline{0}$, $\underline{1}$ and $\underline{\lambda}$ the following $\mathrm{pL}\mu$ formulas:

$$\underline{0} \stackrel{\text{def}}{=} \mu X.X \qquad \underline{1} \stackrel{\text{def}}{=} \nu X.X \qquad \underline{\lambda} \stackrel{\text{def}}{=} \underline{1} +_{\lambda} \underline{0}.$$

Note that $\overline{(\underline{0})} = \underline{1}$ and $\overline{(\underline{1})} = \underline{0}$.

Proposition 3.3.6. *For every PLTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a \in L}\rangle$, $[0,1]$-interpretation $\rho \in \mathcal{V} \to [0,1]^P$ of the variables and process state $p \in P$, the following equalities hold:*

$$[\![\underline{0}]\!]^{\mathcal{L}}_{\rho}(p) = 0 \qquad [\![\underline{1}]\!]^{\mathcal{L}}_{\rho}(p) = 1 \qquad [\![\underline{\lambda}]\!]^{\mathcal{L}}_{\rho}(p) = \lambda$$

We will adopt the following convention, which reduces some annoying bureaucracy when dealing with the $+_{\lambda}$ operator in $\mathrm{pL}\mu^{\odot}_{\oplus}$ formulas.

Convention 3.3.7. We identify the $\mathrm{pL}\mu^{\odot}_{\oplus}$ formulas $F +_{\lambda} G$ and $G +_{1-\lambda} F$. It is trivial to verify that the two formulas are semantically equivalent. An immediate advantage of this convention is that the expected equality $\overline{(\underline{\lambda})} = \underline{1 - \lambda}$ holds.

3.3.2 Fragments of $\mathrm{pL}\mu^{\odot}_{\oplus}$

As we anticipated in Chapter 1, a central contribution of this thesis is the definition of a game semantics for the logic $\mathrm{pL}\mu^{\odot}_{\oplus}$ and thus for all its fragments obtained by dropping some operators. In this section we consider some interesting fragments of the logic $\mathrm{pL}\mu^{\odot}_{\oplus}$. The classification is based on the features of the games which are needed to interpret each sub-logic and on the basis of expressivity considerations. The basic logic $\mathrm{pL}\mu$, discussed in Section 3.2.3, is the simplest such fragment, admitting a game-semantics given in terms of 2-player stochastic games.

3.3.2.1 Logic $\mathrm{pL}\mu \cup \{+_{\lambda}\}$

The logic $\mathrm{pL}\mu \cup \{+_{\lambda}\}$, which we refer to as the *probabilistic modal μ-calculus with convex combinations*, is the fragment of the logic $\mathrm{pL}\mu^{\odot}_{\oplus}$ obtained by removing the set of operators $\{\odot, \cdot, \oplus, \ominus\}$. The logic pL$\mu \cup \{+_\lambda\}$ is a mild extension of the basic logic pLμ and, as we shall see discuss in Section 3.3.3, it is simple to extend the game semantics of pLμ, given in terms of 2-player stochastic parity games, to interpret the new family of operators $\{+_\lambda\}_{\lambda \in (0,1)}$.

These operators are useful for expressing interesting properties of PLTS's, but also of standard LTS's. Consider, for instance, the pL$\mu \cup \{+_\lambda\}$ formula $\nu X.\big(\langle a \rangle X +_{\frac{99}{100}} (\langle a \rangle \underline{1} \wedge [a]\, X)\big)$ whose meaning can be described as the limit probability the controller have of producing an infinite sequence of a-actions when sometimes, based on a small probability, the hostile environment is allowed to choose which a-actions to take.

3.3.2.2 Logic pL$\mu^{\odot}_{\oplus} \setminus \{+_\lambda\}$

To the other side of the spectrum we have the logic pL$\mu^{\odot}_{\oplus} \setminus \{+_\lambda\}$, which we refer to as the *probabilistic modal μ-calculus with independent product and truncated sum*. As the notation suggests, pL$\mu^{\odot}_{\oplus} \setminus \{+_\lambda\}$ is the fragment of the logic pL$\mu^{\odot}_{\oplus}$ obtained by removing the family of operators $\{+_\lambda\}_{\lambda \in (0,1)}$.

As observed in Section 2.2, the pairs of operations $\{\odot, \oplus\}$ and $\{\cdot, \ominus\}$ of type $[0,1]^2 \to [0,1]$, are quantitative generalizations of the standard boolean operations of join and meet in the two element lattice $\{0,1\}$. As a consequence we can state the following proposition.

Proposition 3.3.8. *Let F be a pL$\mu^{\odot}_{\oplus} \setminus \{+_\lambda\}$ formula and $F\!\downarrow$ the Lμ formula obtained from F by replacing every occurrence of the operators $\{\odot, \oplus\}$ and $\{\cdot, \ominus\}$ with $\vee$ and $\wedge$ respectively. For every LTS $\mathcal{L}$, process state p and boolean interpretation of the variables $\rho \in \mathcal{V} \to \{0,1\}^P$, the standard L$\mu$-semantics of $F\!\downarrow$ at p under ρ and the pL$\mu^{\odot}_{\oplus} \setminus \{+_\lambda\}$ semantics of F under ρ, interpreted in the PLTS $\mathcal{L}'$ corresponding to $\mathcal{L}$ (see Convention 3.2.2), coincide at p.*

Proof. It follows trivially from previous observations and from the fact that the inductive definition of denotational semantics for pL$\mu^{\odot}_{\oplus} \setminus \{+_\lambda\}$ preserves two-valuedness. In particular the interpretations of $\langle a \rangle$ and $[a]$ preserve two-valuedness only when interpreted over LTS's. $\square$

In other words when considering non-probabilistic PLTS's (i.e., LTS's) and $\{0,1\}$-valued interpretations of the variables, the logic pL$\mu^{\odot}_{\oplus} \setminus \{+_\lambda\}$ collapses to the standard modal μ-calculus Lμ. Therefore pL$\mu^{\odot}_{\oplus} \setminus \{+_\lambda\}$ can be considered as a conservative $[0,1]$-generalization of Lμ.

The presence, or the absence, of the family of operators $\{+_\lambda\}$ in the fragments of the logic pL$\mu^{\odot}_{\oplus}$ we consider below does not affect the development of their theory and, in particular, the definition of appropriate game semantics. For this reasons we shall implicitly assume that the family of operators $\{+_\lambda\}_{\lambda\in(0,1)}$ is always part of the the syntax of the sub-logics we hereafter discuss, and omit to specify, in their nomenclature, the "*with convex combinations*" description.

3.3.2.3 Logics pL$\mu^{\odot}$ and pL$\mu^{\{0,1\}}$

The logic pL$\mu^{\odot}$, which we refer to as the *probabilistic modal μ-calculus with independent product*, is the fragment of the logic pL$\mu^{\odot}_{\oplus}$ obtained by removing the pair of operators $\{\oplus, \ominus\}$.

The logic pL$\mu^{\odot}$ turns out to be interesting for its expressive power which allows the encoding of *qualitative threshold modalities*, as we now discuss.

Definition 3.3.9. Given a pL$\mu^{\odot}$ formula F, we define the macro formulas $\mathbb{P}_{>0}F$ and $\mathbb{P}_{=1}F$ as follows:

$$\mathbb{P}_{>0}F \stackrel{\text{def}}{=} \mu X.(F \odot X) \qquad \text{and} \qquad \mathbb{P}_{=1}F \stackrel{\text{def}}{=} \nu X.(F \cdot X).$$

Note that, when F is closed, $\overline{\mathbb{P}_{>0}F} = \mathbb{P}_{=1}\overline{F}$ and $\overline{\mathbb{P}_{=1}F} = \mathbb{P}_{>0}\overline{F}$, i.e. the derived formulas constructors $\mathbb{P}_{>0}$ and $\mathbb{P}_{=1}$, if taken as primitive connectives, are De Morgan duals.

For the sake of readability, we extend the negation operator to the derived qualitative modalities.

Definition 3.3.10. Given a closed pL$\mu^{\odot}$ formula F, we define the negation of the formula $\mathbb{P}_{>0}F$, denoted by $\overline{\mathbb{P}_{>0}F}$ or just $\mathbb{P}_{=0}F$, as $\mathbb{P}_{=1}\overline{F}$. Similarly we define the negation of the formula $\mathbb{P}_{=1}F$, denoted by $\overline{\mathbb{P}_{=1}F}$ or just $\mathbb{P}_{<1}F$, as $\mathbb{P}_{>0}\overline{F}$.

The following lemma captures the denotational semantics of the qualitative threshold modalities.

Lemma 3.3.11. *Given a PLTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$, a $[0,1]$-interpretation of the variables $\rho \in \mathcal{V} \to [0,1]^P$ and a pL$\mu^{\odot}$ formula F, the following assertions hold:*

$$[\![\mathbb{P}_{>0}F]\!](p) = \begin{cases} 1 & \text{if } [\![F]\!]^{\mathcal{L}}_{\rho}(p) > 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad [\![\mathbb{P}_{=1}F]\!](p) = \begin{cases} 1 & \text{if } [\![F]\!]^{\mathcal{L}}_{\rho}(p) = 1 \\ 0 & \text{otherwise} \end{cases}$$

for every process-state $p \in P$*.*

Proof. The map $x \mapsto \lambda \odot x$, for a fixed $\lambda \in [0,1]$, has 1 as unique fixed point when $\lambda > 0$, and 0 as the least fixed point when $\lambda = 0$. Similarly for the map $x \mapsto \lambda \cdot x$. The result then follows trivially. □

As an immediate corollary of Lemma 3.3.11 we have the following observation:

Corollary 3.3.12. *The denotational interpretation of an open pL*$\mu^{\odot}$ *formula* F *is, in general, not* ω*-(co)continuous (see Definition 2.1.14) in the free variables, i.e., the denotation of a formula, seen as a map* $\big(\mathcal{V} \to [0,1]^P\big) \to [0,1]$ *where* $\big(\mathcal{V} \to [0,1]^P\big)$ *is ordered pointwise, does not preserve* $\sqsubseteq$*-increasing (decreasing)* ω*-chains.*

Proof. Consider the formula $\mathbb{P}_{=1}X = \nu Y.X \cdot Y$ having just one free variable X. We have that $[\![\mathbb{P}_{=1}X]\!]_\rho(p) = 0$ if $\rho(X)(p) < 1$ and $[\![\mathbb{P}_{=1}X]\!]^{\mathcal{L}}_\rho(p) = 1$ if $\rho(X)(p) = 1$. □

As a consequence, it is not possible to apply, in general, the Kleene fixed point theorem (see Theorem 2.1.15) for characterizing the value of a fixed point formula. This contrasts with the fact that the denotations $[\![F]\!]^{\mathcal{L}}_\rho$ of pLμ formulas, when interpreted over finite PLTS's (in the sense of Definition 3.2.1), are (co)-continuous in the free variables, provided that the interpretation ρ assigns to each variable a (co)continuous function (see[8], e.g., Appendix C in [78]).

As we shall see in Section 7.2, the possibility of encoding the qualitative threshold modalities allows the expression of many interesting properties, as well as the encoding of the *qualitative* fragment of PCTL (see Definition 3.2.8). As a matter of fact, most of the pL$\mu^{\odot}$ formulas used in this thesis to describe properties of PLTS's, are actually definable in the fragment of pL$\mu^{\odot}$ where the use of the operators $\{\odot, \cdot\}$ is restricted to the encodings of the qualitative modalities $\{\mathbb{P}_{>0}, \mathbb{P}_{=1}\}$. We denote this fragment with pL$\mu^{\{0,1\}}$.

3.3.2.4 Logics pL$\mu_\oplus$ and pL$\mu^{[0,1]}$

The logic pL$\mu_\oplus$, which we refer to as the *probabilistic modal* μ*-calculus with truncated sum*, is the fragment of the logic pL$\mu^{\odot}_\oplus$ obtained by removing the pair of operators $\{\cdot, \odot\}$.

[8]The proof, which can be easily be extended to pL$\mu \cup \{+_\lambda\}$-formulas, crucially depends on the fact that *finite* 2-player stochastic parity games are positionally determined (See Theorem 2.3.58). Incidentally, the method adopted in [78] for proving the equivalence of the denotational and game semantics of pLμ is based on the (co)continuity of the denotations, and is thus limited to finite PLTS's.

In the logic pL$\mu_\oplus$ it is possible to encode the qualitative modalities $\{\mathbb{P}_{>0}, \mathbb{P}_1\}$ using an encoding similar discussed earlier, namely defining $\mathbb{P}_{>0}F \stackrel{\text{def}}{=} \mu X.(F \oplus X)$ and $\mathbb{P}_{=1}F \stackrel{\text{def}}{=} \nu X.(F \ominus X)$. However, unlike[9] the logic pL$\mu^\odot$, in pL$\mu_\oplus$ it is also possible to encode *quantitative threshold modalities* and *inequality operators*, as we now discuss.

Definition 3.3.13. Given pL$\mu_\oplus$ formulas F and G, with G closed and F possibly open, we define the macro formulas $F \geq G$ and $F > G$ as follows:

$$F \geq G \stackrel{\text{def}}{=} \mathbb{P}_{=1}(F \oplus \overline{G}) \qquad \text{and} \qquad F > G \stackrel{\text{def}}{=} \mathbb{P}_{>0}(F \ominus \overline{G}).$$

Note that we restricted to closed formulas G in order to apply the negation operator. We also denote with $\mathbb{P}_{\geq\lambda}F$ and $\mathbb{P}_{>\lambda}F$ the pL$\mu^\odot_\oplus$ formulas $F \geq \underline{\lambda}$ and $F > \underline{\lambda}$. Observe that $\underline{\lambda}$ is indeed a closed formula.

For the sake of readability, we extend the negation operator to the derived quantitative threshold modalities.

Definition 3.3.14. Given a closed pL$\mu_\oplus$ formula F, we define the negation of the formula $\mathbb{P}_{\geq\lambda}F$, denoted by $\overline{\mathbb{P}_{\geq\lambda}F}$ or just $\mathbb{P}_{<\lambda}F$, as $\mathbb{P}_{>1-\lambda}\overline{F}$. More concretely, we have that $\mathbb{P}_{<\lambda}F = \mu X.\big(\overline{F} \ominus \underline{\lambda}\big) \oplus X$. Similarly we define the negation of the formula $\mathbb{P}_{>\lambda}F$, denoted by $\overline{\mathbb{P}_{=1}F}$ or just $\mathbb{P}_{\leq\lambda}F$, as $\mathbb{P}_{>1-\lambda}\overline{F}$. More concretely, we have that $\mathbb{P}_{\leq\lambda}F = \nu X.\big(\overline{F} \oplus \underline{\lambda}\big) \ominus X$.

The following lemma captures the denotational semantics of the quantitative threshold modalities.

Lemma 3.3.15. *Given a PLTS $\mathcal{L} = \langle P, \{\stackrel{a}{\longrightarrow}\}_{a\in L}\rangle$, a $[0,1]$-interpretation of the variables $\rho \in \mathcal{V} \to [0,1]^P$, a closed pL$\mu^\odot_\oplus$ formula G and a (possibly open) pL$\mu^\odot_\oplus$ formula F, the following assertions hold:*

$$[\![F \geq G]\!](p) = \begin{cases} 1 & \textit{if } [\![F]\!]^{\mathcal{L}}_\rho(p) \geq [\![G]\!]^{\mathcal{L}}_\rho(p) \\ 0 & \textit{otherwise} \end{cases}$$

$$[\![F > G]\!](p) = \begin{cases} 1 & \textit{if } [\![F]\!]^{\mathcal{L}}_\rho(p) > [\![G]\!]^{\mathcal{L}}_\rho(p) \\ 0 & \textit{otherwise} \end{cases}$$

and in particular

[9] Although we strongly believe that pL$\mu^\odot$ can not encode the quantitative threshold modalities, we could not come up with a proof. We shall go back to this point in Chapter 8.

$$[\![\mathbb{P}_{\geq\lambda}F]\!](p)=\begin{cases}1 & \textit{if } [\![F]\!]^{\mathcal{L}}_{\rho}(p)\geq\lambda\\ 0 & \textit{otherwise}\end{cases}\qquad \textit{and} \qquad [\![\mathbb{P}_{>\lambda}F]\!](p)=\begin{cases}1 & \textit{if } [\![F]\!]^{\mathcal{L}}_{\rho}(p)>\lambda\\ 0 & \textit{otherwise}\end{cases}$$

for every process-state $p \in P$.

Proof. By definition we have that $F \geq G = \mathbb{P}_{=1}(F \oplus \overline{G})$. From the properties of the negation operator, we know that $[\![\overline{G}]\!]^{\mathcal{L}}_{\rho}(p) = 1 - [\![G]\!]^{\mathcal{L}}_{\rho}(p)$ and therefore $[\![F \oplus \overline{G}]\!]^{\mathcal{L}}_{\rho}(p) = \min\{1, [\![F]\!]^{\mathcal{L}}_{\rho}(p) + \big(1 - [\![G]\!]^{\mathcal{L}}_{\rho}(p)\big)\}$. Thus $[\![\overline{G} \oplus F]\!]^{\mathcal{L}}_{\rho}(p) = 1$ if and only if $[\![F]\!]^{\mathcal{L}}_{\rho}(p) \geq [\![G]\!]^{\mathcal{L}}_{\rho}(p)$. The desired result then trivially follows

By definition we have that $F > G = \mathbb{P}_{>0}(F \ominus \overline{G})$. We have that $[\![F \ominus \overline{G}]\!]^{\mathcal{L}}_{\rho}(p) = \max\{0, \big(1 - [\![G]\!]^{\mathcal{L}}_{\rho}(p)\big) + [\![F]\!]^{\mathcal{L}}_{\rho}(p) - 1\}$. Thus $[\![\overline{G} \oplus F]\!]^{\mathcal{L}}_{\rho}(p) > 0$ if and only if $[\![F]\!]^{\mathcal{L}}_{\rho}(p) > [\![G]\!]^{\mathcal{L}}_{\rho}(p)$. □

As we shall see in Section 7.2, the possibility of encoding the quantitative threshold modalities allows the expression of many interesting properties, as well as the encoding of *full* PCTL (see Definition 3.2.8). Most of the interesting properties of PLTS's formalized as pL$\mu_{\oplus}$ formulas, could be actually specified in the fragment of pL$\mu_{\oplus}$ where the use of the operators $\{\oplus, \ominus\}$ is restricted to the encoding of the quantiative threshold operators $\{\mathbb{P}_{\geq\lambda}, \mathbb{P}_{>\lambda}\}$. We denote this fragment with pL$\mu^{[0,1]}$.

3.3.3 Towards a game semantics

In the previous sections we introduced the logic pL$\mu^{\odot}_{\oplus}$ defining its denotational semantics and discussing the expressive power of some of its fragments in terms of the ability to encode useful operators, such as the qualitative and quantitative modalities. However, as discussed in Section 3.2.3, in the $[0, 1]$-predicate approach to probabilistic temporal logics, it is of conceptual importance to offer some interpretation for the meaning of the formulas that goes beyond their mere denotational meaning, which is potentially just a map in $[0, 1]^P$ lacking any useful probabilistic interpretation. As discussed in Section 3.2.4, an important tool for providing such an interpretation is given by *game semantics*: the logic pLμ (a fragment of pL$\mu^{\odot}_{\oplus}$) has an interpretation given in terms of $2\frac{1}{2}$-player parity games which allows one to understand the meaning of a formula at a process-state p in terms of the interactions between the controller (Player 1) and an hostile environment (Player 2) in the context of the stochastic choices modeled by PLTS's (Nature), as for Lμ formulas.

The principal task we undertake in this thesis is to extend the game semantics of pLμ to the richer logics pL$\mu \cup \{+_\lambda\}$, pL$\mu^{\odot}$ and pL$\mu^{\odot}_{\oplus}$, where *extending* means that the games associated with the pLμ fragment of the richer logics should coincide with the $2\frac{1}{2}$-player parity games for pLμ described in Section 3.2.4.

Extending the game semantics of pLμ to pL$\mu \cup \{+_\lambda\}$ is quite simple: at the new kind of states of the form $\langle p, G +_\lambda H\rangle$, corresponding to the new operator $+_\lambda$, Nature moves and chooses the state $\langle p, G\rangle$ with probability λ, and the state $\langle p, H\rangle$ with probability $1-\lambda$, whence from the $2\frac{1}{2}$-player game continues. Indeed pL$\mu \cup \{+_\lambda\}$ seems to be a natural extension of pLμ, importing directly into the syntax of the logic the probabilistic nature of its game interpretation.

Extending the game interpretation of pLμ to the richer logic pL$\mu^{\odot}$ does not seem possible working within the framework of $2\frac{1}{2}$-player games, and in particular not in that of $2\frac{1}{2}$-player parity games. The denotational semantics of the new connectives $\cdot$ and $\odot$, however, suggests an elementary interpretation for their meaning: $G \cdot H$ expresses the probability that both G and H hold when verified *independently*, and $G \odot H$ expresses the probability that at least one of G or H hold when verified independently. To capture formally this intuition, we will introduce a game semantics for the logic in which independent play of many instances of the game is allowed. Our games build on those for pLμ described in Section 3.2.4. Novelty arises in the game interpretation of the game-states $\langle p, H_1 \cdot H_2\rangle$ and $\langle p, H_1 \odot H_2\rangle$: when during the execution of the game one of these kinds of nodes is reached, the game is split into two concurrent and independent sub-games continuing their executions from the states $\langle p, H_1\rangle$ and $\langle p, H_2\rangle$ respectively. The difference between the game-interpretation of product and coproduct operators is that on a product configuration $\langle p, H_1 \cdot H_2\rangle$, Player 1 has to win in both generated sub-games, while on a coproduct configuration $\langle p, H_1 \odot H_2\rangle$ Player 1 needs to win just one of the two generated sub-games.

To illustrate the main ideas, let us consider the PLTS of figure 3.2(a) and the pLμ formula $F = \langle a\rangle\langle a\rangle\underline{1}$ which asserts the possibility of performing two consecutive a-steps. The probability of F being satisfied at p is $\frac{1}{2}$, since after the first a-step, the process $\mathbf{0}$ is reached with probability $\frac{1}{2}$ and no further a-step is possible. Let us consider the pL$\mu^{\odot}$ formula $H = \mu X.F \odot X$. Figure 3.2(b) depicts a play in the game starting from the configuration $\langle p, H\rangle$ (the fixed-point unfolding step is omitted). The branching points represent places where coproduct is the main connective, and each T_i represents play in one of

(a) Example of PLTS (b) Branching play rooted in $\langle p, H\rangle$

Figure 3.2: Illustrative example

the independent subgames for $\langle p, F\rangle$ thereupon generated. We call such a tree, describing play on all independent subgames, a *branching play*. Since all branches are coproducts, and the fixpoint is a least fixpoint, the objective for Player 1 is to win at least one of the games T_i. Since the probability of winning a particular game T_i is $\frac{1}{2}$, and there are infinitely many independent such games, almost surely Player 1 will win one of them. Therefore the game semantics assigns H at p the value 1. Incidentally the formula $H = \mu X.F \odot X$ considered above is $\mathbb{P}_{>0}F$, and indeed $[\![\mathbb{P}_{>0}F]\!]_\rho(p) = 1$.

As a matter of fact, designing a game semantics for the logic pL$\mu^{\{0,1\}}$, i.e., a game interpretation for the qualitative modalities $\mathbb{P}_{>0}$ and $\mathbb{P}_{=1}$, does not seem any simpler than working directly with the more expressive logic pL$\mu^{\odot}$. Moreover, the operational interpretation of the qualitative modalities one gets from the game semantics of pL$\mu^{\odot}$ is quite natural. As in the previous example, the formula $\mathbb{P}_{>0}F$ can be thought as generating countably many concurrent and independent instances of the game associated with F, where Player 1 needs to win in at least one of them. Similarly, the formula $\mathbb{P}_{=1}F$ generates countably many concurrent and independent instances of the game associated with F, and Player 1 needs to win in all of them.

Formalizing the pL$\mu^{\odot}$ games outlined above is a surprisingly technical undertaking. To account for the *branching plays* that arise, we introduce, in Chapter 4, a general notion of *tree game* which is of interest in its own right and itself an inportant contribution of the thesis. Tree games generalize ordinary $2\frac{1}{2}$-player games, and are powerful enough to encode certain classes of games of imperfect information such as Blackwell games, as we shall discuss in Section 4.2. The theory of tree games we are going to develop, will also allow us to formulate other notions that appear in the literature on ordinary stochastic, games such as *qualitative determinacy*, in terms of determinacy of appropriate classes of tree games, as we shall discuss in Section 4.4.

A further level of difficulty arises in expressing when a branching play in a pL$\mu^{\odot}$ game is considered an objective for Player 1. This is delicate because branching plays can contain infinitely many interleaved occurrences of product and coproduct operations. So the simple explanation of the objective of Player 1 being to win both subgames, in the case of a product, and to win at least one subgame, in the case of a coproduct, does not suffice. To account for this, branching plays are themselves considered as ordinary 2-player (parity) games with coproduct nodes as Player 1 nodes, and product nodes as Player 2 nodes. Player 1's goal in the *outer* pL$\mu^{\odot}$ game is to produce a branching play for which, when itself considered as an ordinary parity game, the *inner* game, they have a winning strategy. To formalize the class of tree games whose objective is specified by means of *inner* games, we shall introduce the notion of $2\frac{1}{2}$-*player meta-game* in Chapter 5.

Unlike the operators $\{\odot, \cdot\}$, the denotational semantics of the operators $\{\oplus, \ominus\}$ does not suggest any clear probabilistic interpretation. However the following lemma provides a possible interpretation for these operations in terms of products and coproducts

Lemma 3.3.16. *Let* $\Phi : ([0,1]^2 \to [0,1]) \to ([0,1]^2 \to [0,1])$ *be the functional defined as* $\Phi\big(f\big)(x,y) = f(x \odot y, x \cdot y) \sqcup x$*. Then the following equality holds:* $\oplus = \mathrm{lfp}(\Phi)$*. Moreover* $x \oplus y = \bigsqcup_{n\in\mathbb{N}} a^n_{x,y}$*, where the increasing sequence* $\{a^n_{x,y}\}_{n\in\mathbb{N}}$ *is defined by mutual induction with the decreasing sequence* $\{b^n_{x,y}\}_{n\in\mathbb{N}}$*, as follows:*

- $a^0_{x,y} = x$, $b^0_{x,y} = y$, *and*
- $a^{n+1}_{x,y} = a^n_{x,y} \odot b^n_{x,y}$, $b^{n+1}_{x,y} = a^n_{x,y} \cdot b^n_{x,y}$.

for every $x, y \in [0,1]$.

Proof. Note that functional Φ is monotone, where the order on the space of functionals of the same type of Φ is defined lifting the order on $[0,1]$ pointwise. Hence the existence of a least fixed point is guaranteed by the Knaster-Tarski theorem and $\mathrm{lfp}(\Phi) = \bigsqcap\{f \mid f \sqsupseteq \Phi(f)\}$. Note that $\oplus$ is a fixed point of Φ because $\big((x \odot y) \oplus (x \cdot y)\big) \sqcup x = \min\{1, x + y - (x \cdot y) + (x \cdot y)\} = x \oplus y$. Therefore, to prove the desired result, we just need to show that for any map $f:[0,1]^2 \to [0,1]$, such that $f \sqsupseteq \Phi(f)$, the inequality $\oplus \sqsubseteq f$ holds.

Fix some $f \sqsupseteq \Phi(f)$. By definition of Φ we have that, for every $x, y \in [0,1]$, $f(x,y) \geq x$ and $f(x,y) \geq f(x \odot y, x \cdot y)$. It then easily follows that, for every $n \in \mathbb{N}$, $f(x,y) \geq f(a^n_{x,y}, b^n_{x,y}) \geq a^n_{x,y}$, where the sequences $a^n_{x,y}$ and $b^n_{x,y}$ are defined as in the statement of the lemma. Therefore the inequality $f(x,y) \geq \bigsqcup_n a^n_{x,y}$ holds for every $f \sqsupseteq \Phi(f)$.

To conclude the proof we just need to show that $x \oplus y = \bigsqcup_n a^n_{x,y}$. Let us define the function $\phi : [0,1]^2 \to [0,1]^2$ as $\phi(x,y) = \langle x \odot y, x \cdot y \rangle$. The function ϕ is clearly bounded, pointwise monotone and continuous. Note that for every $n \in \mathbb{N}$, $\phi(a^n_{x,y}, b^n_{x,y}) = \langle a^{n+1}_{x,y}, b^{n+1}_{x,y} \rangle$. Therefore the equality $\{\phi^n(x,y)\}_{n\in\mathbb{N}} = \{\langle a^n_{x,y}, b^n_{x,y} \rangle\}_{n\in\mathbb{N}}$ holds. Given that, for every $n \in \mathbb{N}$, $a^{n+1}_{x,y} \geq a^n_{x,y}$ and $b^{n+1}_{x,y} \leq b^n_{x,y}$, it follows that the sequences $\{a^n_{x,y}\}_{n\in\mathbb{N}}$ and $\{b^n_{x,y}\}_{n\in\mathbb{N}}$ converge, by the monotone convergence theorem. The function ϕ is continuous, hence $\lim_{n\to\infty} \phi^n(x,y)$ is a fixed point of ϕ. From the previous observations we have that $\lim_{n\to\infty} \phi^n(x,y) = \lim_{n\to\infty} \langle a^n_{x,y}, b^n_{x,y} \rangle = \langle \bigsqcup_n a^n_{x,y}, \bigsqcap_n b^n_{x,y} \rangle$, and therefore $\langle \bigsqcup_n a^n_{x,y}, \bigsqcap_n b^n_{x,y} \rangle$ is a fixed point of ϕ. As a last observation note that, for every $n \in \mathbb{N}$, the equality $a^n_{x,y} + b^n_{x,y} = x + y$ holds. To conclude the proof it is sufficient to observe that the set of fixed points of ϕ is precisely $\{\langle w, 0 \rangle \mid w \in [0,1]\} \cup \{\langle 1, z \rangle \mid z \in [0,1]\}$. It is then simple to verify that $\bigsqcup_n a^n_{x,y} = x \oplus y$ as desired, and also that $\bigsqcap_n b^n_{x,y} = x \ominus y$. □

The following dual result follows as a corollary.

Corollary 3.3.17. *Let* $\Psi : ([0,1]^2 \to [0,1]) \to ([0,1]^2 \to [0,1])$ *be the functional defined as* $\Psi(f)(x,y) = f(x \cdot y, x \odot y) \sqcap x$. *Then the following equality holds:* $\ominus = \mathrm{gfp}(\Psi)$. *Moreover, for every* $x, y \in [0,1]$, $x \ominus y = \bigsqcap_{n\in\mathbb{N}} b^n_{x,y}$, *where the decreasing sequence* $\{b^n_{x,y}\}_{n\in\mathbb{N}}$ *is defined by mutual induction with the increasing sequence* $\{a^n_{x,y}\}_{n\in\mathbb{N}}$, *as in Lemma 3.3.16.*

On the basis of these result, we will be able to provide a game interpretation to the operators $\{\oplus, \ominus\}$ as follows: at game states of the form $\langle p, G \oplus H \rangle$ Player 1 can choose to continue the game from one of the states $\langle p, A_n \rangle$, where the $\mathbb{N}$-indexed set of formulas $\{A_n\}_{n\in\mathbb{N}}$ is defined by mutual induction with the $\mathbb{N}$-indexed set $\{B_n\}_{n\in\mathbb{N}}$ as follows: $A_0 = G$, $B_0 = H$, $A_{n+1} = A_n \odot B_n$ and $B_{n+1} = A_n \cdot B_n$. Similarly, at game states of the form $\langle p, G \ominus H \rangle$, Player 2 can choose to continue the game from one of the states $\langle p, B_n \rangle$, where the $\mathbb{N}$-indexed sets of formulas $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ are defined as above.

Thus, tree games will also be used to provide an appropriate game semantics for the full logic $pL\mu^{\odot}_{\oplus}$. This semantics is however, in many ways, less satisfactory than the game semantics for $pL\mu^{\odot}$. Indeed the connectives $\oplus$ and $\ominus$ do not have a clear and direct interpretation, as opposed to the operators $\cdot$ and $\odot$ around which the notion of tree games is conceived, but are instead modeled by a simple, yet not necessarily illuminating, protocol formalized as an appropriate tree game.

One of our main result, proved in Chapter 7, is the equivalence with respect to all models, (i.e., including PLTS's having infinitely many states) of the denotational semantics and the corresponding game semantics for the logics $pL\mu$ (result stated earlier as Theorem 3.2.14), $pL\mu^{\odot}$ and $pL\mu^{\odot}_{\oplus}$. As anticipated in Chapter 1, the proof of equivalence for the logics $pL\mu^{\odot}$ and $pL\mu^{\odot}_{\oplus}$ is carried out in $ZFC + MA_{\aleph_1}$ set theory, due to the measure theoretic complications arising in tree games.

3.4 Summary of the chapter

In this chapter we provided the necessary background on program logics. In Section 3.1 we considered logics for expressing properties of LTS's, and in particular CTL and the modal μ-calculus ($L\mu$). In Section 3.2 we turned our attention to logics for expressing properties of PLTS's. We introduced the logics PCTL and $pL\mu$. We presented the denotational semantics of $pL\mu$, highlighting the conceptual importance of giving a description, going beyond the mere denotational interpretation, of the properties expressed by formulas. We then introduced the game semantics of $pL\mu$ as in [78]. This alternative semantics offers a natural operational interpretation for the properties expressed by $pL\mu$ formulas. Lastly, we discussed the meaning of a few examples of useful $pL\mu$ formulas.

The chapter, although mostly based on ideas and results already appeared in the literature, contains some minor contributions. In Section 3.3 we identified the logic $pL\mu^{\odot}_{\oplus}$ and some of its fragments, including $pL\mu^{\odot}$. These probabilistic μ-calculi are obtained by extending $pL\mu$ with additional connectives. With the possible exception of the family of operators $\{+_{\lambda}\}_{\lambda\in(0,1)}$, the denotational interpretations of the new connectives have been already considered in [56] as *alternative* interpretations for the $pL\mu$ connectives $\{\wedge, \vee\}$. Thus, we took the small but apparently novel step of considering the (co)product operations $\{\odot, \cdot\}$ and the truncated (co)sum operations $\{\oplus, \ominus\}$ in combination with $\vee$ and $\wedge$. In the extended logics it is possible to encode useful operators, such as the qualitative and quantitative threshold modalities. Another minor contribution lies in the informal discussion, carried out in Section 3.3.3, of the ideas which will lead us toward appropriate game semantics for $\text{pL}\mu^{\odot}_{\oplus}$ and its fragments. The result of Lemma 3.3.16 is quite useful in this sense, as it offers an interpretation for the meaning of the $\{\oplus, \ominus\}$ connectives in terms of sequences of (co)product operations. This provides a possible, albeit indirect, reading of the properties associated with $\text{pL}\mu^{\odot}_{\oplus}$ formulas containing truncated (co)sum connectives.

Chapter 4

Tree Games

In this chapter we introduce a new class of games which we call *two player stochastic tree games*, or just $2\frac{1}{2}$-player *tree games*. Two player stochastic tree games generalize standard two player turn based stochastic games, which we introduced in Section 2.3.4, by allowing the execution of a play to be split into concurrent sub-games which continue their execution independently. This is formalized by introducing a new class of nodes in the game arenas, which we call *branching nodes*. When a play reaches one of these nodes, no action from the two players, or from Nature, is performed. Instead the play is automatically split into several subplays, one for each successor node of the current branching node, which continue their execution independently.

The chapter is organized as follows. In Section 4.1 we formally define the class of two player stochastic tree games. In Section 4.2 we provide an expressivity result, showing how the class of Blackwell games, introduced in Section 2.3.2, can be faithfully encoded in terms of two player tree games. In section 4.3 we identify an interesting class of winning sets, which we call *subtree-monotone* winning sets, and study some of their properties. In Section 4.4 we show how 2-player tree games can faithfully model $2\frac{1}{2}$-player tree games. Thus, from a foundational point of view, stochasticity in tree games is not of fundamental importance. This result is used to discuss an interesting open problem in the field of ordinary $2\frac{1}{2}$-player games known as *strong qualitative determinacy*.

The idea of concurrent and independent execution of sub-games, upon which tree games are defined, is inspired by the considerations about the logic $\text{pL}\mu^{\odot}$ discussed in Section 3.3.3. Nevertheles this section, although mainly motivated and inspired by our study of probabilistic μ-calculi, can be read without any knowledge of the topics discussed in Chapter 3. We do assume, on the other hand, knowledge of the basic topics of game theory discussed in Section 2.3.

4.1 Formal definitions

In this section we formally introduce the class of two player stochastic tree games. Most of the definitions and several key concepts coincide or generalize those discussed in Section 2.3.4 for standard $2\frac{1}{2}$-player games.

Two player stochastic tree games are infinite duration games played by Player 1, Player 2 and a third probabilistic agent named *Nature*, on a Arena $\mathcal{A} = \langle (S,E),(S_1,S_2,S_N,B),\pi\rangle$, where (S,E) is a directed graph with *countable* set of vertices S and transition relation $E \subseteq S \times S$, the tuple (S_1,S_2,S_N,B) is a partition of S and $\pi : S_N \rightarrow \mathcal{D}(S)$ assigns a probability distribution to every Nature state.The states in S_1, S_2, S_N and B are called *Player* 1 states, *Player* 2 states, *probabilistic* states and *branching* states respectively. Thus a $2\frac{1}{2}$-player tree game arena generalizes a standard $2\frac{1}{2}$-player game arena, as defined in Section 2.3.4, by allowing the new kind of branching state. We denote with $E(s)$, for $s \in S$, the set $\{s' \mid (s,s') \in E\}$, which we refer to as the set of successor states of s, and as a technical constraint[1], we require that $supp(\pi(s)) \subseteq E(s)$, for every $s \in S_N$.

A tree game played on some arena $\mathcal{A}$, starts at a given state $s_0 \in S$, and proceeds as an ordinary $2\frac{1}{2}$-player game until a branching state is reached, i.e., Player 1 and Player 2 choose how to move on states in S_1 and S_2 respectively, and Nature moves[2], in accordance with the probability distribution $\pi(s)$, a successor when the current state is some state $s \in S_N$. The novelty in tree games arises when the current state is some state $s \in B$, i.e., a branching state of the game. In this case no action is performed by any agent, since the game is split automatically into I-many concurrent sub-games, where $E(s) = \{s_i\}_{i \in I}$, each continuing its execution from the state s_i. Note that if $E(s) = \emptyset$, then no sub-game is generated and the game terminates immediately at the state s.

The result of a play, i.e., the result of the choices made by the two players and Nature at the respective points, is therefore not just a path, as in standard

[1]The requirement $supp(\pi(s)) = E(s)$ might seem more natural and in line with Definition 2.3.40. However, in Section 6.2 and 6.3, it will be technically convenient to work with arenas having the strict inclusion $supp(\pi(s)) \subsetneq E(s)$. We shall discuss these advantages in the relevant sections.

[2]Note that the constraint $supp(\pi(s)) \subseteq E(s)$ guarantees that Nature moves to states $t \in E(s)$ even though, when $supp(\pi(s)) \subsetneq E(s)$, some states in $E(s)$ are never chosen by Nature.

$2\frac{1}{2}$-player games, but a *tree*, having nodes with more than one child occurring at states in the game where the current state was a branching state, and several sub-games were generated. We call this class of games *tree games*, to highlight the nature of its set of outcomes.

To formalize this idea, we need to define the notions of paths and trees in the arena $\mathcal{A}$. Paths in a $2\frac{1}{2}$-player tree game arena and the associated operations are specified as in Definition 2.3.41 for standard $2\frac{1}{2}$-player game arenas. The set $\mathcal{P}_{\mathcal{A}}$ is endowed with a Polish 0-dimensional topology specified as in Definition 2.1.33.

An outcome of the game in $\mathcal{A}$, which we call a *branching play*, is a tree T in $\mathcal{A}$ (see Definition 2.1.35) defined as follows:

Definition 4.1.1 (Branching play in $\mathcal{A}$)**.** A *branching play* in the arena $\mathcal{A}$ is a tree T in $\mathcal{A}$ which is uniquely branching in $S_1 \cup S_2 \cup S_N$ and fully branching in B (see Definition 2.1.37). We denote with $\mathcal{BP}_{\mathcal{A}}$, or just $\mathcal{BP}$ if $\mathcal{A}$ is clear from the context, the set of branching plays T in the arena $\mathcal{A}$.

A branching play T represents a possible execution of the game from the state s labeling the root of T. The nodes of T with more than one child are all labeled with a state $s \in B$ and are the branching points of the game; their children represent the independent instances of play generated at the branching points. The set $\mathcal{BP}_{\mathcal{A}}$ is endowed with a 0-dimensional Polish topology specified as in Definition 2.1.38.

As usual when working with stochastic games, it is useful to look at the possible outcomes of a play up to the behavior of Nature. In the context of standard two player turn based stochastic games we considered, in Section 2.3.4, the notion of Markov play. In the new setting the following definition of Markov branching play is natural:

Definition 4.1.2 (Markov branching play in $\mathcal{A}$)**.** A *branching play* in the arena $\mathcal{A}$ is a tree M in $\mathcal{A}$ which is uniquely branching in $S_1 \cup S_2$ and fully branching in $S_N \cup B$. We denote with $\mathcal{MBP}_{\mathcal{A}}$, or just $\mathcal{MBP}$ if $\mathcal{A}$ is clear from the context, the set of branching plays T in the arena $\mathcal{A}$. We look at a Markov Branching play M, as a tree whose edges are labeled with probabilities. This labeling is given by a function $\pi_M : E_M \rightarrow [0,1]$, where $E_M \subseteq M \times M$, is the set of edges in the tree M, formally defined as $E_M = \{(\vec{s}, \vec{t}) \mid \vec{t} \text{ is a child of } \vec{s} \text{ in } M\}$. The labeling function π_M associated with the Markov branching play M, is defined[3] as follows:

[3]Note even if, in general, $supp(s) \subsetneq E(s)$ (see Footnote 1) the probability distribution $\pi(s)(t)$

$$\pi_M(\vec{s},\vec{t}) = \begin{cases} \pi(s)(t) & \text{if } s = last(\vec{s}),\ s \in S_N, \text{ and } \vec{t} = \vec{s}.\{t\} \\ 1 & \text{otherwise} \end{cases}$$

The set $\mathcal{MBP}_{\mathcal{A}}$ is endowed with a 0-dimensional Polish topology specified as in Definition 2.1.38.

A Markov branching play, is similar to a branching play except that the probabilistic choices of Nature have not been resolved. Such a structure is useful because it is possible to extract from it the probability that Nature, with its probabilistic choices (modeled by the labeling of edges of the Markov branching play), will produce an outcome contained in some given set of branching plays. This is formally captured by the following definition.

Definition 4.1.3 (Probability measure $\mathbb{P}(M)$)**.** Every Markov branching play M determines a probability assignment $\mathbb{P}_M(O_F)$ to every basic clopen set $O_F \subseteq \mathcal{BP}$ of branching plays, for F a finite tree (we can assume that every node of $\vec{s} \in F$ such that $last(\vec{s}) \in S_N$ has at most one child in F, otherwise $O_F = \emptyset$), defined as follows:

$$\mathbb{P}_M(O_F) \stackrel{\text{def}}{=} \begin{cases} \prod\{\pi_M(\vec{s},\vec{t}) \mid \vec{s},\vec{t} \in F \text{ and } \vec{t} \text{ is a child of } \vec{s} \text{ in M}\} & \text{if } F \subseteq M \\ 0 & \text{otherwise} \end{cases}$$

The assignment $\mathbb{P}_M$ on basic clopen sets extends to a unique (Borel) probability measure $\mathbb{P}_M \in \mathcal{M}_1(\mathcal{BP})$, whence to a (unique) complete probability measure on $\mathcal{BP}$ which we also denote with $\mathbb{P}_M$. We denote with $\mathbb{P} : \mathcal{MBP} \rightarrow \mathcal{M}_1(\mathcal{BP})$ the function defined as $\mathbb{P}(M) = \mathbb{P}_M$.

The above definition implements the *probabilistic independence* of the sub-branching plays generated at some branching node in a $2\frac{1}{2}$-player tree game. Importantly, tree games exhibit also an epistemic independence between the sub-branching plays which will be implemented in the notion of strategy.

An important property of the above defined construction is exposed by the following lemma.

Lemma 4.1.4. *The function* $\mathbb{P} : \mathcal{MBP} \rightarrow \mathcal{M}_1(\mathcal{BP})$ *is continuous.*

Proof. By Theorem 2.1.66(5), we just need to show that for each sub-basic open set $U \subseteq \mathcal{M}_1(\mathcal{BP})$, i.e., each set of the form $U = \{\mu \in \mathcal{M}_1(\mathcal{BP}) \mid \mu(O_F) > \lambda\}$ for some finite tree F in $\mathcal{A}$ and $\lambda \in (0,1)$, the set $\mathbb{P}^{-1}(U)$ is open in $\mathcal{MBP}$. The

is well defined on all states $s \in S$, and $\pi(s)(t) = 0$ for all states $t \notin supp(\pi(s))$.

set $\mathbb{P}^{-1}(U)$ consists of all Markov branching plays M such that $\mathbb{P}_M(O_F) > \lambda$, where O_F is the basic open set of branching plays containing the finite tree F. By definition of $\mathbb{P}_M$, we have that $\mathbb{P}_M(O_F) = \mathbb{P}_N(O_F)$ for every $M, N \in \mathbb{P}^{-1}(U)$. Thus either $\mathbb{P}^{-1}(U)$ is empty, hence trivially open, or it is the set of all Markov branching play containing the finite tree F (because $\lambda > 0$ and every $M \in \mathcal{MBP}$ not containing F is such that $\mathbb{P}_M(O_F) = 0$ by definition), i.e., it is the basic open set O_F of Markov branching plays. $\square$

It is appropriate at this time to highlight the fact that, if there are no branching nodes in $\mathcal{A}$, i.e., if $B = \emptyset$, then a $2\frac{1}{2}$-player tree game $\mathcal{A}$ is just a standard $2\frac{1}{2}$-player game arena. This intuition is formalized, in a slightly more general way, as follows:

Definition 4.1.5. A $2\frac{1}{2}$-player tree game arena $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B), \pi \rangle$ is called a *non-branching* $2\frac{1}{2}$*-player tree game arena*, if for every $b \in B$, $|E(b)| \leq 1$.

The notion of *non-branching* $2\frac{1}{2}$*-player tree game arena* captures the collection of $2\frac{1}{2}$-player tree game arenas on which the game is never split into (more than one) concurrent sub-games. The intuition that any such arena is just a standard $2\frac{1}{2}$-player game arena is captured by the following lemma:

Lemma 4.1.6. *If* $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B), \pi \rangle$ *is a non-branching* $2\frac{1}{2}$*-player tree game arena, then the following assertions holds:*

- *The spaces* $\mathcal{P}_{\mathcal{A}}$ *and* $\mathcal{BP}_{\mathcal{A}}$ *are homeomorphic via the function* $f(\vec{s}) = \vec{s}\downarrow$,
- *the definition of Markov branching play coincides with that of Markov play (see Definition 2.3.46).*

As for standard $2\frac{1}{2}$-player games, a $2\frac{1}{2}$-player tree game is specified by a $2\frac{1}{2}$-player tree game arena $\mathcal{A}$ and a payoff function Φ, which maps each possible outcome to a corresponding reward for Player 1 in the real interval $[0, 1]$. Since the outcomes of $2\frac{1}{2}$-player tree games are branching plays, it is natural to give the following definition of two player stochastic tree games.

Definition 4.1.7 (Two player stochastic tree game)**.** A *two player stochastic tree game* (or a $2\frac{1}{2}$-player tree game) is given by a pair $\langle \mathcal{A}, \Phi \rangle$, where $\mathcal{A}$ is a stochastic tree game arena as described above, and $\Phi : \mathcal{BP} \to [0, 1]$, which is called the *payoff function* for Player 1, is a universally measurable function. If Φ is the characteristic function of some (universally measurable) set $X \subseteq \mathcal{BP}$, then we just refer to Φ as the *winning set* for Player 1. A $2\frac{1}{2}$-player tree game $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$ such that the set S_N of probabilistic states in the arena $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B), \pi \rangle$ is empty, is called a *two player tree game*, or just a 2-player tree game.

As usual the intended interpretation is that Player 1 receives the reward $\Phi(T)$, if T is the final outcome a play in $\mathcal{G}$. Player 1 wants to maximize their reward, while Player 2 tries to minimize it. The final outcome T of a play of the game will be determined probabilistically according to the probability measure on the Markov Branching Play generated by the players' moves. Since there can in principle be an arbitrarily complex set of such probability measures, and the function Φ needs to be evaluated (integrated) with respect to all of them, universal measurability is the natural assumption for the payoff function Φ.

Definition 4.1.8 (Expected value of a probability measure over $\mathcal{BP}$)**.** Let $\langle \mathcal{A}, \Phi \rangle$ be a $2\frac{1}{2}$-player tree game, and $\mu \in \mathcal{M}_1(\mathcal{BP})$ a probability measure over branching plays in $\mathcal{A}$. We define the *expected value* of Φ under μ as follows:

$$E(\mu) = \int_{\mathcal{BP}} \Phi \, \mathrm{d}\mu.$$

This is a good definition since Φ is universally measurable, hence μ-measurable. If Φ is a winning set, then the above definition coincides with $E(\mu) = \mu(\Phi)$.

The expected reward assigned to Player 1 when the Markov branching play is the outcome of the game up to the behavior of nature can therefore be defined as follows:

Definition 4.1.9 (Expected value of M)**.** Let $\langle \mathcal{A}, \Phi \rangle$ be a $2\frac{1}{2}$-player tree game, and M a Markov branching play in $\mathcal{A}$. We define, with a slight abuse of notation, the *expected value* of M as follows:

$$E(M) = \int_{\mathcal{BP}} \Phi \, \mathrm{d}\mathbb{P}(M).$$

where $\mathbb{P} : \mathcal{MBP} \to \mathcal{M}_1(\mathcal{BP})$ specified as in Definition 4.1.3.

Up to this point, we have described how the players behave in a two player stochastic tree game, adopting the informal idea of *concurrent* and *independent* execution of the sub-games generated at the branching nodes. To formalize this intuition we define the notion of (deterministic) strategy, which captures the way in which the players can actually behave, exactly as in Definition 2.3.43. Hence both players when acting on a given instance of the game, know all the history of the actions happened on that sub-game, but have no knowledge of the evolution of the other *independent* parallel sub-games. Thus, this form of *epistemic independence* completes the already implemented *stochastic independence*.

Definition 4.1.10 (Topologies on Σ_1 and Σ_2)**.** Let $\Gamma = \{\vec{s_i}\}_{0\leq i\leq n}$ be a collection of n paths in $\mathcal{P}_1^{<\omega}$ and let $\Delta = \{s_i\}_{0\leq i\leq n}$ be a collection of n states in S, for some $n\in\mathbb{N}$. Let us denote with $O_{\Gamma\mapsto\Delta}$ the set of all strategies σ_1 for Player 1 such that $\sigma_1(\vec{s_i}) = s_i$, for every $0\leq i\leq n$. We fix the topology on Σ_1, where the countable basis for the open sets is given by the clopen sets $O_{\Gamma\mapsto\Delta}$, for every pair (Γ,Δ) as defined above. This is a 0-dimensional Polish space. The topology on Σ_2 is defined in a similar way.

Note how the definition of the topologies on Σ_1 and Σ_2 slight differs from the corresponding one, given in the context of standard $2\frac{1}{2}$-player games, in Definition 2.3.44.

As discussed earlier, a Markov branching play M represents the result of a play of the players up to the behavior of Nature. This is made precise by the following definition.

Definition 4.1.11 ($M^{s_0}_{\sigma_1,\sigma_2}$)**.** Given an initial state $s_0\in S$ and a strategy profile $\langle\sigma_1,\sigma_2\rangle$ a unique Markov branching play, denoted by $M^{s_0}_{\sigma_1,\sigma_2}$, is determined:

1. the root of M is the path $\{s_0\}$ of length one,
2. for every $\vec{s}\in M^{s_0}_{\sigma_1,\sigma_2}$, if $last(\vec{s}) = s$ with $s\in S_1$ not a terminal state, then the unique child of $\vec{s}$ in $M^{s_0}_{\sigma_1,\sigma_2}$ is $\vec{s}.\{\sigma_1(\vec{s})\}$,
3. for every $\vec{s}\in M^{s_0}_{\sigma_1,\sigma_2}$, if $last(\vec{s}) = s$ with $s\in S_2$ not a terminal state, then the unique child of $\vec{s}$ in $M^{s_0}_{\sigma_1,\sigma_2}$ is $\vec{s}.\{\sigma_2(\vec{s})\}$.

The Markov branching play $M^{s_0}_{\sigma_1,\sigma_2}$ is then determined uniquely because Markov branching plays branch fully on probabilistic and branching states. Given a state $s_0\in S$, we denote with $\langle \_\,,\_\rangle^{s_0}:\Sigma_1\times\Sigma_2\rightarrow\mathcal{MBP}$, or just with $\langle \_\,,\_\rangle$ if s_0 is clear from the context, the function defined as $\langle\sigma_1,\sigma_2\rangle^{s_0} = M^{s_0}_{\sigma_1,\sigma_2}$.

The following property about the function $\langle \_\,,\_\rangle^{s_0}$ is immediate to verify.

Lemma 4.1.12. *For every state $s\in S$, the function $\langle \_\,,\_\rangle^{s_0}:\Sigma_1\times\Sigma_2\rightarrow\mathcal{MBP}$ is continuous.*

We introduce, in order to improve readability, the following notation.

Definition 4.1.13. We denote with $\mathbb{P}^{s_0}_{\sigma_1,\sigma_2} \in \mathcal{M}_1(\mathcal{BP})$, for every $s \in S$ and every strategy profile $\langle \sigma_1, \sigma_2 \rangle$, the probability measure $\mathbb{P}(\langle \sigma_1, \sigma_2 \rangle^{s_0})$, where the continuous function $\mathbb{P} : \mathcal{MBP} \rightarrow \mathcal{M}_1(\mathcal{BP})$ is specified as in Definition 4.1.3. We also denote, for $s \in S$, with $\mathbb{P}^s : \Sigma_1 \times \Sigma_2 \rightarrow \mathcal{M}_1(\mathcal{BP})$ the continuous function defined as $\mathbb{P}^s = \mathbb{P} \circ \langle \_, \_ \rangle^s$.

For any universally measurable set $X \subseteq \mathcal{BP}$, the value $\mathbb{P}^s_{\sigma_1,\sigma_2}(X)$ is the probability that a branching play $T \in X$ is the outcome of the $2\frac{1}{2}$-player tree game, when the game starts at s and Player 1 and Player 2 follow the deterministic strategies σ_1 and σ_2 respectively. Similarly the value $E(\mathbb{P}^s_{\sigma_1,\sigma_2})$ is the expected reward assigned to Player 1 when the game starts at s and Player 1 and Player 2 follow the deterministic strategies σ_1 and σ_2 respectively.

We now define the notions of lower and upper values of $2\frac{1}{2}$-tree games, when the players use deterministic strategies.

Definition 4.1.14 (Upper and lower values of $\mathcal{G}$ under deterministic strategies)**.** Let $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$ be a $2\frac{1}{2}$-player tree game. We define the lower and upper values of $\mathcal{G}$ at the state s, when players use deterministic strategies, as the values denoted by $\mathsf{VAL}^s_{\downarrow}(\mathcal{G})$ and $\mathsf{VAL}^s_{\uparrow}(\mathcal{G})$ respectively, specified as follows:

$$\mathsf{VAL}^s_{\downarrow}(\mathcal{G}) = \bigsqcup_{\sigma_1} \bigsqcap_{\sigma_2} E(\mathbb{P}^s_{\sigma_1,\sigma_2}) \qquad \mathsf{VAL}^s_{\uparrow}(\mathcal{G}) = \bigsqcap_{\sigma_2} \bigsqcup_{\sigma_1} E(\mathbb{P}^s_{\sigma_1,\sigma_2})$$

As usual $\mathsf{VAL}^s_{\downarrow}(\mathcal{G})$ represents the limit expected reward assigned to Player 1, when the game begins at s, Player 1 chooses its deterministic strategy σ_1 first, and then Player 2 chooses their deterministic strategy σ_2, possibly making their choice based on the strategy σ_1 previously chosen by Player 1. Similarly for $\mathsf{VAL}^s_{\uparrow}(\mathcal{G})$. Clearly, for every s, the following inequality holds: $\mathsf{VAL}^s_{\downarrow}(\mathcal{G}) \leq \mathsf{VAL}^s_{\uparrow}(\mathcal{G})$. In the special case (not true in general) that this inequality is an equality, we say that the game $\mathcal{G}$ at s is *determined under deterministic strategies*, or just *determined*.

The notion of deterministic strategy defined earlier, models the behavior of players which make deterministic moves based on the history of visited states. However, as already discussed in Section 2.3.1, in game theory one often wants to model the behavior of players which can use randomized procedures to make their decisions. This is formalized by the concept of mixed strategy, which captures the ability of the players to randomly choose a deterministic strategy at the very beginning of the game.

The following definition closely follows the corresponding ones given in Section 2.3 for Gale–Stewart and Blackwell games.

Definition 4.1.15 (Mixed strategies)**.** A *mixed strategy* η_1 for Player 1 in $\mathcal{A}$ is defined to be a probability measure $\eta_1 \in \mathcal{M}_1(\Sigma_1)$. Similarly, a mixed strategy σ_2 for Player 2 in $\mathcal{A}$ is defined to be a probability measure $\eta_2 \in \mathcal{M}_1(\Sigma_2)$. A pair $\langle \eta_1, \eta_2 \rangle$ of mixed strategies, one for each player, is called a *mixed strategy profile*, and induces the product measure $\eta_1 \times \eta_2$ on the space $\Sigma_1 \times \Sigma_2$ endowed with the product topology.

The choice of interpreting a mixed strategy profile $\langle \eta_1, \eta_2 \rangle$ as the product measure $\eta_1 \times \eta_2$ captures the intuitive idea that the probabilistic choices made by the two players at the beginning of the game, about the deterministic strategy to use in the rest of the game, are done independently.

Given an initial state s in $\mathcal{A}$, a mixed strategy profile induces in a natural way a probability measure over branching plays, which we formalize as follows.

Definition 4.1.16. For every state $s \in S$, we define the probability measure $\mathbb{P}^s_{\eta_1,\eta_2} \in \mathcal{M}_1(\mathcal{BP})$ over branching plays induced by a mixed strategy profile $\langle \eta_1, \eta_2 \rangle$ when the game starts at s, as follows:

$$\mathbb{P}^s_{\eta_1,\eta_2} = \mathtt{b}\Big((\eta_1 \times \eta_2), \mathbb{P}^s\Big)$$

where $\mathtt{b} : \Big(\mathcal{M}_1(\Sigma_1 \times \Sigma_2) \times \big((\Sigma_1 \times \Sigma_2) \to \mathcal{M}_1(\mathcal{BP})\big)\Big) \to \mathcal{M}_1(\mathcal{BP})$ is the bind operation associated with the Giry monad $\mathcal{M}_1$ (see Theorem 2.1.66(10)), and $\mathbb{P}^s : \Sigma_1 \times \Sigma_2 \to \mathcal{M}_1(\mathcal{BP})$ is specified as in Definition 4.1.13. Equivalently, $\mathbb{P}^s_{\eta_1,\eta_2}$ can be defined more concretely as follows:

$$\mathbb{P}^s_{\eta_1,\eta_2}(A) = \int_{\mathcal{M}_1(\mathcal{BP})} \rho_A \ \mathrm{d}\Big(\mathcal{M}_1\big(\mathbb{P}^s\big)(\eta_1 \times \eta_2)\Big)$$

where $\rho_A(\mu) = \mu(A)$, for every Borel set $A \subseteq \mathcal{BP}$.

Another approach for capturing probabilistic behaviors of the players , widely used in the context of standard $2\frac{1}{2}$-player games, can be formulated in the context of $2\frac{1}{2}$-player tree games as follows:

Definition 4.1.17 (History-based random strategy)**.** A *history-based random strategy* γ_1 for Player 1 is a map $\mathcal{P}_1^{<\omega} \to \mathcal{D}(S) \times \{\bullet\}$ such that $\gamma_1(\vec{s}) = \bullet$ if

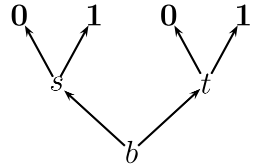
$E(last(\vec{s})) = \emptyset$ and $d \in \mathcal{D}(E(last(\vec{s}))$ otherwise. A *history-based random strategy* γ_2 for Player 2 is defined as a map $\mathcal{P}_2^{<\omega} \rightarrow \mathcal{D}(S) \times \{\bullet\}$ in a similar way. A pair of history-based random strategies $\langle \gamma_1, \gamma_2 \rangle$ for Player 1 and Player 2 respectively is called a history-based random strategy profile.

The intuition about history-based random strategies, is that both players are not committed to choosing randomly, once and for all at the very beginning of the game, which deterministic strategy to use, but they can instead make probabilistic choices, when making their moves, at every step of the game based the full history of previously played moves. In the context of $2\frac{1}{2}$-player games, history-based random strategies capture the same probabilistic behaviors[4] allowed by random strategies. Interestingly, in the context of $2\frac{1}{2}$-player tree games, history-based random strategies allow fewer probabilistic behaviors than mixed strategies. Rather than defining formally the probability measure on $\mathcal{BP}$ induced by a history-based random strategy profile, we show by means of a simple example, why history-based random strategies are not as expressive as mixed strategies.

Example 4.1.18. Let us consider the 2-player tree game arena $\mathcal{A}$ specified by the tuple $\langle (S, E), (S_1, S_2, S_N, B), \pi \rangle$ defined as follows:

- $S = \{b, s, t, \mathbf{0}, \mathbf{1}\}$,
- $E(b) = \{s, t\}$, $E(s) = E(t) = \{\mathbf{0}, \mathbf{1}\}$, and $E(\mathbf{0}) = E(\mathbf{1}) = \emptyset$,
- $S_1 = \{s, t, \mathbf{0}, \mathbf{1}\}$, $S_2 = S_N = \emptyset$ and $B = \{b\}$,
- π is just the empty function since $S_N = \emptyset$

The game played on the arena $\mathcal{A}$, starting at the state b, can be depicted as follows:



The game, starting at b, can be informally described as follows: at the beginning the game is split in two concurrent and independent sub-games, one continuing its execution at the state s and the other at the state t. In both sub-games, Player 1, who is the only participant of the game, chooses whether to move to

[4] See, e.g., Footnote 4 in Section 2.3.1.

the state **0** or **1**. For the point we want to make, we shall only be interested in the dynamics of the game played on the arena $\mathcal{A}$, thus we do not specify any reward function Φ. Suppose now that Player 1, in the game starting at b, wants to behave probabilistically in such a way that with probability $\frac{1}{2}$ they will choose to move to the state **0** in both generated sub-games, and with probability $\frac{1}{2}$ they will choose to move to the state **1**, again in both generated sub-games. This behavior can be captured by the mixed strategy η_1 which chooses to behave as the deterministic strategy σ_0 with probability $\frac{1}{2}$ or as the deterministic strategy σ_1 with probability $\frac{1}{2}$, where the strategy σ_x, for $x \in \{0,1\}$ is defined as: $\sigma_x(\{b.s\}) = x$ and $\sigma_x(\{b.t\}) = x$. On the other hand this probabilistic behavior can not be captured by any history-based random strategy γ_1. Suppose indeed that $\gamma_1(\{b.s\}) = d_1$ and $\gamma_1(\{b.t\}) = d_2$, with $d_1, d_2 \in \mathcal{D}(\{\mathbf{0}, \mathbf{1}\})$, $d_i(\mathbf{0}) = \lambda_i$, $d_i(\mathbf{1}) = 1 - \lambda_i$, with $i \in \{1, 2\}$. Then, intuitively, Player 1 following the strategy γ_1 will choose to move to **0** or **1** from s with probability λ_1 and $1 - \lambda_1$ respectively, and similarly will choose move to **0** or **1** from t with probability λ_2 and $1 - \lambda_2$ respectively. Observe how, crucially, the choices made by Player 1 in the two concurrent sub-games at the states s and t, are made (probabilistically) independently one from the other. Therefore Player 1, following the strategy γ_1, will make different choices in the states s and s with probability $\lambda_1 \cdot (1 - \lambda_2) + \lambda_2 \cdot (1 - \lambda_1)$. Thus, no matter how the strategy γ_1 is specified, Player 1 following γ_1 can not reproduce the same probabilistic behavior as that induced by the mixed strategy η_1.

We will use in the rest on the thesis only mixed-strategies because they are strictly more powerful (it is well known how to model history-based random strategies by means of mixed strategies, see e.g., [74]) and at the same time simpler to analyze for our purposes. Nevertheless history-based random strategies, and their induced probabilistic behaviors, constitute a natural class which, we suggest, might be worth further investigations.

The notions of lower and upper values of a of $2\frac{1}{2}$-player tree game, when players use mixed strategies, can be defined as follows.

Definition 4.1.19 (Upper and lower values of $\mathcal{G}$ under mixed strategies)**.** Let $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$ be a $2\frac{1}{2}$-player tree game. We define the lower and upper values of G at the state s, when players used mixed strategies, as the values denoted by $\mathsf{MVAL}^s_{\downarrow}(G)$ and $\mathsf{MVAL}^s_{\uparrow}(G)$ respectively, specified as follows:

$$\mathsf{MVAL}^s_{\downarrow}(\mathcal{G}) = \bigsqcup_{\eta_1} \bigsqcap_{\eta_2} E(\mathbb{P}^s_{\eta_1,\eta_2}) \qquad \mathsf{MVAL}^s_{\uparrow}(\mathcal{G}) = \bigsqcap_{\eta_2} \bigsqcup_{\eta_1} E(\mathbb{P}^s_{\eta_1,\eta_2})$$

Again, for every s, the following inequality trivially holds: $\mathsf{MVAL}^s_{\downarrow}(\mathcal{G}) \leq \mathsf{MVAL}^s_{\uparrow}(\mathcal{G})$. In the special case (not true in general) that this inequality is an equality, we say that the game $\mathcal{G}$ at s is *determined under mixed strategies.*

We now define the concept of ϵ-optimal (deterministic or mixed) strategy, in accordance with the corresponding notions (see Definition 2.3.24) introduced in the context of Blackwell and standard $2\frac{1}{2}$-player games.

Definition 4.1.20 (ϵ-optimal strategies)**.** Let $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$ be a $2\frac{1}{2}$-player tree game. We say that a deterministic strategy σ_1 for Player 1 in $\mathcal{G}$ is ϵ-optimal, for $\epsilon \geq 0$, if for every state s, the following inequality holds:

$$\textstyle\bigsqcap_{\sigma_2} E(\mathbb{P}^s_{\sigma_1,\sigma_2}) \geq \mathsf{VAL}^s_{\downarrow}(\mathcal{G}) - \epsilon.$$

Similarly we say that a deterministic strategy σ_2 for Player 2 in $\mathcal{G}$ is ϵ-optimal, for $\epsilon \geq 0$, if for every state s, the following inequality holds:

$$\textstyle\bigsqcup_{\sigma_1} E(\mathbb{P}^s_{\sigma_1,\sigma_2}) \leq \mathsf{VAL}^s_{\uparrow}(\mathcal{G}) + \epsilon.$$

In a similar way, we say that a mixed strategy η_1 for Player 1 in $\mathcal{G}$ is ϵ-optimal, for $\epsilon \geq 0$, if for every state s, the following inequality holds:

$$\textstyle\bigsqcap_{\eta_2} E(\mathbb{P}^s_{\eta_1,\eta_2}) \geq \mathsf{MVAL}^s_{\downarrow}(\mathcal{G}) - \epsilon,$$

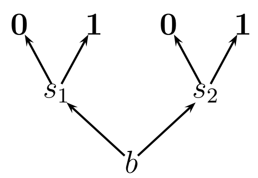
and we say that a mixed strategy η_2 for Player 2 in $\mathcal{G}$ is ϵ-optimal, for $\epsilon \geq 0$, if for every state s, the following inequality holds:

$$\textstyle\bigsqcup_{\eta_1} E(\mathbb{P}^s_{\eta_1,\eta_2}) \leq \mathsf{MVAL}^s_{\uparrow}(\mathcal{G}) + \epsilon.$$

Clearly ϵ-optimal (mixed and deterministic) strategies for Player 1 and Player 2 always exist for every $\epsilon > 0$, but not necessarily so for $\epsilon = 0$.

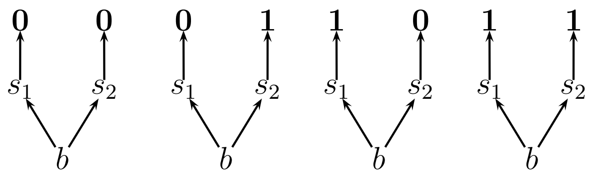
We now propose a few simple examples of a $2\frac{1}{2}$-player tree games, in order to fix some ideas and show that not all tree games are determined under deterministic or mixed strategies.

Example 4.1.21. Let us consider the 2-player tree game arena $\mathcal{A}$ which can be depicted as follows:



where b is a branching state and the state s_1 and s_2 are under the control of Player 1 and Player 2 respectively. Since the states $\mathbf{0}$ and $\mathbf{1}$ are terminal, it is irrelevant to specify their membership in B, S_1 or S_2. The game, starting at b, can be informally described as follows: at the beginning the game is split in two concurrent and independent sub-games, one continuing its execution at the state s_1 and the other at the state s_2. In these sub-games, Player 1 and Player 2 respectively choose, independently to each other, to move on the state $\mathbf{0}$ or $\mathbf{1}$. Note again how this independence is modeled by the chosen notion of (deterministic) strategy. Once the choices are made, the game ends since both $\mathbf{0}$ and $\mathbf{1}$ have no successors.

When the game starts at b, the following four branching plays, denoted by T_{00}, T_{01}, T_{10} and T_{11} respectively, are the possible outcomes of the game:



The branching play records the choices of the players in all generated sub-games, of which, in this case, there are just two, and also records the points in which the game was split into two independent sub-games.

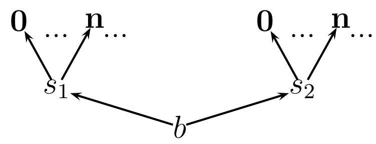
Let us now define an payoff function Φ for the arena $\mathcal{A}$. An interesting example is given by the winning set[5] $\Phi = \{T_{00}, T_{11}\}$. The game $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$ can be described as the simple game, where the two players choose independently to play a even or odd number (represented here by $\mathbf{0}$ and $\mathbf{1}$ respectively). Player 1 wins if the sum of the chosen numbers is even, and Player 2 wins otherwise. This is a well known example in game theory of a game not determined under the use of deterministic strategies. It is trivial to verify that $\mathsf{VAL}^b_{\downarrow}(\mathcal{G}) = 0$ while $\mathsf{VAL}^b_{\uparrow}(\mathcal{G}) = 1$. However, as everyone who played this game would probably know, if the players pick their numbers at random tossing a fair coin, the probability of winning the game is $\frac{1}{2}$, no matter what the other player does. This is formalized in our setting by observing that $\mathsf{MVAL}^b_{\downarrow}(\mathcal{G}) = \mathsf{MVAL}^b_{\uparrow}(\mathcal{G}) = \frac{1}{2}$. It is easy to verify that both players have 0-optimal mixed strategies in $\mathcal{G}$, formalizing the coin tossing strategy informally discussed above.

We now propose a simple modification of the previous game $\mathcal{G}$, which is not

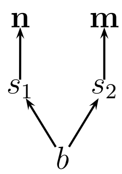
[5]To be precise we should say that Φ is the characteristic function $\Phi : \mathcal{BP} \to \{0, 1\}$ associated of the set $\{T_{00}, T_{11}\}$, but as announced earlier, we will often avoid this level of formality.

determined even under the use of mixed strategies.

Example 4.1.22. Let us consider the arena $\mathcal{A}$ which can be depicted as follows:



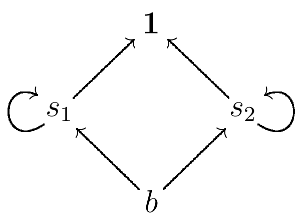
where b is a branching state, Player 1 controls the states $S_1 = \{s_1\} \cup \text{Evens}$ and Player 2 controls the states $S_2 = \{s_2\} \cup \text{Odds}$, where Evens and Odds are the sets of even and odd natural numbers respectively. The set of branching plays in $\mathcal{A}$ of the form $T_{n,m}$, for $n, m \in \mathbb{N}$, and rooted at b can be depicted as follows:



Let us fix as payoff for the game the winning set of branching plays $\Phi = \{T_{n,m} \mid n > m\}$. The game $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$ can be described as the game, where the two players choose independently to play two natural numbers n and m respectively. Player 1 wins if $n > m$ and Player 2 wins otherwise. This is a popular example of game not determined even under the use of mixed strategies. Indeed it is trivial to check that $\mathsf{MVAL}^b_{\downarrow}(\mathcal{G}) = 0$ while $\mathsf{MVAL}^b_{\uparrow}(\mathcal{G}) = 1$.

The previous example shows that there exists a $2\frac{1}{2}$-player tree game not determined under the use of mixed strategies. However, since the game arena $\mathcal{A}$ considered in the previous example is infinite, one might wonder if there exists a $2\frac{1}{2}$-player tree game with a finite arena which is not determined under the use of mixed strategies. We provide a positive answer to this question, by a slight modification of the previous example.

Example 4.1.23. Let us consider the arena $\mathcal{A}$ which can be depicted as follows:



where b is a branching state and the states s_1 and s_2 are under the control of Player 1 and Player 2 respectively. The game played on the arena $\mathcal{A}$, starting at b, can be informally described as follows: at the beginning the game is split in two concurrent and independent sub-games, one continuing its execution at the state s_1 and the other at the state s_2. In these sub-games, Player 1 and Player 2 respectively choose, independently to each other, either to loop $n \in \mathbb{N}$ times through the state s_1 (s_2 respectively) and finally move to the terminal state $\mathbf{1}$, or to loop forever through the state s_1 (s_2 respectively). In the first case we say that Player 1 *played the number* $n+1$, while in the case of an infinite loop we say that Player 1 *played the number* 0, and similarly for Player 2.

By fixing, as winning set for the game, the set of branching plays such that Player 1 plays n and Player 2 played m, with $n>m$, we get a two player tree-game, with a finite arena, which is essentially the game described in the previous example. It follows by the same kind of arguments, that the game is not determined by the use of mixed strategies. We omit the routine details.

The above discussed examples show that there exist 2-player tree games with finite game arenas, which are not determined under deterministic strategies, nor under mixed strategies. Notwithstanding these simple negative examples, the thesis will later contribute some positive determinacy results.

We conclude this section with some remarks on the concept of concurrent and independent execution of sub-games captured by $2\frac{1}{2}$-player tree games. Looking at Player 1 (respectively Player 2) as a human might generate a little bit of confusion, as the idea that Player 1 acts on a given generated sub-game independently on how *they* themself play on the other generated sub-games is a bit odd. Looking at Player 1 as a human might be reasonable in the context of standard $2\frac{1}{2}$-player games[6]. Still, game theorists are used to thinking in a slight different way. We quote a paragraph from [14], which describes David H. Blackwell point of view on the matter[7]:

> Imagine that you are to play the white pieces in a single game of chess, and that you discover you are unable to be present for the occasion. There is available a deputy, who will represent you on the occasion, and who will carry out your instructions exactly, but who is absolutely unable to make any decisions of his own volition. Thus, in order to guarantee that your deputy will be able to conduct the white pieces

[6] Although the idea of a human playing a game of infinite duration is already hard to imagine.

[7] The author found this wonderful passage in M. R. Vervoort PhD's thesis [111].

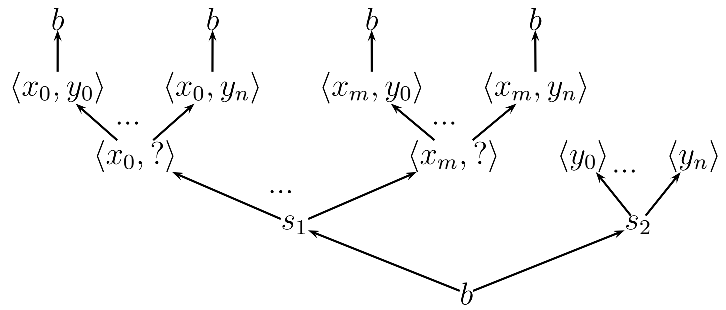
> throughout the game, your instructions to him must envisage every possible circumstance in which he may be required to move, and must specify, for each such circumstance, what his choice is to be. Any such complete set of instructions constitutes what we shall call a strategy.

Therefore we should not look at Player 1 directly as the *chess player*. Rather, Player 1 is just a *deputy* who blindly follows somebody else's instructions. This metaphor fits perfectly with the notion of concurrent and independent execution of sub-games modeled by $2\frac{1}{2}$-player tree games. When, at some branching state, the game is split in concurrent and independent sub-games, Player 1 calls enough new available deputies, communicates them the instructions received from the chess player, and all together they keep playing, independently, in the several sub-games. We suggest that our definition of $2\frac{1}{2}$-player tree game faithfully models this scenario. Also the notion of mixed strategy, formalized in Definition 4.1.15, is compatible with this picture: the chess player chooses randomly a strategy, before the start of the game, and communicate it to the deputies.

4.2 Encoding of Blackwell games

The examples of $2\frac{1}{2}$-player tree games discussed in Section 4.1 show that $2\frac{1}{2}$-player tree games can be used to model simple examples of games, such as the "even-odd number" game of Example 4.1.22. In this section we show that $2\frac{1}{2}$-player tree games can actually model the wide class of Blackwell games, which we introduced in Section 2.3.2. Our results suggest that the simple form of concurrent and independent execution of sub-games, modeled by $2\frac{1}{2}$-player tree games, is surprisingly a powerful abstraction.

Let us fix an arbitrary Blackwell game $\mathsf{B}(X, Y, \phi)$, where $X = \{x_0, \ldots, x_m\}$ and $Y = \{y_0, \ldots, y_n\}$, for $m, n \in \mathbb{N}^+$, and $\phi : (X \times Y)^\omega \to [0, 1]$ is a (universally) measurable payoff function. We are going to construct a two player (non-stochastic) tree game $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$ which encodes $\mathsf{B}(X, Y, \phi)$ in a sense we will make precise later on. The game $\mathcal{G}$, which can be depicted as follows,



is formally defined by the finite arena $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B), \pi \rangle$ specified as:

- $S = \{b, s_1, s_2\} \cup \{\langle x, ?\rangle\}_{x \in X} \cup \{\langle x, y\rangle\}_{x \in X, y \in Y} \cup \{\langle y\rangle\}_{y \in Y}$.

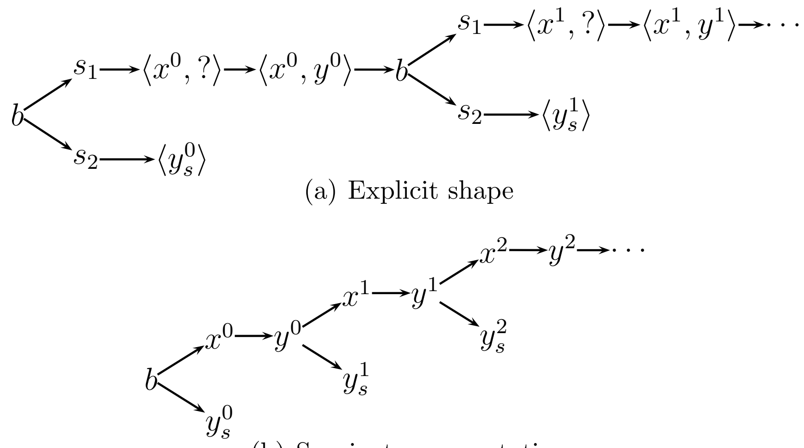
- $E(b) = \{s_1, s_2\}$,
  $E(s_1) = \{\langle x, ?\rangle\}_{x \in X}$,
  $E(s_2) = \{\langle y\rangle\}_{y \in Y}$,
  $E(\langle x, ?\rangle) = \{\langle x, y\rangle\}_{y \in Y}$, for every $x \in X$,
  $E(\langle y\rangle) = \emptyset$, for every $y \in Y$, and
  $E(\langle x, y\rangle) = \{b\}$, for every $x \in X$ and $y \in Y$.

- $S_1 = \{s_1\}$
  $S_2 = \{s_2\} \cup \{\langle x, ?\rangle\}_{x \in X} \cup \{\langle x, y\rangle\}_{x \in X, y \in Y} \cup \{\langle y\rangle\}_{y \in B}$
  $S_N = \emptyset$,
  $B = \{b\}$.

- π is just the empty function since $S_N = \emptyset$.

As a first observation, note that since $S_N = \emptyset$, there are no states under the control of Nature in $\mathcal{A}$. This means that every Markov branching play in $\mathcal{A}$ is actually a branching play, i.e., $\mathcal{MBP} = \mathcal{BP}$. For this reasons we will just denote with T_{σ_1, σ_2} the (Markov) branching play induced by a deterministic strategy profile $\langle \sigma_1, \sigma_2\rangle$ in $\mathcal{A}$ from the state b, i.e., $T_{\sigma_1, \sigma_2} = \langle \sigma_1, \sigma_2\rangle^b$, and we will directly write $\Phi(T^b_{\sigma_1, \sigma_2})$ in place of $E(T^b_{\sigma_1, \sigma_2})$. Secondly, the states of the form $\langle y\rangle$, for $y \in Y$, are terminal because they have no successors in $\mathcal{A}$, and the states of the form $\langle x, y\rangle$, for $x \in X$ and $y \in Y$ have only one successor, namely the state b. These states have been defined to be in S_2, i.e., under the control of Player 2, by convention.

The tree game played on the arena $\mathcal{A}$ starting at b, can be described as follows. At the initial step the game is split in two concurrent and independent sub-games, one continuing its execution from the state s_1 and the other from the state s_2. In the sub-game starting at s_2 Player 2 chooses to move to one of the states $\langle y\rangle$,



(a) Explicit shape

(b) Succinct representation

Figure 4.1: Shape of a branching play T in $\mathcal{A}$ rooted at b

with $y \in Y$, on which the sub-game terminates. We say that Player 2 plays y as their *side move*. In the sub-game starting at s_1 Player 1 has to choose to move to one of the states of the form $\langle x, ?\rangle$, for $x \in X$. We say that Player 1 plays x as their *main move*. Once the state $\langle x, ?\rangle$ is reached, Player 2 has to choose a state $\langle x, y\rangle$, for $y \in Y$. We say that Player 2 plays y as their *main move in response to* x. From the state $\langle x, y\rangle$ Player 2 has only one forced move to the state b, after which the process is repeated again and again as described above.

Given this description, every branching play in the arena $\mathcal{A}$, starting at b, can be depicted as in Figure 4.1(a), or using a more succinct representation, as in Figure 4.1(b), where we denoted with x^n, y^n and y^n_s, for all $n \in \mathbb{N}$, the main move of Player 1, the main move of Player 2 in response of x^n and the side move of Player 2, at the n-th stage of the game, respectively. Note how from the succinct representation one can reconstruct the explicit shape of any branching play T in $\mathcal{A}$ in the obvious way.

The intuition behind the construction of $\mathcal{A}$ is that we want to model a play $((x_0, y_0), \dots, (x_n, y_n), \dots)$ in the Blackwell game $\mathsf{B}(X, Y, \phi)$ as a branching play in $\mathcal{A}$ having $x^n = x_n$ and $y^n = y_n$, using the notation introduced above, for every $n \in \mathbb{N}$. However, while in the Blackwell game $\mathsf{B}(X, Y, \phi)$, at any stage n, Player 1 and Player 2 have to choose concurrently and independently their moves x_{n+1} and y_{n+1}, basing their decision on the previous history of played moves $h = ((x_0, y_0), \dots, (x_n, y_n))$, in the tree game $\mathcal{A}$, Player 1 is forced to choose a move x^{n+1} based on the history h (as in $\mathsf{B}(X, Y, \phi)$), but Player 2, can choose their y^{n+1} move based on h *and* on the move x^{n+1} made by Player 1. This clearly introduces an advantage for Player 2. In order to fix this asymmetry in the roles of the two players, we introduced in $\mathcal{A}$ the structure necessary to model the *side moves* of Player 2. Player 2, when choosing their side move at the n-th stage of the game, can base their decision only on the history $h = ((x^0, y^0), \dots, (x^n, y^n))$, without knowing what Player 1 will play as $n{+}1$-th move. If Player 2 were forced to play in such a way that $y^n = y^n_s$, for all $n \in \mathbb{N}$, we would re-establish the original symmetry between the two players that we had in the Blackwell game $\mathsf{B}(X, Y, \phi)$. An infinite branch in a branching play T in $\mathcal{A}$, for which $y^n = y^n_s$ for all $n \in \mathbb{N}$, thus faithfully models a play in the game $\mathsf{B}(X, Y, \phi)$. This idea will be fully formalized in the rest of this section.

We start by defining the set *Fair* of branching plays in $\mathcal{A}$ on which Player 2 respects the policy of playing identical main moves and side moves, at each stage of the game.

Definition 4.2.1. Let us define the set $\mathit{Fair} \subseteq \mathcal{BP}$ of branching plays in the arena $\mathcal{A}$ as the set of branching plays T that, once represented as in Figure 4.1(a), satisfy the following property: $\forall n \in \mathbb{N}.\ y^n = y^n_s$. If a branching play T is in *Fair* we say that T is a *fair* branching play. It is simple to check that $\overline{\mathit{Fair}}$ is open, being the union of all branching plays having, at some point, distinct main and side moves. Hence *Fair* is a closed set.

As discussed above, each play in $\mathsf{B}(X, Y, \phi)$ will be represented faithfully by the (unique) infinite path $((x^0, y^0), \dots, (x^n, y^n), \dots)$ of a fair branching play. We denote with $\mathit{inf}(T)$ the infinite path in the fair branching play T. Note that inf is a bijection between *Fair* and $(X \times Y)^\omega$. Moreover, fixing the topology on *Fair* as the subspace topology induced by the topology on $\mathcal{BP}$, the map $\mathit{inf} : \mathit{Fair} \to (X \times Y)^\omega$ and its inverse inf^{-1} are continuous, which means that inf is a homeomorphism between $(X \times Y)^\omega$ and *Fair*.

We now specify the payoff function Φ for the arena $\mathcal{A}$.

Definition 4.2.2. The payoff function $\Phi : \mathcal{BP} \to [0, 1]$ for the arena $\mathcal{A}$ is defined as follows:

$$\Phi(T) = \begin{cases} \phi\big(\mathit{inf}(T)\big) & \text{if } T \in \mathit{Fair} \\ 1 & \text{otherwise} \end{cases}$$

Lemma 4.2.3. *The function Φ is universally measurable.*

Proof. We just need to show that, for every rational $\lambda \in [0, 1)$, $\Phi^{-1}(\lambda, 1]$ is universally measurable. We have that

$$\Phi^{-1}\big((\lambda,1]\big) = \overline{\mathit{Fair}} \cup \Phi|_{\mathit{Fair}}^{-1}(\lambda,1],$$

where $\Phi|_{\mathit{Fair}}$ denotes the function Φ restricted to the set *Fair*. This is obvious since $\Phi(T)=1$ for all $T \in \overline{\mathit{Fair}}$, by definition of Φ. As observed above, since *Fair* is a closed set, we have that $\overline{\mathit{Fair}}$ is open. Moreover the fact that $\Phi|_{\mathit{Fair}}^{-1}\big((\lambda,1]\big)$ is universally measurable, follows from the hypothesis that ϕ is univerally measurable, *inf* is continuous as observed before, and the fact that composition of universally measurable functions is universally measurable (see Corollary 2.1.74). □

Remark 4.2.4. In the context of Blackwell games, one often[8] says that ϕ is open[9], closed, $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, etc., if for all rationals $\lambda \in [0,1)$, $\phi^{-1}\big((\lambda,1]\big)$ is open, closed, $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, etc. By inspecting the proof of Lemma 4.2.3, and by recalling that $\overline{\mathit{Fair}}$ is open, one can easily show that ϕ and Φ have the same complexity, except in the simple case when ϕ is closed, in which case Φ would be in $\mathbf{\Delta}^0_2$, the smallest class in the Borel hierarchy which contains all Boolean combinations of closed and open sets. More formally, Φ is $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\beta$ or $\mathbf{\Delta}^0_\beta$ if and only if ϕ is $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\beta$ or $\mathbf{\Delta}^0_\beta$ respectively, for every α and every $\beta > 1$.

Intuitively, it is the choice of the function Φ that discourages Player 2 from playing main moves different from side moves, because any unfair branching play is maximally rewarded in favor of Player 1 by Φ. This suggests that Player 2 does not want to play unfairly, where the notion of unfair strategy for Player 2 is formally defined as follows:

Definition 4.2.5 (Unfair strategy for Player 2 in $\mathcal{A}$)**.** Let $\vec{s}$ be a finite path in $\mathcal{A}$ such that $\mathit{last}(\vec{s})=b$, and let $x \in X$. We say that a strategy σ_2 for Player 2 in $\mathcal{A}$ is $(\vec{s},x)$-*unfair*, if

1. $\sigma_2(\vec{s}.s_2) = \langle y_1 \rangle$, with $y_1 \in Y$
2. $\sigma_2(\vec{s}.s_1.\langle x, ?\rangle) = \langle x, y_2\rangle$, with $y_2 \in Y$
3. $y_1 \neq y_2$.

Note that the set of $(\vec{s},x)$-unfair strategies is a basic open in Σ_2, and it is denoted here by $O_{\vec{s},x}$. We define the set Σ_2^{unfair} of *unfair strategies* for Player 2 in $\mathcal{A}$, as the union of the sets $O_{\vec{s},x}$, for every pair $(\vec{s},x)$ as described above. It follows that Σ_2^{unfair} is open and its Σ_2^{fair}, which we refer to as the set of *fair strategies* for Player 2, is closed.

[8] This convention is, for instance, followed in [74] and [111].
[9] Note that *open payoffs* are not *open functions* in the topological sense.

Note that a branching play resulting from any play in $\mathcal{A}$, where Player 2 follows a fair strategy, is fair. Further, an equivalent characterization for Σ_2^{fair} is as the set of all strategies for Player 2 for which no counter-strategy for Player 1 can induce an unfair branching play. The fact, informally stated earlier, that Player 2 does not want to play unfair strategies, is formalized by the following lemma.

Lemma 4.2.6. *Given an arbitrary strategy* σ_2 *for Player* 2 *in* $\mathcal{G}=\langle\mathcal{A},\Phi\rangle$*, define the strategy* σ_2^f *as follows:*

$$\sigma_2^f(\vec{s}) = \begin{cases} \langle x,y\rangle & \text{if } \vec{s}=\vec{t}.b.s_1.\langle x,?\rangle \text{ for some } x\in X \text{ and } \sigma_2(\vec{s}.b.s_2)=\langle y\rangle \\ \sigma_2(\vec{s}) & \text{if } last(\vec{s})=s_2 \\ b & \text{if } last(\vec{s})=\langle x,y\rangle, \text{ for some } x\in X,\ y\in X \\ \bullet & \text{if } last(\vec{s})=\langle y\rangle, \text{ for some } y\in Y \end{cases}$$

where $\vec{s}\in\mathcal{P}_2^{<\omega}$*. Then for every strategy* σ_1 *for Player* 1 *in* $\mathcal{G}$, $\Phi(T^b_{\sigma_1,\sigma_2})\geq\Phi(T^b_{\sigma_1,\sigma_2^f})$.

Proof. It is easily seen that σ_2^f is well-defined and is always a fair strategy. Let us fix an arbitrary strategy σ_1 for Player 1. We need to show that the inequality $\Phi(T^b_{\sigma_1,\sigma_2})\geq\Phi(T^b_{\sigma_1,\sigma_2^f})$ holds. If $T^b_{\sigma_1,\sigma_2}\in\overline{Fair}$, then the inequality trivially holds, since by definition $\Phi(T^b_{\sigma_1,\sigma_2})=1$. Suppose then that $T^b_{\sigma_1,\sigma_2}\in Fair$. It is easy to see that, in this case, $T^b_{\sigma_1,\sigma_2}=T^b_{\sigma_1,\sigma_2^f}$. This is because a play never reaches an history $\vec{s}$, at some n-th stage of the game, in which σ_2 plays different side and main moves, and the decisions taken by σ_2 and σ_2^f on all histories $\vec{s}$ (of the first two cases presented in the definition above), where σ_2 behaves fairly, coincide. Therefore in this case we simply have $\Phi(T^b_{\sigma_1,\sigma_2})=\Phi(T^b_{\sigma_1,\sigma_2^f})$. ☐

In other words, however Player 1 plays, the fair strategy σ_2^f always performs at least as well, from Player 2's prospective, as σ_2.

Definition 4.2.7. We denote with *fair* : $\Sigma_2\to\Sigma_2^{fair}$ the function defined by $fair(\sigma_2)=\sigma_2^f$. Note that *fair*, restricted to the set Σ_2^{fair} is just the identity function. Moreover the map *fair* is clearly continuous.

We next show that the sets of strategies for Player 1 and Player 2 in $\mathsf{B}(X,Y,\phi)$ coincide with the set of strategies for Player 1 in $\mathcal{G}$ and the set of fair strategies for Player 2 in $\mathcal{G}$ respectively.

Definition 4.2.8. Let Σ_1^{B} and Σ_1 denote the sets of strategies in the Blackwell game $\mathsf{B}(X,Y,\phi)$ and in the tree game $\mathcal{G}=\langle\mathcal{A},\Phi\rangle$ respectively. We define the function $code_1:\Sigma_1^{\mathsf{B}}\to\Sigma_1$ as follows:

$$code_1(\sigma_1^{\mathsf{B}})(\vec{s}) = \langle \sigma_1^{\mathsf{B}}(main(\vec{s})), ?\rangle$$

where $\vec{s} \in \mathcal{P}_1^{<\omega}$, which in the context of the arena $\mathcal{A}$ implies $last(\vec{s}) = s_1$, and $main(\vec{s}) : \mathcal{P}_1^{<\omega} \to (X \times Y)^{<\omega}$ extracts the sequence of main moves played by Player 1 and Player 2 in the history $\vec{s}$.

Lemma 4.2.9. *The function $code_1$ defined above is a homeomorphism between Σ_1^{B} and Σ_1.*

Proof. Straightforward. ☐

Definition 4.2.10. Let Σ_2^{B} denote the set of strategies for Player 2 in the Blackwell game $\mathsf{B}(X, Y, \phi)$. We define the function $code_2 : \Sigma_2^{\mathsf{B}} \to \Sigma_2^{fair}$ as follows:

$$code_2(\sigma_2^{\mathsf{B}})(\vec{s}) = \begin{cases} \langle x, \sigma_2^{\mathsf{B}}(main(\vec{t}))\rangle & \text{if } \vec{s} = \vec{t}.\langle x, ?\rangle \text{ for some } x \in X \\ \langle \sigma_2^{\mathsf{B}}(main(\vec{s}))\rangle & \text{if } last(\vec{s}) = s_2 \\ b & \text{if } last(\vec{s}) = \langle x, y\rangle, \text{ for some } x \in X,\ y \in Y \\ \bullet & \text{if } last(\vec{s}) = \langle y\rangle, \text{ for some } y \in Y \end{cases}$$

where $\vec{s} \in \mathcal{P}_2^{<\omega}$, which in the context of the arena $\mathcal{A}$ implies that $\vec{s}$ is in exactly one of the four cases considered in the definition, and *main* is specified as in Definition 4.2.8. Note that in the last two clauses, the definition is forced by the structure of $\mathcal{A}$, and in the third clause the definition is automatically induced by the first clause, because, as specified by its codomain, $code_2$ has to map strategies in σ_2^{B} into fair strategies.

Lemma 4.2.11. *The function $code_2$ defined above is a homeomorphism between Σ_2^{B} and Σ_2^{fair}.*

Proof. Straightforward. ☐

It is worth observing how the strategy $code_2(\sigma_2^{\mathsf{B}})$, when making a decision after one of Player 1's main moves x (see first clause in Definition 4.2.10), ignores the information about x and just bases their decision on the previous game-history. This, together with the fact that Σ_2^{fair} is homeomorphic (via $code_2$) to Σ_2^{B} provides a formal statement of the fact that Player 2 when playing fairly in $\mathcal{A}$, really does act as if they were playing in $\mathsf{B}(X, Y, \phi)$.

Definition 4.2.12. We define the function $code : (\Sigma_1^{\mathsf{B}} \times \Sigma_2^{\mathsf{B}}) \to (\Sigma_1 \times \Sigma_2^{fair})$ as $code(\sigma_1^{\mathsf{B}}, \sigma_2^{\mathsf{B}}) = \langle code_1(\sigma_1^{\mathsf{B}}), code_2(\sigma_2^{\mathsf{B}})\rangle$. The map *code* is clearly a homeomorphism.

The following equation, which will be useful later on, follows trivially from the definitions of *code* and *inf*.

$$inf \circ \langle \_ \, , \_ \rangle^b \circ code = \langle \_ \, , \_ \rangle^{\mathsf{B}} \tag{4.1}$$

where $\langle \_ \, , \_ \rangle^b$ and $\langle \_ \, , \_ \rangle^{\mathsf{B}}$ are specified as in definitions 4.1.11 and 2.3.22 respectively.

So far we have established a correspondence between deterministic strategy profiles in $\mathsf{B}(X, Y, \phi)$ and fair deterministic strategy profiles in $\mathcal{G}$, via the homeomorphism *code*. We now carry on by providing a correspondence between mixed strategies for Player 1 and Player 2 in the Blackwell game $\mathsf{B}(X, Y, \phi)$ and in $\mathcal{G}$.

Recall, from definitions 2.3.26 and 4.1.15, that a mixed strategy η_1^{B} for Player 1 in $\mathsf{B}(X, Y, \phi)$ is a probability measure over Σ_1^{B}, i.e., $\eta_1^{\mathsf{B}} \in \mathcal{M}_1(\Sigma_1^{\mathsf{B}})$, while a mixed strategy η_1 for Player 1 in $\mathcal{G}$ is a probability measure over Σ_1, i.e., $\eta_1 \in \mathcal{M}_1(\Sigma_1)$.

Definition 4.2.13. We define the function $mcode_1 : \mathcal{M}_1(\Sigma_1^{\mathsf{B}}) \rightarrow \mathcal{M}_1(\Sigma_1)$ from mixed strategies for Player 1 in $\mathsf{B}(X, Y, \phi)$ to mixed strategies for Player 1 in $\mathcal{G}$, as $mcode_1 = \mathcal{M}_1(code_1)$, or equivalently as the probability measure uniquely specified on every Borel set $S \subseteq \Sigma_1$ as follows:

$$mcode_1(\eta_1^{\mathsf{B}})(S) = \eta_1^{\mathsf{B}}\big(code_1^{-1}(S)\big).$$

It follows from the fact that $\mathcal{M}_1$ is a functor on Polish spaces, that $mcode_1$ is a homeomorphism between $\mathcal{M}_1(\Sigma_1^{\mathsf{B}})$ and $\mathcal{M}_1(\Sigma_1)$. Thus we have established a correspondence (via $mcode_1$) between mixed strategies for Player 1 in the two games. We now turn our attention to Player 2's mixed strategies. We first define the notion of fair mixed strategy for Player 2 in the game $\mathcal{G}$ as follows:

Definition 4.2.14 (Fair mixed strategy)**.** We define a *fair mixed strategy* for Player 2 in $\mathcal{G}$ as a probability measure $\eta_2^f \in \mathcal{M}_1(\Sigma_2)$ over the space of mixed strategies Σ_2 such that $\eta_2^f(\Sigma_2^{fair}) = 1$.

In other words a mixed strategy η_2 for Player 2 is fair if it chooses randomly to behave almost surely as a fair strategy. Clearly fair mixed strategies are in 1-1 correspondence with probability measures on the subspace Σ_2^{fair}, thus we just look at η_2^f as an element in $\mathcal{M}_1(\Sigma_2^{fair})$.

Definition 4.2.15. We define the function $mcode_2 : \mathcal{M}_1(\Sigma_2^{\mathsf{B}}) \rightarrow \mathcal{M}_1(\Sigma_2^{fair})$ from mixed strategies for Player 1 in $\mathsf{B}(X, Y, \phi)$ to fair mixed strategies for Player 1 in $\mathcal{G}$, as $mcode_2 = \mathcal{M}_1(code_2)$, or equivalently as the probability measure uniquely specified on every Borel set $S \subseteq \Sigma_2^{fair}$:

$$mcode_2(\eta_2^{\mathsf{B}})(S) = \eta_2^{\mathsf{B}}\big(code_2^{-1}(S)\big).$$

It follows, again from the fact that $\mathcal{M}_1$ is a functor on Polish spaces and the fact that Σ_2 and Σ_2^{fair} are homeomorphic via $code_2$, that $mcode_2$ is a homeomorphism between $\mathcal{M}_1(\Sigma_2^{\mathsf{B}})$ and $\mathcal{M}_1(\Sigma_2^{fair})$.

We now prove the analogue of Lemma 4.2.6 for fair mixed strategies for Player 2 in $\mathcal{A}$, i.e., we formalize the intuition that Player 2 does not want to play unfair mixed strategies.

Lemma 4.2.16. *Let us fix an arbitrary mixed strategy η_2 for Player 2 in $\mathcal{G}$. Then the fair mixed strategy $\eta_2^f = \mathcal{M}_1\big(fair\big)(\eta_2)$ is such that for every mixed strategy η_1 for Player 1 in $\mathcal{G}$, $E(\mathbb{P}^b_{\eta_1,\eta_2^f}) \leq E(\mathbb{P}^b_{\eta_1,\eta_2})$ holds.*

Proof. The mixed strategy $\eta_2^f \in \mathcal{M}_1(\Sigma_2)$ can be more explicitly defined as the unique probability measure over Σ_2 specified by the assignment $\eta_2^f(S) = \eta_2\big(fair^{-1}(S)\big)$ on all Borel subsets S of Σ_2. Thus it is obvious that η_2^f is indeed a fair mixed strategy. Let us fix an arbitrary mixed strategy $\eta_1 \in \mathcal{M}_1(\Sigma_1)$ for Player 1 in $\mathcal{G}$. By definition of η_2^f and application of Lemma 4.2.6, the following inequalities hold:

$$\int_{\Sigma_1\times\Sigma_2} \Phi(T^b_{\sigma_1,\sigma_2})\,\mathrm{d}(\eta_1\times\eta_2^f) = \int_{\Sigma_1\times\Sigma_2} \Phi(T^b_{\sigma_1,fair(\sigma_2)})\,\mathrm{d}(\eta_1\times\eta_2) \leq \int_{\Sigma_1\times\Sigma_2} \Phi(T^b_{\sigma_1,\sigma_2})\,\mathrm{d}(\eta_1\times\eta_2).$$

The desired result then trivially follows from Definition 4.1.8. □

In other words, however Player 1 plays, the fair mixed strategy η_2^f always performs at least as well, from Player 2's prospective, as η_2.

Definition 4.2.17. We define the function *mcode*, mapping Blackwell mixed strategis profiles $\langle \eta_1^{\mathsf{B}}, \eta_2^{\mathsf{B}}\rangle \in \big(\mathcal{M}_1(\Sigma_1^{\mathsf{B}}) \times \mathcal{M}_1(\Sigma_2^{\mathsf{B}})\big)$ to fair mixed strategy profiles $\langle \eta_1, \eta_2\rangle \in \big(\mathcal{M}_1(\Sigma_1) \times \mathcal{M}_1(\Sigma_2^{fair})\big)$ in $\mathcal{G}$, as follows:

$$mcode(\eta_1^{\mathsf{B}}, \eta_2^{\mathsf{B}}) = \langle mcode_1(\sigma_1^{\mathsf{B}}), mcode_2(\sigma_2^{\mathsf{B}})\rangle.$$

Thus *mcode* is clearly a homeomorphism between the two spaces.

We are now finally ready to state our main theorem which formally proves that $\mathcal{G} = \langle \mathcal{A}, \Phi\rangle$ is a faithful encoding of $\mathsf{B}(X, Y, \phi)$.

Theorem 4.2.18. *The following assertions hold:*

1. $\mathsf{MVAL}^b_{\downarrow}(\mathcal{G}) = \bigsqcup_{\eta_1} \bigsqcap_{\eta_2^f} E(\mathbb{P}^b_{\eta_1,\eta_2^f})$,

2. $\mathsf{MVAL}^b_\uparrow(\mathcal{G}) = \bigsqcap_{\eta_2^f} \bigsqcup_{\eta^1} E(\mathbb{P}^b_{\eta_1,\eta_2^f})$,

3. $\mathsf{MVAL}_\downarrow\big(\mathsf{B}(X,Y,\phi)\big) = \mathsf{MVAL}^b_\downarrow(\mathcal{G})$,

4. $\mathsf{MVAL}_\uparrow\big(\mathsf{B}(X,Y,\phi)\big) = \mathsf{MVAL}^b_\uparrow(\mathcal{G})$,

where in the first two assertions η_2^f ranges over the set $\mathcal{M}_1(\Sigma_2^{fair})$ of fair mixed strategies for Player 2 in $\mathcal{G}$.

Proof. Let us recall, from Definition 4.1.19, that $\mathsf{MVAL}^b_\downarrow(\mathcal{G})$ and $\mathsf{MVAL}^b_\uparrow(\mathcal{G})$ are defined as

$$\mathsf{MVAL}^b_\downarrow(\mathcal{G}) = \textstyle\bigsqcup_{\eta_1} \bigsqcap_{\eta_2} E(\mathbb{P}^b_{\eta_1,\eta_2}) \quad \text{and} \quad \mathsf{MVAL}^b_\uparrow(\mathcal{G}) = \bigsqcap_{\eta_2} \bigsqcup_{\eta_1} E(\mathbb{P}^b_{\eta_1,\eta_2})$$

respectively. Then the first two assertions trivially follow from Lemma 4.2.16.

We prove the third assertion by means of the following chain of equalities:

$$\begin{aligned}
\mathsf{MVAL}^b_\downarrow(\mathcal{G}) &\overset{\text{def}}{=}_{4.1.19} \bigsqcup_{\eta_1} \bigsqcap_{\eta_2} \Big(\int_{\Sigma_1 \times \Sigma_2} \Phi(T^b_{\sigma_1,\sigma_2})\, \mathrm{d}(\eta_1 \times \eta_2)\Big) \\
&=_{\mathrm{E}_1} \bigsqcup_{\eta_1} \bigsqcap_{\eta_2^f} \Big(\int_{\Sigma_1 \times \Sigma_2} \Phi \circ \langle _, _ \rangle^b\, \mathrm{d}(\eta_1 \times \eta_2^f)\Big) \\
&=_{\mathrm{E}_2} \bigsqcup_{\eta_1} \bigsqcap_{\eta_2^f} \Big(\int_{\Sigma_1 \times \Sigma_2^{fair}} \Phi \circ \langle _, _ \rangle^b\, \mathrm{d}(\eta_1 \times \eta_2^f)\Big) \\
&=_{\mathrm{E}_3} \bigsqcup_{\eta_1} \bigsqcap_{\eta_2^f} \Big(\int_{\Sigma_1 \times \Sigma_2^{fair}} (\phi \circ \mathit{inf}) \circ \langle _, _ \rangle^b\, \mathrm{d}(\eta_1 \times \eta_2^f)\Big) \\
&=_{\mathrm{E}_4} \bigsqcup_{\eta_1^{\mathsf{B}}} \bigsqcap_{\eta_2^{\mathsf{B}}} \Big(\int_{\Sigma_1^{\mathsf{B}} \times \Sigma_2^{\mathsf{B}}} \phi \circ \langle _\,, _ \rangle^{\mathsf{B}}\, \mathrm{d}(\eta_1^{\mathsf{B}} \times \eta_2^{\mathsf{B}})\Big) \\
&\overset{\text{def}}{=}_{2.3.28} \mathsf{MVAL}_\downarrow\big(\mathsf{B}(X,Y,\phi)\big).
\end{aligned}$$

The equality E_1 follows from the first assertion of the theorem which allow us to restrict to fair mixed strategies η_2^f for Player 2. Equality E_2 holds because every fair mixed strategy σ_2^f assigns probability 1 to the set Σ_2^{fair} of fair deterministic strategies for Player 2. Since we restricted our attention to fair strategies for Player 2, which necessarily induces fair branching plays, the equality E_3 holds by Definition 4.2.2 of the payoff function Φ. Lastly, E_4 follows from the simple high-level observation that given Polish spaces A, A', B and B' and homeomorphisms $f\colon A \to B$ and $g\colon A' \to B'$, the following equality holds:

$$\bigsqcup_{\mu} \bigsqcap_{\nu} \int_{A \times B} h\, \mathrm{d}(\mu \times \nu) = \bigsqcup_{\mu'} \bigsqcap_{\nu'} \int_{A' \times B'} h \circ (f^{-1} \times g^{-1})\, \mathrm{d}(\mu' \times \nu')$$

where $h: A \times B \to [0,1]$ is universally measurable and μ, ν, μ' and ν' range over the set of probability measures over A, B, A' and B' respectively. In our case, A, B, A' and B' are the spaces Σ_1^{B}, Σ_2^{B}, Σ_1 and Σ_2^{fair}, and the homeomorphisms are the maps $code_1$ and $code_2$. The proof then follows taking $h = \phi \circ \langle \_ , \_ \rangle^{\mathsf{B}}$ and from the fact, which is an immediate consequence of Equation 4.1, that $h \circ (code_1^{-1} \times code_2^{-1}) = \phi \circ inf \circ \langle \_ , \_ \rangle^b$.

The proof of the fourth assertion of the theorem is similar. □

Corollary 4.2.19. *If the payoff function of the Blackwell game* $\mathsf{B}(X, Y, \phi)$ *is Borel measurable, then* $\mathsf{MVAL}^b_{\downarrow}(\mathcal{G}) = \mathsf{MVAL}^b_{\uparrow}(\mathcal{G})$.

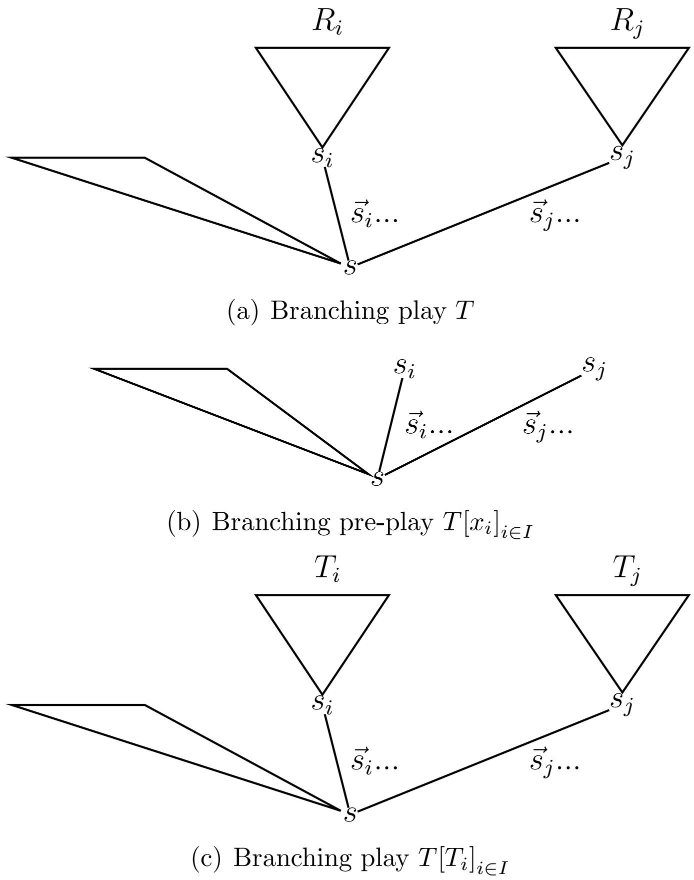
Proof. An immediate consequence of theorems 4.2.18 and 2.3.31. □

4.3 Subtree-monotone winning sets

We introduce in this section useful structures, called branching pre-plays and Markov branching pre-plays, which will be useful for analyzing properties of $2\frac{1}{2}$-player tree games in later sections, and will allow us to identify an interesting class of winning sets for $2\frac{1}{2}$-player tree games.

Definition 4.3.1 (Antichain of finite paths in $\mathcal{A}$). Given a $2\frac{1}{2}$-player tree game arena $\mathcal{A}$, an *antichain* $\mathbb{S}$ *of finite paths in* $\mathcal{A}$ is an antichain (see Definition 2.1.6) in the poset $(\mathcal{P}^{<\omega}_{\mathcal{A}}, \lhd)$, i.e., a possibly empty subset $\mathbb{S} = \{\vec{s}_j\}_{i \in J} \subseteq \mathcal{P}^{<\omega}$, such that for every $i \neq j \in J$, $\vec{s}_i \not\lhd \vec{s}_j$ and $\vec{s}_j \not\lhd \vec{s}_i$. Note that since $\mathcal{P}^{<\omega}$ is countable, so is the index set J of every antichain in $\mathcal{A}$. Thus we shall always assume that $J \subseteq \mathbb{N}$.

Definition 4.3.2 (Branching pre-play). Let $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$ be a $2\frac{1}{2}$-player tree game, with $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B), \pi \rangle$, and let us fix an antichain $\mathbb{S} = \{\vec{s}_j\}_{j \in J}$ of finite paths in $\mathcal{A}$. We denote with s_j the state $last(\vec{s}_j)$, for $j \in J$. Let $T \in \mathcal{BP}$ be a branching play in the arena $\mathcal{A}$, and let $I \subseteq J$ be the index set of all paths $\{\vec{s}_j\}_{j \in J} \cap T$. Let us denote with R_i, for $i \in I$, the set of paths defined as follows: $\{s_i\} \cup \{\vec{r} \mid \exists \vec{t} \in T. \vec{t} = \vec{s}_i.\vec{r}\}$. The set R_i is a branching play rooted at s_i, as it is the sub-branching play of T rooted at the path $\vec{s}_i$. The branching play T can be depicted as in Figure 4.2(a), where the triangle on the left represents the set of paths in T not having any path in $\mathbb{S}$ as prefix, the edge labeled with $\vec{s}_i$ represents the path $\vec{s}_i$ whose last state is s_i, and the triangle labeled with R_i represents the



(a) Branching play T

(b) Branching pre-play $T[x_i]_{i\in I}$

(c) Branching play $T[T_i]_{i\in I}$

Figure 4.2: Branching plays and branching pre-plays

sub-branching play R_i, for $i \in I$ [10]. We define the *branching pre-play* obtained by pruning T with the antichain $\mathbb{S}$, denoted by $T[x_i]_{i\in I}$, as the tree which can be depicted as in Figure 4.2(b), i.e the set of finite paths formally defined as follows:

$$T[x_i]_{i\in I} = T \setminus \Big(\bigcup_{i\in I}\{\vec{t}\in T \mid \vec{s}_i \lhd \vec{t} \ \wedge \vec{t} \neq \vec{s}_i\ \}\Big).$$

In other words $T[x_i]$ is the tree T where the subtrees R_i (except their roots s_i), for $i \in I$, have been removed.

In what follows, we shall consistently use the letters J and I to denote the entire index set of an antichain and the the set of indexes of those paths that lie in an already identified branching play, respectively.

Definition 4.3.3. Let $\mathcal{G} = \langle \mathcal{A}, \Phi\rangle$ be a $2\frac{1}{2}$-player tree game, and $\mathbb{S} = \{\vec{s}_j\}_{j\in J}$ an antichain of finite paths in $\mathcal{A}$. We denote with $\mathcal{BP}|_{\mathbb{S}}$ the set of all branching

[10] The picture is quite simplistic. For example there are, in general, (infinite) paths in T not belonging to any subtree R_i, for $i\in I$, that nonetheless branch away from the path $\vec{s}_i$, for $i\in I$, somewhere between the root s and the last state s_i of $\vec{s}_i$, whereas the picture depicts all such branches as branching away immediately at the root s.

pre-plays obtained by pruning some branching play $T \in \mathcal{BP}$ with the antichain $\mathbb{S}$. We denote with $\pi|_{\mathbb{S}}: \mathcal{BP} \to \mathcal{BP}|_{\mathbb{S}}$ the function mapping a branching play $T \in \mathcal{BP}$ to the corresponding branching pre-play $T[x_i]_{i \in I}$ obtained by pruning T with $\mathbb{S}$.

We now endow the set $\mathcal{BP}|_{\mathbb{S}}$ with a topology, similar to the one defined on the set $\mathcal{BP}$ of branching plays (see Definition 2.1.38).

Definition 4.3.4 (Topology on $\mathcal{BP}|_{\mathbb{S}}$)**.** Given a finite tree F in $\mathcal{A}$, we denote with O_F the set of all branching pre-plays $T[x_i]_{i \in I} \in \mathcal{BP}|_{\mathbb{S}}$ containing the finite set of paths F. The topology on $\mathcal{BP}|_{\mathbb{S}}$ is generated by the basis consisting of all sets O_F, for F a finite tree in $\mathcal{A}$. This is a 0-dimensional Polish space. Note that the map $\pi|_{\mathbb{S}}$ is continuous with respect to this topology.

The notation adopted for branching pre-plays is motivated by the use we make of them. We consider a branching pre-play $T[x_i]_{i \in I} \in \mathcal{BP}|_{\mathbb{S}}$ as a context on which we can plug in, at the holes $i \in I$, other branching plays. We now formalize this idea.

Definition 4.3.5. Let $\mathbb{S} = \{\vec{s}_j\}_{j \in J}$ be an antichain of finite paths in $\mathcal{A}$. We say that a branching play $T \in \mathcal{BP}$ in $\mathcal{A}$ is *compatible with* s_j, with $j \in J$, if and only if $root(T) = last(\vec{s}_j)$. The basic open set of all branching plays compatible with $\vec{s}_j$ is denoted with $\mathcal{BP}_j$. We denote with $\mathcal{BP}^{\mathbb{S}} \subseteq \mathcal{BP}^J$, the set of all J-indexed sequences of branching plays $\{T_j\}_{j \in J}$, such that $T_j \in \mathcal{BP}_j$, and we refer to this set as the set of $\mathbb{S}$-*compatible sequences of branching plays.* Recall that the index-set J is at most countable.

Definition 4.3.6 (Topology of $\mathcal{BP}^{\mathbb{S}}$)**.** Each set $\mathcal{BP}_j$, for $j \in J$ is endowed with the subspace topology from $\mathcal{BP}$. The set $\mathcal{BP}^{\mathbb{S}}$ is then endowed with the product topology from $\mathcal{BP}_j$, for $j \in J$. More concretely, given a K-indexed collection $\vec{F}_k = \{F_k\}_{k \in K}$ of finite trees in $\mathcal{A}$, with K a finite subset of J, we denote with $O_{\vec{F}_k}$ the set of all tuples $\{T_j\}_{j \in J} \in \mathcal{BP}^{\mathbb{S}}$ such that $F_k \subseteq T_k$, for every $k \in K$. This is a 0-dimensional Polish space.

We are now ready to describe how the holes of a branching pre-play can be filled by other branching plays.

Definition 4.3.7. Let $\mathbb{S} = \{\vec{s}_j\}_{j \in J}$ be an antichain of finite paths in $\mathcal{A}$, and let $T[x_i]_{i \in I} \in \mathcal{BP}|_{\mathbb{S}}$, for $I \subseteq J$, be a branching pre-play in $\mathcal{A}$, i.e., a branching pre-play obtained by pruning some $T \in \mathcal{BP}$ with $\mathbb{S}$. Let $\{T_j\}_{j \in J} \in \mathcal{BP}^{\mathbb{S}}$ be a compatible sequence of branching plays. We then denote with $T[T_i]_{i\in I}$, the branching play which can be depicted as in Figure 4.2(c), i.e the branching play formally defined as follows:

$$T[T_i]_{i\in I} = T[x_i]_{i\in I} \cup \big(\bigcup_{i\in I} merge(\vec{s}_i, T_i)\big)$$

where $merge(\vec{s}_i, T_i)$, specified as in Definition 2.1.33, is the set of finite paths obtained by concatenating $\vec{s}_i$ with every path $\vec{t} \in T_i$, by merging the last state of $\vec{s}_i$ with the first state of $\vec{t}$. We denote with $\mathit{fill}_{\mathbb{S}} : \big(\mathcal{BP}|_{\mathbb{S}} \times \mathcal{BP}^{\mathbb{S}}\big) \to \mathcal{BP}$ the operation defined as: $\mathit{fill}_{\mathbb{S}}\big(T[x_i]_{i\in I}, \{T_j\}_{j\in J}\big) = T[T_i]_{i\in I}$. The function $\mathit{fill}_{\mathbb{S}}$ is clearly continuous.

Note that given a branching pre-play $T[x_i]_{i\in I} \in \mathcal{BP}|_{\mathbb{S}}$, there are in general many compatible sequences $\{T_j\}_{j\in J} \in \mathcal{BP}^{\mathbb{S}}$ such that $\mathit{fill}_{\mathbb{S}}\big(\langle T[x_i]_{i\in I}, \{T_j\}_{j\in J}\rangle\big) = T$, for a given $T \in \mathcal{BP}$ (this is always the case when $I \subsetneq J$, and some of the branching plays in $\{T_j\}_{j\in J}$ cannot be filled in $T[x_i]_{i\in I}$). Observe that $\mathit{fill}_{\mathbb{S}}$ is surjective.

Moreover note that:

1. if $T = \mathit{fill}_{\mathbb{S}}(\langle T[x_i]_{i\in I}, \{T_j\}_{j\in J}\rangle)$, then $\pi|_{\mathbb{S}}(T) = T[x_i]_{i\in I}$, i.e., if a branching play T is obtained by filling a branching pre-play $T[x_i]_{i\in I}$ with some compatible branching plays, then the branching pre-play obtained by pruning T with $\mathbb{S}$ is indeed $T[x_i]_{i\in I}$.

2. Fixed a branching pre-play $T[x_i]_{i\in I}$, the branching play $\mathit{fill}_{\mathbb{S}}(\langle T[x_i]_{i\in I}, \{T_j\}_{j\in J}\rangle)$ obtained by filling $T[x_i]_{i\in I}$ with a sequence $\{T_j\}_{j\in J} \in \mathcal{BP}^{\mathbb{S}}$ is determined only by the fillable components (T_i for $i \in I$) and does not depend on the others (T_j for $j \in J \setminus I$).

3. The branching play $\mathit{fill}_{\mathbb{S}}(\langle T[x_i]_{i\in I}, \{T_j\}_{j\in J}\rangle)$ itself uniquely determines, not only the branching pre-play $T[x_i]_{i\in I}$ as observed earlier, but also the I-indexed collection of fillable branching plays T_i.

Given the above, we can unambiguously write $T =_{\mathbb{S}} T[T_i]_{i\in I}$ to denote the decomposition of T into its induced branching pre-play $T[x_i]_{i\in I}$ and its collection of compatible branching plays $\{T_j\}_{i\in J}$, since the branching plays T_j with $j \in J \setminus I$ play no role in the substitution.

We are now ready to introduce the central notion of this section.

Definition 4.3.8 (Subtree-monotone winning sets)**.** Let $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$ be a two player stochastic tree-game, in which the payoff Φ is a winning set, i.e., a (universally meaurable) function $\Phi : \mathcal{BP} \to \{0,1\}$. We say that $\mathcal{G}$, or more precisely the winning set Φ of $\mathcal{G}$, is *subtree-monotone* if, for every antichain of finite paths $\mathbb{S} = \{\vec{s}_j\}_{j\in J}$ in $\mathcal{A}$ and for every branching pre-play $T[x_i]_{i\in I} \in \mathcal{BP}|_{\mathbb{S}}$, the following property is satisfied:

$$\forall i \in I. \big(\Phi(T_i) \leq \Phi(R_i) \;\Rightarrow\; \Phi(T[T_i]_{i\in I}) \leq \Phi(T[R_i]_{i\in I}),$$

for every collection of $\mathbb{S}$-compatible branching plays $\{R_j\}_{i\in J}$ and $\{T_j\}_{i\in J}$ in $\mathcal{BP}^{\mathbb{S}}$.

In other words the winning set Φ of $\mathcal{G}$ satisfies the subtree-monotonicity property when, for every branching play $T =_{\mathbb{S}} T[T_i]_{i\in I}$, if one replaces the sub-branching play T_i with R_i, for $i \in I$, taking care of substituting winning branching plays with winning branching plays, then $T[R_i]_{i\in I}$ is in Φ whenever the original $T[T_i]_{i\in I}$ is.

We now discuss another way to formulate the notion of subtree-monotonicity of a winning set Φ (Lemma 4.3.12 below) which is going to be useful later on. The payoff function $\Phi : \mathcal{BP} \to \{0,1\}$ can be lifted to a function $\hat{\Phi}$ of type $(\mathcal{BP}|_{\mathbb{S}} \times \mathcal{BP}^{\mathbb{S}}) \to \{0,1\}$ using the $\mathit{fill}_{\mathbb{S}}$ map: $\hat{\Phi} = \Phi \circ \mathit{fill}_{\mathbb{S}}$. It follows from the continuity of $\mathit{fill}_{\mathbb{S}}$ and the universal measurability of Φ, that the lifted function $\hat{\Phi}$ is universally measurable. Given a branching pre-play $T[x_i]_{i\in I}$, it is useful to denote with $\hat{\Phi}_{T[\vec{x}_i]} : \mathcal{BP}^{\mathbb{S}} \to \{0,1\}$ the function $\hat{\Phi}_{T[\vec{x}_i]}(\{T_j\}_{j\in J}) = \hat{\Phi}(T[x_i]_{i\in I}, \{T_j\}_{j\in J})$, i.e., the map obtained from $\hat{\Phi}$ by fixing the first component to $T[x_i]_{i\in I}$.

Lemma 4.3.9. *For every branching pre-play $T[x_i]_{i\in I} \in \mathcal{BP}|_{\mathbb{S}}$, the function $\hat{\Phi}_{T[\vec{x}_i]}$ is universally measurable.*

Proof. This follows from the application of Lemma 2.1.76. □

Note that, by previous considerations about the filling operation $\mathit{fill}_{\mathbb{S}}$, the function $\hat{\Phi}_{T[\vec{x}_i]} : \mathcal{BP}^{\mathbb{S}} \to \{0,1\}$ ignores the k-th component of its input $\{T_j\}_{j\in J}$, for $k \in J \setminus I$.

Let us further define the function $\Phi^{\mathbb{S}} : \mathcal{BP}^{\mathbb{S}} \to \{0,1\}^J$ defined by pointwise application of Φ to the J-indexed sequences of compatible branching plays in $\mathcal{BP}^{\mathbb{S}}$, i.e., the map formally specified as follows: $\Phi^{\mathbb{S}}\big(\{T_j\}_{j\in J}\big) = \{\Phi(T_j)\}_{j\in J}$. The space $\{0,1\}^J$ is a pospace (see Definition 2.1.58), where the topology is the product topology (with $\{0,1\}$ endowed with the discrete topology), and the order on J-indexed sequences is is defined pointwise from the order $0 \sqsubseteq 1$.

Observation 4.3.10. Note that the map $\Phi^{\mathbb{S}}$ is, in general, not surjective. However, as we now discuss, $\Phi^{\mathbb{S}}$ factors through a surjective (universally measurable) map to a closed subset $\mathbb{B}^{\mathbb{S}}$ of $\{0,1\}^J$ and a (continuous) map $\gamma^{\mathbb{S}}:\mathbb{B}^{\mathbb{S}}\rightarrow\{0,1\}^J$. Let us define, for every $j\in J$, the set $b_j\subseteq\{0,1\}$ as $b_j=\{0\}$ if $\mathcal{BP}_j\cap\Phi=\emptyset$, $b_j=\{1\}$ if $\mathcal{BP}_j\subseteq\Phi$, and $b_j=\{0,1\}$ otherwise. We denote with $\mathbb{B}^{\mathbb{S}}$ the product space $\prod_j b_j$, where each b_j is given the discrete topology. Note that $\mathbb{B}^{\mathbb{S}}$ is indeed a closed sub-pospace of $\{0,1\}^J$. The function $\phi^{\mathbb{S}}$, of type $\mathbb{B}^{\mathbb{S}}\rightarrow\{0,1\}^J$, is defined as the identity and is trivially continuous, by definition of subspace topology (see Definition 2.1.20). It is easy to verify that $\Phi^{\mathbb{S}}$ ranges over the restricted codomain $\mathbb{B}^{\mathbb{S}}$ and therefore is surjective

Lemma 4.3.11. *The function $\Phi^{\mathbb{S}}$ is universally measurable. Moreover, when the codomain is narrowed to its range $\mathbb{B}^{\mathbb{S}}$, $\Phi^{\mathbb{S}}$ preserves the open sets, i.e., $\Phi^{\mathbb{S}}(U)$ is open in $\mathbb{B}^{\mathbb{S}}$ for every open set $U\in\mathcal{BP}^{\mathbb{S}}$.*

Proof. The universal measurability follows by application of Lemma 2.1.77 and the assumption that Φ is universally measurable. For the second point, since direct images preserve arbitrary unions, we just need to prove that $\Phi^{\mathbb{S}}(U)$ is open for every basic open set $U\subseteq\mathcal{BP}^{\mathbb{S}}$, i.e., every set of the form $U=\{U_j\}_{j\in J}$ with $U_k\subseteq\mathcal{BP}_k$ for some finite set $K\subseteq J$ and $U_j=\mathcal{BP}_j$ for all $j\in J\setminus K$. Since Φ is surjective, as observed earlier, the set $f(U)$ is of the form $\{\Phi(U_k),U_j\}_{k\in K,j\in J\setminus K}$. The desired result then follows from the fact $\Phi(U_k)$ is open for every $k\in K$, because every subset of every subset of b_k is open. □

We can now rephrase Definition 4.3.8 as follows:

Lemma 4.3.12. *Let $\mathcal{G}=\langle\mathcal{A},\Phi\rangle$ be a two player stochastic tree-game, in which $\Phi:\mathcal{BP}\rightarrow\{0,1\}$ is a winning set. Then Φ is subtree-monotone if and only if for every antichain of finite paths $\mathbb{S}=\{\vec{s}_j\}_{j\in J}$ in $\mathcal{A}$ and for every branching pre-play $T[x_i]_{i\in I}\in\mathcal{BP}|_{\mathbb{S}}$, the following property is satisfied:*

$$\hat{\Phi}_{T[\vec{x}_i]}=\phi_{T[\vec{x}_i]}\circ\Phi^{\mathbb{S}}$$

for some monotone map (relative to the pointwise ordering) $\phi_{T[\vec{x}_i]}:\mathbb{B}^{\mathbb{S}}\rightarrow\{0,1\}$.

Proof. We first prove that if $\hat{\Phi}_{T[\vec{x}_i]}=\phi_{T[\vec{x}_i]}\circ\Phi^{\mathbb{S}}$ then Φ is subtree monotone. Consider two $\mathbb{S}$-compatible sequences $\{T_j\}_{j\in J}$ and $\{R_j\}_{j\in J}$ of branching plays and suppose $\Phi(T_i)\leq\Phi(R_i)$, for every $i\in I$. We need to show thet $\Phi(T[T_i]_{i\in I})\leq\Phi(T[R_i]_{i\in I})$. From the assumptions it follows that the i-th component of the sequence $\Phi^{\mathbb{S}}(\{T_j\}_{j\in J})$ is less or equal than the i-th component of $\Phi^{\mathbb{S}}(\{T_j\}_{j\in J})$, for $i\in I$. By previous considerations we know that $\hat{\Phi}_{T[\vec{x}]}$ ignores the j-th component of its input, for $j\in J\setminus I$. It then follows from the assumption that $\hat{\Phi}_{T[\vec{x}_i]}=\phi_{T[\vec{x}_i]}\circ\Phi^{\mathbb{S}}$, that $\phi_{T[\vec{x}]}$ ignores the j-th component of its input, for $j\in J\setminus I$, as well. The desired result then follows by monotonicity of $\phi_{T[\vec{x}]}$.

Suppose now that Φ is subtree monotone. This means that for every branching pre-play $T[x_i]_{i\in I}$ and every $\mathbb{S}$-compatible sequences $\{T_j\}_{j\in J}$ and $\{R_j\}_{j\in J}$ of branching plays such that $\Phi(T_i)\leq\Phi(R_i)$, the inequality $\Phi(T[T_i]_{i\in I})\leq\Phi(T[R_i]_{i\in I})$ holds. Recall that $T[T_i]_{i\in I}$ denotes the branching play $\mathit{fill}_{\mathbb{S}}\big(T[x_i]_{i\in I},\{T_j\}_{j\in J}\big)$ which does not depend on the j-th component of $\{T_j\}_{j\in J}$, for $j\in J\setminus I$. From the definition of $\hat{\Phi}_{T[\vec{x}]}$, we have that $\Phi(T[T_i]_{i\in I})=\hat{\Phi}_{T[\vec{x}]}\big(\{T_j\}_{j\in\ J}\big)$, and $\hat{\Phi}_{T[\vec{x}]}$ ignores the j-th component of its input, for $j\in J\setminus I$. Therefore $\hat{\Phi}_{T[\vec{x}]}:\mathcal{BP}^{\mathbb{S}}\to\{0,1\}$ factors as $\mathcal{BP}^{\mathbb{S}}\xrightarrow{\Phi^{\mathbb{S}}}\mathbb{B}^{\mathbb{S}}\xrightarrow{f}\{0,1\}$ for some monotone f. In other words, the map $\Phi^{\mathbb{S}}$ restricts the information contained in the input $\{T_i,T_j\}_{i\in I,j\in J\setminus I}$ to the relevant one. The proof is concluded by taking $\phi_{T[\vec{x}_i]}=f$. □

Note that Lemma 4.3.12 does not assert any measurability property of $\phi_{T[\vec{x}_i]}$. The following property will be useful in the proof of the main result of this section (Theorem 4.3.17 below).

Lemma 4.3.13. *Given a probability measure $\mu\in\mathcal{M}_1(\mathcal{BP}^{\mathbb{S}})$, let $\phi_{T[\vec{x}_i]}:\mathbb{B}^{\mathbb{S}}\to\{0,1\}$ be specified as in Lemma 4.3.12. The function $\phi_{T[\vec{x}_i]}$ is ν-measurable, where $\nu=\mathcal{M}_1\big(\Phi^{\mathbb{S}}\big)(\mu)$.*

Proof. Since $\{0,1\}$ is endowed with the product topology, it is sufficient to show that the set $C=\phi^{-1}_{T[\vec{x}_i]}(\{1\})$ is ν-measurable. Note that $\Phi^{\mathbb{S}^{-1}}(C)$ is μ-measurable, because $\phi_{T[\vec{x}_i]}\circ\Phi^{\mathbb{S}}=\hat{\Phi}_{T[\vec{x}]}$ is universally measurable, by Lemma 4.3.9. The result then follows from the fact that $\Phi^{\mathbb{S}}:\mathcal{BP}^{\mathbb{S}}\to\mathbb{B}^{\mathbb{S}}$ is surjective, preserving the open sets (Lemma 4.3.11) and by application of Lemma A.5.2, taking $f=\Phi^{\mathbb{S}}$, $X=\Phi^{\mathbb{S}^{-1}}(C)$ and $Y=C$ in the statement of the lemma. □

Let us now turn our attention to Markov branching plays. The same kind of constructions, namely the notion of Markov branching pre-play $M[x_i]_{i\in I}$ (for some $I\subseteq J$), $\mathbb{S}$-compatible sequence of Markov branching plays $\{M_j\}_{j\in J}$, the topological spaces $\mathcal{MBP}|_{\mathbb{S}}$, $\mathcal{MBP}^{\mathbb{S}}$, and the continuous map $\mathit{Mfill}_{\mathbb{S}}:\mathcal{MBP}|_{\mathbb{S}}\times\mathcal{MBP}^{\mathbb{S}}\to\mathcal{MBP}$ can be defined as for their branching play counterparts, just by replacing in the definitions the word "branching play" with "Markov branching play".

Definition 4.3.14. We define the map $\mathbb{P}^{\mathbb{S}}\colon \mathcal{MBP}^{\mathbb{S}} \to \mathcal{M}_1(\mathcal{BP}^{\mathbb{S}})$ as follows:

$$\mathbb{P}^{\mathbb{S}}\big(\{M_j\}_{j\in J}\big) = \times_{j\in J}\big(\mathbb{P}(M_j)\big)$$

where $\times_{j\in J}\big(\mathbb{P}(M_j)\big)$ denotes the product measure having $\mathbb{P}(M_j)$ as j-th component for every $j\in J$.

It follows from Lemma 2.1.77 and from the fact that $\mathbb{P}$ is continuous (see 4.1.4), that $\mathbb{P}^{\mathbb{S}}$ is continuous as well.

Definition 4.3.15. We define $\mathbb{P}|_{\mathbb{S}}\colon \mathcal{MBP}|_{\mathbb{S}} \to \mathcal{M}_1(\mathcal{BP}|_{\mathbb{S}})$ as the probability measure uniquely determined by the assignment $\mathbb{P}|_{\mathbb{S}}\big(M[x_i]_{i\in I}\big)(O_F)$ on basic clopen sets $O_F \subseteq \mathcal{BP}|_{\mathbb{S}}$ defined as:

$$\begin{cases} \prod\{\pi_{M[\vec{x}_i]}(\vec{s},\vec{t}) \mid \vec{s},\vec{t}\in F \text{ and } \vec{t} \text{ is a child of } \vec{s} \text{ in } M[x_i]_{i\in I}]\} & \text{if } F\subseteq M[x_i]_{i\in I} \\ 0 & \text{otherwise} \end{cases}$$

where the function $\pi_{M[\vec{x}_i]}$ labeling the edges in $M[x_i]_{i\in I}$ with probabilities, is induced from the arena $\mathcal{A}$, as described in Definition 4.1.2.

Lemma 4.3.16. *The following equality holds:*

$$\mathcal{M}_1(\mathit{fill}_{\mathbb{S}})\Big(\mathbb{P}|_{\mathbb{S}}\big(M[x_i]_{i\in I}\big)\times\mathbb{P}^{S}\big(\{M_j\}_{j\in J}\}\big)\Big) = \mathbb{P}\Big(M\mathit{fill}_{\mathbb{S}}\big(\langle M[x_i]_{i\in I},\{M_j\}_{j\in J}\rangle\big)\Big).$$

Proof. In what follows we just write $M[M_i]_{i\in I}$, or just M, to denote the Markov branching play $M\mathit{fill}_{\mathbb{S}}\big(\langle M[x_i]_{i\in I},\{M_j\}_{j\in J}\rangle\big)$, which does not depend on the j-th component, for $j\in J\setminus I$, of $\{M_j\}_{j\in J}$. We shall keep the notation consistent by always writing $M[x_i]_{i\in I}$ to denote the Markov branching pre-play of $M[M_i]_{i\in I}$. We just need to prove that for each basic open set $O_F\subseteq\mathcal{BP}$ the following equality holds:

$$\mathcal{M}_1(\mathit{fill}_{\mathbb{S}})\Big(\mathbb{P}|_{\mathbb{S}}(M[x_i]_{i\in I})\times\mathbb{P}^{S}(\{M_j\}_{j\in J}\})\Big)(O_F) = \mathbb{P}\big(M[M_i]_{i\in I}\big)(O_F),$$

or equivalently

$$\Big(\mathbb{P}|_{\mathbb{S}}(M[x_i]_{i\in I})\times\mathbb{P}^{S}(\{M_j\}_{j\in J}\})\Big)\big(\mathit{fill}_{\mathbb{S}}^{-1}(O_F)\big) = \mathbb{P}\big(M[M_i]_{i\in I}\big)(O_F).$$

Recall that O_F is the set of branching plays containing the finite tree F in $\mathcal{A}$. Let us define, from the set (of finite paths in $\mathcal{A}$) F, the sets G and $\{G_j\}_{j\in J}$ as follows: $G=\{\vec{t}\in F \mid \forall\vec{s}_j\in\mathbb{S}.\ \vec{s}_j \not\lhd \vec{t}\}$, and $G_j=\{\vec{t}\in F \mid \vec{s}_j\lhd\vec{t}\}$, for every $j\in J$. Recall from the Definition 2.1.33 that the relation $\lhd$ is not strict, hence $\vec{s}_j$ can be contained in both G and in G_j. Note that, since F is finite, only finitely many of these partitioning sets are non-empty, and G is necessarily not empty. Moreover each of these non-empty sets is $\lhd$-down closed and has a minimal element: the minimal element of G is $root(F)$ and indeed G is a finite tree in $\mathcal{A}$, and the minimal element of G_j is $\vec{s}_j$. Let us define for each G_j the set H_j as follows:

$$H_j = \{last(\vec{s}_j)\} \cup \big\{\vec{r} \mid \vec{t} = \vec{s}_j.\vec{r} \text{ for some } \vec{t} \in G_j\big\}$$

Then H_j is a finite tree in $\mathcal{A}$ rooted at s_j. Therefore O_{H_j} is a basic open subset in $\mathcal{BP}_j$, the space of branching plays in $\mathcal{A}$ rooted at s_j, for every $j \in J$. In particular, if $G_j = \emptyset$ or $G_j = \{\vec{s}_j\}$ for some $\vec{s}_j \in \mathbb{S}$, then $O_{H_j} = \mathcal{BP}_j$, for every $j \in J$. Note that G can not be non-empty because it contains the path $root(F)$. When $root(F) = s_j$, for some $\vec{s}_j = \{s_j\}$ in $\mathbb{S}$, then $G = \{\vec{s}_j\}$ and $O_G = \mathcal{BP}_j$.

Let us now turn our attention to the Markov branching play $M = M[M_i]_{i \in I}$ obtained by filling $M[x_i]_{i\in I}$ with $\{M_i\}_{i\in I}$. It easily follows from earlier definitions of the sets G and H_j, for $j \in J$, that $fill_{\mathbb{S}}^{-1}(O_F) = \{\langle T[x_l]_{l\in L}, \{T_j\}_{j\in J} \mid T[x_l]_{l\in L} \in O_G \text{ and } \forall j \in J.\ T_j \in O_{H_j}\}$, where $T[x_l]_{l\in L}$ (and the associated index set $L \subseteq J$) ranges over Markov branching pre-plays. Therefore, by definitions of $\mathbb{P}|_{\mathbb{S}}$, $\mathbb{P}^{\mathbb{S}}$ and of product measure, we have that the probability assigned by $\mathcal{M}_1(fill_{\mathbb{S}})\Big(\mathbb{P}|_{\mathbb{S}}(M[x_i]_{i\in I}) \times \mathbb{P}^{S}(\{M_j\}_{j\in J})\Big)$ to O_F can be specified as follows:

$$\mathbb{P}|_{\mathbb{S}}\big(M[x_i]_{i\in I}\big)(O_G) \cdot \Big(\prod_{j\in J} \mathbb{P}\big(M_j\big)(O_{H_j})\Big).$$

From the fact that $\mathbb{P}(M)(O_F) = \mathbb{P}\big(M\big)(O_{G\cup(\bigcup_j G_j)})$, by Definition of 4.1.3 we have that $\mathbb{P}(M)(O_F) = \lambda_G \cdot \prod_{j\in J} \lambda_{G_j}$ where

$$\lambda_G = \begin{cases} \prod\{\pi_M(\vec{s},\vec{t}) \mid \vec{s},\vec{t} \in G \text{ and } \vec{t} \text{ is a child of } \vec{s} \text{ in M}\} & \text{if } G \subseteq M \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_{G_j} = \begin{cases} \prod\{\pi_M(\vec{s},\vec{t}) \mid \vec{s},\vec{t} \in G_j \text{ and } \vec{t} \text{ is a child of } \vec{s} \text{ in M}\} & \text{if } G_j \subseteq M \\ 0 & \text{otherwise} \end{cases}$$

It follows by definition of G that $\lambda_G = \mathbb{P}|_{\mathbb{S}}\big(M[x_i]_{i\in I}\big)(O_G)$. Note that if $G \not\subseteq M[x_i]_{i\in I}$ then $\mathbb{P}|_{\mathbb{S}}\big(M[x_i]_{i\in I}\big)(O_G) = 0$ and the desired result immediately follows. In particular, since $M[x_i]_{i\in I}$ does not contain any path $\vec{k} \in \mathbb{S}$ with $k \notin I$, we have that $\mathbb{P}|_{\mathbb{S}}\big(M[x_i]_{i\in I}\big)(O_G) = 0$ if G contains any path $\vec{s}_k \in \mathbb{S}$, for some $k \notin J \setminus I$. Let us then assume that G does not contain any path $\vec{s}_k \in \mathbb{S}$, with $k \in J \setminus L$. For any such $k \in J \setminus L$, by definition of G_k, we have that $G_k = \emptyset$ and as observed earlier this implies $O_{H_k} = \mathcal{BP}_k$. Therefore $\mathbb{P}\big(M_k\big)(O_{H_k}) = \lambda_{G_k}$, for every $k \in J \setminus L$, because the empty product has value 1. To conclude the proof we just need to show that

$\mathbb{P}\big(M_i\big)(O_{H_i}) = \lambda_{G_i}$, for every $i \in I$. This follows immediately from definition of H_i and Definition 4.1.3 of $\mathbb{P}(M_i)$ because M_i is exactly, by Definition 4.3.2, the sub-Markov branching play of M rooted at $\vec{s}_i \in \mathbb{S}$. □

We are now ready to state the main theorem of this section.

Theorem 4.3.17. *Let $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$ be a two player stochastic tree game in which Φ is a winning set. If Φ is subtree-monotone then, for every antichain of finite paths $\mathbb{S} = \{\vec{s}_j\}_{j\in J}$ in $\mathcal{A}$ and for every Markov branching pre-play $M[x_i]_{i\in I} \in \mathcal{BP}|_{\mathbb{S}}$, the following property is satisfied for every $\epsilon \geq 0$:*

$$\forall i \in I.\ \Big(E(M_i) \leq E(N_i) + \tfrac{\epsilon}{\#(i)}\Big) \ \Rightarrow\ E(M[M_i]_{i\in I}) \leq E(M[N_i]_{i\in I}) + \epsilon,$$

for every $\mathbb{S}$-compatible sequences $\{M_j\}_{j\in J}, \{N_j\}_{j\in J} \in \mathcal{MBP}^{\mathbb{S}}$ of Markov branching plays in $\mathcal{A}$, where $\# : \mathbb{N} \to \mathbb{N}$ is specified[11] *as in Definition 2.2.9. For $\epsilon = 0$ the property simplifies as follows:*

$$\forall i \in I.\ \big(E(M_i) \leq E(N_i)\big) \ \Rightarrow\ E(M[M_i]_{i\in I}) \leq E(M[N_i]_{i\in I}),$$

Proof. By definition of $E : \mathcal{MBP} \to [0,1]$ we have that

$$E(M[M_i]_{i\in I}) = \int_{\mathcal{BP}} \Phi \,\mathrm{d}\mathbb{P}(M[M_i]_{i\in I}) \quad \text{and} \quad E(M[N_i]_{i\in I}) = \int_{\mathcal{BP}} \Phi \,\mathrm{d}\mathbb{P}(M[N_i]_{i\in I}).$$

Recall that $M[M_i]_{i\in I} = \mathit{Mfill}\big(M[x_i]_{i\in I}, \{M_i, R_j\}_{i\in I, j\in J\setminus I}\big)$, i.e., it is the Markov branching play obtained by filling the holes in the Markov branching pre-play $M[x_i]_{i\in I}$ with the $\mathbb{S}$-compatible sequence of Markov branching plays $\{M_i, R_j\}_{i,j}$, where the choice of R_j, for $j \in J \setminus I$, is arbitrary because the function *Mfill* ignores the components which can not be filled into $M[x_i]_{i\in I}$. Similarly for $M[N_i]_{i\in I}$. From Lemma 4.3.16, we know that the equalities:

$$\mathbb{P}(M[M_i]_{i\in I}) = \mathcal{M}_1(\mathit{fill}_{\mathbb{S}})\Big(\mathbb{P}|_{\mathbb{S}}\big(M[x_i]_{i\in I}\big) \times \mathbb{P}^{S}\big(\{M_i, R_j\}_{i\in I, j\in J\setminus I}\big)\Big),$$

and,

$$\mathbb{P}(M[N_i]_{i\in I}) = \mathcal{M}_1(\mathit{fill}_{\mathbb{S}})\Big(\mathbb{P}|_{\mathbb{S}}\big(M[x_i]_{i\in I}\big) \times \mathbb{P}^{S}\big(\{N_i, R_j\}_{i\in I, j\in J\setminus I}\big)\Big),$$

hold. In what follows we just write $\{M_i, R_j\}_{i,j}$ and $\{N_i, R_j\}_{i,j}$ to refer to the $\mathbb{S}$-compatible sequences $\{M_i, R_j\}_{i\in I, j\in J\setminus I}$ and $\{N_i, R_j\}_{i\in I, j\in J\setminus I}$ respectively. By definition of $\hat{\Phi} = \Phi \circ \mathit{fill}$ we have that:

$$E(M[M_i]_{i\in I}) = \int_{\mathcal{BP}|_{\mathbb{S}} \times \mathcal{BP}^{\mathbb{S}}} \hat{\Phi}\ \ \mathrm{d}\Big(\mathbb{P}|_{\mathbb{S}}\big(M[x_i]_{i\in I}\big) \times \mathbb{P}^{\mathbb{S}}\big(\{M_i, R_j\}_{i,j}\big)\Big).$$

[11] Recall from Definition 4.3.1, that the index sets J, and therefore also $I \subseteq J$, are always assumed to be a subset of $\mathbb{N}$.

By application of Fubini's theorem and definition of $\hat{\Phi}_{T[\vec{y}_k]}$, for $T[y_k]_{k\in K}\in\mathcal{BP}|_{\mathbb{S}}$, we have that

$$E(M[M_i]_{i\in I})=\int_{T[\vec{y}_k]\in\mathcal{BP}|_{\mathbb{S}}}\Big(\int_{\mathcal{BP}^{\mathbb{S}}}\hat{\Phi}_{T[\vec{y}_k]}\,\mathrm{d}\mathbb{P}^{\mathbb{S}}\big(\{M_i,R_j\}_{i,j}\big)\Big)\,\mathrm{d}\mathbb{P}|_{\mathbb{S}}\big(M[x_i]_{i\in I}\big).$$

By similar considerations, we have that

$$E(M[N_i]_{i\in I})=\int_{T[\vec{y}_k]\in\mathcal{BP}|_{\mathbb{S}}}\Big(\int_{\mathcal{BP}^{\mathbb{S}}}\hat{\Phi}_{T[\vec{y}_k]}\,\mathrm{d}\mathbb{P}^{\mathbb{S}}\big(\{N_i,R_j\}_{i,j}\big)\Big)\,\mathrm{d}\mathbb{P}|_{\mathbb{S}}\big(M[x_i]_{i\in I}\big).$$

Observe that the outer integrals of both equalities coincide in form. Let us consider the inner integrals. Since Φ is by hypothesis subtree-monotone, we know from Lemma 4.3.12 that the function $\hat{\Phi}_{T[\vec{y}_k]}$ is of the form $\phi_{T[\vec{y}]}\circ\Phi^{\mathbb{S}}$. Thus we can rewrite both inner integrals as

$$\int_{\mathcal{BP}^{\mathbb{S}}}\phi_{T[\vec{y}]}\circ\Phi^{\mathbb{S}}\;\mathrm{d}\mathbb{P}^{\mathbb{S}}\big(\{M_l,R_j\}_{i,j}\big)\quad\text{and}\quad\int_{\mathcal{BP}^{\mathbb{S}}}\phi_{T[\vec{y}]}\circ\Phi^{\mathbb{S}}\;\mathrm{d}\mathbb{P}^{\mathbb{S}}\big(\{N_l,R_j\}_{i,j}\big)$$

or equivalently as:

$$\int_{\mathbb{B}^{\mathbb{S}}}\phi_{T[\vec{y}]}\;\mathrm{d}\Big(\mathcal{M}_1(\Phi^{\mathbb{S}})\big(\mathbb{P}^{\mathbb{S}}(\{M_i,R_j\}_{i,j})\big)\Big)\text{ and }\int_{\mathbb{B}^{\mathbb{S}}}\phi_{T[\vec{y}]}\;\mathrm{d}\Big(\mathcal{M}_1(\Phi^{\mathbb{S}})\big(\mathbb{P}^{\mathbb{S}}(\{N_i,R_j\}_{i,j})\big)\Big)$$

respectively, for some monotone $\phi_{T[\vec{y}_k]}:\{0,1\}^J\to\{0,1\}$. By Lemma 4.3.13, this is a valid step because the function $\phi_{[T[\vec{y}]}$ is measurable with respect to the two relevant probability measures.

Recall from definition of $\mathbb{P}^{\mathbb{S}}$ that $\mathbb{P}^{\mathbb{S}}\big(\{M_i,R_j\}_{i,j}\big)=\times\{\mathbb{P}(M_i),\mathbb{P}(R_j)\}_{i\in I,j\in J\setminus I}$. Moreover, by definition of $\Phi^{\mathbb{S}}$, we have that the probability measure

$$\mathcal{M}_1(\Phi^{\mathbb{S}})\Big(\times\{\mathbb{P}(M_i),\mathbb{P}(R_j)\}_{i,j}\Big)\in\mathcal{M}_1(\mathbb{B}^{\mathbb{S}})$$

assigns probability $\mathbb{P}(M_i)(\Phi)$, or equivalently $E(M_i)$, to the basic open subset of $\mathbb{B}^{\mathbb{S}}\subseteq\{0,1\}^J$ having i-th component equal to 1, for every $i\in I$, and probability $\mathbb{P}(R_j)(\Phi)$ to every the basic open set subset of $\{0,1\}^J$ having j-th component equal to 1, for $j\in J\setminus I$. Similar observations applies for the probability measure

$$\mathcal{M}_1(\Phi^{\mathbb{S}})\Big(\times\{\mathbb{P}(N_i),\mathbb{P}(R_j)\}_{i,j}\Big)\in\mathcal{M}_1(\mathbb{B}^{\mathbb{S}}).$$

In other words $\mathcal{M}_1(\Phi^{\mathbb{S}})\big(\mathbb{P}^{\mathbb{S}}(\{M_i,R_j\}_{i,j})\big)$ is the product probability measure over $\mathbb{B}^{\mathbb{S}}$ having as i-th (respectively j-th) component the probability measure over $b_j\subseteq\{0,1\}$ (see Observation 4.3.10) corresponding to the probability of hitting a winning branching play (i.e., in the set Φ) induced via $\mathbb{P}$ by the Markov branching play M_i (respectively R_j). Similarly for $\mathcal{M}_1(\Phi^{\mathbb{S}})\circ\mathbb{P}^{\mathbb{S}}\big(\{N_l,R_j\}\big)$. In particular the two probability measures coincide in the j-th component for $j\in J\setminus I$.

Let us now consider the first assertion of the theorem, which assumes that $E(M_i)\leq E(N_i)+\frac{\epsilon}{\#(i)}$, or equivalently

$$\int_{\mathcal{BP}} \Phi \, \mathrm{d}\mathbb{P}(M_i) \;\leq\; \Big(\int_{\mathcal{BP}} \Phi \, \mathrm{d}\mathbb{P}(N_i)\Big) + \frac{\epsilon}{\#(i)},$$

for every $i \in I$. By application of Lemma A.5.3), it follows that

$$\int_{\mathbb{B}^{\mathbb{S}}} \phi_{T[\vec{x}_k]} \;\, \mathrm{d}\big(\mathcal{M}_1(\Phi^{\mathbb{S}})(\mathbb{P}^{\mathbb{S}}(\{M_i, R_j\}))\big) \;\leq\; \int_{\mathbb{B}^{\mathbb{S}}} \phi_{T[\vec{x}_k]} \;\, \mathrm{d}\big(\mathcal{M}_1(\Phi^{\mathbb{S}})(\mathbb{P}^{\mathbb{S}}(\{N_i, R_j\}))\big) + \epsilon,$$

or equivalently by previous considerations,

$$\int_{\mathcal{BP}^{\mathbb{S}}} \hat{\Phi}_{T[\vec{y}_k]} \, \mathrm{d}\big(\mathbb{P}^{\mathbb{S}}(\{M_i, R_j\})\big) \;\leq\; \int_{\mathcal{BP}^{\mathbb{S}}} \hat{\Phi}_{T[\vec{y}_k]} \, \mathrm{d}\big(\mathbb{P}^{\mathbb{S}}(\{N_i, R_j\})\big) + \epsilon.$$

The desired result $E(M[M_i]_{i\in I}) \leq E(M[N_i]_{i\in I}) + \epsilon$ then trivially follows from the fact, highlighted before, that the two outer integrals coincide in form. □

The result of Theorem 4.3.16 can be understood as follows: given a Markov branching play $M[M_i]_{i\in I}$, if one replaces the sub-Markov branching play M_i with N_i, for $i \in I$, taking care to pick N_i with an expected probability of hitting Φ lower or equal than that of M_i plus $\frac{\epsilon}{\#(i)}$, then the resulting Markov branching play $M[M_i]_{i\in I}$ has an expected probability of hitting Φ lower or equal than that of $M[M_i]_{i\in I}$ plus ϵ. Thus Theorem 4.3.16 establishes a sort of continuity result of the expected value of Markov branching plays: small changes in the (expected value of) sub-Markov branching plays M_i produce small changes in the compound Markov branching play $M[M_i]_{i\in I}$.

As we shall see in the rest of this chapter, subtree monotone winning sets constitute an interesting class of objectives in $2\frac{1}{2}$-player tree games. They might be considered a natural generalization of the notion of prefix-independent winning sets (in ordinary $2\frac{1}{2}$-player games) to the context of $2\frac{1}{2}$-player tree games. In particular, standard $2\frac{1}{2}$-player games with prefix independent winning sets, when considered as $2\frac{1}{2}$-player tree games without branching states (see Definition 4.1.5), are subtree monotone.

Lemma 4.3.18. *Let $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$ a standard $2\frac{1}{2}$-player game with a prefix independent winning set Φ. Then Φ is subtree monotone.*

Proof. Since the game arena $\mathcal{A}$ does not contain branching plays, the set of branching plays in $\mathcal{A}$ coincides with the set of completed paths in $\mathcal{A}$ (see Lemma 4.1.6). Thus the branching pre-play, obtained by pruning a branching play (i.e., a completed path $\vec{r}$) with an antichain $\mathbb{S}$ of finite paths in $\mathcal{A}$, is either:

- a finite path $\vec{s}_i \in \mathbb{S}$, if $\vec{r} = \vec{s}_i.\vec{t}$: in this case the branching pre-play has just one hole waiting to be filled in by a branching play (i.e., a completed path) starting at $last(\vec{s}_i)$;
- or $\vec{r}$ itself: in this case the branching pre-play has no holes.

We need to prove that, given Φ prefix independent, Φ is subtree monotone. In the context of a game arena $\mathcal{A}$ without branching states, all we need to prove is that, for every finite path $\vec{s}$ and for every pair of completed paths $\vec{r}$ and $\vec{t}$, if $\Phi(\vec{r}) \leq \Phi(\vec{t})$ then $\Phi(\vec{s}.\vec{r}) \leq \Phi(\vec{r}.\vec{t})$. This implication trivially follows from Definition 2.3.59 of prefix independent set. □

Another important (informal) viewpoint, which actually originally inspired the definition of subtree-monotonicity, is the following. In a $2\frac{1}{2}$-player tree game $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$ with Φ subtree monotone, both players would not get any advantage if they were allowed to "observe" the execution of other concurrent sub-games which might have been generated during a play. Indeed suppose that $T[T_1, T_2]$ (i.e., a branching play with two sub-branching plays T_1 and T_2), was the final outcome of the game and Player 1, say, observed the execution of the sub-game(s) corresponding to the sub-branching play T_1 in order to improve their play in the sub-game(s) which ended up in T_2. Since Φ is subtree-monotone, Player 1 could have just played their best in the second sub-game(s), ignoring how the first sub-game(s) were progressing, inducing a winning sub-branching play T'_2 as least as good as T_2. The resulting $T[T_1, T'_2]$ is winning whenever $T[T_1, T_2]$ is winning. This is deliberately a very vague property about subtree-monotone winning sets. Formalizing it precisely might be an interesting direction of future work.

We now state two questions about subtree-monotone winning sets.

Question 4.3.19. Every $2\frac{1}{2}$-player tree game $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$ with Φ a Borel subtree monotone winning set is determined under deterministic strategies.

More generally,

Question 4.3.20. $\mathbf{\Gamma}^1_n$-determinacy (see Definition 2.3.17) implies that every $2\frac{1}{2}$-player tree game $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$ with Φ a $\mathbf{\Gamma}^1_n$-measurable subtree monotone winning set is determined[12] under deterministic strategies.

[12]Note that, in accordance with Definition 4.1.7, the winning set of a tree game must be universally measurable. It follows from Theorem 2.3.19 that, under $\mathbf{\Gamma}^1_n$-determinacy, every set in $\mathbf{\Gamma}^1_n$ is universally measurable.

These results, which we were not able to prove in this thesis work, would have important implications in the theory we will develop in later sections.

We conclude this section by presenting a useful generalization of Theorem 4.3.17.

Definition 4.3.21. Let $\mathcal{G}=\langle\mathcal{A},\Phi\rangle$ be a two player stochastic tree-game, in which the payoff Φ is a winning set, i.e., a (universally meaurable) function $\Phi\in\{0,1\}^{\mathcal{BP}}$. For every universally measurable set $\Psi\in\{0,1\}^{\mathcal{BP}}$ and antichain of finite paths $\mathbb{S}$ is $\mathcal{A}$, we say that Φ is $(\mathbb{S},\Psi)$-*subtree monotone* if for every branching pre-play $T[x_i]_{i\in I}\in\mathcal{BP}|_{\mathbb{S}}$, the following property is satisfied:

$$\forall i\in I.\big(\Psi(T_i)\leq\Psi(R_i)\big)\ \Rightarrow\ \Phi(T[T_i]_{i\in I})\leq\Phi(T[R_i]_{i\in I}),$$

for every collection of $\mathbb{S}$-compatible branching plays $\{R_j\}_{i\in J}$ and $\{T_j\}_{i\in J}$ in $\mathcal{BP}^{\mathbb{S}}$.

It then follows that the subtree monotonicity property of Definition 4.3.8 can be rephrased as follows: Φ is subtree monotone if and only if it is $(\mathbb{S},\Phi)$-subtree monotone for every antichain of finite paths $\mathbb{S}$ is $\mathcal{A}$.

Theorem 4.3.22. *Let $\mathcal{G}=\langle\mathcal{A},\Phi\rangle$ be a $2\frac{1}{2}$-player tree game in which Φ is a winning set. If Φ is $(\mathbb{S},\Psi)$-subtree monotone then, for every Markov branching pre-play $M[x_i]_{i\in I}\in\mathcal{BP}|_{\mathbb{S}}$, the following property is satisfied for every $\epsilon\geq 0$:*

$$\forall i\in I.\ \Big(\mathbb{P}_{M_i}(\Psi)\leq\mathbb{P}_{N_i}(\Psi)+\tfrac{\epsilon}{\#(i)}\Big)\ \Rightarrow\ E(M[M_i]_{i\in I})\leq E(M[N_i]_{i\in I})+\epsilon,$$

for every $\mathbb{S}$-compatible sequences $\{M_j\}_{j\in J},\{N_j\}_{j\in J}\in\mathcal{MBP}^{\mathbb{S}}$.

This theorem is proved by checking that the proof of Theorem 4.3.22 applies *mutatis mutandis* to the new situation. Our reason for not proving the result at this greater level of generality is to avoid inessential notational complications in a proof that is already technically quite involved.

4.4 De-randomization of $2\frac{1}{2}$-player tree games

Stochastic games, such as standard $2\frac{1}{2}$-player games or $2\frac{1}{2}$-player tree games, are played by Player 1 and Player 2 together with a third agent, named Nature, which models the randomized choices occurring in the game. As we shall see in this section, the concept of concurrent and independent execution of sub-games is expressive enough to model stochastic behavior without the need of introducing the third agent Nature: we shall prove that 2-player tree games can faithfully model $2\frac{1}{2}$-player tree games.

The idea is quite simple: in a $2\frac{1}{2}$-player tree game, although branching plays should be considered as the final outcomes of the game, Markov branching plays are the objects used to model the stochastic execution of the game induced by the strategies of Player 1 and Player 2 up-to the behavior of Nature. Branching plays and Markov branching plays are quite similar objects, the main difference being that in branching plays only branching game-states branch fully (i.e., they can have more than one child, see Definition 2.1.36), while in Markov branching plays also probabilistic-states branch fully. Given a $2\frac{1}{2}$-player tree game arena $\mathcal{A}=\langle (S,E),(S_1,S_2,S_N,B),\pi\rangle$, the two player tree game arena $\mathcal{A}^d= \langle (S,E),(S_1,S_2,\emptyset,B\cup S_N)\rangle$, obtained by defining the probabilistic states in $\mathcal{A}$ as branching states in $\mathcal{A}^d$, is such that its set of branching plays $\mathcal{BP}_{\mathcal{A}^d}$ coincides with the set of Markov branching plays $\mathcal{MBP}_{\mathcal{A}}$ in $\mathcal{A}$.

Definition 4.4.1. Let $\mathcal{A}=\langle (S,E),(S_1,S_2,S_N,B),\pi\rangle$ be a $2\frac{1}{2}$-player tree game arena. The 2-player tree game arena $\mathcal{A}^d=\langle (S,E),(S_1,S_2,\emptyset,B\cup S_N),\pi'\rangle$, where $\pi':\emptyset\rightarrow\mathcal{D}(S)$ is trivially the empty function, is called the *de-randomization* of $\mathcal{A}$.

Proposition 4.4.2. *Let $\mathcal{A}$ be a $2\frac{1}{2}$-player tree game arena and $\mathcal{A}^d$ its de-randomized 2-player tree game arena. The set of deterministic strategies Σ_1 and Σ_2, for Player 1 and Player 2 respectively, in the two tree game arenas coincide. Moreover the sets $\mathcal{MBP}_{\mathcal{A}}$ of Markov branching plays in $\mathcal{A}$ and $\mathcal{BP}_{\mathcal{A}^d}$ of branching plays in $\mathcal{A}^d$ coincide. In particular, given any deterministic strategy profile $\langle\sigma_1,\sigma_2\rangle\in\Sigma_1\times\Sigma_2$, the Markov branching play M_{σ_1,σ_2} in $\mathcal{A}$ corresponds to the branching play T_{σ_1,σ_2} in $\mathcal{A}^d$.*

Proof. Follows immediately from the definitions of deterministic strategies, branching plays and Markov branching plays: see Section 4.1. □

The result of Proposition 4.4.2 can be informally stated as follows: Player 1 and Player 2 have the same kind of possible behaviors in the tree games played on the arena $\mathcal{A}$ and its derandomization $\mathcal{A}^d$. In order to model a $2\frac{1}{2}$-player tree game $\langle\mathcal{A},\Phi\rangle$ as a 2-player tree game played on the de-randomized arena $\mathcal{A}^d$, we have to find an appropriate payoff function $\Phi^d:\mathcal{BP}_{\mathcal{A}^d}\rightarrow[0,1]$ such that $E(M_{\sigma_1,\sigma_2})= \Phi^d(T_{\sigma_1,\sigma_2})$ for every strategy profile $\langle\sigma_1,\sigma_2\rangle$, where $E(M_{\sigma_1,\sigma_2})$ is the expected payoff associated with the Markov branching play M_{σ_1,σ_2} in $\mathcal{A}$, as specified in Definition 4.1.9.

Lemma 4.4.3. *Let $\langle \mathcal{A}, \Phi \rangle$ be a $2\frac{1}{2}$-player tree game and $\mathcal{A}^d$ the de-randomization of $\mathcal{A}$. The payoff function $\Phi^d : \mathcal{BP}_{\mathcal{A}^d} \rightarrow [0,1]$ specified as $\Phi^d(T_{\sigma_1,\sigma_2}) = E(M_{\sigma_1,\sigma_2})$ is continuous (respectively Borel and universally measurable) if Φ is continuous (respectively Borel and universally measurable).*

Proof. By definition of the function $E : \mathcal{MBP}_{\mathcal{A}} \rightarrow [0,1]$, we have that $\Phi^d(T_{\sigma_1,\sigma_2}) = \int_{\mathcal{BP}_{\mathcal{A}}} \Phi \, \mathrm{d}\mathbb{P}(M_{\sigma_1,\sigma_2})$, where $\mathbb{P} : \mathcal{MBP}_{\mathcal{A}} \rightarrow \mathcal{M}_1(\mathcal{BP}_{\mathcal{A}})$ is the continuous function mapping Markov branching plays in $\mathcal{A}$ to the corresponding probability measures over branching plays in $\mathcal{A}$, as specified in Definition 4.1.3. Equivalently, $\Phi^d = \tilde{\Phi} \circ \mathbb{P}$, where $\tilde{\Phi} : \mathcal{M}_1(\mathcal{BP}_{\mathcal{A}}) \rightarrow [0,1]$ is defined as $\tilde{\Phi}(\mu) = \int_{\mathcal{BP}_{\mathcal{A}}} \Phi \, \mathrm{d}\mu$. Note that $\tilde{\Phi}$ is well defined, since Φ is always assumed to be universally measurable. The desired result then follows from Lemma 2.1.75 □

Thus, the function Φ^d, which assigns to the branching play T_{σ_1,σ_2} in $\mathcal{A}^d$ the same expected payoff assigned to the Markov branching play M_{σ_1,σ_2} in $\mathcal{A}$ by Φ, has, roughly speaking, the same complexity of Φ. Thus we have that the $2\frac{1}{2}$-player tree game $\langle \mathcal{A}, \Phi \rangle$ is equivalent to the 2-player tree game $\langle \mathcal{A}^d, \Phi^d \rangle$ in the following formal sense:

Lemma 4.4.4. *Let $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$ be a $2\frac{1}{2}$-player tree game and $\mathcal{G} = \langle \mathcal{A}^d, \Phi^d \rangle$ its de-randomization. Then the following equalities hold:*

$$\mathsf{VAL}_{\downarrow}(\mathcal{G}) = \mathsf{VAL}_{\downarrow}(\mathcal{G}^d) \quad \textit{and} \quad \mathsf{VAL}_{\uparrow}(\mathcal{G}) = \mathsf{VAL}_{\uparrow}(\mathcal{G}^d)$$

and

$$\mathsf{MVAL}_{\downarrow}(\mathcal{G}) = \mathsf{MVAL}_{\downarrow}(\mathcal{G}^d) \quad \textit{and} \quad \mathsf{MVAL}_{\uparrow}(\mathcal{G}) = \mathsf{MVAL}_{\uparrow}(\mathcal{G}^d).$$

Proof. The equalities follows immediately from Lemma 4.4.2, Lemma 4.4.3 and definitions 4.1.14 and 4.1.19. □

These results show that, from a foundational point of view, we can restrict our attention to the class of 2-player (non-stochastic) tree games, and motivate further research of the concept of concurrent and independent execution of the game, upon which tree games are designed. Stochasticity, anyway, remains an important and intuitive concept, and very often it is more convenient to work with $2\frac{1}{2}$-player tree games, rather that with 2-player tree games with complex payoff functions. This is the case in particular when the objective of the $2\frac{1}{2}$-player tree game is a winning set, rather than a general $[0,1]$-valued payoff function.

In the rest of this section we use the result about de-randomization of $2\frac{1}{2}$-player tree games, and the results of Section 4.3 about subtree monotone tree games, to discuss an open problem in the literature about standard $2\frac{1}{2}$-player games. Although we do not solve the problem, our approach is interesting in several ways which will be discussed at the end of this section.

We start by defining the concept of strong determinacy in standard $2\frac{1}{2}$-player games, as formulated in, e.g., [19] and [20].

Definition 4.4.5 (Strong Determinacy)**.** Let $\mathcal{G}=\langle\mathcal{A},\Phi\rangle$ be a standard $2\frac{1}{2}$-player game, where Φ is a Borel winning set, i.e., a Borel measurable function $\Phi:\mathcal{BP}_{\mathcal{A}}\rightarrow\{0,1\}$. We say that $\mathcal{G}$ is *strongly* $(\rhd\lambda)$*-determined*, if one of the following two mutually exclusive conditions holds:

1. $\exists\sigma_1.\forall\sigma_2.E(M_{\sigma_1,\sigma_2})\rhd\lambda$,
2. $\exists\sigma_2.\forall\sigma_1.E(M_{\sigma_1,\sigma_2})\not\rhd\lambda$

where $\lambda\in[0,1]$, $\rhd\in\{>,\geq\}$, and σ_1 and σ_2 range over the set of deterministic strategies for Player 1 and Player 2 respectively. We say that $\mathcal{G}$ is *strongly determined* if $\mathcal{G}$ is strongly $(\rhd\lambda)$-determined, for all $\lambda\in[0,1]$ and $\rhd\in\{>,\geq\}$. Lastly we say that $\mathcal{G}$ is *qualitatively strongly determined* if $\mathcal{G}$ is strongly (>0)-determined and $(=1)$-determined.

The notion of strong determinacy have not been studied extensively in the literature, despite its naturalness. To the knowledge of the author, the only works which address some questions about strong determinacy for standard $2\frac{1}{2}$-player games are [19] and [20], already cited above. In [7], qualitative determinacy is investigated but in the context of stochastic games with signals, a class of partial information games.

The question of which classes of standard $2\frac{1}{2}$-player games are strongly determined, or qualitative strongly determined is quite subtle. From the determinacy (under deterministic strategies) of standard $2\frac{1}{2}$-player games with Borel winning sets we can immediately deduce that if both players have optimal, i.e., 0-optimal, strategies, then the game is $\rhd\lambda$ determined, for all $\lambda\in[0,1]$ and $\rhd\in\{>,\geq\}$, i.e., the game is strongly determined. However, although each player have a ϵ-optima strategy, for every $\epsilon>0$, optimal strategies do not necessarily exists. Therefore, the only problematic case is $(\rhd\lambda)$-determinacy in standard $2\frac{1}{2}$-player games $\mathcal{G}$ (with Borel winning sets) whose value $\mathsf{VAL}(\mathcal{G})$ is exactly λ. In [19] the authors show that, in general, $2\frac{1}{2}$-player Borel games are not strongly determined. Strong qualitative determinacy, on the other hand, holds in the games considered in [19]: infinite state standard $2\frac{1}{2}$-player reachability games (i.e., with a open winning set), such that every game-state has only finitely many successors. However it is unknown if strong qualitative determinacy holds in more general settings. We now state the problem(s) precisely.

Open problem 4.4.6. Let $\mathcal{G}=\langle\mathcal{A},\Phi\rangle$ be a standard $2\frac{1}{2}$-player game, where the payoff function Φ is a winning set. The state of the following questions, each generalizing the previous, is currently unknown:

1. is $\mathcal{G}$ strongly qualitatively determined when Φ is a reachability winning set, even if the game-states of $\mathcal{G}$ have countably many successors?

2. Is $\mathcal{G}$ strongly qualitatively determined when Φ is a parity (see Definition 2.3.54) winning set?

3. Is $\mathcal{G}$ strongly qualitatively determined when Φ is a prefix independent (see Definition 2.3.59) Borel winning set?

4. Is $\mathcal{G}$ strongly qualitatively determined when Φ is a general Borel winning set?

We could not provide a full answer to the problem. However we are able to prove point 3 under the hypothesis that Question 4.3.19 has a positive answer.

Theorem 4.4.7. *Let $\mathcal{G}=\langle\mathcal{A},\Phi\rangle$ be a standard $2\frac{1}{2}$-player game, with Φ a prefix independent Borel winning set. Assume Question 4.3.19 has a positive answer. Then $\mathcal{G}$ is strongly qualitatively determined.*

Proof. We shall look at $\mathcal{G}$ as a $2\frac{1}{2}$-player tree game without branching states, and in particular we shall refer to paths and Markov plays in $\mathcal{G}$, as branching plays and Markov branching plays respectively. From Proposition 4.3.18 we know that Φ is subtree monotone.

The game $\mathcal{G}$ is strong $(=1)$-determined if either Player 1 has a deterministic strategy σ_1 such that $E(M_{\sigma_1,\sigma_2})=1$ for every strategy σ_2, or if Player 2 has a deterministic strategy σ_2 such that $E(M_{\sigma_1,\sigma_2})<1$ for every strategy σ_1. In other words, either Player 1 has a strategy which ensure that the outcome Markov branching play M is such that $E(M)=1$, or Player 2 has a strategy which ensures that the outcome M is such that $E(M) < 1$. Let us denote with $\mathcal{M}^{=1} \subseteq \mathcal{MBP}_{\mathcal{A}}$ the set of Markov branching plays M in $\mathcal{A}$ such that $E(M) = 1$. Let us consider the de-randomized game arena $\mathcal{A}^d$. The set $\mathcal{BP}_{\mathcal{A}^d}$ of branching plays in $\mathcal{A}^d$ coincides with the set $\mathcal{MBP}_{\mathcal{A}}$ of Markov branching plays in $\mathcal{A}$. Note that the set $\mathcal{M}^{=1}$ is Borel measurable, as the map $E : \mathcal{MBP} \rightarrow [0, 1]$ is Borel measurable whenever Φ is Borel measurable (see Lemma 4.4.3). We will now show that the 2-player tree game $\langle \mathcal{A}^d, \mathcal{M}^{=1} \rangle$ is subtree monotone. This will conclude the proof since every 2-player tree game, with a Borel subtree monotone winning set, is determined under deterministic strategies because, by assumption, Question4.3.19 has a positive answer. We need to show that for every antichain of paths $\mathbb{S}$ in $\mathcal{A}^d$ (and thus also in $\mathcal{A}$) and branching pre-play $M[x_i]_{i \in I}$, and for every $\mathbb{S}$-compatible sequences of branching plays (i.e., Markov branching plays in $\mathcal{A}$) $\{M_i\}_{j \in J}$ and $\{N_j\}_{j \in J}$ such that for every $i \in I$, $M_i \in \mathcal{M}^{=1}$ implies $N_i \in \mathcal{M}^{=1}$, the implication $M[M_i]_{i \in I} \in \mathcal{M}^{=1}$ implies $M[N_i]_{i \in I} \in \mathcal{M}^{=1}$ holds. Suppose $M[M_i]_{i \in I} \in \mathcal{M}^{=1}$. Then it is simple to verify that, for all $i \in I$, $M_i \in \mathcal{M}^{=1}$ observing that every sub-Markov (branching) play M_i is reachable from the root of $M[M_i]_{i \in I}$ by following the path $\vec{s_i} \in \mathbb{S}$, and that each edge in $\vec{s_i}$ has an associated positive probability[13], by Definition 2.3.46. It then follows from the hypothesis, that $N_i \in \mathcal{M}^{=1}$ for every $i \in I$. Since Φ is subtree monotone, it follows from Theorem 4.3.17 that $E(M[M_i]_{i \in I}) \leq E(M[N_i]_{i \in I})$. Thus $M[N_i]_{i \in I} \in \mathcal{M}^{=1}$ as desired.

By Lemma 2.3.60 we know that the complement set $\overline{\Phi}$ is prefix independent. Then the game $\langle \overline{\mathcal{A}}, \overline{\Phi} \rangle$, where the game arena $\overline{\mathcal{A}}$ is obtained from $\mathcal{A}$ just by swapping the role of the two players, is such that $\mathsf{VAL}(\mathcal{G}) = 1 - \mathsf{VAL}(\overline{\mathcal{G}})$. We refer to Lemma 5.1.16 in Section 5.1.1 for a (generalized) proof of this standard fact. Then $\overline{\mathcal{G}}$ is $(= 1)$-determined by previous considerations, and it is immediate to verify that this implies that $\mathcal{G}$ is strong > 0-determined as desired. □

As a consequence of Theorem 4.4.7, if Question 4.3.19 has a positive answer, all $2\frac{1}{2}$-player parity games are qualitatively strongly determined.

This result is quite interesting for at least two reasons: its generality, compared with previous results of [19] and [20], abundantly motivates further research towards a (dis)proof of Question 4.3.19; secondly, it is interesting to observe that the problem of qualitative strong determinacy has been reduced to a problem, determinacy of all 2-player (non-stochastic) tree games with subtree monotone Borel winning sets, which has apparently nothing to do with probability and

[13]Note that, by definition 2.3.40, $supp(s) = E(s)$ for every probabilistic state $s \in S_N$.

stochasticity. This is, arguably, further evidence for the potential importance of tree games in the field of theoretical Game Theory.

4.5 Summary of results

We conclude this chapter with a quick summary of its content.

In Section 4.1 we discussed a novel class of games named $2\frac{1}{2}$*-player tree games*. Tree games generalize standard $2\frac{1}{2}$-player tree games with the concurrent execution of independent sub-games generated at the so called *branching states*. The underlying idea is thus natural and straightforward and, in our opinion, interesting in its own right. In Section 4.1 a few important preliminary results are discussed, namely the fact that certain 2-player tree games are not determined under deterministic strategies, nor under mixed strategies.

Our study of tree games continued in Section 4.2, where we showed that the simple form of concurrency modeled by tree games is sufficient for encoding the class of partial information games of Blackwell games, introduced in Section 2.3.2. This is an important expressiveness result, further supporting the primitive notion of concurrent and independent execution of sub-games.

In Section 4.3 we developed the technical machinery associated with the notion of (Markov) branching pre-play, and used it to identify an important class of winning sets named *subtree monotone*. Beside the concepts introduced, the main result of Section 4.3 is Theorem 4.3.16, which can be read as a sort of continuity property of the expected values of Markov branching plays in terms of the expected values of their sub-Makrov branching plays. This result will have an important role in later chapters.

In Section 4.4 we proved another interesting results about tree games: 2-player tree games can faithfully model $2\frac{1}{2}$-player tree games. Unlike Gale–Stewart games or Blackwell games, whose stochastic counterparts are based on the introduction of a third player named Nature, 2-player tree games can mimic probabilistic behavior just with appropriate payoff functions: this is yet more evidence for the expressive power of the concept of concurrent and independent execution of tree games. The result is used to prove an interesting result: the (open) problem of strong qualitative determinacy in ordinary $2\frac{1}{2}$-player games with prefix independent winning Borel sets, can be reduced to the determinacy of the class of 2-player tree games with Borel winning sets which are subtree monotone (Question 4.3.19).

As already observed, this is interesting in many ways: the problem is reduced to a question about determinacy for games which have apparently nothing to do with probability, and the notion of subtree monotone winning sets seems to capture the relevant properties of Markov runs in games with prefix-indepenent winning sets.

Chapter 5

Two player stochastic meta-games

In sections 4.2 and 4.3 we identified important classes of $2\frac{1}{2}$-player games, namely the encodings of Blackwell games and $2\frac{1}{2}$-player tree games with subtree monotone winning sets. In this chapter we introduce another class of two player stochastic tree games, which we call *two player stochastic meta-games*, and study some of their properties. Two player stochastic meta-games will be used in later sections, to give game-semantics to the probabilistic modal μ-calculi introduced in Section 3.3, and therefore they constitute an important, well motivated, sub-class of $2\frac{1}{2}$-player tree games.

5.1 Formal definitions

Two player stochastic meta-games are $2\frac{1}{2}$-player tree games $\mathcal{G} = \langle \mathcal{A}, \Phi \rangle$ for which the winning set Φ, i.e., the set of branching plays $T \in \mathcal{BP}$ which are considered to be winning for Player 1, is defined indirectly as follows: $T \in \Phi$ if and only if a two player turn based game of infinite duration (see Section 2.3.4) $\mathcal{G}_T$, which is played on the tree structure T, admits a winning strategy for Player 1. Before entering into the details of how the game $\mathcal{G}_T$ is constructed, when Player 1 and Player 2 win in G_T *etcetera*, let us observe that the $2\frac{1}{2}$-player meta-game $\mathcal{G}$ can be thought as a game of transfinite duration, precisely of $\omega + \omega$ duration: in the first ω-moves Player 1, Player 2 and Nature play on $\mathcal{G}$ producing a unique branching play T as outcome of their play; after this stage, the game takes place in T itself interpreted as the arena for a new game $\mathcal{G}_T$, where Player 1 and Player 2 play another ω-sequence of moves. Player 1 wins in $\mathcal{G}$ if and only if Player 1 wins the second stage of the game, i.e., in the game $\mathcal{G}_T$. We often refer to the first stage of the game, played by Player 1, Player 2 and Nature on the arena $\mathcal{A}$, as the *outer-game* of $\mathcal{G}$ and we refer to the second stage of the game, played on T by Player 1 and Player 2, as the *inner-game* of $\mathcal{G}_T$ associated with the branching play T.

We now proceed with the formal definitions:

Definition 5.1.1. A *two player stochastic meta-game specification* is a pair $\langle \mathcal{A}, (Pl, \mathcal{W})\rangle$, where $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B)), \pi\rangle$ is a $2\frac{1}{2}$-player tree game arena and the pair $(Pl, \mathcal{W})$ consists of a function $Pl : B \to \{1, 2\}$ which assigns a *player identifier* to each branching state $b \in B$, and a set $\mathcal{W} \subseteq \mathcal{P}_{\mathcal{A}}$ of completed paths in $\mathcal{A}$, which we often refer to as the *meta-winning set* of the specification. We say that $\langle \mathcal{A}, (Pl, \mathcal{W})\rangle$ is a *Borel* meta-game specification if $\mathcal{W}$ is a Borel set.

It is the player assignment Pl which, together with the meta-winning set $\mathcal{W}$, allows us to define the game $\mathcal{G}_T$ and its associated winning criterion, for every branching play T in a $2\frac{1}{2}$-player tree game arena. This is formalized by the following definition.

Definition 5.1.2 (Inner game $\mathcal{G}_T$)**.** Fix a $2\frac{1}{2}$-player meta-game specification $\langle \mathcal{A}, (Pl, \mathcal{W})\rangle$. The game $\mathcal{G}_T$ is a 2-player game played on the tree T, starting from *root*(T), where Player 1 and Player 2 move at vertices $\vec{s} \in T$ with $Pl(s) = 1$ and $Pl(s) = 2$ respectively, for $s = last(\vec{s})$. By definition 4.1.1, all other vertices $\vec{s}$ (i.e., with $last(\vec{s}) \in S_1 \cup S_2 \cup S_N$) have at most one child $\vec{t}$ in T. At these nodes the game automatically progresses to $\vec{t}$. The result of a play of the two players in $\mathcal{G}_T$ is a sequence $\{\vec{s}_i\}_{i \in I \subseteq \mathbb{N}}$ of nodes in T, either finite and ending in a leaf of T, or infinite. Every such sequence induces a completed path $\vec{s} = \{last(\vec{s}_i)\}_{i \in I \subseteq \mathbb{N}}$ in $\mathcal{A}$. For simplicity we will often refer to $\vec{s}$ as an outcome of the game $\mathcal{G}_T$. Player 1 wins in the game $\mathcal{G}_T$ if and only if the outcome of the play is a completed path $\vec{s} \in \mathcal{W}$. Player 2 wins otherwise.

The notions of strategies available to Player 1 and Player 2 in the game $\mathcal{G}_T$ are defined in the standard way, i.e., as in Definition 2.3.43 of Section 2.3.4. We denote with Σ_1^T and Σ_2^T the sets of strategies for Player 1 and Player 2 in the inner game $\mathcal{G}_T$, respectively. The sets Σ_1^T and Σ_2^T are endowed with a 0-dimensional Polish topology as specified in Definition 2.3.44.

Adopting standard terminology (see Definition 2.3.3) we say that Player 1 has a winning strategy in $\mathcal{G}_T$ if and only if $\exists \sigma_1 \forall \sigma_2 . \vec{s}_{\sigma_1, \sigma_2} \in \mathcal{W}$. Similarly we say that

Player 2 has a winning strategy in $\mathcal{G}_T$ if and only if $\exists\sigma_2\forall\sigma_1.\vec{s}_{\sigma_1,\sigma_2}\notin\mathcal{W}$. The set of winning branching plays associated with a $2\frac{1}{2}$-player meta-game specification is defined as follows.

Definition 5.1.3. Given a two player stochastic meta-game $\langle\mathcal{A},(Pl,\mathcal{W})\rangle$ specification, we denote with $\Phi_{\mathcal{W}}\subseteq\mathcal{BP}$, or just with Φ if $\mathcal{W}$ is clear from the context, the set defined as:

$$\Phi_{\mathcal{W}} = \{T \mid \text{Player 1 has a winning strategy in } \mathcal{G}_T\}.$$

We are now ready to define the class of $2\frac{1}{2}$-player meta-games.

Definition 5.1.4. A $2\frac{1}{2}$-*player meta-game* specified by the pair $\langle\mathcal{A},(Pl,\mathcal{W})\rangle$ is formalized as the $2\frac{1}{2}$-player tree game $\mathcal{G}=\langle\mathcal{A},\Phi_{\mathcal{W}}\rangle$.

Note that not all $2\frac{1}{2}$-player meta-game specifications can be formalized as tree games. Indeed if the set $\Phi_{\mathcal{W}}$ is not universally measurable then, in accordance with Definition 4.1.7, $\Phi_{\mathcal{W}}$ is not a valid winning set. Thus, when working with $2\frac{1}{2}$-player meta-games $\langle\mathcal{A},\Phi_{\mathcal{W}}\rangle$ we shall always implicitly assume the universal measurability of $\Phi_{\mathcal{W}}$. We now discuss the precise strength of this assumption.

We know from Theorem 2.3.15, i.e., from the determinacy (under deterministic strategies) of all standard two player turn based games with Borel winning sets, that either Player 1 or Player 2 have a winning strategy in $\mathcal{G}_T$, for every branching play T of a $2\frac{1}{2}$-player meta-parity game $\langle\mathcal{A},\Phi_{\mathcal{W}}\rangle$ with $\mathcal{W}$ Borel.

Lemma 5.1.5. *Given a $2\frac{1}{2}$-player Borel meta-game specification $\langle\mathcal{A},(Pl,\mathcal{W})\rangle$, the following equality holds:*

$$\mathcal{BP}\setminus\Phi_{\mathcal{W}} = \{T \mid \text{Player 2 has a winning strategy in } \mathcal{G}_T\}.$$

Moreover $\mathcal{BP}\setminus\Phi_{\mathcal{W}}$ is the winning set induced by the specification $\langle\mathcal{A},(\overline{Pl},\overline{\mathcal{W}})\rangle$, where $\overline{Pl}$ is defined as: $\overline{Pl}(s)=1$ if and only if $Pl(s)=2$, for all branching states $b\in B$.

Proof. Note that the set of completed paths contained in T, for any branching play $T\in\mathcal{BP}_{\mathcal{A}}$, is a closed subset of $\mathcal{P}$. Thus the winning set of any inner-game $\mathcal{G}_T$ is Borel. The result then follows immediately from the determinacy (under deterministic strategies) of all 2-player games on graphs with Borel winning sets (Theorem 2.3.15). □

Of course the same property holds for $2\frac{1}{2}$-player meta-games $\langle \mathcal{A}, (Pl, \mathcal{W})\rangle$ with $\mathcal{W} \in \boldsymbol{\Gamma}^1_n$, for $\boldsymbol{\Gamma} \in \{\boldsymbol{\Delta}, \boldsymbol{\Sigma}, \boldsymbol{\Gamma}\}$, under the set-theoretic axiom of $\boldsymbol{\Gamma}^1_n$-determinacy (see Definition 2.3.1).

The next technical lemma provides an upper bound on the descriptive complexity of the the set $\Phi_{\mathcal{W}}$.

Lemma 5.1.6. *Given a two player stochastic meta-game specified by $\langle \mathcal{A}, (Pl, \mathcal{W})\rangle$ with $\mathcal{W} \in \boldsymbol{\Delta}^1_n$, the following assertion is valid in* ZFC*:*

$$\text{ZFC} + \boldsymbol{\Delta}^1_n\text{-}determinacy \vdash \Phi_{\mathcal{W}} \in \boldsymbol{\Delta}^1_{n+1}.$$

where $\boldsymbol{\Delta}^1_n$-determinacy, for $n \in \mathbb{N}^+$, is defined as in Definition 2.3.17.

Proof. Let us denote with $\mathcal{P}^{<\omega}_{B_1}$ the set of finite paths $\vec{s} \in \mathcal{P}^{<\omega}$ in $\mathcal{A}$ such that $last(\vec{s}) \in B$ and $Pl(last(\vec{s})) = 1$. Similarly $\mathcal{P}^{<\omega}_{B_2}$ denotes the set of finite paths $\vec{s} \in \mathcal{P}^{<\omega}$ in $\mathcal{A}$ such that $last(\vec{s}) \in B$ and $Pl(last(\vec{s})) = 2$. Let us consider the set Σ_1 of functions $\mathcal{P}^{<\omega}_{B_1} \rightarrow \mathcal{P}^{<\omega} \cup \{\bullet\}$. This set contains all the strategies available to Player 1 in every game $\mathcal{G}_T$, for $T \in \mathcal{BP}$, seen as functions $f \in \Sigma_1$ restricted to T. Similarly for the set Σ_2 of functions $\mathcal{P}^{<\omega}_{B_2} \rightarrow \mathcal{P}^{<\omega} \cup \{\bullet\}$. We endow Σ_1 with the Baire space-like topology, where for every pair (x, y), with $x \in \mathcal{P}^{<\omega}_{B_1}$ and $y \in \mathcal{P}^{<\omega} \cup \{\bullet\}$, the set $O_{x,y}$ of all functions $f \in \Sigma_1$ such that $f(x) = y$ is a basic open set. This is a 0-dimensional Polish space. Similarly for Σ_2.

Let us now consider the subset of $\mathcal{BP} \times \Sigma_1 \times \Sigma_2$, denoted by $\mathcal{T}$, consisting of all triples (T, σ_1, σ_2) such that the strategies σ_1 and σ_2 are valid strategies in $\mathcal{G}_T$, i.e., $\sigma_1 \in \Sigma^T_1$ and $\sigma_2 \in \Sigma^T_2$. It is easy to see that $\mathcal{T}$ is a closed subset of $\mathcal{BP} \times \Sigma_1 \times \Sigma_2$ (endowed with the product topology). Indeed, the set of triples which do not belong to $\mathcal{T}$ is open, because one can tell if one of the two strategies σ_1 and σ_2 is not valid in the inner game $\mathcal{G}_T$, i.e., it makes choices which are not in T, just by looking at finite information about T, σ_1 and σ_2. Hence $\mathcal{T}$ is a Polish space, as it is a closed subset of the Polish space $\mathcal{BP} \times \Sigma_1 \times \Sigma_2$.

Let us denote with $out : \mathcal{T} \rightarrow \mathcal{P}_{\mathcal{A}}$ the function which maps a triple (T, σ_1, σ_2) to the induced play in $\mathcal{G}_T$, i.e., to the completed path $\vec{s}_{\sigma_1,\sigma_2}$. The function *out* is continuous. In order to determine any finite amount of information about the induced completed path, one just needs to look at finite amount of information about the input triplet.

Let us now consider the set $A \subseteq \mathcal{T}$ defined as the set of triples (T, σ_1, σ_2) such that the game G_T is won by Player 2 when the two players follow the (valid for $\mathcal{G}_T$)

strategies σ_1 and σ_2 respectively, i.e., the set formally defined as $A = out^{-1}(\overline{\mathcal{W}})$. Since *out* is continuous and $\mathcal{W}$ is a $\mathbf{\Delta}^1_n$ set, it follows that $\overline{\mathcal{W}}$ is $\mathbf{\Delta}^1_n$ set and, by application of Theorem 2.1.55, we have that A is a $\mathbf{\Delta}^1_n$ set too.

Let us now define the set $B \subseteq \mathcal{BP} \times \Sigma_1$ as follows:

$$B = \{(T, \sigma_1) \mid \exists \sigma_2 \in \Sigma_2.(T, \sigma_1, \sigma_2) \in A\}.$$

In other words the set B is the set of all pairs (T, σ_1), such that Player 2 has a strategy σ_2 in the game $\mathcal{G}_T$ winning against σ_1, i.e., such that the strategy profile (σ_1, σ_2) induces in $\mathcal{G}_T$ a completed path in $\overline{\mathcal{W}}$. The set B is $\mathbf{\Sigma}^1_n$ set by Proposition 2.1.55. Observe that the set $\overline{B}$ is the set of all pairs (T, σ_1) such that Player 2 does not have a strategy σ_2 for the game $\mathcal{G}_T$ which is winning against the strategy σ_1 for Player 1, or equivalently by $\mathbf{\Delta}^1_n$-determinacy (see Theorem 2.3.17), σ_1 is a winning strategy for Player 1 in $\mathcal{G}_T$. By construction the set $\overline{B}$ is a $\mathbf{\Pi}^1_n$ set.

We can now define the set $\Phi_{\mathcal{W}} \subseteq \mathcal{BP}$ of all branching plays T where Player 1 has a winning strategy in $\mathcal{G}_T$ as follows:

$$\Phi_{\mathcal{W}} = \{T \mid \exists \sigma_1 \in \Sigma_1.(T, \sigma_1) \in \overline{B}\}.$$

It then follows that, by construction, $\Phi_{\mathcal{W}}$ is a $\mathbf{\Sigma}^1_{n+1}$ set.

The result then follows by observing that $\overline{\Phi_{\mathcal{W}}}$ is also a $\mathbf{\Sigma}^1_{n+1}$ set. This is because $\overline{\Phi_{\mathcal{W}}}$ is the winning set associated with the specification $\langle \mathcal{A}, (\overline{Pl}, \overline{\mathcal{W}}) \rangle$, and $\overline{\mathcal{W}} \in \mathbf{\Delta}^1_n$. □

Corollary 5.1.7. *Given any $2\frac{1}{2}$-player meta-game specified by $\langle \mathcal{A}, (Pl, \mathcal{W}) \rangle$, with $\mathcal{W}$ a Borel set, the set $\Phi_{\mathcal{W}}$ is a $\mathbf{\Delta}^1_2$ set.*

Proof. It follows from Theorem 2.3.15 and the fact $\mathbf{\Delta}^1_1$ is the class of Borel sets (Theorem 2.1.53). □

We shall see, in Chapter 6 (Theorem 6.4.3), that the result of the previous corollary is tight, i.e., there exists a $2\frac{1}{2}$-player meta-game specification $\langle \mathcal{A}, (Pl, \mathcal{W}) \rangle$, with $\mathcal{W}$ Borel, such that $\Phi_{\mathcal{W}}$ is neither a $\mathbf{\Sigma}^1_1$ nor a $\mathbf{\Pi}^1_1$ set.

In the rest of this chapter we restrict our attention to $2\frac{1}{2}$-player meta-game Borel specifications $\langle \mathcal{A}, (Pl, \mathcal{W}) \rangle$, i.e., by the previous corollary, to $2\frac{1}{2}$-player meta-games $\langle \mathcal{A}, \Phi_{\mathcal{W}} \rangle$ with $\Phi_{\mathcal{W}} \in \mathbf{\Delta}^1_2$. Hence we just refer to $2\frac{1}{2}$-player meta-games omitting the “Borel” adjective.

From Lemma 5.1.6 we cannot derive that the winning set $\Phi_{\mathcal{W}}$ of a $2\frac{1}{2}$-player meta-game is universally measurable. Indeed it is consistent with ZFC that not all $\mathbf{\Delta}^1_2$ sets are universally measurable (see Theorem 2.1.81).

Definition 5.1.8. Let mG-UM be the following assertion:

"The winning set $\Phi_{\mathcal{W}}$ of any $2\frac{1}{2}$-player meta-game is universally measurable"

We also define mG-UM(Γ) to be the following assertion:

"The winning set $\Phi_{\mathcal{W}}$ of any game $\mathcal{G} \in \Gamma$ is universally measurable"

where Γ is a collection of $2\frac{1}{2}$-player meta-games.

Clearly, given the result of Lemma 5.1.6, the set-theoretic assertion $\mathbf{\Delta}^1_2$-UM defined in the introduction (see Definition 2.1.80) implies mG-UM. However we do not know if the converse implication holds. We leave this as a question:

Question 5.1.9. Does mG-UM imply $\mathbf{\Delta}^1_2$-UM ?

Anyway, since $\mathbf{\Delta}^1_2$-UM is consistent with ZFC, as shown in Theorem 2.1.81, we know that mG-UM is consistent with ZFC as well. Note that if mG-UM(Γ) does not hold, there exist two player stochastic meta-game specifications in Γ which are not definable, in accordance with Definition 4.1.7, as $2\frac{1}{2}$-player tree games because their winning sets are not universally measurable. Therefore when working with some class Γ of $2\frac{1}{2}$-player meta-games, we will always implicitly assume mG-UM(Γ).

Before developing the theory of $2\frac{1}{2}$-player meta-games any further, it is useful to remark the following two points.

Remark 5.1.10. The notion of $2\frac{1}{2}$-player meta-game is not necessarily confined to $2\frac{1}{2}$-player tree games having payoff functions which are winning sets. Indeed one could give a meaningful definition of $2\frac{1}{2}$-player meta-games specified by $\langle \mathcal{A}, (Pl, \phi)\rangle$, with ϕ a Borel-measurable function $\phi : \mathcal{P} \rightarrow [0, 1]$, in terms of $2\frac{1}{2}$-player tree games as well. The inner games $\mathcal{G}_T$ associated with branching plays in $\mathcal{A}$ would then become standard 2-player turn based games with real-valued payoff function ϕ, as defined in Section 2.3.3. One can then generalize the winning set $\Phi_{\mathcal{W}}$ to the real-valued payoff function Φ_ϕ as $\Phi_\phi(T) = \mathsf{VAL}(\mathcal{G}_T)$, where $\mathsf{VAL}(\mathcal{G}_T)$ is well defined because, from Theorem 2.3.38, every 2-player turn based game with Borel-measurable payoff function is determined under deterministic strategies. It is also simple to adapt the proof of Lemma 5.1.6, to prove that Φ_ϕ is a $\mathbf{\Delta}^1_2$-measurable function. Indeed it follows from Theorem 2.3.37, that the set $\Phi^{-1}_\phi\big((\lambda, 1]\big)$, for every $\lambda \in [0, 1)$, can be characterized as the set

$\{T \mid \exists\sigma_1.\forall\sigma_2.\phi(\vec{s}_{\sigma_1,\sigma_2}) > \lambda \text{ in } \mathcal{G}_T\}$ and its complement $\Phi_\phi^{-1}([0,\lambda])$ as the (countable) intersection[1] of the sets $\{T \mid \exists\sigma_2.\forall\sigma_1.\phi(\vec{s}_{\sigma_1,\sigma_2}) < \lambda + \epsilon \text{ in } \mathcal{G}_T\}$, for every rational $\epsilon \in (0,1]$. One can then show that each of these sets is a $\mathbf{\Sigma}^1_2$ set by trivial modifications of the proof of Lemma 5.1.6. By recalling that the collection of $\mathbf{\Sigma}^1_2$ sets is closed under countable intersections, we have that $\Phi_\phi^{-1}([0,\lambda])$ is a $\mathbf{\Sigma}^1_2$ set, hence $\Phi_\phi^{-1}((\lambda,1])$ is also a $\mathbf{\Pi}^1_2$ and this concludes the proof.

Remark 5.1.11. Meta-games whose game arenas are not stochastic, i.e., such that $S_N = \emptyset$, can be reduced to ordinary 2-player games. Indeed consider a meta-game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathcal{W}}\rangle$ specified by $\langle \mathcal{A}, (Pl, \mathcal{W})\rangle$, with $\mathcal{A} = \langle (S,E), (S_1, S_2, \emptyset, B), \pi\rangle$ and π the trivial empty function. The game $\mathcal{G}$ is equivalent to the standard 2-player turn-based game $\mathcal{G}' = \langle \mathcal{A}', \mathcal{W}\rangle$ played on the arena $\mathcal{A}' = \langle (S,E), (S_1 \cup B_1, S_2 \cup B_2)\rangle$, where $B_1 = Pl^{-1}(\{1\})$ and $B_2 = Pl^{-2}(\{2\})$, in the sense that Player 1 has a winning strategy in $\mathcal{G}$ if and only if they have a winning strategy in $\mathcal{G}'$, and similarly for Player 2. Intuitively, the game $\mathcal{G}'$ is played as $\mathcal{G}$ but in just one, rather than two, stages. By Borel determinacy of standard 2-player turn-based games, the game $\mathcal{G}'$ is determined under deterministic strategies. It is easy to see that a winning strategy σ for Player 1, say, in the game $\mathcal{G}'$ provides the information necessary to construct a pair of strategies $\langle \sigma_1^{\mathrm{I}}, \sigma_1^{\mathrm{II}}\rangle$ for Player 1 in $\mathcal{G}$, where σ_1^{I} is a winning strategy for Player 1 in the outer game, and σ_1^{II} a winning strategy for Player 1 in any inner game $\mathcal{G}_T$ associated with a play T in $\mathcal{G}$ compatible with σ_1^{I}. In other words, without stochasticity, a player does not get any advantage when delaying their choices in the outergame.

It will be often convenient to describe a $2\frac{1}{2}$-player meta-game specified by $\langle \mathcal{A}, (Pl, \mathcal{W})\rangle$ and played on the arena $\mathcal{A} = \langle (S,E), (S_1, S_2, S_N, B), \pi\rangle$, by including the information provided by the player assignment Pl directly in the structure of the arena by means of a partition of the set of branching states B. More formally we define the notion of two player stochastic meta-game arena as follows:

Definition 5.1.12 (Two player stochastic meta-game arena)**.** A two player stochastic meta-game arena is a tuple $\mathcal{A} = \langle (S,E), (S_1, S_2, S_N, B_1, B_2), \pi\rangle$. The states in B_1 are called Player 1's branching states, and similarly the states in B_2 are called Player 2's branching states. Clearly any $2\frac{1}{2}$-player meta-game arena induces a unique $2\frac{1}{2}$-player tree game arena and player assignment Pl as follows:

[1]This is necessary as σ_2 might not have a 0-optimal strategy in $\mathcal{G}_T$, for some T with $\mathsf{VAL}(\mathcal{G}_T) = \lambda$, as discussed in Section 2.3.3.

$$\langle (S,E),(S_1,S_2,S_N,B_1,B_2),\pi\rangle \Longleftrightarrow \begin{cases} \langle (S,E),(S_1,S_2,S_N,B_1\cup B_2),\pi\rangle \\ Pl(b)=\begin{cases} 1 & \text{if } b\in B_1 \\ 2 & \text{if } b\in B_2 \end{cases} \end{cases}$$

From now on, we will always denote the two player stochastic meta-game specification $\langle \mathcal{A},(Pl,\mathcal{W})\rangle$, with $\langle \mathcal{B},\mathcal{W}\rangle$, where $\mathcal{B}$ is the two player stochastic meta-game arena corresponding to $\mathcal{A}$ and to the player assignment Pl, as described above.

We introduce the following terminology to classify $2\frac{1}{2}$-player meta-games.

Definition 5.1.13. Let $\mathcal{G}=\langle \mathcal{A},\Phi_{\mathcal{W}}\rangle$ be a $2\frac{1}{2}$-player meta-game with meta-game arena $\mathcal{A}=\langle (S,E),(S_1,S_2,S_N,B_1,B_2),\pi\rangle$. We say that $\mathcal{G}$, or more precisely the arena $\mathcal{A}$ of $\mathcal{G}$, is:

- *finitely branching in A*, for some $A\subseteq S$, if for all $s\in A$, the set $E(s)$ is finite,
- *finitely branching in the player nodes* if $\mathcal{G}$ is finitely branching in $S_1\cup S_2$,
- *finitely branching in the branching nodes* if $\mathcal{G}$ is finitely branching in $B_1\cup B_2$,
- *uniquely branching in A*, for some $A\subseteq S$, if for all $s\in A$, the set $E(s)\leq 1$,
- *finite* if S is a finite set.

Clearly if $\mathcal{A}$ is finite, it is also finitely branching in the branching nodes and finitely branching in the player nodes. Moreover if $\mathcal{A}$ is uniquely branching in both B_1 and B_2, then the arena $\mathcal{A}$ is a non-branching $2\frac{1}{2}$-player tree game arena in the sense of Definition 4.1.5.

Lemma 5.1.14. *Let $\mathcal{G}=\langle \mathcal{A},\Phi_{\mathcal{W}}\rangle$ be a $2\frac{1}{2}$-player meta-game. Then the following assertions hold:*

1. *if $\mathcal{A}$ is uniquely branching in B_1 then $\Phi_{\mathcal{W}}$ is a $\mathbf{\Pi}^1_1$ set.*
2. *if $\mathcal{A}$ is uniquely branching in B_2 then $\Phi_{\mathcal{W}}$ is a $\mathbf{\Sigma}^1_1$ set.*
3. *if $\mathcal{A}$ is a non-branching $2\frac{1}{2}$-player tree game arena, then $\Phi_{\mathcal{W}}$ is a Borel set.*

Hence ZFC $\vdash$ mG-UM$(\Gamma_{B_1\leq 1}\cup\Gamma_{B_2\leq 1})$ *holds, where we denoted by $\Gamma_{B_i\leq 1}$ the collection of $2\frac{1}{2}$-player meta-games uniquely branching in B_i, for $i\in\{1,2\}$.*

Proof. The proof, which follows the same lines of that of Lemma 5.1.6, is based on the fact that if $\mathcal{A}$ is uniquely branching in B_1 then for each branching play $T \in \mathcal{BP}$, Player 1 has a just one valid strategy in the inner game $\mathcal{G}_T$, i.e., Player 1 does not have an active role in $\mathcal{G}_T$. Similarly, if $\mathcal{A}$ is uniquely branching in B_2 then for each branching play $T \in \mathcal{BP}$, Player 2 has a just one valid strategy in the inner game $\mathcal{G}_T$. The third point follows immediately from the fact that $\mathbf{\Delta}^1_1 = \mathbf{\Pi}^1_1 \cap \mathbf{\Sigma}^1_1$ is precisely the collection of all Borel sets (Theorem 2.1.53). Lastly, by application of Theorem 2.1.72, mG-UM($\Gamma_{B_1 \leq 1} \cup \Gamma_{B_2 \leq 1}$) holds. □

As we shall see in Chapter 6 (Lemma 6.1.5) these upper bounds are strict.

We now introduce a useful construction on $2\frac{1}{2}$-player meta-games.

Definition 5.1.15. Let $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathcal{W}} \rangle$, be a two player stochastic meta-game with $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B_1, B_2), \pi \rangle$. We define the *negation of the game* $\mathcal{G}$, denoted by $\neg\mathcal{G}$, as the $2\frac{1}{2}$-player meta-game $\neg\mathcal{G} = \langle \neg\mathcal{A}, \Phi_{\overline{\mathcal{W}}} \rangle$, where the arena $\neg\mathcal{A}$ is defined as $\neg\mathcal{A} = \langle (S, E), S_2, S_1, S_N, B_2, B_1), \pi \rangle$ and $\Phi_{\overline{\mathcal{W}}}$ is the winning set induced by the meta-winning set of completed paths $\overline{\mathcal{W}}$. Thus,

1. if a state s is under the control of Player 1 in $\mathcal{A}$, then the same state is under the control of Player 2 in $\neg\mathcal{A}$,
2. if a state s is under the control of Player 2 in $\mathcal{A}$, then the same state is under the control of Player 1 in $\neg\mathcal{A}$,
3. if a state s is a branching state under the control of Player 1 in $\mathcal{A}$, then the same state is a branching state under the control of Player 2 in $\neg\mathcal{A}$,
4. if a state s is a branching state under the control of Player 2 in $\mathcal{A}$, then the same state is a branching state under the control of Player 1 in $\neg\mathcal{A}$,

and a completed path $\vec{s}$ is winning for Player 1 in any inner-game $\mathcal{G}_T$ of $\neg\mathcal{G}$ if and only if the same completed path is winning for Player 2 in the same inner-game $\mathcal{G}_T$ of $\mathcal{G}$. Observe how the probabilistic states, are identical in the two games $\mathcal{G}$ and $\neg\mathcal{G}$, as it is the function π which assigns them probability distributions over the set of game-states..

We will often refer to the game $\neg\mathcal{G}$ as the *dual* game of $\mathcal{G}$. This choice of terminology is justified by the following lemma.

Lemma 5.1.16. *Let $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathcal{W}} \rangle$ be a $2\frac{1}{2}$-player meta-game with arena $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B_1, B_2), \pi \rangle$ and let $\neg\mathcal{G} = \langle \neg\mathcal{A}, \Phi_{\overline{\mathcal{W}}} \rangle$ be its dual. Then the following assertions*

1. $\Phi_{\overline{\mathcal{W}}} = \overline{\Phi_{\mathcal{W}}}$, *and*

2. *for each* $s \in S$, $\mathsf{VAL}^s_{\downarrow}(\mathcal{G}) = 1 - \mathsf{VAL}^s_{\uparrow}(\neg\mathcal{G})$ *and* $\mathsf{VAL}^s_{\uparrow}(\mathcal{G}) = 1 - \mathsf{VAL}^s_{\downarrow}(\neg\mathcal{G})$

hold.

Proof. Both points are trivial. The main observation is that the graph structure of $\mathcal{A}$ and $\neg\mathcal{A}$ are identical. Therefore the set $\mathcal{P}$ of completed paths, the set $\mathcal{BP}$ of branching plays and the set $\mathcal{MBP}$ of Markov branching plays in the two games coincide. Moreover, since the function π is identical in the two games, the probability measure $\mathbb{P}(M)$ over branching plays induced by a Markov branching play is the same in both games. The result then follows by observing that the role of the two players, both in the outer and in the inner games, have been swapped and the meta-winning set dualized. □

5.2 Prefix independent $2\frac{1}{2}$-player meta-games

In this section we consider the class of $2\frac{1}{2}$-player meta-games $\mathcal{G} = \langle \mathcal{A}, \mathcal{W} \rangle$, which we shall call *prefix independent* $2\frac{1}{2}$*-player meta-games*, whose set of completed paths $\mathcal{W}$ satisfies the prefix independence property, specified as in Definition 2.3.59. As done in the second part of the previous section, we restrict our attention to Borel prefix-independent meta-winning sets $\mathcal{W}$.

Convention 5.2.1. When working with prefix independent $2\frac{1}{2}$-player meta games $\mathcal{G} = \langle \mathcal{A}, \mathcal{W} \rangle$ played on some arena $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B_1, B_2), \pi \rangle$, we will always respect the following convention: for every terminal node $s \in S$, if $\{s\} \in \mathcal{W}$ then $s \in S_2 \cup B_2$, and if $\{s\} \notin \mathcal{W}$ then $s \in S_1 \cup B_1$. Observe that this is not a significant restriction, since the ownership of terminal states have no impact whatsoever in the way the game $\mathcal{G}$ is played.

By Definition 2.3.59 every terminated completed path $\vec{s}$, i.e., every finite path whose last state $s = last(\vec{s})$ is terminal in $\mathcal{A}$, is in $\mathcal{W}$ if and only if $\{s\} \in \mathcal{W}$. Thus, by following Convention 5.2.1, we model the fact that the player who gets stuck in the inner-game loses. Note that, by Lemma 2.3.60, this convention is preserved by the negation operator ($\neg$) on $2\frac{1}{2}$-player meta-games: for every prefix-independent meta-game $\mathcal{G}$ satisfying Convention 5.2.1, the game $\neg\mathcal{G}$ is a prefix-independent meta-game and satisfying Convention 5.2.1.

Our first result about prefix-independent $2\frac{1}{2}$-player meta-games shows that their winning sets satisfy the subtree-monotone property of Definition 4.3.8.

Proposition 5.2.2. *Let $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathcal{W}} \rangle$ a $2\frac{1}{2}$-player meta-game whose set of completed paths $\mathcal{W}$ satisfies the prefix independence property. Then the set $\Phi_{\mathcal{W}}$ of winning branching plays satisfies the subtree-monotone property.*

Proof. Let us fix an arbitrary antichain $\mathbb{S} = \{\vec{s}_j\}_{j\in J}$ of finite paths (see Definition 4.3.1) in $\mathcal{A}$, a branching play $T =_{\mathbb{S}} T[T_i]_{i\in I} \in \mathcal{BP}$, and a collection $\{R_j\}_{j\in j}$ of branching plays compatible with $\mathbb{S}$ (see Definition 4.3.5) such that assumption of Definition 4.3.8 holds, i.e., for all $i \in I$, if $T_i \in \Phi_{\mathcal{W}}$ then $R_i \in \Phi_{\mathcal{W}}$. We need to show that if $T[T_i]_{i\in I} \in \Phi_{\mathcal{W}}$, then $T[R_i]_{i\in I} \in \Phi_{\mathcal{W}}$ as well.

If $T[T_i]_{i\in I} \in \Phi_{\mathcal{W}}$ then, by definition, Player 1 has a winning strategy σ_1 in the inner game $\mathcal{G}_{T[T_i]_{i\in I}}$. Similarly, Player 1 has a winning strategy σ_1^i in each inner games $\mathcal{G}_{R_i}$, for all $i \in I$ such that $T_i \in \Phi_{\mathcal{W}}$. We need to prove that Player 1 has a winning strategy σ_1^* in $\mathcal{G}_{T[R_i]_{i\in I}}$ as well. We prove this by using a strategy stealing argument, i.e., we construct a winning strategy σ_1^* for Player 1 in $\mathcal{G}_{T[R_i]_{i\in I}}$ using the strategies σ_1 and σ_1^i, for all $i \in I$ such that $T_i \in \Phi_{\mathcal{W}}$.

Denote with s_i, for $i \in I$, the state $last(\vec{s}_i)$. Note that, in the inner game $\mathcal{G}_{T[T_i]_{i\in I}}$, if Player 1 uses the strategy σ_1 then, if a state s_i is ever reached following the path $\vec{s}_i \in \mathbb{S}$, it must be the case that $T_i \in \Phi_{\mathcal{W}}$. Suppose otherwise. Then, since Player 2 has a winning strategy in T_i, Player 2 could force the outcome of the inner game to be a completed path with a losing (for Player 1) tail (i.e., a path $\vec{t} = \vec{s}_i.\vec{r}$ with $\vec{r} \notin \mathcal{W}$). By prefix independence of $\mathcal{W}$ this implies that $\vec{t} \notin \mathcal{W}$ and therefore Player 1, using the strategy σ_1 would be losing, which is a contradiction.

The strategy σ_1^* can be described as follows: at the beginning of the game $\mathcal{G}_{T[R_i]_{i\in I}}$ the strategy σ_1^* behaves as σ_1 in $\mathcal{G}_{T[T_i]_{i\in I}}$. This is possible since $\mathcal{G}_{T[T_i]_{i\in I}}$ and $\mathcal{G}_{T[R_i]_{i\in I}}$ are identical up to the states s_i reached following a path $\vec{s}_i \in \mathbb{S}$, or equivalently, they have the same branching pre-play $T[x_i]_{i\in I}$ induced by $\mathbb{S}$. If a state s_i is ever reached following a path $\vec{s}_i \in \mathbb{S}$, with $i \in I$, then T_i is necessarily in $\Phi_{\mathcal{W}}$ by our previous observation. The strategy σ_1^* then plays the rest of the game in $\mathcal{G}_{T[R_i]_{i\in I}}$ as the winning strategy σ_1^i would play in $\mathcal{G}_{R_i}$. Since σ_1^i is winning by hypothesis, the result of the play is a completed path of the form $\vec{t} = \vec{s}_i.\vec{r}$, with $\vec{r} \in \mathcal{W}$, and by the prefix independence of $\mathcal{W}$, $\vec{t} \in \mathcal{W}$ and thus Player 1 wins in $T[R_i]_{i \in I}$. If instead no state s_i is ever reached following some path $\vec{s_i} \in \mathbb{S}$, then the completed path resulting as outcome of the game $\mathcal{G}_{T[R_i]_{i \in I}}$ is winning, because the same completed path is a possible play in $\mathcal{G}_{T[T_i]_{i \in I}}$ under the winning, by assumption, strategy σ_1. □

Observation 5.2.3. Note that the Axiom $\mathbf{\Delta}^1_2$-determinacy (see Definition 2.3.1) implies that:

1. Every $\mathbf{\Delta}^1_2$ set is universally measurable (Theorem 2.3.19), hence mG-UM($\Gamma_{p.i.}$) holds, where $\Gamma_{p.i.}$ denotes the collection of all prefix-independent $2\frac{1}{2}$-player meta-games $\langle \mathcal{A}, \Phi_{\mathcal{W}} \rangle$ with $\mathcal{W}$ Borel.

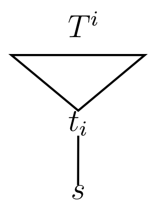
2. If Question 4.3.20 has a positive answer for $\mathbf{\Delta}^1_2$ winning sets, every $\mathcal{G} \in \Gamma_{p.i.}$ is determined under deterministic strategies.

Our next lemma exposes stronger properties of winning sets of prefix independent $2\frac{1}{2}$-player meta-games.

Lemma 5.2.4. *Let $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathcal{W}} \rangle$ be a prefix independent $2\frac{1}{2}$-player meta-game, with $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B_1, B_2), \pi \rangle$. Let $T \in \mathcal{BP}$ be a branching play in $\mathcal{A}$ and let $s \in S$ be the state labeling the root of T. Then the following assertions hold:*

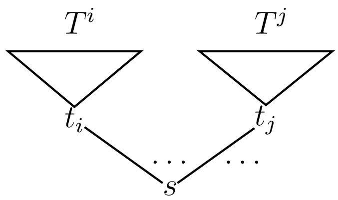
1. *if $E(s) = \emptyset$ then $T \in \Phi_{\mathcal{W}}$ if and only if $s \in \{S_2 \cup B_2\}$,*

2. *if $s \in S_1 \cup S_2 \cup S_N$, with $E(s) = \{t_i\}_{i \in I}$, then s has a unique child labeled with $t_i \in E(s)$ in T. In other words T can be depicted as follows:*



where T^i is the sub-branching play of T rooted at t_i. Then $T \in \Phi_{\mathcal{W}}$ if and only if $T^i \in \Phi_{\mathcal{W}}$.

3. *if $s \in B_1 \cup B_2$ with $E(s) = \{t_i\}_{i \in I}$, then the set of children of s in T is the set of nodes $\{t_i\}_{i \in I}$. In other words T can be depicted as follows:*



where T^i is the sub-branching play of T rooted at t_i. Then the following assertions hold:

- *if $s\in B_1$, then $T\in\Phi_{\mathcal{W}}$ if and only if $\exists i\in I.\ (T_i\in\Phi_{\mathcal{W}})$, and*
- *if $s\in B_2$, then $T\in\Phi_{\mathcal{W}}$ if and only $\forall i\in I.\ (T_i\in\Phi_{\mathcal{W}})$.*

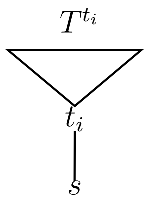
Proof. The first case follows by Convention 5.2.1. The second case is straightforward: in the inner-game $\mathcal{G}_T$, at the first move, the game evolves automatically to the state t_i which is the root of T^i. Since $\mathcal{W}$ is prefix independent, Player 1 have a winning strategy in $\mathcal{G}_T$ if and only if they have a winning strategy in $\mathcal{G}_{T^i}$.

Suppose T is of the third kind, i.e., $s\in B_1\cup B_2$ with $E(s)\neq\emptyset$. We just consider the case $s\in B_1$ because the case $s\in B_2$ is similar. In the inner game $\mathcal{G}_T$, the state s is under the control of Player 1 who can choose to move to some state t_i with $t_i\in E(s)$. The proof goes by a strategy-stealing argument. Suppose $T^i\in\Phi_{\mathcal{W}}$. Let σ_1^i be a winning strategy for Player 1 in the inner-game $\mathcal{G}T_i$. Define the strategy σ_1 in $\mathcal{G}_T$ by $\sigma(\{s\})_1=t_i$ and $\sigma_1(s.\vec{t})=\sigma_1^i(\vec{t})$. This is winning because $\mathcal{W}$ is prefix independent, hence $T\in\Phi_{\mathcal{W}}$. Similarly, assume $T\in\Phi_{\mathcal{W}}$ and let σ_1 be a winning strategy for Player 1 in $\mathcal{G}_T$. Let t_i be the choice made by σ on the first move, i.e., $t_i=\sigma(\{s\})$. Define σ_1^i by $\sigma_1^i(t_i.\vec{t})=\sigma(s.t_i.\vec{t})$. Again, this is winning because $\mathcal{W}$ is prefix independent, hence $T^i\in\Phi_{\mathcal{W}}$. ☐

The previous result expresses a fixed point property of winning sets of prefix independent $2\frac{1}{2}$-player meta-games which we now formalize.

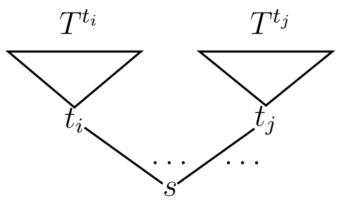
Definition 5.2.5. Given a prefix independent $2\frac{1}{2}$-player meta-game $\mathcal{G}=\langle\mathcal{A},\Phi_{\mathcal{W}}\rangle$ we define the function $\mathbb{W}_{\mathcal{G}}:2^{\mathcal{BP}}\to 2^{\mathcal{BP}}$ as follows: a branching play $T\in\mathcal{BP}$ is in $\mathbb{W}_{\mathcal{G}}(X)$ when the following (mutually disjoint) conditions hold:

1. If $root(T)=s$, with $E(s)=\emptyset$, then $T\in\mathbb{W}_{\mathcal{G}}(X)$ if and only if $s\in S_2\cup B_2$.
2. If $root(T)=s$, with $s\in S_1\cup S_2\cup S_N$ and $E(s)\neq\emptyset$, then T can be depicted as follows:



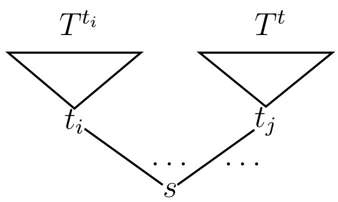
where $E(s)=\{t_i\}_{i\in I}$, and T^{t_i} is a branching play rooted at t_i for some $i\in I$. Then $T\in\mathbb{W}_{\mathcal{G}}(X)$ if and only if $T^{t_i}\in X$.

3. If $root(T)=s$, with $s\in B_1$ and $E(s)\neq\emptyset$, then T can be depicted as follows:



where $E(s)=\{t_i\}_{i\in I}$, and T^{t_i} is a branching play rooted at t_i. Then $T\in\mathbb{W}_{\mathcal{G}}(X)$ if and only if there is some $i\in I$ such that $T^{t_i}\in X$.

4. If $root(T)=s$, with $s\in B_2$ and $E(s)\neq\emptyset$, then T can be depicted as follows:



where $E(s)=\{t_i\}_{i\in I}$, and T^{t_i} is a branching play rooted at t_i, for every $i\in I$. Then $T\in\mathbb{W}_{\mathcal{G}}(X)$ if and only if for all $i\in I$, it holds that $T^{t_i}\in X$.

The function $\mathbb{W}_{\mathcal{G}}$ is clearly monotone with respect to the pointwise (inclusion) order.

Lemma 5.2.6. *Given a prefix independent $2\frac{1}{2}$-player meta-game $\mathcal{G}=\langle\mathcal{A},\Phi_{\mathcal{W}}\rangle$, let $\mathbb{W}_{\mathcal{G}}$ be defined as in Definition 5.2.5. Then $\Phi_{\mathcal{W}}$ is a fixed point of $\mathbb{W}_{\mathcal{G}}$.*

Proof. It follows immediately from Lemma 5.2.4. □

Lemma 5.2.7. *If $X\subseteq\Phi$ then also $\mathbb{W}(X)\subseteq\Phi$.*

Proof. Follows immediately by application of Lemma 5.2.4. □

The following technical result will be useful in Section 6.1.

Lemma 5.2.8. *For every prefix independent $2\frac{1}{2}$-player meta-game $\mathcal{G}=\langle\mathcal{A},\Phi_{\mathcal{W}}\rangle$, if $X\subseteq\mathcal{BP}_{\mathcal{A}}$ is a Borel (universally measurable) set, then $\mathbb{W}_{\mathcal{G}}(X)$ is a Borel (universally measurable) set. Moreover, if $\mathcal{A}$ is finitely branching in B_2 (see Definition 5.1.13) and $X\subseteq\mathcal{BP}_{\mathcal{A}}$ is an open set then $\mathbb{W}(X)$ is an open set.*

Proof. The set $\mathbb{W}(X)$ is the set of branching plays which satisfy the conditions 1-4 of Definition 5.2.5. Let us consider, one by one, the set of branching plays C_n included in $\mathbb{W}(X)$ by the case $n\in\{1,\dots,4\}$ of Definition 5.2.5:

C_1: This is the set of branching plays rooted at some terminal state $s\in S$. Thus, C_1 is the union of the basic open sets $O_{\{s\}}$, for s a terminal state. It then follows that C_1 is open.

C_2: This is the set of branching plays $T^{s.t_i}$ rooted at some non-terminal state $s\in S_1\cup S_2\cup S_N$, such that the root s has as unique child in T the state t_i, for some $t_i\in E(s)$, and the sub-branching play T^{t_i} of $T^{s.t_i}$ rooted at t_i is in X. Let us denote with $\mathcal{BP}_{s.t_i}$ and $\mathcal{BP}_{t_i}$ the set of branching plays rooted at s and having as unique child of s the state t_i, and the set of branching plays rooted at t_i respectively, where $s\in S_1\cup S_2\cup S_N$ and $t_i\in E(s)$. Let us define the function $m_{(s,t_i)}:\mathcal{BP}_{s.t_i}\rightarrow\mathcal{BP}_{t_i}$ defined as $m_{(s,t_i)}(T^{s.t_i})=T^{t_i}$, where T^{t_i} is the sub-branching play of $T^{s.t_i}$ rooted at t_i. It is clear that this function is a homeomorphism between $\mathcal{BP}_{s.t_i}$ and $\mathcal{BP}_{t_i}$. Observe that a branching play in $T\in\mathcal{BP}_{s.t_i}$ is in C_2 if and only if $T\in m^{-1}_{(s,t_i)}(X\cap\mathcal{BP}_{t_i})$. Note that since m is continuous, and $\mathcal{BP}_{t_i}$ open, $T\in m^{-1}_{(s,t_i)}(X\cap\mathcal{BP}_{t_i})$ is Borel (universally measurable, open) if X is Borel ((universally measurable, open). The proof is concluded by observing that C_2 is the union of the sets $T\in m^{-1}_{(s,t_i)}(X\cap\mathcal{BP}^{t_i})$, for every pair (s,t_i) of a state $s\in S_1\cup S_2\cup S_N$ with $E(s)\neq\emptyset$ and $t_i\in E(s)$, and the fact that the collection of such pairs is countable.

C_3: This is the set of branching plays T^s rooted at some non-terminal state $s\in B_1$ such that there exists an $i\in I$, where $E(s)=\{t_i\}_{i\in I}$, and the sub-branching play T^{t_i} of T^s rooted at the child t_i is in X. Let us denote with $\mathcal{BP}_s$ and $\mathcal{BP}_{t_i}$ the set of branching plays rooted at s, and the set of branching plays rooted at t_i respectively, where $s\in B_1$ and $t_i\in E(s)$. Let us define the function $m_s:\mathcal{BP}_s\rightarrow\prod_{i\in I}\mathcal{BP}_{t_i}$, defined as $m_s(T^s)=\{T^{t_i}\}_{i\in I}$, following the (graphical) notation adopted in Definition 5.2.5. Again it is straightforward to check that m_s is a homomorphism between $\mathcal{BP}_s$ and $\prod_{i\in I}\mathcal{BP}_{t_i}$ endowed with the product topology. Observe that a branching play in $T\in\mathcal{BP}_s$ is in C_3 if and only if $T\in m_s^{-1}\big(\bigvee_{i\in I}(X\cap\mathcal{BP}_{t_i})\big)$, where $\bigvee_{i\in I}(X\cap\mathcal{BP}_{t_i})$ denotes the set $\bigcup_{i\in I}\big\{\{T^{t_i}\}_{i\in I}\mid T^{t_i}\in X\cap\mathcal{BP}_{t_i}\big\}$. Thus, since the index-set I is countable and $\mathcal{BP}_{t_i}$ is open for every t_i, it follows that $\bigvee_{i\in I}(X\cap\mathcal{BP}_{t_i})\subseteq\times_{i\in I}\mathcal{BP}_{t_i}$ is Borel (universally measurable, open) if X is (universally measurable, open). From this observation and the fact that m_s is continuous we have that $m_s^{-1}\big(\bigvee_{i\in I}(X\cap\mathcal{BP}_{t_i})\big)$ is Borel (universally measurable, open) if X is Borel. The proof is concluded by observing that C_3 is the countable union of the sets $m_s^{-1}\big(\bigvee_{i\in I}(X\cap\mathcal{BP}_{t_i})\big)$, for every state $s\in B_1$ with $E(s)\neq\emptyset$.

C_4:: This is the set of branching plays T^s rooted at some non-terminal state $s\in B_2$, with $E(s)=\{t_i\}_{i\in I}$, such that for all $i\in I$ the sub-branching play T^{t_i} of T^s rooted at t_i is in X. Let us define the homeomorphism $m_s:\mathcal{BP}_s\rightarrow\prod_{i\in I}\mathcal{BP}_{t_i}$ as in the previous point. In this case we have that branching play in $T\in\mathcal{BP}_s$ is in C_4 if and only if $T\in m_s^{-1}\big(\prod_{i\in I}(X\cap\mathcal{BP}_{t_i})\big)$. By considerations analogous the one of the previous case we have that $m_s^{-1}\big(\prod_{i\in I}(X\cap\mathcal{BP}_{t_i})\big)$ is Borel if X is Borel. By application of Theorem 2.1.73 and Theorem 2.1.72(3) it also follows that $m_s^{-1}\big(\prod_{i\in I}(X\cap\mathcal{BP}_{t_i})\big)$ is universally measurable if X is universally measurable. However observe that $m_s^{-1}\big(\prod_{i\in I}(X\cap\mathcal{BP}_{t_i})\big)$ is open, when X is open, if and only if the I-indexed product is finite. This is the case precisely when $\mathcal{A}$ is finitely branching in B_2. The proof is completed, as in the previous case, observing that C_4 is the union of the sets $T\in m_s^{-1}\big(\prod_{i\in I}(X\cap\mathcal{BP}_{t_i})\big)$, for every state $s\in B_1$ with $E(s)\neq\emptyset$, and the fact that this collection is countable.

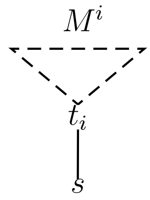
□

We now turn our attention to Markov branching plays in prefix independent $2\frac{1}{2}$-player meta-games. As a consequence of Proposition 5.2.2, we have that every Markov branching play M in a prefix independent $2\frac{1}{2}$-player meta-game satisfies the properties stated in Theorem 4.3.17. We now prove a useful lemma capturing further important properties of the expected values of Markov branching plays in prefix independent $2\frac{1}{2}$-player meta-games.

Lemma 5.2.9. *Let $\mathcal{G}=\langle\mathcal{A},\Phi_{\mathcal{W}}\rangle$ be a prefix independent $2\frac{1}{2}$-player meta-game, with $\mathcal{A}=\langle(S,E),(S_1,S_2,S_N,B_1,B_2),\pi\rangle$. Let $M\in\mathcal{MBP}$ be a Markov branching* *play in $\mathcal{A}$ and let $s \in S$ be the state labeling the root of M. Then the following assertions hold:*

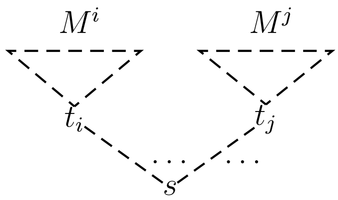
1. *if $E(s) = \emptyset$ then $\mathbb{P}_M(\Phi_{\mathcal{W}}) = 1$ if $s \in \{S_2 \cup B_2\}$ and $\mathbb{P}_M(\Phi_{\mathcal{W}}) = 0$ if $s \in \{S_1 \cup B_1\}$.*

2. *if $s \in S_1 \cup S_2$, with $E(s) = \{t_i\}_{i \in I}$, then s has a unique child labeled with $t_i \in E(s)$ in M. In other words M can be depicted as follows:*



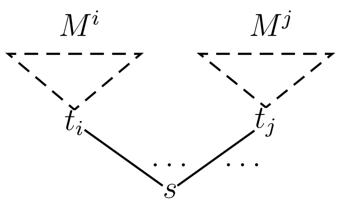
where M^i is the sub-Markov branching play of M rooted at t_i. Then the equality $\mathbb{P}_M(\Phi_{\mathcal{W}}) = \mathbb{P}_{M^i}(\Phi_{\mathcal{W}})$ holds.

3. *if $s \in S_N$, with $E(s) = \{t_i\}_{i \in I}$, then then the set of children of s in M is the set of nodes labeled with $\{t_i\}_{i \in I}$. In other words M can be depicted[2] as follows:*



where M^i, for $i \in I$, is the sub-Markov branching play of M rooted at t_i. Then the equality $\mathbb{P}_M(\Phi_{\mathcal{W}}) = \sum_{i \in I} \big(\pi(s)(t_i) \cdot \mathbb{P}_{M^i}(\Phi_{\mathcal{W}})\big)$ holds.

4. *if $s \in B_1 \cup B_2$ with $E(s) = \{t_i\}_{i \in I}$, then the set of children of s in M is the set of nodes labeled with $\{t_i\}_{i \in I}$. In other words M can be depicted[3] as follows:*



[2] The edge connecting s with t_i has been dashed to highlight that s is a probabilistic node and the edge carries a probability weight, namely $\pi(s)(t_i)$.

[3] The edge connecting s with t_i has not been dashed to highlight that s is a branching node.

where, for each $i \in I$, M^i *is the sub-Markov branching play of* M *rooted at* t_i. *Therefore the edge connecting* s *with* t_i *does not carry any probability weight. The following assertions hold:*

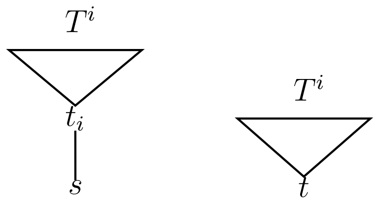
- *if* $s \in B_1$ *the equality* $\mathbb{P}_M(\Phi_{\mathcal{W}}) = \coprod_{i \in I} \mathbb{P}_{M^i}(\Phi_{\mathcal{W}})$ *holds,*
- *if* $s \in B_2$ *the equality* $\mathbb{P}_M(\Phi_{\mathcal{W}}) = \prod_{i \in I} \mathbb{P}_{M^i}(\Phi_{\mathcal{W}})$ *holds.*

where $\prod$ *and* $\coprod$ *refer to (possibly infinitary) product and coproduct operations (see Definition 2.2.3).*

where $\mathbb{P}_M$ *and* $\mathbb{P}_{M^i}$ *denotes the probability measures on the set of branching plays* $\mathcal{BP}$ *in* $\mathcal{G}$ *induced by* M *and* M^i *as described in Definition 4.1.3.*

Proof. The first case follows by Convention 5.2.1. We prove points 3 and 4, because point 2 is simpler and can be proved with the same techniques.

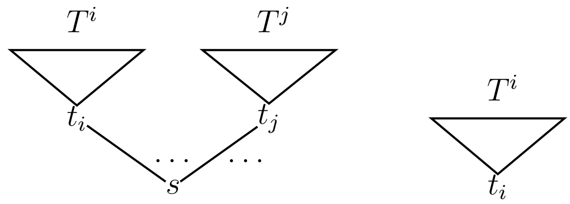
Let us consider the case when M is of the third kind, i.e., when $s \in S_N$. We can restrict attention, for what concerns the probability measure $\mathbb{P}_M$, to the set of branching plays $\cup_{i \in I} \mathcal{BP}_{s.i}$, where $\mathcal{BP}_{s.i}$ denotes the set of branching plays rooted at s and having t_i as immediate children of s, since the set of all other branching plays in $\mathcal{G}$ gets assigned probability 0 by $\mathbb{P}_M$. Similarly we can restrict, for what concerns $\mathbb{P}_{M^i}$, to the set $\mathcal{BP}_i$ of branching plays rooted at t_i. We can depict the branching plays in $\mathcal{BP}_{s.i}$ and the branching plays in $\mathcal{BP}_i$ as follows:



where we use T^i to range over the set of branching plays in $\mathcal{G}$ rooted at t_i. We denote with $T^{s.i}$ the branching play on the left, if T^i is its sub-branching play rooted at t_i. By Lemma 5.2.4 we know that $T^{s.i} \in \Phi_{\mathcal{W}}$ if and only if $T^i \in \Phi_{\mathcal{W}}$.

We now want to prove that $\mathbb{P}_M(\Phi_{\mathcal{W}}) = \sum_{i \in I} \big(\pi(s)(t_i) \cdot \mathbb{P}_{M^i}(\Phi_{\mathcal{W}})\big)$. As a first observation note that for every $i \neq j$, $\mathcal{BP}_{s.i} \cap \mathcal{BP}_{s.j} = \emptyset$. Hence the value $\mathbb{P}_M(\Phi_{\mathcal{W}})$ satisfies the following equality: $\mathbb{P}_M(\Phi_{\mathcal{W}}) = \sum_{i \in I} \mathbb{P}_M(\Phi_{\mathcal{W}} \cap \mathcal{BP}_{s.i})$. Therefore we need to show that, for every $i \in I$, the equality $\mathbb{P}_M(\Phi_{\mathcal{W}} \cap \mathcal{BP}_{s.i}) = \lambda_i \cdot \mathbb{P}_M(\Phi_{\mathcal{W}} \cap \mathcal{BP}_i)$ holds, where $\lambda_i \stackrel{\text{def}}{=} \pi(s)(t_i)$. Define $m : \mathcal{BP}_i \rightarrow \mathcal{BP}_{s.i}$ as $m(T^i) = T^{s.i}$. The map m is easily seen to be a homeomorphism. We now show that $\mathbb{P}_{M^i}(X) = \lambda_i \cdot \mathbb{P}_M(m(X))$, for every measurable X. By regularity of the two measures (see Theorem 2.1.66) we just need to prove that for each basic open set $O_F \subseteq \mathcal{BP}_i$, where F is finite tree rooted at t_i, the equality $\mathbb{P}_{M^i}(O_F) = \lambda_i \cdot \mathbb{P}_M(m(O_F))$ holds. By definition $m(O_F) = O_{s.F}$, where the set $s.F$ is defined as $\{s.\vec{t} \mid \vec{t} \in F\}$. The desired equality then follows from Definition 4.1.3, because the edge connecting s with t is probabilistic and carries the probability $\pi(s)(t_i)$.

Let us consider the case when M is of the fourth kind, i.e., when $s \in B_1 \cup B_2$. For what concerns $\mathbb{P}_M$, we can restrict attention to the set of branching plays $\mathcal{BP}_s$, where $\mathcal{BP}_s$ denotes the set of branching plays rooted at s, since the set of all other branching plays in $\mathcal{G}$ gets assigned probability 0 by $\mathbb{P}_M$. Similarly, when considering $\mathbb{P}_{M^i}$, we can restrict to the set $\mathcal{BP}_i$ of branching plays rooted at t_i. We can depict the branching plays in $\mathcal{BP}_s$ and the branching plays in $\mathcal{BP}_i$ as follows:



where we use T^i to range over the set of branching plays in $\mathcal{G}$ rooted at t_i. We denote with $s[T_i]_{i\in I}$ the branching play on the left. Let $\prod_i \mathcal{BP}_i$ be the product topology. Define $m : \prod_i \mathcal{BP}_i \to \mathcal{BP}_s$ as $m(\{T_i\}_{i\in I}) = s[T_i]_{i\in I}$. It is easy to verify that m is a homeomorphism. Consider the product measure $\prod_{i\in I} \mathbb{P}_{M^i} \in \mathcal{M}_1(\prod_i \mathcal{BP}_i)$. We now show that $\prod_{i\in I} \mathbb{P}_{M^i}(X) = \mathbb{P}_M(m(X))$, for every measurable $X \subseteq \prod_i \mathcal{BP}_i$. Again, by regularity of the two measures, we just need to prove that each basic open set $O \subseteq \prod_i \mathcal{BP}_i$ the equality $\prod_{i\in I} \mathbb{P}_{M^i}(O) = \mathbb{P}_M(m(O))$ holds. The set O is of the form $O_{F_0} \times \ldots \times O_{F_k} \times \prod_{i>k} \mathcal{BP}_i$ with $O_{F_n} \subseteq \mathcal{BP}_n$, for some $k \in \mathbb{N}$ and $0 \leq n \leq k$. As usual, O_{F_n} denotes the basic open set of branching plays containing the finite tree F_n. The desired equality $\prod_{i\in I} \mathbb{P}_{M^i}(O) \stackrel{\text{def}}{=} \prod_{n=0}^{k} \mathbb{P}_{M^i}(O_{F_n}) = \mathbb{P}_M(m(O))$ then follows by definition of m and Definition 4.1.3 of the probability measures $\mathbb{P}_M$ and $\mathbb{P}_{M^i}$.

Suppose first that $s \in B_2$. The desired result is derived as follows:

$$\begin{aligned} \mathbb{P}_M(\Phi_{\mathcal{W}}) &= \mathbb{P}_M(\Phi_{\mathcal{W}} \cap \mathcal{BP}_s) \\ &=_{E_1} \mathbb{P}_M\Big(m\big(\textstyle\prod_{i\in I}(\Phi_{\mathcal{W}} \cap \mathcal{BP}_i)\big)\Big) \\ &= \textstyle\prod_i \mathbb{P}_{M^i}(\Phi_{\mathcal{W}} \cap \mathcal{BP}_i) \\ &= \textstyle\prod_i \mathbb{P}_{M^i}(\Phi_{\mathcal{W}}) \end{aligned}$$

where equation E_1 follows by Lemma 5.2.4. If instead $s \in B_1$ the result is derived as follows:

$$\begin{array}{rcl}
\mathbb{P}_M(\Phi_{\mathcal{W}} \cap \mathcal{BP}_s) & =_{E_1} & \mathbb{P}_M\big(m(\overline{\prod_{i\in I}(\overline{\Phi_{\mathcal{W}} \cap \mathcal{BP}_i})})\big) \\
& =_{E_1} & 1 - \mathbb{P}_M\big(m(\prod_{i\in I}(\overline{\Phi_{\mathcal{W}} \cap \mathcal{BP}_i}))\big) \\
& = & 1 - \prod_{i\in I} \mathbb{P}_{M^i}(\overline{\Phi_{\mathcal{W}} \cap \mathcal{BP}_i}) \\
& = & 1 - \prod_{i\in I} \big(1 - \mathbb{P}_{M^i}(\Phi_{\mathcal{W}} \cap \mathcal{BP}_i)\big) \\
& = & 1 - \prod_{i\in I} \big(1 - \mathbb{P}_{M^i}(\Phi_{\mathcal{W}})\big) \\
& = & \coprod_{i\in I} \mathbb{P}_{M^i}(\Phi_{\mathcal{W}})
\end{array}$$

where equation E_1 follows by Lemma 5.2.4 and equation E_2 holds because m is a homeomorphism, hence $m(\overline{X}) = \overline{m(X)}$ for every measurable X. ☐

We are now ready to prove following theorem which captures a useful property of the lower and upper values of a prefix independent $2\frac{1}{2}$-player meta-game.

Theorem 5.2.10. *Let $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathcal{W}} \rangle$ be a prefix independent $2\frac{1}{2}$-player meta-game, with $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B_1, B_2), \pi \rangle$. Then the functions $\mathsf{VAL}_{\downarrow}(\mathcal{G})$ and $\mathsf{VAL}_{\uparrow}(\mathcal{G})$, defined as*

$$\mathsf{VAL}_{\downarrow}(\mathcal{G}) = \lambda s \in S.\mathsf{VAL}^s_{\downarrow}(\mathcal{G}) \quad \textit{and} \quad \mathsf{VAL}_{\uparrow}(\mathcal{G}) = \lambda s \in S.\mathsf{VAL}^s_{\uparrow}(\mathcal{G}),$$

of type $S \to [0, 1]$, are fixed points of the monotone functional $\mathbb{F}_{\mathcal{G}} : [0, 1]^S \to [0, 1]^S$, defined as follows:

$$\mathbb{F}_{\mathcal{G}}(f)(s) = \begin{cases}
\bigsqcup_{t\in E(s)} f(t) & \textit{if } s \in S_1 \\
\bigsqcap_{t\in E(s)} f(t) & \textit{if } s \in S_2 \\
\sum_{t\in E(s)} \big(\pi(s)(t) \cdot f(t)\big) & \textit{if } s \in S_N \\
\coprod_{t\in E(s)} f(t) & \textit{if } s \in B_1 \\
\prod_{t\in E(s)} f(t) & \textit{if } s \in B_2
\end{cases}$$

Proof. The fact that $\mathbb{F}_{\mathcal{G}}$ is monotone with respect to the pointwise order is trivial. We just prove that $\mathsf{VAL}_{\downarrow}(\mathcal{G})$ is a fixed point of $\mathbb{F}_{\mathcal{G}}$, i.e., that the equality $\mathsf{VAL}_{\downarrow}(\mathcal{G})(s) = \mathbb{F}_{\mathcal{G}}\big(\mathsf{VAL}_{\downarrow}(\mathcal{G})\big)(s)$ holds, for every $s \in S$. The case for $\mathsf{VAL}_{\uparrow}(\mathcal{G})$ can be proved in a similar way. The proof follows in a simple way from Lemma 5.2.9.

We just sketch how to prove the cases for $s \in S_1$, $s \in S_N$ and $s \in B_1$, as the cases for $s \in S_2$ and $s \in B_2$ are similar.

Case $s \in S_1$**:** If $E(s) = \emptyset$ the result follows by Convention 5.2.1. So assume $E(s) \neq \emptyset$. We need to prove that $\mathsf{VAL}_\downarrow(\mathcal{G})(s) = \bigsqcup_{t \in E(s)} \mathsf{VAL}_\downarrow(\mathcal{G})(t)$. We prove this equality showing that the two inequalities

$$\mathsf{VAL}_\downarrow(\mathcal{G})(s) \leq \bigsqcup_{t \in E(s)} \mathsf{VAL}_\downarrow(\mathcal{G})(t) \quad \text{and} \quad \mathsf{VAL}_\downarrow(\mathcal{G})(s) \geq \bigsqcup_{t \in E(s)} \mathsf{VAL}_\downarrow(\mathcal{G})(t)$$

hold. Let us consider the first inequality which is equivalent, by Definition 4.1.14 of $\mathsf{VAL}_\downarrow$, to $\bigsqcup_{\sigma_1} \bigsqcap_{\sigma_2} E(M^s_{\sigma_1,\sigma_2}) \leq \bigsqcup_{t \in E(s)} \big(\bigsqcup_{\tau_1} \bigsqcap_{\tau_2} E(M^t_{\tau_1,\tau_2}) \big)$. The proof goes by a strategy stealing argument. For every σ_1 let n be the state chosen by σ_1 at s, i.e., the state $\sigma_1(\{s\})$. Define τ^n_1 as $\tau_i(t_n.\vec{t}) = \sigma_1(s.t_n.\vec{t_n})$. In other words τ^n_1 (starting from t_n) behaves as σ_n (starting at s) after the first move from s to t_n is done. The equality $\bigsqcap_{\sigma_2} E(M^s_{\sigma_1,\sigma_2}) = \bigsqcap_{\tau_2} E(M^{t_n}_{\tau^n_1,\tau_2})$ is then simple to show by applying the result of Lemma 5.2.9.

Let us consider the second inequality: $\mathsf{VAL}_\downarrow(\mathcal{G})(s) \geq \bigsqcup_{t \in E(s)} \mathsf{VAL}_\downarrow(\mathcal{G})(t)$, which is equivalent, by Definition 4.1.14 of $\mathsf{VAL}_\downarrow$, to the equality $\bigsqcup_{\sigma_1} \bigsqcap_{\sigma_2} E(M^s_{\sigma_1,\sigma_2}) \geq \bigsqcup_{t \in E(s)} \big(\bigsqcup_{\tau_1} \bigsqcap_{\tau_2} E(M^t_{\tau_1,\tau_2}) \big)$. Suppose, by contradiction, that the inequality does not hold and let $t_n \in E(s)$ such that $\bigsqcup_{\sigma_1} \bigsqcap_{\sigma_2} E(M^s_{\sigma_1,\sigma_2}) < \big(\bigsqcup_{\tau_1} \bigsqcap_{\tau_2} E(M^{t_n}_{\tau_1,\tau_2}) \big)$. We derive the desired contradiction using, again, a strategy stealing argument. For every τ_1 define σ_1 as $\sigma_1(\{s\}) = t_n$ and $\sigma_1(s.t_n.\vec{t}) = \tau_1(t_n.\vec{t})$. Again, The equality $\bigsqcap_{\sigma_2} E(M^s_{\sigma_1,\sigma_2}) = \bigsqcap_{\tau_2} E(M^{t_n}_{\tau^n_1,\tau_2})$ is then simple to show by applying the result of Lemma 5.2.9.

This result can be understood as follows. If $\mathsf{VAL}_\downarrow(\mathcal{G})(t_n)$ is λ then, for every $\epsilon > 0$, Player 1 has a ϵ-optimal strategy τ^n_1 which guarantees probability of winning greater than $\lambda - \epsilon$ when the game starts at t_n. When the game starts at s, consider the strategy $s.\tau^n_1$ which chooses to move from s to t_n and play the rest of the game as τ^n_1. Then $s.\tau^n_1$ guarantees probability of winning greater than $\lambda - \epsilon$. Moreover, every strategy σ_1 for the game starting at s is of the form $s.\tau^n_1$. Hence, for every $\epsilon > 0$, there is a $t_n \in E(s)$ such that $\mathsf{VAL}_\downarrow(\mathcal{G})(t_n) \geq \mathsf{VAL}_\downarrow(\mathcal{G})(s) - \epsilon$.

Case $s \in S_N$**:** By definition 4.1.14 of $\mathsf{VAL}_\downarrow$, we need to prove that the equality $\bigsqcup_{\sigma_1} \bigsqcap_{\sigma_2} E(M^s_{\sigma_1,\sigma_2}) \leq \sum_{t \in E(s)} \pi(s)(t) \cdot \big(\bigsqcup_{\tau_1} \bigsqcap_{\tau_2} E(M^t_{\tau_1,\tau_2}) \big)$ holds. Let $E(s) = \{t_i\}_{i \in I}$ and $\lambda_i = \pi(s)(t_i)$. At the state s Nature chooses to move to t_i with probability λ_i. The desired result can be proven as follows. If $\mathsf{VAL}_\downarrow(\mathcal{G})(t_i)$ is γ_i, Player 1 has a ϵ_i optimal strategy τ^i_1, for $\epsilon_i > 0$, which guarantees probability of winning greater than $\gamma_i - \epsilon_i$ when the game starts at t_i. When the game starts at

s, define $\sigma_1^{\sum_i}$ behaving as τ_1^i once the game reaches the state t_i after the initial choice of Nature, for $i \in I$. It is then easy to prove, by application of Lemma 5.2.9, that $\sigma_1^{\sum_i}$ guarantees probability of winning greater than $\big(\sum_i \lambda_i \cdot (\gamma_i - \epsilon_i)\big)$ or, equivalently, $\sum \mathsf{VAL}_{\downarrow}(\mathcal{G})(t_i) - \sum_i \epsilon_i$. Of course $\sum_i \epsilon_i$ can be made as small as desired. Moreover, every strategy σ_1 for the game starting at s is of the form $\sigma_1^{\sum_i}$. Hence, for every $\epsilon_i > 0$, $\sum_i \lambda_i \cdot \mathsf{VAL}_{\downarrow}(\mathcal{G})(t_i) \geq \mathsf{VAL}_{\downarrow}(\mathcal{G})(s) - \epsilon$.

Case $s \in B_1$**:** By definition 4.1.14 of $\mathsf{VAL}_{\downarrow}$, we need to prove that the equality $\bigsqcup_{\sigma_1} \bigsqcap_{\sigma_2} E(M^s_{\sigma_1,\sigma_2}) \leq \coprod_{t \in E(s)} \big(\bigsqcup_{\tau_1} \bigsqcap_{\tau_2} E(M^t_{\tau_1,\tau_2})\big)$ holds. Let $E(s) = \{t_i\}_{i \in I}$. At the state s the game is split in I-many subplays continuing their execution from the states t_i. If $\mathsf{VAL}_{\downarrow}(\mathcal{G})(t_i)$ is γ_i, Player 1 has a ϵ_i optimal strategy τ_1^i, for $\epsilon_i > 0$, which guarantees probability of winning greater than $\gamma_i - \epsilon_i$ when the game starts at t_i. When the game starts at s, define $\sigma_1^{\prod_i}$ behaving as τ_1^i in the subplay continuing its execution from t_i, generated after the first game-step at s. Again, by application of Lemma 5.2.9, it is easy to prove that $\sigma_1^{\prod_i}$ guarantees probability of winning greater than $\coprod_i (\gamma_i - \epsilon_i)$. Clearly $\coprod_i (\gamma_i - \epsilon_i)$ can be made arbitrarily close to $\coprod_i \gamma_i$ by appropriate choices of ϵ_i. As in the previous cases, every strategy σ_1 for the game starting at s is of the form $\sigma_1^{\prod_i}$. Hence, for every $\epsilon_i > 0$, $\coprod_i \mathsf{VAL}_{\downarrow}(\mathcal{G})(t_i) \geq \mathsf{VAL}_{\downarrow}(\mathcal{G})(s) - \epsilon$. □

5.3 Two player stochastic meta-parity games

In this section we focus on a particular class of prefix independent two player stochastic meta-games, which we call *two player stochastic meta-parity games*. This is the class of games we will use to give game semantics to the probabilistic modal μ-calculi introduced in Section 3.3. The adjective parity describes the inner games, which are parity games.

A *parity assignment* for a two player stochastic meta-game arena $\mathcal{A}$, is a function $\mathrm{Pr} : S \to \mathbb{N}$ specified as in Definition 2.3.53. Every parity assignment induces a set $\mathcal{W}_{\mathrm{Pr}}$ of completed paths in $\mathcal{A}$, specified as in Definition 2.3.54. Recall from Definition 2.3.54 that a set $X \subseteq \mathcal{P}$ of completed paths in $\mathcal{A}$ is a *parity set* if $X = \mathcal{W}_{\mathrm{Pr}}$ for some parity assignment in $\mathcal{A}$, and that every every parity set is a prefix independent Borel set (see Proposition 2.3.61).

We now introduce the class of two player stochastic meta-parity games.

Definition 5.3.1. A $2\frac{1}{2}$-player meta-game $\langle \mathcal{A}, \Phi_{\mathcal{W}} \rangle$ is called a $2\frac{1}{2}$*-player meta-parity game* if and only if the meta-winning set $\mathcal{W}$ is a parity set induced by some parity assignment Pr on $\mathcal{A}$, i.e., if $\mathcal{W} = \mathcal{W}_{\mathrm{Pr}}$. We often denote with $\langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ the $2\frac{1}{2}$-player meta-parity game $\langle \mathcal{A}, \Phi_{\mathcal{W}_{\mathrm{Pr}}} \rangle$.

Observation 5.3.2. Consider a $2\frac{1}{2}$-player meta-parity game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$, with $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B_1, B_2), \pi \rangle$. By Definition 2.3.54, the path $\{s\}$ is contained in Φ_{Pr} if and only if $\mathrm{Pr}(s) = 1$. Therefore $\mathcal{G}$ satisfies Convention 5.2.1 if and only if for all terminal states s, if $\mathrm{Pr}(s)$ is odd then $s \in S_2 \cup B_2$ and if $\mathrm{Pr}(s)$ is even then $s \in S_1 \cup B_1$.

We shall always consider $2\frac{1}{2}$-player meta-parity games satisfying Convention 5.2.1.

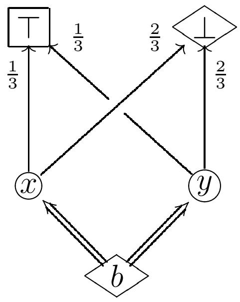
Since every $2\frac{1}{2}$-player meta-parity game is a prefix independent $2\frac{1}{2}$-player meta-game, all the results of Section 5.2 apply. Recall from Proposition 2.3.57, that the complement of a parity winning set $\mathcal{W}_{\mathrm{Pr}}$ is the parity winning set $\mathcal{W}_{\neg \mathrm{Pr}}$, with $\neg \mathrm{Pr}(s) \stackrel{\mathrm{def}}{=} \mathrm{Pr}(s) + 1$. Hence the negation of a $2\frac{1}{2}$-player meta-parity game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathcal{W}_{\mathrm{Pr}}} \rangle$, as specified in Definition 5.1.15, is the $2\frac{1}{2}$-player meta-parity game $\neg\mathcal{G} = \langle \neg\mathcal{A}, \Phi_{\mathcal{W}_{\neg \mathrm{Pr}}} \rangle$, or more succinctly $\neg\mathcal{G} = \langle \neg\mathcal{A}, \Phi_{\neg \mathrm{Pr}} \rangle$. Thus the lower and upper values (under deterministic strategies) of the $2\frac{1}{2}$-player meta-parity game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ and its negation $\neg\mathcal{G} = \langle \neg\mathcal{A}, \Phi_{\neg \mathrm{Pr}} \rangle$ are related as the result of Lemma 5.1.16 shows.

We now discuss some examples of $2\frac{1}{2}$-player meta-parity games to provide some intuitions and fix the main ideas.

Example 5.3.3. Let us consider the $2\frac{1}{2}$-player meta-parity game $\langle \mathcal{A}, \mathrm{Pr} \rangle$ defined as follows:

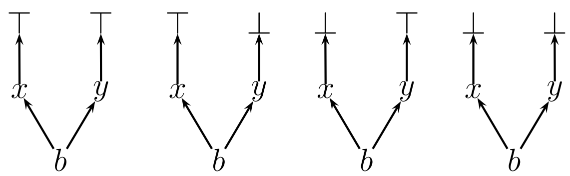
- The arena $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B_1, B_2), \pi \rangle$ is defined as:
  - $S = \{b, x, y, \top, \bot\}$,
  - $E(b) = \{x, y\}$, $E(x) = E(y) = \{\top, \bot\}$, $E(\top) = E(\bot) = \emptyset$,
  - $S_1 = \{\top\}$, $S_2 = \{\bot\}$, $S_N = \{x, y\}$, $B_1 = \{b\}$, $B_2 = \emptyset$,
  - $\pi(x)(\top) = \pi(y)(\top) = \frac{1}{3}$, $\pi(x)(\bot) = \pi(y)(\bot) = \frac{2}{3}$.
- and the priority assignment $\mathrm{Pr} : S \to \mathbb{N}$ is defined as:
  - $\mathrm{Pr}(\top) = 1$, $\mathrm{Pr}(\bot) = 0$ and $\mathrm{Pr}(x) = \mathrm{Pr}(y) = \mathrm{Pr}(b) = 0$.

The $2\frac{1}{2}$-player meta-parity game $\langle \mathcal{A}, \mathrm{Pr} \rangle$ can be depicted as follows:



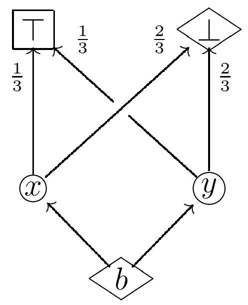
The conventions we will use in depicting our examples are the following. All states in S_1 and in B_1 are labeled with a diamond, but they are distinguishable by the fact that the edges leaving states in S_1 are depicted as single lines, while those leaving states in B_1 are doubled. Similarly all states in S_2 and in B_2 are labeled with a rectangle, but they are distinguishable by the fact that the edges leaving a state in S_2 are depicted single lines, while those leaving a state in B_2 are doubled. Observe that by the Convention 5.2.1 introduced earlier, all terminal states labeled with diamonds have assigned an even priority, while all the terminal states labeled with rectangles have assigned an odd priority. Lastly all probabilistic states $s \in S_N$ are labeled by circles, and the edges connecting s with each $t \in E(s)$ are depicted with single lines labeled with the probability $\pi(s)(t)$.

As a first observation about the game $\mathcal{G}$, note that the only agent taking part in the outer-game is Nature, since S_1 and S_2 just contain terminal states. This means that, from each starting state, there exists a unique Markov branching play M in $\mathcal{G}$. When the game starts at b, the following four branching plays, denoted by $T_{\top\top}$, $T_{\top\bot}$, $T_{\bot\top}$ and $T_{\bot\bot}$, are the possible outcomes of the game generated by the possible behaviors of Nature:

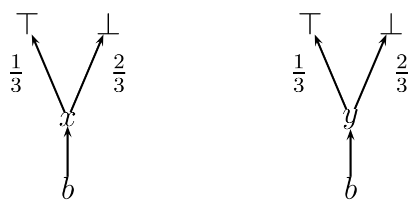


and they get assigned probabilities $\frac{1}{9}$, $\frac{2}{9}$, $\frac{2}{9}$ and $\frac{4}{9}$ respectively, by the probability measure $\mathbb{P}_{M^b}$ associated with the unique Markov branching play M^b rooted at b. Let us consider, for instance, the inner-game $\mathcal{G}_{T_{\top\bot}}$ associated with the branching play $T_{\top\bot}$. Since b is a state in B_1, i.e., under the control of Player 1 in the inner game $\mathcal{G}_{T_{\top\bot}}$, Player 1 can simply choose to move to x at the first move, and then reach the terminal state $\top$, which has assigned odd priority. The path resulting as a play from this strategy is therefore $b.x.\top$, which is winning for Player 1 in the inner game $\mathcal{G}_{T_{\top\bot}}$. This implies that Player 1 has a winning strategy in $\mathcal{G}_{T_{\top\bot}}$, which means that $\mathcal{G}_{T_{\top\bot}} \in \Phi$. By similar arguments it is easy to check that also the inner games associated with $T_{\top\top}$ an $T_{\bot\top}$ are winning for Player 1, i.e., they are in Φ, while the inner game $\mathcal{G}_{T_{\bot\bot}}$ is not. Therefore we have that $\mathbb{P}_{M^b}(\Phi) = \frac{5}{9}$, which is the value of the game $\mathcal{G}$ at b, since there are no strategies to consider in the outer-game.

Example 5.3.4. The second example we consider is a slight variation of the previous one. Let us consider the $2\frac{1}{2}$-player meta-parity game $\langle \mathcal{A}, \Pr \rangle$ which can be depicted as follows:



where $\Pr(\top) = 1$, $\Pr(\bot) = 0$ and $\Pr(x) = \Pr(y) = \Pr(b) = 0$. Compared to the Example 5.3.3, the state b is in S_1 instead of B_1. This modification makes significant changes in the way the game $\mathcal{G}$ is played. When the game starts at b, Player 1 has to make a choice, in the first stage of the game (i.e., in the outer-game of $\mathcal{G}$) between the successor states x and y, from which the rest of the game continues as in Example 5.3.3. Therefore, from the state b, there are two possible Markov branching plays M_x and M_y capturing the outcome of the (outer) game $\mathcal{G}$ up to the behavior of Nature. The Markov branching plays M_x and M_y can be depicted as follows:



Let us consider, for instance, the Markov branching play M_x. The probability measure $\mathbb{P}_{M_x}$ on $\mathcal{BP}$ induced by it, assigns probabilities $\frac{1}{3}$ and $\frac{2}{3}$ respectively to the branching plays $T_{x\top}$ and $T_{x\bot}$ which can be depicted as follows:

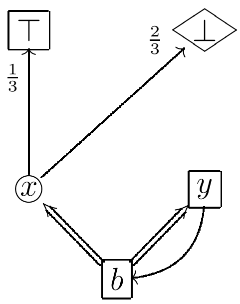
The two inner-games associated with the branching plays $T_{x\top}$ and $T_{x\bot}$ are trivial, as there are no choices to be made by Player 1 or Player 2. And, as it is easy to observe, only the inner game associated with $T_{x\top}$ is winning for Player 1, i.e., $T_{x\top} \in \Phi$ and $T_{x\bot} \notin \Phi$. Therefore we have that $\mathbb{P}_{M_x}(\Phi) = \frac{1}{3}$, and by similar observations, $\mathbb{P}_{M_y}(\Phi) = \frac{1}{3}$ as well. Since M_x and M_y are the only possible Markov branching plays in the game $\mathcal{G}$ at the state b, it follows the the value of the game $\mathcal{G}$ is $\frac{1}{3}$.

Examples 5.3.3 an 5.3.4 are helpful useful because they highlight the main difference between states in S_1 and states in B_1: at all states $s \in S_1$, Player 1 is committed to make an immediate choice among the successor states of the state s, while on all states $s \in B_1$ Player 1 *delays* their choice, which will later be made in the second stage of the game, i.e., in the inner-game resulting from the outer-game of $\mathcal{G}$. Both players prefer to delay their choices, since this allows them to base their decisions in the inner game, on the choices made by Nature in the outer-game.

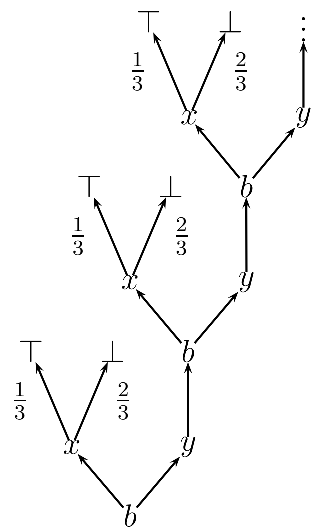
The $2\frac{1}{2}$-player meta-parity game of Example 5.3.3 could be informally described as the game in which Player 1 can toss two (biased) coins, and only *after* the outcome is revealed, they win if they can choose among the two tossed coins, one which turned out to be $\top$. The game proposed in Example 5.3.4, on the other hand, forces Player 1 to choose between one of the two (biased) coins *before* the outcome of the random events is revealed; Player 1 wins if and only if when tossing the chosen coin, $\top$ is the outcome. As a result of the different game-dynamics implemented in the two examples, the value (i.e., the probability of winning for Player 1) of the game of Example 5.3.3 is $\frac{5}{9}$ while the value of the game presented in Example 5.3.4 is just $\frac{1}{3}$.

We now consider two slightly more complex examples, in order to settle the intuitions about the different behaviors of states is S_1 (respectively S_2) and states in B_1 (respectively B_2).

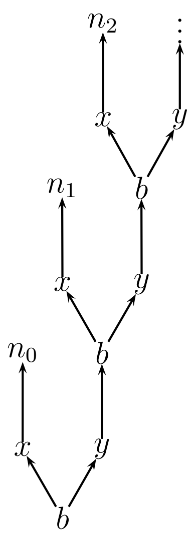
Example 5.3.5. Let us consider the $2\frac{1}{2}$-player meta-parity game $\langle \mathcal{A}, \Pr \rangle$ which can be depicted as follows:



where $\Pr(\top) = 1$, $\Pr(\bot) = 0$ and $\Pr(x) = \Pr(y) = \Pr(b) = 0$. Observe that, as in Example 5.3.3, the only agent taking part in the outer-game is Nature, since S_1 and S_2 just contain terminal states. This means that, from each starting state, there exists a unique Markov branching play M in $\mathcal{G}$. The Markov branching play M^b rooted at b can be depicted as follows:



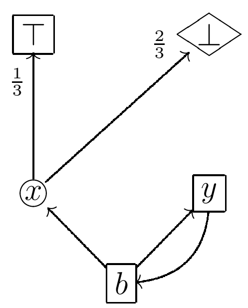
The Markov branching play M^b induces a probability measure $\mathbb{P}_{M^b}$ over the set of branching play. Each branching plays can be depicted as follows:



where the set of labels n_i, for $i \in \mathbb{N}$, range over the two elements set $\{\top, \bot\}$. Let us now consider the inner game $\mathcal{G}_T$ associated with a branching play T rooted at b induced by the behavior of Nature. Since the only nodes in T with more than one successors are labeled by the state b, which is a state in B_2, only Player 2 can actively play in the inner game $\mathcal{G}_T$, by making their choices at the nodes labeled with b. By definition of $\mathcal{W}_{\mathrm{Pr}}$, i.e., the set of paths winning for Player 1 in the inner game $\mathcal{G}_T$, the only completed paths winning for Player 2 in $\mathcal{G}_T$, i.e., those completed paths $\vec{s} \notin W_{\mathrm{Pr}}$, are the terminated paths $\vec{s} \in \mathcal{P}$ such that $last(\vec{s}) = \bot$. This is because any terminated path $\vec{s}$ with $last(\vec{s}) = \top$ is in $\mathcal{W}_{\mathrm{Pr}}$, since $\mathrm{Pr}(\top) = 1$, and because the only infinite path $\vec{s}$ in T just contains states having assigned even priority (namely b and y, which have assigned priority 0), and therefore it is by definition in $\mathcal{W}_{\mathrm{Pr}}$. Therefore Player 2 has a winning strategy in the inner game $\mathcal{G}_T$ associated with the branching play T, i.e., $T \notin \Phi$, if and only if Player 2 can find a path terminating at the state $\bot$ in T. By observing that $\mathbb{P}_{M^b}$ assigns probability 1 to the set of branching plays T having at least one path terminating in the state $\bot$, we can concluded that the value of the game $\mathcal{G}$ is 0, i.e., Player 2 wins almost surely the game $\mathcal{G}$.

The game $\mathcal{G}$ of Example 5.3.5 can be described informally as follows: Player 2 can toss a (biased) coin; if the result turns out to be $\bot$, Player 2 immediately wins; otherwise Player 2 repeats the process and toss another (independent) coin, and so on. Player 2 wins if and only if at some point (i.e., after a finite amount of trials) the outcome of the toss is $\bot$; Player 2 loses otherwise. It is clear that Player 2 wins in the game almost surely because the coin will, with probability 1, turn $\bot$ at some point.

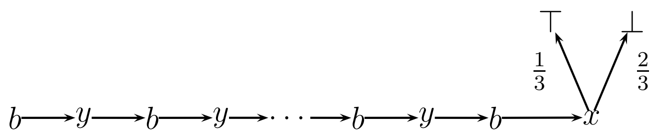
Example 5.3.6. Let us consider the $2\frac{1}{2}$-player meta-parity game $\langle \mathcal{A}, \mathrm{Pr} \rangle$ which can be depicted as follows:



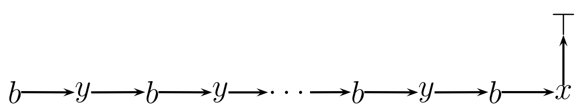
with $\Pr(\top)=1$, $\Pr(\bot)=0$ and $\Pr(x)=\Pr(y)=\Pr(b)=0$. Compared to the game of Example 5.3.5, the state b is in S_2 rather than in B_2, i.e., under the control of Player 2 in the outer game of $\mathcal{G}$ and not in the inner game. This means that both Nature and Player 2 have an active role in the outer game of $\mathcal{G}$. In particular Player 2 has ω-many strategies, which we denote with σ_2^i, for $i \in \mathbb{N}$, and σ_2^ω, to play in $\mathcal{G}$ starting at b. The strategy σ_2^i, from the starting state b, chooses to move to the state y and from there (necessarily) back to b, i-many times, after which it chooses to move to x, from which the rest of the outer-game is independent of the choices of Player 2. The strategy σ_2^ω instead, always chooses to move from b to y, infinitely many times. If Player 2 uses the strategy σ_2^ω the Markov branching play M_ω^b rooted at b, which can be depicted as follows, is generated:

$$b \longrightarrow y \longrightarrow b \longrightarrow y \longrightarrow b \longrightarrow y \longrightarrow \cdots$$

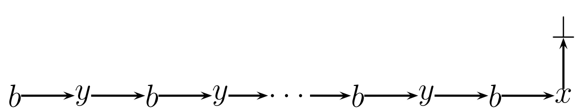
The Markov branching play M_ω^b is also a branching play T_ω^b, which means that it is the result of a play in the outer-game of $\mathcal{G}$ in which Nature didn't have any active role. The resulting inner game $\mathcal{G}_{T_\omega^b}$ associated with T_ω^b is trivial, as neither Player 1 nor Player 2 have an active role in it, and is winning for Player 1, since all states in the infinite path $b.y.b.y\ldots$ are labeled with the even priority 0 by Pr. If Player 2 uses the strategy σ_2^i instead, the Markov branching play M_i^b rooted ab b, which can be depicted as follows, is generated:



The Markov branching play M_i^b induces a probability measure $\mathbb{P}_{M_i^b}$, which assigns probability $\frac{1}{3}$ to the branching play $T_{i,\top}^b$ which can be depicted as follows:



and probability $\frac{2}{3}$ to the branching play $T_{i,\bot}^b$, which can be depicted as follows:



The inner games associated with the branching plays $T^b_{i,\top}$ and $T^b_{i,\bot}$ are both trivial. In particular Player 1 has winning strategy in $\mathcal{G}_{T^b_{i,\top}}$, i.e., $T^b_{i,\top} \in \Phi$, and Player 2 has a winning strategy in $T^b_{i,\bot}$, i.e., $T^b_{i,\bot} \notin \Phi$. Therefore if Player 2 uses the strategy σ^ω_2 they surely lose, i.e., Player 1 surely wins, and if they use the strategy σ^i_2 they win with probability $\frac{2}{3}$, i.e., Player 1 wins with probability $\frac{1}{3}$. From this we can conclude that the value of the game $\mathcal{G}$ is $\frac{1}{3}$.

The game $\mathcal{G}$ of Example 5.3.6 can be described informally as follows: Player 2, at the state b, can either choose to toss a (biased) coin, in which case they win if and only if the outcome of the toss is $\bot$, or to procrastinate the choice (which is modeled in the game, by the possibility of moving to the state y and return, after a step, to the state b). Player 2 loses if they procrastinate their choice infinitely many times. It is clear that Player 2 has no advantage in procrastinating at all, and therefore the whole game just reduces to a (biased) coin toss. Note how the dynamics of the game of Example 5.3.5 differ from those of Example 5.3.6. In the latter, Player 2 is forced to make a choice in the outer game, and this drastically reduces (from 1 to $\frac{2}{3}$) their chances to win.

The examples discussed in this section are intentionally very simple and, as we have shown, determined under deterministic strategies. As we shall see in the next chapter, every $2\frac{1}{2}$-player meta-parity game is determined under mixed strategies under the set-theoretic assumption $\text{MA}_{\aleph_1}$. We shall see more interesting example in Chapter 7, where $2\frac{1}{2}$-player meta-parity games will be used to give game semantics to the probabilistic μ-calculi defined in Section 3.3.

5.4 Summary of results

In this chapter we identified a useful class of $2\frac{1}{2}$-player tree games which we named $2\frac{1}{2}$-player meta-games. In $2\frac{1}{2}$-player meta-games the set of winning branching plays is described, somewhat declaratively, by means of inner games which are standard 2-player turn-based games played on branching plays. As we discussed in Section 5.1, $2\frac{1}{2}$-player meta-games can be understood as games played in two (infinite) stages: the $2\frac{1}{2}$-player outer tree game and the standard 2-player inner game. The key idea is that, in the inner games, both players can base their moves on the choices made by the opponent *and* by Nature in the outer game. As we observed in Remark 5.1.11, if the outer game is not stochastic, then the division of the game in two stages does not provide any gain to the two players, which can indeed play as if they were competing in a standard 2-player game. This fact highlights the important feature of $2\frac{1}{2}$-player meta-games. At player states (i.e., in S_1 or S_2), Player 1 and Player 2 have to make their choices assuming all possible random behaviors of Nature. At branching states (i.e., in B_1 or B_2), Player 1 and Player 2 can instead delay their choices and, only once the moves of Nature become known, declare their choices in the second stage of the game.

In Section 5.1 we discussed how working with $2\frac{1}{2}$-player meta-games (with Borel meta-winning sets) is technically problematic because their winning sets are, in general, $\mathbf{\Delta}^1_2$-sets, hence not necessarily universally measurable. Thus we defined the assertion mG-UM(Γ), for a collection Γ of meta-games, which is the minimal assumption necessary for dealing with the games $\mathcal{G}\in\Gamma$. As we discussed, mG-UM(Γ) is consistent with ZFC because, for instance, it follows from $\mathbf{\Delta}^1_2$-UM which is one of the consequences of Martin's Axiom at $\aleph_1$ (see Theorem 2.1.87).

In Section 5.2 we focused on $2\frac{1}{2}$-player meta-games $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathcal{W}}\rangle$ with $\mathcal{W}$ a (Borel) prefix-independent set. We derived useful properties of this class of meta-games, which will be used later on in Chapter 6. Interestingly, another useful metaphor, for the game interactions, is available for prefix-independent $2\frac{1}{2}$-player meta-games. At player states, Player 1 and Player 2 have to make their choices assuming all possible random behaviors of Nature, as already discussed earlier. At branching states $s\in B_1$, the game can be thought of as generating a collection of independent subplays, one for each successor state of s, and Player 1 wins if and only if the outcome of *at least one* subplay is winning for them (thus ignoring the prefix by which the play arrived at the current state). Similarly, at branching states $s\in B_2$, a collection of independent sub-plays is generated and Player 1 wins if and only if the outcome of *all* subplays is winning for them. These existential and universal conditions match the roles of Player 1 and Player 2 in the inner games. Moreover the winning condition expressed just in terms of the outcomes of the generated sub-plays, and thus not on the previous history of the game, is justified by the prefix-independency of the meta-winning sets.

In Section 5.3 we focused on $2\frac{1}{2}$-player meta-parity games, having a parity meta-winning set. As we shall see in Chapter 7, $2\frac{1}{2}$-player meta-parity games will be used to give a game semantics to the probabilistic modal μ-calculi discussed in Section 3.3. The above discussed metaphor for the interactions occurring in $2\frac{1}{2}$-player meta-parity games matches exactly the informal description of pL$\mu^{\odot}$

games given in Section 3.3.3. The result of Theorem 5.2.10 constrains the *value* of $2\frac{1}{2}$-player meta-parity games in terms of the operations $\{\bigsqcup, \bigsqcap, \sum, \prod, \coprod\}$ which are the basic operators used in the denotational semantics of $\text{pL}\mu^{\odot}$. For $2\frac{1}{2}$-player meta-parity games, the value is indeed well defined because, as we shall see in the next Chapter, every $2\frac{1}{2}$-player meta-parity game is determined under deterministic strategies, at least in ZFC + $\text{MA}_{\aleph_1}$ set theory. Lastly, Remark 5.1.11, which reduces non-stochastic meta-games to ordinary 2-player games, can be thought as a game-counterpart of Proposition 3.3.8, which reduces a $\text{pL}\mu^{\odot}_{\oplus}$ formula to an equivalent, when interpreted on ordinary LTS's, modal μ-calculus formula.

We conclude this summary section by pointing to a possibly fruitful connection between $2\frac{1}{2}$-player meta-parity games and what A. Arnold and D. Niwinski call *game tree languages* in [3]. A *game tree language* $\mathcal{W}_{i,k}$, for $i, k \in \mathbb{N}$ and $i \leq k$, is the set of trees T over the finite alphabet $\{1, 2\} \times \{i, \ldots, k\}$ (see Definition 2.1.41) such that, when interpreted as 2-player parity games (a node $\langle p, j \rangle$ in T, for $p \in \{1, 2\}$ and $j \in \{i, \ldots, k\}$, is under the control of Player p and is labeled with priority j), Player 1 has a winning strategy. There is an obvious relation between the assertion "$\mathcal{W}_{i,k}$ is universally measurable for every $i \leq k$" and mG-UM(Γ_p), where Γ_p is the collection of all $2\frac{1}{2}$-player meta-parity games. Some deep properties of game tree languages are known. For instance, the sets $\mathcal{W}_{0,n}$, for $n \in \mathbb{N}$, form a strictly $\leq_W$-increasing chain in the Wadge order (see Definition 2.1.56). We refer to [3] for a very concise proof of this fact. However, to our knowledge, the measurability problem has not been studied in the literature. We think it is quite plausible that mG-UM(Γ_p) holds in ZFC alone. We leave this as an interesting problem for future work.

Chapter 6

Determinacy of $2\frac{1}{2}$-player meta-parity games

We develop in this Chapter our main result about $2\frac{1}{2}$-player meta-parity games. We prove that every $2\frac{1}{2}$-player meta-parity game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ is determined under deterministic strategies, in the sense of Definition 4.1.14. This is proven by induction on $|\,\mathrm{Pr}\,|$, i.e., on the number of priorities assigned to states in $\mathcal{A}$ by the priority assignment Pr. Proving results about parity games[1] by induction on the number of priorities is not a novel idea. Indeed this (high-level) proof technique is known under the name of *unfolding* or *unravelling method*, ascribable to the proof technique developed in [72] by Donald A. Martin (see, e.g., [60, Ch. 20]), and developed in the context of parity games in the works of Luigi Santocanale [100] and Erich Grädel [46] among others. The unfolding method is also the main technique adopted in [35] to prove the equivalence of the denotational and game semantics of what the authors call *quantitative modal μ-calculus*, a recently introduced fixed-point modal logic for expressing properties of quantitative labeled transition systems[2]. The techniques adopted in [46] and [35] greatly inspired the development of the results of this chapter. However, in the context of $2\frac{1}{2}$-player meta-parity games several complications arise, and our proofs require a significant technical effort. Thus we believe the present proof, beside establishing an impor-

[1] Here the term *parity game* ranges, informally, over any *game* whose winning condition is specified by some sort of parity assignment on the game arena.

[2] The game semantics is given in terms of standard 2-player turn based games with real-valued payoffs. Thus their determinacy follows by standard results (Theorem 2.3.37). The models are (non probabilistic) LTS's whose transitions are decorated with real-values. The calculus of [35] is different from the μ-calculi discussed in Section 3.3. We used the adjective "probabilistic" to highlight this difference.

tant result, constitutes a significant contribution to applicability of the general unfolding method proof technique.

This chapter is organized as follows. In Section 6.1 we prove that every $2\frac{1}{2}$-player meta-parity game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ with $|\,\mathrm{Pr}\,| = 1$ (see Definition 2.3.53) is determined under deterministic strategies. This result is the base of our inductive proof. In Section 6.3 we prove that, under appropriate assumptions coming from the inductive hypothesis, every $2\frac{1}{2}$-player meta-parity game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$, with $|\,\mathrm{Pr}\,| = N + 1$ priorities, is determined under deterministic strategies. This is proven by reducing the problem, using the unfolding method, to the determinacy under deterministic strategy of a $2\frac{1}{2}$-player meta-parity game with just N priorities. In Section 6.2, we develop the technical machinery associated with the unfolding procedure. In Section 6.4 we compose the results of sections 6.1, 6.2 and 6.3 and prove that every $2\frac{1}{2}$-player meta-parity game is determined. Due to the complexity of the winning sets of $2\frac{1}{2}$-player meta-parity games, and the associate measure theoretic issues, our proof is carried out in $\mathrm{ZFC} + \mathrm{MA}_{\aleph_1}$ set theory. The proof is heavily based on the machinery developed on Section 4.3. The technology we introduce for proving the main theorem, i.e., determinacy under deterministic strategies of all $2\frac{1}{2}$-player meta-parity games, will allow other interesting properties of $2\frac{1}{2}$-player meta-parity games to be proven. In Section 6.5 we summarize the results obtained in this chapter.

6.1 $2\frac{1}{2}$-player meta-parity games with one priority

Let us fix an arbitrary $2\frac{1}{2}$-player meta-parity game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$, where the arena $\mathcal{A}$ is specified as $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B_1, B_2), \pi \rangle$, and such that $|\,\mathrm{Pr}\,| = 1$. We denote in this section with p the unique priority assigned by Pr to states in S. We denote with Φ the winning set of the game $\mathcal{G}$, i.e., the set Φ_{Pr}. We denote with $\mathcal{P}^{<\omega}$, $\mathcal{P}$, $\mathcal{BP}$, $\mathcal{MBP}$ the sets of finite paths, completed paths, branching plays and Markov branching plays in $\mathcal{A}$, respectively.

We first prove our results assuming that p is an odd number. The case for p even will be considered at the end of this section.

We want to show that, for every $s \in S$, $\mathsf{VAL}^s_{\downarrow}(\mathcal{G}) = \mathsf{VAL}^s_{\uparrow}(\mathcal{G})$. By Theorem 5.2.10 we know that both $\mathsf{VAL}_{\downarrow}(\mathcal{G}) : S \to [0, 1]$ and $\mathsf{VAL}_{\uparrow}(\mathcal{G}) : S \to [0, 1]$ are fixed points of the functional $\mathbb{F}_{\mathcal{G}}$ specified as in Theorem 5.2.10. Moreover we know that $\mathsf{VAL}_{\downarrow}(\mathcal{G}) \sqsubseteq \mathsf{VAL}_{\uparrow}(\mathcal{G})$. We are going to prove the desired result showing that

$\mathsf{VAL}_{\uparrow}(\mathcal{G}) \sqsubseteq \text{lfp}(\mathbb{F}_{\mathcal{G}})$, where $\text{lfp}(\mathbb{F}_{\mathcal{G}})$ denotes the least fixed point of the functional $\mathbb{F}_{\mathcal{G}}$. This clearly implies the desired result. In order to do this, we obtain an inductive characterization of the winning set Φ which allows us to prove the desired inequality by an inductive argument.

Let $\mathbb{W}_{\mathcal{G}} \in 2^S \to 2^S$ be the monotone functional associated with the $2\frac{1}{2}$-player meta-parity game $\mathcal{G}$, as specified in Definition 5.2.5.

Corollary 6.1.1. *By the Knaster-Tarski theorem, $\mathbb{W}_{\mathcal{G}}$ has least and greatest fixed points. In particular the least fixed point of $\mathbb{W}_{\mathcal{G}}$, denoted by $\text{lfp}(\mathbb{W}_{\mathcal{G}})$, can be characterized as follows:*

$$\text{lfp}(\mathbb{W}_{\mathcal{G}}) = \bigcup_{\alpha} \mathbb{W}^{\alpha}_{\mathcal{G}}$$

where α ranges over the ordinals, and the sets $\mathbb{W}^{\alpha}_{\mathcal{G}}$ are defined, for each ordinal α, as $\mathbb{W}^{\alpha}_{\mathcal{G}} = \bigcup_{\beta<\alpha} \mathbb{W}_{\mathcal{G}}(\mathbb{W}^{\beta}_{\mathcal{G}})$.

The operator $\mathbb{W}_{\mathcal{G}}$ is of our interest because it allows to provide an inductive characterization for the winning set Φ.

Lemma 6.1.2. *The equality $\text{lfp}(\mathbb{W}_{\mathcal{G}}) = \Phi$ holds.*

Proof. Since $\text{lfp}(\mathbb{W}_{\mathcal{G}}) = \bigcup_{\alpha} \mathbb{W}^{\alpha}_{\mathcal{G}}$ we already know, by application of Lemma 5.2.7, that $\text{lfp}(\mathbb{W}_{\mathcal{G}}) \subseteq \Phi$. Therefore we just need to show that $\text{lfp}(\mathbb{W}_{\mathcal{G}}) \supseteq \Phi$. We prove the desired result showing that for every $T \in \overline{\text{lfp}(\mathbb{W}_{\mathcal{G}})}$, Player 2 has a winning strategy in the inner game $\mathcal{G}_T$. Observe that, since Pr assigns the odd priority p to every state $s \in S$, no terminal state in $\mathcal{A}$ is in $S_1 \cup B_1$ by Convention 5.2.1. Moreover it follows by Definition 2.3.54 that the set $\mathcal{W}_{\text{Pr}}$ of winning paths for Player 1 in the inner games $\mathcal{G}_T$, for $T \in \mathcal{BP}$, is exactly the set of terminated paths, i.e., the set of completed paths ending in a terminal state. Thus, every infinite completed path is winning for Player 2. A winning strategy for Player 2 in the inner game $\mathcal{G}_T$ is therefore a strategy that can force a play in $\mathcal{G}_T$ through an infinite path. We now show how player 2 can force the play in the inner-game $\mathcal{G}_T$ to be an infinite path, for any $T \in \overline{\text{lfp}(\mathbb{W}_{\mathcal{G}})}$, by following the rules considered below.

1. Every branching play T' whose $root(T')$ is a terminal state in not in $\overline{\text{lfp}(\mathbb{W}_{\mathcal{G}})}$ by Definition 5.2.5 of $\mathbb{W}_{\mathcal{G}}$. So assume that $root(T)$ is not terminal in the following cases.

2. If $s = root(T) \in S_1 \cup S_2 \cup S_N$, then s has a unique child t_i in T, for some $t_i \in E(s)$. It follows, by definition of $\mathbb{W}_{\mathcal{G}}$, that the branching play T^i, which is the sub-branching play of T rooted at t_i, is in $\overline{\text{lfp}(\mathbb{W}_{\mathcal{G}})}$ as well, because T would be in $\text{lfp}(\mathbb{W}_{\mathcal{G}})$ otherwise. The move in the game $\mathcal{G}_T$ from the state s to the state t_i is forced, and Player 2 can proceed in the rest of the sub-game $\mathcal{G}_{T^i}$, iterating the same protocol, and thus forcing the game into an infinite path.

3. If $s = root(T) \in B_1$, with $E(s) = \{t_i\}_{i \in I}$ then, by the same kind of argument, for each $i \in I$, the sub-branching play T^i of T rooted at t_i is in $\overline{\text{lfp}(\mathbb{W}_{\mathcal{G}})}$. In the game $\mathcal{G}_T$ the state s is under the control of Player 1 who can move to a state t_i, for $i \in I$. If player 1 chooses to move to the state t_i, then Player 2 play in the rest of the sub-game $\mathcal{G}_{T^i}$, iterating the same protocol, and thus forcing the game into an infinite path.

4. If $s = root(T) \in B_2$, with $E(s) = \{t_i\}_{i \in I}$ then there must exist some $i \in I$ such that the sub-branching play T^i of T rooted at t_i is in $\overline{\text{lfp}(\mathbb{W}_{\mathcal{G}})}$. In the game $\mathcal{G}_T$ the state s is under the control of Player 2 who can move to the t_i, and play in the rest of the sub-game $\mathcal{G}_{T^i}$, iterating the same protocol, and thus forcing the game into an infinite path.

□

The following is an immediate consequence of Lemma 6.1.2 and Lemma 5.2.8.

Lemma 6.1.3. *If $\mathcal{A}$ is finitely branching in B_2, then the winning set Φ is open.*

Proof. From Lemma 6.1.2 we know that $\Phi = \bigcup_\alpha \mathbb{W}^\alpha_{\mathcal{G}}$, and from Lemma 5.2.8, $\mathbb{W}^\alpha_{\mathcal{G}}$ is open for every ordinal α. □

Lemma 6.1.3 implies that mG-UM(Γ) (see Definition 5.1.8) holds in ZFC, when Γ is set of all $2\frac{1}{2}$-player meta-parity games with just one odd priority having an arena finitely branching in B_2.

The next lemma provides useful information about the number of iterations necessary to reach the least fixed point of $\mathbb{W}_{\mathcal{G}}$.

Lemma 6.1.4. *If the $2\frac{1}{2}$-player meta-parity game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\text{Pr}} \rangle$ is finitely branching in B_2 (see Definition 5.1.13), then the operator $\mathbb{W}_{\mathcal{G}}$ reaches its fixed point in at most ω iterations, i.e.,* $\text{lfp}(\mathbb{W}_{\mathcal{G}}) = \bigcup_{\alpha<\omega} \mathbb{W}^\alpha_{\mathcal{G}}$. *If $\mathcal{G}$ is not finitely branching in*

B_2, then the operator $\mathbb{W}_{\mathcal{G}}$ reaches its fixed point in at most ω_1 iterations, i.e., $\mathrm{lfp}(\mathbb{W}_{\mathcal{G}})=\bigcup_{\alpha<\omega_1}\mathbb{W}^{\alpha}_{\mathcal{G}}$.

Proof. Let us first consider the case when $\mathcal{G}$ is finitely branching in B_2. We show that for all $T\in\mathcal{BP}$, if $T\in\mathbb{W}^{\omega+1}_{\mathcal{G}}$ then $T\in\mathbb{W}^{\omega}_{\mathcal{G}}$. Assume $T\in\mathbb{W}^{\omega+1}_{\mathcal{G}}$. Then one of the points 1-4 of Definition 5.2.5 hold.

1. In this case $T\in\mathbb{W}_{\mathcal{G}}(\mathbb{W}^{\omega_1}_{\mathcal{G}})$ because T is the branching play $T=\{s\}$ rooted at some terminal state $s\in S$. Thus $T\in\mathbb{W}^{1}_{\mathcal{G}}$ as desired.

2. In this case $T\in\mathbb{W}_{\mathcal{G}}(\mathbb{W}^{\omega}_{\mathcal{G}})$ because T is a branching play rooted at $s\in S_1\cup S_2\cup S_N$, with $E(s)\neq\emptyset$, and the sub-branching play T^{t_i} rooted at the unique child t_i of s in T is in $\mathbb{W}^{\omega}_{\mathcal{G}}$. Since $\mathbb{W}^{\omega}_{\mathcal{G}}=\bigcup_{\alpha<\omega}\mathbb{W}^{\alpha}_{\mathcal{G}}$, we have that $T^{t_i}\in\mathbb{W}^{\alpha}_{\mathcal{G}}$ for some $\alpha<\omega$. But then $T\in\mathbb{W}^{\alpha+1}_{\mathcal{G}}$, and $\alpha+1<\omega$ as desired.

3. In this case $T\in\mathbb{W}_{\mathcal{G}}(\mathbb{W}^{\omega}_{\mathcal{G}})$ because T is a branching play rooted at $s\in B_1$, with $E(s)=\{t_i\}_{i\in I}\neq\emptyset$, and there exists an $i\in I$ such that the sub-branching play T^{t_i} of T is in $\mathbb{W}^{\omega}_{\mathcal{G}}$. Again, as for the previous case, $T^{t_i}\in\mathbb{W}^{\alpha}_{\mathcal{G}}$ for some $\alpha<\omega$ and $T\in\mathbb{W}^{\alpha+1}_{\mathcal{G}}$ as desired.

4. In this case $T\in\mathbb{W}_{\mathcal{G}}(\mathbb{W}^{\omega}_{\mathcal{G}})$ because T is a branching play rooted at $s\in B_2$, with $E(s)=\{t_i\}_{i\in I}\neq\emptyset$, and for all $i\in I$ the sub-branching play T^{t_i} of T is in $\mathbb{W}^{\omega}_{\mathcal{G}}$. By considerations analogous to previous ones, each branching play T^{t_i} is contained in $\mathbb{W}^{\alpha_i}_{\mathcal{G}}$ for some $\alpha_i<\omega$, for every $i\in I$. Since $\mathcal{A}$ is finitely branching in B_2, the set I indexing the successors states of s is finite. Let us define the ordinal $\beta=\bigsqcup_{i\in I}\alpha_i$. Then $\beta<\omega$, since I is finite, and we have $T\in\mathbb{W}^{\beta+1}_{\mathcal{G}}$ as desired.

Let us now consider the general case, when $\mathcal{G}$ is not finitely branching in B_2. The proof technique we adopt is similar. Assume $T\in\mathbb{W}^{\omega_1+1}_{\mathcal{G}}$. The first two points handled as before.

4. In this case $T\in\mathbb{W}_{\mathcal{G}}(\mathbb{W}^{\omega_1}_{\mathcal{G}})$ because T is a branching play rooted at $s\in B_2$, with $E(s)=\{t_i\}_{i\in I}\neq\emptyset$, and for all $i\in I$ the sub-branching play T^{t_i} of T is in $\mathbb{W}^{\omega_1}_{\mathcal{G}}$. By considerations analogous to previous ones, each branching play T^{t_i} is contained in $\mathbb{W}^{\alpha_i}_{\mathcal{G}}$ for some $\alpha_i<\omega_1$, for every $i\in I$. Thus the ordinal $\beta=\bigsqcup_{i\in I}\alpha_i<\omega_1$ is countable, because I is countable. Then we have $T\in\mathbb{W}^{\beta+1}_{\mathcal{G}}$, and $\beta+1<\omega_1$, as desired.

□

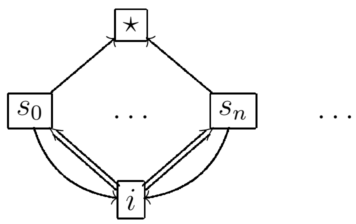
We now prove that if $\mathcal{A}$ is not finitely branching in B_2 then, in general, the winning set Φ is not Borel.

Lemma 6.1.5. *There exists a $2\frac{1}{2}$-player meta-parity game with just one odd priority such that its winning set Φ is not Borel.*

Proof. We provide a concrete example by constructing a $2\frac{1}{2}$-player meta-parity game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ with just one odd priority such that:

1. $\mathcal{BP}_{\mathcal{A}}$ is Borel isomorphic to $\mathcal{T}(\mathbb{N})$ (see Definition 2.1.41), and
2. $\Phi_{\mathrm{Pr}} \subseteq \mathcal{BP}_{\mathcal{A}}$ is Borel isomoprhic to the set of well-founded trees over $\mathcal{T}(\mathbb{N})$.

The result then follows by application of Theorem 2.1.57. The game $\mathcal{G}$ can be depicted as follows:



where the branching state i has countably many successors states $\{s_n\}_{n\in\mathbb{N}}$. We now show that the set $\mathcal{BP}_i$ of branching plays rooted at the state i is homeomorphic to $\mathcal{T}(\mathbb{N})$. The tree over $\mathbb{N}$ corresponding to the branching play $T = \{\vec{s}\}_{i\in I}$ contains the finite sequence $a_0.a_1 \dots a_k$ of natural numbers if and only if T contains the path $i.s_{a_0}.i.s_{a_1}.i \dots i.s_{a_k}.i$. Moreover the sequence $a_0.a_1 \dots a_k$ is a leaf, in the tree corresponding to T, if and only if T contains the set of paths $\{i.s_{a_0}.i.s_{a_1}.i \dots i.s_{a_k}.i.s_n.\star \mid n \in \mathbb{N}\}$.

Define the function function $f : \mathcal{T}(\mathbb{N}) \rightarrow \mathcal{BP}_i$ as follows:

$$f\big(\{\vec{n}_i\}_{i\in I}\big) = \begin{cases} \bigcup_{n\in\mathbb{N}} i.s_n.\star & \text{if } I = \emptyset \\ \big(\bigcup_{i\in I} \vec{t_i}\big) \cup \big(\bigcup_{j\in L} \bigcup_{n\in\mathbb{N}} \vec{t_j}.s_n.\star\big) & \text{otherwise} \end{cases}$$

where $\vec{t_i} = i.s_{a_0}.i.s_{a_1}.i \dots i.s_{a_k}.i$ if $\vec{n}_i = a_0.a_1 \dots a_k$, and $L \subseteq I$ is the set of (indexes of) leaves in $\{\vec{n}_i\}_{i\in I}$. It is easy to check that $f\big(\{\vec{n}_i\}_{i\in I}\big)$ is indeed a branching play in $\mathcal{G}$ and that f is a bijection. It also immediate to see that f is continuous (the inverse image of every sub-basic open set $O_{\vec{s}}$ is a sub-basic open set in $\mathcal{T}(\mathbb{N})$ and its inverse f^{-1} is Borel measurable (the inverse under f^{-1} of a sub-basic open set $\{T \mid \vec{n} \notin T\}$ is closed).

Let us now consider the winning set Φ_{Pr} of the game $\mathcal{G}$. Since an odd priority is assigned to each state $s \in S$ by Pr, the set $\mathcal{W}_{\mathrm{Pr}}$ of winning paths for Player 1 in any inner game $\mathcal{G}_T$, for $T \in \mathcal{BP}_i$, is the set of all terminated completed paths, which in this context, is the set of all completed paths ending in the terminal state $\star$. Moreover, since every branching state in $\mathcal{A}$ is under the control of Player 2, it follows that a branching play $T \in \mathcal{BP}_i$ is winning for Player 1 if and only if there are no infinite paths in the inner game $\mathcal{G}_T$. Given this description for the set Φ, it is simple to see that $T \in \Phi$, if and only if $f^{-1}(T) \in \mathrm{WF}$, where $\mathrm{WF} \subseteq \mathcal{T}(\mathbb{N})$ is the set of all well-founded trees on $\mathbb{N}$. □

Note that the example proposed in Lemma 6.1.5 is uniquely branching in B_1, in the sense of Definition 5.1.13. This means that the upper bound, provided in Lemma 5.1.14, on the complexity of winning sets of $2\frac{1}{2}$-player meta-games uniquely branching in B_1, is strict.

As another consequence of Lemma 6.1.5, we have that the upper bound provided by Lemma 6.1.4 on the number of iterations needed to reach the least fixed point of the operator $\mathbb{W}_{\mathcal{G}}$ is tight.

Corollary 6.1.6. *There exists a $2\frac{1}{2}$-player meta-parity game with just one odd priority such that for every countable ordinal α, $\mathbb{W}^{\alpha}_{\mathcal{G}} \neq \mathbb{W}^{\omega_1}_{\mathcal{G}}$.*

Proof. Consider the game $\mathcal{G}$ defined in the proof of Lemma 6.1.5. Recall from Lemma 6.1.2 that $\Phi = \bigcup_{\beta<\omega_1} \mathbb{W}^{\beta}_{\mathcal{G}}$ where Φ is the winning set of $\mathcal{G}$. For each countable ordinal α, the set $\mathbb{W}^{\alpha}_{\mathcal{G}}$ is Borel by application of Lemma 5.2.8. □

We are now ready to prove the main theorem of this section.

Theorem 6.1.7. *Let, $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ be a $2\frac{1}{2}$-player meta-parity game, with $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B_1, B_2), \pi \rangle$, having just one odd priority, i.e., such that $\min(\mathrm{Pr}) = 1$ and $\min(\mathrm{Pr}) = p$ for some odd number p. Then the following assertions hold*

$$\begin{array}{lll} \mathrm{ZFC} & \vdash & Val^s_{\uparrow}(\mathcal{G}) \leq \mathrm{lfp}(\mathbb{F}_{\mathcal{G}})(s), \text{ if } \mathcal{G} \text{ is finitely branching in } B_2 \\ \mathrm{ZFC} + \mathrm{MA}_{\aleph_1} & \vdash & Val^s_{\uparrow}(\mathcal{G}) \leq \mathrm{lfp}(\mathbb{F}_{\mathcal{G}})(s) \end{array}$$

for every $s \in S$. Moreover if $\mathcal{A}$ is finitely branching in S_2, then Player 2 has a optimal positional strategy (see Definition 2.3.45).

Proof. As a first observation, note that the two assertions are well-defined only under the hypothesis mG-UM($\Gamma_1^{B_2}$) and mG-UM(Γ_1) respectively, with $\Gamma_1^{B_2}$ being the class of all $2\frac{1}{2}$-player meta-parity games finitely branching in B_2 (see Definition 5.1.13) with one odd priority, and Γ_1 the wider class of all $2\frac{1}{2}$-player meta-parity games with one odd priority. However, by applications of Lemma 6.1.3 we know that ZFC $\vdash$ mG-UM(Γ_1^B) holds. Moreover ZFC + $\mathrm{MA}_{\aleph_1}$ $\vdash$ mG-UM(Γ_1) holds, by Lemma 2.1.88 and the observations following Definition 5.1.8. Thus we omitted the hypotheses mG-UM(Γ_1^B) and mG-UM(Γ_1) from the statements.

We will prove the two assertions together in a uniform way, specifying when we use the set-theoretic assumption $\mathrm{MA}_{\aleph_1}$ in the general case, i.e., when $\mathcal{G}$ is not finitely branching in B_2.

Since p is odd, it follows from Convention 5.2.1 that there are no terminal states in $S_1 \cup B_1$ in $\mathcal{G}$. Thus, we can rewrite the definition of $\mathbb{F}_{\mathcal{G}}$ (see Theorem 5.2.10) as follows:

$$\mathbb{F}_{\mathcal{G}}(f)(s) = \begin{cases} 1 & \text{if } E(s)=\emptyset \\ \bigsqcup_{t\in E(s)} f(t) & \text{if } s\in S_1 \\ \bigsqcap_{t\in E(s)} f(t) & \text{if } s\in S_2 \\ \sum_{t\in E(s)} \big(\pi(s)(t)\cdot f(t)\big) & \text{if } s\in S_N \\ \coprod_{t\in E(s)} f(t) & \text{if } s\in B_1 \\ \prod_{t\in E(s)} f(t) & \text{if } s\in B_2 \end{cases}$$

We just denote with $\mathbb{F}$ the functional $\mathbb{F}_{\mathcal{G}}$ in what follows.

Let us fix an arbitrary state $s\in S$. We prove the inequality $Val^s_{\uparrow}(\mathcal{G}) \leq \mathrm{lfp}(\mathbb{F})(s)$ by constructing, for every $\epsilon > 0$ a strategy σ_2 for Player 2 in $\mathcal{G}$ such that the inequality

$$\textstyle\bigsqcup_{\sigma_1} E(M^s_{\sigma_1,\sigma_2}) \leq \mathrm{lfp}(\mathbb{F})(s) + \epsilon$$

holds, for all states $s \in S$. This clearly implies the desired result. Let us an arbitrary $\epsilon > 0$.

Let us first assume that the game arena $\mathcal{A}$ is finitely branching in S_2 (see Definition 5.1.13). In this restricted case, we can define for every non terminal state $s \in S_2$ a successor state $t^s \in E(s)$ which satisfies the following property: $\mathrm{lfp}(\mathbb{F})(t^s) = \bigsqcap_{t\in E(s)} \mathrm{lfp}(\mathbb{F})(t) = \mathrm{lfp}(\mathbb{F})(s)$. In other words t^s is a successor state of

s which minimize lfp($\mathbb{F}$). We then define the strategy σ_2 as $\sigma_2(\vec{s}.s)=t^s$, for every non-terminal state $s\in S_2$, and $\sigma_2(\vec{s}.s)=\{\bullet\}$ for every terminal state $s\in S_2$. In other words the strategy σ_2 always picks the successor state (if there is any) which minimizes the function lfp($\mathbb{F}$). Note that the so defined strategy σ_2 is positional.

We could now prove that σ_2 satisfies the desired inequality. However, because of the restriction on the set of the arenas, namely the finitely branching in S_2 ones, we would have to prove the desired result for the general case separately. In particular the conceptual difference, when working with general arenas, is that Player 2, at some non-terminal state $s\in S_2$, might not have an optimal, in the sense of minimizing lfp($\mathbb{F}$), successor state to pick. Intuitively an ϵ-optimal strategy for Player 2 in such a state would just choose an almost optimal, say $\frac{\epsilon}{2}$-close to lfp($\mathbb{F}$)(s), successor $t\in E(s)$ of s. However after this choice, Player 2 will have, on subsequent choices, to improve their decisions further, i.e., they will have to chose successors closer than $\frac{\epsilon}{2}$ to lfp($\mathbb{F}$). Matters are further complicated by the different behaviors that Player 2 might need to sustain in different subgames. This said, it is clear that an ϵ-optimal strategy for Player 2 in a general arena, constructed following the previous informal discussion, crucially bases its choices on the history of the previously played moves.

Let us consider a numbering of all finite paths in $\mathcal{A}$, i.e., an injective map $e:\mathcal{P}^{<\omega}\rightarrow\mathbb{N}\setminus\{0\}$ which satisfies $e(\vec{s})<e(\vec{t})$ whenever $\vec{s}$ is a proper prefix of $\vec{t}$. Let us denote with σ_2^e any strategy for Player 2 which satisfies the following specification:

$$\sigma_2(\vec{s}) = \begin{cases} \bullet & \text{only if } E(last(\vec{s}))=\emptyset \\ t & \text{only if } E(last(\vec{s}))\neq\emptyset,\ t\in E(s) \text{ and } \mathrm{lfp}(\mathbb{F})(t)\leq \mathrm{lfp}(\mathbb{F})(s)+\frac{\epsilon}{\#(e(\vec{s}))} \end{cases}$$

for all histories $\vec{s}$ ending at some state $s\in S_2$, where the function $\#:\mathbb{N}\rightarrow\mathbb{N}$ is defined as in Definition 2.2.9. By previous observations, a strategy σ_2^e always exists. Note how σ_2^e uses the information contained in the history $\vec{s}$ in a simple, but crucial way. Moreover observe that, if $\mathcal{A}$ is finitely branching in S_2, then the positional strategy we discussed earlier, satisfies the above specification for every numbering e defined as above. This allows us to cover the cases of finitely-branching in S_2 arenas and general arenas at the same time.

Let us define, for every numbering e defined as above and for every state $s\in S$, the numbering $s.e$ as follows:

$$s.e(\vec{s}) == \begin{cases} e(s.\vec{s}) & \text{if } s.\vec{s}\in\mathcal{P}^{<\omega} \\ e(\vec{s}) & \text{if } \mathit{first}(\vec{s})\not\in E(s) \end{cases}$$

It is immediate to observe that, for every $\vec{t}\in\mathcal{P}^{<\omega}$, $e(\vec{t})\leq s.e(\vec{t})$. This definition is useful because it allows us to characterize the behavior of a strategy σ_2^e as

$$\sigma_2^e(\{s\}) = \begin{cases} \bullet & \text{only if } E(last(\vec{s}))=\emptyset \\ t & \text{only if } t\in E(s) \text{ and } \mathrm{lfp}(\mathbb{F})(t)<\mathrm{lfp}(\mathbb{F})(s)+\frac{\epsilon}{\#(e(\{s\}))} \end{cases} \tag{6.1}$$

for all histories $\{s\}$ of length 1 with $s\in S_2$, and as $\sigma_2^e(s.\vec{t})=\sigma_2^{s.e}(\vec{t})$ for all other histories $\vec{s}$ with $last(\vec{s})\in S_2$.

We are now going to show that any strategy σ_2^e which follows the above specification for some numbering e, satisfies the desired inequality: for all $s\in S$, $\bigsqcup_{\sigma_1} E(M^s_{\sigma_1,\sigma_2^e}) \leq \mathrm{lfp}(\mathbb{F})(s)+\epsilon$. We do this by proving a stronger property, namely that for all $s\in S$, $\bigsqcup_{\sigma_1} E(M^s_{\sigma_1,\sigma_2^e}) \leq \mathrm{lfp}(\mathbb{F})(s) + \frac{2\cdot\epsilon}{\#(e(\{s\}))}$.

Let us fix an arbitrary strategy σ_1 for Player 1 in $\mathcal{G}$. We need to show that $E(M^s_{\sigma_1,\sigma_2^e})\leq \mathrm{lfp}(\mathbb{F})(s)+\frac{2\cdot\epsilon}{\#(e(\{s\}))}$. Recall that, by definition, $E(M^s_{\sigma_1,\sigma_2^e})=\mathbb{P}^s_{\sigma_1,\sigma_2^e}(\Phi)$, where $\mathbb{P}^s_{\sigma_1,\sigma_2^e}$ denotes the probability measure over $\mathcal{BP}$ induced by the Markov branching play $M^s_{\sigma_1,\sigma_2^e}$, and $\Phi\subseteq\mathcal{BP}$ is the set of winning branching plays for Player 1 in $\mathcal{G}$.

By Theorem 6.1.2, we know that $\Phi=\bigcup_\alpha \mathbb{W}^\alpha_{\mathcal{G}}$. Let us now consider separately the cases when $\mathcal{A}$ is finitely branching in B_2 and when $\mathcal{A}$ is instead general. If $\mathcal{A}$ is finitely branching in B_2, by Lemma 6.1.4 we know that $\Phi=\bigcup_{\alpha<\omega}\mathbb{W}^\alpha_{\mathcal{G}}$. Therefore by ω-continuity of all probability measures, we have that $\mathbb{P}^s_{\sigma_1,\sigma_2^e}(\Phi)=\bigsqcup_{\alpha<\omega}\mathbb{P}^s_{\sigma_1,\sigma_2^e}(\mathbb{W}^\alpha_{\mathcal{G}})$. If instead the arena $\mathcal{A}$ is not finitely branching in B_2, we know, again from Lemma 6.1.4, that $\Phi=\bigcup_{\alpha<\omega_1}\mathbb{W}^\alpha_{\mathcal{G}}$. It thus follows from $\mathrm{MA}_{\aleph_1}$ that $\mathbb{P}^s_{\sigma_1,\sigma_2^e}(\Phi)=\bigsqcup_{\alpha<\omega_1}\mathbb{P}^s_{\sigma_1,\sigma_2^e}(\mathbb{W}^\alpha_{\mathcal{G}})$ (see Theorem 2.1.87 and Proposition 2.1.88). Therefore, in both cases, the equality

$$\mathbb{P}^s_{\sigma_1,\sigma_2^e}(\Phi)=\bigsqcup_{\alpha<\omega_1}\mathbb{P}^s_{\sigma_1,\sigma_2^e}(\mathbb{W}^\alpha_{\mathcal{G}})$$

holds, even if in the general case we need to invoke $\mathrm{MA}_{\aleph_1}$.

This equality allows us to set up a proof, by transfinite induction over ordinals, of the desired inequality $E(M^s_{\sigma_1,\sigma_2^e})\leq\mathrm{lfp}(\mathbb{F})(s)+\frac{2\cdot\epsilon}{\#(e(\{s\}))}$. We are going to show that for every countable ordinal α, the inequality

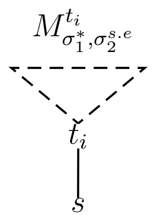
$$\mathbb{P}^s_{\sigma_1,\sigma_2^e}(\mathbb{W}^\alpha_{\mathcal{G}})\leq\mathrm{lfp}(\mathbb{F})(s)+\frac{2\cdot\epsilon}{\#(e(\{s\}))} \tag{6.2}$$

holds. This clearly implies the desired inequality.

Suppose, by induction hypothesis, that the inequality 6.2 holds for every $\beta<\alpha$. Let us consider the possible shapes of the Markov branching play $M^s_{\sigma_1,\sigma_2^e}$.

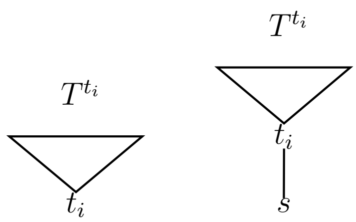
1. If s is a terminal state, i.e., if $E(s)=\emptyset$, then by Convention 5.2.1, s is in the set $S_2\cup B_2$ because $\Pr(s)=p$ is odd. In this case, by definition of $\mathbb{F}$, we have that $\mathbb{F}^\alpha(s)=1$ and the result trivially holds.

2. If $s\in S_1$, then $M^s_{\sigma_1,\sigma_2^e}$ can be depicted as follows:



where t_i is the state chosen by σ_1 at the initial state s, i.e., $t_i=\sigma_1(\{s\})$, and the sub-Markov branching play $M^{t_i}_{\sigma_1^*,\sigma_2^{s.e}}$ of $M^s_{\sigma_1,\sigma_2^e}$ is induced by the strategy $\sigma_2^{s.e}$ (because once reached the state t_i, σ_2^e starts behaving as $\sigma_2^{s.e}$, as observed earlier) and σ_1^*, where the strategy σ_1^* for Player 1 in $\mathcal{G}$ follows the behavior of σ_1 after the state t_i is reached, and is defined as follows: $\sigma_1^*(\vec{s})=\sigma_1(s.\vec{s})$, if the first state of $\vec{s}$ is t_i.

The Markov branching plays $M^{t_i}_{\sigma_1^*,\sigma_2^{s.e}}$ and $M^s_{\sigma_1,\sigma_2^e}$ induce the probability measures $\mathbb{P}^{t_i}_{\sigma_1^*,\sigma_2^{s.e}}$ and $\mathbb{P}^s_{\sigma_1,\sigma_2^e}$ on the set of branching plays $\mathcal{BP}$, respectively. We can restrict attention, when considering $\mathbb{P}^{t_i}_{\sigma_1^*,\sigma_2^{s.e}}$, to the set $\mathcal{BP}_{t_i}$ of branching plays rooted at t_i, since all other sets of branching plays get assigned measure 0 by $\mathbb{P}^{t_i}_{\sigma_1^*,\sigma_2^{s.e}}$. Similarly, we can restrict, when considering $\mathbb{P}^s_{\sigma_1,\sigma_2^e}$, to the set $\mathcal{BP}_{s.t_i}$ of branching plays rooted at s and having t_i as unique child of s. These two classes of branching plays can be depicted, respectively, as follows:



where we use T^{t_i} to range over $\mathcal{BP}_{t_i}$ and just T to range over $\mathcal{BP}_{s.t_i}$. It follows from the definition of $\mathbb{W}_\mathcal{G}$, that $T\in\mathbb{W}^\alpha_\mathcal{G}$ if and only if $T^{t_i}\in\mathbb{W}^\beta_\mathcal{G}$, for some $\beta<\alpha$, with T and T^{t_i} as depicted above. From this observation we have that the following equality holds:

$$\mathbb{P}^{s}_{\sigma_1,\sigma_2^e}(\mathbb{W}^{\alpha}_{\mathcal{G}}) = \mathbb{P}^{t_i}_{\sigma_1^*,\sigma_2^{s.e}}\big(\bigcup_{\beta<\alpha}\mathbb{W}^{\beta}_{\mathcal{G}}\big).$$

By induction hypothesis, we have that $\mathbb{P}^{t_i}_{\sigma_1^*,\sigma_2^{s.e}}\big(\mathbb{W}^{\beta}_{\mathcal{G}}\big) \leq \mathrm{lfp}(\mathbb{F})(t_i) + \frac{2\cdot\epsilon}{\#(s.e(\{t_i\}))}$, for all $\beta<\alpha$. Moreover it follows from the definition of $s.e$ that $s.e(\{t_i\}) = e(s.\{t_i\})$, and by our definition of numbering we have that $e(\{s\}.\{t_i\}) > e(\{s\})$. Therefore $\#(s.e(\{t_i\})) > \#(e(\{s\}))$. In then follows that the inequality

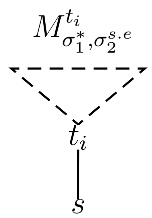
$$\mathbb{P}^{t_i}_{\sigma_1^*,\sigma_2^{s.e}}\big(\bigcup_{\beta<\alpha}\mathbb{W}^{\beta}_{\mathcal{G}}\big) \leq \mathrm{lfp}(\mathbb{F})(t_i) + \frac{2\cdot\epsilon}{\#(e(\{s\}))}$$

holds. Since by definition of $\mathbb{F}$, $\mathrm{lfp}(\mathbb{F})(t_i) \leq \mathrm{lfp}(\mathbb{F})(s)$, we have that the desired inequality 6.2

$$\mathbb{P}^{s}_{\sigma_1,\sigma_2^e}(\mathbb{W}^{\alpha}_{\mathcal{G}}) \leq \mathrm{lfp}(\mathbb{F})(s) + \frac{2\cdot\epsilon}{\#(e(\{s\}))}$$

holds.

3. If $s \in S_2$, then $M^{s}_{\sigma_1,\sigma_2^e}$ can be depicted as follows:



where t_i is the state chosen by σ_2^e at the initial state s (and this implies, by Equation 6.1, that $\mathrm{lfp}(\mathbb{F})(t_i) \leq \mathrm{lfp}(\mathbb{F})(s) + \frac{\epsilon}{\#(e(\{s\}))}$) and the sub-Markov branching play $M^{t_i}_{\sigma_1^*,\sigma_2^{s.e}}$ of $M^{s}_{\sigma_1,\sigma_2^e}$ is induced by the strategy $\sigma_2^{s.e}$ (because once the state t_i is reached, σ_2^e starts behaving as $\sigma_2^{s.e}$, as observed earlier) and σ_1^*, where the strategy σ_1^* for Player 1 in $\mathcal{G}$ follows the behavior of σ_1 after the state t_i is reached, and is defined as follows: $\sigma_1^*(\vec{s}) = \sigma_1(s.\vec{s})$ if the first state of $\vec{s}$ is t_i. By arguments similar to the ones used in the previous case, we have that the equality

$$\mathbb{P}^{s}_{\sigma_1,\sigma_2^e}(\mathbb{W}^{\alpha}_{\mathcal{G}}) = \mathbb{P}^{t_i}_{\sigma_1^*,\sigma_2^{s.e}}\big(\bigcup_{\beta<\alpha}\mathbb{W}^{\beta}_{\mathcal{G}}\big).$$

holds and from the induction hypothesis the inequality

$$\mathbb{P}^{t_i}_{\sigma_1^*,\sigma_2^{s.e}}\big(\bigcup_{\beta<\alpha}\mathbb{W}^{\beta}_{\mathcal{G}}\big) \leq \mathrm{lfp}(\mathbb{F})(t_i) + \frac{2\cdot\epsilon}{\#(s.e(\{t_i\}))}$$

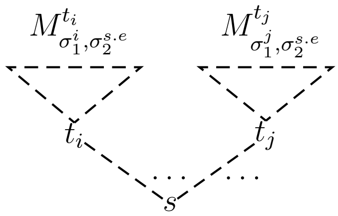
holds as well. Since, as observed above, by definition of σ_2^e we have that $\text{lfp}(\mathbb{F})(t_i) \leq \text{lfp}(\mathbb{F})(s) + \frac{\epsilon}{\#(e(\{s\}))}$, the inequality

$$\mathbb{P}^s_{\sigma_1,\sigma_2^e}(\mathbb{W}^\alpha_\mathcal{G}) \leq \big(\text{lfp}(\mathbb{F})(s) + \tfrac{\epsilon}{\#(e(\{s\}))}\big) + \tfrac{2\cdot\epsilon}{\#(s.e(\{t_i\}))}$$

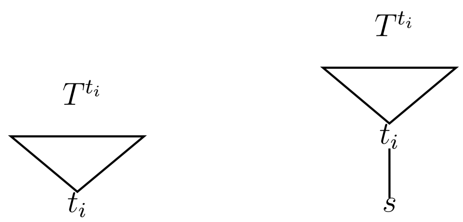
holds. It follows easily from Definition 2.2.9 of $\# : \mathbb{N} \to \mathbb{N}$, $\#(n) = 2^{2^n+1}$, and from the fact that $s.e(\{t_i\}) > e(\{s\})$, that the inequality $\frac{2\cdot\epsilon}{\#(s.e(\{t_i\}))} \leq \frac{\epsilon}{\#(e(\{s\}))}$ holds. Therefore we have the desired inequality 6.2

$$\mathbb{P}^s_{\sigma_1,\sigma_2^e}(\mathbb{W}^\alpha_\mathcal{G}) \leq \big(\text{lfp}(\mathbb{F})(s) + \tfrac{\epsilon}{\#(e(\{s\}))}\big) + \tfrac{\epsilon}{\#(e(\{s\}))}.$$

4. If $s \in S_N$, then $M^s_{\sigma_1,\sigma_2^e}$ can be depicted as follows:



where $E(s) = \{t_i\}_{i\in I}$, and the sub-Markov branching play $M^{t_i}_{\sigma_1^i,\sigma_2^{s.e}}$ of $M^s_{\sigma_1,\sigma_2^e}$, for $i \in I$, is induced by the strategy $\sigma_2^{s.e}$ (because once the state t_i is reached, for any $i \in I$, σ_2^e starts behaving as $\sigma_2^{s.e}$, as observed earlier) and σ_1^i, where the strategy σ_1^i for Player 1 in $\mathcal{G}$ follows the behavior of σ_1 after the state t_i is reached, and is defined as: $\sigma_1^i(\vec{s}) = \sigma_1(s.\vec{s})$ if the first state of $\vec{s}$ is t_i. The Markov branching plays $M^{t_i}_{\sigma_1^i,\sigma_2^{s.e}}$, for $i \in I$, and $M^s_{\sigma_1,\sigma_2^e}$, induce the probability measures $\mathbb{P}^{t_i}_{\sigma_1^i,\sigma_2^{s.e}}$ and $\mathbb{P}^s_{\sigma_1,\sigma_2^e}$ on the set of branching plays $\mathcal{BP}$, respectively. We can restrict attention, when considering $\mathbb{P}^{t_i}_{\sigma_1^i,\sigma_2^{s.e}}$, for $i \in I$, to the set $\mathcal{BP}_{t_i}$ of branching plays rooted at t_i. Similarly, we can restrict, when considering $\mathbb{P}^s_{\sigma_1,\sigma_2^e}$, to the set $\cup_{i\in I}\mathcal{BP}_{s.t_i}$, where $\mathcal{BP}_{s.t_i}$ denotes the set of branching plays rooted at s and having t_i as unique child of s. These classes of branching plays can be depicted, respectively, as follows:



We use T^{t_i} to range over $\mathcal{BP}_{t_i}$ and $T^{s.t_i}$ to range over $\mathcal{BP}_{s.t_i}$. It follows from the definition of $\mathbb{W}_{\mathcal{G}}$, that $T^{s.t_i} \in \mathbb{W}^{\alpha}_{\mathcal{G}}$ if and only if $T^{t_i} \in \mathbb{W}^{\beta}_{\mathcal{G}}$, for some $\beta < \alpha$, with $T^{s.t_i}$ and T^{t_i} as depicted above. From this observation, and from the fact that $\mathbb{P}^s_{\sigma_1,\sigma_2^e}(\mathcal{BP}_{s.t_i}) = \pi(s)(t_i)$ by Definition 4.1.3 of $\mathbb{P}^s_{\sigma_1,\sigma_2^e}$, we have that the following equality holds:

$$\mathbb{P}^s_{\sigma_1,\sigma_2^e}(\mathbb{W}^{\alpha}_{\mathcal{G}} \cap \mathcal{BP}_{s.t_i}) = \pi(s)(t_i) \cdot \mathbb{P}^{t_i}_{\sigma_1^i,\sigma_2^{s.e}}\big(\textstyle\bigcup_{\beta<\alpha} \mathbb{W}^{\beta}_{\mathcal{G}}\big).$$

From this, and from the fact that for all $i, j \in I$, $\mathcal{BP}_{s.t_i} \cap \mathcal{BP}_{s.t_j} = \emptyset$, we have that the following equality holds:

$$\mathbb{P}^s_{\sigma_1,\sigma_2^e}(\mathbb{W}^{\alpha}_{\mathcal{G}}) = \sum_{i\in I} \pi(s)(t_i) \cdot \mathbb{P}^{t_i}_{\sigma_1^i,\sigma_2^{s.e}}\big(\bigcup_{\beta<\alpha} \mathbb{W}^{\beta}_{\mathcal{G}}\big).$$

By induction hypothesis, we know that for each $i \in I$, the inequality

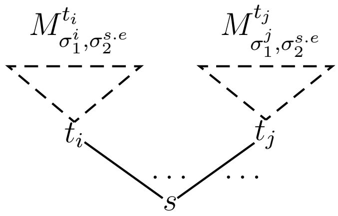
$$\mathbb{P}^{t_i}_{\sigma_1^i,\sigma_2^{s.e}}\big(\textstyle\bigcup_{\beta<\alpha} \mathbb{W}^{\beta}_{\mathcal{G}}\big) \leq \mathrm{lfp}(\mathbb{F})(t_i) + \frac{2\cdot\epsilon}{\#(s.e(\{t_i\}))}$$

holds. We can then derive the desired inequality 6.2 as follows:

$$\begin{aligned}
\mathbb{P}^s_{\sigma_1,\sigma_2^e}(\mathbb{W}^{\alpha}_{\mathcal{G}}) &\leq \sum_{i\in I} \pi(s)(t_i) \cdot \big(\mathrm{lfp}(\mathbb{F})(t_i) + \frac{2\cdot\epsilon}{\#(s.e(\{t_i\}))}\big) \\
&= \sum_{i\in I} \big(\pi(s)(t_i) \cdot \mathrm{lfp}(\mathbb{F})(t_i)\big) + \sum_{i\in I} \big(\pi(s)(t_i) \cdot \frac{2\cdot\epsilon}{\#(s.e(\{t_i\}))}\big) \\
&= \mathrm{lfp}(\mathbb{F})(s) + \sum_{i\in I} \big(\pi(s)(t_i) \cdot \frac{2\cdot\epsilon}{\#(s.e(\{t_i\}))}\big) \\
&\leq \mathrm{lfp}(\mathbb{F})(s) + \sum_{i\in I} \big(\pi(s)(t_i) \cdot \frac{2\cdot\epsilon}{\#(e(\{s\}))}\big) \\
&= \mathrm{lfp}(\mathbb{F})(s) + \tfrac{2\cdot\epsilon}{\#(e(\{s\}))}
\end{aligned}$$

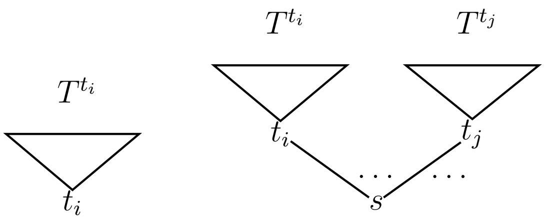
The third equality follows from the definition of $\mathbb{F}$. The fourth inequality follows by observing that $\frac{2\cdot\epsilon}{\#(s.e(\{t_i\}))} \leq \frac{2\cdot\epsilon}{\#(e(\{s\}))}$ holds because $s.e(\{t_i\}) \geq e(\{s\})$. The last equality follows from the fact that $\sum_{i\in I} \pi(s)(t_i) = 1$ and $0 \leq \pi(s)(t_i) \leq 1$.

5. The cases for $s \in B_1 \cup B_2$ are treated in a similar way. We just consider the case $s \in B_1$. In this case $M^s_{\sigma_1,\sigma_2^e}$ can be depicted as follows:



where $E(s)=\{t_i\}_{i\in I}$, and the sub-Markov branching play $M^{t_i}_{\sigma_1^i,\sigma_2^{s.e}}$ of $M^s_{\sigma_1,\sigma_2^e}$, for $i\in I$, is induced by the strategy $\sigma_2^{s.e}$ (because once the state t_i is reached, σ_2^e starts behaving as $\sigma_2^{s.e}$, as observed earlier) and σ_1^i, where the strategy σ_1^i for Player 1 in $\mathcal{G}$ follows the behavior of σ_1 after the state t_i is reached, and is defined as follows: $\sigma_1^i(\vec{s})=\sigma_1(s.\vec{s})$, for all histories $\vec{s}$ starting at t_i.

The Markov branching plays $M^{t_i}_{\sigma_1^i,\sigma_2^{s.e}}$, for $i \in I$, and $M^s_{\sigma_1,\sigma_2^e}$, induce the probability measures $\mathbb{P}^{t_i}_{\sigma_1^i,\sigma_2^{s.e}}$ and $\mathbb{P}^s_{\sigma_1,\sigma_2^e}$ on the set of branching plays $\mathcal{BP}$, respectively. We can restrict attention, when considering $\mathbb{P}^{t_i}_{\sigma_1^i,\sigma_2^{s.e}}$ to the set $\mathcal{BP}_{t_i}$ of branching plays rooted at t_i. Similarly, we can restrict, when considering $\mathbb{P}^s_{\sigma_1,\sigma_2^e}$, to the set $\mathcal{BP}_s$, where $\mathcal{BP}_s$ denotes the set of branching plays rooted at s. These classes of branching plays can be depicted, respectively, as follows:



We use T^{t_i} to range over $\mathcal{BP}_{t_i}$ and T^s to range over $\mathcal{BP}_s$. Since $s\in B_1$ it follows, from the definition of $\mathbb{W}_{\mathcal{G}}$, that $T^s\in\mathbb{W}^{\alpha}_{\mathcal{G}}$ if and only if there exists an $i\in I$ such that $T^i\in\mathbb{W}^{\beta}_{\mathcal{G}}$, for some $\beta<\alpha$, with T^s and T^{t_i} as depicted above. From this observation, and by adapting the result of Lemma 5.2.9, we have that the following equality holds:

$$\mathbb{P}^s_{\sigma_1,\sigma_2^e}(\mathbb{W}^{\alpha}_{\mathcal{G}}) \;=\; \coprod_{i\in I}\mathbb{P}^{t_i}_{\sigma_1^i,\sigma_2^{s.e}}\big(\bigcup_{\beta<\alpha}\mathbb{W}^{\beta}_{\mathcal{G}}\big).$$

By induction hypothesis we know that, for each $i\in I$, the following equality holds:

$$\mathbb{P}^{t_i}_{\sigma_1^i,\sigma_2^{s.e}}\big(\bigcup_{\beta<\alpha}\mathbb{W}^{\beta}_{\mathcal{G}}\big) \;\leq\; \mathrm{lfp}(\mathbb{F})(t_i)+\frac{2\cdot\epsilon}{\#(s.e(\{t_i\}))}.$$

Therefore the following inequality holds:

$$\mathbb{P}^{s}_{\sigma_1,\sigma_2^e}(\mathbb{W}^{\alpha}_{\mathcal{G}}) \;\leq\; \coprod_{i\in I}\big(\mathrm{lfp}(\mathbb{F})(t_i) + \frac{2\cdot\epsilon}{\#(s.e(\{t_i\}))}\big).$$

Let n_i be the (positive) natural number $s.e(\{t_i\}) - e(\{s\})$. Note that, since e and $s.e$ are injective by definition, for every $i \neq j$, the inequality $n_i \neq n_j$ holds. By definition of the function $\# \colon \mathbb{N} \to \mathbb{N}$, specified as $\#(n) = 2^{2^n+1}$, it is simple to prove that $\#(n+m) \geq \#(n)\cdot\#(m)$, for every pair of positive naturals n and m. Therefore we have that $\#(s.e(\{t_i\})) \geq \#(e(\{s\}))\cdot\#(n_i)$. As a consequence, the following inequality holds:

$$\mathbb{P}^{s}_{\sigma_1,\sigma_2^e}(\mathbb{W}^{\alpha}_{\mathcal{G}}) \;\leq\; \coprod_{i\in I}\big(\mathrm{lfp}(\mathbb{F})(t_i) + \frac{2\cdot\epsilon}{\#(e(\{s\}))}\cdot\frac{1}{\#(n_i)}\big)$$

By application of Lemma 2.2.11, we then have that the desired inequality 6.2

$$\begin{aligned}\mathbb{P}^{s}_{\sigma_1,\sigma_2^e}(\mathbb{W}^{\alpha}_{\mathcal{G}}) \;&\leq\; \coprod_{i\in I}\big(\mathrm{lfp}(\mathbb{F})(t_i)\big) + \frac{2\cdot\epsilon}{\#(e(\{s\}))}\\ &=\; \mathrm{lfp}(\mathbb{F})(s) + \tfrac{2\cdot\epsilon}{\#(e(\{s\}))}\end{aligned}$$

holds.

□

As an immediate corollary of Theorem 6.1.7 and Theorem 5.2.10 we have that all $2\frac{1}{2}$-player meta-parity games with just one odd priority are determined under deterministic strategies.

Corollary 6.1.8. *Let, $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}}\rangle$ be a $2\frac{1}{2}$-player meta-parity game, with $\mathcal{A} = \langle (S,E),(S_1,S_2,S_N,B_1,B_2),\pi\rangle$, having just one odd priority, i.e., such that $\mathrm{Pr}(S) = \{p\}$ for some odd $p \in \mathbb{N}$. Then the following assertions hold*

$$\begin{aligned}&\mathrm{ZFC} && \vdash\; Val^s_{\downarrow}(\mathcal{G}) = Val^s_{\uparrow}(\mathcal{G}),\ \textit{if } \mathcal{G} \textit{ is finitely branching in } B_2\\ &\mathrm{ZFC}\ + \mathrm{MA}_{\aleph_1} && \vdash\; Val^s_{\downarrow}(\mathcal{G}) = Val^s_{\uparrow}(\mathcal{G})\end{aligned}$$

for every $s \in S$.

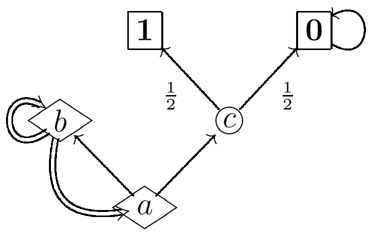
Proof. Both $Val^s_{\downarrow}(\mathcal{G})$ and $Val^s_{\uparrow}(\mathcal{G})$ are fixed points of $\mathbb{F}_{\mathcal{G}}$ (see Theorem 5.2.10). □

Therefore we just write $\mathsf{VAL}(\mathcal{G})$ to denote the lower and upper values of $\mathcal{G}$. We will always assume the set-theoretic assumption $\mathrm{MA}_{\aleph_1}$ when using the value $\mathsf{VAL}(\mathcal{G})$ of a $\mathcal{G}$ having arena $\mathcal{A}$ not finitely branching in B_2.

An important aspect of Theorem 6.1.7 is the existence of optimal positional strategies (see Definition 2.3.45) for Player 2 in every $2\frac{1}{2}$-player meta-parity game with one odd priority and finitely branching in S_2. It is then interesting to seek the weakest conditions under which also Player 1 is guaranteed to have a optimal positional strategy. Unfortunately, as our next lemma shows, these conditions are necessarily very strong.

Lemma 6.1.9. *There exists a $2\frac{1}{2}$-player meta-parity game $\mathcal{G} = \langle \mathcal{A}, \mathrm{Pr} \rangle$ with one odd priority and finite arena $\mathcal{A}$, such that Player 1 does not have an optimal positional strategy.*

Proof. We provide a concrete example. Let $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ which can be depicted as follows:



where $\mathrm{Pr}(s) = 1$ for all states s. In the game $\mathcal{G}$ there are only two positional strategies σ_1^b and σ_1^c available to Player 1: the strategy σ_1^b chooses to move always from the state a to the state b, i.e., $\sigma_1(\vec{s}) = b$, for every history $\vec{s}$ with $last(\vec{s}) = a$; the strategy σ_1^c chooses instead to move from the state a to the state c, i.e., $\sigma_1(\vec{s}) = c$, for every history $\vec{s}$ with $last(\vec{s}) = a$. We leave the reader to verify that $E(M^a_{\sigma_1^b}) = 0$ and $E(M^a_{\sigma_1^c}) = \frac{1}{2}$, i.e., the expected values of the game starting at a when Player 1 plays in accordance with the strategies σ_1^c and σ_1^b are 0 and $\frac{1}{2}$ respectively. However $\mathsf{VAL}^a(\mathcal{G}) = 1$. To see this, by application of Theorem 6.1.7 it is enough to verify that $\mathrm{lfp}(\mathbb{F}_\mathcal{G})(a) = 1$. We just show that $\mathrm{lfp}(\mathbb{F}_\mathcal{G})(a) = \bigsqcup_\alpha \mathbb{F}^\alpha_\mathcal{G}(a) > \frac{1}{2}$ as this is sufficient for the desired result and simple to verify with a few iterations of the functional $\mathbb{F}_\mathcal{G}$: $\mathbb{F}^1_\mathcal{G}(c) = \frac{1}{2}$, $\mathbb{F}^2_\mathcal{G}(a) = \frac{1}{2}$, $\mathbb{F}^3_\mathcal{G}(b) = \frac{1}{2}$, $\mathbb{F}^4_\mathcal{G}(b) = \frac{1}{2} \odot \frac{1}{2} = \frac{3}{4}$ and lastly $\mathbb{F}^5_\mathcal{G}(a) = \frac{3}{4}$.

An optimal strategy for Player 1 in $\mathcal{G}$ requires at least on bit of information. Define σ_1 as $\sigma_1(\{s\}) = b$ and $\sigma_{\vec{s}.s} = c$ for all $\vec{s}$ containing at least one occurrence of

s. In other words, the strategy σ_1 first chooses to move to b, thus generating an infinite sequence of subplays continuing their execution from s, and then always chooses to move to c. Each subplay starting at c ends up in a win for Player 1 with probability $\frac{1}{2}$. Thus Player 1 wins the game almost surely when following the strategy σ_1, because $b \in B_1$ is a state under the control of Player 1 in the inner-game. $\square$

Although positional strategies for Player 1 do not exist in general, we leave open the following interesting problem.

Question 6.1.10. Is there a function $m : \mathbb{N} \to \mathbb{N}$ such that, for every $2\frac{1}{2}$-player meta-parity game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ with finite $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B_1, B_2), \pi \rangle$ and just one odd priority, Player 1 has an optimal strategy requiring only $f(|S \uplus E|)$ bits of memory?

We conclude this section by considering the class of $2\frac{1}{2}$-player meta-parity games with just one *even* priority. By applications of Lemma 5.1.16 and Lemma 2.3.57, the duals of the results concerning $2\frac{1}{2}$-player meta-parity games with just one odd priority hold, if an even priority is considered instead. We summarize in the following lemma the interesting results about $2\frac{1}{2}$-player meta-parity games with just one even priority.

Lemma 6.1.11. *Let $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ be a $2\frac{1}{2}$-player meta-parity game with just one even priority, i.e., such that $|\,\mathrm{Pr}\,| = 1$ and $\max(\mathrm{Pr}) = \min(\mathrm{Pr})$ is even. Then the following assertions holds:*

1. *the winning set Φ_{Pr} of $\mathcal{G}$ can be characterized as the greatest fixed point of the monotone operator $\mathbb{W}_{\mathcal{G}}$ (see Definition 5.2.5), i.e., $\Phi_{\mathrm{Pr}} = \mathrm{gfp}(\mathbb{W}_{\mathcal{G}})$,*

2. *if $\mathcal{A}$ is finitely branching in B_1, then*

   - $\mathrm{gfp}(\mathbb{W}_{\mathcal{G}}) = \bigcap_{\beta<\omega} \mathbb{W}_{\mathcal{G}}{}^{\beta}$, *with* $\mathbb{W}_{\mathcal{G}}{}^{\alpha} = \mathbb{W}_{\mathcal{G}}(\bigcap_{\gamma<\alpha} \mathbb{W}^{\gamma}_{\mathcal{G}})$, *for α, β and γ ordinals.*
   - Φ_{Pr} *is a closed set.*
   - $\mathrm{ZFC} \vdash \ \mathsf{VAL}^{s}_{\downarrow}(\mathcal{G}) = \mathsf{VAL}^{s}_{\uparrow}(\mathcal{G}) = \mathrm{gfp}(\mathbb{F}_{\mathcal{G}})(s)$.

3. *if $\mathcal{A}$ is not finitely branching in B_1, then*

   - $\mathrm{gfp}(\mathbb{W}_{\mathcal{G}}) = \bigcap_{\beta<\omega_1} \mathbb{W}^{\beta}_{\mathcal{G}}$,

- Φ_{Pr} *is, in general, not a Borel set.*
- $\mathrm{ZFC} \ + \mathrm{MA}_{\aleph_1} \vdash \ \mathsf{VAL}^{s}_{\downarrow}(\mathcal{G}) = \mathsf{VAL}^{s}_{\uparrow}(\mathcal{G}) = \mathrm{gfp}(\mathbb{F}_{\mathcal{G}})(s).$

4. If $\mathcal{A}$ is finitely branching in S_1 then Player 1 has an optimal strategy in $\mathcal{G}$,

where the the function $\mathrm{gfp}(\mathbb{F}_{\mathcal{G}})$ denotes the greatest fixed point of the functional $\mathbb{F}_{\mathcal{G}}$ specified as in Theorem 5.2.10.

Proof. The main observation is that $\overline{\Phi_{\mathrm{Pr}}} = \Phi_{\neg \mathrm{Pr}}$, where the assignment $\neg \mathrm{Pr}$, specified as in Definition 2.3.57, assigns odd priority to all states in S. The result follows by routine application of Theorem 5.1.16. □

6.2 Unfolding of $2\frac{1}{2}$-player meta-parity games

In the previous section we proved that any two player stochastic meta-parity game with just one priority is determined under deterministic strategies. In this section we develop the technical machinery which shall allow us, in section 6.3 and 6.4, to prove that every $2\frac{1}{2}$-player meta-parity game with $N+1$ priorities, for $N > 0$, is determined.

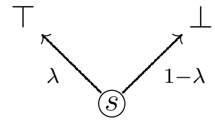
We introduce the following convention on $2\frac{1}{2}$-player meta-parity games to which we shall adhere in the rest of this chapter.

Convention 6.2.1. Let $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ be a two player stochastic meta-parity game, and let $p = \min(\mathrm{Pr})$. The game $\mathcal{G}$ is such that:

1. There is a designated terminal state s (i.e., $E(s) = \emptyset$) in $\mathcal{A}$ and $\mathrm{Pr}(s) = p$. The state s is denoted by $\top$ if p is odd and by $\bot$ if p is even.
2. There is a designated self-loop state t (i.e., $E(t) = \{t\}$) in $\mathcal{A}$ and $\mathrm{Pr}(s) = p$. The state s is denoted by $\top$ if p is even and by $\bot$ if p is odd.

Clearly every $2\frac{1}{2}$-player meta-parity game can be transformed into one satisfying the above convention by adding, if necessary, two new states. Note that the states $\top$ and $\bot$ are winning states for Player 1 and Player 2 respectively, i.e., $\mathsf{VAL}^{\top}_{\downarrow}(\mathcal{G}) = \mathsf{VAL}^{\top}_{\uparrow}(\mathcal{G}) = 1$ and $\mathsf{VAL}^{\bot}_{\downarrow}(\mathcal{G}) = \mathsf{VAL}^{\bot}_{\uparrow}(\mathcal{G}) = 0$, for every $2\frac{1}{2}$-player meta-parity game $\mathcal{G}$.

In $2\frac{1}{2}$-player meta-parity games the payoff function is a winning set, i.e., a function $\Phi : \mathcal{BP} \rightarrow \{0, 1\}$. However, consider any state s in a $2\frac{1}{2}$-player meta-parity game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ which can be depicted as follows:



At the state s the game $\mathcal{G}$ ends in favor of Player 1 and Player 2 with probability λ and $1-\lambda$ respectively. We now introduce some notation for game states of this form.

Definition 6.2.2. Let $\mathcal{G}=\langle\mathcal{A},\Phi_{\mathrm{Pr}}\rangle$ be a two player stochastic meta-parity game, with $\mathcal{A}=\langle(S,E),(S_1,S_2,S_N,B_1,B_2),\pi\rangle$. A state $s\in S_N$ is called a λ-*valued leaf*, for $\lambda\in[0,1]$, if:

- $\mathrm{Pr}(s)=\min(\mathrm{Pr})$,
- $E(s)=\{\top,\bot\}$,
- $\pi(s)(\top)=\lambda$ and $\pi(s)(\bot)=1-\lambda$.

Note that, by definition of tree game arena (see Section 4.1), the set $supp(\pi(s))$ is just required to be a subset of $E(s)$. When $\lambda\in\{0,1\}$, we have indeed a strict inclusion $supp(\pi(s))\subsetneq E(s)$. In this case one of the two states $\{\top,\bot\}$ is not reachable in any play in $\mathcal{G}$, because Nature will never choose it. Nevertheless it is going to be technically convenient to work with $E(s)=\{\top,\bot\}$, i.e., including an edge labeled with probability 0 in the graph structure. We shall remark the advantages of this technical assumption when we use it. Another useful observation is that, for every λ-valued leaf s, there is only one Markov branching play in $\mathcal{G}$ rooted at s. We denote it with M^s_λ. Lastly, the priority assigned to the state s is not important, because once the game reaches the λ-valued leaf s, it progresses to the (morally) terminal states $\top$ and $\bot$ with probability λ and $1-\lambda$ respectively. The choice of assigning minimal priority to λ-valued leaf is arbitrary and technically convenient.

It is clear that $[0,1]$-valued leaves can be used to encode simple forms of $[0,1]$-valued payoffs for $2\frac{1}{2}$-player meta-parity games. Given a $2\frac{1}{2}$-player meta-parity game $\mathcal{G}$ and a terminal state s in $\mathcal{G}$, one can *simulate* a quantitative reward $\lambda\in[0,1]$ for Player 1 when the state s is reached, by changing the game structure of $\mathcal{G}$ and making s a λ-valued leaf[3]. Note that turning a terminal state into a

[3]This fact can be seen as a game-counterpart of Definition 3.3.5, where we used the operator $+_\lambda$ to define the formula $\underline{\lambda}=\overline{1}+_\lambda\overline{0}$ whose semantics is the constant function mapping process states to λ (see Proposition 3.3.6).

λ-valued leaf does not increase, and possibly decreases, the number of priorities used in the game.

Definition 6.2.3 (Leaf monotonicity)**.** Let $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ be a $2\frac{1}{2}$-player meta-parity game with $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B_1, B_2), \pi \rangle$. Let $\{s_n\}_{n \in \mathbb{N}}$ be a collection of states in $\mathcal{G}$ such that s_n is a λ_n-valued leaf, for $\lambda_n \in [0, 1]$. Let $\mathcal{G}'$ be the $2\frac{1}{2}$-player meta-parity game obtained from $\mathcal{G}$ by turning the state s_n into a γ_n-valued leaf, for $\gamma_n \in [0, 1]$ and $n \in \mathbb{N}$. We say that $\mathcal{G}$ is *leaf monotone* if the following implication holds:

$$\forall n.\ (\lambda_n \leq \gamma_n) \implies \mathsf{VAL}^t_{\downarrow}(\mathcal{G}) \leq \mathsf{VAL}^t_{\downarrow}(\mathcal{G}') \text{ and } \mathsf{VAL}^t_{\uparrow}(\mathcal{G}) \leq \mathsf{VAL}^t_{\uparrow}(\mathcal{G}')$$

for every $t \in S$.

Lemma 6.2.4. *Let $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ be a $2\frac{1}{2}$-player meta-parity game with just one priority $p = \min(\mathrm{Pr})$. Then the following assertions hold:*

$$\begin{array}{lcl} \mathrm{ZFC} & \vdash & \textit{If } \mathcal{A} \textit{ is finitely branching in } B_i \textit{ then} \\ & & \mathcal{G} \textit{ satisfies the leaf monotonicity property.} \\ \mathrm{ZFC} + \mathrm{MA}_{\aleph_1} & \vdash & \mathcal{G} \textit{ satisfies the leaf monotonicity property.} \end{array}$$

where $i = 2$ if p is odd, and $i = 1$ otherwise.

Proof. We just consider the case when $\min(\mathrm{Pr})$ is odd. The other case follows by duality. By Theorem 6.1.7 we know that the equalities $\mathsf{VAL}^t(\mathcal{G}) = \bigsqcup_\alpha \mathbb{F}^\alpha_{\mathcal{G}}(t)$ and $\mathsf{VAL}^t(\mathcal{G}') = \bigsqcup_\alpha \mathbb{F}^\alpha_{\mathcal{G}'}(t)$ hold. The desired result can be proved by showing that $\mathbb{F}^\alpha_{\mathcal{G}}(t) \leq \mathbb{F}^\alpha_{\mathcal{G}'}(t)$, for every $t \in S$, by transfinite induction on the ordinals, where the interesting case is for $\alpha = 1$. ☐

The technique we adopt in the next sections for proving that every $2\frac{1}{2}$-player meta-parity game $\mathcal{G}$ with $N+1$ priorities is determined under deterministic strategies is to reduce the problem to the determinacy of a $2\frac{1}{2}$-player meta-parity game $\mathcal{G}'$ having N priorities. In the rest of this section we describe the construction of the game $\mathcal{G}'$.

Let us fix an arbitrary $2\frac{1}{2}$-player meta-parity game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ with $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B_1, B_2), \pi \rangle$ and $|\mathrm{Pr}| = N + 1$, for some $N > 0$. In the rest of this section we denote with $\mathcal{P}^{<\omega}$, $\mathcal{P}$, $\mathcal{BP}$, $\mathcal{MBP}$ and Φ the sets of finite paths, completed paths, branching plays, Markov branching plays and the winning set of the game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ respectively. Let $\mathrm{Pr}(S) = \{p_1, \ldots, p_{N+1}\}$ be the ordered list of priorities assigned to states in $\mathcal{A}$ by Pr, i.e., such that $p_i < p_j$ for all

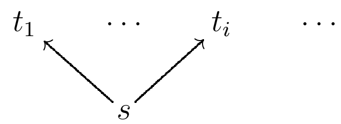
$1 \leq i < j \leq N+1$. Let $S_{\max}$ be the set of states in $\mathcal{A}$ which get assigned maximal priority by Pr, i.e., the set $S_{\max} = \mathrm{Pr}^{-1}(p_{N+1})$.

Remark 6.2.5. Note that, by Convention 6.2.1 and Definition 6.2.2, the states $\top$ and $\bot$ and the λ-valued leaves can not be states in $S_{\max}$, because they are assigned minimal priority by Pr.

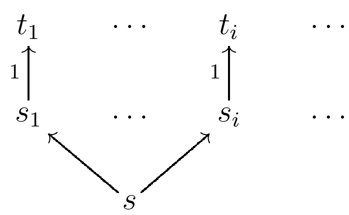
Convention 6.2.6. In what follows we will also assume that every state $s \in S_{\max}$ in $\mathcal{A}$ is in S_N (hence not terminal) and has a unique successor state. This convention, which allows a slightly simpler and more transparent proof, gives no loss of generality. Suppose indeed that the game arena $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B_1, B_2), \pi \rangle$ of $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ is not in this form. Then we can consider the game $\mathcal{G}' = \langle \mathcal{A}', \Phi_{\mathrm{Pr}'} \rangle$, obtained from $\mathcal{G}$ by the following operations:

1. if $s \in S_{\max}$ and $E(s) = \emptyset$, then s is a terminal state. If p_{N+1} is odd then s is a winning state for Player 1 in $\mathcal{G}$. In this case turn s into a probabilistic state in $\mathcal{A}'$ and add one edge (carrying probability 1) from s to the state $\top$. Similarly, if p_{N+1} is even, turn s into a probabilistic state in $\mathcal{A}'$ and add one edge (carrying probability 1) from s to the state $\bot$.

2. if $s \in S_{\max}$ and $E(s) = \{t_i\}_{i \in I}$ with $|E(s)| > 1$, then we could depict this state as follows:



   Add I-many new states $\{s_i\}_{i \in I}$ in $\mathcal{A}'$ and define $E'(s) = \{s_i\}_{i \in I}$, and for each $i \in I$, $E'(s_i) = \{t_i\}$. Moreover we set $s_i \in S'_N$ for every $i \in I$. Thus we can depict s in $\mathcal{A}'$ as follows:



Lastly, we define $\Pr'(s)=p_N$, and $\Pr'(s_i)=p_{N+1}$, for every $i\in I$. This change does not affect the game in any way: the step from s to t_i is simulated by the two steps from s to s_i and from s_i to t_i, and each time the game reaches the state s, then a state of maximal priority p_{N+1} is visited, when the game progresses to s_i, for some $i\in I$.

It is clear that $\mathcal{G}'$ is in the desired form. Moreover it is straightforward to verify that the sets $\mathcal{P}$, $\mathcal{BP}$ and $\mathcal{MBP}$ in $\mathcal{G}$ are homeomorphic to the sets $\mathcal{P}'$, $\mathcal{BP}'$ and $\mathcal{MBP}'$ in $\mathcal{G}'$ and that $Val_{\downarrow}(\mathcal{G})=Val_{\downarrow}(\mathcal{G}')$ and $Val_{\uparrow}(\mathcal{G})=Val_{\uparrow}(\mathcal{G}')$. We omit the routine details.

We are now ready to describe our reduction from $\mathcal{G}$, a $2\frac{1}{2}$-player meta-parity game with $N+1$ priorities, to a $2\frac{1}{2}$-player meta-parity game with just N priorities.

Definition 6.2.7. A function $\rho\colon S_{\max}\rightarrow[0,1]$ is called a *value assignment* to the states in $S_{\max}$.

Definition 6.2.8 (Unfolding of $\mathcal{G}$)**.** Given a value assignment ρ to the states $S_{\max}$, we define the two player stochastic meta-parity game $\mathcal{G}^{-}_{\rho}=\langle\mathcal{A}^{-},\Phi_{\Pr^{-}}\rangle$, which we call the *unfolding of* $\mathcal{G}$ *with* ρ, as the game obtained from $\mathcal{G}$ as follows:

1. remove the unique (by Convention 6.2.6) outgoing edge of s_i, for $s_i\in S_{\max}$.
2. Turn s_i, for every $s_i\in S_{\max}$, into a λ_i-valued leaf with $\lambda_i=\rho(s)$, i.e., add two edges to the states $\top$ and $\bot$ (see Convention 6.2.1), define $\pi'(s)(\top)=\lambda_i$, $\pi'(s)(\top)=1-\lambda_i$ and set $\Pr'(s)=\min(\Pr)$.

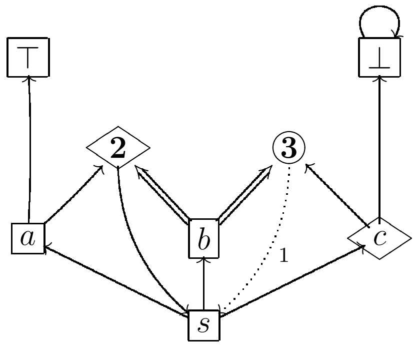
Clearly $\max(\Pr')\leq N$.

Observe that the two games $\mathcal{G}^{-}_{\rho_1}$ and $\mathcal{G}^{-}_{\rho_2}$, for two distinct value assignment ρ_1 and ρ_2, are structurally identical and differ only on the assignments of probability distributions to the states $s\in S_{\max}$.

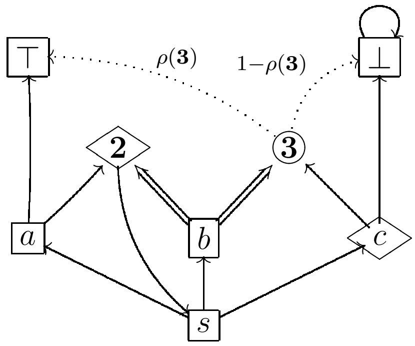
Remark 6.2.9. Note how this structural identity holds because, as observed earlier, the strict inclusion $supp(s_i)\subsetneq E(s_i)$ is allowed in $2\frac{1}{2}$-player tree game arenas. Working with the constraint $supp(s_i)=E(s_i)$ would break the property. Indeed a state $s\in S_{\max}$ could have only one child in $\mathcal{G}^{-}_{\rho}$, say $\top$, when $\rho(s)=1$. This would be, of course, a minor technical issue, but dealing with it would make our argument slightly heavier. This is why we opted for the weak constraint $supp(s_i)\subsetneq E(s_i)$ on two player tree games in Section 4.1.

We now consider an example to illustrate the construction of the game $\mathcal{G}^-_\rho$.

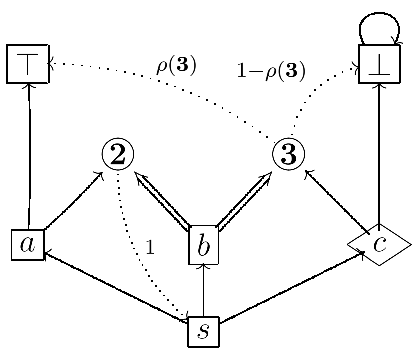
Example 6.2.10. Let us consider the two player (non stochastic) meta-parity game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\text{Pr}} \rangle$, which can depicted as follows:



where $\text{Pr}(\mathbf{3}) = 3$, $\text{Pr}(\mathbf{2}) = 2$ and $\text{Pr}(s) = 1$ for all $s \in \{\top, \bot, s, a, b, c\}$. Thus $S_{\max} = \{\mathbf{3}\}$. Note how $\mathcal{G}$ satisfies conventions 5.2.1, 6.2.6 and 6.2.1. For a value assignment $\rho : \{\mathbf{3}\} \to [0, 1]$, the game $\mathcal{G}^-_\rho$ can be depicted as follows:



with $\text{Pr}(\mathbf{3}) = 1$, $\text{Pr}(\mathbf{2}) = 1$ and $\text{Pr}(s) = 1$ for all $s \in \{\top, \bot, s, a, b, c\}$. Note that $\mathcal{G}^-_\rho$ does not satisfy anymore Convention 6.2.6, because the state **2**, which is of maximal priority in $\mathcal{G}^-_\rho$, is not probabilistic. If desired, the game $\mathcal{G}^-_\rho$ can be transformed into the equivalent game



with $\Pr(\mathbf{3})=1$, $\Pr(\mathbf{2})=2$ and $\Pr(s)=1$ for all $s\in\{\top,\bot,s,a,b,c\}$, which satisfies conventions 5.2.1, 6.2.6 and 6.2.1.

The intuition about the construction of $\mathcal{G}^-_\rho$ is the following: the states $s\in S_{\max}$, which are the most important ones (in the obvious sense) in the game $\mathcal{G}$, are reduced to λ-valued leaves. A play in $\mathcal{G}^-_\rho$ proceeds as in $\mathcal{G}$ until one state $s\in S_{\max}$ is reached. In the game $\mathcal{G}$ the play progresses to the unique successor state of s, while in $\mathcal{G}^-_\rho$ the game ends in favor of Player 1 with probability $\rho(s)$. The main idea is that, by careful choices of ρ, we will be able to simulate a play in the game $\mathcal{G}$ with a (simpler) play in $\mathcal{G}^-_\rho$.

Let us define the set $\mathbb{S}_{\mathsf{m}}$ of finite paths in $\mathcal{G}$ as follows: $\mathbb{S}_{\mathsf{m}}=\{\vec{s}\mid \mathit{last}(\vec{s})\in S_{\max}\}$. It is clear that $\mathbb{S}_{\mathsf{m}}=\{\vec{s_j}\}_{j\in J}$ is an antichain of finite paths in $\mathcal{G}$ (see Definition 4.3.1). We denote with s_j the state $\mathit{last}(\vec{s_j})\in S_{\max}$, for every $j\in J$.

Remark 6.2.11. We now list a few useful properties of the game $\mathcal{G}^-_\rho$.

1. The two $2\frac{1}{2}$-player meta-parity game arena $\mathcal{A}$ and $\mathcal{A}^-$ have the same set of states.

2. Every finite path $\vec{s}=\{s_0,\dots,s_k\}$ in $\mathcal{A}$ such that $s_n\notin S_{\max}$ for all $0\le n<k$, is also a finite path in $\mathcal{A}^-$. In particular all finite paths in the antichain $\mathbb{S}_{\mathsf{m}}$ in $\mathcal{A}$ are finite paths in $\mathcal{A}^-$, and *vice versa*. Therefore $\mathbb{S}_{\mathsf{m}}$ is also antichain of finite paths in $\mathcal{A}^-$.

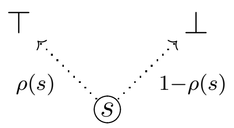
3. Every infinite path $\vec{s}$ in $\mathcal{A}$ without any occurrence of states $s\in S_{\max}$ is also an infinite path in $\mathcal{A}^-$ and *vice versa*.

4. For every $s\in S_{\max}$, there are exactly two[4] branching plays rooted at s in $\mathcal{G}^-_\rho$, which can be depicted as follows:

We denote with $T^s_\top$ the branching play on the left and with $T^s_\bot$ the one on the right, for $s\in S_{\max}$. It is clear, by Convention 6.2.1, that $T^s_\top\in\Phi^-$ and $T^s_\bot\notin\Phi^-$, where $\Phi^-=\Phi_{\Pr^-}$ denotes the winning set of $\mathcal{G}^-_\rho$.

[4]Again, this follows from the fact that $E(s)=\{\top,\bot\}$ even when $\mathit{supp}(s)\subsetneq\{\top,\bot\}$.

5. Since every state $s \in S_{\max}$ is a $\rho(s)$-valued leaf in $\mathcal{A}^-$, as we observed earlier, there is only one Markov branching play rooted at s in $\mathcal{G}_\rho^-$, which can be depicted as follows:



   We denote it with M^s_λ, where $\lambda = \rho(s)$. It is clear that the probability measure over branching plays in $\mathcal{A}^-$ induced M^s_λ, assigns probability λ to the winning branching play $T^s_\top$, i.e., $\mathbb{P}_{M^s_\lambda}(\Phi^-) = \lambda$.

6. As a consequence of the previous observations, note that the sets $\mathcal{P}_{\mathcal{A}^-}$, $\mathcal{BP}_{\mathcal{A}^-}$, $\mathcal{MBP}_{\mathcal{A}^-}$ and $\Phi^- = \Phi_{\mathrm{Pr}^-}$ in the unfolded game $\mathcal{G}^-_\rho$ do not depend on any particular choice of ρ. However the probability measure $\mathbb{P}_M$ over $\mathcal{BP}_{\mathcal{A}^-}$ induced by a Markov branching play $M \in \mathcal{MBP}_{\mathcal{A}^-}$ crucially depends on the value assignment ρ of $\mathcal{G}^-_\rho$.

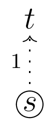
An important consequence of the first observation, is that every branching pre-play $T[x_i]_{i\in I}$ (see Section 4.3) in $\mathcal{A}$ (obtained by pruning some $T \in \mathcal{BP}$ with $\mathbb{S}_{\mathsf{m}}$) is also a branching pre-play in $\mathcal{A}^-$ (obtained by pruning some $T \in \mathcal{BP}_{\mathcal{A}^-}$ with $\mathbb{S}_{\mathsf{m}}$). In particular any branching play $T \in \mathcal{BP}_{\mathcal{A}^-}$ is uniquely of the form $T[T_i]_{i\in I}$ for some branching pre-play $T[x_i]_{i\in I}$ in $\mathcal{A}^-$ (which is also a branching pre-play in $\mathcal{A}$) and some I-indexed collection of compatible branching plays $\{T_i\}_{i\in I}$ in $\mathcal{BP}_{\mathcal{A}^-}$. From the third observation and by Definition 4.3.5, any such T_i is either $T^{s_i}_\top$ or $T^{s_i}_\bot$. This allows us to state the following useful lemma relating winning branching plays in $\mathcal{G}$ and winning branching plays in $\mathcal{G}^-_\rho$, for every value assignment $\rho : S_{\max} \to [0,1]$.

Lemma 6.2.12. *Let $T[T^s_{b_i}]_{i\in I}$ be a branching play in $\mathcal{G}^-_\rho$, with $b_i \in \{\top, \bot\}$ for $i \in I$, and $T[T_i]_{i\in I}$ a branching play in $\mathcal{G}$ having the same branching pre-play $T[x_i]_{i\in I}$. Then the following assertion holds:*

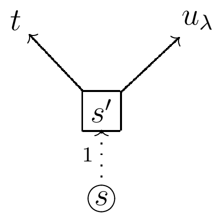
$$\forall i \in I. \big(T_i \in \Phi \text{ iff } b_i = \top\big) \;\Rightarrow\; T[T_i]_{i\in I} \in \Phi \text{ iff } T[T^{s_i}_{b_i}]_{i\in I} \in \Phi^-.$$

Proof. It follows from the fact that winning set of every prefix-independent $2\frac{1}{2}$-player meta game is subtree monotone (see Proposition 5.2.2), and Definition

4.3.8. Technically speaking, we cannot directly apply the property of Definition 4.3.8, because it just relates branching plays in the same game arena, whereas we are comparing branching plays of the two different arenas $\mathcal{A}$ and $\mathcal{A}^-$. This bureaucratic issue is, however, easily circumvented. For instance, instead of working with $\mathcal{A}$, consider the $2\frac{1}{2}$-player meta-parity game arena $\mathcal{B}$ obtained from $\mathcal{A}$ as follows. Every state $s \in S_{\max}$ in $\mathcal{A}$ can be depicted, by Convention 6.2.6, as:



where t is the unique successor state of s. The arena $\mathcal{B}$ is obtained by restructuring the states $s \in S_{\max}$ as follows:



where s' is new state under the control of Player 2, say, and u_λ is $\rho(s)$-valued leaf in $\mathcal{B}$, i.e., it is identical to the state s in $\mathcal{A}^-$. In the arena $\mathcal{B}$ there are more branching plays, and in particular both $T[T^s_{b_i}]_{i\in I}$ and $T[T_i]_{i\in I}$ are branching plays in $\mathcal{B}$ (up to routine modifications). The result then follows immediately by Definition 4.3.8. $\square$

The result of Lemma 6.2.12 can be described as follows. If a branching play $T[T^s_{b_i}]_{i\in I}$ in $\mathcal{G}^-_\rho$ is identical to a branching play $T[T_i]_{i\in I}$ in $\mathcal{G}$ up to the first occurrences of states in $S_{\max}$ (i.e., the two branching plays have the same branching pre-play) and if, for every reached $s_i \in S_{\max}$, the sub-play continuing its execution from s_i ends in a victory for the same player in the two games, then $T[T^s_{b_i}]_{i\in I}$ mimics $T[T_i]_{i\in I}$ is a faithful way, in the sense that $T[T^s_{b_i}]_{i\in I}$ is winning for Player 1 in $\mathcal{G}^-_\rho$ if and only if $T[T_i]_{i\in I}$ is winning for Player 1 in $\mathcal{G}$. This is quite an intuitive result, given that $2\frac{1}{2}$-player meta-parity games are prefix independent.

Lemma 6.2.12 exposes a fixed-point property of the winning set Φ of $\mathcal{G}$ which we now formalize.

Definition 6.2.13. For every set $X \subseteq \mathcal{BP}_\mathcal{A}$ and $j \in J$ (the index set of the antichain $\mathbb{S}_\mathsf{m}$ defined earlier) let us denote with $(\underline{\in}^j X)\!:\!\mathcal{BP}_\mathcal{A} \!\rightarrow\! \mathcal{BP}_{\mathcal{A}^-}$ the function defined as follows:

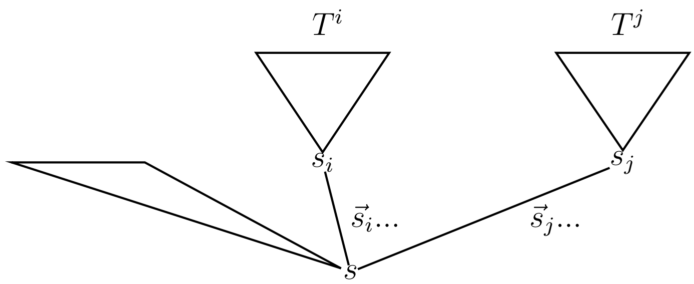
$$(\underline{\in}^j X)(T) = \begin{cases} T^{s_j}_\top & \text{if } T \in X \\ T^{s_j}_\bot & \text{otherwise} \end{cases}$$

where $T^{s_j}_\top$ and $T^{s_j}_\bot$ are the only two branching plays in $\mathcal{BP}_{\mathcal{A}^-}$ rooted at s_j, as observed in Remark 6.2.11. We just write $T\underline{\in}^j X$ for $(\underline{\in}^j X)(T)$.

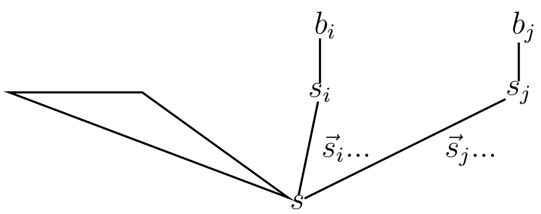
Definition 6.2.14. We define the function $\mathbb{W}^-_\mathcal{G}\!:\!2^{\mathcal{BP}_\mathcal{A}} \rightarrow 2^{\mathcal{BP}_\mathcal{A}}$ as follows:

$$\mathbb{W}^-_\mathcal{G}(X) = \big\{ T[T_i]_{i\in I} \mid T[T_i \underline{\in}^i X]_{i\in I} \in \Phi^- \big\}$$

Note that this is a good definition since, as observed earlier, every branching play $T \in \mathcal{BP}_\mathcal{A}$, is uniquely of the form $T[T_i]_{i\in I}$ for some branching pre-play $T[x_i]_{i\in I}$ obtained by pruning T with $\mathbb{S}_\mathsf{m}$ and some I-indexed collection $\{T_i\}_{i\in I}$ of branching plays, with each T_i rooted at s_i, for some index set I. Furthermore $T[T_i\underline{\in}^i X]_{i\in I}$ uniquely defines a branching play in $\mathcal{BP}_{\mathcal{A}^-}$, as the branching pre-play $T[x_i]_{i\in I}$ is, as observed earlier, also a branching-pre play in $\mathcal{A}^-$, and the I-indexed collection of branching plays $\{T_i\underline{\in}^i X\}_{i\in I}$ is compatible with $\mathbb{S}_\mathsf{m}$, because $root(T_i) = s_i$, by definition of $(\underline{\in}^i X)$. More informally, a branching play $T[T_i]_{i\in I} \in \mathcal{BP}_\mathcal{A}$, which can be depicted as follows



is in $\mathbb{W}^-_\mathcal{G}(X)$ if and only if the branching play $T[T_i\underline{\in}^i X]_{i\in I} \in \mathcal{BP}_{\mathcal{A}^-}$, which can be depicted as follows



where $b_i = \top$ if $T_i \in X$ and $b_i = \bot$ otherwise, is in Φ^-. Note in particular that, for every $T \in \mathcal{BP}_\mathcal{A}$ rooted at some state $s_j = last(\vec{s}_j)$, for some $\vec{s}_j \in \mathbb{S}_\mathsf{m}$, the following assertion holds: $T \in \mathcal{W}^-_\mathcal{G}(X)$ if and only if $T \in \Phi$.

Lemma 6.2.15. *The function $\mathbb{W}^{-}_{\mathcal{G}}$ is monotone.*

Proof. Fix $X \subseteq Y \subseteq \mathcal{BP}_{\mathcal{A}}$. Assume $T[T_i]_{i\in I} \in \mathbb{W}^{-}_{\mathcal{G}}(X)$, i.e., $T[T_i \underline{\in}^i X]_{i\in I} \in \Phi^{-}$. We need to prove that $T[T_i \underline{\in}^i Y]_{i\in I} \in \Phi^{-}$ too. The result then follows from the fact that Φ^{-} is a subtree monotone winning set (see Lemma 5.2.2) and by Definition 4.3.8 of subtree monotonicity. □

Lemma 6.2.16. *The winning set Φ of the $2\frac{1}{2}$-player meta-parity game $\mathcal{G}$ is a fixed point of $\mathbb{W}^{-}_{\mathcal{G}}$.*

Proof. We need to prove that $\mathbb{W}^{-}_{\mathcal{G}}(\Phi) \stackrel{\text{def}}{=} \{T[T_i]_{i\in I} \mid T[T_i \underline{\in}^i \Phi]_{i\in I} \in \Phi^{-}\} = \Phi$. This follows immediately from Lemma 6.2.12. □

Corollary 6.2.17. *For every $X \subseteq \mathcal{BP}_{\mathcal{A}}$ such that $X \subseteq \Phi$, the inclusion $\mathbb{W}^{-}_{\mathcal{G}}(X) \subseteq \Phi$ holds.*

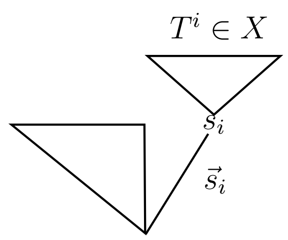
The following property of the function $\mathbb{W}^{-}_{\mathcal{G}}$ will be useful in the next section.

Lemma 6.2.18. *If $X \subseteq \mathcal{BP}_{\mathcal{A}}$ is a Borel (universally measurable) set then, if Φ^{-} is a Borel (universally measurable) set, also $\mathbb{W}^{-}_{\mathcal{G}}(X)$ is a Borel (universally measurable) set.*

Proof. For notational convenience, define $R_X : \mathcal{BP}_{\mathcal{A}} \to \mathcal{BP}_{\mathcal{A}^{-}}$ as $R(T[T_i]_{i\in I}) = T[T_i \underline{\in}^i X]_{i\in I}$. By definition of $\mathbb{W}^{-}_{\mathcal{G}}$, the set $\mathbb{W}^{-1}_{\mathcal{G}}(X)$ is Borel (universally measurable) if and only if $R^{-1}_X(\Phi^{-})$ is Borel (universally measurable).

We show that the inverse image under R_X of any sub-basic open sets $O_{\vec{s}\downarrow}$ in $\mathcal{BP}_{\mathcal{A}^{-}}$ (see Proposition 2.1.39) is a Borel (universally measurable) subset of $\mathcal{BP}$. This will immediately prove the desired result for the "Borel"case. The "universally measurable" case follows by application of Theorem 2.1.73. There are three interesting cases: $\vec{s} = \vec{s}_i.\top$, $\vec{s} = \vec{s}_i.\bot$, for $\vec{s}_i \in \mathbb{S}_{\mathsf{m}}$, and $\vec{s}$ not of the two previous shapes.

If $\vec{s} = \vec{s}_i.\top$ then $R^{-1}_X(O_{\vec{s}})$ is the set of all branching plays T in $\mathcal{A}$ which can be depicted as follows:



In other words $R_X^{-1}(O_{\vec{s}})$ is the set of all branching plays containing the path $\vec{s}_i$ and such that the sub-branching play T^i rooted at $\vec{s}_j$ is in X. Let use define the function $f: O_{\vec{s}_i} \rightarrow O_{\{s_i\}}$, from branching plays in $\mathcal{BP}$ containing the path $\vec{s}_i$ to branching plays in arena rooted at the state s_i, as $f(T) = T^i$, where T and T^i are as in the image above. It is immediate to verify that f is continuous, where $O_{\vec{s}_i}$ and $O_{\{s_i\}}$ are endowed with the subspace topologies. By the previous observation, we have that $R_X^{-1}(O_{\vec{s}}) = f^{-1}(X \cap O_{\{s_i\}})$. Then, by continuity of f, the set $R_X^{-1}(O_{\vec{s}})$ is Borel if X is Borel and, by application of Theorem 2.1.73, $R_X^{-1}(O_{\vec{s}})$ is universally measurable if X is universally measurable.

The case for $\vec{s} = \vec{s}_i.\bot$ is treated similarly, observing that $\overline{X}$ is Borel (universally measurable) if X is Borel (universally measurable).

If $\vec{s}$ is not in the two previous forms, then it follows by definition of R_X that $R_X^{-1}(O_{\vec{s}}) = O_{\vec{s}}$, i.e., $R_X^{-1}(O_{\vec{s}})$ is the set of all branching plays containing $\vec{s}$. This set is open. □

Another useful consequence of the observations of Remark 6.2.11 is the following.

Lemma 6.2.19. *Let $M[M^{s_i}_{\lambda_i}]_{i\in I}$ be a Markov branching play in $\mathcal{G}^-_\rho$ and $M[M_i]_{i\in I}$ a Markov branching play in $\mathcal{G}$ having the same Markov branching pre-play $M[x_i]_{i\in I}$ induced by $\mathbb{S}_{\mathsf{m}}$. Then, for every $\epsilon > 0$, the following assertions hold:*

$$\forall i \in I.\ \left(\mathbb{P}_{M_i}(\Phi) \leq \lambda_i + \tfrac{\epsilon}{\#(i)}\right) \ \Rightarrow\ \mathbb{P}_{M[M_i]}(\Phi) \leq \mathbb{P}_{M[M^{s_i}_{\lambda_i}]}(\Phi^-) + \epsilon,$$

and

$$\forall i \in I.\ \left(\mathbb{P}_{M_i}(\Phi) \geq \lambda_i - \tfrac{\epsilon}{\#(i)}\right) \ \Rightarrow\ \mathbb{P}_{M[M_i]}(\Phi) \geq \mathbb{P}_{M[M^{s_i}_{\lambda_i}]}(\Phi^-) - \epsilon.$$

Proof. The proof follows again (up to the same routine observations made in the proof of Lemma 6.2.12) from the fact that winning set of every prefix-independent $2\frac{1}{2}$-player meta game is subtree monotone (see Proposition 5.2.2), and Theorem 4.3.17. □

The result of Lemma 6.2.19 can be described as follows. If a play in $\mathcal{G}$ is played by the two players as a play in $\mathcal{G}^-_\rho$ up to the first occurrences of states in $S_{\max}$, and if each sub-play continuing its execution from $s_i \in S_{\max}$ in $\mathcal{G}$ ends up in a victory for Player 1 with probability close enough to $\rho(s_i)$ (which is, by previous observations, the probability of a sub-play continuing its execution from

s_i in $\mathcal{G}^-_\rho$ ending in a victory for Player 1) then the play in $\mathcal{G}$ is guaranteed to be winning for Player 1 with the same probability, up to $\pm\epsilon$, of the play in $\mathcal{G}^-_\rho$.

These results suggest that $\mathcal{G}^-_\rho$, the unfolding of $\mathcal{G}$ with ρ, can be used to simulate plays in $\mathcal{G}$ with appropriate choices of value assignments ρ. We now make this intuition precise.

Definition 6.2.20. Given a function $f : S \to [0,1]$, we define the value assignment $\rho_f : S_{\max} \to [0,1]$ as $\rho_f(s) = f(t)$, where t is the unique (in accordance with Convention 6.2.6) successor state of s in $\mathcal{A}$.

Definition 6.2.21. Let $\mathbb{H}_\mathcal{G} : [0,1]^S \to [0,1]^S$ be the functional defined as follows: $\mathbb{H}_\mathcal{G}(f)(s) = \mathsf{VAL}^s(\mathcal{G}^-_{\rho_f})$.

Remark 6.2.22. Note that $\mathbb{H}_\mathcal{G}$ is well defined only if the game $\mathcal{G}^-_\rho$ is determined under deterministic strategies, for every value assignment ρ. In the rest of this section we work under the hypothesis that $\mathcal{G}^-_\rho$ is determined under deterministic strategies and satisfied the leaf monotonicity property of Definition 6.2.3.

From this definition we clearly have that a function $f : S \to [0,1]$ is a fixed point of the functional $\mathbb{H}_\mathcal{G}$ if and only if $\mathsf{VAL}(\mathcal{G}^-_{\rho_f}) = f$.

Lemma 6.2.23. *The functional* $\mathbb{H}_\mathcal{G}$ *is monotone.*

Proof. Fix two functions $f, g : S \to [0,1]$ such that $f \sqsubseteq g$. Then $\rho_f \sqsubseteq \rho_g$ holds. Note that $\mathcal{G}^-_{\rho_f}$ and $\mathcal{G}^-_{\rho_g}$ are identical, except that every state $s \in S_{\max}$ is a $\rho_f(s)$-leaf in $\mathcal{G}^-_{\rho_f}$ and a $\rho_g(s)$-leaf in $\mathcal{G}^-_{\rho_g}$. Since $\mathcal{G}^-_{\rho_f}$ and $\mathcal{G}^-_{\rho_g}$ satisfy the leaf monotonicity property, by Remark 6.2.22, it follows that $\mathsf{VAL}^s(\mathcal{G}^-_{\rho_f}) \leq \mathsf{VAL}^s(\mathcal{G}^-_{\rho_g})$, for every $s \in S$. □

Theorem 6.2.24. *The functions* $\mathsf{VAL}_\downarrow(\mathcal{G}) : S \to [0,1]$ *and* $\mathsf{VAL}_\uparrow(\mathcal{G}) : S \to [0,1]$ *are fixed points of the functional* $\mathbb{H}_\mathcal{G}$.

Proof. We just prove that $\mathsf{VAL}_\uparrow(\mathcal{G})$ is a fixed point of $\mathbb{H}_\mathcal{G}$. The proof for $\mathsf{VAL}_\downarrow(\mathcal{G})$ is similar. Let us denote with f the function $\mathsf{VAL}_\uparrow(\mathcal{G}) = \lambda s.\mathsf{VAL}^s_\uparrow(\mathcal{G})$. We then need to show that the equality $\mathsf{VAL}(\mathcal{G}^-_{\rho_f}) = f$, or equivalently, $\mathsf{VAL}_\uparrow(\mathcal{G}^-_{\rho_f}) = f$ holds. We prove this by showing that the two inequalities $\mathsf{VAL}_\uparrow(\mathcal{G}^-_{\rho_f}) \leq f$ and $\mathsf{VAL}_\uparrow(\mathcal{G}^-_{\rho_f}) \geq f$ hold. We just consider the inequality $\mathsf{VAL}_\uparrow(\mathcal{G}^-_{\rho_f}) \geq \mathsf{VAL}_\uparrow(\mathcal{G})$. The proof readily applies to the oder inequality.

By definitions, we need to prove that, for every $s \in S$, the inequality

$$\textstyle\bigsqcap_{\tau_2}\bigsqcup_{\tau_1}\mathbb{P}_{M^s_{\tau_1,\tau_2}}(\Phi^-)\geq\bigsqcap_{\sigma_2}\bigsqcup_{\sigma_1}\mathbb{P}_{M^s_{\sigma_1,\sigma_2}}(\Phi)$$

holds, where $\mathbb{P}_{M^s_{\tau_1,\tau_2}}$ is the probability measure associated with the Markov branching play in $M^s_{\tau_1,\tau_2}$ in $\mathcal{G}^-_{\rho_f}$ and, similarly, $\mathbb{P}_{M^s_{\sigma_1,\sigma_2}}$ is the probability measure associated with $M^s_{\sigma_1,\sigma_2}$ in $\mathcal{G}$. It is enough to show that for any strategy τ_2 for Player 2 in $\mathcal{G}^-_{\rho_f}$ and for every $\epsilon>0$, there exists a strategy σ_2 for Player 2 in $\mathcal{G}$ such that the inequality $\bigsqcup_{\tau_1}\mathbb{P}_{M^s_{\tau_1,\tau_2}}(\Phi^-)\geq\bigsqcup_{\sigma_1}\mathbb{P}_{M^s_{\sigma_1,\sigma_2}}(\Phi)-\epsilon$ holds for every $s\in S$. The strategy σ_2 is constructed from τ_2 as follows:

$$\sigma_2(\vec{s})=\begin{cases}\tau_2(\vec{s}) & \text{if } \vec{s} \text{ does not contain any state in } S_{\max}\\ \sigma^j_2(\vec{r}) & \text{if } \vec{s}=\vec{s}_j.\vec{r} \text{ with } \vec{s}_j\in\mathbb{S}_{\mathsf{m}}\end{cases}$$

where σ^j_2 is a $\frac{\epsilon}{\#(j)}$-optimal strategy for Player 2 in $\mathcal{G}$, which exists since $\epsilon>0$. To prove that the desired inequality $\bigsqcup_{\tau_1}\mathbb{P}_{M^s_{\tau_1,\tau_2}}(\Phi^-)\geq\bigsqcup_{\sigma_1}\mathbb{P}_{M^s_{\sigma_1,\sigma_2}}(\Phi)-\epsilon$ holds, we just need to show that for each strategy σ_1 for Player 1 in $\mathcal{G}$ there exists a strategy τ_1 for Player 1 in $\mathcal{G}^-_{\rho_f}$ such that the inequality

$$\mathbb{P}_{M^s_{\tau_1,\tau_2}}(\Phi^-)\geq\mathbb{P}_{M^s_{\sigma_1,\sigma_2}}(\Phi)-\epsilon \tag{6.3}$$

holds. The strategy τ_1 is constructed from σ_1 as follows: $\tau_1(\vec{s})=\sigma_1(\vec{s})$ if $\vec{s}$ does not contain states in $S_{\max}$. If a play eventually reaches a state $s\in S_{\max}$ in $\mathcal{G}^-_{\rho_f}$, the rest of the play is independent on the strategy τ_1. Furthermore we define the strategy σ^j_1, for $j\in J$, following the behavior of σ_1 when a state $s\in S_{\max}$ is reached following some path $\vec{s}_j\in\mathbb{S}_{\mathsf{m}}$, as follows: $\sigma^j_1(\vec{r})=\sigma_1(\vec{s}_j.\vec{r})$, for every path $\vec{r}$ such that $\vec{s}_j.\vec{r}$ is a valid path in $\mathcal{A}$. We shall not be interested in the choices of σ^j_1 on histories $\vec{r}$ not of this form, thus we avoid a complete specification. Hence the strategy σ_1 can be characterized as follows:

$$\sigma_1(\vec{s})=\begin{cases}\tau_1(\vec{s}) & \text{if } \vec{s} \text{ does not contain any state in } S_{\max}\\ \sigma^j_1(\vec{r}) & \text{if } \vec{s}=\vec{s}_j.\vec{r} \text{ with } \vec{s}_j\in\mathbb{S}_{\mathsf{m}}\end{cases}$$

Given the definitions of σ_2, σ^j_2, τ_1 and σ^j_1 and the characterization of σ_1 discussed above, it is clear that the Markov branching play M_{τ_1,τ_2} in $\mathcal{G}^-_{\rho_f}$ and the Markov branching play M_{σ_1,σ_2} are identical up to the first occurrences of states in $S_{\max}$. This means that M_{τ_1,τ_2} and M_{σ_1,σ_2} induce the same Markov branching pre-play $M[x_i]_{i\in I}$. Moreover, we have that $M_{\tau_1,\tau_2}=_{\mathbb{S}_{\mathsf{m}}}M[M^{s_i}_{\lambda_i}]_{i\in I}$ and $M_{\sigma_1,\sigma_2}=_{\mathbb{S}_{\mathsf{m}}}M[M^{s_i}_{\sigma^i_1,\sigma^i_2}]_{i\in I}$, where:

1. the Markov branching play $M^{s_i}_{\lambda_i}$ is the unique Markov branching play rooted at $s_i\in S_{\max}$ in $\mathcal{G}^-_{\rho_f}$. It models a play where Player 1 wins with probability $\lambda_i=\rho_f(s_i)=f(s_i)=\mathsf{VAL}^{s_i}_{\uparrow}(\mathcal{G})$. More formally, $\mathbb{P}_{M^{s_i}_{\lambda_i}}(\Phi^-)=\mathsf{VAL}^{s_i}_{\uparrow}(\mathcal{G})$.

2. the Markov branching play $M^{s_i}_{\sigma_1^i,\sigma_2^i}$ is rooted as s_i and is induced by the strategy profile (σ_1^i, σ_2^i). Since the strategy σ_2^i is $\frac{\epsilon}{\#(i)}$-optimal by definition, we have that $\mathbb{P}_{M^{s_i}_{\sigma_1^i,\sigma_2^i}}(\Phi) \leq \mathsf{VAL}^{s_i}_{\uparrow}(\mathcal{G}) + \frac{\epsilon}{\#(i)}$.

Therefore $\mathbb{P}_{M^{s_i}_{\sigma_1^i,\sigma_2^i}}(\Phi) \leq \mathsf{VAL}^{s_i}_{\uparrow}(\mathcal{G}) + \frac{\epsilon}{\#(i)}$, for every $i \in I$. The inequality 6.3

$$\mathbb{P}_{M_{\tau_1,\tau_2}}(\Phi^-) + \epsilon \geq \mathbb{P}_{M_{\sigma_1,\sigma_2}}(\Phi)$$

then follows by Lemma 6.2.19 □

The result of Theorem 6.2.24 can be described as follows. Playing the game $\mathcal{G}^-_\rho$ is like playing the game $\mathcal{G}$ until some state $s \in S_{\max}$ is reached. Once such a state is reached, the game $\mathcal{G}^-_\rho$ ends up, by means of the choice made by Nature, in favour of Player 1 with probability $\rho(s)$, and in favour of Player 2 with probability $1-\rho(s)$. In the game $\mathcal{G}$ instead, the play progresses to the unique (by Convention 6.2.6) successor state t of s. From t, Player 1 will be able to force the game to end up in their favour with limit probability $\mathsf{VAL}^t_{\downarrow}(\mathcal{G})$ and similarly, Player 2 will be able to bound the limit probability, of the game ending up in favour of Player 1, to $\mathsf{VAL}^t_{\uparrow}(\mathcal{G})$. Therefore, if $\rho = \rho_{\mathsf{VAL}_{\downarrow}(\mathcal{G})}$, i.e., if at the state s Nature chooses to end up the game $\mathcal{G}^-_\rho$ if favour of Player 1 with probability $\mathsf{VAL}^t_{\downarrow}(\mathcal{G})$, Player 1 can force a winning outcome in the two games with the same limit probabilities. Similarly, if $\rho = \rho_{\mathsf{VAL}_{\uparrow}(\mathcal{G})}$, i.e., if at the state s Nature chooses to end up the game $\mathcal{G}^-_\rho$ if favour of Player 1 with probability $\mathsf{VAL}^t_{\uparrow}(\mathcal{G})$, Player 2 can bound, with the same limit probabilities, the possibility of the game ending up in favour of Player 1 in the two games.

Note how the results of Lemma 6.2.16 and Theorem 6.2.24 constitute useful improvements of the similar results of Lemma 5.2.6 and Theorem 5.2.10.

6.3 $2\frac{1}{2}$-player meta-parity games with $N+1$ priorities

In Section 6.1 we proved that any two player stochastic meta-parity game with just one priority is determined under deterministic strategies. In Section 6.2 we showed how, from every $2\frac{1}{2}$-player meta-parity game with $N+1$ priorities, one can construct the associated game $\mathcal{G}^-_\rho$, having only N priorities, by means of a procedure which we call *unfolding*.

In this section we will prove than every two player stochastic meta-parity game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\Pr} \rangle$ with $N+1$ priorities, i.e., such that $|\Pr| = N+1$, is determined under deterministic strategies assuming that the function $\mathbb{H}_{\mathcal{G}}$ specified in Definition 6.2.21 is well defined (see Remark 6.2.22).

The general shape of our proof closely resembles the proof, given in Section 6.1, of determinacy of $2\frac{1}{2}$-player meta-parity game with just one priority. The main difference is that we base our arguments on top of the results of Lemma 6.2.16 and Theorem 6.2.24 obtained in Section 6.2, whereas in Section 6.1 we invoked the results of Lemma 5.2.6 and Theorem 5.2.10, obtained in Section 5.2.

Let us fix an arbitrary $2\frac{1}{2}$-player meta-parity game $\mathcal{G}=\langle\mathcal{A},\Phi_{\mathrm{Pr}}\rangle$ having $N+1$ priorities, i.e., such that $|\mathrm{Pr}| = N+1$ for some $N > 1$, and satisfying, without loss of generality, Convention 5.2.1, Convention 6.2.1 and Convention 6.2.6. Following the same notational choices adopted in the previous section, we denote with $\mathcal{P}^{<\omega}$, $\mathcal{P}$, $\mathcal{BP}$, $\mathcal{MBP}$ and Φ the sets of finite paths, completed paths, branching plays, Markov branching plays and the winning set of $\mathcal{G}$, respectively. Let $\mathrm{Pr}(S) = \{p_1,\dots,p_{N+1}\}$ be the ordered list of priorities assigned to states in $\mathcal{A}$ by Pr. We denote with $S_{\max}$ the set $\mathrm{Pr}^{-1}(p_{N+1})$. The antichain $\mathbb{S}_{\mathsf{m}}$ of finite paths in $\mathcal{A}$ is defined, as in the previous section, as $\mathbb{S}_{\mathsf{m}}=\{\vec{s}\mid \mathit{last}(\vec{s})\in S_{\max}\}$. We denote with s_j the state $\mathit{last}(\vec{s}_j)\in S_{\max}$, for every $j\in J$.

Remark 6.3.1. Without any loss of generality we will assume that p_N is odd if p_{N+1} is even, and similarly p_N is even if p_{N+1} is odd. Suppose indeed that p_N and p_{N+1} are both even or odd numbers, and consider the priority assignment Pr' defined as follows:

$$\mathrm{Pr}'(s)=\begin{cases}\mathrm{Pr}(s) & \text{if } \mathrm{Pr}(s)<p_{N+1}\\ p_N & \text{if } \mathrm{Pr}(s)=p_{N+1}\end{cases}$$

Since p_N and p_{N+1} are both even (or odd) priorities greater than all other priorities assigned to states in $\mathcal{A}$, it is the immediate to verify that $\mathcal{W}_{\mathrm{Pr}}=\mathcal{W}_{\mathrm{Pr}'}$, where the sets of winning paths $\mathcal{W}_{\mathrm{Pr}}$ and $\mathcal{W}_{\mathrm{Pr}'}$ induced by the priority assignments Pr and Pr' respectively, are specified as in Definition 2.3.54. It then follows from Definition 5.3.1, that $\langle\mathcal{A},\Phi_{\mathrm{Pr}}\rangle$ and $\langle\mathcal{A},\Phi_{\mathrm{Pr}'}\rangle$ are identical and the game $\langle\mathcal{A},\Phi_{\mathrm{Pr}'}\rangle$ uses just N priorities.

We assume from now on that p_{N+1} is odd, and thus p_N is even. The case for max(Pr) even will be considered at the end of this section. Given a value assignment $\rho\colon S_{\max}\to[0,1]$ (see Definition 6.2.7), we denote with $\mathcal{G}^-_\rho=\langle\mathcal{A}^-,\mathrm{Pr}^-\rangle$ the unfolding of $\mathcal{G}$ with ρ (see Definition 6.2.8). We shall often denote, for notational convenience, the sets $\mathcal{P}_{\mathcal{A}^-}$, $\mathcal{BP}_{\mathcal{A}^-}$, $\mathcal{MBP}_{\mathcal{A}^-}$ and Φ_{Pr^-} in $\mathcal{G}^-_\rho$ with $\mathcal{P}^-$, $\mathcal{BP}^-$,

$\mathcal{MBP}^-$ and Φ^-, respectively. As observed in Remark 6.2.11, these sets do not depend on the particular choice of ρ.

Our first result provides an inductive characterization of the winning set Φ of $\mathcal{G}$. From Lemma 6.2.15, we know that the function $\mathbb{W}^-_{\mathcal{G}}$ is monotone.

Corollary 6.3.2. *By the Knaster-Tarski theorem, $\mathbb{W}^-_{\mathcal{G}}$ has least and greatest fixed points. In particular the least fixed point of $\mathbb{W}^-_{\mathcal{G}}$, denoted by $\mathrm{lfp}(\mathbb{W}^-_{\mathcal{G}})$, can be characterized as follows:*

$$\mathrm{lfp}(\mathbb{W}^-_{\mathcal{G}}) = \bigcup_{\alpha} {\mathbb{W}^-_{\mathcal{G}}}^{\alpha}$$

where α ranges over the ordinals, and the sets ${\mathbb{W}^-_{\mathcal{G}}}^{\alpha}$ are defined as ${\mathbb{W}^-_{\mathcal{G}}}^{\alpha} = \bigcup_{\beta<\alpha} \mathbb{W}^-_{\mathcal{G}}({\mathbb{W}^-_{\mathcal{G}}}^{\alpha})$.

Lemma 6.3.3. *The equality $\Phi = \mathrm{lfp}(\mathbb{W}^-_{\mathcal{G}})$ holds.*

Proof. We know, from Lemma 6.2.16, that Φ is a fixed point of $\mathbb{W}^-_{\mathcal{G}}$. Hence $\mathrm{lfp}(\mathbb{W}^-_{\mathcal{G}}) \subseteq \Phi$. We prove the desired result by showing that for every $T \in \mathcal{BP}_{\mathcal{A}}$, if $T \not\in \mathrm{lfp}(\mathbb{W}^-_{\mathcal{G}})$ then $T \not\in \Phi$. This is done by constructing a winning strategy σ^T_2 for Player 2 in the inner-game $\mathcal{G}_T$ associated with T.

For every branching play $T =_{\mathbb{S}_\mathsf{m}} T[T_i]_{i\in I}$ in $\mathcal{A}$ let us denote with T^- the branching play $T[T_i \trianglelefteq^i \mathrm{lfp}(\mathbb{W}^-_{\mathcal{G}})]_{i\in I}$ in $\mathcal{A}^-$ as specified in Definition 6.2.14. By Definition 6.2.14 of $\mathbb{W}^-_{\mathcal{G}}$, we know that $T \not\in \mathrm{lfp}(\mathbb{W}^-_{\mathcal{G}})$ if and only if $T^- \not\in \Phi^-$. A branching play $T^- \in \mathcal{A}^-$ is not in Φ^- if and only if Player 2 has a winning strategy in the inner-game $\mathcal{G}_{T^-}$ associated with T^-. Let us denote this strategy with $\tau_2^{T^-}$.

For every branching play $T \in \mathcal{BP}_{\mathcal{A}}$ such that $T \not\in \mathrm{lfp}(\mathbb{W}^-_{\mathcal{G}})$ we define the strategy σ^T_2 as follows. The strategy σ^T_2 behaves like $\tau_2^{T^-}$ until a state s_i, for $s_i = \mathit{last}(\vec{s}_i)$ with $\vec{s}_i \in \mathbb{S}_\mathsf{m}$, is reached. Note that this is a good definition since T and T^- are identical up to this kind of states, as observed in Remark 6.2.11. If eventually a state s_i is reached following the path $\vec{s}_i$, then T_i (the sub-branching play of $T = T[T_i]_{i\in I}$) is necessarily not contained in $\mathrm{lfp}(\mathbb{W}^-_{\mathcal{G}})$. This is because, otherwise the game $\mathcal{G}_{T^-}$, played by Player 2 in accordance with the strategy $\tau_2^{T^-}$, would have ended in a state s_i having as unique successor the terminal state $T_i \trianglelefteq^i \mathrm{lfp}(\mathbb{W}^-_{\mathcal{G}}) \stackrel{\text{def}}{=} \top$. Thus Player 2 would be losing playing in accordance with a winning strategy. A contradiction. We define the strategy σ^T_2 to keep playing in the sub-game $\mathcal{G}_{T'}$ as the strategy $\sigma_2^{T'}$ (note the inductive definition), forgetting about the past history.

We now prove that, for every branching play $T \notin \text{lfp}(\mathbb{W}^-_{\mathcal{G}})$, the strategy σ^T_2 is winning for Player 2 in the inner game $\mathcal{G}_T$. The game $\mathcal{G}_T$, played by Player 2 in accordance with σ^T_2, ends up in a completed path $\vec{s}$ in T such that:

1. Either $\vec{s}$ contains infinitely many occurrences of states s_j. In this case Player 2 wins because $\Pr(s_j) = p_{N+1}$ by definition of $\mathbb{S}_{\mathsf{m}}$, and p_{N+1} is odd.
2. Or $\vec{s}$ contains only a finite number of occurrences of states s_i, i.e., it is of the form $\vec{s}_{i_1}.\dots.\vec{s}_{i_k}.\vec{t}$ where $\vec{s}_{i_1},\dots,\vec{s}_{i_k} \in \mathbb{S}_{\mathsf{m}}$ and $\vec{t}$ does not contain occurrences of states s_i. Let s_k be the last state of $\vec{s}_{i_k}$. Then, by definition of σ^T_2, the completed path $\vec{t}$ is also a possible outcome of a play in $\mathcal{G}_{T^-_k}$ played in accordance with the strategy $\tau_2^{T^-_k}$, for some branching play $T_k \notin \text{lfp}(\mathbb{W}^-_{\mathcal{G}})$ rooted at s_k. Since $\tau_2^{T^-_k}$ is by construction a winning strategy for Player 2 in $\mathcal{G}_{T^-_k}$, the path $\vec{t}$ is necessarily winning for Player 2. It then follows from the fact that parity winning sets are prefix independent, that also $\vec{s}_{i_1}.\dots.\vec{s}_{i_k}.\vec{t}$ is a winning path for Player 2 in $\mathcal{G}_T$.

Therefore σ^T_2 is a winning strategy for Player 2 in the inner game $\mathcal{G}_T$, for every $T \notin \text{lfp}(\mathbb{W}^-_{\mathcal{G}})$, as dedired. $\square$

Note that, even if $\Phi = \bigcup_\alpha {\mathbb{W}^-_{\mathcal{G}}}^\alpha$ is subtree monotone by Proposition 5.2.2, each set ${\mathbb{W}^-_{\mathcal{G}}}^\beta$, with ${\mathbb{W}^-_{\mathcal{G}}}^\beta \subsetneq \Phi$, is easily seen to be not subtree monotone. However the following weaker property holds.

Lemma 6.3.4. *For every ordinal α, the set ${\mathbb{W}^-_{\mathcal{G}}}^\alpha$ is $\big(\mathbb{S}_{\mathsf{m}}, \bigcup_{\beta<\alpha} {\mathbb{W}^-_{\mathcal{G}}}^\beta\big)$-subtree monotone (see Definition 4.3.21).*

Proof. For notational convenience let us denote the set ${\mathbb{W}^-_{\mathcal{G}}}^\alpha$ with $\mathbb{U}^\alpha$, for every ordinal α. We need to prove that given two branching plays $T =_{\mathbb{S}_{\mathsf{m}}} T[T_i]$ and $T =_{\mathbb{S}_{\mathsf{m}}} T[R_i]$ in $\mathcal{A}$ having the same branching pre-play $T[x_i]_{i\in I}$, the following implication holds:

$$\forall i.\big(T_i \in \textstyle\bigcup_{\beta<\alpha} \mathbb{U}^\beta \;\Rightarrow\; R_i \in \bigcup_{\beta<\alpha} \mathbb{U}^\beta\big) \;\Rightarrow\; \big(T[T_i]_{i\in I} \in \mathbb{U}^\alpha \;\Rightarrow T[R_i]_{i\in I} \in \mathbb{U}^\alpha\big).$$

By definition of $\mathbb{U}^\alpha$, we know that $T[T_i]_{i\in I} \in \mathbb{U}^\alpha$ if and only if $T[T_i \underline{\in}^i \bigcup_{\beta<\alpha} \mathbb{U}^\beta] \in \Phi^-$. The desired result the follows immediately by the fact that Φ^- is subtree monotone by Proposition 5.2.2. $\square$

As a result, the following lemma, analogous to Lemma 6.2.19, holds.

Lemma 6.3.5. *Let $M[M^{s_i}_{\lambda_i}]_{i\in I}$ be a Markov branching play in $\mathcal{G}^-_\rho$ and $M[M_i]_{i\in I}$ a Markov branching play in $\mathcal{G}$ having the same Markov branching pre-play $M[x_i]_{i\in I}$ induced by $\mathbb{S}_{\mathsf{m}}$. Then, for every $\epsilon > 0$, the following assertions holds for every ordinal α:*

$$\forall i \in I.\ \big(\mathbb{P}_{M_i}\big(\bigcup_{\beta<\alpha}\mathbb{U}^\beta\big) \leq \lambda_i + \tfrac{\epsilon}{\#(i)}\big) \ \Rightarrow\ \mathbb{P}_{M[M_i]}(\mathbb{U}^\alpha) \leq \mathbb{P}_{M[M^s_{\lambda_i}]}(\Phi^-) + \epsilon,$$

and

$$\forall i \in I.\ \big(\mathbb{P}_{M_i}\big(\bigcup_{\beta<\alpha}\mathbb{U}^\beta\big) \geq \lambda_i - \tfrac{\epsilon}{\#(i)}\big) \ \Rightarrow\ \mathbb{P}_{M[M_i]}(\mathbb{U}^\alpha) \geq \mathbb{P}_{M[M^s_{\lambda_i}]}(\Phi^-) - \epsilon.$$

where $\mathbb{U}^\gamma$ denotes the set $\mathbb{W}_{\mathcal{G}}^{-\gamma}$ for every ordinal γ.

Proof. The proof follows from Theorem 4.3.22. Again, as for the proofs of Lemma 6.2.19 and Lemma 6.2.12, we have to deal with the bureaucratic issue that Theorem 4.3.22 only relates properties of Markov branching plays in the same $\mathcal{A}$. This issue can be easily circumvented, as discussed in the proof of Lemma 6.2.12). ☐

We now prove a result about $\mathbb{W}^-_{\mathcal{G}}$, similar to the one of Lemma 6.1.4 about $\mathbb{W}_{\mathcal{G}}$, which provides bounds the number of iterations necessary to reach the least fixed point of $\mathbb{W}^-_{\mathcal{G}}$.

Lemma 6.3.6. *The operator $\mathbb{W}^-_{\mathcal{G}}$ reaches its fixed point in at most ω_1 iterations, i.e.,* $\mathrm{lfp}(\mathbb{W}^-_{\mathcal{G}}) = \bigcup_{\alpha<\omega_1}\mathbb{W}_{\mathcal{G}}^{-\alpha}$.

Proof. For notational convenience we just write $\mathbb{U}^\alpha$ to denote $\mathbb{W}_{\mathcal{G}}^{-\alpha}$, for every ordinal α. Fix any $T[T_i]_{i\in I} \in \mathbb{U}^{\omega_1+1}$. We need to prove that $T[T_i]_{i\in I} \in \mathbb{U}^{\omega_1}$. By definition of $\mathbb{W}^-_{\mathcal{G}}$ we have that $T[T_i]_{i\in I} \in \mathbb{U}^{\omega_1+1}$ if and only if $T[T_i \underline{\in}^i \mathbb{U}^{\omega_1}]_{i\in I} \in \Phi^-$. Recall that the index set I is necessarily countable, since every antichain of paths in $\mathcal{A}$ is countable. Let $K \subseteq I$, be the collection of indexes associated with sub-branching plays T_k such that $T_k \in \mathbb{U}^{\omega_1}$. Since $\mathbb{U}^{\omega_1} = \bigcup_{\alpha<\omega_1}\mathbb{U}^\alpha$, each branching play T_k is in $\mathbb{U}^{\beta(k)}$, for some countable ordinal $\beta(k) < \omega_1$. Let $\beta = \bigsqcup_{k\in K}\beta(k)$ be the supremum of the $\{\beta(k)\}_{k\in K}$ collection of countable ordinals. The ordinal β is countable. It follows that $T[T_i]_{i\in I} \in \mathbb{W}^-_{\mathcal{G}}(\mathbb{U}^\beta) = \mathbb{U}^{\beta+1}$, since $T[T_i \underline{\in}^i \mathbb{U}^{\omega_1}]_{i\in I}$ and $T[T_i \underline{\in}^i \mathbb{U}^\beta]_{i\in I}$ are identical by construction, because the two functions $\underline{\in}^i\mathbb{U}^{\omega_1}$ and $\underline{\in}^i\mathbb{U}^\beta$ (see Definition 6.2.14) coincide on all branching plays T_i, for $i \in I$. The proof is concluded by observing that $\beta + 1 < \omega_1$ because β is a countable ordinal and ω_1 is the smallest uncountable ordinal. ☐

In Section 6.1 we proved that, when considering two player stochastic meta-parity games with just one odd (even) priority finitely branching in B_2 (B_1) (see Definition 5.1.13), the least fixed point of the operator $\mathbb{W}_{\mathcal{G}}$ (which played in Section 6.1 a role similar to the operator $\mathbb{W}^{-}_{\mathcal{G}}$ considered in this section) is reached in just ω steps. We will show that this is not the case when considering two player stochastic meta-parity games using more than one priority.

We first prove that there exists a two player stochastic meta-parity game with a finite arena and just two priorities whose winning set is not Borel.

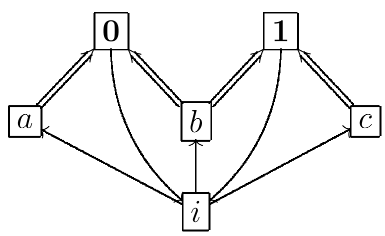
Lemma 6.3.7. *There exists a two player stochastic meta-parity game $\mathcal{G}=\langle\mathcal{A},\Phi_{\mathrm{Pr}}\rangle$ with finite arena $\mathcal{A}$ and just two priorities, i.e., $|\,\mathrm{Pr}\,|=2$, such that Φ_{Pr}, the winning set of the game $\mathcal{G}$, is $\mathbf{\Pi}^1_1$-complete.*

Proof. We provide a concrete example by constructing a $2\frac{1}{2}$-player meta-parity game $\mathcal{G}=\langle\mathcal{A},\Phi_{\mathrm{Pr}}\rangle$ with just two priorities such that:

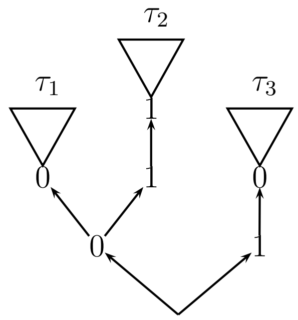
1. $\mathcal{BP}_{\mathcal{A}}$ is homeomorphic to the set $\mathsf{NT}\subseteq\mathcal{T}(\{0,1\})$, which is clearly closed, of non-terminating trees over $\{0,1\}$ (see Definition 2.1.41), and

2. $\Phi_{\mathrm{Pr}}\subseteq\mathcal{BP}_{\mathcal{A}}$ is homeomorphic to the set $A\subseteq\mathsf{NT}$ of non-termintating trees not containing infinite branches with infinitely many 1's.

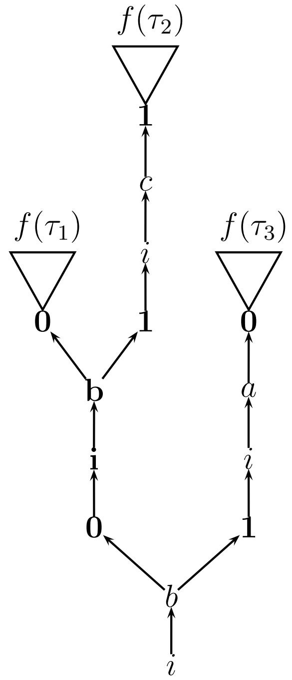
The result then follows by application of Theorem 2.1.57. The game $\mathcal{G}$ can be depicted (without displaying the states $\top$ and $\bot$ which are isolated) as follows:



with $\mathrm{Pr}(\mathbf{1})=1$ and $\mathrm{Pr}(s)=0$ for all other states. We now define a bijective map $f:\mathsf{NT}\to\mathcal{BP}_i$, where $\mathcal{BP}_i$ is the set of branching plays in $\mathcal{G}$ rooted at i. Rather than providing a formal definition, we find it clearer to describe it by means of an example, from which the formal definition is evident. The function f maps the following tree $\tau\in\mathsf{NT}$



where τ_1, τ_2 and τ_3 are subtrees of τ (i.e., they are themselves trees in NT), to the branching play



It is clear that f is continuous, and indeed it is easy to prove that the spaces NT and $\mathcal{BP}_i$ are homeomorphic via f.

Let us now go back to the game $\mathcal{G}$. Since every branching node in $\mathcal{G}$ is under the control of Player 2, it follows that a branching play $T \in \mathcal{BP}_i$ is winning for Player 2, i.e., Player 2 has a winning strategy in the inner game $\mathcal{G}_T$, if and only if T contains an infinite path having infinitely many occurrences of the state $\mathbf{1}$. This is because the state $\mathbf{1}$ is the only state which get assigned an odd priority by Pr in $\mathcal{G}$, and by Definition 2.3.54 of $\mathcal{W}_{\mathrm{Pr}}$, every infinite path containing only finitely many states of odd priority is winning for Player 1. After this observation is clear that Φ_{Pr}, the set of branching plays winning for Player 1, is the set of branching plays not containing infinite paths with infinitely many occurrences of $\mathbf{1}$'s, i.e., $\Phi_{\mathrm{Pr}} = f(A)$. This concludes the proof. ☐

Lemma 6.3.7 allows us to prove that, in general, the upper bound on the number of iterations required to reach the least fixed point of $\mathbb{W}^{-}_{\mathcal{G}}$ provided in Lemma 6.3.6 is tight, even when considering two player stochastic meta-parity games with finite arenas.

Lemma 6.3.8. *There exists a 2-player (non stochastic) meta-parity game* $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ *with finite arena* $\mathcal{A}$ *and just two priorities, i.e.,* $|\,\mathrm{Pr}\,| = 2$*, such that for every countable ordinal* α*,* $\mathbb{W}^{-\alpha}_{\mathcal{G}} \subsetneq \mathbb{W}^{-\omega_1}_{\mathcal{G}}$.

Proof. Let us consider the game $\mathcal{G}$ proposed in Lemma 6.3.7, and let us denote with Φ its winning set Φ_{Pr}. By Definition 6.2.8, we have that $\mathcal{G}^{-}_{\rho}$ has a finite arena (thus trivially finitely branching in the branching nodes) and just one even priority. Thus, the winning set Φ^{-} of the game $\mathcal{G}^{-}_{\rho}$ is, by application of Lemma 6.1.11, a Borel set. Hence, by application of Lemma 6.2.18, for every countable ordinal α, the set $\mathbb{W}^{-\alpha}_{\mathcal{G}}$ is Borel. From the result of Lemma 6.3.7 we know that Φ is not a Borel set, hence not of the form $\mathbb{W}^{-\alpha}_{\mathcal{G}}$ for any countable ordinal α. ◻

Even if Lemma 6.3.7 and Lemma 6.3.8 imply that an analogue of Lemma 6.1.3 for two player stochastic meta-parity games with more than one priority does not hold, we can prove the following weaker property.

Lemma 6.3.9. *If* $\mathcal{A}$ *is a standard two player stochastic game arena, in the sense of Definition 4.1.5, then* $\Phi = \mathbb{W}^{-\omega}_{\mathcal{G}}$.

Proof. Following the same argument used in the proof of Lemma 4.3.18, we know that a branching pre-play, obtained by pruning a branching play (i.e., a completed path $\vec{r}$) with an antichain $\mathbb{S}$ of finite paths in $\mathcal{A}$, is either:

- a finite path $\vec{s_i} \in \mathbb{S}$, if $\vec{r} = \vec{s_i}.\vec{t}$: in this case the branching pre-play has just one hole waiting to be filled in by a branching play (i.e., a completed path) starting at $last(\vec{s_i})$;
- or $\vec{r}$ itself: in this case the branching pre-play has no holes.

Thus we have that $\mathbb{W}^{-n}_{\mathcal{G}}$, for $0 < n < \omega$ is the set of all completed paths $\vec{s}$ in $\mathcal{A}$ such that

1. $\vec{s}$ does not contain states in $S_{\max}$ and $\vec{s}$ is winning in $\mathcal{G}^{-}_{\rho}$ and, by Remark 6.2.11, also in $\mathcal{G}$, or

2. $\vec{s}=\vec{s_i}.\vec{t}$, where $\vec{s_i}\in\mathbb{S}_{\mathsf{m}}$ and $\vec{t}\in\mathbb{W}_{\mathcal{G}}^{-n-1}$.

It follows that $\mathbb{W}_{\mathcal{G}}^{-\omega}$ is the set of completed paths having a winning tail without occurrences of states in $s\in S_{\max}$. This is precisely the winning set $\Phi_{\mathrm{Pr}}=\mathcal{W}_{\mathrm{Pr}}$, because every completed path with infinitely many occurrences of states $s\in S_{\max}$, i.e., states labeled with maximal odd priority p_{N+1}, is losing for Player 1 by Definition 2.3.54. □

We are now ready to prove the main technical result of this section.

Theorem 6.3.10. *Let $\mathcal{G}=\langle\mathcal{A},\Phi_{\mathrm{Pr}}\rangle$ be a $2\frac{1}{2}$-player meta-parity game, with $\mathcal{A}=\langle(S,E),(S_1,S_2,S_N,B_1,B_2),\pi\rangle$, having $N+1$ priorities and maximal odd priority, i.e., such that $|\mathrm{Pr}|=N+1$ and $\max(\mathrm{Pr})$ is odd. Assume the function $\mathbb{H}_{\mathcal{G}}$ of Definition 6.2.21 is well defined (see Remark 6.2.22). Then the following assertions hold for every $s\in S$:*

$$\begin{array}{lll}\mathrm{ZFC} & \vdash & Val^s_{\uparrow}(\mathcal{G})\leq\mathrm{lfp}(\mathbb{H}_{\mathcal{G}})(s),\ \textit{if } \mathcal{A} \textit{ is a standard } 2\tfrac{1}{2}\textit{-player arena}\\ \mathrm{ZFC}\ +\mathrm{MA}_{\aleph_1} & \vdash & Val^s_{\uparrow}(\mathcal{G})\leq\mathrm{lfp}(\mathbb{H}_{\mathcal{G}})(s).\end{array}$$

Proof. The proof technique we adopt closely resembles the one used to prove Theorem 6.1.7. As a first observation, note that the two assertions are well-defined only under the hypothesis mG-UM(Γ_{N+1}) and mG-UM(Δ_{N+1}) (see Definition 5.1.8) respectively, with Γ_{N+1} being the class of all $2\frac{1}{2}$-player meta-parity games with a standard $2\frac{1}{2}$-player arena (see Definition 4.1.5), $N+1$ priorities and maximal odd priority, and Δ_{N+1} the wider class of all $2\frac{1}{2}$-player meta-parity games with $N+1$ priorities and maximal odd priority. However, by application of Lemma 5.1.14 we know that ZFC $\vdash$ mG-UM(Γ_{N+1}). Moreover ZFC + $\mathrm{MA}_{\aleph_1}$ $\vdash$ mG-UM(Δ_{N+1}) holds, by Lemma 2.1.88 and the observations following Definition 5.1.8. Thus we omitted the hypotheses mG-UM(Γ_{N+1}) and mG-UM(Δ_{N+1}) from the statements.

We will prove the two assertions together in a uniform way, specifying when we use the set-theoretic assumption $\mathrm{MA}_{\aleph_1}$ in the general case, i.e., when $\mathcal{A}$ is not a standard $2\frac{1}{2}$-player arena.

Let us fix an arbitrary state $s\in S$. We prove the inequality $Val^s_{\uparrow}(\mathcal{G})\leq\mathrm{lfp}(\mathbb{H}_{\mathcal{G}})(s)$ holds by constructing, for every $\epsilon>0$ a strategy σ_2^{ϵ} for Player 2 in $\mathcal{G}$ such that the inequality

$$\bigsqcup_{\sigma_1} E(M^s_{\sigma_1,\sigma_2^{\epsilon}})\leq\mathrm{lfp}(\mathbb{H}_{\mathcal{G}})(s)+\epsilon$$

holds. This clearly implies the desired result. In what follows we simply denote with $\mathbb{H}$ the function $\mathbb{H}_{\mathcal{G}}$, and with ρ the value assignment $\rho_{\mathrm{lfp}(\mathbb{H})}$ (see Definition 6.2.20). As observed after Definition 6.2.21, the equality $\mathrm{lfp}(\mathbb{H}) = \mathsf{VAL}(\mathcal{G}^{-}_{\rho})$ holds.

The strategy σ^{ϵ}_{2} is constructed using the collection of δ-optimal strategies τ^{δ}_{2}, for $\delta > 0$ for Player 2 in the game $\mathcal{G}^{-}_{\rho}$. Thus the strategy τ^{δ}_{2} is such that, for every strategy τ_1 for Player 1 in $\mathcal{G}^{-}_{\rho}$ and every state $s \in S$, the following inequality holds:

$$\bigsqcup_{\tau_1} E(M^{s}_{\tau_1,\tau^{\delta}_{2}}) \leq \mathrm{lfp}(\mathbb{H})(s) + \delta.$$

The strategy σ^{ϵ}_{2} is defined, for every $\epsilon > 0$, as follows:

$$\sigma^{\epsilon}_{2}(\vec{s}) = \begin{cases} \tau^{\frac{\epsilon}{2}}_{2}(\vec{s}) & \text{if } \vec{s} \text{ does not contain states in } S_{\max} \\ \\ \sigma^{\frac{\epsilon}{2}\cdot\frac{1}{\#(j)}}_{2}(\vec{t}) & \text{if } \vec{s} = \vec{s}_j.\vec{t} \text{ for some } \vec{s}_j \in \mathbb{S}_{\mathsf{m}} \end{cases}$$

for every finite path with last state in S_2, where the function $\# : \mathbb{N} \to \mathbb{N}$ is specified as in Definition 2.2.9. Note how this inductive definition is well specified, since by Convention 6.2.6, $last(\vec{s}_j) \notin S_2$ and every finite path $\vec{s}$ not containing any state in $S_{\max}$ is necessarily not of the form $\vec{s} = \vec{s}_j.\vec{t}$, for $\vec{s}_j \in \mathbb{S}_{\mathsf{m}}$.

We are now going to show that, for every $\epsilon > 0$, the strategy σ^{ϵ}_{2} satisfies the desired inequality: for all $s \in S$, $\bigsqcup_{\sigma_1} E(M^{s}_{\sigma_1,\sigma^{\epsilon}_{2}}) \leq \mathrm{lfp}(\mathbb{H})(s) + \epsilon$. Let us fix an arbitrary strategy σ_1 for Player 1 in $\mathcal{G}$. We need to show that the equality

$$E(M^{s}_{\sigma_1,\sigma^{\epsilon}_{2}}) \leq \mathrm{lfp}(\mathbb{H})(s) + \epsilon \tag{6.4}$$

holds. Recall that, by definition, $E(M^{s}_{\sigma_1,\sigma^{\epsilon}_{2}}) = \mathbb{P}^{s}_{\sigma_1,\sigma^{\epsilon}_{2}}(\Phi)$, where $\mathbb{P}^{s}_{\sigma_1,\sigma^{\epsilon}_{2}}$ denotes the probability measure over $\mathcal{BP}$ induced by the Markov branching play $M^{s}_{\sigma_1,\sigma^{\epsilon}_{2}}$, and Φ denotes the set of winning branching plays for Player 1 in $\mathcal{G}$, i.e., the set Φ_{Pr}.

By Theorem 6.3.3, we know that $\Phi = \bigcup_{\alpha} \mathbb{W}^{-\alpha}_{\mathcal{G}}$. In what follows, for the sake of readability, we just denote with $\mathbb{U}^{\alpha}$ the set $\mathbb{W}^{-\alpha}_{\mathcal{G}}$, for every ordinal α.

Let us now consider separately the cases when $\mathcal{A}$ is a standard $2\frac{1}{2}$-player arena and when $\mathcal{A}$ is general instead. If $\mathcal{A}$ is a standard $2\frac{1}{2}$-player arena we know, from Lemma 6.3.9, that $\Phi = \bigcup_{\alpha<\omega} \mathbb{U}^{\alpha}$. Therefore by ω-continuity of all probability measures, we have that $\mathbb{P}^{s}_{\sigma_1,\sigma^{\epsilon}_{2}}(\Phi) = \bigsqcup_{\alpha<\omega} \mathbb{P}^{s}_{\sigma_1,\sigma^{\epsilon}_{2}}(\mathbb{U}^{\alpha})$. If instead the arena $\mathcal{A}$ is not a standard $2\frac{1}{2}$-player arena we know, from Lemma 6.3.6, that $\Phi = \bigcup_{\alpha<\omega_1} \mathbb{U}^{\alpha}$. It thus follows from $\mathrm{MA}_{\aleph_1}$ that $\mathbb{P}^{s}_{\sigma_1,\sigma^{\epsilon}_{2}}(\Phi) = \bigsqcup_{\alpha<\omega_1} \mathbb{P}^{s}_{\sigma_1,\sigma^{\epsilon}_{2}}(\mathbb{U}^{\alpha})$ (see Theorem 2.1.87 and Proposition 2.1.88). Therefore we have that in both cases, $\mathbb{P}^{s}_{\sigma_1,\sigma^{\epsilon}_{2}}(\Phi) = \bigsqcup_{\alpha<\omega_1} \mathbb{P}^{s}_{\sigma_1,\sigma^{\epsilon}_{2}}(\mathbb{U}^{\alpha})$, even if in the general case we need to invoke $\mathrm{MA}_{\aleph_1}$.

This equality allows us to set up a proof by transfinite induction for the desired inequality 6.4. We are going to show that for every countable ordinal α, the inequality

$$\mathbb{P}^s_{\sigma_1,\sigma_2^\epsilon}(\mathbb{U}^\alpha) \leq \text{lfp}(\mathbb{H})(s) + \epsilon \tag{6.5}$$

holds.

Suppose, by induction hypothesis, that inequality 6.5 holds for every ordinal $\beta < \alpha$. Let us define, for every $\vec{s_j} \in \mathbb{S}_{\mathsf{m}}$, the strategy σ_1^j which follows the behavior of σ_1 on histories having $\vec{s_j}$ as prefix, i.e., the strategy formally defined as follows:

$$\sigma_1^j(\vec{s}) = \begin{cases} \sigma_1(\vec{t}) & \text{if } \vec{t} = \vec{s_j}.\vec{s} \\ \sigma_1(\vec{s}) & \text{otherwise} \end{cases}$$

for every $\vec{s}$ with last state in S_1. Moreover we define the strategy τ_1, which behaves in $\mathcal{G}_\rho^-$ as σ_1 until a state in $S_{\max}$ is reached, as follows: $\tau_1(\vec{s}) = \sigma_1(\vec{s})$ if $\vec{s}$ does not contains states in $S_{\max}$. Note that once a state $s \in S_{\max}$ is reached in $\mathcal{G}_\rho^-$, the rest of the play does not depend on Player 1's choices, because s is a $\rho(s)$-valued leaf in $\mathcal{G}_\rho^-$ (see Definition 6.2.8). Therefore we can characterize the strategy σ_1 as follows:

$$\sigma_1(\vec{s}) = \begin{cases} \tau_1(\vec{s}) & \text{if } \vec{s} \text{ does not contain states in } S_{\max} \\ \sigma_1^j(\vec{t}) & \text{if } \vec{s} = \vec{s_j}.\vec{t} \text{ for some } \vec{s_j} \in \mathbb{S}_{\mathsf{m}} \end{cases}$$

Given this characterization for σ_1 and the definition of the strategies σ_2^ϵ, we have that for every $s \in S$, $M^s_{\sigma_1,\sigma_2^\epsilon} =_{\mathbb{S}_{\mathsf{m}}} M[M_i]_{i\in I}$, where the sub-Markov branching play M_i, rooted at s_i, is induced by the strategy profile $(\sigma_1^i, \sigma_2^{\frac{\epsilon}{2}\cdot\frac{1}{\#(i)}})$ for every $i \in I$. By induction hypothesis we know that, for every $\beta < \alpha$, the inequality 6.5

$$\mathbb{P}_{M_i}\big(\mathbb{U}^\beta\big) = \mathbb{P}^{s_i}_{\sigma_1^i,\sigma_2^{\gamma_i}}\big(\mathbb{U}^\beta\big) \;\leq\; \text{lfp}(\mathbb{H})(s_i) + \big(\frac{\epsilon}{2}\cdot\frac{1}{\#(i)}\big) \tag{6.6}$$

holds, where γ_i is a shorthand for $\frac{\epsilon}{2}\cdot\frac{1}{\#(i)}$. Furthermore note that $M[M^{s_i}_{\lambda_i}]$ (for $\lambda_i = \rho(s_i)$), the unique Markov branching play in $\mathcal{G}_\rho^-$ having the same Markov branching play of $M^s_{\sigma_1,\sigma_2^\epsilon}$ (see Remark 6.2.11), is induced by the strategy profile $(\tau_1, \tau_2^{\frac{\epsilon}{2}})$. Given this last observation, it follows from the fact that $\tau_2^{\frac{\epsilon}{2}}$ is an $\frac{\epsilon}{2}$-optimal strategy for Player 2 in $\mathcal{G}_\rho^-$, that the following inequality holds:

$$E(M[M^{s_i}_{\lambda_i}]_{i\in I}) \;\leq\; \mathsf{VAL}^s(\mathcal{G}_\rho^-) + \tfrac{\epsilon}{2},$$

or equivalently, from definitions of $\mathbb{H}$ and ρ,

$$E(M[M^{s_i}_{\lambda_i}]_{i\in I}) \;\leq\; \text{lfp}(\mathbb{H})(s) + \frac{\epsilon}{2} \tag{6.7}$$

Recall, from Remark 6.2.11, that the expected value $E(M^{s_i}_{\lambda_i})$ of the sub-Markov branchig play $M^{s_i}_{\lambda_i}$ in the game $\mathcal{G}^-_\rho$ is $\lambda_i = \rho(s_i) = \mathrm{lfp}(\mathbb{H})(s_i)$. Thus, by equation 6.6, the following equality holds:

$$\mathbb{P}_{M_i}\big(\bigcup_{\beta<\alpha}\mathbb{U}^\beta\big) \;\leq\; E(M^{s_i}_{\lambda_i}) + \big(\tfrac{\epsilon}{2}\cdot\tfrac{1}{\#(i)}\big) \;=\; \lambda_i + \big(\tfrac{\epsilon}{2}\cdot\tfrac{1}{\#(i)}\big).$$

for every $i \in I$. By application of Lemma 6.3.5, the following equality

$$\mathbb{P}^s_{\sigma_1,\sigma_2^\epsilon}\big(\mathbb{U}^\alpha\big) \;\leq\; E(M[M^{s_i}_{\lambda_i}]_{i\in I}) + \tfrac{\epsilon}{2},$$

holds. Hence, Given equation 6.7, the desired inequality 6.5

$$\mathbb{P}^s_{\sigma_1,\sigma_2^\epsilon}\big(\mathbb{U}^\alpha\big) \;\leq\; \mathrm{lfp}(\mathbb{H})(s) + \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2}.$$

and this concludes the proof. □

Corollary 6.3.11. *Let $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}}\rangle$ be a $2\frac{1}{2}$-player meta-parity game, with $\mathcal{A} = \langle (S,E),(S_1,S_2,S_N,B_1,B_2),\pi\rangle$, having $N+1$ priorities and maximal odd priority, i.e., such that $|\,\mathrm{Pr}\,| = N+1$ and $\max(\mathrm{Pr})$ is odd. Assume the function $\mathbb{H}_\mathcal{G}$ of Definition 6.2.21 is well defined (see Remark 6.2.22). Then the following assertions hold:*

ZFC	$\vdash$	*$\mathcal{G}$ is determined, if $\mathcal{A}$ is a standard $2\frac{1}{2}$-player arena*
ZFC $+\,\mathrm{MA}_{\aleph_1}$	$\vdash$	*$\mathcal{G}$ is determined.*

Proof. By Theorem 6.2.24 we know that $\mathsf{VAL}_\downarrow(\mathcal{G})$ and $\mathsf{VAL}_\uparrow(\mathcal{G})$ are both fixed points of $\mathbb{H}_\mathcal{G}$. The result then trivially follows from Theorem 6.3.10. □

We now prove that every $2\frac{1}{2}$-player meta-parity game with $N+1$ priorities and maximal odd priority satisfy the leaf monotonicity property of Definition 6.2.3.

Lemma 6.3.12. *Let $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}}\rangle$ be a $2\frac{1}{2}$-player meta-parity game, with $\mathcal{A} = \langle (S,E),(S_1,S_2,S_N,B_1,B_2),\pi\rangle$, having $N+1$ priorities and maximal odd priority, i.e., such that $|\,\mathrm{Pr}\,| = N+1$ and $\max(\mathrm{Pr})$ is odd. Assume the function $\mathbb{H}_\mathcal{G}$ of Definition 6.2.21 is well defined (see Remark 6.2.22), and that $\mathcal{G}^-_\rho$ satisfies the leaf monotonicity property for every value assignment ρ. Then the following assertions hold:*

ZFC	$\vdash$	*If $\mathcal{A}$ is a standard $2\frac{1}{2}$-player arena, then $\mathcal{G}$ satisfies the leaf monotonicity property.*
ZFC $+\,\mathrm{MA}_{\aleph_1}$	$\vdash$	*$\mathcal{G}$ satisfies the leaf monotonicity property.*

Proof. Let $\{s_n\}_{n\in\mathbb{N}}$ be a collection of λ_n-valued leaves in $\mathcal{G}$, with $\lambda_n \in [0,1]$. Let $\{\gamma_n\}_{n\in\mathbb{N}}$ such that $\gamma_n \geq \lambda_n$ and let $\mathcal{L}$ be the game $\mathcal{G}$ where the leaf s_n is turned in a γ_n-valued leaf, for every $n \in \mathbb{N}$. We need to prove that $\mathsf{VAL}(\mathcal{G}) \leq \mathsf{VAL}(\mathcal{L})$. Let us consider the unfolded games $\mathcal{G}^-_\rho$ and $\mathcal{L}^-_\rho$, for some value assignment ρ. Note that every state s_n is a λ_n-valued (γ_n-valued) leaf in $\mathcal{G}^-_\rho$ ($\mathcal{L}^-_\rho$). This is because the unfolding of a game can only introduce new leaves. Moreover note that $\mathcal{L}^-_\rho$ is precisely the game $\mathcal{G}^-_\rho$ where the every s_n leaf is turned in a γ_n-valued leaf. By hypothesis $\mathcal{G}^-_\rho$ and $\mathcal{L}^-_\rho$ satisfies the leaf monotonicity property. Therefore the inequality $\mathsf{VAL}(\mathcal{G}^-_\rho) \leq \mathsf{VAL}(\mathcal{L}^-_\rho)$ holds for every value assignment ρ. Let us define $f = \mathrm{lfp}(\mathbb{H}_\mathcal{G})$ and $g = \mathrm{lfp}(\mathbb{H}_\mathcal{L})$. By application of Theorem 6.2.24 and Corollary 6.3.11 we have that $\mathsf{VAL}(\mathcal{G}) = f$ and $\mathsf{VAL}(\mathcal{L}) = g$. We also know, by the observation following Definition 6.2.21, that $f = \mathsf{VAL}(\mathcal{G}^-_{\rho_f})$ and $g = \mathsf{VAL}(\mathcal{L}^-_{\rho_g})$. The result then follows. ☐

We conclude this section by considering the class of $2\frac{1}{2}$-player meta-parity games with $N+1$ priorities, having maximal *even* priority. By applications of Lemma 5.1.16 and Lemma 2.3.57, the duals of the results concerning $2\frac{1}{2}$-player meta-parity games having maximal odd priority hold, if an even priority is considered instead. We summarize in the following lemma the interesting results about $2\frac{1}{2}$-player meta-parity games having maximal even priority.

Lemma 6.3.13. *Let $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ be a $2\frac{1}{2}$-player meta-parity game, with $\mathcal{A} = \langle (S,E), (S_1, S_2, S_N, B_1, B_2), \pi \rangle$, having $N+1$ priorities and maximal even priority, i.e., such that $|\,\mathrm{Pr}\,| = N+1$ and $\max(\mathrm{Pr})$ is even. Then the following assertions holds:*

1. *the winning set Φ_{Pr} of $\mathcal{G}$ can be characterized as the greatest fixed point of the operator $\mathbb{W}^-_\mathcal{G} : 2^{\mathcal{BP}} \to 2^{\mathcal{BP}}$, i.e., $\Phi_{\mathrm{Pr}} = \mathrm{gfp}(\mathbb{W}^-_\mathcal{G})$,*
2. *if $\mathcal{A}$ is a standard $2\frac{1}{2}$-player game arena and the function $\mathbb{H}_\mathcal{G}$ of Definition 6.2.21 is well defined, then*
   - $\mathrm{gfp}(\mathbb{W}^-_\mathcal{G}) = \bigcap_{\beta<\omega} \mathbb{W}^{-\beta}_\mathcal{G}$, *with* $\mathbb{W}^{-\alpha}_\mathcal{G} = \mathbb{W}^-_\mathcal{G}(\bigcap_{\gamma<\alpha} \mathbb{W}^{-\gamma}_\mathcal{G})$, *for α, β and γ ordinals.*
   - $\mathrm{ZFC} \vdash \ \mathsf{VAL}^s_\downarrow(\mathcal{G}) = \mathsf{VAL}^s_\uparrow(\mathcal{G}) = \mathrm{gfp}(\mathbb{H}_\mathcal{G})(s)$.
   - *If $\mathcal{G}^-_\rho$ satisfies the leaf monotonicity property for every ρ, then* $\mathrm{ZFC} \vdash$ *$\mathcal{G}$ satisfies the open leaf property.*

3. *if $\mathcal{A}$ is not a standard $2\frac{1}{2}$-player game arena and the function $\mathbb{H}_{\mathcal{G}}$ of Definition 6.2.21 is well defined, then*
   - $\mathrm{gfp}(\mathbb{W}^{-}_{\mathcal{G}}) = \bigcap_{\beta<\omega_1} \mathbb{W}^{-\beta}_{\mathcal{G}}$,
   - $\mathrm{ZFC} + \mathrm{MA}_{\aleph_1} \vdash \mathsf{VAL}^{s}_{\downarrow}(\mathcal{G}) = \mathsf{VAL}^{s}_{\uparrow}(\mathcal{G}) = \mathrm{gfp}(\mathbb{H}_{\mathcal{G}})(s)$.
   - *If $\mathcal{G}^{-}_{\rho}$ satisfies the leaf monotonicity property for every ρ,* $\mathrm{ZFC} + \mathrm{MA}_{\aleph_1} \vdash$ *$\mathcal{G}$ satisfies the open leaf property.*

Proof. As is the proof of Lemma 6.1.11, the main observation is that $\overline{\Phi_{\mathrm{Pr}}} = \Phi_{\neg\,\mathrm{Pr}}$, where the assignment $\neg\,\mathrm{Pr}$, specified as in Definition 2.3.57, assigns odd priority to all states in S. The result follows by routine application of Theorem 5.1.16. □

6.4 Conclusion of the inductive proof

In this section we summarize and conclude the inductive proof carried out in the previous sections.

Theorem 6.4.1. *For every $2\frac{1}{2}$-player meta-parity game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ with a standard $2\frac{1}{2}$-player arena $\mathcal{A}$, the following assertions hold:*

1. $\mathsf{VAL}_{\downarrow}(\mathcal{G}) = \mathsf{VAL}_{\uparrow}(\mathcal{G})$,
2. *if $|\,\mathrm{Pr}\,| = 1$, then*
   (a) *if* $\max(\mathrm{Pr})$ *is odd, then* $\mathsf{VAL}(\mathcal{G}) = \mathrm{lfp}(\mathbb{F}_{\mathcal{G}})$,
   (b) *if* $\max(\mathrm{Pr})$ *is even, then* $\mathsf{VAL}(\mathcal{G}) = \mathrm{gfp}(\mathbb{F}_{\mathcal{G}})$,
3. *if $|\,\mathrm{Pr}\,| > 1$, then*
   (a) *if* $\max(\mathrm{Pr})$ *is odd, then* $\mathsf{VAL}(\mathcal{G}) = \mathrm{lfp}(\mathbb{H}_{\mathcal{G}})$,
   (b) *if* $\max(\mathrm{Pr})$ *is even, then* $\mathsf{VAL}(\mathcal{G}) = \mathrm{gfp}(\mathbb{H}_{\mathcal{G}})$,
4. *$\mathcal{G}$ satisfies the leaf monotonicity property of Definition 6.2.3.*

Proof. In accordance with Definition 4.1.5 and Definition 5.1.13, a standard $2\frac{1}{2}$-player arena is both finitely branching in B_1 and in B_2. It then follows by Corollary 6.1.8 and Lemma 6.1.11 that if $\mathcal{G}$ has only one priority, i.e., if $|\,\mathrm{Pr}\,| = 1$, then it is determined under deterministic strategies in ZFC-alone and, similarly, we know from Lemma 6.2.4 that $\mathcal{G}$ satisfies the leaf monotonicity property. Suppose by induction that both properties hold for every $2\frac{1}{2}$-player meta-parity game $\mathcal{G}$ with a standard $2\frac{1}{2}$-player arena $\mathcal{A}$ having $N>1$ priorities. Consider a $2\frac{1}{2}$-player meta-parity game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ with a standard $2\frac{1}{2}$-player arena $\mathcal{A}$ and $N+1$ priorities. It is immediate to observe that, for every value assignment ρ (see Definition 6.2.7), the game $\mathcal{G}_\rho^-$ is a $2\frac{1}{2}$-player meta-parity game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ with a standard $2\frac{1}{2}$-player arena and just N priorities. Therefore the function $\mathbb{H}_{\mathcal{G}}$ is well defined (see Remark 6.2.22) by inductive hypothesis, and $\mathcal{G}_\rho^-$ satisfies the open monotonicity property for every value assignment ρ. The desired result then follow by application of Corollary 6.3.11, Lemma 6.3.12 and Lemma 6.3.13. □

Recall from Definition 4.1.5 that $2\frac{1}{2}$-player meta-parity games with standard $2\frac{1}{2}$-player game arenas are just ordinary $2\frac{1}{2}$-player parity games. Thus the first point of Theorem 6.4.1 follows already from the determinacy results of [74], as discussed in Section 2.3.4.1. However our proof is interesting because we characterize the values of a $2\frac{1}{2}$-player parity game as least (greatest) fixed point of appropriate operators. This useful result will be exploited in the next Chapter.

Theorem 6.4.2 ($\mathrm{MA}_{\aleph_1}$). *For every $2\frac{1}{2}$-player meta-parity game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$, the following assertions hold:*

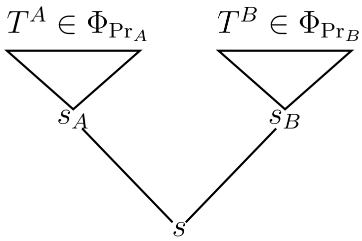
1. $\mathsf{VAL}_{\downarrow}(\mathcal{G}) = \mathsf{VAL}_{\uparrow}(\mathcal{G})$,
2. *if* $|\,\mathrm{Pr}\,| = 1$, *then*
   (a) *if* $\max(\mathrm{Pr})$ *is odd, then* $\mathsf{VAL}(\mathcal{G}) = \mathrm{lfp}(\mathbb{F}_{\mathcal{G}})$,
   (b) *if* $\max(\mathrm{Pr})$ *is even, then* $\mathsf{VAL}(\mathcal{G}) = \mathrm{gfp}(\mathbb{F}_{\mathcal{G}})$,
3. *if* $|\,\mathrm{Pr}\,| > 1$, *then*
   (a) *if* $\max(\mathrm{Pr})$ *is odd, then* $\mathsf{VAL}(\mathcal{G}) = \mathrm{lfp}(\mathbb{H}_{\mathcal{G}})$,
   (b) *if* $\max(\mathrm{Pr})$ *is even, then* $\mathsf{VAL}(\mathcal{G}) = \mathrm{gfp}(\mathbb{H}_{\mathcal{G}})$,
4. *$\mathcal{G}$ satisfies the leaf monotonicity property of Definition 6.2.3.*

Proof. Similar to the proof of Theorem 6.4.1. □

The following is a useful result which proves that the upper-bound on the complexity of winning sets in $2\frac{1}{2}$-player meta-games given in Corollary 5.1.7 is strict.

Theorem 6.4.3. *There exists a $2\frac{1}{2}$-player meta-parity game $\mathcal{G} = \langle \mathcal{A}, \Phi_{\mathrm{Pr}} \rangle$ whose winning set Φ_{Pr} is neither a $\mathbf{\Sigma}^1_1$ nor a $\mathbf{\Pi}^1_1$ set.*

Proof. We now, by Lemma 6.3.7, that there exists a finite $2\frac{1}{2}$-player meta-parity game $\mathcal{G}_A = \langle \mathcal{A}_A, \Phi_{\mathrm{Pr}_A} \rangle$ whose winning set (and in particular the set of winning branching plays rooted at a designated state s_A) is $\mathbf{\Pi}^1_1$-complete. Thus, by Lemma 5.1.16 and the fact that $2\frac{1}{2}$-player meta-parity games are closed under the negation operation of Definition 5.1.15, there exists a $2\frac{1}{2}$-player game $\mathcal{G}_B = \langle \mathcal{A}_B, \Phi_{\mathrm{Pr}_B} \rangle$ whose winning set (and in particular the set of branching plays rooted at a designated state s_B) is $\mathbf{\Sigma}^1_1$-complete. Suppose without loss of generality that the set of states of the two arenas $\mathcal{A}_A$ and $\mathcal{A}_B$ are disjoint. Consider the game arena $\mathcal{A}$ obtained by merging $\mathcal{A}_A$ and $\mathcal{A}_B$ and adding a new branching state s under the control of Player 2 having s_A and s_B as successor states. Define $\mathrm{Pr}(s) = \mathrm{Pr}_A(s)$ if s is in $\mathcal{A}_A$, $\mathrm{Pr}(s) = \mathrm{Pr}_B(s)$ if s in is $\mathcal{A}_B$ and $\mathrm{Pr}(s_0) = 0$. Then the set of winning branching plays rooted at s can be depicted as follows:



It is then trivial to prove that $\Phi_{\mathrm{Pr}_A} \leq_W \Phi_{\mathrm{Pr}}$ and $\Phi_{\mathrm{Pr}_A} \leq_W \Phi_{\mathrm{Pr}}$, where $\leq_W$ is the Wadge order specified as in Definition 2.1.56. The result then follows. ☐

6.5 Summary of results

We conclude this chapter by summarizing and commenting on the obtained results.

All the results of this chapter converges into Theorem 6.4.2, which states that every $2\frac{1}{2}$-player meta-parity game is determined under deterministic strategies. Moreover the value $\mathsf{VAL}(\mathcal{G})$, of a $2\frac{1}{2}$-player meta-parity game $\mathcal{G}$, is the least (greatest) fixed point on an appropriate monotone endomap. This is one of the main results of the thesis, and constitutes an important starting point, for future work, towards the study of other theoretically interesting classes of $2\frac{1}{2}$-player meta-games such as prefix independent $2\frac{1}{2}$-player meta-games or, more generally, the larger class (see Proposition 5.2.2) of all $2\frac{1}{2}$-player tree games with subtree-monotone winning sets (see, e.g., Question 4.3.19).

However note that the general result has been proven only working within ZFC set theory extended with Martin's Axiom at $\aleph_1$ ($\mathrm{MA}_{\aleph_1}$), thus not in a conventional mathematical setting. This is quite uncommon practice in theoretical computer science, as opposed to several fields of mathematics where consistency results are often discussed. It is thus quite important to discuss why, and how, the non-standard axiom $\mathrm{MA}_{\aleph_1}$ is used in our proof. There are essentially two distinct uses we make of $\mathrm{MA}_{\aleph_1}$. The first one is to ensure that the assertion mG-UM(Γ_p) holds (see Definition 5.1.8), where Γ_p is the set of all $2\frac{1}{2}$-player meta-parity games. Indeed, as a consequence of lemmas 5.1.6 and 2.1.88, $\mathrm{MA}_{\aleph_1}$ is sufficient to prove that the winning set of every $2\frac{1}{2}$-player meta-game is universally measurable. As already discussed in last paragraph of Section 5.4, we do not know if mG-UM(Γ_p) holds in ZFC alone, even though we think this is quite plausible. The second use we make of $\mathrm{MA}_{\aleph_1}$ is required by the kind of proof technique we developed, which is based on a transfinite induction up to the first uncountable ordinal ω_1. The proof is based on an inductive characterization of the winning set Φ_{Pr} of a $2\frac{1}{2}$-player meta-parity game as the ω_1-limit of a chain of approximants ($\mathbb{W}^{\alpha}_{\mathcal{G}}$ and $\mathbb{W}^{-\alpha}_{\mathcal{G}}$ in Lemma 6.1.2 and Lemma 6.1.2, respectively). We crucially use $\mathrm{MA}_{\aleph_1}$ to obtain the equality $\mu(\Phi_{\mathrm{Pr}}) = \bigsqcup_{\alpha<\omega_1} \mu\big(\mathbb{W}^{-\alpha}_{\mathcal{G}}\big)$, where μ is the probability measure associated with a Markov branching play. This equality is indeed one of the consequences $\mathrm{MA}_{\aleph_1}$ (see Proposition 2.1.88).

While the validity of mG-UM(Γ_p) is necessary, simply to make sense of the general notion of $2\frac{1}{2}$-player meta-parity games, the condition of ω_1-continuity on probability measures is, in principle, just required to support the inductive proof we developed. In particular, from our result, one should not derive any sort of (necessary) connection between determinacy of $2\frac{1}{2}$-player meta-parity games and the set theoretic axiom $\mathrm{MA}_{\aleph_1}$, or the negation of the *Continuum Hypothesis*, one of its consequences. As a matter of fact, if Question 4.3.20 has a positive answer for $\mathbf{\Delta}^1_2$ sets, then every prefix independent $2\frac{1}{2}$-player meta-game, and thus every $2\frac{1}{2}$-player meta-parity game, is determined under the set-theoretic assumption of $\mathbf{\Delta}^1_2$-determinacy. It is known that ZFC + $\mathbf{\Delta}^1_2$-determinacy does not imply CH.

Even though the axiom $\mathrm{MA}_{\aleph_1}$ is required in our proof, in the general case, we identified classes of $2\frac{1}{2}$-player meta-parity games for which Theorem 6.4.2 could be proved in ZFC alone:

1. $2\frac{1}{2}$-player meta-parity games with just one odd (even) priority (see Theorem 6.1.7) finitely branching in B_2 (B_1). These games could be named $2\frac{1}{2}$-player *meta-reachability* (*meta-safety*) games.

2. standard $2\frac{1}{2}$-player parity games.

In particular the second point, stated formally as Theorem 6.4.1, constitutes an interesting result, because our proof does not invoke the general determinacy axioms of [74], and provide useful information about the values of the games.

An important observation about $2\frac{1}{2}$-player meta-parity games is that they are not, in general, determined under positional strategies, even when the $2\frac{1}{2}$-player game arena is finite (see Lemma 6.1.9). This result contrasts with the corresponding one for standard $2\frac{1}{2}$-player parity games, which are indeed positionally determined when played on finite arenas (see, e.g., [78] or [117]).

Perhaps the lack of positional determinacy, together with the technical efforts necessary to deal with $2\frac{1}{2}$-player meta-parity games, constitute a manifestation of the complexity of this class of games. Formally, this is partially captured by some of our side-results obtained from the inductive characterization of the winning sets Φ_{Pr}. In particular, in Theorem 6.4.3, we gave an example of 2-player meta-parity game whose winning set is strictly in $\mathbf{\Delta}^1_2$ in the Projective hierarchy. Clearly $\mathbf{\Delta}^1_2$ sets can be extremely complex (see, e.g., Lemma 2.1.81), and in particular are far more complex than standard $2\frac{1}{2}$-player parity winning sets which are $\mathbf{\Delta}^0_3$ sets (see Lemma 2.3.56), quite low in the Borel hierarchy.

Chapter 7

Game Semantics

In Chapter 4 we defined the class of $2\frac{1}{2}$-player games, and in Chapter 5 we identifed an interesting sub-class of tree games named $2\frac{1}{2}$-player meta-games. In Chapter 6 we proved that every $2\frac{1}{2}$-player meta-parity game is determined. We shall now use these results to define a *game semantics*, in terms of $2\frac{1}{2}$-player meta-parity games, for the probabilistic modal μ-calculi of Section 3.3.

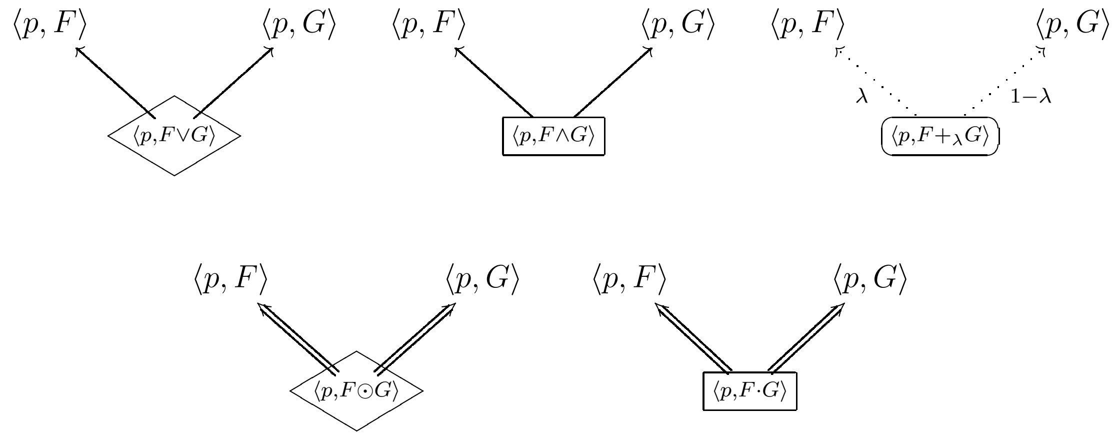
This chapter is organized as follows. In Section 7.1 we define the game semantics for $\text{pL}\mu^{\odot}_{\oplus}$ and its fragments and prove that the denotational and game semantics coincide on all models. In Section 7.2 we discuss a few examples of interesting $\text{pL}\mu^{\odot}$ formulas, using the denotational or the game semantics to explain their meaning, thus taking advantage of the results of Section 7.1. The examples show how important properties of PLTS's can be expressed in our new logic $\text{pL}\mu^{\odot}$, or actually in its fragment $\text{pL}\mu^{\{0,1\}}$, and provide important justifications for all our work. Some examples will also be used to expose interesting properties of the logic, such as the failure of the so called *finite model property*. We conclude Section 7.2 by proving that the *qualitative* fragment of PCTL and *full* PCTL (see Definition 3.2.8) can be encoded in the logics $\text{pL}\mu^{\{0,1\}}$ and $\text{pL}\mu^{[0,1]}$ respectively. In Section 7.3 we summarize and comment on the results obtained in this chapter.

7.1 Formal definitions and main result

As for the logics $\text{L}\mu$ and $\text{pL}\mu$ (see sections 3.1.2 and 3.2.4), the game semantics of the logic $\text{pL}\mu^{\odot}_{\oplus}$ and its fragments is defined, given a PLTS $\mathcal{L}=\langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$, an interpretation of the variables ρ and a formula F, by constructing a game $\mathcal{G}(F,\rho)$

and defining $(\!|F|\!)^{\mathcal{L}}_{\rho}: P \to [0,1]$ as assigning to the state $p \in P$ the value of the game $\mathcal{G}(F,\rho)$ at the game state $\langle p, F\rangle$. In the case of the logic Lμ, $\mathcal{G}(F,\rho)$ is an ordinary 2-player parity game. In the case of the logic pLμ (and as we shall see also of pL$\mu \cup \{+_\lambda\}$), $\mathcal{G}(F,\rho)$ is an ordinary $2\frac{1}{2}$-player parity game. In the case of pL$\mu^{\odot}_{\oplus}$ and its fragments pL$\mu^{\odot}$, pL$\mu_{\oplus}$, pL$\mu^{\{0,1\}}$ and pL$\mu^{[0,1]}$, $\mathcal{G}(F,\rho)$ is a $2\frac{1}{2}$-player meta-parity game.

The class of $2\frac{1}{2}$-player meta-parity games has been designed to match the informal and intuitive description of the game semantics for pL$\mu^{\odot}$ given in Section 3.3.3. The result of Theorem 5.2.10 immediately suggests the game interpretation of the pL$\mu^{\odot}$ operators $\{\vee, \wedge, +_\lambda, \odot, \cdot\}$ as Player 1 nodes (S_1), Player 2 nodes (S_2), probabilistic nodes (S_N), Player 1 branching nodes (B_1) and Player 2 branching nodes (B_2), respectively. These states can be represented as follows:

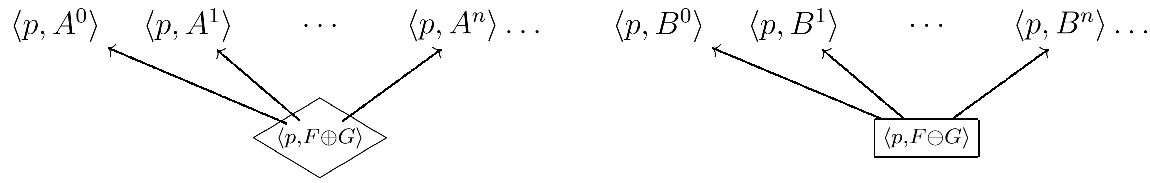


Note that, in the particular case when $F = G$, the game states $\langle p, F \odot G\rangle$ and $\langle p, F \cdot G\rangle$ have only one child: $\langle p, F\rangle$. This implies, in light of the result of Theorem 5.2.10, that the game interpretation of the formula $F \odot F$ does not agree with the corresponding denotational meaning. This small issue can be easily circumvented by, e.g., working with $2\frac{1}{2}$-player meta-parity games whose transition relations are multisets. In what follows we solve this problem in a different, and possibly simpler, way by restricting our attention to a class of formulas where this problem cannot arise.

Definition 7.1.1. A pL$\mu^{\odot}_{\oplus}$ formula F is in *product simple form* if every subformula $G \in Sub(F)$ of the form $G_1 \star G_2$, with $\star \in \{\odot, \cdot, \oplus, \ominus\}$, is such that $G_1 \neq G_2$, i.e., the two subformulas G_1 and G_2 are syntactically different. Every pL$\mu^{\odot}_{\oplus}$ formula F can be transformed in a semantically equivalent formula in product simple form by, e.g., inductively replacing a subformula of the form $G_1 \star G_1$ with the (semantically) equivalent formula $G_1 \star (G_1 \vee G_1)$.

In what follows we restrict our attention, without any loss of generality, to normal pL$\mu^{\odot}_{\oplus}$ formulas (see Definition 3.1.4) in product simple form.

The game interpretation of the modalities $\{\langle a\rangle, [a]\}$ is given as in pLμ games (see Section 3.2.4) and the role of the fixed point operators $\{(\mu X.), (\nu X.)\}$ is captured in the game semantics by appropriate parity assignments. As discussed in Section 3.3.3, although the pL$\mu^{\odot}_{\oplus}$ operators $\{\oplus, \ominus\}$ do not seem to have a clear probabilistic meaning, they satisfy the useful property expressed by Lemma 3.3.16. This allows us to interpret the operators $\{\oplus, \ominus\}$ as game states under the control of Player 1 and Player 2 respectively, depicted as follows:



where the formulas A^n and B^n are defined by mutual induction as follows: $A^0 = F$, $B^0 = G$, $A^{n+1} = A^n \odot B^n$ and $B^{n+1} = A^n \cdot B^n$. It then follows that the game associated with a pL$\mu^{\odot}_{\oplus}$ formula F contains, in general, states of the form $\langle p, G\rangle$ with $G \notin Sub(F)$. In the following definition we describe the set of formulas, denoted by $Sub^{\oplus,\ominus}(F)$, which can appear in the game associated with a pL$\mu^{\odot}_{\oplus}$ formula F.

Definition 7.1.2. Given a pL$\mu^{\odot}_{\oplus}$ formula F we denote with $Sub^{\oplus,\ominus}(F)$ the smallest set of pL$\mu^{\odot}_{\oplus}$ formulas containing $Sub(F)$ and such that if $G \oplus H$ or $G \ominus H$ are in $Sub^{\oplus,\ominus}(F)$ then the set of formulas $\{A^{G,H}_n, B^{G,H}_n\}_{n\in\mathbb{N}}$ is contained in $Sub^{\oplus,\ominus}(F)$, where the $\mathbb{N}$-indexed set of formulas $\{A^{G,H}_n\}_{n\in\mathbb{N}}$ is defined by mutual induction with $\{B^{G,H}_n\}_{n\in\mathbb{N}}$ as follows: $A^{G,H}_0 = G$, $B^{G,H}_0 = H$, $A^{G,H}_{n+1} = A^{G,H}_n \odot B^{G,H}_n$ and $B^{G,H}_{n+1} = A^{G,H}_n \cdot B^{G,H}_n$. It is going to be useful to the define the following partial order on $Sub^{\oplus,\ominus}(F)$:

1. $G, H \sqsubseteq G \star H$, for $\star \in \{\vee, \wedge, \odot, \cdot\}$,

2. $A^{G,H}_n, B^{G,H}_n \sqsubseteq G \star H$, for all $n \in \mathbb{N}$ and $\star \in \{\oplus, \ominus\}$.

3. $G \sqsubseteq \langle a\rangle G$ and $G \sqsubseteq [a]\, G$,

4. $G \sqsubseteq \mu X.G$ and $G \sqsubseteq \mu X.G$.

It is clear that $(Sub^{\oplus,\ominus}(F), \sqsubseteq)$ is well-founded (see Definition 2.1.7). Note that if F is in product simple form, then $Sub^{\oplus,\ominus}(F)$ does not contain formulas of the form $G \cdot G$ or $G \odot G$.

We are now ready to discuss how the game $\mathcal{G}(F, \rho)$ associated with a PLTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a \in L}\rangle$, a pL$\mu^{\odot}_{\oplus}$ formula F and a $[0,1]$-interpretation of the variables ρ, is constructed.

Definition 7.1.3 (pL$\mu^{\odot}_{\oplus}$ Games)**.** Let $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a \in L}\rangle$ be a PLTS, F a pL$\mu^{\odot}_{\oplus}$ formula and ρ a $[0,1]$-interpretation of the variables. The game $\mathcal{G}(F, \rho)$ is a $2\frac{1}{2}$-player meta-parity game $\langle \mathcal{A}, \Phi_{\mathrm{Pr}}\rangle$, with $\mathcal{A} = \langle (S, E), (S_1, S_2, S_N, B_1, B_2), \pi\rangle$, defined as follows. The set S is defined as $S = \{\top, \bot\} \cup \big((P \cup \mathcal{D}(\mathcal{L})) \times Sub^{\oplus,\ominus}(F)\big)$, where $\mathcal{D}(\mathcal{L})$ is the set of probability distributions in $\mathcal{L}$ (see Definition 3.2.3) and $\{\top, \bot\}$ a distinct pair of states which we introduce for technical convenience. Thus note that, in general, the set S is countably infinite even when $\mathcal{L}$ is finite (see Definition 3.2.1). The game states $\{\top, \bot\}$ are terminal in $\mathcal{G}(F, \rho)$, i.e., $E(\top) = E(\bot) = \emptyset$. As in pL$\mu$ games, the states of the form $\langle d, G\rangle$, for $d \in \mathcal{D}(\mathcal{L})$, are in S_N, i.e., under the control of Nature which moves to the state $\langle q, G\rangle$ with probability $d(q)$. This is formalized by specifying $\pi(\langle d, G\rangle)(\langle q, G\rangle) = d(q)$. The states of the form $\langle p, G\rangle$, for $p \in P$ and $G \in Sub^{\oplus,\ominus}(F)$, are specified as follows:

I. $\langle p, X\rangle$, with $X \in free(F)$, is in S_N, i.e., it is under the control of Nature which moves to the state $\top$ with probability $\rho(X)(p)$ and to the state $\bot$ with probability $1 - \rho(X)(p)$. Thus, $\pi(\langle p, X\rangle)(\top) = \rho(X)(p)$ and $\pi(\langle p, X\rangle)(\bot) = 1 - \rho(X)(p)$.

II. $\langle p, G_1 \vee G_2\rangle$ is in S_1 and $E(\langle p, G_1 \vee G_2\rangle) = \{\langle p, G_1\rangle, \langle p, G_2\rangle\}$.

III. $\langle p, G_1 \wedge G_2\rangle$ is in S_2 and $E(\langle p, G_1 \wedge G_2\rangle) = \{\langle p, G_1\rangle, \langle p, G_2\rangle\}$.

IV. $\langle p, \langle a\rangle G\rangle$ is in S_1 and $E(\langle p, \langle a\rangle G\rangle) = \{\langle d, G\rangle \mid p \xrightarrow{a} d\}$. Note that if $p \stackrel{a}{\not\longrightarrow}$, then $\langle p, \langle a\rangle G\rangle$ is a terminal state.

V. $\langle p, [a]\, G\rangle$ is in S_2 and $E(\langle p, [a]\, G\rangle) = \{\langle d, G\rangle \mid p \xrightarrow{a} d\}$.

VI. $\langle p, \sigma X.G\rangle$, for $\sigma \in \{\mu, \nu\}$, is in S_N and has only one successor state $\langle p, G\rangle$, i.e., $E(\langle p, \sigma X.G\rangle) = \{\langle p, G\rangle\}$. Hence $\pi(\langle p, \sigma X.G\rangle)(\langle p, G\rangle) = 1$.

VII. $\langle p, X\rangle$, with $X \in bound(F)$, is in S_N and has only one successor state $\langle p, G\rangle$, where G is the unique (since F is normal) subformula of F bound by a fixed point operator ($\sigma X.$) in F, for $\sigma \in \{\mu, \nu\}$. Hence, $\pi(\langle p, X\rangle)(\langle p, G\rangle) = 1$.

VIII. $\langle p, G_1 +_\lambda G_2\rangle$ is in S_N, i.e., it is under the control of Nature which move to the states $\langle p, G_1\rangle$ and $\langle p, G_2\rangle$ with probability λ and $1-\lambda$ respectively. This is formalized by specifying the function π as: $\pi(\langle p, G_1 +_\lambda G_2\rangle)(\langle p, G_1\rangle) = \lambda$ and $\pi(\langle p, G_1 +_\lambda G_2\rangle)(\langle p, G_2\rangle) = 1 - \lambda$.

IX. $\langle p, G_1 \odot G_2\rangle$ is in B_1, i.e., it is a branching state under the control of Player 1. The relation E is specified as $E(\langle p, G_1 \odot G_2\rangle) = \{\langle p, G_1\rangle, \langle p, G_2\rangle\}$.

X. $\langle p, G_1 \cdot G_2\rangle$ is in B_2 and $E(\langle p, G_1 \odot G_2\rangle) = \{\langle p, G_1\rangle, \langle p, G_2\rangle\}$.

XI. $\langle p, G_1 \oplus G_2\rangle$ is in S_1 and $E(\langle p, G_1 \oplus G_2\rangle) = \{\langle p, A_n^{G_1,G_2}\rangle\}_{n\in\mathbb{N}}$ where the formulas $A_n^{G_1,G_2}$ are specified as in Definition 7.1.2.

XII. $\langle p, G_1 \ominus G_2\rangle$ is in S_2 and $E(\langle p, G_1 \ominus G_2\rangle) = \{\langle p, B_n^{G_1,G_2}\rangle\}_{n\in\mathbb{N}}$ where the formulas $B_n^{G_1,G_2}$ are specified as in Definition 7.1.2.

Lastly, the parity assignment $\mathrm{Pr}: S \to \mathbb{N}$, inducing the parity set $\mathcal{W}_{\mathrm{Pr}}$ of completed paths in $\mathcal{A}$ as specified in Definition 2.3.54, is defined as follows:

1. $\mathrm{Pr}(\top) = 1$ and $\mathrm{Pr}(\bot) = 0$. This assignment models the fact that $\top$ and $\bot$ are terminal states winning for Player 1 and Player 2 respectively.

2. $\mathrm{Pr}(\langle p, \langle a\rangle G\rangle) = 0$ and $\mathrm{Pr}(\langle p, [a]\, G\rangle) = 1$. This assignment models the fact that if Player 1 gets stuck at states of the form $\langle p, \langle a\rangle G\rangle$ then they lose. Similarly for Player 2 at states $\langle p, [a]\, G\rangle$.

3. Let $\alpha : bound(F) \to \mathbb{N} \setminus \{0, 1\}$ an assignment of natural numbers (greater than 1) to the variables bound in F such that: $\alpha(X)$ is odd if X is bound in F by a least fixed point operator ($\mu X.$), $\alpha(X)$ is even if X is bound in F by a greatest fixed point operator ($\nu X.$) and $\alpha(X) > \alpha(Y)$ if X subsumes Y in F. We then specify the priority assigned to game states of the form $\langle p, X\rangle$, with $X \in bound(F)$, as follows: $\mathrm{Pr}(\langle p, X\rangle) = \alpha(X)$.

4. $Pr(s) = 0$, for all other game states.

Therefore a completed path is in $\mathcal{W}_{\mathrm{Pr}}$ if it is finite and ending in the terminal state $\top$ or in a terminal state $\langle p, [a]\, G\rangle$ (i.e., with $p \not\xrightarrow{a}$), or an infinite sequence of configurations whose dominant variable X (i.e., the unique variable bound in F occurring infinitely often in states of the form $\langle p, X\rangle$, for $p\in P$, and subsuming all other bound variables occurring infinitely often) is bound by a greatest fixed point in F.

The rules I-VII corresponds to the the pLμ fragment of pL$\mu^{\odot}_{\oplus}$ and indeed coincide with the description of pLμ games given in Section 3.2.3. Rule VIII describes the game interpretation of the $+_\lambda$ operator. Rules IX and X describe the pL$\mu^{\odot}$ operators $\{\odot, \cdot\}$. As discussed in Section 5.4, the branching states of the form $\langle p, G_1 \odot G_2\rangle$ can be understood as generating two independent subgames with the requirement for Player 1 to win in both subgames. Similarly, the branching states of the form $\langle p, G_1 \cdot G_2\rangle$ can be understood as generating two independent subgames with the requirement for Player 1 to win in at least one subgame. Lastly, rules XI and XII describe the game interpretation of the $\{\oplus, \ominus\}$ operators.

Observation 7.1.4. If F is a pL$\mu^{\odot}$ formula, we can restrict the set of game states in $\mathcal{G}(F,\rho)$ to the set $\{\top,\bot\} \cup \big((P \cup \mathcal{D}(\mathcal{L})) \times Sub(F)\big)$ which is finite when $\mathcal{L}$ is finite. This is because states of the form $\langle q, G\rangle$ with $G \notin Sub(F)$ and $G = A_n$ or $G = B_n$ (as specified in Definition 7.1.2) are not reachable in any play from states $\langle p, F\rangle$, with $p \in P$. Thus pL$\mu^{\odot}$ games enjoy the nice property of being finite when interpreted over finite PLTS's.

Observation 7.1.5. Following the previous observation, if F is a pL$\mu \cup \{+_\lambda\}$ formula, the sets of branching plays B_1 and B_2 in the game $\mathcal{G}(F,\rho)$ are empty. Therefore $\mathcal{G}(F,\rho)$ is a $2\frac{1}{2}$-player meta-parity game with a standard $2\frac{1}{2}$-player arena (see Definition 4.1.5), i.e., it is an ordinary $2\frac{1}{2}$-player parity game.

Observation 7.1.6. For every pL$\mu^{\odot}_{\oplus}$ formula F, the branching states $s\in B_1 \cup B_2$ in $\mathcal{G}(F,\rho)$ have exactly two successor states. Thus $\mathcal{G}(F,\rho)$ is a $2\frac{1}{2}$-player meta-parity game finitely branching in the branching nodes (see Definition 5.1.13).

Observation 7.1.7. The parity assignment Pr for the game $\mathcal{G}(F,\rho)$ has been defined in order to be as transparent as possible. Of course, it is possible to economize on the number of priorities used in the game. For instance, rather than defining $\Pr(\top) = 1$, one could add a self-loop to the state $\top$ (i.e., define $E(\top) = \{\top\}$) and consider the parity assignment $\Pr(\top) = 0$ instead. Clearly the two games are equivalent (see Definition 2.3.54). Also note that, with this transformation, the game $\mathcal{G}(F,\rho)$ would satisfy Convention 6.2.1. It is also immediate to verify that every game $\mathcal{G}(\sigma X.G, \rho)$, for $\sigma \in \{\mu, \nu\}$, satisfies Convention 5.2.1 (by specification of Pr in Definition 7.1.3) and Convention 6.2.6 (by point VII in Definition 7.1.3).

Observation 7.1.8. Given a PLTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a \in L}\rangle$, a $[0,1]$-interpretation $\rho : \mathcal{V} \to [0,1]$ and a $\text{pL}\mu^{\odot}_{\oplus}$ formula of the form $F = \mu X.G$ let $\mathcal{G}(F, \rho)$ be the associated $2\frac{1}{2}$-player meta-parity game. As a first observation, every state of the form $\langle p, Y\rangle$, with $Y \neq X$ a free variable in F, is a $\rho(Y)(p)$-valued leaf in the sense of Definition 6.2.2. This follows from point I in Definition 7.1.3. Moreover, note that the states of maximal priority in $\mathcal{G}(F, \rho)$ are necessarily the states of the form $\langle p, X\rangle$ with $p \in P$. Thus a value assignment for $\mathcal{G}(F, \rho)$ (see Definition 6.2.7) is a function of type $\{\langle p, X\rangle \mid p \in P\} \to [0,1]$ or, equivalently, a function in $[0,1]^P$. It follows from the previous points that, given any $f : P \to [0,1]$, the $\text{pL}\mu^{\odot}_{\oplus}$ game $\mathcal{G}(G, \rho[f/X])$ is precisely the *unfolding* of $\mathcal{G}(F, \rho)$ with f (see Definition 6.2.8).

Definition 7.1.9 (Game semantics)**.** Given a PLTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a \in L}\rangle$, a $[0,1]$-interpretation $\rho \in \mathcal{V} \to [0,1]^P$ and a $\text{pL}\mu^{\odot}_{\oplus}$ formula F, we define the *game semantics* of F over $\mathcal{L}$ as the map $(\!| F |\!)^{\mathcal{L}}_{\rho} \in [0,1]^P$, defined as follows:

$$(\!| F |\!)^{\mathcal{L}}_{\rho}(p) = \mathsf{VAL}\big(\mathcal{G}(F, \rho)\big)(\langle p, F\rangle.$$

Note that, in general, the function $(\!| F |\!)^{\mathcal{L}}_{\rho}$ is well defined only under the set-theoretic assumption $\text{MA}_{\aleph_1}$ (see Theorem 6.4.2). However, by observation 7.1.5 and application of Theorem 6.4.1, the function $(\!| F |\!)^{\mathcal{L}}_{\rho}$ is always well defined if F is a $\text{pL}\mu \cup \{+_\lambda\}$ formula.

We are now ready to prove the main result of this section[1].

Theorem 7.1.10. *Given any PLTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a \in L}\rangle$ and $[0,1]$-interpretation $\rho : \mathcal{V} \to [0,1]^P$ of the variables into $\mathcal{L}$, the following assertions hold:*

$$\begin{array}{lcl} \text{ZFC} & \vdash & [\![F]\!]^{\mathcal{L}}_{\rho} = (\!| F |\!)^{\mathcal{L}}_{\rho} \\ \text{ZFC} + \text{MA}_{\aleph_1} & \vdash & [\![G]\!]^{\mathcal{L}}_{\rho} = (\!| G |\!)^{\mathcal{L}}_{\rho} \end{array}$$

where F ranges over $\text{pL}\mu \cup \{+_\lambda\}$ formulas, and G over arbitrary $\text{pL}\mu^{\odot}_{\oplus}$ formulas.

Proof. The two assertions are proven in a uniform way. The only difference is that for $\text{pL}\mu \cup \{+_\lambda\}$ formulas, in light of Observation 7.1.5, we shall apply the

[1]The result, restricted to $\text{pL}\mu^{\odot}$ formulas, was announced at the 14th International Conference on *Foundations of Software Science and Computation Structures* (FoSSaCS) [83].

results of Theorem 6.4.1, while in the general case we shall apply the results of Theorem 6.4.2 which hold under the set-theoretic assumption $\text{MA}_{\aleph_1}$.

The proof proceeds by well-founded induction on $H \in Sub^{\oplus,\ominus}(G)$ (see Definition 7.1.2), showing that for every subformula $H \in Sub^{\oplus,\ominus}(G)$, the desired equality $[\![H]\!]_\rho = (\!| H |\!)_\rho$ holds for every $[0,1]$-interpretation of the variables ρ.

The case for $H = X$, with X a free variable, is trivial. For the case $H = H_1 \star H_2$, with $\star \in \{\vee, \wedge, +_\lambda, \odot, \cdot\}$, we know by induction hypothesis that, for every $p \in P$, $\mathsf{VAL}\big(\mathcal{G}(H_i,\rho)\big)(\langle p, H_i\rangle) = [\![H_i]\!]_\rho(p)$. Observe that a play in the game $\mathcal{G}(H,\rho)$ starting at $\langle p, H_i\rangle$ is identical to a play in $\mathcal{G}(H_i,\rho)$ starting at $\langle p, H_i\rangle$. The desired result then follows by application of Theorem 5.2.10.

The cases for $H = \langle a\rangle H_1$ and $H = [a]\, H_1$ can be proved with the same kind of argument. For example, in the case of $H = \langle a\rangle H_1$, one first shows, using the inductive hypothesis and of Theorem 5.2.10, that $\mathsf{VAL}\big(\mathcal{G}(H,\rho)\big)(\langle d, H_1\rangle) = \sum d(q) \cdot [\![H_1]\!]_\rho(q)$, for every probability distribution d. Then, again by applying Theorem 5.2.10, we have that $\mathsf{VAL}\big(\mathcal{G}(H,\rho)\big)(\langle p, H\rangle) = \bigsqcup_{p \xrightarrow{a} d} d(q) \cdot [\![H_1]\!]_\rho(q)$, and the result follows.

The cases for $H = H_1 \oplus H_2$ and $H = H_1 \ominus H_2$ can be proved with the same methodology by applying the result of Lemma 3.3.16. For example, in the case of $H = H_1 \oplus H_2$, the proof goes by showing, by applying the inductive hypothesis, that the value of the game $\mathcal{G}(H,\rho)$ at the states $\langle p, A_n^{H_1,H_2}\rangle$, i.e., the states reachable from $\langle p, H_1 \odot H_2\rangle$, is $a^n_{x,y}$ (see Lemma 3.3.16), where $x = [\![H_1]\!]_\rho(p)$ and $y = [\![H_2]\!]_\rho(p)$. By application of Theorem 5.2.10 we have that $\mathsf{VAL}\big(\mathcal{G}(H,\rho)\big)(\langle p, H_1 \odot H_2\rangle) = \bigsqcup_n a^n_{x,y}$. The result then follows by application of Lemma 3.3.16.

The interesting cases are $H = \mu X.H_1$ and $H = \nu X.H_1$. We just show how to prove the desired result for $H = \mu X.H_1$, because the case for $H = \nu X.H_1$ can be proved with the same technique.

Since, for every $p \in P$, the state $\langle p, H\rangle$ of the game $\mathcal{G}(H,\rho)$ has a unique successor $\langle p, H_1\rangle$, we know by application of Theorem 5.2.10 that the following equality holds:

$$(\!| H |\!)_\rho(p) \stackrel{\text{def}}{=} \mathsf{VAL}\big(\mathcal{G}(H,\rho)\big)(\langle p, H\rangle) = \mathsf{VAL}\big(\mathcal{G}(H,\rho)\big)(\langle p, H_1\rangle) \tag{7.1}$$

holds. Moreover, since the state $\langle p, H\rangle$, once left after the first move, cannot be reached any more in any play in $\mathcal{G}(H,\rho)$, we can ignore the state $\langle p, H\rangle$ from our analysis. This allows us to have, in the two games $\mathcal{G}(H,\rho)$ and $\mathcal{G}(H_1, \rho[f/X])$, the same set S of game-states, for every function $f \in [0,1]^P$. In what follows we use f and g to range over the function spaces $[0,1]^P$ an $[0,1]^S$ respectively. Given any function $g \in [0,1]^S$, we denote with $\hat{g} \in [0,1]^P$ the function defined as $\hat{g}(p) = g(\langle p, H_1\rangle)$. Thus, given Equation 7.1, the following equalites hold:

$$(\!| H |\!)_\rho = \widehat{\mathsf{VAL}\big(\mathcal{G}(H,\rho)\big)} \quad \text{and} \quad (\!| H_1 |\!)_{[f/X]\rho} = \mathsf{VAL}\big(\widehat{\mathcal{G}(H_1,\rho}[f/X])\big) \tag{7.2}$$

The states of maximal priority in $\mathcal{G}(H,\rho)$ are of the form $\langle p, X\rangle$, for $p \in P$. It then follows the every value assignment for $\mathcal{G}(H,\rho)$ (see Definition 6.2.7) is a function of type $\{\langle p, X\rangle \mid p \in P\} \to [0,1]$. In what follows, for notational convenience, we equate the two function spaces $P \to [0,1]$ and $\{\langle p, X\rangle \mid p \in P\} \to [0,1]$. Thus, in accordance with the convention above, we use the letter f to range over value assignments in $\mathcal{G}(H,\rho)$. Note that, for every $g \in [0,1]^S$, the value assignment f_g (see Definition 6.2.20) is precisely the function $\hat{g}$, because the unique successor state of $\langle p, X\rangle$ in $\mathcal{G}(H,\rho)$ is the state $\langle p, H_1\rangle$, for every $p \in P$.

From the previous discussion, Observation 7.1.8, and Definition 6.2.21 of $\mathbb{H}_{\mathcal{G}} : [0,1]^S \to [0,1]^S$, the following equalities holds:

$$\mathbb{H}_{\mathcal{G}(F,\rho)}(g) \stackrel{\text{def}}{=} \mathsf{VAL}\big(\mathcal{G}(H,\rho)^-_{\hat{g}}\big) = \mathsf{VAL}\big(\mathcal{G}(H_1,\rho[\hat{g}/X])\big). \tag{7.3}$$

We know, by application of Theorem 6.4.2, that the equality

$$\mathsf{VAL}\big(\mathcal{G}(H,\rho)\big) \;=\; \mathrm{lfp}\big(\lambda g \in [0,1]^S .\ \mathsf{VAL}\big(\mathcal{G}(H_1,\rho[\hat{g}/X])\big)\big)$$

holds. Equivalently, by the Knaster-Tarski theorem, the following equality holds.

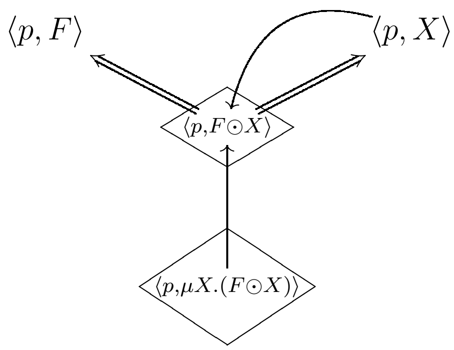
$$\mathsf{VAL}\big(\mathcal{G}(H,\rho)\big) \;=\; \bigsqcup_\alpha \ \mathsf{VAL}\big(\mathcal{G}(H_1,\rho[f^\alpha/X])\big)$$

where $f^\alpha = \bigsqcup_{\beta<\alpha} \mathsf{VAL}\big(\widehat{\mathcal{G}(H_1,\rho}[f^\beta/X])$, or equivalently, given Equation 7.2, $f^\alpha = \bigsqcup_{\beta<\alpha} (\!| H_1 |\!)_{\rho[f^\beta/X]}$. In then follows from Equation 7.1 that the equality $(\!| H |\!)_\rho = \bigsqcup_\alpha (\!| H_1 |\!)_{\rho[f^\beta/X]}$ holds. By induction hypothesis, we know that $(\!| H_1 |\!)_{\rho[f^\alpha/X]} = [\![H_1]\!]_{\rho[f^\alpha/X]}$ for every f^α. The desired result $(\!| H |\!)_\rho = [\![H]\!]_\rho$ the follows by the definition of the denotational semantics of $\mathrm{pL}\mu^{\odot}_{\oplus}$. □

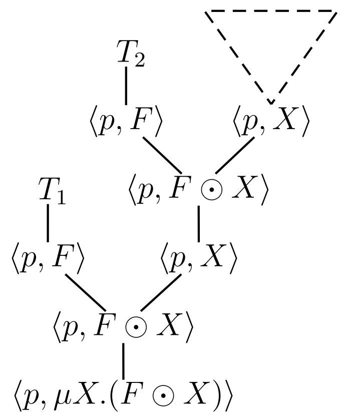
Since $\mathrm{pL}\mu$ is a fragment of $\mathrm{pL}\mu \cup +_\lambda$, as an immediate consequence of Theorem 7.1.10, we have that Theorem 3.2.14 holds, i.e., the denotational and game semantics for $\mathrm{pL}\mu$ coincide on all models. Therefore our result settles a problem left open in [78], where the equivalence of the denotational and game semantics of $\mathrm{pL}\mu$ was proven only with respect to finite models (see Theorem 3.2.13).

Theorem 7.1.10 is a fundamental result providing a strong semantical basis for the logics pLμ, pL$\mu^{\odot}$ and pL$\mu^{\odot}_{\oplus}$. Indeed one can reason about these logics using the different tools offered by the denotational and game semantics. One of the most important advantages of having an adequate game semantics for pL$\mu^{\odot}_{\oplus}$ is the possibility of visualizing the $2\frac{1}{2}$-player meta-parity game associated to a formula in order to grasp its meaning.

As an important example of this approach, we now discuss the meaning of the derived qualitative threshold modalities $\mathbb{P}_{>0}$ and $\mathbb{P}_{=1}$ (see Section 3.3.2.3), using the game semantics of the logic pL$\mu^{\odot}$. Let us consider the game $\mathcal{G}(\mathbb{P}_{>0}F, \rho)$ associated with a PLTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$, a $[0,1]$-interpretation ρ and the pL$\mu^{\odot}$ formula $\mathbb{P}_{>0}F \stackrel{\text{def}}{=} \mu X.(F \odot X)$. The game $\mathcal{G}(\mu X.(F \odot X), \rho)$, at the state $\langle p, \mu X.(F \odot X)\rangle$ for some $p \in P$ can be depicted as follows:



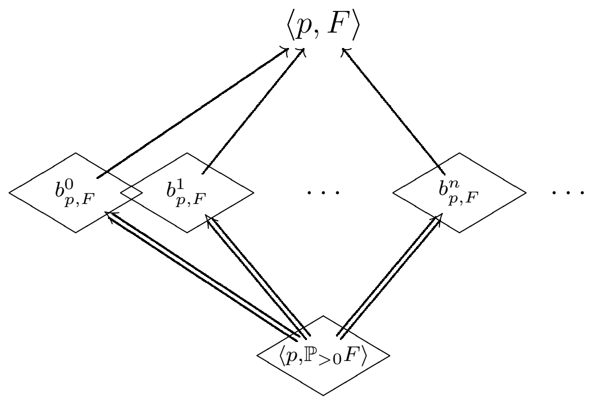
After an initial unfolding step from $\langle p, \mu X.(F \odot X)\rangle$ to $\langle p, F \odot X\rangle$, the game is split in two concurrent sub-games, one continuing its execution from the state $\langle p, F\rangle$ (this sub-game can be considered an instance of the game $\mathcal{G}(F, \rho)$ starting at $\langle p, F\rangle$) and the other from the state $\langle p, X\rangle$. In order to win the game $\mathcal{G}(\mu X.(F \odot X), \rho)$, Player 1 has to win in at least one of the two generated sub-games, thus either in the instance of $\mathcal{G}(F, \rho)$ or in the sub-games continuing at $\langle p, X\rangle$. This second sub-game, however, after an unfolding step, progresses to the game state $\langle p, F \odot X\rangle$, where the protocol is repeated generating yet another two sub-games. The infinite execution of the game leads to the generation of infinitely many instances of the game $\mathcal{G}(F, \rho)$. A branching play T in $\mathcal{G}(\mu X.(F \odot X), \rho)$ can be depicted as follows:



where $T_1, T_2, \ldots$, represent the branching plays corresponding to the plays in each generated instance of the game $\mathcal{G}(F, \rho)$. Since the variable X unfolded infinitely often in the rightmost path in T is bound by a least fixed point in $\mu X.(F \odot X)$, and since the $\odot$ nodes are Player 1 choices in the inner game $\mathcal{G}_T$, we have that T is a winning branching play for Player 1 if and only if there exists some $n \in \mathbb{N}$ such that T_n, the outcome of the n-th generated instance of $\mathcal{G}(F, \rho)$, is winning for Player 1.

From the previous discussion, it follows that the game $\mathcal{G}(\mu X.(F \odot X), \rho)$, at the state $\langle p, \mu X.F \odot X\rangle$ can be simply described as follows: generate an infinite number of instances of the game $\mathcal{G}(F, \rho)$ at the state $\langle p, F\rangle$; Player 1 wins if at least one of the infinitely many generated instances of $\mathcal{G}(F, \rho)$ ends up in a winning branching play and Player 2 wins otherwise. It is then quite clear that if $(\!| F |\!)^{\mathcal{L}}_{\rho}(p) > 0$ (or equivalently $[\![F]\!]^{\mathcal{L}}_{\rho}(p) > 0$ by Theorem 7.1.10), then the probability that at least one (and in fact countably many) of the infinite instances of $\mathcal{G}(F, \rho)$ will result in a win for Player 1, is 1. Similarly, if $(\!| F |\!)^{\mathcal{L}}_{\rho}(p) = 0$, then the probability that at least one of the infinite instances of $\mathcal{G}(F, \rho)$ will result in a win for Player 1, is 0. The game semantics for pL$\mu^{\odot}$ thus offers a straightforward interpretation for the probabilistic qualitative modality $\mathbb{P}_{>0}$ exploiting the simple idea that an event (which we can, at some extent, see as *a* pL$\mu^{\odot}$ property) has probability greater than zero if and only if, when repeated infinitely many time, it almost surely occurs at least once. An analogous straightforward interpretation can be given to the other qualitative threshold modality $\mathbb{P}_{=1}F \stackrel{\text{def}}{=} \nu X.(F \cdot X)$: generate an infinite number of instances of the game $\mathcal{G}(F, \rho)$ at the state $\langle p, F\rangle$; Player 1 wins if all of them end up in a winning branching play for Player 1, and Player 2 wins otherwise.

It is trivial, given the previous discussion, to define a game semantics directly for the $\text{pL}\mu^{\{0,1\}}$ fragment of $\text{pL}\mu^{\odot}$ (consisting of $\text{pL}\mu \cup \{+_\lambda\}$ extended with the two modalities $\mathbb{P}_{>0}$ and $\mathbb{P}_{=1}$) in terms of $2\frac{1}{2}$-player meta-parity games. States of the form $\langle p, \mathbb{P}_{>0}\rangle$ and $\langle p, \mathbb{P}_{=1}\rangle$ are branching states under the control of Player 1 and Player 2 respectively, having an $\mathbb{N}$-indexed collection of successor states which we denote with $b^n_{p,F}$, for $n \in \mathbb{N}$. Each state state $b^n_{p,F}$ has, as unique successor, the state $\langle p, F\rangle$. The game $\mathcal{G}(\mathbb{P}_{>0}F, \rho)$ at the state $\langle p, \mathbb{P}_{>0}F\rangle$ can be depicted as follows:



The use of the game-states $\{b^n_{p,F}\}_{n\in\mathbb{N}}$ is technically necessary as they allow us to represent the desired branching state generating countably many sub-games as continuing their execution from the same state $\langle p, F\rangle$.

The straightforward interpretation of the qualitative probabilistic modalities in terms of $\text{pL}\mu^{\odot}$ games is, in our opinion, an important feature of the whole game semantics for $\text{pL}\mu^{\odot}$ which is simple enough to be understood at an elementary level of abstraction, and yet very expressive, as we shall see in Section7.2.

The game semantics for the full logic $\text{pL}\mu^{\odot}_{\oplus}$, on the other hand, is less satisfactory because the interpretation of the operators $\{\oplus, \ominus\}$ is not necessarily very illuminating. Indeed the games associated to the quantitative threshold modalities $\mathbb{P}_{>\lambda}F$ and $\mathbb{P}_{\geq\lambda}F$ (see Section 3.3.1) are harder to understand, and do not necessarily offer an intuitive operational interpretation. On the other hand, the game-semantics of $\text{pL}\mu^{\odot}_{\oplus}$ is an important witness for the expressiveness of the new class of $2\frac{1}{2}$-player meta-parity games introduced in Section 5.3, and constitutes a formal operational interpretation of a very powerful logic which, as we shall see in Section 7.2, counts the full logic PCTL as one of its fragments.

7.2 Examples of pL$\mu^{\odot}$ and pL$\mu^{\odot}_{\oplus}$ formulas

In this Chapter we discuss a few examples of interesting pL$\mu^{\odot}$ and pL$\mu^{\odot}_{\oplus}$ formulas, using both the denotational and the game semantics to explain their meaning, thus taking advantage of the results of Chapter 7. The examples show how important properties of PLTS's can be expressed in the logic pL$\mu^{\odot}$, or actually in its fragment pL$\mu^{\{0,1\}}$. Some examples will also be used to expose interesting properties of the logic, such as the failure of the so called *finite model property.* We are also going to show, at the end of this section, how the qualitative fragment of PCTL (see Section 3.2.2) can be seen as a fragment of the logic pL$\mu^{\odot}$, and similarly, how full PCTL is a fragment of the logic pL$\mu^{\odot}_{\oplus}$. These results demonstrate the expressivity of the quantitative logics pL$\mu^{\odot}$ and pL$\mu^{\odot}_{\oplus}$ and constitute a contribution to the development of the *quantitative* approach to probabilistic temporal logics, as discussed in Section 3.2.3.

We start with some simple example.

Example 7.2.1. Consider the following pL$\mu^{\odot}$ formulas:

1. $G_1 \stackrel{\text{def}}{=} \mathbb{P}_{=1}(\nu X.\langle a\rangle X)$
2. $G_2 \stackrel{\text{def}}{=} \mu X.\big(G_1 \vee \langle b\rangle X\big)$
3. $G_3 \stackrel{\text{def}}{=} \mathbb{P}_{=1}\big(\mu X.(\langle a\rangle X \vee H)\big)$
4. $G_4 \stackrel{\text{def}}{=} \mathbb{P}_{=1}\big(\mu X.(\langle a\rangle X \vee \mathbb{P}_{=1}H)\big)$
5. $G_5 \stackrel{\text{def}}{=} \mathbb{P}_{=1}\big(\mathbb{P}_{>0}(\mathbb{P}_{=1}G_4)\big)$

where the sub-formula H of G_3 and G_4 is defined as $\nu X.\langle b\rangle X$, and can be understood as the formula F_2 discussed in Section 3.2.5.

Let us first consider the formula G_1. Its semantics is, quite intuitive, the map assigning to each state p, value 1 if the state p satisfies the pLμ formula $\nu X.\langle a\rangle X$ with probability 1, and value 0 otherwise. This example is trivial, but shows that pL$\mu^{\odot}$ formulas of the form $\mathbb{P}_{=1}F$ and $\mathbb{P}_{>1}F$ always describe *sets* of process states, and therefore it makes sense to say that such a formula *holds*, or *does not hold*, at a given process state p. Recall from the examples of Section 3.2.5, that the logic pLμ cannot express such properties.

The above considerations are useful for understanding the formula G_2 whose interpretation at a state p corresponds to the limit probability of reaching, by

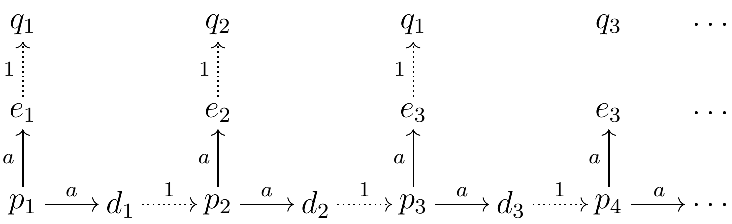


Figure 7.1: Example of PLTS with infinitely many states

means of b-actions, a state where the formula G_1 holds, i.e., a state q where the formula $\nu X.\langle a\rangle X$ holds almost surely. Thus pL$\mu^{\odot}$ formulas are still interpreted as the limit probability of satisfying a property, but in describing this property we can use boolean predicates over states, i.e., set of states, defined using the qualitative threshold operators. As we shall see later in this section, one can specify all the properties definable in qualitative PCTL as pL$\mu^{\odot}$-formulas.

Another interesting property is expressed by the formula G_3 which holds at a state p if the limit probability of satisfying the property associated with the pLμ formula $\mu X.(\langle a\rangle X \vee H)$ is 1. The interpretation of $\mu X.(\langle a\rangle X \vee H)$ at a state q is the limit probability the controller has of reaching some state, by means of a finite number of a-steps, and from there producing an infinite sequence of b-actions. Here it is important to stress the concept of *limit* probability. Indeed the formula G_3 might hold at a given state p of a PLTS $\mathcal{L}$ even if there is no single process state satisfying the sub-formula H with probability 1. For example consider the PLTS of Figure 7.1, and assume that for each $n\in\mathbb{N}^+$, $[\![H]\!](q_n)=1-\frac{1}{n}$ and $[\![H]\!](p_n)=0$. Clearly no state in the PLTS satisfies the formula H with probability 1, although, for every $\epsilon>0$ there exists a process state q_n which statisfies H with probability greater than $1-\epsilon$. Thus the formula $\mu X.(\langle a\rangle X \vee H)$ at the state p_n, for every $n\in N$, holds with probability 1: Controller can always reach a state, by means of a-actions, and from there satisfy the property expressed by H with probability as close to 1 as desired. Thus $[\![G_2]\!](p_n)=1$, for every process state p_n in the PLTS of Figure 7.1. Note that the game associated with G_3 is a classical example of $2\frac{1}{2}$-player meta-parity game where Player 1 does not have optimal strategies.

However one might want to describe a slightly stronger property, namely the possibility of reaching, by mean of a-actions, a process state where H actually holds with probability 1. The subformula $\mu X.(\langle a\rangle X \vee \mathbb{P}_{=1}H)$ of G_4 expresses exactly this property. In particular $[\![\mu X.(\langle a\rangle X \vee \mathbb{P}_{=1}H)]\!](p_n) = 0$, and thus $[\![G_4]\!](p_n) = 0$, for every process state p_n in the PLTS of Figure 7.1. The examples G_3 and G_4 show how the logic pL$\mu^{\odot}$ is capable of expressing capable of expressing subtle distinctions in the encoding of qualitative properties.

We use the last example, given by the formula G_5, to discuss an important property of the qualitative threshold modalities $\mathbb{P}_{>0}$ and $\mathbb{P}_{=1}$ which is useful to simplify and make some formulas more readable . As is it easy to observe, the formula G_5 is semantically equivalent to the formula G_4. Indeed it holds, in general, that the pL$\mu^{\odot}$ formula $\mathbb{P}_{c_1}\mathbb{P}_{c_2}\ldots\mathbb{P}_{c_n}F$ is semantically equivalent to the simpler pL$\mu^{\odot}$ formula $\mathbb{P}_{c_n}F$, where $c_1,\ldots,c_n \in \{=1,>0\}$. The qualitative threshold modalities share several important properties with the modalities of the S5 modal logic [25]. Beside the above mentioned way of simplifying sequences of modalities, also the inequalities $[\![\mathbb{P}_{=1}F]\!](p)\leq[\![F]\!](p)\leq[\![\mathbb{P}_{>0}F]\!](p)$ trivially hold for every process state p. We leave for future work further investigations on the above mentioned connections between the modalities of the S5 modal logic and the qualitative threshold modalities.

In the previous examples, we used the probabilistic threshold modalities to build formulas of the form $\mathbb{P}_{=1}F$ or $\mathbb{P}_{>0}F$ where F is a closed formula. This kind of formula is indeed very simple to understand: one starts by interpreting the meaning of F and then the meaning of $\mathbb{P}_{=1}F$ or $\mathbb{P}_{>0}F$ is simply the set of process states satisfying the qualitative probabilistic constraint. However interesting formulas can also be built from the qualitative threshold modalities using open formulas, which can then be bound by further fixed-point operators. We now discuss a few such examples.

Example 7.2.2. Consider the following pL$\mu^{\odot}$ formulas:

1. $H_1 = \nu X.\big(\mathbb{P}_{>0}(\langle a\rangle X)\big)$,
2. $H_2 = \mu X.\big(\mathbb{P}_{=1}([a]\,X)\big)$,
3. $H_3 = \mu X.\big((\mathbb{P}_{>0}\langle a\rangle X) \vee \mathbb{P}_{=1}H\big)$, for some closed pL$\mu^{\odot}$ formula H,
4. $H_4 = \nu X.\big((\mathbb{P}_{>0}\langle a\rangle X) \wedge \mathbb{P}_{=1}H\big)$, for some closed pL$\mu^{\odot}$ formula H,

The interpretation of the formula H_1 might seem a bit obscure at a first sight but it is actually rather simple. Given a PLTS $\mathcal{L}$, the interpretation of the formula H_1 at a process state p_1 is 1 if there exists an infinite a-run starting at p_1 in $\mathcal{L}$ (see Definition 3.2.5), i.e., a sequence of the form

$$p_1 \xrightarrow{a} d_1 \rightsquigarrow p_2 \xrightarrow{a} d_2 \rightsquigarrow p_3 \xrightarrow{a} \dots$$

where p_{n+1} has positive probability in d_n, for every $n \in \mathbb{N}$; the interpretation of H_1 at p_1 is 0 otherwise. Thus the semantics of H_1 is a boolean predicate, i.e., a set of processes. To see that this is indeed the interpretation of the formula, we consider the game-theoretic interpretation of the qualitative threshold modality $\mathbb{P}_{>0}$ discussed in Section 7.1. The game $\mathcal{G}(H_1)$ (the interpretation ρ is not important as the formula H_1 is closed) starts at the game-state $\langle p_1, \nu X.\mathbb{P}_{>0}\langle a\rangle X\rangle$ and after an unfolding step reaches the game-state $\langle p_1, \mathbb{P}_{>0}\langle a\rangle X\rangle$ which is a branching state under the control of Player 1 in the $2\frac{1}{2}$-player meta-parity game $\mathcal{G}(H_1)$. From this game-state, a countably infinite number of sub-games continuing their execution from the game-state $\langle p_1, \langle a\rangle X\rangle$ is generated, and Player 1 needs to win just one of them in order to win in the whole game $\mathcal{G}(H_1)$. The most transparent interpretation is then, arguably, to think about the game state $\langle p_1, \mathbb{P}_{>0}\langle a\rangle X\rangle$ as a point on which Player 1 has an infinite number of trials as their disposal to try to satisfy the property $\langle a\rangle X$ at the state p. The formula $\langle a\rangle X$ expresses the possibility of making an a-action and reach a state p_2 satisfying the formula X, which is unfolded again as $\mathbb{P}_{>0}\langle a\rangle X$. Thus Player 1, having an infinite number of trials at their disposal at the game state $\langle p, \langle a\rangle X\rangle$, can choose a given transition $p_1 \xrightarrow{a} d_1$ in $\mathcal{L}$ in all sub-games; in this way Player 1, with probability 1, will reach their preferred state $p_2 \in supp(d_1)$ for continuing the next iteration of the fixed-point game $\mathcal{G}(H_1)$. Clearly if an infinite path $p_1 \xrightarrow{a} d_1 \rightsquigarrow p_2 \xrightarrow{a} d_2 \rightsquigarrow p_3 \xrightarrow{a} \dots$ exists, then Player 1 will be able to win, almost surely, thanks to the infinitely many opportunities they have of satisfying each $\langle a\rangle X$-step of the game. On the other hand if there is no such an infinite path, Player 1 has no possibility of winning, because in all generated sub-games they will reach at some point a process state p_n without enabled a-transition, i.e., such that $p \not\xrightarrow{a}$.

The formula H_2 expresses the dual property "there is no infinite a-run". The interpretation can be shown to be correct, either by observing that, $H_2 = \overline{H_1}$, i.e., H_2 is the negation of the formula H_1, or by an argument analogous to the one above, where the role of Player 1 is replaced by that of Player 2. These two examples are useful because they show how the logic pL$\mu^{\odot}$ is capable of expressing properties of the labeled graph structure underlying a given PLTS $\mathcal{L}$, i.e., the one obtained by ignoring the probability values labeling the probabilistic steps $d \overset{\lambda>0}{\rightsquigarrow} p$ from probability distributions $d \in \mathcal{D}(P)$ to process states $p \in P$ in $\mathcal{L}$.

The slightly more complicated formula H_3 expresses the existence of a finite a-

run $p_1 \xrightarrow{a} d_1 \rightsquigarrow p_2 \xrightarrow{a} d_2 \rightsquigarrow p_3 \xrightarrow{a} \dots p_n$ leading to a process state p_n satisfying the formula $\mathbb{P}_1 H$. Note how the qualitative threshold modality ensures that the sub-formula $\mathbb{P}_1 H$ can be indeed understood as a boolean predicate, i.e., a set of process states. Furthermore, if H has already a boolean predicate interpretation (e.g. $H = \langle a \rangle \underline{1}$ or $H = \mathbb{P}_{>0} H'$) then $[\![\mathbb{P}_{=1} H]\!] = [\![H]\!]$.

Similarly, the formula H_4 expresses the existence of an infinite a-run $p_1 \xrightarrow{a} d_1 \rightsquigarrow p_2 \xrightarrow{a} d_2 \rightsquigarrow \dots$ such that every process state p_n appearing in the run satisfies the property $\mathbb{P}_{=1} H$.

The examples discussed above supply plenty of evidence that the logic $pL\mu^{\odot}$ can express many interesting properties of PLTS's. Moreover we argue that the game semantics in terms of $2\frac{1}{2}$-player meta-parity games, one of the main results of this thesis, is sometimes helpful for discussing, or understanding, the meaning of pL$\mu^{\odot}$ formulas and, more importantly, allows one to interpret the denotational semantics as the (limit) probability of a property holding.

We now consider a few other examples which we will use to establish some fundamental properties of the logic pL$\mu^{\odot}$, for example, the failure of the finite model property.

Example 7.2.3. Let us consider the following pLμ formulas:

1. $\lessdot_a F = \mathbb{P}_{>0}\big(\langle a \rangle \underline{1} \wedge [a]\, F\big)$,
2. $S(F) = \lessdot_a F \wedge \lessdot_a \overline{F}$,
3. $INF = \mathbb{P}_{>0}\Big(\nu X.\big(S(\langle a \rangle \underline{1}) \wedge \langle a \rangle X\big)\Big)$,

where F is some closed pL$\mu^{\odot}$ formulas without occurrences of the $+_\lambda$ operator.

Let us first discuss the meaning of the formula $\lessdot_a F$. First of all, since it is a formula whose main connective is a threshold modality, its semantics is a boolean predicate, i.e., a set of process states. A process state p satisfies $\lessdot_a F$ if:

i. p satisfies $\langle a \rangle \underline{1}$ with positive probability, i.e., there exists an a-transition $p \xrightarrow{a} d$ in the PLTS,

ii. p satisfies $[a]\, F$ with positive probability, i.e., $\displaystyle\bigsqcap_{p \xrightarrow{a} e} \big(\sum_{q \in supp(e)} e(q) \cdot [\![F]\!](q)\big) > 0$.

In particular, and this is an equivalent characterization of the property if p has only finitely many a-successor probability distributions, for every $p \xrightarrow{a} e$, there exists a process state $q_e \in supp(e)$ such that $[\![F]\!](q_e) > 0$.

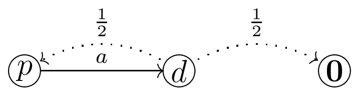


Figure 7.2: Example of PLTS satisfying the formula $S(\langle a\rangle \underline{1})$

Thus the modality $\lhd_a$ expresses a combination of existential and universal properties.

Let us now consider the formula $S(F)$, where the letter S stands for *stochastic*. This formula is interesting because every process state p in a PLTS which is non probabilistic (i.e., a LTS seen as a PLTS as discussed in Definition 3.2.2), cannot satisfy $S(F)$, while there exists a PLTS which satisfies, for example, $S(\langle a\rangle \underline{1}) = \lhd_a(\langle a\rangle \underline{1}) \wedge \lhd_a([a]\,\underline{0})$, with probability 1. Let us consider some LTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$ (seen as a PLTS) and a process state $p \in P$. Since $\mathcal{L}$ is not probabilistic we have that the $[\![F]\!](p) \in \{0,1\}$ for every process state $p \in P$ (see Lemma 3.3.8). Thus the meaning of the formula $\lhd_a F$ at a process state $p \in P$ can be simplified as follows: $[\![\lhd_a(F]\!](p) = 1$ if:

i. there exists $p \xrightarrow{a} \delta_q$, and

ii. for all $p \xrightarrow{a} \delta_q$, with $supp(\delta_q) = \{q\}$, $[\![F]\!](q) = 1$.

Similarly the formula $\lhd_a(\overline{F})$ holds at a state p if:

i. there exists $p \xrightarrow{a} \delta_q$, and

ii. for all $p \xrightarrow{a} \delta_q$, with $supp(\delta_q) = \{q\}$, $[\![F]\!](q) = 0$.

It is then clear that no process state p in a LTS $\mathcal{L}$ can satisfy the formula $S(F)$, for any closed formula F without occurrences of the $+_\lambda$ operator. On the other hand it is immediate to verify that the process state p of the PLTS of Figure 7.2 satisfies $S(\langle a\rangle \underline{1})$, i.e., it satisfies both formulas $\lhd_a(\langle a\rangle \underline{1})$ and $\lhd_a([a]\,\underline{0})$: for every reachable distribution d, some elements in $supp(d)$ can perform a-actions, and some can not.

Lastly, we now discuss the meaning of the formula INF, which is necessarily a boolean-predicate since its outermost connective is the qualitative threshold modality $\mathbb{P}_{>0}$. As we shall show, the name of the formula is motivated by the fact that there is no finite PLTS $\mathcal{L}$ (see Definition 3.2.1) and process state p in $\mathcal{L}$ such that $[\![INF]\!](p) = 1$, but there exists an infinite PLTS, and a process

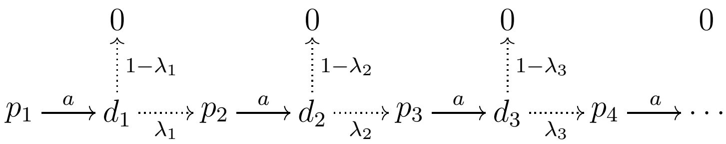


Figure 7.3: Example of PLTS with infinitely many states satisfying INF

state p in it, such that $[\![INF]\!](q) = 1$. The interpretation of the sub-formula $\nu X.\big(S(\langle a\rangle\underline{1}) \wedge \langle a\rangle X\big)$ at a process state p can be described as the limit probability that the controller has of producing an infinite sequence of a-actions, always reaching states satisfying the formula $S(\langle a\rangle\underline{1})$. Recall that $S(\langle a\rangle\underline{1})$ holds at a state p if all (and at least one) of its a-successor distributions $p \xrightarrow{a} d$ assign probability greater than 0 to process states not allowing a-transitions, and probability greater than 0 to process states allowing a-transitions. Let us now fix a *finite* PLTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a\in L}\}\rangle$, and let P_a and $P_{\neg a}$ be the partition of P defined as $P_a = \{p \in P \mid \exists d.\ p \xrightarrow{a} d\}$ and $P_{\neg a} = \{p \in P \mid \neg(\exists d.\ p \xrightarrow{a} d)\}$. We have that, for every process state $p \in P$ satisfying $S(\langle a\rangle\underline{1})$, and for every $p \xrightarrow{a} d$, $0 < d(P_a) < 1$. Let λ be the greatest probability assigned by the finite set of a-successors of states satisfying $S(\langle a\rangle\underline{1})$ to the set P_a, i.e., $\lambda = \bigsqcup\{d(P_a) \mid [\![S(\langle a\rangle\underline{1})]\!](p) = 1 \textit{ and } p \xrightarrow{a} d\}$. Note that, since $\mathcal{L}$ is finite, λ is necessarily smaller than 1. Now it is clear that no process state $p \in P$ can satisfy the formula INF. After every a-action, the controller will reach a state satisfying $S(\langle a\rangle\underline{1})$ with probability at most λ. Thus the probability of having reached, after n steps of a-actions, a state not satisfying $S(\langle a\rangle\underline{1})$ is, at least, probability $1 - \lambda^n$. It follows immediately that the probability of producing an infinite sequence of a-actions always staying in states satisfying $S(\langle a\rangle\underline{1})$ is 0 in every finite $\mathcal{L}$. On the other hand, the PLTS of Figure 7.3, where $\mathbf{0}$ is a process state such that $\mathbf{0} \not\xrightarrow{a}$, satisfies the property of the formula $\nu X.\big(S(\langle a\rangle\underline{1}) \wedge \langle a\rangle X\big)$ with probability $\prod_{n\in\mathbb{N}} \lambda_n$, which can be made positive by an appropriate choice of probabilities.

The previous examples allow us to state a few basic, yet fundamental, properties about the satisfiability problem(s) associated with the logic pL$\mu^{\odot}$.

Definition 7.2.4. Given a closed pL$\mu^{\odot}$ formula F, we say that F is λ-*satisfiable*, for $\lambda \in (0, 1]$, if there exists a PLTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$ and a process state $p \in P$, such that $[\![F]\!]^{\mathcal{L}}(p) \geq \lambda$. For every PLTS $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$, we say that $\mathcal{L}$ λ-

satisfies F, if there exists a process state $p \in P$ such that $[\![F]\!]^{\mathcal{L}}(p) \geq \lambda$. If the value λ is omitted, it is assumed to be 1.

As an immediate observation, note that we can just focus, for many purposes, on the simpler notion of 1-satisfiability. Indeed a closed pL$\mu^{\odot}$ formula F is λ-satisfiable, for some $\lambda > 0$, if and only if $\mathbb{P}_{>0}F$ is 1-satisfiable.

Proposition 7.2.5. *The following assertions hold:*

1. *There exists a closed pL$\mu^{\odot}$ formula F which is* 1*-satisfiable, but no LTS* $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$ *(seen as a PLTS)* 1*-satisfies it.*

2. *There exists a closed pL$\mu^{\odot}$ formula F which is* 1*-satisfiable, but no (possibly infinite) LTS or finite PLTS* $\mathcal{L} = \langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$ 1*-satisfies it.*

Proof. Consider the formulas $S(\langle a\rangle\underline{0})$ and *INF* of Example 7.2.3 for point 1 and 2 respectively. □

The first assertion of the Proposition above might be considered an expressivity result: the properties expressible in pL$\mu^{\odot}$ can distinguish between probabilistic and non-probabilistic systems. This is quite a good property for a temporal probabilistic logic like pL$\mu^{\odot}$. The second assertion is quite interesting as it exposes an important point on which the theory of pL$\mu^{\odot}$ deviates significantly from that of the standard modal μ-calculus Lμ. Indeed, every Lμ-formula satisfies the *finite model property*: if F is satisfiable, then it is satisfiable by a finite model [63].

We just settled a couple of basic questions about the satisfiability of pL$\mu^{\odot}$ formulas. Other interesting problems, which we leave open, are listed below.

Question 7.2.6. Does the finite model property, with respect to 1-satisfiability, holds for pLμ formulas?

Question 7.2.7. Let F be a closed pL$\mu^{\odot}$ formula.

1. Does the finite model property, with respect to 1-satisfiability, holds for pLμ formulas?

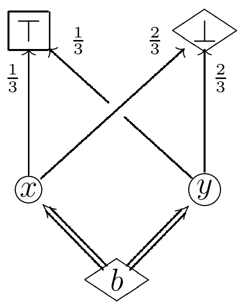
2. Suppose F is 1-satisfiable. Is there always a PLTS $\langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$ which λ-satisfies F, such that all the probabilities appearing[2] in $\mathcal{L}$ are rational?

[2]We say that a probability λ *appears* in $\mathcal{L}$ if there is a process state $p \in P$, a transition $p \xrightarrow{a} d$ and a process state $q \in supp(d)$ such that $d(q) = \lambda$.

3. is the problem of deciding if F is 1-satisfiable decidable for every $pL\mu^{\odot}$ formula F?

The previous considerations on satisfiability, and the examples of $pL\mu^{\odot}$ formulas provided so far are actually expressible in the fragment $pL\mu^{\{0,1\}}$ of the logic (see Section 3.3.2.3). As we already discussed, developing the game semantics of the logic $pL\mu^{\{0,1\}}$ does not seem any easier than than directly working with the more expressive logic $pL\mu^{\odot}$, and that is one of the reasons we focused on $pL\mu^{\odot}$. However there is another important point. The modal-free[3] fragment of the logic $pL\mu^{\odot}$ can be considered as the language of all *finite* $2\frac{1}{2}$-player meta-parity games: branching nodes under the control of Player 1 and Player 2, are modeled by the connectives $\odot$ and $\cdot$ respectively. Clearly branching states with n children can be modeled as a sequence of binary branching states. We now provide a translation of some of the finite $2\frac{1}{2}$-player meta-parity games discussed in this thesis into $pL\mu^{\odot}$ closed modal-free formulas.

Example 7.2.8. Let us consider the $2\frac{1}{2}$-player meta-parity game of Example 5.3.3, which can be depicted as follows:

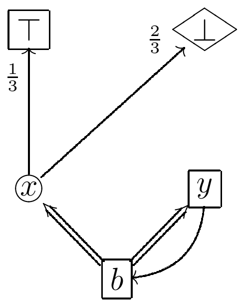


It is immediate to verify that $\mathcal{G}$ is equivalent to the logical game $\mathcal{G}(F)$ associated with the $pL\mu^{\odot}$ formula $F = (\underline{1} +_{\frac{1}{3}} \underline{0}) \odot (\underline{1} +_{\frac{1}{3}} \underline{0})$. Even if in this case the game $\mathcal{G}$ is structurally identical to $\mathcal{G}(F)$, by "equivalent" we refer to a weaker relation of similarity in the way the two games are played. Our purpose here is to just give illustrative examples without elaborating on the notion of equivalence underlying them.

Example 7.2.9. Let us consider the $2\frac{1}{2}$-player meta-parity game $\mathcal{G}$ of Example

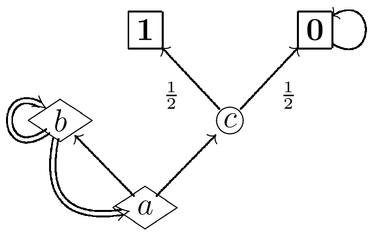
[3]Here by *modal-free fragment* we mean the logic $pL\mu^{\odot}$ without the connectives $\langle a\rangle$ and $[a]$.

5.3.5, which can be depicted as follows:



We can model the presence of the loops in the game graph by using fixed point operators. Indeed it is simple to verify that the game $\mathcal{G}$ is equivalent to the logical game $\mathcal{G}(F)$ associated with the pL$\mu^{\odot}$ formula $F = \mu X.\big((\underline{1} +_{\frac{1}{3}} \underline{0}) \odot X\big)$.

Example 7.2.10. Let us consider the $2\frac{1}{2}$-player meta-parity game $\mathcal{G}$ of Lemma 6.1.9, which can be depicted as follows:

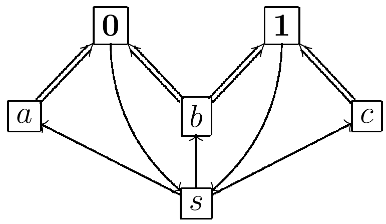


The game $\mathcal{G}$ is described by the pL$\mu^{\odot}$ formula $F = \mu X.\Big(\big(\mu Z.(X \odot Z)\big) \vee (\underline{1} +_{\frac{1}{2}} \underline{0})\Big)$, or equivalently, $F = \mu X.\Big((\mathbb{P}_{>0} X) \vee \underline{\tfrac{1}{2}}\Big)$. Recall from Lemma 6.1.9 that this is an example of $2\frac{1}{2}$-player meta-parity where Player 1 does not have positional optimal strategies.

Proposition 7.2.11. *There exists a (modal-free) closed* pL$\mu^{\{0,1\}}$ *formula F whose logical $2\frac{1}{2}$-player meta-parity game $\mathcal{G}(F)$ is not determined under positional strategies.*

Proof. Take $F = \mu X.\Big(\big(\mu Z.(X \odot Z)\big) \vee (\underline{0} +_{\frac{1}{2}} \underline{1})\Big)$ as in the previous example and apply Lemma 6.1.9. □

Example 7.2.12. Let us consider the $2\frac{1}{2}$-player meta-parity game $\mathcal{G}$ of Example 6.2.10, which can be depicted as follows:



The pL$\mu^{\odot}$ formula describing $\mathcal{G}$ is $F = \mu X.\nu Y.\big(Y \wedge (Y \cdot X) \wedge X\big)$. Note how we modeled the loops from the states **0** to s and from **1** to s, with greatest and least fixed point operators respectively. This is because the node **0** is labeled in $\mathcal{G}$ with an even priority, and **1** is labeled with an odd priority. Moreover since the priority assigned to **1** is greater than the priority assigned to **0**, we imposed by the alternation of fixed points the fact that X subsumes Y in F (see Definition 3.1.5). Note how the Player 2 state s having three children has been modeled by the sub-formula $X \cdot (X \odot Y) \cdot Y$. We did not parenthesise the formula to highlight that the operators $\vee$ and $\wedge$ are associative and commutative, and therefore can be freely used to model n-ary states. Similarly, the operators $\{\odot, \cdot\}$ can model n-ary branching states.

Recall from Lemma 6.3.7 and Lemma 6.3.8 that this is an example of $2\frac{1}{2}$-player meta-parity game with a non-Borel winning set which requires ω_1-iterations of the $\mathbb{W}_R$ function (see Definition 6.2.14) to be computed.

The fact that the modal-free fragment of the logic pL$\mu^{\odot}$ can be considered the language of finite $2\frac{1}{2}$-player meta-parity games can be exploited to investigate the model checking problem from a different viewpoint. Indeed given a *finite* PLTS $\mathcal{L}$, a $[0,1]$-interpretation of the variables ρ and a pL$\mu^{\odot}$ formula F, there exists another (modal) closed pL$\mu^{\odot}$ formula $F^{\rho}_{\mathcal{MC}}$ whose game interpretation is equivalent to the logical game $\mathcal{G}(F,\rho)$. Moreover, it is immediate to verify that if F is a pL$\mu^{\{0,1\}}$ formula, then also $F^{\rho}_{\mathcal{MC}}$ is a pL$\mu^{\{0,1\}}$ formula. Therefore we can reduce the model checking problem of finite structures for the logics pL$\mu^{\odot}$ and pL$\mu^{\{0,1\}}$ to the following:

Question 7.2.13. Given a closed modal-free pL$\mu^{\odot}$ (resp. pL$\mu^{\{0,1\}}$) formula F is it possible to compute its associated[4] value $[\![F]\!]$? A weaker property, still

[4]A closed modal-free formula F is such that $[\![F]\!]^{\mathcal{L}}_{\rho}(p) = [\![F]\!]^{\mathcal{L}'}_{\rho'}(p')$, for every PLTS's $\mathcal{L}$ and $\mathcal{L}'$, $[0,1]$-interpretations ρ and ρ' and process states p and p'. Therefore it make sense to just write $[\![F]\!]$.

practically sufficient for many purposes, is the following: given rationals $\lambda_1 < \lambda_2$, is it decidable whether $[\![F]\!] \in (\lambda_1, \lambda_2)$?

Solving the above presented model checking problem for the logic $\text{pL}\mu^{\odot}$ and its fragment $\text{pL}\mu^{\{0,1\}}$ looks quite challenging: positional strategies do not exist in general, and the failure of the finite model property, albeit not necessarily an issue, seems to suggest that novel ingenuous techniques might be needed to settle the question in a positive, or negative, way.

Up to this point we have only discussed examples, and properties, of $\text{pL}\mu^{\odot}$ formulas. We now provide a few examples of interesting $\text{pL}\mu^{\odot}_{\oplus}$ formulas.

Example 7.2.14. Let us consider the following $\text{pL}\mu^{\odot}_{\oplus}$ formulas:

1. $J_1 = \mu X.\big((\mathbb{P}_{\geq\frac{1}{2}}H) \vee \langle a \rangle X\big)$,
2. $J_2 = \mu X.\big((H > G) \vee \langle a \rangle X\big)$,
3. $J_3 = \mathbb{P}_{>\frac{1}{2}}\Big(\nu X.\big((H > G) \wedge \langle a \rangle X\big)\Big)$,

where the formulas H and G are closed $\text{pL}\mu^{\odot}_{\oplus}$ formulas, and the quantitative probabilistic modalities are specified as in Section 3.3.2.4.

The interpretation of the formula J_1 at a state p, expresses the limit probability the controller has of producing a finite sequence of a-actions eventually reaching a state q that has a probability of at least $\frac{1}{2}$ of satisfying the formula H.

The interpretation of the formula J_2 at a state p, expresses the limit probability the controller has of producing a finite sequence of a-actions eventually reaching a state q at which the property H is more likely to be satisfied than the property G. This kind of formula is potentially useful for specifying properties of, say, gambling systems.

Lastly, the formula J_3 *holds* at a state p, and the terminology is justified since the outermost connective is a threshold modality, if the limit probability the controller has of producing an infinite a-computation consisting of states where H is more likely than G, is greater than $\frac{1}{2}$.

We now conclude this chapter by discussing how the logic PCTL and its qualitative fragment $\text{PCTL}^{\{0,1\}}$ (see Section 3.2.2) can be encoded into the logics $\text{pL}\mu^{\odot}_{\oplus}$ and $pL\mu^{\odot}$, or actually into their fragments $\text{pL}\mu^{[0,1]}$ and $\text{pL}\mu^{\{0,1\}}$, respectively (see Section 3.3). The encoding is formalized as a function $\mathbb{E}$, mapping PCTL formulas to $\text{pL}\mu^{[0,1]}$ formulas, and in particular, $\text{PCTL}^{\{0,1\}}$ formulas to $\text{pL}\mu^{\{0,1\}}$ formulas, defined in the same style as the encoding of CTL into Lμ (see Definition 3.1.17). In order to present the encoding in a natural way, it is convenient to assume that, for each variable $X \in \mathcal{V}$, where $\mathcal{V}$ is the countable set of logical variables used in PCTL and pL$\mu^{\odot}_{\oplus}$ formulas, there exists a variable $\overline{X}$ such that for every $[0,1]$-interpretation $\rho \in \mathcal{V} \rightarrow [0,1]^P$ into some PLTS $\mathcal{L}$, $\rho(\overline{X})(p) = 1 - \rho(X)(p)$, for every process state $p \in P$. It is then natural to extend the negation operator (see Definition 3.3.3) to open Lμ formulas by defining $\overline{(X)} = \overline{X}$. It is clear that with this convention, the property expressed by Proposition 3.3.4 holds for arbitrary open pL$\mu^{\odot}_{\oplus}$ formulas.

Definition 7.2.15. We define the encoding $\mathbb{E}$ from PCTL formulas to L$\mu^{\odot}_{\oplus}$ formulas, by induction on the structure of the PCTL formula F as follows:

1. $\mathbb{E}(X) = X$,
2. $\mathbb{E}(\mathrm{tt}) = \underline{1}$,
3. $\mathbb{E}(F \vee G) = \mathbb{E}(F) \vee \mathbb{E}(G)$,
4. $\mathbb{E}(\neg F) = \overline{\mathbb{E}(F)}$,
5. $\mathbb{E}(\exists(\circ F)) = \mathbb{P}_{>0}\big(\langle \bullet \rangle \mathbb{E}(F)\big)$,
6. $\mathbb{E}(\exists(F\,\mathcal{U}\,G)) = \mu X.\Big(\mathbb{E}(G) \vee \big(\mathbb{E}(F) \wedge \mathbb{P}_{>0}(\langle \bullet \rangle X)\big)\Big)$,
7. $\mathbb{E}(\exists(\mathcal{A}F)) = \nu X.\Big(\mathbb{E}(F) \wedge \big(\mathbb{P}_{>0}(\langle a \rangle X) \vee [a]\,\underline{0}\big)\Big)$,
8. $\mathbb{E}(\mathbb{P}_{\star\lambda} \circ F) = \mathbb{P}_{\star\lambda}\big(\langle \bullet \rangle \mathbb{E}(F)\big)$,
9. $\mathbb{E}(\mathbb{P}_{\star\lambda} F \mathcal{U} G) = \mathbb{P}_{\star\lambda}\Big(\mu X.\Big(\mathbb{E}(G) \vee \big(\mathbb{E}(F) \wedge \langle \bullet \rangle X\big)\Big)\Big)$,
10. $\mathbb{E}(\mathbb{P}_{\star\lambda} \mathcal{A} F) = \mathbb{P}_{\star\lambda}\Big(\nu X.\Big(\mathbb{E}(F) \wedge \big(\langle a \rangle X \vee [a]\,\underline{0}\big)\Big)\Big)$,

where $\star \in \{\geq, >\}$ and $\lambda \in [0,1]$. Note that if F is a qualitative PCTL formula then $\mathbb{E}(F)$ is a pL$\mu^{\{0,1\}}$ formula.

The following Theorem states that the above defined encoding is correct.

Theorem 7.2.16. *Given a PLTS $\mathcal{L} = \langle P, \xrightarrow{\bullet} \rangle$ with one label, a boolean interpretation $\rho \in \mathcal{V} \rightarrow 2^P$ of the variables into $\mathcal{L}$, and a PCTL formula F, the following equality holds: $\|F\|^{\mathcal{L}}_{\rho}(p) = [\![\mathbb{E}(F)]\!]^{\mathcal{L}}_{\rho}(p)$, for every $p \in P$.*

Proof. The proof is by induction on the structure of F. The cases for $F=X$, $F=G\vee H$ and $F=\neg G$ trivial. Let us now consider the cases for $F=\exists\phi$, for ϕ a path formula.

1. Case $F=\exists(\circ G)$. By definition $\|F\|^{\mathcal{L}}_{\rho}=1$ if and only if there exists a run $p \xrightarrow{\bullet} d \rightsquigarrow q\ldots$ such that $\|G\|^{\mathcal{L}}_{\rho}(q)=1$. In other words $\|F\|^{\mathcal{L}}_{\rho}(p)=1$ if and only if there exists a transition $p \xrightarrow{a} d$ and $q \in \mathit{supp}(d)$ such that $\|G\|^{\mathcal{L}}_{\rho}(q)=1$, i.e., by induction hypothesis, if and only if $[\![\langle\bullet\rangle\mathbb{E}(G)]\!]^{\mathcal{L}}_{\rho}(p)>0$. It the follows that $\|F\|^{\mathcal{L}}_{\rho}(p)=[\![\mathbb{E}(F)]\!]^{\mathcal{L}}_{\rho}(p)=1$ as desired.

2. Case $F=\exists(G\mathcal{U}H)$. By induction hypothesis we know that the interpretations of $[\![\mathbb{E}(G)]\!]^{\mathcal{L}}_{\rho}$ and $[\![\mathbb{E}(H)]\!]^{\mathcal{L}}_{\rho}$ are the sets of states satisfying the PCTL formulas G and H respectively. The PCTL formula F holds at a process state p if and only if there exists a run $p \xrightarrow{\bullet} d \rightsquigarrow p_1 \xrightarrow{\bullet} d_1 \rightsquigarrow p_2 \xrightarrow{\bullet} d_3 \rightsquigarrow p_3\ldots$ with some $n\in\mathbb{N}$ such that p_n satisfies H and for all $i<n$, p_i satisfies G. To show that the formula $\mathbb{E}(F)$ captures exactly this property, it is enough to use the same arguments adopted in Example 7.2.2, which explains the role of the probabilistic threshold modality $\mathbb{P}_{>0}$ in the subformula $\mathbb{P}_{>0}(\langle\bullet\rangle X)$ of $\mathbb{E}(F)$. The result then follows by standard arguments, similar to those required to prove Theorem 3.1.18.

3. Case $F=\exists(\mathcal{A}G)$. The proof for this case is similar to the previous one. Indeed, $\mathbb{E}(F)$ is similar to the encoding of the CTL formula into Lμ, specified in Definition 3.1.17, as $\nu X.\Big(\mathbb{E}(F)\wedge\big(\langle a\rangle X\vee[a]\,(\mu Z.Z)\big)\Big)$. Here, the crucial difference is played by the use of the qualitative threshold modality in the subformula $\mathbb{P}_{>0}(\langle\bullet\rangle X)$. By applying the arguments described in Example 7.2.2, it is easy to verify that Player 1 wins the logical pL$\mu^{\odot}_{\oplus}$ game $\mathcal{G}(\mathbb{E}(F),\rho)$ iff there exist a run $p \xrightarrow{\bullet} d_1 \rightsquigarrow p_1 \xrightarrow{\bullet} \ldots$, either finite (i.e., reaching a state p_n such that $p_n \not\xrightarrow{\bullet}$) or infinite, where every process state p_i satisfies the pL$\mu^{\odot}_{\oplus}$ formula $\mathbb{E}(G)$, or equivalently by induction hypothesis, the PCTL formula G. This is precisely the interpretation of the PCTL formula F.

We now discuss the last three cases, dealing with the $\mathbb{P}_{\star\lambda}$ operators of PCTL, for $\star\in\{>,\geq\}$.

1. Case $F=\mathbb{P}_{\geq\lambda}(\circ G)$. The PCTL formula F holds at a process state p, if and only if the supremum probability assigned by the probability measures

$\mathbb{P}_M$ over runs, associated with Markov runs $M \in MRun(p)$ starting at p, to runs having their second state satisfying G is greater or equal than λ. By induction hypothesis we have that $\|G\|^{\mathcal{L}}_{\rho} = [\![G]\!]^{\mathcal{L}}_{\rho}$, thus we can restate the previous property as follows: $\|F\|^{\mathcal{L}}_{\rho}(p) = 1$ if and only if the inequality $\Big(\bigsqcup_{p \xrightarrow{\bullet} d} d\big(\{q \in supp(d) \mid [\![G]\!]^{\mathcal{L}}_{\rho}(q) = 1\}\big)\Big) \geq \lambda$, i.e., if and only if the inequality $[\![\langle \bullet \rangle G]\!]^{\mathcal{L}}_{\rho}(p) \geq \lambda$ holds. The desired result then follows by the denotational interpretation of the quantitative threshold modality $\mathbb{P}_{\geq\lambda}$. The case for $F = \mathbb{P}_{>\lambda} G$ is identical.

2. Case $F = \mathbb{P}_{\geq\lambda}(G\mathcal{U}H)$. The PCTL formula F holds at a process state p, if and only if if the supremum probability assigned by probability measures $\mathbb{P}_M$ over runs, associated with Markov runs $M \in MRun(p)$, to runs satisfying the path formula $G\mathcal{U}H$ is greater or equal than λ. Recall that a run $p \xrightarrow{\bullet} d \rightsquigarrow p_1 \xrightarrow{\bullet} d_1 \rightsquigarrow p_2 \xrightarrow{\bullet} d_3 \rightsquigarrow p_3 \dots$ satisfies $G\mathcal{U}H$ if there exists some $n \in \mathbb{N}$ such that p_n satisfies H and for all $i < n$, p_i satisfies G. By induction hypothesis we know that $\|G\|^{\mathcal{L}}_{\rho}$ and $\|H\|^{\mathcal{L}}_{\rho}$ coincide with $[\![G]\!]^{\mathcal{L}}_{\rho}$ and $[\![H]\!]^{\mathcal{L}}_{\rho}$ respectively. All we need to show then is that the interpretation of the sub-formula $F' = \mu X.\Big(\mathbb{E}(G) \vee \big(\mathbb{E}(F) \wedge \langle \bullet \rangle X\big)\Big)$ at a process state p is precisely the limit probability of runs satisfying $G\mathcal{U}H$; the proof then follows immediately by the denotational interpretation of the quantitative threshold modality $\mathbb{P}_{\geq\lambda}$. Note that the above mentioned fixed-point sub-formula F' is the the encoding into Lμ of the CTL formula $\exists(G\mathcal{U}H)$. Our proof thus matches the standard proof of correctness for Theorem 3.1.18. The game associated with F', after an unfolding step to the state $\langle p, \mathbb{E}(G) \vee \big(\mathbb{E}(F) \wedge \langle \bullet \rangle X\big)\rangle$ can be described as follows:

- if $[\![G]\!]^{\mathcal{L}}_{\rho}(p) = 1$, then Player 1 just chooses the left conjunct and wins with (limit) probability 1 in the subsequent part of the game;
- otherwise, by induction hypothesis, $[\![G]\!]^{\mathcal{L}}_{\rho}(p) = 0$. Thus Player 1 in order to avoid an immediate loss chooses to move to the right disjunct. By a dual argument, Player 2 will move to the left conjunct if $[\![H]\!]^{\mathcal{L}}_{\rho}(p) = 0$ or to the right conjunct otherwise, again, to avoid an immediate loss;
- when the game reaches the state $\langle p, \langle \bullet \rangle X\rangle$, Player 1 has to choose a transition $p \xrightarrow{a} d$ (and loses if such transition does not exists). The

game then reaches the state $\langle X, q\rangle$ with probability $d(q)$, and after an unfolding step, the new stata $\langle q, \mathbb{E}(G) \vee \big(\mathbb{E}(F) \wedge \langle\bullet\rangle X\big)\rangle$ where the dynamics just described get repeated.

We can simplify this description of the game just looking at the behaviors of Player 1 as follows. The game starts at the state p; if $[\![G]\!]^{\mathcal{L}}_{\rho}(p)=1$ then Player 1 wins with probability 1. Suppose instead $[\![G]\!]^{\mathcal{L}}_{\rho}(p) = 0$. If $[\![H]\!]^{\mathcal{L}}_{\rho}(p) = 0$ then Player 1 loses with probability 1. Otherwise Player 1 has to choose a transition $p \xrightarrow{a} d$ (and they lose if there is no such transition) and the game reaches the state $q \in \mathit{supp}(d)$ with probability $d(q)$ and the game progresses from there. Lastly, since the outermost fixed point operator is a least fixed point, every infinite play of the game is losing for Player 1. It is now clear that there exists a one-to-one correspondence between strategies σ_1 for Player 1 in the above described simplified game and Markov runs M in $\mathcal{L}$. Moreover, the probability measures over runs in $\mathcal{L}$ induced by M (see Definition 3.2.7) coincide with the probability measure over Markov branching plays induced by the strategy σ_1 in the game (see Definition 4.1.3 and Definition 7.1.3). The result then simply follows.

3. Case $F = \mathbb{P}_{\geq\lambda}(\mathcal{A}G)$. This case can be proved, following the same lines used in the previous case, showing that $[\![\nu X.\mathbb{E}(F) \wedge \big(\langle a\rangle X \vee [a]\,\underline{0}\big)]\!]^{\mathcal{L}}_{\rho}(p)$ is the upper limit probability $\mathbb{P}_M\big(\{\vec{r} \mid \|\mathcal{A}F\|^{\mathcal{L}}_{\rho}(\vec{r})=1\}\big)$, for $M \in \mathit{MRun}(p)$, of runs satisfying the PCTL path formula $\mathcal{A}G$. Again this is shown by identifying a correspondence between Markov runs in $\mathcal{L}$ and strategies for Player 1 in the logical game $\mathcal{G}(\mathbb{E}(F), \rho)$.

□

Note how the encoding of PCTL formulas of the form $\exists(\phi)$, exploits the possibility, discussed in Example 7.2.2, of specifying temporal properties of the labeled graph underlying a given PLTS as $\mathrm{pL}\mu^{\{0,1\}}$ formulas. On the other hand, the encoding of PCTL formulas of the form $\mathbb{P}_{\star\lambda}(\phi)$ is quite straightforward, being of the form $\mathbb{P}_{\star\lambda}F'$, where F' is the encoding in $\mathrm{L}\mu$ of the CTL formula $\exists(\phi)$.

7.3 Summary of results

In Section 7.1 we defined the game semantics, given in term of $2\frac{1}{2}$-player meta-parity games, of the logic $\mathrm{pL}\mu^{\odot}_{\oplus}$ and its fragments. The proof of equivalence of the denotational and game semantics for $\mathrm{pL}\mu^{\odot}_{\oplus}$ (Theorem 7.1.10) is the main result of this chapter. Since the proof technique we adopted is based on the results of Chapter 6, the equivalence is valid in $\mathrm{ZFC} + \mathrm{MA}_{\aleph_1}$ set theory. For the fragment $\mathrm{pL}\mu\{+_\lambda\}$ of the logic, however, the equivalence holds in ZFC alone, because the games used to interpret $\mathrm{pL}\mu \cup \{+_\lambda\}$ are standard parity games, and the results of Chapter 6 concerning parity games do not depend on Martin's Axiom at $\aleph_1$.

With the techniques developed in Chapter 6, it is possible to prove that the equivalence between game and denotational semantics holds in ZFC alone for other interesting classes of $\mathrm{pL}\mu^{\odot}_{\oplus}$ formulas. We now list, without providing formal proofs, some of the results we have obtained in this direction.

Theorem 7.3.1. *For the following classes of* $\mathrm{pL}\mu^{\odot}_{\oplus}$ *formulas, the game and denotational semantics can be proved to coincide in* ZFC *alone:*

1. *the set of* $\mathrm{pL}\mu^{\odot}$ *formulas without nested occurrences of fixed-point operators,*
2. *the set of* $\mathrm{pL}\mu^{[0,1]}$ *formulas generated by the following grammar:*

$$F, G, C \;::=\; X \mid F \star G \mid \langle a \rangle F \mid [a]\, F \mid \mu X.F \mid \nu X.F \mid \mathbb{P}_{>\lambda} C \mid \mathbb{P}_{\geq\lambda} C$$

where $\star \in \{\vee, \wedge, +_\lambda\}$*,* $\lambda \in [0,1]$ *and* C *is a closed formula.*

We leave for future research the identification of further interesting classes of $\mathrm{pL}\mu^{\odot}_{\oplus}$ formulas whose proof of equivalence can be carried in in ZFC alone.

In Section 7.1 we discussed how the game semantics of $\mathrm{pL}\mu^{\odot}$ offers a straightforward interpretation of the qualitative threshold modalities $\mathbb{P}_{>0}$ and $\mathbb{P}_{=1}$. This is one of the pleasant features of the game semantics of $\mathrm{pL}\mu^{\odot}$.

In Section 7.2 we presented a few examples of $\mathrm{pL}\mu^{\odot}$ and $\mathrm{pL}\mu^{\odot}_{\oplus}$ formulas formalizing interesting properties of PLTS's. These examples allowed us to prove some basic results about the logic $\mathrm{pL}\mu^{\odot}$, such as the failure of the finite model property and the fact that some formulas can be satisfied only by probabilistic systems.

The last result of this chapter is the encoding of the qualitative fragment of PCTL and of full PCTL in the logics $\mathrm{pL}\mu^{\{0,1\}}$ and $\mathrm{pL}\mu^{[0,1]}$ respectively. It would be interesting to extend these expressivity results to richer logics for PLTS's, such as, e.g., PCTL* [9].

Chapter 8

Conclusions and future work

8.1 Conclusions

In this thesis we have defined, following the $[0,1]$-*valued* or *quantitative* approach to probabilistic temporal logics, the three logics $\mathrm{pL}\mu \cup \{+_\lambda\}$, $\mathrm{pL}\mu^{\odot}$ and $\mathrm{pL}\mu^{\odot}_{\oplus}$, each extending the base logic $\mathrm{pL}\mu$ of [78, 29, 56] with additional operators. The denotational semantics for these logics is defined straightforwardly. Our main contribution is the definition and analysis of appropriate game semantics.

The game semantics for $\mathrm{pL}\mu \cup \{+_\lambda\}$, which is given in terms of standard 2-*player stochastic parity games*, has been defined as a simple generalization, required for interpreting the new operators $+_\lambda$, of the games for $\mathrm{pL}\mu$ introduced in [78]. Our main result concerning the logic $\mathrm{pL}\mu \cup \{+_\lambda\}$ is that the game semantics and the denotational semantics coincide on all models, i.e., on all PLTS's as defined in Section 3.2.1. Our proof, which is valid in ZFC, constitutes an interesting, though unsurprising, generalization of the main result of [78], where the equivalence of the game and denotational semantics for $\mathrm{pL}\mu$ was proven only with respect to *finite* models.

The logic $\mathrm{pL}\mu^{\odot}$ is obtained extending $\mathrm{pL}\mu$ with the operators of product $(\cdot)$ and coproduct $(\odot)$. These operators have been already investigated in [56], but as alternative interpretations for the $\mathrm{pL}\mu$ connectives $\{\wedge, \vee\}$, thus not as operations to consider in combination as in $\mathrm{pL}\mu^{\odot}$. The game semantics of $\mathrm{pL}\mu^{\odot}$, defined in terms of 2-*player stochastic meta-parity games* offers a clear operational interpretation to the new operators. At a configuration $F \cdot G$, the game splits in two concurrent and independent instances of the game and Player 1 is required to win in both of them. Similarly, at a configuration $F \odot G$, the game splits in two concurrent and independent instances of the game and Player 1 is required to win in at least one them. Our game semantics also offers a straightforward interpretation of the derived qualitative modalities $\mathbb{P}_{>0}$ and $\mathbb{P}_{=1}$.

One of the primary interests in a game semantics for pL$\mu^{\odot}$, and more generally for all logics having a $[0,1]$-valued semantics with an intended probabilistic reading, is to offer an accessible and clear interpretation for the *property described by a formula*. We suggest that our game semantics, built on top of the elementary idea of concurrent execution of independent sub-istances of the game, succeeds in this task. As discussed in Chapter 7.2, the logic pL$\mu^{\odot}$ is quite expressive as it can express interesting combinations of *quantitative* and *qualitative* specifications which, for instance, allow the encoding of the the qualitative fragment of PCTL. An interesting property, also discussed in Chapter 7.2, is that certain pL$\mu^{\odot}$ formulas can be satisfied only by infinite systems. The construction of such formulas exploits in a crucial way the probabilistic behaviors modeled by PLTS's. Thus, we suggest that the *failure* of the *finite model property* should be considered as an interesting expressivity result about pL$\mu^{\odot}$ rather than a negative result.

The strongest logic considered in this thesis is pL$\mu^{\odot}_{\oplus}$, obtained by extending pL$\mu^{\odot}$ with the operators of truncated sum ($\oplus$) and its dual operation ($\ominus$). Our interest in the logic pL$\mu^{\odot}_{\oplus}$ is motivated by the possibility of encoding the full logic PCTL, as discussed in Chapter 7.2. Our main result about pL$\mu^{\odot}_{\oplus}$ is that it is possible to define an adequate game semantics in terms of $2\frac{1}{2}$-player meta-parity games. As for pL$\mu^{\odot}$, the proof of equivalence with the denotational semantics is carried out in ZFC + $\mathrm{MA}_{\aleph_1}$. This is an interesting result because, whereas $2\frac{1}{2}$-player meta-parity games are clearly designed to match the game-interpretation of the operations of product and coproduct, their ability to encode $\oplus$ and $\ominus$ is far more subtle. Indeed, the game semantics we obtain captures the meaning of the operators of $\oplus$ and $\ominus$ by means of infinitary protocols, involving operations of product and coproduct. These protocols, however, do not necessarily offer a clear and illuminating interpretation for the two connectives, and therefore the game semantics for pL$\mu^{\odot}_{\oplus}$ is perhaps less satisfactory than that given for pL$\mu^{\odot}$. Having a game semantics for pL$\mu^{\odot}_{\oplus}$, and its fragment PCTL, is anyway an interesting result, which also serves as an illuminating illustration of the expressivity of the class of $2\frac{1}{2}$-player meta-parity games.

Our study of the logics pLμ and its extensions pL$\mu \cup \{+_{\lambda}\}$, pL$\mu^{\odot}$ and pL$\mu^{\odot}_{\oplus}$, constitutes an interesting contribution to the field of temporal logics for non-deterministic and probabilistic systems and, in particular, to the applicability of the *quantitative approach* to program logics, which is recently finding interesting applications also in the development of (non-probabilistic) logics for expressing properties of *quantitative* and *hybrid* systems [35, 36].

Notwithstanding our primary focus on probabilistic logics, the present thesis contains also some contributions of independent interest in the field of game theory. Indeed, in order to define the class of $2\frac{1}{2}$-player meta-parity games, used to provide game semantics to $\mathrm{pL}\mu^{\odot}$ and $\mathrm{pL}\mu^{\odot}_{\oplus}$, we found it useful to identify a general class of games capturing the primitive concept of *concurrent* and *independent* execution of sub-games. This led to the definition of the class of *2-player tree games*. We showed that, although simple, the kind of *imperfect information* formalized by tree games is sufficient for encoding in a satisfactory way the well-known class of Blackwell games. We also identified an interesting kind of winning set, which we named *subtree monotone*, that enjoys a useful subsitutivity property. Indeed $2\frac{1}{2}$-player meta-parity games are subtree monotone, and the interesting open problem of *qualitative determinacy* is reducible to a determinacy problem for 2-player tree games with subtree monotone winning sets. Moreover we observed that 2-player (non-stochastic) tree games can model stochasticity just by means of appropriate payoff functions, whereas Gale–Stewart and Blackwell games require the introduction of a third agent (*Nature*) to mimic the probabilistic choices.

We identified the class of $2\frac{1}{2}$-player *meta*-games, in which the set of winning branching plays is defined by means of *inner games*. We proceeded with a systematic analysis of those $2\frac{1}{2}$-player meta-games whose inner-games are specified as ordinary 2-player games on trees with *prefix-independent* winning sets. Prefix independent winning sets are used in many applications of game theory in computer science, and in particular in verification. Thus our general study may be of practical interest. However we are primarily interested in one particular class of $2\frac{1}{2}$-player meta-games, namely the class having inner-games specified as ordinary 2-player parity games. We named this class of games as *$2\frac{1}{2}$-player meta-parity games*. The main technical achievement of the thesis is certainly the proof of determinacy of $2\frac{1}{2}$-player meta-parity games, carried out in ZFC set theory extended with Martin's Axiom at $\aleph_1$ ($\mathrm{MA}_{\aleph_1}$). We believe our proof is technically interesting for the following reasons:

1. It is an interesting and natural example of determinacy for a class of games which can be considered of *imperfect information*. This kind of result does

not abound in game theory, as imperfect information often turned out to be mathematically intractable.

2. The fact that the winning set of a $2\frac{1}{2}$-player meta-parity game is, in general, not a Borel set, makes our proof quite intriguing. Indeed it is known that ZFC, extended with instances of Martin's Axiom, cannot prove determinacy results beyond Borel determinacy. For example it is not possible to prove that all Gale–Stewart games with $\mathbf{\Sigma}^1_1$-winning sets are determined in $\mathrm{ZFC} + \mathrm{MA}_{\aleph_1}$ [59]. Hence, it is somewhat surprising that $2\frac{1}{2}$-player meta-parity games which, as we showed, can have $\mathbf{\Pi}^1_1$-complete and $\mathbf{\Sigma}^1_1$-complete winning sets, can be proved to be determined in $\mathrm{ZFC} + \mathrm{MA}_{\aleph_1}$ set theory.

3. We are not aware of any other result in theoretical computer science whose proof is (or at least was originally) carried out in proper extensions of ZFC set theory. Thus, our proof of determinacy carried out in $\mathrm{ZFC} + \mathrm{MA}_{\aleph_1}$, is perhaps noteworthy as being a first example of this kind of result.

Another important, and somewhat unexpected, result about $2\frac{1}{2}$-player meta-parity games is the fact that, even when considering *finite* game-arenas, positional strategies are not sufficient for reaching the optimal value of the game: $2\frac{1}{2}$-player meta-parity games are not *positionally determined*.

8.2 Future work

Several technical theoretical questions have been left open in this thesis, and can be grouped in two general directions for mathematical research. The first is aimed at removing the non-standard axiom $\mathrm{MA}_{\aleph_1}$ from the proofs of our results. Specific questions of this kind include:

1. Is the assertion mG-UM, which asserts the universal measurability of $2\frac{1}{2}$-player meta-game winning sets (see Definition 5.1.8), and in particular mG-UM(Γ_p), with Γ_p the class of $2\frac{1}{2}$-player meta-parity games, provable in ZFC alone?

   Even though we think it is quite likely that mG-UM, in its generality, coincides with the assertion $\mathbf{\Delta}^1_2$-UM, in which case mG-UM would not be probable in ZFC alone (see Theorem 2.1.81), we think it is plausible that mG-UM(Γ_p) holds in ZFC alone.

2. Assuming mG-UM(Γ_p) as above, is it possible to remove the use of $\mathrm{MA}_{\aleph_1}$ from the proof of determinacy of $2\frac{1}{2}$-player meta-parity games?

   As discussed in Section 6.5, this is not obvious, because the ω_1-additivity properties of measure used in our proof, which are consequences of $\mathrm{MA}_{\aleph_1}$, do not follow from mG-UM(Γ_p).

The second is aimed at extending knowledge of determinacy results for varieties of tree games. Specific questions of this kind include:

1. Does Question 4.3.19 has a positive answer, i.e., Is the class of meta-games with subtree monotone (Borel) winning sets determined?

   As discussed in Section 4.4, a positive answer to the this question would settle the interesting problem of *qualitative determinacy* in standard $2\frac{1}{2}$-player games with prefix-independent (Borel) winning sets.

2. Can our determinacy theorem be extended to all prefix independent $2\frac{1}{2}$-player meta-games?

3. It the determinacy of *all* $2\frac{1}{2}$-player tree games (with arbitrary meta-winning sets) consistent with ZF (without the Axiom of Choice)?

Beside these abstract theoretical questions, our work opens the door to other interesting research directions. Providing verification methods for probabilistic concurrent systems using the logic $\mathrm{pL}\mu^{\odot}$ or, possibly, using $\mathrm{pL}\mu^{\odot}_{\oplus}$, is an interesting area for further research. We now list some the possible goals.

1. Model Checking: given a finite PLTS $\langle P, \{\xrightarrow{a}\}_{a\in L}\}$, and a $\mathrm{pL}\mu^{\odot}_{\oplus}$ formula F, is it possible to compute (or at least approximate) the value $[\![F]\!](p)$? Similarly for $\mathrm{pL}\mu^{\odot}$ formulas. Note that Model checking decision procedures for $\mathrm{pL}\mu$ are known (see, e.g., [116]).

2. Satisfiability problem: given a $\mathrm{pL}\mu^{\odot}_{\oplus}$ formula F, is it possible to verify automatically if there exists a PLTS $\langle P, \{\xrightarrow{a}\}_{a\in L}\}$ which satisfies it with, say, probability 1? Similarly for $\mathrm{pL}\mu^{\odot}$ and the other probabilistic μ-calculi discussed in this thesis.

3. Axiomatization: is it possible to find an appropriate system of (in)equalities such that if $[\![F]\!](p) \leq [\![G]\!](p)$, for every PLTS $\langle P, \{\xrightarrow{a}\}_{a\in L}\}\rangle$ and $p \in P$, then $F \leq G$ is derivable by (in)equational reasoning?

Related to the last point, a promising direction for future research is the development of *proof systems for verification*. In the past 15 years, several proof systems have been proposed for reasoning over modal μ-calculus (Lμ) properties. In [104, 105] the author introduces a sound and complete sequent based proof system for proving Hennessy-Milner properties for processes described by a class of well behaved process calculi. At the same time a sequent based proof system for CCS (see., e.g., [79]) processes and general Lμ properties is introduced in [27]. The judgments of these systems are of the following form

$$t_1 : F_1, \dots, t_n : F_n \vdash v_1 : G_1, \dots, v_n : G_m$$

where t, v range over process-terms described by appropriate operational rules (GSOS [16] or CCS [79]), and F, G range over Hennessy-Milner or Lμ formulas. The intended interpretation is that if the the LTS's corresponding to $t_1, \dots, t_n$ satisfy the formulas $F_1, \dots, F_n$, respectively, then for some $i \in \{1 \dots, m\}$, the LTS's corresponding to v_i satisfies the formula G_i. These papers provided evidence for the many advantages offered by the use of sequents as basic judgments in the proposed proof systems (see [105] for a detailed overview).

During our research we have developed [80] a sequent based proof system for verifying pLμ properties of PTLS's specified by PGSOS operation rules (see, e.g., [6]). As in the above mentioned systems, the judgments are of the form

$$t_1 : F_1, \dots, t_n : F_n \vdash v_1 : G_1, \dots, v_n : G_m$$

but are interpreted as follows:

$$[\![F_1]\!](t_1) \cdot \ \dots \ \cdot [\![F_n]\!](t_n) \ \leq \ [\![G_1]\!](v_1) \odot \ \dots \ \odot [\![G_n]\!](v_m).$$

Preliminary results show that interesting properties of systems can be derived as judgments of this kind. We suggest that further investigations in this direction might be useful for developing (non automatic) verification methods for infinite-state probabilistic systems.

Other interesting problems are related to expressivity questions for the logics discussed in this thesis. For example it would be interesting to verify if the qualitative fragment of PCTL* (see, e.g., [9]) could be encoded into pL$\mu^{\odot}$. Similarly for full PCTL* and pL$\mu^{\odot}_{\oplus}$. Another interesting expressivity question is the following. We often declared that the qualitative modality $\mathbb{P}_{>0}$ is expressible in pL$\mu^{\odot}$ but not in pLμ and, similarly, the quantitative modality $\mathbb{P}_{>\frac{2}{3}}$ is expressible in pL$\mu^{\odot}_{\oplus}$ but not in pL$\mu^{\odot}$. Although we are quite confident about the validity of these assertions, we do not know how to prove them. As a last, but not least, direction for future work, we suggest that studying the equivalence induced by the logic pL$\mu^{\odot}$ or pL$\mu^{\odot}_{\oplus}$ on states of PLTS's (naturally defined as $p \equiv q$ if and only if for all pL$\mu^{\odot}$ closed formulas F the equality $[\![F]\!](p) = [\![F]\!](q)$ holds) could be quite interesting. One hopes to identify fruitful connections between the behavioral equivalences developed for PLTS's (see, e.g., [93]), and equivalences induced by strong logics for expressing concrete properties of systems.

Appendix A

Appendix

A.1 Proofs of Section 2.1.4

This section contains proofs of results discussed in Section 2.1.4.

Definition A.1.1. Given a topological space $(X, \mathcal{T})$, we say that a function $f : X \to [0, 1]$ is *lower semicontinuous* if $f^{-1}\big((\lambda, 1]\big) \in \mathcal{T}$, for every rational number $\lambda \in [0, 1]$. Clearly every continuous function is lower semicontinuous.

Lemma A.1.2. *Given a topological space* $(X, \mathcal{T})$*, if* $f : X \to [0, 1]$ *and* $(1 - f)$*, defined as* $(1 - f)(x) = 1 - f(x)$*, are both lower semicontinuous, then they are both continuous.*

Proof. We just need to prove that f is continuous, i.e., we need to show that $f^{-1}\big((\lambda_1, \lambda_2)\big) = f^{-1}\big((\lambda_1, 1]\big) \cap f^{-1}\big([0, \lambda_2)\big)$ is open, for every open interval with rational endpoints $\lambda_1, \lambda_2 \in [0, 1]$. The proof follows by observing that $f^{-1}\big([0, \lambda_2)\big) = (1 - f)^{-1}\big((\lambda_2, 1]\big)$. □

Proposition A.1.3. *Given a topological space* $(X, \mathcal{T})$*, any pointwise supremum of lower semicontinuous functions is lower semicontinuous.*

Proposition A.1.4. *Let* $(X, \mathcal{T})$ *be a Polish space and* $\mathcal{B}$ *a basis for* $\mathcal{T}$*. For every basic open set* $U \in \mathcal{B}$*, the characteristic function* $\chi_U : X \to \{0, 1\}$ *of* U *is the pointwise supremum of a countable increasing sequence* $\{f_n\}$ *of continuous functions* $f_n : X \to [0, 1]$*. It follows that also the characteristic function* χ_V*, with* V *an arbitrary open sets, is the pointwise supremum of a countable sequence of continuous functions.*

Proposition A.1.5. *Given a topological space $(X,\mathcal{T})$ and finitely many lower semicontinuous functions $\{f_0,\dots,f_k\}$, any function defined as $f(x)=\sum_{i=0}^{k}\lambda_i\cdot f_i(x)$, with $\lambda_i\geq 0$, is lower semicontinuous.*

Proposition A.1.6. *Given a Polish space $(X,\mathcal{T})$ and a basis $\mathcal{B}$ for $\mathcal{T}$, any continuous function $f:X\to[0,1]$ is the supremum limit of an increasing sequence $\{g_i\}_{i\in\mathbb{N}}$ of lower semicontinuous functions $g_i(x)=\sum_{j=0}^{n_i}\lambda_i^j\cdot\chi_{U_i^j}(x)$ where, for every $i\in\mathbb{N}$,*

1. *$n_i\in\mathbb{N}$,*
2. *$\lambda_i^j\geq 0$,*
3. *the sets $\{U_i^j\}_{0\leq j\leq n_i}$ are open sets.*

Lemma A.1.7. *Let $(X,\mathcal{T})$ be a Polish space. The sets $U_{O,\lambda}=\{\mu \mid \mu(O)>\lambda\}$, where $O\in\mathcal{T}$ and $\lambda\in[0,1]$ is a rational number, form a subbasis for the weak topology on $\mathcal{M}_1(X)$.*

Proof. Let $\mathcal{W}$ be the weak topology on $\mathcal{M}_1(X)$, i.e., the coarser topology such that the map $\mu\mapsto\int_X f\ \mathrm{d}\mu$ is continuous for every continuous $f:X\to[0,1]$. Let $\mathcal{S}$ be the topology generated by the sets of the form $U_{O,\lambda}$. We need to show that $\mathcal{S}=\mathcal{T}$, i.e., we need to show that:

1. Every set $U_{O,\lambda}$ is in $\mathcal{W}$, and
2. every function $\mu\mapsto\int_X f\ \mathrm{d}\mu$, for $f:X\to[0,1]$ continuous, is continuous with respect to the space $(\mathcal{M}_1,\mathcal{S})$.

For the first point, consider an arbitrary open set O. From Proposition A.1.4, we know that $\chi_O=\bigsqcup\{f_n\}_{n\in\mathbb{N}}$ for a countable increasing sequence of continuous functions $f_n:X\to[0,1]$. Therefore we need to show that:

$$\begin{aligned}\{\mu \mid \mu(O)>\lambda\} &= \{\mu \mid \textstyle\int_X(\bigsqcup_i f_n)\ \mathrm{d}\mu>\lambda\}\\ &= \{\mu \mid \textstyle\bigsqcup_n\left(\int_X f_n\ \mathrm{d}\mu\right)>\lambda\}\\ &= \textstyle\bigcup_n\{\mu \mid \int_X f_n\ \mathrm{d}\mu>\lambda\}\end{aligned}$$

is open. The result then follows from the fact that $\{\mu \mid \int_X f_n\ \mathrm{d}\mu>\lambda\}\in\mathcal{W}$ by definition, since f_n is continuous and $(\lambda,1]$ is open.

For the second point, let us fix a continuous function $f:X\to[0,1]$. By Proposition A.1.6, we know that

$$f = \bigsqcup_{i\in\mathbb{N}} g_i$$

where each g_i is of the form $g_i = \sum_{j=0}^{n_i} \lambda_i^j \cdot \chi_{U_i^j}$ as specified in Proposition A.1.6. We need to show that the map $\mu \mapsto \int_X f \; \mathrm{d}\mu$ is continuous with respect to the space $(\mathcal{M}_1, \mathcal{S})$. The following equalities hold:

$$\begin{aligned}
\int_X f \; \mathrm{d}\mu &= \int_X \big(\bigsqcup_i g_i\big) \; \mathrm{d}\mu \\
&= \bigsqcup_i \int_X g_i \; \mathrm{d}\mu \\
&= \bigsqcup_i \int_X \big(\sum_{j=0}^{n_i} \lambda_i^j \cdot \chi_{U_i^j}\big) \; \mathrm{d}\mu \\
&= \bigsqcup_i \Big(\sum_{j=0}^{n_i} \lambda_i^j \cdot \big(\int_X \chi_{U_i^j} \; \mathrm{d}\mu\big)\Big) \\
&= \bigsqcup_i \Big(\sum_{j=0}^{n_i} \lambda_i^j \cdot \mu(U_i^j)\Big).
\end{aligned}$$

The function $\mu \mapsto \mu(U_i^j)$ is lower semicontinuous, relative to $(\mathcal{M}_1, \mathcal{S})$ by definition of $\mathcal{S}$. Hence by Proposition A.1.5 and Proposition A.1.3, the function $\hat{f}$ defined as $\hat{f}(\mu) = \int_X f \; \mathrm{d}\mu$ is lower semicontinuous as well. Since f is continuous, so is $(1-f)$, and with the same technique adopted above, one can show that the map $\widehat{1-f}$ defined as $\mu \mapsto \int_X (1-f) \; \mathrm{d}\mu$ is lower semicontinuous. The desired result then follows, by application of Lemma A.1.2, by observing that $(1-\hat{f}) = \widehat{1-f}$. ☐

Lemma A.1.8. *Let $(X, \mathcal{T})$ be a 0-dimensional Polish space and $\mathcal{B}$ a countable basis of clopen sets for $\mathcal{T}$, such that for every $A, B \in \mathcal{B}$, the set $A \setminus B$ is expressible as a disjoint union of clopen sets in $\mathcal{B}$. The sets $U_{B,\lambda} = \{\mu \mid \mu(B) > \lambda\}$, where $B \in \mathcal{B}$ and $\lambda \in [0,1]$ is a rational number, form a subbasis for the weak topology on $\mathcal{M}_1(X)$.*

Proof. As a first observation, note that if $U \in \mathcal{T}$ is a disjoint union of basic clopen sets in $\mathcal{B}$, and $B \in \mathcal{B}$, then the set $U \cup B$ can also be expressed as a disjoint union of basic clopen sets in $\mathcal{B}$. It then follows that finite unions of basic clopen sets can be expressed as disjoint unions of basic clopen sets. Thus, every open set U is expressible as $U = \bigcup_n U^n$, where each U^n is a disjoint union of basic clopen sets in $\mathcal{B}$.

Let $\mathcal{S}_0$ be the topology on $\mathcal{M}_1(X)$ generated by the sets $U_{B,\lambda}$. We prove the result by showing that every set of the form $\{\mu \mid \mu(O) > \lambda\}$, for $O \in \mathcal{T}$ is open with respect to $\mathcal{S}_0$. This, by application of Lemma A.1.7, will conclude the proof.

Let $O = \bigcup_n O_n$, where, for every $n \in \mathbb{N}$, the set O_n is a disjoint union of basic clopen sets in $\mathcal{B}$. Note that the equality $\{\mu \mid \mu(O) > \lambda\} = \bigcup_n \{\mu \mid \mu(U_n) > \lambda\}$ holds. Hence we just need to prove that $\{\mu \mid \mu(U_n) > \lambda\}$ is open with respect to $\mathcal{S}_0$. We know that U_n is of the form $U_n = \bigcup_m B^n_m$, where the clopen sets $B^n_m \in \mathcal{B}$, for $m \in \mathbb{N}$, are pairwise disjoint. By the same argument as above, we just need to show that, for every $k \in \mathbb{N}$, the set $\{\mu \mid \mu(\bigcup_{i=0}^k B^n_n) > \lambda\}$ is open in $\mathcal{S}_0$. Since the sets B^n_i are disjoint, we have that $\mu(\bigcup_{i=0}^k B^n_i) = \sum_{i=0}^k \mu(B^n_i)$. Let $L^k_\lambda \subseteq [0,1]^k$ be the set $\{\langle \gamma_1, \dots, \gamma_k \rangle \mid \sum_{i=0}^k \gamma_i > \lambda\}$. The following equality

$$\{\mu \mid \sum_{i=0}^{k} \mu(\bigcup_{i=0}^{k} B^n_i) > \lambda\} = \bigcup_{\vec{\gamma} \in L^k_\lambda} \{\mu \mid \mu(B_1) > \gamma_1 \wedge \cdots \wedge \mu(B_k) > \gamma_k\}$$

holds. The desired result then follows straightforwardly. □

A.2 Proofs of Section 2.1.5

Proposition A.2.1. *Let $(X, \mathcal{T})$ be a Polish space and μ a complete probability measure on X. We define, for every $\epsilon > 0$, the partially ordered set $(\mathcal{P}_\epsilon, \leq)$ specified as $\mathcal{P}_\epsilon = \{U \in \mathcal{T} \mid \mu(U) < \epsilon\}$ and $U \leq V$ if $V \subseteq U$. Then $(\mathcal{P}_\epsilon, \leq)$ satisfies the countable chain condition.*

Proof. Let $\mathcal{B} = \{B_n\}_n$ be a countable basis for X. Suppose, by contradiction, that A is an uncountable antichain in $\mathcal{P}_\epsilon$. Then $A \subseteq \mathcal{P}_\epsilon$ is uncountable and, for every $U, V \in A$, the open set U is not compatible with V, i.e., there is no open set $W \in \mathcal{P}_\epsilon$ such that $A, B \subseteq W$ or, equivalently, $\mu(U \cup V) \geq \epsilon$. Since $A = \bigcup_n \{U \in A \mid \mu(A) < (1 - \frac{1}{n})\epsilon\}$, there exists some $\delta = \frac{1}{n}\epsilon$ such that the set $\mathcal{E} = \{U \in A \mid \mu(U) < \epsilon - \delta\}$ is uncountable. For each $U \in \mathcal{E}$ let F_U be a finite union of basic open sets contained in U such that $\mu(U \setminus F_U) < \frac{\delta}{2}$. Note that the set $\{F_U \mid U \in \mathcal{E}\}$ is countable, because there are only countably many finite unions of basic open sets. We derive a contradiction with the assumption that $\mathcal{E}$ is uncountable, by showing that if $U \neq V \in \mathcal{E}$, then $F_U \neq F_V$. If $U, V \in \mathcal{E}$ and $V \neq U$, then U and V are incompatible, i.e., $\mu(U \cup V) > \epsilon$. But $\mu(F_U \cup F_V) \geq \mu(U \cup V) - \frac{\delta}{2} - \frac{\delta}{2} \geq \epsilon - \delta$. Since $\mu(F_U) \leq \mu(U) < \epsilon - \delta$, it follows that $F_U \neq F_V$.

□

Theorem A.2.2 ($\mathrm{MA}_{\aleph_1}$)**.** *Let $(X, \mathcal{T})$ be a Polish space. Then for every probability measure $\mu \in \mathcal{M}_1(X)$, the σ-ideal NULL_μ is ω_1-additive.*

Proof. Let $\{N_\alpha\}_{\alpha<\omega_1}$ be a collection of μ-null sets, and let $N = \bigcup_{\alpha<\omega_1} N_\alpha$. We prove that N is μ-null by showing that, for every $\epsilon > 0$, there exists an open set $G \in \mathcal{T}$ such that $G \supseteq N$ and $\mu(G) \leq \epsilon$.

Fix some $\epsilon > 0$. Define the poset $(\mathcal{P}_\epsilon, \leq)$ as $\mathcal{P}_\epsilon = \{U \in \mathcal{T} \mid \mu(U) < \epsilon\}$ and $U \leq V$ if $V \subseteq U$. For every $\alpha < \omega_1$, define the set $D_\alpha = \{U \in \mathcal{P}_\epsilon \mid U \leq N_\alpha\}$. Clearly D_α is down-closed. We now prove that D_α is dense in $\mathcal{P}_\epsilon$. We need to show that for every $U \in \mathcal{P}_\epsilon$, there is some $V \in D_\alpha$ such that $U \geq V$ (i.e., $U \subseteq V$). Since $\mu(N_\alpha) = 0$, by regularity of μ (see Theorem 2.1.66) there is some $V' \in D_\alpha$ such that $\mu(V') < \epsilon - \mu(U)$. Then $V = V' \cup U$ satisfies the desired property. Define $\mathcal{D} = \{D_\alpha\}_{\alpha<\omega_1}$. Since $\mathcal{P}_\epsilon$ satisfies the countable chain condition (Proposition A.2.1) It then follows from $\mathrm{MA}_{\aleph_1}$ that there is a $\mathcal{D}$-generic filter G in $\mathcal{P}_\epsilon$. Clearly $\bigcup G$ is an open set and $N \subseteq G$. Assume by contradiction that $\mu(G) > \epsilon$. Then there exists $U_0, \ldots U_k \in G$ such that $\mu(\bigcup_{i=0}^k U_i) > \epsilon$. Since G is a filter on $\mathcal{P}_\epsilon$, the set $\bigcup_{i=0}^k U_i$ is in G and thus in $\mathcal{P}_\epsilon$. This provides the desired contradiction. □

A.3 Proofs of Section 2.2

This section contains proofs of results discussed in Section 2.2.

Lemma A.3.1. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers $x_i \in [0,1]$, and let $\epsilon \in (0,1]$. Then the following inequality holds:*

$$\prod_{i\in\mathbb{N}} (x_i + \frac{\epsilon}{2^{2^i+1}}) \leq (\prod_{i\in\mathbb{N}} x_i) + \epsilon$$

Proof. We show that the inequality holds by proving the equivalent inequality:

$$\prod_{i\in\mathbb{N}} (x_i + \frac{\epsilon}{2^{2^i+1}}) \leq (\prod_{i\in\mathbb{N}} x_i) + \sum_{i\in\mathbb{N}} \frac{\epsilon}{2^{i+1}}.$$

We do this by induction on the naturals, showing that for each $n \in \mathbb{N}$ the inequality

$$\prod_{i=0}^{n} (x_i + \frac{\epsilon}{2^{2^i+1}}) \leq (\prod_{i=0}^{n} x_i) + \sum_{i=0}^{n} \frac{\epsilon}{2^{i+1}}$$

holds. For the case $n = 0$, we need to show that $x_0 + \frac{\epsilon}{2^{2^0+1}} \leq x_0 + \frac{\epsilon}{2^1}$, and this is trivial. Let us suppose, by induction hypothesis, that the desired inequality holds for n. Then we have that the following inequality holds:

$$\begin{aligned}\prod_{i=0}^{n+1}(x_i + \frac{\epsilon}{2^{2^i+1}}) &= (\prod_{i=0}^{n}(x_i + \frac{\epsilon}{2^{2^i+1}})) \cdot (x_{n+1} + \frac{\epsilon}{2^{2^{n+1}+1}}) \\ &\leq (\prod_{i=0}^{n} x_i + \sum_{i=0}^{n}\frac{\epsilon}{2^{i+1}}) \cdot (x_{n+1} + \frac{\epsilon}{2^{2^{n+1}+1}})\end{aligned}$$

Since all the summands in the last expression are real numbers in $[0,1]$, we have that the following inequality holds:

$$\begin{aligned}(\prod_{i=0}^{n} x_i + \sum_{i=1}^{n}\frac{\epsilon}{2^{i+1}}) \cdot (x_{n+1} + \frac{\epsilon}{2^{2^{n+1}+1}}) &\leq (\prod_{i=0}^{n} x_i \cdot x_{n+1}) + (\sum_{i=1}^{n}\frac{\epsilon}{2^{i+1}}) + \frac{\epsilon}{2^{2^{n+1}+1}} + \frac{\epsilon}{2^{2^{n+1}+1}} \\ &= (\prod_{i=0}^{n+1} x_i) + (\sum_{i=1}^{n}\frac{\epsilon}{2^{i+1}}) + \frac{\epsilon}{2^{2^{n+1}}}\end{aligned}$$

By observing that $\frac{\epsilon}{2^{2^{n+1}}} \leq \frac{\epsilon}{2^{(n+1)+1}}$ holds, for every natural number n, we get the desired inequality:

$$\begin{aligned}\prod_{i=0}^{n+1}(x_i + \frac{\epsilon}{2^{2^i+1}}) &\leq \prod_{i=0}^{n+1} x_i + \sum_{i=1}^{n}\frac{\epsilon}{2^{i+1}} + \frac{\epsilon}{2^{(n+1)+1}} \\ &= \prod_{i=0}^{n+1} x_i + \sum_{i=1}^{n+1}\frac{\epsilon}{2^{i+1}}.\end{aligned}$$

□

Lemma A.3.2. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers $x_i \in [0,1]$, and let $\epsilon \in (0,1]$. Then the following inequality holds:*

$$\prod_{i\in\mathbb{N}}(x_i - \frac{\epsilon}{2^{2^i+1}}) \geq (\prod_{i\in\mathbb{N}} x_i) - \epsilon$$

Proof. We show that the inequality holds by proving the following property:

$$\prod_{i\in\mathbb{N}}(x_i - \frac{\epsilon}{2^{i+1}}) \geq (\prod_{i\in\mathbb{N}} x_i) - \epsilon$$

which clearly implies the desired one, since for all $n \in \mathbb{N}$, $\frac{\epsilon}{2^{2^n+1}} \leq \frac{\epsilon}{2^{n+1}}$ holds. We prove this inequality by showing that the equivalent inequality

$$\prod_{i\in\mathbb{N}}(x_i - \frac{\epsilon}{2^{i+1}}) \geq (\prod_{i\in\mathbb{N}} x_i) - \sum_{i\in\mathbb{N}}\frac{\epsilon}{2^{i+1}}.$$

holds. This is proven, as for the previous case, by induction on the naturals, showing that for each $n \in \mathbb{N}$ the inequality

$$\prod_{i=0}^{n}(x_i - \frac{\epsilon}{2^{i+1}}) \geq (\prod_{i=0}^{n} x_i) - \sum_{i=0}^{n}\frac{\epsilon}{2^{i+1}}.$$

holds. The case for $n = 0$ is trivial. Let us suppose, by induction hypothesis, that the desired inequality holds for some n. Then we have that the following inequality holds:

$$\begin{aligned}\prod_{i=0}^{n+1}(x_i - \frac{\epsilon}{2^{i+1}}) &= \Big(\prod_{i=0}^{n}(x_i - \frac{\epsilon}{2^{i+1}})\Big) \cdot \big(x_{n+1} - \frac{\epsilon}{2^{(n+1)+1}}\big) \\ &\geq \Big(\prod_{i=0}^{n} x_i - \sum_{i=0}^{n} \frac{\epsilon}{2^{i+1}}\Big) \cdot \big(x_{n+1} - \frac{\epsilon}{2^{(n+1)+1}}\big)\end{aligned}$$

Let us denote with α, β, and γ, the values $\prod_{i=0}^{n} x_i$, $\sum_{i=0}^{n} \frac{\epsilon}{2^{i+1}}$ and $\frac{\epsilon}{2^{(n+1)+1}}$ respectively. Then $\prod_{i=0}^{n+1}(x_i - \frac{\epsilon}{2^{i+1}}) \geq (\alpha - \beta) \cdot (x_{n+1} - \gamma)$. By observing that $\alpha, \beta, x_{n+1}, \gamma \in [0, 1]$, we get the desired inequality as follows:

$$\begin{aligned}\prod_{i=0}^{n+1}(x_i - \frac{\epsilon}{2^{i+1}}) &\geq (\alpha - \beta) \cdot (x_{n+1} - \gamma) \\ &= (\alpha \cdot x_{n+1}) - (\beta \cdot x_{n+1}) - (\alpha \cdot \gamma) + (\beta \cdot \gamma) \\ &\geq (\alpha \cdot x_{n+1}) - \beta - \gamma \\ &= \Big(\prod_{i=0}^{n} x_i \cdot x_{n+1}\Big) - \Big(\sum_{i=0}^{n} \frac{\epsilon}{2^{i+1}}\Big) - \frac{\epsilon}{2^{(n+1)+1}} \\ &= \prod_{i=0}^{n+1} x_i - \sum_{i=0}^{n+1} \frac{\epsilon}{2^{i+1}}\end{aligned}$$

□

Lemma A.3.3. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers $x_i \in [0, 1]$, and let $\epsilon \in (0, 1]$. Then the following inequalities hold:*

$$\coprod_{i\in\mathbb{N}}(x_i + \frac{\epsilon}{\#(i)}) \leq \Big(\coprod_{i\in\mathbb{N}} x_i\Big) + \epsilon \quad \textit{and} \quad \prod_{i\in\mathbb{N}}(x_i - \frac{\epsilon}{\#(i)}) \geq \Big(\prod_{i\in\mathbb{N}} x_i\Big) - \epsilon.$$

Proof. We just prove the first inequality. The proof for the second one can be carried out in a similar way.

By De Morgan duality of the product and coproduct operations, we know that following equality holds:

$$\coprod_{i\in\mathbb{N}}(x_i + \frac{\epsilon}{\#(i)}) = 1 - \prod_{i\in\mathbb{N}}\big(1 - (x_i + \frac{\epsilon}{\#(i)})\big).$$

Let us denote with y_i the value $1 - x_i$. We can then rewrite the previous equality as follows:

$$\coprod_{i\in\mathbb{N}}(x_i + \frac{\epsilon}{\#(i)}) = 1 - \prod_{i\in\mathbb{N}}\big(y_i - \frac{\epsilon}{\#(i)}\big).$$

By lemma 2.2.10 we know that the inequality

$$\prod_{i\in\mathbb{N}}(y_i - \frac{\epsilon}{\#(i)}) \geq (\prod_{i\in\mathbb{N}} y_i) - \epsilon$$

holds, and therefore we have the following inequality:

$$\coprod_{i\in\mathbb{N}}(x_i + \frac{\epsilon}{\#(i)}) \leq 1 - (\prod_{i\in\mathbb{N}} y_i) + \epsilon$$

from which, by De Morgan duality, we get the desired inequality:

$$\coprod_{i\in\mathbb{N}}(x_i + \frac{\epsilon}{\#(i)}) \leq (\coprod_{i\in\mathbb{N}} x_i) + \epsilon$$

□

A.4 Proofs of Section 2.3.1

This section contains proofs of results discussed in Section 2.3.1.

Theorem A.4.1. *There exists a universally measurable set $A \subseteq \mathbb{N}^\omega$, such that $\mathsf{GS}(\mathbb{N}, A)$ is determined under mixed strategies, with $\mathsf{MVAL}\big(\mathsf{GS}(\mathbb{N}, A)\big) = 0$, but is not determined under deterministic strategies.*

Proof. Let B be an uncountable (i.e., $|B| > \aleph_0$) universally null subset of $\mathbb{N}^\omega$. Recall that a set B is universally null if it is μ-measurable and $\mu(B) = 0$ for every atomless Borel probability measure on $\mathbb{N}^\omega$. The existence of such a set can be proven in ZFC, see e.g. [103] and [48]. Moreover every universally null set is universally measurable, see e.g. [39, 211X(e) in §21]. The set B can not be a Perfect set, i.e., can not have a closed subset C homeomorphic to the Cantor space, which is 2^ω endowed with the product topology (see e.g. [60, 6.2]). Suppose indeed that B is Perfect, and therefore there exists a continuous map $f : 2^\omega \to \mathbb{N}^\omega$ such that $f(2^\omega) = C$. Then we can take the uniform measure μ on 2^ω, which is atomless, and push it to a probability measure μ' on $\mathbb{N}^\omega$ specified by the assignment $\mu'(X) = \mu\big(f^{-1}(X \cap C)\big)$, on all Borel sets $X \subseteq \mathbb{N}^\omega$. Then μ' is atomless and $\mu'(B) = 1$. A contradiction. Let $\mathsf{GS}(\mathbb{N}, A)$ be the Gale-Stewart "Perfect set game" associated with B, as defined in e.g. [60] or [59]. It is known (see e.g. [60, §21] or [59, §33.9]) that Player 1 has a deterministic winning strategy in $\mathsf{GS}(\mathbb{N}, A)$ if and only if B is a Perfect set, and Player 2 has a deterministic winning strategy if and only if B is countable. Hence $\mathsf{GS}(\mathbb{N}, A)$ is not determined by deterministic strategies. However, from the fact that B is universally null, it is possible to show (see [74]) that $\mathsf{GS}(\mathbb{N}, A)$ is determined under mixed strategies with $\mathsf{MVAL}\big(\mathsf{GS}(\mathbb{N}, A)\big) = 0$. □

A.5 Proofs of Section 4.3

This section contains proofs of results discussed in Section 4.3.

Lemma A.5.1. *Let $(X, \sqsubseteq)$ be a compact pospace and μ, ν complete Borel probability measures on X. If, for some $\epsilon \geq 0$, the inequality $\mu(U) \leq \nu(U) + \epsilon$ holds for every upper-closed open set $U \subseteq X$, then $\mu(W) \leq \nu(W) + \epsilon$ holds for every upper-closed μ-ν-measurable set $W \subseteq X$.*

Proof. It is enough to prove the results for W upper-closed and compact. Indeed, since every probability measure on Polish spaces is regular, for a general W the following inequalities hold:

$$\begin{array}{rcl} \mu(W) & = & \bigsqcup\{\mu(K) \mid K \subseteq W\} \\ & = & \bigsqcup\{\mu(K)\!\uparrow \mid K \subseteq W\} \\ & \leq & \bigsqcup\{\nu(K)\!\uparrow + \epsilon \mid K \subseteq W\} \\ & = & \bigsqcup\{\nu(K)\!\uparrow \mid K \subseteq W\} + \epsilon \\ & = & \nu(W) + \epsilon \end{array}$$

where K ranges over compact subsets of X, and the second inequality follows from the fact that the upper-closure of a compact set is compact (see Theorem 2.1.60). So let us consider a compact upper-closed set W. We want to show that $\mu(W) \leq \nu(W) + \epsilon$, or equivalently that the following inequality holds:

$$\prod_{U \supseteq W} \mu(U) \leq \Big(\prod_{U \supseteq W} \nu(U)\Big) + \epsilon$$

where U ranges over, not necessarily upper-closed, open subsets of X. For an arbitrary $\lambda > 0$, take an open set $U \supseteq W$ such that $\mu(U) < \mu(W) + \lambda$ and $\nu(U) < \nu(W) + \lambda$. Note that if $x \notin U$, then $x\!\downarrow \cap W = \emptyset$, because W is upper-closed. Let us consider the closed, hence compact, set $X \setminus U$. By previous considerations, and since the down-closure of a compact set is compact, we have that W and $(X \setminus U)\downarrow$ are disjoint compact sets, which are respectively upper-closed and down-closed. By the order normality Lemma 2.1.61, we know that there exist an upper-closed open set $U' \supseteq W$ and a down-closed open set $U'' \supseteq (X \setminus U)\downarrow$, such that $U' \cap U'' = \emptyset$. Observe that U' is an upper-closed open set such that $W \subseteq U' \subseteq U$, $\mu(U') < \mu(W) + \lambda$ and $\nu(U') < \nu(W) + \lambda$. Then, the following inequalities hold:

$$\begin{array}{rcl} \mu(W) & \leq & \mu(U') \\ & \leq & \nu(U') + \epsilon \\ & \leq & \nu(W) + \lambda + \epsilon \end{array}$$

and this concludes the proof, since λ is arbitrarily small. □

Lemma A.5.2. *Let A and B be Polish spaces, $f : A \to B$ a surjective map preserving the open sets, i.e., such that $f(U)$ is open for every open set $U \subseteq A$, and $\mu \in \mathcal{M}_1(A)$ a complete Borel probability measure over A. For every $Y \subseteq B$, if $f^{-1}(Y)$ is μ-measurable then Y is $\big(\mathcal{M}_1(f)\big)(\mu)$-measurable.*

Proof. Let us just denote with $f[\mu]$ the probability measure $\big(\mathcal{M}_1(f)\big)(\mu)$. We first want to prove that $f[\mu]^*(Y) \leq \mu^*\big(f^{-1}(Y)\big)$, where ν^*, for a probability measure ν, denotes the corresponding outer-measure. Let us fix an arbitrary $\epsilon > 0$. We need to find an open cover of Y having $f[\mu]$-measure less than $\mu\big(f^{-1}(Y)\big) + \epsilon$. Fix an open set $U \supseteq f^{-1}(Y)$ such that $\mu(U) < \mu\big(f^{-1}(Y)\big) + \epsilon$. Then $f\big(f^{-1}(Y)\big) \subseteq f(U)$, or equivalently since f is surjective, $Y \subseteq f(U)$. By hypothesis the set $f(U)$ is open and $f[\mu]\big(f(U)\big) = \mu(U)$ by definition of $f[\mu]$. Hence the desired inequality open cover is $f(U)$.

We now prove that $f[\mu]_*(Y) \geq \mu_*\big(f^{-1}(Y)\big)$, where ν_*, for a probability measure ν, denotes the corresponding inner-measure. This, by the regularity of all complete probability measures on Polish spaces, will conclude the proof. Let us consider the set $\overline{Y} = B \setminus Y$. Clearly $f^{-1}(\overline{Y})$ is μ-measurable, since $f^{-1}(Y)$ is μ-measurable by hypothesis. With the same method adopted above, we can prove that $f[\mu]^*(\overline{Y}) \leq \mu^*\big(f^{-1}(\overline{Y})\big)$. Since by hypothesis the map f is surjective, we have that $f^{-1}(\overline{Y}) = \overline{f^{-1}(Y)}$, and therefore the desired inequality $f[\mu]_*(Y) \geq \mu_*\big(f^{-1}(Y)\big)$ holds. □

Lemma A.5.3. *Let $\{\mu_n\}_{n\in\mathbb{N}}$ and $\{\nu_n\}_{n\in\mathbb{N}}$ be two $\mathbb{N}$-indexed collections of probability measures over the pospace $2 = \{0, 1\}$, which is endowed with the discrete topology and ordered by $0 \sqsubseteq 1$. Let $\mu = \prod_{n\in\mathbb{N}} \mu_n$ and $\nu = \prod_{n\in\mathbb{N}} \nu_n$ be the corresponding product measures on the product pospace 2^ω, which is ordered pointwise. If $\mu_n(\{1\}) \leq \nu_n(\{1\}) + \frac{\epsilon}{\#(n)}$ holds for every $n \in \mathbb{N}$ for some $\epsilon \geq 0$, where the map $\# : \mathbb{N} \to \mathbb{N}$ is specified as in Definition 2.2.9, then for every upper-closed μ-ν-measurable set $W \subseteq 2^\omega$ the following inequality holds: $\mu(W) \leq \nu(W) + \epsilon$.*

Proof. We first establish the result when the set W is an upper-closed open set. Let us define the $\mathbb{N}$-indexed collection $\{\nu'_n\}_{n\in\mathbb{N}}$ of probability measures over 2^ω defined as follows: $\nu'_n = (\prod_{i<n} \nu_i) \times (\prod_{j\geq n} \mu_i)$. In other words ν'_n is the product measure having ν_i as i-th component, for $i < n$, and μ_j as j-th component for $j \geq n$. Clearly $\nu'_0 = \mu$. Moreover the sequence $\{\nu'_n\}_{n\in\mathbb{N}}$ converges in measure to

ν: for every basic open set U, there exists a $m \in \mathbb{N}$ such that $\nu_n(U) = \nu(U)$, for all $n \geq m$. Thus, by regularity of probability measures on Polish spaces (see Theorem 2.1.66), for every Borel set $A \subseteq 2^\omega$ and $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $|\nu'_n(A) - \nu(A)| < \epsilon$.

We now prove by induction, that the inequality $\mu(W) \leq \nu'_n(W) \cdot (1 + \coprod_{i<n} \frac{\epsilon}{\#(i)})$ holds for every $n \in \mathbb{N}$. The case for $n = 0$ trivially holds, since $\mu = \nu'_0$ as observed before. Let us assume that the property holds for $n \geq 0$. Since the open set W is upper-closed, if it contains a sequence having a 0 in $(n+1)$-th position, than it also contain the same sequence where the 0 digit is replaced by a 1. More formally the set W is of the form $(\{0,1\} \times R) \cup (\{1\} \times T)$, for disjoint open sets $R, T \subseteq \prod_{j \neq n} \{0,1\}$ i.e., sets of ω-sequences lacking the n-th component. Note that $\nu'_n\big(\{0,1\} \times R)\big) = \nu'_{n+1}\big(\{0,1\} \times R)\big)$, because the probability measures ν'_n and ν'_{n+1} differs only on the n-th component. Moreover, by the hypothesis on the probability measures μ_{n+1} and ν_{n+1}, we have that the inequality $\nu'_n\big(\{1\} \times T) \leq \nu'_{n+1}\big(\{1\} \times T) \cdot (1 + \frac{\epsilon}{\#(n)})$ holds. Therefore the following inequalities holds:

$$\begin{array}{rcl}
\nu'_n(W) & = & \nu'_n\big((\{0,1\} \times R) \cup (\{1\} \times T)\big) \\
& = & \nu'_n\big(\{0,1\} \times R\big) + \nu'_n\big(\{1\} \times T\big) \\
& = & \nu'_{n+1}\big(\{0,1\} \times R\big) + \nu'_n\big(\{1\} \times T\big) \\
& \leq & \nu'_{n+1}\big(\{0,1\} \times R\big) + \nu'_{n+1}\big(\{1\} \times T\big) \cdot (1 + \frac{\epsilon}{\#(n)}) \\
& \leq & \Big(\nu'_{n+1}\big(\{0,1\} \times R\big) + \nu'_{n+1}\big(\{1\} \times T\big)\Big) \cdot (1 + \frac{\epsilon}{\#(n)}) \\
& = & \nu'_{n+1}\big(W\big) \cdot (1 + \frac{\epsilon}{\#(n)})
\end{array}$$

By induction hypothesis we know that $\mu(W) \leq \nu'_n(W) \cdot (1 + \coprod_{i<n} \frac{\epsilon}{\#(i)})$. Therefore the following inequalities

$$\begin{array}{rcl}
\mu(W) & \leq & \nu'_n(W) \cdot \big(1 + \coprod_{i<n} \frac{\epsilon}{\#(i)}\big) \\
& \leq & \nu'_{n+1}(W) \cdot (1 + \frac{\epsilon}{\#(n)}) \cdot \big(1 + \coprod_{i<n} \frac{\epsilon}{\#(i)}\big) \\
& = & \nu'_{n+1}(W) \cdot \big(1 + \frac{\epsilon}{\#(n)}) + \coprod_{i<n} \frac{\epsilon}{\#(i)} + (\frac{\epsilon}{\#(n)} \cdot \coprod_{i<n} \frac{\epsilon}{\#(i)})\big) \\
& = & \nu'_{n+1}(W) \cdot \big(1 + (\frac{\epsilon}{\#(n)} \odot \coprod_{i<n} \frac{\epsilon}{\#(i)})\big) \\
& = & \nu'_{n+1}(W) \cdot \big(1 + \displaystyle\coprod_{i<n+1} \frac{\epsilon}{\#(i)}\big)
\end{array}$$

hold as desired. By Lemma 2.2.11 we know that $\lim_{n\to\infty} \big(\displaystyle\coprod_{i<n+1} \frac{\epsilon}{\#(i)}\big) \leq \epsilon$. Hence, since the sequence $\{\nu'_n\}_{n\in\mathbb{N}}$ converges in measure to ν, we have that the desired inequality $\mu(W) \leq \nu(W) + \epsilon$ holds, and this concludes the proof for the case when W is a upper-closed open set. To extend the proof to arbitrary μ-ν-measurable upper-closed set W, we just need to invoke the result of Lemma A.5.1. □

A.6 Symbol List

The following table contains references and brief descriptions of the symbols appearing the most often in the thesis.

Symbol	Page	Description
$\mathcal{D}(X)$	18	Discrete probability distributions over the set X.
lfp, gfp	20	Least and greatest fixed point operators.
$\boldsymbol{\Sigma}$, $\boldsymbol{\Pi}$, $\boldsymbol{\Delta}$	30, 32	Classes of sets in Polish spaces.
μ, μ^*, μ_*	35	Probability (outer, inner) measure.
NULL_μ, MEAS_μ	35	Collections of μ-null and μ-measurable subsets of a space.
$\mathcal{M}_1(X)$, $\mathcal{M}_1(f)$	36	Probability Monad.
$\mathrm{UM}(X)$	39	Collection of universally measurable subsets of X.
CH	42	Continuum Hypothesis.
$\boldsymbol{\Delta}^1_2$-UM	42	Statement about $\boldsymbol{\Delta}^1_2$ sets.
$\mathrm{MA}_\kappa, \mathrm{MA}_{\aleph_1}$	44	Martin's Axiom at the cardinals κ, $\aleph_1$.
$\#: \mathbb{N} \to \mathbb{N}$	50	Fast growing function.
$\mathsf{GS}(X, A)$, $\mathsf{GS}(X, \phi)$	51	Gale–Stewart game.
$\mathsf{B}(X, Y, \phi)$	58	Blackwell game.
$\mathcal{P}_\mathcal{A}$, $\mathcal{P}^{<\omega}_\mathcal{A}$, $\mathcal{P}^t_\mathcal{A}$	71	Sets of paths in the arena $\mathcal{A}$.
$\vec{s}$, $\vec{t}$	71	Symbol ranging over paths.
$\lhd$	71	Prefix relation on paths.
Σ_1, Σ_2	72	Sets of strategies for Player 1 and Player 2.
σ_1, σ_2, τ_1, τ_2	72	Symbols ranging over strategies.
Pr	75	Priority assignment of a $2\frac{1}{2}$-player meta-game.
$\mathcal{L}$	93	Probabilistic Labeled Transision System (PLTS).
PCTL	95	Probabilistic Computation Tree Logic

pLμ, pL$\mu^{\odot}$, pL$\mu^{\odot}_{\oplus}$, pL$\mu^{\{0,1\}}$	108	Fragments of pL$\mu^{\odot}_{\oplus}$
$\mathcal{A}$	124	Game arena.
$\langle (S,E),(S_1,S_2,S_N,B),\pi \rangle$	124	Structure of a tree game arena.
$\pi : S \to \mathcal{D}(S)$	124	Probabilistic transition function.
$\mathcal{BP}_{\mathcal{A}}$	125	Set of branching plays in the arena $\mathcal{A}$.
T	125	Symbol ranging over branching plays.
$\mathcal{MBP}_{\mathcal{A}}$	125	Set of Markov branching plays in the arena $\mathcal{A}$.
M	125	Symbol ranging over Markov branching plays.
$\mathbb{P}_M$	126	Probability measure over branching plays induced by M.
Φ	127	Payoff function of a tree game.
$E(M)$	128	Expected value (payoff) associated with M.
$M^s_{\sigma_1,\sigma_2}$	129	Markov branching play induced by $\langle \sigma_1, \sigma_2 \rangle$.
$\mathbb{P}^s_{\sigma_1,\sigma_2}$	130	Probability measure over branching plays induced by $M^s_{\sigma_1,\sigma_2}$.
$\mathsf{VAL}_{\downarrow}$, $\mathsf{VAL}_{\uparrow}$, $\mathsf{MVAL}_{\downarrow}$, $\mathsf{MVAL}_{\uparrow}$	130	Values of a tree game.
$\mathbb{S}$	148	Antichain of finite paths.
$T[x_i]_{i\in I}$, $T[T_i]_{i\in I}$	148	Empty and filled branching pre-play.
$\langle (S,E),(S_1,S_2,S_N,B_1,B_2),\pi \rangle$	175	Structure of a $2\frac{1}{2}$-player meta-game arena.
$\mathcal{W}_{\mathrm{Pr}}$, Φ_{Pr}	190	Set of completed paths induced by Pr.
Φ_{Pr}, Φ_{Pr}	190	Payoff function induced by Pr.

Bibliography

[1] M. Alvarez-Manilla, A. Jung, and K. Keimel. The probabilistic powerdomain for stably compact spaces. *Theoretical Computer Science*, 328:221–244, 2004.

[2] A. Arnold and D. Niwinski. *Rudiments of μ-calculus*. Studied in Logic. North-Holland, 2001.

[3] A. Arnold and D. Niwinski. Continuous separation of game languages. *Fundamenta Informaticae*, 81:19–28, 2008.

[4] C. Baier and J. P. Katoen. *Principles of Model Checking*. The MIT Press, 2008.

[5] C. Baier and M. Kwiatkowska. Model checking for a probabilistic branching time logic with fairness. *Distributed Computing*, 11:125–155, 1998.

[6] F. Bartels. GSOS for probabilistic transition systems. In *Electronic Notes in Theoretical Computer Science, Volume 65, Issue 1*, 2002.

[7] N. Bertrand, B. Genest, and H. Gimbert. Qualitative determinacy and deciability of stochastic games with signals. In *Proceedings of the 24th Annual IEEE Symposium on Logic In Computer Science*, 2009.

[8] D. P. Bertsekas and S. Shreve. *Stochastic Optimal Control: The Discrete-Time Case*. Athena Scientific; 1st edition, 2007.

[9] A. Bianco and L. de Alfaro. Model checking of probabilistic and nondeterministic systems. In *Foundations of Software Technology and Theoretical Computer Science*, number 1026 in Lecture Notes in Computer Science, pages 499–513. Springer-Verlag, 1995.

[10] D. Blackwell. Infinite G_δ games with imperfect information. *Matematyki Applicationes Mathematicae*, Hugo Steinhaus Jubilee Volume X, 1969.

[11] D. Blackwell. Operator solution of infinite G_δ games of imperfect information. In T. Anderson, K. Athreya, and D. Iglehart, editors, *Papers in Honor of S. Karlin*, pages 83,87. Academic Press, New York, 1989.

[12] D. Blackwell. Games with infinitely many moves and slightly imperfect information. In R. J. Nowakowski, editor, *Games of no chance, Combinatorial games at MSRI, Workshop, July 11–21, 1994 in Berkeley, CA*, volume 29 of *Mathematical Sciences Research Institute Publications*, pages 407–408, 1997.

[13] D. Blackwell and P. Diaconis. A Non-Measurable Tail Set. In *Statistics, Probability and Game Theory: Papers in Honor of David Blackwell*, volume 30 of *Lecture Notes-Monograph Series*, 1996.

[14] D. Blackwell and M. Girshick. *Theory of Games and Statistical Decisions*. Dover Publications Inc., New York,, 1954.

[15] A. Blass. Combinatorial cardinal characteristics of the *continuum*. In *Handbook of Set Theory*. Springer, 2010.

[16] B. Bloom, S. Istrail, and A. R. Meyer. Bisimulation can't be traced. *Journal of the ACM (JACM)*, 42:232–268, 1995.

[17] J. C. Bradfield. Fixpoint alternation: Arithmetic, transition systems, and the binary tree. *Theoretical Informatics and Applications*, 1999.

[18] J. C. Bradfield and C. Stirling. Modal logics and mu-calculi: an introduction. In *Handbook of Process Algebra*. Elsevier, 2001.

[19] T. Brázdil, V. Brozek, A. Kucera, and J. Obdrzálek:. Qualitative reachability in stochastic BPA games. *Information and Computation*, 209, 2011.

[20] V. Brozek. Optimal strategies in infinite-state stochastic reachability games. In *GandALF 2011*, number 54 in EPTCS, 2001.

[21] K. Chatterjee. *Stochastic ω-Regular Games*. PhD thesis, University of California, Berkeley, 2007.

[22] K. Chatterjee, L. Doyen, and T. A. Henzinger. A survey of stochastic games with `lim-sup` and `lim-inf` objective. In *Proceedings of the 36th Internatilonal Colloquium on Automata, Languages and Programming*. Springer-Verlag Berlin, 2009.

[23] E. Clarke, E. A. Emerson, and P. A. Sistla. Automatic verification of finite-state concurrent systems using temporal logic specifications. In *Proc. 10th ACM Symposium on Principles of Programming Languages*, 1983.

[24] E. M. Clarke, O. Grumberg, and D. A. Peled. *Model Checking*. The MIT Press, 2000.

[25] N. B. Cocchiarella and M. A. Freund. *Modal Logic: an introduction to its syntax and semantics*. Oxford University Press, 2008.

[26] M. Dam. CTL* and ECTL* as fragments of the modal μ-calculus. In *Theoretical Computer Science, Volume 126, Issue 1*, pages 77,96. Elsevier Science B.V., 1994.

[27] M. Dam and D. Gurov. Compositional verification of ccs processes. In *PSI 1999*, pages 247–256, London, UK, 2000. Springer-Verlag.

[28] L. de Alfaro, M. Faella, and M. Stoelinga. Linear and branching system metrics. *Logical Methods in Computer Science*, 5(2), 2009.

[29] L. de Alfaro and R. Majumdar. Quantitative solution of omega-regular games. *Journal of Computer and System Sciences, Volume 68 , Issue 2*, pages 374 – 397, 2004.

[30] Y. Deng and R. van Glabbeek. Characterising probabilistic processes logically. In *Logic for programming, artificial intelligence and reasoning*, volume 6397 of *Lecture Notes in Computer Science*, 2010.

[31] J. Desharnais, V. Gupta, R. Jagadeesan, and P. Panangaden. Weak bisimulation is sound and complete for PCTL. In *Proceedings of the 13nd International Conference on Concurrency Theory*, volume 2421 of *Lecture Notes in Computer Science*, pages 355–370, 2002.

[32] E. A. Emerson and C. S. Jutla. Tree automata, mu-calculus and determinacy. In *Proceedings of the 32nd annual Symposium on Foundations of Computer Science*, pages 368–377. IEEE Computer Society, 1991.

[33] E. A. Emerson, C. S. Jutla, and P. A. Sistla. On model checking for fragments of μ-calculus. In *Procs of CAV, LNCS*, volume 697, pages 385–396, 1993.

[34] E. A. Emerson and C.-L. Lei. Modalities for model checking: branching time logic strikes back. *Science of Computer Programming*, 9, 1987.

[35] D. Fischer, E. Grädel, and Ł. Kaiser. Model checking games for the quantitative μ-calculus. *Theory of Computing Systems*, 47(3), 2010.

[36] D. Fischer and Ł. Kaiser. Model checking the quantitative mu-calculus on linear hybrid systems. In *Proceedings of the 38th International Colloquium on Automata, Languages and Programming*, 2011.

[37] C. Freiling. Axioms of symmetry: throwing darts at the real number line. *Journal of Symbolic Logic*, 51(1), 1986.

[38] D. H. Fremlin. *Consequences of Martin's axiom.* Cambridge tracts in mathematics, 1984.

[39] D. H. Fremlin. *Measure Theory*, volume 2. Torres Fremlin, 2001.

[40] D. Gale and F. M. Stewart. Infinite games with perfect information. In *Contributions to the theory of games*, volume 28 of *Annals of Mathematical Studies*, pages 245–266. Princeton University Press, 1953.

[41] M. Gehrke, C. Walker, and E. Walker. A mathematical setting for fuzzy logics. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, pages 223–228, 1997.

[42] M. Gehrke, C. Walker, and E. Walker. Averaging operators on the unit interval. *International Journal of Intelligent Systems*, 14:883–898, 1999.

[43] H. Gimbert and W. Zielonka. When can you play positionally? In *Mathematical Foundations of Computer Science (MFCS)*, 2004.

[44] M. Giry. A categorical approach to probability theory. *Categorical Aspects of Topology and Analisis*, 1982.

[45] R. J. V. Glabbeek, S. A. Smolka, and B. Steffen. Reactive, generative and stratified models of probabilistic processes. *Information and Computation*, 121:130–141, 1990.

[46] E. Grädel. *Finite Model Theory and Its Application.* Texts in Theoretical Computer Science. Springer, 2007.

[47] E. Grädel and I. Walukiewicz:. Positional determinacy of games with infinitely many priorities. In *Logical Methods in Computer Science (LMCS)*, 2006.

[48] E. Grzegorek and C. Ryll-Nardzewski. On universal null sets. In *Proceedings of the American Mathematical Society*, volume 81, pages 613–617, 1981.

[49] P. Hájek. *Metamathematics of Fuzzy Logic.* Trends in Logic. Springer, November 2001.

[50] H. Hansson. *Time and Probability in Formal Design of Distributed Systems*, volume 1 of *Real-Time Safety Critical Systems.* Elsevier, 1994.

[51] H. Hansson and B. Jonsson. A calculus for communicating systems with time and probabilities. *Proceedings 11th IEEE Real-Time Systems Symposium (RTSS)*, 1990.

[52] M. Hennessy and R. Milner. Algebraic laws for nondeterminism and concurrency. *Journal of the ACM (JACM)*, 32:137–161, 1985.

[53] O. M. Herescu and C. Palamiessi. Probabilistic asynchronous π-calculus. In *Proceedings of FOSSACS 2000*, pages 146–160, 2000.

[54] H. Hermanns, A. Parma, R. Segala, B. Wachter, and L. Zhang. Probabilistic logical characterization. *Information and Computation*, 209(2), 2011.

[55] J. Hintikka. *Logic, Language, Games and Information: Kantian Themes in the Philosophy of Logic.* Clarendon Press, Oxford, 1973.

[56] M. Huth and M. Kwiatkowska. Quantitative analysis and model checking. In *Proceedings of the 12th Annual IEEE Symposium on Logic In Computer Science*, page 111, Washington, DC, USA, 1997. IEEE Computer Society.

[57] D. Ikegami. *Games in Set Theory and Logic.* PhD thesis, Institute for Logic, Language and Computation, Universiteit van Amsterdam, 2010.

[58] D. Janin and I. Walukiewicz. On the expressive completeness of the propositional mu-calculus with respect to monadic second order logic. *Lecture Notes in Computer Science*, 1119:263–277, 1996.

[59] T. Jech. *Set Thery.* Springer Monographs in Mathematics. Springer, 2002.

[60] A. S. Kechris. *Classical Descriptive Set Theory.* Graduate Texts in Mathematics. Springer Verlag, 1994.

[61] B. Knaster and A. Tarski. Un théorème sur les fonctions d'ensembles. *Ann. Soc. Polon. Math.*, 6:133–134, 1928.

[62] D. Kozen. Results on the propositional mu-calculus. In *Theoretical Computer Science*, pages 333–354, 1983.

[63] D. Kozen. A finite model theorem for the propositional μ-calculus. *Studia Logica*, 47(3):233–241, 1988.

[64] S. A. Kripke. Semantical analysis of modal logic I: Normal propositional calculi. *Zeit. Math. Logik. Ground.*, 9:67–96, 1963.

[65] M. Kwiatkowska, G. Norman, and D. Parker. Prism 4.0: Verification of probabilistic real-time systems. In Springer, editor, *In Proceedings of the 23rd International Conference on Computer Aided Verification*, 2011.

[66] M. Kwiatkowska, G. Norman, D. Parker, and M. Vigotti. Probabilistic mobile ambients. *Theoretical Computer Science*, pages 12–13, 2009.

[67] K. G. Larsen and A. Skou. Bisimulation through probabilistic testing. *Information and Computation*, 94:1–28, 1991.

[68] B. Löwe. Consequences of Borel determinacy. *Bulletin of Irish Mathematical Society*, 49:43–69, 2002.

[69] P. Maddy. Believing the axioms I-II. *The Journal of Symbolic Logic*, 53(2), 1988.

[70] A. P. Maitra and W. D. Sudderth. Finitely additive stochastic games with borel measurable payoffs. *International Journal of Game Theory*, 27(2):257–267, 1998.

[71] D. A. Martin. Measurable cardinals and analytic games. *Fundamenta Mathematicae*, 66:287–291, 1970.

[72] D. A. Martin. Borel determinacy. *Annals of Mathematics*, 2(102):363–371, 1975.

[73] D. A. Martin. A purely inductive proof of Borel determinacy. In A. Nerode and R. Shore, editors, *Recursion theory*, volume 42 of *Proceedings of Symposia in Pure Mathematics*, pages 303–308. American Mathematical Society, 1985.

[74] D. A. Martin. The determinacy of Blackwell games. In *Journal of Symbolic Logic Volume 63, Issue 4, 1565-1581*, 1998.

[75] D. A. Martin and R. M. Solovay. Internal Cohen extensions. *Ann. Math. Logic*, 2:143–178, 1970.

[76] D. A. Martin and J. R. Steel. A proof of projective determinacy. *Journal of the American Mathematical Society*, 2(1):71–125, 1989.

[77] A. McIver and C. Morgan. *Abstraction, Refinement and Proof for Probabilistic Systems.* Springer, 2005.

[78] A. McIver and C. Morgan. Results on the quantitative μ-calculus qMμ. *ACM Trans. Comput. Logic*, 8(1):3, 2007.

[79] R. Milner. *A Calculus of Communicating Systems.* Springer-Verlag New York, Inc., 1982.

[80] M. Mio. A proof system for reasoning about probabilistic concurrent processes. Proof Systems for Program Logics (PSPL) 2010, a LICS 2010 affiliated workshop.

[81] M. Mio. The equivalence of denotational and game semantics for the probabilistic μ-calculus. In *Proceedings of 7th FICS Workshop, Brno*, 2010.

[82] M. Mio. The equivalence of denotational and game semantics for the probabilistic μ-calculus. *Theoretical Informatics and Applications (to appear)*, 2011.

[83] M. Mio. Probabilistic Modal -Calculus with Independent Product. In *Foundations of Software Science and Computation Structures*, volume 6604 of *Lecture Notes in Computer Science*, pages 290–304. Springer-Verlag Berlin, 2011.

[84] C. Morgan and A. McIver. A probabilistic temporal calculus based on expectations. In *In Lindsay Groves and Steve Reeves, editors, Proc. Formal Methods*. Springer Verlag, 1997.

[85] J. R. Munkres. *Topology (2nd edition).* Prentice Hall, 2000.

[86] J. Mycielski and H. Steinhaus. A mathematical axiom contradicting the axiom of choice. *Bulletin of the Polish Academy of Sciences*, 10:1–3, 1962.

[87] J. Mycielski and S. Swierczkowski. On the Lebesgue measurability and the axiom of determinateness. *Fundamenta Mathematicae*, 54:67–71, 1964.

[88] L. Nachbin. *Order and Topology.* Von Nostrand, Princeton, N.J., 1965.

[89] J. V. Neumann. Zur theorie der gesellschaftsspiele. *Mathematische Annalen*, 100:295–320, 1928.

[90] R. D. Nicola and F. Vaandrager. Action versus state based logics for transition systems. *Lecture Notes in Computer Science*, 469, 1990.

[91] P. Panangaden. *Labelled Markov processes.* Imperial College Press, 2009.

[92] J. B. Paris. $ZF \vdash \Sigma^0_4$-determinateness. *Journal of Symbolic Logic*, 37:661–667, 1972.

[93] A. Parma and R. Segala. Logical characterizations of bisimulations for discrete probabilistic systems. In *Proceedings of the 10th International Conference on Foundations of Software Science and Computational Structures (FOSSACS)*, Lecture Notes in Computer Science, pages 287–201, 2007.

[94] K. R. Parthasarathy. *Probability measures on metric spaces.* Academic Press, 1967.

[95] G. D. Plotkin. A structural approach to operational semantics. *Journal of Logic and Algebraic Programming*, 60-61:17–139, 1981.

[96] G. D. Plotkin. The origins of structural operational semantics. *Journal of Logic and Algebraic Programming*, 60-61:3–15, 2004.

[97] A. Pnueli. The temporal logic of programs. In *Proceedings of the 18th IEEE Symposium on Foundations of Computer Science*, pages 46–67, 1977.

[98] A. Prior. *Time and Modality.* Oxford Clarendon Press, 1957.

[99] M. O. Rabin. Decidability of second-order theories and automata on infinite trees. *Transactions of American Mathematical Society*, 141:1–35, 1969.

[100] L. Santocanale. μ-bicomplete categories and parity games. *Theoretical Informatics and Applications*, 36:195–227, September 2002.

[101] R. Segala. *Modeling and Verification of Randomized Distributed Real-Time Systems.* PhD thesis, Laboratory for Computer Science, Massachusetts Institute of Technology, 1995.

[102] R. Segala and A. Turrini. Comparative analysis of bisimulation relations on alternating and non-alternating probabilistic models. In *Proceedings of the Second International Conference on the Quantitative Evaluation of Systems*, 2005.

[103] W. Sierpinski and E. Szpilrajn-Marczewski. Remarque sur le problème de la mesure. *Fundamenta Mathematicae*, 26, 1936.

[104] A. K. Simpson. Compositionality via cut-elimination: Hennessy-Milner logic for an arbitrary GSOS. In *Logic in Computer Science*, pages 420–430. IEEE Computer Society Press, 1995.

[105] A. K. Simpson. Sequent calculi for process verification: Hennessy-Milner logic for an arbitrary GSOS. *Journal of Logic and Algebraic Programming*, pages 287–322, 2004.

[106] C. Stirling. *Modal and temporal logics for processes.* Springer (Texts in Computer Science), 2001.

[107] T. Tao. *An introduction to measure theory.* Graduate Studies in Mathematics. American Mathematical Society, 2011.

[108] A. Tarski. A lattice-theoretical fixedpoint theorem and its applications. *Pacific Journal of Mathematics*, 5(2):285–309, 1955.

[109] M. Y. Vardi and P. Wolper. Yet another process logic. *Lecture Notes in Computer Science*, 164:501–512, 1984.

[110] M. R. Vervoort. Blackwell games. In L. S. T.S. Ferguson and J. MacQueen, editors, *Papers in Honor of David Blackwell*, pages 369–390. IMS Lecture Notes - Monograph Series 30, 1996.

[111] M. R. Vervoort. *Games, Walks and Grammars: Problem's I've Worked On.* PhD thesis, Institute for Logic, Language and Computation, Universiteit van Amsterdam, 2000.

[112] J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1944.

[113] C. Walker and E. Walker. Inequalities in De Morgan Systems (I-II). In *IEEE Proceedings of the International Conference on Fuzzy Systems*, pages 607–615, 2002.

[114] I. Walukiewicz. Completeness of Kozen's axiomatisation of the propositional μ-calculus. *Information and Computation*, pages 14–24, 1995.

[115] P. Wolfe. The strict determinateness of certain infinite games. *Pacific Journal of Mathematics*, 5:841–847, 1955.

[116] W. Zielonka. Infinite games on finitely coloured graphs with applications to automata on infinite trees. *Theoretical Computer Science*, 200:135–183, 1998.

[117] W. Zielonka. Perfect-information stochastic parity games. In *Foundations of Software Science and Computation Structures*, volume 2987 of *Lecture Notes in Computer Science*, pages 499–513. Springer, 2004.