SEQUENTIAL ESTIMATION PROCEDURES

FOR BINARY RESPONSE

by

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DECLARATION

The following record of research work is submitted as a thesis for the degree of Doctor of Philosophy in the University of Edinburgh, having being submitted for no other degree. Except where acknowledgement is made, the work is original.

Dedicated to Inge Gerstl

A fool...is a man who never tried an experiment in his life.

ERASMUS DARWIN 1792.

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ABSTRACT

This thesis contains the results of a study of various sequential strategies for choosing stimulus levels for experiments with binary response.

Several estimators of the ED50 that have been suggested for use when an Up and Down rule is operated are compared by means of small sample calculation of bias and mean square error and calculation of their asymptotic distributions. All these estimators are asymptotically equivalent to either an estimator suggested Dixon and Mood or one suggested by Wetherill. Simple expressions for asymptotic mean and variance of Dixon and Mood's estimator have been known for some time. Similar expressions for Wetherill's estimator are derived. An alternative to Wetherill's estimator is suggested which has the same asymptotic bias but lower asymptotic variance. This last estimator is compared with the others using similar calculations. An estimator of scale is also suggested and its properties investigated. Properties of all the estimators for larger sample sizes are examined by means of simulation and compared with those of maximum likelihood estimates.

Properties of these estimators when an Up and Down transform rule is operated are compared in the same way as before. A procedure for estimating the scale parameter by using two Up and Down transform rules is examined in detail. An estimator of scale that has been suggested for use when this procedure is operated is criticised and an alternative is suggested.

The use of a two interval forced choice procedure is discussed. This procedure is often used in psychometric studies in conjunction with an Up and Down transform rule. Calculation of asymptotic distributions of estimates and small sample simulations indicate that estimators have large bias and high variability.

Properties of the Robbins-Monro stochastic approximation procedure and various variants upon it are compared by means of simulation. An attempt is made to compare these procedures with the Up and Down rule. Difficulties in making such a comparison are discussed. Modification of these procedures for estimating stimulus levels other than the ED50 are compared. The use of maximum likelihood estimates is discussed. Simulations indicate that maximum likelihood estimation is not successful. Finally Venter's and Anbar's procedures are investigated. Though these stochastic approximation procedures give asymptotically fully efficient estimators of ED50 they have important defects in small samples.

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1. INTRODUCTION

1.1 THE BINARY RESPONSE PROBLEM AND STANDARD MODELS

In many experiments a stimulus or dose is administered to a subject who can respond in only one of two ways. Such responses are termed binary or quantal, and the response types can be labelled positive and negative. A particularly important example of such experiments is in bio-assay in which a dose of a drug is administered to a laboratory animal which either dies or survives. Many other examples of such experiments exist such as experiments for testing detonators of explosive or in psychological experiments where a subject gives a yes or no response to a stimulus.

Suppose that for each subject there exists a tolerance level above which a positive response is given and otherwise a negative response is given. Variation among subjects is often expressed by a probability density function f(x) for the tolerance level. The probability that a subject gives a positive response at dose level x is then

$$\int_{-\infty}^{x} f(y)dy. \qquad 1.1.1$$

The function of x in Formula 1.1.1 is called the response curve and is the cumulative distribution function of the tolerance distribution. In psychological experiments where the same subject is used throughout an experiment a function called the psychometric function is often assumed to exist describing the probability of positive response as a function of stimulus level. I will from now on denote the probability of positive response at level x by $F(x)$.

In bio-assay if units of log. dose are used it is often reasonable to assume that the tolerance distribution is normal, the response curve then taking the form

$$F(x) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x} \exp(-(y-\mu)^2/2\sigma^2)\ dy. \qquad 1.1.2$$

Another response curve that is often assumed in such circumstances is the logistic where now

$$F(x) = (1.0+\exp(-\beta(x-\mu)))^{-1}. \qquad 1.1.3$$

This response curve corresponds to a tolerance distribution with slightly heavier tails than the normal. In practice there is little to choose between the two forms (see Finney (1971), page 49) and many observations are required to distinguish between them. The logistic curve is often preferred for ease of calculation and also to some extent because in non-sequential experiments there are two statistics sufficient for the two parameters. These are by no means the only forms for response curves that have been studied (for example in Davis (1965a) and (1965b) linear, exponential and reversed exponential curves are discussed), but they are certainly the most widely used.

Often the main object of an experiment is to provide an estimate of the dose level for which the probability of positive response takes a particular value p. Commonly p equals 0.5 and the level to be estimated is the median of the underlying tolerance distribution. This level is sometimes called the ED50 or LD50 (the 50 per cent effective or lethal dose) or simply the $L_{1/2}$ level. The level for which the probability of positive response is p is often called the L_p level. Usually the experimenter would also like some estimate of a scale or slope parameter for the response curve.

1.2 ANALYSIS AND DESIGN OF EXPERIMENTS

Finney in 'Probit Analysis' (1971) gives much of the history of probit estimation. This method essentially provides maximum likelihood estimates of parameters under the assumption of a normal tolerance distribution. The data are plotted on normal probability paper and a line is visually fitted to give initial estimates of μ and σ. The likelihood equations are non-linear in the parameter values. The solution of the equations can be approached by Newton-Raphson iterations using as starting values the initial estimates. The log. likelihood is the function to be maximised and iterations are performed in terms of parameters a and β, where $a=-\mu/\sigma$ and $\beta=1/\sigma$. In the matrix of second derivatives one can either use the actual proportions of positive and negative responses or replace them by the expected proportions given the current parameter estimates. Garwood (1941) discusses the merits of these two procedures. With modern computing facilities there is little to choose between them. For the logistic curve these procedures coincide as the second derivatives are functions of only the numbers of observations at each level and the parameter values.

In non-sequential experiments several alternatives to the maximum likelihood estimator of the ED50 have been suggested. Cornfield and Mantel (1950) describe the method of Spearman (1908) and Kärber (1931) for estimating the mean of the tolerance distribution and they also describe a similar approximation to the second moment discussed in Churchman and Epstein (1946). They suggest that these estimators should be used to give starting values for iteration to maximum likelihood estimates. The Spearman-Karber and other alternative estimators are discussed and compared in Finney (1950) and (1952b) for assumed probit and logistic response curves. The conclusion of both of these papers is that of these alternatives to the maximum likelihood estimator only the Spearman-Kärber and moving average method (see Thompson (1947)) should ever be employed as the others have no theoretical or practical advantages. A further alternative estimator is Berkson's minimum transform chi-squared estimator (see Berkson (1944)). To obtain this estimator first order Taylor expansions are made of the likelihood equations and then the resulting linear equations are solved to give estimates of the parameters (this method is described more fully in Section 2.1). Using this method estimates of parameters are obtained, without iteration, which have the same asymptotic properties as maximum likelihood estimates. In Berkson (1956) mean square errors of the minimum transform chi squared estimator and the maximum likelihood estimator are compared in small sample experiments. The results showed that there was a m.s.e. advantage for the minimum transform chi squared estimators. However Cramer (1964) in similar calculations when dose levels are not symmetrically placed about μ found an advantage in a m.s.e. sense for the maximum likelihood estimators. Berkson (1980) contains a discussion of the merits of these procedures. There are further alternatives to maximum likelihood estimation for use in sequential experiments which will be discussed later along with descriptions of the corresponding sequential methods.

Apart from the problem of analysing results there is also the problem of experimental design. Suppose in a design observations are made at k levels: $x_1,\ldots,x_k$. Suppose further F(x) equals $G((x-\mu)/\sigma)$ for some known function G. If $\hat{\mu}$ and $\hat{\sigma}$ are m.l.e.'s of μ and σ then $n^{1/2}(\hat{\mu}-\mu)$ and $n^{1/2}(\hat{\sigma}-\sigma)$, where n is the number of observations, converge in probability to a bivariate normal distribution with variance-covariance matrix

$$\begin{bmatrix} \sigma^2((\sum_i w_i\lambda_i)^{-1}+(\bar{d}-\mu)^2 S) & -\sigma^3(\bar{d}-\mu)S \\ -\sigma^3(\bar{d}-\mu)S & \sigma^4 S \end{bmatrix}, \qquad 1.2.3$$

where λ_i is the proportion of observations made at level x_i, the w_i are weights equal to $\sigma^2(f(x_i))^2/(F(x_i)(1-F(x_i)))$, $\bar{d}=\sum_i w_i\lambda_i x_i/\sum w_i\lambda_i$ and $S^{-1}=\sum_i w_i\lambda_i(x_i-\bar{d})^2$. If σ is known the $\sigma^2(\bar{d}-\mu)^2S$ term in the variance expression for μ is dropped. The lower bound on the asymptotic variance for μ is $\sigma^2\pi/2n$; for the logistic this lower bound is $4/n\beta^2$. Suppose in an experiment equal numbers of observations are made at two points symmetrically placed about μ, by making the distance between these two points arbitrarily low these bounds can be approached (of course in practice μ is unknown and such an experiment cannot be set up). For a logistic response curve the lower bound on the asymptotic variance of the m.l.e. of β is $2.28\beta^2/n$ (see Wetherill (1963)) this bound being attained when equal numbers of observations are made at levels for which the probabilities of positive response are close to 0.085 and 0.915. These lower bounds on asymptotic variance cannot of course be attained simultaneously. For logistic response Wetherill (1963) suggests a design criterion of minimising the product of the asymptotic variance expressions for parameters μ and β. The design that achieves the minimum is such that observations are placed in equal numbers at levels for which the probabilities of positive response are close to 0.176 and 0.824, that is close to

$$\mu \pm 1.55/\beta. \qquad 1.2.4$$

The asymptotic variance expressions for $\hat{\mu}$ and $\hat{\beta}$ then equal $6.90/\beta^2 n$ and $2.89\beta^2/n$. Another criterion is to minimise the determinant of the variance-covariance matrix (this is discussed at length in Abdelbasit (1980)). With this criterion terms involving $(\bar{d}-\mu)$ cancel. If one is particularly interested in estimating the parameter μ the contribution to the asymptotic variance expression for μ from the $\sigma^2(\bar{d}-\mu)^2 S$ term is important. In such circumstances I would be reluctant to use this criterion. Another criterion suggested by Finney (see Finney (1952a), pages 218-222) is to use a design which minimises the length of a Fieller's theorem 95 per cent confidence interval for the ED50. Davis (1965a) and (1965b) uses this criterion in non-sequential experiments with 12 subjects (Professor Finney has told me that he would not seriously consider a proposal for a quantal response experiment with so few subjects). With this criterion the design alters as the number of proposed observations is increased with all observations eventually being required to be made close to the generally unknown value for μ.

Whatever criteria are used all 'optimal' designs depend on unknown parameter values. Sequential methods for choosing levels, which are designed to overcome disastrous effects of bad initial estimates of parameters, have long been sought. Methods fall into three main categories: methods using variants of the Up and Down rule, Stochastic approximation methods and Bayesian methods. In the following chapters I have made a study of the first two of these categories.

1.3 VARIANTS OF THE UP AND DOWN RULE

One of the earliest references to the Up and Down rule is in Dixon and Mood (1948). They remark that they first came across this procedure for choosing testing levels in 1943 at the Explosive Research Laboratory in Bruceton, Pennsylvania. The rule is essentially very simple, the first observation is made at at a level guessed to be close to the ED50 level, the level y_t that is visited after (t-1) observations is related to y_{t-1} by the formula

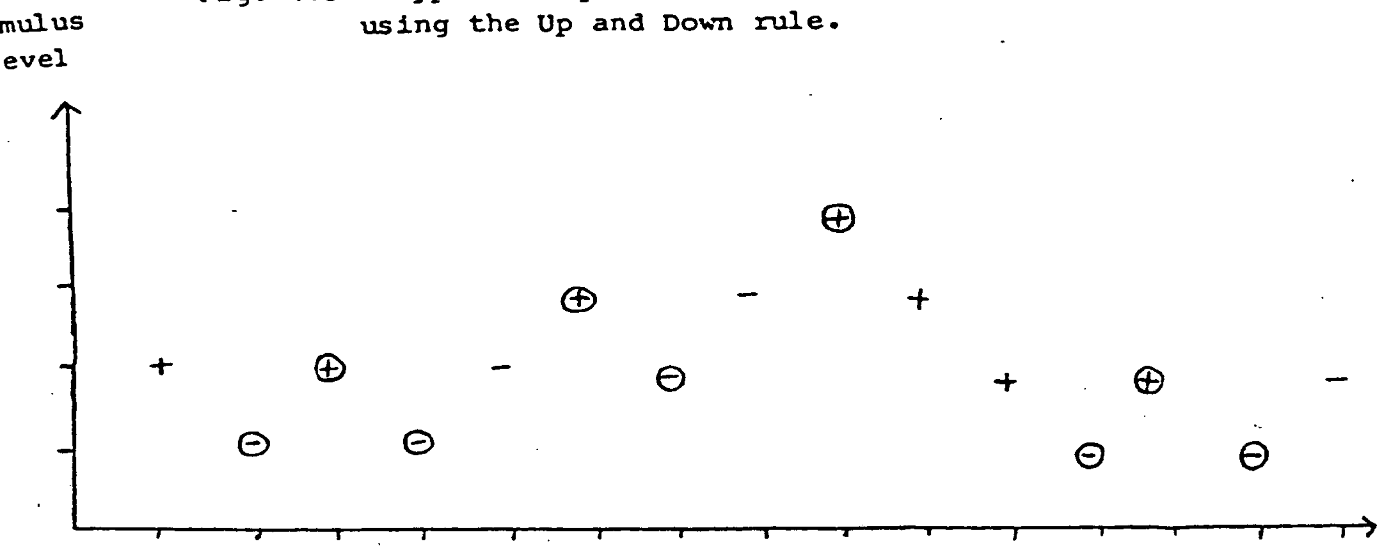
$$y_t = y_{t-1} + \alpha_{t-1} d, \qquad 1.3.1$$

where d is some fixed step size and α_t takes the value 1 if the response at observation t is negative and -1 if it is positive. The strategy was devised to concentrate observations around the ED50 level. The results of such experiments can be analysed by maximum likelihood estimation. The levels visited form a Markov chain for which any level that can be visited equals x_i for some integer i and constant x_0 where

$$x_i = x_0 + (id). \qquad 1.3.2$$

Providing regularity conditions for application of a theorem in Billingsley (1961) hold one can show that the asymptotic properties of the m.l.e.'s are similar to those of m.l.e.'s in non-sequential experiments only each proportion λ_i in the variance-covariance matrix is replaced by the equilibrium probability π_i of being at x_i. In Appendix 5 these regularity conditions are given, many response curves satisfy the conditions and it is easy to verify that they are satisfied by probit or logistic response curves. Alternative estimators of the ED50 that have been proposed are described in Section 2.1. The alternative estimators in greatest use are asymptotically equivalent to the mean of all levels visited (see Dixon and Mood and Brownlee, Hodges and Rosenblatt (1953)) or the mean of the peaks and valleys in the sequence of levels visited (see Wetherill, Chen and Vasudeva (1966)). By a peak I mean an observation for which the response type changes from being previously negative to positive and by a valley I mean an observation where there is change from positive to negative. Fig.

Fig. 1.3.1 Typical sequence of levels visited using the Up and Down rule.



O denotes peak or valley in the sequence

1.3.1 illustrates a typical result of operating the Up and Down rule. Both estimators are in general biased but under certain circumstances the bias is very small. In many studies of the Up and Down rule and its variants maximum likelihood estimation of parameters is not considered in small sample calculations or simulations and much emphasis is placed on estimating the ED50 of the response curve with little or no consideration of the problem of estimating slope or scale parameters. The availability of easily formed alternative estimators of the ED50 has discouraged use of maximum likelihood estimation. In small samples the maximum likelihood equations often have a degenerate solution. One can see this as a failing of the maximum likelihood technique but I consider that a more natural conclusion is that sample sizes considered are often too small.

In Chapters 2,3 and 4 I discuss in detail the small sample and asymptotic properties of estimators of location, scale and slope for several applications of the Up and Down rule.

1.4 STOCHASTIC APPROXIMATION PROCEDURES

Robbins and Monro (1951) described a method for sequentially choosing levels at which to test in quantal response experiments. The rule for choosing levels is similar to the Up and Down rule but the step size is not held constant. The level y_t visited after (t-1) trials (with possibly more than one observation per trial) is related to y_{t-1} by the formula

$$y_t = y_{t-1} + a_{t-1}(p-P(y_{t-1})), \qquad 1.4.1$$

where $p \in (0,1)$, $P(y_t)$ is the proportion of positive responses observed at trial t and $\{a_t\}$ is a sequence of constants. Robbins and Monro suggested that a_t be set equal to c/t and they proved that

$$\lim_{t\to\infty}(E((y_t-L_p)^2)) = 0.0. \qquad 1.4.2$$

The conditions they place on the response curve F for 1.4.2 to hold are that F is monotonic with derivative greater than zero at L_p. Hodges and Lehmann (1956) modified a result in Chung (1954) to find the asymptotic distribution of y_t. If g_p is the slope of the response curve at L_p then, providing $c>1/2g_p$, y_t has asymptotic normality with mean L_p and variance

$$c^2p(1-p)/((2g_pc-1)mt), \qquad 1.4.3$$

where m is the number of observations per trial. The asymptotic variance expression in 1.4.3 is minimised when $c=1/g_p$. For the logistic response curve

$$g_p = \beta(1-p)p. \qquad 1.4.4$$

So for the logistic the expression in 1.4.3 is minimised when p=0.5 and $c=4/\beta$. The asymptotic variance of y_t is then $4/\beta^2mt$ which is the lower bound in non-sequential experiments on the asymptotic variance of the m.l.e. of μ.

Several modifications of the Robbins-Monro procedure have been suggested. Kesten (1958) suggests a procedure designed to accelerate convergence to the ED50 (i.e. it is used when $p=0.5$). Again the a_t are set equal to c/t but here step sizes are only changed when the two previous steps have been in opposite directions (the first and second steps in the process being c and $c/2$). Davis (1965a), (1965b) and (1971) report results of simulations using this procedure. In Davis's work a non rigorous development in Cochran and Davis (1963) is cited to illustrate why the expression in Formula 1.4.3 is the correct variance expression. He uses this argument to conjecture that the last level visited in Kesten's procedure is asymptotically normal, providing $c>1/4g_{1/2}$, with mean equal to the ED50 and variance

$$c^2/((4g_{1/2}c-1)mn), \qquad 1.4.5$$

where n is the number of steps taken. I have not been able to find a rigorous proof of this result. The value of c minimising the expression in Formula 1.4.5 is $1/2g_{1/2}$; that is half the value with the original procedure. A modification very similar to Kesten's that I have suggested is to decrease the step size at each change in response rather than to wait until the next step.

In Chapter 5 I make some comparisons between these and other stochastic approximation procedures. I have also tried to make comparisons between these procedures and procedures using the Up and Down rule. Davis makes similar comparisons in his work for experiments using 12 observations. I found that it is difficult to determine which step sizes with the Up and Down rule and values for c with the Robbins-Monro procedure are comparable as such comparability depends very much on the number of observations made in an experiment.

2. USE OF THE UP AND DOWN RULE TO ESTIMATE THE ED50

2.1 POSSIBLE ESTIMATORS OF THE ED50 VALUE

One of the principal parameters of interest in many problems in bio-assay is the ED50 (i.e. the stimulus level at which the probabilities of positive and negative response are both 0.5). This is of course the median of any assumed underlying tolerance distribution and will correspond to the mean if such a distribution is symmetric.

For use with the Up and Down rule, various alternatives to maximum likelihood estimation have been proposed. Dixon and Mood (1948) suggest an estimator which I will call E_{DM}. It is derived from taking a linear approximation to one of the likelihood equations and is asymptotically equivalent to the mean of levels visited (see Appendix 2). If at the end of the experiment positive responses have been less frequent than negative then

$$E_{DM} = \sum_i n_i x_i / \sum_i n_i - (d/2), \qquad 2.1.1$$

where n_i is the number of positive responses at level x_i and d is the distance between adjacent levels (index 'i' denotes the position of the level along the stimulus axis). Otherwise

$$E_{DM} = \sum_i m_i x_i / \sum_i m_i + (d/2), \qquad 2.1.2$$

where m_i is the number of negative responses at x_i. If the numbers of positive and negative responses are equal the expressions in Formulae 2.1.1 and 2.1.2 both equal the mean of the levels visited.

Brownlee, Hodges and Rosenblatt (1953) consider using simply the mean of levels visited as an estimate of ED50 with the modification that the starting level is not included in the mean (they argue that this level is completely determined by the experimenter), instead they include the level that would have been visited if the experiment had continued for one more step. I will call this estimator E_B. Suppose after n observations the sequence of levels visited is $y_1, \ldots, y_n$ and that the level that would have been visited after one more step is y_{n+1}, then

$$E_B = \sum_{T=2}^{n+1} y_T / n. \qquad 2.1.3$$

Brownlee et al give recursive formulae which allow calculation of the bias and m.s.e. of E_B in relatively large 'small samples' (see Tsutakawa (1967a) where samples of size 30 are considered). Instead of investigating every possible outcome, the number of which rises exponentially with the number of observations, one only has to evaluate a number of terms rising quadratically. In many papers investigating the Up and Down rule the estimator E_B is used rather than E_{DM} (see Choi (1971), Cochran and Davis (1964), Davis (1965a), (1965b) and (1971), Hsi(1969), Wetherill (1963), Wetherill, Chen and Vasudeva (1966) and Tsutakawa (1967a)). It is difficult to see why the properties of E_{DM} have been so seldom studied as in simulations it is only slightly more troublesome to calculate than E_B. One disadvantage that E_{DM} has over E_B is that recursive formulae such as those for E_B cannot be used to obtain exact values of m.s.e. and bias. So for example calculations made for E_B in Tsutakawa and Hsi would be much more difficult for E_{DM}. The shortcomings of E_B for extreme starts and small step sizes were realised by Brownlee et al (i.e. that bias of the estimator becomes large), they suggested a further modification of ignoring the first run of constant response type by forming a 'delayed' estimator. I will call this estimator E_{BD}.

$$E_{BD} = \sum_{T=T'}^{n+1} y_T / (n-T'+2), \qquad 2.1.4$$

where at the T'th. response the response type first changes (if $T'=2$ then of course $E_{BD} = E_B$). Davis considered E_{BD} in detail. From calculations in the next section it appears that E_{DM} and E_{BD} have similar small sample behaviour. As with E_{DM} the bias and m.s.e. of E_{BD} cannot be calculated using recursive formulae. I believe that E_{DM} merits further consideration, first because it is derived directly from Dixon and Mood's approximate formulae and second because it behaves reasonably well for small step size and extreme starting level without any special modification.

In the paper of Wetherill et al an estimator which they term $\bar{w}$ is suggested. For convenience I will call this estimator E_{WE}. In the sequence of positive and negative responses, whenever there is a change in response type an intuitive estimate of the ED50 is the level midway between the consecutive levels at which this change takes place. E_{WE} is simply the mean of all such estimates arising from a staircase and so

$$E_{WE} = 0.5\sum_{T\in\tilde{T}}(y_T+y_{T-1})/m, \qquad 2.1.5$$

where $\tilde{T}$ is the set of T such that responses at y_T and y_{T-1} have opposite sign and m is the number of times response type changes. Choi considers a further estimator that he terms $\tilde{w}$ which is the mean of peaks and valleys in a sequence (where a level at which response changes from negative to positive is a peak, if the change is from positive to negative it is a valley). I will call this estimator E_{PV}.

$$E_{PV} = \sum_{T\in\tilde{T}} y_T/m. \qquad 2.1.6$$

It is easy to see that this estimator is asymptotically equivalent to E_{WE}. E_{WE} equals E_{PV} if the number of changes in response type is even and otherwise

$$E_{PV} = E_{WE} \pm (0.5d/m). \qquad 2.1.7$$

The sign is positive if there are more peaks than valleys, negative otherwise.

All these estimators are in general asymptotically biased but if the underlying tolerance distribution is symmetric and stimulus levels are symmetrically placed about the ED50 then all the biases are zero. In Section 2.3 some values of biases of these estimators are given (see Tables 2.3.1 and 2.3.2); for small step sizes the biases are very small.

One alternative to maximum likelihood that has been widely used in non-sequential experiments is minimum logit chi squared estimation (here one assumes a logistic response curve). Estimators from this procedure have the same asymptotic distribution as the maximum likelihood estimators nd are obtained explicitly, values of m.l.e.'s must be approached by iteration. The m.l.e.'s for logistic response curve are the solutions of the equations

$$\sum_i (n_i+m_i)(\tilde{F}_i - F_i) = 0.0, \qquad 2.1.8$$

$$\sum_i (n_i+m_i)x_i(\tilde{F}_i - F_i) = 0.0, \qquad 2.1.9$$

where $F_i = (1.0+\exp(-(a+\beta x_i))^{-1}$ and $\tilde{F}_i$ is $n_i/(n_i+m_i)$. For minimum logit chi squared estimation $(\tilde{F}_i - F_i)$ is approximated using a first order Taylor expansion by

$$\tilde{F}_i(1-\tilde{F}_i)(\log(\tilde{F}_i/(1-\tilde{F}_i))-a-\beta x_i). \qquad 2.1.10$$

With this approximation the equations become linear in parameters a and β. Berkson has suggested that in some circumstances these estimates are preferable to m.l.e.'s. There is a problem as to how to treat levels at which responses are all of the same type, $\log(\tilde{F}_i/(1-\tilde{F}_i))$ cannot then be evaluated. If such levels are ignored, estimates do not exist when there is only one level of mixed response. Berkson (1957) suggests use of what he calls a '1/2n' rule but this is not really appropriate for use in Up and

Down experiments. The procedure to adopt in such circumstances will be discussed in the next section. The ED50 equals $-a/\beta$ so providing the min. logit chi estimates of a and β exist and the estimate of β is not zero an estimate of the ED50 can be formed.

I suggest two further estimators, which I will call E_{WE}^{*} and E_{PV}^{*}, as alternatives to E_{WE} and E_{PV}. E_{WE} and E_{PV} can be written in the following forms

$$E_{WE} = \left(\sum_i((x_i-(d/2))P_i \quad +\sum_i(x_i+(d/2))V_i)\right)/\sum_i(P_i+V_i), \qquad 2.1.11$$

$$E_{PV} = \sum_i x_i(P_i+V_i)\ /\sum_i(P_i+V_i), \qquad 2.1.12$$

where P_i is the proportion of observations for which peaks are recorded at x_i and V_i is the proportion for which valleys are recorded at x_i. It is easy to see that the equilibrium probability of being at level x_i and observing a positive response after moving up from the level below (i.e. of observing a peak) is $\pi_{i-1}(1-F_{i-1})F_i$, where π_i is the equilibrium probability of being at level x_i and F_i is the probability of positive response at this level. P_i is an estimate of this quantity which is asymptotically unbiased as the number of observations increases. Another estimate of this quantity which is also asymptotically unbiased is $n_i m_{i-1}/((n_i+m_i)n)$ (i.e. π_{i-1} is estimated by $(n_{i-1}+m_{i-1})/n$, F_i by $n_i/(n_i+m_i)$ and $(1-F_{i-1})$ by $m_{i-1}/(n_{i-1}+m_{i-1})$). V_i provides an estimate of $\pi_{i+1}(1-F_i)F_{i+1}$; an alternative estimate of this quantity is $n_{i+1}m_i/((n_i+m_i)n)$. The estimators corresponding to E_{WE} and E_{PV} if these alternatives to P_i and V_i are used will be denoted by E_{WE}^{*} and E_{PV}^{*} where

$$E_{WE}^{*} = \sum_i ((x_i-(d/2))\tilde{P}_i + (x_i+(d/2))\tilde{V}_i)/\sum_i(\tilde{P}_i+\tilde{V}_i), \qquad 2.1.13$$

$$E_{PV}^{*} = \sum_i x_i(\tilde{P}_i+\tilde{V}_i)/\sum_i(\tilde{P}_i+\tilde{V}_i), \qquad 2.1.14$$

$\tilde{P}_i$ and $\tilde{V}_i$ equal $n_i m_{i-1}/(n_i+m_i)$ and $n_{i+1} m_i/(n_i+m_i)$ respectively (the factor n cancels in 2.1.13 and 2.1.14. $\tilde{P}_i$ and $\tilde{V}_i$ are set equal to zero if (n_i+m_i) is zero). These estimators have the same asymptotic expectation as E_{WE} and E_{PV} but lower asymptotic variance (see the argument in Theorem 3 of Appendix 4).

2.2 COMPARISON OF ESTIMATORS OF THE ED50

In the previous section many estimators of the ED50 for use in Up and Down experiments are described. I have made a comparison of these estimators for small samples. The two most commonly used forms for the response curve are the logistic and probit. In these calculations the logistic form is assumed as it is easier to program and in practice there is little to choose between the two forms.

In practice the value of the slope parameter β will not be known. Often in problems in bio-assay there is a rough prior estimate of β from experiments on a standard preparation. The problem of estimating β is discussed in the next chapter. Here it is assumed that the prior estimate of β differs from the true value by no more than a factor of two.

Values of bias and m.s.e. have been calculated for several estimators. Experiments consisted of 12 observations (as in Davis (1965a), (1965b) and (1971)) with the Up and Down rule being operated. Dixon and Mood suggest a step spacing equal to the standard deviation of the underlying tolerance distribution. For the logistic response curve this standard deviation is $\pi/(3.0^{1/2}\beta)$. In the experiments step sizes were set equal to 0.5(0.5)2.0. The slope parameter, β, is set equal to $\pi/3.0^{1/2}$ so that the standard deviation of the tolerance distribution is 1.0. Starting levels were set equal to 0.00(0.25)4.00 relative to μ (as in Tsutakawa (1967a)). E_B, E_{BD} and E_{DM} all have the same asymptotic normal distribution as the mean of the levels visited (I will call the mean level estimator E_M; details of its asymptotic distribution are given in the next section). E_{PV} has the same asymptotic normal distribution as E_{WE} (see Appendix 3); E_{PV}^* has the same asymptotic normal distribution as E_{WE}^* (see Appendix 4). In the 12 step experiments the estimators which are calculated are E_M, E_B, E_{BD}, E_{DM}, E_{WE}, E_{PV}, E_{WE}^*, E_{PV}^* and the minimum logit chi squared estimator of μ. As in Davis' work outcomes of probability less than 10.0^{-8} are automatically excluded (the number of possible outcomes is only 4096 so this seems reasonable). For some of the outcomes some of the estimators as defined do not exist. E_M and E_B always exist; E_{BD}, E_{DM}, E_{WE}, E_{PV}, E_{WE}^* and E_{PV}^* all exist if and only if there is more than one type of response.

Tables 2.2.1 to 2.2.4 give values of m.s.e.'s of E_M, E_B, E_{BD}, E_{DM}, E_{WE}, E_{PV}, E_{WE}^*, and E_{PV}^*. These tables also contain asymptotic theory predicted values for m.s.e.'s. Tables 2.2.5 to 2.2.8 give all the analogous values of biases. The relationships between these estimators is also illustrated graphically. Figs. 2.2.1 to 2.2.4 illustrate values of m.s.e. of E_M, E_B, E_{BD} and E_{DM}. Figs. 2.2.5 to 2.2.8 illustrate values of m.s.e. of E_{WE} and E_{WE}^* (values for E_{PV} and E_{PV}^* are not illustrated as they are often very close to values for E_{WE} and E_{WE}^*). The probability of outcomes of individual probability less than 10.0^{-8} is always less than 2.0×10.0^{-6}. The probability that all outcomes are of the same type (i.e. that only E_M and E_B exist) is always less than 10.0^{-3} and

Table 2.2.1 100×m.s.e. of estimators in 12 step experiments for step size 0.5 and $\beta=\pi/3.0^{1/2}$

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E^*_{WE}	E^*_{PV}	A^*_{WE}
0.00	8.74	9.77	9.69	9.64	12.55	10.43	11.23	13.30	10.11	10.43	12.59
0.25	8.99	9.85	9.91	10.17	12.55	10.80	11.54	13.30	10.28	10.58	12.59
0.50	9.77	10.10	10.53	11.49	12.55	11.74	12.37	13.30	10.84	11.05	12.59
0.75	11.21	10.58	11.48	13.08	12.55	13.07	13.57	13.30	12.00	12.02	12.59
1.00	13.47	11.42	12.71	14.65	12.55	14.82	15.16	13.30	13.75	13.62	12.59
1.25	16.80	12.80	14.16	16.28	12.55	17.00	17.15	13.30	15.68	15.48	12.59
1.50	21.56	14.92	15.74	18.06	12.55	19.36	19.37	13.30	17.40	17.12	12.59
1.75	28.14	18.08	17.40	19.87	12.55	21.74	21.64	13.30	19.10	18.66	12.59
2.00	36.99	22.65	19.12	21.70	12.55	24.27	24.10	13.30	21.21	20.64	12.59
2.25	48.66	29.05	20.93	23.75	12.55	27.16	26.90	13.30	23.62	23.06	12.59
2.50	63.77	37.74	22.80	26.14	12.55	30.22	29.95	13.30	25.91	25.36	12.59
2.75	82.99	49.25	24.73	28.73	12.55	33.32	33.12	13.30	28.13	27.45	12.59
3.00	106.99	64.24	26.79	31.56	12.55	36.76	36.68	13.30	30.94	30.09	12.59
3.25	136.56	83.36	29.04	34.99	12.55	40.89	40.97	13.30	34.65	33.79	12.59
3.50	172.56	107.27	31.49	39.24	12.55	45.53	45.88	13.30	38.86	38.08	12.59
3.75	215.86	136.77	34.13	44.21	12.55	50.47	51.25	13.30	43.34	42.47	12.59
4.00	267.27	172.72	36.99	50.07	12.55	56.14	57.46	13.30	48.80	47.68	12.59

Table 2.2.2 100×m.s.e. of estimators in 12 step experiments for step size 1.0 and $\beta=\pi/3.0^{1/2}$

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E^*_{WE}	E^*_{PV}	A^*_{WE}
0.00	12.85	13.90	13.55	13.37	15.20	14.49	16.29	16.32	14.22	14.58	15.76
0.25	12.86	13.92	13.65	13.57	15.19	14.55	16.22	15.91	14.18	14.63	15.26
0.50	12.97	13.96	13.93	14.18	15.19	14.97	16.33	15.49	14.23	14.85	14.76
0.75	13.28	13.99	14.34	15.12	15.19	15.90	16.92	15.91	14.48	15.16	15.26
1.00	13.90	13.98	14.81	16.21	15.20	16.87	17.65	16.32	14.78	15.30	15.76
1.25	14.93	13.92	15.28	17.21	15.19	17.40	18.10	15.91	15.13	15.28	15.26
1.50	16.41	13.92	15.75	17.92	15.19	17.78	18.50	15.49	15.84	15.55	14.76
1.75	18.38	14.11	16.26	18.30	15.19	18.49	19.25	15.91	17.03	16.36	15.26
2.00	20.90	14.61	16.84	18.45	15.20	19.46	20.16	16.32	18.26	17.43	15.76
2.25	24.15	15.51	17.49	18.65	15.19	20.45	20.95	15.91	19.16	18.46	15.26
2.50	28.43	16.88	18.19	19.15	15.19	21.60	21.84	15.49	19.83	19.48	14.76
2.75	34.06	18.76	18.89	20.06	15.19	22.99	23.02	15.91	20.40	20.37	15.26
3.00	41.28	21.19	19.51	21.23	15.20	24.13	24.13	16.32	20.68	20.69	15.76
3.25	50.23	24.37	20.03	22.34	15.19	24.61	24.79	15.91	20.74	20.46	15.26
3.50	61.03	28.60	20.49	23.10	15.19	24.79	25.26	15.49	21.15	20.38	14.76
3.75	73.74	34.19	20.99	23.45	15.19	25.31	26.03	15.91	22.21	21.00	15.26
4.00	88.54	41.37	21.60	23.56	15.20	26.26	27.06	16.32	23.55	22.16	15.76

Note: A_{DM}, A_{WE} and A^*_{WE} denote columns for asymptotic predicted m.s.e.'s of E_{DM}, E_{WE} and E^*_{WE} respectively.

Table 2.2.3 100×m.s.e. of estimators in 12 step experiments for step size 1.5 and $\beta = \pi/3.0^{1/2}$.

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E_{WE}^*	E_{PV}^*	A_{WE}^*
0.00	16.70	17.56	17.17	16.96	18.34	19.59	22.72	22.45	19.03	19.25	22.20
0.25	16.52	17.45	17.09	16.88	18.22	18.73	21.70	20.59	18.24	18.53	20.18
0.50	16.17	17.25	16.98	16.81	17.98	17.12	19.63	17.04	16.72	17.23	16.36
0.75	15.98	17.19	17.11	17.08	17.86	16.65	18.51	15.36	16.07	16.90	14.57
1.00	16.21	17.35	17.60	17.87	17.98	18.27	19.48	17.04	16.99	18.11	16.36
1.25	16.80	17.57	18.29	19.02	18.22	21.01	21.77	20.59	18.62	19.84	20.18
1.50	17.56	17.57	18.85	20.13	18.34	22.79	23.39	22.45	19.51	20.62	22.20
1.75	18.48	17.24	19.08	20.94	18.22	22.44	23.06	20.59	19.06	19.86	20.18
2.00	19.84	16.75	19.06	21.43	17.98	20.83	21.59	17.04	18.10	18.39	16.36
2.25	21.90	16.46	19.06	21.77	17.86	19.96	20.93	15.36	18.05	17.68	14.57
2.50	24.63	16.60	19.33	22.05	17.98	20.98	22.20	17.04	19.61	18.56	16.36
2.75	27.68	17.11	19.85	22.19	18.22	23.16	24.61	20.59	21.96	20.42	20.18
3.00	30.82	17.80	20.37	22.03	18.34	24.69	26.27	22.45	23.46	21.75	22.20
3.25	34.29	18.67	20.71	21.60	18.22	24.52	26.03	20.59	23.23	21.68	20.18
3.50	38.78	19.98	20.95	21.22	17.98	23.46	24.65	17.04	22.12	21.04	16.36
3.75	44.91	22.00	21.37	21.34	17.86	23.37	24.07	15.36	21.76	21.37	14.57
4.00	52.82	24.70	22.11	22.23	17.98	25.24	25.49	17.04	22.89	23.25	16.36

Table 2.2.4 100×m.s.e. of estimators in 12 step experiments for step size 2.0 and $\beta = \pi/3.0^{1/2}$.

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E_{WE}^*	E_{PV}^*	A_{WE}^*
0.00	21.85	22.45	22.14	21.98	22.93	28.05	32.98	33.17	26.90	27.00	33.09
0.25	21.32	22.00	21.69	21.49	22.49	26.14	30.95	29.77	25.18	25.31	29.51
0.50	19.97	20.89	20.57	20.26	21.36	21.63	25.97	22.28	21.07	21.32	21.73
0.75	18.53	19.74	19.44	19.06	20.17	17.27	20.71	15.59	17.08	17.56	14.91
1.00	17.88	19.29	19.11	18.76	19.66	15.77	18.00	13.00	15.62	16.44	12.31
1.25	18.47	19.86	19.93	19.81	20.17	18.26	19.31	15.59	17.54	18.76	14.91
1.50	19.98	21.08	21.57	21.84	21.36	23.73	24.01	22.28	21.66	23.23	21.73
1.75	21.53	22.15	23.26	23.93	22.49	29.41	29.51	29.77	25.57	27.37	29.51
2.00	22.45	22.45	24.31	25.29	22.93	32.25	32.59	33.17	27.13	29.09	33.09
2.25	22.75	21.75	24.36	25.62	22.49	30.80	31.46	29.77	25.71	27.70	29.51
2.50	23.15	20.30	23.48	25.17	21.36	26.21	27.08	22.28	22.29	24.02	21.73
2.75	24.67	18.79	22.18	24.52	20.17	21.44	22.43	15.59	19.03	20.01	14.91
3.00	27.96	18.09	21.32	24.37	19.66	19.34	20.48	13.00	18.05	17.89	12.31
3.25	32.76	18.64	21.54	25.07	20.17	21.14	22.54	15.59	20.32	18.93	14.91
3.50	37.92	20.10	22.77	26.31	21.36	25.93	27.71	22.28	24.98	22.64	21.73
3.75	42.17	21.62	24.31	27.34	22.49	31.08	33.36	29.77	29.71	26.91	29.51
4.00	45.07	22.52	25.33	27.47	22.93	33.66	36.51	33.17	31.97	29.20	33.09

Table 2.2.5 100×bias of estimators in 12 step experiments for step size 0.5 and $\beta = \pi/3.0^{1/2}$.

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.25	5.24	3.18	3.20	4.66	0.00	4.04	3.59	3.73	3.44	0.00
0.50	10.67	6.52	6.25	8.63	0.00	7.81	6.96	7.37	6.75	0.00
0.75	16.44	10.20	9.08	11.72	0.00	11.30	10.10	10.85	9.96	0.00
1.00	22.67	14.37	11.65	14.25	0.00	14.60	13.07	13.92	12.89	0.00
1.25	29.51	19.16	13.92	16.50	0.00	17.56	15.75	16.39	15.25	0.00
1.50	37.08	24.65	15.89	18.47	0.00	20.06	18.03	18.38	17.09	0.00
1.75	45.45	30.93	17.62	20.19	0.00	22.25	20.02	20.30	18.88	0.00
2.00	54.65	38.09	19.20	21.87	0.00	24.38	21.94	22.26	20.82	0.00
2.25	64.75	46.15	20.64	23.69	0.00	26.42	23.75	24.06	22.61	0.00
2.50	75.81	55.14	21.97	25.51	0.00	28.20	25.33	25.62	24.07	0.00
2.75	87.82	65.09	23.24	27.29	0.00	29.90	26.80	27.28	25.61	0.00
3.00	100.78	76.04	24.53	29.26	0.00	31.78	28.39	29.34	27.63	0.00
3.25	114.72	87.98	25.82	31.58	0.00	33.78	30.05	31.64	29.93	0.00
3.50	129.65	100.89	27.10	34.10	0.00	35.70	31.59	33.89	32.12	0.00
3.75	145.54	114.79	28.42	36.75	0.00	37.71	33.16	36.29	34.39	0.00
4.00	162.39	129.69	29.89	39.82	0.00	40.20	35.07	39.39	37.40	0.00

Table 2.2.6 100×bias of estimators in 12 step experiments for step size 1.0 and $\beta = \pi/3.0^{1/2}$.

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.25	2.69	0.99	1.27	2.78	-0.01	2.05	1.00	1.71	1.51	-0.23
0.50	5.62	2.01	2.44	5.32	0.00	4.05	2.18	3.39	2.79	0.00
0.75	8.92	3.06	3.39	7.28	0.01	5.63	3.25	4.85	3.59	0.23
1.00	12.59	4.26	4.09	8.43	0.00	6.55	3.96	6.02	4.00	0.00
1.25	16.60	5.79	4.66	8.79	-0.01	7.21	4.53	7.14	4.51	-0.23
1.50	20.90	7.85	5.23	8.66	0.00	8.10	5.27	8.39	5.43	0.00
1.75	25.48	10.51	5.87	8.50	0.01	9.26	6.08	9.51	6.57	0.23
2.00	30.37	13.70	6.52	8.68	0.00	10.40	6.72	10.28	7.61	0.00
2.25	35.74	17.37	7.09	9.40	-0.01	11.42	7.25	10.80	8.44	-0.23
2.50	41.72	21.44	7.50	10.51	0.00	12.29	7.80	11.19	8.92	0.00
2.75	48.37	25.85	7.68	11.55	0.01	12.69	8.17	11.32	8.78	0.23
3.00	55.62	30.62	7.66	12.07	0.00	12.52	8.17	11.19	8.15	0.00
3.25	63.38	35.91	7.60	11.94	-0.01	12.19	8.07	11.22	7.68	-0.23
3.50	71.55	41.84	7.69	11.44	0.00	12.30	8.21	11.69	7.91	0.00
3.75	80.09	48.45	7.99	11.03	0.01	12.93	8.58	12.43	8.77	0.23
4.00	89.01	55.68	8.42	11.15	0.00	13.79	8.93	13.15	9.88	0.00

Note: A_{DM} and A_{WE} denote columns for asymptotic predicted biases of E_{DM} and E_{WE} respectively.

Table 2.2.7 100×bias of estimators in 12 step experiments for step size 1.5 and $\beta=\pi/3.0^{1/2}$.

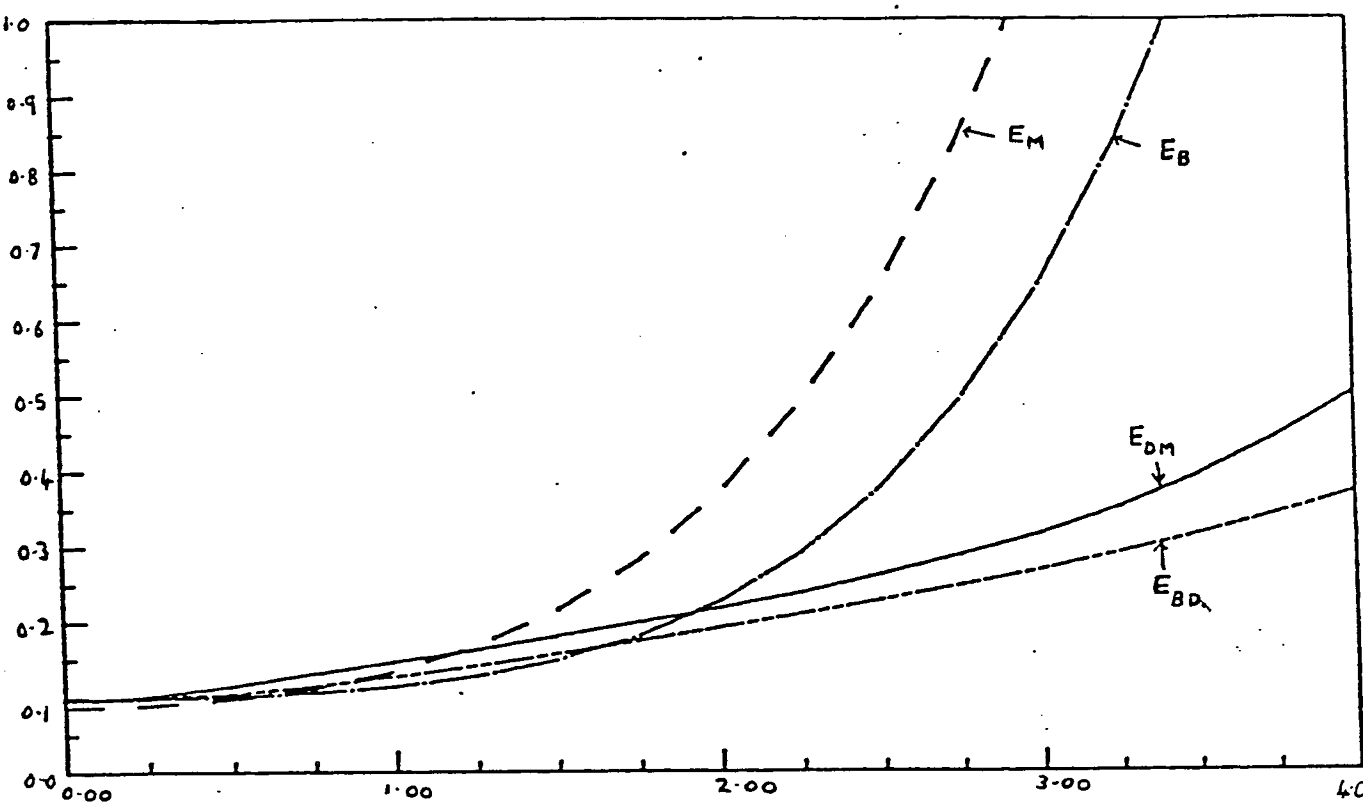
Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.25	1.11	0.01	0.27	1.33	-0.42	-0.14	-1.90	-0.19	-0.22	-2.40
0.50	2.86	0.42	0.91	3.12	-0.42	1.13	-2.10	0.89	0.70	-2.37
0.75	5.42	1.21	1.85	5.33	0.00	3.75	-0.48	3.18	2.56	0.00
1.00	8.52	1.98	2.60	7.39	0.42	6.24	1.50	5.36	3.96	2.37
1.25	11.74	2.38	2.70	8.61	0.42	7.03	2.25	6.13	3.69	2.40
1.50	14.99	2.49	2.11	8.74	0.00	5.94	1.49	5.50	1.94	0.00
1.75	18.50	2.86	1.38	8.08	-0.42	4.44	0.49	4.71	0.20	-2.40
2.00	22.55	4.08	1.11	7.23	-0.42	4.25	0.71	5.04	-0.12	-2.37
2.25	27.05	6.27	1.50	6.55	0.00	5.70	2.26	6.58	1.15	0.00
2.50	31.66	9.10	2.24	5.97	0.42	7.60	3.83	8.23	2.87	2.37
2.75	36.04	12.14	2.83	5.35	0.42	8.48	4.01	8.74	3.81	2.40
3.00	40.26	15.26	3.10	4.82	0.00	8.05	2.64	7.99	3.75	0.00
3.25	44.78	18.69	3.28	4.89	-0.42	7.37	1.04	7.10	3.66	-2.40
3.50	50.11	22.67	3.69	5.92	-0.42	7.75	0.71	7.29	4.51	-2.37
3.75	56.35	27.14	4.30	7.75	0.00	9.29	1.88	8.59	6.09	0.00
4.00	63.25	31.72	4.67	9.67	0.42	10.77	3.37	9.82	7.04	2.37

Table 2.2.8 100×bias of estimators in 12 step experiments for step size 2.0 and $\beta=\pi/3.0^{1/2}$.

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.25	-1.31	-1.93	-1.76	-1.15	-2.14	-4.07	-6.52	-3.84	-3.82	-6.93
0.50	-1.21	-2.60	-2.26	-0.94	-3.00	-5.20	-9.71	-4.91	-4.91	-9.44
0.75	0.95	-1.55	-1.03	1.20	-2.11	-2.38	-8.42	-2.28	-2.42	-6.44
1.00	4.79	0.69	1.37	4.76	0.00	3.12	-3.88	2.82	2.33	0.00
1.25	9.23	2.92	3.68	8.49	2.11	8.74	1.25	7.92	6.75	6.44
1.50	13.15	3.93	4.59	10.98	3.00	11.78	4.27	10.58	8.34	9.44
1.75	15.98	3.22	3.45	11.32	2.14	10.67	3.60	9.56	5.96	6.93
2.00	18.00	1.33	0.69	9.66	0.00	6.08	-0.14	5.69	0.66	0.00
2.25	20.18	-0.39	-2.34	7.11	-2.14	0.90	-4.25	1.52	-4.77	-6.93
2.50	23.53	-0.58	-4.15	5.11	-3.00	-1.66	-5.82	-0.33	-7.55	-9.44
2.75	28.40	1.38	-3.97	4.52	-2.11	-0.27	-3.75	1.19	-6.58	-6.44
3.00	34.32	5.08	-2.11	5.16	0.00	4.06	0.79	5.18	-2.83	0.00
3.25	40.26	9.43	0.30	6.06	2.11	8.90	5.34	9.47	1.54	6.44
3.50	45.22	13.28	1.97	6.10	3.00	11.70	7.34	11.75	4.22	9.44
3.75	48.78	16.06	2.14	4.75	2.14	10.98	5.40	10.73	3.94	6.93
4.00	51.39	18.06	1.04	2.52	0.00	7.29	0.28	7.03	1.26	0.00

Fig. 2.2.1 M.s.e.'s of estimators of the ED50 in 12 step experiments with step size 0.5 ($\beta=\pi/3.0^{1/2}$).

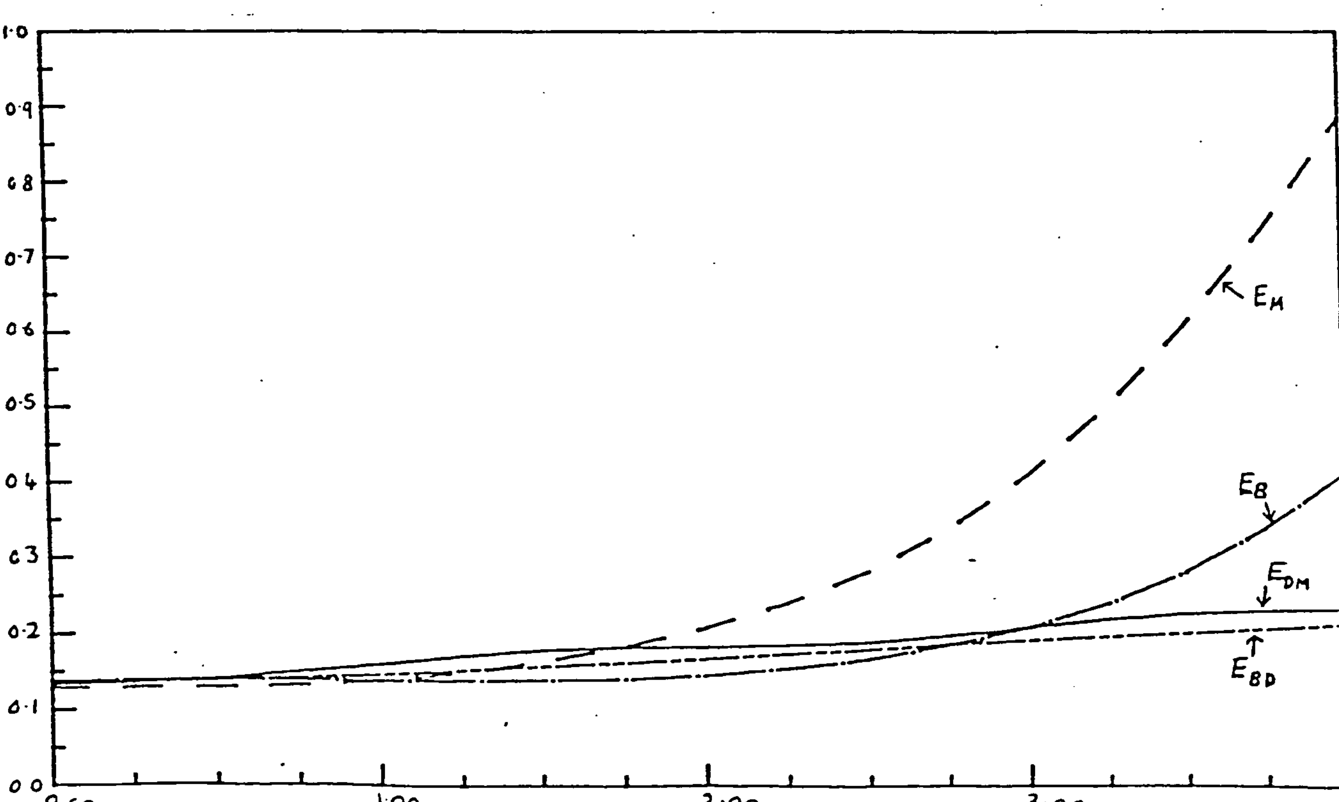
m.s.e.



starting level

Fig. 2.2.2 As in Fig. 2.2.1 only with step size 1.0.

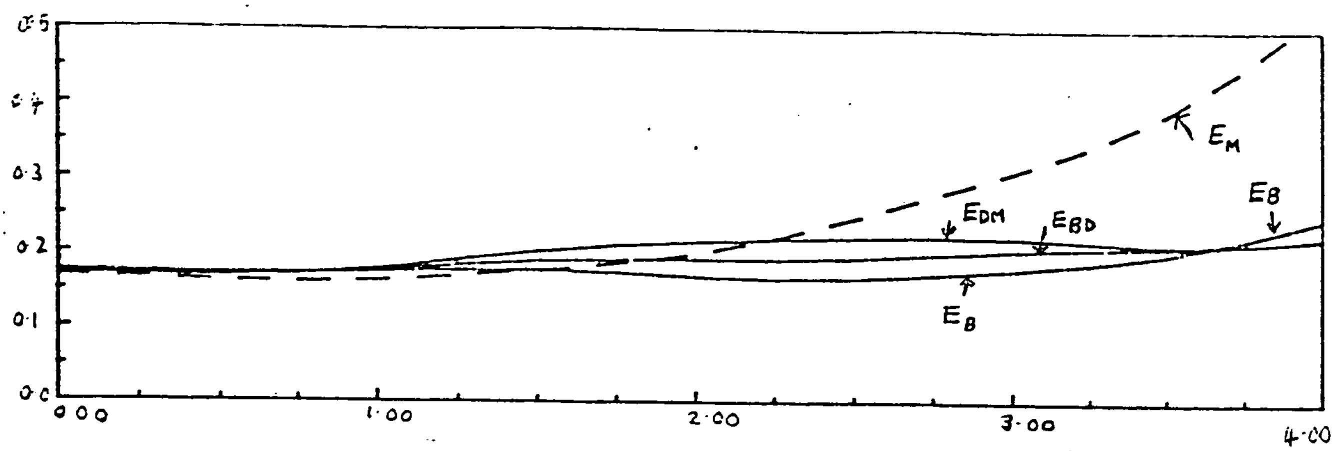
m.s.e.



starting level

m.s.e.

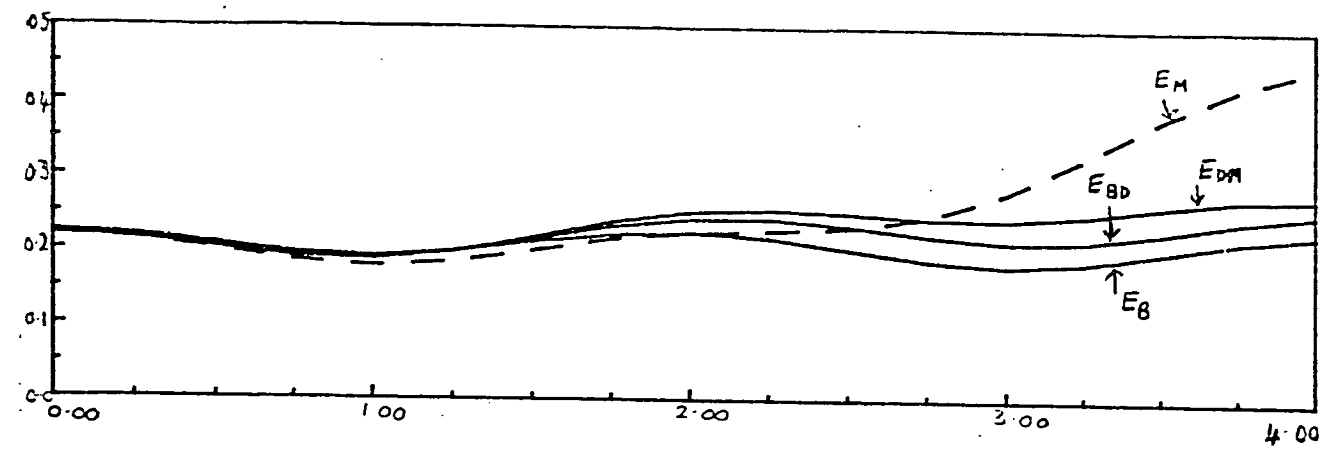
Fig. 2.2.3 As in Fig. 2.2.1 only with step size 1.5.



starting level

m.s.e.

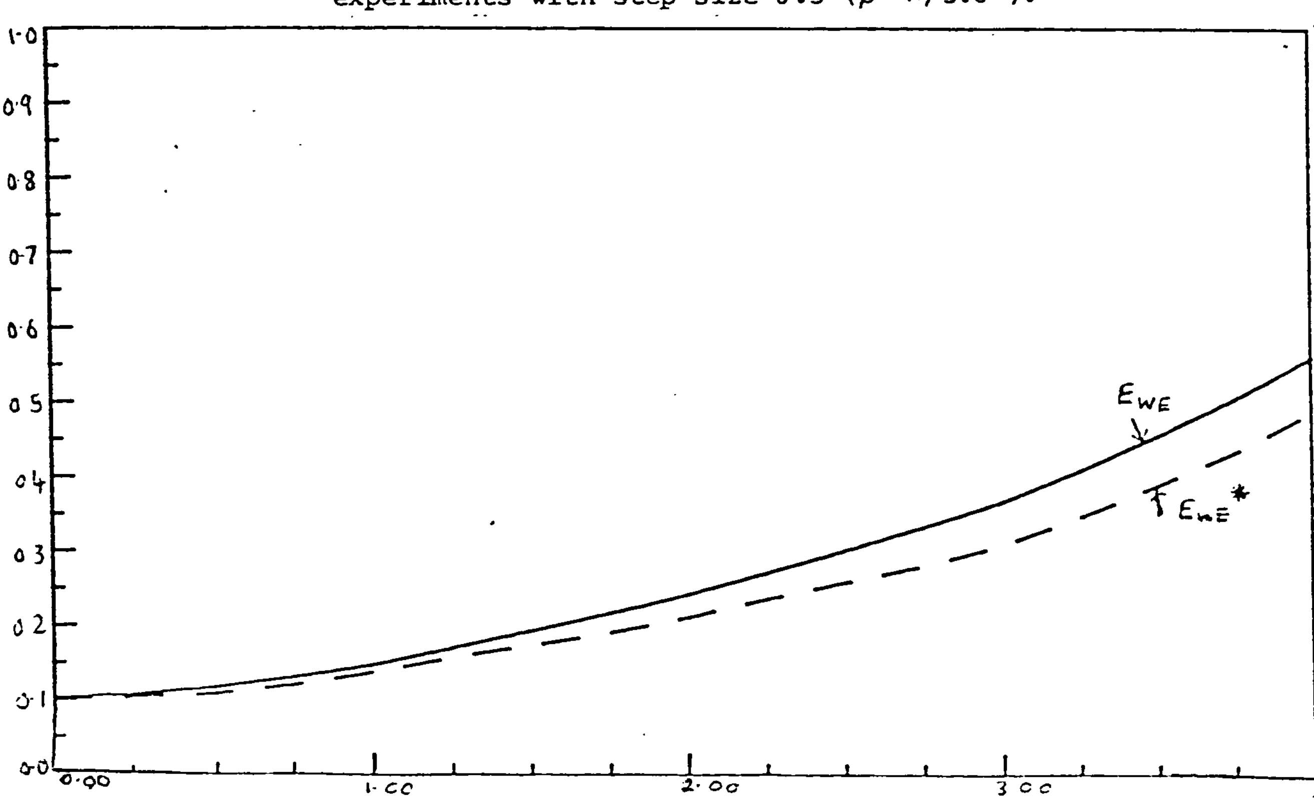
Fig. 2.2.4 As in Fig. 2.2.1 only with step size 2.0.



starting level

m.s.e.

Fig. 2.2.5 M.s.e.'s of E_{WE} and E_{WE}^* in 12 step experiments with step size 0.5 ($\beta = \pi/3.0^{1/2}$).



starting level

Fig. 2.2.6 As in Fig. 2.2.5 only with step size 1.0.

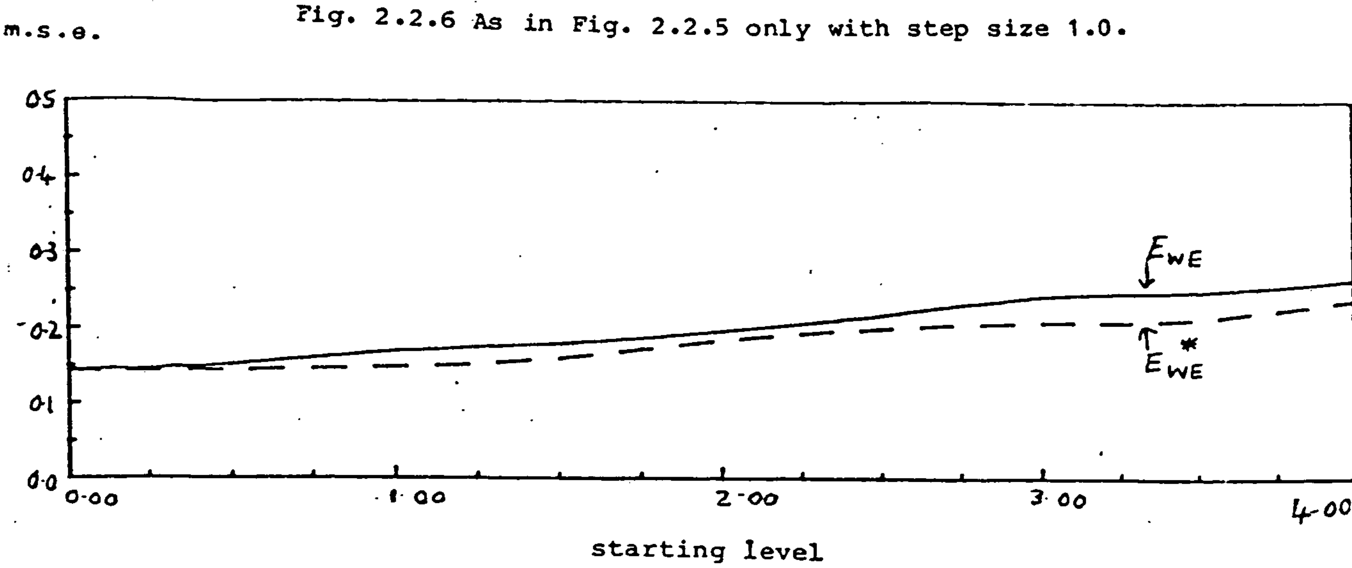


Fig. 2.2.7 As in Fig. 2.2.5 only with step size 1.5.

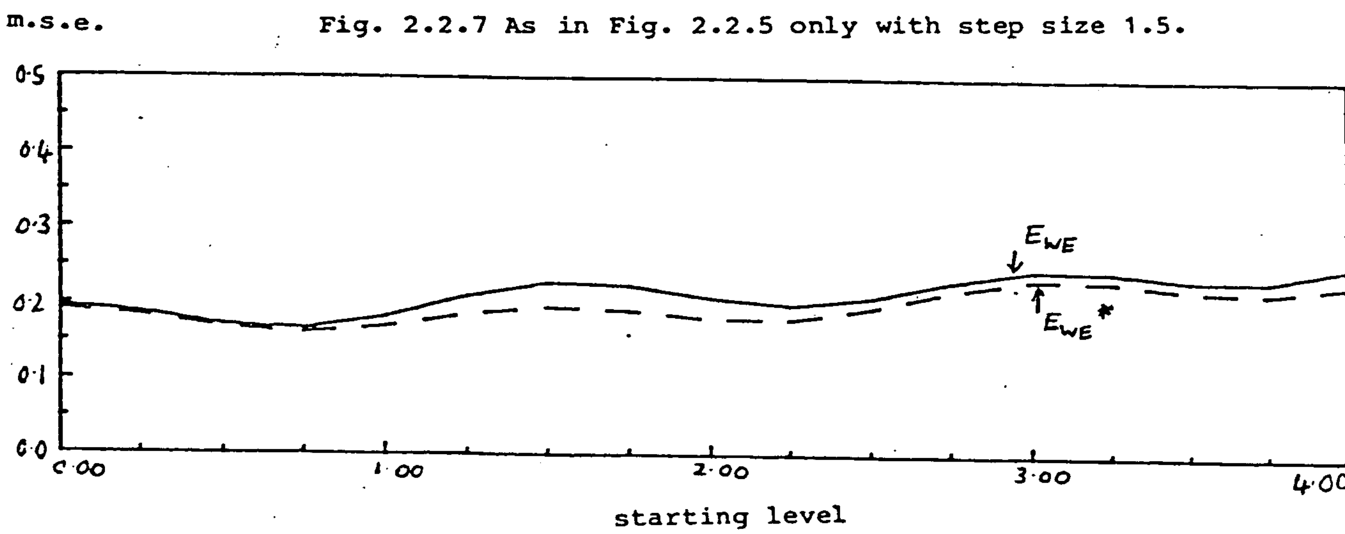
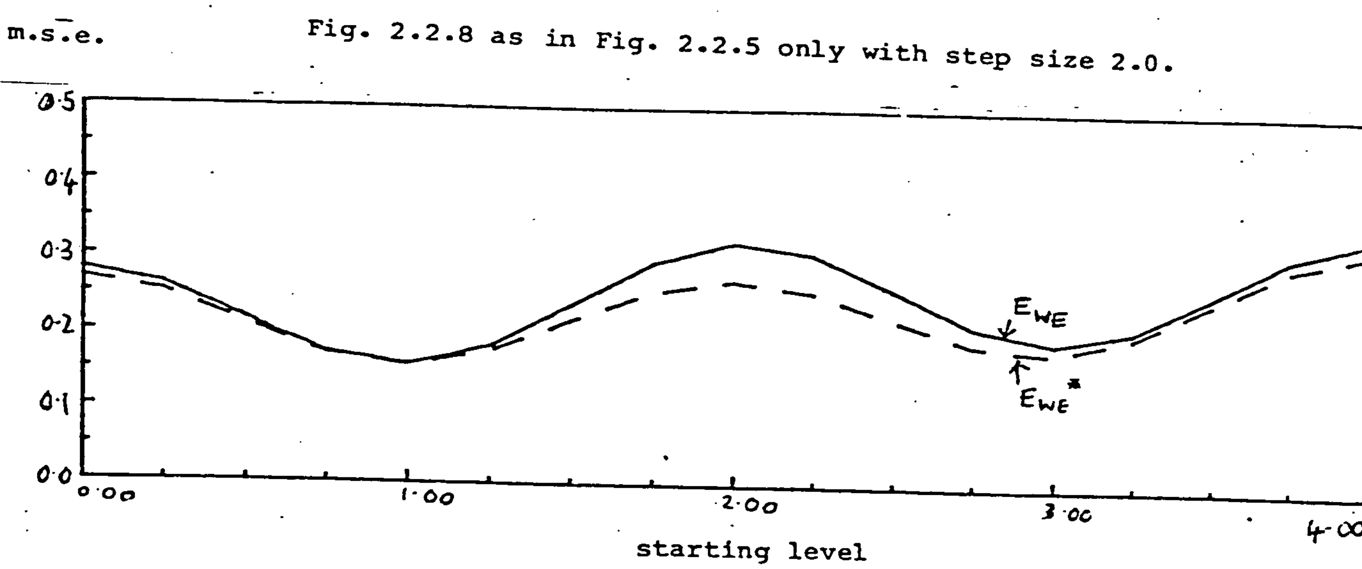


Fig. 2.2.8 as in Fig. 2.2.5 only with step size 2.0.



except for step size 0.5 and starts 2.75(0.25)4.00 this probability is less than 10.0^{-8}.

One interesting point to note is that the m.s.e.'s of E_{WE}^* and E_{PV}^* are always less than corresponding values for E_{WE} and E_{PV}. This is what asymptotic theory suggests. It could be argued that the reductions in m.s.e. are not large enough to justify the extra calculation required, though now such objections carry less weight than in the past. Results for E_{WE}^* do clarify the relationship between E_{WE} and E_{DM}. The m.s.e.'s and biases of E_{DM} and E_{WE}^* are close for step size 0.5; that is an estimator which is in a m.s.e. sense slightly better than E_{WE} has very similar behaviour to that of E_{DM}. For step size 1.0 the relationship between E_{DM} and E_{WE}^* is not so close but they have roughly similar biases and m.s.e.'s. For step size 1.5 the biases of E_{DM} and E_{WE}^* are not similar and m.s.e.'s are only roughly comparable in magnitude. For step size 2.0 the biases of E_{DM} and E_{WE}^* are again not similar and now it is clear that the m.s.e.'s of E_{WE}^* are much more dependent on phasing than those of E_{DM} (where by phasing I mean the distance of the nearest level above μ from μ divided by the step size). This is what one would expect from the asymptotic theory for these estimators. In general one can say that for the smallest step size the estimators E_{WE}, E_{PV}, E_{WE}^* and E_{PV}^* have higher or similar m.s.e.'s to those of E_{DM} but as step sizes increases these m.s.e.'s eventually become heavily dependent on phasing and oscillate above and below values for E_{DM}. These results suggest to me that E_{DM} is preferable to all of these estimators.

If one is intent upon using one of the estimators E_{WE}, E_{PV}, E_{WE}^* or E_{PV}^* it seems sensible to choose E_{WE}^* or E_{PV}^*. The m.s.e.'s of E_{WE}^* and E_{PV}^* are close with often m.s.e.'s of E_{WE}^* being slightly smaller than those of E_{PV}^* for starts close to μ and slightly bigger for distant starts. The expectation of E_{PV}^* is always less than that of E_{WE}^*, usually E_{WE}^* and E_{PV}^* are both positively biased (all starting levels are above μ) and then the bias of E_{PV}^* is smaller than that of E_{WE}^*. The m.s.e. of E_{PV} is usually greater than that of E_{WE}, however E_{PV} has always lower expectation and usually smaller bias. In Choi (1971) there are similar results. Choi asserts that in experiments in which a fixed odd number of peaks and valleys are obtained that the variance of E_{WE} is less than or equal to that of E_{PV} and he shows that for starts above μ the expection of E_{PV} is less than that of E_{WE} (if the number of peaks and valleys is a fixed even number then E_{PV} always equals E_{WE}).

Suppose one is to use one of the estimators E_M, E_B, E_{BD} or E_{DM}. The estimator E_M has, as would be expected, large bias and high m.s.e. for distant starting levels. For starts close to μ this estimator has the lowest m.s.e. but this advantage is never very great. It would appear unwise to use E_M unless one is sure that the starting value is close to μ. The estimator E_B has similar advantages and defects as E_M. This estimator often has lower m.s.e. than E_{BD} or E_{DM} but again this advantage is never great and its m.s.e. is very high for step size 0.5 and starts beyond 2.00. The estimators E_{BD} and E_{DM} have m.s.e.'s that are always roughly similar. Both estimators are such that bad starting values do not greatly inflate m.s.e.'s even for the smallest step size. For starts close to μ the estimator E_{DM} has slightly lower m.s.e. than E_{BD} but eventually as the distance of starting level from μ is increased E_{BD} has the lower m.s.e. (E_{DM} has m.s.e. less than E_{BD} for a range of starts increasing with step size). The expectation of E_{BD} is always less than that of E_{DM} and usually the bias of E_{BD} is less than that of E_{DM}. This suggests to me for such small scale experiments, where reduction of bias and m.s.e. due to bad starting values is very important, that the estimator E_{BD} should be preferred.

Appendix 1 gives conditions under which the m.l.e. of μ exists when trials are made according to the Up and Down rule and the response is logistic. In such small experiments there appeared no point in looking in great detail at the possibility of using maximum likelihood estimation as there is for all conditions a high probability that the maximum likelihood equations have a degenerate solution (for example for step size 1.0 the probability that the m.l.e. of β is infinite ranges between 0.407 and 0.574 for starts at 0.50 and 4.00 respectively). These difficulties can be seen as an indication that too few experimental units have been considered or that maximum likelihood estimation is inadequate for small samples. There are similar difficulties in trying to form minimum logit chi squared estimates. The minimum logit chi squared estimates of a ($a=-\beta\mu$) and β have the form

$$\bar{\beta} = \sum_i v_i \log(n_i/m_i)(x_i-\bar{x}) / \sum_i v_i (x_i-\bar{x})^2, \qquad 2.2.1$$

$$\bar{a} = \sum_i v_i \log(n_i/m_i) / \sum_i v_i - \bar{\beta}\bar{x}, \qquad 2.2.2$$

where n_i and m_i are the number of positive and negative responses at x_i, v_i is $n_i m_i/(n_i+m_i)$ and $\bar{x}$ is the weighted mean of the x_i with weights proportional to v_i. These expressions exist if and only if there are two levels of mixed response type (where levels for which the response is of only one type are ignored). The probability that the expressions do not exist is slightly higher than the probability that the m.l.e. of β is infinite (for example for step size 1.0 this probability ranges from 0.410 to 0.599 for starts at 0.5 and 4.00 respectively). The minimum logit chi squared estimate of μ cannot be formed if the estimate of β is 0.00. As before paths of probability less than 10.0^{-8} are not included. With such high probabilities of experiments yielding no estimates it is impossible to make a useful comparison between the estimators previously discussed and the minimum logit chi squared estimator of μ (or with the m.l.e. of μ). Berkson (1957) suggests the use of a '1/2n' rule for levels where only one response type is recorded (if n_i positive and no negative responses are recorded at x_i then he replaces n_i by $(n_i-0.5)$ and m_i by 0.5, if there are m_i negative and no positive responses he replaces n_i by 0.5 and m_i by $(m_i-0.5)$), this rule seems somewhat arbitrary but the probability that a path will give an estimate is much higher (for example when step size is 1.00 it is greater than 0.9975 for all starts). However using this rule did not give satisfactory results, the estimator of μ has much higher m.s.e. than values for the other estimators. These results are not surprising as for such small samples often there will only be one observation at some of the more extreme levels and application of the '1/2n' rule makes no sense as it makes such observations at the extremes seem close to μ (Berkson of course

Table 2.2.9 Mean and m.s.e. of min. logit chi squared estimates of μ when the step size is 1.0

'1/2n' rule used

	Starting level 0.0	1.0	2.0	3.0	4.0
mean	0.000	0.108	0.029	0.234	-0.192
m.s.e.	0.338	0.354	0.657	3.421	72.946

'1/2n' rule used but levels visited only once are ignored

	Starting level 0.0	1.0	2.0	3.0	4.0
mean	0.000	0.031	-0.010	0.036	-0.018
m.s.e.	0.193	0.392	6.036	4.335	5.661

only proposed that the rule should be used in non-sequential experiments in which all the 'n' are large). I tried to overcome this difficulty by ignoring levels visited only once but otherwise using the '1/2n' rule. Again the probability of paths from which an estimate of μ can be formed is much greater than when the '1/2n' rule is not used, unfortunately the m.s.e.'s are usually well above values for the other estimators. Table 2.2.9 gives some results for step size 1.0 using both procedures.

Davis in (1965a) and (1965b) does discuss minimum transform chi squared estimates of parameters in non-sequential experiments involving 12 observations. He encountered similar problems and resorted to putting a lower bound on estimates of slope equal to 0.2 times the true value. The probability of an 'unacceptable' estimate of slope was often very high. In Davis (1971) he ommits discussion of this estimator.

As the probabilities of experiments for which the maximum likelihood equations have a degenerate solution can be very high it would be surprising if minimum logit chi squared estimation had given satisfactory results. In the next chapter simulated experiments consisting of greater numbers of observations are described and a more useful comparison of maximum likelihood, minimum logit chi squared and alternative methods of estimation can be made.

2.3 DESCRIPTION OF SOME ASYMPTOTIC THEORY

In the previous section various estimators of the ED50 have been compared. Expressions for asymptotic bias and variance of estimators allow one to compare estimators as the number of observations tend to infinity and also indicate to what extent small sample results conform to asymptotic theory.

In the Up and Down experiments the sequence of levels visited can be thought of as a Markov chain. The equilibrium probability π_i of being at level x_i can be obtained by solving all the equations of the form

$$\pi_i(1.0-F_i) = \pi_{i-1}F_{i-1}, \qquad 2.3.1$$

subject to the condition $\sum_i \pi_i = 1.0$; F_i is probability of positive response at level x_i. When the response curve is logistic then π_i is proportional to

$$\exp(-\beta(x_i-\mu-(d/2))^2/2d)+\exp(-\beta(x_i-\mu+(d/2))^2/2d), \qquad 2.3.2$$

where d is the distance between adjacent levels (for derivation of 2.3.2 see Appendix 6). The first term in Formula 2.3.2 is the contribution from positive responses, the second that from negative responses. So the positive responses are arranged asymptotically at stimulus levels in proportion to the value at each stimulus level of a normal density with mean $\mu+(d/2)$ and variance d/β; the negative responses are arranged in proportion to a normal density with mean $\mu-(d/2)$ and variance d/β. This suggests that μ_1 or μ_2 could be used as rough estimates of μ, where μ_1 is the mean of levels of positive response minus d/2 and μ_2 is the mean of levels of negative response plus d/2. Expressions for the Dixon and Mood mean are given in Formulae 2.1.1 and 2.1.2. E_{DM} equals μ_1 if positive responses are less frequent and equals μ_2 otherwise. The expressions for the equilibrium probabilities of positive and negative responses for a logistic curve can be seen as further motivation for use of E_{DM} as a rough estimate of μ. The asymptotic expectation of the Dixon and Mood estimator is

$$\sum_i (x_i-(d/2))u_i/\sum_i u_i, \qquad 2.3.3$$

where u_i equals $\exp(-\beta(x_i-\mu-(d/2))^2/2d)$. This of course is the asymptotic expectation of E_M, E_B and E_{BD} as they are all asymptotically equivalent to E_{DM}. The asymptotic variance expression of all these estimators has a more complicated form. Tsutakawa (1967a,b) uses a central limit theorem in Chung (1960) to derive the expressions for asymptotic expectation and variance of these estimators. The asymptotic mean and variance of these estimators are M and V/n (where n is the number of observations)

$$M = \sum_i \pi_i x_i, \qquad 2.3.4$$

$$V = \sum_i \pi_i E_i^2(-1+(2/\rho_i))+4\sum_{\substack{0<i<j \\ j<i<0}}\sum \pi_j E_i E_j/\rho_i - 2\pi_0 E_0^2, \qquad 2.3.5$$

where E_i equals x_i-M and ρ_i is the probability that the process starting at x_i reaches x_0 before returning to x_i (ρ_0 is defined as 1.0). The expressions to be evaluated are infinite sums which have no closed form. The approximation of setting π_i equal to zero for $|i|>40$ is made in all following calculations; as $|i|\to\infty$ the π_i tend to zero rapidly (π_i is soon dominated by a term which is a multiple of $\exp(-d\beta i^2/2)$) and the approximation will give evaluation of the sums well within the desired level of accuracy.

From symmetry one can deduce that the biases of E_{DM} for μ/d equal to x and -x will be for all x of the same magnitude but of opposite sign. If μ/d equals k+x for some integer k then the bias will be the same as when μ/d equals x (in this case the scale has been translated without the phasing of the levels being altered). If one knows the bias for $\mu/d \in [0.0,0.5]$ one can deduce the bias for all μ/d values. Calculations reveal that the bias of the E_{DM} is very small for a wide range of β and μ values. Table 2.3.1 contains values of asymptotic expectation of E_{DM}/d for μ/d values of 0.00(0.05)0.50 and βd values of 2.25(0.25)4.00 (values of βd of 0.25(0.25)2.00 were also considered but to three decimals the bias of E_{DM}/d was zero). The biases for μ/d equal to 0.50 and 0.00 are always zero as then the possible stimulus levels are symmetrically placed about μ.

The bias is towards the midpoint of the two possible stimulus levels falling on either side of μ (when μ is actually at a possible stimulus level the bias is zero). Taking the limit as $\beta\to\infty$ in Formula 2.3.3 then the term for which $(x_i-(d/2)-\mu)^2$ is a minimum will eventually dominate and the expression for the

Table 2.3.1 Values of asymptotic expectation of E_{DM} /d.

μ/d	βd 2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00
0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
0.400	0.401	0.401	0.402	0.403	0.405	0.407	0.410	0.413
0.300	0.301	0.302	0.303	0.306	0.308	0.312	0.316	0.321
0.250	0.251	0.252	0.253	0.256	0.259	0.263	0.267	0.273
0.200	0.201	0.202	0.203	0.206	0.208	0.212	0.217	0.222
0.100	0.101	0.101	0.102	0.103	0.105	0.108	0.110	0.113
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 2.3.2 Values of asymptotic expectation of E_{WE}/d.

μ/d	βd 1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00
0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
0.400	0.400	0.401	0.402	0.405	0.408	0.411	0.415	0.420	0.424	0.429	0.434
0.300	0.301	0.302	0.304	0.308	0.312	0.318	0.325	0.332	0.340	0.348	0.356
0.250	0.251	0.252	0.254	0.258	0.263	0.269	0.276	0.284	0.293	0.302	0.311
0.200	0.201	0.202	0.204	0.208	0.212	0.218	0.225	0.232	0.241	0.250	0.259
0.100	0.100	0.101	0.102	0.105	0.108	0.111	0.116	0.121	0.126	0.132	0.138
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 2.3.3 Asymptotic variance expressions (μ/d=0.0).

Asymptotic Variance of:	βd 0.25	0.50	0.75	1.00
$(\hat{\mu}-\mu)\beta n^{1/2}$	4.251	4.504	4.762	5.025
$(E_{DM}-M_{DM})\beta n^{1/2}$	4.253	4.514	4.781	5.056
$(E^*_{WE}-M_{WE})\beta n^{1/2}$	4.259	4.527	4.799	5.076
$(E_{WE}-M_{WE})\beta n^{1/2}$	4.429	4.769	5.071	5.357

asymptotic expectation of E_{DM} will tend to the midpoint of the two levels on either side of μ (providing μ is not at a stimulus level).

The asymptotic expectations of E_{WE}, E_{PV}, E_{WE}^* and E_{PV}^* are all the same and equal the following expression (see Appendices 3 and 4)

$$\sum_i \pi_i x_i (F_i (1.0-F_{i-1})+(1.0-F_i) F_{i+1}) / \sum_i \pi_i (F_i (1.0-F_{i-1})+(1.0-F_i) F_{i+1}). \quad 2.3.6$$

Values of the asymptotic expectation of E_{WE}/d were calculated for the same μ/d and βd values as used in calculations for the expectation of E_{DM}/d (again one only need consider $\mu/d \in [0.0,0.5]$ as biases outside this range can be deduced from values within this range in the same way as values for biases of E_{DM}). Table 2.3.2 gives some values of the asymptotic expectation of E_{WE}/d (for βd values of 0.25(0.25)1.25 the bias is zero to three decimals). As with E_{DM} the biases are towards the midpoint of the two levels either side of μ, the biases being larger than corresponding values for E_{DM}. The estimators are asymptotically normal with the asymptotic variance expression for E_{WE} and E_{PV} given in Appendix 3 and that for E_{WE}^* and E_{PV}^* in Theorem 3 of Appendix 4.

The m.l.e.'s of parameters μ and β will be asymptotically unbiased and have an asymptotic bivariate normal distribution. This follows from a theorem in Billingsley (1961). The regularity conditions required to apply this theorem are given in Appendix 5 and it is easy to verify that they are satisfied for the Markov chain generated from using the Up and Down rule with logistic response. The theorem gives expressions for the asymptotic variances and covariances of the m.l.e.'s, which are just as for non-sequential experiments only with proportions of observations at levels used replaced by equilibrium probabilities of being at the levels (see Section 1.3). That is if $\hat{\mu}$ and $\hat{\beta}$ are m.l.e.'s of μ and β from n observations then $n^{1/2}(\hat{\mu}-\mu)$ and $n^{1/2}(\hat{\beta}-\beta)$ are asymptotically bivariate normal with the following variance-covariance matrix.

$$\begin{bmatrix} ((1/\sum_i \pi_i w_i)+(\bar{x}-\mu)^2 S)/\beta^2 & (\bar{x}-\mu)S/\beta \\ (\bar{x}-\mu)S/\beta & S \end{bmatrix}, \qquad 2.3.7$$

where w_i is the logit weight associated with observations at x_i (i.e. w_i equals $F_i(1.0-F_i)$), $\bar{x}$ is the weighted mean of the x_i with weights proportional to $\pi_i w_i$ and

$$S = (\sum_i \pi_i w_i (x_i-\bar{x})^2)^{-1}. \qquad 2.3.8$$

Tsutakawa (1967a) gives a similar expression for probit response where probit weights are used instead of the logit weights. If μ is estimated conditional on a known β then the $(\bar{x}-\mu)^2 S/\beta^2$ term in the variance expression for μ is dropped.

I calculated values for the asymptotic variance of $n^{1/2}(\hat{\mu}-\mu)\beta$, and in Fig. 2.3.1 some of these values are illustrated. I also calculated values for the asymptotic variance of $n^{1/2}(E_{DM}-M_{DM})\beta$, $n^{1/2}(E_{WE}-M_{WE})\beta$ and $n^{1/2}(E_{WE}^{*}-M_{WE})\beta$ (where M_{DM} and M_{WE} are the

Fig. 2.3.1 Asymptotic variance of $n^{1/2}(\hat{\mu}-\mu)\beta$.

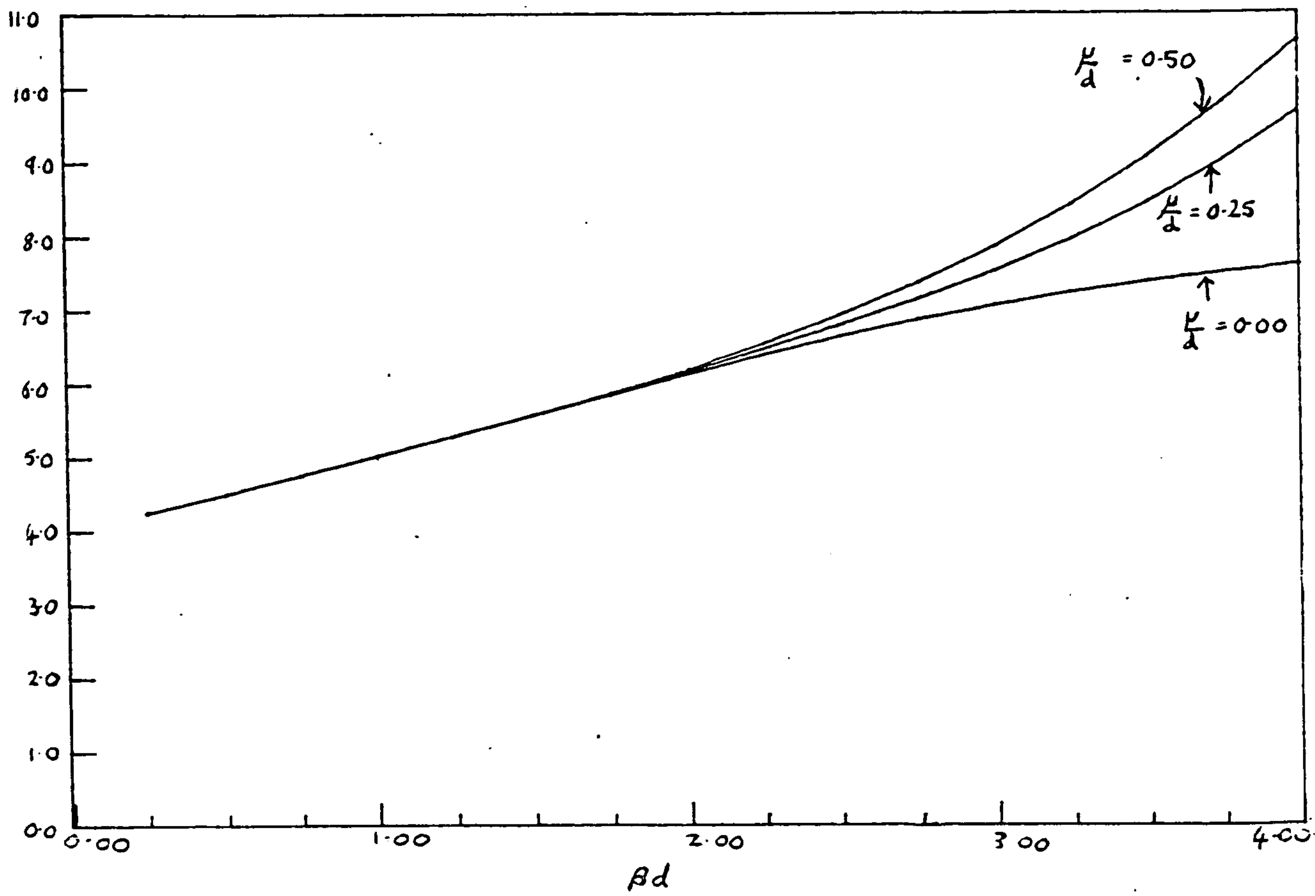


Fig. 2.3.2 Asymptotic variance of $n^{1/2}(E_{OM} - M_{DM})\beta$.

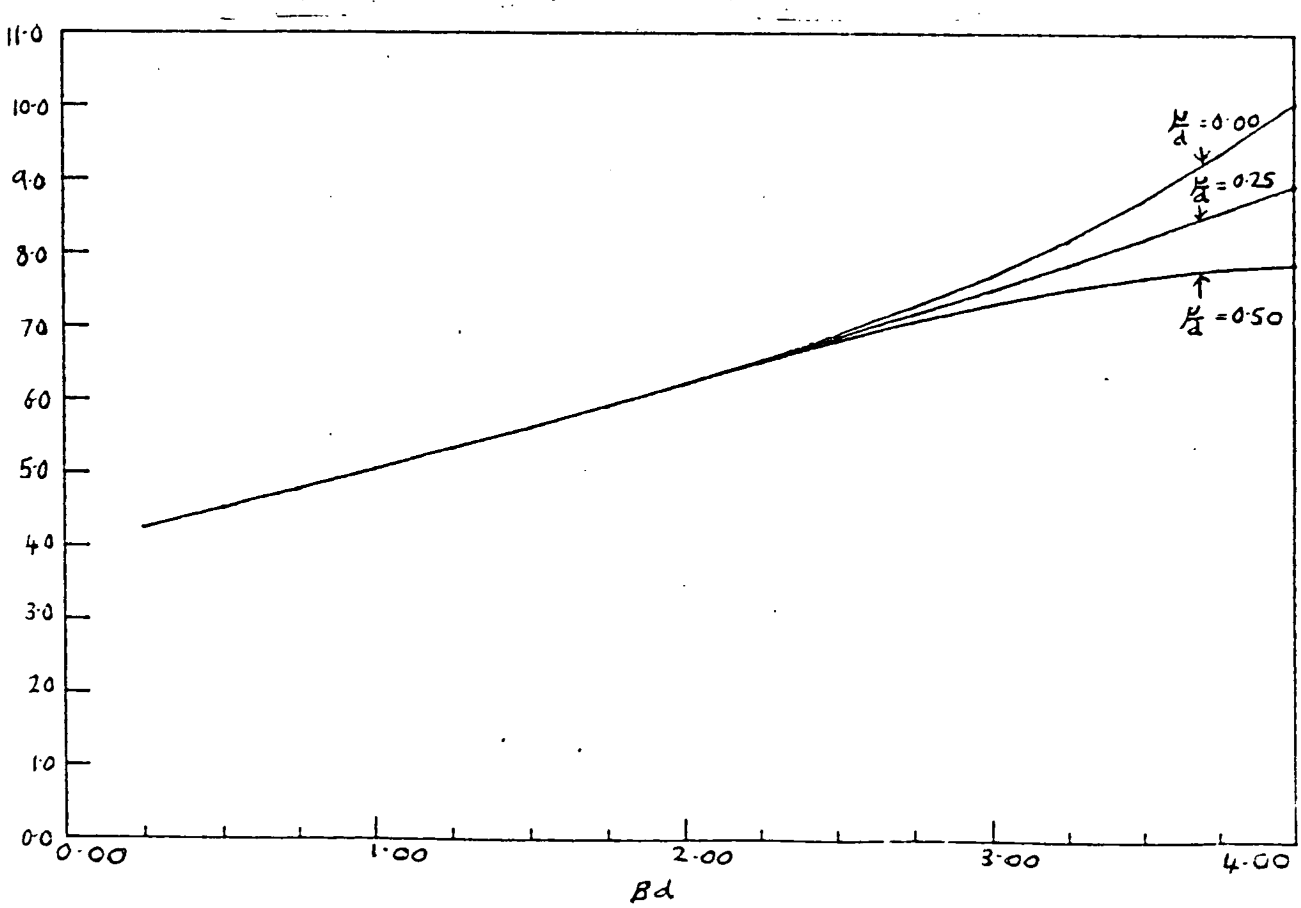
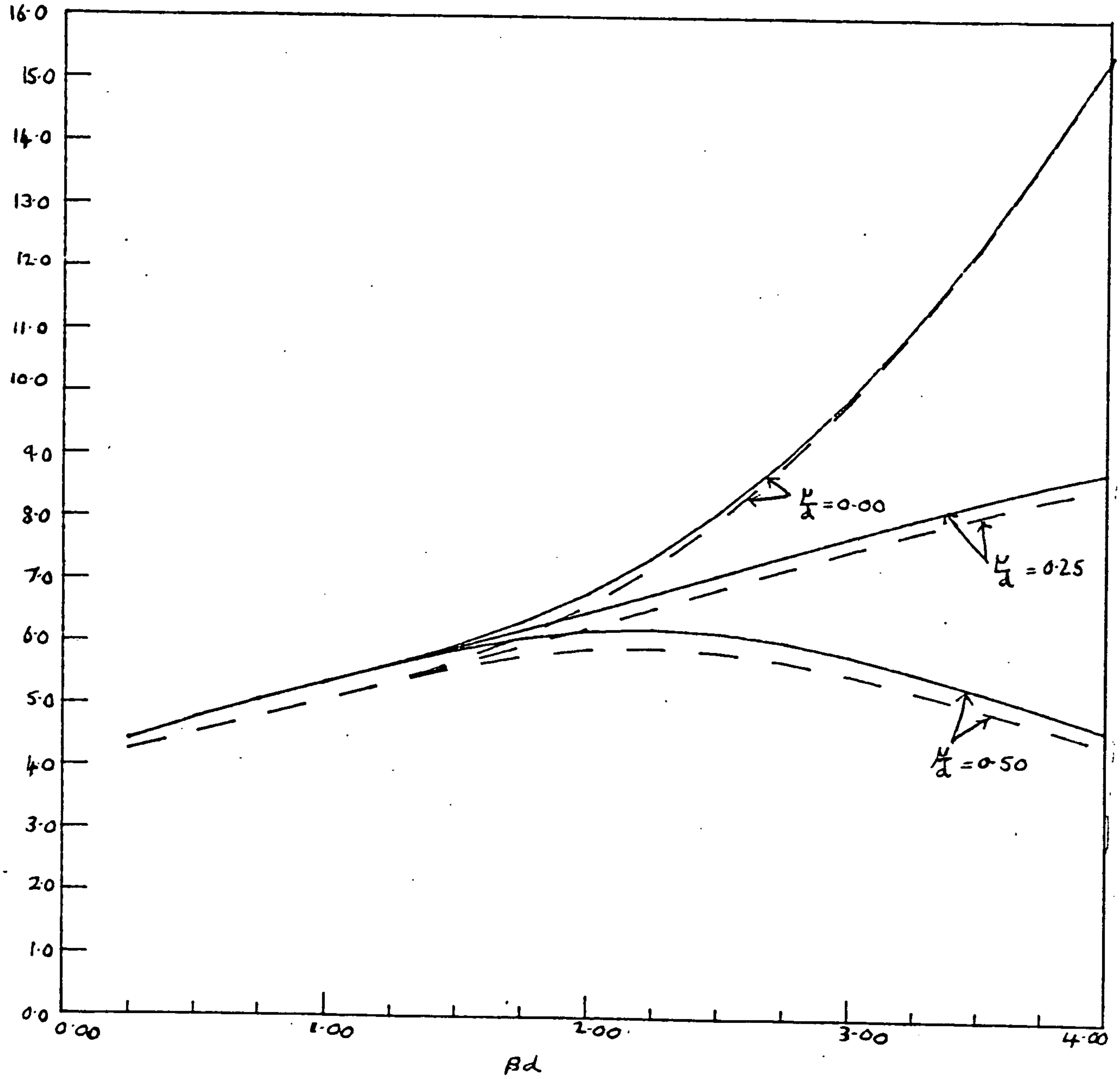


Fig. 2.3.3 Asymptotic variance of $n^{1/2}(E_{WE} - M_{WE})\beta$ (dashed lines link values for $n^{1/2}(E^{*}_{WE} - M_{WE})\beta$).



asymptotic expectations of E_{DM} and E_{WE} respectively). Some of these values are illustrated in Figs. 2.3.2 and 2.3.3. The asymptotic variance expression for $n^{1/2}(\hat{\mu}-\mu)\beta$ hardly changes with phasing for $\beta d<2.00$ but above this phasing begins to have a more marked effect, with higher values for μ/d equal to 0.5 and lower if μ/d equals 0.0 (this is as could be expected as if the step size is large in the former case all possible stimulus levels yield little information but in the latter at least at the level μ the logit weight of observations is at its highest possible value). The asymptotic variance expression for $n^{1/2}(E_{DM}-M_{DM})\beta$ also hardly changes with phasing for $\beta d<2.00$ but again above this phasing begins to have a marked effect, with now lower values for μ/d equal to 0.5 levels and higher if μ/d equals 0.0 (this is the opposite pattern to that for $\hat{\mu}$. This is not however surprising, as observations at levels which yield little information have low variability and those which yield much information have high variability; when the step size is large and μ is midway between levels observations have very low variability and E_{DM} has lower variance than when μ is at a level). Fig. 2.3.3 illustrates values of asymptotic variance of $n^{1/2}(E_{WE}-M_{WE})\beta$, analogous values for $n^{1/2}(E_{WE}^{*}-M_{WE})\beta$ are also illustrated by the points joined by dashed lines. For these estimators phasing begins to have a marked effect for $\beta d>1.25$. The dependence on phasing is similar to that for E_{DM} but the effect of phasing is greater. The difference between the variance expressions for E_{WE} and E_{WE}^{*} is not a high proportion of the variance expression for E_{WE}.

For small step sizes the values of asymptotic variance expressions for $\hat{\mu}$, E_{DM} and E_{WE}^{*} are close with that for μ being lowest and that for E_{WE}^{*} highest; the value for E_{WE} is then some way above the other values. Table 2.3.3 gives values of asymptotic variance of $n^{1/2}(\hat{\mu}-\mu)\beta$ and the analogous expressions for E_{DM}, E_{WE}, and E_{WE}^{*} for μ/d equal to zero and βd values 0.25(0.25)1.00, for these βd values the effect of phasing on all the values is very small. For small step size all these variances, except those for E_{WE}, are approximately equal to $4.0+\beta d$.

The high dependence on phasing of the asymptotic variance expressions for E_{WE} and E_{WE}^{*} for large step sizes does much to explain the large oscillations in m.s.e.'s of these estimators observed for the experiments of the previous section for the largest step size (see Tables 2.2.1 to 2.2.4). The similar m.s.e.'s of E_{WE} and E_{WE}^{*} for small step size are not surprising as asymptotic theory predicts that the estimators have similar variance for small step size. The results of the previous section together with these asymptotic calculations tend to suggest that one should use the E_{BD} or E_{DM} in preference to E_{WE} or E_{WE}^{*} but there is little to choose between E_{DM} and E_{WE}^{*} for low step size.

As step size becomes smaller the asymptotic variance expression for μ decreases and so one could say that it is best to use as small a step size as possible. However there are two important disadvantages in using a small step size. One is that if a starting level distant from μ is chosen it will take many observations before anything like the asymptotic distribution of levels is achieved and in small samples estimates will have high m.s.e.; the other is that in the absence of precise knowledge about β one will wish to obtain an estimate from the results of the experiment but for small step sizes the asymptotic variance of the m.l.e. of β becomes very large (see Fig. 3.1.8).

However observations are placed the asymptotic variance expression for $n^{\frac{1}{2}}(\hat{\mu}-\mu)\beta$ is bounded below by 4.0 (this correspond to the limit as d tends to zero when observations are made in equal numbers at μ+d and μ-d). Of course μ is not known so one cannot place observations at short distances either side of μ to approach this limit, also if an estimate of β is required it would be a mistake to place all observations close to μ. In the next chapter a comparison of the asymptotic properties of the Up and Down method and non-sequential methods is made.

3. ESTIMATION OF BOTH SCALE AND LOCATION PARAMETERS

3.1 COMPARISON OF UP AND DOWN AND NON-SEQUENTIAL DESIGNS

It has been suggested that the slope parameter β of a logistic response curve cannot be satisfactorily estimated using sequential strategies (see the conclusion of Wetherill (1963)). In the following a comparison is made between the Up and Down design and several commonly used non-sequential designs; one would expect that some non-sequential design would give better results, at least for estimation of slope, than the Up and Down design.

In the Up and Down design the possible stimulus levels form a lattice of equispaced dose levels; the non-sequential designs considered are those where equal numbers of observations are placed at k consecutive lattice points, with k equal to 2,3,4 or 5. One would wish to place dose levels symmetrically about μ as the tolerance distribution is symmetric, but in practice the value of μ is not known. For the Up and Down rule values for μ/d considered are 0.00(0.05)0.50. For the non-sequential designs I placed the centre point(s) of the design at the lattice point(s) nearest to μ (one of the principal advantages of the Up and Down rule is that it is a strategy that is able to adjust the testing levels when the initial estimate of μ is poor, so in a sense such comparisons are favourable to non-sequential strategies). For the non-sequential designs the points are symmetrically placed about μ for μ/d equal to 0.5 when there is an even number of design points, and for μ/d

equal to 0.0 for an odd number. As in the small sample experiments of Section 2.2 the value of β is set equal to $\pi^{1/2}/3.0$ so that the variance of the tolerance distribution is normalised to unity. Calculations were made for step sizes 0.25(0.25)3.00.

For convenience the asymptotic variances of $n^{1/2}(\hat{\mu}-\mu)$ and $n^{1/2}(\hat{\beta}-\beta)$, where $\hat{\mu}$ and $\hat{\beta}$ are m.l.e.'s of μ and β, will be denoted by $V(\mu)$ and $V(\beta)$. The comparison between values of $V(\mu)$ for the Up and Down and non-sequential designs is favourable to the Up and Down design. Fig. 3.1.1 illustrates values of $V(\mu)\beta^2$ for the all the designs when the step size is 0.25. Although each non-sequential design has a lower value for $V(\mu)\beta^2$ than that for the Up and Down design for some phasing of levels, non has a lower value for all phasings. The Up and Down design has the advantage that the value of $V(\mu)\beta^2$ is almost independent of phasing; for the other designs this is certainly not the case. The principal reason for this low dependence of $V(\mu)$ when the Up and Down rule is used is that $\bar{x}$ in the matrix in 2.3.7 is very close to μ whatever the phasing. The contribution to $V(\mu)$ from the $(\bar{x}-\mu)^2 S/\beta^2$ term is always calculated to be zero and the covariance of $n^{1/2}(\hat{\mu}-\mu)$ and $n^{1/2}(\hat{\beta}-\beta)$ is zero to 5 decimals. As step size increases magnitudes of covariances increases but they are low over a wide range of step sizes. Values for covariances are given in Table 3.1.1. The average of $V(\mu)$ over phasings is lowest for the Up and Down design (see Fig. 3.1.7). For the other step sizes considered the value of $V(\mu)$ for the Up and Down design is always lowest or second lowest among these designs. For step sizes 0.50 and 0.75 the value is lower for the 3 point design for μ/d values up to 0.25 and 0.20

Fig. 3.1.1 $V(\mu)\beta^2$ for step size 0.25, $\beta = \pi/3.0^{1/2}$.

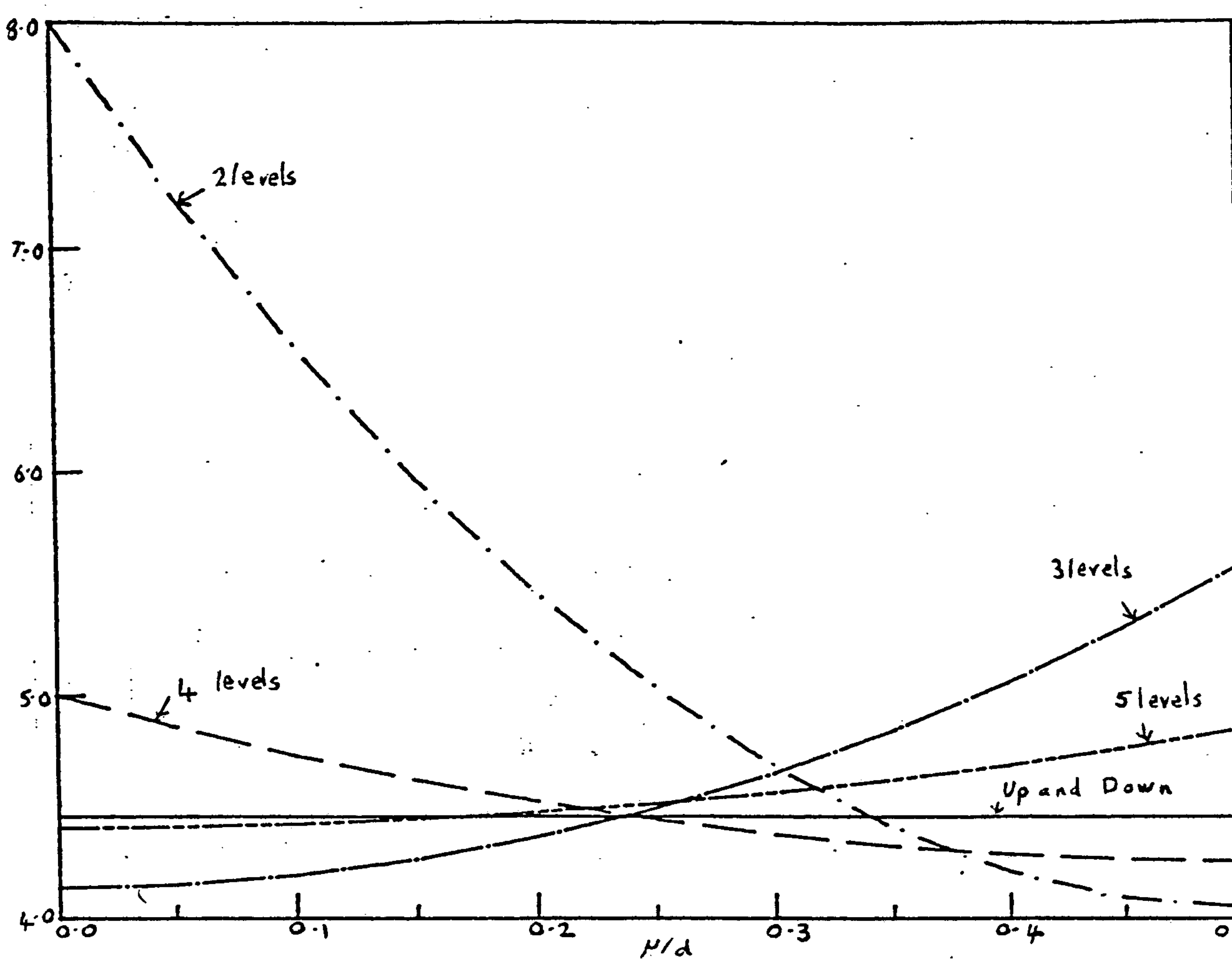


Fig. 3.1.2 $V(\mu)\beta^2$ for step size 0.50, $\beta = \pi/3.0^{1/2}$.

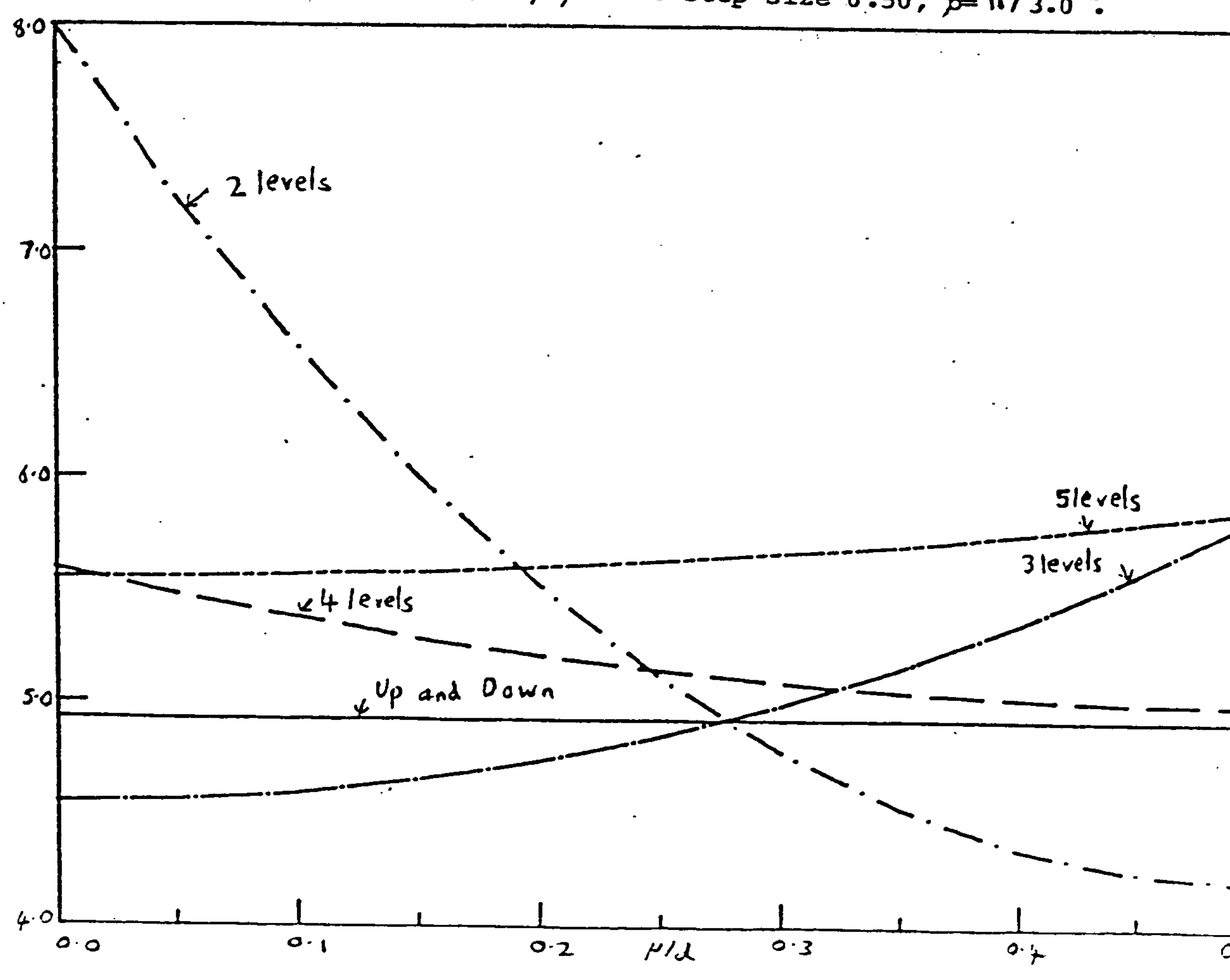


Fig. 3.1.3 $V(\mu)\beta^2$ for step size 0.75, $\beta=\pi/3.0^{1/2}$.

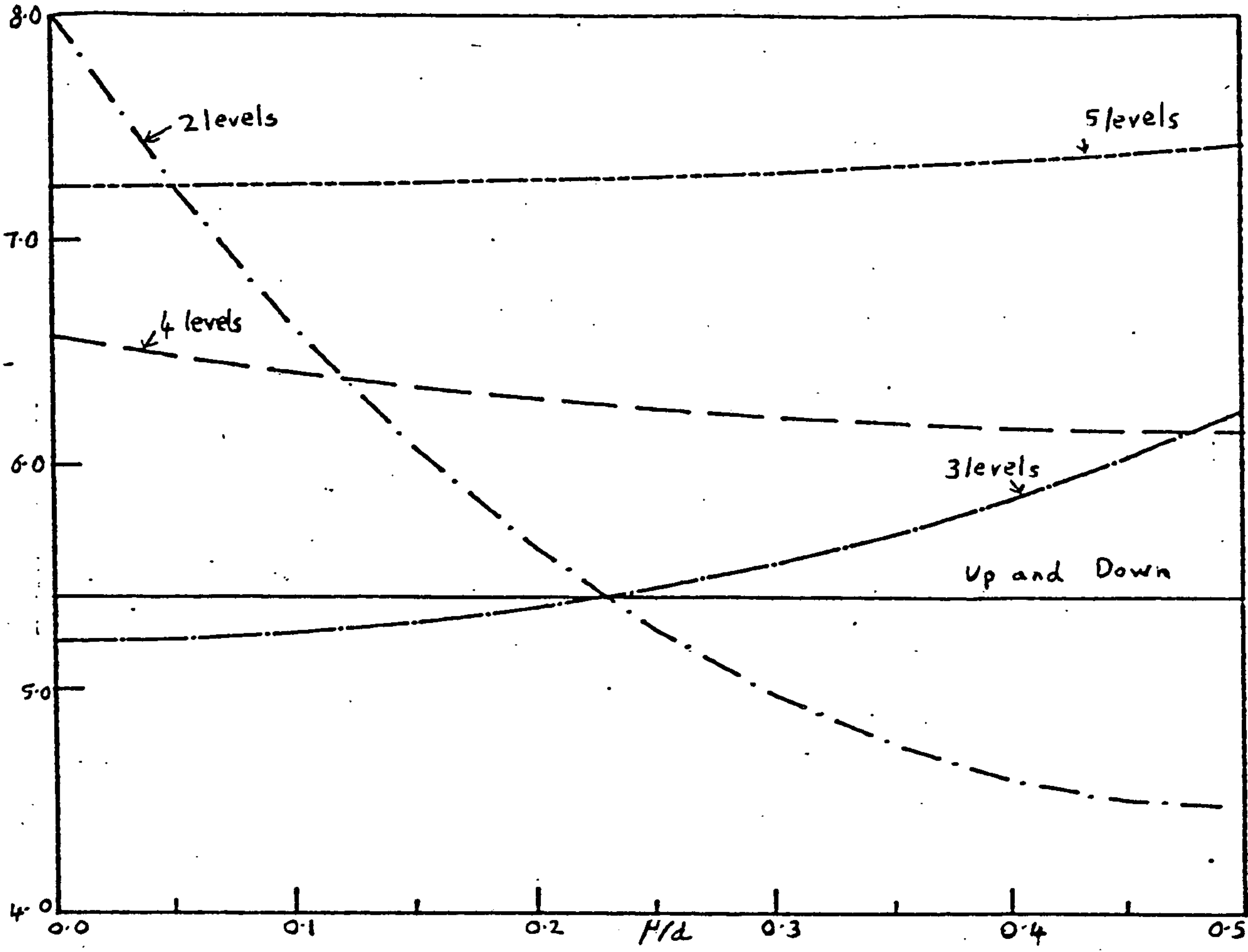


Table 3.1.1 Minus asympt. covariances of $n^{1/2}(\hat{\mu}-\mu)$ and $n^{1/2}(\hat{\beta}-\beta)$ for $\beta=\pi/3.0^{1/2}$

Step size	Phasing										
	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
1.00	0.00	0.01	0.02	0.03	0.03	0.03	0.03	0.03	0.02	0.01	0.00
1.25	0.00	0.05	0.09	0.12	0.14	0.14	0.13	0.11	0.08	0.04	0.00
1.50	0.00	0.13	0.24	0.32	0.37	0.37	0.34	0.28	0.20	0.10	0.00
1.75	0.00	0.28	0.51	0.67	0.74	0.73	0.66	0.53	0.37	0.19	0.00
2.00	0.00	0.52	0.94	1.20	1.29	1.24	1.08	0.86	0.59	0.30	0.00
2.25	0.00	0.89	1.58	1.95	2.03	1.90	1.62	1.26	0.86	0.44	0.00
2.50	0.00	1.44	2.49	2.98	3.02	2.75	2.30	1.77	1.19	0.60	0.00
2.75	0.00	2.25	3.78	4.40	4.32	3.84	3.16	2.39	1.60	0.80	0.00
3.00	0.00	3.42	5.60	6.32	6.05	5.25	4.25	3.18	2.11	1.05	0.00

Covariances to 2 decimals are 0.00 for step sizes 0.25, 0.50 and 0.75. Values are negative for phases between 0.00 and 0.50 and positive for phases between 0.50 and 1.00.

respectively but for μ/d values beyond that the 2 point design is lower. For these step sizes the Up and Down design can be seen as a compromise between the 2 and 3 point designs which works well over the whole range of phasings (see Figs. 3.1.2 and 3.1.3). Again the average of $V(\mu)$ over phasings is lowest for the Up and Down design. For step size 1.00 and above only the 2 point design has a lower value for $V(\mu)$ than that for the Up and Down design. The average over phasings of $V(\mu)$ is now always lowest for the 2 point design. As the step size increases the 2 point design has the lowest value of $V(\mu)$ over a wide range of the μ/d values and for step sizes above 2.00 is lower for all μ/d values considered except 0.00.

In the previous section it is noted that for small step sizes the value of $V(\mu)\beta^2$ using the Up and Down rule is approximately equal to $4.0+\beta d$ (see the values of $V(\mu)\beta^2$ in Table 2.3.3). In the 2 step design with observations placed at $\mu \pm l_0$ a value of $V(\mu)$ equal to V_0 is achieved when

$$l_0 = \log((1+h)/(1-h))/\beta, \quad 3.1.1$$

where

$$h = (1-(4/V_0\beta^2))^{1/2}. \quad 3.1.2$$

If $V_0\beta^2$ is $4.0+\beta d$ then for small values of βd the value of l_0 is approximately $(d/b)^{1/2}$. In terms of $V(\mu)$ the Up and Down design is roughly equivalent to a non-sequential design with observations in equal numbers at $\mu \pm (d/b)^{1/2}$. It is not surprising that for the smallest step $V(\mu)$ is less for the non-sequential designs than for the Up and Down design, providing design points are close to being symmetrically placed about μ, as testing levels are O(d) away from μ not $O(d^{1/2})$. However it must be remembered that ensuring approximate symmetry of levels becomes difficult for small step sizes without a very good prior estimate of μ.

For large step sizes the sequence of levels visited following an Up and Down rule, whenever μ/d is not close to 0.0, will typically consist of alternations between the two lattice points nearest to μ with occasional visits to more distant levels that yield less information, this explains the slight advantage the 2 point design has in such circumstances over the Up and Down design. Designs with very large step sizes are not of much interest as then corresponding values of $V(\mu)$ are very large.

The comparison between designs based on values of $V(\mu)$ alone can be very misleading; if values of $V(\beta)$ are considered quite a different pattern emerges. For estimation of β or β^{-1} the Up and Down rule is not so satisfactory; the value of $V(\beta)$ only has the lowest or second lowest value for step sizes 0.25 and 0.50 with certain values of μ/d and for step sizes 1.5 and above. However the performance of the Up and Down design is not so poor as might first be thought. The lowest ratio of $V(\beta)$ for a non-sequential design against $V(\beta)$ for the Up and Down design is for the 5 point design with phasing of 0.0 and step size 0.5. The relative efficiency of the Up and Down design in estimating β under these conditions is 68.2 percent; for most phasings the Up and Down design has 80 to 90 percent efficiency in estimating β relative to the best non-sequential design. These gains in efficiency are not very large and there is usually a corresponding drop in efficiency in estimating μ. The rapid rise in $V(\beta)$ as step size decreases below about 1.0, for all designs, should discourage any experimenter from using what he guesses to be a very small step size in order to make a small asymptotic gain in estimating μ.

When step sizes are small the values of $V(\mu)$ and $V(\beta)$ for the Up and Down design are almost independent of phasing, but in the 2 and 3 point designs $V(\mu)$ does vary over fairly wide ranges even for small step sizes. Figs. 3.1.4, 3.1.5 and 3.1.6. illustrate how $V(\mu)$ varies with phasing for the Up and Down, 2 point and 3 point designs for all step sizes considered. The values of $V(\mu)$ for the 4 and 5 point design do not depend so much on phasing as those for the 2 and 3 point designs but the values of $V(\mu)$ do rise very rapidly as step size increases. The value for $V(\mu)\beta^2$ with the 2 point design and μ/d equal to 0.0 is always equal to 8.00 whatever the value of d; it is easy to show that in general $V(\mu)$ equals $2/w_0$ where w_0 is the weight associated with observations at μ. In the 2 point designs with observations made at $\mu+\theta d$ and $\mu-(1-\theta)d$, where respective weights of observations are w_1 and w_2

$$V(\mu)\beta^2 = 2((w_1+w_2)^{-1}+((\theta w_1-(1-\theta)w_2)^2)/w_1 w_2(w_1+w_2)), \qquad 3.1.3$$

and so for small step size $V(\mu)\beta^2$ is approximately $(1+(2\theta-1)^2)/w_0$ (this approximation holds very well for step size 0.25).

Fig. 3.1.4 $V(\mu)\beta^2$ for the Up and Down rule, $\beta = \pi/3.0^{1/2}$.

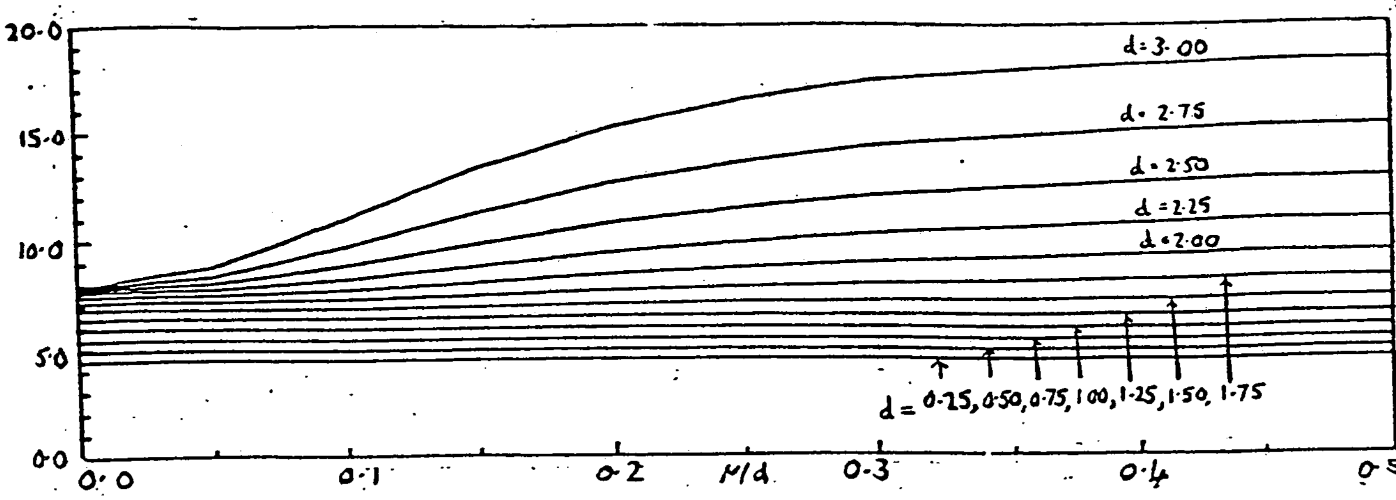


Fig. 3.1.5 $V(\mu)\beta^2$ for the 2 point design, $\beta = \pi/3.0^{1/2}$

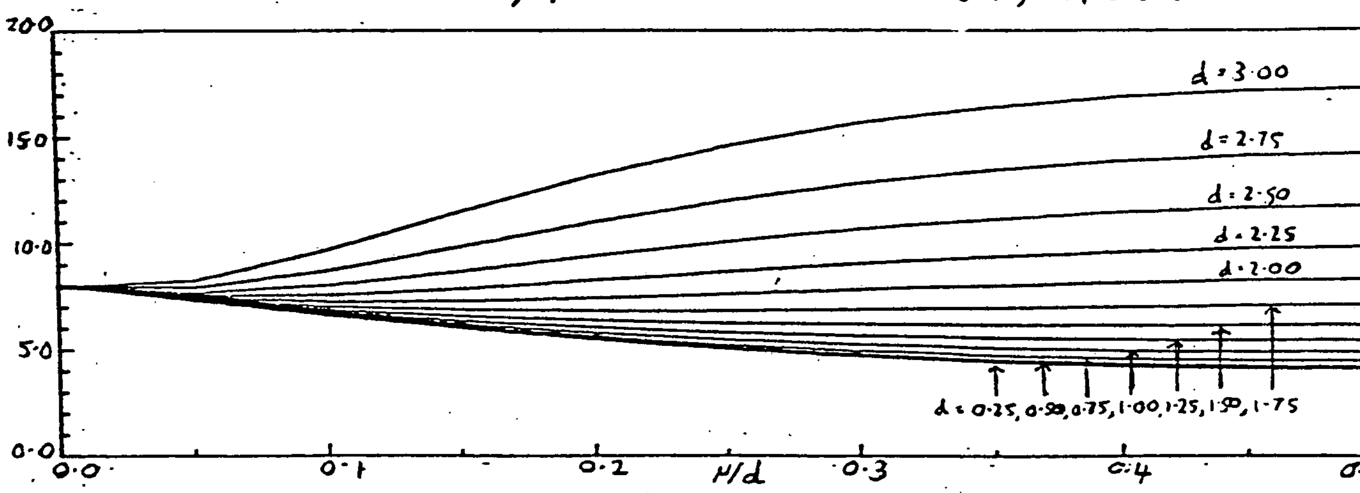


Fig. 3.1.6 $V(\mu)\beta^2$ for the 3 point design, $\beta = \pi/3.0^{1/2}$

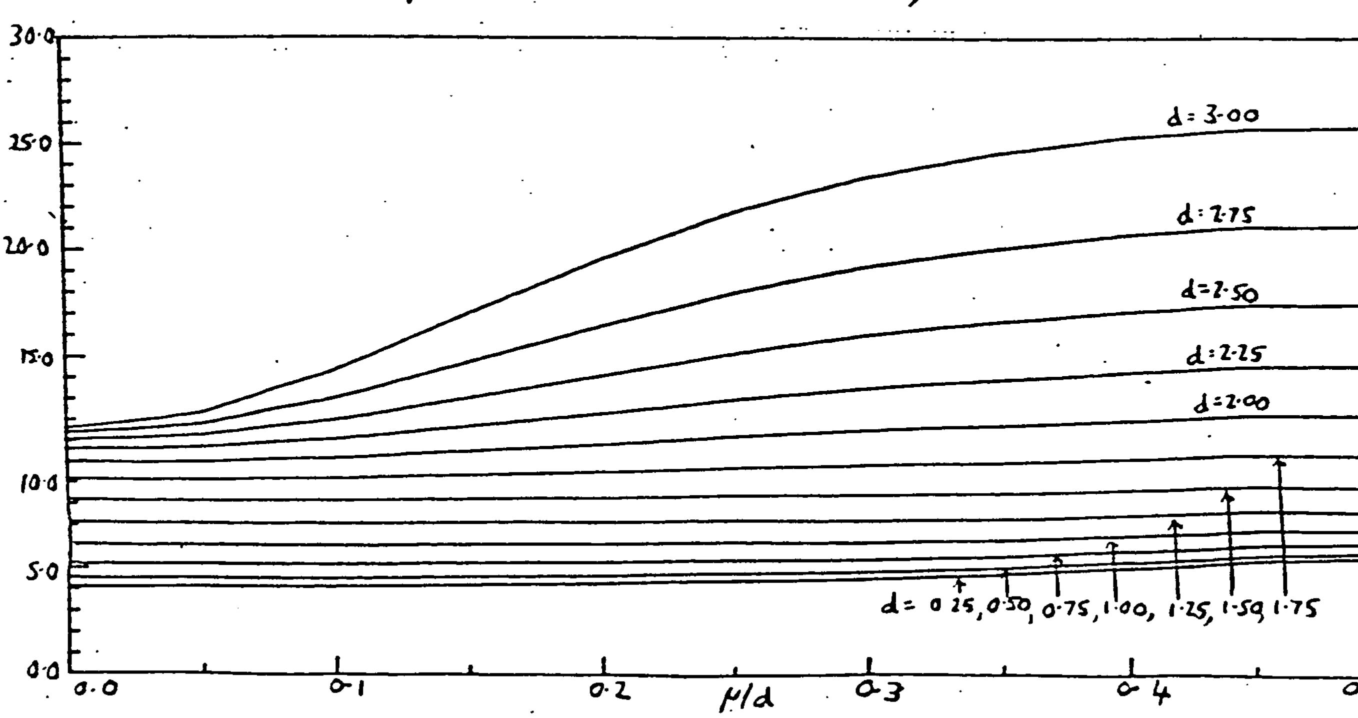


Fig. 3.1.7 $V(\mu)\beta^2$ averaged over μ=0.00(0.05)0.50, $\beta=\pi/3.0^{1/2}$.

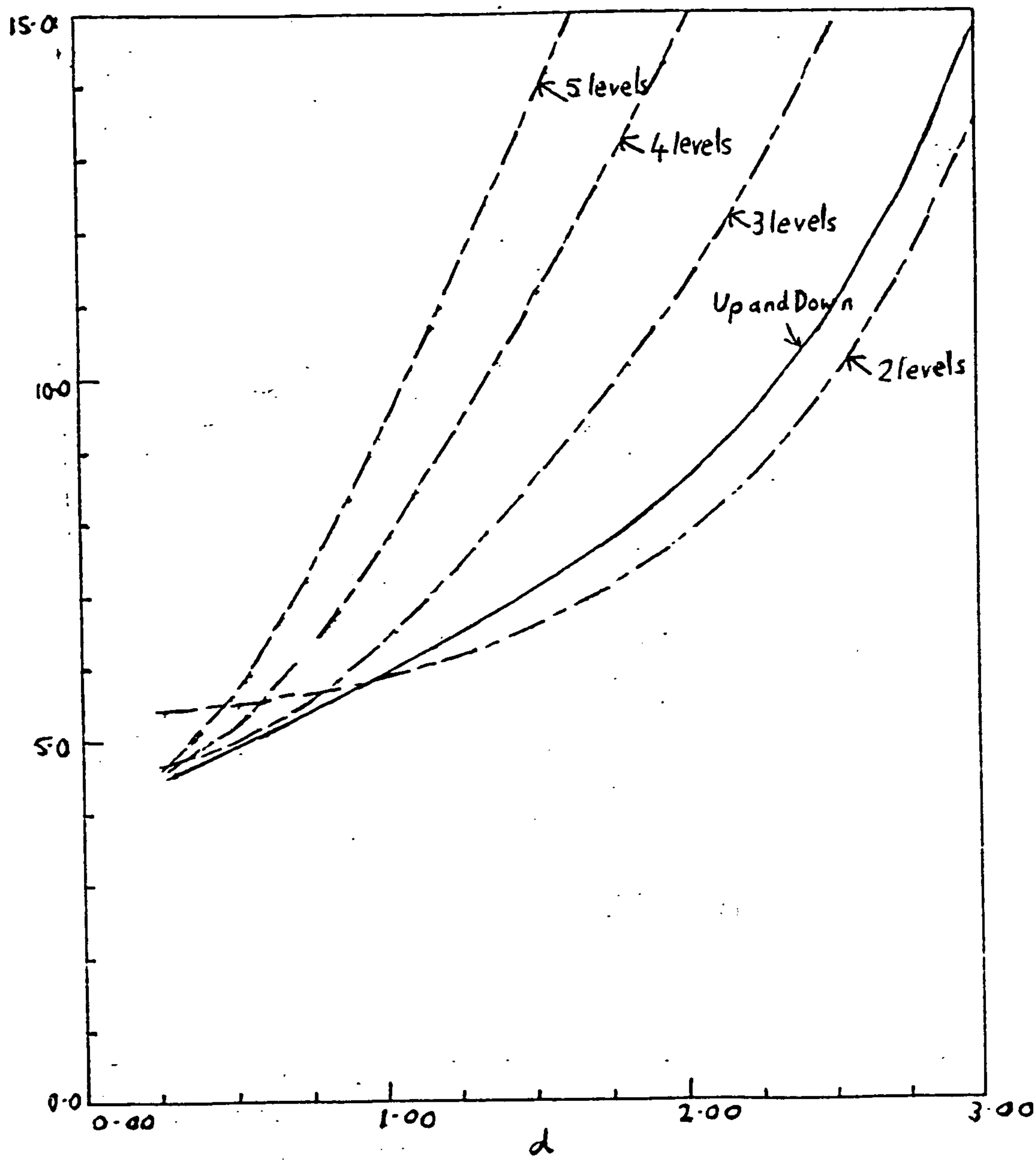
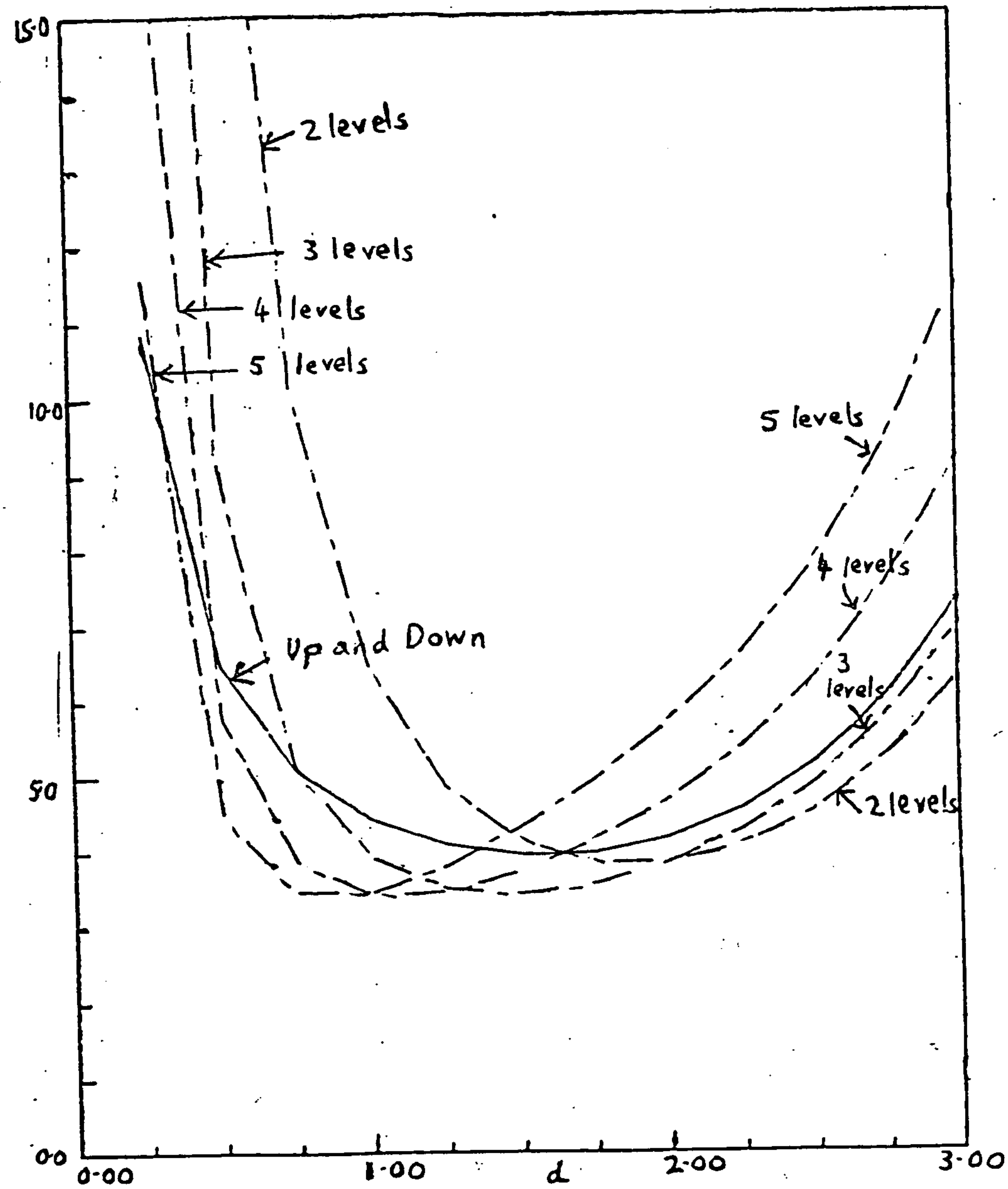


Fig. 3.1.8 $V(\beta)/\beta^2$ averaged over $\mu=0.00(0.05)0.50$, $\beta=\pi/3.0^{1/2}$



Figs. 3.1.7 and 3.1.8 illustrate values of $V(\mu)\beta^2$ and $V(\beta)/\beta^2$ after taking averages over the values of μ/d (i.e. μ/d equal to 0.00(0.05)0.50). For step sizes above 1.00 the average of $V(\mu)$ is lowest for the 2 point design and for step sizes below it is lowest for the Up and Down design. However any advantage of the 2 point design over the Up and Down design for the larger step sizes in estimating μ is small. The value of $V(\beta)$ for the 2 point design rises a long way above the value for the Up and Down design for small step sizes, so in the absence of a good initial estimate of β (and hence of relative step size) it seems reasonable to always use the Up and Down design. Although the value of $V(\beta)$ for the Up and Down design is sometimes some distance above the corresponding values for some non-sequential designs it is never considerably higher.

In Wetherill (1963) a lower bound is given for $V(\beta)/\beta^2$ of 2.28 (see Section 1.2). For step sizes of 1.00 to 2.25 the average of $V(\beta)/\beta^2$ for the Up and Down design is less than 5.00, though the Up and Down design is not close to fully efficient in estimating β it does have for a wide range of step sizes around 50 percent asymptotic efficiency. Even with the most efficient design for estimating β one would expect to make around 900 observations before the standard error of β is down to 5 percent of the magnitude of β. The Up and Down design undoubtedly will give unsatisfactory estimates of β in small samples but so will any other design.

3.2 A POSSIBLE ESTIMATOR OF $1/\beta$ FOR LOGISTIC RESPONSE

Dixon and Mood (1948), in addition to an approximate estimator of the parameter μ, gave an estimator for σ of an assumed underlying normal tolerance distribution. In Appendix 2 the steps taken to arrive at E_{DM} as an approximate estimator μ are given, the same approximations can be made for both the normal and logistic response curves. Suppose at level x_i that n_i positive and m_i negative responses have been recorded. If $d < 2\sigma$ the estimator for σ they suggest is

$$1.620((v/d)+0.029d), \qquad 3.2.1$$

where if positive responses are less frequent than negative

$$v = (\sum_i n_i x_i^2/\sum_i n_i) - (\sum_i n_i x_i/\sum_i n_i)^2, \qquad 3.2.2$$

and if negative responses are less frequent

$$v = (\sum_i m_i x_i^2/\sum m_i) - (\sum_i m_i x_i/\sum m_i)^2. \qquad 3.2.3$$

Using a theorem on page 87 of Chung (1960) it follows that the expression in Formula 3.2.2 converges with probability one to

$$\sum_i \Pi_i F_i x_i^2/\sum_i \Pi_i F_i - (\sum_i \Pi_i F_i x_i/\sum_i \Pi_i F_i)^2, \qquad 3.2.4$$

and that the expression in Formula 3.2.3 converges with probability one to

$$\sum_i \pi_i(1-F_i)x_i^2/\sum_i \pi_i(1-F_i) - (\sum_i \pi_i(1-F_i)x_i/\sum_i \pi_i(1-F_i))^2. \qquad 3.2.5$$

It is easy to show that these two limits are equal. So v converges with probability one to the expression in 3.2.4.

The estimator of σ given in Formula 3.2.1 was suggested for the normal response curve because, if this limit is substituted for v in the formula, then the resulting expression is very close to σ providing $\sigma/d > 0.5$. For the logistic response curve the limit is

$$\sum_i (x_i-\mu_o)^2 \exp(-\beta(x_i-\mu-(d/2))^2/2d)/\sum_i \exp(-\beta(x_i-\mu-(d/2))^2/2d), \qquad 3.2.6$$

where μ_o is the asymptotic expectation of the Dixon and Mood estimator. In Appendix 7 I show that as d tends to zero the expression in 3.2.6 divided by d tends to $1/\beta$. So the limit in probability of v/d is arbitrarily close to $1/\beta$ for sufficiently small d. The estimator of $1/\beta$ that I suggest is $1/\tilde{\beta}$ where

$$1/\tilde{\beta} = v/d. \qquad 3.2.7$$

The limit with probability one of $1/\tilde{\beta}$ is in fact very close to $1/\beta$ for $\beta d < 2.0$. In Table 3.2.1 limits for $1/\tilde{\beta}d$ are given for βd equal to 1.75(0.25)4.00 and μ/d equal to 0.0(0.1)0.5 and 0.25. Calculations were also made for βd equal to 0.25(0.25)1.50, then the biases to 3 decimals were zero. This estimator has been derived in a similar way to the estimator of σ given by Dixon and

Table 3.2.1 Limits with probability one of $1/\tilde{\beta}d$.

βd	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00
$1/\beta d$	0.571	0.500	0.444	0.400	0.364	0.333	0.308	0.286	0.267	0.250
μ/d										
0.00	0.572	0.501	0.447	0.405	0.372	0.346	0.325	0.309	0.296	0.286
0.10	0.572	0.501	0.446	0.404	0.370	0.343	0.322	0.304	0.290	0.279
0.20	0.572	0.500	0.445	0.401	0.366	0.337	0.313	0.293	0.275	0.261
0.25	0.571	0.500	0.444	0.400	0.364	0.333	0.308	0.286	0.266	0.249
0.30	0.571	0.500	0.444	0.399	0.361	0.330	0.302	0.279	0.257	0.239
0.40	0.571	0.499	0.442	0.396	0.357	0.324	0.294	0.267	0.243	0.221
0.50	0.571	0.499	0.422	0.395	0.356	0.322	0.291	0.263	0.238	0.215

Table 3.2.2 Mean and m.s.e of $1/\tilde{\beta}$ in 12 step experiments together with asymptotic predicted variance of the m.l.e. of $1/\beta$ ($\beta=\pi/3.0^{1/2}$; i.e. $1/\beta=0.5513$ to 4 decimals).

	Starting level				
	0.0	1.0	2.0	3.0	4.0
Step size 0.5					
mean	0.300	0.373	0.423	0.405	0.274
m.s.e.	0.110	0.126	0.242	0.409	0.438
Asympt. Var. m.l.e.	0.164	0.164	0.164	0.164	0.164
Step size 1.0					
mean	0.374	0.415	0.422	0.419	0.401
m.s.e.	0.086	0.093	0.113	0.142	0.161
Asympt. Var. m.l.e.	0.114	0.114	0.114	0.114	0.114
Step size 1.5					
mean	0.420	0.411	0.451	0.439	0.397
m.s.e.	0.069	0.091	0.097	0.095	0.120
Asympt. Var. m.l.e.	0.116	0.092	0.092	0.116	0.092
Step size 2.0					
mean	0.477	0.379	0.488	0.421	0.477
m.s.e.	0.051	0.111	0.067	0.122	0.073
Asympt. Var. m.l.e.	0.152	0.070	0.152	0.070	0.152

Mood and will be used to give starting values for iterations to m.l.e.'s in the simulations of the next section. I calculated $1/\tilde{\beta}$ in the experiments described in Section 2.2. The actual value of $1/\beta$ in these experiments is $3^{1/2}/\pi$ which equals 0.5513 to 4 decimals. Often there were marked negative biases in the estimates. Some of the results are given in Table 3.2.2. The results are not encouraging as the m.s.e.'s are very high given the actual magnitude of $1/\beta$. However for many starting values the m.s.e.'s are lower than the asymptotic predicted variances of the m.l.e. of $1/\beta$.

From Appendix 7 it follows that for the normal response curve the limit with probability one of v/d is, for sufficiently small d, arbitrarily close to $(\pi/8)^{1/2}$. This suggests an estimator for σ of

$$(8/\pi)^{1/2} v/d. \qquad 3.2.8$$

The value of $(8/\pi)^{1/2}$ is 1.596 to 3 decimals. This estimator is close to that suggested by Dixon and Mood for a wide range of values of v/d.

3.3 RESULTS OF SOME SIMULATIONS

In Section 2.2 m.s.e.'s and means of various estimators of μ are calculated for 12 step experiments. The sample size is too small to make a valid comparison of these estimators with the maximum likelihood estimator (the probability that the maximum likelihood equations have a degenerate solution is always high). Also the sample size is so small that no useful estimates of slope or scale could be expected. Beyond 12 steps it becomes rapidly less practicable to calculate m.s.e.'s and biases of estimators by looking at each possible outcome. Except for E_M and E_B, no recursive formulae exist for calculating biases and m.s.e.'s. I investigated the small sample properties of estimators for larger numbers of steps by means of simulation.

I first simulated 24 step experiments. Again I restricted attention to a logistic response curve and set β equal to $\pi/3.0^{1/2}$. As before I considered step sizes of 0.5(0.5)2.0 and starting levels were set at 0.00(0.25)4.00 relative to μ. For each set of conditions 2000 simulations were made. Tables 3.3.1 to 3.3.4 give values of m.s.e.'s of E_M, E_B, E_{BD}, E_{DM}, E_{WE}, E_{PV}, E^*_{WE} and E^*_{PV}. Tables 3.3.5 to 3.3.8 give corresponding values for bias. E_M, E_B, E_{BD} and E_{DM} are asymptotically equivalent, as are E_{WE} and E_{PV} and also E^*_{WE} and E^*_{PV}. These tables contain asymptotic predicted values of m.s.e. and bias. Values of m.s.e. for E_M, E_B, E_{BD} and E_{DM} are also illustrated in Figs. 3.3.1 to 3.3.4; values for E_{WE} and E^*_{WE} are illustrated in Figs. 3.3.5 to 3.3.8 (for the sake

Table 3.3.1 100×m.s.e. of estimators in 24 step experiments for step size 0.5 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E^*_{WE}	E^*_{PV}	A^*_{WE}
0.00	5.09	5.30	5.27	5.35	6.27	5.65	5.83	6.65	5.41	5.49	6.30
0.25	5.29	5.49	5.50	5.63	6.27	5.92	6.07	6.65	5.54	5.60	6.30
0.50	5.56	5.68	5.79	6.06	6.27	6.30	6.43	6.65	5.78	5.81	6.30
0.75	5.87	5.75	6.01	6.39	6.27	6.61	6.69	6.65	6.11	6.11	6.30
1.00	6.40	5.87	6.23	6.75	6.27	6.89	6.94	6.65	6.45	6.41	6.30
1.25	7.34	6.30	6.71	7.14	6.27	7.57	7.56	6.65	6.92	6.84	6.30
1.50	8.57	6.95	7.31	7.70	6.27	8.41	8.38	6.65	7.48	7.38	6.30
1.75	10.22	7.70	7.65	7.92	6.27	8.65	8.58	6.65	7.93	7.79	6.30
2.00	12.38	8.71	7.90	8.08	6.27	8.91	8.81	6.65	8.18	8.03	6.30
2.25	15.54	10.56	8.61	8.58	6.27	9.77	9.63	6.65	8.77	8.60	6.30
2.50	19.12	12.59	9.02	8.99	6.27	10.55	10.41	6.65	9.14	8.97	6.30
2.75	24.00	15.45	9.15	9.07	6.27	10.55	10.39	6.65	9.34	9.13	6.30
3.00	29.85	18.98	9.18	9.04	6.27	10.49	10.33	6.65	9.34	9.14	6.30
3.25	37.53	23.99	9.66	9.48	6.27	11.13	10.91	6.65	9.80	9.60	6.30
3.50	46.27	29.70	10.07	9.86	6.27	11.91	11.70	6.65	10.09	9.87	6.30
3.75	57.25	37.11	10.57	10.30	6.27	12.37	12.16	6.65	10.69	10.45	6.30
4.00	70.79	46.45	10.69	10.46	6.27	12.48	12.28	6.65	10.90	10.65	6.30

Table 3.3.2 100×m.s.e. of estimators in 24 step experiments for step size 1.0 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E^*_{WE}	E^*_{PV}	A^*_{WE}
0.00	6.84	7.04	6.91	7.03	7.60	7.46	7.86	8.16	7.30	7.43	7.88
0.25	7.04	7.27	7.17	7.28	7.60	7.52	7.88	7.95	7.39	7.54	7.63
0.50	6.80	7.06	7.02	7.14	7.60	7.38	7.68	7.75	6.98	7.12	7.38
0.75	6.95	7.16	7.23	7.38	7.60	7.85	8.04	7.95	7.09	7.23	7.63
1.00	7.17	7.26	7.46	7.78	7.60	8.30	8.48	8.16	7.41	7.52	7.88
1.25	7.77	7.58	7.94	8.42	7.60	8.87	8.97	7.95	7.96	7.99	7.63
1.50	7.67	7.20	7.71	8.22	7.60	8.43	8.57	7.75	7.72	7.66	7.38
1.75	8.12	7.18	7.82	8.24	7.60	8.44	8.57	7.95	8.01	7.88	7.63
2.00	8.67	7.13	7.74	8.06	7.60	8.63	8.72	8.16	8.21	8.03	7.88
2.25	9.59	7.54	8.17	8.38	7.60	8.88	8.93	7.95	8.55	8.41	7.63
2.50	10.44	7.69	8.13	8.26	7.60	8.98	8.99	7.75	8.22	8.13	7.38
2.75	12.12	8.33	8.37	8.47	7.60	9.56	9.49	7.95	8.38	8.30	7.63
3.00	13.72	8.82	8.57	8.73	7.60	9.95	9.92	8.16	8.59	8.53	7.88
3.25	16.30	9.94	9.00	9.34	7.60	10.40	10.34	7.95	9.08	8.96	7.63
3.50	18.36	10.55	8.84	9.25	7.60	9.87	9.89	7.75	8.93	8.76	7.38
3.75	21.79	12.06	8.99	9.36	7.60	10.01	10.05	7.95	9.33	9.08	7.63
4.00	25.57	13.76	8.81	9.07	7.60	10.03	10.08	8.16	9.38	9.11	7.88

Note: A_{DM}, A_{WE} and A^*_{WE} denote columns for asymptotic predicted m.s.e.'s of E_{DM}, E_{WE} and E^*_{WE} respectively

Table 3.3.3 100xm.s.e. of estimators in 24 step experiments for step size 1.5 ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_{β}	$E_{\beta D}$	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E^*_{WE}	E^*_{PV}	A^*_{WE}
0.00	8.52	8.68	8.54	8.62	9.17	10.19	10.89	11.22	9.89	9.95	11.10
0.25	8.80	9.02	8.88	8.91	9.11	9.88	10.58	10.32	9.68	9.79	10.12
0.50	8.40	8.68	8.60	8.59	8.99	8.43	9.03	8.55	8.25	8.42	8.21
0.75	8.18	8.51	8.48	8.47	8.93	7.88	8.32	7.68	7.60	7.83	7.28
1.00	8.44	8.73	8.77	8.88	8.99	8.90	9.15	8.55	8.15	8.38	8.21
1.25	8.58	8.81	8.98	9.11	9.11	10.40	10.49	10.32	9.26	9.49	10.12
1.50	8.85	8.85	9.17	9.44	9.17	11.21	11.33	11.22	9.93	10.16	11.10
1.75	9.05	8.81	9.26	9.64	9.11	10.74	10.82	10.32	9.65	9.87	10.12
2.00	9.21	8.62	9.20	9.71	8.99	9.58	9.73	8.55	8.71	8.85	8.21
2.25	9.84	8.54	9.21	9.82	8.93	9.04	9.19	7.68	8.47	8.37	7.28
2.50	10.26	8.49	9.28	9.85	8.99	9.60	9.83	8.55	9.04	8.78	8.21
2.75	10.74	8.34	9.13	9.56	9.11	10.28	10.51	10.32	9.95	9.57	10.12
3.00	11.81	8.65	9.31	9.64	9.17	11.38	11.64	11.22	10.82	10.41	11.10
3.25	12.60	9.10	9.89	10.03	9.11	11.28	11.59	10.32	10.87	10.59	10.12
3.50	13.65	9.25	9.73	9.74	8.99	10.03	10.30	8.55	9.63	9.48	8.21
3.75	15.03	9.51	9.48	9.41	8.93	9.28	9.40	7.68	8.79	8.74	7.28
4.00	17.11	10.25	9.76	9.70	8.99	10.44	10.43	8.55	9.29	9.32	8.21

Table 3.3.4 100xm.s.e. of estimators in 24 step experiments for step size 2.0 ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_{β}	$E_{\beta D}$	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E^*_{WE}	E^*_{PV}	A^*_{WE}
0.00	11.24	11.37	11.26	11.27	11.47	15.13	16.22	16.59	14.63	14.65	16.55
0.25	11.27	11.39	11.28	11.33	11.27	14.34	15.55	15.13	13.96	14.00	14.99
0.50	10.13	10.36	10.26	10.22	10.73	10.85	12.06	11.59	10.70	10.80	11.31
0.75	9.45	9.84	9.75	9.60	10.11	8.23	9.25	8.00	8.22	8.40	7.66
1.00	8.94	9.36	9.30	9.19	9.83	6.92	7.41	6.50	6.93	7.18	6.16
1.25	9.38	9.76	9.77	9.74	10.11	8.38	8.40	8.00	8.05	8.34	7.66
1.50	10.22	10.27	10.39	10.63	10.73	11.81	11.59	11.59	10.58	10.78	11.31
1.75	11.06	11.12	11.39	11.57	11.27	15.16	14.85	15.13	13.41	13.67	14.99
2.00	11.14	11.08	11.51	11.79	11.47	16.17	16.23	16.59	14.16	14.57	16.55
2.25	11.06	11.14	11.76	11.91	11.27	15.07	15.21	15.13	13.51	14.18	14.99
2.50	10.58	10.39	11.21	11.40	10.73	12.26	12.47	11.59	11.11	11.86	11.31
2.75	10.84	9.80	10.72	11.11	10.11	9.62	9.82	8.00	8.90	9.37	7.66
3.00	11.65	9.35	10.23	10.82	9.83	8.16	8.36	6.50	7.76	7.74	6.16
3.25	12.99	9.64	10.44	11.19	10.11	9.28	9.44	8.00	9.15	8.66	7.66
3.50	14.91	10.49	11.21	11.98	10.73	12.62	12.82	11.59	12.18	11.38	11.31
3.75	15.91	10.79	11.42	12.04	11.27	15.02	15.29	15.13	14.44	13.59	14.99
4.00	16.74	11.33	12.12	12.66	11.47	16.69	17.19	16.59	15.99	15.30	16.55

Table 3.3.5 100×bias of estimators in 24 step experiments for step size 0.5 ($\beta = \pi / 3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E^*_{WE}	E^*_{PV}	A_{WE}
0.00	-0.36	-0.43	-0.43	-0.30	0.00	-0.25	-0.24	-0.39	-0.39	0.00
0.25	2.29	1.22	1.30	2.05	0.00	1.99	1.77	1.46	1.26	0.00
0.50	4.52	2.37	2.38	3.47	0.00	3.40	3.02	2.91	2.56	0.00
0.75	7.90	4.70	4.35	5.36	0.00	5.56	5.01	5.22	4.73	0.00
1.00	11.09	6.86	5.65	6.51	0.00	7.14	6.47	6.56	5.98	0.00
1.25	14.91	9.68	7.19	7.71	0.00	8.92	8.12	8.06	7.39	0.00
1.50	18.16	11.84	7.45	7.79	0.00	9.45	8.59	8.46	7.73	0.00
1.75	22.66	15.29	8.54	8.60	0.00	10.49	9.55	9.64	8.87	0.00
2.00	27.47	19.07	9.42	9.32	0.00	11.59	10.60	10.41	9.62	0.00
2.25	32.60	23.20	10.03	9.72	0.00	12.26	11.20	10.86	10.04	0.00
2.50	37.69	27.21	9.75	9.57	0.00	12.28	11.21	10.70	9.85	0.00
2.75	44.02	32.49	10.48	10.20	0.00	12.83	11.71	11.54	10.67	0.00
3.00	50.55	37.98	10.85	10.52	0.00	13.40	12.25	11.85	10.98	0.00
3.25	57.68	44.12	11.33	10.91	0.00	14.14	12.92	12.22	11.33	0.00
3.50	64.85	50.20	11.05	10.75	0.00	13.99	12.77	12.02	11.08	0.00
3.75	72.88	57.18	11.27	10.94	0.00	14.10	12.83	12.36	11.41	0.00
4.00	81.80	65.06	12.14	11.84	0.00	15.24	13.93	13.25	12.30	0.00

Table 3.3.6 100×bias of estimators in 24 step experiments for step size 1.0 ($\beta = \pi / 3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E^*_{WE}	E^*_{PV}	A_{WE}
0.00	-0.28	-0.33	-0.36	-0.26	0.00	-0.24	-0.23	-0.34	-0.34	0.00
0.25	0.48	-0.48	-0.30	0.56	-0.01	0.66	0.16	-0.19	-0.39	-0.23
0.50	1.93	0.08	0.41	1.77	0.00	1.70	0.87	0.55	0.09	0.00
0.75	4.17	1.20	1.51	3.31	0.01	3.05	1.95	2.05	1.26	0.23
1.00	5.47	1.20	1.29	3.29	0.00	2.78	1.62	2.28	1.16	0.00
1.25	7.82	2.35	2.01	3.82	-0.01	3.34	2.15	3.42	2.05	-0.23
1.50	9.18	2.56	1.48	2.84	0.00	2.92	1.66	3.28	1.78	0.00
1.75	11.63	4.05	1.96	2.87	0.01	3.72	2.29	3.89	2.38	0.23
2.00	14.67	6.27	2.89	3.63	0.00	4.64	2.97	4.60	3.22	0.00
2.25	16.90	7.60	2.59	3.33	-0.01	5.02	3.16	4.12	2.82	-0.23
2.50	20.05	9.86	3.05	3.93	0.00	5.60	3.67	4.31	3.00	0.00
2.75	24.01	12.70	3.80	5.04	0.01	6.35	4.42	5.16	3.73	0.23
3.00	27.11	14.51	3.05	4.57	0.00	5.46	3.65	4.66	3.06	0.00
3.25	31.22	17.42	3.26	4.76	-0.01	5.27	3.57	5.19	3.44	-0.23
3.50	34.68	19.73	2.55	3.68	0.00	4.50	2.85	4.72	2.92	0.00
3.75	39.35	23.44	3.11	3.89	0.01	5.29	3.56	5.32	3.59	0.23
4.00	44.23	27.50	3.75	4.48	0.00	5.92	3.99	5.76	4.21	0.00

Note: A_{DM} and A_{WE} denote columns for asymptotic predicted biases of E_{DM} and E_{WE} respectively.

Table 3.3.7 100×bias of estimators in 24 step experiments for step size 1.5 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_β	$E_{\beta D}$	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
0.00	-0.29	-0.24	-0.28	-0.32	0.00	-0.40	-0.39	-0.45	-0.46	0.00
0.25	-0.89	-1.43	-1.29	-0.81	-0.42	-1.90	-2.79	-2.34	-2.43	-2.40
0.50	0.62	-0.65	-0.36	0.74	-0.42	-0.77	-2.24	-1.43	-1.65	-2.37
0.75	2.00	-0.23	0.21	1.95	0.00	1.63	-0.31	0.56	0.04	0.00
1.00	3.67	0.19	0.65	3.09	0.42	3.77	1.64	2.44	1.49	2.37
1.25	5.61	0.82	1.17	4.01	0.42	4.59	2.40	3.48	2.06	2.40
1.50	6.72	0.46	0.46	3.59	0.00	2.56	0.51	2.21	0.33	0.00
1.75	8.42	0.56	0.10	3.15	-0.42	0.83	-0.97	1.18	-1.13	-2.40
2.00	10.16	0.84	-0.38	2.32	-0.42	0.29	-1.31	0.95	-1.68	-2.37
2.25	13.01	2.59	0.49	2.64	0.00	2.52	0.98	3.19	0.45	0.00
2.50	14.64	3.41	0.26	1.63	0.42	3.47	1.75	4.01	1.35	2.37
2.75	16.90	5.00	0.60	1.38	0.42	3.92	1.89	4.36	1.93	2.40
3.00	19.68	7.22	1.40	1.92	0.00	3.48	1.03	3.59	1.51	0.00
3.25	20.86	7.82	0.29	0.76	-0.42	1.71	-1.16	1.25	-0.49	-2.40
3.50	23.93	10.16	0.82	1.55	-0.42	2.01	-1.04	1.28	-0.21	-2.37
3.75	27.35	12.62	1.31	2.62	0.00	3.93	0.76	2.82	1.37	0.00
4.00	30.89	14.92	1.44	3.52	0.42	5.59	2.49	4.11	2.46	2.37

Table 3.3.8 100×bias of estimators in 24 step experiments for step size 2.0 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_β	$E_{\beta D}$	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
0.00	-0.42	-0.23	-0.24	-0.52	0.00	-0.56	-0.59	-0.68	-0.71	0.00
0.25	-2.44	-2.69	-2.59	-2.39	-2.14	-5.88	-7.10	-5.90	-5.92	-6.93
0.50	-2.49	-3.28	-3.09	-2.32	-3.00	-7.36	-9.40	-7.35	-7.39	-9.44
0.75	-1.59	-2.98	-2.66	-1.45	-2.11	-4.85	-7.66	-5.22	-5.40	-6.44
1.00	2.13	-0.06	0.34	2.12	0.00	1.54	-1.68	0.74	0.30	0.00
1.25	5.13	1.73	2.22	4.72	2.11	7.26	3.81	5.91	5.03	6.44
1.50	7.32	2.40	2.90	6.19	3.00	10.00	6.56	8.29	6.83	9.44
1.75	8.67	2.11	2.46	6.27	2.14	8.76	5.47	7.40	5.35	6.93
2.00	8.11	-0.30	-0.37	3.92	0.00	2.68	-0.23	2.27	-0.37	0.00
2.25	8.76	-1.50	-2.19	2.21	-2.14	-2.77	-5.11	-2.17	-5.36	-6.93
2.50	9.92	-2.23	-3.73	0.59	-3.00	-5.60	-7.40	-4.58	-8.23	-9.44
2.75	12.66	-0.93	-3.28	0.59	-2.11	-3.52	-5.06	-2.49	-6.39	-6.44
3.00	16.41	1.87	-1.39	1.84	0.00	1.44	-0.04	2.33	-1.62	0.00
3.25	19.54	4.17	-0.14	2.30	2.11	6.21	4.62	6.68	2.77	6.44
3.50	22.48	6.62	1.29	2.83	3.00	8.74	6.73	8.89	5.20	9.44
3.75	24.49	8.37	1.75	2.57	2.14	7.79	5.18	7.86	4.55	6.93
4.00	25.10	8.62	0.38	0.66	0.00	2.93	-0.30	2.97	0.17	0.00

Fig. 3.3.1 M.s.e.'s of various estimators of μ in 2000 simulated 24 step experiments with step size 0.5 ($\beta=\pi/3.0^{1/2}$).

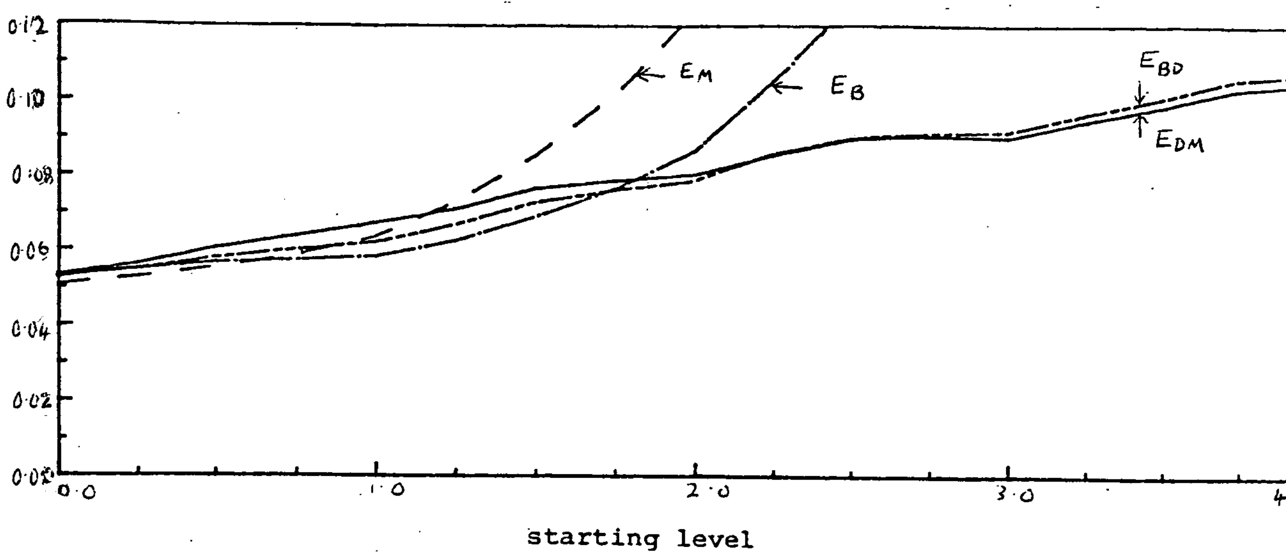


Fig. 3.3.2 As in Fig. 3.3.1 only with step size 1.0.

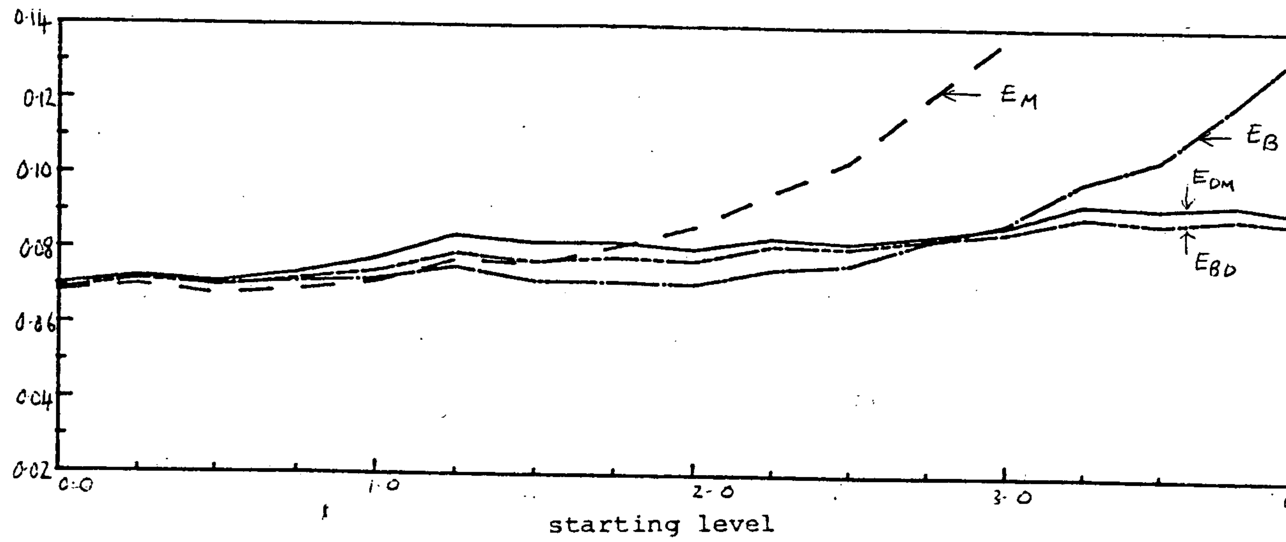


Fig. 3.3.3 As in Fig. 3.3.1 only with step size 1.5.

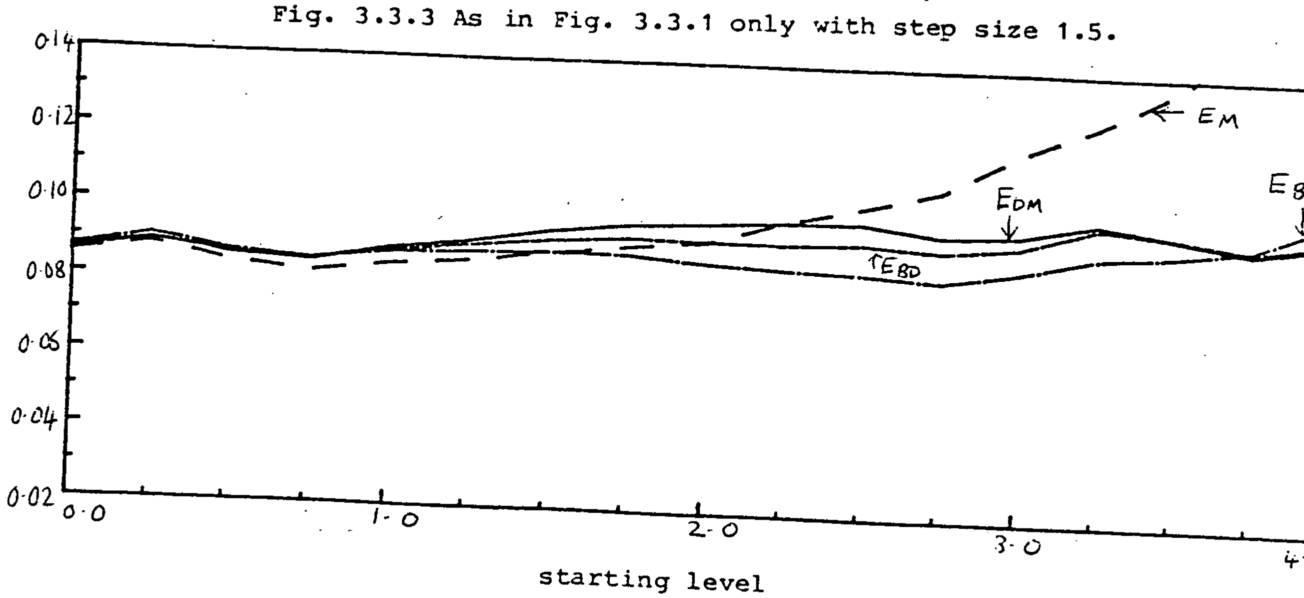


Fig. 3.3.4 As in Fig. 3.3.1 only with step size 2.0.

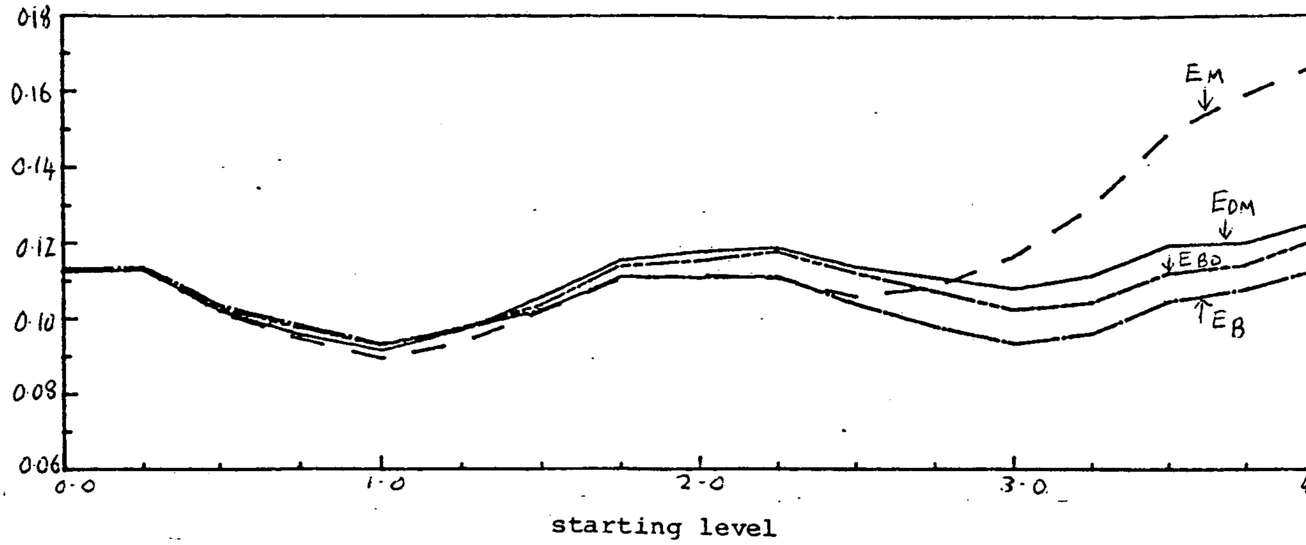


Fig. 3.3.5 M.s.e.'s of E_{WE} and E_{WE}^* in 2000 simulated 24 step experiments with step size 0.5 ($\beta = \pi/3.0^{1/2}$).

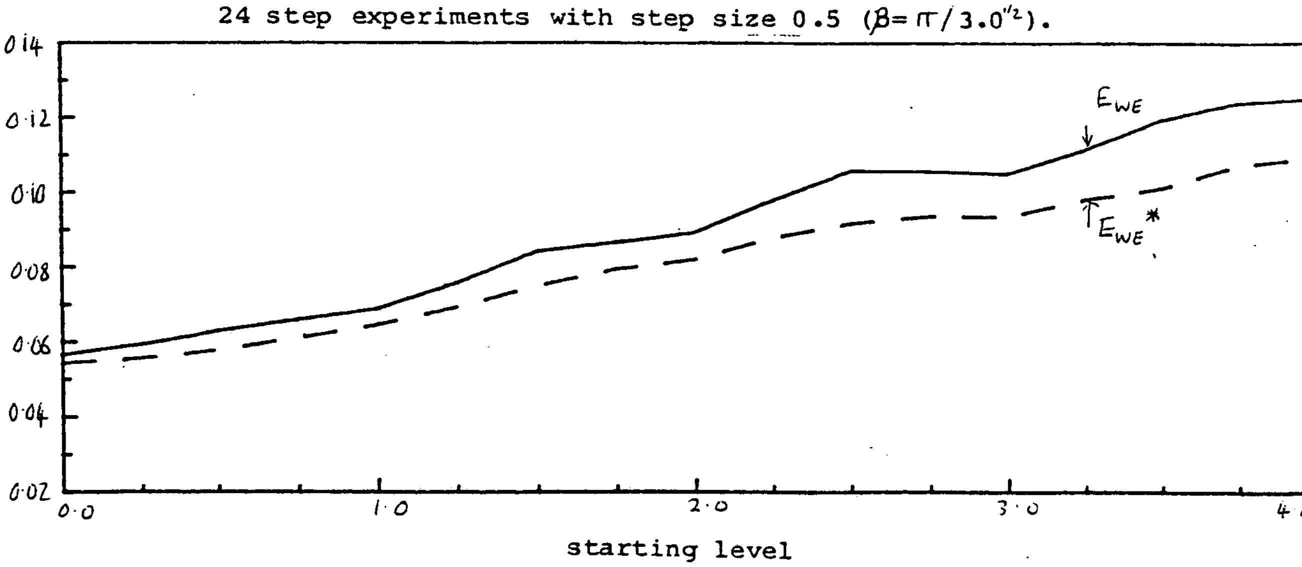


Fig. 3.3.6 As in Fig. 3.3.5 only with step size 1.0

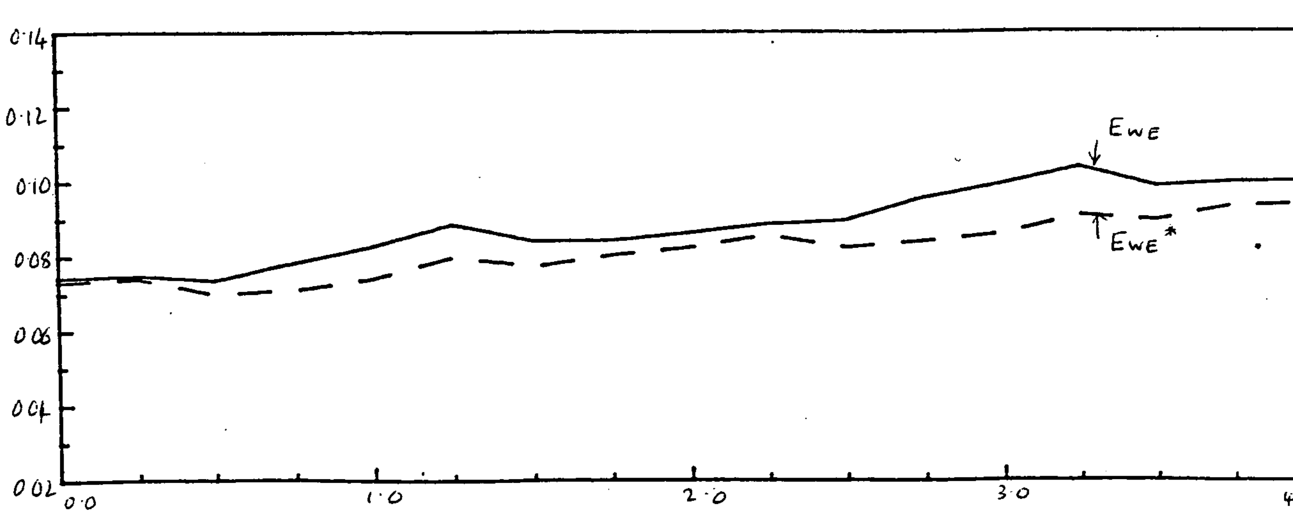


Fig. 3.3.7 As in Fig. 3.3.5 only with step size 1.5.

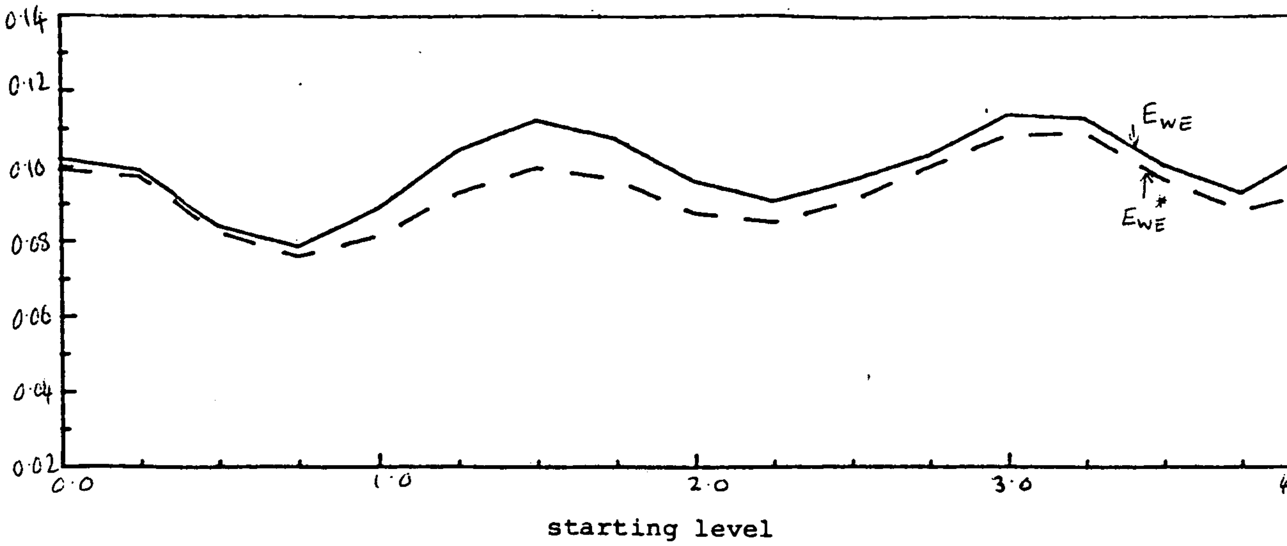
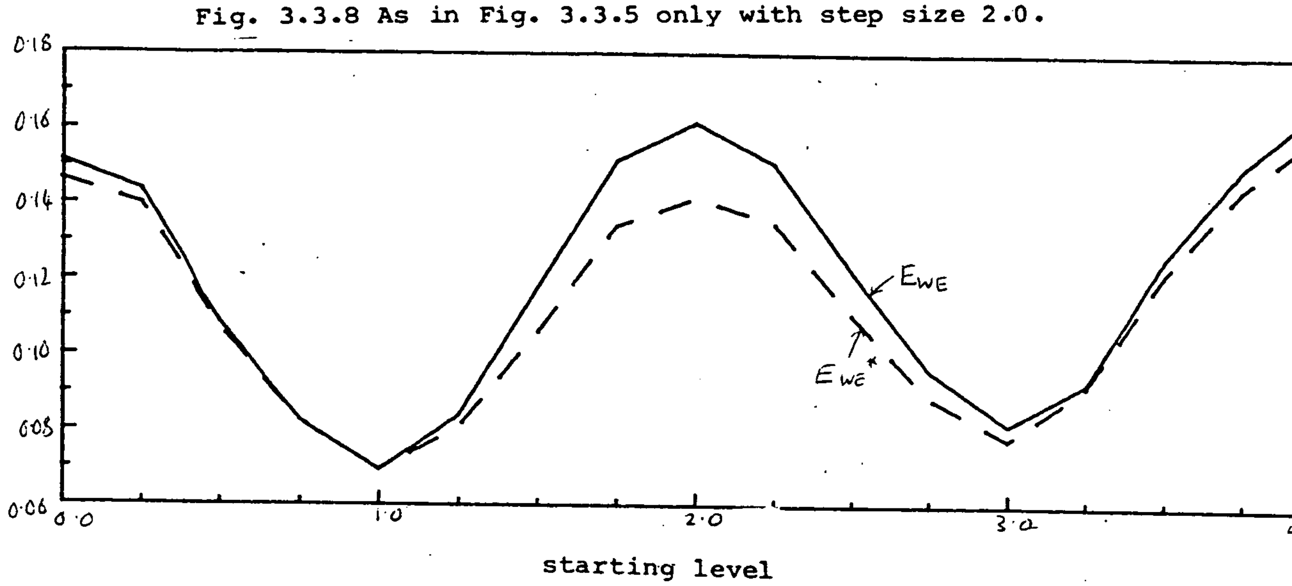


Fig. 3.3.8 As in Fig. 3.3.5 only with step size 2.0.



of clarity values for E_{PV} and E^*_{PV} are not illustrated on these graphs as they are usually very close to values for E_{WE} and E^*_{WE}).

The figures illustrate very clearly the relations between these estimators. These are much the same as for 12 step experiments (in Figs. 2.2.1 to 2.2.8 analogous values of m.s.e. for 12 steps are illustrated). One point to notice is that with just one exception the m.s.e. of E_{WE} is greater than that of E^*_{WE} (for step size 2.00 and starting level 1.00 the m.s.e. of E_{WE} is 0.0692 and that of E^*_{WE} 0.0693). The m.s.e. of E_{PV} is always greater than that of E^*_{PV}. For step size 0.5 the m.s.e.'s of E_{DM} and E^*_{WE} are close (as in the 12 step experiments) and biases are roughly similar. So again a close connection between E_{DM} and E_{WE} is apparent from the very similar behaviour of E_{DM} and the alternative to E_{WE} of E^*_{WE}. As step size is increased this connection breaks down. For step size 2.00 the m.s.e.'s of E_{WE}, E^*_{WE}, E_{PV} and E^*_{PV} are oscillating above and below corresponding values for E_{DM}. This is what one would expect from asymptotic theory.

Among the estimators E_{WE}, E^*_{WE}, E_{PV}, and E^*_{PV} it seems again sensible to use E^*_{WE} or E^*_{PV}. As in 12 step experiments E^*_{WE} and E^*_{PV} have similar m.s.e.'s. The mean of E^*_{PV} is always less than that of E^*_{WE} and often E^*_{PV} has smaller bias. E_{PV} often has m.s.e above that of E_{WE} but smaller bias.

The relations between E_M ,E_B , E_{BD} and E_{DM} are much the same as in the 12 step experiments. Again E_M and E_B have high m.s.e.'s and large biases for step size 0.5 and distant starting levels. The estimators E_{BD} and E_{DM} have similar m.s.e.'s and are more robust than the other two estimators against bias due to bad starting levels. In the 12 step experiments E_{BD} had a very obvious advantage over E_{DM} in that E_{BD} almost always had smaller bias. In these 24 step experiments E_{BD} often has smaller bias than E_{DM} but biases due to bad starting levels are in any case less pronounced. As E_{BD} and E_{DM} are asymptotically equivalent it is not surprising that in the experiments with larger numbers of observations there is less to choose between them.

I used E_{DM} and $1/\tilde{\beta}$ to give starting values of μ and $1/\beta$ for iteration to maximum likelihood estimates (recall that $1/\tilde{\beta}$ is the approximate estimator of $1/\beta$ described in the previous section). Newton-Raphson iterations were performed in terms of parameters a and β ($a=-\mu/\beta$) with the function to be maximised being the log. likelihood. After each Newton-Raphson step I formed provisional estimates of μ and $1/\beta$. Iterations were terminated when the difference in both these estimates before and after a step was less than 0.5×10^{-4} (this is an arbitrary criterion but appeared reasonable considering the magnitude of standard errors of estimators). I did not start iteration when a degenerate response curve fitted the observed responses (this happens if there has been only one type of response or if after the first reversal of sign of response only two or three levels are subsequently visited). I also stopped iterations if the determinant of the matrix to be inverted in each iteration became less than 10^{-8} in magnitude (typically in such cases one change in response would allow the responses to be fitted by a degenerate curve). Unfortunately the numbers of experiments for which iterations were either not started or terminated before the convergence criterion was satisfied are large for the larger step sizes. In Fig. 3.3.9 the number of experiments for which estimates could be formed is plotted against starting level. For most of the experiments that did not give estimates the likelihood equations had a degenerate solution. It is clear that for the larger step sizes the number of experiments discarded reaches maxima for phasing of 0.0 and minima for phasing of 0.5. This is not surprising as if phasing is close to 0.0 one level is close to μ but the the two adjacent levels are about one step size distant from μ and so if the step size is large the probability of there being only one level of mixed response in small sample experiments is high; if phasing is close to 0.5 there are two levels which are only about half a step size distant from μ and providing the step size is not very large the probability of there being only at most one level of mixed response is much reduced. Only a small proportion of the simulated experiments were discarded because the determinant of the matrix to be inverted became too small (for step size 0.5 the highest number of such discards at any level is only 20, for step size 1.0 it is 8, for step size 1.5 it is 23 and for step step size 2.0 it is 41). So providing m.l.e.'s exist there are relatively few problems encountered in the iterations. When there is convergence the average number of iterative steps taken is always between 3 and 4. Fig. 3.3.10 is a plot of the m.s.e.'s of E_{DM} and $\hat{\mu}$, the m.l.e. of μ, for step size 0.5 (it must be remembered that these m.s.e.'s are not directly comparable as in the simulations E_{DM} always existed

Fig. 3.3.9 Numbers of 24 observation experiments out of 2000 which are not discarded in the course of attempts to find m.l.e.'s.

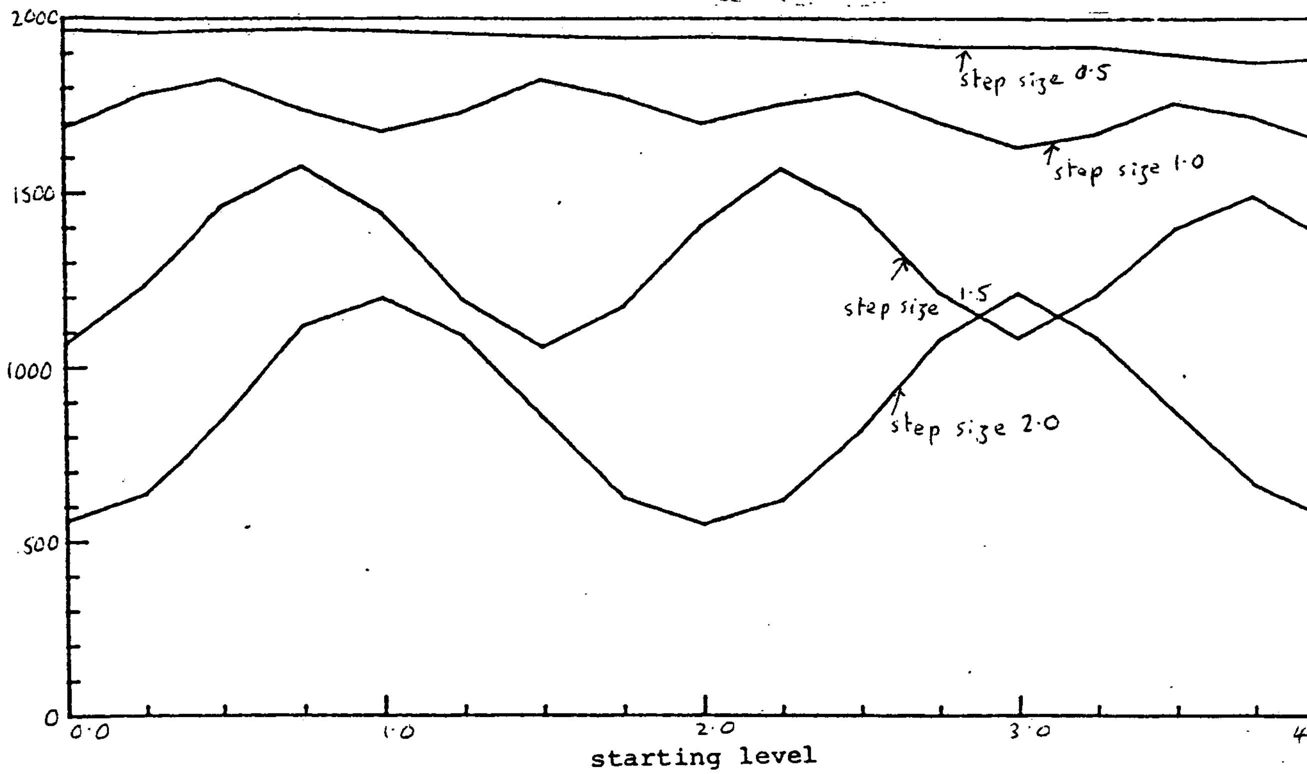


Fig. 3.3.10 M.s.e.'s of E_{DM} and $\hat{\mu}$ (the m.l.e. of μ) in 24 step experiments with step size 0.5.

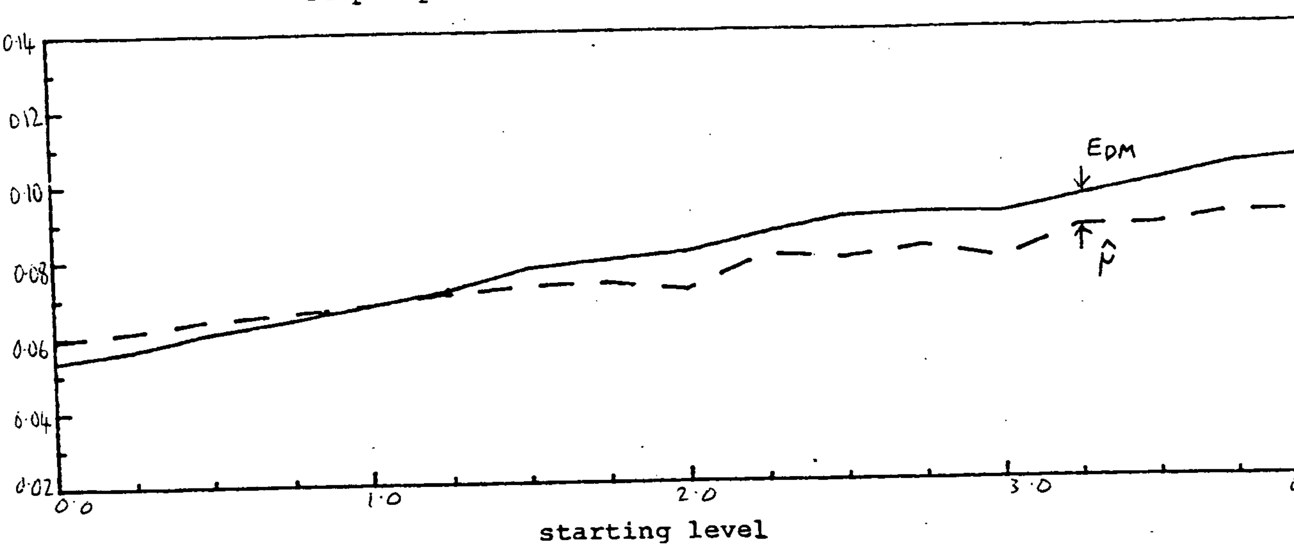


Fig. 3.3.11 M.s.e.'s of E_{DM}, $\hat{\mu}$ and μ^*($\mu^*=\hat{\mu}$ if $\hat{\mu}$ can be found, otherwise $\mu^*=E_{DM}$) in 24 step experiments with step size 1.0.

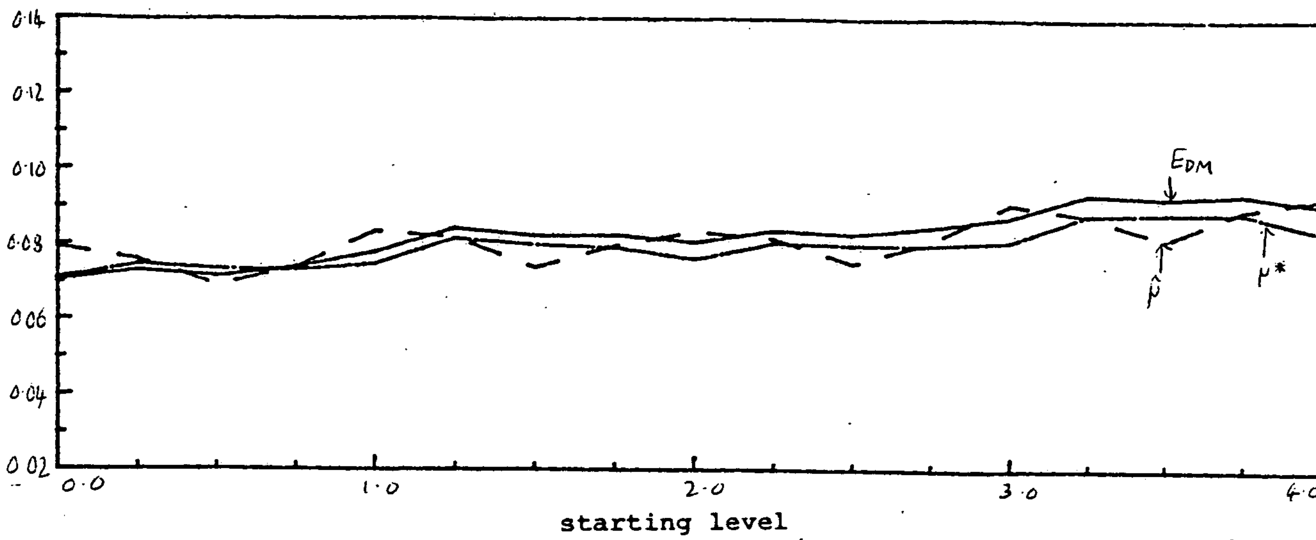


Fig. 3.3.12 As in Fig. 3.3.11 only with step size 1.5.

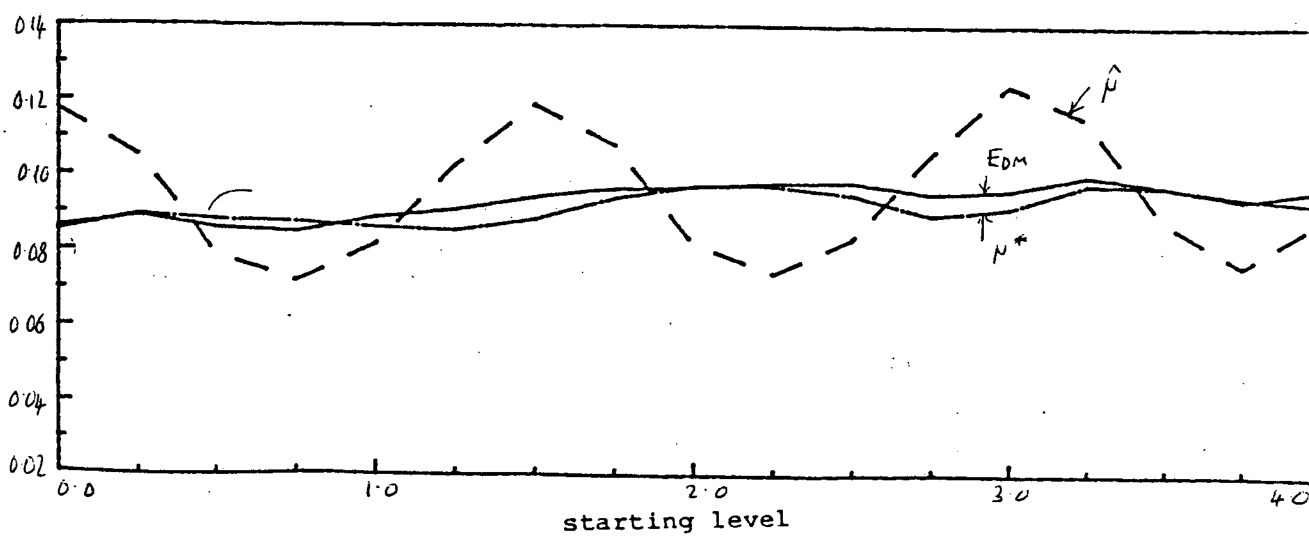
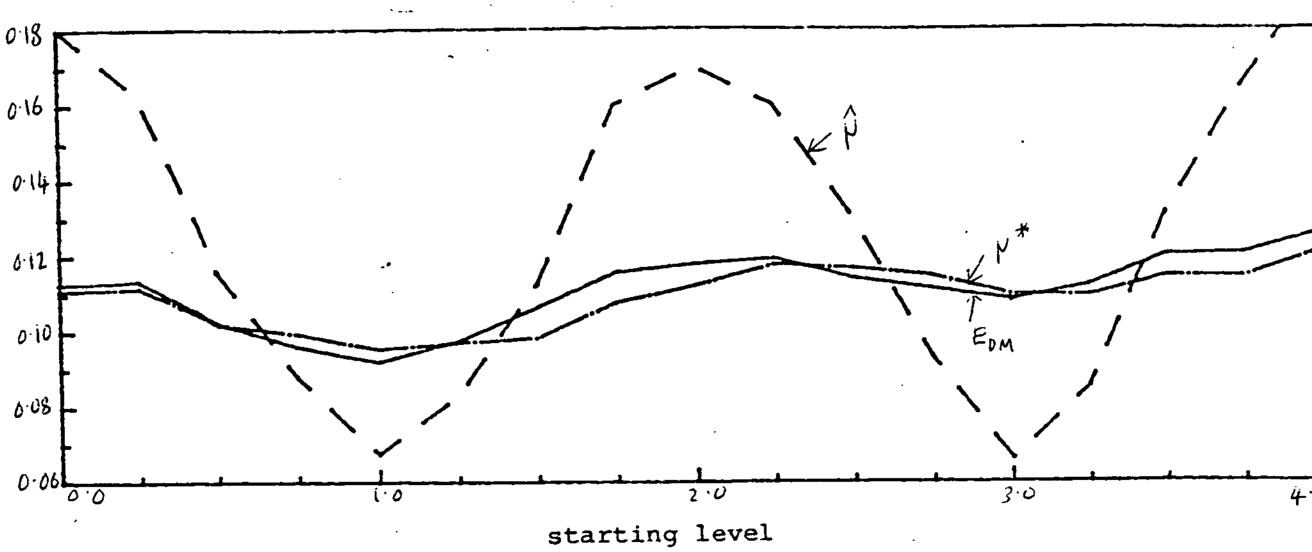
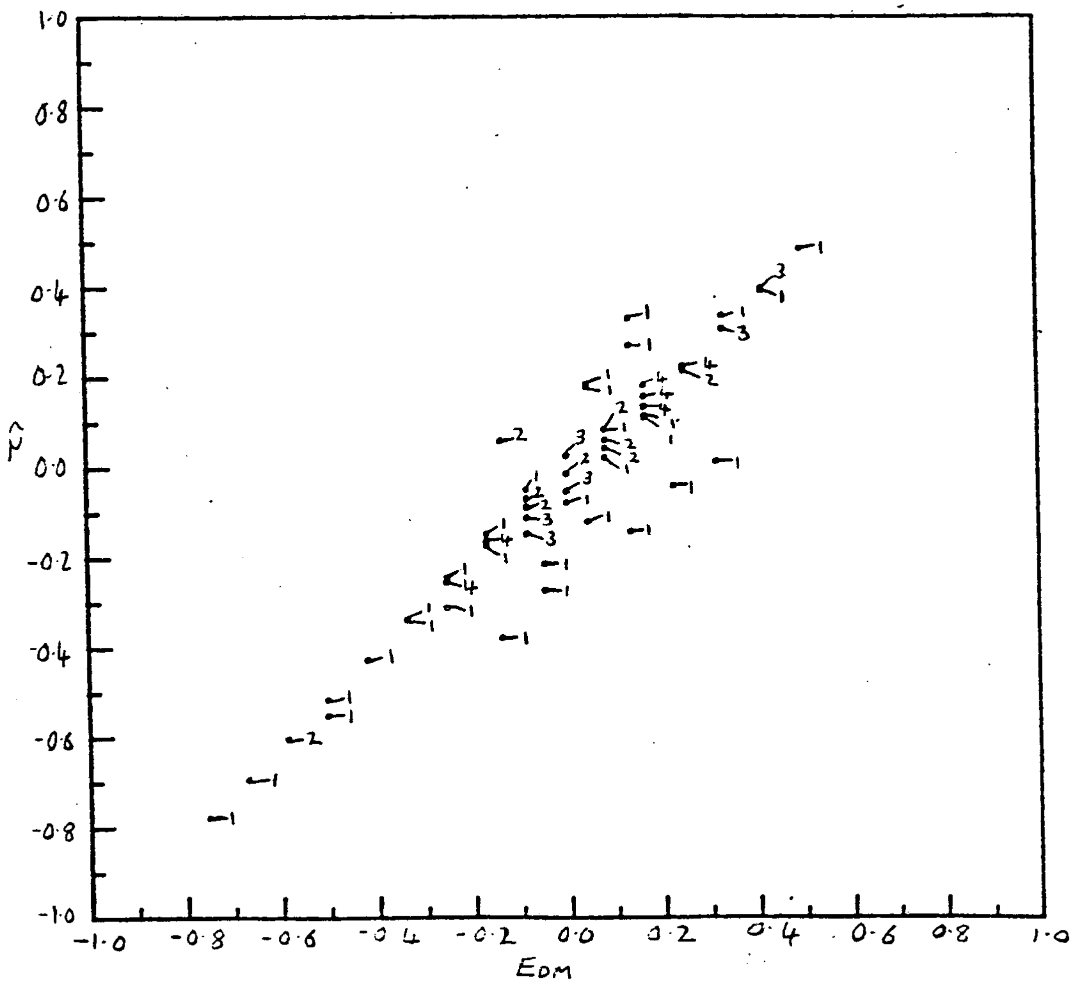


Fig. 3.3.13 As in Fig. 3.3.11 only with step size 2.0.



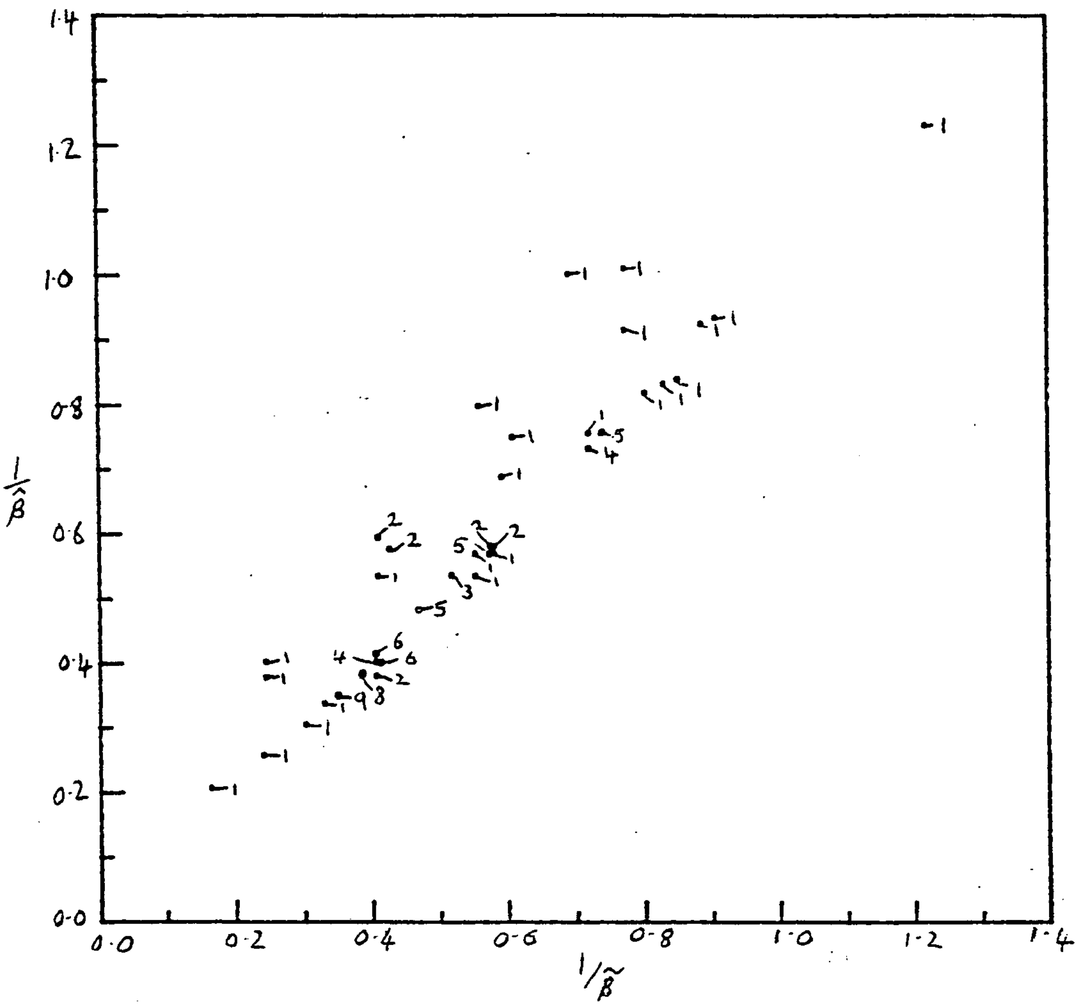
but often $\hat{\mu}$ did not). Figs. 3.3.11 to 3.3.13 are the same plots for step sizes 1.0, 1.5 and 2.0 only the m.s.e.'s of an estimator that I will call μ^* are also plotted. I define μ^* as being equal to $\hat{\mu}$ if it can be found and otherwise equal to E_{DM} (for step size 0.5 few experiments are discarded and m.s.e.'s of $\hat{\mu}$ when it exists and μ^* are very close). The m.s.e.'s of μ^* and E_{DM} are close for all step sizes. There is in fact a high correlation between $\hat{\mu}$ and E_{DM}; in simulations this correlation is at least 0.86 for step size 0.5 and at least 0.95 for step sizes 1.0, 1.5 and 2.0. Fig. 3.3.14 is a plot of $\hat{\mu}$ against E_{DM} for step size 1.0 and starting level 0.0 from the first 100 experiments simulated (only 88 of these experiments gave values for $\hat{\mu}$). Some of the points are multiple points or are very close to some other point (numbers next to points give the number of times any point is recorded). Most points are such that E_{DM} and $\hat{\mu}$ are close though there are a few outlying values. In all the simulations linear regression coefficients for the regression of $\hat{\mu}$ on E_{DM} were calculated using the admittedly arbitrary criterion of least squares; always intercepts were close to 0.0 and slopes were close to 1.0. The correlations between $1/\hat{\beta}$, the m.l.e. of $1/\beta$, and $1/\tilde{\beta}$ are also high, being at least 0.83 for step size 0.5, 0.93 for step size 1.0, 0.87 for step size 1.5 and 0.81 for step size 2.00. Fig. 3.3.15 is similar to Fig. 3.3.14 only values of $1/\hat{\beta}$ are plotted against values of $1/\tilde{\beta}$. There are several points for which $1/\hat{\beta}$ is some way above $1/\tilde{\beta}$ but usually values of the two estimates are close. For all simulations linear regression coefficients were calculated; these regression coefficients are not usually as close to 0.0 and 1.0 as corresponding values for the regression of $\hat{\mu}$ on

Fig 3.3.14 Plot of $\hat{\mu}$ against E_{DM} for simulated experiments where the step size is 1.0 and the starting level is 0.0.



Note: Numbers in Figs. 3.3.14 indicate multiplicity of each point

Fig 3.3.15 Plot of $1/\hat{\beta}$ against $1/\tilde{\beta}$ for simulated experiments where the step size is 1.0 and the starting level is 0.0.



E_{DM}.

The estimator μ^* has an advantage over the alternative estimators previously discussed in that for the smaller step sizes it often has much smaller bias. Table 3.3.9 gives values for bias and m.s.e. of this estimator. It seems curious that a composite estimator such as μ^* behaves so well. However there is no reason to suppose that the m.l.e. if it exists can be significantly bettered by any alternative and when it does not exist then there is some justification for using E_{DM} as it is an approximation to the conditional m.l.e. of μ given β, which is independent of the usually unknown value of β (see Appendix 2 and Dixon and Mood (1948)). Of course how well this approximation holds depends on β; the approximation breaks down if β is very large or if β is close to 0.0 and the numbers of positive and negative responses are not equal.

Clearly as the number of observations increases μ^* is asymptotically equivalent to $\hat{\mu}$. Table 3.3.10 gives values for the asymptotic variances of estimators (this table is similar to Table 2.3.3; again M_{DM} and M_{WE} denote asymptotic means of E_{DM} and E_{WE}). Asymptotically the distribution of all these estimators depends on the starting level only through the phasing of levels so from Table 3.3.10 asymptotic variances of the estimators under all the conditions simulated can be deduced (asymptotic biases are given in Tables 3.3.5 to 3.3.8). For step sizes 0.5 and 1.0 the asymptotic variance expression for E_{DM} is slightly above that of $\hat{\mu}$, the expression for E_{WE}^* is slightly above that of E_{DM} and the expression

Table 3.3.9 100×m.s.e. and 100×bias of μ^* in 24 step experiments ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

	Step size							
	0.5		1.0		1.5		2.0	
Start	100 × bias	100 × m.s.e.	100 × bias	100 × m.s.e.	100 × bias	100 × m.s.e.	100 × bias	100 × m.s.e.
0.00	-0.48	5.93	-0.38	7.09	-0.27	8.51	-0.22	11.10
0.25	-0.21	6.10	-0.84	7.44	-1.25	8.96	-2.51	11.15
0.50	-0.40	6.39	-0.70	7.33	-0.60	8.82	-2.85	10.20
0.75	0.11	6.57	0.00	7.30	-0.42	8.75	-2.45	9.94
1.00	0.23	6.82	-0.38	7.46	-0.14	8.61	0.47	9.54
1.25	0.67	6.98	0.17	8.15	0.86	8.56	2.40	9.71
1.50	0.33	7.17	-1.04	7.98	0.78	8.84	3.54	9.82
1.75	0.72	7.30	-1.22	7.91	0.57	9.39	4.13	10.77
2.00	0.92	7.10	-0.55	7.61	-0.32	9.74	2.09	11.22
2.25	0.81	7.91	-0.95	8.04	-0.23	9.74	0.58	11.75
2.50	0.53	7.81	-0.61	7.97	-1.20	9.53	-1.06	11.67
2.75	0.92	8.13	0.11	7.99	-1.09	8.98	-1.30	11.46
3.00	0.98	7.83	-0.26	8.08	-0.42	9.17	-0.29	10.96
3.25	0.90	8.58	0.29	8.80	-1.48	9.80	0.27	10.92
3.50	0.84	8.62	-0.83	8.83	-1.08	9.73	0.95	11.43
3.75	0.88	9.08	-0.79	8.86	-0.62	9.48	1.13	11.44
4.00	1.21	8.92	-0.27	8.35	-0.18	9.32	-0.45	12.10

Table 3.3.10 Asymptotic variance expressions for various starts and step sizes when $\beta = \pi/3.0^{1/2}$.

Variance of:	$n^{1/2}(\hat{\mu} - \mu)\beta$	$n^{1/2}(E_{DM} - M_{DM})\beta$	$n^{1/2}(E_{WE}^{*} - M_{WE})\beta$	$n^{1/2}(E_{WE} - M_{WE})\beta$
Step size 0.5				
Start				
0.00	4.927	4.953	4.972	5.252
0.25	4.927	4.953	4.972	5.251
Step size 1.0				
0.00	5.920	6.000	6.220	6.443
0.25	5.935	5.999	6.023	6.279
0.50	5.949	5.997	5.828	6.117
Step size 1.5				
Start				
0.00	6.833	7.239	8.763	8.863
0.25	6.973	7.191	7.942	8.104
0.50	7.204	7.096	6.439	6.706
0.75	7.301	7.049	5.751	6.063
Step size 2.0				
0.00	7.442	9.054	13.065	13.096
0.25	7.913	8.859	11.460	11.565
0.50	8.723	8.397	8.227	8.445
0.75	9.254	7.946	4.860	5.130
1.00	9.423	7.762	5.724	5.989

for E_{WE} is some way above all of these. So for these step sizes $\hat{\mu}$ has an asymptotic advantage over the other estimators in terms of variance. For step sizes 1.5 and 2.0 this is not always the case but the other estimators then often have high asymptotic bias and rates of change of bias have higher magnitude than corresponding values for the smaller step sizes.

In Figs. 3.3.16 to 3.3.19 m.s.e.'s of the m.l.e. of μ conditional on the true value of β are illustrated, in the simulations this estimator always existed (it exists providing there is one level of mixed response). In practice β would usually be unknown and it would be impossible to form this estimator. It is encouraging that the m.s.e.'s of E_{BD} and E_{PM} are close to values for this estimator for all the conditions simulated. These results confirmed my view that E_{BD} or E_{PM} should be used if approximate estimates of μ are required.

Also illustrated in these figures are m.s.e.'s of $\tilde{\mu}$, the minimum logit chi squared estimator of μ. These estimates have been calculated using Berkson's '1/2n' rule, but any levels visited only once are ignored (to me it seems unreasonable to give high weight to levels visited only once which is what happens if the '1/2n' rule is used for such levels). The results are better than in the 12 step experiments. Such estimates existed for all the simulated experiments. Looking in more detail at the results for step size 0.5 it was apparent that values of the estimate from a few experiments were inflating m.s.e.'s by large amounts. To counter this I excluded experiments for which the estimate of β was

Fig. 3.3.16 M.s.e.'s of min. logit χ^2 estimates of μ, m.l.e.'s of μ for known β and of E_{DM} in 24 step experiments with step size 0.5

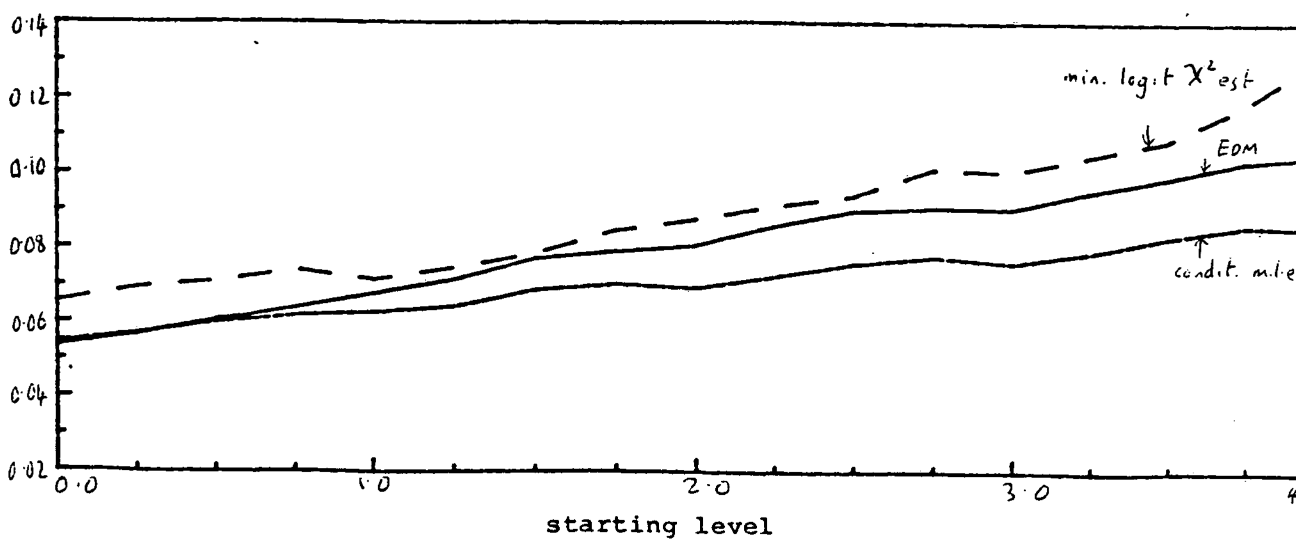


Fig. 3.3.17 As in Fig. 3.3.16 only with step size 1.0.

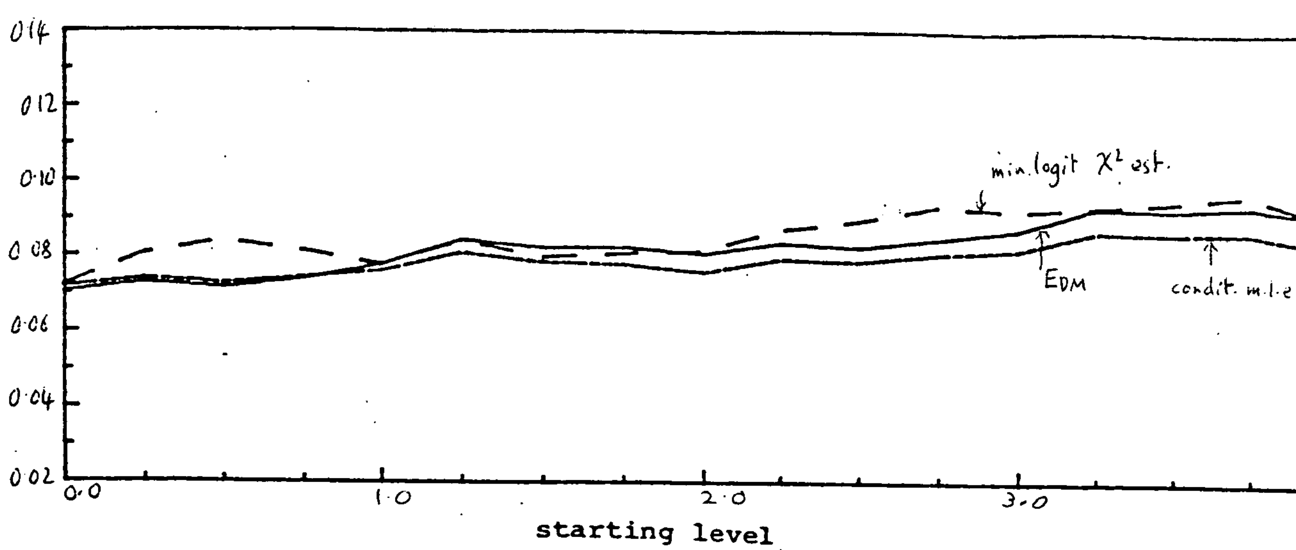


Fig. 3.3.18 As in Fig. 3.3.16 only with step size 1.5.

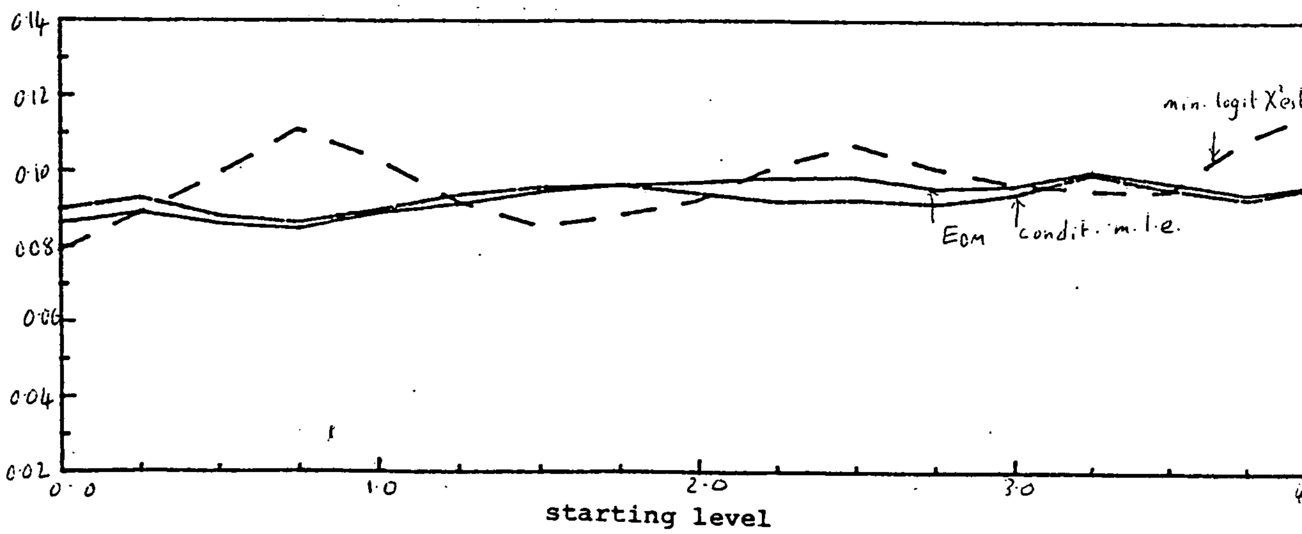
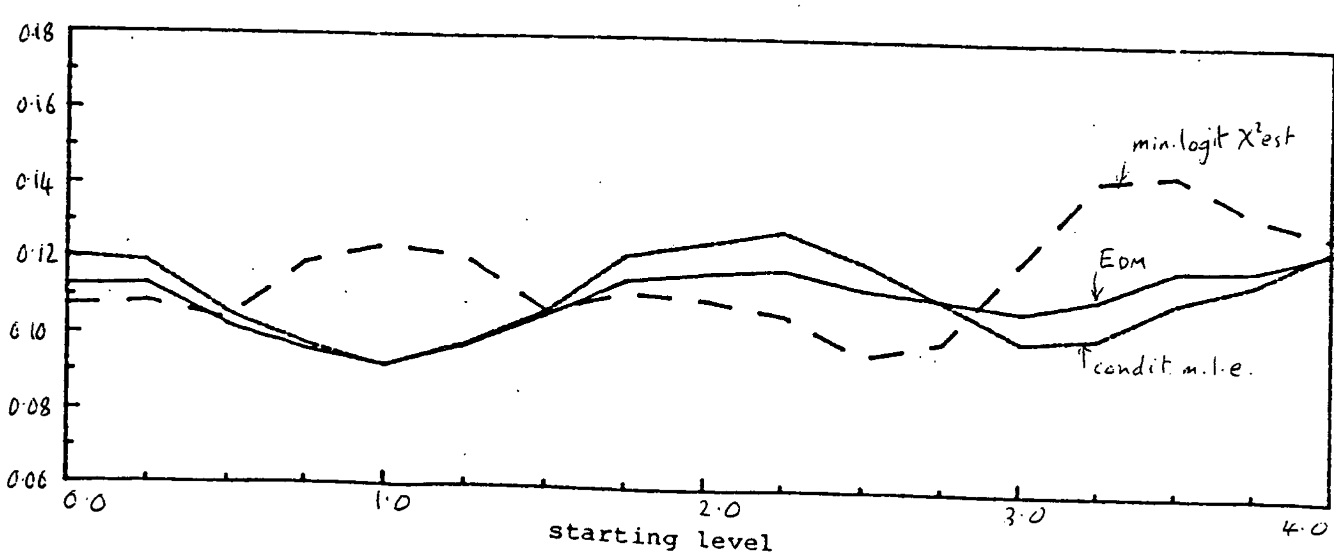


Fig. 3.3.19 As in Fig. 3.3.16 only with step size 2.0.



less than 0.2 (these bad estimates of μ were associated with low values of β). Only at most 11 out of 2000 experiments were in this way discarded. The bias of $\bar{\mu}$ is usually less than that of E_{DM} but I could see no reason for preferring it to μ^*. Berkson (1980) makes a case for using minimum chi squared estimates instead of maximum likelihood estimates. In this paper he refers to Wetherill (1963) and Little (1964), who compared these two estimation procedures when the Up and Down rule is operated. These papers corroborate some of Berkson's results but both authors indicate that they have no strong preferences between procedures. Wetherill uses the '1/2n' with no special procedure for dealing with levels visited only once; Little calculates estimates conditional on the value of slope.

So far I have not discussed in detail $1/\tilde{\beta}$ that I used to start iterations. Tables 3.3.11 to 3.3.14 give values for the mean and m.s.e. of $1/\tilde{\beta}$. I have also given analogous values for an estimator $1/\beta^*$ (I define $1/\beta^*$ to equal $1/\hat{\beta}$ if it can be found and otherwise to equal $1/\tilde{\beta}$) and $1/\bar{\beta}$, the minimum logit chi squared estimator of $1/\beta$ (where the '1/2n' rule is used, levels visited only once are ignored and for step size 0.5 some experiments were discarded because the estimate of β is less than 0.2). The estimators $1/\tilde{\beta}$ and $1/\beta^*$ always have some negative bias ($1/\beta$ = 0.5513 to 4 decimals). For all step sizes except 0.5 they have roughly similar m.s.e.'s and biases. For step size 0.5 the m.s.e.'s of $1/\tilde{\beta}$ rapidly mount as the starting level is increased beyond 2.00 (though the bias actually decreases as starting level increases). The estimator $1/\bar{\beta}$ always has positive bias. It also has much

Table 3.3.11 Mean and 100×m.s.e. of estimators of $1/\beta$ in 24 step experiments for step size 0.5 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

	$1/\tilde{\beta}$		$1/\beta^*$		$1/\bar{\beta}$		
Start	mean	100 × m.s.e.	mean	100 × m.s.e.	mean	100 × m.s.e.	Asympt. Var. $1/\hat{\beta}$
0.00	0.410	6.44	0.453	7.92	0.708	25.37	8.22
0.25	0.421	6.50	0.454	7.34	0.707	23.52	8.22
0.50	0.434	6.29	0.452	7.04	0.704	21.99	8.22
0.75	0.458	6.30	0.460	6.84	0.726	23.00	8.22
1.00	0.469	6.95	0.461	8.28	0.742	25.28	8.22
1.25	0.485	7.55	0.461	8.07	0.773	25.01	8.22
1.50	0.490	8.07	0.454	7.01	0.791	25.64	8.22
1.75	0.500	8.59	0.452	6.47	0.816	29.94	8.22
2.00	0.509	9.88	0.454	6.43	0.845	38.61	8.22
2.25	0.521	11.43	0.456	7.76	0.866	43.18	8.22
2.50	0.516	12.38	0.446	6.89	0.861	42.90	8.22
2.75	0.523	14.03	0.444	7.01	0.875	48.44	8.22
3.00	0.524	15.36	0.443	6.83	0.891	54.36	8.22
3.25	0.530	16.74	0.443	7.77	0.894	52.63	8.22
3.50	0.523	16.07	0.433	7.04	0.900	54.85	8.22
3.75	0.527	17.40	0.431	7.28	0.900	57.53	8.22
4.00	0.528	18.39	0.432	7.10	0.925	66.85	8.22

Table 3.3.12 Mean and 100×m.s.e. of estimators of $1/\beta$ in 24 step experiments for step size 1.0 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

	$1/\tilde{\beta}$		$1/\beta^*$		$1/\bar{\beta}$		
Start	mean	100 × m.s.e.	mean	100 × m.s.e.	mean	100 × m.s.e.	Asympt. Var. $1/\hat{\beta}$
0.00	0.458	4.80	0.493	5.20	0.730	15.26	5.72
0.25	0.462	4.56	0.494	4.81	0.749	15.24	5.62
0.50	0.467	4.71	0.494	4.83	0.770	15.07	5.52
0.75	0.479	5.02	0.495	5.12	0.765	16.08	5.62
1.00	0.485	4.92	0.489	5.14	0.738	14.85	5.72
1.25	0.495	4.80	0.494	4.95	0.761	14.81	5.62
1.50	0.507	5.21	0.501	5.06	0.806	15.97	5.52
1.75	0.508	5.54	0.500	5.04	0.829	17.65	5.62
2.00	0.499	5.78	0.494	5.11	0.822	19.57	5.72
2.25	0.493	5.68	0.491	4.82	0.826	20.53	5.62
2.50	0.490	5.89	0.489	4.77	0.849	21.28	5.52
2.75	0.496	6.46	0.488	5.13	0.867	24.51	5.62
3.00	0.495	6.60	0.480	5.28	0.857	25.65	5.72
3.25	0.499	5.97	0.482	4.96	0.851	24.43	5.62
3.50	0.505	6.29	0.487	5.05	0.868	24.91	5.52
3.75	0.504	6.47	0.486	5.06	0.870	25.35	5.62
4.00	0.495	6.50	0.483	5.18	0.851	25.81	5.72

Table 3.3.13 Mean and 100×m.s.e. of estimators of $1/\beta$ in 24 step experiments for step size 1.5 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	$1/\tilde{\beta}$ mean	$1/\tilde{\beta}$ 100 × m.s.e.	$1/\beta^*$ mean	$1/\beta^*$ 100 × m.s.e.	$1/\bar{\beta}$ mean	$1/\bar{\beta}$ 100 × m.s.e.	Asympt. Var. $1/\hat{\beta}$
0.00	0.486	3.74	0.507	4.08	0.786	12.97	5.80
0.25	0.487	3.93	0.513	4.32	0.809	14.42	5.35
0.50	0.474	4.49	0.504	4.64	0.826	16.84	4.61
0.75	0.474	4.95	0.505	4.97	0.845	19.58	4.29
1.00	0.481	4.78	0.504	4.69	0.837	18.20	4.61
1.25	0.493	4.35	0.503	4.34	0.806	14.77	5.35
1.50	0.507	4.36	0.508	4.35	0.794	13.58	5.80
1.75	0.514	4.75	0.511	4.52	0.814	14.96	5.35
2.00	0.510	5.15	0.508	4.64	0.860	18.26	4.61
2.25	0.509	5.25	0.508	4.62	0.916	21.84	4.29
2.50	0.515	5.29	0.511	4.76	0.947	24.90	4.61
2.75	0.513	4.82	0.507	4.29	0.924	24.06	5.35
3.00	0.509	4.65	0.506	4.17	0.889	23.33	5.80
3.25	0.502	4.70	0.507	4.33	0.874	21.34	5.35
3.50	0.482	5.30	0.497	4.86	0.886	22.40	4.61
3.75	0.477	5.81	0.495	5.22	0.932	26.09	4.29
4.00	0.477	5.44	0.491	4.92	0.956	27.45	4.61

Table 3.3.14 Mean and 100×m.s.e. of estimators of $1/\beta$ in 24 step experiments for step size 2.0 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	$1/\tilde{\beta}$ mean	$1/\tilde{\beta}$ 100 × m.s.e.	$1/\beta^*$ mean	$1/\beta^*$ 100 × m.s.e.	$1/\bar{\beta}$ mean	$1/\bar{\beta}$ 100 × m.s.e.	Asympt. Var. $1/\hat{\beta}$
0.00	0.541	2.74	0.552	3.14	0.911	16.30	7.61
0.25	0.522	3.09	0.536	3.49	0.911	16.83	6.64
0.50	0.488	4.29	0.509	4.60	0.899	18.95	4.97
0.75	0.461	5.60	0.491	5.73	0.890	22.76	3.85
1.00	0.440	6.34	0.475	6.32	0.864	23.59	3.49
1.25	0.453	5.81	0.482	5.81	0.879	22.51	3.85
1.50	0.490	4.99	0.508	5.11	0.905	20.72	4.97
1.75	0.527	3.93	0.533	3.97	0.917	18.40	6.64
2.00	0.554	3.57	0.554	3.48	0.921	17.92	7.61
2.25	0.546	3.91	0.545	3.72	0.926	18.13	6.64
2.50	0.517	4.54	0.518	4.23	0.942	20.51	4.97
2.75	0.491	6.00	0.498	5.51	0.974	26.28	3.85
3.00	0.476	6.53	0.488	5.94	1.001	31.87	3.49
3.25	0.477	5.75	0.486	5.24	1.022	34.36	3.85
3.50	0.516	5.03	0.517	4.58	1.045	36.32	4.97
3.75	0.544	3.96	0.541	3.56	1.024	32.85	6.64
4.00	0.551	3.51	0.548	3.17	0.989	28.69	7.61

higher m.s.e. than the other two estimators. The magnitude of the m.s.e.'s of these estimators relative to the magnitude of $1/\beta$ indicates that these estimators of $1/\beta$ are still not very useful but the m.s.e.'s of $1/\tilde{\beta}$ and $1/\beta^*$ are often close to the asymptotic predicted values for the variance of $1/\beta$.

When a normal tolerance distribution is assumed one would usually in addition to an estimate of μ want an estimate of the scale parameter σ. For the logistic $1/\beta$ is a scale parameter and of course has the same units as μ. It seemed natural to me that one would wish to estimate this quantity. However in studies using the logistic it has been the slope parameter β that that has been estimated (for example see Wetherill (1963)). I repeated calculations using $\tilde{\beta}$ and β^* as estimators of β. In a few experiments there was only one level at which the less frequent response is recorded and no estimate of β could be formed because $1/\tilde{\beta}$ equals 0.0 (the numbers of such experiments increase with step size: such experiments accounted for only at most 1 out of 2000 experiments for step size 0.5, 4 out of 2000 for step size 1.0, 23 out of 2000 for step size 1.5 and 63 out of 2000 for step size 2.0). Iterations were stopped when changes in estimates of μ and β before and after a step were less than 0.5×10.0^{-4}. From asymptotic theory I expected that the variance of the estimators would be approximately $\pi^4/9.0$ times the variance of corresponding estimators of $1/\beta$ (i.e. variances should be about 11 times larger). However I found that m.s.e.'s are for the smallest step sizes much in excess of what I expected from results for estimators of $1/\beta$. For step size 0.5 the m.s.e. of β is at least 2.79, for step size 1.0

at least 1.83, for step size 1.5 at least 0.99 and for step size 2.0 at least 0.39 (m.s.e.'s for $\tilde{\beta}$ and β^* are usually close). It is quite common that estimates of $1/\beta$ are some way below the actual values of $1/\beta$. These estimates do not greatly inflate m.s.e.'s of the estimators of $1/\beta$ but their reciprocals give very poor estimates for β. For this small sample size it is apparent that there are estimates of scale that behave well compared to the best performance one could expect from asymptotic theory but the same is not true for the slope.

I also simulated 48 and 96 step experiments under the same set of conditions. Appendix 9 contains tables summarising my results. Much of what I have said concerning 12 and 24 step experiments applies equally well to 48 and 96 step experiments. There are a number of points I particularly want to stress:

(1) The m.s.e.'s of E^*_{WE} and $E^*_{\rho V}$ are always less than those of E_{WE} and $E_{\rho V}$ respectively.

(2) The m.s.e.'s of $E_{\beta D}$ and E_{DM} are close and both estimators are robust against bad starting values for small step sizes (unlike E_M and E_β). The biases of $E_{\beta D}$ and E_{DM} have similar behaviour for step sizes 0.5 and 1.0; for step sizes 1.5 and 2.0 biases of both estimators are small.

(3) The m.s.e.'s of E_{DM} and E_{WE}^{*} are close for step size 0.5 but as step size is increased m.s.e.'s of E_{WE}^{*} (and E_{WE}) begin to oscillate above and below values for E_{DM} in accordance with asymptotic theory.

(4) The m.s.e.'s of μ^{*} are similar to those of E_{BD} and E_{DM} but μ^{*} has the advantage of having low bias for all the conditions simulated.

(5) The estimators $1/\tilde{\beta}$ and $1/\beta^{*}$ have similar behaviour except for step size 0.5 when often $1/\tilde{\beta}$ has smaller bias but higher m.s.e.

From (1) it is reasonable to conclude that E_{WE}^{*} and E_{PV}^{*} should be used in preference to E_{WE} and E_{PV}. From (2) I conclude that E_{BD} and E_{DM} are estimators with much the same properties providing the number of observations is reasonably large. Brownlee et al (1953) suggested the estimator E_{BD} to overcome the difficulties they encountered in using E_{B}; these results suggest that they could just as well have suggested a return to the original estimator E_{DM}. From (3) I conclude that at least for the conditions I have considered E_{WE} has no special advantage over E_{BD} or E_{DM} for small step sizes and it has definite disadvantages for large step sizes. From (4) one can see that there is often a close relationship between $\hat{\mu}$ and the estimators E_{BD} and E_{DM}. For reduction of bias it seems wise to use $\hat{\mu}$ if it exists. The results confirm my belief that if an approximate estimator of μ is required E_{BD} or E_{DM} should be used. From (5) one sees that there is also a relationship between $1/\tilde{\beta}$ and $1/\hat{\beta}$. As the m.s.e.'s of $1/\tilde{\beta}$ and $1/\beta^*$ are high compared to the magnitude of $1/\beta$ (even for 96 step experiments) I would be reluctant to make much use of an estimate of $1/\beta$ arising from one experiment. The estimator $1/\tilde{\beta}$ does appear to be of some use for giving a starting value for iterations to the maximum likelihood estimates.

As one would expect for these experiments in which more steps are taken the asymptotic theory is more closely obeyed. For the 48 step experiments a large proportion of experiments did not yield m.l.e.'s for step sizes 1.5 and 2.0; for 96 step experiments there are only large proportions of such experiments for step size 2.0.

The m.s.e.'s of $\tilde{\beta}$ and β^* are usually close. As before m.s.e.'s of $\tilde{\beta}$ and β^* are often higher than one would expect. Even for 96 step experiments when the step size is 0.5 the m.s.e.'s of β^* are about 50 per cent above asymptotic expected values and for step size 2.0 m.s.e.'s are very variable with often values inflated by results from a few experiments. This suggests that if estimates are pooled from several experiments one should pool estimates of $1/\beta$ and not estimates of β.

4 USE OF UP AND DOWN TRANSFORM RULES

4.1 DESCRIPTION AND SOME PROPERTIES OF THE UDTR RULE

In Chapters 2 and 3 the problem of estimating parameters μ, β and $1/\beta$, where the response curve is logistic, has been considered. Often an experimenter wishes to estimate the stimulus level at which the probability of positive response takes some general value p. This level can be denoted as the ED(100p) level, or using another common notation as the L_p level. If the response curve has parametric form $F(\beta(x-\mu))$ for some known function F then

$$L_p = \mu + (k/\beta), \qquad 4.1.1$$

where $F(k) = p$; for logistic response $k = \log(p/(1-p))$.

When the Up and Down rule is used with logistic response, the asymptotic correlation between m.l.e.'s of μ and β is small for small step sizes (see Table 3.1.1). The value of $V(\mu)$ is close to its lower bound of $4/\beta^2$ but the value of $V(\beta)$ is large (see Figs. 3.1.7 and 3.1.8). As step size decreases the m.l.e. of μ is at least asymptotically approaching full efficiency. However as step size decreases the asymptotic variances of m.l.e.'s of all other L_p levels increase without bound. For small step sizes observations are eventually made close to μ. One has to rely very heavily upon the assumed form for the response curve to obtain any estimate of a L_p level for p not close to 0.5. Bartlett (1946) emphasises the importance of estimating an extreme percentage point from observations made in the neighbourhood of the point and suggests an inverse sampling procedure for use in non-sequential experiments.

Wetherill (1963) gave a strategy called 'Routine 15' for estimating general percentage points which uses the Up and Down rule on a transformed response curve, it has been called the Up and Down Transform rule (UDTR rule). To operate the UDTR rule one must first specify some $q \in (0.0, 1.0)$ and integer n_o . Tests continue at a level until either:

(1) The proportion of positive responses is less than q, in which case the stimulus level is increased by one step.

(2) The proportion of positive responses based on n_o or more trials exceeds q, in which case the stimulus level is decreased by one step.

(Responses at previous visits to a level are not used in forming proportions.)

If after n_o trials the proportion of positive responses equals q, then one more observation is made to determine the direction of the next step. Alternatively one can base the UDTR rule on the proportion of negative responses. Wetherill in 'Sequential Methods in Statistics' (1966, Page 184) gives an example of an UDTR rule where $n_o = 4$ and $q = 0.75$. With this rule moves up are made after sequences of responses −,+−,++− or +++−− and moves down after ++++ or +++−+. If the response curve has the form $F(\beta(x-\mu))$ then the probability that a move down will be made at the next change in level given that the level just entered is x is G(x) where

$$G(x) = F(\beta(x-\mu))^4 (2-F(\beta(x-\mu))). \qquad 4.1.2$$

The sequence of levels visited can be viewed as being generated by an Up and Down rule with the response curve G (where the number of observations made at each visit to a level is ignored). The value of $F(\beta(x-\mu))$ which gives equal probability of moving up or down will be the root of the equation $(2z^4-z^6) = 0.5$. This equation has just one root in (0.0,1.0) which is close to 0.8.

A simple example of an UDTR rule is when $n_o = 2$ and q takes any value in (0.5,1.0), here a move up is made after sequences of responses - or +- and down after ++. The sequence of levels visited can in this case be viewed as being generated using the Up and Down rule on the square of the original response curve. This rule has been extensively used in psychometric studies; its properties will be discussed in the remainder of this section. Wetherill, Chen and Vasudeva (1966) suggest when an UDTR rule is used that Wetherill's estimator E_{WE} (where moves down are taken as positive responses and moves up as negative responses) can be used to estimate L_p, where p is the probability of positive response that gives equal probability of moving up or down (so in the last example of an UDTR rule, where $p = 0.5^{1/2}$, E_{WE} would be used to estimate $L_{1/\sqrt{2}}$). One could in the same way use E_{DM}, or any of the other estimators discussed in Section 2.1, to provide estimates of $L_{1/\sqrt{2}}$.

Often the assumed response curve arises from a symmetric tolerance distribution, when an UDTR rule is used the transformed response curve no longer has this symmetry. Using the Up and Down rule E_{DM} and E_{WE} are, for logistic response, estimators of μ of small asymptotic bias for small step sizes (see Tables 2.3.1 and 2.3.2), and if stimulus levels are symmetrically placed about μ the biases are zero. The asymptotic biases of E_{DM} and E_{WE} for the UDTR rule for estimating $L_{1/\sqrt{2}}$ (described in the previous paragraph), with logistic response, are for small step sizes, much larger than corresponding biases for estimates when the Up and Down rule is used to give estimates of μ. This is not surprising, as in the special case where the response curve is logistic, one can show that the asymptotic biases of E_{DM} and E_{WE} divided by d tend to zero as d tends to zero. For the UDTR rule I have only been able to show that the biases divided by $d^{1/2}$ tend to zero as d tends to zero. These results follow from results in Appendix 6. Fig. 4.1.1 illustrates values of the asymptotic bias of E_{DM} in estimating $L_{1/\sqrt{2}}$ for βd = 0.25(0.25)4.00 and μ/d = 0.00(0.25)0.75. Fig. 4.1.2 illustrates analogous biases of E_{WE}. The bias for small values of βd is smaller for E_{WE} than E_{DM}, but the maximum bias for high values of βd is greater for E_{WE}.

Wetherill <u>et al</u> describe some simulations operating this UDTR rule, using E_{WE} as an estimator of $L_{1/\sqrt{2}}$. These simulations indicate that this estimator sometimes has m.s.e. below that of the m.l.e. of $L_{1/\sqrt{2}}$ (their results will be discussed in more detail in the next section).

For convenience I will define $V(L_{1/\sqrt{2}})$ to equal the asymptotic

Fig. 4.1.1 (Asymptotic bias of E_{DM})/d when estimating $L_{1/\sqrt{2}}$.

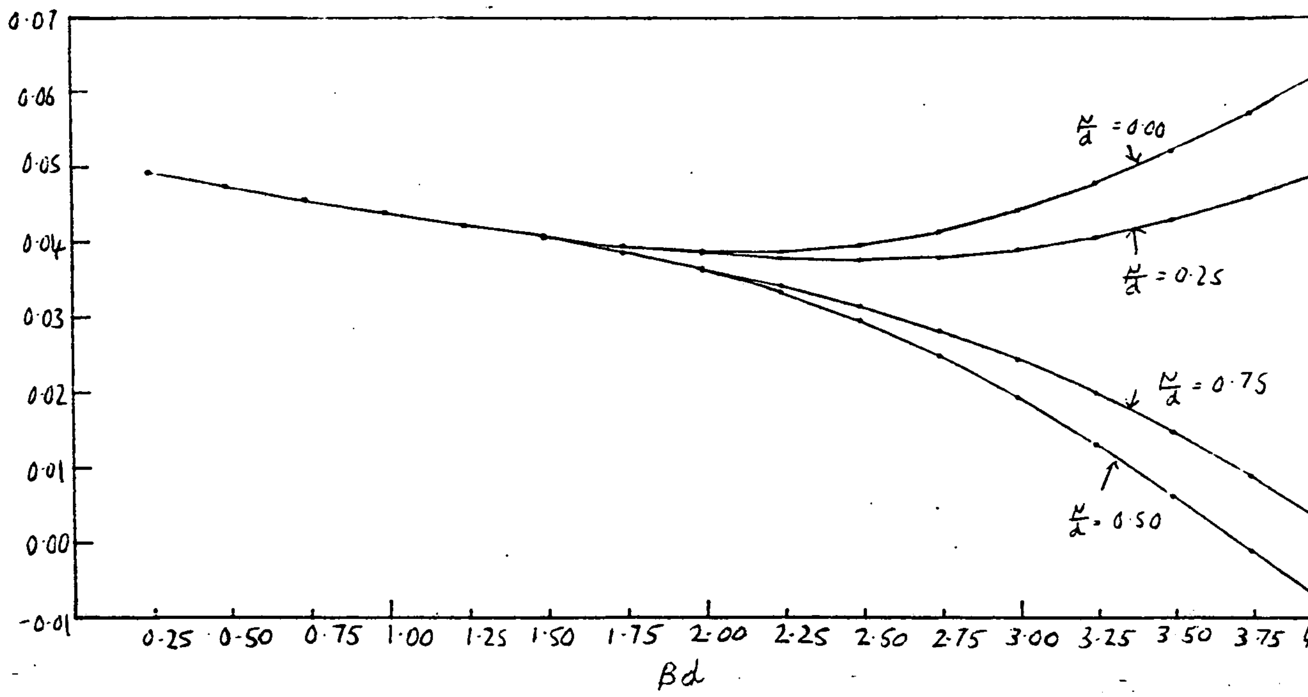
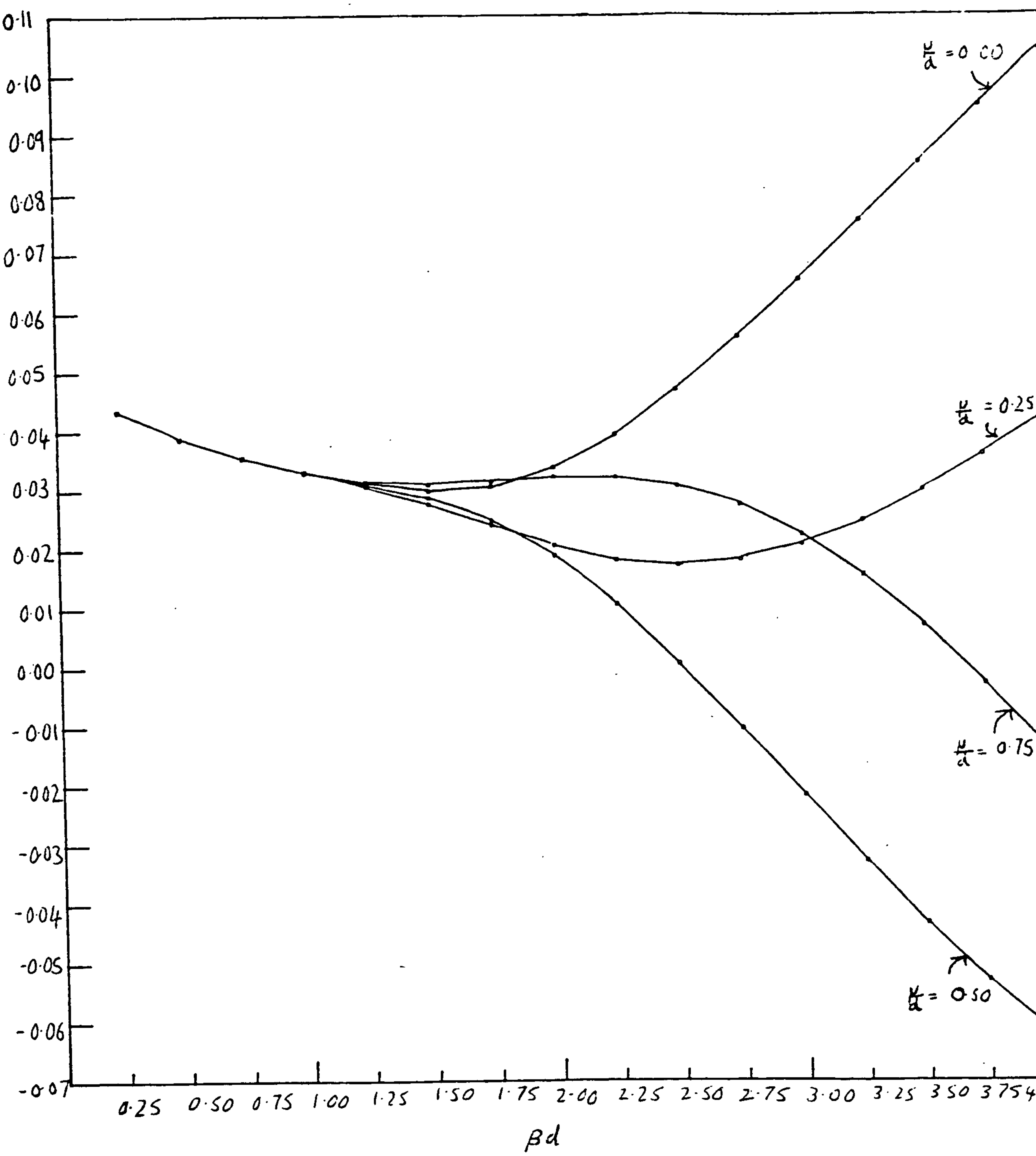


Fig. 4.1.2 (Asymptotic bias of E_{WE})/d when estimating $L_{1/\sqrt{2}}$.



variance of $(\hat{L}_{1/\sqrt{2}} - L_{1/\sqrt{2}})n^{1/2}$, where $\hat{L}_{1/\sqrt{2}}$ is the m.l.e. of $L_{1/\sqrt{2}}$ and $n^{1/2}$ is the number of observations. Fig. 4.1.3 illustrates values of $V(L_{1/\sqrt{2}})\beta^2$ for $\mu/d = 0.00(0.25)0.75$ and $\beta d = 0.25(0.25)4.00$. Fig. 4.1.4 illustrates values of the analogous variance expression for E_{DM}, that is the asymptotic variance of $n^{1/2}(E_{DM} - M_{DM})\beta$; where M_{DM} is the asymptotic expectation of E_{DM}. Fig. 4.1.5 illustrates values for the asymptotic variance expressions of $n^{1/2}(E_{WE} - M_{WE})\beta$ and $n^{1/2}(E^*_{WE} - M_{WE})\beta$, where M_{WE} is the asymptotic expectation of E_{WE} (points joined by dashed lines correspond to expressions for E^*_{WE}). The value of $V(L_{1/\sqrt{2}})\beta^2$ hardly changes with phase for $\beta d \leqslant 2.0$, but above this phase begins to have a marked effect with higher values for $\mu/d = 0.25$ and lower for $\mu/d = 0.75$. The asymptotic variance of $n^{1/2}(E_{DM} - M_{DM})\beta$ also hardly changes with phase for $\beta d \leqslant 2.0$, but above this phase begins to have a marked effect with now higher values for $\mu/d = 0.75$ and lower for $\mu/d = 0.75$. So the dependence of the asymptotic variance expression for E_{DM} on phase is quite different from that for $\hat{L}_{1/\sqrt{2}}$. This is no surprise as using the Up and Down rule the dependence of asymptotic variance expressions for E_{DM} and $\hat{\mu}$ are also quite different (see Figs. 2.3.1 and 2.3.2). The asymptotic variances of $n^{1/2}(E_{WE} - M_{WE})\beta$ and $n^{1/2}(E^*_{WE} - M_{WE})\beta$ have a similar dependence on phase as the asymptotic variance of $n^{1/2}(E_{DM} - M_{DM})\beta$. This dependence on phase is small for $\beta d \leqslant 1.25$ but becomes very large for $\beta d = 4.0$. From this one can anticipate that for large step sizes E_{WE} and E^*_{WE} will have properties more dependent on phase than $\hat{L}_{1/\sqrt{2}}$ and E_{DM}. The drop in variance in using E^*_{WE} rather than E_{WE} is not a high proportion of the variance of E_{WE}.

Fig. 4.1.3 $V(L_{1/\sqrt{2}})\beta^2$ when the UDTR for $L_{1/\sqrt{2}}$ is operated with logistic response.

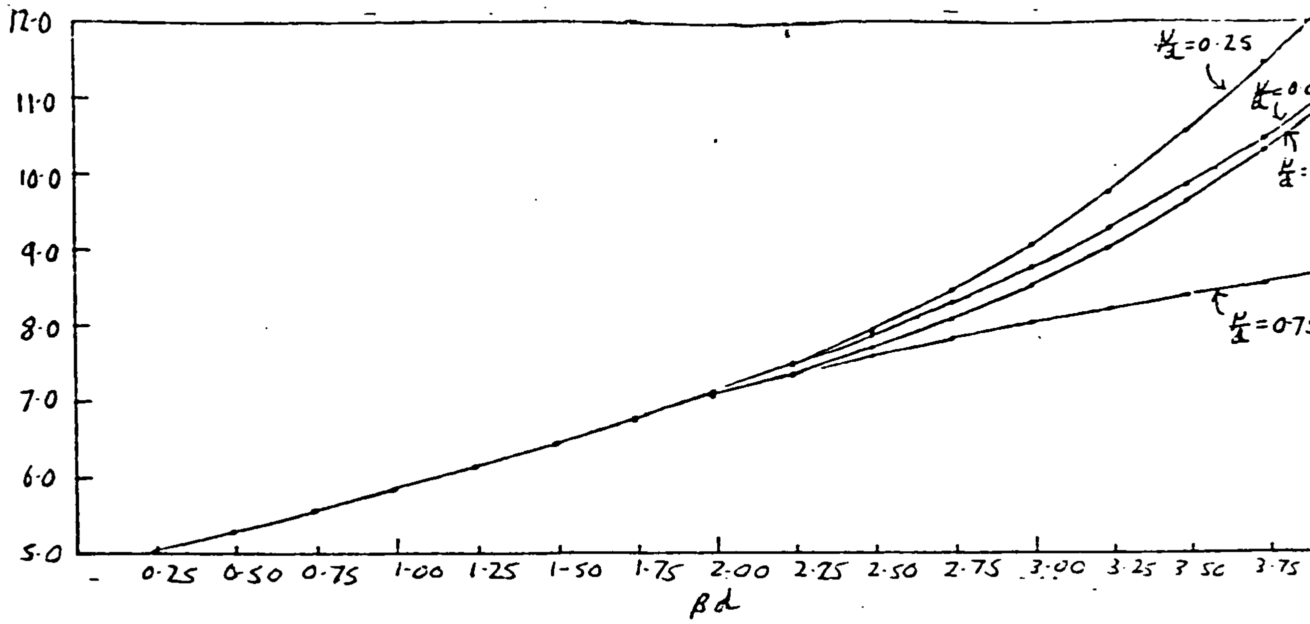


Fig. 4.1.4 Asymptotic variance of $(E_{DM}-M_{DM})\beta n^{1/2}$ when the UDTR for $L_{1/\sqrt{2}}$ is operated (M_{DM} is the asymptotic mean of E_{DM}).

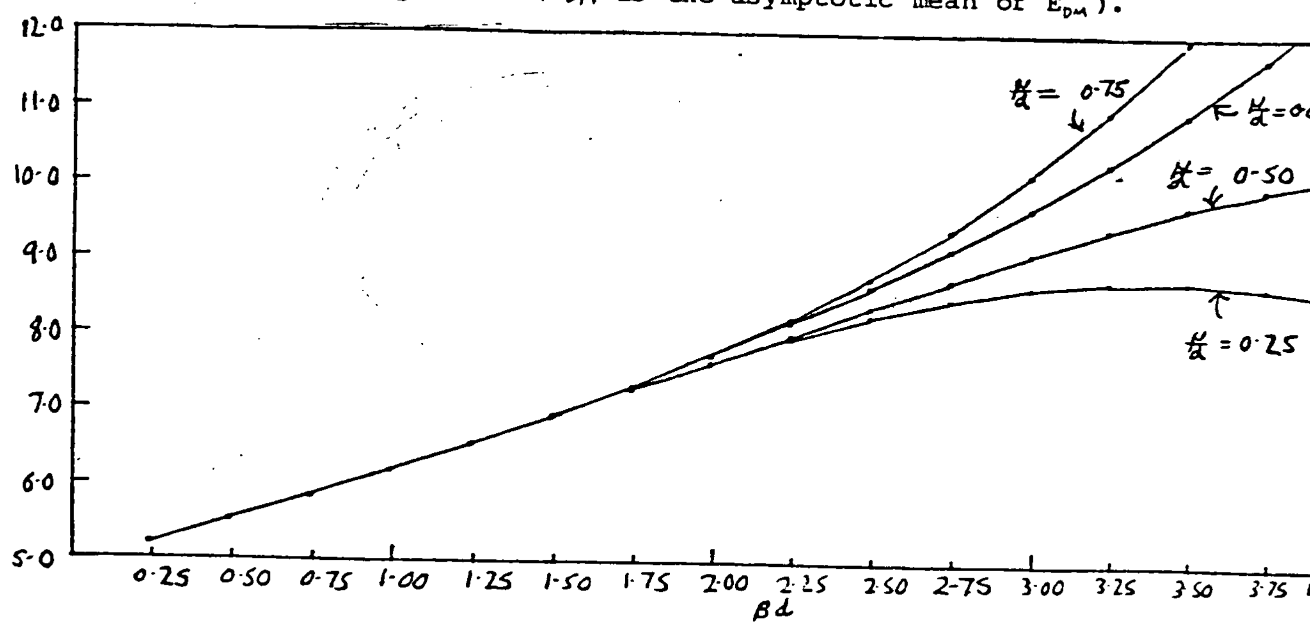
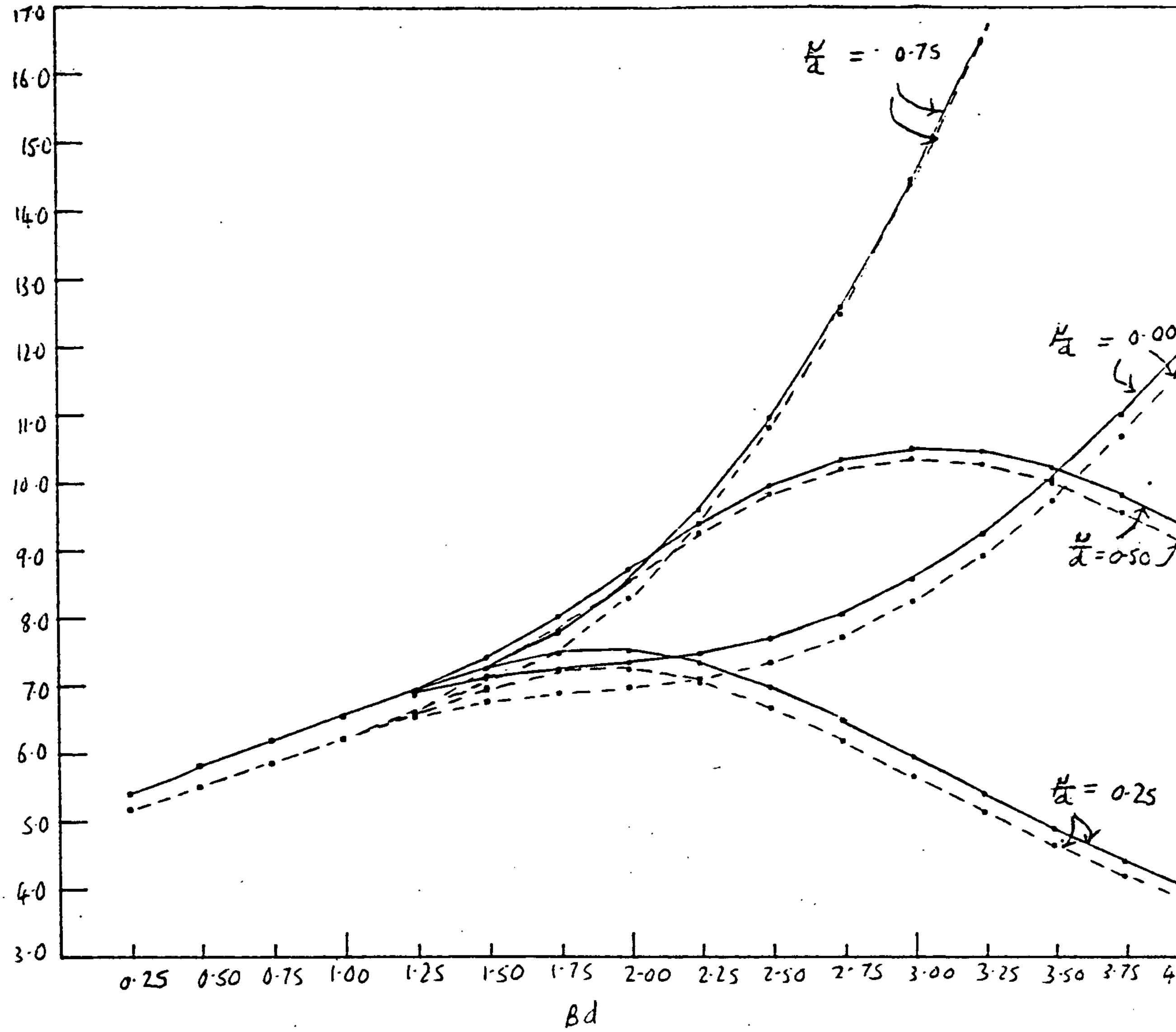


Fig. 4.1.5 Asymptotic variance of $(E_{WE}-M_{WE})\beta n^{1/2}$ when the UDTR for $L_{1/\bar{n}}$ is operated (M_{WE} is the asymptotic mean of E_{WE}). The broken lines join corresponding values for $(E^*_{WE}-M_{WE})\beta n^{1/2}$.



Note: For $\beta d=4.00$ and $\mu/d=0.75$ the asymptotic variances of $(E_{WE}-M_{WE})\beta n^{1/2}$ and $(E^*_{WE}-M_{WE})\beta n^{1/2}$ are 23.35 and 23.33 respectively.

For small step size the asymptotic variance expression for E_{DM} and E_{WE}^* are close with the values for E_{DM} being slightly lower. The asymptotic variances for E_{WE} are some way above these values and the asymptotic variances for $\hat{L}_{1/\sqrt{2}}$ are some way below. Table 4.1.1 gives values for $V(L_{1/\sqrt{2}})\beta^2$ and the analogous variance expressions for E_{DM}, E_{WE} and E_{WE}^*. In this table values are given for $\beta d = 0.25(0.25)0.75$ and $\mu/d = 0.0$, for these βd values the dependence of these expressions on μ/d is small.

Table 4.1.2 gives some values of $V(\hat{L}_{1/\sqrt{2}})\beta^2$, $V(\mu)\beta^2$ and $V(\beta)/\beta^2$ for both the UDTR and Up and Down rule. For values of βd of 0.25(0.25)1.00 the dependence on phase of these expressions is small. For small values of βd the value of $V(L_{1/\sqrt{2}})$ is much greater for the Up and Down rule than the UDTR rule. However with the UDTR rule, for all sets of parameter values considered, the asymptotic correlation between m.l.e.'s of μ and β is negative and as could be expected $V(\mu)$ is always greater for the UDTR rule than for the Up and Down rule. For the smaller values of βd the values of $V(\beta)$ are greater for the UDTR rule than for the Up and Down rule, but for βd values 2.75 and above $V(\beta)$ is for some μ/d values smaller for the UDTR rule. However there is never any great gain in efficiency in estimating β using the UDTR rule.

In Section 3.2 a possible estimator of $1/\beta$ was discussed equal to the variance of the levels of less frequent response type divided by step size. From Appendix 7 it follows that when the UDTR rule is used, for d sufficiently small, the limit with probability one of this quantity is arbitrarily close to $1/4\lambda_0$,

Table 4.1.1 Asymptotic variance expressions (μ/d=0.0).

Asymptotic Variance of:	βd 0.25	0.50	0.75	1.00
$n^{1/2}(\hat{L}_{1/\sqrt{2}}-L_{1/\sqrt{2}})\beta$	5.089	5.356	5.630	5.913
$n^{1/2}(E_{DM}-M_{DM})\beta$	5.278	5.594	5.923	6.264
$n^{1/2}(E^*_{WE}-M_{WE})\beta$	5.294	5.632	5.978	6.331
$n^{1/2}(E_{WE}-M_{WE})\beta$	5.522	5.943	6.313	6.664

Table 4.1.2 Values of $V(L_{1/\sqrt{2}})\beta^2$, $V(\mu)\beta^2$ and $V(\beta)/\beta^2$ for the UDTR and Up and Down rules (μ/d=0.0).

	βd 0.25	0.50	0.75	1.00
$V(L_{1/\sqrt{2}})\beta^2$ for UDTR rule	5.089	5.356	5.630	5.913
$V(L_{1/\sqrt{2}})\beta$ for Up & Down	18.240	12.294	10.503	9.757
$V(\mu)\beta^2$ for UDTR rule	22.135	13.582	10.894	9.679
$V(\mu)\beta^2$ for Up & Down	4.250	4.504	4.762	5.025
$V(\beta)/\beta^2$ for UDTR rule	24.551	13.367	9.528	7.679
$V(\beta)/\beta^2$ for Up & Down	18.008	10.029	7.290	6.091

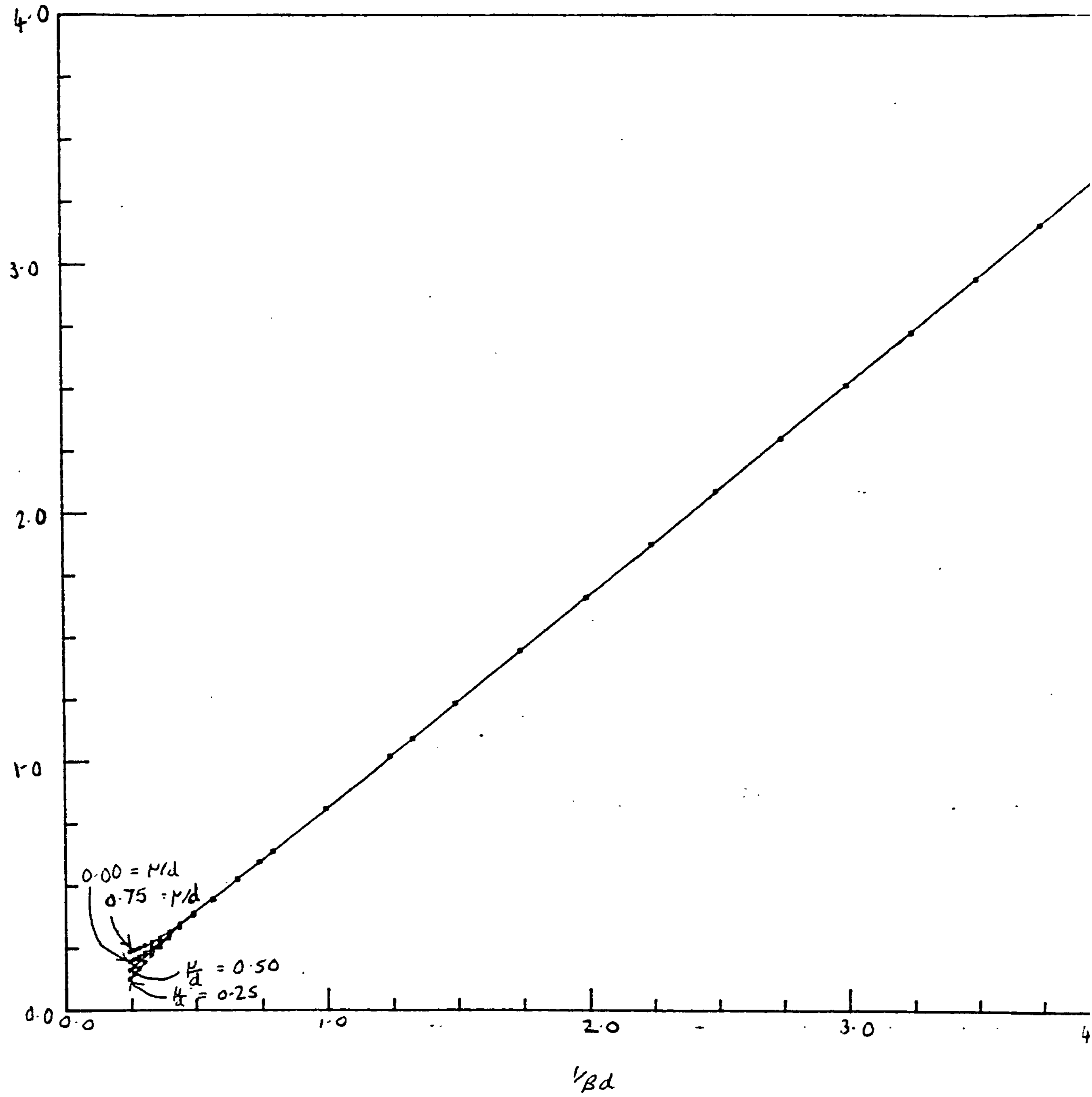
where λ_0 is the slope of the transformed response curve at $L_{1/\sqrt{2}}$. The value of λ_0 is $\beta(1 - (0.5)^{1/2})$, so this result suggests a possible of $1/\beta$, that I will call $1/\tilde{\tilde{\beta}}$ where

$$1/\tilde{\tilde{\beta}} = 4(1 - (0.5)^{1/2})v/d, \qquad 4.1.3$$

and v is the variance of the levels of less frequent response type (where again moves down are taken as positive responses and moves up as negative responses). A more detailed definition of v is given in Section 3.2. In Fig. 4.1.6 values of the limit with probability one of v/d are illustrated. For $\beta d \leqslant 2.0$ the values of these limits are very close to $1/(4(1 - (0.5)^{1/2})\beta)$; that is for $\beta d \leqslant 2.0$ the limit with probability one of $1/\tilde{\tilde{\beta}}$ is close to $1/\beta$. Whether the estimator $1/\tilde{\tilde{\beta}}$ is of much use is questionable as for small values of βd the asymptotic variance of even the m.l.e. of $1/\beta$ is relatively high. The uses of this estimator will be discussed in the next section.

Wetherill et al suggest that in order to estimate the slope two UDTR rules should be operated, one designed to concentrate observations about some level L_p and the other about L_{1-p} (i.e. roles of positive and negative responses interchanged). One could for example use the UDTR rules that are designed to concentrate observations close to the $L_{1/\sqrt{2}}$ and $L_{1-1/\sqrt{2}}$ levels. Wetherill et al also suggest that both UDTR staircases should be stopped after a fixed number of changes in response type. If $w_{1/\sqrt{2}}$ and $w_{1-1/\sqrt{2}}$ are estimates of $L_{1/\sqrt{2}}$ and $L_{1-1/\sqrt{2}}$ based on using E_{WE} for each staircase then an estimate of $1/\beta$ suggested in Wetherill et al is

Fig. 4.1.6 Plot of limit with probability one of v/d against 1/βd.



$$(w_{1/\sqrt{2}} - w_{1-1/\sqrt{2}})/2\bar{k}, \qquad 4.1.4$$

where $\bar{k} = \log((2)^{1/2} + 1)$. One could also form an estimate of μ equal to

$$(w_{1/\sqrt{2}} + w_{1-1/\sqrt{2}})/2. \qquad 4.1.5$$

Estimates of μ and $1/\beta$ can be obtained by maximum likelihood estimation, values of $V(\mu)\beta^2$ and $V(\beta)/\beta^2$ are illustrated in Figs. 4.1.7 and 4.1.8 respectively. Points joined by unbroken lines correspond to variance expressions when two UDTR rules are used; those joined by broken lines correspond to expressions when the Up and Down rule is used. There is some loss in efficiency in estimates of μ using this new procedure but asymptotically at least there is protection against poor estimates of $1/\beta$ if a small step size is used. As with calculations made for the Up and Down rule these asymptotic values should be interpreted with care. The asymptotic variance of μ decreases with step size, but as step sizes become smaller, larger samples will be required before anything close to the asymptotic distribution of observations is achieved. Another point to remember is that if a small step size is used most observations will eventually be made close to the $L_{1/\sqrt{2}}$ and $L_{1-1/\sqrt{2}}$ levels; estimates of μ and $1/\beta$ will not then be robust against departures of the model from the assumed form.

At best one would hope that $w_{1/\sqrt{2}}$ and $w_{1-1/\sqrt{2}}$ have properties similar to those predicted by asymptotic theory for the m.l.e.'s of

Fig. 4.1.7 $V(\mu)\beta^2$ when the two UDTR's for $L_{1/\sqrt{2}}$ and $L_{1-1/\sqrt{2}}$ are used (values for the Up and Down rule are joined by dashed lines).

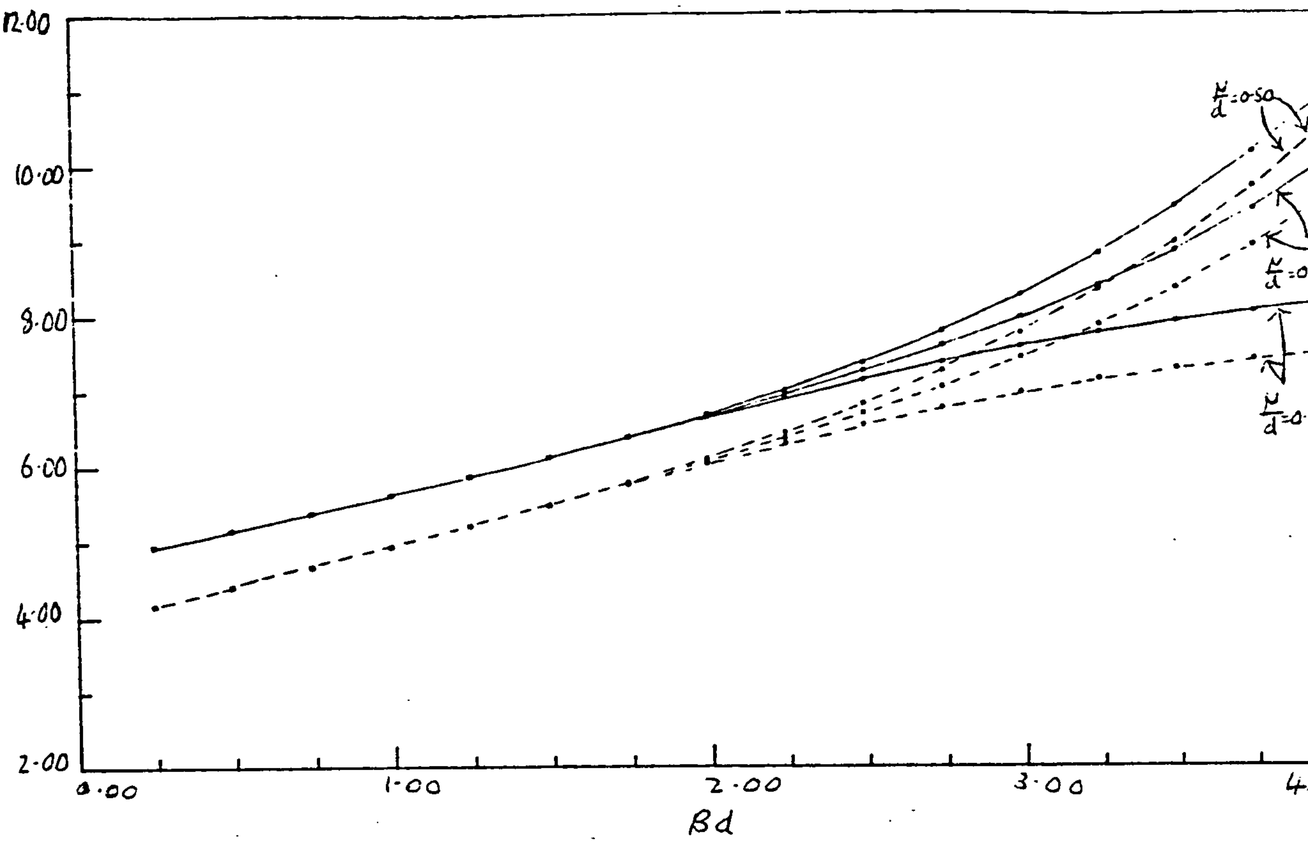
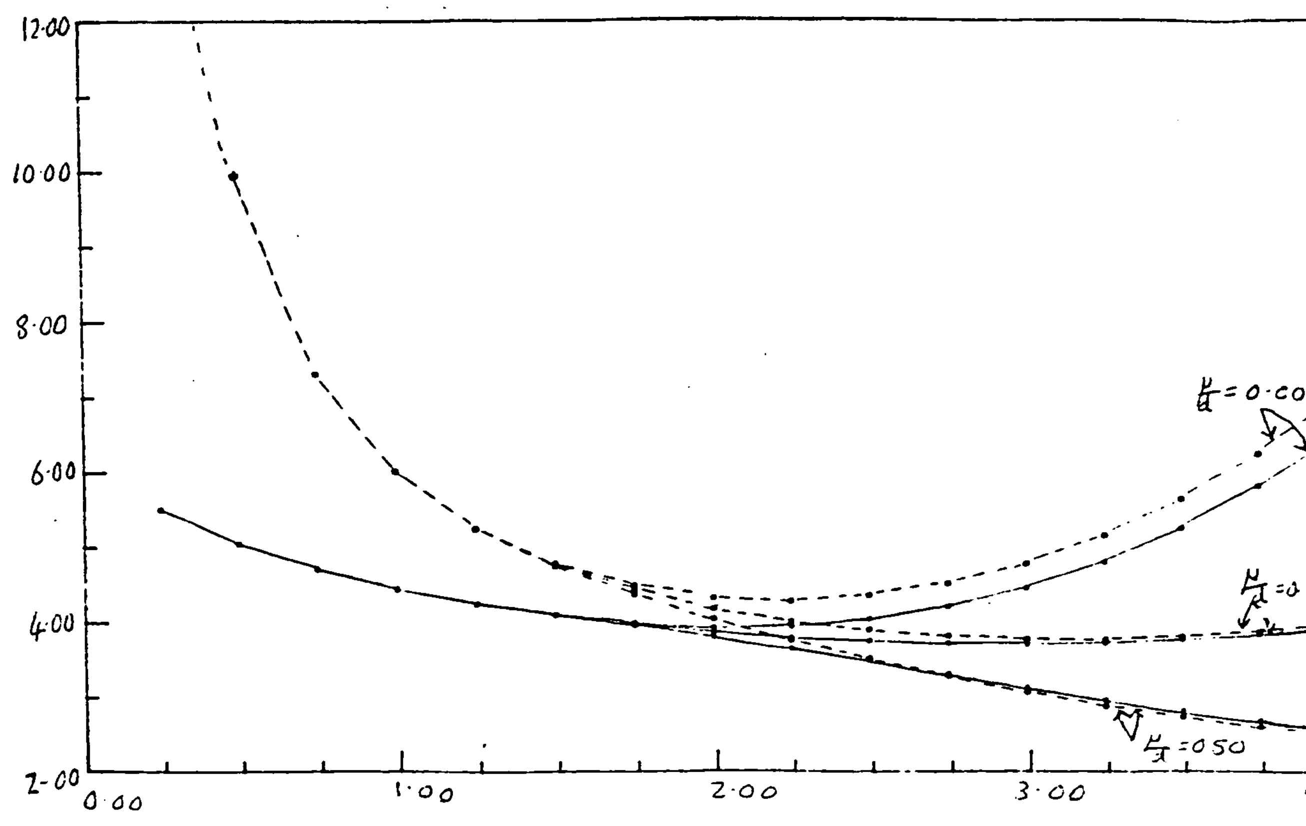


Fig. 4.1.8 $V(\beta)/\beta^2$ when the two UDTR's for $L_{1/\sqrt{2}}$ and $L_{1-1/\sqrt{2}}$ are used (values for the Up and Down rule are joined by dashed lines).



$L_{1/\sqrt{2}}$ and $L_{1-1/\sqrt{2}}$ from the two respective staircases. Suppose the m.l.e. of $L_{1/\sqrt{2}}$ from the staircase designed to place observations around the $L_{1/\sqrt{2}}$ level is $\hat{L}_{1/\sqrt{2}}$, and the m.l.e. of $L_{1-1/\sqrt{2}}$ from the other staircase is $\hat{L}_{1-1/\sqrt{2}}$. These two estimators can be combined to give an estimator of $1/\beta$ of the form.

$$(\hat{L}_{1/\sqrt{2}} - \hat{L}_{1-1/\sqrt{2}})/2\bar{k}. \qquad 4.1.6$$

As the staircases are independent the variance of such an estimator is simply

$$((\text{variance of } \hat{L}_{1/\sqrt{2}}) + (\text{variance of } \hat{L}_{1-1/\sqrt{2}}))/4\bar{k}^2 \qquad 4.1.7$$

The estimator of $1/\beta$ given in Formula 4.1.6 is asymptotically unbiased but does not have full efficiency relative to the m.l.e. of $1/\beta$. Suppose equal numbers of observations are made in each staircase. Expressions for the variances of the m.l.e.'s of μ and β are given by the formulae in matrix 2.3.7 (there is sufficient regularity to apply results in Billingsley (1961)). If a small step size is chosen all observations are asymptotically made close to the $L_{1/\sqrt{2}}$ and $L_{1-1/\sqrt{2}}$ levels. The estimator in Formula 4.1.6 will then be of high efficiency relative to the m.l.e. of $1/\beta$. However the efficiency rapidly drops as step size increases. Table 4.1.3 gives values of this asymptotic efficiency for phases of 0.00,0.25 and 0.50 (from symmetry the efficiency for phasing 0.75 is the same as for 0.25). What these calculations are indicating is that, unless a very small step size is chosen, one cannot expect an estimator such as that in Formula 4.1.4 to have variance close to

Table 4.1.3 Efficiency of the estimator in Formula 4.1.6 relative to the m.l.e. of $1/\beta$.

μ/d \ βd	0.25	0.50	0.75	1.00	1.50	2.00	3.00	4.00
0.00	0.852	0.744	0.661	0.595	0.498	0.435	0.400	0.457
0.25	0.852	0.744	0.661	0.595	0.498	0.430	0.343	0.294
0.50	0.852	0.744	0.661	0.595	0.498	0.425	0.292	0.187

Table 4.1.4 Efficiency of the estimator in Formula 4.1.8 relative to the m.l.e. of μ.

μ/d \ βd	0.25	0.50	0.75	1.00	1.50	2.00	3.00	4.00
0.00	0.989	0.980	0.973	0.966	0.954	0.938	0.870	0.735
0.25	0.989	0.980	0.973	0.966	0.954	0.945	0.937	0.950
0.50	0.989	0.980	0.973	0.966	0.955	0.952	0.974	0.995

Table 4.1.5 Efficiency of the estimator analogous to $1/\beta'$, based on m.l.e.'s from both staircases, relative to the m.l.e. of $1/\beta$.

μ/d \ βd	0.25	0.50	0.75	1.00	1.50	2.00	3.00	4.00
0.00	0.980	0.966	0.955	0.947	0.935	0.929	0.937	0.961
0.25	0.980	0.966	0.955	0.947	0.936	0.928	0.911	0.902
0.50	0.980	0.966	0.955	0.947	0.936	0.929	0.907	0.873

that of the m.l.e. of $1/\beta$ from the two staircases. An estimate of the μ level can be formed equal to

$$(\hat{L}_{1/\sqrt{2}} + \hat{L}_{1-1/\sqrt{2}})/2. \qquad 4.1.8$$

The efficiency of this estimator relative to the m.l.e. of μ is high for the parameter values considered. Table 4.1.4 gives values of asymptotic efficiency for for phases 0.00, 0.25 and 0.50. This suggests that an estimator of the form of the expression in Formula 4.1.5 could possibly give estimates of μ whose variance is close to that of the m.l.e. of μ.

If one proposes to form an estimate of $1/\beta$ from the two staircases, without using maximum likelihood estimation, then there are serious objections to the use of an expression such as that in Formula 4.1.4. The use of an expression such as that in Formula 4.1.5 may provide useful estimates of μ. In forming Wetherill's estimate of $1/\beta$ one is ignoring any possible information available for estimating $1/\beta$ from the individual staircases. The estimate of $1/\beta$ given in 4.1.3 can be calculated for both staircases. Suppose these estimates are respectively $1/\beta_1$ and $1/\beta_2$ and that

$$1/\beta_3 = (A_{1/\sqrt{2}} - A_{1-1/\sqrt{2}})/2\bar{k}, \qquad 4.1.9$$

where $A_{1/\sqrt{2}}$ and $A_{1-1/\sqrt{2}}$ are the Dixon and Mood estimates of $L_{1/\sqrt{2}}$ and $L_{1-1/\sqrt{2}}$ (i.e. $1/\beta_3$ is an estimate similar to Wetherill's only based on using E_{DM}). It is not at clear how such estimates of $1/\beta$ should be combined. Suppose in the staircase for $L_{1/\sqrt{2}}$ that there are r_1 responses of the less frequent type, at levels $y_{11}, y_{12}, \ldots, y_{1r_1}$ (here by response I mean a move up or down). Suppose that $z_{1k} = y_{1k} \pm d/2$, where the sign is negative if moves down are most frequent and positive otherwise. By definition $A_{1/\sqrt{2}} = \sum_{k=1}^{r_1} z_{1k} / r_1$ and $1/\beta_1 = \lambda \sum_{k=1}^{r_1} (z_{1k} - A_{1/\sqrt{2}})^2 / r_1 d$, where $\lambda = 4(1 - (0.5)^{1/2})$. Suppose for the other staircase there are r_2 responses of the less frequent type and that $z_{21}, z_{22}, \ldots, z_{2r_2}$ are defined in the same way as the z_{1k} of the first staircase. Define V to equal

$$\sum_{i=1}^{2} \sum_{k=1}^{r_i} (z_{ik} - \bar{z})^2 / (r_1 + r_2), \qquad 4.1.10$$

where $\bar{z} = \sum_{i=1}^{2} \sum_{k=1}^{r_i} z_{ik} / (r_1 + r_2)$. This is of course a variance expression for all the z_{ik}. Rearranging it follows that

$$V = (d/\lambda)((\theta_1/\beta_1) + (\theta_2/\beta_2)) + \theta_1 \theta_2 4(\bar{k}/\beta_3)^2, \qquad 4.1.11$$

where $\theta_1 = r_1/(r_1 + r_2)$ and $\theta_2 = r_2/(r_1 + r_2)$. For d sufficiently small the limits with probability one of all the β_i are arbitrarily close to β. One can set V equal to

$$(d/\lambda)((\theta_1/\beta) + (\theta_2/\beta)) + \theta_1 \theta_2 4(\bar{k}/\beta)^2. \qquad 4.1.12$$

to obtain a quadratic in $1/\beta$. The quadratic has only one positive root; I will call this root $1/\beta'$. It can be used to provide an estimate of $1/\beta$. The form of this estimator is somewhat complicated but simulations in Section 4.2 indicate that under certain conditions it has much lower m.s.e. than $1/\beta$. For small step sizes $1/\beta'$ is close to $1/\beta_3$; for large step sizes $1/\beta'$ is close to $((\theta_1/\beta_1)+(\theta_2/\beta_2))$.

I used $1/\beta'$ as an estimator because I could see of no other natural way of combining the estimates of $1/\beta$. One can make some justication for using $1/\beta'$ by calculating asymptotic efficiencies of an analogous estimator based on m.l.e.'s from both staircases. Suppose one operates both staircases so that equal numbers of observations are made in each. Asymptotically θ_1 and θ_2 will tend in probability to 0.5. Suppose that $\hat{\beta}_1$ and $\hat{\beta}_2$ are the m.l.e.'s of β from the two staircases and $1/\hat{\beta}_3$ equals the expression in 4.1.6. If one substitutes these $\hat{\beta}_i$ for the β_i in the expression for $1/\beta'$, then the resulting expression is an asymptotically unbiased estimator of $1/\beta$. This estimator is asymptotically equivalent to its first order Taylor expansion in terms of $1/\hat{\beta}_1$, $1/\hat{\beta}_2$ and $1/\hat{\beta}_3$. Using this expansion one can calculate the aymptotic efficiency of the estimator relative to the m.l.e. of $1/\beta$. Values of the efficiency are given in Table 4.1.5. The efficiencies are high compared to the values in Table 4.1.3.

4.2 RESULTS OF SOME SIMULATIONS

An UDTR rule which is in common use is that designed to centre observations close to the $L_{1/\sqrt{2}}$ level of the response curve. This rule has been described in Section 4.1 (i.e. after - or +- responses move up, after ++ move down). Experiments were simulated with this rule operating on the logistic curve. The set of conditions considered was similar to those used in the calculations of Section 2.2 and the simulations of Section 3.3. Starting levels were at -2.00(0.25)2.00 relative to the $L_{1/\sqrt{2}}$ level (the logistic tolerance distribution is symmetric so in Section 2.2 and 3.3 one only had to consider starting levels above μ; with the UDTR rule being used starts both above and below $L_{1/\sqrt{2}}$ must be considered). The value of β was again set equal to $\pi/3.0^{1/2}$ and step sizes used were 0.5(0.5)2.0. For each set of conditions 2000 experiments consisting of 24 observations were simulated. The estimators E_M, E_B, E_{BD}, E_{DM}, E_{WE}, E_{PV}, E_{WE}^* and E_{PV}^* were calculated. For a rough estimate of $1/\beta$ the estimator $1/\tilde{\tilde{\beta}}$ was used (see Formula 4.1.3).

The m.s.e.'s and biases of these estimators of $L_{1/\sqrt{2}}$ are given in Tables 4.2.1 to 4.2.8. My conclusions concerning the relative merits of estimators are much the same as those made in Section 3.3. The estimators E_M and E_B have similar m.s.e.'s to those for E_{BD} and E_{DM} for the larger step sizes, but have the disadvantage of larger m.s.e. and bias for the smaller step sizes and distant starts. The estimator E_{BD} usually has slightly smaller m.s.e. and smaller bias than E_{DM} but these differences are never very great.

Table 4.2.1 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 24 observation UDTR experiments for step size 0.5 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E_{WE}^*	E_{PV}^*	A_{WE}^*
-2.00	18.12	10.34	8.35	8.98	7.82	9.92	9.85	8.31	8.85	8.67	7.88
-1.75	14.26	8.85	8.84	9.53	7.82	10.31	10.32	8.31	9.35	9.17	7.88
-1.50	10.84	7.36	8.20	8.86	7.82	9.67	9.69	8.31	8.66	8.52	7.88
-1.25	9.08	6.98	7.99	8.75	7.82	9.18	9.26	8.31	8.43	8.33	7.88
-1.00	7.35	6.37	7.32	8.03	7.82	8.26	8.42	8.31	7.60	7.54	7.88
-0.75	6.68	6.45	7.11	7.84	7.82	7.94	8.23	8.31	7.31	7.31	7.88
-0.50	6.28	6.52	6.82	7.35	7.82	7.57	7.95	8.31	6.99	7.09	7.88
-0.25	6.10	6.63	6.68	6.98	7.82	7.26	7.68	8.31	6.83	7.03	7.88
0.00	6.23	6.85	6.75	6.92	7.82	7.29	7.74	8.31	6.92	7.13	7.88
0.25	6.60	7.07	7.08	7.32	7.82	7.55	7.96	8.31	7.22	7.40	7.88
0.50	7.31	7.39	7.61	8.14	7.82	8.35	8.70	8.31	7.85	7.94	7.88
0.75	8.64	8.01	8.50	9.34	7.82	9.48	9.68	8.31	8.82	8.80	7.88
1.00	10.65	8.93	9.68	10.74	7.82	11.21	11.29	8.31	10.18	10.03	7.88
1.25	13.44	10.27	11.08	12.23	7.82	13.08	13.10	8.31	11.79	11.56	7.88
1.50	16.83	11.59	11.68	12.88	7.82	13.96	13.88	8.31	12.56	12.25	7.88
1.75	22.41	14.38	13.09	14.35	7.82	15.87	15.63	8.31	14.08	13.71	7.88
2.00	29.56	18.06	14.25	15.51	7.82	17.51	17.23	8.31	15.42	15.01	7.88

Table 4.2.2 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 24 observation UDTR experiments for step size 1.0 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E_{WE}^*	E_{PV}^*	A_{WE}^*
-2.00	10.82	7.81	9.37	9.83	9.57	10.87	11.19	10.62	9.77	9.74	10.37
-1.75	9.78	8.11	9.64	10.08	9.57	10.66	11.04	9.78	9.70	9.72	9.44
-1.50	8.92	8.19	9.48	10.05	9.64	10.34	10.75	9.44	9.54	9.63	8.99
-1.25	8.98	8.79	9.71	10.19	9.64	10.56	10.95	10.28	9.78	9.86	9.90
-1.00	7.99	8.26	8.80	9.24	9.57	9.91	10.43	10.62	9.08	9.21	10.37
-0.75	8.23	8.90	9.08	9.50	9.57	9.55	10.25	9.78	9.05	9.35	9.44
-0.50	8.20	8.95	8.91	9.34	9.64	9.32	10.17	9.44	8.85	9.29	8.99
-0.25	8.68	9.45	9.30	9.58	9.64	9.89	10.84	10.28	9.34	9.82	9.90
0.00	8.32	8.97	8.76	9.07	9.57	9.63	10.55	10.62	9.01	9.44	10.37
0.25	9.00	9.53	9.37	9.77	9.57	10.03	10.86	9.78	9.38	9.78	9.44
0.50	9.52	9.86	9.83	10.24	9.64	10.31	11.09	9.44	9.80	10.14	8.99
0.75	10.01	10.17	10.36	10.82	9.64	11.03	11.56	10.28	10.51	10.71	9.90
1.00	10.23	9.77	10.18	10.84	9.57	11.49	11.91	10.62	10.65	10.64	10.37
1.25	11.55	10.29	10.99	11.82	9.57	12.05	12.27	9.78	11.14	10.94	9.44
1.50	12.64	10.27	11.08	12.07	9.64	12.32	12.46	9.44	11.35	11.03	8.99
1.75	14.07	10.71	11.76	12.63	9.64	13.32	13.46	10.28	12.27	11.92	9.90
2.00	15.70	10.58	11.33	12.22	9.57	13.44	13.48	10.62	12.26	11.81	10.37

Note: A_{DM}, A_{WE} and A_{WE}^* denote columns for asymptotic predicted m.s.e.'s of E_{DM}, E_{WE} and E_{WE}^* respectively

Table 4.2.3 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 24 observation UDTR experiments for step size 1.5 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E^*_{WE}	E^*_{PV}	A^*_{WE}
-2.00	11.12	10.35	11.94	12.26	11.95	12.13	12.84	10.79	11.11	11.43	10.35
-1.75	11.62	11.49	12.80	13.21	12.20	15.38	16.16	14.68	13.81	14.02	14.42
-1.50	10.93	11.18	11.98	12.46	11.87	15.14	15.89	16.12	13.75	13.84	16.04
-1.25	9.97	10.57	10.90	11.29	11.28	12.47	13.11	13.44	11.84	12.00	13.29
-1.00	9.86	10.92	10.97	11.28	11.01	10.25	11.12	9.66	10.10	10.55	9.35
-0.75	9.96	11.33	11.20	11.29	11.35	9.20	10.31	8.43	9.28	10.03	8.00
-0.50	10.74	12.07	11.90	11.95	11.95	11.02	12.39	10.79	10.62	11.50	10.35
-0.25	11.57	12.57	12.39	12.65	12.20	14.25	15.86	14.68	13.11	14.00	14.42
0.00	11.07	11.96	11.78	12.01	11.87	14.45	16.02	16.12	13.30	14.09	16.04
0.25	10.95	11.81	11.66	11.79	11.28	13.27	14.55	13.44	12.42	13.21	13.29
0.50	10.86	11.53	11.38	11.57	11.01	10.79	11.94	9.66	10.50	11.15	9.35
0.75	11.09	11.34	11.27	11.68	11.35	9.89	10.68	8.43	9.75	10.12	8.00
1.00	13.09	12.81	12.89	13.78	11.95	12.90	13.44	10.79	12.40	12.47	10.35
1.25	13.63	13.13	13.52	14.27	12.20	15.90	16.34	14.68	14.73	14.61	14.42
1.50	13.56	12.58	13.27	13.93	11.87	16.71	17.18	16.12	15.19	14.98	16.04
1.75	13.55	11.77	12.72	13.37	11.28	15.02	15.36	13.44	13.92	13.63	13.29
2.00	14.48	11.55	12.51	13.42	11.01	12.78	12.97	9.66	12.17	11.80	9.35

Table 4.2.4 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 24 observation UDTR experiments for step size 2.0 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E^*_{WE}	E^*_{PV}	A^*_{WE}
-2.00	15.11	14.92	16.08	16.61	16.00	23.04	23.88	25.34	20.64	20.72	25.32
-1.75	14.22	13.67	14.18	15.16	14.57	19.68	19.99	21.38	18.24	17.95	21.23
-1.50	12.29	12.69	12.87	13.50	12.68	14.40	14.68	14.15	13.96	13.98	13.85
-1.25	10.73	11.97	11.90	12.03	11.44	9.36	10.11	8.14	9.62	9.98	7.84
-1.00	10.27	12.50	12.25	11.83	11.86	7.63	8.91	6.42	8.06	9.14	6.13
-0.75	12.32	15.09	14.85	14.03	13.69	11.21	13.12	9.75	11.21	12.88	9.39
-0.50	14.00	16.41	16.27	15.52	15.57	16.06	18.19	16.39	15.15	16.85	15.98
-0.25	15.82	17.50	17.35	17.16	16.40	21.58	23.99	23.01	19.76	21.37	22.79
0.00	15.03	16.37	16.27	16.16	16.00	22.02	24.43	25.34	20.10	21.50	25.32
0.25	14.12	15.46	15.37	15.21	14.57	20.17	22.19	21.38	18.36	19.79	21.23
0.50	12.34	13.41	13.28	13.29	12.68	14.70	16.59	14.15	13.95	15.27	13.85
0.75	11.37	12.19	12.05	11.92	11.44	9.57	10.93	8.14	9.83	10.71	7.84
1.00	12.13	12.07	11.95	12.32	11.86	8.21	8.85	6.42	8.61	8.84	6.13
1.25	15.19	14.33	14.35	15.20	13.69	12.26	12.14	9.75	12.12	11.80	9.39
1.50	17.93	16.58	16.85	17.96	15.57	18.54	17.95	16.39	17.60	16.98	15.98
1.75	18.74	17.30	17.93	18.98	16.40	23.56	23.40	23.01	21.81	21.13	22.79
2.00	17.79	16.49	17.56	18.44	16.00	24.89	25.59	25.34	22.58	22.15	25.32

Table 4.2.5 100×bias of estimators of $L_{1/\sqrt{2}}$ in 24 observation UDTR experiments for step size 0.5 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E^*_{WE}	E^*_{PV}	A_{WE}
-2.00	-37.28	-23.75	-6.38	-7.75	2.27	-9.96	-8.20	-8.34	-7.06	1.75
-1.75	-30.61	-18.53	-5.92	-7.32	2.27	-9.26	-7.61	-7.89	-6.61	1.75
-1.50	-24.32	-13.74	-5.35	-6.66	2.27	-8.63	-7.00	-7.17	-5.93	1.75
-1.25	-19.46	-10.48	-5.55	-6.94	2.27	-8.56	-7.11	-7.22	-6.03	1.75
-1.00	-14.25	-6.90	-4.39	-5.96	2.27	-6.92	-5.60	-5.93	-4.85	1.75
-0.75	-9.85	-4.21	-3.31	-4.99	2.27	-5.42	-4.35	-4.60	-3.65	1.75
-0.50	-6.15	-2.20	-2.08	-3.80	2.27	-3.67	-2.91	-3.15	-2.47	1.75
-0.25	-2.30	-0.03	-0.15	-1.40	2.27	-1.22	-0.89	-0.67	-0.25	1.75
0.00	2.22	2.80	2.82	2.27	2.27	2.30	2.25	2.56	2.63	1.75
0.25	5.65	4.33	4.49	4.85	2.27	4.56	4.06	4.69	4.40	1.75
0.50	9.79	6.60	6.52	7.64	2.27	7.20	6.38	7.13	6.57	1.75
0.75	15.00	10.06	9.28	10.67	2.27	10.64	9.45	10.34	9.54	1.75
1.00	20.44	13.62	11.34	12.89	2.27	13.28	11.87	12.76	11.77	1.75
1.25	25.96	17.26	12.51	14.03	2.27	14.78	13.11	14.23	13.13	1.75
1.50	32.44	21.83	13.82	15.19	2.27	16.65	14.81	15.70	14.53	1.75
1.75	40.31	27.70	15.43	16.88	2.27	18.45	16.37	17.31	16.08	1.75
2.00	48.58	33.96	16.39	17.85	2.27	19.78	17.53	18.35	17.00	1.75

Table 4.2.6 100×bias of estimators of $L_{1/\sqrt{2}}$ in 24 observation UDTR experiments for step size 1.0 ($\beta=\pi/3.0^{1/2}$; based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E^*_{WE}	E^*_{PV}	A_{WE}
-2.00	-20.06	-5.68	0.86	-1.78	3.87	-3.25	-0.62	-2.25	0.25	2.51
-1.75	-15.55	-2.64	1.65	-0.86	3.99	-2.76	-0.24	-1.51	0.97	2.39
-1.50	-11.76	-0.38	2.05	-0.46	4.01	-1.75	0.68	-0.81	1.60	3.19
-1.25	-9.84	-0.31	0.72	-1.76	3.89	-2.46	-0.14	-1.60	0.71	3.31
-1.00	-6.69	1.23	1.36	-1.29	3.87	-2.02	0.04	-1.17	0.96	2.51
-0.75	-3.32	2.99	2.67	0.16	3.99	-0.29	1.38	0.60	2.45	2.39
-0.50	-0.89	3.84	3.42	1.02	4.01	1.32	2.48	2.01	3.39	3.19
-0.25	0.48	3.25	3.01	1.32	3.89	1.70	2.21	2.28	3.10	3.31
0.00	2.57	3.57	3.60	2.58	3.87	2.20	2.01	2.86	3.04	2.51
0.25	5.22	4.56	4.81	4.51	3.99	3.65	2.64	4.23	3.84	2.39
0.50	7.56	5.16	5.56	6.04	4.01	5.52	3.98	5.71	4.81	3.19
0.75	9.53	5.19	5.53	6.77	3.89	6.46	4.41	6.47	5.09	3.31
1.00	12.63	6.56	6.52	8.10	3.87	7.12	4.68	7.34	5.63	2.51
1.25	16.37	8.55	7.72	9.44	3.99	8.56	5.81	8.98	7.08	2.39
1.50	20.04	10.48	8.43	10.18	4.01	9.85	6.80	10.33	8.34	3.19
1.75	22.82	11.25	7.36	9.10	3.89	9.17	5.97	9.75	7.65	3.31
2.00	27.60	14.21	8.09	10.00	3.87	9.70	6.33	10.31	8.08	2.51

Note: A_{DM} and A_{WE} denote columns for asymptotic predicted biases of E_{DM} and E_{WE} respectively.

Table 4.2.7 100×bias of estimators of $L_{1/\sqrt{2}}$ in 24 observation UDTR experiments for step size 1.5 ($\beta=\pi/3.0^{1/2}$; based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
-2.00	-12.69	2.53	5.28	1.73	6.31	1.94	5.28	2.73	6.31	8.31
-1.75	-10.54	2.96	4.37	0.71	5.16	0.73	4.13	1.64	5.25	6.92
-1.50	-8.40	3.35	3.71	0.22	3.95	-1.84	1.46	-0.55	3.05	2.06
-1.25	-6.14	3.96	3.67	0.22	3.89	-3.56	-0.69	-1.96	1.52	-1.15
-1.00	-2.72	6.06	5.63	1.85	5.03	-0.76	1.71	0.59	3.76	0.41
-0.75	0.02	7.12	6.63	3.24	6.25	3.05	5.07	3.85	6.55	5.02
-0.50	2.15	7.46	7.05	4.10	6.31	6.09	7.58	6.44	8.43	8.31
-0.25	2.67	6.09	5.83	3.45	5.16	5.24	5.88	5.89	7.08	6.92
0.00	3.02	4.57	4.52	3.05	3.95	2.36	2.14	3.52	3.93	2.06
0.25	4.35	4.09	4.23	3.60	3.89	0.81	-0.53	1.99	1.56	-1.15
0.50	6.05	4.26	4.61	4.59	5.03	1.78	-0.42	2.78	1.68	0.41
0.75	8.89	5.33	5.83	6.69	6.25	5.78	2.98	6.10	4.35	5.02
1.00	12.31	7.01	7.55	9.11	6.31	9.83	6.48	9.79	7.65	8.31
1.25	13.50	6.45	6.80	8.66	5.16	9.49	5.65	9.55	7.01	6.92
1.50	14.72	5.66	5.43	7.81	3.95	6.44	2.40	6.99	4.18	2.06
1.75	17.30	6.41	5.31	7.80	3.89	4.76	0.49	5.76	2.90	-1.15
2.00	21.05	8.66	6.33	8.83	5.03	5.66	1.21	7.06	4.18	0.41

Table 4.2.8 100×bias of estimators of $L_{1/\sqrt{2}}$ in 24 observation UDTR experiments for step size 2.0 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
-2.00	-12.40	3.07	3.63	-0.88	2.74	-3.58	0.65	-1.81	3.15	1.30
-1.75	-12.80	0.96	0.75	-3.76	-0.09	-10.50	-6.59	-8.16	-3.30	-8.07
-1.50	-9.19	3.11	2.64	-2.10	0.51	-9.56	-5.93	-7.34	-2.71	-9.64
-1.25	-4.56	6.37	5.84	1.33	4.16	-4.15	-1.14	-2.69	1.81	-3.64
-1.00	2.75	12.30	11.84	7.36	8.79	5.74	8.42	6.45	10.50	6.21
-0.75	7.67	15.58	15.20	11.03	11.75	14.01	16.35	14.07	17.52	15.16
-0.50	8.04	14.05	13.79	10.20	11.22	15.67	17.51	15.57	18.20	18.32
-0.25	5.25	9.28	9.14	6.24	7.45	10.35	11.17	10.91	12.61	12.75
0.00	1.89	3.84	3.76	1.87	2.74	1.28	0.88	2.81	3.38	1.30
0.25	0.00	-0.06	-0.03	-0.96	-0.09	-5.95	-7.66	-3.90	-4.47	-8.07
0.50	2.00	0.10	0.22	0.28	0.51	-6.58	-9.04	-4.63	-6.14	-9.64
0.75	6.17	2.77	3.02	3.95	4.16	-1.67	-4.93	-0.39	-2.51	-3.64
1.00	12.06	7.36	7.82	9.31	8.79	7.12	3.15	7.74	5.21	6.21
1.25	17.12	10.84	11.46	13.34	11.75	15.33	10.87	15.15	12.11	15.16
1.50	19.21	11.22	11.86	14.18	11.22	18.59	13.54	18.05	14.79	18.32
1.75	18.37	8.37	8.71	11.33	7.45	14.64	9.23	14.49	10.86	12.75
2.00	15.69	3.81	3.37	6.22	2.74	5.39	-0.35	6.24	2.53	1.30

There is not a great deal to choose between estimators E_{WE}, $E_{\rho V}$, E_{WE}^* and $E_{\rho V}^*$. For step sizes 0.5 and 1.0 the m.s.e.'s for E_{WE}^* and $E_{\rho V}^*$ are always less than corresponding values for E_{WE} and $E_{\rho V}$ respectively. For step sizes 1.5 and 2.0 this is not always the case but m.s.e.'s of E_{WE}^* and $E_{\rho V}^*$ are never much greater than corresponding m.s.e.'s for E_{WE} and $E_{\rho V}$. For step size 0.5 the m.s.e.'s of E_{BD}, E_{DM}, E_{WE}, $E_{\rho V}$, E_{WE}^* and $E_{\rho V}^*$ are all close. Fig. 4.2.1 illustrates how the m.s.e. of E_{DM} is always slightly less than that of E_{WE}, how the m.s.e. of E_{WE}^* is always slightly less than or equal to that of E_{DM} and how the m.s.e. of E_{BD} is slightly less than that of E_{WE}^*. The pattern is roughly similar for step size 1.0 (see Fig 4.2.2). However for step sizes 1.5 and 2.0 the m.s.e.'s of E_{WE}, $E_{\rho V}$, E_{WE}^* and $E_{\rho V}^*$ oscillate above and below corresponding values for E_{BD} and E_{DM} (see Figs. 4.2.3 and 4.2.4). The dependence on phase of these oscillation is much as one would expect from asymptotic theory. It appears that for small step sizes there is not much to choose between estimators (except E_M and E_B then have large biases and m.s.e.'s for starts not close to $L_{1/\sqrt{2}}$). For the larger step sizes the behaviour of E_{WE}, $E_{\rho V}$, E_{WE}^* and $E_{\rho V}^*$ is very dependent on phase. For this reason I would recommend use of E_{BD} or E_{DM}. This is the same conclusion as I reached in Sections 2.2 and 3.3.

Table 4.2.9 contains values of expectation and m.s.e. of $1/\tilde{\beta}$ (the value of $1/\beta$ is 0.5513 to four decimals). This estimator always has some negative bias. For step sizes 1.5 and 2.0 the m.s.e. oscillates with phase, with minima roughly when the $L_{1/\sqrt{2}}$ level is at a stimulus level and maxima when it is midway between

Fig. 4.2.1 M.s.e.'s of estimators of $L_{1/\sqrt{2}}$ in 24 observation UDTR experiments with step size 0.5 ($\beta=\pi/3.\sigma^{1/2}$).

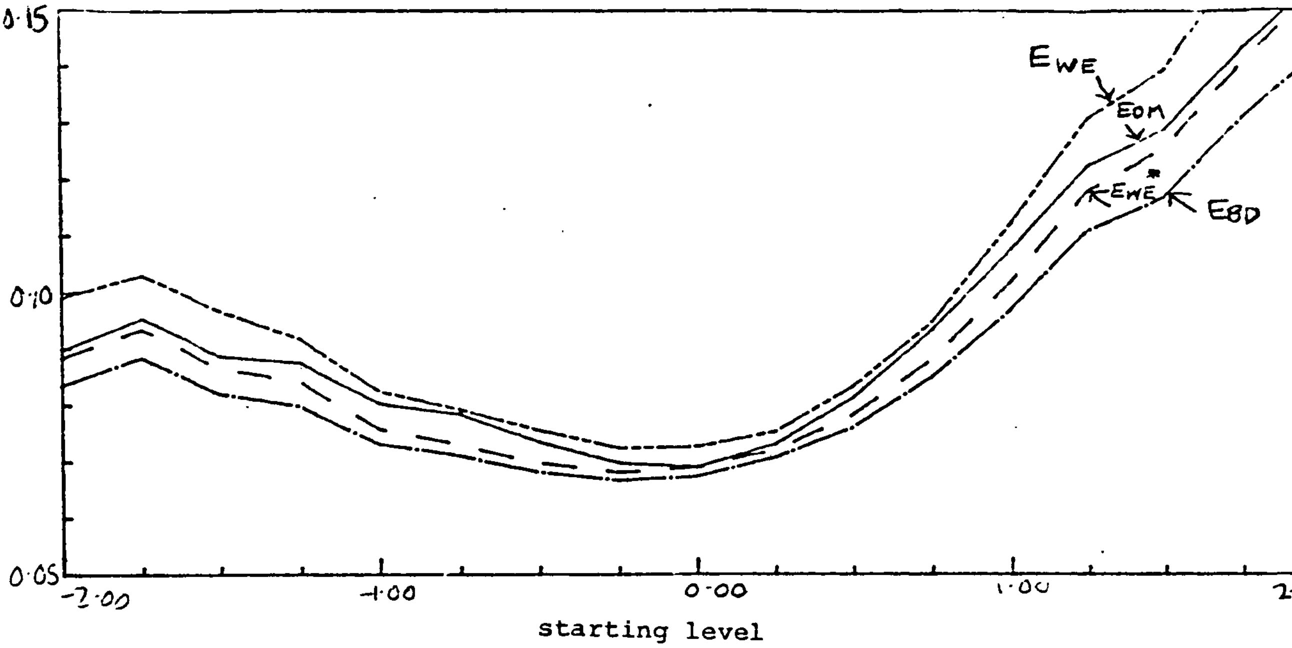


Fig. 4.2.2 As in Fig. 4.2.1 only with step size 1.0.

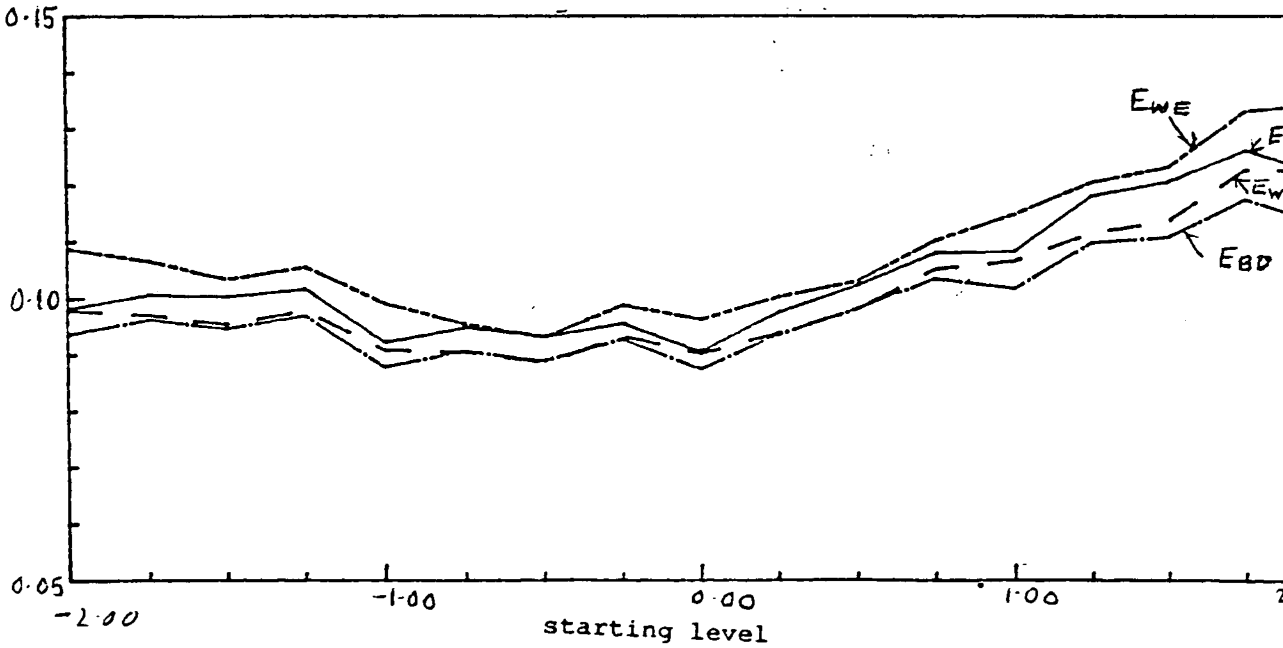


Fig. 4.2.3 As in Fig. 4.2.1 only with step size 1.5.

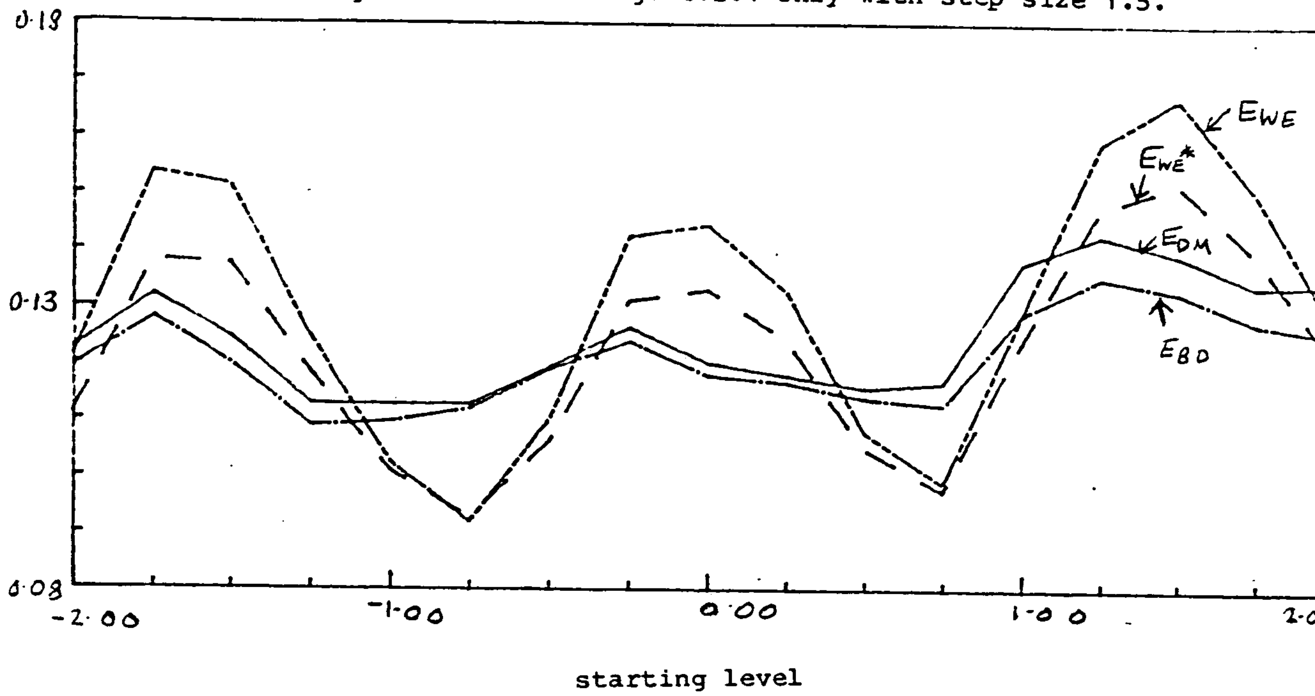


Fig. 4.2.4 As in Fig. 4.2.1 only with step size 2.0.

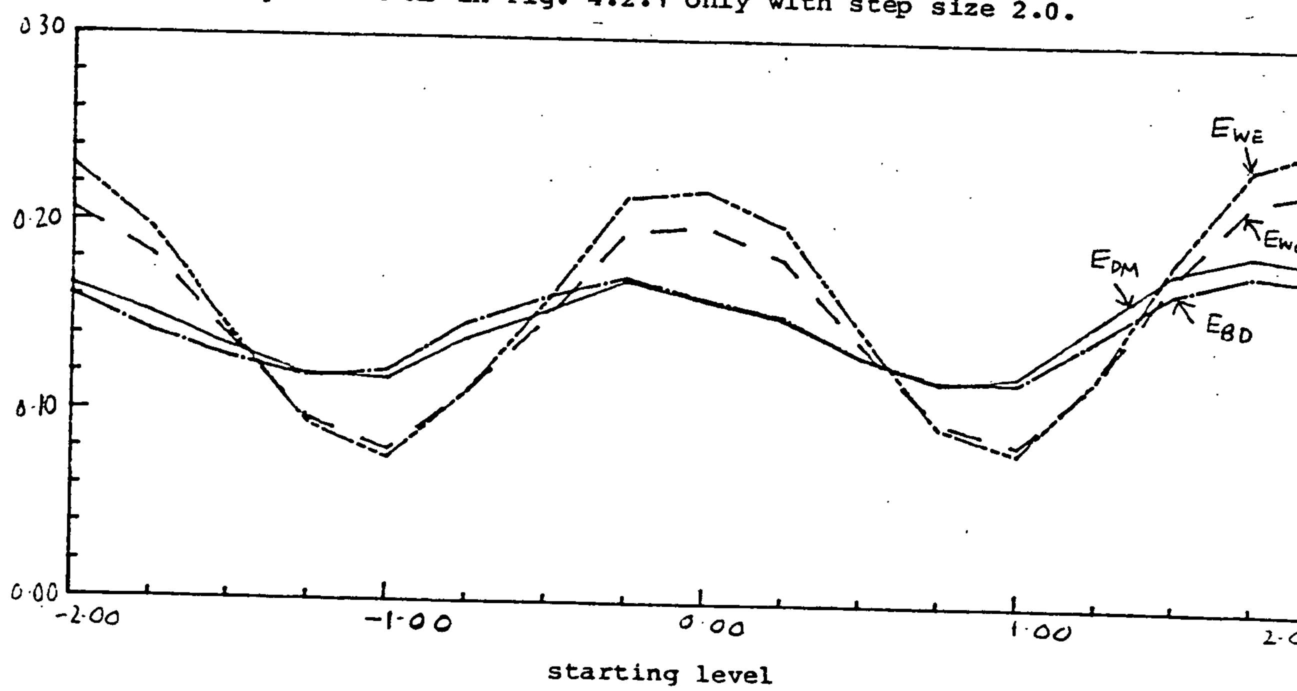


Table 4.2.9 Mean and 100×m.s.e. of $1/\tilde{\tilde{\beta}}$ in 24 observation UDTR experiments ($\beta = \pi / 3.0^{1/2}$, based on 2000 simulations).

	Step size							
	0.5		1.0		1.5		2.0	
Start	mean	100 × m.s.e.	mean	100 × m.s.e.	mean	100 × m.s.e.	mean	100 × m.s.e
−2.00	0.430	13.20	0.438	8.05	0.444	7.67	0.543	3.56
−1.75	0.413	11.81	0.436	8.11	0.465	5.88	0.499	4.74
−1.50	0.430	11.31	0.435	7.79	0.466	5.50	0.428	7.13
−1.25	0.425	10.59	0.439	7.30	0.446	6.45	0.371	10.07
−1.00	0.421	10.19	0.432	7.12	0.418	8.12	0.354	11.86
−0.75	0.403	8.72	0.429	7.42	0.413	8.26	0.394	10.32
−0.50	0.390	8.82	0.423	7.18	0.441	7.32	0.471	7.83
−0.25	0.363	9.10	0.418	6.94	0.459	5.54	0.525	4.51
0.00	0.355	8.65	0.414	6.93	0.466	4.88	0.546	3.23
0.25	0.359	8.76	0.418	7.22	0.444	5.96	0.503	4.17
0.50	0.376	8.73	0.419	7.29	0.419	7.51	0.439	6.36
0.75	0.393	9.30	0.427	7.12	0.413	8.43	0.373	9.80
1.00	0.421	10.48	0.438	7.42	0.437	7.89	0.355	11.85
1.25	0.427	12.47	0.440	8.18	0.464	6.36	0.390	10.67
1.50	0.437	14.80	0.442	8.50	0.473	5.78	0.454	8.59
1.75	0.453	18.86	0.446	9.22	0.463	7.10	0.518	5.38
2.00	0.464	21.78	0.443	9.08	0.435	8.39	0.541	3.90

stimulus levels. The estimator $1/\tilde{\tilde{\beta}}$ is not very accurate but with such a small sample much greater accuracy could not be expected.

The values of E_{DM} and $1/\tilde{\tilde{\beta}}$ provided starting values for $L_{1/\sqrt{2}}$ and $1/\beta$ in Newton-Raphson iterations to find m.l.e.'s of $1/\beta$ and $L_{1/\sqrt{2}}$. Iterations were performed using parameters a and β (where $a=-\beta\mu$) but the criterion for stopping iterations was that the change in estimates of $1/\beta$ and $L_{1/\sqrt{2}}$ should be less than 0.5×10.0^{-4}. Iterations were not started if a degenerate curve with β infinite would fit the observed responses. If the determinant of the matrix that is inverted at each iterative step became less than 10.0^{-8} iterations were abandoned (usually this happened when just one change of response would allow the observations to be fitted by a degenerate curve). Iterations were also abandoned if there had not been convergence after 10 iterations. In experiments where m.l.e.'s were eventually obtained on average between 3 and 5 iterative steps were taken. Using the UDTR rule there is a possibility that $1/\tilde{\tilde{\beta}}$ is 0.0, iterations were not started in such circumstances (starting values for a and β could not then be easily found), such outcomes were rare for all the conditions simulated. In the 24 step experiments, the m.s.e. of m.l.e.'s of $L_{1/\sqrt{2}}$ and $1/\beta$ are somewhat misleading measures of dispersion. For step size 0.5 the m.s.e.'s of these estimates are for many starting values above corresponding values for alternative estimators. However these m.s.e.'s are inflated by results coming from a small proportion of the simulated experiments. Discarding the few experiments for which the magnitude of the m.l.e. of $1/\beta$ was greater than 2.00 often gave rise to considerable drops in m.s.e.'s. The few experiments discarded usually make a disproportionate contribution to m.s.e.

Table 4.2.10 gives, for the various sets of conditions, the numbers of experiments which were not discarded. For all the step sizes large proportions of experiments were discarded; for step sizes 1.5 and 2.0 often more than half the experiments were not be used. For the larger step sizes most discards were made because there is only a degenerate solution to the maximum likelihood equations. The number of experiments discarded for other reasons is always less than 15 percent of the total number (often considerably less). Such discards were usually made because the determinant of the matrix to be inverted became too small.

I considered an estimator of $L_{1/\sqrt{2}}$ equal to the m.l.e. of $L_{1/\sqrt{2}}$ in the experiments which are not discarded and otherwise equal to E_{DM}. I also considered an estimator of $1/\beta$ equal to the m.l.e. of $1/\beta$ in the experiments which are not discarded and is otherwise equal to $1/\tilde{\tilde{\beta}}$. I will call these estimators $L^*_{1/\sqrt{2}}$ and $1/\beta^*$. Values of m.s.e. and bias of these estimators are given in Tables 4.2.11 to 4.2.14. For step sizes 1.0 and 1.5, $L^*_{1/\sqrt{2}}$ and $1/\beta^*$ always have smaller m.s.e. than E_{DM} and $1/\beta$. For $L^*_{1/\sqrt{2}}$ the drop in m.s.e. was usually less than 10 percent but for $1/\beta^*$ it was often around 25 percent. For step sizes 0.5 and 2.0, $L^*_{1/\sqrt{2}}$ and $1/\beta^*$ often have smaller m.s.e.'s than E_{DM} and $1/\tilde{\tilde{\beta}}$ but the differences are not so great. One advantage $L^*_{1/\sqrt{2}}$ has over E_{DM} is than the bias of $L^*_{1/\sqrt{2}}$ is often much less. When so many experiments are discarded how these results are interpreted is open to question but it does indicate that for these step sizes there is some gain in efficiency in using, whenever

Table 4.2.10 Numbers of 24 observation UDTR experiments out of 2000 where m.l.e.'s of parameters can be obtained ($\beta = \pi/3.0^{1/2}$).

Start	Step size 0.5	1.0	1.5	2.0
-2.00	1658	1507	932	562
-1.75	1676	1522	902	758
-1.50	1717	1490	1037	889
-1.25	1771	1476	1220	900
-1.00	1798	1578	1227	727
-0.75	1850	1595	1098	593
-0.50	1835	1494	923	463
-0.25	1808	1418	854	420
0.00	1794	1482	968	507
0.25	1826	1589	1175	691
0.50	1814	1521	1240	888
0.75	1829	1449	1127	873
1.00	1801	1503	965	786
1.25	1772	1528	911	620
1.50	1731	1481	972	476
1.75	1680	1371	1120	430
2.00	1670	1413	1191	497

Table 4.2.11 Values of mean and m.s.e. of $L^*_{1/\sqrt{2}}$ and $1/\beta^*$ in 24 observation UDTR experiments for step size 0.5 ($\beta=\pi/3.0^{1/2}$).

Start	$L^*_{1/\sqrt{2}}$ 100 × mean	$L^*_{1/\sqrt{2}}$ 100 × m.s.e.	$A_{L_{1/\sqrt{2}}}$	$1/\beta^*$ mean	$1/\beta^*$ 100 × m.s.e.	$A_{1/\beta}$
-2.00	-2.45	8.56	7.35	0.417	8.28	10.44
-1.75	-2.58	9.06	7.35	0.414	8.09	10.44
-1.50	-1.79	8.51	7.35	0.430	8.01	10.44
-1.25	-2.50	8.44	7.35	0.435	8.13	10.44
-1.00	-1.94	7.90	7.35	0.438	8.14	10.44
-0.75	-2.35	7.81	7.35	0.437	7.97	10.44
-0.50	-2.29	7.47	7.35	0.436	8.51	10.44
-0.25	-1.69	7.29	7.35	0.417	8.49	10.44
0.00	-0.18	7.19	7.35	0.412	7.96	10.44
0.25	-0.39	7.63	7.35	0.416	8.50	10.44
0.50	0.33	8.09	7.35	0.418	8.47	10.44
0.75	1.36	8.58	7.35	0.413	8.36	10.44
1.00	2.22	9.24	7.35	0.417	8.49	10.44
1.25	1.90	9.75	7.35	0.413	9.61	10.44
1.50	2.70	10.14	7.35	0.405	9.99	10.44
1.75	3.00	11.29	7.35	0.404	11.88	10.44
2.00	2.81	12.50	7.35	0.401	11.86	10.44

Table 4.2.12 Values of mean and m.s.e. of $L^*_{1/\sqrt{2}}$ and $1/\beta^*$ in 24 observation UDTR experiments for step size 1.0 ($\beta=\pi/3.0^{1/2}$).

Start	$L^*_{1/\sqrt{2}}$ 100 × mean	$L^*_{1/\sqrt{2}}$ 100 × m.s.e.	$A_{L_{1/\sqrt{2}}}$	$1/\beta^*$ mean	$1/\beta^*$ 100 × m.s.e.	$A_{1/\beta}$
-2.00	-1.38	9.10	8.72	0.460	6.07	6.70
-1.75	-0.45	9.53	8.76	0.455	6.50	6.54
-1.50	-0.31	9.54	8.77	0.454	6.72	6.68
-1.25	-2.09	9.52	8.73	0.469	6.16	6.85
-1.00	-1.79	8.58	8.72	0.476	5.99	6.70
-0.75	-0.63	8.93	8.76	0.473	6.51	6.54
-0.50	-0.15	8.64	8.77	0.460	6.76	6.68
-0.25	-0.99	9.11	8.73	0.461	6.36	6.85
0.00	-0.60	8.61	8.72	0.469	6.14	6.70
0.25	-0.32	8.90	8.76	0.478	6.86	6.54
0.50	0.34	8.99	8.77	0.466	6.85	6.68
0.75	0.02	9.55	8.73	0.463	6.48	6.85
1.00	0.39	9.49	8.72	0.472	6.34	6.70
1.25	1.38	10.05	8.76	0.465	6.69	6.54
1.50	1.87	9.98	8.77	0.456	7.33	6.68
1.75	0.55	10.95	8.73	0.455	7.16	6.85
2.00	0.76	10.39	8.72	0.457	6.53	6.70

Note: $A_{L_{1/\sqrt{2}}}$ and $A_{1/\beta}$ denote columns for asymptotic predicted variances of $L^*_{1/\sqrt{2}}$ and $1/\beta^*$ respectively.

Table 4.2.13 Values of mean and m.s.e. of $L_{1/\sqrt{2}}^*$ and $1/\beta^*$ in 24 observation UDTR experiments for step size 1.5 ($\beta=\pi/3.0^{1/2}$).

Start	$L_{1/\sqrt{2}}^*$ 100 × mean	100 × m.s.e.	$A_{L_{1/\sqrt{2}}}$	$1/\beta^*$ mean	100 × m.s.e.	$A_{1/\beta}$
-2.00	0.97	11.28	10.55	0.454	6.80	6.09
-1.75	-0.38	12.45	10.12	0.489	5.11	6.81
-1.50	-1.28	11.78	9.97	0.507	4.56	6.28
-1.25	-1.27	10.48	10.25	0.494	5.41	5.26
-1.00	0.36	9.88	10.60	0.463	7.47	4.79
-0.75	1.60	9.80	10.75	0.444	7.93	5.13
-0.50	2.44	10.66	10.55	0.460	7.14	6.09
-0.25	1.52	11.88	10.12	0.489	5.28	6.81
0.00	0.17	11.53	9.97	0.511	4.46	6.28
0.25	-0.66	10.79	10.25	0.507	5.70	5.26
0.50	-0.21	9.91	10.60	0.474	7.10	4.79
0.75	1.63	9.68	10.75	0.452	8.19	5.13
1.00	3.68	11.42	10.55	0.464	7.96	6.09
1.25	3.17	12.64	10.12	0.489	5.95	6.81
1.50	1.50	12.97	9.97	0.507	4.87	6.28
1.75	0.87	11.93	10.25	0.498	5.43	5.26
2.00	1.74	11.07	10.60	0.469	6.71	4.79

Table 4.2.14 Values of mean and m.s.e. of $L_{1/\sqrt{2}}^*$ and $1/\beta^*$ in 24 observation UDTR experiments for step size 2.0 ($\beta=\pi/3.0^{1/2}$).

Start	$L_{1/\sqrt{2}}^*$ 100 × mean	100 × m.s.e.	$A_{L_{1/\sqrt{2}}}$	$1/\beta^*$ mean	100 × m.s.e.	$A_{1/\beta}$
-2.00	-2.05	16.20	10.77	0.573	3.31	7.85
-1.75	-5.39	14.85	11.53	0.543	4.27	5.68
-1.50	-3.90	12.55	12.61	0.482	6.63	4.24
-1.25	-0.26	10.57	13.47	0.419	9.43	3.77
-1.00	5.97	10.52	13.97	0.382	11.65	4.14
-0.75	9.98	12.79	13.92	0.409	10.20	5.37
-0.50	8.74	14.44	12.99	0.478	7.56	7.40
-0.25	4.65	16.67	11.40	0.545	4.67	8.89
0.00	-0.22	15.83	10.77	0.579	3.38	7.85
0.25	-3.81	14.67	11.53	0.553	4.35	5.68
0.50	-3.69	12.35	12.61	0.499	6.26	4.24
0.75	0.16	10.19	13.47	0.424	9.34	3.77
1.00	5.84	10.06	13.97	0.390	11.48	4.14
1.25	9.92	12.71	13.92	0.412	10.78	5.37
1.50	10.88	15.77	12.99	0.464	8.53	7.40
1.75	8.09	17.83	11.40	0.530	5.18	8.89
2.00	2.21	17.96	10.77	0.566	3.68	7.85

possible, maximum likelihood estimation.

I made some further simulations of 48 and 96 observation experiments under the same sets of conditions. Values of m.s.e.'s and biases of estimators are given in Appendix 10. The results of these simulations were broadly similar to those for the 24 observation experiments. For these larger numbers of observations the estimators conform more closely to asymptotic theory and there is less to choose between asymptotically equivalent estimators. For the smaller step sizes properties of E_{BD}, E_{DM}, E_{WE}, E_{PV}, E_{WE}^* and E_{PV}^* are similar. For the larger step sizes m.s.e.'s for E_{WE}, E_{PV}, E_{WE}^* and E_{PV}^* oscillate above and below corresponding values for E_{BD} and E_{DM}. I would again recommend that E_{BD} or E_{DM} should be used rather than one of the estimators related to E_{WE}. It was possible to make a more direct comparison between these estimators and the m.l.e. of $L_{1/2}$ for experiments as fewer experiments were discarded and the m.s.e.'s of m.l.e.'s are not inflated by a few bad values. The bias of $L_{1/2}^*$ is often much less than that of E_{DM} and it always has smaller m.s.e. The bias of $1/\beta^*$ is often less than that of $1/\tilde{\beta}$ and the m.s.e. of $1/\beta^*$ is often substantially smaller than that of $1/\tilde{\beta}$. This again suggests that there is some gain to be made in using m.l.e.'s. It should be remembered that, even for 96 observation experiments, large numbers of experiments were still discarded when the step size is 2.0.

In Wetherill et al some results of simulations of experiments using an UDTR rule are given. These results indicate that E_{WE} can have smaller m.s.e. than the m.l.e. of the level the UDTR is designed to estimate. In my simulations I have only seen a marked advantage of this kind in 24 observation experiments with step size 0.5 (corresponds to spacing in Wetherill et al's work of 0.9069 as they set β=1.0) and only then if one includes estimates from a few experiments which greatly inflate the m.s.e.'s. Wetherill et al remark that in there simulations

'Sometimes patterns of results occur which give the impression of a very flat response curve and maximum likelihood then extrapolates and gives estimates well outside the range of levels used'.

It seems likely that with such experiments contributing to m.s.e. that the m.s.e. alone will not be a useful measure of dispersion. In making a comparison of maximum likelihood estimation with the other procedures it is crucial to decide upon how one should treat experiments for which a degenerate curve fits the observed responses or for which outlying estimates of parameters are produced. The m.s.e.'s of the m.l.e.'s are sometimes grossly inflated by results coming from a small proportion of experiments. It cannot be right to compare m.l.e.'s with alternative estimators using these m.s.e.'s. My procedure of discarding some experiments in calculation m.l.e.'s was an attempt to make a more useful comparison.

In Section 4.1 the use of two UDTR rules to give an estimate of slope was discussed. I have simulated some experiments consisting of two staircases, both of 24 observations, one being designed to concentrate observations around the $L_{1/\sqrt{2}}$ levels and the other around the $L_{1-1/\sqrt{2}}$ level. The response curve was again logistic with β equal to $\pi/3.0^{1/2}$. The starting levels for both staircases were chosen to be at the same level, the levels were at 0.00(0.25)4.00 relative to μ (with two such complementary UDTR's being operated from symmetry there is no need to consider starting values below μ). Step sizes were set at 0.5(0.5)2.0 with 2000 experiments simulated for each set of conditions.

Estimates of $L_{1/\sqrt{2}}$ and $L_{1-1/\sqrt{2}}$ can be formed from the two staircases using E_{DM}. Suppose these estimates equal $A_{1/\sqrt{2}}$ and $A_{1-1/\sqrt{2}}$. The design using two UDTR rules was suggested very much with the problem of estimation of $1/\beta$ in mind. I formed an estimate of $1/\beta$ equal to

$$(A_{1/\sqrt{2}} - A_{1-1/\sqrt{2}})/2\bar{k}, \qquad 4.2.1$$

where $\bar{k}$ equals $\log(2^{1/2}+1)$ for the logistic curve (i.e. this estimator is $1/\beta_3$, see Formula 4.1.9). This is much the same as the estimator suggested by Wetherill <u>et</u> <u>al</u> only I have used estimates of $L_{1/\sqrt{2}}$ and $L_{1-1/\sqrt{2}}$ based on using E_{DM} rather than E_{WE}. Another estimator of $1/\beta$, that I call $1/\beta'$, is suggested at the end of Section 4.1. This is an estimator which combines this estimator of $1/\beta$ with the estimates of $1/\beta$ derived from both staircases using Formula 4.1.3. Its form depends to a great extent on the particular response curve that is assumed but the estimator in Formula 4.2.1 also depends on the assumed response curve through the value of $\bar{k}$.

Values of the mean and m.s.e. of $1/\beta_3$ and $1/\beta'$ are given in

Tables 4.2.15 and 4.2.16. Clearly the estimator $1/\beta'$ has a large advantage over $1/\beta_3$ in that its m.s.e. is usually much smaller. I used as an estimator for μ the mean of $A_{1/\sqrt{2}}$ and $A_{1-1/\sqrt{2}}$. I will for convenience call this estimator μ' (this estimator has a similar form to the expression in Formula 4.1.5). Values of the mean and m.s.e. of μ' are given in Table 4.2.17.

The estimators μ' and $1/\beta'$ were used as starting values for μ and $1/\beta$ in Newton-Raphson iterations to find the m.l.e.'s of μ and $1/\beta$. The convergence criterion was the same as in the iterations described in Section 3.3. The same criteria for discarding experiments were used with the additional criterion that an experiment would be discarded if the value of $1/\beta$ is 0.0. Table 4.2.18 gives, for the various sets of conditions, the numbers of experiments for which m.l.e.'s could be formed. Again for the larger step sizes most discards are made because there is only a degenerate solution to the likelihood equations (discards for other reasons are always amount to less than 6 percent of the total number of experiments). I considered estimators μ^{**} and $1/\beta^{**}$ equal to the m.l.e.'s if they could be found but otherwise equal to μ' and $1/\beta'$. Values of mean and m.s.e. of μ^{**} and $1/\beta^{**}$ are given in Tables 4.2.19 and 4.2.20. For step size 0.5 the m.s.e.'s of μ^{**} and $1/\beta^{**}$ are slightly less than corresponding values for μ' and $1/\beta'$ for starts close to μ but much less for distant starts. For the other step sizes m.s.e.'s of μ^{**} and $1/\beta^{**}$ are slightly less than those for μ' and $1/\beta'$ over the whole range of starts. These experiments were specifically designed to provide estimates of $1/\beta$; values of m.s.e.'s of $1/\beta_3$, $1/\beta'$ and $1/\beta^{**}$ are illustrated in Figs. 4.2.5 to

Table 4.2.15 Values of mean and m.s.e. of $1/\beta_3$ when 2 UDTR's of 24 observations are operated ($\beta=\pi/3.0^{1/2}$).

	Step size							
	0.5		1.0		1.5		2.0	
Start	mean	100 × m.s.e.	mean	100 × m.s.e.	mean	100 × m.s.e.	mean	100 × m.s.e.
0.00	0.507	4.90	0.562	5.99	0.596	8.00	0.659	10.77
0.25	0.515	4.62	0.558	5.97	0.600	7.73	0.645	10.14
0.50	0.529	4.64	0.563	5.86	0.586	7.54	0.599	8.73
0.75	0.538	4.87	0.570	6.00	0.576	7.42	0.560	8.31
1.00	0.548	4.93	0.580	6.25	0.580	7.20	0.547	8.45
1.25	0.566	5.31	0.584	6.65	0.597	7.66	0.558	8.37
1.50	0.574	5.79	0.594	6.61	0.615	8.53	0.603	8.91
1.75	0.581	6.10	0.593	6.89	0.619	8.42	0.647	10.31
2.00	0.586	6.24	0.600	7.11	0.609	8.11	0.670	11.19
2.25	0.593	6.76	0.593	7.16	0.597	8.09	0.671	10.95
2.50	0.600	7.31	0.598	7.14	0.603	8.27	0.620	9.27
2.75	0.608	7.88	0.605	7.36	0.609	8.65	0.584	9.05
3.00	0.611	7.96	0.602	7.50	0.617	8.92	0.565	8.80
3.25	0.627	8.97	0.596	7.67	0.611	8.75	0.567	9.51
3.50	0.623	9.20	0.599	8.01	0.607	8.79	0.612	9.73
3.75	0.637	10.20	0.604	8.34	0.603	8.66	0.658	11.14
4.00	0.651	11.57	0.605	8.42	0.602	8.79	0.674	11.76

Table 4.2.16 Values of mean and m.s.e. of $1/\hat{\beta}'$ when 2 UDTR's of 24 observations are operated ($\beta = \pi/3.0''$).

	Step size							
	0.5		1.0		1.5		2.0	
Start	mean	100 m.$\hat{s}$.e.	mean	100 m.$\hat{s}$.e.	mean	100 m.$\hat{s}$.e.	mean	100 m.$\hat{s}$.e.
0.00	0.477	3.29	0.509	2.87	0.525	2.86	0.563	2.14
0.25	0.480	3.12	0.507	2.95	0.526	2.90	0.550	2.44
0.50	0.490	3.05	0.508	2.90	0.519	3.08	0.520	3.00
0.75	0.499	3.05	0.512	2.89	0.510	3.26	0.489	3.91
1.00	0.512	2.95	0.520	2.87	0.516	3.03	0.483	4.30
1.25	0.531	3.13	0.524	3.04	0.523	2.86	0.490	3.98
1.50	0.540	3.38	0.534	3.05	0.534	3.03	0.520	3.15
1.75	0.547	3.69	0.534	3.10	0.534	3.05	0.551	2.40
2.00	0.550	4.00	0.539	3.19	0.530	3.25	0.564	2.36
2.25	0.557	4.40	0.535	3.47	0.525	3.46	0.560	2.53
2.50	0.564	4.87	0.538	3.52	0.529	3.32	0.526	2.98
2.75	0.570	5.40	0.540	3.67	0.534	3.11	0.503	3.99
3.00	0.572	5.50	0.542	3.61	0.539	3.21	0.493	4.42
3.25	0.586	6.27	0.537	3.77	0.532	3.21	0.497	4.25
3.50	0.583	6.53	0.539	3.91	0.529	3.62	0.526	3.28
3.75	0.595	7.44	0.542	3.94	0.524	3.90	0.558	2.64
4.00	0.607	8.33	0.543	4.00	0.528	3.71	0.570	2.51

Table 4.2.17 Values of mean and m.s.e. of μ' when 2 UDTR's of 24 observations are operated ($\beta = \pi/3.0''^{2}$).

Step size

Start	0.5		1.0		1.5		2.0	
	100 x mean	100 x m.s.e.	100 x mean	100 x m.s.e.	100 x mean	100 x m.s.e.	100 x mean	100 x m.s.e.
0.00	0.26	3.58	0.38	4.52	0.39	5.80	0.92	7.49
0.25	1.94	3.78	0.92	4.62	-0.48	5.67	-1.82	7.14
0.50	4.36	3.90	1.38	4.47	-0.02	5.49	-2.82	6.85
0.75	6.15	4.09	2.67	4.95	1.08	5.71	-1.95	6.55
1.00	8.23	4.72	3.80	4.93	2.19	5.75	0.87	6.24
1.25	9.44	5.13	4.37	5.20	2.62	6.06	3.39	6.72
1.50	10.93	5.64	4.43	5.26	3.77	6.46	5.31	7.18
1.75	11.44	6.14	5.82	5.55	3.00	6.33	4.87	7.87
2.00	12.30	6.54	6.05	5.62	2.89	6.38	3.90	8.21
2.25	13.02	6.93	5.71	5.90	3.88	6.33	1.28	7.84
2.50	13.70	7.52	5.88	5.73	4.38	6.43	-0.04	7.79
2.75	14.59	7.88	6.71	6.14	4.55	6.49	0.90	7.12
3.00	15.88	8.38	6.95	6.09	4.33	6.67	3.38	6.86
3.25	16.06	9.08	6.45	6.16	3.90	6.84	5.36	7.44
3.50	16.80	9.71	6.66	6.30	3.95	6.87	6.50	8.04
3.75	17.87	10.88	7.21	6.81	4.42	7.00	5.93	8.37
4.00	19.77	11.67	7.09	6.90	5.22	7.10	4.44	8.47

Table 4.2.18 Numbers of experiments operating 2 UDTR's of 24 observations for which m.l.e.'s are obtained.

	Step size			
Start	0.5	1.0	1.5	2.0
0.00	1997	1953	1593	966
0.25	1997	1964	1694	1083
0.50	1998	1989	1855	1358
0.75	1999	1974	1902	1587
1.00	1996	1960	1877	1702
1.25	1995	1961	1731	1632
1.50	1994	1987	1621	1419
1.75	1998	1972	1695	1151
2.00	1986	1955	1839	992
2.25	1990	1954	1903	1109
2.50	1985	1983	1843	1335
2.75	1986	1955	1697	1570
3.00	1987	1937	1592	1673
3.25	1972	1931	1640	1586
3.50	1964	1970	1812	1353
3.75	1961	1947	1867	1113
4.00	1954	1926	1816	961

Table 4.2.19 Values of mean and m.s.e. of $1/\beta^{**}$ when 2 UDTR's of 24 observations are operated ($\beta = \pi/3.0''$).

	Step size							
	0.5		1.0		1.5		2.0	
Start	mean	100 × m.s.e.	mean	100 × m.s.e.	mean	100 × m.s.e.	mean	100 × m.s.e.
0.00	0.504	2.99	0.517	2.48	0.523	2.42	0.559	1.66
0.25	0.504	2.99	0.518	2.48	0.529	2.42	0.549	1.91
0.50	0.505	2.88	0.520	2.37	0.531	2.46	0.529	2.39
0.75	0.505	2.83	0.519	2.35	0.527	2.47	0.511	3.01
1.00	0.506	2.76	0.516	2.37	0.525	2.36	0.512	3.34
1.25	0.511	2.82	0.517	2.43	0.523	2.24	0.512	3.08
1.50	0.508	2.78	0.522	2.34	0.525	2.29	0.529	2.43
1.75	0.509	2.88	0.518	2.43	0.526	2.34	0.549	1.84
2.00	0.503	2.96	0.517	2.41	0.526	2.38	0.557	1.65
2.25	0.504	2.90	0.515	2.54	0.525	2.45	0.550	1.71
2.50	0.503	3.07	0.514	2.50	0.521	2.42	0.527	2.18
2.75	0.504	3.16	0.513	2.61	0.519	2.25	0.514	2.97
3.00	0.499	3.06	0.512	2.50	0.520	2.25	0.509	3.31
3.25	0.503	3.30	0.509	2.61	0.516	2.36	0.507	3.15
3.50	0.494	3.34	0.512	2.62	0.518	2.57	0.523	2.36
3.75	0.494	3.35	0.511	2.66	0.518	2.70	0.546	1.77
4.00	0.491	3.48	0.507	2.66	0.517	2.61	0.555	1.62

Table 4.2.20 Values of mean and m.s.e. of $\hat{\mu}^{**}$ when 2 UDTR's of 24 observations are operated ($\beta=\pi/3.0^{1/2}$).

	Step size							
	0.5		1.0		1.5		2.0	
Start	100 $\hat{}$ mean	100 $\hat{}$ m.s.e.	100 $\hat{}$ mean	100 $\hat{}$ m.s.e.	100 $\hat{}$ mean	100 $\hat{}$ m.s.e.	100 $\hat{}$ mean	100 $\hat{}$ m.s.e.
0.00	0.13	3.41	0.44	3.92	0.18	4.63	0.59	6.06
0.25	0.04	3.52	0.05	4.02	-0.10	4.54	-1.38	5.85
0.50	0.62	3.54	-0.24	3.99	0.13	4.73	-1.55	5.64
0.75	0.56	3.59	-0.04	4.13	-0.32	5.06	-0.97	5.78
1.00	0.97	3.78	0.39	4.04	-0.51	4.97	-0.22	6.01
1.25	0.86	3.92	0.43	4.32	-0.48	4.85	0.38	5.87
1.50	1.10	4.02	0.06	4.47	0.44	4.96	1.78	5.53
1.75	0.56	4.19	0.65	4.47	0.54	5.12	1.96	6.07
2.00	0.71	4.21	0.77	4.38	0.39	5.48	1.38	6.78
2.25	0.55	4.42	0.19	4.64	0.12	5.76	0.01	6.61
2.50	0.33	4.81	0.00	4.59	-0.24	5.36	-0.79	6.77
2.75	0.36	4.76	0.35	4.60	0.04	4.98	-0.25	6.52
3.00	0.91	4.69	0.65	4.68	0.25	5.16	0.63	6.70
3.25	-0.38	5.20	0.27	4.88	0.53	5.45	1.05	6.40
3.50	-0.04	5.38	0.15	5.05	0.41	5.66	1.99	6.11
3.75	-0.38	5.73	0.57	5.03	-0.07	5.93	2.35	6.37
4.00	0.06	5.80	0.32	5.09	-0.19	5.62	1.62	6.95

Fig. 4.2.5 M.s.e.'s of estimators of $1/\beta$ in experiments using 2 UDTR rules of 24 observations, with step size 0.5.

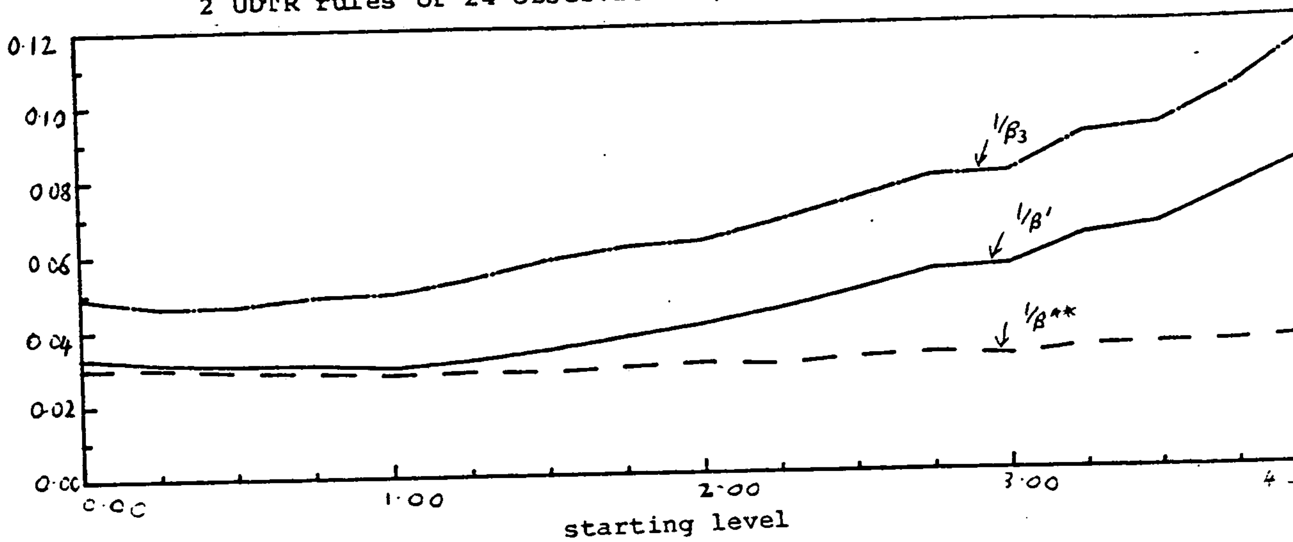


Fig. 4.2.6 As in Fig. 4.2.5 only with step size 1.0.

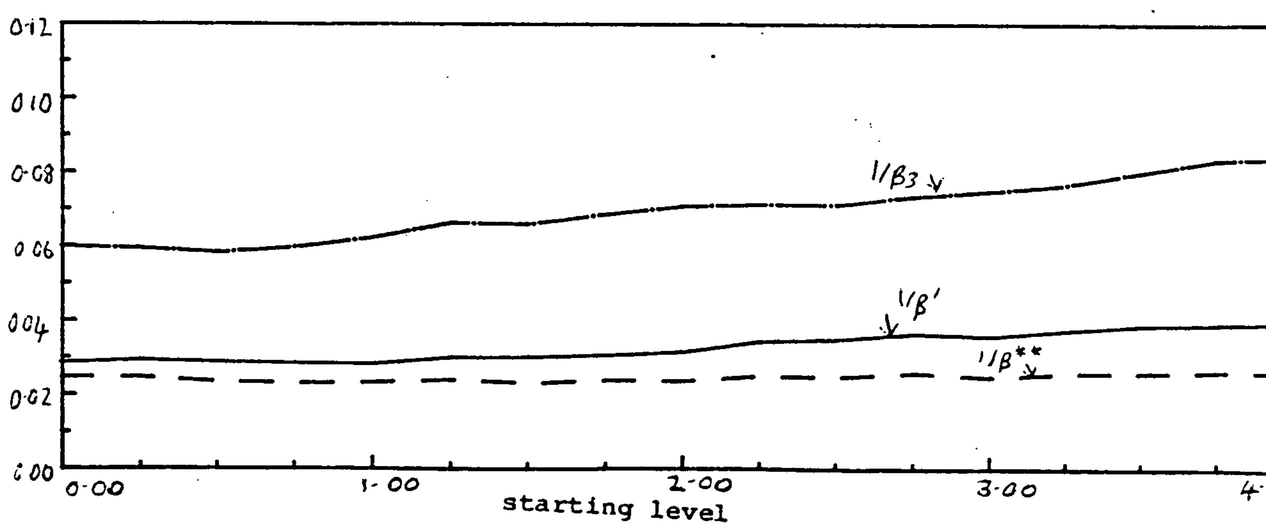


Fig. 4.2.7 As in Fig. 4.2.5 only with step size 1.5.

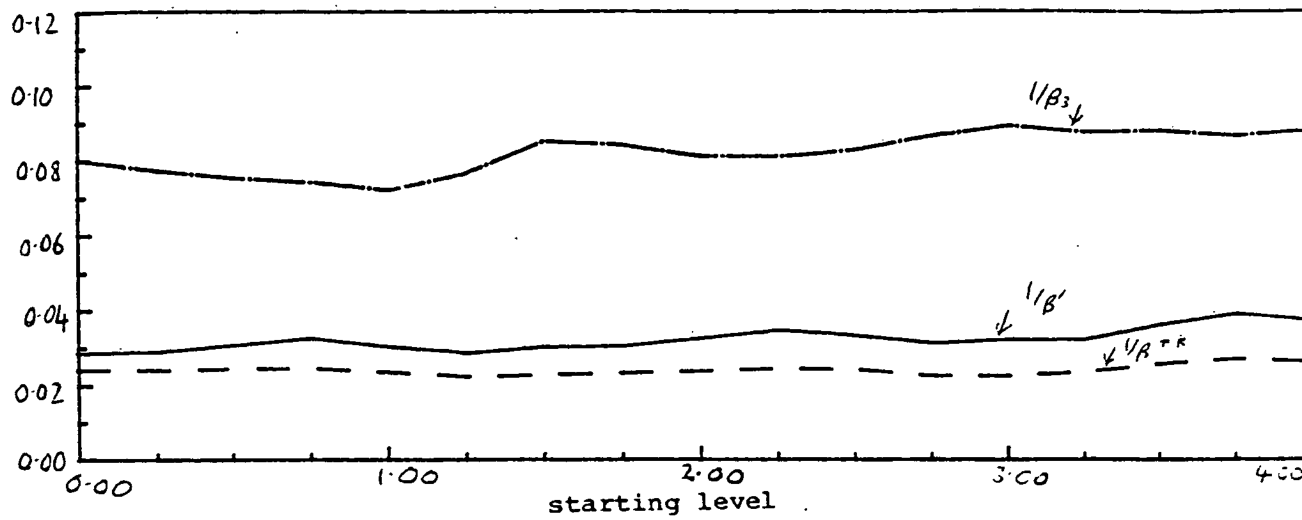
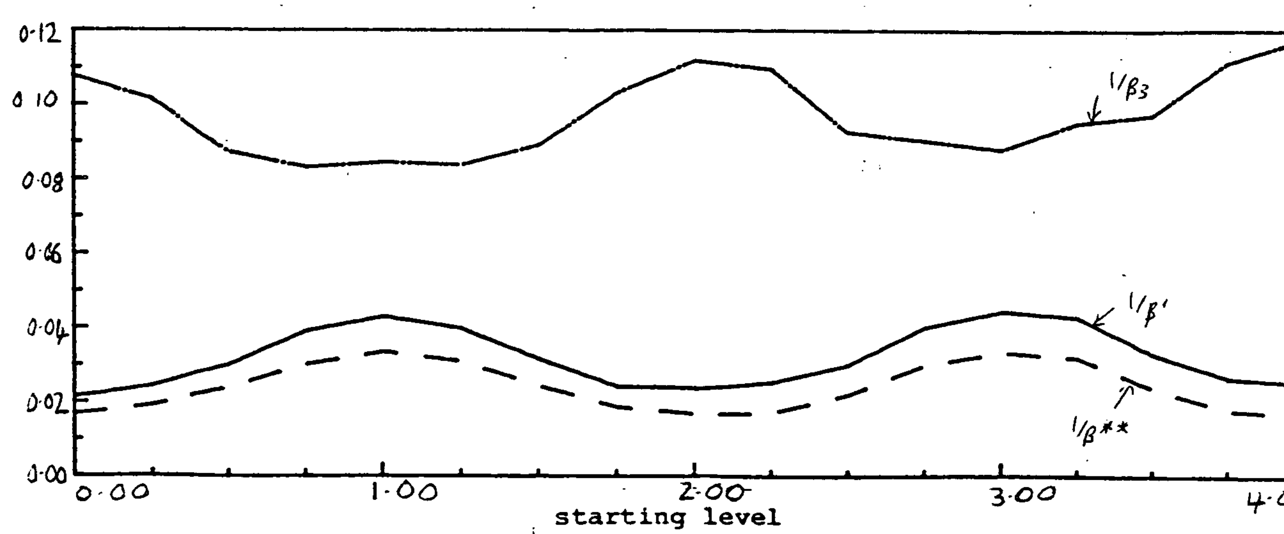


Fig. 4.2.8 As in Fig. 4.2.5 only with step size 2.0.



4.2.8. Although one may not be so interested in estimating μ it is also worth noting that for the smaller step sizes μ^{**} has a clear advantage over μ' in that it has smaller bias. So there appears to be some advantage to be gained from calculating m.l.e.'s (though again this must be qualified as some experiments have been discarded when the m.l.e.'s are formed). It is interesting to compare Table 4.2.19 with Table 12 of Appendix 9 which gives values of m.s.e. and mean of the estimator $1/\beta^*$ in 48 step Up and Down experiments ($1/\beta^*$ also equals the m.l.e. of $1/\beta$ if it can be found). For step sizes 0.5 and 1.0 the m.s.e. of $1/\beta^{**}$ is less than that of $1/\beta^*$, but for step sizes 1.5 and 2.0 it is often higher (i.e. using two UDTR's is not giving better estimates of $1/\beta$ than those obtained from using an Up and Down rule for the same total number of observations). Such a comparison is to some extent unfair on the procedure of using two UDTR's as in my simulations I have started both staircases at the same level; in practice one would start the staircase for $L_{1/\sqrt{2}}$ some way above the start for the staircase for $L_{1-1/\sqrt{2}}$.

The main conclusion that can be drawn from the simulations using two UDTR rules is that experimenters should be very wary of using estimators of $1/\beta$ such as $1/\beta_3$ (i.e. of a form such as that in Formula 4.2.1) and should instead be prepared to carry out maximum likelihood estimation. How seriously my suggested alternative estimator, $1/\beta'$, should be taken is questionable but in the simulations it did have m.s.e. close to that of the m.l.e. of $1/\beta$.

4.3 PROPERTIES OF THE UDTR RULE IN 2IFC EXPERIMENTS

In psychometric studies if an UDTR rule is used it is very often when a two interval forced choice (2IFC) procedure is operated. This procedure is described in Rose, Teller and Rendleman (1970). In such experiments the stimulus to be detected is presented in one of two intervals. The subject is asked to choose the interval in which he judges the stimulus is most likely to have been present; the subject must make a choice even if he has no idea as to which interval contains the stimulus. It has been argued that this procedure is better than the yes-no procedure, where a stimulus is presented to a subject who must say whether or not it has been detected. The argument is that with the yes-no procedure the subject is free to set his own criterion for giving a positive response but in the 2IFC experiment the criterion is brought under the control of the experimenter.

If the probability of detection of the stimulus is given by $G(x)$, where x is the stimulus level, then the probability of a correct choice at level x in the 2IFC experiment is given by

$$(1-G(x))/2 + G(x) = (1+G(x))/2. \qquad 4.3.1$$

If the stimulus is detected it is assumed that the subject will always choose the correct interval and of course if the signal is not detected the subject will have probability 0.5 of making the correct choice.

In other disciplines it is difficult to see applications for this procedure. For example in bio-assay there is a clear physical response to the stimulus by the subject, there is no sensible way and certainly no reason to operate a procedure such as the 2IFC in these circumstances.

If a 2IFC procedure is adopted then the Up and Down rule cannot be used as whatever the response curve is there will always be probability of at least 0.5 of making a correct choice. The $L_{\sqrt{2}-1}$ of the response curve G corresponds to the $L_{1/\sqrt{2}}$ of the response curve in the 2IFC experiments. Experimenters have often been satisfied to obtain an estimate of the $L_{1/\sqrt{2}}$ level of the 2IFC response curve by means of the UDTR rule designed to concentrate observations around this level discussed in the Section 4.1. They often cite a paper by Wetherill and Levitt (1965), which describes the UDTR rule, but has no discussion of its suitability for use in 2IFC experiments. The stimulus level they try to estimate is the $L_{\sqrt{2}-1}$ level of the function G.

Rose et al compare the yes-no procedure where the Up and Down rule is operated with the 2IFC procedure where the UDTR rule is used. In this section and Section 4.4 these strategies will for convenience be referred to as Routine A and Routine B respectively. They made this comparison by means of simulation, where usually a linear response curve was assumed, though some simulations were made assuming a normal response curve. They concluded that statistical properties of estimators are better with Routine A than Routine B providing subjects are ideal in the sense that they say they detect the stimulus if and only if they actually do. The calculations of this section allow one to make a similar comparison of asymptotic properties of these strategies when the underlying response curve is logistic; Section 4.4 contains results of a simulation study designed to compare small sample properties. Rose et al's results are discussed in more detail at the end of Section 4.4.

Fig. 4.3.1 illustrates values of $V(L_{\sqrt{2}-1})\beta^2$ for Routine B with underlying logistic response, that is the values of the asymptotic variance of $n^{1/2}(\hat{L}_{\sqrt{2}-1} - L_{\sqrt{2}-1})\beta$, where $\hat{L}_{\sqrt{2}-1}$ is the m.l.e. of $L_{\sqrt{2}-1}$ and n is the number of observations (there is sufficient regularity to use results in Billingsley (1961)). The value of $V(\mu)\beta^2$ is illustrated in Fig. 4.3.2. The values of $V(\mu)$ and $V(L_{\sqrt{2}-1})$ for Routine B are close over a wide range of βd values (this is not surprising as the μ and $L_{\sqrt{2}-1}$ levels only differ by $0.3466/\beta$). The dashed lines in Fig. 4.3.2 join points representing values of $V(\mu)\beta^2$ if Routine A is used with ideal subjects (Fig. 2.3.1 illustrates these values on a more appropriate scale). There is a considerable loss in efficiency in estimating μ by maximum likelihood estimation if Routine B is used rather than Routine A. The subjects will not usually be ideal but these asymptotic calculations must raise the question as to whether experimenters should be using the 2IFC procedure at all unless they have serious doubts about the reliability of subjects. Also it seems unlikely that methods developed for use with the normal or logistic response curves, with the Up and Down or UDTR rules being operated, will be directly applicable to experiments where the 2IFC procedure is used, where in effect a natural responsiveness of 0.5 has been

Fig. 4.3.1 Values of $V(L_{\sqrt{2}-1})\beta^2$ for Routine B.

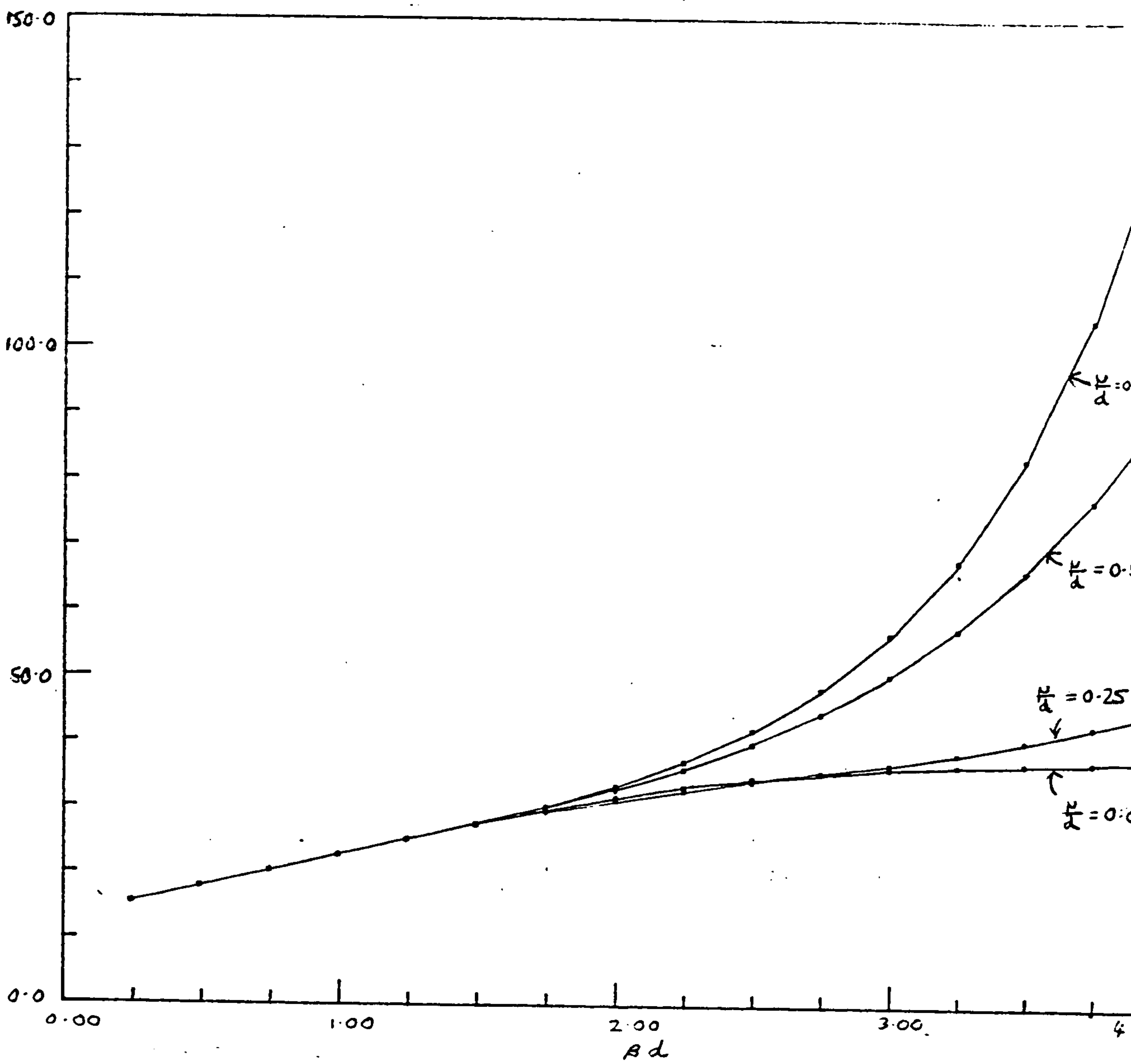


Fig. 4.3.2 Values of $V(\mu)\beta^2$ for Routine B (analogous values for Routine A are joined by dashed lines).

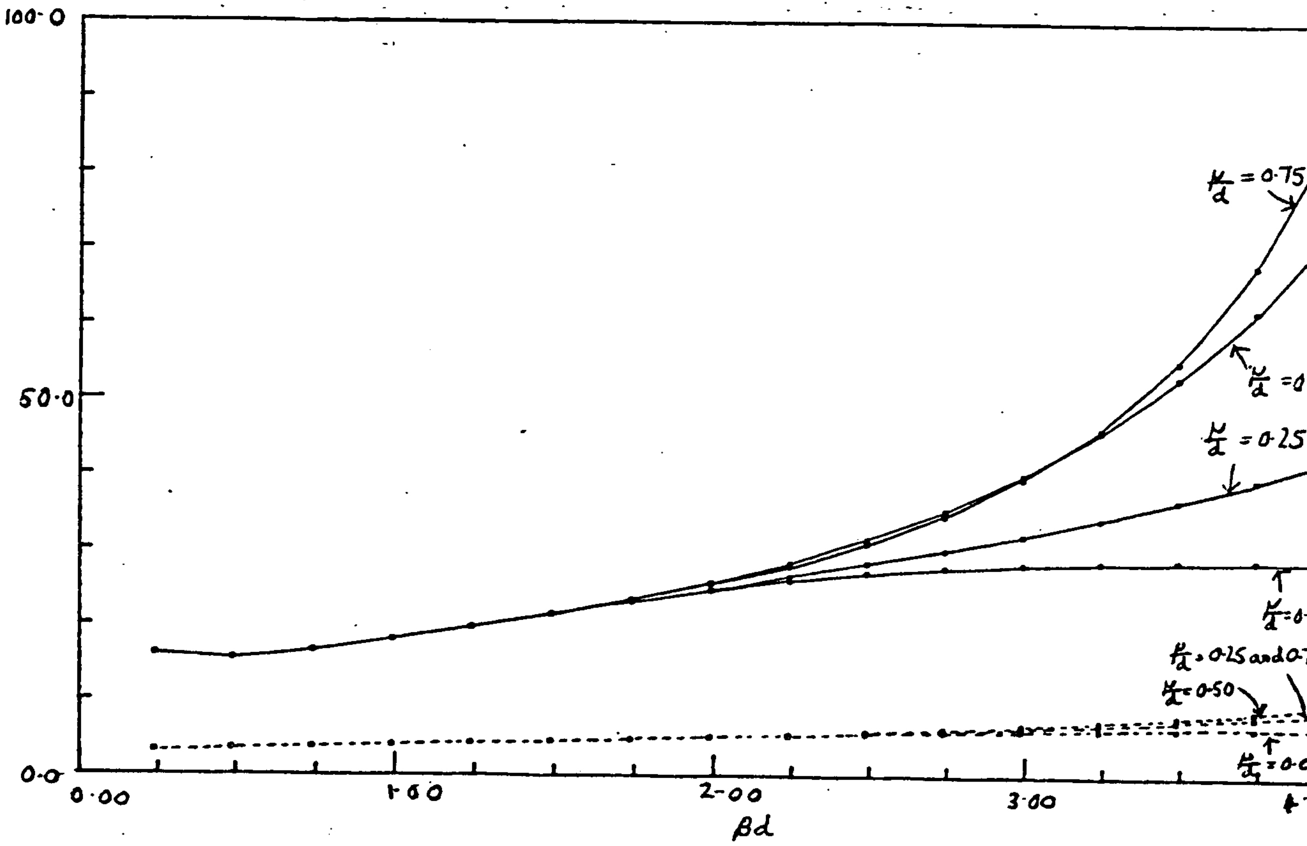


Table 4.3.1 Values of $V(\beta)/\beta^2$ for $\mu/d = 0.0$.

	βd						
	0.25	0.50	1.00	1.50	2.00	3.00	4.00
Routine A	18.008	10.288	6.091	4.858	4.410	4.809	7.067
Routine B	51.427	32.299	23.080	20.531	20.246	24.183	35.287

introduced.

The values of asymptotic variance for the m.l.e. of β with Routine B compare unfavourably with the corresponding values when Routine A is used. Table 4.3.1 gives some values of $V(\beta)/\beta^2$ for both routines when the phasing of levels is 0.0. Values vary with phase but under the conditions for which calculations are made (i.e. phases 0.00(0.25)0.75 and values for βd of 0.25(0.25)4.00) there was always a considerable advantage in terms of asymptotic efficiency in using Routine A rather than Routine B.

The weights, w_i, entering into the asymptotic variance expression of matrix 2.3.7 are

$$w_i = (dG(x)/dx)^2/\beta^2(G(x)(1-G(x)))\Big|_{x=x_i}. \qquad 4.3.2$$

If the 2IFC procedure is used then the response curve is of the form (1+G(x))/2 and new values for w_i are

$$w_i = (dG(x)/dx)^2/\beta^2((1+G(x))(1-G(x)))\Big|_{x=x_i}. \qquad 4.3.3$$

So with this new response curve a factor of G(x)/(1+G(x)) has been introduced. This factor is less than or equal to 0.5; the weight that is attached to observations in a 2IFC experiment is considerably less than that for the yes-no procedure. Fig. 4.3.3 illustrates the values for these weights using the two procedures when the response curve is logistic. Use of Routine A and Routine B give rise to different asymptotic distributions of design points

Fig. 4.3.3 Weight attached to observations using the yes-no and 2IFC procedures for the logistic curve.

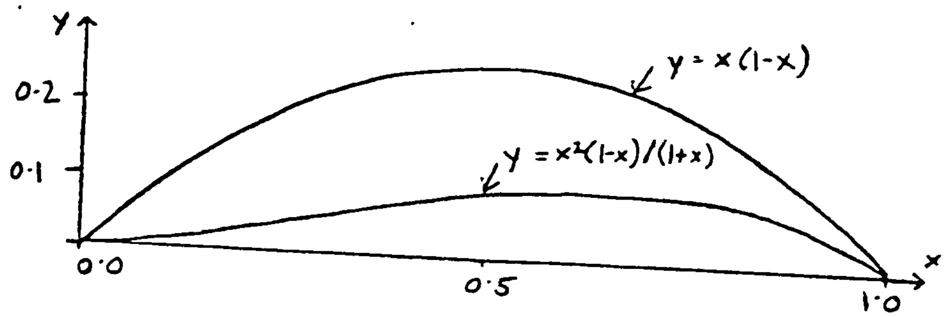
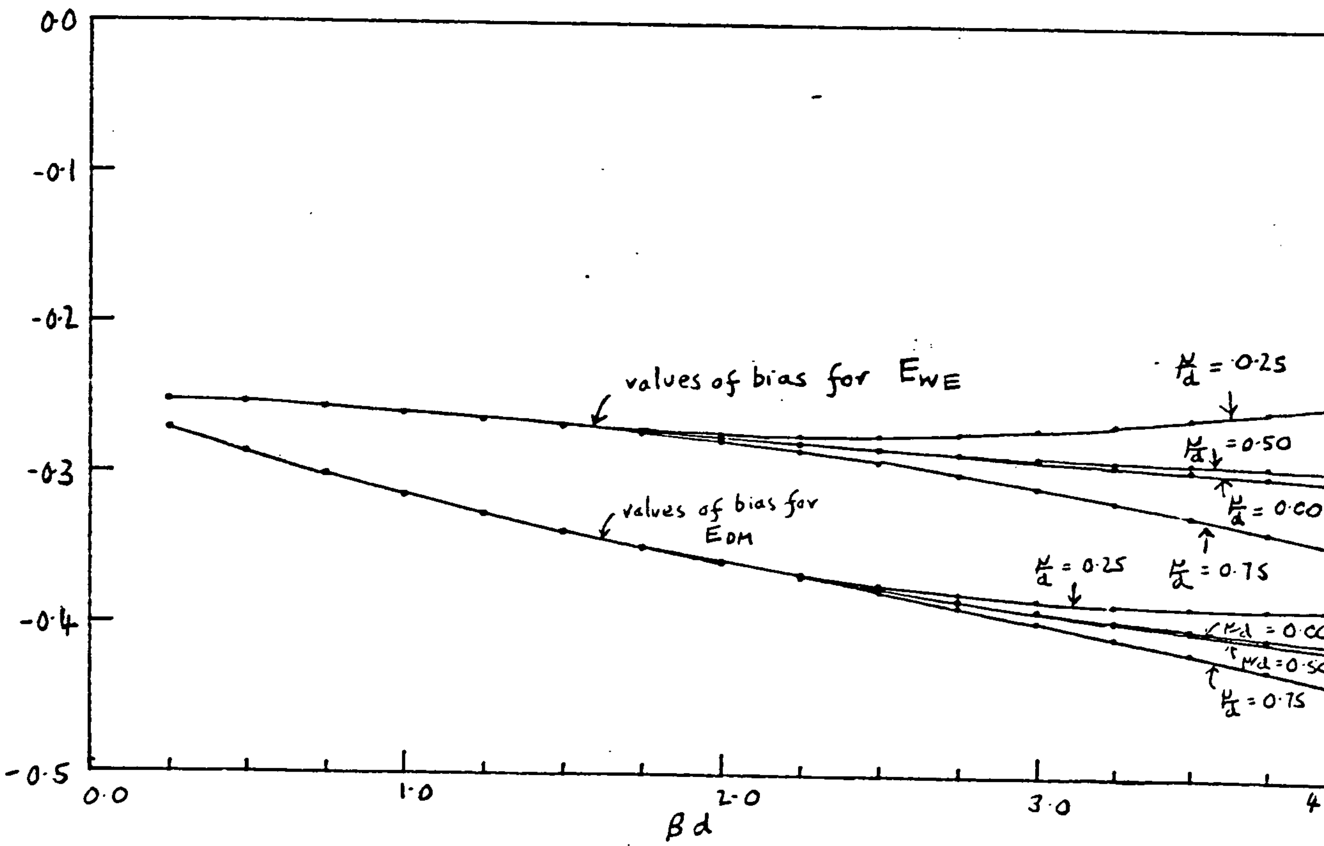


Fig. 4.3.4 (Asymptotic bias)/d of E_{OM} and E_{WE} when estimating L_{n-1}.

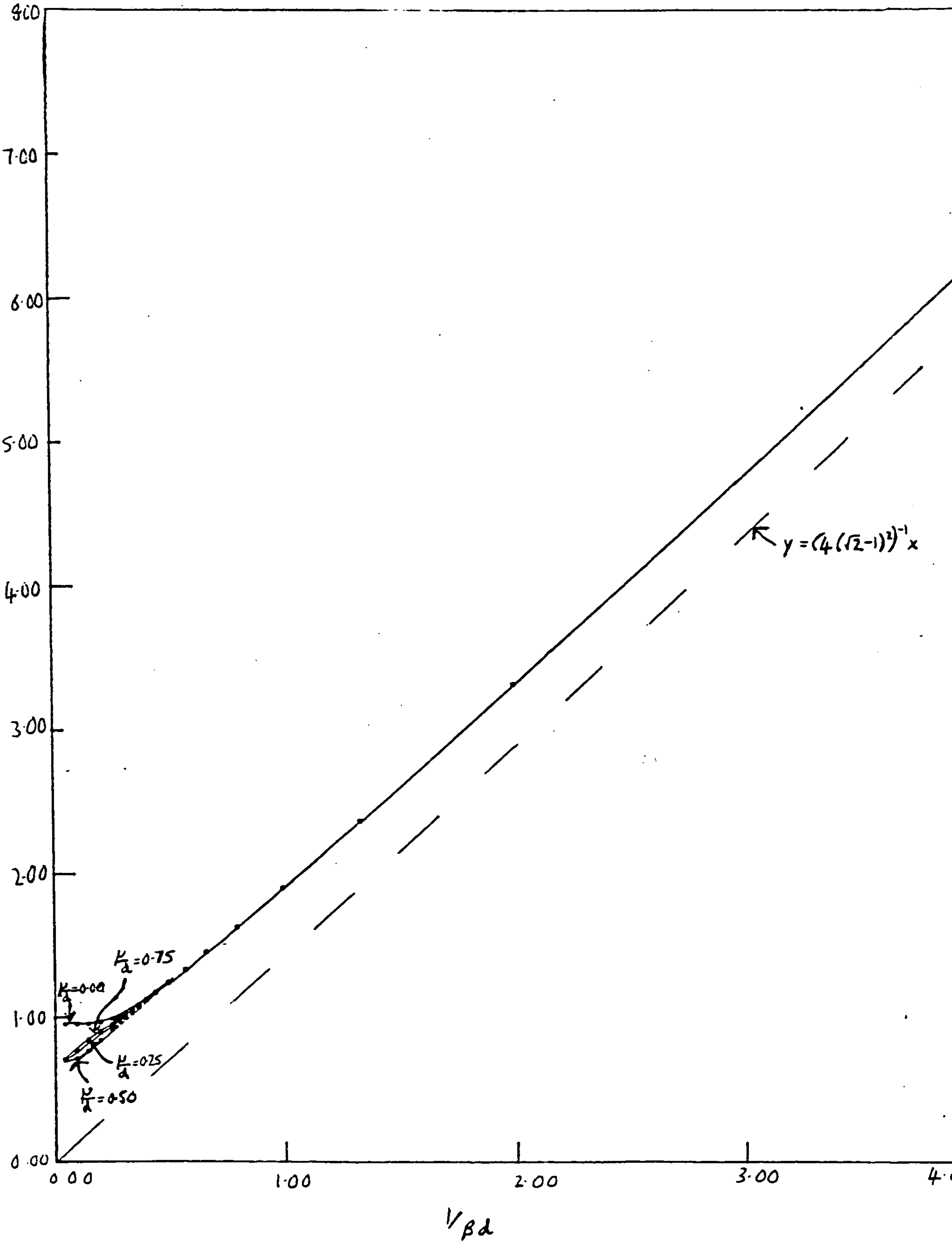


but it is not at all surprising that asymptotic variances of m.l.e.'s of μ and β are much larger with Routine B than Routine A.

With Routine B experimenters often use either E_{WE} or some estimator asymptotically equivalent to E_{DM} to give an estimate of the $L_{1/\sqrt{2}}$ level of the 2IFC response curve. Fig. 4.3.4 illustrates some values of asymptotic bias for these estimators for logistic response. The bias is negative for both estimators, with bias of E_{WE} being smaller than that of E_{DM} for the parameter values considered. Figs. 4.1.1 and 4.1.2 give analogous values for bias when the UDTR rule is used with the yes-no procedure, these values for bias are much smaller than those illustrated in Fig. 4.3.4. Tables 2.3.1 and 2.3.2 give analogous values for bias using Routine A, here values for bias are very small for a wide range of parameter values.

As in the previous situations considered the limit with probability one of the variance of the levels of frequent response type (i.e. where - and +- responses count as a negative response and ++ as a positive response) is close to being linearly related to $1/\beta$, providing $\beta d \leqslant 2.0$. This relation is illustrated in Fig. 4.3.5 for underlying logistic response. The slope of the square of the 2IFC response curve at its $L_{1/\sqrt{2}}$ level is equal to $\beta(2^{1/2}-1)^2$. From Appendix 7 it follows that, for d sufficiently small, the this limit divided by d is arbitrarily close to $1/(4(2^{1/2}-1)^2\beta)$. This suggests that the points in Fig. 4.3.5 should be fitted by a line with slope $1/(4(2^{1/2}-1)^2)$. The dashed line in Fig. 4.3.5 corresponds to such a line through the origin.

Fig. 4.3.5 Plot of limit with probability one of v/d against 1/βd.



A line with this slope but intercept on the y-axis of 0.5 fits the calculated values closely for $\beta d \leqslant 2.0$. This suggests that one could estimate $1/\beta$ by $1/\tilde{\tilde{\beta}}$ where

$$1/\tilde{\tilde{\beta}} = ((v/d)-(d/2))(8^{1/2}-2)^2, \qquad 4.3.4$$

where v is the variance of the less frequent response type. If $\beta d \leqslant 2.0$ the asymptotic bias of such an estimator is small. However the values for the asymptotic variance of the m.l.e. of $1/\beta$ suggest that such an estimator will have low precision.

Fig. 4.3.6 illustrates values of the asymptotic variance expressions for $n^{1/2}(E_{DM}-M_{DM})\beta$. For Routine A or the UDTR rule with the yes-no procedures these expressions were, for small step sizes, close to the corresponding values of $V(\mu)\beta^2$ and $V(L_{1/\sqrt{2}})\beta^2$ (i.e. E_{DM} and the corresponding m.l.e.'s had similar variances). For Routine B, for the βd values I consider, such values are usually substantially above corresponding values of $V(L_{\sqrt{2}-1})\beta^2$ (i.e. there appears to be no close relationship between E_{DM} and the m.l.e. of $L_{\sqrt{2}-1}$). One could argue that there may be close relationship between E_{DM} and the m.l.e. of $L_{\sqrt{2}-1}$ for smaller step sizes, but then the asymptotic variances for the m.l.e. of β will be enormous. Figs. 4.3.7 and 4.3.8 illustrates values of asymptotic variance expressions for $n^{1/2}(E_{WE}-M_{WE})\beta$ and $n^{1/2}(E_{WE}^*-M_{WE})\beta$ respectively, these are also not close to corresponding values $V(L_{\sqrt{2}-1})\beta^2$. The effect of phasing on variance of estimators E_{WE} and E_{WE}^* is, for the larger values of βd considered, much smaller for Routine B than that for Routine A or the UDTR rule operated with

Fig. 4.3.6 Asymptotic variance of $(E_{DM}-M_{DM})\beta n^{1/2}$ for Routine B.

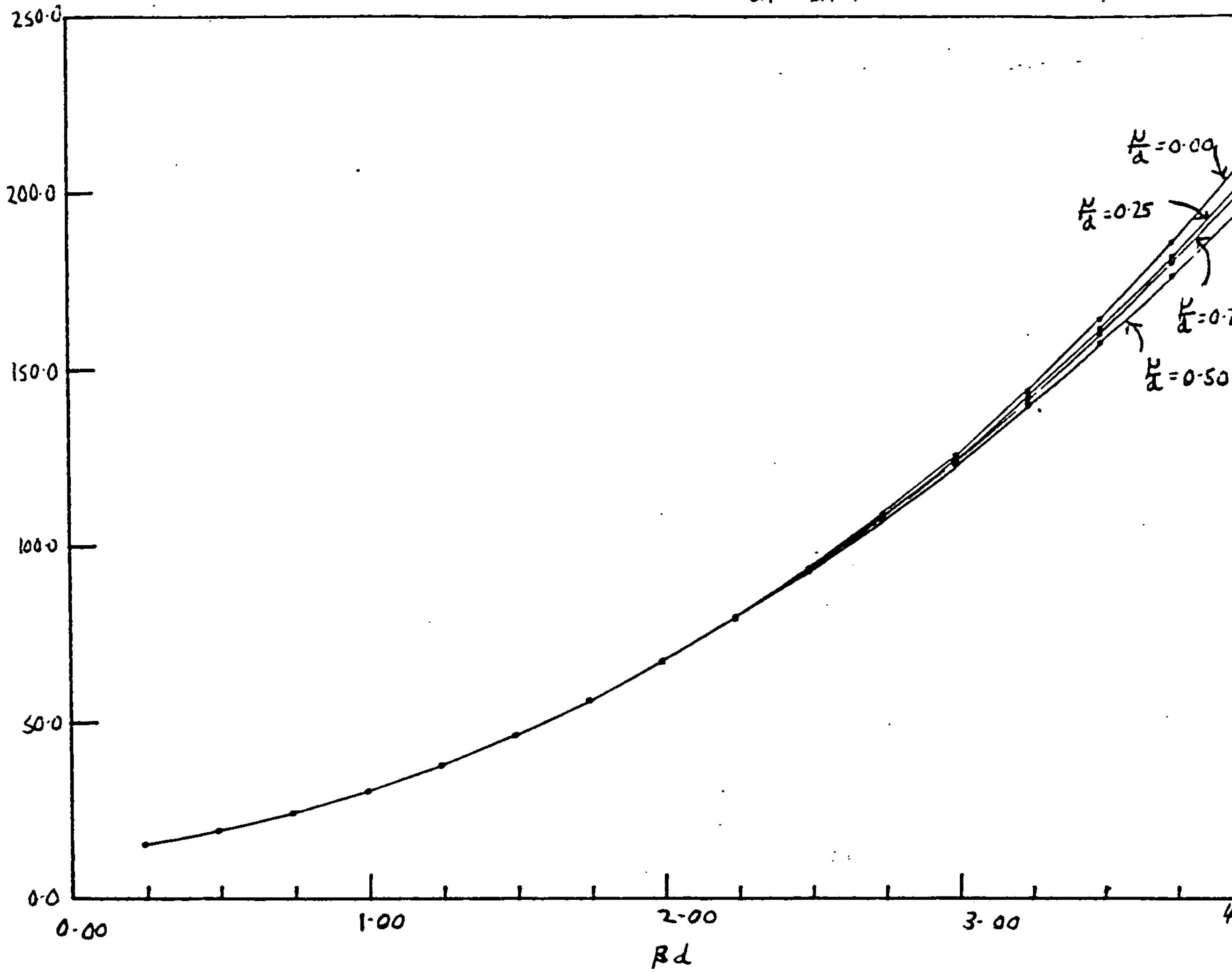


Fig. 4.3.7 Asymptotic variance of $(E_{WE}-M_{WE})\beta n^{1/2}$ for Routine B.

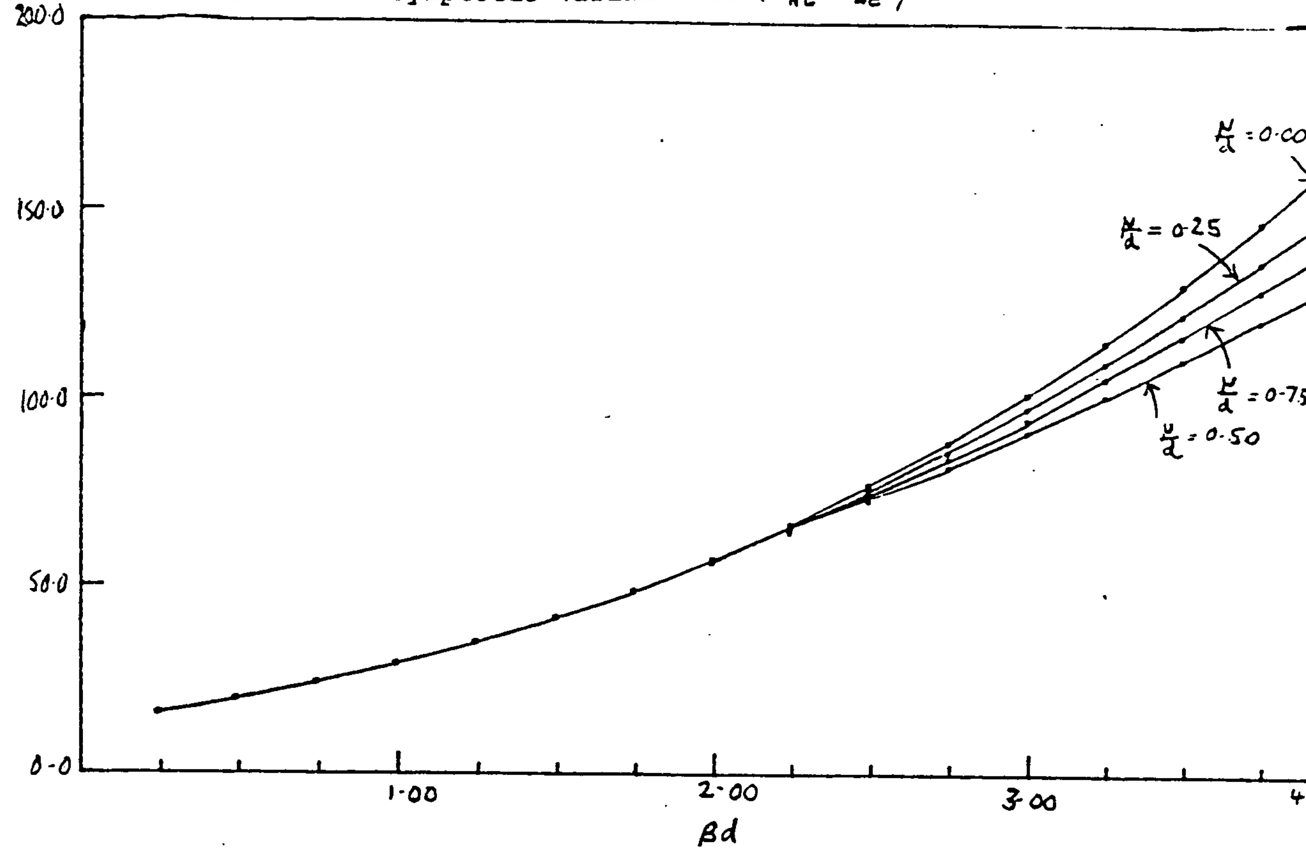
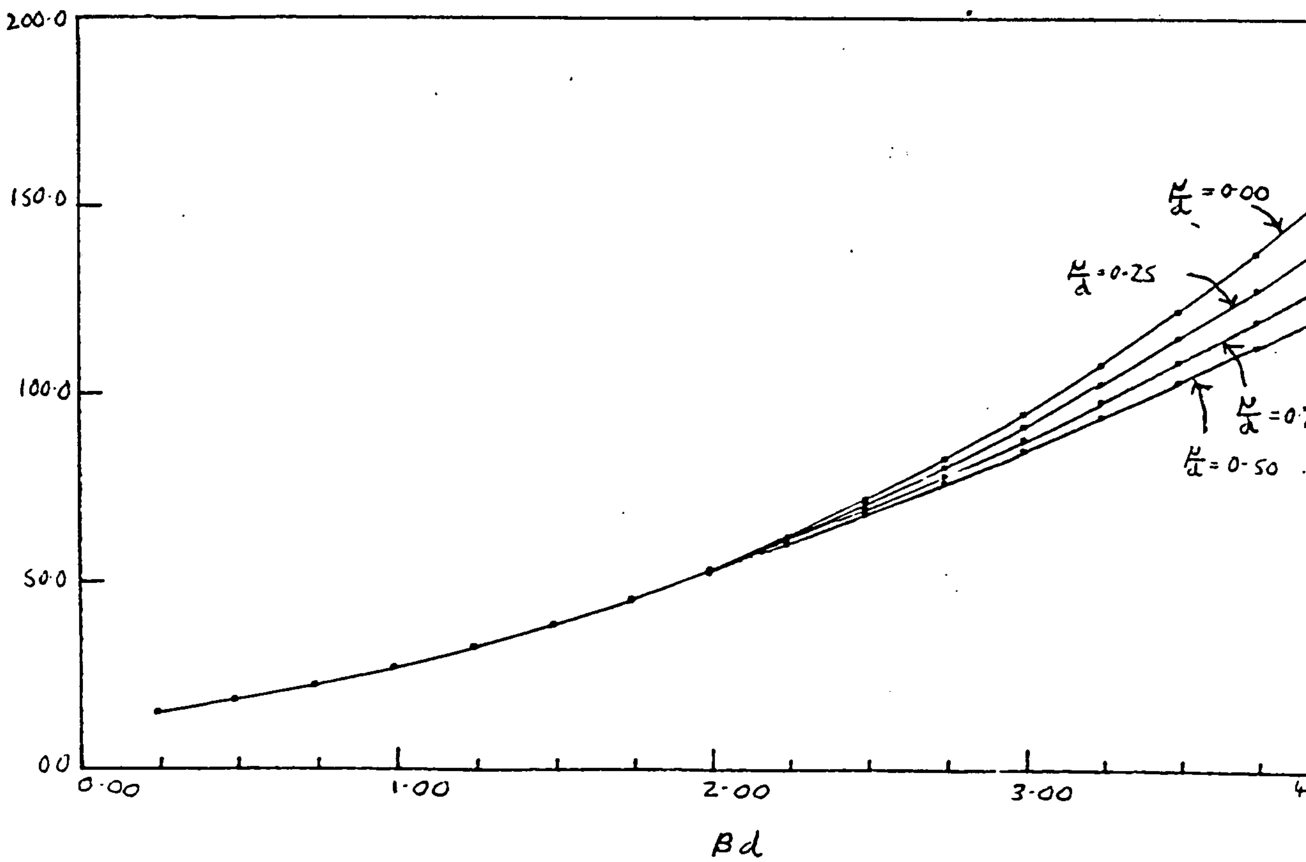


Fig. 4.3.8 Asymptotic variance of $(E^{*}_{WE}-M_{WE})\beta n^{1/2}$ for Routine B.



the yes-no procedure (see Figs. 2.3.3 and 4.1.5).

The main conclusions at the end of this section are:

(1) Estimators asymptotically equivalent to E_{DM}, E_{WE} or E_{WE}^{*}, have apparently no close relationship with the m.l.e. of $L_{\Omega-1}$ for the conditions considered.

(2) Using Routine B rather than Routine A with ideal subjects can increase the asymptotic variance of m.l.e.'s dramatically.

4.4 RESULTS OF SIMULATIONS OF 2IFC EXPERIMENTS

I made some simulations of Routine B in experiments consisting of 24 observations. I assumed that the underlying response curve was logistic with β equal to $\pi/3.0^{1/2}$. The values of E_M, E_θ, $E_{\theta D}$, E_{DM}, E_{WE}, E_{PV}, E_{WE}^* and E_{PV}^* were calculated and used to estimate $L_{\sqrt{2}-1}$. Starting levels were set to be at -2.00(0.25)2.00 relative to $L_{\sqrt{2}-1}$ and step sizes 0.5(0.5)2.0 were used. Again 2000 experiments were simulated for each set of conditions. Values of m.s.e and bias of these estimators are given in Tables 4.4.1 to 4.4.8. The m.s.e.'s and biases are relatively large compared to corresponding values for Routine A. To see this one should compare these tables with Tables 3.3.1 to 3.3.8 (in this section values in the tables are multiplied by 10 not 100). These values are also, for most sets of conditions, much larger than corresponding values for experiments where the UDTR rule is used with the yes-no procedure (see Tables 4.2.1 to 4.2.8). All the estimators have similar m.s.e.'s and biases. This was certainly not the case in the simulations of Sections 3.3 and 4.2 where, for step sizes 1.5 and 2.0, the m.s.e.'s of E_{WE}, E_{PV}, E_{WE}^* and E_{PV}^* oscillated above and below the m.s.e.'s for $E_{\theta D}$ and E_{DM}. That there are no such large oscillations is not surprising as the dependence of asymptotic variance and bias on phase is relatively small (see Figs. 4.3.6, 4.3.7 and 4.3.8). The m.s.e.'s of the estimators are relatively stable for all step sizes as starting levels are increased from $L_{\sqrt{2}-1}$ to 2.00 above $L_{\sqrt{2}-1}$, but values rapidly rise as the starting level is dropped to 2.00 below $L_{\sqrt{2}-1}$. In fact the m.s.e.'s of all the

Table 4.4.1 10×m.s.e. of estimators of $L_{\sqrt{2}-1}$ in 24 observation experiments using Routine B for step size 0.5.

Start	E_M	E_β	$E_{\beta D}$	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E_{WE}^*	E_{PV}^*	A_{WE}^*
-2.00	11.55	9.98	9.73	11.24	4.06	11.17	11.21	3.86	10.71	10.68	3.67
-1.75	8.86	7.77	7.68	8.85	4.06	8.77	8.81	3.86	8.43	8.42	3.67
-1.50	6.77	6.08	6.10	6.95	4.06	6.83	6.88	3.86	6.60	6.59	3.67
-1.25	5.19	4.81	4.92	5.51	4.06	5.42	5.48	3.86	5.25	5.25	3.67
-1.00	3.97	3.83	3.94	4.33	4.06	4.27	4.33	3.86	4.13	4.14	3.67
-0.75	3.10	3.14	3.26	3.49	4.06	3.46	3.53	3.86	3.37	3.39	3.67
-0.50	2.59	2.76	2.87	2.96	4.06	2.97	3.05	3.86	2.91	2.94	3.67
-0.25	2.11	2.33	2.43	2.45	4.06	2.53	2.63	3.86	2.44	2.48	3.67
0.00	1.83	2.09	2.19	2.20	4.06	2.25	2.34	3.86	2.20	2.24	3.67
0.25	1.67	1.92	2.05	2.08	4.06	2.10	2.18	3.86	2.07	2.10	3.67
0.50	1.59	1.81	2.01	2.08	4.06	2.05	2.13	3.86	2.03	2.07	3.67
0.75	1.64	1.74	1.99	2.14	4.06	2.12	2.18	3.86	2.05	2.07	3.67
1.00	1.81	1.77	2.10	2.32	4.06	2.31	2.36	3.86	2.21	2.21	3.67
1.25	2.11	1.90	2.31	2.55	4.06	2.55	2.59	3.86	2.44	2.43	3.67
1.50	2.46	2.00	2.42	2.69	4.06	2.79	2.82	3.86	2.60	2.58	3.67
1.75	3.05	2.24	2.53	2.82	4.06	3.00	3.01	3.86	2.74	2.70	3.67
2.00	3.80	2.62	2.78	3.07	4.06	3.34	3.34	3.86	3.02	2.99	3.67

Table 4.4.2 10×m.s.e. of estimators of $L_{\sqrt{2}-1}$ in 24 observation experiments using Routine B for step size 1.0.

Start	E_M	E_β	$E_{\beta D}$	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E_{WE}^*	E_{PV}^*	A_{WE}^*
-2.00	11.58	10.65	10.71	12.02	8.94	11.61	11.75	7.37	11.30	11.28	7.00
-1.75	9.85	9.26	9.36	10.40	8.94	10.03	10.18	7.39	9.77	9.75	7.02
-1.50	8.58	8.31	8.42	9.14	8.93	8.88	9.03	7.35	8.69	8.68	6.97
-1.25	7.50	7.45	7.64	8.16	8.93	8.02	8.19	7.33	7.83	7.84	6.95
-1.00	6.69	6.87	7.06	7.36	8.94	7.27	7.44	7.37	7.16	7.18	7.00
-0.75	5.82	6.14	6.32	6.53	8.94	6.44	6.61	7.39	6.38	6.42	7.02
-0.50	5.38	5.84	6.01	6.10	8.93	6.03	6.23	7.35	6.00	6.07	6.97
-0.25	4.85	5.39	5.51	5.61	8.93	5.59	5.81	7.33	5.55	5.62	6.95
0.00	4.57	5.17	5.31	5.35	8.94	5.30	5.54	7.37	5.31	5.39	7.00
0.25	4.28	4.93	5.21	5.24	8.94	5.17	5.43	7.39	5.18	5.28	7.02
0.50	3.93	4.64	5.01	5.03	8.93	4.88	5.14	7.35	4.95	5.06	6.97
0.75	3.72	4.40	4.89	4.98	8.93	4.82	5.11	7.33	4.83	4.93	6.95
1.00	3.43	4.13	4.69	4.78	8.94	4.67	4.92	7.37	4.63	4.72	7.00
1.25	3.16	3.74	4.41	4.51	8.94	4.36	4.60	7.39	4.31	4.38	7.02
1.50	3.16	3.72	4.62	4.73	8.93	4.54	4.78	7.35	4.53	4.60	6.97
1.75	3.09	3.52	4.62	4.74	8.93	4.59	4.82	7.33	4.55	4.60	6.95
2.00	3.16	3.48	4.80	4.94	8.94	4.80	5.04	7.37	4.73	4.77	7.00

Note: A_{DM}, A_{WE} and A_{WE}^* denote columns for asymptotic predicted m.s.e.'s of E_{DM}, E_{WE} and E_{WE}^* respectively

Table 4.4.3 10×m.s.e. of estimators of $L_{\sqrt{2}-1}$ in 24 observation experiments using Routine B for step size 1.5.

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E^*_{WE}	E^*_{PV}	A^*_{WE}
-2.00	15.72	15.23	15.37	16.53	16.85	15.54	15.78	12.20	15.57	15.53	11.58
-1.75	14.18	14.00	14.16	15.11	16.88	14.40	14.65	12.43	14.42	14.38	11.82
-1.50	13.06	13.19	13.35	14.01	16.98	13.56	13.80	12.83	13.55	13.53	12.22
-1.25	12.00	12.36	12.66	13.15	17.04	12.78	13.06	12.99	12.79	12.79	12.35
-1.00	10.96	11.54	11.82	12.08	17.01	11.65	11.94	12.75	11.80	11.84	12.10
-0.75	10.02	10.77	10.98	11.19	16.92	10.59	10.90	12.35	10.84	10.89	11.71
-0.50	9.72	10.64	10.89	11.01	16.85	10.43	10.78	12.20	10.65	10.74	11.58
-0.25	9.10	10.14	10.37	10.48	16.88	10.08	10.47	12.43	10.21	10.32	11.82
0.00	8.58	9.73	9.96	9.99	16.98	9.77	10.21	12.83	9.86	10.01	12.22
0.25	8.32	9.59	9.89	9.97	17.04	9.64	10.15	12.99	9.82	10.01	12.35
0.50	8.03	9.48	10.01	10.00	17.01	9.46	10.01	12.75	9.84	10.07	12.10
0.75	7.87	9.41	10.14	10.13	16.92	9.52	10.13	12.35	9.84	10.09	11.71
1.00	7.14	8.75	9.67	9.66	16.85	9.01	9.60	12.20	9.33	9.57	11.58
1.25	6.57	8.10	9.09	9.14	16.88	8.55	9.13	12.43	8.74	8.95	11.82
1.50	6.33	7.93	9.10	9.09	16.98	8.68	9.24	12.83	8.71	8.92	12.22
1.75	5.88	7.40	8.79	8.82	17.04	8.42	8.98	12.99	8.44	8.62	12.35
2.00	5.78	7.36	9.11	9.15	17.01	8.47	9.04	12.75	8.71	8.90	12.10

Table 4.4.4 10×m.s.e. of estimators of $L_{\sqrt{2}-1}$ in 24 observation experiments using Routine B for step size 2.0.

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E^*_{WE}	E^*_{PV}	A^*_{WE}
-2.00	21.95	22.09	22.35	23.37	28.44	22.53	22.86	20.62	22.61	22.53	19.66
-1.75	20.73	21.10	21.32	22.25	28.86	21.32	21.70	21.21	21.57	21.49	20.19
-1.50	19.38	20.00	20.15	20.79	28.95	19.78	20.14	20.80	20.14	20.07	19.78
-1.25	18.15	19.02	19.34	19.91	28.66	18.48	18.86	19.71	19.00	18.93	18.73
-1.00	17.02	18.21	18.53	18.79	28.17	17.20	17.56	18.61	17.87	17.86	17.69
-0.75	15.87	17.29	17.58	17.73	27.75	16.24	16.67	18.07	16.88	16.91	17.20
-0.50	15.46	17.06	17.44	17.62	27.66	16.26	16.79	18.39	16.86	16.98	17.52
-0.25	14.93	16.70	17.07	17.27	27.95	16.41	17.06	19.43	16.74	16.90	18.52
0.00	14.70	16.66	17.07	17.09	28.44	16.59	17.33	20.62	16.83	17.06	19.66
0.25	14.33	16.47	16.93	16.99	28.86	16.37	17.25	21.21	16.69	17.00	20.19
0.50	13.80	16.22	16.69	16.57	28.95	15.58	16.54	20.80	16.27	16.67	19.78
0.75	13.44	16.09	16.86	16.64	28.66	15.47	16.53	19.71	16.18	16.64	18.73
1.00	12.63	15.41	16.59	16.36	28.17	14.90	16.00	18.61	15.67	16.13	17.69
1.25	12.15	15.02	16.35	16.23	27.75	14.56	15.65	18.07	15.34	15.78	17.20
1.50	11.46	14.36	16.05	15.96	27.66	14.43	15.48	18.39	15.11	15.54	17.52
1.75	10.69	13.51	15.28	15.25	27.95	14.06	15.07	19.43	14.46	14.85	18.52
2.00	10.29	13.21	15.20	15.09	28.44	14.29	15.29	20.62	14.50	14.89	19.66

Table 4.4.5 10×bias of estimators of L_{Γ_2-1} in 24 observation experiments using Routine B for step size 0.5.

Start	E_M	E_B	E_{BP}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
-2.00	-8.98	-7.80	-7.09	-7.60	-1.52	-7.74	-7.65	-7.45	-7.38	-1.27
-1.75	-7.55	-6.50	-5.94	-6.40	-1.52	-6.48	-6.40	-6.26	-6.20	-1.27
-1.50	-6.26	-5.37	-4.96	-5.37	-1.52	-5.36	-5.28	-5.22	-5.16	-1.27
-1.25	-5.08	-4.34	-4.07	-4.40	-1.52	-4.38	-4.30	-4.28	-4.22	-1.27
-1.00	-4.02	-3.42	-3.26	-3.55	-1.52	-3.49	-3.42	-3.44	-3.38	-1.27
-0.75	-3.03	-2.59	-2.51	-2.74	-1.52	-2.67	-2.61	-2.65	-2.59	-1.27
-0.50	-2.32	-2.07	-2.05	-2.16	-1.52	-2.11	-2.07	-2.11	-2.07	-1.27
-0.25	-1.47	-1.38	-1.40	-1.42	-1.52	-1.38	-1.35	-1.39	-1.37	-1.27
0.00	-0.76	-0.84	-0.87	-0.79	-1.52	-0.77	-0.77	-0.80	-0.79	-1.27
0.25	-0.05	-0.31	-0.34	-0.19	-1.52	-0.15	-0.18	-0.23	-0.25	-1.27
0.50	0.46	0.02	-0.06	0.19	-1.52	0.22	0.16	0.12	0.08	-1.27
0.75	1.25	0.63	0.48	0.77	-1.52	0.85	0.76	0.70	0.64	-1.27
1.00	1.97	1.16	0.87	1.18	-1.52	1.31	1.19	1.15	1.07	-1.27
1.25	2.70	1.70	1.18	1.48	-1.52	1.68	1.53	1.45	1.36	-1.27
1.50	3.42	2.24	1.40	1.74	-1.52	1.98	1.80	1.75	1.65	-1.27
1.75	4.32	2.95	1.73	2.06	-1.52	2.34	2.14	2.08	1.96	-1.27
2.00	5.17	3.61	1.85	2.20	-1.52	2.53	2.31	2.23	2.11	-1.27

Table 4.4.6 10×bias of estimators of L_{Γ_2-1} in 24 observation experiments using Routine B for step size 1.0.

Start	E_M	E_B	E_{BP}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
-2.00	-7.32	-6.14	-5.82	-6.32	-3.46	-5.98	-5.83	-6.03	-5.89	-2.67
-1.75	-6.39	-5.36	-5.14	-5.60	-3.47	-5.25	-5.12	-5.31	-5.16	-2.69
-1.50	-5.67	-4.82	-4.69	-5.05	-3.47	-4.72	-4.59	-4.80	-4.65	-2.70
-1.25	-4.93	-4.26	-4.21	-4.51	-3.46	-4.20	-4.07	-4.30	-4.17	-2.68
-1.00	-4.35	-3.84	-3.84	-4.04	-3.46	-3.76	-3.65	-3.85	-3.73	-2.67
-0.75	-3.73	-3.39	-3.41	-3.55	-3.47	-3.28	-3.19	-3.40	-3.30	-2.69
-0.50	-3.20	-3.03	-3.06	-3.09	-3.47	-2.89	-2.81	-2.98	-2.90	-2.70
-0.25	-2.64	-2.66	-2.67	-2.63	-3.46	-2.43	-2.40	-2.57	-2.52	-2.68
0.00	-2.26	-2.46	-2.46	-2.34	-3.46	-2.14	-2.15	-2.30	-2.29	-2.67
0.25	-1.91	-2.28	-2.28	-2.09	-3.47	-1.92	-2.00	-2.10	-2.12	-2.69
0.50	-1.48	-2.04	-2.05	-1.77	-3.47	-1.58	-1.71	-1.79	-1.87	-2.70
0.75	-1.00	-1.75	-1.80	-1.46	-3.46	-1.28	-1.45	-1.52	-1.64	-2.68
1.00	-0.63	-1.56	-1.66	-1.30	-3.46	-1.08	-1.32	-1.32	-1.47	-2.67
1.25	-0.09	-1.17	-1.37	-1.01	-3.47	-0.73	-1.01	-0.99	-1.16	-2.69
1.50	0.33	-0.95	-1.31	-0.92	-3.47	-0.61	-0.91	-0.87	-1.06	-2.70
1.75	0.83	-0.66	-1.23	-0.83	-3.46	-0.49	-0.82	-0.76	-0.98	-2.68
2.00	1.29	-0.38	-1.21	-0.83	-3.46	-0.46	-0.82	-0.72	-0.95	-2.67

Note: A_{DM} and A_{WE} denote columns for asymptotic predicted biases of E_{DM} and E_{WE} respectively.

Table 4.4.7 10×bias of estimators of $L_{\sqrt{2}-1}$ in 24 observation experiments using Routine B for step size 1.5.

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
-2.00	-7.91	-6.84	-6.73	-7.17	-5.64	-6.44	-6.25	-6.73	-6.51	-4.16
-1.75	-7.22	-6.32	-6.27	-6.65	-5.59	-5.93	-5.75	-6.21	-6.00	-4.02
-1.50	-6.61	-5.87	-5.86	-6.16	-5.60	-5.53	-5.35	-5.78	-5.58	-4.07
-1.25	-6.02	-5.43	-5.46	-5.71	-5.67	-5.18	-5.02	-5.41	-5.22	-4.26
-1.00	-5.51	-5.11	-5.15	-5.29	-5.71	-4.81	-4.66	-5.04	-4.87	-4.40
-0.75	-4.95	-4.71	-4.73	-4.82	-5.70	-4.33	-4.19	-4.60	-4.45	-4.35
-0.50	-4.69	-4.66	-4.67	-4.63	-5.64	-4.15	-4.06	-4.44	-4.32	-4.16
-0.25	-4.34	-4.50	-4.48	-4.38	-5.59	-3.85	-3.81	-4.17	-4.09	-4.02
0.00	-3.89	-4.21	-4.17	-3.99	-5.60	-3.48	-3.50	-3.81	-3.78	-4.07
0.25	-3.67	-4.15	-4.11	-3.89	-5.67	-3.47	-3.58	-3.79	-3.81	-4.26
0.50	-3.49	-4.17	-4.14	-3.82	-5.71	-3.44	-3.63	-3.78	-3.87	-4.40
0.75	-3.19	-4.05	-4.04	-3.69	-5.70	-3.26	-3.53	-3.65	-3.80	-4.35
1.00	-2.76	-3.81	-3.84	-3.42	-5.64	-2.94	-3.29	-3.36	-3.57	-4.16
1.25	-2.20	-3.43	-3.51	-3.06	-5.59	-2.55	-2.94	-2.96	-3.22	-4.02
1.50	-1.86	-3.26	-3.42	-2.95	-5.60	-2.42	-2.85	-2.78	-3.07	-4.07
1.75	-1.46	-3.01	-3.29	-2.83	-5.67	-2.30	-2.78	-2.65	-2.95	-4.26
2.00	-1.14	-2.92	-3.37	-2.87	-5.71	-2.32	-2.81	-2.68	-3.00	-4.40

Table 4.4.8 10×bias of estimators of $L_{\sqrt{2}-1}$ in 24 observation experiments using Routine B for step size 2.0.

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
-2.00	-8.87	-7.91	-7.89	-8.27	-7.77	-7.28	-7.03	-7.67	-7.38	-5.42
-1.75	-8.48	-7.68	-7.69	-8.04	-8.03	-7.21	-6.98	-7.57	-7.29	-5.99
-1.50	-8.15	-7.56	-7.58	-7.81	-8.26	-7.15	-6.91	-7.47	-7.22	-6.44
-1.25	-7.68	-7.23	-7.25	-7.47	-8.33	-6.79	-6.59	-7.13	-6.88	-6.50
-1.00	-7.22	-6.96	-6.98	-7.07	-8.19	-6.33	-6.14	-6.73	-6.51	-6.15
-0.75	-6.64	-6.56	-6.55	-6.56	-7.94	-5.69	-5.53	-6.14	-5.95	-5.61
-0.50	-6.11	-6.21	-6.19	-6.10	-7.70	-5.12	-5.01	-5.62	-5.46	-5.18
-0.25	-5.78	-6.07	-6.02	-5.85	-7.63	-4.89	-4.83	-5.40	-5.29	-5.09
0.00	-5.60	-6.04	-5.98	-5.76	-7.77	-4.90	-4.92	-5.38	-5.33	-5.42
0.25	-5.56	-6.17	-6.10	-5.84	-8.03	-5.17	-5.32	-5.60	-5.63	-5.99
0.50	-5.51	-6.34	-6.26	-5.88	-8.26	-5.34	-5.59	-5.80	-5.92	-6.44
0.75	-5.40	-6.41	-6.36	-5.94	-8.33	-5.43	-5.78	-5.90	-6.10	-6.50
1.00	-5.06	-6.22	-6.22	-5.75	-8.19	-5.08	-5.52	-5.62	-5.88	-6.15
1.25	-4.58	-5.92	-5.94	-5.44	-7.94	-4.59	-5.09	-5.19	-5.50	-5.61
1.50	-3.95	-5.47	-5.54	-4.99	-7.70	-4.02	-4.59	-4.65	-5.00	-5.18
1.75	-3.38	-5.08	-5.21	-4.64	-7.63	-3.64	-4.23	-4.23	-4.62	-5.09
2.00	-3.06	-4.93	-5.15	-4.57	-7.77	-3.66	-4.30	-4.17	-4.58	-5.42

estimators are above the asymptotic predicted m.s.e.'s for the low starting levels, but below these values for the high starting levels. In using the UDTR rule with the yes-no procedure there appeared, for small step sizes, to be a slight advantage in using starting levels below the $L_{1/\sqrt{2}}$ level in the sense of giving smaller m.s.e for estimators (see Figs. 4.2.1 and 4.2.2). For the 2IFC procedure starting levels above $L_{\sqrt{2}-1}$ seem preferable. Much of the large m.s.e. for low starting values is due to bias, but there is also a marked increase in the variability of estimators. The probability of taking a step down using Routine B is not tending to 0.00 as stimulus level decreases but is bounded below by 0.25. Even if the starting level is far below the $L_{\sqrt{2}-1}$ level there is still a relatively high probability of staying close to the starting level, even when a moderately large sample size is used. For all the step sizes asymptotic biases bear little relation to the actual biases, however the agreement with asymptotic theory is closer for the larger step sizes. Which of these estimators one should prefer is not at all obvious. The estimator E_M often has the smallest m.s.e. but there is not a great deal to choose between estimators.

I also simulated some experiments consisting of 48 and 96 observations under the same set of conditions. Values of m.s.e and bias of E_{DM}, E_{WE} and E_{WE}^* are given in Appendix 11. One interesting point to note is that often E_{WE}^* has a slightly larger m.s.e. than E_{WE} (contrary to what asymptotic theory predicts). E_{WE}^* usually has larger bias than E_{WE}, and the contribution to the m.s.e. from bias is large in these experiments. The m.s.e.'s of E_{WE} and E_{WE}^* are usually less than those of E_{DM}; this is what asymptotic theory predicts.

These estimators not only have large biases but also have large variability. The step sizes I have considered may be inappropriate; their use was motivated by a recommendation for step size for the yes-no procedure. However it must be remembered that if smaller step sizes are used the asymptotic variance of the m.l.e. of β will be very large.

I used $1/\hat{\tilde{\beta}}$ (as defined in Formula 4.3.4) to estimate $1/\beta$. Values of m.s.e. and mean of $1/\beta$ are given in Table 4.4.9. This estimator is useless in 24 observation experiments, not only because it has large m.s.e., but it also has marked negative bias. For 48 and 96 observation experiments the estimator has for each step size smaller bias for most starts (see Appendix 11). However the variability of the estimator is always high, even in the 96 observation experiments.

It may be as psychologists suggest that Routine A cannot be sensibly used. However if Routine B is used it appears that larger sample sizes than those common in psychometric studies are needed to give estimates with acceptable precision. Also it appears that if possible maximum likelihood estimation should be used to derive estimates, as the approximate estimators may have little relation to the quantities they pupport to estimate.

Some difficulties were encountered when attempts were made to

Table 4.4.9 Mean and 10×m.s.e. of $1/\tilde{\tilde{\beta}}$ in 24 observation experiments using Routine B ($\beta = \pi/3.0^{1/2}$, with 2000 simulations).

	Step size							
	0.5		1.0		1.5		2.0	
Start	mean	10 × m.s.e.	mean	10 × m.s.e.	mean	10 × m.s.e.	mean	10 × m.s.e
-2.00	0.462	3.15	0.289	3.05	0.176	4.84	0.133	7.31
-1.75	0.387	2.60	0.245	3.04	0.153	4.92	0.118	7.45
-1.50	0.323	2.23	0.204	3.03	0.141	4.96	0.093	7.84
-1.25	0.262	2.06	0.179	3.06	0.122	5.08	0.048	7.94
-1.00	0.214	2.02	0.154	3.15	0.101	5.24	0.032	8.26
-0.75	0.176	2.09	0.136	3.33	0.080	5.29	0.012	8.24
-0.50	0.152	2.19	0.121	3.45	0.073	5.43	0.013	8.01
-0.25	0.134	2.31	0.115	3.39	0.079	5.33	0.042	7.94
0.00	0.134	2.33	0.116	3.47	0.085	5.33	0.065	7.84
0.25	0.134	2.35	0.118	3.42	0.084	5.16	0.062	7.49
0.50	0.146	2.36	0.121	3.49	0.087	5.46	0.060	8.37
0.75	0.160	2.41	0.121	3.45	0.078	5.46	0.035	8.30
1.00	0.171	2.37	0.131	3.58	0.072	5.46	0.010	8.15
1.25	0.181	2.48	0.123	3.41	0.071	5.25	0.006	8.15
1.50	0.187	2.65	0.125	3.48	0.071	5.17	-0.004	7.87
1.75	0.188	2.76	0.130	3.51	0.069	5.21	0.009	7.55
2.00	0.193	2.83	0.121	3.54	0.065	5.18	0.027	7.36

calculate m.l.e.'s of parameters. The system of equations to be solved is relatively simple. If there are n_i positive and m_i negative responses at x_i and the probability of positive response at this level is $(1+G(x_i))/2$, then the likelihood of the observations is

$$\prod_i ((1+G(x_i))/2)^{n_i} ((1-G(x_i))/2)^{m_i}. \qquad 4.4.1$$

If the response curve is logistic the derivatives of the log likelihood, l, with respect to parameters a and β (where $a=-\mu\beta$) are

$$\frac{\partial l}{\partial a} = \sum_i (n_i(1-G(x_i)) - m_i G(x_i)) - \sum_i n_i(1-G(x_i))/(1+G(x_i)), \qquad 4.4.2$$

$$\frac{\partial l}{\partial \beta} = \sum_i x_i(n_i(1-G(x_i)) - m_i G(x_i)) - \sum_i x_i n_i(1-G(x_i))/(1+G(x_i)). \qquad 4.4.3$$

The second derivatives are

$$\frac{\partial^2 l}{\partial a^2} = \sum_i z_i, \qquad 4.4.4$$

$$\frac{\partial^2 l}{\partial a \partial \beta} = \sum_i x_i z_i, \qquad 4.4.5$$

$$\frac{\partial^2 l}{\partial \beta^2} = \sum_i x_i^2 z_i, \qquad 4.4.6$$

where

$$z_i = -(n_i+m_i)G(x_i)(1-G(x_i)) + 2\, n_i G(x_i)(1-G(x_i))/(1+G(x_i))^2. \qquad 4.4.7$$

The matrix of second derivatives is not in general negative definite. With the yes-no procedure the corresponding matrix is always negative definite; so if finite maximum likelihood estimates of a and β exist, they are unique and Newton-Raphson iterations converge to these values. With the 2IFC procedure there exists a degenerate solution to the likelihood equations with G(x) equal to 0.0 for all x. There can also exist another degenerate solution. Suppose the highest level for which some negative response is recorded is x_K. If $n_K > m_K$, then G(x) taking the value $(n_K - m_K)/(n_K + m_K)$ at x_K, 1.0 above x_K and 0.0 below, satisfies the likelihood equations. The corresponding value of z_K is $-(n_K - m_K)^2 m_K/(n_K(n_K + m_K))$, all the other z_i are 0.0. So the matrix of second derivatives of l is negative definite. Moving towards this degenerate solution one is approaching a local maxima for l.

I tried to obtain m.l.e.'s of parameters in the 24 observations experiments using a simple Newton-Raphson iterative algorithm. I performed these iterations in terms of the parameters a and β, using the actual values as starting values. Iterations often broke down because the matrix to be inverted at each iterative step became less than 10.0^{-8} in magnitude. I tried to start iterations using different parameterisations but similar problems were encountered. The problems in all these iterations arose because the iterations were moving towards a degenerate solution of the likelihood equations. I tried to overcome such problems by using a modification of the Newton-Raphson procedure contained in the NAG library called NAG routine E04LAF. This uses the method described in Gill and Murray (1976). I again used as starting values for iterations the actual values of a and β. The routine indicates that various problems have arisen in iterations by means of the value of an integer IFAIL. If the value of IFAIL was 0 on exit then no apparent problems had arisen. The value 5 for IFAIL was common. When IFAIL is 5,6,7 or 8 this indicates that there is some uncertainty as to whether at exit a maximum has been reached, the value 5 represents the lowest level of uncertainty. Results were accepted if IFAIL is 0; it also appeared reasonable to accept results if IFAIL is 5. Much the same problems arose using this routine as before, in that iterations often began to move towards degenerate solutions of the likelihood equations. With this routine one can set upper and lower bounds on the possible parameter values. The value of β is 1.814 to 3 decimals. I decided to place upper and lower bounds on β of 10.0 and 0.5. These bounds were chosen arbitrarily but seemed reasonable considering the actual value of β. The number of experiments for which IFAIL is 5 remains at around a quarter of those simulated for all except step size 2.0 where it is around a half. I formed estimates of $L_{\sqrt{2}-1}$ and $1/\beta$, that I call $L^*_{\sqrt{2}-1}$ and $1/\beta^*$, which equal the m.l.e.'s of $L_{\sqrt{2}-1}$ and $1/\beta$ if neither bound on β was attained, but otherwise equal E_{DM} and $1/\tilde{\tilde{\beta}}$. Unfortunately the proportion of experiments for which one of the bounds on β was reached is often very high. Table 4.4.10 gives the number of experiments for which neither bound is reached. Values of m.s.e. and mean of $L^*_{\sqrt{2}-1}$ and $1/\beta^*$ are given in Tables 4.4.11 and 4.4.12. How one interprets these results in such circumstances is not at all clear. For step sizes 0.5, 1.0 and 1.5 the m.s.e. of $L^*_{\sqrt{2}-1}$ is smaller than that of E_{DM} for the lowest start but eventually becomes larger for some higher start. For step size 0.5 the bias of $L^*_{\sqrt{2}-1}$ is always smaller

Table 4.4.10 Numbers of 24 observation experiments using Routine B where bounds on β are not attained in the course of iterations with E04LAF ($\beta = \pi/3.0^{1/2}$, with 2000 simulations).

	Step size			
	0.5	1.0	1.5	2.0
Start				
-2.00	841	826	632	756
-1.75	916	829	608	970
-1.50	957	835	606	1072
-1.25	1060	894	633	1132
-1.00	1124	886	629	958
-0.75	1160	878	667	840
-0.50	1199	883	669	795
-0.25	1236	966	667	763
0.00	1281	972	658	805
0.25	1265	979	697	987
0.50	1276	966	701	1053
0.75	1265	1002	755	955
1.00	1253	999	743	888
1.25	1259	937	729	847
1.50	1228	909	680	816
1.75	1187	914	630	840
2.00	1179	921	619	942

Table 4.4.11 10×Mean and 10×m.s.e. of $L^*_{\Omega-1}$ in 24 observation experiments using Routine B ($\beta = \pi/3.0''$, with 2000 simulations).

	Step size							
	0.5		1.0		1.5		2.0	
Start	10 × mean	10 × m.s.e.	10 × mean	10 × m.s.e.	10 × mean	10 × m.s.e.	10 × mean	10 × m.s.e
-2.00	-5.955	10.41	-5.045	10.76	-6.157	15.32	-6.026	20.12
-1.75	-4.910	7.80	-4.455	9.47	-5.760	14.20	-5.220	18.44
-1.50	-3.932	6.99	-4.068	8.45	-5.413	13.22	-4.698	17.66
-1.25	-3.081	5.78	-3.583	7.62	-4.991	12.37	-4.279	18.02
-1.00	-2.434	4.21	-3.287	7.18	-4.632	11.37	-4.372	17.83
-0.75	-1.836	3.70	-2.960	6.49	-4.202	10.79	-4.208	17.10
-0.50	-1.615	3.41	-2.622	6.14	-4.147	10.95	-4.266	16.66
-0.25	-1.177	3.12	-2.360	6.02	-3.990	10.58	-4.182	15.72
0.00	-0.830	3.04	-2.195	5.93	-3.721	10.16	-4.087	15.09
0.25	-0.539	2.96	-2.147	5.98	-3.714	10.10	-4.001	14.89
0.50	-0.528	3.09	-2.036	5.90	-3.708	10.19	-3.986	14.96
0.75	-0.313	3.04	-1.802	5.76	-3.617	10.46	-4.230	15.10
1.00	-0.115	3.25	-1.863	5.81	-3.551	10.30	-4.165	15.75
1.25	-0.017	3.40	-1.666	5.45	-3.251	9.88	-4.195	15.83
1.50	0.095	3.36	-1.648	5.74	-3.231	9.99	-4.014	15.42
1.75	0.132	3.60	-1.665	5.84	-3.166	9.54	-3.671	14.46
2.00	0.018	4.12	-1.741	6.25	-3.256	9.86	-3.469	14.09

Table 4.4.12 Mean and 10×m.s.e. of $1/\beta^*$ in 24 observation experiments using Routine B ($\beta=\pi/3.0^{1/2}$, with 2000 simulations).

	Step size							
	0.5		1.0		1.5		2.0	
Start	mean	10 × m.s.e.	mean	10 × m.s.e.	mean	10 × m.s.e.	mean	10 × m.s.e
-2.00	0.502	2.70	0.438	2.77	0.360	4.77	0.304	6.77
-1.75	0.443	2.20	0.407	2.77	0.332	4.79	0.327	6.55
-1.50	0.398	1.84	0.377	2.80	0.332	4.89	0.323	6.69
-1.25	0.358	1.66	0.373	2.79	0.322	4.97	0.298	6.48
-1.00	0.338	1.60	0.354	2.87	0.298	5.09	0.248	7.11
-0.75	0.318	1.65	0.340	3.05	0.287	5.02	0.204	7.43
-0.50	0.314	1.70	0.328	3.07	0.281	5.30	0.193	7.36
-0.25	0.306	1.78	0.336	2.96	0.287	5.27	0.211	7.32
0.00	0.315	1.78	0.339	3.10	0.292	5.29	0.226	7.22
0.25	0.310	1.83	0.345	3.08	0.305	5.06	0.273	6.55
0.50	0.321	1.87	0.341	3.13	0.302	5.28	0.292	7.41
0.75	0.333	1.93	0.343	2.98	0.304	5.19	0.256	7.23
1.00	0.332	1.94	0.353	3.13	0.299	5.28	0.216	7.12
1.25	0.332	2.05	0.330	3.05	0.293	5.08	0.214	7.34
1.50	0.329	2.14	0.321	3.10	0.278	5.00	0.197	7.17
1.75	0.323	2.28	0.323	3.17	0.262	5.11	0.197	6.88
2.00	0.324	2.32	0.313	3.18	0.252	5.06	0.217	6.29

than that of E_{DM}, but is sometimes larger for step sizes 1.0 and 1.5. For step size 2.0 the $L^*_{\sqrt{2}-1}$ always has smaller m.s.e. and bias than E_{DM}. The m.s.e. of $1/\beta^*$ is always smaller than that of $1/\tilde{\tilde{\beta}}$; this is mainly on account of smaller bias. The estimator $1/\beta^*$ itself has a large negative bias and could not be reasonably used to estimate $1/\beta$. Results of these calculations are inconclusive and it cannot be said that use of maximum likelihood estimation significantly improves the quality of estimates.

I made similar calculations for the 48 and 96 observation experiments. The results are of greater interest as, at least for the smaller step sizes, fewer problems were encountered in iterations. Tables analogous to Tables 4.4.9 to 4.4.12 are contained in Appendix 11. For the 48 observation experiments, with step sizes 1.0, 1.5 and 2.0, the biases of E_{DM}, E_{WE} and E^*_{WE} are larger than those of $L^*_{\sqrt{2}-1}$. This is not always the case for step size 0.5, but for low starts the biases of $L^*_{\sqrt{2}-1}$ are smaller than those of the other estimators. For step sizes 1.5 and 2.0 the m.s.e.'s of E_{DM}, E_{WE} and E^*_{WE} are larger than those of $L^*_{\sqrt{2}-1}$. For the step sizes 0.5 and 1.0 the m.s.e.'s of $L^*_{\sqrt{2}-1}$ at least do not reach as high levels of those of the other estimators for low starts. For 96 observation experiments the biases of E_{DM}, E_{WE} and E^*_{WE} are always larger than those of $L^*_{\sqrt{2}-1}$. For all except step size 0.5 the m.s.e.'s of $L^*_{\sqrt{2}-1}$ are smaller than those of the other estimators. For step size 0.5 the m.s.e.'s of $L^*_{\sqrt{2}-1}$ are slightly larger than those of the other estimators for some high starts. These results indicate that $L^*_{\sqrt{2}-1}$ has some advantages over alternative estimators in that bias and m.s.e. are often smaller. This is not surprising as the other estimators have relatively large asymptotic biases but $L_{\Gamma_{2}-1}^{*}$ is asymptotically unbiased. For both the 48 and 96 observation experiments the m.s.e.'s of $1/\beta^{*}$ are always smaller than those of $1/\tilde{\beta}$. Also the biases of $1/\beta^{*}$ are usually smaller than those of $1/\tilde{\tilde{\beta}}$. Although $L_{\Gamma_{2}-1}^{*}$ and $1/\beta^{*}$ appear to have some advantages over alternative estimators they have relatively high variability compared to analogous estimators from Routine A (see tables for mean and m.s.e. of μ^{*} and $1/\beta^{*}$ in Appendix 9).

It is not possible, for a number of reasons, to make a direct comparison of results in this section with those in Rose et al. The assumed forms for the response curve are different and the number of observations per experiment were set at different numbers. More importantly Rose et al used estimates based on the mean and median of stimulus levels presented, ignoring the first 10 trials, and also averaged results over starting levels (i.e. several possible starting levels covering a wide range of stimulus intensities were assumed and results were pooled over these levels). They identify three main failings of forced choice estimates as opposed to yes-no estimates as being that: forced choice estimates are more variable than yes-no estimates, properties of forced choice estimates are a function of the stimulus spacing but those of the yes-no estimates are not, and in the yes-no procedure it is sometimes possible to identify a staircase run that is very likely to lead to a biased estimate but it is much less likely that such an identification can be made if a forced choice procedure is used. Certainly my results also support the first of these statements. Rose et al state that in their experiments the standard deviation of the forced choice estimates is on average 2.6 times as large as that of yes-no estimates, that is the variance is around 7.0 times larger. Looking in more detail at their results it is apparent that the ratio of standard deviations depends to a large extent on step size, with larger step sizes usually giving higher ratios. Comparing Table. 4.4.1 and Table. 3.3.1 one sees that, for the logistic curve ($\beta = \pi/3.0^{1/2}$) in 24 observation experiments for the range of starting values considered, the ratio of the m.s.e.'s of E_{DM}, using Routine A and Routine B, is for step sizes 2.0 and 1.5 at least 10.0, for step size 1.0 at least around 4.0, and for step size 0.5 at least around 2.0, with often these ratios being considerably larger. This is similar to the pattern in Rose et al's results. The asymptotic variances of m.l.e.'s illustrated in Figs. 4.3.1 and 4.3.2 also indicate that estimates using Routine B will in large samples be more variable than those from Routine A. The second assertion Rose et al make is mistaken as even when Routine A is used the properties of estimators will depend on step size. They also say that their estimators using Routine A are unbiased. In general such estimators will be biased if stimulus levels are placed asymmetrically about μ. However, from both their results and mine, it is clear that for a wide range of conditions bias of estimators and effect of step size are much smaller using Routine A rather than Routine B. They observed marked negative bias in estimates of $L_{\sqrt{\frac{1}{2}}}$ from Routine B for large step sizes which is in line with my findings for step sizes 1.5 and 2.0. The asymptotic biases of the E_{DM} and E_{WE} are indeed negative over a wide range of conditions (see Fig. 4.3.4). The third assertion that Rose et al make (i.e.

that it is easier with Routine A than with Routine B to identify a run which is likely to give a biased estimator) seems reasonable enough; though any criteria for making such judgements will for both Routines be somewhat subjective.

At the end of this study any conclusions about the usefulness of Routine B cannot be encouraging. The bias and variability of approximate estimates of parameters are large. Asymptotic calculations and simulations suggest that a large sample will be required for maximum likelihood estimation to give useful estimates.

5. STOCHASTIC APPROXIMATION METHODS

5.1 ESTIMATION USING THE ROBBINS-MONRO METHOD

In Chapters 2, 3 and 4 I have discussed variants of the Up and Down procedure. In Section 3.1 I made a comparison between asymptotic properties of the Up and Down design and non-sequential designs. The Up and Down procedure has often been compared with the Robbins-Monro procedure (for example in Davis (1965a), (1965b) and (1971), Cochran and Davis (1964) and Wetherill (1963) and (1966)). This procedure and Kesten's modification of it are described in Section 1.4. Davis's work provides a very detailed comparison between methods but with the limitation that only 12 subjects are used in each design with 1,2,3 or 4 animals per trial in 12,6,4 or 3 trials respectively. Davis was able with such small numbers of observations to calculate exact values of m.s.e.'s and biases. I performed similar calculations to obtain the results in Section 2.2. As is usual when operating the Robbins-Monro procedure Davis took as his estimate the level that would have been visited had one more observation been made. It is no surprise in these circumstances that delayed versions of the Robbins-Monro and Up and Down procedures are recommended for use under all conditions considered, as they are specially designed to reduce bias due to bad starting levels which is pronounced in such small scale experiments (see recommendations at the end of Davis (1971)). What Davis terms the delayed Up and Down design is the use of the Up and Down procedure with the estimator E_{8D}; the delayed Robbins-Monro design uses the Robbins-Monro procedure with the modification of following the Up and Down rule until the first change of response type. Davis used several response curves including the logistic and all were normalised so that the tolerance distribution had unit variance. He found that, providing starting levels were within distance 2.0 of the ED50 and step sizes in the Up and Down procedure are between 0.5 and 2.0, the two delayed procedures were about equally good in terms of the m.s.e.'s of the corresponding estimators (he made comparisons between procedures using the same multiples of his recommended step size and c values). Calculated values in Davis (1965b) provide a check on some results in Section 2.2. Much of the motivation for Davis's work is contained in Cochran and Davis (1964). In the discussion at the end of this paper Marvin Schneiderman raises the problem of estimating points other than the ED50. Wetherill (1963) performed a simulation study of the Robbins-Monro method, some of the results of which are reproduced in 'Sequential Methods in Statistics' (1966), using the method to estimate percentage points other than the ED50 gave very disappointing results with estimators subject to substantial bias. An explanation for this behaviour is given in Section 10.2 of Wetherill (1966). Wetherill does cite Kesten (1958) and Davis (1963) for examples of modifications to this procedure which may overcome such difficulties but he makes no simulations using these modified procedures.

For the Up and Down procedure a step size recommended by Dixon and Mood (1948) and Brownlee _et al_ (1953) is the standard deviation of the tolerance distribution underlying the response (values for asymptotic variances in Section 3.1 for logistic response indicate that this is a sensible step size use). For the Robbins-Monro and delayed Robbins-Monro procedures a value of c equal to $1/g_{1/2}$ will give the lowest asymptotic variance for the estimator ($g_{1/2}$ is the slope of the response curve at the ED50 and c is the step multiplier, see Section 1.4). Davis discusses Kesten's modification of the procedure where the step size is changed only when the two previous steps have been in opposite directions (the first two steps being of length c and c/2). His conjectures concerning the asymptotic variance of the estimator from this procedure (see Section 1.4) suggest that c should then be chosen equal to $1/2g_{1/2}$. For convenience I will refer to the Robbins-Monro, delayed Robbins-Monro and Kesten procedures as Procedures 1,2 and 3 respectively. I have considered an alternative procedure where the step size is changed if the next step to be taken is in the opposite direction to the previous; this I will call Procedure 4 (here the step size is changed one step earlier than would be the case for Procedure 3). Using the same arguments one would again try to set c equal to $1/2g_{1/2}$. Usually the value of $g_{1/2}$ is not known exactly and one must use some prior estimate for $g_{1/2}$ in deciding upon an appropriate value for c.

To investigate the behaviour of these procedures when p=0.5 (i.e. the ED50 is estimated) I simulated 24 step experiments where the response curve is logistic. I set β equal to $\pi/3.0^{1/2}$ so that comparisons with previous simulations using the Up and Down rule would be easy. For Procedures 1 and 2 the value of c minimising the asymptotic variance of the estimator is $4.0/\beta$ and for

Procedures 3 and 4 it is $2.0/\beta$. I considered values of c equal to 0.5(0.5)2.0 times these values. Starting levels were set equal to 0.00(0.25)4.00 and 2000 simulations were made for each set of conditions. Figs. 5.1.1 to 5.1.4 illustrate m.s.e.'s of estimators. Values of m.s.e. of estimators are also given in Tables 5.1.1 to 5.1.4. and the corresponding values for bias are given in Tables 5.1.5 to 5.1.8.

For c equal to 0.5 times the recommended value the differences between procedures are very obvious. When Procedure 1 is used the m.s.e.'s rise very rapidly as the starting level is made more distant from μ. The m.s.e.'s and biases of estimators when Procedure 2 is used are much lower for the distant starts. Procedure 3 has much the same defects as Procedure 1; this is somewhat surprising as Procedure 3 is supposed to accelerate convergence. Davis had similar results in his 12 step experiments. He decided to compare Procedure 3 with Procedures 1 and 2 using the same value of c (i.e. twice the value asymptotic theory would suggest). If I followed his example I would compare values in Tables 5.1.1 and 5.1.5 for Procedures 1 and 2 with values in Tables 5.1.2 and 5.1.6 for Procedure 3. I can see some merit in his suggestion, but I prefer comparisons between procedures under conditions for which they have similar asymptotic properties. For the conditions he considered the larger value of c used in Procedure 3 did make the m.s.e.'s more comparable to those for Procedure 2 but in his conclusion he still thought it best to use Procedure 2. It encouraged me to see that m.s.e.'s and biases of estimators using my alternative procedure, Procedure 4, are

Fig. 5.1.1 M.s.e's of estimators from Procedures 1 to 4 with c equal to 0.5 times the asymptotic optimal values.

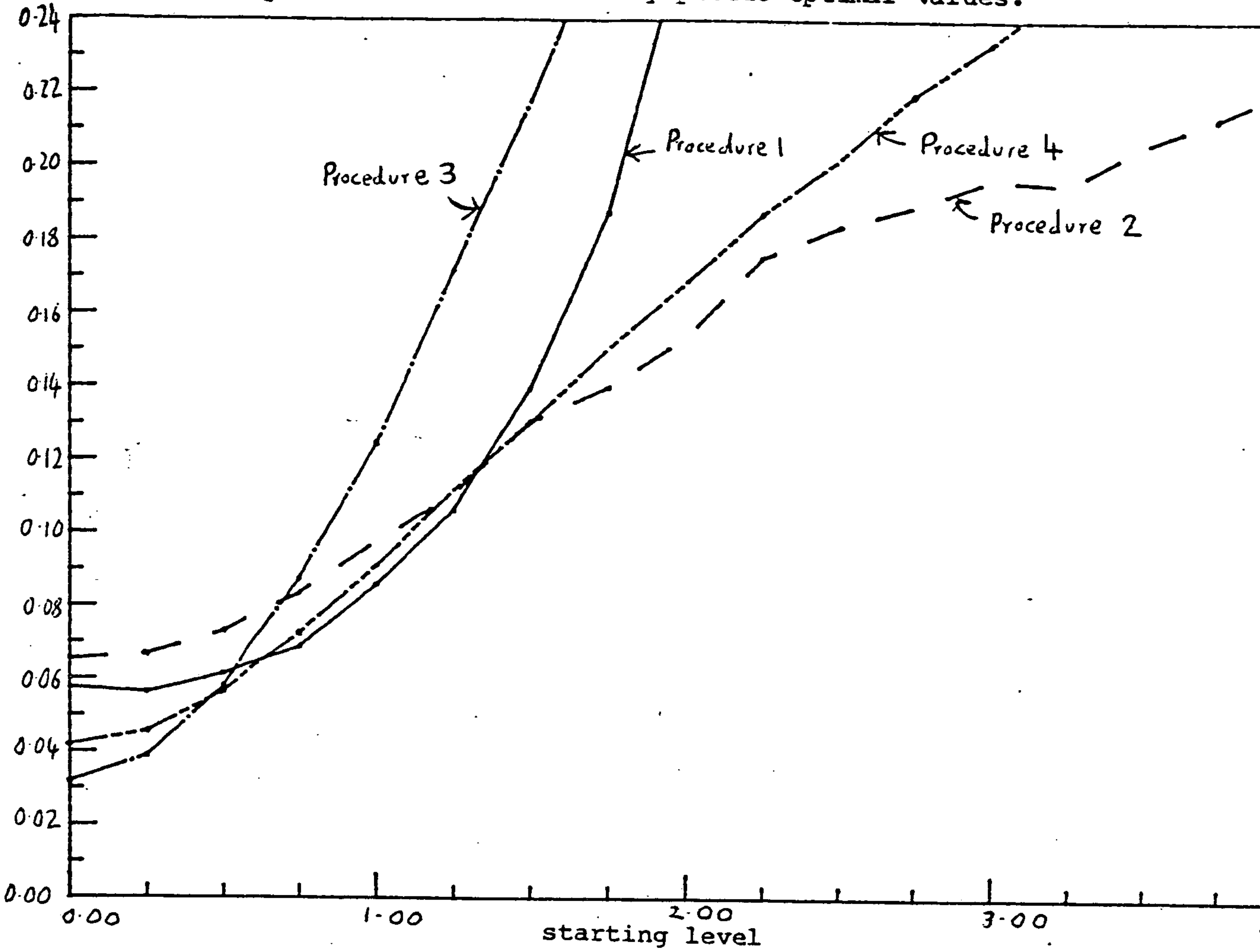


Fig. 5.1.2 As in Fig. 5.1.1 only with the optimal c values.

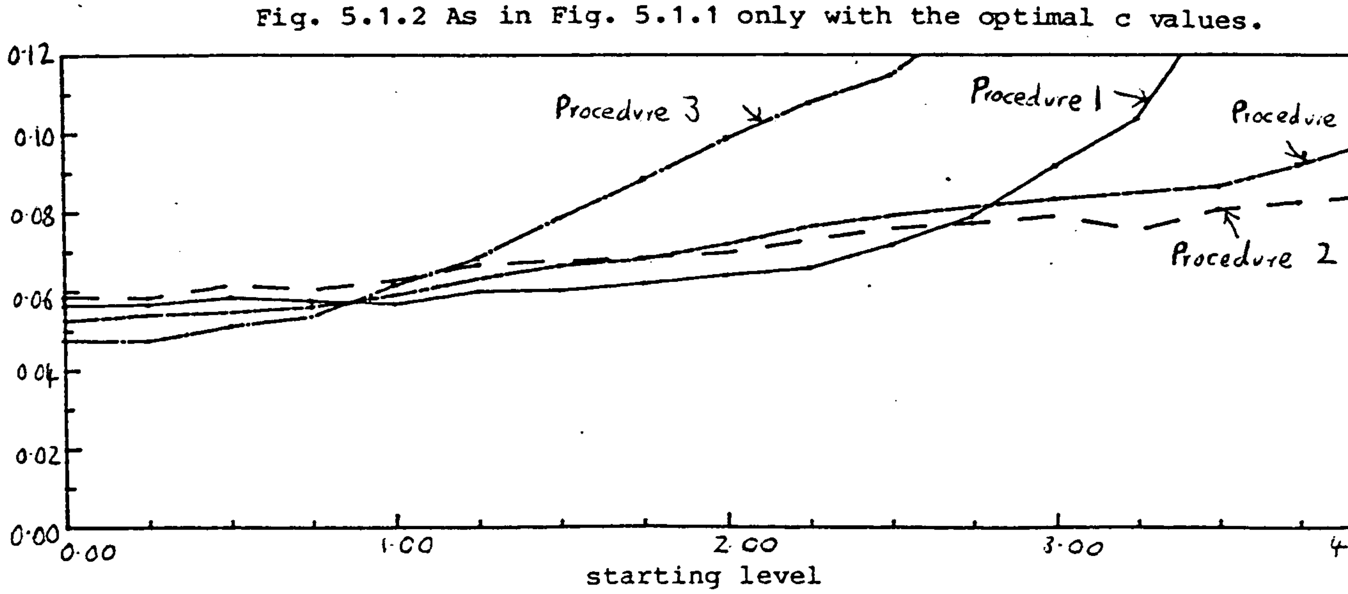


Fig. 5.1.3 As in Fig. 5.1.1 only with c equal to 1.5 times the optimal values.

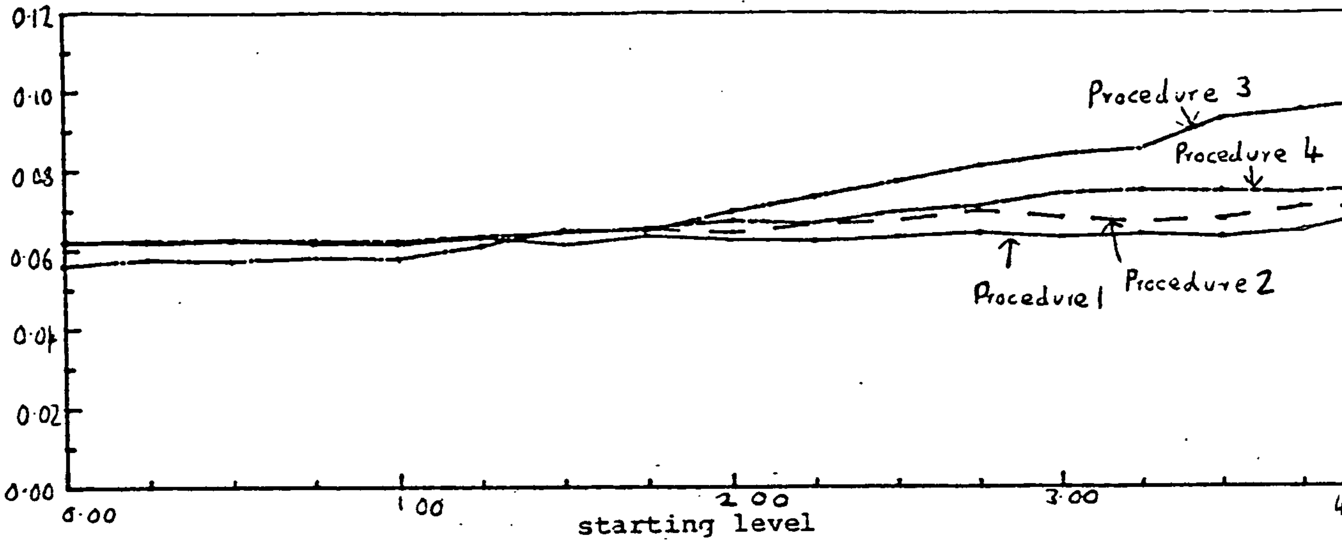


Fig 5.1.4 As in Fig. 5.1.1 only with c equal to 2.0 times the optimal values.

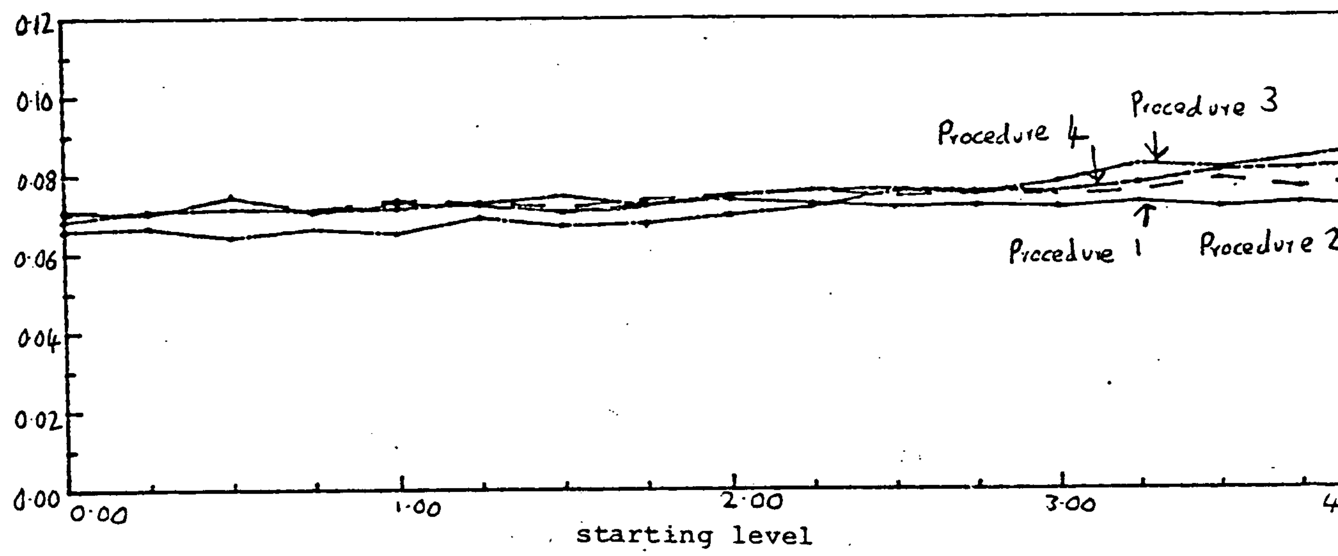


Table 5.1.1 100×m.s.e. of estimators in 24 step experiments using stochastic approximation procedures with c equal to 0.5 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	5.75	6.53	3.18	4.19
0.25	5.62	6.68	3.89	4.56
0.50	6.14	7.30	5.81	5.66
0.75	6.86	8.34	8.74	7.24
1.00	8.56	9.76	12.43	9.08
1.25	10.63	11.05	17.14	11.16
1.50	13.97	13.04	21.76	13.03
1.75	18.74	13.99	26.93	15.04
2.00	26.70	15.36	32.88	16.85
2.25	38.98	17.52	38.49	18.71
2.50	56.36	18.35	44.11	20.17
2.75	81.35	18.91	51.90	21.97
3.00	116.98	19.66	60.57	23.38
3.25	162.67	19.51	70.79	24.90
3.50	221.77	20.59	82.34	27.22
3.75	294.13	21.33	96.90	29.70
4.00	379.97	22.23	115.81	31.50

Table 5.1.2 100×m.s.e. of estimators in 24 step experiments using stochastic approximation procedures with c equal to the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	5.66	5.88	4.76	5.26
0.25	5.67	5.84	4.75	5.41
0.50	5.84	6.16	5.11	5.47
0.75	5.76	6.05	5.37	5.61
1.00	5.68	6.29	6.15	5.91
1.25	5.99	6.68	6.85	6.32
1.50	6.03	6.78	7.87	6.65
1.75	6.21	6.88	8.84	6.83
2.00	6.42	6.99	9.87	7.21
2.25	6.59	7.31	10.78	7.65
2.50	7.18	7.60	11.48	7.92
2.75	7.90	7.73	12.93	8.14
3.00	9.19	7.92	14.00	8.35
3.25	10.37	7.55	14.96	8.51
3.50	13.26	8.07	16.15	8.66
3.75	17.81	8.27	17.52	9.20
4.00	24.53	8.44	18.99	9.84

Table 5.1.3 100×m.s.e. of estimators in 24 step experiments using stochastic approximation procedures with c equal to 1.5 times the asymptotic optimal values ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	6.22	6.19	5.62	6.20
0.25	6.16	6.18	5.75	6.25
0.50	6.23	6.28	5.72	6.25
0.75	6.15	6.17	5.80	6.25
1.00	6.13	6.24	5.78	6.23
1.25	6.30	6.29	6.09	6.36
1.50	6.12	6.45	6.50	6.43
1.75	6.34	6.51	6.49	6.56
2.00	6.23	6.41	6.95	6.71
2.25	6.20	6.62	7.32	6.66
2.50	6.29	6.71	7.72	6.94
2.75	6.41	6.97	8.12	7.11
3.00	6.31	6.80	8.40	7.43
3.25	6.38	6.69	8.54	7.49
3.50	6.31	6.78	9.32	7.47
3.75	6.48	7.08	9.53	7.44
4.00	6.90	6.97	9.73	7.54

Table 5.1.4 100×m.s.e. of estimators in 24 step experiments using stochastic approximation procedures with c equal to 2.0 times the asymptotic optimal values ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	7.08	7.10	6.59	6.84
0.25	7.00	7.04	6.65	7.09
0.50	7.46	7.43	6.41	7.14
0.75	7.04	7.14	6.63	7.10
1.00	7.28	7.37	6.51	7.14
1.25	7.22	7.32	6.92	7.29
1.50	7.07	7.18	6.70	7.48
1.75	7.24	7.37	6.77	7.22
2.00	7.36	7.50	6.97	7.47
2.25	7.27	7.59	7.17	7.61
2.50	7.16	7.43	7.55	7.63
2.75	7.21	7.49	7.51	7.56
3.00	7.15	7.46	7.80	7.56
3.25	7.29	7.56	8.24	7.76
3.50	7.15	7.87	8.14	8.07
3.75	7.26	7.63	8.38	8.13
4.00	7.11	7.87	8.65	8.22

Table 5.1.5 100×bias of estimators in 24 step experiments using stochastic approximation procedures with c equal to 0.5 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	-0.30	-0.43	0.01	-0.13
0.25	3.59	2.53	7.23	4.64
0.50	7.39	5.33	14.64	9.97
0.75	12.16	7.96	21.82	13.78
1.00	16.91	10.25	28.07	17.64
1.25	22.66	12.21	34.51	20.63
1.50	29.41	13.59	39.37	22.98
1.75	37.35	14.22	44.50	25.38
2.00	47.28	16.02	49.48	26.70
2.25	59.19	17.20	53.76	28.80
2.50	72.87	17.21	58.01	30.57
2.75	88.81	17.60	63.37	31.95
3.00	107.27	17.82	68.77	32.58
3.25	127.08	17.69	74.86	33.90
3.50	148.63	18.72	81.37	35.66
3.75	171.33	18.84	89.57	37.16
4.00	194.83	18.59	99.91	38.64

Table 5.1.6 100 bias of estimators in 24 step experiments using stochastic approximation procedures with c equal to the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	-0.14	-0.14	-0.28	-0.80
0.25	0.43	0.33	2.18	0.88
0.50	0.55	0.14	4.89	2.60
0.75	1.41	0.62	7.20	3.28
1.00	2.47	1.18	9.68	4.35
1.25	3.57	1.92	12.01	4.96
1.50	4.57	2.02	14.11	5.82
1.75	5.53	1.99	15.90	6.06
2.00	7.10	1.50	17.34	7.02
2.25	8.94	2.14	18.89	7.05
2.50	11.79	2.91	19.68	7.23
2.75	15.03	2.83	21.52	7.46
3.00	18.73	3.15	22.75	7.57
3.25	22.85	2.82	23.62	7.72
3.50	28.91	2.47	24.75	8.53
3.75	36.09	2.02	26.63	8.54
4.00	44.88	2.37	28.18	8.90

Table 5.1.7 100×bias of estimators in 24 step experiments using stochastic approximation procedures with c equal to 1.5 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	-0.19	-0.24	-0.29	-0.52
0.25	-0.57	-0.51	0.57	-0.21
0.50	-0.16	-0.25	1.33	0.48
0.75	-0.04	0.01	2.38	0.57
1.00	0.57	0.33	3.50	1.19
1.25	0.10	-0.16	4.84	1.16
1.50	0.71	0.17	5.54	1.42
1.75	0.86	0.47	6.05	1.74
2.00	0.89	-0.11	6.88	2.03
2.25	1.04	0.15	7.69	2.10
2.50	1.58	-0.15	7.90	2.37
2.75	2.24	-0.18	8.57	2.09
3.00	2.50	0.09	8.74	1.83
3.25	3.45	0.20	9.54	2.19
3.50	4.24	-0.12	10.03	2.36
3.75	5.53	-0.06	10.48	2.11
4.00	6.79	0.43	11.14	2.77

Table 5.1.8 100×bias of estimators in 24 step experiments using stochastic approximation procedures with c equal to 2.0 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	-0.25	-0.31	-0.25	-0.23
0.25	-0.32	-0.36	-0.00	-0.43
0.50	-0.47	-0.39	-0.12	-0.33
0.75	-0.22	-0.37	0.28	-0.09
1.00	0.08	-0.06	1.03	-0.11
1.25	-0.34	-0.17	1.81	0.55
1.50	0.16	0.24	1.98	-0.22
1.75	-0.10	-0.23	2.72	0.48
2.00	-0.10	-0.21	2.90	0.07
2.25	-0.15	-0.28	2.91	0.31
2.50	-0.48	-0.44	3.74	0.88
2.75	-0.18	-0.39	3.64	0.34
3.00	-0.25	-0.21	3.69	0.87
3.25	-0.16	-0.64	3.92	1.04
3.50	0.57	-0.56	4.12	1.03
3.75	0.67	-0.34	4.36	0.34
4.00	0.58	-0.20	4.97	0.33

comparable to those for Procedure 2. The m.s.e.'s with Procedure 4 do rise some way above those for Procedure 2 for distant starts but not to the heights reached using Procedures 1 and 3. The bias is usually higher with Procedure 4 than 2 but again it does not rise to the much higher levels reached using Procedures 1 and 3.

When c equals the asymptotic optimal value, asymptotic theory predicts a value for the variance of the estimators of 0.0507. Close to μ this asymptotic prediction is fairly closely obeyed (m.s.e.'s with Procedure 3 are then a little below this value and with the other procedures they are a little above). Much of what was said in the previous paragraph applies equally well here. Procedures 1 and 3 have bad behaviour for distant starts. Estimators from Procedures 2 and 4 have similar m.s.e.'s that do not rise so high as those for the other procedures. The biases using Procedure 2 are lower than using Procedure 4.

When c equals 1.5 times the asymptotic optimal value the asymptotic predicted variance for the estimators is 0.0570. The m.s.e.'s of Procedures 1,2 and 4 are now close with Procedure 2 having a slight advantage in terms of bias for the distant starts. These m.s.e.'s are always above 0.0570 but not greatly above. The biases using Procedure 3 are usually the highest and the m.s.e.'s rise some way above those for the other procedures for distant starts.

When c equals 2.0 times the asymptotic optimal value the predicted variance of estimators is equal to 0.0675. There is now little to choose between procedures. The biases are always small and the m.s.e.'s are usually a little way above the predicted variance. The bias is greatest for distant starts when Procedure 3 is used.

I also simulated experiments of 48 and 96 steps under the same set of conditions. The values of m.s.e.'s and biases from these simulations are contained in Appendix 12. Much of what I have said concerning 24 step experiments also applies to 48 and 96 step experiments. The asymptotic theory that applies for multiples of 1.0(0.5)2.0 of the recommended value of c is more closely obeyed. There appears to be no good reason in these experiments to follow the expedient suggested by Davis of comparing Procedure 3 with the other procedures using the same value for c (asymptotic theory suggests that the c value for Procedure 3 should be half the c value used in Procedures 1 and 2). For c equal to 0.5 times the recommended value the m.s.e.'s using Procedure 4 are below those for Procedure 2 but the biases are larger. The m.s.e.'s and biases using Procedures 1 and 3 are above values for Procedures 2 and 4 for distant starts (though in 96 step experiments the m.s.e. for Procedure 3 does not rise to very high values for distant starts). For c equal to the recommended step size m.s.e.'s for Procedures 2 and 4 are similar but the bias with Procedure 4 is higher. Again Procedures 1 and 3 have some disadvantage in that they have higher m.s.e.'s and biases for distant starts but this disadvantage is less than it was for 24 step experiments. For multiples of the recommended value for c of 1.5 and 2.0 there is little to choose between procedures as biases are low and m.s.e.'s are similar (the bias is often lowest for Procedure 2 and highest for Procedure 3).

The main conclusions from these simulations can be summarised as follows:

(1) Procedure 2 is to be preferred to Procedure 1 as for distant starts m.s.e.'s and biases using Procedure 1 are often much higher than those using Procedure 2 but they take similar values for starts close to μ.

(2) There is some uncertainty over what value of c using Procedure 3 should be used in comparison with Procedures 1 and 2. I can see no evidence that Procedure 3 has any particular advantages over the other procedures and in some respects it compares very badly with Procedure 2 in that it has similar defects to Procedure 1.

(3) The modification of Procedure 3 that I suggest, Procedure 4, appears much more effective than Procedure 3 in accelerating convergence. The behaviour of estimators using Procedure 4 is often similar to that of estimators using Procedure 2. Procedure 2 usually has an advantage in that m.s.e.'s are similar for these procedures but biases are lower with Procedure 2. For the lowest c value considered estimators from Procedure 4 often have lower m.s.e. than those from Procedure 2. From the asymptotic theory alone I believe that one should be careful to avoid such a low value for c. If c is less than or equal to half the asymptotic optimal value, one cannot show that the estimators are

asymptotically normal; although the estimators tend in mean square to μ the convergence is not as fast as $O(1/n)$, where n is the number of observations (see Hodges and Lehmann (1956)).

(4) From the results of simulations of 48 and 96 step experiments it seems sensible to use a value of c of about 1.5 times the recommended value because then:

(a) The dependence of m.s.e.'s on starting levels is small.

(b) There is little to choose between procedures.

(c) There is only a low possible loss in efficiency relative to what could be expected with the asymptotic optimal c.

(d) One has some protection against choosing a value of c which is less than or equal to half the asymptotic optimal value if the initial estimate of β is too high.

So far I have not made any comparison between these stochastic approximation procedures and the Up and Down procedure. In such comparisons there is immediately the problem of what sets of conditions are comparable as the asymptotic properties of the Up and Down procedure are quite different. Davis having decided upon what were appropriate c values for the stochastic approximation procedures and step size for the Up and Down procedure made comparisons between experiments where the same multiples of these recommended values were used. He recommends the delayed forms of the procedures and notes that estimators from these procedures have similar m.s.e.'s over a wide range of conditions. For the experiments with 24 observations there are difficulties in making comparisons. With Procedures 2 and 4, for c value 0.5 times the asymptotic optimal value, the m.s.e.'s of estimators are, for distant starts, well above m.s.e.'s for $E_{\beta D}$ and E_{DM} for step size 0.5 with the Up and Down procedure (see Tables 3.3.1 and 5.1.1). The m.s.e.'s with Procedures 2 and 4, for c value, 2.0 times the asymptotic optimal value are always below values for $E_{\beta D}$ and E_{DM} for step size 2.0 (see Tables 3.3.4 and 5.1.4). I simulated Up and Down experiments for step sizes 0.25 and 0.75 and starts at 0.00(0.25)4.00 making 2000 simulations per set of conditions. Tables 5.1.9 and Table 5.1.10 give values of m.s.e.'s and biases of E_{DM} for step sizes 0.25(0.25)1.00 (values for step sizes 0.50 and 1.00 come from results in Section 3.3). The m.s.e.'s and biases of E_{DM} for step sizes 0.25 and 0.50 are close to m.s.e.'s and biases using Procedure 4 for c values 0.50 and 1.00 times the asymptotic optimal value. The m.s.e.'s of E_{DM} for step sizes 0.75 and 1.00 are fairly close to values of m.s.e.'s using Procedure 4 for c values 1.50 and 2.00 times the asymptotic optimal value (though now the bias of E_{DM} is usually some way above the bias of the estimator from Procedure 4). For this number of observations the stochastic approximation procedures are more closely comparable to the Up and Down procedure if one compares results for the stochastic approximation procedures with c equal to k times the asymptotic optimal value with the Up and Down procedure for step size k/2 times the recommended value (where k=0.5(0.5)2.0).

Table 5.1.9 100×m.s.e. of E_{DM} in 24 step experiments ($\beta = \pi / 3.0^{1/2}$, based on 2000 simulations).

Step Size / Start	0.25	0.50	0.75	1.00
0.00	4.16	5.35	6.29	7.03
0.25	4.60	5.63	6.30	7.28
0.50	5.41	6.06	6.54	7.14
0.75	6.62	6.39	6.91	7.38
1.00	7.58	6.75	7.46	7.78
1.25	9.08	7.14	7.37	8.42
1.50	9.93	7.70	7.37	8.22
1.75	11.63	7.92	7.75	8.24
2.00	12.64	8.08	7.80	8.06
2.25	14.19	8.58	8.26	8.38
2.50	15.67	8.99	8.54	8.26
2.75	17.56	9.07	8.57	8.47
3.00	19.22	9.04	8.61	8.73
3.25	21.98	9.48	8.61	9.34
3.50	24.35	9.86	8.81	9.25
3.75	27.75	10.30	9.04	9.36
4.00	31.88	10.46	9.18	9.07

Table 5.1.10 100×bias of E_{DM} in 24 step experiments ($\beta = \pi / 3.0^{1/2}$, based on 2000 simulations).

Step Size / Start	0.25	0.50	0.75	1.00
0.00	0.10	-0.30	-0.47	-0.26
0.25	4.18	2.05	1.43	0.56
0.50	8.48	3.47	2.54	1.77
0.75	11.46	5.36	3.63	3.31
1.00	14.14	6.51	3.44	3.29
1.25	16.40	7.71	4.68	3.82
1.50	18.54	7.79	4.68	2.84
1.75	20.45	8.60	5.43	2.87
2.00	22.08	9.32	5.27	3.63
2.25	23.97	9.72	6.09	3.33
2.50	25.89	9.57	5.50	3.93
2.75	28.14	10.20	6.29	5.04
3.00	29.94	10.52	5.92	4.57
3.25	32.88	10.91	6.18	4.76
3.50	35.07	10.75	6.10	3.68
3.75	37.79	10.94	6.87	3.89
4.00	41.31	11.84	5.98	4.48

This result is not very surprising as in the stochastic approximation procedures the average step size decreases with n but for the Up and Down procedure remains fixed. One might hope that, by setting the ratio of the fixed step size to c value equal to some decreasing function of n, one could obtain close comparability between the Up and Down procedure and one of the stochastic approximation procedures for any n. Consideration of the asymptotic properties of estimators shows two reasons why this is not possible. One is that whatever step size is chosen with the Up and Down procedure the m.l.e. of μ is asymptotically normal with asymptotic variance tending to zero as $O(1/n)$ but if the c value is chosen too low in the stochastic approximation procedures then (as I have already remarked) the corresponding estimators cannot be shown to be asymptotically normal and have asymptotic variances tending to zero at a rate slower than $O(1/n)$. So the asymptotic properties of estimators from the respective procedures are quite different for small step sizes and c values. The other reason is that the asymptotic variance expression for the m.l.e. of μ with the Up and Down procedure approaches the lower bound of $4/\beta^2 n$ as the step size decreases but with the stochastic approximation procedures this lower bound is attained only for c equal to $1/g_{1/2}$. For comparability between procedures for any n one would need the ratio of c to step size to depend not only on n but also on the c value. The step size in the Up and Down procedure would have to tend to zero as n increases for $c<1/2g_{1/2}$ or $c=1/g_{1/2}$ but to some finite limit otherwise.

For experiments involving 48 and 96 observations I tried to find some basis for comparability between the Up and Down procedure and the other procedures. I simulated Up and Down experiments with various step sizes to try to find a step size for which the m.s.e.'s of E_{DM} were similar to m.s.e.'s of the estimator from Procedure 4 with the asymptotic optimal c value. I finally settled on step sizes 0.4 and 0.3 as giving roughly comparable results for 48 and 96 observations respectively. I then in addition simulated experiments for multiples of these step sizes of 0.5, 1.5 and 2.0. Values of m.s.e. and bias of E_{DM} from these simulations are given in Tables 17 to 20 of Appendix 12. Using these lower step sizes in making comparisons does help to make the Up and Down procedure more comparable with the other procedures but as n increases the comparability across the range of multiples of step size begins to break down (e.g. in the 96 step experiments a larger step size than 0.60 appears to be needed for comparability with the stochastic approximation procedures with c equal to 2.0 times the asymptotic optimal value).

I have already given reasons (see point (4) of my conclusions for Procedures 1 to 4) for using a c value above the asymptotic optimal value. If such a c value is used the asymptotic variance of the estimator will be above its lower bound and some value of step size with the Up and Down procedure will be such that the m.l.e. has the same asymptotic variance. The stochastic approximation procedures have an advantage in that asymptotically unbiased estimators are easily obtained without making strong assumptions about the form of the response curve. However the procedures are more complicated to operate than the Up and Down procedure (a check has to to be made after each visit to a level to determine whether the step size should be changed), also, because observations are eventually concentrated close to one level, one cannot expect accurate internal estimates of slope even for very large n, and a further disadvantage is that there are potentially disasterous consequences if the c value is set too low. Of course estimates of μ from the Up and Down procedure can be seriously biased if the step size is low and the starting level is distant but such biases can be substantially reduced by forming the m.l.e. (see Section 3.3).

It is not at all clear whether one should use stochastic approximation or the Up and Down procedure; it is even difficult to determine under what conditions they should be compared. In Section 5.3 I discuss in more detail the relative merits of all procedures. I also consider variants of the stochastic approximation procedures for which estimators have full asymptotic efficiency and the possibility of using m.l.e.'s from Robbins-Monro experiments. In Section 5.2 I consider the problem of estimating levels other than the ED50 using stochastic approximation.

5.2 RESULTS OF SIMULATIONS OF EXPERIMENTS DESIGNED TO FIND STIMULUS LEVELS OTHER THAN THE ED50

In the simulations of Section 5.1 the value of p in the stochastic approximation procedures was set equal to 0.5 (see Section 1.4) and so it was the ED50 (alternatively termed the $L_{1/2}$ level) that was estimated. In the work of Davis (1965a), (1965b) and (1971) attention has been restricted to $p = 0.5$, but in Wetherill (1963) some small sample simulations using the Robbins-Monro procedure are made with $p \neq 0.5$. Conclusions in Section 10.2 of Wetherill (1966) concerning these simulations are not encouraging; his simulations indicate that

'away from the immediate neighbourhood of $L_{1/2}$, the process leads to small sample estimates which frequently have large biases, and in addition, the sample variances are greatly in excess of those predicted from asymptotic theory.'

The delayed Robbins-Monro procedure and the Kesten version of the Robbins-Monro procedure were both devised with improvement in small sample estimation of $L_{1/2}$ in mind. In this Section I consider whether these modifications have any merit for $p \neq 0.5$. For estimating a general L_p level the levels visited, $\{y_t\}$, are related by

$$y_{t+1} = y_t + c(p-p_t)/t, \qquad 5.2.1$$

where $p_t = 1$ or 0 according to whether the response is positive or negative. Providing the value of c is greater than $1/2g_p$ (where g_p is the slope of the response curve at L_p) then the estimator from the Robbins-Monro procedure (i.e. the level that would have been visited following one more step) is asymptotically normal with variance given by

$$c^2 p(1-p)/(2cg_p - 1)n, \qquad 5.2.2$$

where n is the number of observations. Suppose the response type changes at observation t_0. What I term the delayed procedure is where the levels visited are linked by the equations

$$y_{t+1} = y_t + c(p-p_t) \quad \text{for } t < t_0. \qquad 5.2.3$$

$$y_{t+1} = y_t + c(p-p_t)/(t+2-t_0) \quad \text{for } t \geqslant t_0. \qquad 5.2.4$$

The asymptotic properties of this procedure are the same as the original procedure. Similar arguments to those in Davis's work suggest that, for the procedure with Kesten's modification a value of c of $k(p^2+(1-p)^2)/g_p$ will give an estimator with the same asymptotic variance expression as that for the unmodified procedure with c equal to k/g_p (where $k > 0.5$). The $(p^2+(1-p)^2)$ term appears because it is the limit in probability of the ratio of the number of changes in response type to n (this follows using similar arguments to those in Appendix 8).

A modification of the Robbins-Monro procedure which I have considered is to operate the procedure on a transformed curve (I discuss operating the Up and Down rule on transformed curves in

Section 4.1). I thought that this could possibly be successful as the suggestion in Wetherill (1966) of operating the Up and Down rule on a transformed curve to obtain estimates of levels other than the $L_{1/2}$ is certainly successful for some sets of conditions (see results in Section 4.2). A simple example of such a procedure would be to visit levels, $\{y_t\}$, making at most two observations per visit according to the following rule

$$y_{t+1} = y_t - c(z_t - 0.5)/t, \qquad 5.2.5$$

$$\text{where } z_t = \begin{cases} 1.0 \text{ if at y a ++ response is recorded,} \\ 0.0 \text{ if at y a - or +- response are recorded.} \end{cases}$$

That is moves are made between levels following much the same rules as the UDTR rule designed to give estimates of $L_{1/\sqrt{2}}$(see description in Section 4.1) but the size of the steps decreases throughout the experiment. In Appendix 8 some asymptotic properties of the estimator from this procedure are derived. The level that would have been visited had one more observation been taken provides an estimate of the $L_{1/\sqrt{2}}$ level of the response curve. The value of c minimising the asymptotic variance expression for this estimator is $(0.5)^{1/2}/g_{1/\sqrt{2}}$ (i.e. the slope at $L_{1/\sqrt{2}}$ of the transformed response curve). Operating on the transformed response curve does not greatly reduce asymptotic efficiency. If c values chosen for this procedure and the Robbins-Monro procedure with $p = (0.5)^{1/2}$ are $k(0.5)^{1/2}/g_{1/\sqrt{2}}$ and $k/g_{1/\sqrt{2}}$ respectively $(k > 0.5)$ then the ratio of the asymptotic variance expressions is 0.9706 to four decimals (see Appendix 8). Of course one can use any of the modifications of the

Robbins-Monro procedure that I have discussed on the transformed curve. This procedure can be adapted to provide estimators of any L_p by replacing 0.5 by p^2 in Formula 5.2.5.

I simulated stochastic approximation procedures which provide estimates of $L_{1/\sqrt{2}}$ (i.e. the level estimated in the simulations of Section 4.2). I will use the terminology of the Section 5.1 for the different procedures considered. I considered Procedures 1 to 4 operating on the untransformed response curve and the same procedures on the transformed response curve. I will call Procedures 1 to 4 operating on the transformed curve Procedures 5 to 8 respectively. I simulated experiments for c values equal to 0.5(0.5)2.0 times the asymptotic optimal values ; 24 observations were made in each experiment. The response curve was logistic with β equal to $\pi/3.0^{1/2}$. Starting levels were chosen at -2.00(0.25)2.00 relative to the position of $L_{1/\sqrt{2}}$. Again 2000 simulations were made per set of conditions. Values of m.s.e.'s and biases of estimators are given in Tables 5.2.1 to 5.2.8. The m.s.e.'s of estimators from Procedures 1,4,5 and 8 are illustrated in Figs. 5.2.1 to 5.2.4; these are the procedures I consider to be of greatest interest. My reasons for considering Procedures 2,3,6 and 7 to be of less interest are as follows:

(1) Procedure 2 has poor performance in that for all except the smallest c value m.s.e.'s with this procedure are always in excess of corresponding values with Procedure 1. Even for the smallest c value the m.s.e. with Procedure 2 is only lower than that with Procedure 1 for starts at 1.25(0.25)2.00.

Table 5.2.1 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 24 step experiments using stochastic approximation procedures with c equal to 0.5 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	9.41	10.74	16.32	11.22	33.88	11.41	25.47	13.41
-1.75	8.23	10.80	14.08	10.43	21.83	11.16	20.53	12.44
-1.50	7.17	10.01	11.46	9.38	14.28	10.67	16.92	11.47
-1.25	6.77	9.22	9.15	8.57	10.38	10.49	13.29	10.22
-1.00	6.36	8.45	7.45	7.51	7.85	9.75	10.46	8.93
-0.75	6.43	8.26	5.77	6.54	6.40	8.58	7.77	7.24
-0.50	6.71	8.14	4.79	5.77	5.80	7.70	6.28	5.70
-0.25	7.26	8.37	4.45	5.17	5.71	7.19	5.99	4.85
0.00	8.19	9.03	4.71	5.50	6.17	7.33	5.71	4.46
0.25	8.99	10.01	5.65	5.72	6.46	7.44	6.36	5.21
0.50	10.65	11.96	7.58	6.58	8.00	9.15	7.60	7.36
0.75	12.00	13.17	9.88	7.47	9.85	11.17	8.87	10.42
1.00	14.22	14.75	13.16	8.93	13.43	14.34	10.15	14.71
1.25	18.57	16.80	17.07	10.31	18.54	17.04	11.85	19.26
1.50	25.89	18.87	21.11	11.61	26.57	19.51	13.66	24.51
1.75	39.23	21.27	25.39	13.04	40.02	22.40	14.69	30.47
2.00	58.17	22.43	31.71	14.15	60.50	24.56	16.34	37.35

Table 5.2.2 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 24 step experiments using stochastic approximation procedures with c equal to the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	6.50	9.73	6.21	7.23	5.88	8.25	9.95	7.65
-1.75	6.74	9.61	6.00	7.18	5.74	8.03	9.44	7.71
-1.50	7.18	10.10	5.95	7.01	5.90	8.19	9.00	7.23
-1.25	7.26	10.04	6.03	6.63	6.20	8.06	8.86	7.14
-1.00	8.00	10.97	6.04	6.85	6.42	7.94	8.47	6.81
-0.75	8.53	10.51	6.41	6.70	6.47	7.65	8.60	6.63
-0.50	8.80	10.41	6.63	6.90	6.82	7.68	8.97	6.33
-0.25	9.10	9.96	6.75	7.32	7.12	7.60	9.13	6.33
0.00	9.41	9.82	7.36	7.22	7.03	7.20	9.71	6.52
0.25	10.20	10.32	7.27	7.12	7.42	7.58	10.45	6.50
0.50	10.95	11.36	8.05	7.55	7.94	8.20	11.26	7.08
0.75	11.41	11.84	8.44	7.82	7.96	8.38	11.30	7.91
1.00	11.95	12.61	9.48	7.77	8.70	9.02	12.94	9.20
1.25	12.22	12.94	10.39	8.01	9.60	10.03	12.98	10.29
1.50	12.33	13.17	11.23	8.42	10.19	10.76	14.28	11.41
1.75	13.61	14.13	12.53	9.22	10.88	11.50	14.77	12.71
2.00	14.07	14.20	13.62	9.51	12.03	12.05	16.18	13.65

Table 5.2.3 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 24 step experiments using stochastic approximation procedures with c equal to 1.5 times the asymptotic optimal values ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	8.43	11.59	7.71	8.53	7.00	8.74	11.28	8.54
-1.75	8.66	11.56	7.77	8.57	7.50	8.38	11.63	8.03
-1.50	9.17	12.06	7.95	8.64	7.79	8.57	11.55	7.90
-1.25	9.21	11.50	8.31	8.48	7.96	8.66	11.70	7.95
-1.00	9.06	10.34	8.54	8.63	7.96	8.56	11.97	7.84
-0.75	9.31	10.14	8.84	8.86	8.15	8.58	12.52	8.06
-0.50	9.86	10.39	8.78	8.97	7.95	8.33	12.22	7.61
-0.25	9.92	10.29	8.91	9.17	8.22	8.24	12.54	7.91
0.00	9.88	10.09	9.18	8.85	8.16	8.20	13.69	7.93
0.25	10.44	10.59	9.13	8.83	8.06	8.18	13.45	7.79
0.50	11.83	11.95	9.34	9.14	8.38	8.51	14.38	8.24
0.75	11.81	12.07	9.56	9.12	8.34	8.40	14.89	8.15
1.00	12.28	12.53	9.97	9.46	8.56	8.62	15.33	8.63
1.25	12.07	12.68	10.34	9.20	8.76	8.88	16.81	9.65
1.50	12.04	12.59	10.79	9.63	8.64	9.01	16.73	9.96
1.75	12.18	13.26	11.60	9.80	9.09	9.44	18.43	10.26
2.00	11.76	12.22	12.02	9.83	9.29	9.56	18.10	11.00

Table 5.2.4 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 24 step experiments using stochastic approximation procedures with c equal to 2.0 times the asymptotic optimal values ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	9.75	11.69	10.14	10.42	8.99	9.60	14.52	9.34
-1.75	9.83	11.40	10.07	10.58	9.14	9.70	14.00	9.45
-1.50	10.13	11.48	10.82	10.45	9.35	9.84	14.25	9.37
-1.25	10.23	11.26	10.75	10.79	9.29	9.77	15.09	9.46
-1.00	10.46	11.24	11.14	10.98	9.59	9.73	15.86	9.08
-0.75	10.86	11.31	11.08	10.96	9.42	9.57	15.47	9.01
-0.50	10.69	10.92	11.06	10.85	9.37	9.39	15.86	9.24
-0.25	10.97	10.96	11.02	11.17	9.40	9.47	16.69	9.38
0.00	10.92	10.98	11.08	10.82	9.31	9.42	16.66	9.09
0.25	11.93	11.94	11.33	10.82	9.39	9.44	16.81	9.03
0.50	12.61	12.79	11.29	10.82	9.37	9.45	17.73	9.32
0.75	12.83	12.97	10.99	11.53	9.44	9.43	18.38	9.03
1.00	12.67	12.90	11.58	11.20	9.49	9.54	18.26	9.29
1.25	12.54	12.88	11.61	11.25	9.66	9.61	19.20	10.29
1.50	12.40	13.21	12.31	11.40	9.54	9.68	20.08	10.05
1.75	12.52	12.94	12.43	11.10	9.83	9.95	21.98	10.66
2.00	12.40	12.79	12.78	11.47	9.58	9.97	20.98	10.96

Table 5.2.5 100×bias of estimators of $L_{1/\sqrt{2}}$ in 24 step experiments using stochastic approximation procedures with c equal to 0.5 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	-22.07	-7.72	-32.65	-20.38	-56.78	-10.44	-43.20	-23.95
-1.75	-17.99	-6.90	-29.94	-19.64	-44.32	-9.72	-37.80	-23.02
-1.50	-13.94	-5.66	-26.02	-17.69	-33.88	-9.66	-33.13	-22.30
-1.25	-10.74	-4.80	-21.72	-15.22	-26.13	-9.80	-28.32	-20.47
-1.00	-7.26	-3.27	-17.21	-13.62	-19.18	-8.44	-23.53	-18.29
-0.75	-4.25	-1.22	-12.21	-10.62	-13.00	-6.79	-17.61	-14.51
-0.50	-1.37	1.05	-7.01	-7.92	-7.89	-4.66	-12.20	-9.92
-0.25	1.74	3.47	-1.98	-4.52	-2.95	-1.33	-7.78	-5.14
0.00	4.94	6.29	3.71	-1.06	1.12	1.65	-3.08	0.88
0.25	8.50	8.92	8.99	2.16	6.04	5.32	0.82	7.00
0.50	13.11	11.99	15.04	5.18	11.75	9.64	4.13	13.70
0.75	18.23	14.08	19.87	8.05	18.01	12.94	5.99	20.40
1.00	24.51	15.27	25.03	10.31	25.76	16.26	8.73	26.00
1.25	33.15	17.20	30.19	12.46	34.84	18.62	10.40	30.90
1.50	43.97	18.96	34.40	13.98	45.75	20.59	13.55	35.99
1.75	57.94	19.92	38.66	15.83	59.43	22.60	15.51	40.57
2.00	73.42	20.24	44.54	16.56	75.37	23.59	20.69	46.06

Table 5.2.6 100×bias of estimators of $L_{1/\sqrt{2}}$ in 24 step experiments using stochastic approximation procedures with c equal to the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	2.70	7.94	-4.49	-1.61	-9.29	1.56	-13.71	-4.89
-1.75	3.99	8.48	-3.06	-1.53	-6.14	1.22	-12.50	-4.64
-1.50	4.64	8.28	-1.83	-1.59	-4.31	1.54	-10.77	-4.30
-1.25	5.87	8.71	-0.48	-1.70	-2.36	1.15	-10.08	-4.29
-1.00	7.76	10.04	0.76	-0.53	-1.64	0.95	-7.07	-3.26
-0.75	8.48	9.93	2.54	0.09	0.16	1.79	-5.60	-2.06
-0.50	9.24	10.47	3.73	0.63	1.25	2.27	-4.50	-1.30
-0.25	9.46	10.52	4.73	1.15	1.94	2.54	-3.68	0.88
0.00	10.02	11.03	6.06	1.68	2.49	2.56	-2.25	1.91
0.25	11.03	12.28	7.95	2.49	3.66	3.72	-1.29	3.84
0.50	10.90	12.53	9.54	2.96	5.24	5.00	-0.60	6.16
0.75	11.16	12.43	11.10	3.54	6.04	5.61	0.83	8.20
1.00	11.39	12.74	12.97	4.24	7.74	6.61	0.34	9.72
1.25	11.63	12.26	13.93	4.49	9.38	7.33	1.68	11.60
1.50	13.14	12.31	15.69	5.03	11.49	7.36	0.61	13.22
1.75	15.85	13.02	17.04	5.46	14.31	8.35	1.57	14.73
2.00	18.72	13.61	18.71	5.89	18.24	8.01	0.95	15.41

Table 5.2.7 100×bias of estimators of $L_{1/\sqrt{2}}$ in 24 step experiments using stochastic approximation procedures with c equal to 1.5 times the asymptotic optimal values ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	7.17	10.26	4.08	1.97	1.08	3.09	-4.66	0.36
-1.75	8.00	10.52	4.21	2.35	1.58	3.36	-4.44	0.70
-1.50	8.73	10.72	5.36	2.89	1.80	3.06	-3.52	0.34
-1.25	8.52	9.79	5.57	2.67	1.90	3.39	-2.96	0.70
-1.00	9.36	9.98	6.68	3.00	2.68	3.56	-3.04	0.55
-0.75	10.46	10.84	6.64	3.36	3.56	4.11	-2.06	1.28
-0.50	10.91	11.20	7.11	3.48	3.16	3.53	-1.41	1.33
-0.25	11.79	11.99	6.47	3.05	3.02	3.12	-1.82	2.33
0.00	11.77	12.05	7.42	3.21	2.98	3.06	-1.55	3.02
0.25	12.00	12.67	8.08	3.46	3.60	3.60	-0.83	3.63
0.50	12.29	13.21	8.50	4.06	3.49	3.54	-0.60	4.23
0.75	11.12	11.87	9.12	3.04	4.15	4.11	-1.44	4.70
1.00	10.81	12.12	9.60	3.84	4.63	4.58	0.00	5.51
1.25	10.32	11.11	10.92	4.04	4.77	4.60	0.61	6.53
1.50	9.64	10.97	11.01	4.55	5.47	5.07	-0.61	7.01
1.75	9.72	10.85	12.14	4.56	5.69	5.08	-0.60	7.09
2.00	9.73	10.73	12.00	4.75	6.17	4.81	-0.34	7.84

Table 5.2.8 100 bias of estimators of $L_{1/\sqrt{2}}$ in 24 step experiments using stochastic approximation procedures with c equal to 2.0 times the asymptotic optimal values ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	7.40	8.58	7.25	3.73	2.63	3.80	-1.14	2.49
-1.75	7.49	8.27	7.16	3.89	3.02	3.61	-0.36	2.14
-1.50	8.36	8.91	8.14	3.88	3.78	4.12	-0.78	2.08
-1.25	8.69	9.04	8.03	3.70	2.95	3.63	-0.57	1.88
-1.00	9.70	9.95	7.59	4.09	3.62	3.76	-0.33	2.55
-0.75	10.63	10.78	7.82	3.98	3.57	3.82	-0.46	2.64
-0.50	11.06	11.17	7.52	3.71	3.83	3.89	-0.16	3.00
-0.25	11.45	11.39	8.22	3.85	3.30	3.42	0.19	3.10
0.00	11.69	11.83	8.07	3.73	3.45	3.60	-0.18	3.14
0.25	12.29	12.35	8.23	4.86	4.17	4.26	-0.68	3.14
0.50	12.71	12.84	8.36	4.53	3.48	3.52	0.25	3.61
0.75	11.32	11.57	8.67	4.51	3.74	3.77	0.07	3.72
1.00	10.16	10.44	9.07	4.15	3.69	3.60	-0.26	4.30
1.25	9.75	10.44	9.46	4.58	3.74	3.96	0.97	5.16
1.50	9.12	10.19	9.53	4.11	3.83	3.66	1.86	4.81
1.75	9.15	9.85	10.21	4.76	4.76	4.33	0.53	5.47
2.00	8.53	9.76	9.97	4.20	4.67	4.01	-0.31	5.43

Fig. 5.2.1 M.s.e.'s of estimators from Procedures 1,4,5 and 8 with c equal to 0.5 times the asymptotic optimal values.

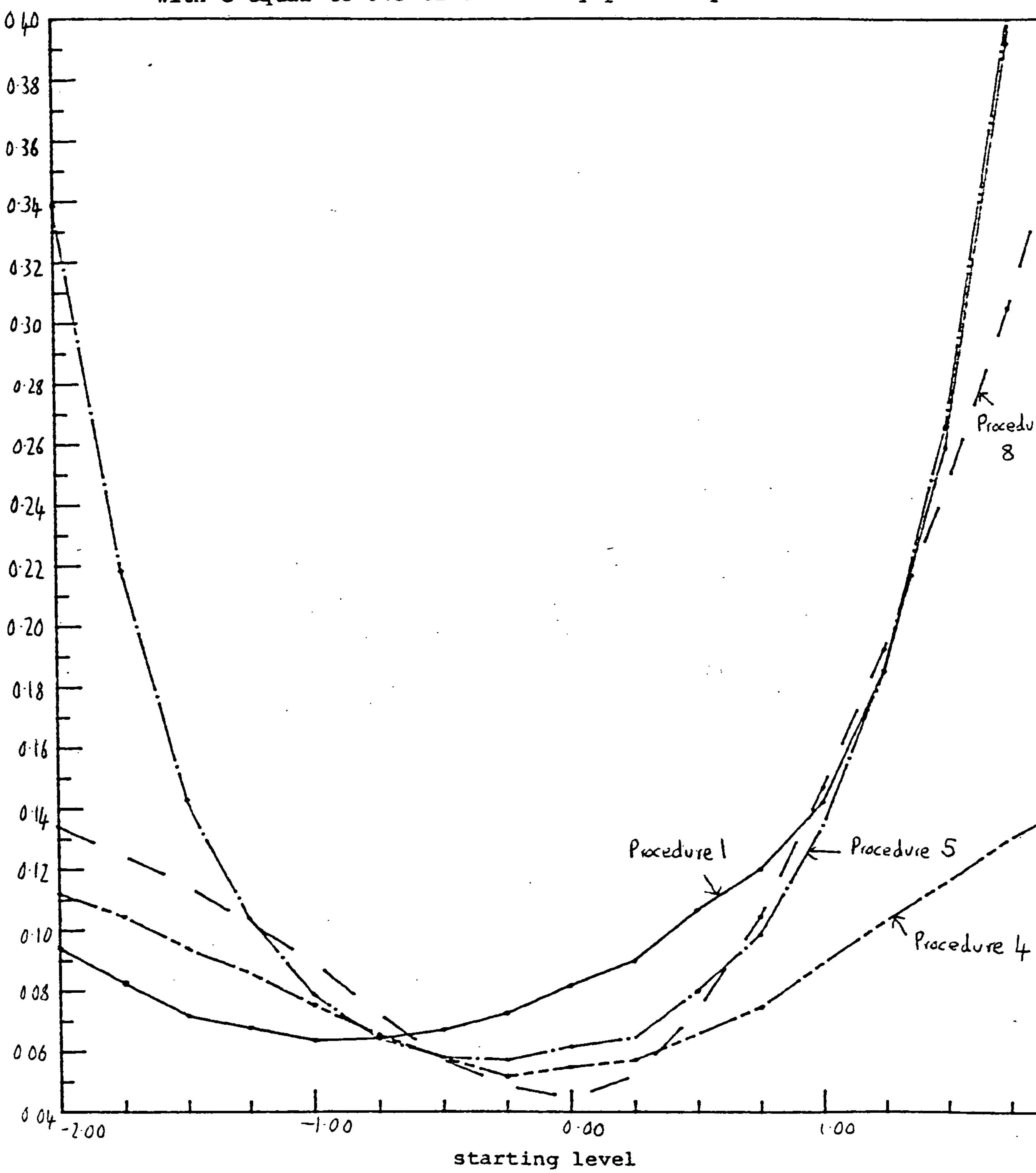


Fig. 5.2.2 As in Fig. 5.2.1 only with the optimal c values.

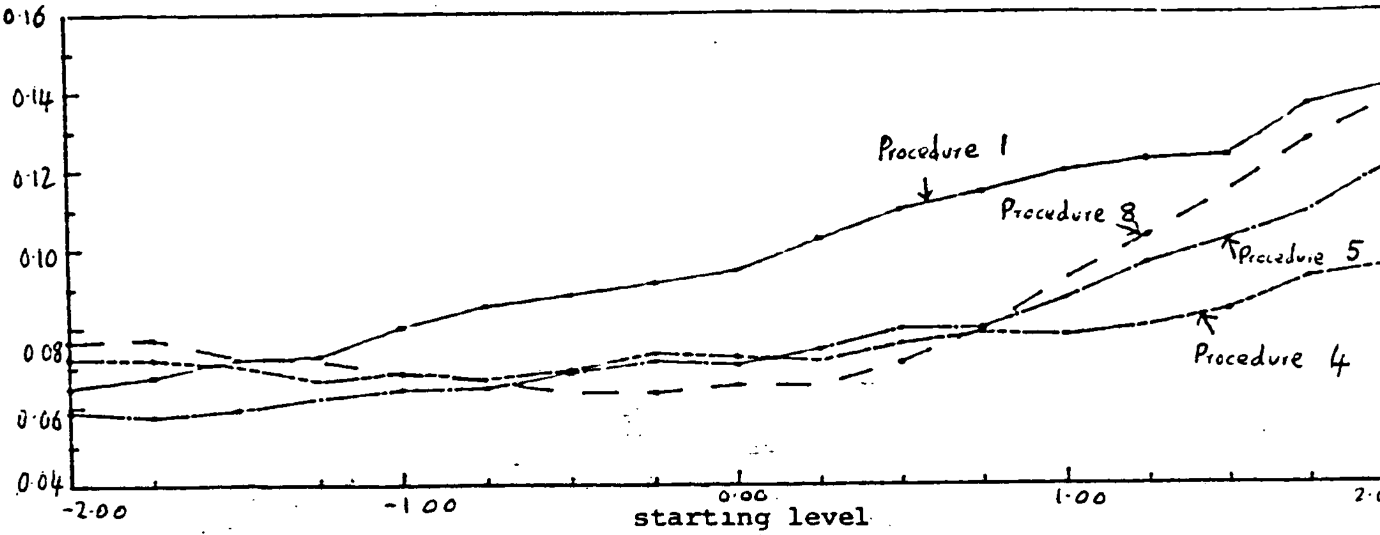


Fig. 5.2.3 As in Fig. 5.2.1 only with c equal to 1.5 times the optimal values.

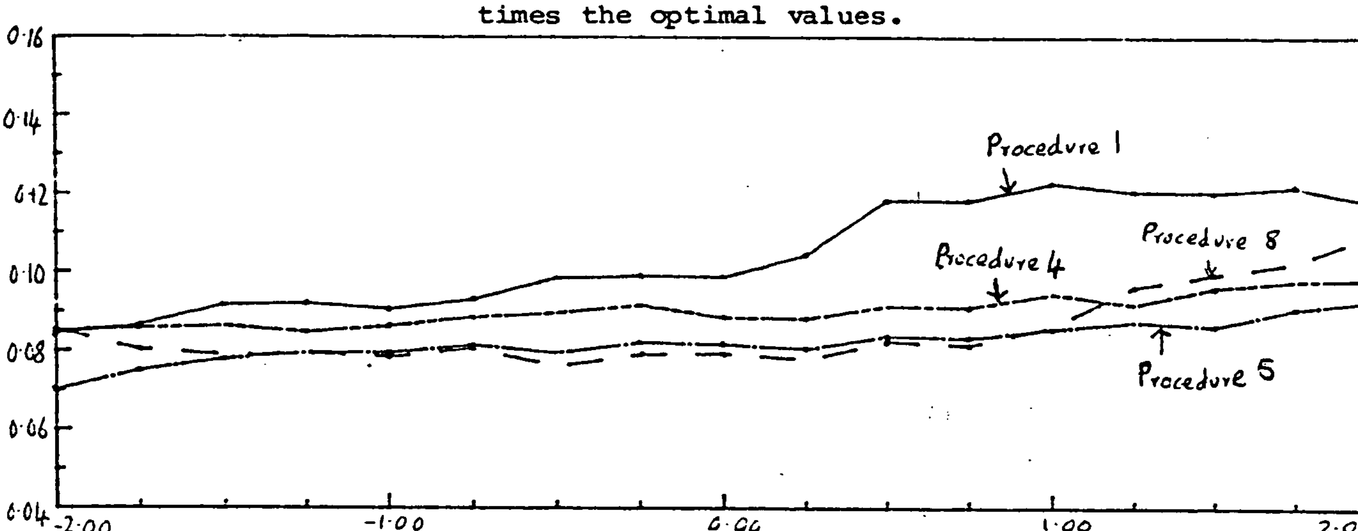
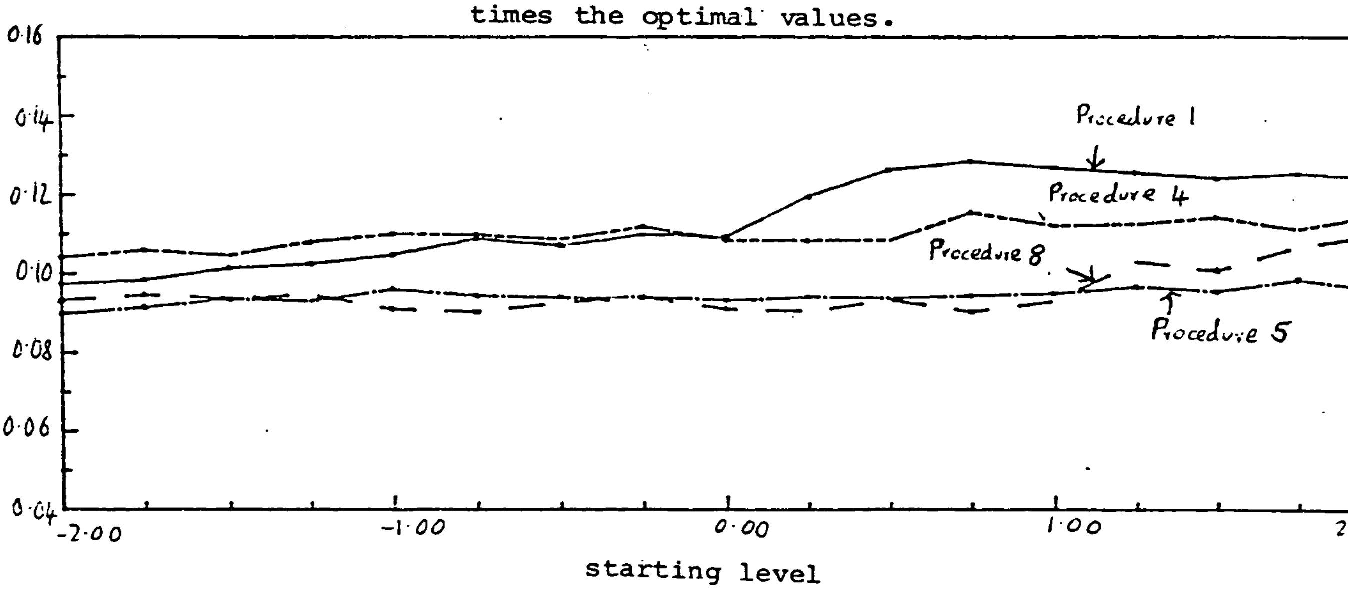


Fig. 5.2.4 As in Fig. 5.2.1 only with c equal to 2.0 times the optimal values.



(2) Procedure 3 deserves greater consideration than Procedure 2. The reason I do not recommend its use is that it has certain disadvantages when compared with Procedure 4. The biases of estimators from Procedure 4 are usually smaller than those of estimators from Procedure 3 (they are always smaller for the two larger c values). For the smallest c values the m.s.e.'s with Procedure 3 are smaller than corresponding m.s.e's with Procedure 4 for starts at -1.00(0.25)0.25. However m.s.e.'s with Procedure 3 become much larger than those with Procedure 4 for high starts. For the other c values m.s.e.'s with Procedure 3 are smaller than those with Procedure 4 for low starts but larger for high starts. The m.s.e.'s with Procedure 4 depend less upon the start than m.s.e.'s with Procedure 3 and have a lower average value over the range of starts.

(3) Values of m.s.e.'s with Procedure 6 are usually higher than those using Procedure 5. This is true for all starts for multiples of the asymptotic optimal c value of 1.0 and 1.5. For the largest c value the m.s.e.'s are smaller with Procedure 6 only for starts at 1.25 and 1.75 (where m.s.e.'s with the two procedures are close). For the smallest c value the performance of Procedure 6 relative to Procedure 5 is better, but m.s.e.'s are still less with Procedure 5 for starts at -1.25(0.25)1.00.

(4) Procedure 7 has very poor performance in that m.s.e.'s with this procedure are often substantially above values with Procedure 5 and always above values with Procedure 4.

A direct comparison between Procedures 1 and 5 (i.e. between the Robbins-Monro procedure for $L_{1/2}$ operating on the untransformed and the analogous procedure operating on the transformed curve) indicates that Procedure 5 has some advantages. The m.s.e.'s using Procedure 5 are less than those using Procedure 1 for:

(1) Starts -0.75(0.25)1.25, when c values equal 0.5 times the asymptotic optimal values.

(2) All starts considered, when c values equal 1.0, 1.5 and 2.0 times the asymptotic optimal values.

The biases using Procedure 5 are less than those using Procedure 1 for:

(1) Starts 0.00(0.25)0.75, when c values equal 0.5 times the asymptotic optimal values.

(2) Starts -1.50(0.25)2.00, when c values equal the asymptotic optimal values.

(3) All starts, when c values equal 1.5 and 2.0 times the asymptotic optimal values.

The m.s.e.'s with Procedure 5 are only much greater than those with Procedure 1, when starts are low and c values are equal to 0.5

times the asymptotic optimal values. For multiples of asymptotic optimal c values of 1.0, 1.5 and 2.0, asymptotic theory predicts variances for the procedures on the untransformed curve of 0.0612, 0.0688 and 0.0815 respectively. For the procedures on the transformed curve corresponding values are 0.0630, 0.0709 and 0.0840. For Procedure 1 the m.s.e.'s are always above these predicted variances. For Procedure 5 the m.s.e.'s are sometimes below these predicted variances (i.e. for low starts and multiples of the asymptotic optimal c values of 1.0 and 1.5) and certainly the departure from asymptotic theory is not so great as for Procedure 1.

One interesting point to note is that, for multiples of asymptotic optimal c values of 1.5 and 2.0, the biases with all procedures, except Procedure 7, are always positive. This accords with results in Wetherill (1963) for Procedure 1. Wetherill argues that such biases are, to a large extent, due to experiments where an initial negative response is recorded; even starting close to L_ρ many steps must be taken before one is again close to L_ρ.

Procedure 4 also has some definite advantages over Procedure 1. For multiples of asymptotic optimal c values of 1.0, 1.5 and 2.0, the biases of estimators with Procedure 4 are always less than with Procedure 1. For low starts the m.s.e.'s are slightly greater with Procedure 4 than with Procedure 1 but the m.s.e.'s do not become so large for high starting levels. For the smallest c values and the lowest start, the m.s.e. with Procedure 4 is some way above that with Procedure 1 but for the high starts the m.s.e.'s with Procedure 4 are much smaller.

How well Procedure 5 compares with Procedure 4 depends very much on the multiples of asymptotic optimal c values considered. For multiples of 1.5 and 2.0 the m.s.e.'s are always smaller with Procedure 5 than with Procedure 4. The biases with Procedure 5 are smaller than those for Procedure 4 for low starts but greater for high starts. The average value of the biases over the range of starts is roughly the same for both procedures. For c values equal to the asymptotic optimal values, if m.s.e. is used as a criterion, there is not much to choose between Procedures 4 and 5. Biases are greater with Procedure 5 than with Procedure 4. Procedure 5 compares very unfavourably with Procedure 4 for the lowest c values considered. Values of m.s.e. and bias are often much larger with Procedure 5 than with Procedure 4 (values are similar for starts close to $L_{\frac{1}{2}}$). If one uses c values that one guesses to be above the asymptotic optimal values, then Procedure 5 is preferable. If smaller c values are used Procedure 4 appears best.

For multiples of the asymptotic optimal c values of 1.0,1.5 and 2.0 the m.s.e.'s with Procedure 8 are slightly less than with Procedure 5 for most starts. For distant starts the m.s.e.'s with Procedure 8 are higher than for Procedure 5. Procedure 8 is more complicated to operate than Procedure 5 and never, using m.s.e. as a criterion, has a great advantage over Procedure 5. For these c values I would prefer to use Procedure 5 rather than Procedure 8. For the lowest c value the m.s.e.'s using Procedures 5 and 8 are again close for most starts but now for distant starts it is the m.s.e.'s with Procedure 5 which are much higher.

As could be anticipated from asymptotic theory, the behaviour of all procedures is relatively poor for the smallest c values considered. Usually the slope of the response curve will not be known precisely. It seems sensible to choose c values which one guesses to be above asymptotic optimal values (this is a precaution against choosing c values that are too small). If such a c value is chosen, one would expect estimators from using Procedures 5 and 8 to have the lowest m.s.e.'s.

I repeated simulations of Procedures 1 to 8 under the same conditions but with 48 and 96 observations per experiment. Again 2000 simulations were for each set of conditions. Values of m.s.e.'s and biases of the estimators are given in Tables 1 to 16 of Appendix 13. Many of the remarks I made for 24 observation experiments apply equally well for 48 and 96 observations. For the larger numbers of observations, the distinctions between procedures is less marked than for 24 observations. There is still little to be lost if attention is restricted to Procedures 1,4,5 and 8. The other procedures have either, similar properties to, or compare unfavourably with, at least one of these procedures. There are several points I wish to stress:

(1) For the smallest c values, using m.s.e. as a criterion, Procedure 4 has good behaviour in that m.s.e.'s over for all starts

are relatively small.

(2) For c values equal to the asymptotic optimal values the m.s.e.'s using Procedures 4,5 and 8 are fairly close but biases are usually less with Procedure 4.

(3) For c values equal to 1.5 and 2.0 times the asymptotic optimal values:

(a) The m.s.e.'s using Procedures 5 and 8 for all starts are relatively small. There is no strong reason for using Procedure 8 rather than the less complicated Procedure 5.

(b) In the 48 observation experiments, the m.s.e.'s with Procedure 4 are only less than those with Procedure 1 for starts at 1.00(0.25)2.00. In the 96 observation experiments, these m.s.e.'s are never less. The advantages Procedure 4 had over Procedure 1 for 24 observation experiments no longer exist.

(c) In the 96 observation experiments m.s.e.'s using Procedure 1 are close to those for Procedures 5 and 8, and are often slightly lower (this is what one would expect from asymptotic theory). However biases with Procedure 1 are higher than those with Procedures 5 and 8.

Providing one chooses a c value a little way above the asymptotic optimal value it appears that Procedure 5 (i.e. the Robbins-Monro procedure operating on the transformed curve) compares well with the other procedures. However if a large number of observations is made (for example 96) then m.s.e.'s with Procedure 1 are close to m.s.e.'s with Procedure 5. From asymptotic theory it follows that Procedure 1 must eventually be more efficient than Procedure 5. For small c values Procedure 4 appears to have some advantages. However one would try in any case to avoid using small c values.

I remarked earlier in this section that Procedure 5 could be adapted to provide estimates of any L_{p^2} (i.e. by replacing 0.5 by p^2 in Formula 5.2.5). I decided to simulate some more experiments where the $L_{0.9}$ level is to be estimated. One cannot expect estimates with much accuracy for this extreme level without making large numbers of observations. I simulated 96 observation experiments, using Procedures 1,4,5 and 8 (c values used in Procedures 4 and 8 were $((0.9)^2+(0.1)^2)$ times corresponding values used in Procedures 1 and 5). Starts were at -2.00(0.25)2.00 relative to $L_{0.9}$. In all other respects conditions for simulations were as before. Values of m.s.e. and bias of estimators from these simulations are given in Tables 17 to 20 of Appendix 13. For multiples of asymptotic optimal c values of 1.0,1.5 and 2.0, asymptotic theory predicts variances of estimators using Procedures 1 and 4 of 0.0352,0.0396 and 0.0469 respectively. For Procedures 5 and 8 analogous values are 0.0353,0.0397 and 0.0470. The m.s.e.'s using all the procedures are well above these values. For all except the smallest c values, biases are positive for all procedures. The biases with Procedure 1 are then much higher for low starts than for high, this seems a somewhat curious result. It is not so surprising when one considers that an initial negative response (which Wetherill argues accounts for such bias) is extremely unlikely for the high starts but is fairly likely for the low starts.

The biases and m.s.e.'s can be enormous with Procedure 1 and it is clear that Procedure 5 is in these circumstances much to be preferred (the m.s.e.'s and biases are usually much smaller with Procedure 5 than with Procedure 1). The only conditions for which the m.s.e. is higher with Procedure 5 than with Procedure 1 is for the lowest c value and starts at 1.75 and 2.00. Procedure 4 is also to be preferred to Procedure 1 (the m.s.e.'s and biases with Procedure 4 are always smaller than those for Procedure 4). However it is Procedure 8 which has the best behaviour among these procedures. The m.s.e.'s with this procedure are less than those for the other procedures, with exceptions for the lowest start and the smallest and largest c values (then the m.s.e. with procedure 5 is lower). For all except the lowest c values the bias is always smallest with Procedure 8.

In the simulations to find $L_{1/\sqrt{2}}$ there was not a great deal to choose between Procedures 5 and 8. For the simulations to find $L_{0.9}$ Procedure 8 has better behaviour. The number of sets of condition that one can consider in any simulation study will always be limited. The results of this section indicate that it is a good idea to operate the stochastic approximation procedures on transformed curves (even though asymptotically there is a small drop in efficiency in using such procedures). What I have called Procedure 8 (which is my modification of Kesten's procedure operating on the transformed curve) has worked relatively well under all the conditions simulated.

5.3 ALTERNATIVE STOCHASTIC APPROXIMATION PROCEDURES

In Sections 5.1 and 5.2 I simulated experiments using variants of the Robbins-Monro procedure. The estimator used was always the level that would have been visited had one more observation been taken. Asymptotic properties of such estimators are described in Section 1.4 (one can establish these properties under the condition that the response curve is monotonic with derivative greater than zero at the level to be estimated). Suppose one wants an estimate of the L_ρ level and that the slope of the response curve at L_ρ is g_ρ. At the start of an experiment one must choose a value for a positive constant c. There is an optimal value of c for which the estimators from these procedures are asymptotically normal with mean L_ρ and variance $p(1-p)/(g_\rho^2 n)$ (where n is the number of observations). For other values of c either the asymptotic variance expression is higher or the estimator is not asymptotically normal (this is for c less than or equal to half the optimal value; the estimator then has mean squared error tending to zero at a rate slower than $O(1/n)$). The optimal c value depends upon the generally unknown value of g_ρ (for the unmodified and delayed Robbins-Monro procedure the optimal c value equals $1/g_\rho$).

It would obviously be preferable to obtain estimates from stochastic approximation procedures which are less dependent on the value of g_ρ. One could try to use different estimators with the same procedures; for example if one assumes a parametric form for the response curve the maximum likelihood estimates could be calculated. Alternatively one could try to devise new stochastic approximation procedures. Venter (1967) discusses a procedure in which observations are made in pairs, $y_r \pm c_r$, where y_r is an estimate of $L_{1/2}$ after 2r observations. In his procedure all the c_r are positive and $c_r r^{\gamma}$ tends to c as r increases for some $c > 0.0$ and $\gamma \in (0.0, 0.5)$. The y_r are determined by the recursive relation

$$y_{r+1} = y_r - d_r A_r^{-1} Z_r, \qquad 5.3.1$$

where d_r is a sequence of positive numbers satisfying $d_r = 1/r\ (1+O(1/r^{1/2}))$. The value of Z_r equals 0.5 if the responses at $y_r \pm c_r$ are positive, -0.5 if they are negative and 0.0 if they are of opposite sign. A_r is an estimator of $g_{1/2}$ which is determined as follows. Let

$$B_r = \sum_{k=1}^{r} W_k / 2c_k r, \qquad 5.3.2$$

where W_k equals 1.0 if the responses at $y_k + c_k$ and $y_k - c_k$ are respectively positive and negative, 0.0 if they are of the same sign and -1.0 if they are respectively negative and positive. The expectation of W_k is $G(y_k + c_k) - G(y_k - c_k)$, where G is the response curve. All the $W_k/2c_k$ terms provide crude estimates of slope; the bias is small if the levels are close to $L_{1/2}$. In this procedure one requires positive lower and upper bounds on $g_{1/2}$, say k_1 and k_2. A_r is defined to equal B_r truncated by k_1 and k_2; that is $A_r = B_r$ if $B_r \in (k_1, k_2)$, $A_r = k_1$ if $B_r \leqslant k_1$ and $A_r = k_2$ if $B_r \geqslant k_2$. Venter suggested that γ be set equal to 0.25 and that a moderate value of c be used. Providing the second derivative of the response curve at $L_{1/2}$ is 0.0 (as it is for logistic and normal response curves), then $(y_r - L_{1/2})n^{1/2}$ is asymptotically normal with mean 0.0 and variance $0.25/g_{1/2}^2$ $(n = 2r)$. The estimator y_r has asymptotic variance equal to that using the Robbins-Monro procedure with the optimal c value. Anbar (1977) suggests another procedure with similar properties. In Anbar's procedure observations are made one at a time. The levels visited are determined by the same recursive relations as for the Robbins-Monro procedure but the value of c is altered throughout the experiment. For the first two changes in level c values are chosen arbitrarily. Suppose z_t equals 0.5 or -0.5 according to whether the response at y_t is positive or negative. The expectation of z_t is $G(y_t)-0.5$; this is approximately equal to $g_{1/2}(y_t - 0.5)$ for y_t close to $L_{1/2}$. The c value used after n observations $(n > 2)$ is $\tilde{A}_{n-1}^{-1}$, where $\tilde{A}_n$ is an estimate of $g_{1/2}$ determined as follows. Let

$$\tilde{B}_n = \sum_{t=1}^{n}(y_t - \bar{y})z_t \Big/ \sum_{t=1}^{n}(y_t - \bar{y})^2, \qquad 5.3.3$$

where $\bar{y} = \sum_{t=1}^{n} y_t/n$. $\tilde{A}_n$ equals $\tilde{B}_n$ truncated at k_1 and k_2. Anbar (1978) shows that $(y_t - L_{1/2})n^{1/2}$ tends in distribution to a normal with mean 0.0 and variance $0.25/g_{1/2}^2$. So with both Venter's and Anbar's procedures estimators can be obtained which are asymptotically normal and have asymptotic variances equal to the lowest possible asymptotic variance using the Robbins-Monro procedure.

Wetherill (1963) tried to obtain maximum likelihood estimates from simulated experiments where the Robbins-Monro procedure had been used. His principal object was to try to obtain estimates of β (his simulations were also made assuming a logistic response curve). He remarks that 'many iterations are required and the project has been dropped'. I decided to perform similar calculations for simulated experiments (described in Section 5.1), where the unmodified Robbins-Monro procedure had been used for 24 steps. I used the Robbins-Monro estimator as a starting value for μ in Newton-Raphson iterations; I took the actual value of β as a starting value for β. The function to be maximised was the log likelihood; I worked in terms of parameters a and β, where $a = -\mu/\beta$. I stopped iterations when the change in estimates of μ between steps was less than 0.5×10^{-4}. I discarded experiments if the determinant of the matrix to be inverted at each step in iteration became less than 10^{-8}. For the lowest c value, equal to 0.5 times the optimal value, a large proportion of experiments are discarded. For the most distant starting level the number of discards is 1343 out of 2000; this is clearly unacceptable. However the number of such discards only starts to rise rapidly for starts beyond 2.25 (at this level 3 out of 2000 experiments are discarded). For starting levels below 1.75 no discards were made at all (so it appears that then the probability of discarding is very small). For the other c values only at most 10 discards out of 2000 were made for any set of conditions. For most sets of condition it was possible to satisfy my convergence criterion in a large proportion of the experiments. However I encountered further difficulties, some experiments were giving estimates of μ that grossly inflated the m.s.e. of the m.l.e. of μ. These poor estimates of μ came from experiments where the final estimate of β

is negative or very small. I decided to discard experiments for which estimates of β were less than half the actual value. In the remaining experiments the iterations converged quite rapidly; usually on average between 4 and 5 iterative steps were taken. I performed similar calculations to obtain m.l.e.'s of μ conditional on the true value of β. It is easy to deduce that these conditional estimates have the same asymptotic properties as the Robbins-Monro estimates for optimal c (though if one knew β one would know the value of $g_{1/2}$ and so be able to use the optimal c). I discarded experiments if in the iterations, the second derivative of the log likelihood with respect to the parameter a became less than 10^{-8} in magnitude (this happened if and only if all responses had been positive). In Table 5.3.1 I give the total numbers of discarded experiments when iterations are made to the m.l.e. of μ, with and without conditioning on β. For the smallest c value so many discards are made for distant starts that calculated m.s.e.'s of the m.l.e. of μ are of little value. In comparing m.s.e.'s of m.l.e.'s of μ with those of the Robbins-Monro estimators one must remember that the Robbins-Monro estimator can be obtained in all experiments. Figs. 5.3.1 to 5.3.4 illustrate m.s.e.'s of the Robbins-Monro estimator, the m.l.e. of μ and the m.l.e. of μ conditional on β. In these simulations maximum likelihood estimation has not in general proved a useful alternative to using the Robbins-Monro estimator. One could argue for the largest c value considered that, as there are relatively few discards and the m.s.e. of the m.l.e. of μ is always less than that of the Robbins-Monro estimator, the m.l.e. of μ is then preferable but any advantage is never great. I calculated biases and m.s.e.'s of

Table 5.3.1 Number of discards out of 2000 simulations, in maximum likelihood estimation from 24 observation Robbins-Monro experiments, where the response curve is logistic.

Start	Multiple of optimal c: 0.5	1.0	1.5	2.0	0.5*
0.00	10	36	44	30	0
0.25	28	35	40	28	0
0.50	51	37	48	29	0
0.75	85	50	49	36	0
1.00	118	59	61	37	0
1.25	143	65	47	42	0
1.50	181	68	45	44	0
1.75	220	80	48	48	0
2.00	254	91	52	42	0
2.25	297	105	50	38	0
2.50	351	104	65	33	1
2.75	443	113	71	31	13
3.00	631	128	59	30	75
3.25	867	136	56	27	242
3.50	1134	158	62	38	510
3.75	1380	170	78	51	1142
4.00	1563	211	83	50	1385

* This column is for estimation when one knows β. With the optimal c only one experiment is discarded (this is for start at 4.00). For multiples of 1.5 and 2.0 none of the experiments are discarded.

Fig. 5.3.1 M.s.e.'s of the Robbins-Monro estimator, the m.l.e. estimator of μ and the m.l.e. of μ conditional on β, in 24 step experiments with c equal to 0.5 times the optimal value.

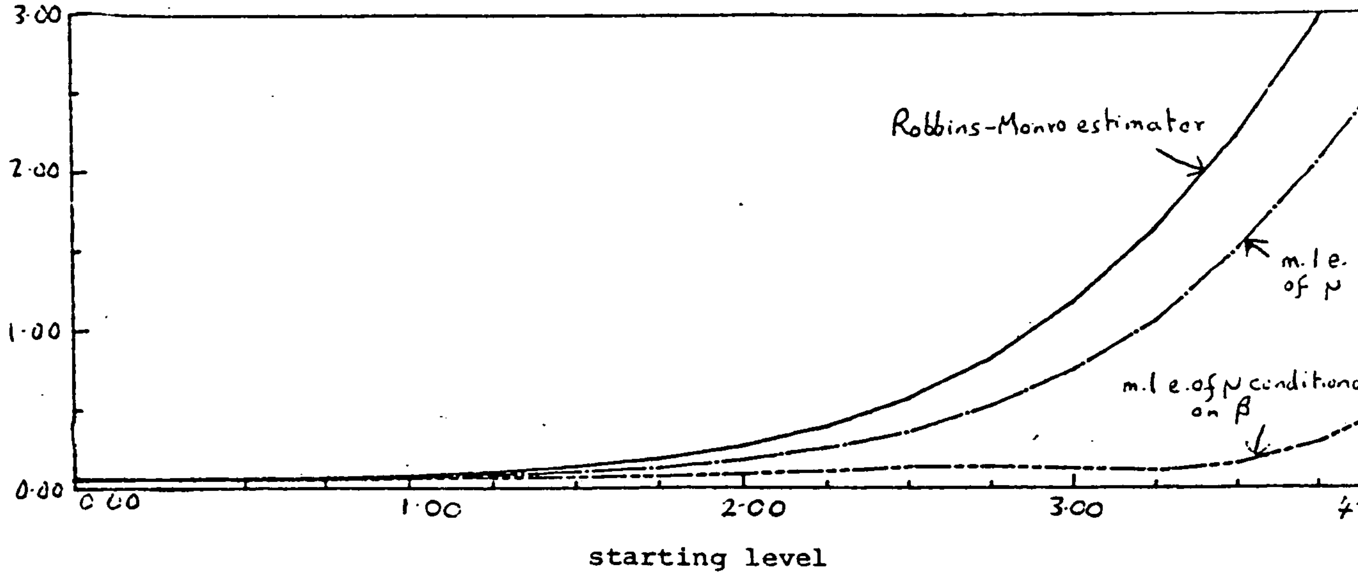


Fig. 5.3.2 As in Fig 5.3.1 only with the optimal c value.

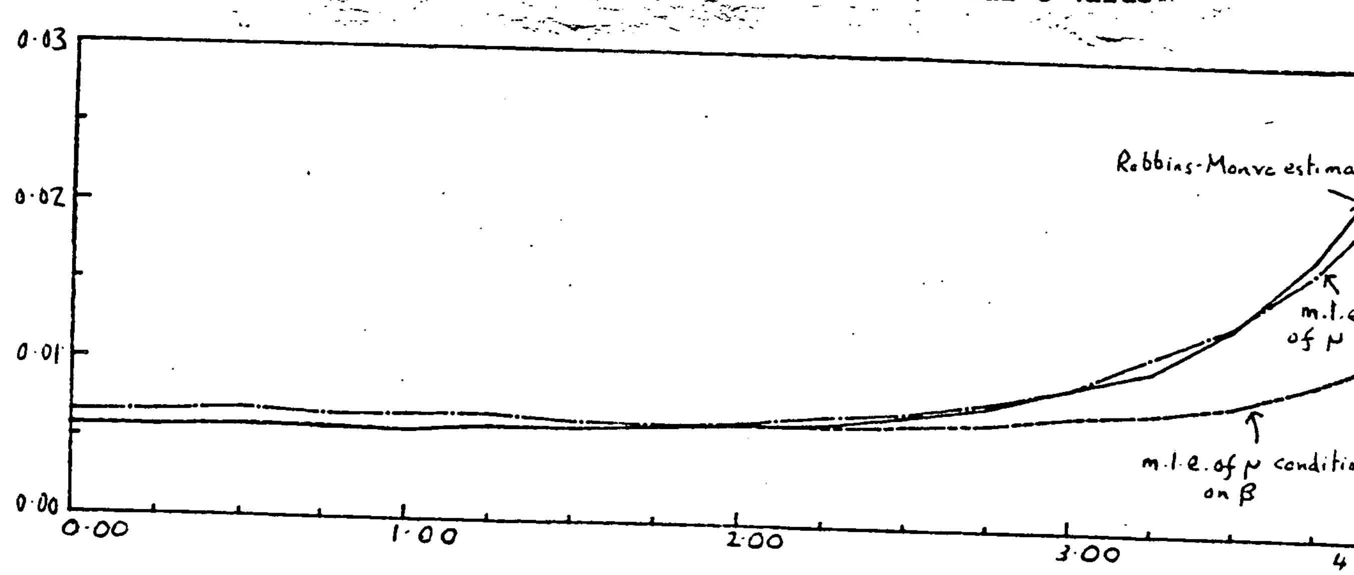
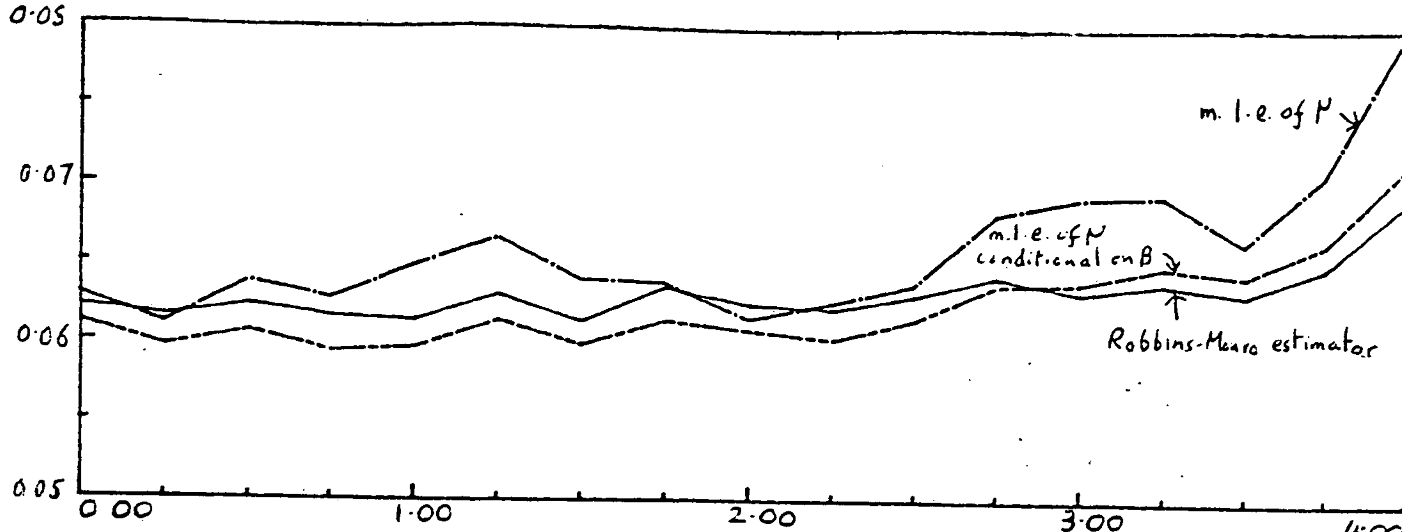
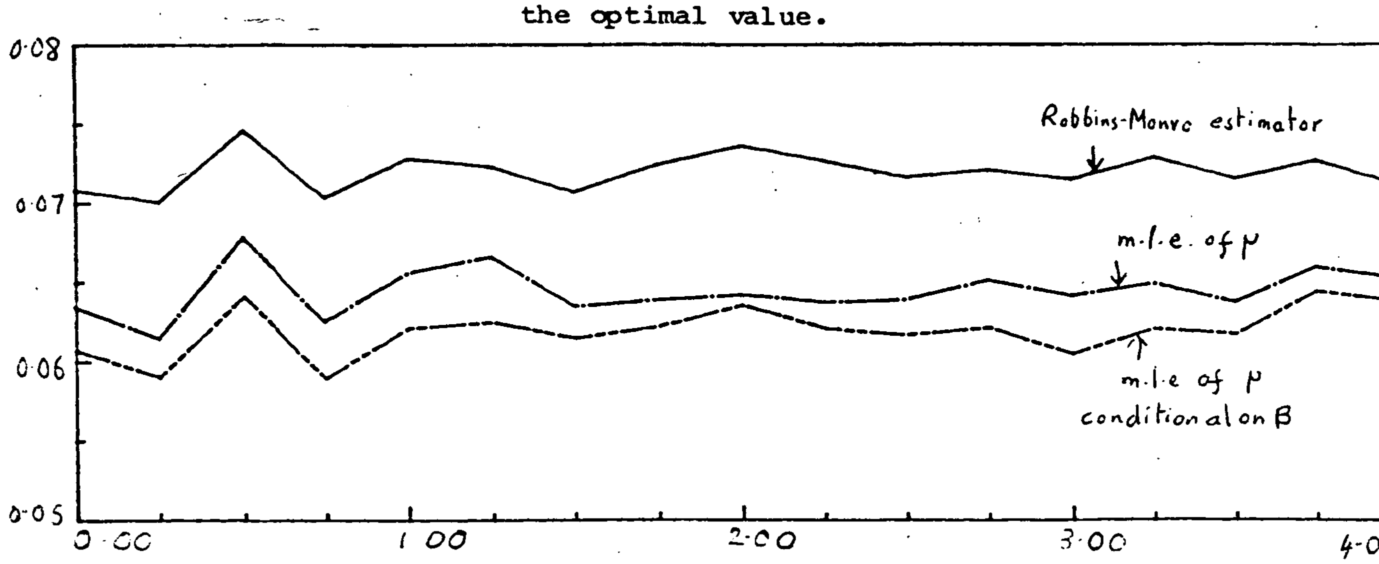


Fig. 5.3.3 As in Fig 5.3.1 only with c equal to 1.5 times the optimal value.



starting level

Fig. 5.3.4 As in Fig 5.3.1 only with c equal to 2.0 times the optimal value.



starting level

$1/\beta$; these estimates were negatively biased often having expectation around half the true value. It appears that no useful information about β or $1/\beta$ can be obtained by maximum likelihood estimation. I have not encountered the same problems with convergence as Wetherill (1963) but on the basis of this small study I cannot recommend use of maximum likelihood estimation with the Robbins-Monro procedure.

Anbar (1977) simulated experiments using both his and Venter's procedures. He used a normal response curve, with $\sigma = 1.0$. His experiments consisted of 12 or 24 observations. In Venter's procedure he set γ equal to 0.25 (as suggested by Venter) and c equal to 0.5,1.3,1.7,2.1 or 2.9. He truncated B_r and $\widetilde{B}_n$ at 0.5 and 1.5 times the actual value of $g_{1/2}$. Both Venter and Anbar suggest a possible modification to the expressions for B_r and $\widetilde{B}_n$ (see Formulae 5.3.2 and 5.3.3), they say that one could ignore the first m-1 terms in the summations (where m may depend on n) in order to avoid large deviations due to results from the first few observations. In Anbar's simulations he considered m equal to 1,2 or 3. For $r < m$, A_r was set equal to $g_{1/2}$; for $n \leqslant m+1$, $\widetilde{A}_{n-1}$ was also set equal to $g_{1/2}$. Even with such small numbers of observations his results indicated that his procedure works well; the m.s.e.'s of estimates were usually close to asymptotic predicted variances. Venter's procedure appears to work well for starts close to $L_{1/2}$ but m.s.e.'s rise very rapidly as starting levels were made more distant (this is particularly true for the larger c values). I decided to simulate experiments using these procedures under the conditions considered in Section 5.1. For each set of conditions I simulated 2000 experiments; each experiment consisted of 24 observations. I set the constant c in Venter's procedure equal to 0.5(0.5)2.0. As in Anbar's simulations B_r and $\tilde{B}_n$ were truncated at 0.5 and 1.5 times the actual value of $g_{1/2}$ and m was set equal to 1,2 or 3. Fig. 5.3.5 illustrates m.s.e.'s of estimates using Anbar's and Venter's procedures (m = 1). The asymptotic predicted variances of these estimators is 0.0507. This graph is not unlike Fig. 2 of Anbar (1977) (in this graph he plots m.s.e.'s for m = 1, n = 24). The results of all the simulations were much as would be expected from Anbar's results (I also simulated experiments with exactly the same conditions as in Anbar's paper and found my results were very similar to his). It appeared that Anbar's procedure had much to commend it. In this procedure the values of $\tilde{A}_n$ provide estimates of $g_{1/2}$. I calculated the mean and mean square error of the final estimates of $g_{1/2}$ derived from $\tilde{A}_n$. I found that the estimates of $g_{1/2}$ were often substantially biased, positively for starts close to $L_{1/2}$ and negatively for distant starts. Often for starts close to $L_{1/2}$ the expectations were close to the upper truncation level and for distant start close to the lower truncation level. For example, when m = 1 in the simulations with logistic response, expectations were 0.605 and 0.228 for starts 0.00 and 4.00; the truncation levels are 0.680 and 0.227. I found that when I repeated simulations with truncation levels at $0.25 \times g_{1/2}$ and $3.0 \times g_{1/2}$ the results were often quite different. Fig. 5.3.6 illustrates values of m.s.e.'s of estimates analogous to those in Fig. 5.3.5, when these broader truncation levels are used.

Anbar's and Venter's procedures have asymptotic properties

Fig. 5.3.5 M.s.e.'s of estimators using Venter's procedure (with γ =0.25 and c=0.5,1.0,1.5 and 2.0) and Anbar's procedure, in 24 step experiments with truncation at 0.5 and 1.5 times $g_{1/2}$.

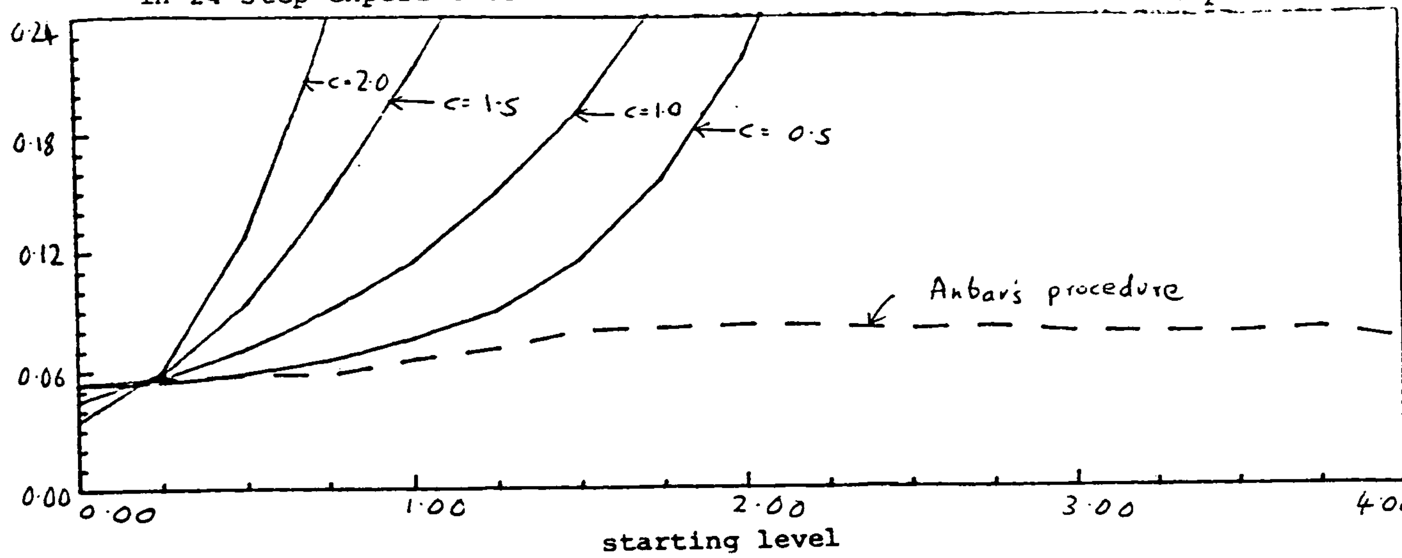
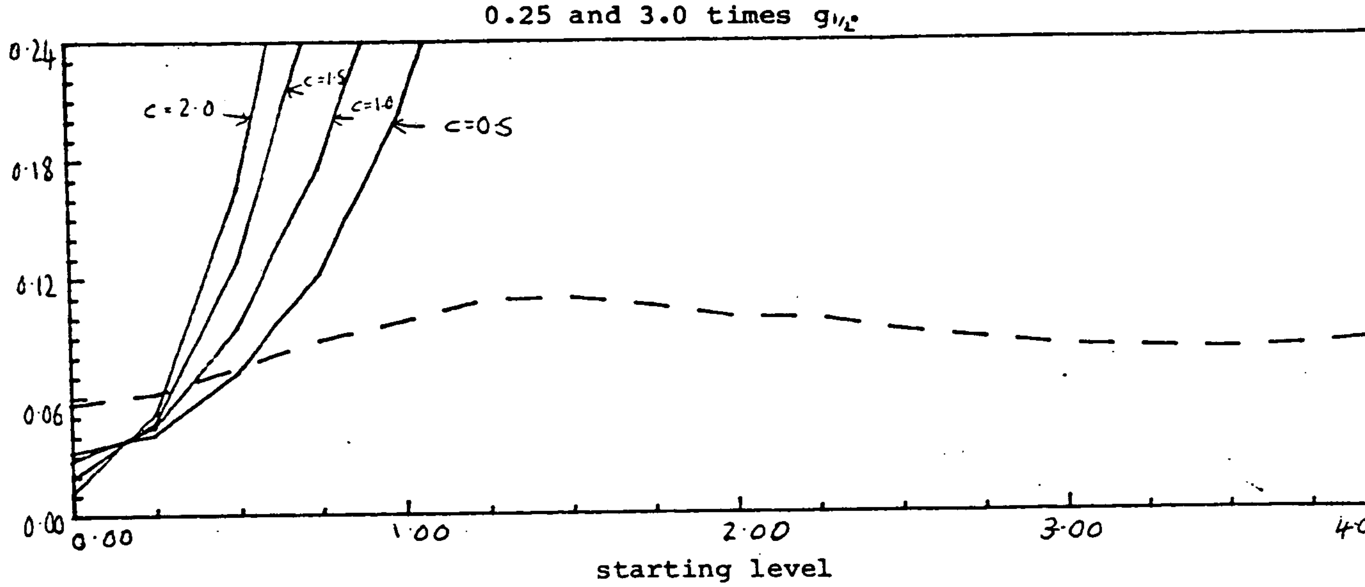


Fig. 5.3.6 As in Fig. 5.3.5 only with truncation at 0.25 and 3.0 times $g_{1/2}$.



Note: For m=2 and 3 the m.s.e.'s also depend very much on the truncation levels used. The relative performance of Anbar's procedure with the wider truncation levels to the original procedure is worse for m=2 and 3 than for m=1.

independent of initial estimates of $g_{1/2}$ but in small samples it appears that properties of estimators depend to a large extent upon what truncation levels are used. These levels are set with reference to initial estimates of $g_{1/2}$. The results are not surprising when one considers that, even in non-sequential experiments and those using the Up and Down rule, one will often need large numbers of observations before $g_{1/2}$ can be determined at all accurately. In Anbar's and Venter's procedures asymptotic optimality depends upon using internal estimates of $g_{1/2}$. In these and other stochastic approximation methods observation are eventually made close to one level and one cannot expect to obtain good estimates of $g_{1/2}$.

Stochastic approximation procedures and variants of the Up and Down procedure differ in that:

(1) With the variants of the Up and Down procedures it is often necessary to make a large number of observations before accurate m.l.e.'s of slope or scale parameters can be obtained but at least such estimates have asymptotic normality with variance tending to zero as O(1/n). With stochastic approximation procedures it is not at all clear how one should obtain estimates of these parameters. If one requires an estimate of slope or scale, to use a procedure designed to concentrate all observations about one level must surely be unwise.

(2) If the initial estimates of slope are poor, the asymptotic properties of estimates from the procedures described in Section

5.1 can be very bad. If the value of c is chosen to be less than or equal to half the optimal value the estimates are no longer asymptotically normal. In this section I have considered procedures where the asymptotic properties of estimates do not depend on these initial estimates. However these procedures use internal estimates of slope which are very poor in small samples. With the variants of the Up and Down procedure one chooses a step size with reference to some initial estimate of slope. Small sample properties of estimates can be bad if this estimate is poor but m.l.e.'s of location parameters will always be asymptotically normal with variance tending to zero as $O(1/n)$.

The stochastic approximation procedures have the advantage that asymptotically unbiased estimators can be obtained without assuming a parametric form for the response curve. However if estimates of slope parameters are required these procedures appear to be of little use. If one uses one of the procedures discussed in Section 5.1, without having a good estimate of slope, one might unwittingly choose a c value for which the Robbins-Monro estimator is not asymptotically normal.

APPENDIX 1

A NOTE ON THE EXISTENCE OF MAXIMUM LIKELIHOOD ESTIMATES FOR THE UP AND DOWN PROCEDURE WITH LOGISTIC RESPONSE

Suppose at the end of an experiment n_i positive responses and m_i negative responses have been recorded at level x_i and that the observations are recorded in some known sequence. Suppose further that the probability of positive response at level x_i is given by $F(\beta x_i+a)$; where F is a known function taking values in (0.0,1.0) with upper and lower asymptotes at 1.0 and 0.0 as its argument tends to positive and negative infinity respectively. The logistic form for the response curve (see (8)) satisfies these conditions. The likelihood of observations is

$$\prod_i F_i^{n_i}(1-F_i)^{m_i} \quad \text{where } F_i=F(\beta x_i+a). \tag{1}$$

If the observations are not recorded in a sequence or one only knows a set of possible sequences that could give rise to such results the likelihood of observations takes the above form but is multiplied by some function of the n_i and m_i.

As (βx_i+a) increases $(1-F_i)$ tends to zero and as it decreases F_i tends to zero. From this it follows that for any level of mixed response (i.e. $n_i > 0$ and $m_i > 0$),

$$F_i^{n_i}(1-F_i)^{m_i} < \varepsilon \quad \text{for } |\beta x_i+a| > K, \tag{2}$$

for any positive ε providing K is sufficiently large. As all the terms in the product in (1) are bounded by 1.0 one can also ensure that the likelihood is arbitrarily small. In searching for maximum likelihood estimates one can restrict attention to a region for which $|\beta x_i+a| \leqslant K$ (K can be chosen so that the likelihood outside of this region is less than some known value taken by the likelihood). Suppose z_2 is a level of mixed response and that positive responses are recorded at levels z_1 and z_3 which are above and below z_2. The region in which one should search for m.l.e.'s of a and β can be restricted to

$$|\beta z_2+a| \leqslant K_2, \tag{3}$$

for K_2 sufficiently large as z_2 is a level of mixed response. At z_1 and z_3 one only knows that there is a positive response and so one can only restrict the region in which to search to

$$\beta z_1+a \geqslant K_1, \tag{4}$$

$$\beta z_3+a \geqslant K_3, \tag{5}$$

for K_1 and K_3 sufficiently low. All the K_i have been chosen so that outside of the regions defined by (3),(4) and (5) the likelihood is always less than some known value taken by the likelihood, so it is easy to see that the intersection of these regions is not empty. The inequalities (4) and (5) can be put in the forms

$$\beta(z_1-z_2)+(\beta z_2+a) \geqslant K_1, \tag{6}$$

$$\beta(z_3-z_2)+(\beta z_2+a) \geqslant K_3. \tag{7}$$

By assumption (z_1-z_2) and (z_3-z_2) have opposite signs, also βz_2+a is bounded. From (6) and (7) it is clear that β is bounded which in turn implies that a is bounded, so the intersection is a bounded closed set. The problem of maximising the likelihood is equivalent to that of maximising the log likelihood. For the logistic response curve

$$F_i=(1+\exp(-(\beta x_i+a)))^{-1}, \tag{8}$$

and if L is the log likelihood then

$$\frac{\partial^2 L}{\partial a^2} = -\sum_i (n_i+m_i)F_i(1-F_i), \tag{9}$$

$$\frac{\partial^2 L}{\partial a \partial \beta} = -\sum_i x_i(n_i+m_i)F_i(1-F_i), \tag{10}$$

$$\frac{\partial^2 L}{\partial \beta^2} = -\sum_i x_i^2(n_i+m_i)F_i(1-F_i). \tag{11}$$

The matrix of second derivatives is a negative definite matrix so the log likelihood is concave and will have a maximum within the closed bounded set. Such a maximum will be the unique solution of the likelihood equations.

From this one deduces that if there is one level of mixed response with levels for which there are positive responses above and below, then finite m.l.e. of a and β exist. By similar arguments the same is true if there are negative reponses above and below a level of mixed response or if one only knows that there are two levels of mixed response. The remaining cases possible using the Up and Down rule are that all responses except possibly the last are of the same type or there is a level of mixed response with only responses of opposite sign above and below. For these cases a degenerate curve with β infinite fits the observed response rates exactly. From these results one deduces that unique finite m.l.e.'s of a and β exist providing that after the initial run of constant response type more than three levels are visited (in this condition one includes the level that would have been visited had one more step been taken).

The ED50 of the logistic response curve equals $-a/\beta$ and so the m.l.e. of the ED50 exists providing the m.l.e. of β is not zero. In the following I derive a condition that is satisfied if and only if the m.l.e. of β is zero. Suppose in the Up and Down experiment the sequence of levels visited is $y_1,\ldots,y_n$ and that $s_1,\ldots,s_n$ are such that s_t equals 1 if the response at y_t is positive and equals -1 otherwise. Cornfield and Mantel (1950) show that in non-sequential experiments $\sum_{t=1}^{n} y_t s_t$ and $\sum_{t=1}^{n} s_t$ are sufficient statistics for β and a. Davis (1970) shows in Up and Down experiments that

$$\sum_{t=1}^{n} y_t s_t/2 = \left(\sum_{t=1}^{n} s_t/2\right)y_1 + (dn/4) - \left(\sum_{t=1}^{n} s_t/2\right)^2 d, \qquad (12)$$

where d is the step size. Davis (1970) incorrectly concludes that only degenerate curves fit results from Up and Down experiments. In his argument he assumed $\sum_{t=1}^{n} y_t s_t$ and $\sum_{t=1}^{n} s_t$ are sufficient statistics as in the non-sequential experiments; this is not true, because the set of levels at which tests are made is not fixed over possible outcomes. If the m.l.e.'s of a and β were both zero then from the likelihood equations it is easy to deduce that both $\sum_{t=1}^{n} y_t s_t$ and $\sum_{t=1}^{n} s_t$ would have to be zero which is inconsistent with the identity in (12). So if the m.l.e. of β is zero, the m.l.e. of a is not zero and no finite m.l.e. of the ED50 can exist. From the likelihood equations the m.l.e. of β is zero if and only if there exists a λ such that

$$\sum_{t=1}^{n}(((1+s_t)/2)-\lambda) = 0, \tag{13}$$

$$\sum_{t=1}^{n} y_t(((1+s_t)/2)-\lambda) = 0, \tag{14}$$

where the m.l.e. of a equals $\log(\lambda/1-\lambda)$. Unless all the s_t have the same sign (in which case no m.l.e.'s exist) the solution of (13) is always in (0.0,1.0) and $\log(\lambda/1-\lambda)$ is well defined. Substituting for $\sum_{t=1}^{n} y_t s_t$ in (13) using (12) the condition for the m.l.e. of β to be zero becomes

$$(dn/4) - v^2/d + (y_1 - (\sum_{t=1}^{n} y_t/n))v = 0, \tag{15}$$

where v equal $\sum_{t=1}^{n} s_t/2$. For example when in the sequence (n/2)-1 responses of the same sign are followed by (n/2)+1 responses of the opposite sign then (15) is satisfied and the m.l.e. of β equals zero.

There are many possible circumstances for which the m.l.e. of β is zero, but in the simulations of Section 3.3 such experiments were not encountered. These simulations were of 24 step experiments with 2000 simulations; the probability that the m.l.e. of β is zero must be very low for the conditions simulated.

APPENDIX 2

CONSTRAINTS ON RESPONSES USING THE UP AND DOWN RULE AND A NOTE ON DIXON AND MOOD'S ESTIMATOR

Suppose an Up and Down rule is operated between levels x_i, distance d apart, that is for some x_i

$$x_i = x_0 + (id) \quad \text{where i is an integer.} \qquad (1)$$

Fig.1 is a representation of a possible sequence of levels visited operating the Up and Down rule. As moves are made, a path is taken which can be represented by a directed graph each arc representing one move between levels (Fig.2 is such a graph corresponding to the experiment whose results are represented in Fig.1). The graph in Fig.2 can be constructed simply from the numbers of positive and negative responses at each level (for the level x_i I denote these as n_i and m_i respectively). The graph has an Eulerian chain (i.e. there exists a path visiting each directed arc once and only once) as the graph has been traced out in a continuous chain in the course of the experiment.

If one is given only the values of n_i and m_i for each level then each Eulerian chain within the directed graph corresponds to a possible Up and Down sequence from which the n_i and m_i could have been generated. Each distinct possible Up and Down sequence will

Fig. 1 A possible Up and Down sequence.

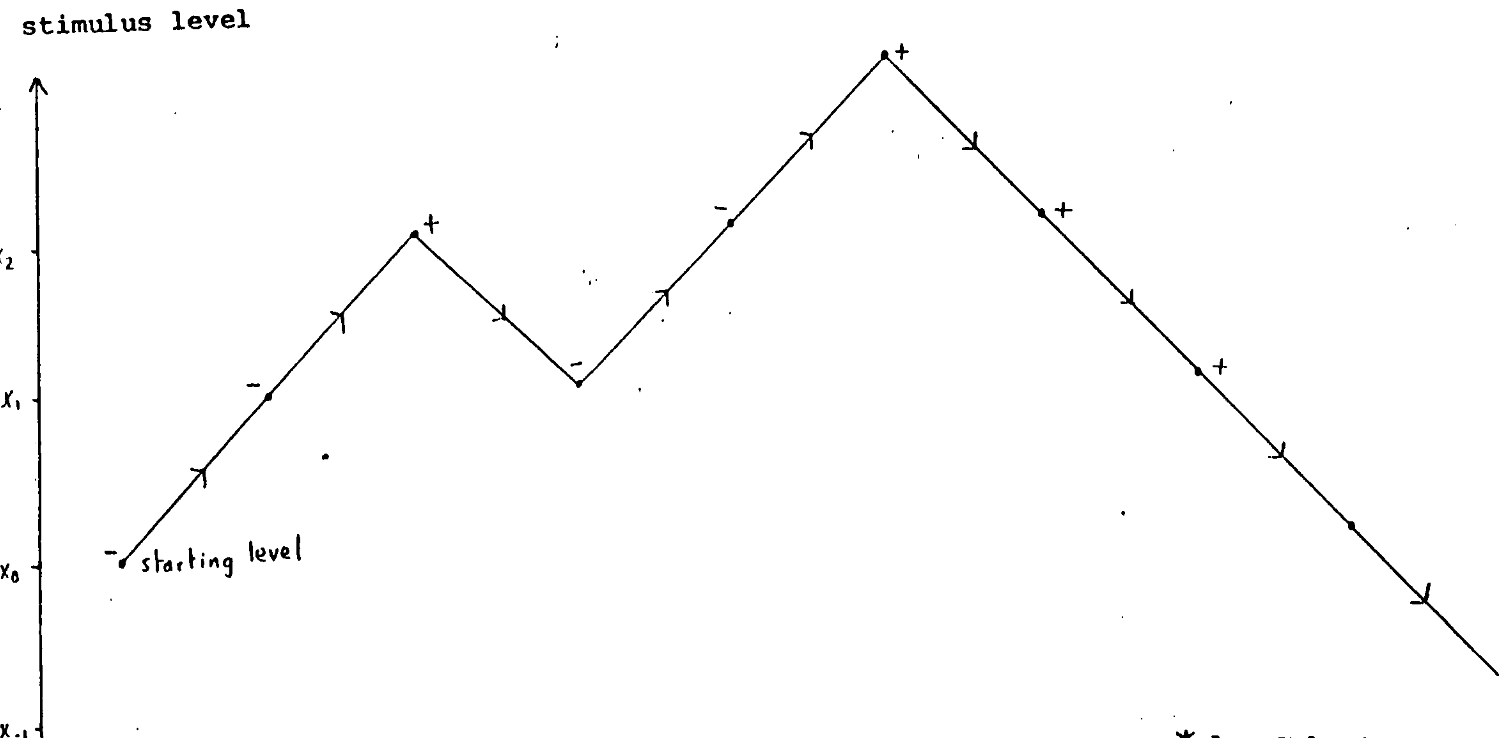
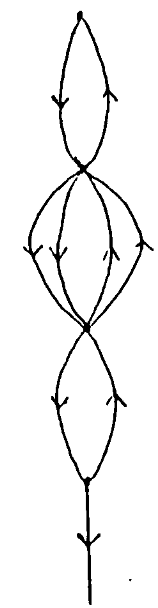


Fig. 2 The directed graph corresponding to the sequence in Fig. 1.*



* Any Eulerian chain in the graph in Fig. 2 corresponds to a possible Up and Down sequence of equal probability to that represented in Fig. 1.

be equiprobable under the assumption that the response curve is constant throughout the experiment.

The number of arcs of the directed graph out of level x_i is $n_i + m_i$ and the number of arcs in is $n_{i+1} + m_{i-1}$. From an elementary result in graph theory (see Theorem 3.6 of Busacker and Saaty) a directed graph has an Eulerian chain if and only if all vertices have equal numbers of arcs directed in as out or there are just two vertices for which this is not so one of which has one more arc directed out than in (the source) and another with one more arc directed in than out (the sink). Translating these conditions into conditions on the n_i and m_i one has $(n_i + m_i - n_{i+1} - m_{i-1})$ equal to zero for all i or equal to zero except at the source and the sink where it takes values 1 and -1 respectively. The n_i and m_i are zero for levels sufficiently high or low. From this one can easily deduce that

(a) If there is no source and sink, n_i equals m_i for all i.

(b) If the source is at level k_1 and the sink is at k_2 where $k_1 < k_2$, then $n_i - m_{i-1}$ equals -1 for $k_1 < i < k_2 + 1$ and is zero otherwise.

(c) If the source and sink are as in (b) but $k_2 < k_1$, then $n_i - m_{i-1}$ equals 1 for $k_2 < i < k_1 + 1$ and is zero otherwise.

The source is the starting point and the sink is the level that would have been reached if the experiment had continued for just one more step (when no source and sink exist these two levels are the same).

The Dixon and Mood estimator is the mean of the levels of the less frequent response type plus or minus half a step size depending upon whether the negative or positive responses are less frequent. If positive responses are less frequent then the estimator equals

$$\sum_i (n_i + m_i - (m_i - n_{i+1}))x_i / \sum_i (n_i + m_i - (m_i - n_{i+1})). \qquad (2)$$

If negative responses are more frequent then the estimator equals

$$\sum_i (n_i + m_i - (n_i - m_{i-1}))x_i / \sum_i (n_i + m_i - (n_i - m_{i-1})), \qquad (3)$$

The estimator weights levels in proportion to the number of visits to a level subtracting 1 from this number if the number of negative responses at a level is greater than the number of positive responses at the next higher level or if the number of positive responses at a level is greater than the number of negative responses at the next lower level. From theorems in Chung (1960) it is possible to deduce the asymptotic distribution of the mean of the levels visited (see Tsutakawa (1967a) and (1967b)). Whenever one returns to the starting level the Dixon and Mood estimator equals the mean of the levels visited as then n_{i+1} equals m_i for all i. It is easy to show from the arguments used in Chung that the Dixon and Mood estimator is asymptotically equivalent to the mean of the levels visited.

Suppose the response curve is of the form $F(\beta(x-\mu))$, where $F(0.0)$ is 0.5, then the likelihood equations are

$$\sum_i n_i((dF_i/dz)/F_i) - m_i((dF_{i-1}/dz)/(1-F_{i-1})) = 0, \qquad (4)$$

$$\sum_i n_i(x_i(dF_i/dz)/F_i) - m_{i-1}(x_{i-1}(dF_{i-1}/dz)/(1-F_{i-1})) = 0, \qquad (5)$$

where dF_i/dz is the derivative of the F at $\beta(x_i-\mu)$. A linear approximation to the expression on the left in equation (4) from expanding about μ is

$$\sum_i (n_i(2\lambda-4\lambda^2\beta(x_i-\mu)) - m_{i-1}(2\lambda+4\lambda^2\beta(x_{i-1}-\mu))), \qquad (6)$$

where λ is the derivative of F at 0.0 (if the second derivative of F at zero is zero, as is the case for the logistic and probit response curves, second order terms in a quadratic approximation vanish).

In Dixon and Mood (1948) a further approximation is made, m_{i-1} is replaced by n_i in (6) if the positive responses are less frequent otherwise m_{i-1} replaces n_i. The resulting expression is then zero when μ equals the Dixon and Mood estimator.

APPENDIX 3

THE ASYMPTOTIC DISTRIBUTION OF WETHERILL'S ESTIMATOR

In the following the asymptotic distribution of Wetherill's estimator, E_{WE}, will be derived. The argument is similar to that used by Tsutakawa (1967b) in deriving the asymptotic distribution of the sample average estimator (which is asymptotically equivalent to Brownlee _et al_'s and Dixon and Mood's estimators).

The sequence of levels visited in operating the Up and Down rule can be thought of as states visited in a Markov chain with transition probabilities of moving from x_i to x_{i+1} or x_{i-1} being $1-F_i$ and F_i respectively (where F_i is the probability of positive response at x_i). If the response curve is monotonic increasing and takes values above and below 0.5 then the states form a positive class with some equilibrium distribution $\{\pi_i\}$.

Suppose that h is some function on the state space and the first n states visited are $y_1, \ldots, y_n$. Results on pages 82, 83 and 94 of Chung (1960) together can be used to show that the asymptotic distribution of

$$\sum_{t=1}^{n} h(y_t)/n, \tag{1}$$

is N(M,V/n) where

$$M = \sum_j \Pi_j h(x_j), \qquad (2)$$

$$V = \sum_j \Pi_j h_j^2 + 2\sum_j \Pi_j h_j \sum_{k \neq i} \Pi_k h_k (m_{ji} + m_{ik} - m_{jk}), \qquad (3)$$

$h_j = (h(x_j)-M)$, m_{jk} is the mean first passage time from x_j to x_k and i is any integer. The sums in (3) must be absolutely convergent.

Tsutakawa simplified the expression for V to give an expression for the asymptotic variance of the sample average estimator (see Tsutakawa (1967a)); he took as his function h the identity function.

The distribution of E_{WE} cannot be found by considering a sum such as that in (1) as the state space does not include information on whether a level visited is a peak or a valley. It is useful to consider the following Markov chain in which the state of being at level x_i is further subdivided into states (x_i, λ), where λ = 1,2,3,4.

State $(x_i, 1)$ is entered when level x_i is reached from a valley at x_{i-1}.

State $(x_i, 2)$ is entered when level x_i is reached from x_{i-1} but not from a valley.

State $(x_i, 3)$ is entered when level x_i is reached from a peak at x_{i+1}.

State $(x_i, 4)$ is entered when level x_i is reached from x_{i+1} but not from a peak.

In other words the new states designate not only the current level but also the previous two levels visited. Clearly the sequence of states visited form a Markov chain. I will denote the equilibrium probability of being in state (x_i, λ) by $\pi_{i\lambda}$. To reach state $(x_i, 1)$ one must be at level x_i two steps before and take one step down followed by one step up; the equilibrium probability of being at x_i is π_i and there is probability of moving into state $(x_i, 1)$ after two steps of $F_i(1-F_{i-1})$ so the equilibrium probability of being in state $(x_i, 1)$ is given by

$$\pi_{i1} = F_i(1-F_{i-1})\pi_i. \tag{4}$$

By similar arguments the equilibrium probabilities of being at $(x_i, 2)$, $(x_i, 3)$ and $(x_i, 4)$ are

$$\pi_{i2} = (1-F_{i-2})(1-F_{i-1})\pi_{i-2}, \tag{5}$$

$$\pi_{i3} = (1-F_i)F_{i+1}\pi_i, \tag{6}$$

$$\pi_{i4} = F_{i+2}F_{i+1}\pi_{i+2}. \tag{7}$$

Let g be a function of the x_i and λ such that $g(x_i, \lambda)$ equals $x_i - m$ if λ equals 1 or 3 and is zero otherwise where

$$m = \sum_{\substack{j \\ \lambda=1,3}} \pi_{j\lambda} x_j \Big/ \sum_{\substack{j \\ \lambda=1,3}} \pi_{j\lambda}. \tag{8}$$

Suppose that (y_T, λ_T), for T=1,...,n, are the first n states visited in this Markov chain (here n+1 observations are made; y_T equals the level that would be moved to following T+1 observations). The value of m is chosen so that $\sum_{\lambda}\sum_{i}\pi_{i\lambda}\, g(x_i, \lambda)$ is zero. Using the results in Chung it follows that $\sum_{T=1}^{n} g(y_T, \lambda_T)/n$ has an asymptotic N(0,U/n) distribution. If the mean first passage time from state (x_j, λ_1) to (x_k, λ_2) is denoted by $m_{j\lambda_1,k\lambda_2}$ and $e_k = x_k - m$ then

$$U = \sum_{\substack{j \\ \lambda=1,3}} \pi_{j\lambda} e_j^{\,2} + 2\sum_{\substack{j \\ \lambda_1=1,3}} \pi_{j\lambda_1} e_j \sum_{\substack{k \\ \lambda_2=1,3 \\ (k,\lambda_2)\neq(i,\lambda)}} \pi_{k\lambda_2} e_k v_{j\lambda_1,k\lambda_2}, \tag{9}$$

where

$$v_{j\lambda_1,k\lambda_2} = m_{j\lambda_1,i\lambda} + m_{i\lambda,k\lambda_2} - m_{j\lambda_1,k\lambda_2}. \tag{10}$$

Again absolute convergence of terms in the summations is required. Any i and λ such that (x_i, λ) is a possible state can be used in (9); in the following I will set (i,λ) equal to (0,1). If $\sum_{T=1}^{n} g(y_T, \lambda_T)/n$ is divided by the proportion of times equals 1 or 3 (i.e. by the proportion of peaks or valleys) for the first n states visited the resulting expression equals

$$\left(\sum_j x_j\,(\theta_{j+1} + \delta_{j-1})/\sum_j(\theta_j + \delta_j)\right) - m, \tag{11}$$

where θ_j is number of peaks at x_j and δ_j is number of valleys. From a result on page 87 of Chung it follows that

$$\lim_{n\to\infty} \text{ with probability one of } \sum_j(\theta_j + \delta_j)/n = \tilde{m}, \tag{12}$$

where

$$\tilde{m} = \sum_{\substack{j \\ \lambda = 1,3}} \pi_{j\lambda}. \qquad (13)$$

From this one can deduce, providing terms in summations are absolutely convergent, that the expression in (11) will have an asymptotic $N(0, U/\tilde{m}^2 n)$ distribution.

E_{WE} and E_{PV} are defined in Formulae 2.1.11 and 2.1.12 respectively. E_{WE} is equal to

$$\sum_j (x_j + d/2)(\theta_{j+1} + \delta_j) / \sum_j (\theta_j + \delta_j), \qquad (14)$$

and E_{PV} is equal to

$$\sum_j x_j (\theta_j + \delta_j) / \sum_j (\theta_j + \delta_j). \qquad (15)$$

When E_{WE}-m is subtracted from the expression in (11) the remainder is $d/2 \sum_j (-\theta_j + \delta_j) / \sum_j (\theta_j + \delta_j)$; when E_{PV}-m is subtracted the remainder is twice this quantity. The total number of peaks differs from the total number of valleys by at most 1 (i.e. $|\sum(\theta_j - \delta_j)|$ is 1 or 0). From (12) it follows that $n / \sum_j (\theta_j + \delta_j)$ has a limit with probability one of $1/\tilde{m}$; a trivial consequence of this is that $n^{1/2} / \sum_j (\theta_j + \delta_j)$ has a limit with probability one of zero. From these observations it immediately follows that $n^{1/2}$ times the differences between the expression in (11) and E_{WE}-m and between this expression and E_{PV}-m both tend with probability one to zero, and so E_{WE}-m and E_{PV}-m both have the asymptotic distribution of the expression in (11) (i.e. a $N(0,U/\tilde{m}^2 n)$ distribution). The expression for U can be simplified; to do this one must express the $m_{j\lambda_1 k\lambda_2}$ for λ_1 and λ_2 equal to 1 or 3 in terms of m_{jk}. I will consider separately cases where (x_k, λ_2) can be reached from (x_j, λ_1) in a minimum of one, two or more than two steps.

Starting in state $(x_j, 1)$ one can move in one step to $(x_{j-1}, 3)$ with probability F_j and to $(x_{j+1}, 2)$ with probability $1-F_j$; starting in state $(x_j, 3)$ one can move in one step to $(x_{j+1}, 1)$ with probability $1-F_j$ and to $(x_{j-1}, 4)$ with probability F_j. From these observations it is easy to deduce that

$$m_{j1,j-1\,3} = 1 + (1-F_j)(m_{j+1\,2,j-1\,3}), \qquad (16)$$

$$m_{j3,j+1\,1} = 1 + F_j(m_{j-1\,4,j+1\,3}). \qquad (17)$$

Starting from state (x_j, λ_1) then, whatever the value of λ_1, it will take at least two steps to enter either state $(x_j, 1)$ or $(x_j, 3)$. The probability of moving to any state from (x_j, λ_1) after two steps is independent of λ_1 as λ_1 only gives information about the two steps made before entering (x_j, λ_1). It follows that $m_{j\lambda_1,j1}$ and $m_{j\lambda_1,j3}$ are independent of λ_1, that is

$$m_{j\lambda_1,j\lambda_2} = m_{j\lambda_2,j\lambda_2} \quad \text{for } \lambda_2 = 1 \text{ or } 3. \qquad (18)$$

Suppose $m_{j\lambda_1,k\lambda_2}$ is required where $\lambda_2 = 1$ or 3. Suppose further that (x_k, λ_2) cannot be reached from (x_j, λ_1) in one step and also that $j \neq k$ (it is easy to see that this second condition implies that (x_k, λ_2) cannot be reached in two steps). Two steps previous to being in state (x_k, λ_2) one must be at level x_k so one must pass through some state (x_k, λ_3), where $\lambda_3 \neq \lambda_2$, in the sequence of states visited in moving from (x_j, λ_1) to (x_k, λ_2). The first passage time consists of the first passage time from (x_j, λ_1) to any state (x_k, λ_3) (it is known that λ_3 cannot equal λ_2) plus the first passage time from (x_k, λ_3) to (x_k, λ_2). It is clear that the first of these times has mean m_{jk} (as it is just a first passage time from x_j to x_k) and from (18) the second has mean $m_{k\lambda_2, k\lambda_2}$ so

$$m_{j\lambda_1, k\lambda_2} = m_{jk} + m_{k\lambda_2, k\lambda_2}. \qquad (19)$$

Using Formula (19) in (16) and (17) one can deduce that

$$m_{j1, j-1\,3} = m_{j\,j-1} + (1-F_j)(m_{j-1\,3, j-1\,3}), \qquad (20)$$

$$m_{j3, j+1\,1} = m_{j\,j+1} + F_j\,(m_{j+1\,1, j+1\,1}) \quad . \qquad (21)$$

All the $m_{j\lambda, j\lambda}$ are equal to $1/\pi_{j\lambda}$ and so can easily be calculated. Using (18) (19) (20) and (21) the value of $m_{j\lambda_1, k\lambda_2}$, when λ_1 and λ_2 equal 1 or 3, can be found in terms of m_{jk} and $\pi_{k\lambda_2}$. When Formulae (18) and (19) apply throughout the expression in (10) with $(i, \lambda) = (0, 1)$

$$v_{j\lambda_1 k\lambda_2} = m_{01, 01} + (1-\delta_{j0})m_{j0} + (1-\delta_{0k})m_{0k} - (1-\delta_{jk})m_{jk}, \qquad (22)$$

where δ_{jk} is the Kronecker delta. In general a correction will have to be made to the right hand side in (22) whenever (20) or (21) have to be used in calculation. In the following I shall calculate the value of U assuming (22) always holds and then show what correction must be made to give the correct value.

The m_{oio1} term when substituted into the expression (9) for U will vanish as $\sum_{\substack{j\\ \lambda=1,3}} \pi_{j\lambda} e_j$ equals zero. The value of $((1-\delta_{jo})m_{jo}+(1-\delta_{oK})m_{oK}-(1-\delta_{jK})m_{jK})$ equals

$$
\begin{array}{ll}
0 & \text{if } j=0 \text{ or } k=0 \\
0 & \text{if } j>0>k \text{ or } k>0>j \text{ as then } m_{jo}+m_{oK}=m_{jK} \\
m_{jo}+m_{oj} & \text{if } 0<j<k \text{ or } 0>j>k \text{ as then } m_{oj}+m_{jK}=m_{oK} \\
m_{Ko}+m_{oK} & \text{if } 0<k<j \text{ or } 0>k>j \text{ as then } m_{jK}+m_{Ko}=m_{jo} \\
m_{jo}+m_{oj} & \text{if } j=k \text{ but } j\neq 0.
\end{array}
$$

So the contribution to U from this term is

$$\sum_{\substack{j\\ \lambda=1,3}} \pi_{j\lambda} e_j^2 + 4\sum_{\substack{o<j<K\\ K<j<o\\ \lambda_1=1,3\ \lambda_2=1,3}} \pi_{j\lambda_1}\pi_{K\lambda_2} e_j e_K (m_{jo}+m_{oj}) + 2\sum_{\substack{\lambda_1=1,3\ \lambda_2=1,3\\ j\neq 0}} \pi_{j\lambda_1}\pi_{j\lambda_2} e_j^2 (m_{jo}+m_{oj}). \quad (23)$$

Harris (1952) shows that for i not equal to zero $(m_{io}+m_{oi})$ equals $1/\pi_i\rho_i$ where ρ_i is the probability that starting at x_i one reaches x_o before returning to x_i. There exist recurrence relations for calculating the ρ_i and so it is possible to evaluate terms in (23). For i > 1 the relations are

$$1/\rho_{i+1} = (1-F_i)/(F_{i+1}\rho_i) + 1/F_{i+1}, \quad (24)$$

$$1/\rho_{-(i+1)} = F_{-i}/((1-F_{-(i+1)})\rho_{-i}) + 1/(1-F_{-(i+1)}), \quad (25)$$

where ρ_1 equals F_1 and ρ_{-1} equals $1-F_{-1}$.

The first term in (10), when $(i,\lambda) = (0,1)$, must be calculated using (21) if $(j,\lambda_1) = (-1,3)$. The second term must be calculated using (20) if $(k,\lambda_2) = (-1,3)$. The third term must be calculated using (20) and (21) when one of the pair (j,λ_1) and (k,λ_2) takes value $(i+1,1)$ and the other the value $(i,3)$. The correction to U due to the first term is

$$-2\pi_{-1\,3}(1-F_{-1})m_{01,01}\, e_{-1}\sum_{\substack{\lambda_2=1,3\\(k,\lambda_2)\neq(0,1)}}\pi_{k\lambda_2}e_k. \qquad (26)$$

As $\sum_{\substack{k\\ \lambda=1,3}}\pi_{k\lambda}e_k$ is zero the summation in (26) is just $-\pi_{01}e_0$; also the correction to U due to the second term vanishes as $\sum_{\substack{k\\ \lambda=1,3}}\pi_{k\lambda_2}e_k$ enters as a factor into this correction. The correction in U due to the third term is

$$2\left(\sum_j \pi_{j1}\pi_{j-1\,3}F_j\, m_{j-1\,3,j-1\,3}\, e_j e_{j-1} + \sum_{j\neq -1}\pi_{j3}\pi_{j+1\,1}(1-F_j)m_{j+1\,1,j+1\,1}\, e_{j+1}e_j\right). \qquad (27)$$

The term missing in (27) because the second summation does not include $j = -1$ is just the expression in (26). So dropping this restriction gives an expression for the total correction. This expression can be simplified by using (4) and (6) and also the identity $m_{j\lambda,j\lambda} = 1/\pi_{j\lambda}$ to the following

$$4\sum_j \pi_j F_j^2(1-F_{j-1})e_j e_{j-1}. \qquad (28)$$

Using all these results one can deduce (again making use of (4) and (6)) that E_{WE} and $E_{\rho V}$ have an asymptotic $N(m,U^*/n)$ distribution with U^* equal to

$$\Big(\sum_j \pi_j e_j W_j (1+(2W_j/\rho_j)) + 4\sum_{\substack{0<i<j \\ j<i<0}} \pi_j e_i e_j W_i W_j/\rho_i - 2\pi_0 e_0^2 W_0 + 4\sum_j \pi_j e_j e_{j-1} Z_j\Big) \Big/ \Big(\sum \pi_j W_j\Big)^2 \qquad (29)$$

where W_j equals $(F_j(1-F_{j-1}) + (1-F_j)F_{j+1})$, Z_j equals $F_j^2(1-F_{j-1})$ and ρ_0 is defined as 1. Note that m equals $\sum_j \pi_j W_j x_j / \sum_j \pi_j W_j$ and $\tilde{m}$ equals $\sum_j \pi_j W_j$.

Conditions on the response curve in Tsutakawa (1967b) are certainly satisfied by a monotonic increasing response curve taking values above and below 0.5. Under these conditions Tsutakawa deduced from relations (24) and (25) that $\inf_j . \rho_j > 0$. This is enough to show that the terms in all the summations are absolutely convergent as the $\pi_{j\lambda}$ tend to zero exponentially as $|j| \to \infty$.

The e_j equal $x_j - m$, if m is replaced by $\sum \pi_i x_i$ throughout (29) and W_i and Z_i are replaced by -1 and 0 respectively then one has the analogous asymptotic variance expression for the mean level estimator (see Tsutakawa (1967a) and Formula 2.3.5). So a program for calculating the asymptotic distribution of E_{WE} can be easily adapted to find that of all the estimators asymptotically equivalent to the mean of levels visited.

APPENDIX 4

CONDITIONAL DISTRIBUTIONS AND SOME ASYMPTOTIC PROPERTIES OF PEAKS AND VALLEYS IN AN UP AND DOWN SEQUENCE

This Appendix contains three theorems. The first and second are concerned with the distribution of peaks and valleys in an Up and Down sequence given the starting level and the numbers of positive and negative responses at each level. The third gives the asymptotic distribution of E^*_{WE} and $E^*_{\rho V}$ (for definition of these estimators see Formulae 2.1.13 and 2.1.14 respectively); this theorem makes use of the result proved in the first.

In Appendix 2 a directed graph corresponding to an Up and Down sequence was discussed. Each Eulerian chain in this graph corresponds to a possible Up and Down sequence and all such distinct sequences will be equiprobable. This graph can be drawn if and only if the values of n_i and m_i for each level x_i are known (where n_i and m_i are numbers of positive and negative responses respectively). The Dixon and Mood estimator is a function of the n_i and m_i. Wetherill's estimator is not a function of the n_i and m_i; its values for different Eulerian chains in the graph are not in general equal.

For all except the first visit to a level a peak is recorded if and only if the response at the current visit and previous visit are both positive, and a valley is recorded if and only if they are both negative. So with the exception of peaks and valleys possibly recorded at the first visit to a level the number of peaks and of valleys at a level equal the number of agreements in positive and negative sign respectively in the sequence of responses at the level.

Usually the directed graph will have a source and a sink. At the source there is one more directed arc out than in, at the sink one more arc in than out, at all other levels the number of directed arcs into a level equals the number of directed arcs out. Any Eulerian chain must start at the source and end at the sink. If the level that would have been visited operating the Up and Down rule for one more step is the starting level no such identifiable source and sink exist; in such cases I designate the source and sink as both equal to the starting level.

Theorem 1

Given the values of n_i and m_i for all levels, and the value of the starting level, then the distributions of the number of peaks and the number of valleys at a level are independent of the number of peaks or valleys at any other level.

Proof

If the starting level is given, it is always possible to identify the source and sink of the directed graph. Above the sink one must finally depart from each level by moving down (the last response is positive); below the sink one must depart by moving up (the last response is negative). Above the source one must initially enter each level from below; below the source one must enter each level from above. There is a peak at the first visit to a level if and only if the level is first entered from below (i.e. the level is above the source) and the first response is positive. There is a valley at the first visit to a level if and only if the level is first entered from above (i.e. the level is below the source) and the first response is negative. The number of peaks between the second and last visits to the level is the number of positive agreements in the sequence of responses at the level; the number of valleys is the number of negative agreements in sign. So given the source, the numbers of peaks and of valleys at a level are functions of the sequence of responses at that level. Given the sink, the last response at all levels except the sink is fixed.

Suppose K visits are made to a level where $K>2$, suppose further that responses at the (j-1)th and jth visits ($1<j<K$) are of opposite sign. Suppose that at the (j-1)th visit the response is positive and at the jth visit negative. Between the (j-1)th and the jth visit levels passed through are all below the level, and between the jth and (j+1)th visit they are all above the level. One can interchange the responses at the (j-1)th and jth visit without altering the sequence of responses at any other level by simply interchanging these paths above and below the level (see Figs. 1 and 2). If the last two responses at the sink are of

Fig. 1 An example of a possible Up and Down sequence where at the j-1th. and jth. visits to level x_k responses are positive and negative respectively.

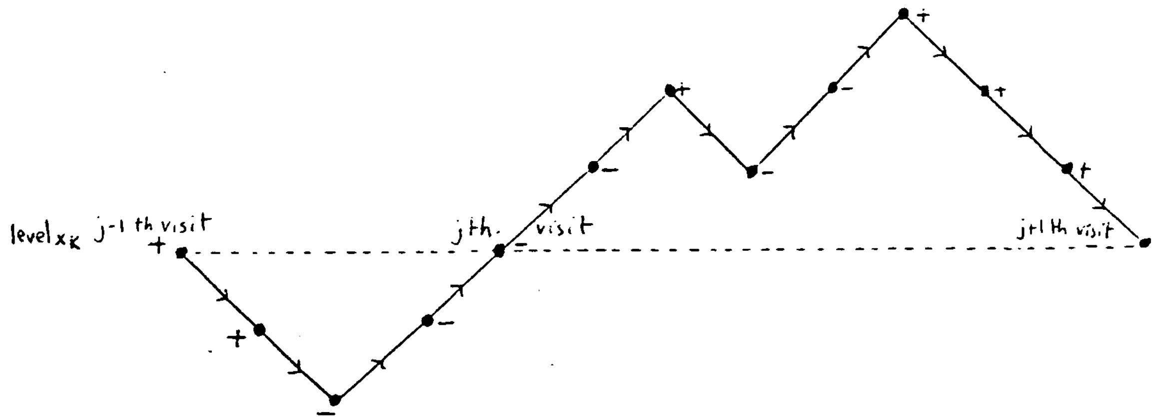
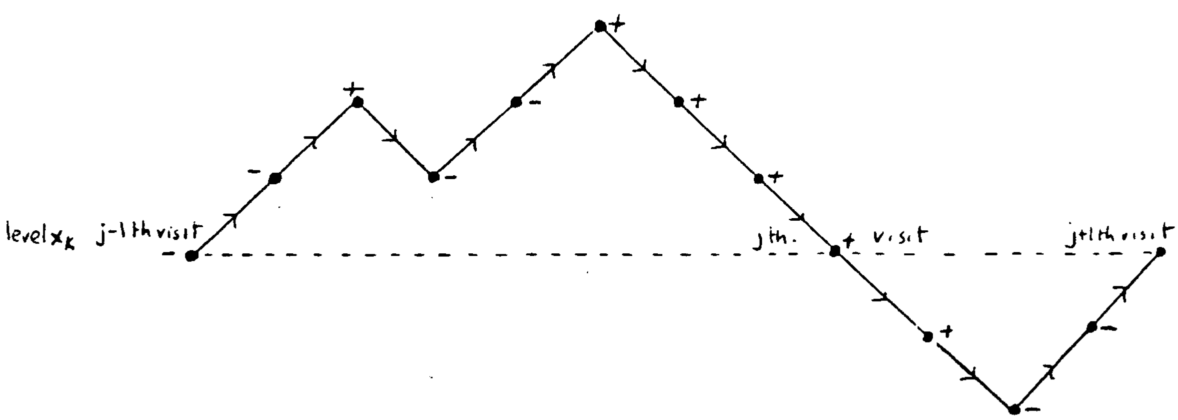


Fig. 2 A possible Up and Down sequence where the sequences of responses at all levels except x_k are the same as in Fig. 1, but the signs of the responses at the j-1th. and jth. visit to x_k are interchanged (i.e. paths above and below x_k are interchanged.



opposite sign one can also interchange these responses in the same way without altering the sequence of responses at any other level as there is a path from the last response at the sink returning back to the sink. As one interchange between adjacent responses at a level can be made any number of such interchanges can be made. From this it follows that the sequence of responses at a level may be permuted among themselves without altering the sequence of responses at any other level, with the only restriction being that at all levels except the sink the sign of the last response is fixed. From the remarks made at the end of the last paragraph it immediately follows that the numbers of peaks and of valleys at a level are conditionally independent of those at other levels.

If one considers using Wetherill's estimator one may also consider replacing the numbers of peaks and of valleys entering into this estimator by their conditional expectations given the values of n_i , m_i and the starting level.

Clearly if n_i+m_i equals 1 then there is a peak at the level if and only if n_i equals 1 and it is the highest level reached; there is a valley if and only if m_i equals 1 and it is the lowest level reached.(this is assuming that the level is not the source).

Theorem 2

Suppose that δ equals zero at the sink and one otherwise. The expected number of peaks at the level x_i given the n_i, m_i and the starting level is

$$(n_i(n_i-1) + m_{i-1}(m_{i-1}-1))/(2(n_i+m_i-\delta) \qquad \text{if } n_i+m_i > 1,$$

and the expected number of valleys is

$$(m_i(m_i-1) + n_{i+1}(n_{i+1}-1))/(2(n_i+m_i-\delta) \qquad \text{if } n_i+m_i > 1.$$

Proof

Consider first the case of sequences of n positive and m negative responses. The number of such sequences is just the number of ways of choosing n out of n+m, that is using a common notation

$$\binom{n+m}{n}. \tag{1}$$

The expected number of agreements in positive sign in the sequence is just

$$\sum_{j=1}^{n+m-1} p_j, \tag{2}$$

where p_j is the proportion of sequences for which the jth and (j+1)th responses are both positive. The number of sequences in which any specified pair of responses are positive is the number of ways of choosing the rest of the sequence, that is the number of ways of choosing n-2 from n+m-2. Hence the proportion of sequences for which the jth and j+1th response are positive is

$$\binom{n+m-2}{n-2} \Big/ \binom{n+m}{n} = \frac{n(n-1)}{(n+m)(n+m-1)}. \qquad (3)$$

Substituting back into (2) one deduces that the expected number of agreements in positive sign is

$$n(n-1)/(n+m). \qquad (4)$$

In a similar way one can show that the proportion of sequences for which any specified response is positive equals

$$n/(n+m). \qquad (5)$$

Each level is in one of five categories: it can be above, between or below the source and sink and also at the source or the sink. Suppose that more than one visit is made to the level x_i.

If the level x_i is above the source and sink then n_i equals m_{i-1}, the level is first entered from below and the last response is positive. The expected number of peaks at the first visit is just the proportion of times a sequence of n_i-1 positives and m_i negatives start with a positive, from (5) this equals $(n_i-1)/(n_i+m_i-1)$. The expected number of peaks at the last visit to the level is the proportion of times such a sequence ends with a positive (as the last response is positive) which is again equal to $(n_i-1)/(n_i+m_i-1)$. The expected number of peaks at the remaining visits to the level is the expected number of positive agreements in sign in this sequence, which from (4) equals $(n_i-1)(n_i-2)/(n_i+m_i-1)$. So combining these results the expected number of peaks at this level is

$$n_i(n_i-1)/(n_i+m_i-1). \qquad (6)$$

If the level is between the source and the sink there are two cases to consider. If the sink is below the source n_i equals $m_{i-1}+1$, the first visit is from above (so there is no peak at the first visit) and the last response is positive. The expected number of peaks at the level is just as before only there is no contribution to the expectation from peaks at the first visit. The expected number of peaks is now

$$((n_i(n_i-1)) + ((n_i-1)(n_i-2))/(2(n_i+m_i-1)). \qquad (7)$$

If the source is below the sink then n_i equals $m_{i-1}-1$, the first visit to the level is from below and the last response is negative (so there is no peak at the last visit). The expected number of peaks is the proportion of sequences of n_i positives and m_i-1 negatives which start with a positive (which from (5) is $n_i/(n_i+m_i-1)$) plus the expected number of positive agreements in sign in this sequence (which from (4) is $n_i(n_i-1)/(n_i+m_i-1)$). So the expected number of peaks is

$$((n_i(n_i-1)) + ((n_i+1)n_i))/(2(n_i+m_i-1)). \qquad (8)$$

Below the source and sink n_i equals m_{i-1}, the level is first entered from above and the last response is negative. As no peaks can be recorded at the first or last visits the expected number of peaks is just the expected number of agreements in sign in a sequence of n_i positives and m_i -1 negatives, that is from (4)

$$n_i(n_i-1)/(n_i+m_i-1). \qquad (9)$$

The source is above, below or at the sink. If the source is above the sink then at the source n_i equals $m_{i-1}+1$ and the last response is positive. The same arguments as used in the paragraph before (7) can be applied and the expression in (7) is the expected number of peaks. At the sink n_i equals m_{i-1} and the level is first entered from above (so there is no peak at the first visit), the expected number of peaks is the expected number of positive agreements in sign in such a sequence which from (4) equals

$$n_i(n_i-1)/(n_i+m_i) \qquad (10)$$

If the source is below the sink then at the source n_i equals

m_{i-1} and the last response is negative. The same arguments as used in the paragraph before (9) can be applied to show the expected number of peaks is given by the expression in (9). At the sink n_i equals $m_{i-1}-1$ and the level is first entered from below. The expected number of peaks is just equal to the expression in (10) plus the proportion of sequences of n_i positives and m_i negatives which start with a positive (which from (5) is $n_i/(n_i+m_i)$), this equals

$$((n_i(n_i-1)) + ((n_i+1)n_i))/(n_i+m_i). \tag{11}$$

If the source and sink coincide n_i equals m_{i-1}; at the start no peaks are recorded so the expected number of peaks is using the arguments in the paragraph before (10) equal to the expression in (10).

These results together prove the first part of Theorem 2; clearly the second part of the Theorem follows using similar arguments.

The alternatives to Wetherill's estimator E_{WE} and Choi's closely related estimator $E_{\rho\gamma}$, termed E_{WE}^{*} and $E_{\rho\gamma}^{*}$, are such that the actual numbers of peaks and of valleys at x_i in the expressions for E_{WE} and $E_{\rho\gamma}$ are replaced by $n_i m_{i-1}/(n_i+m_i)$ and $n_{i+1} m_i/(n_i+m_i)$ respectively.

Suppose the response curve is monotonic increasing taking values above and below 0.5, then results in Appendix 3 hold and E_{WE} has an asymptotic $N(m,U^*/n)$ distribution where m and U^* are given in Formulae (8) and (29) respectively of Appendix 3 and n is the number of observations.

Theorem 3

With the conditions given in the last paragraph the asymptotic distribution of E_{WE}^* and E_{PV}^* is $N(m,(U^*-V^*)/n)$ where (using the notation of Appendix 3) V^* equals

$$\sum_j \pi_j 4(F_j(1-F_j))^2 (x_j-m)^2 / \tilde{m}^2 ; \qquad (12)$$

$\tilde{m}$ is $\sum_{\substack{i \\ \lambda=1,3}} \pi_{i\lambda}$ and m equals $\sum_{\substack{i \\ \lambda=1,3}} \pi_{i\lambda} x_i/\tilde{m}$ ($\pi_{i\lambda}$ is the equilibrium probability of being in state (x_i,λ); states $(x_i,1),(x_i,2),(x_i,3)$ and $(x_i,4)$ are defined on page 233 in Appendix 3).

Proof

I will consider the following estimator E'_{PV}

$$E'_{PV} = \sum_j x_j (\theta'_j + \delta'_j)/(\sum_j(\theta'_j+\delta'_j)), \qquad (13)$$

where θ'_j is the expected number of peaks at x_j given the n_i, m_i and starting level, and δ'_j is the conditional expected number of valleys. This is similar to the expression for E_{PV} given in (15) of Appendix 3, only numbers of peaks and valleys are replaced by their conditional expectations (expressions for θ_j' and δ_j' can be obtained using Theorem 2). In Appendix 3 θ_j and δ_j denote the number of peaks and valleys respectively at x_j; I will use the same terminology here.

The proportion of peaks and valleys recorded has an asymptotic $N(\tilde{m},\tilde{U}/n)$ distribution; where $\tilde{U}$ is equal to an expression such as that in (9) of Appendix 3, but now with summations being taken over all λ, with e_j equal to $1-\tilde{m}$ for λ equal to 1 or 3 and equal to $-\tilde{m}$ otherwise (it is easy to show by the same arguments used at the end of Appendix 3 that these sums are absolutely convergent). The expression in (9) of Appendix 3 is still made up of absolutely convergent sums if all the e_j are replaced by 1; if all the h_j in (3) of Appendix 3 are replaced by 1 the sums are still absolutely convergent (again one can use arguments in Appendix 3). These two conditions are enough to ensure that a theorem on page 97 of Chung can be applied and that

$$\lim_{n \to \infty}(nE(\text{prop. of peaks and valleys} - \tilde{m})^2) = \tilde{U}. \qquad (14)$$

So $\sum_j(\theta_j+\delta_j)/n$ tends in mean square to $\tilde{m}$. The variance of $\sum_j(\theta_j'+\delta_j')/n$ is always less than that of $\sum_j(\theta_j+\delta_j)/n$ as the latter is the conditional expectation of the former. So it follows that $\sum_j(\theta_j'+\delta_j')/n$ also tends in mean square to $\tilde{m}$ and hence in probability to $\tilde{m}$.

Define m_K, B_K and B_K' as the following

$$m_K = \sum_{|j|\leq K} x_j(\pi_{j+1,1}+\pi_{j-1,3}) / \sum_{|j|\leq K}(\pi_{j+1,1}+\pi_{j-1,1}), \quad (15)$$

$$B_K = \sum_{|j|\leq K}(x_j - m_K)(\theta_j + \gamma_j)/n, \quad (16)$$

$$B'_K = \sum_{|j|\leq K}(x_j - m_K)(\theta'_j + \gamma'_j)/n. \quad (17)$$

Consider now

$$n^{1/2}(\sum(\theta'_j + \gamma'_j)/n)(E\rho'_V - m), \quad (18)$$

which equals

$$n^{1/2}B'_K + (n^{-1/2}(\sum_{|j|>K}(x_j - m)(\theta'_j + \gamma'_j) + \sum_{|j|\leq K}(m_K - m)(\theta'_j + \gamma'_j))). \quad (19)$$

The terms in the brackets in (19) equal the conditional expectation given the n_i, m_i and starting level of (using the notation of Appendix 3)

$$\sum_{T=1}^{n} p(y_T, \lambda_T)/n, \quad (20)$$

where (y_T, λ_T), $T=1,\ldots,n$, are the first n states visited in the Markov chain described in Appendix 3 and p is a function of the x_j and λ such that

$$p(x_{j+1}, 1) = x_j - m \quad \text{if } |j| > k,$$

$$p(x_{j+1}, 1) = m_K - m \quad \text{if } |j| \leq k,$$

$$p(x_{j-1}, 3) = x_j - m \quad \text{if } |j| > k,$$

$$p(x_{j-1}, 3) = m_K - m \quad \text{if } |j| \leq k,$$

$$p(x_i, \lambda) = 0 \quad \text{if } \lambda = 2 \text{ or } 4.$$

The value of m_k has been chosen so that $\sum_{j,\lambda}\pi_{j\lambda}p(x_j,\lambda)$ is zero. Now using similar arguments to those in Appendix 3 one can show that $\sum_\tau p(y_\tau,\lambda_\tau)/n$ has an asymptotic $N(0,\delta_k)$ distribution where

$$\delta_k = \sum_{j,\lambda}\pi_{j\lambda}p(x_j,\lambda)^2 + 2\sum_{j,\lambda_1}\pi_{j\lambda_1}p(x_j,\lambda_1)\sum_{(k,\lambda_2)\neq(i,\lambda)}\pi_{k\lambda_2}p(x_k,\lambda_2)v_{j\lambda_1,k\lambda_2}, \tag{21}$$

(see (10) of Appendix 3 and paragraph preceding (9) for definition of $v_{j\lambda_1,k\lambda_2}$). The summations are absolutely convergent for all k. The value of m_k tends to m as $k\to\infty$, it is easy to deduce that the expression in (21) tends to zero as $k\to\infty$. Also conditions for the application of the theorem on page 97 of Chung are satisfied and

$$\lim_{n\to\infty}\text{it}(nE(\sum_{\tau=1}^{n}p(y_\tau,\lambda_\tau)/n)^2) = \delta_k. \tag{22}$$

I have already noted that the terms in brackets in (19) equal the conditional expectation given the n_i, m_i and starting level of $\sum_{\tau=1}^{n}p(y_\tau,\lambda_\tau)/n$. This conditional expectation has lower variance than $\sum_{\tau=1}^{n}p(y_\tau,\lambda_\tau)/n$ and so it follows that the square of these terms have expectation arbitrarily close to or below δ_k for n sufficiently large. By choosing k sufficiently large δ_k can be made arbitrarily small. So for k and n sufficiently large one can ensure that the expression in (18) is arbitrarily close to $n^{1/2}B_k'$ with arbitrarily high probability.

Suppose q is a function of x_j and λ such that

$$q(x_{j+1},1) = x_j - m_k \quad \text{if } |j| \leq k,$$

$$q(x_{j-1},3) = x_j - m_K \quad \text{if } |j| \leq k.$$

The asymptotic distribution of B_K is $N(0,U_K/n)$, where U_K is similar to the expression in (21) only q replaces p throughout. The sums are absolutely convergent and it is not difficult to see that as $k \to \infty$ the value of U_K tends to U, where U/n is the asymptotic variance of $\sum_j (x_j - m)(\theta_j + \delta_j)/n$ (an equation for U is given in (9) of Appendix 3). The characteristic function of $n^{1/2} B_K$ can be written in the form

$$\sum_S \exp(itn^{1/2} B_K')(\sum_{B_K} \exp(itn^{1/2}(B_K - B_K'))p(B_K | S))p(S), \tag{23}$$

where $i^2 = -1$, S denotes the set of values of n_i, m_i and starting level, $p(B_K | S)$ is the probability of observing B_K given S, and p(S) is the probability of a particular set S of n_i, m_i and starting level. The inner summation in (23) is the characteristic function of $n^{1/2}(B_K - B_K')$ given S. From Theorem 1 conditional on S the values of $(\theta_j + \delta_j)$ are independent and so this characteristic function equals

$$\prod_{|j| \leq k} E(\exp(it(x_j - m_K)c_j)), \tag{24}$$

where

$$c_j = ((\theta_j + \delta_j) - (\theta_j' + \delta_j'))/n^{1/2}. \tag{25}$$

The value of $(\theta_j + \delta_j)$ differs from the number of agreements in sign in the sequence of responses at this level by at most one (this is when there is a peak or a valley at the first visit). The number of runs of positive and negative responses plus the number of agreements in sign equals the total number of observations at a level. So c_j is asymptotically equivalent to

$$(-r_j + E(r_j \mid S))/n^{1/2}, \tag{26}$$

where r_j is the number of runs at level x_j. The distribution of the number of runs in a sequence of positive and negative responses has been much discussed in the past. In Wald and Wolfowitz (1940), expressions are given for the mean and variance of the number of runs in a randomly permuted sequence of n positive and m negative responses. The expected number of runs is

$$(2mn/(m+n)) + 1, \tag{27}$$

and the variance of the number of runs is

$$2mn(2mn-(m+n))/((m+n)^2(m+n-1)). \tag{28}$$

Suppose that

$$0 < \alpha_1 < (n/(m+n)) < \alpha_2 < 1, \tag{29}$$

for some fixed α_1 and α_2. For (m+n) sufficiently large the expression in (28) divided by (m+n) is less than

$$4((\alpha_1+1)/2)^2((2-\alpha_2)/2)^2, \tag{30}$$

so using Chebishev's inequality it follows that the probability that the number of runs differs from the expression in (27) by more than $\alpha_0(m+n)^{1/2}$ is arbitrarily small for $(m+n)$ and α_0 sufficiently large. Suppose $(2mn/(m+n))+1+(w(m+n)^{1/2})$ is a possible number of runs where $|w|<\alpha_0$. Following the method in Theorem 1 of Wald and Wolfowitz one can show that the probability of this number of runs is proportional to

$$\exp((-w^2/(8((\xi(1-\xi))^2)+O((m+n)^{-1/2})), \tag{31}$$

where ξ equals $n/(n+m)$. With the bounds in (29) the $O((m+n)^{-1/2})$ term is bounded by an $O((m+n)^{-1/2})$ term depending only on α_1 and α_2.

I define sets ω_{α_0} and $\bar{\omega}_{\alpha_0}$ such that

$$\omega_{\alpha_0} = \{ w : |w|<\alpha_0,\ w=(j-1-(2mn/(m+n)))/(m+n)^{1/2},\ j \text{ is an integer} \}$$

$$\bar{\omega}_{\alpha_0} = \{ w : |w|\geq\alpha_0,\ w=(j-1-(2mn/(m+n)))/(m+n)^{1/2},\ j \text{ is an integer} \}$$

Suppose the number of runs in the sequence is r. The characteristic function of $(r-E(r))/(m+n)^{1/2}$ is of the form

$$\Big(\sum_{w\in\omega_{\alpha_0}} f_1(w)f_2(w)\text{Prob}(w\in\omega_{\alpha_0})/\sum_{w\in\omega_{\alpha_0}} f_2(w)\Big)+(E(f_1(w)\,|\,w\in\bar{\omega}_{\alpha_0})\text{Prob}(w\in\bar{\omega}_{\alpha_0})), \tag{32}$$

where $f_1(w)$ equals $\exp(itw)$ and $f_2(w)$ equals the expression in (31). For $(m+n)$ and α_0 sufficiently large, $\text{Prob}(w\in\bar{\omega}_{\alpha_0})$ is arbitrarily small. As the derivative with respect to of a normal density with variance $4((\xi(1-\xi))^2)$ is bounded it is possible, providing the inequalities in (29) hold, to construct uniform bounds on the modulus of the difference between the expression in (32) and the characteristic function for a $N(0,4(\xi(1-\xi))^2)$ distribution, which tend to zero as (m+n) increases.

Providing π_i is not zero, then as n increases $n_i/(n_i+m_i)$ tends in probability to F_i. So whenever F_i is between one and zero, the condition in (29) holds with arbitrarily high probability, for α_1 and α_2 arbitrarily close to F_i, for n sufficiently large. Also the value of $(n_i+m_i)/n$ tends in probability to π_i. The values of n_i and m_i will increase above any bound with arbitrarily high probability. Conditioning on S fixes the sign of the last response at all levels except the sink. It is clear that this restriction will not affect the asymptotic distribution of the expression in (26). Combining these results it follows if n is large enough that, outside of a set of S of arbitrarily low probability, the characteristic function of the expression in (26) is arbitrarily close to that of a $N(0,\pi_j 4((F_j(1-F_j))^2)$ distribution. From this and Formula (24) it follows that, for n large enough, the inner summation in expression (23) is, with arbitrarily high probability, arbitrarily close to that of the characteristic function of a $N(0,V_k)$ distribution, where

$$V_K = 4\sum_{|j|\leq K}\pi_j((F_j(1-F_j))^2)(x_j-m_K)^2. \tag{33}$$

The expression in (23) tends to the characteristic function of a

$N(0,U_K)$ distribution. It is easy to deduce that as n increases the characteristic function of $n^{1/2}B_K'$ tends to that of a $N(0,U_K-V_K)$ distribution.

I have already shown that for k large enough the expression in (18) is, with arbitrarily high probability, arbitrarily close to $n^{1/2}B_K'$ (see paragraph following formula (22)). So it follows that by choosing k and n large enough the characteristic function of the expression in (18) is arbitrarily close to that of a $N(0,U_K-V_K)$. I have also already shown that $\sum_j(\theta_j'+\gamma_j')/n$ tends in probability to $\tilde{m}$. So $n^{1/2}(E_{PV}'-m)$ has a characteristic function, for large enough k and n, which is arbitrarily close to that of a $N(0,(U_K-V_K)/\tilde{m}^2)$ distribution. As k increases $U_K/\tilde{m}^2$ and $V_K/\tilde{m}^2$ tend to U^* and V^*; so $n^{1/2}(E_{PV}'-m)$ has an asymptotic $N(0,U^*-V^*)$ distribution.

The differences between θ_j' and $n_j m_{j-1}/(n_j+m_j)$ and between and $n_{j+1}m_j/(n_j+m_j)$ are bounded and equal zero when a level has never been visited. Replacing θ_j' and γ_j' by $n_j m_{j-1}/(n_j+m_j)$ and $n_{j+1}m_j/(n_j+m_j)$ respectively will not alter the asymptotic distribution of the estimator. That is E_{PV}^* has the same asymptotic distribution as E_{PV}'. The total numbers of valleys and peaks differ by at most one; so the expected total numbers given S differ by at most one. It follows that E_{WE}^* also has the same asymptotic distribution as E_{PV}^* and E_{PV}'.

APPENDIX 5

A NOTE ON THE REGULARITY CONDITIONS REQUIRED FOR APPLICATION OF BILLINGSLEY'S THEOREM

Suppose that the Up and Down rule is operated and that the probabilities of moving up or down a step, given the current level is x, are $(1-F(x,\underset{\sim}{\theta}))$ and $F(x,\underset{\sim}{\theta})$ respectively where F is some known function and $\underset{\sim}{\theta}$ is some vector of parameters (i.e. the response curve takes the form $F(x,\underset{\sim}{\theta})$). Billingsley (1961) states a theorem which gives among other results the asymptotic distribution of the maximum likelihood estimator of $\underset{\sim}{\theta}$. This theorem is Theorem 2.2 on page 13 of his monograph. When this theorem holds the maximum likelihood estimator of $\underset{\sim}{\theta}$ will have asymptotic normality with an asymptotic variance covariance matrix similar to that when a non-sequential design is used but proportions of observations made at each level are replaced by equilibrium probabilities of being at a level.

Billingsley's results were for a time-discrete Markov process. The conditions he requires can be somewhat simplified for the Markov chain generated by use of the Up and Down rule. These conditions are then:

(A) For each possible $\underset{\sim}{\theta}$ there exists a unique equilibrium distribution for stimulus levels.

(B) The set of x for which $F(x,\underset{\sim}{\theta}) \in (0,1)$ does not depend on $\underset{\sim}{\theta}$.

(C) First, second and third partial derivatives of $F(x,\underset{\sim}{\theta})$ with respect to the $\underset{\sim}{\theta}$ parameters exist and are continuous for all $\underset{\sim}{\theta}$.

(D) For any possible $\underset{\sim}{\theta}$ there exist a neighbourhood N of $\underset{\sim}{\theta}$ for which

$$E_\theta(\sup_{\theta' \in N} |\partial^3 H(x,\underset{\sim}{\theta})/\partial\theta_i \partial\theta_j \partial\theta_k|) < \infty, \tag{1}$$

where E_θ denotes the expectation for the equilibrium distribution of x and $H(x,\underset{\sim}{\theta})$ is a random variable such that

$$H(x,\underset{\sim}{\theta}) = \log(F(x,\underset{\sim}{\theta})) \text{ with probability } F(x,\underset{\sim}{\theta}), \tag{2}$$

$$H(x,\underset{\sim}{\theta}) = \log(1-F(x,\underset{\sim}{\theta})) \text{ with probability } (1-F(x,\underset{\sim}{\theta})). \tag{3}$$

If $F(x,\underset{\sim}{\theta})$ equals 1.0 or 0.0 then $H(x,\underset{\sim}{\theta})$ is set equal to 0.0.

(E) The following inequality must be satisfied

$$E_\theta((\partial H(x,\underset{\sim}{\theta})/\partial\theta_i)^2) < \infty. \tag{4}$$

(F) There exists no linear combination v of the elements of $\underset{\sim}{\theta}$ such that the derivative of $F(x,\underset{\sim}{\theta})$ with respect to v is zero for all x (i.e. there is no redundancy in the parameterisation).

Conditions (B) to (F) are restatements of Billingsley's Condition 1.1.

From Theorem 1.3 on page 7 of Billingsley the first part of his Condition 1.2 holds if the Markov chain generated by use of the Up and Down rule is irreducible and each state is recurrent and non null. The existence of a unique equilibrium distribution ensures that this is the case (see Condition (A)). Billingsley notes that the second part of Condition 1.2 can be replaced by what I call Condition (E).

It is relatively easy to check whether Conditions (A),(B),(C) and (F) hold but it is more difficult to check (D) and (E). Tsutakawa (1967b) shows that providing the response curve is truncated so that there are only finitely many possible levels then only Conditions (A),(B),(C) and (F) are needed (Condition (E) is automatically satisfied) and (C) can be relaxed in that existence of third order partial derivatives is not required.

For logistic and normal tolerance distributions the response curve can be written in the form $F(\beta(z-\mu))$. It is easy to see that conditions (A),(B),(C) and (F) are satified. Condition (E) can be restated as the following:

$$E_\theta(z^2 w(z)) < \infty, \qquad (5)$$

$$E_\theta(|zw(z)|) < \infty, \qquad (6)$$

$$E_\theta(w(z)) < \infty, \qquad (7)$$

where w(z) are the logit or probit weights associated with observations at z. It is easy to show that then inequalities hold as these weights are bounded. One can use similar arguments to show that Condition (D) holds. In the proofs one has to show that there are bounds on

$$(dF(x)/dx)^{3/2}/(1-F(x)), \qquad (8)$$

$$\text{and } (dF(x)/dx)^{3/2}/F(x). \qquad (9)$$

For the logistic response curve the expressions in (8) and (9) equal

$$F(x)^{3/2}(1-F(x))^{1/2}, \qquad (10)$$

$$F(x)^{1/2}(1-F(x))^{3/2}, \qquad (11)$$

and it is immediately obvious that they are bounded. For the probit response curve dF(x)/dx equals $\exp(-x^2/2)/(2\pi)^{1/2}$ and clearly as x decreases the expression in (8) tends to zero. From a result in Abromowitz and Stegun (1965) it follows that

$$1-F(x) = \exp(-x^2/2)(1+(\theta/x^2))/((2\pi)^{1/2}x) \quad \text{for some } |\theta| < 1.0. \qquad (12)$$

Using this result one can then show that the expression in (8) tends to zero as x increases. The expression in (8) is continuous so it follows that it is bounded. A similar argument shows that the expression in (9) is bounded.

APPENDIX 6

NOTE ON SOME APPROXIMATIONS TO THE ASYMPTOTIC EXPECTATIONS OF ESTIMATORS FOR THE UP AND DOWN PROCEDURE

Suppose that a response curve is of the form G(x) where

$$G(x) = F(\beta(x-\mu)) \quad \text{and } \beta > 0.0. \tag{1}$$

F is some known function having limits, as its argument increases and decreases, above and below 0.5 respectively. F only takes values between 0.0 and 1.0 for one set of consecutive stimulus levels. Suppose further that

$$F(0.0) = 0.5, \tag{2}$$

$$dF(z)/dz\big|_{z=0.0} = k_1 \quad \text{where } k_1 > 0.0, \tag{3}$$

$$d^2F(z)/dz^2\big|_{z=0.0} = k_2. \tag{4}$$

If the Up and Down rule is operated with step size d one can assume, without loss of generality, that the possible stimulus levels, $\{x_i\}$, are given by

$$x_i = \mu + (i+\theta)d \quad \text{for some } \theta \in [0.0, 1.0). \tag{5}$$

The sequence of levels visited can be viewed as a Markov chain. Suppose the equilibrium probability of being at x_i is π_i. The equations to be solved to find the π_i are of the form

$$\pi_i \, G(x_i) = \pi_{i-1} \, (1-G(x_{i-1})). \qquad (6)$$

These equations have a solution

$$\pi_i = \pi_0 \prod_{j=1}^{i} (1-G(x_{j-1}))/G(x_j) \qquad i > 0, \qquad (7)$$

$$\pi_i = \pi_0 \prod_{j=i}^{-1} G(x_{j+1})/(1-G(x_j)) \qquad i < 0, \qquad (8)$$

where π_0 can take any value ($\pi_i = 0.0$ if and only if $G(x_{i+1})$ or $(1-G(x_{i-1}) = 0.0)$. I have assumed that $G(x_1) \neq 0.0$ and $G(x_{-1}) \neq 1.0$; from the continuity of G at μ this is bound to be the case for d sufficiently small. Providing $\sum_j \pi_j$ is convergent, the π_i can be normalised so that they sum to 1.0. These normalised π_i are the unique equilibrium probabilities for the process. This sum is convergent because the conditions on the limits of F ensure that, for some $\xi \in (0.0, 1.0)$ and i sufficiently large,

$$(1-G(x_{i-1}))/G(x_i) < \xi, \qquad (9)$$

$$G(x_{-i+1}) \; (1-G(x_{-i}) < \xi. \qquad (10)$$

Clearly from (9) and (10) it follows that the π_i can be dominated for $|i|$ sufficiently large by terms decreasing exponentially in $|i|$.

In general there will be no explicit expression for the products on the right of (7) and (8). In the following theorem I derive an expression which allows one, when the value of d is small, to make an approximation to these products

Theorem 1

If $|i| \leq c/d^{1/2}$ for some $c > 0.0$, and providing d is sufficiently small then

$$\pi_i/\pi_c = \exp(-(4k_1\beta(x_i-\mu)^2/2d) + \varepsilon_i), \qquad (11)$$

where

$$\varepsilon_i / d^{1/2} < K_c, \qquad (12)$$

and K_c is a constant depending upon c. That is for small d the equilibrium probabilities are roughly proportional to the density for a normal distribution with mean μ and variance $d/4k_1\beta$.

Proof

For convenience I will define a function H(z) which equals $\log((1-G(z))/G(z))$.

If $\pi_i \neq 0.0$ and $i > 0$, then from (7) it follows that

$$\log(\pi_i/\pi_c) = \sum_{j=c}^{i-1} H(x_j) + \log(G(x_i)/G(x_c)). \qquad (13)$$

The first and second derivatives of H are given by

$$dH(z)/dz = -dG(z)/dz/((1-G(z))G(z)), \tag{14}$$

$$d^2H(z)/dz^2 = -d^2G(z)/dz^2/((1-G(z))G(z)) + (1-2G(z))(dH(z)/dz)^2 \tag{15}$$

From (1),(2),(3) and (14) it follows that the derivative at μ of H is $-4k_1\beta$. From (1),(2),(4) and (15) it follows that the second derivative of H at μ is $-4k_2\beta^2$. Making first order Taylor series expansions of all the $H(x_j)$ terms in (13) about μ one obtains the following

$$\log(\pi_i/\pi_0) = -4k_1\beta\sum_{j=0}^{i-1}(x_j-\mu) + \sum_{j=0}^{i-1}(x_j-\mu)^2(v_j/2) + \log(G(x_i)/G(x_0)) \tag{16}$$

where v_j equals the value of the second derivative of H for some stimulus level between x_j and μ. Here one assumes that the second derivative of H exists for all levels between x_{i-1} and μ. For convenience I will call the first, second and third terms on the right of (16), A_1, A_2 and A_3 respectively. From (5)

$$A_1 = -4k_1\beta\sum_{j=0}^{i-1}(j+\theta)d, \tag{17}$$

which simplifies to

$$A_1 = -4k_1\beta((x_i-\mu)^2/2d) + 2k_1\beta(i+\theta^2)d. \tag{18}$$

Suppose $0 < i \leqslant c/d^{1/2}$, then for d sufficiently small the second term in (18) is less than $3k_1\beta cd^{1/2}$. From (5)

$$0.0 < (x_i-\mu) \leqslant cd^{1/2} + \theta d. \tag{19}$$

From continuity of G and the second derivative of H at μ it follows that, for sufficiently small d, values of $G(x_{i-1})$ and $G(x_{i+1})$ are arbitrarily close to 0.5 and values of $d^2H(z)/dz^2$ for z in (μ, x_i) are arbitrarily close to $-4k_2\beta^2$. So in particular for sufficiently small d

$$0.25 < G(x_{i-1}) \text{ and } G(x_{i+1}) < 0.75, \tag{20}$$

and

$$\sup_{z\in(\mu,x_i)} |d^2H(z)/dz^2| < 8|k_2|\beta^2. \tag{21}$$

From (20) it follows that $\pi_i > 0.0$, and from (21) it follows that

$$|A_2| \leqslant 4|k_2|\beta^2 \sum_{j=0}^{m-1} (j+\theta)^2 d^2, \tag{22}$$

where m is the integer part of $c/d^{1/2}$; the expression in (22) equals

$$4|k_2|\beta^2((m(m-1)(2m-1)/6) + \theta m(m-1) + \theta^2 m)d\ . \tag{23}$$

This in turn is less than $4|k_2|\beta^2((c^3 d^{1/2}/3)+\theta c^2 d+\theta^2 c d^{3/2})$. For d sufficiently small this is less than $2|k_2|\beta^2 c^3 d^{1/2}$; this provides a bound for $|A_2|$. From making a Taylor series expansion

$$A_3 = dG(z)/dz/G(z)\Big|_{z=z_0}(x_0-\mu) - dG(z)/dz/G(z)\Big|_{z=z_i}(x_i-\mu), \tag{24}$$

for some z_0 and z_i in (μ, x_0) and (μ, x_i) respectively (providing d is sufficiently small for the derivative of logG(z) to exist in (μ, x_i)). For d sufficiently small both the dG(z)/dz/G(z) terms are can be made arbitrarily close to $2k_1\beta$. The $(x_i-\mu)$ and $(x_0-\mu)$ terms are bounded by $(cd^{1/2} + \theta d)$. It is easy to show from (24) that, for sufficiently small d, $|A_3|$ is less than $5k_1\beta c d^{1/2}$. Combining these results it follows that for sufficiently small d

$$|\log(\pi_i/\pi_0) + 4k_1\beta((x_i-\mu)^2/2d)| < d^{1/2}(8k_1\beta c + 2|k_2|\beta^2 c^3). \quad (25)$$

It is a trivial matter to show that, for sufficiently small d, (25) holds for i=0. One can show using similar arguments that (25) holds for sufficiently small d if $0 > i \geq -c/d^{1/2}$. The theorem immediately follows from this inequality.

When the response curve is logistic one can find an explicit expression for π_i/π_0. There is some simplification because $H(z)$ equals $-\beta(z-\mu)$. The second term in (16) is 0.0. The value of k is 0.25. From (16) and (18) (i.e. the expression for the first term on the right in (16)), it follows that

$$\log(\pi_i G(x_i)/\pi_0 G(x_0)) = -\beta((x_i-\mu) - (x_i-\mu)d + (\theta-\theta^2)d^2)/2d. (26)$$

From this it follows, after some simplification, that

$$\pi_i \propto (\exp(-\beta(x_i-\mu-(d/2))^2/2d) + \exp(-\beta(x_i-\mu+(d/2))^2/2d)). \quad (27)$$

The first term on the right in (27) corresponds to the contribution to π_i from the positive responses, the second term is the contribution from negative responses.

The estimators E_M , E_β , $E_{\beta D}$, and E_{DM} , described in Section 2.1, all have asymptotic expectation

$$\sum \pi_i x_i . \tag{28}$$

If the response curve is such that $F(z) = (1-F(-z))$ (as are the logistic and normal response curves) and the stimulus levels are symmetrically placed, then the asymptotic bias of these estimators is 0.0. However in general there is some bias. In the following theorem I show, as d tends to 0.0, this bias tends to 0.0 faster than $O(d^{1/2})$.

Theorem 2.

Suppose that F satisfies the same conditions as before. Suppose that F also satisfies the following:

(a) F(z) takes values above and below 0.5 according to whether z is positive or negative.

(b) There exists some $\varepsilon > 0.0$ such that if $|z| < \varepsilon$, then $F(z) = F(y)$ implies $z = y$.

A consequence of (a) is that μ is the unique ED50 for G. F is continuous in $[-\delta', \delta']$ for sufficiently small δ', so one can assume without loss of generality that F is continuous for $|z| < \varepsilon$ (simply replace ε by $\inf.(\varepsilon, \delta')$).

With these conditions it follows that

$$\underset{d \to 0 \cdot 0}{\text{limit}} (\sum \pi_i (x_i - \mu)/d^{1/2}) = 0.0. \tag{29}$$

Proof

First I will show that

$$\underset{d \to 0 \cdot c}{\text{limit}} \sum_i \pi_i (x_i - \mu)/\pi_o = 0.0. \tag{30}$$

Consider

$$\sum_{|i| \leq m} \pi_i (x_i - \mu)/\pi_c , \tag{31}$$

where m is the integer part of $c/d^{1/2}$, for some $c > 0.0$. From Theorem 1 the expression in (31) equals

$$\sum_{|i| \leq m} z_i d^{1/2} \exp(-2k_1 \beta z_i^2 + \varepsilon_i), \tag{32}$$

where z_i equals $(x_i - \mu)/d^{1/2}$ and $\varepsilon_i/d^{1/2}$ is bounded. This is a step function approximation to the integral

$$\int_{-c}^{c} z \exp(-2k_1 \beta z^2) \, dz. \tag{33}$$

The range of integration in (33) is finite and the

$z_i \exp(-2k_1\beta z_i^2 + \varepsilon_i)$ terms in (32) are bounded. It is a simple matter to apply Lebesgue's dominated convergence theorem to show that the summation in (32) tends, as d tends to 0.0, to the integral in (33) (for a statement of this theorem see Bartle (1966), page 44). This integral equals 0.0, so it follows that the expression in (31) tends to 0.0 as d tends to 0.0.

Consider

$$\sum_{i>m} \pi_i (x_i - \mu)/\pi_0 . \qquad (34)$$

Let $\delta = (1-G(x_m))/G(x_{m+1})$; from (a) $\delta < 1.0$. For d sufficiently small $\beta(x_{m+1}-\mu) < \varepsilon$. From (b) (where ε is chosen sufficiently small so that F(z) is continuous for $|z| < \varepsilon$) it follows that F must be monotonic increasing between 0.0 and $\beta(x_{m+1}-\mu)$. Also from (b) it follows that $G(x_{i+1})$ can take no value between 0.5 and $G(x_{m+1})$ for $i > m$, and from (a) it follows $G(x_{i+1}) > 0.5$. So it follows that $G(x_{i+1}) > G(x_{m+1})$, similarly $G(x_i) > G(x_m)$. From these results it follows that

$$(1-G(x_i)/G(x_{i+1}) < \delta \quad \text{for } i > m. \qquad (35)$$

From (6) it follows that

$$\pi_i \leqslant \pi_m \delta^{i-m} \quad \text{for } i \geqslant m. \qquad (36)$$

So the expression in (34) is bounded by

$$\sum_{i>m} \pi_m \delta^{i-m}(m+i+\theta)d/\pi_c, \quad (37)$$

which equals

$$\pi_m d(((1-\delta)(m+\theta)) + 1)\delta/((1-\delta)^2\pi_c). \quad (38)$$

For d sufficiently small it follows, from making a first order Taylor expansion of δ, that

$$\delta = 1 - 4k_1\beta(m+\theta)d + O(d). \quad (39)$$

From this result and Theorem 1 it follows that, as d tends to 0.0, the expression in (38) tends to

$$\exp(-2k_1\beta c^2)((4k_1\beta)^{-1} + (4k_1\beta c)^{-2}). \quad (40)$$

For c sufficiently large this expression is arbitrarily small, and so $\sum_{i>m}\pi_i(x_i-\mu)/\pi_c$ is also arbitrarily small (as the expression in (38) bounds that in (34)). By similar arguments it follows that $\sum_{i<-m}\pi_i(x_i-\mu)/\pi_0$ is also arbitrarily small for c sufficiently large. I have already shown that $\sum_{|i|\leq m}\pi_i(x -\mu)/\pi_c$ tends to 0.0 as d tends to 0.0 for any c. An immediate consequence of these results is that (30) is true.

One can use similar arguments to show that

$$\lim_{d\to 0.0} \sum \pi_i d^{1/2}/\pi_c = (\pi/2k_1\beta)^{1/2}, \quad (41)$$

$\sum_i \pi_i d^{1/2}/\pi_0$ (i.e. $d^{1/2}/\pi_0$) is a step function approximation to $\int \exp(-2k_i \beta z^2)\,dz$. Theorem 2 follows from (30) and (41).

When the response curve is logistic one can show that the asymptotic bias of the estimators tends at a faster rate to 0.0 as d decreases.

Theorem 3

For the logistic response curve

$$\lim_{d \to 0.0} \left(\sum_i \pi_i (x_i - \mu)/d\right) = 0.0. \qquad (42)$$

That is $d^{1/2}$ in (29) is replaced by d.

Proof

For convenience I will define a function h where

$$h(z) = \exp(-\beta(z-\mu-(d/2))^2/2d) + \exp(-\beta(z-\mu+(d/2))^2/2d). \qquad (43)$$

From (27) it follows that π_i is proportional to $h(x_i)$. Consider

$$\sum h(x_i)(x_i-\mu)/d^{1/2}. \qquad (44)$$

Using the identities $h(-z+2\mu) = h(z)$ and $-x_{-i+1}+2\mu = x_i-((1+2\theta)d)$, and rearranging terms it follows that this expression equals

$$\sum_{i=1}^{\infty}((h(x_i)(x_i-\mu) - h(x_i-\lambda d)(x_i-\lambda d-\mu))/d)d^{1/2}, \tag{45}$$

where $\lambda = (1+2\theta)$. This in turn equals

$$\sum_{i=1}^{\infty}(\lambda d(h(z)z)/dz\Big|_{z=z_i})d^{1/2}, \tag{46}$$

for some z_i between x_i and $x_i-\lambda d$. This is a step function approximation to the integral

$$\int_0^{\infty} 2\lambda(1-\beta z^2)\exp(-\beta z^2/2)\ dz, \tag{47}$$

which equals 0.0. A function taking the value $4\beta z^2\exp(-\beta z^2/4)$ for $z^2 > 1/\beta$, and 2.0 otherwise can be used in the dominated convergence theorem to show that, as d tends to 0.0, the limit of (44) is 0.0. From Theorem 1 it follows that the constant of proportionality between π_i/π_0 and $h(x_i)$ tends to 0.5 as d tends to 0.0, and so

$$\lim_{d\to 0.0} \sum \pi_i(x_i-\mu)/d^{1/2}\pi_0 = 0.0. \tag{48}$$

From (48) and (41) Theorem 3 immediately follows.

The value of all the $G(x_i)$ depends on β and d only through βd, so the bias of the estimators as a proportion of d only depends on

βd. Suppose $B(\beta,d)$ is the bias for slope β and step size d, then

$$B(\beta_1, d)\beta_1 = B(\beta_2, \beta_1 d/\beta_2)\beta_2. \qquad (49)$$

From (49) the following corollary to Theorems 2 and 3 follows.

Corollary 1

Under the conditions for Theorem 1

$$\lim_{\beta \to 0.0} (B(\beta,d)\beta^{1/2}) = 0.0. \qquad (50)$$

For the logistic response curve

$$\lim_{\beta \to 0.0} (B(\beta,d)) = 0.0. \qquad (51)$$

In Section 2.1 further estimators, E_{NE}, E_{PV}, E_{WE}^* and E_{PV}^*, of μ are described. They have asymptotic bias

$$\sum \pi_i w(x_i)(x_i-\mu)/\sum \pi_i w(x_i), \qquad (52)$$

where $w(x_i) = ((G(x_i))^2 + (1-G(x_i))^2)$.

Corollary 2

The results in Theorems 2 and 3, and in Corollary 1 apply equally well to the expression in (52). There are bounds on $w(x_i)$, also $w(x_i)$ tends to a non-zero limit as x_i tends to μ; this is enough to ensure that Theorem 1 still holds. Theorem 2 still holds; in the proof, terms involving h(z) in (44),(45) and (46) must be replaced by h(z)w(z). Both w(z) and its derivative are bounded and there is no difficulty in again applying the dominated convergence theorem. Corollary 1 again holds for the same reasons as before.

APPENDIX 7

NOTE ON THE LIMIT WITH PROBABILITY ONE OF $1/\tilde{\beta}$

In Section 3.2 I suggested an estimator of $1/\beta$, that I term $1/\tilde{\beta}$, for use when the response curve is logistic (see Formulae 3.2.2, 3.2.3 and 3.2.7 and paragraphs preceding these expressions). This estimator equals the variance of levels of the less frequent response type divided by step size (see Section 3.2). Suppose the response curve is of the form G(x); where $G(x)=F(\beta(x-\mu))$ for some known function F ($\beta > 0.0$), and F satisfies all the conditions in Theorem 2 of Appendix 6. From a theorem on page 87 of Chung (1960) one can deduce that (using the notation of Appendix 6) as the number of observations increases this estimator converges with probability one to

$$((\sum_i \pi_i G(x_i)x_i^2 / \sum_i \pi_i G(x_i)) - (\sum_i \pi_i G(x_i)x_i / \sum_i \pi_i G(x_i))^2)/d. \qquad (1)$$

I will assume, without loss of generality that possible stimulus levels, $\{x_i\}$, are given by

$$x_i = \mu + (i+\theta)d \quad \text{for some } \theta \in [0.0, 1.0). \qquad (2)$$

Theorem

With these conditions the expression in (1) tends to $1/4k_1\beta$ as d tends to 0.0, where k_1 is the derivative of F at 0.0 (by assumption $k_1 > 0.0$). For the logistic response curve $k_1 = 0.25$

and so this limit is $1/\beta$.

Proof

The function G is bounded having a limit of 0.5 as its argument tends to μ. Following similar arguments to those used in Theorem 2 of Appendix 6 one can show that

$$\lim_{d \to 0.0} \sum_i \pi_i G(x_i)(x_i - \mu)/\pi_c = 0.0, \tag{3}$$

$$\lim_{d \to 0.0} \sum_i \pi_i 2G(x_i) d^{1/2}/\pi_c = (\pi/2k_1\beta)^{1/2}. \tag{4}$$

From (3) and (4) it follows that

$$\lim_{d \to 0.0} (\Sigma \pi_i G(x_i)(x_i - \mu)/\Sigma \pi_i G(x_i))^2/d = 0.0. \tag{5}$$

So the limit of the expression in (1) as d tends to 0.0 will equal the limit as d tends to 0.0 of

$$\sum_i \pi_i G(x_i)(x_i - \mu)^2 / (\sum_i \pi_i G(x_i) d), \tag{6}$$

providing such limits exists. From (4) this equals

$$\lim_{d \to 0.0} (2k_1\beta/\pi)^{1/2} \sum_i \pi_i 2G(x_i)(x_i - \mu)^2/d^{1/2}\pi_0. \tag{7}$$

Consider

$$\sum_{1 \le i \le m} \pi_i 2G(x_i)(x_i - \mu)^2/d^{1/2}\pi_0, \tag{8}$$

where m is the integer part of $c/d^{1/2}$, for some $c > 0.0$. From Theorem 1 of Appendix 6 this equals

$$\sum_{|i| \le m} z_i^2 \, 2G(x_i) \exp(-2k_1 \beta z_i^2 + \varepsilon_i) d^{1/2}, \tag{9}$$

where $z_i = (x_i - \mu)/d^{1/2}$ and $\varepsilon_i/d^{1/2}$ is bounded. For d sufficiently small, the $2G(x_i)$ and ε_i terms in (9) are arbitrarily close to 1.0 and 0.0 respectively. The summation in (9) is a step function approximation to the integral

$$\int_{-c}^{c} z^2 \exp(-2k_1 \beta z^2) \, dz. \tag{10}$$

The range of integration is finite and the $z_i^2 2G(x_i) \exp(-2k_1 \beta z_i^2 + \varepsilon_i)$ are bounded. It is easy to apply Lebesgue's dominated convergence theorem (see Bartle (1966), page 44). As d tends to 0.0 the expression in (9) tends to the integral in (10). This integral in turn is arbitrarily close to $(2\pi)^{1/2}/(4k_1\beta)^{3/2}$ for c sufficiently large (i.e. it is arbitrarily close to the integral from $-\infty$ to ∞).

Consider

$$\sum_{i > m} \pi_i \, 2G(x_i)(x_i - \mu) / d^{1/2} \pi_0. \tag{11}$$

$G(x_i)$ is bounded by 1.0, so the expression in (11) is bounded by

$$\sum_{i > m} \pi_i \, 2(x_i - \mu)^2 / d^{1/2} \pi_0. \tag{12}$$

Let $\delta = (1-G(x_m))/G(x_{m+1})$. From Condition (a) of Appendix 6, $\delta < 1.0$. From the argument preceding (36) of Appendix 6 it follows, for sufficiently small d, that

$$\pi_i \leqslant \delta^{i-m}\pi_m \text{ for } i \geqslant m. \tag{13}$$

From (13) it follows that the expression in (11) is bounded by

$$\sum_{i>m} \delta^{i-m} 2(m+i+\theta)^2 d^{3/2}/\pi_c. \tag{14}$$

This expression equals

$$\pi_m((2\delta^2/(1-\delta)^3)+(2(m+\theta)+1)\delta/(1-\delta)^2)+((m+\theta)^2\delta/(1-\delta)))d^{3/2}/\pi_0. \tag{15}$$

From a first order Taylor series expansion, for d sufficiently small

$$\delta = 1 - 4k_1\beta(x_i-\mu)d + O(d). \tag{16}$$

From this result and Theorem 1 of Appendix 6 it follows that, as d tends to 0.0, the expression in (15) tends to

$$\exp(-2k_1\beta c^2)((2/(4k_1\beta c)^3)+(2/(4k_1\beta)^2 c)+(c/4k_1\beta)). \tag{17}$$

For c sufficiently large this is arbitrarily small. So for c sufficiently large, $\sum_{i>m} \pi_i 2G(x_i)(x_i-\mu)^2/d^{1/2}\pi_0$ is arbitrarily small (as the expression in (15) bounds that in (11)). By similar arguments one can show that $\sum_{i<-m} \pi_i 2G(x_i)(x_i-\mu)^2/d^{1/2}\pi_0$ is arbitrarily small for c large enough. I have already shown that $\sum_{|i|\leq m} \pi_i 2G(x_i)(x_i-\mu)^2/d^{1/2}\pi_0$ tends to the integral in (10) as d tends to 0.0, which is arbitarily close to $(2\pi)^{1/2}/(4k_1\beta)^{3/2}$ for c sufficiently large. From these results it follows that

$$\lim_{d \to 0.0} \sum \pi_i 2G(x_i)(x_i-\mu)^2/d^{1/2}\pi_0 = (2\pi)^{1/2}/(4k_1\beta)^{3/2}. \qquad (18)$$

So the limit in (7) equals $1/4k_1\beta$, but this is also the limit as d tends to 0.0 of the expression in (1) and so the theorem is proved.

APPENDIX 8

ASYMPTOTIC PROPERTIES OF THE ROBBINS-MONRO PROCEDURE OPERATING ON A TRANSFORMED RESPONSE CURVE

Suppose that a response curve takes the form F(x), where x is the stimulus level and F is a strictly increasing continuous function taking values in (0.0,1.0) with a non zero derivative, $g_{1/\sqrt{2}}$, at $L_{1/\sqrt{2}}$ $(F(L_{1/\sqrt{2}})=2^{-1/2})$. Suppose further that a Robbins-Monro procedure is operated, but moves down are made after two positive responses and moves up are made after either a negative response or a positive followed by a negative response. The sequence of levels visited $\{y_t\}$ are related by the equation

$$y_{t+1} = y_t - c(z_t-0.5)/t, \qquad (1)$$

where z_t equals 1.0 with probability $F(y_t)^2$ and 0.0 with probability $(1-(F(y_t))^2)$. The sequence of levels visited can be viewed as a Robbins-Monro process operating on the transformed response curve $F(x)^2$. From standard results (see Section 1.4) y_t tends in mean square to $L_{1/\sqrt{2}}$ and providing $c>1/(8^{1/2}g_{1/\sqrt{2}})$ (i.e. is greater than half the inverse of the slope of the response curve $F(x)^2$ at $L_{1/\sqrt{2}}$) then y_t has an asymptotic normal distribution with mean $L_{1/\sqrt{2}}$ and variance

$$c^2/4(8^{1/2}g_{1/\sqrt{2}}c-1)t). \qquad (2)$$

The number of observations made per visit to a level is not fixed. If u_t is the number of observations made at y_t then u_t equals 2 with probability $F(y_t)$ and 1 with probability $(1-F(y_t))$ (two observations are made if and only if the first response at the level is positive). After T levels have been visited the ratio of the number of observations to the number of levels visited is $\sum_{t=1}^{T} u_t/T$. The expectation of u_t given y_t is

$$2F(y_t) + (1-F(y_t)). \tag{3}$$

So the expectation of $\sum_{t=1}^{T} u_t/T$ is

$$1+(\sum_{t=1}^{T} E(F(y_t))/T). \tag{4}$$

From the definition of u_t it follows that the conditional expectation of u_t^2 given y_t is

$$4F(y_t) + (1-F(y_t)). \tag{5}$$

The expectations of the expressions in (3) and (5) are the unconditional expectations of u_t and u_t^2. It follows that the variance of u_t is

$$E(F(y_t))(1-E(F(y_t))). \tag{6}$$

As t increases, y_t tends in mean square to $L_{\frac{1}{2}}$. The function F is bounded and continuous, so $E(F(y_t))$ tends to $2^{-1/2}$ as t increases. From this it follows that, as t increases, the expression in (4) (i.e. the expectation of $\sum_{t=1}^{T} u_t/T$) tends to $(1+2^{-1/2})$ and that the expression in (6) tends to $2^{-1/2}(1-2^{-1/2})$. The covariance of u_i and u_j is the expectation of

$$(u_i-1-E(F(y_i)))(u_j-1-E(F(y_j))). \tag{7}$$

Suppose i>j; then the conditional expectation of the expression in (7) given y_i, y_j and u_j is

$$(F(y_i)-E(F(y_i)))(u_j-1-E(F(y_j))). \tag{8}$$

The $(u_j-1-E(F(y_j)))$ term is bounded in modulus by 1 and so the covariance between u_i and u_j (which is the expectation of the expression in (8)) is bounded in modulus by

$$E|F(y_i)-E(F(y_i))|. \tag{9}$$

The term in the expectation in (9) tends, as i increases, in probability to 0 and is bounded by 1. So it follows that the expectation of this quantity tends to 0 as i increases. From this it follows that the covariance between u_i and u_j tends, as i increases, to 0 uniformly for any j<i. As in addition the variances of the u_t are bounded it follows that the variance of $\sum_{t=1}^{T} u_t/T$ tends to 0 as T increases. The expectation of $\sum_{t=1}^{T} u_t/T$ tends to $1+2^{-1/2}$ so the limit in probability of $\sum_{t=1}^{T} u_t/T$ is also $1+2^{-1/2}$. If n is the number of observations and T levels have been visited then $(\sum_{t=1}^{T} u_T - n)$ equals 1

or 0, and so n/T also tends in probability to $1+2^{-1/2}$. So it follows from (2) that, providing $c>1/(8^{1/2}g_{1/\sqrt{2}})$, y_t has an asymptotic normal distribution with mean $L_{1/\sqrt{2}}$ and variance

$$(1+2^{-1/2})c^2/4((8^{1/2}g_{1/\sqrt{2}}c-1)n). \quad (10)$$

If the Robbins-Monro procedure for estimating $L_{1/\sqrt{2}}$ on the untransformed curve is used, providing $c>1/2g_{1/\sqrt{2}}$, the estimator has asymptotic normality with mean $L_{1/\sqrt{2}}$ and variance

$$c^2(1-2^{-1/2})2^{-1/2}/(2g_{1/\sqrt{2}}c-1)n. \quad (11)$$

If c values of $k/(2^{1/2}g_{1/\sqrt{2}})$ and $k/g_{1/\sqrt{2}}$ are substituted into expressions (10) and (11) respectively (k>0.5) then the expression in (11) divided by that in (10) is

$$(1-2^{-1/2})2^{-1/2}8/(1+2^{-1/2})^2, \quad (12)$$

which equals 0.9706 to four decimals. So there is little loss in asymptotic efficiency in operating the Robbins-Monro procedure on the transformed curve. One can adapt the procedure operating on the transformed curve to obtain an estimate of a general L_ρ by substituting p^2 for 0.5 in (1). Using similar arguments to those for when p equalled $2^{-1/2}$ it follows that for the same multiples of optimal c values for the Robbins-Monro procedure on the transformed and untransformed curves the ratio of asymptotic variance expressions is

$$4p/(1+p)^2. \qquad (13)$$

So for $p>0.5$ the efficiency of the procedure on the transformed curve relative to that on the untransformed curve is greater than 8/9.

APPENDIX 9 TABLES TO ACCOMPANY SECTION 3.3

Table 1 100×m.s.e. of estimators in 48 step experiments for step size 0.5 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E^*_{WE}	E^*_{PV}	A^*_{WE}
0.00	2.72	2.77	2.77	2.78	3.14	2.93	2.97	3.33	2.77	2.79	3.15
0.25	2.91	2.97	2.97	2.99	3.14	3.13	3.16	3.33	2.95	2.96	3.15
0.50	2.90	2.90	2.94	3.00	3.14	3.15	3.18	3.33	2.93	2.94	3.15
0.75	3.06	2.99	3.07	3.16	3.14	3.29	3.31	3.33	3.07	3.06	3.15
1.00	3.14	2.98	3.06	3.18	3.14	3.33	3.34	3.33	3.10	3.08	3.15
1.25	3.45	3.17	3.26	3.37	3.14	3.55	3.54	3.33	3.29	3.26	3.15
1.50	3.69	3.24	3.27	3.36	3.14	3.63	3.61	3.33	3.33	3.29	3.15
1.75	4.24	3.56	3.48	3.53	3.14	3.81	3.79	3.33	3.56	3.52	3.15
2.00	4.75	3.77	3.50	3.49	3.14	3.83	3.81	3.33	3.54	3.49	3.15
2.25	5.61	4.33	3.80	3.73	3.14	4.11	4.07	3.33	3.79	3.74	3.15
2.50	6.49	4.80	3.81	3.71	3.14	4.18	4.15	3.33	3.80	3.74	3.15
2.75	7.89	5.67	3.97	3.84	3.14	4.31	4.28	3.33	3.95	3.89	3.15
3.00	9.30	6.49	3.85	3.69	3.14	4.17	4.13	3.33	3.84	3.78	3.15
3.25	11.17	7.73	4.03	3.85	3.14	4.35	4.30	3.33	3.97	3.92	3.15
3.50	13.34	9.13	4.02	3.84	3.14	4.42	4.38	3.33	3.96	3.91	3.15
3.75	16.25	11.12	4.08	3.95	3.14	4.46	4.42	3.33	4.08	4.02	3.15
4.00	19.52	13.35	3.94	3.81	3.14	4.29	4.25	3.33	3.95	3.89	3.15

Table 2 100×m.s.e. of estimators in 48 step experiments for step size 1.0 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E^*_{WE}	E^*_{PV}	A^*_{WE}
0.00	3.44	3.49	3.46	3.47	3.80	3.76	3.87	4.08	3.63	3.66	3.94
0.25	3.58	3.62	3.60	3.63	3.80	3.78	3.88	3.98	3.67	3.70	3.81
0.50	3.45	3.51	3.51	3.52	3.80	3.60	3.67	3.87	3.42	3.45	3.69
0.75	3.51	3.58	3.60	3.62	3.80	3.84	3.89	3.98	3.58	3.61	3.81
1.00	3.59	3.59	3.64	3.72	3.80	4.05	4.08	4.08	3.72	3.74	3.94
1.25	3.87	3.79	3.88	3.99	3.80	4.19	4.20	3.98	3.88	3.88	3.81
1.50	3.66	3.48	3.59	3.71	3.80	3.78	3.80	3.87	3.57	3.54	3.69
1.75	3.90	3.61	3.75	3.85	3.80	4.02	4.04	3.98	3.82	3.77	3.81
2.00	4.01	3.56	3.67	3.77	3.80	4.03	4.06	4.08	3.87	3.81	3.94
2.25	4.26	3.69	3.80	3.88	3.80	4.06	4.08	3.98	3.91	3.87	3.81
2.50	4.51	3.77	3.84	3.89	3.80	4.04	4.05	3.87	3.81	3.77	3.69
2.75	4.91	3.94	3.93	3.92	3.80	4.30	4.28	3.98	3.95	3.93	3.81
3.00	5.37	4.09	3.98	4.00	3.80	4.52	4.50	4.08	4.07	4.05	3.94
3.25	6.23	4.56	4.26	4.30	3.80	4.68	4.65	3.98	4.26	4.22	3.81
3.50	6.59	4.51	3.96	4.02	3.80	4.21	4.21	3.87	3.94	3.88	3.69
3.75	7.46	4.93	4.04	4.11	3.80	4.34	4.34	3.98	4.13	4.06	3.81
4.00	8.46	5.39	3.96	3.99	3.80	4.39	4.39	4.08	4.16	4.08	3.94

Note: A_{DM}, A_{WE} and A^*_{WE} denote columns for asymptotic predicted m.s.e.'s of E_{DM}, E_{WE} and E^*_{WE} respectively.

Table 3 100×m.s.e. of estimators in 48 step experiments for step size 1.5 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E^*_{WE}	E^*_{PV}	A^*_{WE}
0.00	4.40	4.48	4.46	4.41	4.58	5.37	5.55	5.61	5.25	5.28	5.55
0.25	4.51	4.58	4.55	4.54	4.56	5.10	5.28	5.19	5.03	5.07	5.09
0.50	4.32	4.40	4.37	4.36	4.50	4.14	4.30	4.30	4.10	4.15	4.13
0.75	4.32	4.40	4.39	4.39	4.46	3.88	3.97	3.84	3.72	3.77	3.64
1.00	4.29	4.36	4.38	4.39	4.50	4.35	4.37	4.30	4.05	4.09	4.13
1.25	4.30	4.35	4.39	4.42	4.56	5.15	5.11	5.19	4.77	4.81	5.09
1.50	4.36	4.32	4.39	4.47	4.58	5.52	5.52	5.61	5.15	5.19	5.55
1.75	4.54	4.44	4.55	4.65	4.56	5.23	5.24	5.19	4.93	5.00	5.09
2.00	4.56	4.31	4.46	4.60	4.50	4.51	4.56	4.30	4.19	4.22	4.13
2.25	4.78	4.34	4.51	4.66	4.46	4.20	4.24	3.84	3.93	3.87	3.64
2.50	4.69	4.15	4.32	4.44	4.50	4.35	4.36	4.30	4.17	4.05	4.13
2.75	5.11	4.38	4.53	4.64	4.56	5.17	5.19	5.19	5.06	4.91	5.09
3.00	5.44	4.56	4.66	4.75	4.58	5.65	5.71	5.61	5.55	5.43	5.55
3.25	5.52	4.59	4.75	4.80	4.56	5.35	5.44	5.19	5.26	5.21	5.09
3.50	5.68	4.55	4.65	4.65	4.50	4.48	4.56	4.30	4.39	4.37	4.13
3.75	6.15	4.71	4.65	4.63	4.46	4.21	4.22	3.84	4.03	4.01	3.64
4.00	6.73	4.90	4.69	4.67	4.50	4.83	4.77	4.30	4.44	4.42	4.13

Table 4 100×m.s.e. of estimators in 48 step experiments for step size 2.0 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E^*_{WE}	E^*_{PV}	A^*_{WE}
0.00	5.66	5.68	5.66	5.67	5.73	7.96	8.27	8.29	7.76	7.76	8.27
0.25	5.69	5.74	5.72	5.71	5.66	7.61	7.94	7.80	7.48	7.50	7.74
0.50	5.18	5.27	5.24	5.20	5.41	5.91	6.27	6.24	5.83	5.86	6.10
0.75	4.88	4.98	4.95	4.92	5.08	4.15	4.47	4.21	4.14	4.21	4.04
1.00	4.67	4.76	4.74	4.73	4.92	3.34	3.44	3.25	3.24	3.29	3.08
1.25	4.80	4.84	4.85	4.87	5.08	4.32	4.18	4.21	3.99	4.01	4.04
1.50	5.25	5.23	5.27	5.33	5.41	6.33	6.08	6.24	5.83	5.82	6.10
1.75	5.57	5.55	5.62	5.67	5.66	7.78	7.57	7.80	7.25	7.26	7.74
2.00	5.70	5.66	5.77	5.84	5.73	8.34	8.31	8.29	7.77	7.87	8.27
2.25	5.63	5.72	5.88	5.88	5.66	7.72	7.80	7.80	7.33	7.59	7.74
2.50	5.26	5.23	5.46	5.48	5.41	6.15	6.29	6.24	5.77	6.07	6.10
2.75	5.29	5.03	5.28	5.36	5.08	4.52	4.62	4.21	4.25	4.45	4.04
3.00	5.34	4.63	4.83	4.98	4.92	3.63	3.67	3.25	3.40	3.36	3.08
3.25	5.93	4.80	4.92	5.14	5.08	4.39	4.37	4.21	4.32	4.05	4.04
3.50	6.82	5.34	5.39	5.60	5.41	6.37	6.31	6.24	6.27	5.87	6.10
3.75	7.22	5.58	5.60	5.76	5.66	7.88	7.84	7.80	7.65	7.28	7.74
4.00	7.31	5.77	5.90	5.99	5.73	8.38	8.49	8.29	8.14	7.93	8.27

Table 5 100·bias of estimators in 48 step experiments for step size 0.5 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
0.00	0.49	0.44	0.44	0.53	0.00	0.46	0.47	0.44	0.44	0.00
0.25	1.42	0.87	0.94	1.31	0.00	1.22	1.11	0.97	0.87	0.00
0.50	2.86	1.80	1.85	2.37	0.00	2.38	2.20	2.07	1.89	0.00
0.75	4.58	3.00	2.86	3.32	0.00	3.53	3.27	3.26	3.01	0.00
1.00	6.27	4.15	3.61	3.92	0.00	4.27	3.95	4.00	3.68	0.00
1.25	7.83	5.20	4.00	4.11	0.00	4.79	4.42	4.40	4.04	0.00
1.50	9.67	6.52	4.36	4.35	0.00	5.28	4.89	4.94	4.55	0.00
1.75	11.97	8.30	4.95	4.74	0.00	5.88	5.46	5.53	5.13	0.00
2.00	14.37	10.16	5.34	4.99	0.00	6.21	5.78	5.77	5.36	0.00
2.25	16.68	11.97	5.34	4.87	0.00	6.19	5.73	5.68	5.25	0.00
2.50	19.35	14.12	5.41	4.89	0.00	6.36	5.91	5.83	5.40	0.00
2.75	22.65	16.89	5.84	5.26	0.00	6.80	6.32	6.29	5.85	0.00
3.00	25.95	19.66	6.00	5.36	0.00	6.90	6.42	6.33	5.88	0.00
3.25	29.20	22.41	5.79	5.11	0.00	6.77	6.28	6.06	5.61	0.00
3.50	32.79	25.48	5.66	5.01	0.00	6.69	6.19	6.02	5.56	0.00
3.75	36.97	29.13	5.89	5.20	0.00	6.89	6.39	6.31	5.86	0.00
4.00	41.36	32.99	6.05	5.37	0.00	7.07	6.56	6.40	5.93	0.00

Table 6 100×bias of estimators in 48 step experiments for step size 1.0 ($\beta=\pi/3.0^{1/2}$ based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
0.00	0.38	0.34	0.35	0.40	0.00	0.48	0.48	0.37	0.37	0.00
0.25	0.63	0.15	0.26	0.70	-0.01	0.71	0.46	0.08	-0.04	-0.23
0.50	1.34	0.38	0.58	1.28	0.00	1.34	0.93	0.58	0.33	0.00
0.75	2.33	0.80	1.00	1.91	0.01	1.83	1.31	1.36	0.96	0.23
1.00	3.07	0.93	1.03	2.02	0.00	1.72	1.16	1.56	1.00	0.00
1.25	4.51	1.77	1.66	2.54	-0.01	2.18	1.61	2.26	1.56	-0.23
1.50	5.38	2.05	1.60	2.24	0.00	2.20	1.61	2.51	1.73	0.00
1.75	6.50	2.69	1.73	2.12	0.01	2.68	2.04	2.82	2.05	0.23
2.00	8.08	3.88	2.28	2.48	0.00	3.17	2.41	3.04	2.32	0.00
2.25	8.98	4.33	1.92	2.12	-0.01	2.99	2.16	2.43	1.74	-0.23
2.50	10.45	5.33	1.97	2.35	0.00	3.23	2.35	2.52	1.84	0.00
2.75	12.31	6.61	2.18	2.70	0.01	3.41	2.54	2.94	2.22	0.23
3.00	13.94	7.63	1.98	2.59	0.00	3.07	2.25	2.82	2.03	0.00
3.25	16.17	9.27	2.28	2.86	-0.01	3.09	2.34	3.13	2.27	-0.23
3.50	18.06	10.56	2.08	2.53	0.00	2.86	2.14	3.22	2.32	0.00
3.75	20.20	12.23	2.12	2.39	0.01	3.18	2.43	3.38	2.52	0.23
4.00	22.85	14.48	2.68	2.79	0.00	3.71	2.87	3.58	2.79	0.00

Note: A_{DM} and A_{WE} denote columns for asymptotic predicted biases of E_{DM} and E_{WE} respectively.

Table 7 100×bias of estimators in 48 step experiments for step size 1.5 ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_β	$E_{\beta\nu}$	$E_{\nu M}$	$A_{\nu M}$	E_{WE}	$E_{\rho\nu}$	E^*_{WE}	$E^*_{\rho\nu}$	A_{WE}
0.00	0.66	0.60	0.60	0.70	0.00	0.68	0.69	0.58	0.59	0.00
0.25	-0.22	-0.57	-0.46	-0.13	-0.42	-1.64	-2.04	-1.98	-2.03	-2.40
0.50	0.31	-0.33	-0.14	0.38	-0.42	-1.25	-1.98	-1.82	-1.97	-2.37
0.75	1.48	0.38	0.62	1.46	0.00	1.43	0.47	0.60	0.30	0.00
1.00	2.41	0.65	0.92	2.11	0.42	3.41	2.35	2.60	2.09	2.37
1.25	3.48	1.03	1.27	2.67	0.42	3.92	2.86	3.28	2.54	2.40
1.50	3.93	0.71	0.79	2.34	0.00	1.84	0.86	1.65	0.67	0.00
1.75	4.71	0.73	0.55	2.06	-0.42	0.01	-0.83	0.18	-1.01	-2.40
2.00	5.58	0.89	0.36	1.69	-0.42	-0.26	-0.97	0.13	-1.22	-2.37
2.25	7.21	1.97	1.00	2.04	0.00	1.92	1.23	2.34	0.95	0.00
2.50	8.11	2.44	0.96	1.62	0.42	3.64	2.87	3.89	2.54	2.37
2.75	9.54	3.51	1.40	1.76	0.42	4.06	3.16	4.26	3.02	2.40
3.00	10.82	4.51	1.70	1.91	0.00	2.79	1.68	2.76	1.70	0.00
3.25	10.84	4.25	0.59	0.74	-0.42	0.32	-0.96	0.03	-0.86	-2.40
3.50	12.20	5.32	0.73	0.99	-0.42	0.32	-1.09	-0.20	-0.98	-2.37
3.75	14.20	6.85	1.28	1.81	0.00	2.58	1.11	1.79	1.04	0.00
4.00	16.16	8.15	1.52	2.45	0.42	4.42	2.99	3.59	2.76	2.37

Table 8 100 bias of estimators in 48 step experiments for step size 2.0 ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_β	$E_{\beta\nu}$	$E_{\nu M}$	$A_{\nu M}$	E_{WE}	$E_{\rho\nu}$	E^*_{WE}	$E^*_{\rho\nu}$	A_{WE}
0.00	0.65	0.65	0.63	0.65	0.00	0.65	0.64	0.59	0.59	0.00
0.25	-1.60	-1.79	-1.73	-1.55	-2.14	-5.63	-6.20	-5.73	-5.75	-6.93
0.50	-2.13	-2.55	-2.42	-2.04	-3.00	-7.77	-8.78	-7.89	-7.94	-9.44
0.75	-1.53	-2.28	-2.09	-1.45	-2.11	-5.30	-6.66	-5.71	-5.87	-6.44
1.00	1.15	0.05	0.30	1.15	0.00	0.97	-0.63	0.25	-0.03	0.00
1.25	4.17	2.50	2.79	3.98	2.11	7.55	5.83	6.47	5.99	6.44
1.50	5.63	3.14	3.45	5.04	3.00	10.16	8.48	9.11	8.36	9.44
1.75	5.64	2.28	2.52	4.40	2.14	8.04	6.49	7.23	6.18	6.93
2.00	4.57	0.27	0.31	2.43	0.00	2.02	0.67	1.71	0.34	0.00
2.25	3.92	-1.32	-1.58	0.60	-2.14	-4.00	-5.08	-3.74	-5.38	-6.93
2.50	4.00	-2.15	-2.81	-0.68	-3.00	-6.77	-7.61	-6.29	-8.16	-9.44
2.75	5.95	-0.89	-1.98	-0.08	-2.11	-4.28	-4.97	-3.73	-5.70	-6.44
3.00	8.93	1.62	0.08	1.65	0.00	1.42	0.77	1.88	-0.11	0.00
3.25	11.62	3.87	1.84	3.00	2.11	7.07	6.36	7.30	5.33	6.44
3.50	13.59	5.60	3.05	3.78	3.00	9.93	9.01	10.02	8.16	9.44
3.75	14.32	6.13	2.91	3.29	2.14	8.39	7.23	8.41	6.73	6.93
4.00	13.56	5.22	1.20	1.34	0.00	2.63	1.16	2.62	1.21	0.00

Table 9 Numbers of 48 step experiments out of 2000 where m.l.e.'s of parameters can be obtained ($\beta = \pi/3.0^{1/2}$).

Start	Step size 0.5	1.0	1.5	2.0
0.00	2000	1953	1587	963
0.25	1999	1976	1699	1079
0.50	1999	1994	1882	1388
0.75	2000	1970	1955	1685
1.00	1999	1943	1863	1796
1.25	1999	1970	1674	1674
1.50	1999	1984	1567	1383
1.75	2000	1972	1660	1075
2.00	1999	1953	1857	948
2.25	1999	1977	1937	1071
2.50	1998	1994	1867	1336
2.75	2000	1959	1717	1634
3.00	1999	1934	1594	1774
3.25	2000	1959	1692	1652
3.50	1999	1982	1872	1376
3.75	2000	1961	1946	1092
4.00	1999	1938	1848	965

Table 10 100×m.s.e. and 100×bias of μ^* in 48 step experiments ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

	Step size							
	0.5		1.0		1.5		2.0	
Start	100 × bias	100 × m.s.e.	100 × bias	100 × m.s.e.	100 × bias	100 × m.s.e.	100 × bias	100 × m.s.e.
0.00	0.35	2.90	0.28	3.45	0.50	4.24	0.58	5.30
0.25	0.08	3.11	0.00	3.65	−0.06	4.43	−1.02	5.32
0.50	0.30	3.04	−0.12	3.60	0.15	4.52	−1.02	4.98
0.75	0.60	3.11	−0.11	3.64	0.02	4.77	−1.00	5.40
1.00	0.67	3.04	−0.04	3.60	−0.48	4.46	−0.17	5.65
1.25	0.52	3.22	0.50	3.89	0.10	4.18	0.98	5.17
1.50	0.57	3.15	0.09	3.65	0.40	4.15	1.83	4.87
1.75	0.78	3.29	−0.10	3.71	0.79	4.45	2.07	5.10
2.00	0.76	3.17	0.40	3.54	0.68	4.60	1.09	5.47
2.25	0.52	3.42	0.03	3.75	0.39	4.82	−0.04	5.54
2.50	0.48	3.34	−0.12	3.82	−0.53	4.42	−0.67	5.31
2.75	0.81	3.47	−0.05	3.82	−0.19	4.30	−0.09	5.71
3.00	0.74	3.32	−0.02	3.79	0.55	4.40	0.39	5.72
3.25	0.50	3.52	0.45	4.14	0.01	4.59	0.47	5.36
3.50	0.34	3.48	0.09	3.90	0.16	4.73	1.35	5.18
3.75	0.53	3.61	−0.03	3.92	−0.05	4.95	1.68	5.23
4.00	0.57	3.49	0.57	3.72	−0.45	4.68	0.63	5.54

Table 11 100×m.s.e. and mean of $1/\tilde{\beta}$ in 48 step experiments ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

	Step size							
	0.5		1.0		1.5		2.0	
Start	mean	100 × m.s̃.e.	mean	100 × m.s̃.e.	mean	100 × m.s̃.e.	mean	100 × m.s̃.e.
0.00	0.478	3.56	0.506	2.59	0.528	2.07	0.573	1.60
0.25	0.481	3.55	0.509	2.47	0.525	2.16	0.558	1.80
0.50	0.491	3.64	0.511	2.75	0.513	2.35	0.522	2.28
0.75	0.501	3.42	0.513	2.84	0.506	2.56	0.487	3.03
1.00	0.513	3.72	0.517	2.73	0.513	2.59	0.469	3.50
1.25	0.521	4.05	0.526	2.69	0.528	2.38	0.484	3.08
1.50	0.524	4.29	0.532	2.79	0.537	2.30	0.523	2.50
1.75	0.530	4.25	0.532	2.89	0.537	2.44	0.560	2.01
2.00	0.538	4.80	0.528	2.83	0.530	2.64	0.581	1.85
2.25	0.542	5.23	0.529	2.80	0.525	2.90	0.571	2.07
2.50	0.545	5.59	0.525	3.16	0.532	2.83	0.538	2.49
2.75	0.544	5.66	0.524	3.31	0.538	2.46	0.502	3.28
3.00	0.548	5.94	0.527	3.36	0.539	2.36	0.488	3.61
3.25	0.548	6.21	0.531	3.17	0.534	2.40	0.497	3.24
3.50	0.548	6.36	0.536	3.27	0.519	2.63	0.536	2.63
3.75	0.547	6.43	0.531	3.13	0.510	2.81	0.566	2.08
4.00	0.547	6.47	0.525	3.12	0.514	2.82	0.579	1.85

Table 12 100 m.s.e. and mean of $1/\hat{\beta}^*$ in 48 step experiments ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

<u>Step size</u>

	0.5		1.0		1.5		2.0	
Start	mean	100 m.$\hat{s}$.e.	mean	100 m.$\hat{s}$.e.	mean	100 m.$\hat{s}$.e.	mean	100 m.$\hat{s}$.e.
0.00	0.500	3.69	0.521	2.74	0.530	2.26	0.569	1.58
0.25	0.499	3.69	0.524	2.50	0.533	2.23	0.559	1.79
0.50	0.499	3.77	0.527	2.73	0.533	2.20	0.539	2.14
0.75	0.500	3.43	0.520	2.88	0.532	2.17	0.526	2.46
1.00	0.503	3.55	0.516	2.87	0.530	2.37	0.521	2.61
1.25	0.502	3.67	0.523	2.69	0.529	2.37	0.523	2.51
1.50	0.499	3.67	0.529	2.62	0.529	2.36	0.537	2.29
1.75	0.498	3.38	0.526	2.73	0.529	2.41	0.557	1.87
2.00	0.502	3.57	0.521	2.70	0.531	2.36	0.571	1.62
2.25	0.501	3.63	0.525	2.51	0.533	2.43	0.563	1.83
2.50	0.499	3.66	0.525	2.72	0.534	2.47	0.540	2.21
2.75	0.496	3.53	0.519	2.89	0.529	2.33	0.525	2.66
3.00	0.498	3.54	0.515	3.03	0.530	2.30	0.523	2.69
3.25	0.496	3.63	0.519	2.78	0.533	2.25	0.521	2.58
3.50	0.495	3.61	0.526	2.72	0.531	2.28	0.540	2.31
3.75	0.492	3.59	0.521	2.69	0.530	2.23	0.558	1.81
4.00	0.493	3.64	0.515	2.73	0.527	2.42	0.569	1.59

Table 13 100×m.s.e. of estimators in 96 step experiments for step size 0.5 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E_{WE}^*	E_{PV}^*	A_{WE}^*
0.00	1.49	1.51	1.50	1.50	1.57	1.59	1.60	1.66	1.52	1.52	1.57
0.25	1.63	1.64	1.64	1.66	1.57	1.75	1.76	1.66	1.65	1.65	1.57
0.50	1.58	1.59	1.60	1.62	1.57	1.69	1.70	1.66	1.60	1.60	1.57
0.75	1.53	1.52	1.54	1.56	1.57	1.63	1.63	1.66	1.54	1.53	1.57
1.00	1.58	1.54	1.57	1.59	1.57	1.68	1.67	1.66	1.59	1.59	1.57
1.25	1.77	1.69	1.72	1.75	1.57	1.86	1.85	1.66	1.74	1.73	1.57
1.50	1.76	1.65	1.67	1.68	1.57	1.79	1.78	1.66	1.68	1.67	1.57
1.75	1.81	1.64	1.61	1.62	1.57	1.72	1.71	1.66	1.63	1.62	1.57
2.00	1.97	1.73	1.65	1.65	1.57	1.77	1.76	1.66	1.69	1.68	1.57
2.25	2.32	2.00	1.85	1.84	1.57	2.00	2.00	1.66	1.86	1.85	1.57
2.50	2.48	2.06	1.80	1.78	1.57	1.93	1.92	1.66	1.81	1.80	1.57
2.75	2.74	2.19	1.75	1.71	1.57	1.85	1.84	1.66	1.73	1.72	1.57
3.00	3.12	2.43	1.76	1.72	1.57	1.87	1.86	1.66	1.77	1.76	1.57
3.25	3.75	2.88	1.92	1.87	1.57	2.07	2.06	1.66	1.91	1.90	1.57
3.50	4.25	3.20	1.90	1.84	1.57	2.04	2.03	1.66	1.90	1.88	1.57
3.75	4.89	3.60	1.80	1.74	1.57	1.90	1.88	1.66	1.78	1.77	1.57
4.00	5.72	4.18	1.81	1.76	1.57	1.92	1.91	1.66	1.82	1.80	1.57

Table 14 100×m.s.e. of estimators in 96 step experiments for step size 1.0 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E_{WE}^*	E_{PV}^*	A_{WE}^*
0.00	1.82	1.84	1.83	1.83	1.90	1.99	2.02	2.04	1.94	1.94	1.97
0.25	1.87	1.88	1.87	1.88	1.90	1.98	2.00	1.99	1.90	1.91	1.91
0.50	1.86	1.88	1.88	1.88	1.90	1.88	1.90	1.94	1.81	1.82	1.85
0.75	1.93	1.95	1.96	1.96	1.90	2.06	2.07	1.99	1.96	1.97	1.91
1.00	1.94	1.95	1.96	1.98	1.90	2.10	2.11	2.04	1.98	1.99	1.97
1.25	1.89	1.86	1.89	1.91	1.90	2.02	2.02	1.99	1.89	1.88	1.91
1.50	1.89	1.85	1.87	1.91	1.90	1.94	1.94	1.94	1.85	1.84	1.85
1.75	1.96	1.90	1.93	1.96	1.90	2.07	2.08	1.99	1.98	1.96	1.91
2.00	1.95	1.84	1.87	1.88	1.90	2.06	2.06	2.04	1.98	1.96	1.97
2.25	2.05	1.90	1.93	1.94	1.90	2.06	2.06	1.99	1.96	1.95	1.91
2.50	2.13	1.94	1.96	1.97	1.90	1.98	1.97	1.94	1.90	1.89	1.85
2.75	2.25	2.02	2.02	2.03	1.90	2.15	2.14	1.99	2.04	2.04	1.91
3.00	2.36	2.06	2.04	2.05	1.90	2.21	2.21	2.04	2.08	2.07	1.97
3.25	2.50	2.08	1.98	1.99	1.90	2.13	2.12	1.99	1.98	1.97	1.91
3.50	2.62	2.11	1.96	1.98	1.90	2.03	2.03	1.94	1.93	1.92	1.85
3.75	2.85	2.22	1.99	2.01	1.90	2.12	2.12	1.99	2.04	2.02	1.91
4.00	3.09	2.32	1.95	1.95	1.90	2.15	2.15	2.04	2.06	2.04	1.97

Note: A_{DM}, A_{WE} and A_{WE}^* denote columns for asymptotic predicted m.s.e.'s of E_{DM}, E_{WE} and E_{WE}^* respectively.

Table 15 100×m.s.e. of estimators in 96 step experiments for step size 1.5 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E^*_{WE}	E^*_{PV}	A^*_{WE}
0.00	2.38	2.40	2.39	2.38	2.29	2.87	2.92	2.81	2.85	2.86	2.77
0.25	2.30	2.31	2.30	2.31	2.28	2.63	2.68	2.62	2.59	2.60	2.57
0.50	2.27	2.28	2.27	2.28	2.25	2.17	2.21	2.18	2.12	2.13	2.09
0.75	2.21	2.22	2.21	2.23	2.23	1.91	1.93	1.92	1.82	1.83	1.82
1.00	2.30	2.31	2.31	2.32	2.25	2.25	2.24	2.18	2.13	2.13	2.09
1.25	2.36	2.36	2.37	2.39	2.28	2.81	2.79	2.62	2.65	2.65	2.57
1.50	2.36	2.37	2.39	2.40	2.29	2.87	2.86	2.81	2.80	2.82	2.77
1.75	2.27	2.27	2.30	2.32	2.28	2.56	2.57	2.62	2.51	2.55	2.57
2.00	2.32	2.28	2.31	2.35	2.25	2.25	2.27	2.18	2.16	2.19	2.09
2.25	2.31	2.22	2.26	2.30	2.23	2.03	2.04	1.92	1.90	1.89	1.82
2.50	2.31	2.18	2.22	2.25	2.25	2.25	2.25	2.18	2.14	2.09	2.09
2.75	2.57	2.37	2.40	2.43	2.28	2.80	2.81	2.62	2.76	2.70	2.57
3.00	2.64	2.40	2.43	2.45	2.29	2.94	2.95	2.81	2.90	2.87	2.77
3.25	2.54	2.30	2.34	2.35	2.28	2.67	2.70	2.62	2.61	2.61	2.57
3.50	2.59	2.31	2.34	2.35	2.25	2.24	2.27	2.18	2.17	2.17	2.09
3.75	2.68	2.30	2.28	2.29	2.23	1.99	1.98	1.92	1.90	1.89	1.82
4.00	2.93	2.45	2.39	2.39	2.25	2.35	2.32	2.18	2.23	2.21	2.09

Table 16 100×m.s.e. of estimators in 96 step experiments for step size 2.0 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E^*_{WE}	E^*_{PV}	A^*_{WE}
0.00	2.99	2.99	2.99	2.99	2.87	4.24	4.33	4.15	4.20	4.20	4.14
0.25	2.88	2.89	2.88	2.89	2.85	4.07	4.17	4.14	4.03	4.04	4.11
0.50	2.74	2.76	2.75	2.74	2.75	3.40	3.53	3.57	3.38	3.39	3.50
0.75	2.42	2.44	2.43	2.43	2.56	2.16	2.28	2.31	2.13	2.15	2.23
1.00	2.47	2.48	2.48	2.49	2.46	1.69	1.71	1.62	1.61	1.62	1.54
1.25	2.58	2.55	2.56	2.60	2.56	2.43	2.33	2.31	2.25	2.23	2.23
1.50	2.85	2.80	2.81	2.86	2.75	3.69	3.55	3.57	3.49	3.45	3.50
1.75	3.00	2.96	2.98	3.02	2.85	4.35	4.25	4.14	4.16	4.13	4.11
2.00	2.96	3.00	3.02	3.02	2.87	4.24	4.23	4.15	4.14	4.18	4.14
2.25	2.82	2.93	2.98	2.94	2.85	4.02	4.07	4.14	3.95	4.08	4.11
2.50	2.69	2.80	2.86	2.83	2.75	3.46	3.53	3.57	3.37	3.54	3.50
2.75	2.51	2.54	2.61	2.61	2.56	2.32	2.37	2.31	2.20	2.32	2.23
3.00	2.55	2.38	2.43	2.47	2.46	1.74	1.75	1.62	1.62	1.61	1.54
3.25	2.87	2.50	2.51	2.57	2.56	2.40	2.38	2.31	2.32	2.18	2.23
3.50	3.31	2.82	2.80	2.86	2.75	3.65	3.60	3.57	3.60	3.41	3.50
3.75	3.52	3.01	2.98	3.02	2.85	4.41	4.38	4.14	4.37	4.21	4.11
4.00	3.38	2.98	3.01	3.04	2.87	4.31	4.34	4.15	4.26	4.20	4.14

Table 17 100×bias of estimators in 96 step experiments for step size 0.5 ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
0.00	0.28	0.28	0.27	0.28	0.00	0.38	0.37	0.33	0.33	0.00
0.25	0.90	0.63	0.68	0.83	0.00	0.88	0.82	0.72	0.67	0.00
0.50	1.39	0.88	0.91	1.12	0.00	1.12	1.02	0.98	0.89	0.00
0.75	2.34	1.56	1.50	1.69	0.00	1.78	1.64	1.70	1.57	0.00
1.00	3.11	2.07	1.79	1.91	0.00	2.20	2.04	2.07	1.91	0.00
1.25	4.03	2.72	2.10	2.14	0.00	2.53	2.34	2.36	2.18	0.00
1.50	4.76	3.20	2.09	2.04	0.00	2.45	2.25	2.38	2.19	0.00
1.75	6.04	4.22	2.51	2.37	0.00	2.92	2.71	2.86	2.65	0.00
2.00	7.17	5.08	2.62	2.40	0.00	3.10	2.88	2.98	2.77	0.00
2.25	8.48	6.13	2.75	2.49	0.00	3.23	3.00	3.01	2.80	0.00
2.50	9.68	7.09	2.66	2.34	0.00	3.04	2.81	2.86	2.65	0.00
2.75	11.39	8.53	2.94	2.57	0.00	3.34	3.11	3.17	2.96	0.00
3.00	12.96	9.83	2.92	2.52	0.00	3.38	3.15	3.17	2.95	0.00
3.25	14.81	11.42	3.04	2.63	0.00	3.54	3.31	3.23	3.01	0.00
3.50	16.50	12.87	2.90	2.45	0.00	3.29	3.05	3.05	2.83	0.00
3.75	18.64	14.75	3.04	2.59	0.00	3.42	3.18	3.25	3.03	0.00
4.00	20.75	16.58	3.01	2.55	0.00	3.46	3.22	3.27	3.05	0.00

Table 18 100×bias of estimators in 96 step experiments for step size 1.0 ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
0.00	0.27	0.28	0.29	0.26	0.00	0.36	0.35	0.27	0.27	0.00
0.25	0.51	0.31	0.37	0.52	-0.01	0.52	0.38	0.17	0.11	-0.23
0.50	0.77	0.32	0.41	0.72	0.00	0.73	0.50	0.38	0.26	0.00
0.75	0.99	0.26	0.37	0.78	0.01	0.85	0.57	0.64	0.44	0.23
1.00	1.39	0.36	0.43	0.88	0.00	0.74	0.44	0.67	0.39	0.00
1.25	2.28	0.94	0.89	1.31	-0.01	1.03	0.74	1.10	0.75	-0.23
1.50	2.65	1.02	0.79	1.08	0.00	1.09	0.79	1.25	0.87	0.00
1.75	3.23	1.36	0.87	1.04	0.01	1.34	1.01	1.51	1.12	0.23
2.00	4.10	2.03	1.22	1.30	0.00	1.62	1.25	1.57	1.21	0.00
2.25	4.69	2.41	1.20	1.27	-0.01	1.65	1.23	1.36	1.01	-0.23
2.50	5.31	2.77	1.08	1.25	0.00	1.64	1.19	1.36	1.02	0.00
2.75	5.99	3.17	0.92	1.17	0.01	1.58	1.15	1.44	1.09	0.23
3.00	6.84	3.72	0.87	1.15	0.00	1.34	0.94	1.30	0.92	0.00
3.25	8.21	4.78	1.28	1.54	-0.01	1.53	1.16	1.61	1.18	-0.23
3.50	9.04	5.32	1.06	1.25	0.00	1.44	1.09	1.64	1.19	0.00
3.75	10.14	6.19	1.12	1.21	0.01	1.64	1.28	1.84	1.41	0.23
4.00	11.49	7.34	1.42	1.43	0.00	1.89	1.48	1.83	1.43	0.00

Note: A_{DM} and A_{WE} denote columns for asymptotic predicted biases of E_{DM} and E_{WE} respectively.

Table 19 100×bias of estimators in 96 step experiments for step size 1.5 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_β	$E_{\beta D}$	E_{DM}	A_{DM}	E_{WE}	$E_{\rho V}$	E_{WE}^*	$E_{\rho V}^*$	A_{WE}
0.00	0.50	0.52	0.54	0.49	0.00	0.47	0.46	0.47	0.47	0.00
0.25	-0.06	-0.17	-0.11	-0.05	-0.42	-1.78	-2.01	-1.94	-1.97	-2.40
0.50	0.01	-0.28	-0.18	0.04	-0.42	-1.73	-2.13	-2.08	-2.16	-2.37
0.75	0.79	0.28	0.41	0.78	0.00	0.79	0.28	0.29	0.15	0.00
1.00	1.35	0.53	0.68	1.21	0.42	2.84	2.28	2.44	2.20	2.37
1.25	2.13	0.97	1.09	1.75	0.42	3.29	2.74	3.00	2.65	2.40
1.50	2.10	0.55	0.60	1.32	0.00	0.99	0.50	0.92	0.45	0.00
1.75	2.09	0.15	0.07	0.79	-0.42	-1.28	-1.71	-1.14	-1.73	-2.40
2.00	2.54	0.23	-0.02	0.61	-0.42	-1.30	-1.66	-1.11	-1.77	-2.37
2.25	3.38	0.78	0.30	0.79	0.00	0.79	0.45	1.04	0.35	0.00
2.50	4.09	1.29	0.54	0.86	0.42	2.81	2.42	3.00	2.33	2.37
2.75	5.13	2.17	1.12	1.27	0.42	3.34	2.89	3.52	2.90	2.40
3.00	5.60	2.50	1.08	1.16	0.00	1.51	0.95	1.57	1.04	0.00
3.25	5.47	2.23	0.39	0.43	-0.42	-0.85	-1.49	-0.91	-1.36	-2.40
3.50	5.96	2.55	0.24	0.35	-0.42	-0.98	-1.69	-1.24	-1.63	-2.37
3.75	7.22	3.59	0.79	1.02	0.00	1.39	0.65	0.95	0.59	0.00
4.00	8.27	4.33	1.00	1.41	0.42	3.33	2.62	2.97	2.57	2.37

Table 20 100×bias of estimators in 96 step experiments for step size 2.0 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_β	$E_{\beta D}$	E_{DM}	A_{DM}	E_{WE}	$E_{\rho V}$	E_{WE}^*	$E_{\rho V}^*$	A_{WE}
0.00	0.46	0.49	0.49	0.45	0.00	0.44	0.42	0.45	0.44	0.00
0.25	-1.61	-1.66	-1.62	-1.61	-2.14	-5.98	-6.31	-6.05	-6.06	-6.93
0.50	-2.45	-2.61	-2.54	-2.41	-3.00	-8.48	-9.04	-8.58	-8.62	-9.44
0.75	-1.70	-2.01	-1.91	-1.67	-2.11	-5.82	-6.53	-6.04	-6.11	-6.44
1.00	0.63	0.12	0.25	0.63	0.00	0.57	-0.25	0.16	0.02	0.00
1.25	3.21	2.43	2.58	3.12	2.11	7.07	6.21	6.46	6.24	6.44
1.50	4.51	3.35	3.51	4.24	3.00	9.93	9.07	9.42	9.07	9.44
1.75	3.89	2.30	2.42	3.31	2.14	7.49	6.70	7.13	6.62	6.93
2.00	2.19	0.11	0.14	1.14	0.00	0.87	0.18	0.78	0.11	0.00
2.25	0.85	-1.72	-1.85	-0.79	-2.14	-5.61	-6.16	-5.39	-6.20	-6.93
2.50	0.52	-2.51	-2.83	-1.80	-3.00	-8.12	-8.54	-7.87	-8.79	-9.44
2.75	1.80	-1.59	-2.13	-1.20	-2.11	-5.47	-5.81	-5.18	-6.16	-6.44
3.00	4.35	0.72	-0.05	0.72	0.00	0.62	0.30	0.84	-0.14	0.00
3.25	6.81	2.97	1.97	2.53	2.11	6.74	6.39	6.89	5.91	6.44
3.50	8.29	4.33	3.05	3.42	3.00	9.60	9.15	9.74	8.82	9.44
3.75	8.41	4.36	2.76	2.95	2.14	7.82	7.24	7.94	7.11	6.93
4.00	6.94	2.79	0.78	0.84	0.00	1.41	0.68	1.50	0.80	0.00

Table 21 Numbers of 96 step experiments out of 2000 where m.l.e.'s of parameters can be obtained ($\beta=\pi/3.0^{1/2}$).

Start	Step size 0.5	1.0	1.5	2.0
0.00	2000	1999	1906	1463
0.25	2000	1998	1949	1548
0.50	2000	2000	1988	1788
0.75	2000	2000	1998	1946
1.00	2000	1997	1984	1978
1.25	2000	2000	1954	1943
1.50	2000	2000	1904	1813
1.75	2000	1997	1939	1569
2.00	2000	1999	1989	1453
2.25	2000	1998	1998	1570
2.50	2000	2000	1989	1787
2.75	2000	2000	1951	1933
3.00	2000	1997	1904	1984
3.25	2000	2000	1949	1948
3.50	2000	2000	1989	1793
3.75	2000	1997	1998	1580
4.00	2000	1999	1983	1456

Table 22 100·m.s.e. and 100·bias of $\hat{\mu}$ in 96 step experiments ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Step size

Start	0.5 100 bias	0.5 100 m.s.e.	1.0 100 bias	1.0 100 m.s.e.	1.5 100 bias	1.5 100 m.s.e.	2.0 100 bias	2.0 100 m.s.e.
0.00	0.27	1.54	0.28	1.83	0.47	2.22	0.46	2.60
0.25	0.24	1.67	0.30	1.88	0.57	2.24	0.06	2.50
0.50	0.12	1.62	0.07	1.89	0.40	2.37	0.34	2.61
0.75	0.38	1.55	-0.20	1.96	0.15	2.38	0.47	2.87
1.00	0.34	1.57	-0.15	1.92	-0.45	2.39	0.05	3.21
1.25	0.38	1.70	0.34	1.87	-0.09	2.26	-0.14	2.95
1.50	0.21	1.64	0.07	1.89	0.20	2.20	0.26	2.61
1.75	0.47	1.57	-0.01	1.92	0.53	2.21	0.60	2.53
2.00	0.38	1.59	0.33	1.84	0.51	2.41	0.23	2.64
2.25	0.40	1.75	0.34	1.91	0.02	2.42	0.01	2.56
2.50	0.26	1.70	0.09	1.94	-0.57	2.30	0.35	2.70
2.75	0.48	1.63	-0.16	2.00	-0.12	2.29	0.50	2.98
3.00	0.35	1.64	-0.10	1.98	0.52	2.24	0.07	3.13
3.25	0.42	1.78	0.41	1.94	0.63	2.26	-0.52	2.92
3.50	0.25	1.75	0.12	1.95	0.44	2.43	0.02	2.64
3.75	0.41	1.65	0.07	1.96	0.21	2.43	0.80	2.56
4.00	0.32	1.68	0.40	1.90	-0.40	2.44	0.50	2.63

Table 23 100×m.s.e. and mean of $1/\tilde{\beta}$ in 96 step experiments ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

	Step size							
	0.5		1.0		1.5		2.0	
Start	mean	100 × m.s.e.	mean	100 × m.s.e.	mean	100 × m.s.e.	mean	100 × m.s.e.
0.00	0.512	2.00	0.527	1.43	0.543	1.07	0.587	0.90
0.25	0.516	1.90	0.528	1.42	0.538	1.11	0.571	0.92
0.50	0.519	1.98	0.530	1.48	0.528	1.23	0.536	1.14
0.75	0.524	1.94	0.530	1.49	0.523	1.36	0.499	1.58
1.00	0.529	2.03	0.533	1.45	0.529	1.36	0.483	1.93
1.25	0.535	2.02	0.539	1.44	0.540	1.17	0.501	1.71
1.50	0.536	2.16	0.542	1.49	0.549	1.15	0.538	1.24
1.75	0.540	2.15	0.538	1.52	0.547	1.25	0.573	1.04
2.00	0.544	2.28	0.538	1.51	0.537	1.34	0.591	1.00
2.25	0.547	2.24	0.538	1.48	0.531	1.43	0.578	1.04
2.50	0.546	2.38	0.538	1.55	0.537	1.36	0.542	1.23
2.75	0.547	2.36	0.537	1.58	0.544	1.23	0.507	1.75
3.00	0.549	2.48	0.537	1.56	0.550	1.15	0.491	1.96
3.25	0.551	2.44	0.542	1.52	0.544	1.18	0.504	1.64
3.50	0.550	2.64	0.544	1.57	0.531	1.28	0.543	1.26
3.75	0.549	2.65	0.539	1.57	0.525	1.42	0.575	1.05
4.00	0.550	2.69	0.538	1.55	0.530	1.41	0.591	0.98

Table 24 100×m.s.e. and mean of $1/\hat{\beta}$ in 96 step experiments ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Step size

	0.5		1.0		1.5		2.0	
Start	mean	100 × m.s.e.	mean	100 × m.s.e.	mean	100 × m.s.e.	mean	100 × m.s.e.
0.00	0.524	2.05	0.533	1.49	0.534	1.34	0.560	0.84
0.25	0.525	1.94	0.537	1.43	0.536	1.29	0.551	0.92
0.50	0.523	2.00	0.539	1.45	0.540	1.14	0.540	1.09
0.75	0.523	1.92	0.535	1.49	0.542	1.10	0.538	1.04
1.00	0.524	1.99	0.532	1.50	0.540	1.25	0.539	1.03
1.25	0.525	1.90	0.537	1.42	0.534	1.30	0.540	1.14
1.50	0.522	1.97	0.540	1.40	0.534	1.39	0.543	1.15
1.75	0.522	1.89	0.534	1.47	0.537	1.39	0.552	1.01
2.00	0.524	1.95	0.533	1.49	0.540	1.21	0.560	0.87
2.25	0.525	1.83	0.536	1.40	0.542	1.13	0.552	0.98
2.50	0.522	1.92	0.538	1.41	0.540	1.21	0.539	1.15
2.75	0.522	1.85	0.534	1.47	0.533	1.32	0.539	1.18
3.00	0.523	1.90	0.531	1.51	0.533	1.35	0.540	1.05
3.25	0.524	1.82	0.537	1.41	0.535	1.29	0.537	1.07
3.50	0.522	1.95	0.539	1.39	0.540	1.14	0.542	1.14
3.75	0.521	1.91	0.533	1.46	0.542	1.11	0.551	0.99
4.00	0.522	1.93	0.532	1.48	0.539	1.25	0.560	0.84

APPENDIX 10 TABLES TO ACCOMPANY SECTION 4.2

Table 1 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 48 observation UDTR experiments for step size 0.5 ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E_{WE}^*	E_{PV}^*	A_{WE}^*
-2.00	5.93	4.11	3.95	4.04	3.94	4.38	4.39	4.17	3.99	4.10	3.96
-1.75	5.10	3.87	4.03	4.14	3.94	4.50	4.50	4.17	4.12	4.22	3.96
-1.50	4.23	3.51	3.88	4.01	3.94	4.31	4.33	4.17	3.95	4.06	3.96
-1.25	3.89	3.48	3.82	3.97	3.94	4.19	4.21	4.17	3.89	3.99	3.96
-1.00	3.49	3.40	3.71	3.87	3.94	4.00	4.04	4.17	3.71	3.85	3.96
-0.75	3.47	3.53	3.71	3.87	3.94	3.95	4.01	4.17	3.72	3.88	3.96
-0.50	3.43	3.58	3.66	3.76	3.94	3.89	3.98	4.17	3.64	3.81	3.96
-0.25	3.45	3.62	3.61	3.69	3.94	3.82	3.91	4.17	3.60	3.77	3.96
0.00	3.54	3.71	3.66	3.70	3.94	3.88	3.97	4.17	3.64	3.72	3.96
0.25	3.72	3.81	3.78	3.89	3.94	3.98	4.06	4.17	3.81	3.83	3.96
0.50	3.86	3.82	3.86	4.00	3.94	4.12	4.17	4.17	3.86	3.76	3.96
0.75	4.27	4.02	4.15	4.37	3.94	4.54	4.54	4.17	4.23	4.01	3.96
1.00	4.87	4.34	4.47	4.70	3.94	4.91	4.89	4.17	4.53	4.08	3.96
1.25	5.52	4.63	4.74	4.93	3.94	5.25	5.22	4.17	4.82	4.32	3.96
1.50	6.62	5.21	5.12	5.24	3.94	5.61	5.55	4.17	5.20	4.54	3.96
1.75	7.91	5.86	5.39	5.46	3.94	5.96	5.88	4.17	5.54	4.84	3.96
2.00	9.99	7.03	5.76	5.71	3.94	6.34	6.23	4.17	5.84	4.91	3.96

Table 2 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 48 observation UDTR experiments for step size 1.0 ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E_{WE}^*	E_{PV}^*	A_{WE}^*
-2.00	4.68	4.32	4.88	4.96	4.86	5.32	5.44	5.34	5.04	5.29	5.22
-1.75	4.49	4.34	4.79	4.91	4.86	5.00	5.11	4.92	4.67	4.87	4.75
-1.50	4.37	4.50	4.87	4.95	4.90	4.83	4.96	4.77	4.60	4.77	4.54
-1.25	4.38	4.61	4.83	4.91	4.89	5.08	5.23	5.19	4.84	5.06	5.01
-1.00	4.23	4.50	4.62	4.70	4.86	5.03	5.19	5.34	4.76	4.99	5.22
-0.75	4.30	4.60	4.61	4.70	4.86	4.67	4.85	4.92	4.48	4.69	4.75
-0.50	4.47	4.79	4.75	4.80	4.90	4.68	4.87	4.77	4.48	4.66	4.54
-0.25	4.61	4.86	4.81	4.86	4.89	5.05	5.26	5.19	4.79	4.97	5.01
0.00	4.51	4.73	4.67	4.70	4.86	5.08	5.26	5.34	4.83	5.05	5.22
0.25	4.65	4.77	4.74	4.80	4.86	4.82	4.98	4.92	4.64	4.80	4.75
0.50	4.86	4.89	4.90	4.99	4.90	4.88	4.95	4.77	4.68	4.69	4.54
0.75	5.08	4.97	5.02	5.19	4.89	5.36	5.40	5.19	5.11	5.08	5.01
1.00	5.35	5.06	5.17	5.38	4.86	5.66	5.68	5.34	5.38	5.36	5.22
1.25	5.58	5.04	5.21	5.42	4.86	5.47	5.49	4.92	5.15	5.05	4.75
1.50	6.08	5.26	5.42	5.65	4.90	5.57	5.51	4.77	5.28	4.99	4.54
1.75	6.55	5.40	5.56	5.77	4.89	6.02	5.95	5.19	5.67	5.36	5.01
2.00	7.19	5.61	5.67	5.89	4.86	6.30	6.22	5.34	5.93	5.65	5.22

Note: A_{DM}, A_{WE} and A_{WE}^* denote columns for asymptotic predicted m.s.e.'s of E_{DM}, E_{WE} and E_{WE}^* respectively

Table 3 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 48 observation UDTR experiments for step size 1.5 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E_{WE}^*	E_{PV}^*	A_{WE}^*
-2.00	5.17	5.70	6.19	6.11	6.18	5.76	6.10	5.74	5.57	5.37	5.52
-1.75	5.47	5.98	6.30	6.25	6.23	7.43	7.71	7.58	6.99	7.19	7.45
-1.50	5.39	5.71	5.88	5.94	6.01	7.54	7.76	8.08	7.06	7.56	8.04
-1.25	5.30	5.67	5.74	5.79	5.71	6.57	6.73	6.73	6.24	6.66	6.65
-1.00	5.34	5.90	5.89	5.86	5.63	5.16	5.34	4.83	5.04	5.38	4.67
-0.75	5.57	6.14	6.07	6.00	5.87	4.65	4.97	4.34	4.63	4.67	4.13
-0.50	6.09	6.63	6.55	6.48	6.18	6.02	6.40	5.74	5.84	5.57	5.52
-0.25	5.81	6.24	6.18	6.08	6.23	7.27	7.61	7.58	6.86	6.97	7.45
0.00	5.51	5.77	5.73	5.70	6.01	7.32	7.68	8.08	6.87	7.40	8.04
0.25	5.56	5.71	5.68	5.73	5.71	6.51	6.81	6.73	6.19	6.75	6.65
0.50	5.70	5.84	5.82	5.83	5.63	5.21	5.39	4.83	5.05	5.43	4.67
0.75	6.15	6.05	6.06	6.19	5.87	4.87	4.93	4.34	4.79	4.69	4.13
1.00	6.73	6.47	6.54	6.70	6.18	6.34	6.25	5.74	6.15	5.54	5.52
1.25	6.85	6.49	6.61	6.80	6.23	7.97	7.87	7.58	7.58	7.27	7.45
1.50	6.66	6.13	6.29	6.51	6.01	7.96	8.01	8.08	7.55	7.77	8.04
1.75	6.62	5.88	6.08	6.32	5.71	7.04	7.14	6.73	6.68	7.02	6.65
2.00	7.35	6.24	6.43	6.68	5.63	5.99	6.01	4.83	5.78	5.90	4.67

Table 4 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 48 observation UDTR experiments for step size 2.0 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E_{WE}^*	E_{PV}^*	A_{WE}^*
-2.00	7.50	7.88	8.17	8.13	8.04	11.85	12.07	12.68	11.16	12.41	12.67
-1.75	6.84	6.81	6.93	7.12	7.29	10.30	10.25	11.02	9.59	9.94	10.94
-1.50	6.41	6.60	6.61	6.73	6.34	7.79	7.70	7.54	7.40	7.34	7.39
-1.25	5.68	6.22	6.16	6.12	5.81	4.56	4.67	4.13	4.56	4.80	3.98
-1.00	5.88	6.91	6.80	6.52	6.32	3.87	4.29	3.40	4.01	3.75	3.26
-0.75	7.23	8.27	8.16	7.87	7.54	6.40	7.06	6.02	6.32	4.40	5.84
-0.50	8.54	9.48	9.40	9.07	8.41	10.16	10.85	9.87	9.72	7.35	9.67
-0.25	8.46	9.12	9.06	8.83	8.48	11.93	12.46	12.32	11.31	11.21	12.21
0.00	7.71	8.12	8.09	7.97	8.04	11.61	12.10	12.68	10.88	12.34	12.67
0.25	6.87	7.09	7.07	7.08	7.29	10.13	10.72	11.02	9.48	10.36	10.94
0.50	6.14	6.40	6.39	6.32	6.34	7.33	7.88	7.54	6.97	7.21	7.39
0.75	5.93	6.01	5.99	5.99	5.81	4.44	4.76	4.13	4.45	4.80	3.98
1.00	6.85	6.54	6.54	6.75	6.32	4.11	4.08	3.40	4.21	3.86	3.26
1.25	8.62	7.86	7.90	8.33	7.54	6.79	6.43	6.02	6.72	4.46	5.84
1.50	9.98	9.08	9.21	9.58	8.41	10.88	10.28	9.87	10.36	7.31	9.67
1.75	9.94	9.23	9.43	9.67	8.48	12.89	12.46	12.32	12.31	11.48	12.21
2.00	8.84	8.23	8.46	8.64	8.04	12.21	12.21	12.68	11.59	12.53	12.67

Table 5 100×bias of estimators of $L_{1/\sqrt{2}}$ in 48 observation UDTR experiments for step size 0.5 ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
-2.00	-17.47	-10.43	-1.26	-1.46	2.27	-3.07	-2.30	-2.26	-1.57	1.75
-1.75	-14.27	-8.07	-1.58	-1.75	2.27	-3.22	-2.47	-2.50	-1.82	1.75
-1.50	-10.69	-5.26	-1.01	-1.26	2.27	-2.65	-1.92	-1.87	-1.20	1.75
-1.25	-8.43	-3.88	-1.45	-1.75	2.27	-2.99	-2.30	-2.28	-1.62	1.75
-1.00	-5.15	-1.41	-0.23	-0.68	2.27	-1.68	-1.06	-1.07	-0.46	1.75
-0.75	-3.17	-0.33	-0.01	-0.55	2.27	-1.24	-0.73	-0.63	-0.10	1.75
-0.50	-0.86	1.18	1.13	0.46	2.27	0.07	0.42	0.55	0.95	1.75
-0.25	0.43	1.55	1.41	0.89	2.27	0.74	0.92	1.08	1.29	1.75
0.00	3.00	3.27	3.24	3.02	2.27	2.74	2.71	3.05	3.08	1.75
0.25	4.61	3.96	4.03	4.21	2.27	3.83	3.62	4.05	3.90	1.75
0.50	6.57	5.07	5.11	5.53	2.27	5.05	4.65	5.16	4.86	1.75
0.75	8.67	6.22	5.94	6.45	2.27	6.24	5.71	6.25	5.83	1.75
1.00	11.73	8.42	7.41	7.86	2.27	7.86	7.21	7.88	7.36	1.75
1.25	14.09	9.84	7.68	7.89	2.27	8.28	7.58	8.24	7.67	1.75
1.50	17.54	12.38	8.63	8.65	2.27	9.38	8.61	9.21	8.59	1.75
1.75	20.82	14.71	8.86	8.66	2.27	9.69	8.88	9.49	8.84	1.75
2.00	25.60	18.61	10.24	9.89	2.27	10.98	10.13	10.70	10.04	1.75

Table 6 100×bias of estimators of $L_{1/\sqrt{2}}$ in 48 observation UDTR experiments for step size 1.0 ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}^*
-2.00	-7.51	-0.17	3.03	1.99	3.87	0.42	1.65	1.00	2.30	2.51
-1.75	-5.57	0.92	2.95	1.93	3.99	0.05	1.24	0.80	2.10	2.39
-1.50	-3.18	2.50	3.61	2.54	4.01	1.55	2.73	2.13	3.40	3.19
-1.25	-1.99	2.79	3.17	2.16	3.89	1.34	2.46	1.94	3.15	3.31
-1.00	-0.54	3.43	3.39	2.24	3.87	1.08	2.08	1.71	2.84	2.51
-0.75	0.72	3.82	3.54	2.50	3.99	1.22	2.05	1.90	2.88	2.39
-0.50	2.14	4.39	4.11	3.13	4.01	2.69	3.26	3.10	3.82	3.19
-0.25	2.98	4.35	4.15	3.45	3.89	3.20	3.45	3.67	4.10	3.31
0.00	3.88	4.41	4.38	3.90	3.87	3.01	2.91	3.53	3.64	2.51
0.25	4.97	4.62	4.76	4.65	3.99	3.23	2.81	3.87	3.69	2.39
0.50	6.39	5.16	5.43	5.64	4.01	5.04	4.31	5.32	4.86	3.19
0.75	7.32	5.15	5.41	5.94	3.89	5.46	4.51	5.69	5.00	3.31
1.00	8.64	5.67	5.75	6.40	3.87	5.28	4.14	5.62	4.79	2.51
1.25	10.25	6.36	6.07	6.85	3.99	5.39	4.17	5.84	4.91	2.39
1.50	12.54	7.78	6.91	7.66	4.01	7.05	5.73	7.29	6.25	3.19
1.75	14.14	8.45	6.75	7.43	3.89	7.13	5.73	7.51	6.43	3.31
2.00	16.33	9.79	7.00	7.71	3.87	6.91	5.47	7.46	6.36	2.51

Note: A_{DM} and A_{WE} denote columns for asymptotic predicted biases of E_{DM} and E_{WE} respectively.

Table 7 100×bias of estimators of $L_{1/\sqrt{2}}$ in 48 observation UDTR experiments for step size 1.5 ($\beta = \pi/3.0^{1/2}$; based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
-2.00	-1.83	5.73	7.01	5.45	6.31	6.54	8.18	6.80	8.61	8.31
-1.75	-1.73	5.03	5.59	3.99	5.16	4.71	6.28	5.09	6.95	6.92
-1.50	-1.91	3.99	4.05	2.39	3.95	0.47	2.05	1.14	2.98	2.06
-1.25	-0.64	4.30	4.06	2.44	3.89	-1.84	-0.39	-0.86	0.89	-1.15
-1.00	1.62	5.83	5.51	3.84	5.03	-0.06	1.19	0.97	2.58	0.41
-0.75	3.81	7.16	6.84	5.38	6.25	4.50	5.53	5.00	6.34	5.02
-0.50	5.61	8.10	7.84	6.59	6.31	8.20	8.97	8.33	9.33	8.31
-0.25	4.79	6.46	6.32	5.26	5.16	6.95	7.21	7.12	7.75	6.92
0.00	4.08	4.92	4.88	4.10	3.95	3.02	2.83	3.61	3.82	2.06
0.25	4.44	4.27	4.34	4.11	3.89	0.17	-0.46	1.04	0.85	-1.15
0.50	5.99	5.11	5.29	5.28	5.03	1.48	0.42	2.36	1.84	0.41
0.75	8.23	6.42	6.71	7.17	6.25	5.97	4.66	6.35	5.49	5.02
1.00	10.23	7.51	7.89	8.61	6.31	9.88	8.38	9.90	8.77	8.31
1.25	10.24	6.71	7.02	7.82	5.16	9.11	7.34	9.12	7.83	6.92
1.50	9.93	5.57	5.59	6.48	3.95	5.02	3.16	5.51	4.13	2.06
1.75	10.80	5.47	5.06	6.17	3.89	2.01	0.08	2.76	1.35	-1.15
2.00	13.43	7.34	6.33	7.43	5.03	3.39	1.42	4.24	2.84	0.41

Table 8 100×bias of estimators of $L_{1/\sqrt{2}}$ in 48 observation UDTR experiments for step size 2.0 ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
-2.00	-3.90	3.85	3.99	1.82	2.74	-0.29	1.75	0.62	3.13	1.30
-1.75	-5.74	0.97	0.74	-1.43	-0.09	-8.63	-6.72	-7.22	-4.80	-8.07
-1.50	-3.90	2.01	1.67	-0.52	0.51	-9.24	-7.43	-7.80	-5.46	-9.64
-1.25	0.42	5.68	5.32	3.20	4.16	-3.52	-1.93	-2.43	-0.18	-3.64
-1.00	6.95	11.40	11.09	9.12	8.79	6.59	8.08	7.14	9.09	6.21
-0.75	10.41	14.09	13.87	12.00	11.75	14.93	16.16	14.90	16.56	15.16
-0.50	10.70	13.54	13.39	11.75	11.22	17.84	18.75	17.47	18.76	18.32
-0.25	6.83	8.81	8.73	7.35	7.45	12.02	12.31	12.04	12.87	12.75
0.00	3.31	4.39	4.36	3.34	2.74	2.50	2.18	3.26	3.60	1.30
0.25	0.40	0.45	0.46	0.03	-0.09	-6.45	-7.35	-5.26	-5.46	-8.07
0.50	1.35	0.59	0.66	0.64	0.51	-7.92	-9.26	-6.60	-7.18	-9.64
0.75	5.48	4.00	4.15	4.55	4.16	-2.27	-3.99	-1.29	-2.13	-3.64
1.00	11.37	9.06	9.31	10.10	8.79	7.39	5.48	7.86	6.66	6.21
1.25	15.50	12.31	12.66	13.71	11.75	15.91	13.86	15.91	14.41	15.16
1.50	16.34	12.31	12.73	13.87	11.22	19.45	17.19	18.96	17.22	18.32
1.75	13.45	8.53	8.83	10.01	7.45	14.19	11.66	14.06	12.20	12.75
2.00	10.59	4.84	4.80	5.99	2.74	4.90	2.26	5.58	3.71	1.30

Table 9 Mean and 100× m.s.e. of $1/\tilde{\tilde{\beta}}$ in 48 observation UDTR experiments ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

	Step size							
	0.5		1.0		1.5		2.0	
Start	mean	100× m.s.e.	mean	100× m.s.e.	mean	100× m.s.e.	mean	100× m.s.e
-2.00	0.492	6.74	0.496	4.16	0.496	4.25	0.600	2.13
-1.75	0.491	6.05	0.491	4.12	0.523	3.15	0.553	2.15
-1.50	0.494	5.96	0.488	4.14	0.520	2.62	0.476	3.59
-1.25	0.489	5.41	0.498	3.98	0.489	3.11	0.410	5.91
-1.00	0.485	5.63	0.487	3.95	0.460	4.42	0.396	7.03
-0.75	0.475	5.04	0.483	3.92	0.464	4.64	0.434	5.81
-0.50	0.463	5.23	0.482	4.08	0.494	4.14	0.511	4.07
-0.25	0.453	5.00	0.484	3.97	0.518	3.04	0.582	2.54
0.00	0.449	5.35	0.481	3.85	0.520	2.68	0.600	2.01
0.25	0.453	5.01	0.476	3.96	0.493	3.12	0.554	1.98
0.50	0.459	5.22	0.478	3.91	0.462	4.31	0.478	3.43
0.75	0.477	5.21	0.490	3.88	0.462	4.71	0.415	5.57
1.00	0.491	5.71	0.493	3.86	0.493	4.25	0.395	6.97
1.25	0.506	6.40	0.495	4.10	0.524	3.16	0.434	6.10
1.50	0.516	7.30	0.495	4.29	0.529	2.82	0.509	4.23
1.75	0.533	8.61	0.507	4.60	0.505	3.15	0.584	2.89
2.00	0.533	9.13	0.505	4.60	0.476	4.51	0.608	2.46

Table 10 Numbers of 48 observation UDTR experiments out of 2000 where m.l.e.'s of parameters can be obtained ($\beta=\pi/3.0^{1/2}$).

Start	Step size 0.5	1.0	1.5	2.0
-2.00	1969	1927	1603	992
-1.75	1978	1956	1492	1305
-1.50	1979	1930	1607	1501
-1.25	1994	1906	1791	1437
-1.00	1990	1944	1773	1301
-0.75	1995	1966	1727	1174
-0.50	1995	1939	1591	969
-0.25	1995	1906	1434	823
0.00	1995	1921	1541	918
0.25	1997	1958	1772	1234
0.50	1993	1932	1786	1471
0.75	1992	1913	1731	1483
1.00	1997	1925	1604	1319
1.25	1995	1953	1493	1191
1.50	1987	1936	1554	998
1.75	1992	1897	1736	885
2.00	1992	1916	1780	941

Table 11 Values of mean and m.s.e. of $L_{1/\sqrt{2}}^*$ and $1/\beta^*$ in 48 observation UDTR experiments for step size 0.5 ($\beta=\pi/3.0^{1/2}$).

Start	$L_{1/\sqrt{2}}^*$ 100 × mean	$L_{1/\sqrt{2}}^*$ 100 × m.s.e.	$A_{L_{1/\sqrt{2}}}$	$1/\beta^*$ mean	$1/\beta^*$ 100 × m.s.e.	$A_{1/\beta}$
-2.00	-0.16	3.91	3.68	0.496	3.89	5.22
-1.75	-0.75	3.96	3.68	0.497	3.90	5.22
-1.50	-0.12	3.82	3.68	0.500	3.98	5.22
-1.25	-0.84	3.78	3.68	0.500	4.03	5.22
-1.00	-0.02	3.77	3.68	0.499	4.27	5.22
-0.75	-0.47	3.79	3.68	0.498	4.46	5.22
-0.50	0.00	3.71	3.68	0.496	4.64	5.22
-0.25	-0.65	3.64	3.68	0.495	4.69	5.22
0.00	0.25	3.58	3.68	0.493	4.62	5.22
0.25	-0.15	3.70	3.68	0.494	4.72	5.22
0.50	0.10	3.62	3.68	0.489	4.90	5.22
0.75	-0.17	3.78	3.68	0.491	4.72	5.22
1.00	0.53	3.78	3.68	0.492	4.69	5.22
1.25	-0.03	4.00	3.68	0.488	4.70	5.22
1.50	0.27	4.03	3.68	0.489	4.86	5.22
1.75	-0.38	4.31	3.68	0.494	5.21	5.22
2.00	0.68	4.25	3.68	0.488	4.76	5.22

Table 12 Values of mean and m.s.e. of $L_{1/\sqrt{2}}^*$ and $1/\beta^*$ in 48 observation UDTR experiments for step size 1.0 ($\beta=\pi/3.0^{1/2}$).

Start	$L_{1/\sqrt{2}}^*$ 100 × mean	$L_{1/\sqrt{2}}^*$ 100 × m.s.e.	$A_{L_{1/\sqrt{2}}}$	$1/\beta^*$ mean	$1/\beta^*$ 100 × m.s.e.	$A_{1/\beta}$
-2.00	-0.04	4.48	4.36	0.521	3.08	3.35
-1.75	-0.42	4.48	4.38	0.519	2.85	3.27
-1.50	-0.14	4.54	4.38	0.516	3.13	3.34
-1.25	-0.43	4.43	4.36	0.520	3.12	3.42
-1.00	-0.06	4.23	4.36	0.519	3.09	3.35
-0.75	-0.35	4.31	4.38	0.520	2.97	3.27
-0.50	-0.47	4.39	4.38	0.517	3.28	3.34
-0.25	-0.52	4.28	4.36	0.518	3.44	3.42
0.00	-0.14	4.21	4.36	0.521	3.16	3.35
0.25	-0.45	4.27	4.38	0.521	3.12	3.27
0.50	-0.26	4.36	4.38	0.515	3.15	3.34
0.75	-0.38	4.41	4.36	0.515	3.19	3.42
1.00	-0.17	4.36	4.36	0.521	3.17	3.35
1.25	-0.25	4.43	4.38	0.518	3.11	3.27
1.50	0.04	4.59	4.38	0.513	3.32	3.34
1.75	-0.17	4.61	4.36	0.514	3.41	3.42
2.00	0.21	4.61	4.36	0.516	3.29	3.35

Note: $A_{L_{1/\sqrt{2}}}$ and $A_{1/\beta}$ denote columns for asymptotic predicted variances of $L_{1/\sqrt{2}}^*$ and $1/\beta^*$ respectively.

Table 13 Values of mean and m.s.e. of $L_{1/\sqrt{2}}^*$ and $1/\beta^*$ in 48 observation UDTR experiments for step size 1.5 ($\beta=\pi/3.0^{1/2}$).

Start	$L_{1/\sqrt{2}}^*$ 100 x mean	100 x m.s.e.	$A_{L_{1/\sqrt{2}}}$	$1/\beta^*$ mean	100 x m.s.e.	$A_{1/\beta}$
-2.00	1.14	5.18	5.28	0.514	3.27	3.04
-1.75	0.59	5.23	5.06	0.533	2.44	3.40
-1.50	-0.31	5.12	4.98	0.541	2.08	3.14
-1.25	-0.16	5.14	5.12	0.531	2.32	2.63
-1.00	0.48	5.32	5.30	0.514	3.35	2.39
-0.75	0.56	5.42	5.37	0.509	3.54	2.56
-0.50	1.53	5.33	5.28	0.517	3.46	3.04
-0.25	1.11	5.02	5.06	0.531	2.61	3.40
0.00	0.40	4.91	4.98	0.543	2.19	3.14
0.25	0.02	5.02	5.12	0.537	2.53	2.63
0.50	0.09	5.13	5.30	0.520	3.33	2.39
0.75	0.37	5.30	5.37	0.509	3.57	2.56
1.00	1.49	5.27	5.28	0.518	3.43	3.04
1.25	1.56	5.29	5.06	0.532	2.53	3.40
1.50	0.59	5.36	4.98	0.543	2.24	3.14
1.75	0.08	5.29	5.12	0.533	2.28	2.63
2.00	0.73	5.57	5.30	0.515	3.31	2.39

Table 14 Values of mean and m.s.e. of $L_{1/\sqrt{2}}^*$ and $1/\beta^*$ in 48 observation UDTR experiments for step size 2.0 ($\beta=\pi/3.0^{1/2}$).

Start	$L_{1/\sqrt{2}}^*$ 100 x mean	100 x m.s.e.	$A_{L_{1/\sqrt{2}}}$	$1/\beta^*$ mean	100 x m.s.e.	$A_{1/\beta}$
-2.00	0.16	7.14	5.38	0.602	1.48	3.93
-1.75	-2.29	6.25	5.76	0.575	1.66	2.84
-1.50	-1.42	6.19	6.30	0.527	2.87	2.12
-1.25	1.29	5.93	6.74	0.473	5.01	1.88
-1.00	5.00	6.10	6.99	0.456	6.30	2.07
-0.75	6.76	6.32	6.96	0.477	5.32	2.69
-0.50	7.25	6.91	6.49	0.528	3.67	3.70
-0.25	4.27	7.38	5.70	0.588	2.17	4.44
0.00	0.92	6.95	5.38	0.606	1.58	3.93
0.25	-1.89	6.22	5.76	0.578	1.70	2.84
0.50	-1.38	5.82	6.30	0.529	2.83	2.12
0.75	1.25	5.69	6.74	0.483	4.66	1.88
1.00	4.59	6.07	6.99	0.457	6.17	2.07
1.25	6.92	6.46	6.96	0.478	5.55	2.69
1.50	7.75	6.92	6.49	0.527	3.75	3.70
1.75	5.28	7.69	5.70	0.586	2.15	4.44
2.00	1.93	7.38	5.38	0.608	1.74	3.93

Table 15 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 96 observation UDTR experiments for step size 0.5 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E_{WE}^*	E_{PV}^*	A_{WE}^*
-2.00	2.42	2.03	2.11	2.12	1.99	2.22	2.23	2.10	2.11	2.11	1.99
-1.75	2.07	1.84	1.96	1.99	1.99	2.11	2.12	2.10	1.97	1.97	1.99
-1.50	2.03	1.91	2.06	2.09	1.99	2.16	2.17	2.10	2.07	2.06	1.99
-1.25	1.84	1.80	1.92	1.95	1.99	2.06	2.07	2.10	1.92	1.92	1.99
-1.00	1.90	1.92	2.02	2.05	1.99	2.14	2.15	2.10	2.02	2.03	1.99
-0.75	1.81	1.86	1.91	1.94	1.99	2.02	2.04	2.10	1.90	1.91	1.99
-0.50	1.94	2.01	2.02	2.04	1.99	2.10	2.12	2.10	2.03	2.04	1.99
-0.25	1.87	1.92	1.92	1.93	1.99	2.01	2.03	2.10	1.90	1.91	1.99
0.00	2.01	2.06	2.05	2.05	1.99	2.13	2.15	2.10	2.04	2.05	1.99
0.25	1.95	1.96	1.96	1.98	1.99	2.05	2.06	2.10	1.95	1.95	1.99
0.50	2.04	2.02	2.04	2.06	1.99	2.12	2.13	2.10	2.03	2.03	1.99
0.75	2.13	2.04	2.08	2.12	1.99	2.21	2.20	2.10	2.08	2.07	1.99
1.00	2.43	2.28	2.32	2.36	1.99	2.40	2.38	2.10	2.32	2.30	1.99
1.25	2.54	2.28	2.29	2.32	1.99	2.44	2.42	2.10	2.29	2.27	1.99
1.50	2.88	2.47	2.41	2.42	1.99	2.55	2.53	2.10	2.44	2.40	1.99
1.75	3.19	2.61	2.42	2.40	1.99	2.61	2.58	2.10	2.43	2.39	1.99
2.00	3.80	3.03	2.65	2.59	1.99	2.81	2.77	2.10	2.66	2.62	1.99

Table 16 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 96 observation UDTR experiments for step size 1.0 ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E_{WE}^*	E_{PV}^*	A_{WE}^*
-2.00	2.24	2.30	2.52	2.52	2.50	2.68	2.72	2.70	2.57	2.60	2.64
-1.75	2.25	2.35	2.51	2.51	2.51	2.53	2.57	2.49	2.41	2.44	2.40
-1.50	2.34	2.49	2.61	2.60	2.53	2.53	2.58	2.44	2.44	2.48	2.32
-1.25	2.35	2.50	2.56	2.57	2.52	2.62	2.67	2.65	2.52	2.55	2.56
-1.00	2.31	2.47	2.49	2.49	2.50	2.63	2.68	2.70	2.55	2.58	2.64
-0.75	2.30	2.44	2.43	2.43	2.51	2.41	2.46	2.49	2.33	2.36	2.40
-0.50	2.45	2.58	2.56	2.55	2.53	2.53	2.59	2.44	2.42	2.46	2.32
-0.25	2.43	2.52	2.50	2.50	2.52	2.54	2.59	2.65	2.44	2.48	2.56
0.00	2.46	2.53	2.51	2.51	2.50	2.67	2.70	2.70	2.57	2.60	2.64
0.25	2.48	2.50	2.49	2.51	2.51	2.48	2.51	2.49	2.40	2.41	2.40
0.50	2.65	2.63	2.65	2.66	2.53	2.60	2.61	2.44	2.51	2.51	2.32
0.75	2.69	2.61	2.64	2.68	2.52	2.74	2.72	2.65	2.63	2.60	2.56
1.00	2.75	2.61	2.65	2.69	2.50	2.84	2.81	2.70	2.74	2.71	2.64
1.25	2.81	2.60	2.64	2.71	2.51	2.67	2.65	2.49	2.56	2.52	2.40
1.50	3.03	2.74	2.76	2.83	2.53	2.74	2.71	2.44	2.65	2.60	2.32
1.75	3.16	2.76	2.77	2.82	2.52	2.89	2.84	2.65	2.77	2.71	2.56
2.00	3.38	2.86	2.82	2.87	2.50	3.02	2.97	2.70	2.91	2.85	2.64

Note: A_{DM}, A_{WE} and A_{WE}^* denote columns for asymptotic predicted m.s.e.'s of E_{DM}, E_{WE} and E_{WE}^* respectively

Table 17 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 96 observation UDTR experiments for step size 1.5 ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E_{WE}^*	E_{PV}^*	A_{WE}^*
-2.00	2.98	3.38	3.55	3.48	3.29	3.35	3.50	3.21	3.26	3.42	3.10
-1.75	2.89	3.19	3.28	3.23	3.25	3.91	4.04	4.03	3.76	3.88	3.96
-1.50	2.97	3.21	3.25	3.21	3.08	4.11	4.18	4.06	3.92	3.96	4.04
-1.25	2.71	2.90	2.91	2.88	2.93	3.26	3.30	3.37	3.14	3.13	3.33
-1.00	2.81	3.07	3.06	2.99	2.94	2.51	2.57	2.42	2.45	2.49	2.34
-0.75	3.00	3.28	3.25	3.17	3.13	2.37	2.46	2.30	2.34	2.45	2.19
-0.50	3.34	3.58	3.54	3.48	3.29	3.38	3.51	3.21	3.31	3.44	3.10
-0.25	3.14	3.28	3.26	3.23	3.25	3.93	4.01	4.03	3.77	3.87	3.96
0.00	3.15	3.24	3.23	3.20	3.08	4.08	4.15	4.06	3.88	3.94	4.04
0.25	2.96	2.98	2.97	3.00	2.93	3.31	3.39	3.37	3.18	3.21	3.33
0.50	3.03	3.05	3.05	3.04	2.94	2.54	2.58	2.42	2.48	2.51	2.34
0.75	3.32	3.27	3.28	3.29	3.13	2.47	2.44	2.30	2.45	2.44	2.19
1.00	3.71	3.56	3.59	3.66	3.29	3.56	3.46	3.21	3.47	3.39	3.10
1.25	3.55	3.35	3.40	3.47	3.25	4.23	4.14	4.03	4.06	3.96	3.96
1.50	3.57	3.34	3.39	3.46	3.08	4.32	4.29	4.06	4.15	4.09	4.04
1.75	3.31	3.01	3.05	3.13	2.93	3.41	3.45	3.37	3.31	3.28	3.33
2.00	3.59	3.18	3.20	3.27	2.94	2.70	2.69	2.42	2.63	2.60	2.34

Table 18 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 96 observation UDTR experiments for step size 2.0 ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	A_{WE}	E_{WE}^*	E_{PV}^*	A_{WE}^*
-2.00	3.86	4.09	4.16	4.12	4.06	6.18	6.27	6.35	5.92	5.97	6.34
-1.75	3.48	3.48	3.50	3.53	3.64	5.64	5.57	5.83	5.32	5.15	5.80
-1.50	3.10	3.13	3.13	3.17	3.17	4.20	4.12	4.23	3.98	3.79	4.16
-1.25	2.89	3.12	3.10	3.04	2.99	2.18	2.19	2.13	2.14	2.10	2.06
-1.00	3.36	3.85	3.82	3.61	3.54	2.00	2.14	1.89	2.05	2.23	1.82
-0.75	4.31	4.84	4.80	4.56	4.46	4.25	4.49	4.16	4.23	4.54	4.07
-0.50	4.80	5.24	5.20	5.00	4.84	6.70	6.92	6.61	6.48	6.79	6.51
-0.25	4.54	4.77	4.76	4.66	4.52	6.88	7.01	6.97	6.62	6.81	6.92
0.00	4.03	4.14	4.13	4.09	4.06	6.09	6.20	6.35	5.84	5.93	6.34
0.25	3.57	3.63	3.62	3.63	3.64	5.63	5.81	5.83	5.32	5.40	5.80
0.50	3.13	3.16	3.15	3.17	3.17	4.14	4.37	4.23	3.92	4.01	4.16
0.75	3.12	3.07	3.07	3.10	2.99	2.19	2.31	2.13	2.15	2.19	2.06
1.00	3.89	3.75	3.76	3.81	3.54	2.14	2.05	1.89	2.17	2.12	1.82
1.25	4.97	4.65	4.69	4.79	4.46	4.43	4.17	4.16	4.41	4.22	4.07
1.50	5.45	5.03	5.09	5.24	4.84	6.92	6.55	6.61	6.70	6.42	6.51
1.75	5.17	4.78	4.84	4.98	4.52	7.28	7.02	6.97	7.04	6.80	6.92
2.00	4.52	4.25	4.31	4.40	4.06	6.49	6.47	6.35	6.23	6.15	6.34

Table 19 100×bias of estimators of $L_{1/\sqrt{2}}$ in 96 observation UDTR experiments for step size 0.5 ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
-2.00	-7.82	-4.23	0.45	0.43	2.27	-0.53	-0.17	-0.20	0.16	1.75
-1.75	-5.92	-2.77	0.51	0.52	2.27	-0.46	-0.10	-0.10	0.26	1.75
-1.50	-4.33	-1.58	0.53	0.50	2.27	-0.41	-0.06	-0.08	0.27	1.75
-1.25	-2.91	-0.61	0.60	0.51	2.27	-0.43	-0.10	-0.04	0.30	1.75
-1.00	-1.48	0.41	0.99	0.78	2.27	0.02	0.32	0.32	0.64	1.75
-0.75	-0.30	1.14	1.28	1.02	2.27	0.43	0.67	0.78	1.04	1.75
-0.50	0.60	1.62	1.57	1.23	2.27	0.84	1.01	1.12	1.31	1.75
-0.25	1.68	2.24	2.16	1.91	2.27	1.59	1.67	1.83	1.94	1.75
0.00	2.52	2.66	2.67	2.50	2.27	2.19	2.16	2.35	2.36	1.75
0.25	3.44	3.12	3.17	3.23	2.27	2.80	2.67	3.01	2.93	1.75
0.50	4.33	3.60	3.63	3.76	2.27	3.33	3.12	3.48	3.32	1.75
0.75	5.62	4.41	4.29	4.48	2.27	4.18	3.91	4.31	4.09	1.75
1.00	6.78	5.16	4.66	4.76	2.27	4.63	4.31	4.73	4.47	1.75
1.25	8.20	6.11	5.05	5.06	2.27	5.11	4.77	5.22	4.93	1.75
1.50	9.91	7.39	5.53	5.41	2.27	5.59	5.23	5.67	5.35	1.75
1.75	11.64	8.64	5.74	5.50	2.27	5.79	5.41	5.87	5.54	1.75
2.00	13.57	10.14	5.95	5.58	2.27	6.01	5.62	6.01	5.67	1.75

Table 20 100×bias of estimators of $L_{1/\sqrt{2}}$ in 96 observation UDTR experiments for step size 1.0 ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_B	E_{BD}	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
-2.00	-1.56	2.16	3.73	3.25	3.87	1.76	2.34	2.13	2.80	2.51
-1.75	-0.66	2.64	3.64	3.12	3.99	1.48	2.05	1.86	2.52	2.39
-1.50	0.28	3.19	3.72	3.18	4.01	2.34	2.90	2.61	3.27	3.19
-1.25	1.04	3.50	3.66	3.10	3.89	2.43	2.97	2.76	3.39	3.31
-1.00	1.86	3.88	3.82	3.26	3.87	1.97	2.45	2.34	2.91	2.51
-0.75	2.41	3.99	3.84	3.29	3.99	1.93	2.33	2.26	2.76	2.39
-0.50	3.15	4.34	4.18	3.65	4.01	3.15	3.41	3.41	3.79	3.19
-0.25	3.54	4.28	4.17	3.73	3.89	3.42	3.52	3.71	3.93	3.31
0.00	4.16	4.45	4.43	4.14	3.87	3.12	3.04	3.41	3.47	2.51
0.25	4.40	4.25	4.32	4.23	3.99	2.90	2.66	3.18	3.08	2.39
0.50	5.02	4.47	4.62	4.64	4.01	4.08	3.69	4.23	4.01	3.19
0.75	5.73	4.73	4.88	5.03	3.89	4.51	4.02	4.76	4.43	3.31
1.00	6.57	5.11	5.18	5.42	3.87	4.19	3.62	4.49	4.08	2.51
1.25	7.23	5.33	5.22	5.54	3.99	4.10	3.49	4.40	3.93	2.39
1.50	8.12	5.83	5.40	5.73	4.01	5.10	4.45	5.31	4.81	3.19
1.75	9.06	6.31	5.48	5.79	3.89	5.40	4.72	5.65	5.11	3.31
2.00	10.22	7.02	5.66	5.93	3.87	4.96	4.25	5.27	4.72	2.51

Note: A_{DM} and A_{WE} denote columns for asymptotic predicted biases of E_{DM} and E_{WE} respectively.

Table 21 100×bias of estimators of $L_{1/\sqrt{2}}$ in 96 observation UDTR experiments for step size 1.5 ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_β	$E_{\beta D}$	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}^*
-2.00	2.43	6.32	6.94	6.09	6.31	7.52	8.29	7.68	8.63	8.31
-1.75	1.80	5.24	5.49	4.65	5.16	6.02	6.78	6.12	7.06	6.92
-1.50	1.43	4.38	4.38	3.58	3.95	1.72	2.48	2.08	3.01	2.06
-1.25	1.77	4.32	4.18	3.34	3.89	-1.22	-0.53	-0.69	0.21	-1.15
-1.00	3.44	5.63	5.45	4.59	5.03	0.42	1.01	0.90	1.73	0.41
-0.75	5.17	6.96	6.79	5.96	6.25	4.90	5.36	5.26	5.98	5.02
-0.50	6.21	7.55	7.41	6.69	6.31	8.44	8.78	8.60	9.13	8.31
-0.25	5.23	6.12	6.05	5.45	5.16	7.28	7.39	7.35	7.68	6.92
0.00	4.34	4.74	4.71	4.34	3.95	2.92	2.80	3.21	3.31	2.06
0.25	4.21	4.21	4.24	4.05	3.89	-0.29	-0.66	0.16	0.07	-1.15
0.50	5.43	5.06	5.15	5.09	5.03	0.97	0.39	1.40	1.14	0.41
0.75	7.21	6.43	6.59	6.68	6.25	5.45	4.74	5.80	5.40	5.02
1.00	8.44	7.20	7.39	7.65	6.31	9.15	8.37	9.28	8.76	8.31
1.25	7.82	6.13	6.30	6.63	5.16	8.18	7.29	8.21	7.58	6.92
1.50	7.23	5.05	5.09	5.50	3.95	3.86	2.94	4.16	3.46	2.06
1.75	7.49	4.91	4.72	5.20	3.89	0.73	-0.22	1.20	0.51	-1.15
2.00	9.21	6.25	5.77	6.24	5.03	1.98	1.00	2.45	1.76	0.41

Table 22 100×bias of estimators of $L_{1/\sqrt{2}}$ in 96 observation UDTR experiments for step size 2.0 ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Start	E_M	E_β	$E_{\beta D}$	E_{DM}	A_{DM}	E_{WE}	E_{PV}	E_{WE}^*	E_{PV}^*	A_{WE}
-2.00	-0.09	3.81	3.84	2.81	2.74	1.08	2.08	1.52	2.79	1.30
-1.75	-2.67	0.76	0.62	-0.48	-0.09	-8.04	-7.12	-7.25	-6.00	-8.07
-1.50	-1.60	1.39	1.20	0.09	0.51	-9.27	-8.40	-8.57	-7.38	-9.64
-1.25	2.43	5.11	4.92	3.82	4.16	-3.45	-2.69	-2.84	-1.69	-3.64
-1.00	8.00	10.34	10.19	9.11	8.79	6.62	7.29	6.88	7.92	6.21
-0.75	11.23	13.17	13.06	12.02	11.75	15.05	15.62	15.17	16.04	15.16
-0.50	10.92	12.44	12.36	11.46	11.22	17.96	18.36	17.79	18.49	18.32
-0.25	7.46	8.51	8.47	7.72	7.45	12.82	12.94	12.78	13.22	12.75
0.00	3.35	3.88	3.86	3.37	2.74	2.33	2.15	2.66	2.83	1.30
0.25	0.05	0.14	0.14	-0.13	-0.09	-7.30	-7.80	-6.61	-6.70	-8.07
0.50	0.83	0.48	0.50	0.47	0.51	-8.81	-9.52	-8.15	-8.44	-9.64
0.75	4.79	4.13	4.19	4.32	4.16	-2.97	-3.87	-2.37	-2.79	-3.64
1.00	10.16	9.11	9.23	9.52	8.79	6.88	5.89	7.18	6.62	6.21
1.25	13.56	12.10	12.29	12.68	11.75	15.44	14.38	15.54	14.83	15.16
1.50	13.51	11.61	11.84	12.32	11.22	18.53	17.37	18.33	17.51	18.32
1.75	10.53	8.16	8.34	8.86	7.45	13.67	12.42	13.59	12.68	12.75
2.00	6.97	4.10	4.12	4.67	2.74	3.54	2.27	3.87	2.94	1.30

Table 23 Mean and 100×m.s.e. of $1/\tilde{\tilde{\beta}}$ in 96 observation UDTR experiments ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

	Step size							
	0.5		1.0		1.5		2.0	
Start	mean	100 × m.s.e.	mean	100 × m.s.e.	mean	100 × m.s.e.	mean	100 × m.s.e
-2.00	0.516	3.15	0.521	2.14	0.521	2.28	0.624	1.53
-1.75	0.523	3.02	0.515	2.21	0.546	1.67	0.578	1.25
-1.50	0.518	3.06	0.515	2.15	0.546	1.44	0.499	1.92
-1.25	0.520	2.86	0.521	2.12	0.519	1.75	0.434	3.53
-1.00	0.511	2.84	0.515	2.08	0.487	2.46	0.416	4.46
-0.75	0.512	2.73	0.509	2.18	0.488	2.53	0.455	3.45
-0.50	0.502	2.82	0.512	2.16	0.520	2.23	0.536	2.13
-0.25	0.499	2.73	0.515	2.07	0.543	1.58	0.605	1.59
0.00	0.495	2.87	0.511	2.06	0.546	1.42	0.624	1.48
0.25	0.500	2.71	0.505	2.19	0.519	1.73	0.577	1.18
0.50	0.499	2.69	0.508	2.17	0.486	2.39	0.497	1.91
0.75	0.511	2.77	0.517	2.13	0.486	2.55	0.436	3.42
1.00	0.517	3.02	0.520	2.07	0.520	2.28	0.415	4.42
1.25	0.528	3.07	0.515	2.25	0.547	1.66	0.457	3.52
1.50	0.532	3.27	0.518	2.19	0.550	1.48	0.535	2.27
1.75	0.540	3.47	0.526	2.26	0.524	1.76	0.605	1.75
2.00	0.540	3.69	0.524	2.20	0.493	2.41	0.626	1.58

Table 24 Numbers of 96 observation UDTR experiments out of 2000 where m.l.e.'s of parameters can be obtained ($\beta=\pi/3.0''$).

Start	Step size 0.5	1.0	1.5	2.0
-2.00	2000	2000	1920	1457
-1.75	2000	1999	1866	1739
-1.50	2000	1994	1926	1898
-1.25	2000	1997	1972	1756
-1.00	2000	1998	1965	1614
-0.75	2000	1999	1956	1636
-0.50	2000	1995	1931	1505
-0.25	2000	1998	1841	1325
0.00	2000	1997	1908	1417
0.25	2000	2000	1976	1710
0.50	2000	1998	1964	1883
0.75	2000	2000	1956	1767
1.00	2000	1997	1921	1609
1.25	2000	1999	1850	1616
1.50	2000	1994	1908	1485
1.75	2000	1996	1972	1324
2.00	2000	1996	1972	1415

Table 25 Values of mean and m.s.e. of $L^*_{1/\sqrt{2}}$ and $1/\beta^*$ in 96 observation UDTR experiments for step size 0.5 ($\beta=\pi/3.0^{1/2}$).

Start	$L^*_{1/\sqrt{2}}$ 100 × mean	$L^*_{1/\sqrt{2}}$ 100 × m.s.e.	$A_{L1/\sqrt{2}}$	$1/\beta^*$ mean	$1/\beta^*$ 100 × m.s.e.	$A_{1/\beta}$
-2.00	-0.05	2.05	1.84	0.521	2.14	2.61
-1.75	-0.06	1.91	1.84	0.526	2.18	2.61
-1.50	0.01	2.03	1.84	0.524	2.22	2.61
-1.25	-0.14	1.86	1.84	0.526	2.25	2.61
-1.00	0.02	1.98	1.84	0.521	2.23	2.61
-0.75	-0.03	1.88	1.84	0.526	2.35	2.61
-0.50	-0.01	1.99	1.84	0.521	2.43	2.61
-0.25	-0.05	1.84	1.84	0.524	2.43	2.61
0.00	0.06	1.96	1.84	0.521	2.51	2.61
0.25	-0.06	1.84	1.84	0.523	2.43	2.61
0.50	-0.03	1.87	1.84	0.517	2.32	2.61
0.75	-0.05	1.83	1.84	0.521	2.43	2.61
1.00	0.07	2.01	1.84	0.518	2.46	2.61
1.25	0.02	1.93	1.84	0.521	2.45	2.61
1.50	0.18	1.97	1.84	0.519	2.32	2.61
1.75	0.02	1.94	1.84	0.521	2.43	2.61
2.00	0.08	2.10	1.84	0.518	2.37	2.61

Table 26 Values of mean and m.s.e. of $L^*_{1/\sqrt{2}}$ and $1/\beta^*$ in 96 observation UDTR experiments for step size 1.0 ($\beta=\pi/3.0^{1/2}$).

Start	$L^*_{1/\sqrt{2}}$ 100 × mean	$L^*_{1/\sqrt{2}}$ 100 × m.s.e.	$A_{L1/\sqrt{2}}$	$1/\beta^*$ mean	$1/\beta^*$ 100 × m.s.e.	$A_{1/\beta}$
-2.00	0.32	2.23	2.18	0.539	1.59	1.68
-1.75	-0.03	2.26	2.19	0.538	1.50	1.63
-1.50	-0.17	2.34	2.19	0.536	1.57	1.67
-1.25	-0.14	2.27	2.18	0.539	1.65	1.71
-1.00	0.20	2.20	2.18	0.535	1.66	1.68
-0.75	-0.12	2.17	2.19	0.536	1.59	1.63
-0.50	-0.10	2.29	2.19	0.538	1.68	1.67
-0.25	-0.11	2.17	2.18	0.537	1.73	1.71
0.00	0.20	2.18	2.18	0.536	1.71	1.68
0.25	-0.30	2.18	2.19	0.537	1.62	1.63
0.50	-0.33	2.29	2.19	0.534	1.63	1.67
0.75	-0.06	2.24	2.18	0.536	1.71	1.71
1.00	0.23	2.19	2.18	0.537	1.69	1.68
1.25	-0.04	2.20	2.19	0.536	1.63	1.63
1.50	-0.14	2.30	2.19	0.535	1.64	1.67
1.75	-0.07	2.26	2.18	0.537	1.73	1.71
2.00	0.29	2.27	2.18	0.534	1.69	1.68

Note: $A_{L1/\sqrt{2}}$ and $A_{1/\beta}$ denote columns for asymptotic predicted variances of $L^*_{1/\sqrt{2}}$ and $1/\beta^*$ respectively.

Table 27 Values of mean and m.s.e. of $L_{1/\sqrt{2}}^*$ and $1/\beta^*$ in 96 observation UDTR experiments for step size 1.5 ($\beta=\pi/3.0^{1/2}$).

Start	$L_{1/\sqrt{2}}^*$ 100 x mean	$L_{1/\sqrt{2}}^*$ 100 x m.s.e.	$A_{L1/\sqrt{2}}$	$1/\beta^*$ mean	$1/\beta^*$ 100 x m.s.e.	$A_{1/\beta}$
-2.00	0.11	2.76	2.64	0.538	1.62	1.52
-1.75	0.13	2.52	2.53	0.539	1.52	1.70
-1.50	0.23	2.55	2.49	0.542	1.41	1.57
-1.25	0.28	2.54	2.56	0.543	1.33	1.31
-1.00	0.25	2.80	2.65	0.539	1.45	1.20
-0.75	-0.10	2.81	2.69	0.538	1.50	1.28
-0.50	0.22	2.72	2.64	0.540	1.65	1.52
-0.25	0.47	2.46	2.53	0.538	1.54	1.70
0.00	0.38	2.50	2.49	0.542	1.44	1.57
0.25	0.26	2.57	2.56	0.543	1.39	1.31
0.50	0.01	2.78	2.65	0.538	1.46	1.20
0.75	-0.28	2.85	2.69	0.537	1.51	1.28
1.00	0.15	2.76	2.64	0.540	1.64	1.52
1.25	0.51	2.51	2.53	0.538	1.52	1.70
1.50	0.38	2.59	2.49	0.541	1.40	1.57
1.75	0.41	2.57	2.56	0.541	1.38	1.31
2.00	0.13	2.86	2.65	0.538	1.43	1.20

Table 28 Values of mean and m.s.e. of $L_{1/\sqrt{2}}^*$ and $1/\beta^*$ in 96 observation UDTR experiments for step size 2.0 ($\beta=\pi/3.0^{1/2}$).

Start	$L_{1/\sqrt{2}}^*$ 100 x mean	$L_{1/\sqrt{2}}^*$ 100 x m.s.e.	$A_{L1/\sqrt{2}}$	$1/\beta^*$ mean	$1/\beta^*$ 100 x m.s.e.	$A_{1/\beta}$
-2.00	0.62	3.13	2.69	0.582	0.88	1.96
-1.75	-0.36	2.86	2.88	0.559	1.00	1.42
-1.50	0.28	3.14	3.15	0.538	1.28	1.06
-1.25	1.09	3.26	3.37	0.516	2.25	0.94
-1.00	2.80	3.45	3.49	0.499	3.20	1.03
-0.75	2.91	3.34	3.48	0.514	2.58	1.34
-0.50	3.27	2.96	3.25	0.544	1.62	1.85
-0.25	2.94	3.26	2.85	0.580	1.03	2.22
0.00	0.82	3.14	2.69	0.585	0.89	1.96
0.25	-0.52	2.90	2.88	0.560	1.00	1.42
0.50	-0.03	3.06	3.15	0.537	1.35	1.06
0.75	0.87	3.29	3.37	0.518	2.22	0.94
1.00	2.59	3.52	3.49	0.498	3.18	1.03
1.25	3.08	3.43	3.48	0.514	2.72	1.34
1.50	3.53	3.06	3.25	0.542	1.70	1.85
1.75	3.13	3.37	2.85	0.578	1.07	2.22
2.00	1.12	3.25	2.69	0.585	0.88	1.96

APPENDIX 11 TABLES TO ACCOMPANY SECTION 4.4

Table 1 10×m.s.e. of estimators of $L_{\sqrt{2}-1}$ in 48 observation experiments using Routine B for step size 0.5.

Start	E_{DM}	A_{DM}	E_{WE}	A_{WE}	E^*_{WE}	A^*_{WE}
-2.00	4.55	2.15	4.58	2.01	4.38	1.91
-1.75	3.78	2.15	3.78	2.01	3.60	1.91
-1.50	3.20	2.15	3.19	2.01	3.05	1.91
-1.25	2.75	2.15	2.74	2.01	2.62	1.91
-1.00	2.30	2.15	2.28	2.01	2.20	1.91
-0.75	1.97	2.15	1.96	2.01	1.90	1.91
-0.50	1.75	2.15	1.77	2.01	1.71	1.91
-0.25	1.54	2.15	1.56	2.01	1.52	1.91
0.00	1.46	2.15	1.46	2.01	1.44	1.91
0.25	1.42	2.15	1.40	2.01	1.39	1.91
0.50	1.40	2.15	1.40	2.01	1.37	1.91
0.75	1.33	2.15	1.31	2.01	1.28	1.91
1.00	1.35	2.15	1.33	2.01	1.29	1.91
1.25	1.45	2.15	1.44	2.01	1.37	1.91
1.50	1.52	2.15	1.51	2.01	1.44	1.91
1.75	1.52	2.15	1.50	2.01	1.45	1.91
2.00	1.53	2.15	1.54	2.01	1.48	1.91

Table 2 10×m.s.e. of estimators of $L_{\sqrt{2}-1}$ in 48 observation experiments using Routine B for step size 1.0.

Start	E_{DM}	A_{DM}	E_{WE}	A_{WE}	E^*_{WE}	A^*_{WE}
-2.00	6.00	5.07	5.50	4.04	5.60	3.86
-1.75	5.49	5.07	5.05	4.06	5.16	3.87
-1.50	4.95	5.07	4.48	4.04	4.67	3.85
-1.25	4.60	5.07	4.11	4.03	4.31	3.84
-1.00	4.30	5.07	3.94	4.04	4.08	3.86
-0.75	4.06	5.07	3.74	4.06	3.89	3.87
-0.50	3.93	5.07	3.63	4.04	3.76	3.85
-0.25	3.72	5.07	3.36	4.03	3.54	3.84
0.00	3.50	5.07	3.16	4.04	3.36	3.86
0.25	3.51	5.07	3.20	4.06	3.39	3.87
0.50	3.54	5.07	3.14	4.04	3.38	3.85
0.75	3.38	5.07	2.97	4.03	3.21	3.84
1.00	3.29	5.07	2.96	4.04	3.14	3.86
1.25	3.27	5.07	2.92	4.06	3.09	3.87
1.50	3.43	5.07	3.06	4.04	3.20	3.85
1.75	3.49	5.07	3.07	4.03	3.24	3.84
2.00	3.46	5.07	3.01	4.04	3.22	3.86

Note: A_{DM}, A_{WE} and A^*_{WE} denote columns for asymptotic predicted m.s.e.'s of E_{DM}, E_{WE} and E^*_{WE} respectively

Table 3 10×m.s.e. of estimators of $L_{\sqrt{2}-1}$ in 48 observation experiments using Routine B for step size 1.5.

Start	E_{DM}	A_{DM}	E_{WE}	A_{WE}	E_{WE}^*	A_{WE}^*
-2.00	9.74	10.01	7.96	6.96	8.80	6.66
-1.75	9.16	10.00	7.54	7.02	8.28	6.72
-1.50	8.73	10.06	7.51	7.24	8.06	6.94
-1.25	8.60	10.13	7.48	7.40	8.07	7.08
-1.00	8.14	10.14	7.03	7.34	7.62	7.01
-0.75	8.07	10.08	6.72	7.12	7.48	6.80
-0.50	7.77	10.01	6.51	6.96	7.14	6.66
-0.25	7.47	10.00	6.22	7.02	6.87	6.72
0.00	7.30	10.06	6.21	7.24	6.84	6.94
0.25	7.30	10.13	6.28	7.40	6.92	7.08
0.50	7.25	10.14	6.14	7.34	6.81	7.01
0.75	7.34	10.08	5.98	7.12	6.77	6.80
1.00	7.13	10.01	5.77	6.96	6.51	6.66
1.25	6.69	10.00	5.37	7.02	6.11	6.72
1.50	6.72	10.06	5.63	7.24	6.22	6.94
1.75	6.88	10.13	5.79	7.40	6.39	7.08
2.00	6.85	10.14	5.64	7.34	6.25	7.01

Table 4 10×m.s.e. of estimators of $L_{\sqrt{2}-1}$ in 48 observation experiments using Routine B for step size 2.0.

Start	E_{DM}	A_{DM}	E_{WE}	A_{WE}	E_{WE}^*	A_{WE}^*
-2.00	15.01	17.24	12.38	11.78	13.57	11.30
-1.75	15.19	17.65	12.81	12.40	14.11	11.89
-1.50	14.76	17.89	12.45	12.47	13.67	11.96
-1.25	14.06	17.80	11.34	11.97	12.81	11.48
-1.00	13.57	17.44	10.76	11.20	12.21	10.74
-0.75	13.32	17.02	10.39	10.61	11.79	10.17
-0.50	12.67	16.80	9.84	10.54	11.18	10.10
-0.25	12.85	16.88	10.10	11.01	11.46	10.55
0.00	12.80	17.24	10.42	11.78	11.70	11.30
0.25	13.03	17.65	10.76	12.40	12.16	11.89
0.50	13.22	17.89	10.92	12.47	12.27	11.96
0.75	13.28	17.80	10.66	11.97	12.10	11.48
1.00	12.67	17.44	9.90	11.20	11.35	10.74
1.25	12.53	17.02	9.55	10.61	10.97	10.17
1.50	12.07	16.80	9.04	10.54	10.48	10.10
1.75	11.66	16.88	8.92	11.01	10.28	10.55
2.00	11.64	17.24	9.37	11.78	10.54	11.30

Table 5 10×bias of estimators of $L\sqrt{1-\,}$ in 48 observation experiments using Routine B for step size 0.5.

Start	E_{DM}	A_{DM}	E_{WE}	E_{WE}^*	A_{WE}
-2.00	-4.35	-1.52	-4.45	-4.38	-1.27
-1.75	-3.73	-1.52	-3.78	-3.73	-1.27
-1.50	-3.27	-1.52	-3.26	-3.24	-1.27
-1.25	-2.90	-1.52	-2.85	-2.84	-1.27
-1.00	-2.44	-1.52	-2.33	-2.36	-1.27
-0.75	-1.96	-1.52	-1.87	-1.90	-1.27
-0.50	-1.65	-1.52	-1.55	-1.59	-1.27
-0.25	-1.30	-1.52	-1.21	-1.26	-1.27
0.00	-0.97	-1.52	-0.87	-0.95	-1.27
0.25	-0.68	-1.52	-0.59	-0.69	-1.27
0.50	-0.48	-1.52	-0.38	-0.48	-1.27
0.75	-0.25	-1.52	-0.10	-0.24	-1.27
1.00	-0.07	-1.52	0.11	-0.03	-1.27
1.25	0.05	-1.52	0.29	0.12	-1.27
1.50	0.11	-1.52	0.39	0.22	-1.27
1.75	0.20	-1.52	0.54	0.34	-1.27
2.00	0.30	-1.52	0.69	0.47	-1.27

Table 6 10×bias of estimators of $L\sqrt{1-\,}$ in 48 observation experiments using Routine B for step size 1.0.

Start	E_{DM}	A_{DM}	E_{WE}	E_{WE}^*	A_{WE}
-2.00	-4.64	-3.46	-4.15	-4.39	-2.67
-1.75	-4.27	-3.47	-3.81	-4.04	-2.69
-1.50	-3.93	-3.47	-3.46	-3.72	-2.70
-1.25	-3.72	-3.46	-3.24	-3.48	-2.68
-1.00	-3.48	-3.46	-2.99	-3.24	-2.67
-0.75	-3.28	-3.47	-2.81	-3.07	-2.69
-0.50	-3.09	-3.47	-2.67	-2.92	-2.70
-0.25	-2.87	-3.46	-2.44	-2.69	-2.68
0.00	-2.65	-3.46	-2.19	-2.49	-2.67
0.25	-2.55	-3.47	-2.13	-2.43	-2.69
0.50	-2.50	-3.47	-2.05	-2.38	-2.70
0.75	-2.36	-3.46	-1.93	-2.26	-2.68
1.00	-2.20	-3.46	-1.73	-2.07	-2.67
1.25	-2.09	-3.47	-1.62	-1.95	-2.69
1.50	-2.10	-3.47	-1.61	-1.94	-2.70
1.75	-2.13	-3.46	-1.60	-1.92	-2.68
2.00	-2.06	-3.46	-1.47	-1.82	-2.67

Note: A_{DM} and A_{WE} denote columns for asymptotic predicted biases of E_{DM} and E_{WE} respectively.

Table 7 10 bias of estimators of $L_{\sqrt{2}-1}$ in 48 observation experiments using Routine B for step size 1.5.

Start	E_{DM}	A_{DM}	E_{WE}	E_{WE}^*	A_{WE}
-2.00	-6.12	-5.64	-5.07	-5.58	-4.16
-1.75	-5.83	-5.59	-4.75	-5.25	-4.02
-1.50	-5.53	-5.60	-4.54	-5.01	-4.07
-1.25	-5.49	-5.67	-4.58	-5.06	-4.26
-1.00	-5.33	-5.71	-4.49	-4.95	-4.40
-0.75	-5.11	-5.70	-4.27	-4.75	-4.35
-0.50	-4.91	-5.64	-4.00	-4.49	-4.16
-0.25	-4.79	-5.59	-3.81	-4.32	-4.02
0.00	-4.55	-5.60	-3.61	-4.12	-4.07
0.25	-4.58	-5.67	-3.72	-4.23	-4.26
0.50	-4.60	-5.71	-3.80	-4.31	-4.40
0.75	-4.55	-5.70	-3.72	-4.25	-4.35
1.00	-4.35	-5.64	-3.44	-3.99	-4.16
1.25	-4.12	-5.59	-3.15	-3.72	-4.02
1.50	-4.01	-5.60	-3.05	-3.59	-4.07
1.75	-4.13	-5.67	-3.20	-3.75	-4.26
2.00	-4.14	-5.71	-3.27	-3.78	-4.40

Table 8 10×bias of estimators of $L_{\sqrt{2}-1}$ in 48 observation experiments using Routine B for step size 2.0.

Start	E_{DM}	A_{DM}	E_{WE}	E_{WE}^*	A_{WE}
-2.00	-7.66	-7.77	-6.08	-6.77	-5.42
-1.75	-7.75	-8.03	-6.42	-7.09	-5.99
-1.50	-7.83	-8.26	-6.70	-7.32	-6.44
-1.25	-7.65	-8.33	-6.52	-7.15	-6.50
-1.00	-7.36	-8.19	-6.08	-6.76	-6.15
-0.75	-7.06	-7.94	-5.57	-6.29	-5.61
-0.50	-6.67	-7.70	-5.02	-5.77	-5.18
-0.25	-6.63	-7.63	-4.92	-5.70	-5.09
0.00	-6.51	-7.77	-4.96	-5.70	-5.42
0.25	-6.66	-8.03	-5.36	-6.09	-5.99
0.50	-6.96	-8.26	-5.84	-6.53	-6.44
0.75	-6.98	-8.33	-5.90	-6.57	-6.50
1.00	-6.74	-8.19	-5.50	-6.21	-6.15
1.25	-6.49	-7.94	-5.01	-5.77	-5.61
1.50	-6.16	-7.70	-4.47	-5.29	-5.18
1.75	-5.94	-7.63	-4.22	-5.05	-5.09
2.00	-5.91	-7.77	-4.35	-5.11	-5.42

Table 9 Mean and 10× m.s.e. of $1/\approx{\beta}$ in 48 observation experiments using Routine B ($\beta = \pi/3.0^{1/2}$, with 2000 simulations).

	Step size							
	0.5		1.0		1.5		2.0	
Start	mean	10× m.s.e.	mean	10× m.s.e.	mean	10× m.s.e.	mean	10× m.s.e
-2.00	0.615	3.25	0.436	3.33	0.382	6.23	0.377	9.10
-1.75	0.537	2.54	0.400	3.05	0.354	5.92	0.375	9.35
-1.50	0.465	1.98	0.369	3.27	0.343	5.60	0.356	9.06
-1.25	0.404	1.76	0.349	3.34	0.343	6.06	0.317	9.16
-1.00	0.357	1.62	0.335	3.20	0.335	6.58	0.307	10.07
-0.75	0.331	1.66	0.319	3.35	0.320	6.63	0.290	10.03
-0.50	0.303	1.63	0.309	3.41	0.315	6.18	0.292	9.93
-0.25	0.292	1.78	0.310	3.38	0.314	5.92	0.324	9.22
0.00	0.290	1.75	0.308	3.15	0.307	5.73	0.333	9.38
0.25	0.292	1.87	0.310	3.25	0.315	5.82	0.339	9.31
0.50	0.302	1.78	0.310	3.56	0.322	6.51	0.340	9.21
0.75	0.314	1.89	0.309	3.63	0.311	6.56	0.309	10.25
1.00	0.322	1.85	0.315	3.34	0.308	6.69	0.290	10.55
1.25	0.342	1.97	0.314	3.43	0.300	6.37	0.285	10.52
1.50	0.349	1.90	0.317	3.42	0.304	5.73	0.292	10.67
1.75	0.359	2.07	0.319	3.54	0.317	5.85	0.302	9.49
2.00	0.358	2.03	0.322	3.29	0.320	6.72	0.315	9.03

Table 10 Numbers of 48 observation experiments using Routine B where bounds on β are not attained in the course of iterations with E04LAF ($\beta = \pi/3.0$" with 2000 simulations).

Start	Step size 0.5	1.0	1.5	2.0
-2.00	1563	1329	1088	821
-1.75	1563	1315	1019	1198
-1.50	1623	1351	920	1410
-1.25	1612	1393	899	1553
-1.00	1633	1347	956	1435
-0.75	1653	1287	1112	1239
-0.50	1677	1377	1147	1092
-0.25	1677	1441	1083	949
0.00	1740	1397	929	1053
0.25	1736	1348	858	1316
0.50	1738	1397	962	1457
0.75	1724	1455	1104	1361
1.00	1747	1425	1170	1384
1.25	1742	1362	1135	1243
1.50	1767	1392	1027	1173
1.75	1738	1422	923	1282
2.00	1720	1408	971	1294

Table 11 10 Mean and 10 m.s.e. of $L^{*}_{\sqrt{\lambda-1}}$ in 48 observation experiments using Routine B ($\beta = \pi/3.0^{1/2}$, with 2000 simulations).

	Step size							
	0.5		1.0		1.5		2.0	
Start	10 × mean	10 × m.s.e.	10 × mean	10 × m.s.e.	10 × mean	10 × m.s.e.	10 × mean	10 × m.s.e
-2.00	-1.789	3.15	-2.478	4.59	-4.008	7.98	-4.820	10.70
-1.75	-1.458	2.76	-2.307	4.13	-3.926	7.83	-3.525	9.07
-1.50	-1.190	2.42	-2.108	3.79	-3.979	7.52	-2.608	8.73
-1.25	-1.073	2.20	-1.919	3.67	-3.979	6.83	-2.114	9.77
-1.00	-0.887	1.93	-1.879	3.55	-3.659	6.14	-2.125	10.67
-0.75	-0.704	1.82	-1.915	3.37	-3.254	6.26	-2.793	11.12
-0.50	-0.624	1.74	-1.721	3.20	-3.140	6.56	-3.290	10.10
-0.25	-0.494	1.58	-1.483	3.16	-3.178	6.46	-3.581	9.00
0.00	-0.388	1.59	-1.586	3.23	-3.416	6.51	-3.155	8.34
0.25	-0.387	1.65	-1.685	3.23	-3.633	6.28	-2.394	7.76
0.50	-0.391	1.71	-1.597	3.05	-3.415	5.83	-1.987	8.28
0.75	-0.317	1.60	-1.471	3.08	-3.142	6.02	-2.567	9.56
1.00	-0.327	1.68	-1.413	3.17	-2.962	6.19	-2.379	9.54
1.25	-0.348	1.82	-1.521	3.08	-2.945	6.15	-2.969	9.49
1.50	-0.371	1.93	-1.521	3.17	-3.042	6.17	-3.004	8.48
1.75	-0.385	1.99	-1.528	3.37	-3.337	5.98	-2.746	7.65
2.00	-0.357	1.96	-1.520	3.45	-3.137	5.70	-2.541	6.98

Table 12 Mean and 10×m.s.e. of $1/\beta^*$ in 48 observation experiments using Routine B ($\beta=\pi/3.0^{1/2}$, with 2000 simulations).

Step size

	0.5		1.0		1.5		2.0	
Start	mean	10 × m.s.e.	mean	10 × m.s.e.	mean	10 × m.s.e.	mean	10 × m.s.e
-2.00	0.532	1.99	0.561	2.55	0.572	5.39	0.466	7.09
-1.75	0.505	1.74	0.549	2.42	0.541	5.20	0.473	6.47
-1.50	0.489	1.55	0.533	2.56	0.526	4.67	0.466	5.42
-1.25	0.467	1.44	0.529	2.65	0.529	5.68	0.491	6.39
-1.00	0.456	1.26	0.523	2.45	0.515	5.57	0.466	6.26
-0.75	0.459	1.37	0.513	2.77	0.535	5.75	0.447	6.39
-0.50	0.456	1.32	0.513	2.67	0.538	5.31	0.436	6.54
-0.25	0.451	1.38	0.517	2.64	0.522	4.84	0.405	6.45
0.00	0.458	1.28	0.524	2.53	0.505	4.93	0.391	7.48
0.25	0.467	1.44	0.524	2.68	0.503	5.23	0.396	6.11
0.50	0.471	1.38	0.516	2.79	0.511	5.67	0.412	5.46
0.75	0.468	1.46	0.524	2.84	0.524	5.64	0.442	7.53
1.00	0.472	1.41	0.521	2.56	0.536	5.43	0.435	5.50
1.25	0.482	1.53	0.520	2.79	0.532	5.44	0.434	5.78
1.50	0.481	1.37	0.514	2.62	0.516	4.64	0.413	5.20
1.75	0.478	1.51	0.513	2.70	0.511	5.25	0.410	5.80
2.00	0.474	1.48	0.516	2.55	0.500	5.80	0.393	5.39

Table 13 10×m.s.e. of estimators of $L_{\sqrt{2}-1}$ in 96 observation experiments using Routine B for step size 0.5.

Start	E_{DM}	A_{DM}	E_{WE}	A_{WE}	E_{WE}^*	A_{WE}^*
-2.00	1.98	1.19	1.91	1.08	1.94	1.04
-1.75	1.82	1.19	1.73	1.09	1.77	1.04
-1.50	1.59	1.19	1.50	1.08	1.54	1.04
-1.25	1.49	1.19	1.41	1.09	1.43	1.04
-1.00	1.27	1.19	1.19	1.08	1.22	1.04
-0.75	1.19	1.19	1.12	1.09	1.15	1.04
-0.50	1.08	1.19	1.02	1.08	1.04	1.04
-0.25	1.04	1.19	0.99	1.09	1.01	1.04
0.00	0.94	1.19	0.90	1.09	0.92	1.04
0.25	0.96	1.19	0.92	1.09	0.93	1.04
0.50	0.87	1.19	0.83	1.08	0.84	1.04
0.75	0.88	1.19	0.84	1.09	0.85	1.04
1.00	0.86	1.19	0.81	1.08	0.83	1.04
1.25	0.93	1.19	0.87	1.09	0.88	1.04
1.50	0.87	1.19	0.82	1.08	0.83	1.04
1.75	0.92	1.19	0.86	1.09	0.86	1.04
2.00	0.87	1.19	0.82	1.08	0.82	1.04

Table 14 10×m.s.e. of estimators of $L_{\sqrt{2}-1}$ in 96 observation experiments using Routine E for step size 1.0.

Start	E_{DM}	A_{DM}	E_{WE}	A_{WE}	E_{WE}^*	A_{WE}^*
-2.00	3.56	3.13	2.93	2.38	3.21	2.28
-1.75	3.41	3.13	2.79	2.39	3.08	2.30
-1.50	3.14	3.13	2.56	2.38	2.82	2.29
-1.25	3.09	3.13	2.49	2.37	2.78	2.28
-1.00	2.91	3.13	2.36	2.38	2.63	2.28
-0.75	2.90	3.13	2.35	2.39	2.63	2.30
-0.50	2.73	3.13	2.21	2.38	2.47	2.29
-0.25	2.67	3.13	2.16	2.37	2.42	2.28
0.00	2.56	3.13	2.09	2.38	2.34	2.28
0.25	2.58	3.13	2.12	2.39	2.36	2.30
0.50	2.43	3.13	1.96	2.38	2.21	2.29
0.75	2.52	3.13	2.02	2.37	2.29	2.28
1.00	2.43	3.13	1.99	2.38	2.21	2.28
1.25	2.54	3.13	2.06	2.39	2.31	2.30
1.50	2.43	3.13	1.95	2.38	2.19	2.29
1.75	2.41	3.13	1.90	2.37	2.14	2.28
2.00	2.43	3.13	1.93	2.38	2.18	2.28

Note: A_{DM}, A_{WE} and A_{WE}^* denote columns for asymptotic predicted m.s.e.'s of E_{DM}, E_{WE} and E_{WE}^* respectively

Table 15 10×m.s.e. of estimators of $L_{\sqrt{\Omega}-1}$ in 96 observation experiments using Routine B for step size 1.5.

Start	E_{DM}	A_{DM}	E_{WE}	A_{WE}	E_{WE}^{*}	A_{WE}^{*}
-2.00	6.73	6.60	4.86	4.35	5.62	4.19
-1.75	6.39	6.56	4.57	4.32	5.32	4.17
-1.50	6.15	6.60	4.57	4.45	5.25	4.30
-1.25	6.17	6.67	4.66	4.61	5.36	4.45
-1.00	6.04	6.70	4.58	4.64	5.26	4.47
-0.75	5.98	6.67	4.40	4.50	5.11	4.35
-0.50	5.73	6.60	4.09	4.35	4.79	4.19
-0.25	5.61	6.56	3.99	4.32	4.68	4.17
0.00	5.43	6.60	3.98	4.45	4.64	4.30
0.25	5.54	6.67	4.18	4.61	4.81	4.45
0.50	5.46	6.70	4.13	4.64	4.77	4.47
0.75	5.57	6.67	4.10	4.50	4.79	4.35
1.00	5.46	6.60	3.90	4.35	4.57	4.19
1.25	5.24	6.56	3.72	4.32	4.38	4.17
1.50	5.23	6.60	3.87	4.45	4.46	4.30
1.75	5.38	6.67	4.03	4.61	4.64	4.45
2.00	5.42	6.70	4.02	4.64	4.67	4.47

Table 16 10×m.s.e. of estimators of $L_{\sqrt{\Omega}-1}$ in 96 observation experiments using Routine B for step size 2.0.

Start	E_{DM}	A_{DM}	E_{WE}	A_{WE}	E_{WE}^{*}	A_{WE}^{*}
-2.00	10.94	11.63	7.75	7.36	8.98	7.12
-1.75	11.17	12.05	8.22	7.99	9.49	7.74
-1.50	11.36	12.36	8.43	8.31	9.75	8.06
-1.25	11.25	12.37	8.15	8.10	9.47	7.85
-1.00	10.60	12.08	7.36	7.49	8.65	7.26
-0.75	10.26	11.66	6.84	6.88	8.09	6.66
-0.50	9.72	11.36	6.29	6.61	7.51	6.39
-0.25	9.65	11.35	6.32	6.80	7.57	6.57
0.00	9.76	11.63	6.80	7.36	8.00	7.12
0.25	10.13	12.05	7.38	7.99	8.61	7.74
0.50	10.32	12.36	7.58	8.31	8.87	8.06
0.75	10.38	12.37	7.42	8.10	8.73	7.85
1.00	10.03	12.08	6.86	7.49	8.14	7.26
1.25	9.76	11.66	6.41	6.88	7.65	6.66
1.50	9.43	11.36	6.10	6.61	7.30	6.39
1.75	9.29	11.35	6.10	6.80	7.29	6.57
2.00	9.51	11.63	6.63	7.36	7.76	7.12

Table 17 10×bias of estimators of $L_{\sqrt{2}-1}$ in 96 observation experiments using Routine B for step size 0.5.

Start	E_{DM}	A_{DM}	E_{WE}	E_{WE}^*	A_{WE}
-2.00	-2.86	-1.52	-2.80	-2.91	-1.27
-1.75	-2.64	-1.52	-2.54	-2.65	-1.27
-1.50	-2.36	-1.52	-2.23	-2.34	-1.27
-1.25	-2.21	-1.52	-2.06	-2.16	-1.27
-1.00	-1.94	-1.52	-1.76	-1.87	-1.27
-0.75	-1.77	-1.52	-1.58	-1.69	-1.27
-0.50	-1.55	-1.52	-1.38	-1.49	-1.27
-0.25	-1.44	-1.52	-1.26	-1.39	-1.27
0.00	-1.22	-1.52	-1.06	-1.17	-1.27
0.25	-1.13	-1.52	-0.96	-1.09	-1.27
0.50	-0.94	-1.52	-0.76	-0.90	-1.27
0.75	-0.91	-1.52	-0.71	-0.85	-1.27
1.00	-0.80	-1.52	-0.58	-0.72	-1.27
1.25	-0.80	-1.52	-0.55	-0.70	-1.27
1.50	-0.68	-1.52	-0.39	-0.55	-1.27
1.75	-0.72	-1.52	-0.41	-0.58	-1.27
2.00	-0.65	-1.52	-0.33	-0.49	-1.27

Table 18 10×bias of estimators of $L_{\sqrt{2}-1}$ in 96 observation experiments using Routine B for step size 1.0.

Start	E_{DM}	A_{DM}	E_{WE}	E_{WE}^*	A_{WE}
-2.00	-4.03	-3.46	-3.40	-3.69	-2.67
-1.75	-3.86	-3.47	-3.21	-3.53	-2.69
-1.50	-3.73	-3.47	-3.10	-3.41	-2.70
-1.25	-3.61	-3.46	-2.98	-3.28	-2.68
-1.00	-3.43	-3.46	-2.79	-3.11	-2.67
-0.75	-3.39	-3.47	-2.74	-3.08	-2.69
-0.50	-3.28	-3.47	-2.67	-2.99	-2.70
-0.25	-3.18	-3.46	-2.57	-2.88	-2.68
0.00	-3.10	-3.46	-2.48	-2.81	-2.67
0.25	-3.02	-3.47	-2.40	-2.74	-2.69
0.50	-2.96	-3.47	-2.36	-2.70	-2.70
0.75	-2.91	-3.46	-2.32	-2.64	-2.68
1.00	-2.81	-3.46	-2.19	-2.52	-2.67
1.25	-2.82	-3.47	-2.18	-2.53	-2.69
1.50	-2.80	-3.47	-2.16	-2.51	-2.70
1.75	-2.76	-3.46	-2.11	-2.44	-2.68
2.00	-2.75	-3.46	-2.06	-2.41	-2.67

Note: A_{DM} and A_{WE} denote columns for asymptotic predicted biases of E_{DM} and E_{WE} respectively.

Table 19 10×bias of estimators of $L_{\sqrt{2}-1}$ in 96 observation experiments using Routine B for step size 1.5.

Start	E_{DM}	A_{DM}	E_{WE}	E_{WE}^*	A_{WE}^*
-2.00	-5.89	-5.64	-4.64	-5.15	-4.16
-1.75	-5.71	-5.59	-4.40	-4.92	-4.02
-1.50	-5.55	-5.60	-4.31	-4.82	-4.07
-1.25	-5.55	-5.67	-4.38	-4.90	-4.26
-1.00	-5.50	-5.71	-4.44	-4.94	-4.40
-0.75	-5.41	-5.70	-4.30	-4.81	-4.35
-0.50	-5.25	-5.64	-4.06	-4.57	-4.16
-0.25	-5.22	-5.59	-3.94	-4.47	-4.02
0.00	-5.09	-5.60	-3.87	-4.39	-4.07
0.25	-5.11	-5.67	-3.97	-4.49	-4.26
0.50	-5.12	-5.71	-4.10	-4.61	-4.40
0.75	-5.09	-5.70	-4.03	-4.54	-4.35
1.00	-4.99	-5.64	-3.82	-4.33	-4.16
1.25	-4.87	-5.59	-3.63	-4.15	-4.02
1.50	-4.84	-5.60	-3.62	-4.14	-4.07
1.75	-4.88	-5.67	-3.72	-4.24	-4.26
2.00	-4.94	-5.71	-3.87	-4.38	-4.40

Table 20 10×bias of estimators of $L_{\sqrt{2}-1}$ in 96 observation experiments using Routine B for step size 2.0.

Start	E_{DM}	A_{DM}	E_{WE}	E_{WE}^*	A_{WE}^*
-2.00	-7.72	-7.77	-5.77	-6.48	-5.42
-1.75	-7.83	-8.03	-6.15	-6.84	-5.99
-1.50	-8.03	-8.26	-6.58	-7.22	-6.44
-1.25	-8.04	-8.33	-6.58	-7.22	-6.50
-1.00	-7.72	-8.19	-6.10	-6.75	-6.15
-0.75	-7.45	-7.94	-5.57	-6.26	-5.61
-0.50	-7.16	-7.70	-5.09	-5.80	-5.18
-0.25	-7.12	-7.63	-5.00	-5.75	-5.09
0.00	-7.15	-7.77	-5.21	-5.95	-5.42
0.25	-7.30	-8.03	-5.65	-6.35	-5.99
0.50	-7.54	-8.26	-6.12	-6.80	-6.44
0.75	-7.64	-8.33	-6.22	-6.88	-6.50
1.00	-7.43	-8.19	-5.83	-6.49	-6.15
1.25	-7.18	-7.94	-5.34	-6.01	-5.61
1.50	-6.91	-7.70	-4.87	-5.58	-5.18
1.75	-6.80	-7.63	-4.70	-5.44	-5.09
2.00	-6.90	-7.77	-4.96	-5.67	-5.42

Table 21 Mean and 10× m.s.e. of $1/\tilde{\beta}$ in 96 observation experiments using Routine B ($\beta = \pi/3.0^{1/2}$, with 2000 simulations).

	Step size							
	0.5		1.0		1.5		2.0	
Start	mean	10 × m.s.e.	mean	10 × m.s.e.	mean	10 × m.s.e.	mean	10 × m.s.e
-2.00	0.625	2.27	0.514	3.01	0.507	5.81	0.541	8.83
-1.75	0.568	1.76	0.496	2.80	0.493	5.44	0.550	8.61
-1.50	0.519	1.45	0.465	2.65	0.477	5.09	0.531	9.13
-1.25	0.479	1.27	0.454	2.71	0.483	5.29	0.502	9.01
-1.00	0.446	1.18	0.441	2.58	0.467	5.49	0.469	9.20
-0.75	0.424	1.20	0.440	2.67	0.465	5.80	0.464	9.27
-0.50	0.408	1.16	0.428	2.72	0.462	5.72	0.472	9.16
-0.25	0.405	1.24	0.432	2.71	0.466	5.39	0.505	8.80
0.00	0.398	1.23	0.429	2.65	0.458	5.25	0.521	9.11
0.25	0.404	1.28	0.437	2.76	0.466	5.43	0.531	9.25
0.50	0.402	1.24	0.419	2.72	0.462	5.42	0.518	9.23
0.75	0.414	1.30	0.431	2.92	0.461	5.89	0.492	8.71
1.00	0.418	1.26	0.432	2.81	0.458	5.57	0.466	9.04
1.25	0.433	1.36	0.440	2.75	0.461	5.43	0.462	9.21
1.50	0.432	1.25	0.433	2.78	0.459	5.29	0.474	9.25
1.75	0.441	1.38	0.435	2.73	0.463	5.14	0.496	9.05
2.00	0.439	1.32	0.436	2.75	0.460	5.30	0.510	8.86

Table 22 Numbers of 96 observation experiments using Routine B where bounds on β are not attained in the course of iterations with E04LAF ($\beta = \pi/3.0^{1/2}$, with 2000 simulations).

Start	Step size 0.5	1.0	1.5	2.0
-2.00	1898	1785	1629	978
-1.75	1917	1659	1542	1313
-1.50	1910	1685	1314	1662
-1.25	1899	1804	1127	1833
-1.00	1924	1775	1226	1800
-0.75	1925	1660	1445	1671
-0.50	1925	1684	1634	1494
-0.25	1931	1816	1584	1271
0.00	1931	1790	1330	1367
0.25	1937	1682	1141	1608
0.50	1931	1708	1235	1736
0.75	1949	1826	1452	1653
1.00	1940	1823	1649	1682
1.25	1943	1670	1612	1599
1.50	1940	1691	1365	1574
1.75	1944	1808	1158	1634
2.00	1951	1815	1214	1578

Table 23 10×Mean and 10×m.s.e. of $L^*_{\sqrt{2}-1}$ in 96 observation experiments using Routine B ($\beta = \pi/3.0^{1/2}$, with 2000 simulations).

Step size

	0.5		1.0		1.5		2.0	
Start	10 × mean	10 × m.s.e.	10 × mean	10 × m.s.e.	10 × mean	10 × m.s.e.	10 × mean	10 × m.s.e
-2.00	-0.391	1.07	-0.855	1.94	-1.634	3.25	-4.122	6.04
-1.75	-0.362	1.07	-1.090	1.88	-2.001	3.77	-2.460	4.26
-1.50	-0.297	1.01	-0.959	1.53	-2.637	4.02	-0.878	3.70
-1.25	-0.343	1.02	-0.724	1.62	-2.988	3.67	-0.262	4.60
-1.00	-0.233	0.92	-0.760	1.68	-2.597	2.78	-0.059	4.95
-0.75	-0.246	0.91	-1.038	1.72	-2.025	2.83	-0.832	5.45
-0.50	-0.215	0.88	-0.931	1.41	-1.377	2.73	-1.666	5.50
-0.25	-0.244	0.88	-0.678	1.57	-1.783	3.40	-2.605	4.69
0.00	-0.197	0.87	-0.806	1.63	-2.449	3.62	-2.111	3.48
0.25	-0.219	0.90	-0.924	1.66	-2.794	3.31	-1.183	3.22
0.50	-0.181	0.86	-0.865	1.36	-2.491	2.65	-0.495	3.49
0.75	-0.206	0.88	-0.641	1.47	-1.920	2.61	-0.859	3.88
1.00	-0.164	0.89	-0.626	1.56	-1.295	2.55	-0.784	4.46
1.25	-0.195	0.97	-0.971	1.67	-1.543	3.03	-1.242	4.64
1.50	-0.156	0.93	-0.815	1.35	-2.261	3.56	-1.399	3.83
1.75	-0.245	0.97	-0.604	1.43	-2.684	3.27	-1.520	3.19
2.00	-0.174	0.93	-0.686	1.62	-2.388	2.44	-1.515	2.75

Table 24 Mean and 10×m.s.e. of $1/\beta^*$ in 96 observation experiments using Routine B ($\beta = \pi/3.0^{1/2}$, with 2000 simulations).

Step size

Start	0.5 mean	0.5 10×m.s.e.	1.0 mean	1.0 10×m.s.e.	1.5 mean	1.5 10×m.s.e.	2.0 mean	2.0 10×m.s.e
-2.00	0.525	1.04	0.565	1.50	0.594	2.91	0.565	5.64
-1.75	0.523	1.03	0.571	1.61	0.608	3.14	0.492	4.42
-1.50	0.522	1.00	0.554	1.51	0.599	3.44	0.465	3.28
-1.25	0.516	0.93	0.560	1.46	0.575	3.55	0.503	3.21
-1.00	0.509	0.85	0.558	1.40	0.561	3.52	0.492	2.61
-0.75	0.512	0.90	0.565	1.69	0.582	3.39	0.522	3.34
-0.50	0.515	0.90	0.550	1.48	0.572	2.46	0.530	4.03
-0.25	0.515	0.84	0.560	1.53	0.605	3.07	0.523	4.44
0.00	0.518	0.89	0.563	1.27	0.588	3.22	0.452	4.29
0.25	0.520	0.96	0.562	1.64	0.565	3.36	0.430	3.95
0.50	0.518	0.92	0.550	1.60	0.563	3.45	0.430	2.81
0.75	0.518	0.87	0.556	1.47	0.574	3.26	0.460	2.47
1.00	0.516	0.87	0.553	1.26	0.562	2.04	0.490	2.30
1.25	0.520	0.93	0.563	1.61	0.587	2.17	0.509	2.47
1.50	0.517	0.82	0.541	1.33	0.583	2.93	0.498	2.23
1.75	0.522	0.85	0.550	1.33	0.563	3.27	0.473	3.33
2.00	0.515	0.79	0.554	1.20	0.538	2.75	0.437	3.55

APPENDIX 12 TABLES TO ACCOMPANY SECTION 5.1

Table 1 100×m.s.e. of estimators in 48 step experiments using stochastic approximation procedures with c equal to 0.5 times the asymptotic optimal values ($\beta=\pi/3.0^{1/2}$; based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	3.38	3.81	2.05	2.51
0.25	3.33	3.84	2.39	2.66
0.50	3.52	4.05	3.37	3.14
0.75	4.04	4.72	5.04	4.10
1.00	4.80	5.34	6.80	5.01
1.25	6.14	6.24	9.03	5.88
1.50	7.81	7.08	10.83	6.71
1.75	10.99	8.06	12.87	7.46
2.00	15.16	9.02	14.59	8.09
2.25	21.99	10.38	16.19	8.64
2.50	32.07	11.23	17.73	9.16
2.75	47.18	11.99	19.55	9.29
3.00	69.31	12.65	21.29	9.36
3.25	98.98	12.44	22.98	9.69
3.50	139.30	13.03	24.47	10.08
3.75	191.34	12.55	26.05	10.49
4.00	256.55	12.57	28.85	10.69

Table 2 100×m.s.e. of estimators in 48 step experiments using stochastic approximation procedures with c equal to the asymptotic optimal values ($\beta=\pi/3.0^{1/2}$; based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	2.68	2.71	2.45	2.58
0.25	2.65	2.69	2.44	2.60
0.50	2.63	2.71	2.47	2.62
0.75	2.71	2.79	2.59	2.59
1.00	2.63	2.71	2.77	2.69
1.25	2.70	2.91	3.09	2.84
1.50	2.74	2.88	3.37	2.92
1.75	2.87	3.04	3.67	2.98
2.00	2.99	3.11	3.90	3.12
2.25	3.04	3.18	4.08	3.25
2.50	3.20	3.26	4.14	3.29
2.75	3.50	3.37	4.39	3.35
3.00	3.91	3.44	4.68	3.36
3.25	4.26	3.46	4.63	3.33
3.50	5.11	3.68	4.71	3.29
3.75	6.31	3.54	4.81	3.39
4.00	8.12	3.55	5.13	3.57

Table 3 100×m.s.e. of estimators in 48 step experiments using stochastic approximation procedures with c equal to 1.5 times the asymptotic optimal values ($\beta = \pi / 3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	2.93	2.95	2.75	2.86
0.25	2.87	2.89	2.78	2.81
0.50	2.83	2.85	2.73	2.83
0.75	2.85	2.88	2.77	2.83
1.00	2.86	2.83	2.75	2.92
1.25	2.88	2.92	2.83	2.83
1.50	2.87	2.91	2.94	2.92
1.75	2.95	2.97	3.00	3.01
2.00	2.85	2.98	3.09	3.05
2.25	2.90	2.90	3.18	3.04
2.50	2.97	3.02	3.26	3.08
2.75	2.98	3.01	3.37	3.10
3.00	2.96	3.01	3.29	3.16
3.25	2.92	3.07	3.29	3.17
3.50	2.97	3.01	3.43	3.11
3.75	3.01	3.11	3.53	3.16
4.00	3.07	3.11	3.55	3.13

Table 4 100×m.s.e. of estimators in 48 step experiments using stochastic approximation procedures with c equal to 2.0 times the asymptotic optimal values ($\beta = \pi / 3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	3.46	3.46	3.27	3.44
0.25	3.50	3.50	3.26	3.33
0.50	3.41	3.42	3.21	3.43
0.75	3.42	3.43	3.27	3.34
1.00	3.40	3.41	3.25	3.39
1.25	3.49	3.48	3.26	3.35
1.50	3.43	3.46	3.25	3.44
1.75	3.48	3.54	3.29	3.42
2.00	3.45	3.48	3.43	3.46
2.25	3.48	3.44	3.44	3.53
2.50	3.50	3.49	3.43	3.57
2.75	3.48	3.52	3.48	3.51
3.00	3.46	3.59	3.51	3.60
3.25	3.47	3.60	3.57	3.56
3.50	3.48	3.51	3.56	3.58
3.75	3.56	3.52	3.62	3.55
4.00	3.51	3.58	3.70	3.64

Table 5 100×bias of estimators in 48 step experiments using stochastic approximation procedures with c equal to 0.5 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	0.19	0.29	0.23	0.42
0.25	2.87	2.22	5.64	3.75
0.50	5.71	4.21	10.99	7.49
0.75	9.47	6.51	16.27	10.76
1.00	12.95	8.30	20.28	13.00
1.25	17.08	9.93	24.47	14.76
1.50	21.57	11.02	27.10	16.49
1.75	28.00	11.98	29.85	17.45
2.00	34.95	13.14	32.10	18.16
2.25	43.71	13.72	33.92	19.12
2.50	54.31	13.90	35.63	20.14
2.75	66.99	14.70	37.30	20.15
3.00	82.03	14.99	39.25	20.39
3.25	98.76	14.71	40.59	20.70
3.50	117.55	14.82	41.85	20.98
3.75	138.06	14.25	43.28	21.44
4.00	160.01	14.64	45.14	21.71

Table 6 100×bias of estimators in 48 step experiments using stochastic approximation procedures with c equal to the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	0.29	0.22	0.14	0.22
0.25	0.48	0.39	1.58	0.74
0.50	0.94	0.58	2.73	1.57
0.75	1.08	0.96	4.27	2.12
1.00	1.54	0.98	5.67	2.92
1.25	2.19	1.45	6.80	3.33
1.50	2.87	1.80	7.71	3.49
1.75	3.80	1.97	8.53	3.77
2.00	4.25	1.70	9.06	4.02
2.25	5.20	2.03	9.66	3.93
2.50	6.66	2.08	9.79	3.81
2.75	8.14	2.24	10.43	3.94
3.00	10.27	2.26	10.64	4.19
3.25	12.34	2.24	10.69	4.06
3.50	15.40	2.30	10.98	4.22
3.75	19.15	1.94	11.09	4.07
4.00	23.50	2.10	11.64	4.23

Table 7 100×bias of estimators in 48 step experiments using stochastic approximation procedures with c equal to 1.5 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	0.19	0.14	0.30	0.19
0.25	0.10	0.08	0.33	0.24
0.50	-0.04	-0.09	0.71	0.54
0.75	0.14	0.09	1.12	0.45
1.00	0.42	0.29	1.67	0.93
1.25	0.38	0.29	2.27	0.97
1.50	0.44	0.28	2.66	0.90
1.75	0.77	0.55	2.79	1.01
2.00	0.74	0.46	3.07	1.03
2.25	0.94	0.43	3.30	1.08
2.50	1.09	0.39	3.17	1.25
2.75	1.29	0.23	3.53	1.01
3.00	1.30	0.30	3.57	1.10
3.25	1.53	0.53	3.43	1.03
3.50	2.05	0.57	3.75	0.92
3.75	2.34	0.60	3.84	1.22
4.00	2.93	0.47	3.79	0.88

Table 8 100×bias of estimators in 48 step experiments using stochastic approximation procedures with c equal to 2.0 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	0.02	-0.01	0.05	0.27
0.25	0.05	0.07	0.13	-0.06
0.50	0.03	0.02	0.07	-0.06
0.75	0.04	0.06	0.14	0.13
1.00	0.20	0.17	0.57	0.16
1.25	0.18	0.19	0.97	0.35
1.50	0.28	0.20	0.81	0.30
1.75	0.20	0.20	1.29	0.63
2.00	0.08	0.14	1.37	0.39
2.25	0.19	0.06	1.15	0.37
2.50	0.21	0.10	1.39	0.53
2.75	0.26	0.12	1.12	0.30
3.00	0.22	0.14	1.26	0.53
3.25	0.30	0.06	1.26	0.34
3.50	0.31	-0.25	1.56	0.38
3.75	0.16	0.20	1.41	0.54
4.00	0.32	0.09	1.33	0.29

Table 9 100× m.s.e. of estimators in 96 step experiments using stochastic approximation procedures with c equal to 0.5 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	1.89	2.11	1.31	1.52
0.25	1.90	2.15	1.47	1.61
0.50	2.01	2.29	1.92	1.83
0.75	2.22	2.58	2.66	2.19
1.00	2.56	2.88	3.50	2.59
1.25	3.14	3.19	4.41	2.90
1.50	4.02	3.59	5.24	3.27
1.75	5.66	4.17	6.04	3.48
2.00	7.99	4.68	6.65	3.85
2.25	11.69	5.25	7.06	4.09
2.50	16.99	5.62	7.66	4.30
2.75	25.55	6.26	8.08	4.23
3.00	38.17	6.77	8.61	4.31
3.25	55.83	7.13	8.90	4.36
3.50	81.36	7.20	8.98	4.37
3.75	116.58	7.54	9.19	4.44
4.00	163.20	7.70	9.85	4.52

Table 10 100×m.s.e. of estimators in 96 step experiments using stochastic approximation procedures with c equal to the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	1.37	1.39	1.31	1.36
0.25	1.38	1.37	1.32	1.35
0.50	1.35	1.36	1.32	1.34
0.75	1.37	1.39	1.36	1.38
1.00	1.39	1.40	1.39	1.39
1.25	1.38	1.41	1.43	1.41
1.50	1.41	1.42	1.51	1.39
1.75	1.43	1.47	1.59	1.46
2.00	1.45	1.49	1.64	1.45
2.25	1.46	1.50	1.66	1.48
2.50	1.50	1.54	1.74	1.49
2.75	1.57	1.56	1.75	1.53
3.00	1.62	1.54	1.79	1.52
3.25	1.77	1.57	1.80	1.55
3.50	1.97	1.56	1.82	1.53
3.75	2.30	1.58	1.82	1.53
4.00	2.81	1.57	1.86	1.59

Table 11 100×m.s.e. of estimators in 96 step experiments using stochastic approximation procedures with c equal to 1.5 times the asymptotic optimal values ($\beta = \pi / 3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	1.52	1.52	1.49	1.52
0.25	1.52	1.51	1.47	1.50
0.50	1.51	1.51	1.45	1.47
0.75	1.52	1.53	1.47	1.50
1.00	1.51	1.50	1.46	1.48
1.25	1.51	1.51	1.51	1.51
1.50	1.51	1.51	1.49	1.51
1.75	1.52	1.50	1.52	1.52
2.00	1.50	1.53	1.51	1.53
2.25	1.51	1.52	1.53	1.53
2.50	1.51	1.52	1.53	1.53
2.75	1.51	1.53	1.56	1.54
3.00	1.53	1.54	1.55	1.54
3.25	1.54	1.54	1.56	1.57
3.50	1.52	1.53	1.56	1.53
3.75	1.55	1.55	1.65	1.56
4.00	1.54	1.55	1.62	1.57

Table 12 100×m.s.e. of estimators in 96 step experiments using stochastic approximation procedures with c equal to 2.0 times the asymptotic optimal values ($\beta = \pi / 3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	1.77	1.77	1.73	1.75
0.25	1.77	1.77	1.72	1.74
0.50	1.76	1.76	1.70	1.75
0.75	1.79	1.78	1.70	1.74
1.00	1.76	1.77	1.70	1.76
1.25	1.78	1.80	1.71	1.76
1.50	1.77	1.77	1.72	1.76
1.75	1.80	1.79	1.74	1.74
2.00	1.76	1.77	1.71	1.78
2.25	1.77	1.78	1.72	1.76
2.50	1.80	1.79	1.74	1.75
2.75	1.78	1.81	1.74	1.75
3.00	1.76	1.78	1.76	1.77
3.25	1.78	1.79	1.76	1.78
3.50	1.79	1.79	1.78	1.77
3.75	1.75	1.79	1.77	1.78
4.00	1.78	1.81	1.81	1.80

Table 13 100×bias of estimators in 96 step experiments using stochastic approximation procedures with c equal to 0.5 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	0.10	0.27	0.17	0.23
0.25	1.91	1.47	3.99	2.58
0.50	3.76	2.69	7.59	5.07
0.75	6.10	3.96	11.20	7.04
1.00	8.69	5.38	14.01	8.66
1.25	11.60	6.38	16.52	9.73
1.50	15.02	7.47	18.32	10.85
1.75	19.67	8.17	19.96	11.31
2.00	25.03	8.88	21.25	12.11
2.25	31.59	9.17	21.90	12.86
2.50	39.26	9.41	22.96	13.14
2.75	49.01	10.23	23.56	12.83
3.00	60.62	10.44	24.46	13.15
3.25	73.91	10.66	24.69	13.37
3.50	89.69	10.82	24.70	13.27
3.75	107.63	11.01	24.93	13.49
4.00	127.53	10.90	25.73	13.54

Table 14 100×bias of estimators in 96 step experiments using stochastic approximation procedures with c equal to the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	0.14	0.13	0.10	0.18
0.25	0.35	0.22	0.77	0.43
0.50	0.51	0.43	1.41	0.77
0.75	0.50	0.43	2.20	1.01
1.00	0.71	0.57	2.80	1.41
1.25	0.92	0.58	3.38	1.48
1.50	1.32	0.70	3.74	1.72
1.75	1.56	0.68	4.21	1.92
2.00	2.08	0.70	4.57	1.86
2.25	2.57	0.96	4.67	1.87
2.50	3.27	1.02	4.80	1.86
2.75	4.01	1.01	5.02	2.03
3.00	4.94	0.98	5.04	2.03
3.25	6.11	0.91	5.03	1.98
3.50	7.81	0.99	5.21	2.16
3.75	9.76	0.98	5.14	2.04
4.00	12.09	0.91	5.35	2.09

Table 15 100×bias of estimators in 96 step experiments using stochastic approximation procedures with c equal to 1.5 times the asymptotic optimal values ($\beta = \pi / 3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	0.21	0.21	0.22	0.29
0.25	0.11	0.12	0.30	0.18
0.50	0.18	0.18	0.40	0.30
0.75	0.22	0.21	0.55	0.31
1.00	0.20	0.19	0.72	0.48
1.25	0.23	0.29	1.04	0.40
1.50	0.27	0.24	0.98	0.43
1.75	0.38	0.38	1.22	0.50
2.00	0.42	0.33	1.33	0.47
2.25	0.49	0.43	1.30	0.57
2.50	0.45	0.14	1.31	0.61
2.75	0.51	0.25	1.43	0.62
3.00	0.61	0.19	1.36	0.59
3.25	0.68	0.32	1.43	0.52
3.50	0.76	0.27	1.43	0.48
3.75	0.92	0.29	1.63	0.65
4.00	1.08	0.29	1.52	0.65

Table 16 100×bias of estimators in 96 step experiments using stochastic approximation procedures with c equal to 2.0 times the asymptotic optimal values ($\beta = \pi / 3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4
Start				
0.00	0.11	0.10	0.13	0.27
0.25	0.19	0.18	0.16	0.16
0.50	0.24	0.22	0.18	0.21
0.75	0.26	0.25	0.27	0.23
1.00	0.27	0.28	0.24	0.27
1.25	0.10	0.08	0.42	0.33
1.50	0.27	0.28	0.43	0.27
1.75	0.26	0.27	0.49	0.32
2.00	0.28	0.27	0.56	0.25
2.25	0.22	0.22	0.42	0.35
2.50	0.21	0.28	0.45	0.30
2.75	0.21	0.15	0.57	0.30
3.00	0.25	0.28	0.51	0.39
3.25	0.28	0.25	0.48	0.25
3.50	0.25	0.15	0.57	0.28
3.75	0.31	0.35	0.67	0.29
4.00	0.29	0.35	0.58	0.26

Table 17 100*m.s.e. of E_{DM} in 48 step experiments ($\beta = \pi / 3.0^{1/2}$, based on 2000 simulations).

Step Size	0.20	0.40	0.60	0.80
Start				
0.00	2.30	2.62	3.00	3.18
0.25	2.42	2.68	3.14	3.35
0.50	2.69	2.84	3.09	3.30
0.75	3.17	3.02	3.18	3.44
1.00	3.48	3.08	3.25	3.55
1.25	3.90	3.29	3.33	3.49
1.50	4.25	3.39	3.44	3.48
1.75	4.55	3.39	3.52	3.64
2.00	4.89	3.67	3.53	3.67
2.25	5.16	3.91	3.58	3.71
2.50	5.56	3.76	3.71	3.92
2.75	5.65	3.78	3.75	3.92
3.00	6.13	3.95	3.89	3.82
3.25	6.38	3.90	3.78	3.81
3.50	6.73	3.93	3.66	3.80
3.75	7.26	4.14	3.87	3.95
4.00	7.54	4.08	3.91	4.12

Table 18 100×bias of E_{DM} in 48 step experiments ($\beta = \pi / 3.0^{1/2}$, based on 2000 simulations).

Step Size	0.20	0.40	0.60	0.80
Start				
0.00	0.30	0.52	0.47	0.27
0.25	3.34	1.69	1.14	1.09
0.50	5.47	2.73	1.79	1.41
0.75	7.47	3.97	2.69	1.90
1.00	8.93	4.40	3.47	2.39
1.25	9.96	4.97	3.75	2.58
1.50	11.22	5.45	3.56	3.00
1.75	11.68	5.63	3.37	3.01
2.00	12.44	5.74	4.01	2.83
2.25	13.46	6.27	4.37	3.14
2.50	13.97	6.46	4.49	2.85
2.75	14.31	6.13	4.14	3.08
3.00	15.19	6.65	4.04	3.56
3.25	15.29	6.67	4.33	3.24
3.50	16.29	6.58	4.50	3.00
3.75	17.08	6.67	4.53	3.08
4.00	17.45	6.89	4.36	3.06

Table 19 100×m.s.e. of E_{DM} in 96 step experiments ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Step Size	0.15	0.30	0.45	0.60
Start				
0.00	1.27	1.34	1.46	1.58
0.25	1.29	1.41	1.57	1.73
0.50	1.40	1.41	1.57	1.65
0.75	1.57	1.56	1.50	1.65
1.00	1.67	1.51	1.63	1.63
1.25	1.81	1.57	1.65	1.67
1.50	1.94	1.64	1.58	1.80
1.75	1.99	1.62	1.57	1.67
2.00	2.11	1.73	1.78	1.70
2.25	2.20	1.68	1.76	1.71
2.50	2.28	1.81	1.65	1.79
2.75	2.40	1.77	1.74	1.88
3.00	2.49	1.72	1.81	1.82
3.25	2.53	1.78	1.73	1.73
3.50	2.64	1.71	1.71	1.74
3.75	2.73	1.91	1.88	1.87
4.00	2.81	1.75	1.83	1.91

Table 20 100×bias of E_{DM} in 96 step experiments ($\beta=\pi/3.0^{1/2}$, based on 2000 simulations).

Step Size	0.15	0.30	0.45	0.60
Start				
0.00	0.10	0.04	0.18	0.26
0.25	2.03	0.94	0.85	0.72
0.50	3.45	1.72	1.25	0.87
0.75	4.54	2.47	1.62	1.24
1.00	5.54	2.59	2.14	1.78
1.25	6.19	3.25	2.18	1.88
1.50	6.83	3.24	2.16	1.75
1.75	7.33	3.50	2.39	1.58
2.00	7.61	3.60	2.60	1.88
2.25	7.92	3.50	2.68	2.19
2.50	8.40	4.11	2.79	2.25
2.75	8.67	3.76	2.83	1.98
3.00	9.11	3.92	2.70	1.83
3.25	9.09	4.01	2.58	2.12
3.50	9.14	4.02	2.65	2.26
3.75	9.43	4.18	2.93	2.37
4.00	9.77	3.90	2.83	2.19

APPENDIX 13 <u>TABLES TO ACCOMPANY SECTION 5.2</u>

Table 1 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 48 step experiments using stochastic approximation procedures with c equal to 0.5 times the asymptotic optimal values ($\beta = \pi / 3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	4.84	6.27	7.37	5.33	18.75	6.56	8.73	6.10
-1.75	4.38	6.09	6.39	5.05	12.05	6.43	7.51	6.12
-1.50	3.90	5.85	5.53	4.66	8.00	6.18	6.76	5.66
-1.25	3.78	5.31	4.56	4.27	5.71	5.91	5.73	5.18
-1.00	3.68	4.88	3.79	3.81	4.31	5.38	4.72	4.70
-0.75	3.78	4.96	3.10	3.54	3.58	4.75	3.92	4.01
-0.50	3.92	4.82	2.71	3.14	3.36	4.37	3.45	3.33
-0.25	4.26	5.00	2.58	2.86	3.35	4.15	3.23	2.84
0.00	4.83	5.40	2.66	2.87	3.64	4.28	3.35	2.67
0.25	5.38	5.84	3.18	3.04	4.00	4.60	3.64	3.15
0.50	6.28	6.90	4.05	3.42	4.74	5.24	3.93	4.26
0.75	7.49	8.06	4.91	3.78	5.64	6.16	4.48	5.89
1.00	8.83	8.89	6.15	4.13	7.72	7.85	4.65	7.94
1.25	12.20	10.98	7.56	4.69	11.13	10.16	5.11	10.33
1.50	16.94	12.43	8.68	4.98	15.96	11.94	5.29	12.15
1.75	25.31	14.57	9.98	5.49	24.00	13.80	5.68	14.23
2.00	38.21	15.49	11.63	5.63	36.67	15.69	6.23	16.68

Table 2 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 48 step experiments using stochastic approximation procedures with c equal to the asymptotic optimal values ($\beta = \pi / 3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	3.33	4.53	3.19	3.40	2.97	3.88	4.84	3.53
-1.75	3.42	4.45	3.14	3.47	3.04	3.87	4.91	3.52
-1.50	3.52	4.64	3.18	3.47	3.06	3.85	4.78	3.48
-1.25	3.58	4.82	3.31	3.35	3.09	3.70	4.77	3.45
-1.00	3.81	4.84	3.27	3.33	3.19	3.68	4.72	3.37
-0.75	4.05	4.80	3.35	3.39	3.27	3.62	5.04	3.27
-0.50	4.20	4.97	3.34	3.39	3.38	3.63	5.05	3.27
-0.25	4.18	4.76	3.46	3.46	3.39	3.48	5.16	3.24
0.00	4.32	4.65	3.48	3.42	3.46	3.48	5.38	3.34
0.25	4.64	4.76	3.62	3.43	3.42	3.48	5.53	3.35
0.50	4.79	4.92	3.88	3.56	3.60	3.67	5.71	3.47
0.75	4.86	5.06	4.02	3.65	3.73	3.80	6.01	3.74
1.00	5.17	5.35	4.13	3.69	3.88	3.90	6.35	4.08
1.25	5.83	6.04	4.44	3.75	4.30	4.42	6.39	4.55
1.50	6.23	6.44	4.62	3.84	4.54	4.65	6.89	4.77
1.75	6.90	7.01	4.81	3.94	4.97	5.03	6.93	5.08
2.00	7.39	7.23	5.08	4.05	5.26	5.22	6.97	5.25

Table 3 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 48 step experiments using stochastic approximation procedures with c equal to 1.5 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	3.89	4.73	4.21	4.33	3.63	4.02	6.57	3.90
-1.75	3.94	4.50	4.17	4.37	3.71	3.91	6.49	3.84
-1.50	4.07	4.74	4.36	4.31	3.71	3.94	6.33	3.94
-1.25	4.03	4.78	4.32	4.48	3.85	3.97	6.46	3.88
-1.00	3.94	4.51	4.42	4.37	3.75	3.86	6.85	3.75
-0.75	4.11	4.67	4.45	4.34	3.82	3.84	6.79	3.76
-0.50	4.20	4.59	4.45	4.31	3.77	3.81	7.14	3.72
-0.25	4.24	4.45	4.38	4.44	3.73	3.77	7.21	3.81
0.00	4.24	4.43	4.44	4.26	3.79	3.77	7.43	3.79
0.25	4.26	4.28	4.55	4.40	3.79	3.80	7.43	3.76
0.50	4.37	4.38	4.51	4.47	3.81	3.82	7.51	3.84
0.75	4.42	4.45	4.81	4.46	3.86	3.90	8.12	3.88
1.00	4.61	4.65	4.78	4.52	3.82	3.85	7.89	4.05
1.25	4.91	5.06	4.82	4.56	3.97	4.01	7.98	4.13
1.50	5.02	5.15	4.98	4.63	4.08	4.05	8.41	4.30
1.75	5.22	5.41	5.07	4.62	4.13	4.19	8.71	4.39
2.00	5.35	5.58	5.13	4.62	4.17	4.17	8.48	4.33

Table 4 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 48 step experiments using stochastic approximation procedures with c equal to 2.0 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	4.57	4.96	5.42	5.35	4.44	4.61	8.18	4.55
-1.75	4.48	4.93	5.39	5.37	4.51	4.56	8.38	4.43
-1.50	4.48	4.95	5.37	5.38	4.50	4.57	8.42	4.49
-1.25	4.67	5.17	5.45	5.26	4.50	4.50	8.74	4.41
-1.00	4.61	5.03	5.39	5.38	4.37	4.39	8.36	4.25
-0.75	4.51	4.61	5.44	5.41	4.54	4.55	8.91	4.30
-0.50	4.57	4.70	5.45	5.38	4.52	4.51	8.89	4.38
-0.25	4.55	4.56	5.49	5.41	4.48	4.52	8.88	4.55
0.00	4.59	4.58	5.56	5.27	4.59	4.59	9.11	4.46
0.25	4.70	4.68	5.48	5.48	4.50	4.51	9.18	4.46
0.50	4.73	4.72	5.43	5.42	4.42	4.43	9.49	4.41
0.75	4.70	4.68	5.64	5.38	4.48	4.48	9.75	4.45
1.00	4.66	4.76	5.58	5.39	4.49	4.52	9.82	4.48
1.25	4.78	4.89	5.59	5.63	4.53	4.56	10.10	4.68
1.50	4.91	5.01	5.76	5.57	4.41	4.50	10.11	4.70
1.75	5.15	5.20	5.69	5.57	4.52	4.52	10.48	4.69
2.00	5.13	5.27	5.86	5.52	4.60	4.68	10.36	4.82

Table 5 100×bias of estimators of $L_{1/\sqrt{2}}$ in 48 step experiments using stochastic approximation procedures with c equal to 0.5 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	-14.44	-4.21	-20.38	-12.93	-41.77	-7.01	-21.80	-14.82
-1.75	-11.92	-3.83	-18.54	-12.12	-32.17	-6.83	-19.74	-14.86
-1.50	-9.00	-2.70	-16.44	-11.17	-24.45	-6.46	-17.90	-14.10
-1.25	-6.48	-1.95	-13.74	-9.74	-18.43	-6.16	-15.35	-13.11
-1.00	-3.83	-0.75	-10.49	-8.35	-13.15	-5.44	-12.84	-11.77
-0.75	-1.88	0.62	-6.98	-6.39	-8.67	-4.10	-10.06	-9.64
-0.50	-0.01	1.76	-3.82	-4.64	-4.84	-2.39	-7.02	-6.57
-0.25	2.22	3.68	-0.63	-2.52	-1.72	-0.46	-4.52	-2.91
0.00	4.88	6.00	3.21	-0.18	1.86	2.28	-2.40	1.14
0.25	7.65	8.07	7.00	1.77	5.52	5.10	-0.74	5.70
0.50	10.58	10.10	10.32	3.69	8.99	7.65	0.28	10.19
0.75	14.33	11.57	13.17	5.22	13.22	9.82	0.96	14.70
1.00	19.00	12.89	15.86	6.19	18.98	12.55	1.26	18.30
1.25	25.84	14.89	18.55	7.80	26.07	15.08	1.38	21.82
1.50	34.29	15.78	20.43	8.14	34.28	16.43	1.33	24.22
1.75	45.19	17.39	22.09	8.85	44.91	18.57	1.90	26.65
2.00	58.39	18.15	24.52	9.20	57.67	19.48	2.40	29.87

Table 6 100×bias of estimators of $L_{1/\sqrt{2}}$ in 48 step experiments using stochastic approximation procedures with c equal to the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	2.82	5.77	-0.69	-0.30	-4.36	1.37	-4.59	-1.90
-1.75	3.33	5.98	-0.08	-0.07	-2.86	1.03	-4.08	-1.76
-1.50	3.83	6.02	0.70	0.26	-1.55	1.59	-3.61	-1.70
-1.25	4.34	6.27	1.29	0.11	-0.36	1.58	-3.77	-1.44
-1.00	5.15	6.58	1.87	0.28	0.35	1.72	-2.74	-1.16
-0.75	5.73	6.69	2.65	0.67	1.03	1.78	-2.51	-0.53
-0.50	6.17	6.98	2.80	0.80	1.24	1.72	-2.03	-0.03
-0.25	6.23	6.92	3.47	1.08	1.95	2.07	-1.76	0.89
0.00	6.60	7.16	3.82	1.34	2.13	2.09	-1.55	1.72
0.25	7.33	7.85	4.95	1.39	2.63	2.68	-0.96	2.69
0.50	7.39	8.00	5.68	1.96	3.31	3.17	-0.91	3.66
0.75	7.47	7.97	6.19	2.27	3.91	3.87	-0.93	4.70
1.00	7.30	8.17	6.54	2.29	4.46	4.20	-1.09	5.68
1.25	8.29	8.54	7.33	2.43	5.70	4.62	-0.73	6.74
1.50	9.11	8.87	7.82	2.43	6.84	4.99	-1.19	7.22
1.75	10.42	9.36	8.23	2.62	8.64	5.61	-1.41	8.27
2.00	12.25	9.04	8.53	2.69	10.46	5.76	-1.23	7.92

Table 7 100×bias of estimators of $L_{1/\sqrt{2}}$ in 48 step experiments using stochastic approximation procedures with c equal to 1.5 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	3.99	5.45	3.06	1.38	1.26	2.03	-1.04	0.85
-1.75	4.14	5.34	3.48	1.50	1.33	2.08	-0.86	0.94
-1.50	4.59	5.60	3.75	1.87	1.40	1.87	-0.60	0.57
-1.25	4.87	5.74	3.91	2.06	1.88	2.00	-0.92	1.16
-1.00	4.84	5.37	3.78	2.06	1.97	2.12	-0.50	1.07
-0.75	5.51	5.90	4.24	1.83	2.15	2.22	-0.21	1.18
-0.50	5.48	5.79	4.06	2.02	1.87	1.92	-0.59	1.12
-0.25	5.70	5.93	4.08	1.61	1.81	1.84	-0.57	1.90
0.00	5.57	5.77	4.19	1.80	2.08	2.11	-0.06	1.77
0.25	5.75	5.95	4.33	2.09	2.19	2.26	0.08	2.03
0.50	6.08	6.38	4.46	2.29	2.18	2.28	-0.29	2.49
0.75	5.83	6.04	4.92	1.78	2.43	2.36	-0.76	2.51
1.00	5.58	6.16	5.03	2.07	2.60	2.49	0.01	2.62
1.25	5.96	6.27	5.06	2.22	2.59	2.56	-0.48	3.13
1.50	5.62	6.17	5.07	2.42	2.86	2.81	-0.28	3.33
1.75	5.87	6.22	5.35	2.17	3.04	2.87	-0.31	3.42
2.00	5.81	6.16	5.37	2.36	3.40	2.80	-0.11	3.70

Table 8 100×bias of estimators of $L_{1/\sqrt{2}}$ in 48 step experiments using stochastic approximation procedures with c equal to 2.0 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	3.49	4.12	4.02	2.46	1.87	2.24	0.07	1.53
-1.75	3.66	4.19	4.44	2.23	1.99	2.04	0.16	1.65
-1.50	3.97	4.42	3.99	2.39	2.28	2.26	0.41	1.45
-1.25	3.99	4.35	4.49	2.33	2.09	2.28	0.47	1.78
-1.00	4.33	4.60	4.07	2.62	2.04	2.03	0.57	1.50
-0.75	4.34	4.41	4.35	2.11	2.07	2.14	0.19	1.65
-0.50	4.60	4.69	4.42	2.52	2.05	1.99	0.38	1.74
-0.25	4.53	4.49	4.64	2.38	2.19	2.19	0.49	2.21
0.00	4.68	4.58	4.10	2.36	2.33	2.36	0.58	1.75
0.25	4.78	4.81	4.40	2.27	2.17	2.23	0.78	1.81
0.50	5.03	5.18	4.26	2.31	1.84	1.84	0.25	1.89
0.75	4.81	4.83	4.72	2.39	2.05	2.09	0.30	2.14
1.00	4.57	4.64	4.55	2.20	1.81	1.80	0.20	2.26
1.25	4.71	5.02	4.87	2.53	2.27	2.17	0.14	2.15
1.50	4.39	4.59	4.45	2.30	1.94	2.04	0.82	2.39
1.75	4.70	4.99	4.61	2.50	2.11	1.92	1.03	2.39
2.00	4.45	4.79	4.88	2.54	2.25	2.27	0.10	2.74

Table 9 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 96 step experiments using stochastic approximation procedures with c equal to 0.5 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	2.63	3.60	3.40	2.52	10.24	3.63	3.18	3.12
-1.75	2.36	3.34	3.07	2.42	6.74	3.70	2.88	3.13
-1.50	2.24	3.22	2.67	2.32	4.47	3.43	2.72	3.01
-1.25	2.12	3.12	2.33	2.18	3.25	3.27	2.45	2.81
-1.00	2.13	2.98	2.00	2.05	2.42	2.98	2.15	2.56
-0.75	2.13	2.92	1.72	1.83	2.08	2.68	1.96	2.27
-0.50	2.29	2.84	1.52	1.71	1.97	2.52	1.83	1.97
-0.25	2.57	3.00	1.48	1.59	1.98	2.40	1.81	1.64
0.00	2.78	3.11	1.50	1.58	2.11	2.39	1.85	1.55
0.25	3.09	3.31	1.65	1.61	2.26	2.55	1.87	1.78
0.50	3.44	3.73	1.99	1.68	2.60	2.85	2.01	2.34
0.75	4.05	4.31	2.38	1.84	3.14	3.36	2.19	3.23
1.00	5.10	5.05	2.85	1.96	4.24	4.20	2.23	4.10
1.25	7.16	6.31	3.20	2.09	6.02	5.40	2.30	5.21
1.50	10.12	7.36	3.61	2.25	8.67	6.33	2.36	5.88
1.75	14.82	8.24	3.89	2.37	13.08	7.43	2.40	6.68
2.00	22.46	8.92	4.21	2.46	20.11	8.58	2.60	7.21

Table 10 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 96 step experiments using stochastic approximation procedures with c equal to the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	1.65	2.04	1.68	1.76	1.55	1.76	2.75	1.77
-1.75	1.71	2.03	1.70	1.77	1.56	1.81	2.70	1.75
-1.50	1.73	2.01	1.71	1.76	1.58	1.77	2.71	1.74
-1.25	1.74	2.07	1.71	1.75	1.60	1.79	2.74	1.71
-1.00	1.80	2.20	1.73	1.75	1.62	1.78	2.75	1.68
-0.75	1.85	2.15	1.74	1.73	1.61	1.73	2.83	1.67
-0.50	1.86	2.08	1.77	1.73	1.65	1.73	2.79	1.67
-0.25	1.85	2.00	1.81	1.75	1.69	1.70	2.88	1.66
0.00	1.96	2.06	1.78	1.76	1.70	1.74	2.91	1.68
0.25	2.06	2.10	1.88	1.74	1.72	1.74	2.91	1.64
0.50	2.10	2.19	1.90	1.80	1.72	1.76	3.02	1.73
0.75	2.17	2.28	1.96	1.79	1.72	1.75	3.09	1.79
1.00	2.28	2.41	2.00	1.83	1.74	1.73	3.30	1.91
1.25	2.44	2.51	2.02	1.84	1.93	1.95	3.23	2.00
1.50	2.68	2.80	2.11	1.88	2.01	1.99	3.33	2.11
1.75	2.82	2.85	2.12	1.89	2.10	2.09	3.27	2.14
2.00	3.06	2.98	2.17	1.92	2.18	2.11	3.31	2.32

Table 11 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 96 step experiments using stochastic approximation procedures with c equal to 1.5 times the asymptotic optimal values ($\beta = \pi / 3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	1.91	2.15	2.18	2.24	1.85	1.91	3.83	1.92
-1.75	1.85	1.96	2.22	2.22	1.87	1.92	3.82	1.93
-1.50	1.91	2.02	2.21	2.23	1.88	1.92	3.83	1.87
-1.25	1.89	1.95	2.23	2.23	1.88	1.89	3.86	1.87
-1.00	1.95	2.04	2.24	2.23	1.87	1.87	3.89	1.94
-0.75	1.93	2.00	2.26	2.21	1.89	1.91	3.99	1.85
-0.50	1.98	2.07	2.30	2.23	1.86	1.87	4.02	1.85
-0.25	2.01	2.08	2.24	2.22	1.89	1.90	3.97	1.95
0.00	1.98	2.09	2.25	2.20	1.88	1.90	4.09	1.90
0.25	1.96	2.00	2.28	2.23	1.88	1.88	4.11	1.90
0.50	1.98	2.04	2.26	2.26	1.88	1.88	4.19	1.90
0.75	1.99	2.11	2.32	2.25	1.91	1.91	4.25	1.93
1.00	2.03	2.02	2.34	2.27	1.88	1.89	4.20	1.95
1.25	2.04	2.07	2.34	2.28	1.87	1.89	4.27	1.98
1.50	2.09	2.12	2.37	2.28	1.89	1.86	4.32	2.00
1.75	2.12	2.14	2.36	2.23	1.93	1.91	4.38	2.00
2.00	2.17	2.19	2.43	2.29	1.94	1.95	4.37	2.05

Table 12 100×m.s.e. of estimators of $L_{1/\sqrt{2}}$ in 96 step experiments using stochastic approximation procedures with c equal to 2.0 times the asymptotic optimal values ($\beta = \pi / 3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	2.22	2.25	2.71	2.71	2.20	2.25	4.78	2.23
-1.75	2.17	2.19	2.75	2.74	2.25	2.24	4.81	2.23
-1.50	2.21	2.24	2.76	2.74	2.18	2.22	4.99	2.19
-1.25	2.22	2.26	2.72	2.74	2.24	2.26	4.96	2.25
-1.00	2.17	2.24	2.76	2.74	2.20	2.18	4.95	2.17
-0.75	2.24	2.26	2.75	2.75	2.22	2.24	5.00	2.23
-0.50	2.19	2.23	2.77	2.73	2.23	2.23	5.13	2.23
-0.25	2.20	2.25	2.75	2.71	2.23	2.23	5.03	2.26
0.00	2.18	2.17	2.78	2.73	2.22	2.22	5.05	2.24
0.25	2.18	2.17	2.75	2.70	2.20	2.20	5.11	2.22
0.50	2.21	2.22	2.74	2.69	2.20	2.20	5.13	2.23
0.75	2.21	2.19	2.74	2.70	2.24	2.23	5.23	2.24
1.00	2.24	2.22	2.76	2.77	2.21	2.22	5.31	2.28
1.25	2.22	2.23	2.85	2.77	2.23	2.23	5.47	2.27
1.50	2.25	2.29	2.82	2.79	2.20	2.23	5.35	2.30
1.75	2.27	2.26	2.86	2.84	2.21	2.22	5.40	2.30
2.00	2.26	2.26	2.85	2.75	2.23	2.23	5.49	2.33

Table 13 100×bias of estimators of $L_{1/\sqrt{2}}$ in 96 step experiments using stochastic approximation procedures with c equal to 0.5 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	-10.12	-2.18	-13.23	-8.28	-30.48	-5.09	-10.65	-10.07
-1.75	-8.13	-2.37	-12.09	-7.96	-23.66	-5.04	-9.96	-10.13
-1.50	-6.42	-2.18	-10.77	-7.41	-17.88	-4.70	-9.29	-9.61
-1.25	-4.66	-1.31	-9.08	-6.73	-13.46	-4.68	-8.09	-9.00
-1.00	-2.61	-0.26	-7.03	-5.59	-9.29	-3.63	-6.74	-8.13
-0.75	-1.02	0.76	-4.58	-4.30	-6.18	-2.88	-5.31	-6.86
-0.50	0.32	1.59	-2.42	-3.07	-3.71	-1.85	-3.85	-4.90
-0.25	1.81	2.86	-0.18	-1.73	-1.31	-0.39	-2.75	-2.21
0.00	3.46	4.28	2.22	-0.51	1.06	1.26	-1.70	0.75
0.25	5.44	5.85	4.42	0.99	3.64	3.41	-1.14	3.98
0.50	7.50	6.98	6.49	1.91	6.26	5.13	-0.73	7.24
0.75	10.19	8.18	8.21	2.88	9.47	7.04	-0.47	10.27
1.00	14.00	9.70	9.76	3.68	13.64	8.99	-0.43	12.65
1.25	19.35	11.16	10.98	4.42	18.73	10.79	-0.48	14.88
1.50	25.78	12.20	11.98	4.62	24.84	11.94	-0.71	16.45
1.75	33.99	12.92	12.57	4.95	32.60	13.49	-0.30	17.74
2.00	44.24	13.38	13.53	5.07	42.23	14.33	-0.50	18.88

Table 14 100×bias of estimators of $L_{1/\sqrt{2}}$ in 96 step experiments using stochastic approximation procedures with c equal to the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	1.84	3.44	0.19	0.03	-2.24	0.84	-1.67	-0.78
-1.75	2.13	3.52	0.45	0.15	-1.35	0.75	-1.63	-0.74
-1.50	2.36	3.43	0.88	0.23	-0.64	0.79	-1.41	-0.65
-1.25	2.65	3.60	1.04	0.09	-0.08	0.88	-1.13	-0.57
-1.00	3.11	3.90	1.39	0.39	0.32	0.98	-1.15	-0.52
-0.75	3.38	3.97	1.72	0.49	0.70	1.08	-0.95	-0.09
-0.50	3.56	4.00	1.82	0.57	0.80	1.02	-0.82	0.12
-0.25	3.74	4.04	1.97	0.57	1.02	1.14	-0.78	0.58
0.00	3.73	4.11	2.14	0.64	1.23	1.28	-0.86	1.03
0.25	4.11	4.42	2.72	0.75	1.36	1.35	-0.65	1.56
0.50	4.12	4.56	2.99	0.92	1.69	1.64	-0.71	2.02
0.75	4.13	4.40	3.10	1.01	2.10	2.06	-0.71	2.60
1.00	4.40	4.70	3.41	1.20	2.39	2.08	-0.63	3.12
1.25	4.59	4.79	3.68	1.26	3.20	2.55	-0.76	3.54
1.50	5.39	5.31	3.80	1.11	3.77	2.74	-0.81	3.75
1.75	5.81	5.04	3.87	1.05	4.52	2.99	-0.89	4.29
2.00	6.73	5.24	4.08	1.39	5.34	2.91	-0.92	4.42

Table 15 100×bias of estimators of $L_{1/\sqrt{2}}$ in 96 step experiments using stochastic approximation procedures with c equal to 1.5 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	2.01	2.66	1.70	0.81	0.83	1.09	-0.56	0.76
-1.75	2.01	2.45	1.83	0.71	0.79	1.13	-0.41	0.52
-1.50	2.19	2.55	1.95	0.93	0.83	1.00	-0.34	0.51
-1.25	2.30	2.55	2.05	1.03	1.07	1.09	-0.17	0.65
-1.00	2.26	2.44	2.10	0.94	1.11	1.14	-0.18	0.72
-0.75	2.54	2.66	2.08	0.83	1.01	1.06	-0.20	0.81
-0.50	2.61	2.72	2.15	1.08	1.03	1.06	-0.53	0.74
-0.25	2.78	2.89	2.15	0.89	1.01	1.03	-0.43	0.99
0.00	2.71	2.85	2.10	0.99	1.03	1.05	-0.29	0.88
0.25	2.81	2.94	2.11	0.93	1.10	1.09	-0.37	1.13
0.50	2.71	2.90	2.19	1.10	1.06	1.05	-0.30	1.34
0.75	2.57	2.81	2.21	0.89	1.11	1.11	-0.43	1.35
1.00	2.55	2.77	2.38	0.98	1.25	1.15	-0.30	1.38
1.25	2.67	2.69	2.30	1.01	1.14	1.17	-0.23	1.60
1.50	2.76	3.08	2.21	1.08	1.45	1.43	-0.18	1.55
1.75	2.65	2.80	2.53	0.92	1.45	1.34	-0.23	1.68
2.00	2.68	2.86	2.40	1.11	1.50	1.25	-0.57	1.74

Table 16 100×bias of estimators of $L_{1/\sqrt{2}}$ in 96 step experiments using stochastic approximation procedures with c equal to 2.0 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

Procedure	1	2	3	4	5	6	7	8
Start								
-2.00	1.50	1.65	1.87	0.94	0.97	1.02	0.21	0.80
-1.75	1.71	1.81	1.94	1.02	0.97	0.91	0.04	0.89
-1.50	1.66	1.73	1.97	0.97	0.94	1.02	0.08	0.77
-1.25	1.69	1.80	1.91	0.84	1.03	1.12	0.32	0.78
-1.00	1.86	1.99	1.98	0.87	1.02	1.03	-0.05	0.88
-0.75	1.75	1.81	1.90	1.02	0.97	1.00	-0.10	1.00
-0.50	1.91	1.98	1.99	1.01	0.98	0.94	0.10	0.82
-0.25	1.90	1.94	1.91	1.00	0.85	0.82	-0.02	1.09
0.00	1.85	1.84	1.81	0.89	1.01	1.02	0.03	0.97
0.25	1.91	1.95	1.95	0.97	0.90	0.90	0.12	1.01
0.50	2.01	2.05	1.99	0.92	1.06	1.06	-0.16	0.96
0.75	1.94	1.95	2.00	0.98	0.97	0.97	0.01	1.09
1.00	1.83	1.88	2.11	1.03	0.87	0.88	0.02	1.01
1.25	1.88	1.97	1.99	0.97	1.01	0.98	0.01	1.04
1.50	1.97	1.93	2.01	1.00	1.05	1.06	0.09	1.32
1.75	1.83	1.98	1.96	0.98	1.04	1.02	0.11	1.17
2.00	1.90	1.93	2.02	0.97	1.04	1.10	0.01	1.25

Table 17 100xm.s.e. and 100xbias of estimators of $L_{c.q}$ in 96 step experiments using Procedures 1,4,5 and 8 with c equal to 0.5 times the asymptotic optimal values ($\beta = \pi / 3.0^{1/2}$, based on 2000 simulations).

Procedure	100 x m.s.e.				100 x bias			
	1	4	5	8	1	4	5	8
Start								
-2.00	8.65	5.09	3.23	3.60	14.40	3.80	-3.18	-3.05
-1.75	10.74	5.09	3.45	3.44	18.92	3.92	-1.47	-2.42
-1.50	13.23	5.10	3.66	3.31	22.60	3.97	0.28	-2.49
-1.25	18.15	5.21	4.19	3.29	25.89	3.88	2.29	-2.07
-1.00	22.30	5.18	4.55	3.30	27.01	3.64	4.15	-1.72
-0.75	26.21	5.20	5.12	3.34	26.72	4.24	5.81	-1.08
-0.50	27.84	5.29	6.12	3.33	24.60	4.24	7.26	0.12
-0.25	27.90	5.32	6.97	3.42	22.14	4.49	7.87	0.66
0.00	26.03	5.35	7.91	3.58	18.79	4.48	9.18	1.64
0.25	24.73	5.49	9.03	3.72	17.84	4.40	10.93	2.47
0.50	24.76	5.49	10.29	4.07	18.01	4.97	13.44	3.59
0.75	24.40	5.64	11.34	4.33	19.92	5.03	17.06	4.10
1.00	22.87	5.54	13.78	4.39	22.60	4.82	23.59	4.71
1.25	23.77	5.60	18.06	4.73	28.92	4.72	32.33	5.15
1.50	28.21	5.82	25.65	4.94	38.12	5.23	43.33	5.78
1.75	35.76	5.95	38.46	5.43	49.87	5.21	57.26	5.82
2.00	49.90	5.71	59.32	5.68	64.71	5.00	74.08	6.00

Table 18 100xm.s.e. and 100 bias of estimators of $L_{c.q}$ in 96 step experiments using Procedures 1,4,5 and 8 with c equal to the asymptotic optimal values ($\beta = \pi / 3.0^{1/2}$, based on 2000 simulations).

Procedure	100 x m.s.e.				100 x bias			
	1	4	5	8	1	4	5	8
Start								
-2.00	92.60	9.35	5.76	5.07	86.21	7.60	9.82	3.65
-1.75	120.89	9.47	6.46	5.14	94.37	7.53	11.27	3.88
-1.50	145.91	9.53	6.87	5.27	97.90	7.75	13.25	3.91
-1.25	158.35	9.37	7.92	5.25	93.53	7.31	15.23	3.98
-1.00	157.66	9.26	9.14	5.35	83.98	7.40	17.19	4.16
-0.75	147.84	9.45	10.42	5.34	72.77	7.19	18.62	4.06
-0.50	131.15	9.44	11.44	5.28	61.52	7.31	19.70	4.32
-0.25	110.20	9.11	12.18	5.19	50.51	7.55	18.85	4.31
0.00	86.51	9.29	13.65	5.20	40.81	7.29	18.35	4.22
0.25	72.02	9.25	13.94	5.21	34.37	7.68	17.07	4.30
0.50	62.40	9.36	14.06	5.42	29.81	7.42	15.49	4.48
0.75	56.45	9.22	14.20	5.59	26.97	7.52	14.60	4.70
1.00	45.66	9.48	14.24	5.49	22.47	7.65	13.91	4.60
1.25	38.65	9.55	13.67	5.74	19.67	7.96	13.97	4.89
1.50	34.72	9.49	14.28	5.58	18.41	7.61	14.84	4.69
1.75	29.11	9.32	13.86	5.76	17.47	7.62	15.79	4.92
2.00	23.94	9.60	14.08	5.74	17.46	7.67	18.41	5.19

Table 19 100×m.s.e. and 100×bias of estimators of $L_{0.9}$ in 96 step experiments using Procedures 1,4,5 and 8 with c equal to 1.5 times the asymptotic optimal values ($\beta = \pi/3.0^{1/2}$, based on 2000 simulations).

	100 × m.s.e.				100 × bias			
Procedure	1	4	5	8	1	4	5	8
Start								
-2.00	482.78	14.31	7.75	7.01	197.05	10.83	12.86	5.94
-1.75	527.83	14.53	7.97	7.20	196.51	11.16	14.54	6.11
-1.50	547.44	13.94	9.14	7.33	187.99	11.02	17.69	5.75
-1.25	525.26	14.40	10.21	7.36	168.07	10.87	20.29	5.61
-1.00	468.81	14.05	12.23	7.48	141.66	10.44	23.25	6.13
-0.75	398.73	14.36	14.79	7.27	115.71	10.00	25.13	5.89
-0.50	324.80	13.88	16.63	7.06	92.67	10.42	25.73	5.54
-0.25	253.28	14.00	18.03	6.97	73.08	9.73	24.39	5.68
0.00	186.88	13.62	19.31	7.09	57.49	10.53	22.49	5.46
0.25	146.91	13.48	19.46	7.21	47.82	10.15	20.30	5.77
0.50	120.58	13.95	19.35	7.37	40.51	10.40	18.08	6.09
0.75	102.51	14.09	18.69	7.21	35.40	10.20	16.42	6.07
1.00	80.76	13.95	17.81	7.26	29.91	10.31	15.41	5.93
1.25	65.02	13.85	16.28	7.31	24.52	10.25	13.54	6.30
1.50	56.92	13.92	15.36	7.33	21.94	9.90	12.42	6.24
1.75	44.18	14.21	14.55	7.43	18.48	10.32	12.27	5.87
2.00	34.02	14.00	13.04	7.75	16.18	10.16	11.46	6.12

Table 20 100×m.s.e. and 100×bias of estimators of $L_{0.9}$ in 96 step experiments using Procedures 1,4,5 and 8 with c equal to 2.0 times the asymptotic values ($\beta = \pi/3.0^{1/2}$; based on 2000 simulations).

	100 × m.s.e.				100 × bias			
Procedure	1	4	5	8	1	4	5	8
Start								
-2.00	1233.36	20.82	8.20	9.08	314.02	14.27	14.74	7.02
-1.75	1259.14	19.95	9.44	9.27	302.21	13.49	18.03	7.14
-1.50	1230.30	19.69	11.43	9.26	279.93	13.69	22.48	6.79
-1.25	1119.25	20.11	14.37	9.28	242.88	14.20	27.27	7.51
-1.00	953.37	19.06	17.77	9.23	199.00	12.98	30.60	6.87
-0.75	777.99	18.86	21.79	9.31	158.76	13.25	32.53	7.34
-0.50	609.26	18.45	24.81	9.14	123.50	12.83	31.99	6.78
-0.25	458.42	18.40	26.82	9.03	95.61	13.01	29.11	6.78
0.00	325.56	18.73	27.51	8.96	72.62	13.13	26.08	6.56
0.25	247.31	18.02	26.80	9.00	59.71	12.71	22.72	7.07
0.50	198.23	18.63	25.56	9.23	51.13	13.34	19.76	7.24
0.75	164.54	18.54	23.90	9.03	44.30	12.97	17.40	6.74
1.00	125.66	18.81	21.82	9.16	35.99	12.87	15.61	6.76
1.25	100.32	18.58	19.25	9.38	30.00	13.23	13.70	7.06
1.50	86.31	19.00	18.32	9.18	26.38	12.34	12.87	7.05
1.75	65.79	18.50	16.67	9.14	21.67	12.78	11.50	6.48
2.00	48.44	18.82	14.45	9.13	17.84	13.22	10.98	6.74

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