On Multiparameter Quantum SL_n and Quantum Skew-symmetric Matrices

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Abstract

Since its beginning in the early 1980s the subject of Quantum Groups has expanded to include many areas of mathematics. We will be concerned with studying two particular quantized coordinate rings from an algebraic perspective.

The first quantized coordinate ring under investigation is Multiparameter Quantum SL_n. In Chapter 2, inspired by an observation in a paper by Dipper and Donkin, we tackle the problem of defining a quantum analogue of SL_n in the Multiparameter Quantum Matrices setting when the quantum determinant is not central. We construct a candidate for this algebra in a natural way using the process of Noncommutative Dehomogenisation. We go on to show that the object defined has many appropriate properties for such an analogue and observe that our new algebra can also be obtained via a process known as *twisting*. Finally we see what our definition means in the particular case of Dipper-Donkin Quantum Matrices and also look at the Standard Quantum Matrices case.

In Chapter 3 we move on to our other object of study, Quantum Skew-symmetric Matrices. This was defined, along with the concept of q-Pfaffians, in a paper by Strickland in 1996. We show that this algebra is an iterated skew polynomial ring, and we are able to read off many results by applying the machinery detailed in a book by Brown and Goodearl. We go on to show that a q-Laplace expansion of q-Pfaffians holds and that the highest-length q-Pfaffian is central. Finally we show that a factor of Quantum Skew-symmetric Matrices is isomorphic to $G_q(2,n)$.

Quantum Skew-symmetric Matrices are also mentioned in a 1996 paper by Noumi. In Chapter 4 we recall his definition of the algebra and of q-Pfaffians. These definitions are different to those of Strickland. We show that, when q is not a root of unity, these contrasting definitions are in fact the same. Using Noumi's definition we show that another Laplace-type expansion, the natural q-analogue of a classical result, holds for q-Pfaffians.

In the final chapter we investigate the commutation relations between the q-Pfaffians. In proving the centrality of the highest-length q-Pfaffian in Chapter 3 we determine some specific commutation relations; these are used in Chapter 5 to establish more general results. We observe that the set of q-Pfaffians has the structure of a partially ordered set and show that our relations are well-behaved with respect to this structure.

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Contents

Chapter 1

Introduction

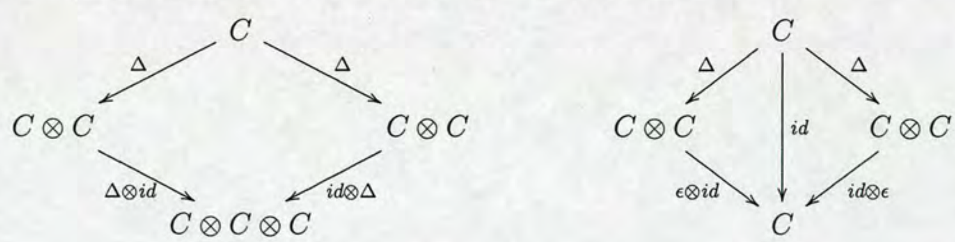
The main objects of study in this thesis are two algebras that can be thought of as quantized coordinate rings. The purpose of this first chapter is to introduce the terminology that we will use to describe and investigate them. We claim no originality for any of the material in this chapter, it is all well known. Indeed we also claim no originality for the presentation, most of the material having been already presented in perfect clarity in [4] - a book which we take as our main source of inspiration. Our sole purpose is to fix our notation and gather together the language and results that will frame the rest of the thesis.

Throughout this thesis, if not explicitly stated, K will denote a fixed base field and tensor products, $\otimes$, will be over K. All rings and algebras will be assumed to contain a unit element.

1.1 Coalgebras, Bialgebras, and Hopf Algebras

Many of the algebras that we will encounter will possess, or will be proved to possess, the structure of either a coalgebra, bialgebra, or a Hopf algebra. In this section we will give definitions of these, and related, concepts - material that can all be found in, for example, [1], [4], [8], and [22].

Definition 1.1.1. *A* coalgebra (C, Δ, ϵ) *is a* K*-vector space* C*, together with* K*-linear maps* $\Delta : C \longrightarrow C \otimes C$*, called the* comultiplication*, and* $\epsilon : C \longrightarrow K$*, called the* counit*, such that the following diagrams commute:*



Remark 1.1.2. *If we think of an algebra as being a K-vector space A, together with K-linear maps $\mu : A \otimes A \longrightarrow A$, $\eta : K \longrightarrow A$ representing multiplication ($a \otimes b \mapsto ab$) and unit ($\alpha \mapsto \alpha \cdot 1$) respectively, then the commutative diagrams in the previous definition are exactly those that one obtains by "dualizing" the commutative diagrams involving μ and η that represent the usual algebra axioms for A.*

Notation 1.1.3. *We will follow the* Sweedler notation *convention and write*

$$\Delta(c) = \sum_c c_1 \otimes c_2$$

for an element $c \in C$.

Definition 1.1.4. *An element g in a coalgebra C is said to be* grouplike *if $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$.*

Definition 1.1.5. *A map $f : C \longrightarrow D$ between coalgebras C and D is said to be a* coalgebra morphism *if it is a linear map such that $\Delta_D \circ f = (f \otimes f) \circ \Delta_C$ and $\epsilon_D \circ f = \epsilon_C$.*

Definition 1.1.6. *A subspace I of a coalgebra C is a* coideal *if $\Delta(I) \subseteq C \otimes I + I \otimes C$ and $\epsilon(I) = 0$. It is easy to see that C/I is then a coalgebra with the induced comultiplication and counit.*

A coalgebra that also has the structure of an algebra is called a bialgebra if the two structures are well-behaved with respect to each other, that is:

Definition 1.1.7. *A* bialgebra $(B, \mu, \eta, \Delta, \epsilon)$ *is a K-vector space B, together with K-linear maps $\mu, \eta, \Delta, \epsilon$ such that (B, μ, η) is an algebra, (B, Δ, ϵ) is a coalgebra and the following equivalent conditions hold:*

1. *Δ and ϵ are algebra morphisms, or*
2. *μ and η are coalgebra morphisms.*

Definition 1.1.8. *A map $f : B \longrightarrow D$ between bialgebras B and D is a* bialgebra morphism *if it is both an algebra morphism and a coalgebra morphism.*

Definition 1.1.9. *A subspace I of a bialgebra B is a* biideal *if it is both a coideal with respect to the coalgebra structure of B and an ideal with respect to the algebra structure. In this case it follows that B/I is a bialgebra.*

Definition 1.1.10. *A* Hopf algebra $(H, \mu, \eta, \Delta, \epsilon, S)$ *is a K-vector space H with K-linear maps $\mu, \eta, \Delta, \epsilon$ making $(H, \mu, \eta, \Delta, \epsilon)$ into a bialgebra, together with a K-linear map $S : H \longrightarrow H$ called the* antipode*, such that, in Sweedler notation,*

$$\sum_h S(h_1)h_2 = \epsilon(h) \cdot 1_H = \sum_h h_1 S(h_2)$$

for all $h \in H$. Let us define the convolution product $*$, *of two maps $f, g \in Hom_K(H, H)$ to be*

$$(f * g)(h) = \sum_h f(h_1)g(h_2)$$

for $h \in H$. Then the previous condition on S may be written as,

$$S * id = \eta \circ \epsilon = id * S.$$

We will often say "H is a Hopf algebra" instead of "$(H, \mu, \eta, \Delta, \epsilon, S)$ is a Hopf algebra" taking the various maps as implied.

The following well-known lemma is useful when working with Hopf algebras:

Lemma 1.1.11. *The antipode, S, of a Hopf algebra, H, is an algebra anti-morphism, that is, $S(ab) = S(b)S(a)$ for all $a, b \in H$.*

Definition 1.1.12. *A map $f : H \longrightarrow G$ between Hopf algebras H and G is a* Hopf algebra morphism *if it is a bialgebra morphism such that $f \circ S_H = S_G \circ f$.*

In Chapter 2 we will have cause to show that certain maps between Hopf algebras are Hopf algebra morphisms. To reduce the amount of work needed to be done in these instances we will call upon a result that we have found only in [8]. Being less well-known we provide a specific reference:

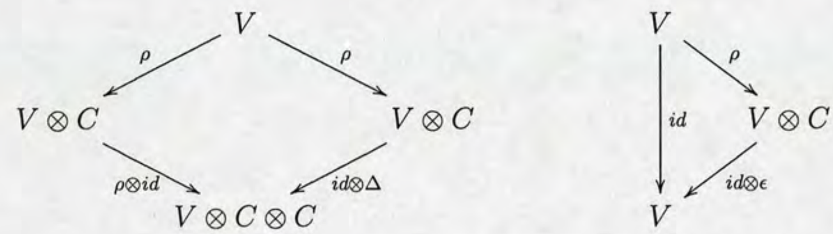
Proposition 1.1.13. [8, Proposition 4.2.5] *Let $f : H \longrightarrow G$ be a map between Hopf algebras H and G. Suppose f is a bialgebra morphism. Then it follows that f is a Hopf algebra morphism.*

Finally the definitions we have seen for coideals and biideals have their appropriate Hopf-analogue:

Definition 1.1.14. *A subspace I of a Hopf algebra H is a* Hopf ideal *if it is a biideal such that $S(I) \subseteq I$. In this case H/I is a Hopf algebra.*

Dual to the notion of modules of algebras is the concept of comodules of coalgebras:

Definition 1.1.15. *Let C be a coalgebra. A* right C-comodule *is a K-vector space V with a linear map $\rho : V \longrightarrow V \otimes C$, called the* coaction, *such that the following diagrams commute:*



Just as when we talk about the "representation theory" of an algebra we are referring to its modules, when we talk of the "corepresentation theory" of an object with a coalgebra structure we refer to its comodules.

To end this section we define a structure that we will encounter in Chapter 3. Suppose we have an algebra H and an H-module, A. If H is in fact a bialgebra and A is in fact an algebra, then it is desirable that these extra structures interact in a "nice" way, that is:

Definition 1.1.16. *Let H be a bialgebra and A an algebra. We say that A is a* left H-module algebra *if A is a left H-module such that*

1. $h(ab) = \sum_h (h_1(a))(h_2(b))$ *for $h \in H$ and $a, b \in A$; and*
2. $h(1_A) = \epsilon(h).1_A$.

1.2 Skew Polynomial Rings

In this section we give another set of definitions and results that will form part of the fundamental language that we use throughout this thesis. All this material and much more can be found in [17].

Let R be a ring and let σ be an automorphism of R.

Definition 1.2.1. *An additive map $\delta : r \longrightarrow R$ is called a* σ-derivation *(more precisely a* left σ-derivation*) on R if $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$ for all $r, s \in R$.*

Let δ be a σ-derivation on R. Then we can form the *skew polynomial ring* $R[x; \sigma, \delta]$ where $xr = \sigma(r)x + \delta(r)$ for $r \in R$. This can be constructed explicitly as a subring of a certain ring of endomorphisms, but we are only concerned with the properties that it possesses:

Definition 1.2.2. *Let R be a ring, let σ be an automorphism of R, and let δ be a σ-derivation on R. Then $T = R[x; \sigma, \delta]$ means*

1. *T is a free left R-module with basis $\{x^n : n \in \mathbb{N}\}$.*
2. *$xr = \sigma(r)x + \delta(r)$ for all $r \in R$.*

Remark 1.2.3. *The above definition would work just as well if σ were only an endomorphism of R, but we will always require σ to be an automorphism. Also we note here, that if R is also a K-algebra then we will always assume that σ is a K-algebra automorphism and δ is K-linear.*

When working with skew polynomial rings we will always write elements with left-hand coefficients, that is for $f \in R$ we will write it in the form

$$f = r_n x^n + r_{n-1}x^{n-1} + \cdots r_1 x + r_0$$

where $r_n, \ldots, r_0 \in R$ and we will say that f is of *degree* n (assuming $r_n \neq 0$).

Definition 1.2.4. *If our $\delta = 0$ then we may localize $R[x; \sigma]$ at the set consisting of the powers of x and form the* skew-Laurent ring $R[x, x^{-1}; \sigma]$.

We now give two key results concerning skew polynomial rings.

Lemma 1.2.5. *If R is a domain then the skew polynomial ring $R[x; \sigma, \delta]$ is a domain.*

Theorem 1.2.6. *If R is noetherian then so is $R[x; \sigma, \delta]$.*

Many of the algebras that we encounter are not only skew polynomial rings but are in fact *iterated skew polynomial rings over K*. We now give a precise definition of what this means.

Definition 1.2.7. *We say A is an* iterated skew polynomial ring over R *and write*

$$A = R[x_1; \sigma_1, \delta_1][x_2; \sigma_2, \delta_2] \cdots [x_n; \sigma_n, \delta_n]$$

if $R[x_1; \sigma_1, \delta_1]$ is a skew polynomial ring and for each $A_i = R[x_1; \sigma_1, \delta_1] \cdots [x_i; \sigma_i, \delta_i]$, A_i is a skew polynomial ring over A_{i-1} ($i = 2, \ldots, n$).

Remark 1.2.8. *Of course a quick induction extends the previous two results to the case of iterated skew polynomial rings.*

1.3 Quantum Groups

The area of Quantum Groups that we will be interested in is that of "quantized coordinate rings" viewed from an algebraic perspective. By a quantized coordinate ring we mean a noncommutative "deformation" of the coordinate ring of an algebraic group or a related algebraic variety. The simplest example is that of the quantum plane. The classical coordinate ring of the plane, $\mathcal{O}(K^2)$, is just $K[x,y]$, polynomials in two commuting variables. For a nonzero element $q \in K^{\times}$ we define the *quantized coordinate ring of the plane* (or *quantum plane* for short) to be

$$\mathcal{O}_q(K^2) = K\langle x, y : xy = qyx \rangle.$$

We do not just allow any noncommutative version of a classical object but usually restrict ourselves to ones which retain certain "nice" properties from the classical case. With the quantum plane it is not hard to see that it is an iterated skew polynomial ring over K and so is a noetherian domain, properties that hold for the classical coordinate ring of the plane. Also setting $q = 1$ brings us back to the classical case - another feature of desirable quantum analogues.

The standard example of a quantized coordinate ring that we will keep in mind throughout this thesis is that of *Quantum Matrices* [32], [33]. In the classical case the coordinate ring of $n \times n$ matrices is just the polynomial ring generated by the n^2 coordinate functions, t_{ij} say, that pick out the ij-th entry of a matrix. The standard quantum version of this is defined below. Throughout what follows q will be a nonzero element of our base field K and we shall write $\hat{q} := (q - q^{-1})$.

Definition 1.3.1. *Let n be a positive integer. The* coordinate ring of quantum $n \times n$ matrices, *$\mathcal{O}_q(M_n)$ is the K-algebra generated by the n^2 indeterminates $\{t_{ij} : i, j = 1, ..., n\}$, subject to the following relations:*

$$\begin{aligned} t_{ij}t_{il} &= qt_{il}t_{ij}, \\ t_{ij}t_{kj} &= qt_{kj}t_{ij}, \\ t_{il}t_{kj} &= t_{kj}t_{il}, \\ t_{ij}t_{kl} &= t_{kl}t_{ij} + \hat{q}t_{il}t_{kj}, \end{aligned}$$

for $1 \leq i < k \leq n$ and $1 \leq j < l \leq n$. The language we have established in the previous two sections now comes into play. It is known (see for example [4, Theorem I.2.7]) that $\mathcal{O}_q(M_n)$ is an iterated skew polynomial algebra over K and hence a noetherian domain. It also possesses a bialgebra structure with the natural comultiplication and counit:

$$\Delta(t_{ik}) = \sum_{j=1}^{n} t_{ij} \otimes t_{jk}$$
$$\epsilon(t_{ik}) = \delta_{ik}.$$

There is a distinguished element of this algebra known as the *quantum determinant*, denoted by $\det_q$,

$$\det_q = \sum_{\pi \in S_n} (-q)^{l(\pi)} t_{1,\pi(1)} t_{2,\pi(2)} \cdots t_{n,\pi(n)}$$

where $l(\pi)$, is the length of the permutation π. Note that if we set $q = 1$ then we have the classical definition of the determinant. By [33, Theorem 4.6.1] we know that $\det_q$ is in the centre of $\mathcal{O}_q(M_n)$. So analogously to the classical case we may define $\mathcal{O}_q(GL_n)$ and $\mathcal{O}_q(SL_n)$ as follows:

$$\mathcal{O}_q(GL_n) := \mathcal{O}_q(M_n)[\det_q^{-1}] \qquad \text{and} \qquad \mathcal{O}_q(SL_n) := \frac{\mathcal{O}_q(M_n)}{\langle \det_q - 1 \rangle}.$$

Now it can be worked out that $\det_q$ is a grouplike element of $\mathcal{O}_q(M_n)$, that is $\Delta(\det_q) = \det_q \otimes \det_q$ and $\epsilon(\det_q) = 1$, and so the bialgebra structure of $\mathcal{O}_q(M_n)$ induces bialgebra structures on $\mathcal{O}_q(GL_n)$ and $\mathcal{O}_q(SL_n)$. Furthermore from [33, Theorem 5.3.2] we know that $\mathcal{O}_q(GL_n)$ and $\mathcal{O}_q(SL_n)$ are Hopf algebras with antipode given by

$$S(t_{ij}) = (-q)^{i-j}[\,\tilde{j} \mid \tilde{i}\,]\det_q^{-1}$$

where $\tilde{i} = \{1, ..., n\} \setminus \{i\}$. We will now explain what we mean by $[\,\tilde{j} \mid \tilde{i}\,]$. Just as in [16], for index sets I, J with $|I| = |J|$, we use the notation $[I|J]$ to denote the quantum determinant of $\mathcal{O}_q(M_{I,J})$ the quantum matrix subalgebra of $\mathcal{O}_q(M_n)$ generated by the elements t_{ij} with $i \in I, j \in J$. We will call $[I|J]$ the *quantum minor* with rows I and columns J. This leads us on to the definition of our final example of a quantized coordinate ring, namely the quantum grassmanian [23]. Firstly we note that Definition 1.3.1 can be naturally extended to enable us to define *quantum* $m \times n$ *matrices*, $\mathcal{O}_q(M_{mn})$. The *quantum* $m \times n$ *grassmanian*, $G_q(m, n)$ is then defined to be the subalgebra of $\mathcal{O}_q(M_{mn})$ generated by the $m \times m$ quantum minors of $\mathcal{O}_q(M_{mn})$.

Before leaving this section we should point out that the original objects to be defined in Quantum Groups were not quantized coordinate rings but rather *quantized enveloping algebras*. These are noncommutative deformations of the universal enveloping algebra of a Lie algebra. We will not discuss these here, although we shall encounter an example of such an object in Chapter 3.

1.4 Noncommutative Properties

This section is a mixed bag of definitions. For completeness we collect here the definitions of various "noncommutative properties" that we refer to later in the thesis. We begin by presenting the definition of Gelfand-Kirillov dimension, a useful tool when dealing with noncommutative algebras. The standard reference for all things Gelfand-Kirillov is [26].

Definition 1.4.1. *Let A be a finitely generated K-algebra. Let V be a finite dimensional K-subspace of A containing 1_A such that V generates A as an algebra and let V^n denote the linear span of all products of at most n elements of V. The* Gelfand-Kirillov dimension *of A is defined to be:*

$$GKdim(A) = \limsup_{n\to\infty} \frac{\log dim_K(V^n)}{\log n}.$$

Remark 1.4.2. *In [26, Lemma 1.1] it is shown that the above definition is independent of the choice of V.*

We will use GKdim more than once to show that a map under consideration is an isomorphism. To do this we will make use of the following two results which we record here:

Lemma 1.4.3. [26, Lemma 3.1] *If B is a subalgebra or a homomorphic image of a K-algebra A, then $GKdim(B) \leq GKdim(A)$.*

Proposition 1.4.4. [26, Proposition 3.15] *Let I be an ideal of a K-algebra A, and assume that I contains a right regular element or a left regular element of A. Then*

$$GKdim(A/I) + 1 \leq GKdim(A).$$

Next we deal with various homological properties that are considered "nice" for a noncommutative ring to satisfy. They can be viewed as the appropriate noncommutative analogues of homological conditions used in commutative algebra. Requiring a noncommutative noetherian ring to have finite injective or global dimensions turns out to be too lenient to be useful and so an extra condition is imposed, see for example [28], [35], [36]. Our sources for the presentation of these definitions are the survey paper [6] and [4, Appendix I.15].

Let R be a noetherian ring.

Definition 1.4.5. *The* grade *of a finitely generated R-module M is defined to be*

$$j(M) := \inf\{i \geq 0 : Ext^i_R(M, R) \neq 0\}.$$

Definition 1.4.6. *Note that for a left (right) R-module M and for $i \geq 0$, $Ext^i_R(M,R)$ is a right (left) R-module via the right (left) action on R. We say that a noetherian ring R satisfies the* Auslander condition *if $j(N) \geq i$ for all finitely generated R-submodules $N \subseteq Ext^i_R(M,R)$ for every finitely generated right or left R-module M and for all $i \geq 0$.*

Definition 1.4.7. *A noetherian ring R is* Auslander-Gorenstein *if it satisfies the Auslander condition and has finite right and left injective dimension.*

Definition 1.4.8. *A noetherian ring R is* Auslander-regular *if it is Auslander-Gorenstein and has finite global dimension.*

Definition 1.4.9. *An algebra A is said to be* Cohen-Macaulay *if*

$$j(M) + GKdim(M) = GKdim(A) < \infty$$

for every nonzero finitely generated A-module M.

Finally, we briefly record the definition of the noncommutative Nullstellensatz given in [4]:

Definition 1.4.10. [4, Definition II.7.14] *Let A be a noetherian K-algebra. We say that A* satisfies the Nullstellensatz over *K provided A is a Jacobson ring and that the endomorphism ring of every irreducible A-module is algebraic over K.*

1.5 Skew-symmetric Matrices and Pfaffians

In the final three chapters of this thesis we will be concerned with investigating the quantum analogue of the coordinate ring of skew-symmetric matrices. In this section we recall the basic setup of the classical situation and recall some known identities.

An $n \times n$ matrix over K, $A = (a_{ij})$ say, is skew-symmetric if $A^t = -A$. In that case we have $a_{ji} = -a_{ij}$ for $i \leq j$ and in particular $a_{ii} = 0$, that is A has zeros on its main diagonal. So we can see that a skew-symmetric matrix A is completely determined by its upper-triangular elements, and we shall write A as

$$A = \begin{pmatrix} a_{12} & \cdots & a_{1n} \\ & \ddots & \vdots \\ & & a_{n-1,n} \\ & & \end{pmatrix}$$

The coordinate ring of skew-symmetric matrices, $\mathcal{O}(Sk_n)$, is therefore commuting polynomials in the $n(n-1)/2$ upper-triangular coordinate functions. This coordinate ring is a representation for the general linear group GL_n. If we think of our a_{ij} as the coordinate functions on a general skew-symmetric matrix and arrange them in our matrix A, then the action of GL_n on $\mathcal{O}(Sk_n)$ is given by

$$X(A) = XAX^t, \quad \text{for } X \in GL_n.$$

We see $\mathcal{O}(Sk_n)$ in this context in [2]. Crucial to the understanding of $\mathcal{O}(Sk_n)$ in [2] are the *Pfaffians* of skew-symmetric matrices - a concept related to the determinant. If n is odd then the determinant of a skew-symmetric matrix is zero. So from now on we restrict ourselves to the cases where n is even:

$$\det \begin{pmatrix} & a_{12} \\ & \end{pmatrix} = a_{12}^2,$$

$$\det \begin{pmatrix} & a_{12} & a_{13} & a_{14} \\ & & a_{23} & a_{24} \\ & & & a_{34} \\ & & & \end{pmatrix} = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2.$$

We see that the determinant in these two examples is the square of a polynomial in the entries of the matrix. In 1849 Cayley proved that this held in general, and the polynomial in question is the Pfaffian of the skew-symmetric matrix (for a historical overview of Pfaffians we refer the reader to [25]). We will now give a precise definition of a Pfaffian and then some Pfaffian identities. The following material comes from [13, Appendix D], [24], and [19].

Let A be a $n \times n$ skew-symmetric matrix with $n = 2m$.

Definition 1.5.1. *Let Γ_n be the subset of S_n consisting of elements σ such that $\sigma(2i-1) < \sigma(2i)$ for $i = 1, ..., m$ and $\sigma(2i-1) < (2i+1)$ for $i = 1, ..., m-1$. Then the Pfaffian of A, which we denote $\mathrm{Pf}(A)$, is*

$$\mathrm{Pf}(A) = \sum_{\sigma \in \Gamma_n} (-1)^{l(\sigma)} a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(n-1)\sigma(n)}$$

where $l(\sigma)$ is the length of the permutation σ.

Since all the a_{ij} commute we have the following equivalent definition:

Definition 1.5.2. *Let $\Omega_n := \{\sigma \in S_n : \sigma(2i-1) < \sigma(2i) \text{ for } i = 1, ..., m\}$. Then*

$$\mathrm{Pf}(A) = \frac{1}{m!} \sum_{\sigma \in \Omega_n} (-1)^{l(\sigma)} a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(n-1)\sigma(n)}.$$

Since $a_{ji} = -a_{ij}$ for $i < j$ we can relax the restrictions on the permutations still further and so we also have the equivalent definition:

Definition 1.5.3.

$$\mathrm{Pf}(A) = \frac{1}{2^m m!} \sum_{\sigma \in S_n} (-1)^{l(\sigma)} a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(n-1)\sigma(n)}.$$

There is also a recursive definition of the Pfaffian. For this we require the notion of a *sub-Pfaffian* . By $\mathrm{Pf}_{i_1,\dots,i_r}(A)$ we mean the Pfaffian of the $r \times r$ skew-symmetric matrix formed by the entries of A which belong to the rows and columns $i_1, \dots, i_r$.

Definition 1.5.4. *If $n = 2$ $\mathrm{Pf}(A) := a_{12}$ and if $n > 2$ then*

$$\mathrm{Pf}(A) := \sum_{r=2}^{n} (-1)^{r-2} a_{1r} \mathrm{Pf}_{2,\dots,\check{r},\dots,n}(A)$$

where $\check{r}$ means "remove r from the list".

This last definition is in fact a special case of a more general result. Being a related concept, we would hope to have Pfaffian versions of the Laplace expansion of determinants. The following result shows that we can expand the Pfaffian along any row:

Theorem 1.5.5. *For fixed $i, k = 1, ..., n$ we have,*

$$\delta_{ik}\mathrm{Pf}(A) = \sum_{k<j} (-1)^{k+j-1} a_{ij} \mathrm{Pf}_{1,\dots,\check{k},\dots,\check{j},\dots,n}(A) + \sum_{k>j} (-1)^{k+j} a_{ij} \mathrm{Pf}_{1,\dots,\check{k},\dots,\check{j},\dots,n}(A).$$

Unlike determinants there is not a simple identity for expanding along a fixed set of rows, however we do have the following:

Theorem 1.5.6. *Let $1 \le d \le m$.*

$$\binom{m}{d} \mathrm{Pf}(A) = \sum_{\substack{\sigma \in S_n \\ \sigma(1)<\cdots<\sigma(2d) \\ \sigma(2d+1)<\cdots<\sigma(n)}} (-q)^{l(\sigma)} \mathrm{Pf}_{\sigma(1)\dots\sigma(2d)}(A) \mathrm{Pf}_{\sigma(2d+1)\dots\sigma(n)}(A).$$

In later chapters we will establish q-analogues of these identities.

Chapter 2

Multiparameter Quantum SL_n

Let K be a field. Let λ be a nonzero element of K with $\lambda \neq -1$, and let $\mathbf{p}$ be a multiplicatively antisymmetric $n \times n$ matrix over K. In this chapter we are concerned with the multiparameter deformation of the coordinate ring of $n \times n$ matrices $\mathcal{O}_{\lambda,\mathbf{p}}(M_n)$ [3]. This is the K-algebra generated by n^2 indeterminates $\{x_{ij} : i, j = 1, ..., n\}$, subject to the following relations:

$$x_{lm}x_{ij} = p_{li}p_{jm}x_{ij}x_{lm} + (\lambda - 1)p_{li}x_{im}x_{lj} \qquad \text{for } l > i, m > j \qquad (2.0.1)$$

$$x_{lm}x_{ij} = \lambda p_{li}p_{jm}x_{ij}x_{lm} \qquad \text{for } l > i, m \leq j \qquad (2.0.2)$$

$$x_{lm}x_{lj} = p_{jm}x_{lj}x_{lm} \qquad \text{for } m > j \qquad (2.0.3)$$

$$\text{where } \; i, j, l, m = 1, ..., n.$$

The algebra $\mathcal{O}_{\lambda,\mathbf{p}}(M_n)$ is an $\mathbb{N}$-graded noetherian domain [4, Theorem I.2.7]. It is also a bialgebra with the natural comultiplication and counit:

$$\Delta(x_{ik}) = \sum_{j=1}^{n} x_{ij} \otimes x_{jk}$$

$$\epsilon(x_{ik}) = \delta_{ik}.$$

The *Quantum Determinant* is [4, Definitions I.2.3] the element

$$D_{\lambda,\mathbf{p}} = \sum_{\pi \in S_n} \left(\prod_{\substack{1 \leq i < j \leq n \\ \pi(i) > \pi(j)}} (-p_{\pi(i),\pi(j)}) \right) x_{1,\pi(1)}x_{2,\pi(2)} \cdots x_{n,\pi(n)}.$$

It is known (e.g. [15, Section 1.3]) that $D_{\lambda,\mathbf{p}}$ is a normal element of $\mathcal{O}_{\lambda,\mathbf{p}}(M_n)$, satisfying the following relations, for all i, j,

$$D_{\lambda,\mathbf{p}}x_{ij} = \lambda^{j-i} \left(\prod_{l=1}^{n} p_{jl}p_{li} \right) x_{ij}D_{\lambda,\mathbf{p}}.$$

In the literature $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$ is only defined when $D_{\lambda,\mathbf{p}}$ is central. When it is merely normal, factoring out $\langle D_{\lambda,\mathbf{p}} - 1 \rangle$ from $\mathcal{O}_{\lambda,\mathbf{p}}(M_n)$ leaves us with the coordinate ring of a torus. Such a degenerate factor does not provide us with a useful analogue of $\mathcal{O}(SL_n)$. A quantum analogue of a classical object should retain certain features from the classical case. In particular, we expect the quantum analogue to be "of the same size" as the classical object, or more specifically we want it to have the same *Gelfand-Kirillov Dimension* (for details of this concept the reader is referred to [26]). The coordinate ring of a torus certainly does not have the same Gelfand-Kirillov Dimension as $\mathcal{O}(SL_n)$. So, motivated by work in [10, Section 5] in which the corepresentation theory of $\mathcal{O}_q(GL_n)$ and $\mathcal{O}_q(SL_n)$ are linked via a certain Hopf algebra embedding, we look for another candidate for *Multiparameter Quantum* SL_n. This is precisely the objective of this chapter.

Remark 2.0.7. *We note that* Dipper Donkin Quantum SL_n *will be a particular instance of this work since* $\mathcal{O}_{DD,q}(M_n) = \mathcal{O}_{q,(1)}(M_n)$ *where* (1) *is the* $n \times n$ *matrix with all entries* 1. *This answers the problem of a lack of suitable* $\mathcal{O}_{DD,q}(SL_n)$ *that was observed in [9].*

2.1 Noncommutative Dehomogenisation

We recall the work done in [23, Section 3]:

Given a commutative $\mathbb{N}$-graded algebra R, and a homogeneous degree one nonzero-divisor $x \in R$, one usually defines the *dehomogenisation* of R at x to be the factor algebra $\frac{R}{\langle x-1 \rangle}$ [5, Appendix 16.D]. This definition is unsuitable in a noncommutative algebra if x is merely normal rather than central (the factor algebra often being too small to be useful). However, in the commutative case, an alternative approach is to observe that the localised algebra $S := R[x^{-1}]$ is $\mathbb{Z}$-graded, $S = \oplus_{i \in \mathbb{Z}} S_i$, and that $S_0 \cong \frac{R}{\langle x-1 \rangle}$.

The authors of [23] then make the following definition ([23, Definition 3.1]):

Definition 2.1.1. *Let* $R = \oplus R_i$ *be an* $\mathbb{N}$*-graded* k*-algebra and let* x *be a regular homogeneous normal element of* R *of degree one. Then the* dehomogenisation *of* R *at* x*, written* $\mathrm{Dhom}(R, x)$*, is defined to be the zero degree subalgebra* S_0 *of the* $\mathbb{Z}$*-graded algebra* $S := R[x^{-1}]$.

Remark 2.1.2. *The central idea of this chapter is to use this work to define our* $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$*. However, we observe that in our case the regular homogeneous normal element under consideration,* $D_{\lambda,\mathbf{p}}$*, is of degree* n *and not of degree one.*

Before ending this section we note a result from [23] that we shall need later,

Lemma 2.1.3. *Let R and x be as above. Then R is a domain if and only if* $\mathrm{Dhom}(R, x)$ *is a domain. Moreover, if R is noetherian then* $\mathrm{Dhom}(R, x)$ *is noetherian.*

2.2 Skew-Laurent Extensions and Hopf Algebras

The following results concerning the creation of new Hopf algebras and bialgebras from existing ones will be necessary in the next section:

Lemma 2.2.1. *Let H be a Hopf algebra and let σ be a Hopf algebra automorphism of H. Then the* skew-Laurent extension *$H[t, t^{-1}; \sigma]$ is also a Hopf algebra.*

Proof. We extend the algebra morphisms Δ and ϵ, and the algebra antimorphism S on H, (the comultiplication, counit and antipode maps of H, respectively), to maps on $H[t, t^{-1}; \sigma]$ in the obvious way; that is, for $\sum_i h^{(i)} t^i \in H[t, t^{-1}; \sigma]$,

$$\Delta\left(\sum_i h^{(i)} t^i\right) := \sum_i \left[\sum_{h^{(i)}} h_1^{(i)} t^i \otimes h_2^{(i)} t^i\right],$$

$$\epsilon\left(\sum_i h^{(i)} t^i\right) := \sum_i \epsilon(h^{(i)}),$$

$$S\left(\sum_i h^{(i)} t^i\right) := \sum_i t^{-i} S(h^{(i)}).$$

It is now a matter of checking whether these definitions give $H[t, t^{-1}; \sigma]$ a Hopf algebra structure. The fact that $H[t, t^{-1}; \sigma]$ is a coalgebra follows immediately from the coalgebra properties of H. To show that $H[t, t^{-1}; \sigma]$ is also a bialgebra is not as straightforward. We must show that our extended Δ and ϵ are algebra morphisms of $H[t, t^{-1}; \sigma]$.

Let

$$u = \sum_i h^{(i)} t^i, \ v = \sum_j g^{(j)} t^j \in H[t, t^{-1}; \sigma].$$

We show that our extended Δ is an algebra morphism. Now,

$$\begin{aligned}\Delta(uv) &= \Delta\left(\left(\sum_i h^{(i)} t^i\right)\left(\sum_j g^{(j)} t^j\right)\right)\\ &= \Delta\left(\sum_{i,j} h^{(i)} t^i g^{(j)} t^j\right)\\ &= \Delta\left(\sum_{i,j} h^{(i)} \sigma^i(g^{(j)}) t^{i+j}\right),\end{aligned}$$

by definition of $H[t,t^{-1};\sigma]$. Applying our extended definition of Δ to the RHS gives,

$$\begin{aligned}\Delta(uv) &= \sum_{i,j}\left[\sum_{h^{(i)}\sigma^i(g^{(j)})}(h^{(i)}\sigma^i(g^{(j)}))_1 t^{i+j}\otimes(h^{(i)}\sigma^i(g^{(j)}))_2 t^{i+j}\right]\\ &= \sum_{i,j}\sum_{h^{(i)}\sigma^i(g^{(j)})}[(h^{(i)}\sigma^i(g^{(j)}))_1\otimes(h^{(i)}\sigma^i(g^{(j)}))_2](t^{i+j}\otimes t^{i+j})\\ &= \sum_{i,j}[\Delta(h^{(i)}\sigma^i(g^{(j)}))(t^{i+j}\otimes t^{i+j})],\end{aligned}$$

by definition of Δ on H. Since H is a Hopf algebra, Δ is an algebra morphism on H, hence,

$$\Delta(uv) = \sum_{i,j}[\Delta(h^{(i)})\Delta(\sigma^i(g^{(j)}))(t^{i+j}\otimes t^{i+j})].$$

Now σ is, in particular, a coalgebra morphism on H, so it follows that,

$$\begin{aligned}\Delta(uv) &= \sum_{i,j}[\Delta(h^{(i)})(\sigma^i\otimes\sigma^i)(\Delta(g^{(j)}))(t^{i+j}\otimes t^{i+j})]\\ &= \sum_{i,j}[\Delta(h^{(i)})\left(\sum_{g^{(j)}}\sigma^i(g_1^{(j)})\otimes\sigma^i(g_2^{(j)})\right)(t^{i+j}\otimes t^{i+j})]\\ &= \sum_{i,j}[\Delta(h^{(i)})\left(\sum_{g^{(j)}}\sigma^i(g_1^{(j)})t^i\otimes\sigma^i(g_2^{(j)})t^i\right)(t^j\otimes t^j)]\\ &= \sum_{i,j}[\Delta(h^{(i)})\left(\sum_{g^{(j)}}t^i g_1^{(j)}\otimes t^i g_2^{(j)}\right)(t^j\otimes t^j)]\\ &= \sum_{i,j}[\Delta(h^{(i)})(t^i\otimes t^i)\left(\sum_{g^{(j)}}g_1^{(j)}\otimes g_2^{(j)}\right)(t^j\otimes t^j)]\\ &= \left(\sum_i\left[\sum_{h^{(i)}}h_1^{(i)}t^i\otimes h_2^{(i)}t^i\right]\right)\left(\sum_j\left[\sum_{g^{(j)}}g_1^{(j)}t^j\otimes g_2^{(j)}t^j\right]\right)\\ &= \Delta\left(\sum_i h^{(i)}t^i\right)\Delta\left(\sum_j g^{(j)}t^j\right)\\ &= \Delta(u)\Delta(v),\end{aligned}$$

where the above deductions have used the definitions of Δ, $H[t,t^{-1};\sigma]$, and simple rearranging. So we have shown that Δ is an algebra morphism. The proof that ϵ is an algebra morphism is similar.

Finally we must show that our extended S is an antipode for $H[t,t^{-1};\sigma]$. It suffices to show that $S * \mathrm{id} = \mathrm{id} * S = \eta\epsilon$. We show that $\mathrm{id} * S = \eta\epsilon$, the other case being similar. Now by definition of the convolution product,

$$(\mathrm{id} * S)\left(\sum_i h^{(i)}t^i\right) = \sum_i \left[\sum_{h^{(i)}} (h_1^{(i)}t^i)S(h_2^{(i)}t^i)\right].$$

Applying our extended definition of S we deduce that,

$$\begin{aligned}(\mathrm{id} * S)\left(\sum_i h^{(i)}t^i\right) &= \sum_i \left[\sum_{h^{(i)}} (h_1^{(i)}t^i)(t^{-i}S(h_2^{(i)}))\right] \\ &= \sum_i \left[\sum_{h^{(i)}} (h_1^{(i)}S(h_2^{(i)}))\right] \\ &= \sum_i [\epsilon(h^{(i)})],\end{aligned}$$

where the last equality holds since S is an antipode for H. Finally by our extended definition of ϵ we have,

$$(\mathrm{id} * S)\left(\sum_i h^{(i)}t^i\right) = \epsilon\left(\sum_i h^{(i)}t^i\right),$$

and so we are done. □

Lemma 2.2.2. *Let R be a bialgebra and let σ be a bialgebra automorphism of R. Then the* skew-Laurent extension $R[t, t^{-1}; \sigma]$ *and the* skew extension $R[t; \sigma]$ *are bialgebras.*

Proof. The proof is similar to the proof of the previous result. □

2.3 Constructing a Candidate for $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$

We noted in a previous section that $D_{\lambda,\mathbf{p}}$ was of degree n, thus preventing us from using directly the *Noncommutative Dehomogenisation* of [23] to define $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$. Instead we make the following construction. For the sake of clarity let us set $A_n := \mathcal{O}_{\lambda,\mathbf{p}}(M_n)$. At this point we must suppose there exist

$$\mu \in K \text{ and } \mathbf{q} = (q_{ij}) \in M_n(K) \text{ such that } \lambda = \mu^n \text{ and } p_{ij} = q_{ij}^n \tag{2.3.1}$$

with $\mathbf{q}$ multiplicatively antisymmetric. Define a map

$$\sigma : A_n \longrightarrow A_n \text{ by } \sigma(x_{ij}) = \mu^{j-i}\left(\prod_{k=1}^n q_{jk}q_{ki}\right)x_{ij}. \tag{2.3.2}$$

extending in the natural way.

Lemma 2.3.1. *The map σ is a bialgebra automorphism.*

Proof. To prove that σ is a well-defined algebra morphism it suffices to show that it respects the relations (2.0.1), (2.0.2), and (2.0.3). We deal with (2.0.1), the other cases being similar. Let $l > i,\ m > j$. Then,

$$\begin{aligned}\sigma(x_{lm}x_{ij}) &= \sigma(x_{lm})\sigma(x_{ij}) \\ &= \mu^{m+j-l-i}\left(\prod_{k=1}^{n} q_{mk}q_{kl}q_{jk}q_{ki}\right) x_{lm}x_{ij} \\ &= \mu^{m+j-l-i}\left(\prod_{k=1}^{n} q_{mk}q_{kl}q_{jk}q_{ki}\right)(p_{li}p_{jm}x_{ij}x_{lm} + (\lambda-1)p_{li}x_{im}x_{lj}),\end{aligned}$$

where the last equality holds by (2.0.1). Thus,

$$\begin{aligned}\sigma(x_{lm}x_{ij}) = \mu^{m+j-l-i}(\prod_{k=1}^{n} q_{mk}q_{kl}q_{jk}q_{ki})p_{li}p_{jm}x_{ij}x_{lm} \\ + (\lambda-1)p_{li}\mu^{m+j-l-i}(\prod_{k=1}^{n} q_{mk}q_{ki}q_{jk}q_{kl})x_{im}x_{lj}.\end{aligned}$$

By definition of σ it follows that,

$$\sigma(x_{lm}x_{ij}) = p_{li}p_{jm}\sigma(x_{ij})\sigma(x_{lm}) - (\lambda-1)p_{li}\sigma(x_{im})\sigma(x_{lj}),$$

and since σ is, by construction K-linear, we have,

$$\sigma(x_{lm}x_{ij}) = \sigma(p_{li}p_{jm}x_{ij}x_{lm} + (\lambda-1)p_{li}x_{im}x_{lj}).$$

Hence σ is indeed an algebra morphism. Our next task is to show that σ is a coalgebra morphism. Now,

$$\begin{aligned}\epsilon(\sigma(x_{ij})) &= \epsilon(\mu^{j-i}\left(\prod_{k=1}^{n} q_{jk}q_{ki}\right) x_{ij}) \\ &= \mu^{j-i}\left(\prod_{k=1}^{n} q_{jk}q_{ki}\right)\epsilon(x_{ij}) \\ &= \mu^{j-i}\left(\prod_{k=1}^{n} q_{jk}q_{ki}\right)\delta_{ij}.\end{aligned}$$

Since δ_{ij} is nonzero only when $i = j$, we may deduce that

$$\epsilon(\sigma(x_{ij})) = \mu^{i-i}\left(\prod_{k=1}^{n} q_{ik}q_{ki}\right)\delta_{ij}.$$

But $q_{ik}q_{ki} = 1$ by definition of $\mathbf{q}$, so,

$$\begin{aligned}\epsilon(\sigma(x_{ij})) &= \delta_{ij} \\ &= \epsilon(x_{ij}).\end{aligned}$$

Also,

$$\begin{aligned}(\sigma \otimes \sigma)(\Delta(x_{ij})) &= \sum_{r=1}^{n} \sigma(x_{ir}) \otimes \sigma(x_{rj}) \\ &= \sum_{r=1}^{n} (\mu^{r-i} \left(\prod_{k=1}^{n} q_{rk}q_{ki} \right) x_{ir}) \otimes (\mu^{j-r} \left(\prod_{k=1}^{n} q_{jk}q_{kr} \right) x_{rj}) \\ &= \sum_{r=1}^{n} \mu^{j-i} \left(\prod_{k=1}^{n} q_{rk}q_{kr}q_{ki}q_{jk} \right) (x_{ir} \otimes x_{rj}),\end{aligned}$$

and since $q_{rk}q_{kr} = 1$,

$$\begin{aligned}(\sigma \otimes \sigma)(\Delta(x_{ij})) &= \mu^{j-i} \left(\prod_{k=1}^{n} q_{jk}q_{ki} \right) \sum_{r=1}^{n} x_{ir} \otimes x_{rj} \\ &= \mu^{j-i} \left(\prod_{k=1}^{n} q_{jk}q_{ki} \right) \Delta(x_{ij}) \\ &= \Delta(\mu^{j-i} \left(\prod_{k=1}^{n} q_{jk}q_{ki} \right) x_{ij}) \\ &= \Delta(\sigma(x_{ij})).\end{aligned}$$

So σ is also a coalgebra morphism. Hence σ is a bialgebra morphism.
Finally we note that σ is clearly an automorphism of A_n. □

So by Lemma 2.2.2 we may form the bialgebra $A_n[u; \sigma]$. We note that the automorphism σ has been constructed so that u^n commutes like $D_{\lambda,\mathbf{p}}$. Now A_n is $\mathbb{N}$-graded so clearly if we let u have degree one then $A_n[u; \sigma]$ is also $\mathbb{N}$-graded.

Let $B := A_n[u; \sigma]/\langle u^n - D_{\lambda,\mathbf{p}} \rangle$. We are, in essence, adjoining an n^{th} root of $D_{\lambda,\mathbf{p}}$ to A_n. Now, by definition, u is a grouplike element of $A_n[u; \sigma]$. It is also the case that $D_{\lambda,\mathbf{p}}$ is grouplike (this follows from the argument on page 890 of [3] concerning the invariance of comultiplication under so-called *twists* and the well known result that the standard quantum determinant is grouplike [32, (1.11)]). So it follows that $\langle u^n - D_{\lambda,\mathbf{p}} \rangle$ is a homogeneous biideal of $A_n[u; \sigma]$. Hence B is a $\mathbb{N}$-graded bialgebra.

Now, by definition of σ, the element u is normal in $A_n[u; \sigma]$ and hence u (or more precisely the element $u + \langle u^n - D_{\lambda,\mathbf{p}} \rangle$; we shall henceforth abuse notation and refer to this element as u) is normal in B.

We now go on to show that u is regular in B. However, before we do, we require the following result which shows us that u and $D_{\lambda,\mathbf{p}}$ commute.

Lemma 2.3.2. $\sigma(D_{\lambda,\mathbf{p}}) = D_{\lambda,\mathbf{p}}$.

Proof. Since σ is an algebra morphism it follows that,

$$\sigma(D_{\lambda,\mathbf{p}}) = \sum_{\pi \in S_n} \left(\prod_{\substack{1 \le i < j \le n \\ \pi(i) > \pi(j)}} (-p_{\pi(i),\pi(j)}) \right) \left(\prod_{r=1}^{n} \sigma(x_{r,\pi(r)}) \right),$$

by the definition of σ we have,

$$\sigma(D_{\lambda,\mathbf{p}}) = \sum_{\pi \in S_n} \left(\prod_{\substack{1 \le i < j \le n \\ \pi(i) > \pi(j)}} (-p_{\pi(i),\pi(j)}) \right) \left(\prod_{r=1}^{n} \mu^{\pi(r)-r} (\prod_{k=1}^{n} q_{\pi(r),k} q_{kr}) x_{r,\pi(r)} \right),$$

which can be written equivalently as,

$$\sigma(D_{\lambda,\mathbf{p}}) =$$

$$\sum_{\pi \in S_n} \left(\prod_{\substack{1 \le i < j \le n \\ \pi(i) > \pi(j)}} (-p_{\pi(i),\pi(j)}) \right) \mu^{\sum_{r=1}^{n}(\pi(r)-r)} \left(\prod_{k=1}^{n} (\prod_{r=1}^{n} q_{\pi(r),k} q_{kr}) \right) \left(\prod_{r=1}^{n} x_{r,\pi(r)} \right).$$

Now $\sum_{r=1}^{n} \pi(r) = \sum_{r=1}^{n} r$ since $\pi \in S_n$ is a bijection. So $\mu^{\sum_{r=1}^{n}(\pi(r)-r)} = 1$. Also, π being a bijection, together with the fact that $\mathbf{q}$ is multiplicatively antisymmetric, allows us to deduce that $\prod_{k=1}^{n}(\prod_{r=1}^{n} q_{\pi(r),k} q_{kr}) = 1$. Hence,

$$\begin{aligned} \sigma(D_{\lambda,\mathbf{p}}) &= \sum_{\pi \in S_n} \left(\prod_{\substack{1 \le i < j \le n \\ \pi(i) > \pi(j)}} (-p_{\pi(i),\pi(j)}) \right) \left(\prod_{r=1}^{n} x_{r,\pi(r)} \right) \\ &= D_{\lambda,\mathbf{p}}. \end{aligned}$$

□

Lemma 2.3.3. *The element u is regular in B.*

Proof. We first require the fact that elements of B may be expressed as polynomials in u over A_n of degree less than or equal to $n - 1$.

Let $f \in A_n[u;\sigma]$. Then $f = \sum_{i=0}^{m} f_i u^i$ for some $m \in \mathbb{N}$ and $f_i \in A_n$. Suppose $m \geq n$. Then,

$$\begin{aligned} f &= \sum_{i=0}^{m} f_i u^i \\ &= f_m u^{m-n}(u^n - D_{\lambda,\mathbf{p}}) + (f_{m-n} + f_m D_{\lambda,\mathbf{p}})u^{m-n} + \sum_{\substack{i=0 \\ i \neq m-n}}^{m-1} f_i u^i \end{aligned}$$

since u and $D_{\lambda,\mathbf{p}}$ commute by Lemma 2.3.2. So modulo $\langle u^n - D_{\lambda,\mathbf{p}} \rangle$ the element f is equivalent to a polynomial of degree $m-1$. Hence, by induction, we are done.

Now suppose there exists $g \in B$ such that $ug = 0$ in B. By the above we may write $g = \sum_{i=0}^{n-1} g_i u^i$, say. So we have, by the definition of B,

$$\begin{aligned} ug &\in \langle u^n - D_{\lambda,\mathbf{p}} \rangle \\ u\left(\sum_{i=0}^{n-1} g_i u^i\right) &\in \langle u^n - D_{\lambda,\mathbf{p}} \rangle \\ \sum_{i=0}^{n-1} \sigma(g_i) u^{i+1} &\in \langle u^n - D_{\lambda,\mathbf{p}} \rangle. \end{aligned}$$

By consideration of degree in u it follows that,

$$\sum_{i=0}^{n-1} \sigma(g_i) u^{i+1} = b(u^n - D_{\lambda,\mathbf{p}}) \quad \text{for some } b \in A_n.$$

Comparing coefficients of u^0 yields that $bD_{\lambda,\mathbf{p}} = 0$, and so, since A_n is a domain, we have that $b = 0$. It follows that $g = 0$. Hence u is not a left zero divisor. Similarly u is not a right zero divisor, and so we are done. □

We have just shown that the normal element u is regular in B. We also have, by definition, that u is of degree one. So u is such that we may use the method of *Noncommutative Dehomegenisation* to consider the algebra $C :=\mathrm{Dhom}(B, u)$. It is this algebra, C, which we propose as a candidate for $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$.

Definition 2.3.4. *Given that K is such that (2.3.1) holds,*

$$\mathcal{O}_{\lambda,\mathbf{p}}(SL_n) := \mathrm{Dhom}\left(\frac{\mathcal{O}_{\lambda,\mathbf{p}}(M_n)[u;\sigma]}{\langle u^n - D_{\lambda,\mathbf{p}} \rangle}, u\right).$$

The rest of the chapter is concerned with proving results concerning the properties of C.

2.4 Properties of our $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$

We note that, as an algebra, C is generated by $\{x_{ij}u^{-1} : i,j = 1,...,n\}$. This can easily be seen by examining the definition of B and $\mathrm{Dhom}(B,u)$. Let $c_{ij} = x_{ij}u^{-1}$. Then, using relations (2.0.1), (2.0.2), and (2.0.3), together with (2.3.2), it may be calculated that, for $i,j,l,m = 1,...,n$,

$$c_{lm}c_{ij} = \mu^{i+m-j-l}\left(\prod_{k=1}^{n} q_{kj}q_{ik}q_{mk}q_{kl}\right)p_{li}p_{jm}c_{ij}c_{lm} + (\lambda-1)p_{li}\mu^{i-l}\left(\prod_{k=1}^{n} q_{ik}q_{kl}\right)c_{im}c_{lj},$$

$$\text{for } l > i,\ m > j, \tag{2.4.1}$$

$$c_{lm}c_{ij} = \lambda p_{li}p_{jm}\mu^{i+m-j-l}\left(\prod_{k=1}^{n} q_{kj}q_{ik}q_{mk}q_{kl}\right)c_{ij}c_{lm},$$

$$\text{for } l > i,\ m \leq j, \tag{2.4.2}$$

$$c_{lm}c_{lj} = p_{jm}\mu^{m-j}\left(\prod_{k=1}^{n} q_{kj}q_{mk}\right)c_{ij}c_{lm},$$

$$\text{for } m > j. \tag{2.4.3}$$

Now by Lemma 2.2.2 since A_n is a bialgebra and $\langle u^n - D_{\lambda,\mathbf{p}}\rangle$ is a biideal it follows that $B[u^{-1}] = A_n[u,u^{-1};\sigma]/\langle u^n - D_{\lambda,\mathbf{p}}\rangle$ is a bialgebra (we note that $B[u^{-1}] = A_n[u,u^{-1};\sigma]/\langle u^n - D_{\lambda,\mathbf{p}}\rangle$ by [17, Exercises 9I and 9L]). Now, as in Lemma 2.2.1, we may extend Δ and ϵ as defined on A_n to $A_n[u,u^{-1};\sigma]$ and hence to $B[u^{-1}]$. So, in particular, we have,

$$\begin{aligned}\Delta(c_{ij}) &= \Delta(x_{ij}u^{-1}) \\ &= \sum_{k=1}^{n} x_{ik}u^{-1} \otimes x_{kj}u^{-1} \\ &= \sum_{k=1}^{n} c_{ik} \otimes c_{kj},\end{aligned}$$

and,

$$\begin{aligned}\epsilon(c_{ij}) &= \epsilon(x_{ij}u^{-1}) \\ &= \epsilon(x_{ij}) \\ &= \delta_{ij}.\end{aligned}$$

Hence C is a subbialgebra of $B[u^{-1}]$ (it is sufficient to check closure on the generators since Δ and ϵ are algebra morphisms on $B[u^{-1}]$).

Proposition 2.4.1. *C is a Hopf algebra.*

Proof. We have just shown above that C is a subbialgebra of $B[u^{-1}]$. We noted that

$$B[u^{-1}] = A_n[u, u^{-1}; \sigma]/\langle u^n - D_{\lambda,\mathbf{p}}\rangle,$$

and so the bialgebra structure of $B[u^{-1}]$ comes from the bialgebra structure of A_n. Likewise, we will show that C is a sub-Hopf-algebra of $B[u^{-1}]$, the Hopf algebra structure of which comes from the Hopf algebra structure of $\mathcal{O}_{\lambda,\mathbf{p}}(GL_n) = A_n[D_{\lambda,\mathbf{p}}^{-1}]$. This can be seen more clearly upon inspection of the following commutative diagram

$$\begin{array}{ccccccc}
A_n & \hookrightarrow & A_n[u;\sigma] & \twoheadrightarrow & \frac{A_n[u;\sigma]}{\langle u^n - D_{\lambda,\mathbf{p}}\rangle} & = & B \\
\| & & \downarrow & & \downarrow & & \\
A_n & \hookrightarrow & A_n[u,u^{-1};\sigma] & \twoheadrightarrow & \frac{A_n[u,u^{-1};\sigma]}{\langle u^n - D_{\lambda,\mathbf{p}}\rangle} & = & B[u^{-1}] \\
\downarrow & & \downarrow & & \downarrow \varphi & & \\
A_n[D_{\lambda,\mathbf{p}}^{-1}] & \hookrightarrow & \left(A_n[D_{\lambda,\mathbf{p}}^{-1}]\right)[u,u^{-1};\sigma] & \twoheadrightarrow & \frac{\left(A_n[D_{\lambda,\mathbf{p}}^{-1}]\right)[u,u^{-1};\sigma]}{\langle u^n - D_{\lambda,\mathbf{p}}\rangle} & &
\end{array}$$

A little thought yields that φ is in fact an isomorphism (with u^{-n} being sent to $D_{\lambda,\mathbf{p}}^{-1}$ under φ). So we have that,

$$B[u^{-1}] \cong \frac{\left(A_n[D_{\lambda,\mathbf{p}}^{-1}]\right)[u,u^{-1};\sigma]}{\langle u^n - D_{\lambda,\mathbf{p}}\rangle}.$$

Now in [3, Theorem 3] the authors prove that $A_n[D_{\lambda,\mathbf{p}}^{-1}]$ is a Hopf algebra with antipode $\widehat{S} : A_n[D_{\lambda,\mathbf{p}}^{-1}] \longrightarrow A_n[D_{\lambda,\mathbf{p}}^{-1}]$ defined by

$$\begin{aligned}
\widehat{S}(x_{ij}) &= (\prod_{m=1}^{i-1}(-p_{im}))(\prod_{s=1}^{j-1}(-p_{sj}))[\,\tilde{j} \mid \tilde{i}\,]D_{\lambda,\mathbf{p}}^{-1}, \\
\widehat{S}(D_{\lambda,\mathbf{p}}^{-1}) &= D_{\lambda,\mathbf{p}}
\end{aligned}$$

We are using here the terminology of *quantum minors*, $[\,\tilde{j} \mid \tilde{i}\,]$, as in [16]. That is, $\tilde{i} = \{1, ..., n\} \setminus \{i\}$ and $[I \mid J]$ denotes, in our case, the *multiparameter quantum determinant* of the matrix subalgebra generated by the elements x_{rs} with $r \in I$ and $s \in J$, where I and J are index sets of the same cardinality. More extensive definitions of these terms are given in [3, Theorem 3] but with $[\,\tilde{j} \mid \tilde{i}\,]$ denoted by $U_{\tilde{j}\tilde{i}}$. Since $A_n[D_{\lambda,\mathbf{p}}^{-1}]$ is a Hopf algebra it follows by Lemma 2.2.1 that $\left(A_n[D_{\lambda,\mathbf{p}}^{-1}]\right)[u,u^{-1};\sigma]$ is a Hopf algebra. It is not hard to see that $\langle u^n - D_{\lambda,\mathbf{p}}\rangle$ is a Hopf ideal of $\left(A_n[D_{\lambda,\mathbf{p}}^{-1}]\right)[u,u^{-1};\sigma]$, and so it follows that $\left(A_n[D_{\lambda,\mathbf{p}}^{-1}]\right)[u,u^{-1};\sigma]/\langle u^n - D_{\lambda,\mathbf{p}}\rangle$, and hence $B[u^{-1}]$, is a Hopf algebra. Keeping in mind that "u behaves like a n^{th}-root of $D_{\lambda,\mathbf{p}}$", it is easy to see that the antipode for $B[u^{-1}]$, coming from $\widehat{S}$, is $S : B[u^{-1}] \longrightarrow B[u^{-1}]$ defined by

$$\begin{aligned} S(x_{ij}) &= (\prod_{m=1}^{i-1}(-p_{im}))(\prod_{s=1}^{j-1}(-p_{sj}))[\,\tilde{j} \mid \tilde{i}\,]u^{-n}, \\ S(u) &= u^{-1}, \\ S(u^{-1}) &= u, \end{aligned}$$

<u>Claim:</u> S induces an antipode on C.
<u>Proof:</u> The bialgebra C is a subbialgebra of $B[u^{-1}]$ so it suffices to show that C is closed under S. Since S is an algebra antiendomorphism it suffices to check this on the generators of C. Now,

$$\begin{aligned} S(c_{ij}) &= S(x_{ij}u^{-1}) \\ &= S(u^{-1})S(x_{ij}) \\ &= uS(x_{ij}). \end{aligned}$$

By definition of S and $[\,\tilde{j} \mid \tilde{i}\,]$, we may write,

$$S(x_{ij}) = \eta_{ij} \sum_{\pi} \tau_\pi x_{1,\pi(1)} \dots x_{i-1,\pi(i-1)} x_{i+1,\pi(i+1)} \dots x_{n,\pi(n)} u^{-n},$$

for some $\eta_{ij}, \tau_\pi \in K$, where the above sum runs over all bijections

$$\pi : \{1, ..., i-1, i+1, ..., n\} \longrightarrow \{1, ..., j-1, j+1, ..., n\}.$$

Hence,

$$S(c_{ij}) = u\eta_{ij} \sum_{\pi} \tau_\pi x_{1,\pi(1)} \dots x_{i-1,\pi(i-1)} x_{i+1,\pi(i+1)} \dots x_{n,\pi(n)} u^{-n}.$$

We note that in each term of the above sum we have a product of exactly $n-1$ x_{lm}'s. It is clear from the definition of $A_n[u, u^{-1}; \sigma]$ that we may "move along powers of u past the x_{lm}" to obtain,

$$S(c_{ij}) = \eta_{ij} \sum_{\pi} \tau_\pi \prod_{\substack{r=1 \\ r \neq i}}^{n} (\kappa_r^\pi x_{r,\pi(r)} u^{-1}),$$

for some $\kappa_r^\pi \in K$. But this is just

$$S(c_{ij}) = \eta_{ij} \sum_{\pi} \tau_\pi \prod_{r \neq i} (\kappa_r^\pi c_{r,\pi(r)}).$$

Hence C is a Hopf algebra. □

Proposition 2.4.2. *C is Noetherian.*

Proof. Since $\mathcal{O}_{\lambda,\mathbf{p}}(M_n)$ is Noetherian, so is $\mathcal{O}_{\lambda,\mathbf{p}}(M_n)[u;\sigma]/\langle u^n - D_{\lambda,\mathbf{p}}\rangle$. Hence C is Noetherian by [23, Corollary 3.3]. □

Lemma 2.4.3. *$A_n/A_nD_{\lambda,\mathbf{p}}$ is a domain.*

Proof. This is [20, Example 3]. □

Proposition 2.4.4. *C is a domain.*

Proof. Now by Lemma 2.1.3 it suffices to prove that B is a domain. Now, as seen earlier, any element of $B = A_n[u;\sigma]/\langle u^n - D_{\lambda,\mathbf{p}}\rangle$ can be thought of as a polynomial in u over A_n of degree less than or equal to $n-1$. Let us write $\delta = D_{\lambda,\mathbf{p}}$ for convenience. Let $0 \neq f, g \in A_n[u;\sigma]/\langle u^n - \delta\rangle$. Say,

$$f = \sum_{i=0}^{n-1} f_i u^i \ , \ g = \sum_{j=0}^{n-1} g_j u^j$$

where $f_i,\ g_j \in A_n$ and are not all zero. Suppose

$$fg = 0 \ \text{ in } \ A_n[u;\sigma]/\langle u^n - \delta\rangle.$$

We will show that this leads to a contradiction. Now we have,

$$\left(\sum_{i=0}^{n-1} f_i u^i\right)\left(\sum_{j=0}^{n-1} g_j u^j\right) \in \langle u^n - \delta\rangle$$

that is,

$$\sum_{i,j=0}^{n-1} f_i u^i g_j u^j = \sum_{i,j=0}^{n-1} f_i \sigma^i(g_j) u^{i+j} \in \langle u^n - \delta\rangle.$$

By consideration of degree in u, we have,

$$\sum_{i,j=0}^{n-1} f_i \sigma^i(g_j) u^{i+j} = \sum_{s=0}^{n-2} r_s u^s (u^n - \delta),$$

for some $r_s \in A_n$. Since u and δ commute (by Lemma 2.3.2),

$$\sum_{i,j=0}^{n-1} f_i \sigma^i(g_j) u^{i+j} = \sum_{s=0}^{n-2} (r_s u^{n+s} - r_s \delta u^s),$$

which can be rewritten as,

$$\sum_{m=0}^{2n-2} \left[\sum_{i+j=m} f_i \sigma^i(g_j)\right] u^m = \sum_{m=n}^{2n-2} r_{m-n} u^m + \sum_{m=0}^{n-2} (-r_m \delta) u^m.$$

Comparing coefficients we have,

$$(A) \qquad \sum_{i+j=n-1} f_i\sigma^i(g_j) = 0$$

$$(B) \qquad \sum_{i+j=s} f_i\sigma^i(g_j) = -\sum_{i+j=s+n} f_i\sigma^i(g_j)\delta, \qquad s = 0, ..., n-2.$$

We may as well assume there exist i, j such that $\delta \nmid f_i$ and $\delta \nmid g_j$ (in A_n) since otherwise, using the fact that A_n is a domain, we could replace the problem with

$$\sum_{i+j=n-1} f'_i\sigma^i(g'_j) = 0$$

$$\sum_{i+j=s} f'_i\sigma^i(g'_j) = -\sum_{i+j=s+n} f'_i\sigma^i(g'_j)\delta \qquad s = 0, ..., n-2$$

where $\deg(f'_i) < \deg(f_i)$ or $\deg(g'_j) < \deg(g_j)$. Iterating this process we would eventually come to a stage where we have an $f_i^{(t)}$ and a $g_j^{(t)}$ not divisible by δ. Let k be minimal such that $\delta \nmid f_k$ and let l be minimal such that $\delta \nmid g_l$. Suppose $\delta \mid f_k\sigma^k(g_l)$. Then since $A_n/A_n\delta$ is a domain by Lemma 2.4.3, $\delta \mid f_k$ or $\delta \mid \sigma^k(g_l)$. So by choice of k we must have $\delta \mid \sigma^k(g_l)$. Now by Lemma 2.3.2 we have that $\delta = \sigma^k(\delta) \mid \sigma^k(g_l)$; that is, $\sigma^k(g_l) = y\sigma^k(\delta)$, for some $y \in A_n$. Since σ is an automorphism it follows that $\delta \mid g_l$, which contradicts our choice of l. Hence

$$\delta \nmid f_k\sigma^k(g_l). \tag{2.4.4}$$

This will be the main tool to show that our supposition that B is not a domain leads to a contradiction.

We will require that $k + l - n \geq 0$. Let us prove this now. First, suppose $k + l \leq n - 2$. Now, for $s = k + l$, (B) says,

$$\delta \mid f_0\sigma^0(g_{k+l}) + ... + f_{k-1}\sigma^{k-1}(g_{l+1}) + f_k\sigma^k(g_l) + f_{k+1}\sigma^{k+1}(g_{l-1}) + ... + f_{k+l}\sigma^{k+l}(g_0).$$

By definition of k and l, along with the fact that σ is an automorphism with $\sigma(\delta) = \delta$, we know that $\delta \mid f_i$ for $i = 0, ..., k-1$ and $\delta \mid \sigma^t(g_j)$ for $j = 0, ..., l-1$ and for any t. So we can deduce that $\delta \mid f_k\sigma^k(g_l)$ which contradicts (2.4.4). Hence $k + l > n - 2$. Now let us suppose that $k + l = n - 1$. By (A),

$$f_k\sigma^k(g_l) = -\sum_{i+j=n-1 \ st \ (i,j)\neq(k,l)} f_i\sigma^i(g_j). \tag{2.4.5}$$

By choice of k and l,

$$\delta \mid f_i \qquad \forall\, i < k = n - 1 - l, \tag{2.4.6}$$

$$\delta \mid g_j \qquad \forall\, j < l = n - 1 - k. \tag{2.4.7}$$

Consider $f_i\sigma^i(g_j)$ such that $i+j=n-1$ with $i \neq k$. First suppose $i<k$. Then $\delta \mid f_i\sigma^i(g_j)$ by (2.4.6). Next, suppose $i>k$. Then $j=n-1-i<n-1-k=l$ and so by (2.4.7) we have $\delta \mid g_j$. Hence $\delta=\sigma^i(\delta) \mid \sigma^i(g_j)$, and so $\delta \mid f_i\sigma^i(g_j)$. Thus by (2.4.5) we have $\delta \mid f_k\sigma^k(g_l)$ which contradicts (2.4.4). So we may deduce that $k+l \geq n$.

We are now in a position to consider $f_i\sigma^i(g_j)$ such that $i+j=k+l-n \geq 0$. Suppose $i \geq k$. Then $j=k+l-n-i \leq k+l-n-k=l-n$. But $l \leq n-1$, so $j \leq -1$ which is clearly false. Hence $i<k$. Suppose $j \geq l$. Then, as above, it follows that $i \leq -1$ which again is clearly false. Hence $j<l$. So by choice of k and l, we have, $\delta \mid f_i$ and $\delta \mid g_j$. It follows that $\delta^2 \mid f_i\sigma^i(g_j)$. For $s=k+l-n$ (B) says that,

$$\sum_{i+j=k+l-n} f_i\sigma^i(g_j) = -\sum_{i+j=k+l} f_i\sigma^i(g_j)\delta.$$

Now, we have just shown that $\delta^2 \mid f_i\sigma^i(g_j)\ \forall i+j=k+l-n$, so we can deduce that

$$\delta^2 \mid \sum_{i+j=k+l} f_i\sigma^i(g_j)\delta.$$

Since A_n is a domain we have that

$$\delta \mid f_0\sigma^0(g_{k+l})+...+f_{k-1}\sigma^{k-1}(g_{l+1})+f_k\sigma^k(g_l)+f_{k+1}\sigma^{k+1}(g_{l-1})+...+f_{k+l}\sigma^{k+l}(g_0).$$

Since we know that $\delta \mid f_i \forall i<k$ and $\delta \mid g_j \forall j<l$ it follows that $\delta \mid f_k\sigma^k(g_l)$ which is a contradiction by (2.4.4). Hence $f=0$ or $g=0$, and so B is a domain. □

2.5 A Link with $\mathcal{O}_{\lambda,\mathbf{p}}(GL_n)$

Now in [10, Section 5] a link between the corepresentation theory of $\mathcal{O}_q(SL(n))$ and $\mathcal{O}_q(GL(n))$ is given by the authors showing that one can find a Hopf algebra embedding,

$$\mathcal{O}_q(GL(n)) \hookrightarrow \mathcal{O}_q(SL(n))[z,z^{-1}].$$

It is a further argument in favour of our candidate for $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$ that we can find a similar relationship between it and $\mathcal{O}_{\lambda,\mathbf{p}}(GL_n)$ (which, as alluded to earlier, is a Hopf algebra by [3, Theorem 3]). Reflecting the fact that in our case we have *normal* rather than *central* $D_{\lambda,\mathbf{p}}$ the embedding is not into Laurent polynomials over $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$, rather it is into *Skew*-Laurent polynomials over $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$.

Now by [23, Lemma 3.2],

$$C[z, z^{-1}; \widehat{\sigma}] \cong B[u^{-1}],$$

where $\widehat{\sigma}$ is the automorphism on C induced by the automorphism of $B[u^{-1}]$ given by $b \mapsto ubu^{-1}$ for $b \in B$. One can see that,

$$\widehat{\sigma} : C \longrightarrow C \text{ is given by } \quad c_{ij} \mapsto \mu^{j-i}(\prod_{k=1}^{n} q_{jk}q_{ki})c_{ij}. \tag{2.5.1}$$

That is,

$$\widehat{\sigma}(c_{ij}) = \sigma(x_{ij})u^{-1}.$$

Lemma 2.5.1. *$\widehat{\sigma} : C \longrightarrow C$ is a Hopf algebra morphism.*

Proof. By [8, Proposition 4.2.5] it suffices to show that $\widehat{\sigma}$ is a bialgebra morphism. Now we know that $\sigma : A_n \longrightarrow A_n$ is a bialgebra morphism. One may check that this can be extended to a bialgebra morphism $\sigma' : A_n[u, u^{-1}; \sigma] \longrightarrow A_n[u, u^{-1}; \sigma]$ by setting $\sigma'(xu^s) := \sigma(x)u^s$ for $x \in A_n$. Now by Lemma 2.3.2 $\sigma(D_{\lambda,\mathbf{p}}) = D_{\lambda,\mathbf{p}}$, so

$$\sigma'(u^n - D_{\lambda,\mathbf{p}}) = u^n - \sigma(D_{\lambda,\mathbf{p}}) = u^n - D_{\lambda,\mathbf{p}}.$$

Hence σ' factors to give a bialgebra morphism σ'' on

$$B[u^{-1}] = A_n[u, u^{-1}; \sigma]/\langle u^n - D_{\lambda,\mathbf{p}}\rangle.$$

On C, we observe that $\sigma'' = \widehat{\sigma}$, and so we are done. □

We are now in a position to form the Hopf algebra $C[z, z^{-1}; \widehat{\sigma}]$ by Lemma 2.2.1.

Henceforth we shall abuse notation and write σ for $\widehat{\sigma}$. Before establishing the Hopf algebra embedding we require the following lemma,

Lemma 2.5.2.

$$\sum_{\pi \in S_n} \left(\prod_{\substack{1 \leq i < j \leq n \\ \pi(i) > \pi(j)}} (-p_{\pi(i),\pi(j)}) \right) \prod_{r=1}^{n} \sigma^{r-1}(c_{r,\pi(r)}) = 1.$$

Proof. Now, by definition of the c_{ij},

$$\sum_{\pi \in S_n} \left(\prod_{\substack{1 \leq i < j \leq n \\ \pi(i) > \pi(j)}} (-p_{\pi(i),\pi(j)}) \right) \prod_{r=1}^{n} \sigma^{r-1}(c_{r,\pi(r)})$$

$$= \sum_{\pi \in S_n} \left(\prod_{\substack{1 \leq i < j \leq n \\ \pi(i) > \pi(j)}} (-p_{\pi(i),\pi(j)}) \right) \prod_{r=1}^{n} \sigma^{r-1}(x_{r,\pi(r)}u^{-1}),$$

and by definition of σ this gives,

$$\sum_{\pi\in S_n}\left(\prod_{\substack{1\le i<j\le n\\ \pi(i)>\pi(j)}}(-p_{\pi(i),\pi(j)})\right)\prod_{r=1}^{n}\sigma^{r-1}(c_{r,\pi(r)})$$
$$=\sum_{\pi\in S_n}\left(\prod_{\substack{1\le i<j\le n\\ \pi(i)>\pi(j)}}(-p_{\pi(i),\pi(j)})\right)\prod_{r=1}^{n}\sigma^{r-1}(x_{r,\pi(r)})u^{-1}.$$

Since, for $x\in A_n$, $xu^{-1}=u^{-1}\sigma(x)$, we can see that,

$$\prod_{r=1}^{n}\sigma^{r-1}(x_{r,\pi(r)})u^{-1}=\left(\prod_{r=1}^{n}x_{r,\pi(r)}\right)u^{-n}, \tag{2.5.2}$$

and using this to rewrite the previous equation gives,

$$\sum_{\pi\in S_n}\left(\prod_{\substack{1\le i<j\le n\\ \pi(i)>\pi(j)}}(-p_{\pi(i),\pi(j)})\right)\prod_{r=1}^{n}\sigma^{r-1}(c_{r,\pi(r)})$$
$$=\sum_{\pi\in S_n}\left(\prod_{\substack{1\le i<j\le n\\ \pi(i)>\pi(j)}}(-p_{\pi(i),\pi(j)})\right)\left(\prod_{r=1}^{n}x_{r,\pi(r)}\right)u^{-n}$$
$$=D_{\lambda,\mathbf{p}}u^{-n}$$
$$=1.$$

□

Proposition 2.5.3. *There exists a Hopf algebra embedding,*

$$\mathcal{O}_{\lambda,\mathbf{p}}(GL_n)\hookrightarrow\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)[z,z^{-1};\sigma].$$

Proof. Define an algebra morphism

$$\psi: A_n[D_{\lambda,\mathbf{p}}^{-1}]\longrightarrow C[z,z^{-1};\sigma]\ \text{ by }\ \ x_{ij}\mapsto c_{ij}z\ \ \text{ and }\ \ D_{\lambda,\mathbf{p}}^{-1}\mapsto z^{-n}.$$

We first show that this is indeed a well-defined algebra morphism. That is, we show ψ respects the relations of $A_n[D_{\lambda,\mathbf{p}}^{-1}]$. Firstly,

$$\psi(D_{\lambda,\mathbf{p}})=\psi\left(\sum_{\pi\in S_n}\left(\prod_{\substack{1\le i<j\le n\\ \pi(i)>\pi(j)}}(-p_{\pi(i),\pi(j)})\right)\prod_{r=1}^{n}x_{r,\pi(r)}\right)$$
$$=\sum_{\pi\in S_n}\left(\prod_{\substack{1\le i<j\le n\\ \pi(i)>\pi(j)}}(-p_{\pi(i),\pi(j)})\right)(\prod_{r=1}^{n}c_{r,\pi(r)}z).$$

Similar to (2.5.2) in the proof of the previous lemma, because of the definition of $C[z, z^{-1}; \sigma]$, we can "move the z's to the right" in the above product, giving us,

$$\begin{aligned}\psi(D_{\lambda,\mathbf{p}}) &= \sum_{\pi \in S_n} \left(\prod_{\substack{1 \leq i < j \leq n \\ \pi(i) > \pi(j)}} (-p_{\pi(i),\pi(j)}) \right) (\prod_{r=1}^{n} \sigma^{r-1}(c_{r,\pi(r)})) z^n \\ &= z^n,\end{aligned}$$

where the last equality holds by the previous lemma. We now show that ψ respects relation (2.0.1). For $l > i,\ m > j$, we have

$$\begin{aligned}\psi(x_{lm}x_{ij} - p_{li}p_{jm}x_{ij}x_{lm} - &(\lambda - 1)p_{li}x_{im}x_{lj}) \\ &= c_{lm}zc_{ij}z - p_{li}p_{jm}c_{ij}zc_{lm}z - (\lambda - 1)p_{li}c_{im}zc_{lj}z \\ &= \Big(c_{lm}\sigma(c_{ij}) - p_{li}p_{jm}c_{ij}\sigma(c_{lm}) - (\lambda - 1)p_{li}c_{im}\sigma(c_{lj})\Big)z^2 \\ &= 0\end{aligned}$$

where the last equality is a consequence of relation (2.4.1). Similarly ψ respects relations (2.0.2) and (2.0.3).

By [8, Proposition 4.2.5], to show ψ is a Hopf algebra morphism it suffices to show that it is a bialgebra morphism. So it remains to show that it is a coalgebra morphism. The task of checking that $(\psi \otimes \psi) \circ \Delta = \Delta \circ \psi$ and $\epsilon \circ \psi = \epsilon$ is a straightforward case of writing down definitions. Hence ψ is a Hopf algebra morphism.

We now show that ψ is an embedding. Suppose $\text{Ker}\psi \neq 0$. Let $g \in \text{Ker}\psi \backslash \{0\}$. By multiplying by some power of $D_{\lambda,\mathbf{p}}$ if necessary we may as well assume that $g \in A_n$. Now A_n is $\mathbb{N}$-graded by total degree and $C[z, z^{-1}; \sigma]$ is $\mathbb{Z}$-graded by degree in z. We observe that $\psi|_{A_n}$ is homogeneous (since $x_{ij} \mapsto c_{ij}z$), and so we may as well assume that g is homogeneous, say of degree s. So,

$$g = \sum_{k=1}^{N} \lambda_k \left[\prod_{r=1}^{s} x_{i_{k_r} j_{k_r}} \right] \quad \text{where } N \in \mathbb{N}, \lambda_k \in K \text{ and } i_{k_r}, j_{k_r} = 1, ..., n.$$

Then,

$$\begin{aligned}0 &= \psi(g) \\ &= \sum_{k=1}^{N} \lambda_k \left[\prod_{r=1}^{s} \psi(x_{i_{k_r} j_{k_r}}) \right] \\ &= \sum_{k=1}^{N} \lambda_k \left[\prod_{r=1}^{s} c_{i_{k_r} j_{k_r}} z \right].\end{aligned}$$

Now, again similar to (2.5.2), we "move the z's over to the right", giving,

$$0 = \sum_{k=1}^{N} \lambda_k \left[\prod_{r=1}^{s} \sigma^{r-1}(c_{i_{k_r} j_{k_r}}) \right] z^s.$$

Therefore, since $C[z, z^{-1}; \sigma]$ is a domain,

$$\begin{aligned} 0 &= \sum_{k=1}^{N} \lambda_k \left[\prod_{r=1}^{s} \sigma^{r-1}(c_{i_{k_r} j_{k_r}}) \right] \\ &= \sum_{k=1}^{N} \lambda_k \left[\prod_{r=1}^{s} \sigma^{r-1}(x_{i_{k_r} j_{k_r}}) u^{-1} \right] \\ &= \sum_{k=1}^{N} \lambda_k \left[\prod_{r=1}^{s} x_{i_{k_r} j_{k_r}} \right] u^{-s}, \end{aligned}$$

Hence,

$$\sum_{k=1}^{N} \lambda_k \left[\prod_{r=1}^{s} x_{i_{k_r} j_{k_r}} \right] = 0$$

That is, $g = 0$, which is a contradiction. Thus $\text{Ker}\psi = 0$, and so we are done. □

2.6 An Unexpected Isomorphism

We now observe that $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$ is in-fact the usual Multiparameter Quantum SL_n for a different choice of the parameters λ and $\mathbf{p}$. Given λ, $\mathbf{p}$, μ, and $\mathbf{q}$ as above, then we have,

Proposition 2.6.1. *Let* $\mathbf{r} = (p_{ij}\mu^{j-i} \prod_{k=1}^{n} q_{jk}q_{ki})$. *We note that* $D_{\lambda,\mathbf{r}}$ *is central in* $\mathcal{O}_{\lambda,\mathbf{r}}(M_n)$, *so the usual* $\mathcal{O}_{\lambda,\mathbf{r}}(SL_n)$ *is defined, and we have*

$$\mathcal{O}_{\lambda,\mathbf{r}}(SL_n) = \frac{\mathcal{O}_{\lambda,\mathbf{r}}(M_n)}{\langle D_{\lambda,\mathbf{r}} - 1 \rangle} \cong \mathcal{O}_{\lambda,\mathbf{p}}(SL_n).$$

Proof. Consideration of the defining relations for $\mathcal{O}_{\lambda,\mathbf{r}}(M_n)$ and the relations (2.4.1), (2.4.2), (2.4.3) for $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$ certainly yields that $x_{ij} \mapsto c_{ij}$ is a surjective Hopf algebra morphism $\mathcal{O}_{\lambda,\mathbf{r}}(M_n) \twoheadrightarrow \mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$.

Before we proceed we require the following result, which says that $D_{\lambda,\mathbf{r}}$ gets sent to 1 under this map,

Claim:

$$\sum_{\pi \in S_n} \left(\prod_{\substack{1 \leq i < j \leq n \\ \pi(i) > \pi(j)}} (-p_{\pi(i),\pi(j)} \mu^{\pi(j)-\pi(i)} \prod_{k=1}^{n} q_{\pi(j),k} q_{k,\pi(i)}) \right) \prod_{r=1}^{n} c_{r,\pi(r)} = 1.$$

Proof: By Lemma 2.5.2 if we can show

$$\sum_{\pi\in S_n}\left(\prod_{\substack{1\le i<j\le n\\ \pi(i)>\pi(j)}}(-p_{\pi(i),\pi(j)}\mu^{\pi(j)-\pi(i)}\prod_{k=1}^{n}q_{\pi(j),k}q_{k,\pi(i)})\right)\prod_{r=1}^{n}c_{r,\pi(r)}$$

$$=\sum_{\pi\in S_n}\left(\prod_{\substack{1\le i<j\le n\\ \pi(i)>\pi(j)}}(-p_{\pi(i),\pi(j)})\right)\prod_{r=1}^{n}\sigma^{r-1}(c_{r,\pi(r)})$$

we are done. Fix $\pi\in S_n$. Then one may see it suffices to show

$$\prod_{\substack{1\le i<j\le n\\ \pi(i)>\pi(j)}}(\mu^{\pi(j)-\pi(i)}\prod_{k=1}^{n}q_{\pi(j),k}q_{k,\pi(i)})=\prod_{r=1}^{n}(\mu^{\pi(r)-r}\prod_{k=1}^{n}q_{\pi(r),k}q_{kr})^{r-1}.$$

Now if we write out the product $\prod_{r=1}^{n}(\mu^{\pi(r)-r}\prod_{k=1}^{n}q_{\pi(r),k}q_{kr})^{r-1}$ in full, without any rearranging or simplification, then we will be presented with a product of elements of the following forms μ^a, μ^{-b}, $(\prod_{k=1}^{n}q_{ck})$, and $(\prod_{k=1}^{n}q_{kd})$ where $a,b,c,d\in\{1,...,n\}$. Similarly for the other side of the desired equality. Fix $s\in\{1,...,n\}$. Let Z be the number of times μ^s occurs on the right-hand side of the equation, Y the number of times μ^{-s} occurs, X the number of times $(\prod_{k=1}^{n}q_{sk})$ occurs and W the number of times $(\prod_{k=1}^{n}q_{ks})$ does. Let A,B,C,D be the similar numbers for the left-hand side of the equation. Since $q_{ij}q_{ji}=1$ if we can show that $Z-Y=A-B$ and $X-W=C-D$ then we are done. We first observe that $Z=X, Y=W, A=C, B=D$ so it suffices to show that $Z-Y=A-B$.

First let us consider $\prod_{r=1}^{n}(\mu^{\pi(r)-r}\prod_{k=1}^{n}q_{\pi(r),k}q_{kr})^{r-1}$. Clearly $Y=s-1$ and $Z=\pi^{-1}(s)-1$. So $Z-Y=\pi^{-1}(s)-s$.

Now let us consider $\prod_{\substack{1\le i<j\le n\\ \pi(i)>\pi(j)}}(\mu^{\pi(j)-\pi(i)}\prod_{k=1}^{n}q_{\pi(j),k}q_{k,\pi(i)})$. A little thought yields that A is the number of i such that $1\le i<\pi^{-1}(s)$ and $\pi(i)>s$, and B is the number of j such that $\pi^{-1}(s)<j\le n$ and $s>\pi(j)$. We define the following subsets of {1,...,n},

$$a=\{m<\pi^{-1}(s)|\pi(m)>s\}$$
$$b=\{m>\pi^{-1}(s)|\pi(m)<s\}$$
$$c=\{m>\pi^{-1}(s)|\pi(m)>s\}$$
$$d=\{m<\pi^{-1}(s)|\pi(m)<s\}.$$

Since π is a bijection it follows that,

$$|a| + |d| = \pi^{-1}(s) - 1$$
$$|b| + |c| = n - \pi^{-1}(s)$$
$$|a| + |c| = n - s$$
$$|b| + |d| = s - 1$$

and hence $A - B = |a| - |b| = (\pi^{-1}(s) - 1 - |d|) - (s - 1 - |d|) = \pi^{-1}(s) - s$. So the claim is proved.

Hence, since $D_{\lambda,\mathbf{r}}$ and 1 are grouplike, we can deduce that our map passes to a surjective Hopf algebra morphism

$$\gamma : \mathcal{O}_{\lambda,\mathbf{r}}(SL_n) \twoheadrightarrow \mathcal{O}_{\lambda,\mathbf{p}}(SL_n).$$

We shall argue that γ must in-fact be an isomorphism via consideration of *Gelfand-Kirillov Dimension* [26]. We shall denote Gelfand-Kirillov Dimension by GKdim.

Suppose γ is not an isomorphism. Then $Ker\gamma$ is a nonempty ideal of $\mathcal{O}_{\lambda,\mathbf{r}}(SL_n)$. Since $\mathcal{O}_{\lambda,\mathbf{r}}(SL_n)$ is a domain [29, Corollary], $Ker\gamma$ will contain a regular element so by [26, Proposition 3.15],

$$\text{GKdim}(\mathcal{O}_{\lambda,\mathbf{r}}(SL_n)/Ker\gamma) < \text{GKdim}(\mathcal{O}_{\lambda,\mathbf{r}}(SL_n)).$$

Now certainly $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$ is a homomorphic image of $\mathcal{O}_{\lambda,\mathbf{r}}(SL_n)/Ker\gamma$ and so by [26, Lemma 3.1] we have,

$$\text{GKdim}(\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)) \leq \text{GKdim}(\mathcal{O}_{\lambda,\mathbf{r}}(SL_n)/Ker\gamma).$$

So,

$$\text{GKdim}(\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)) < \text{GKdim}(\mathcal{O}_{\lambda,\mathbf{r}}(SL_n)).$$

From [29, Corollary] we know $\text{GKdim}(\mathcal{O}_{\lambda,\mathbf{r}}(SL_n)) = n^2 - 1$, it follows that,

$$\text{GKdim}(\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)) < n^2 - 1.$$

Now from Proposition 2.5.3 we know,

$$\mathcal{O}_{\lambda,\mathbf{p}}(GL_n) \hookrightarrow \mathcal{O}_{\lambda,\mathbf{p}}(SL_n)[z, z^{-1}; \sigma],$$

in other words $\mathcal{O}_{\lambda,\mathbf{p}}(GL_n)$ is isomorphic to a subalgebra of $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)[z, z^{-1}; \sigma]$, so by [26, Lemma 3.1] we have,

$$\text{GKdim}(\mathcal{O}_{\lambda,\mathbf{p}}(GL_n)) \leq \text{GKdim}(\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)[z, z^{-1}; \sigma]).$$

From [39, Example 7.4] we know $\mathrm{GKdim}(\mathcal{O}_{\lambda,\mathbf{p}}(GL_n)) = n^2$ giving,

$$n^2 \leq \mathrm{GKdim}(\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)[z, z^{-1}; \sigma]).$$

Now [27, Proposition 1] states that if σ is a *locally algebraic* automorphism then we may deduce that

$$\mathrm{GKdim}(\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)[z, z^{-1}; \sigma]) = \mathrm{GKdim}(\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)) + 1.$$

In this case, given that $\mathrm{GKdim}(\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)) < n^2 - 1$, we would then have

$$n^2 \leq \mathrm{GKdim}(\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)[z, z^{-1}; \sigma]) = \mathrm{GKdim}(\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)) + 1 < (n^2 - 1) + 1 = n^2.$$

that is $n^2 < n^2$ which is a contradiction, and hence γ would be an isomorphism. So if we can show that σ is *locally algebraic* then we are done.

In [27] an automorphism, σ, on a K-algebra A is said to be *locally algebraic* if every finite dimensional K-subspace of A which contains the identity is contained in a σ-stable finite dimensional K-subspace.

To show that σ is a *locally algebraic* automorphism on $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$ it clearly suffices to show that every $a \in \mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$ is contained in a σ-stable finite dimensional K-subspace of $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$. Let $a \in \mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$. Then a can be written as a finite K-linear sum of finite monomials in the c_{ij}, the generators of $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$, say

$$a = \sum_{m=1}^{s} \beta_m \prod_{t=1}^{l_m} c_{i_{m_t} j_{m_t}}$$

where $\beta_m \in K$. Clearly a is contained in the K-linear span of the monomials $\prod_{t=1}^{l_m} c_{i_{m_t} j_{m_t}}$, $m = 1, ..., s$. The K-linear span of a finite number of elements of $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$ is a finite dimensional K-subspace. It is also σ-stable since, by definition of σ, $\sigma(c_{ij}) = \alpha_{ij} c_{ij}$ for some $\alpha_{ij} \in K$, and so $\sigma(\prod_{t=1}^{l_m} c_{i_{m_t} j_{m_t}}) = \mu \prod_{t=1}^{l_m} c_{i_{m_t} j_{m_t}}$ for some $\mu \in K$. So σ is as required, and we are done. □

Remark 2.6.2. *We point out that the last result shows our candidate for $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$ has the correct GK-dimension in terms of the desired properties for a potential $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$ expressed at the beginning of this chapter. So our $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$ is an affine noetherian domain, with a Hopf algebra structure, the same GK-dimension as its classical analogue, and the embedding of the last section shows that its (co)representation theory is linked to that of $\mathcal{O}_{\lambda,\mathbf{p}}(GL_n)$ in an appropriate way.*

2.7 Twisting

The question of how $\mathcal{O}_{\lambda,\mathbf{r}}(M_n)$ and $\mathcal{O}_{\lambda,\mathbf{p}}(M_n)$ are related now poses itself. The answer is via *twisting by 2-cocycles.* For a general discussion of this process we refer the reader to [3, Section 3], [15, 1.5 (d)], and [4, I.12.15-16]. We now give the details of this relationship.

We note that $\mathcal{O}_{\lambda,\mathbf{p}}(M_n)$ can be given a bigrading (namely a $\mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^n_{\geq 0}$-bigrading) under which x_{ij} has bidegree $(\varepsilon_i, \varepsilon_j)$ (where $\varepsilon_1, ..., \varepsilon_n$ is the standard basis for $\mathbb{Z}^n_{\geq 0}$).

Define a map $c : \mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^n_{\geq 0} \longrightarrow K^\times$ by

$$c((a_1, ..., a_n), (b_1, ..., b_n)) = \prod_{i>j} (\mu^{i-j} \prod_{k=1}^{n} q_{ik}q_{kj})^{a_i b_j},$$

which on the basis elements of $\mathbb{Z}^n_{\geq 0}$ is,

$$c(\varepsilon_i, \varepsilon_j) = \begin{cases} \mu^{i-j} \prod_{k=1}^{n} q_{ik}q_{kj} & i > j \\ 1 & i \leq j \end{cases}. \tag{2.7.1}$$

Now c is clearly a 2-cocycle (in-fact it is clearly bilinear), that is,

$$c(x, y + z)c(y, z) = c(x + y, z)c(x, y) \quad \forall x, y, z \in \mathbb{Z}^n_{\geq 0}.$$

Also $c(\underline{0}, \underline{0}) = 1$, where $\underline{0}$ is the identity element of $\mathbb{Z}^n_{\geq 0}$. So all the conditions of [3], for using c to twist the multiplication, are met. We now twist $\mathcal{O}_{\lambda,\mathbf{p}}(M_n)$ simultaneously on the left by c^{-1} and on the right by c to get $\mathcal{O}_{\lambda,\mathbf{p}}(M_n)'$ canonically isomorphic to $\mathcal{O}_{\lambda,\mathbf{p}}(M_n)$ as vector spaces via $a \leftrightarrow a'$, but with multiplication given by

$$a'b' = c(u_1, v_1)^{-1}c(u_2, v_2)(ab)'$$

for homogeneous elements $a, b \in \mathcal{O}_{\lambda,\mathbf{p}}(M_n)$ of bidegrees (u_1, u_2) and (v_1, v_2) respectively. Given (2.7.1) it is routine to go through the relations and check that

$$\mathcal{O}_{\lambda,\mathbf{p}}(M_n)' \cong \mathcal{O}_{\lambda,\mathbf{r}}(M_n).$$

So, given a choice of parameters making $D_{\lambda,\mathbf{p}}$ non-central in $\mathcal{O}_{\lambda,\mathbf{p}}(M_n)$, our construction of $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$ via *Noncommutative Dehomogenisation* is equivalent to the process of taking $\mathcal{O}_{\lambda,\mathbf{p}}(M_n)$, *twisting* its multiplication to get $\mathcal{O}_{\lambda,\mathbf{r}}(M_n)$, a choice of parameters for which the quantum determinant, $D_{\lambda,\mathbf{r}}$, is central in $\mathcal{O}_{\lambda,\mathbf{r}}(M_n)$, and then forming the usual $\mathcal{O}_{\lambda,\mathbf{r}}(SL_n)$.

2.8 The Dipper Donkin Case

We remarked at the beginning of this chapter that *Dipper Donkin Quantum* SL_n would be a particular instance of the work done. In-fact the original motivation for this work was the statement in [9] that *Dipper Donkin Quantum* SL_n did not exist. It was only after a candidate for $\mathcal{O}_{DD,q}(SL_n)$ was constructed that the work was extended to the more general setting of Multiparameter Quantum Matrices.

We shall now run through the particular case of $\mathcal{O}_{DD,q}(SL_n)$. As observed before $\mathcal{O}_{DD,q}(M_n) = \mathcal{O}_{q,(1)}(M_n)$, where (1) is the $n \times n$ matrix with all entries 1. So the relations for $\mathcal{O}_{DD,q}(M_n)$ are:

$$\begin{aligned} x_{lm}x_{ij} &= x_{ij}x_{lm} + (q-1)x_{im}x_{lj} && \text{for } l > i, m > j \\ x_{lm}x_{ij} &= qx_{ij}x_{lm} && \text{for } l > i, m \leq j \\ x_{lm}x_{lj} &= x_{lj}x_{lm} && \text{for } m > j \\ & && \text{where } i,j,l,m = 1,...,n. \end{aligned}$$

The *Dipper Donkin Quantum Determinant* is

$$\delta_{DD} = D_{q,(1)} = \sum_{\pi \in S_n} (-1)^{l(\pi)} x_{1,\pi(1)} x_{2,\pi(2)} \dots x_{n,\pi(n)}$$

where $l(\pi) = \#\{i < j : \pi(i) > \pi(j)\}$. The commutation relations for δ_{DD} are

$$\delta_{DD} x_{ij} = q^{j-i} x_{ij} \delta_{DD}.$$

Given the existence of an n-th root of q in our base field K, say $p \in K$ with $p^n = q$, we may apply the construction of Section 2.3 to produce a candidate for $\mathcal{O}_{DD,q}(SL_n)$. By Proposition 2.6.1 we have that

$$\mathcal{O}_{DD,q}(SL_n) \cong \mathcal{O}_{q,(p^{j-i})}(SL_n).$$

Now the standard single parameter deformation of the coordinate ring of $n \times n$ matrices, denoted $\mathcal{O}_q(M_n)$, has central quantum determinant and so $\mathcal{O}_q(SL_n)$ is defined in the natural way. One can easily observe that $\mathcal{O}_q(M_n) = \mathcal{O}_{\lambda,\mathbf{p}}(M_n)$ with

$$\lambda = q^{-2} \quad \text{and} \quad p_{ij} = \begin{cases} q, & i > j; \\ 1, & i = j; \\ q^{-1}, & i < j. \end{cases}$$

Writing out the relations it is not hard to see that in the 2×2 case we have,

$$\mathcal{O}_{DD,q}(SL_2) \cong \mathcal{O}_{q,(p^{j-i})}(SL_2) = \mathcal{O}_{p^{-1}}(SL_2),$$

that is, *Dipper Donkin Quantum* SL_2, with parameter q, is just the *Standard Quantum* SL_2, with parameter p^{-1}, where $p^2 = q$.

2.9 The Standard Quantum Matrices Case

It would obviously be desirable for our construction of $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$ to be applicable, in general, to the case of central quantum determinant and for it to produce in this case the usual Multiparameter Quantum SL_n. That is, given $\mathcal{O}_{\lambda,\mathbf{p}}(M_n)$ with parameters λ and $\mathbf{p}$ such that $D_{\lambda,\mathbf{p}}$ is central, we would always like to be able to find parameters μ and $\mathbf{q}$ satisfying (2.3.1) such that our construction yields an algebra isomorphic to $\mathcal{O}_{\lambda,\mathbf{r}}(M_n)/\langle D_{\lambda,\mathbf{r}}-1\rangle$. Now by Proposition 2.6.1 our $\mathcal{O}_{\lambda,\mathbf{p}}(SL_n)$ is isomorphic to the standard $\mathcal{O}_{\lambda,\mathbf{r}}(SL_n)$ where $\mathbf{r} = (p_{ij}\mu^{j-i}\prod_{k=1}^{n} q_{jk}q_{ki})$. So we would require our parameters μ and $\mathbf{q}$ to be such that $\mu^{j-i}\prod_{k=1}^{n} q_{jk}q_{ki} = 1$ for $1 \leq i, j \leq n$. Unfortunately, one cannot always find suitable μ and $\mathbf{q}$. We provide an example of when such parameters do not exist in the case $n = 3$.

Let $K = \mathbb{C}$, $\lambda = \frac{1}{2}$, $\mathbf{p} = \begin{pmatrix} 1 & \frac{1}{2} & \omega^2 \\ 2 & 1 & \frac{1}{2}\omega^2 \\ \omega & 2\omega & 1 \end{pmatrix}$, where ω is a primitive cube root of unity. One can check that with this choice of parameters $\lambda^{j-i}\left(\prod_{l=1}^{n} p_{jl}p_{li}\right) = 1$ and so $D_{\lambda,\mathbf{p}}$ is central. Suppose there is a choice of cube roots of $\frac{1}{2}, 2, \omega, 2\omega$, say $\mu, \alpha, \beta, \gamma$ obeying the required relations, ie

$$\mu\alpha^2\beta\gamma^{-1} = 1$$
$$\mu^2\alpha\beta^2\gamma = 1$$
$$\mu\alpha^{-1}\beta\gamma^2 = 1$$

The product of the first two of these equations gives $\mu^3\alpha^3\beta^3 = 1$. In other words $\frac{1}{2}2\omega = 1$, that is $\omega = 1$. But $\omega \neq 1$. So we have our contradiction.

However, all is not lost, as our construction when applied to *Standard Quantum Matrices* does produce the usual *Quantum* SL_n. We shall now illustrate why this is true.

We observed in the previous section that $\mathcal{O}_q(M_n) = \mathcal{O}_{\lambda,\mathbf{p}}(M_n)$ with

$$\lambda = q^{-2} \quad \text{and} \quad p_{ij} = \begin{cases} q, & i > j; \\ 1, & i = j; \\ q^{-1}, & i < j. \end{cases}$$

Assuming that we may find a $p \in K$ with $p^n = q$ then we may define a μ and $\mathbf{q}$ suitable for the construction of Section 2.3 as follows,

$$\mu = p^{-2} \text{ and } \mathbf{q} = (q_{ij}) \text{ where } q_{ij} = \begin{cases} p, & i > j; \\ 1, & i = j; \\ p^{-1}, & i < j. \end{cases}$$

To show that our construction agrees with the usual definition of $\mathcal{O}_q(SL_n)$ it suffices to show, as observed above, that $\mu^{j-i}\prod_{k=1}^{n} q_{jk}q_{ki} = 1$ for $1 \leq i, j \leq n$. Now, keeping in mind that $q_{ii} = q_{jj} = 1$,

$$\begin{aligned}\mu^{j-i}\prod_{k=1}^{n} q_{jk}q_{ki} &= p^{2(i-j)}(\prod_{k=1}^{j-1} q_{jk})(\prod_{k=j+1}^{n} q_{jk})(\prod_{k=1}^{i-1} q_{ki})(\prod_{k=i+1}^{n} q_{ki}) \\ &= p^{2(i-j)}(p^{j-1})(p^{-(n-j)})(p^{-(i-1)})(p^{n-i}) \\ &= p^{2i-2j+j-1-n+j-i+1+n-i} \\ &= 1,\end{aligned}$$

as required. So our construction does produce the usual $\mathcal{O}_q(SL_n)$ when applied to $\mathcal{O}_q(M_n)$.

Chapter 3

Quantum Skew-symmetric Matrices

Let K be a field. Let $q \in K^\times$. Unless specifically stated otherwise K is the ground field and q is as above throughout the rest of this thesis.

The definition of "Quantum Skew-symmetric Matrices" that we use in this chapter comes from a paper by Strickland [38]. Strickland is concerned with establishing a quantum version of the first and second fundamental theorems of invariant theory for the symplectic group. In the classical setting (see [7, Section 6]) the symplectic group acts on n copies of an even dimensional vector space endowed with a symplectic form, and the ring of invariants is found to be generated by coordinate functions of an $n \times n$ skew-symmetric matrix with relations determined by ideals of Pfaffians of a certain size (depending on the value of $\dim V$). In looking for a quantum analogue of this situation Strickland is naturally led to define "Quantum Skew-symmetric Matrices" and the notion of a "quantum Pfaffian".

It is well known that, under certain conditions on q, the representation theory of $U_q(\mathfrak{gl}_n)$ "mirrors" the classical case (this result is stated explicitly in the following chapter, see Theorem 4.2.4). Now $\mathcal{O}(Sk_n)$ is a representation of $U(\mathfrak{gl}_n)$, and so Strickland's definition of a quantum analogue is motivated by the representation theory of $U_q(\mathfrak{gl}_n)$. We need not concern ourselves with the details of this here, it is enough for our purposes that "Quantum Skew-symmetric Matrices" are given in [38] in terms of generators and relations, and that we have an action of $U_q(\mathfrak{gl}_n)$. We note that in Strickland's paper the ground field of her algebra is taken to be rational functions in a variable q over a field of characteristic zero. The definitions and preliminary results that we extract from [38] all remain valid with our choice of K and q. These same definitions and results were later obtained by Kamita [21, Section 3] (in Kamita's paper there are some notational differences that should be noted: q^{-1} is used in place of q and $U_q(\mathfrak{gl}_n)$ is presented differently, notably the roles of E_i and F_i are reversed).

3.1 Definitions and Preliminaries

$\mathcal{O}_q(Sk_n)$ is the K-algebra generated by $\{a_{ij} : 1 \leq i < j \leq n\}$ subject to the following relations ([38, (1.1)], but we use simpler notation):

$$a_{ij}a_{it} = qa_{it}a_{ij} \qquad \text{for } i < j < t \qquad (3.1.1)$$

$$a_{ij}a_{jt} = qa_{jt}a_{ij} \qquad \text{for } i < j < t \qquad (3.1.2)$$

$$a_{ij}a_{sj} = qa_{sj}a_{ij} \qquad \text{for } i < s < j \qquad (3.1.3)$$

$$a_{ij}a_{st} = a_{st}a_{ij} \qquad \text{for } i < s < t < j \qquad (3.1.4)$$

$$a_{ij}a_{st} = a_{st}a_{ij} + \hat{q}a_{it}a_{sj} \qquad \text{for } i < s < j < t \qquad (3.1.5)$$

$$a_{ij}a_{st} = a_{st}a_{ij} + \hat{q}a_{is}a_{jt} - q\hat{q}a_{it}a_{js} \qquad \text{for } i < j < s < t \qquad (3.1.6)$$

where $\hat{q} := (q - q^{-1})$.
It will prove useful to also define

$$a_{ii} := 0 \quad \text{and} \quad a_{ji} := -qa_{ij}, \text{ for } j > i,$$

and view all the a_{ij} as the n^2 coordinate functions on a generic $n \times n$ *Quantum Skew-symmetric Matrix.* However when referring to the *generators* of $\mathcal{O}_q(Sk_n)$ we will, of course, only be referring to the a_{ij} with $i < j$.

When q is a non-root of unity we may define $U_q(\mathfrak{gl}_n)$ to be the K-algebra generated by $E_1, F_1, ..., E_{n-1}, F_{n-1}$ and $L_1^{\pm 1}, ..., L_n^{\pm 1}$ with the following relations [31], [14]:

$$\begin{aligned}
&L_iL_j = L_jL_i,\\
&L_iE_j = q^{\kappa(i,j)}E_jL_i,\\
&L_iF_j = q^{-\kappa(i,j)}F_jL_i,\\
&E_iF_j - F_jE_i = \delta_{ij}\hat{q}^{-1}(L_iL_{i+1}^{-1} - L_i^{-1}L_{i+1}),\\
&E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 = 0 \quad \text{for } |i-j| = 1,\\
&E_iE_j = E_jE_i \quad \text{for } |i-j| > 1,\\
&F_i^2F_j - (q+q^{-1})F_iF_jF_i + F_jF_i^2 = 0 \quad \text{for } |i-j| = 1,\\
&E_iE_j = E_jE_i \quad \text{for } |i-j| > 1,
\end{aligned}$$

where

$$\kappa(i,j) = \begin{cases} 1 & \text{if } j = i \\ -1 & \text{if } j = i-1 \\ 0 & \text{otherwise.} \end{cases}$$

It is endowed with a Hopf algebra structure with comultiplication given by

$$\Delta(L_i^{\pm 1}) = L_i^{\pm 1} \otimes L_i^{\pm 1} \tag{3.1.7}$$

$$\Delta(E_i) = E_i \otimes 1 + L_i L_{i+1}^{-1} \otimes E_i \tag{3.1.8}$$

$$\Delta(F_i) = F_i \otimes L_i^{-1} L_{i+1} + 1 \otimes F_i \tag{3.1.9}$$

and with counit and antipode as follows,

$$\epsilon(E_i) = \epsilon(F_i) = 0, \qquad \epsilon(L_i) = 1,$$

$$S(L_i) = L_i^{-1}, \quad S(E_i) = -L_i^{-1} L_{i+1} E_i, \quad \text{and} \quad S(F_i) = -F_i L_i L_{i+1}^{-1}.$$

Remark 3.1.1. *There are many different presentations of $U_q(\mathfrak{gl}_n)$ that are used in the literature, with the same symbols sometimes used to represent different elements, and with no standard way of defining the Hopf algebra structure. It is important, therefore, for us to define what we mean by $U_q(\mathfrak{gl}_n)$. Although Strickland does not explicitly state which version of $U_q(\mathfrak{gl}_n)$ is used in [38], it can be deduced from the proof of [38, Proposition 1.2] and then verified that it is the version given above.*

There is an action of $U_q(\mathfrak{gl}_n)$ on $\mathcal{O}_q(Sk_n)$ defined, for $i < j$, as follows [38, (1.2)]:

$$E_s a_{ij} = \begin{cases} 0 & \text{if } i, j \neq s+1 \text{ or } i = j-1 = s \\ a_{i-1,j} & \text{if } i = s+1 \\ a_{i,j-1} & \text{if } j = s+1 \text{ and } i \neq s \end{cases} \tag{3.1.10}$$

$$F_s a_{ij} = \begin{cases} 0 & \text{if } i, j \neq s+1 \text{ or } i = j-1 = s \\ a_{i+1,j} & \text{if } i = s \text{ and } j \neq s+1 \\ a_{i,j+1} & \text{if } j = s \end{cases} \tag{3.1.11}$$

$$L_s a_{ij} = \begin{cases} a_{ij} & \text{if } i, j \neq s \\ q a_{ij} & \text{if } i = s \text{ or } j = s. \end{cases} \tag{3.1.12}$$

This action, defined on the generators of $\mathcal{O}_q(Sk_n)$, extends to an action on the whole algebra, making $\mathcal{O}_q(Sk_n)$ into a $U_q(\mathfrak{gl}_n)$-module algebra. In particular, the action of $U_q(\mathfrak{gl}_n)$ on products is given by,

$$u(ab) = \sum_u (u_1(a))\,(u_2(b)),$$

for $u \in U_q(\mathfrak{gl}_n), a, b \in \mathcal{O}_q(Sk_n)$. So, for example, the action of the F_i on products is given by

$$F_i(ab) = F_i(a) L_i^{-1} L_{i+1}(b) + a F_i(b) \tag{3.1.13}$$

for $a, b \in \mathcal{O}_q(Sk_n)$. This $U_q(\mathfrak{gl}_n)$-action on products will be used repeatedly in proofs throughout the rest of this thesis.

In [38, (1.6)] the author makes the following definition:

Definition 3.1.2. *Let $1 \leq i_1 < i_2 < \cdots < i_{2h} \leq n$ for some $1 \leq h \leq \lfloor n/2 \rfloor$. We define the* Quantum Pfaffian $[i_1 i_2 ... i_{2h}]$ *inductively as follows:*
If $h = 1$,

$$[i_1 i_2] := a_{i_1 i_2}$$

and if $h > 1$,

$$[i_1 i_2 ... i_{2h}] := \sum_{r=2}^{2h} (-q)^{r-2} a_{i_1 i_r} [i_2 ... \check{i_r} ... i_{2h}]$$

where $\check{i_r}$ means "remove i_r from the list".

Remark 3.1.3. *This is the natural quantum analogue of the recursive definition of the classical Pfaffian (see Definition 1.5.4).*

Example 3.1.4. *In the case $n = 4$ the 4×4 q-Pfaffian is,*

$$[1234] = a_{12}a_{34} - qa_{13}a_{24} + q^2 a_{14}a_{23}.$$

We make the following notational definition $\mathrm{Pf}_q(k) := [1...k]$. When dealing with $\mathcal{O}_q(Sk_n)$ we shall shorten $\mathrm{Pf}_q(n)$ to Pf_q. Also, when convenient, we shall make use of the convention that odd length q-Pfaffians are zero. The use of the term *length* in reference to a q-Pfaffian has the obvious meaning, so that in general $[i_1 \cdots i_m]$ has *length* m and Pf_q is the highest-length q-Pfaffian in $\mathcal{O}_q(Sk_n)$.

We have previously stated that the $U_q(\mathfrak{gl}_n)$-action will be utilised throughout the proof of many upcoming results. Vital to this method will be the knowledge of the $U_q(\mathfrak{gl}_n)$-action on q-Pfaffians. This following result, [38, Lemma 1.4], is, therefore, key:

Lemma 3.1.5. *For $E_s, F_s \in U_q(\mathfrak{gl}_n)$ and for a q-Pfaffian $[i_1 i_2 ... i_{2h}] \in \mathcal{O}_q(Sk_n)$, we have,*

$$E_s([i_1 i_2 ... i_{2h}]) = \begin{cases} [i_1 ... i_{t-1}\ s\ i_{t+1} ... i_{2h}] & \text{if } s+1 = i_t \text{ for some } t \text{ and } s \neq i_{t-1} \\ 0 & \text{otherwise} \end{cases}$$

and

$$F_s([i_1 i_2 ... i_{2h}]) = \begin{cases} [i_1 ... i_{t-1}\ s+1\ i_{t+1} ... i_{2h}] & \text{if } s = i_t \text{ for some } t \text{ and } s+1 \neq i_{t+1} \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.1.6. *Although not included in the above lemma we note that trivially we also have,*

$$L_s([i_1 i_2 ... i_{2h}]) = \begin{cases} q[i_1 ... i_{2h}] & \text{if } s = i_t \text{ for some } t \\ [i_1 ... i_{2h}] & \text{otherwise.} \end{cases}$$

3.2 $\mathcal{O}_q(Sk_n)$ is an Iterated Skew Polynomial Ring

There is a general consensus in the literature that quantized coordinate rings are desired to be affine noetherian domains (see, for example, [4]). Now $\mathcal{O}_q(Sk_n)$ has been defined in terms of generators and relations, so it is certainly affine. We will show that $\mathcal{O}_q(Sk_n)$ is also both a domain and noetherian, along the way proving that it is an iterated skew polynomial ring.

Definition 3.2.1. *We will say that a monomial $a_{i_1j_1}...a_{i_mj_m}$ is* ordered *if $i_k < j_k$ for $k = 1, ..., m$ and $(i_1, j_1) \leq \cdots \leq (i_m, j_m)$ where we are using $\leq$ to denote the usual lexicographic ordering. We shall refer to such monomials as* ordered monomials.

In [38, Proposition 1.1] the author states that the set of ordered monomials is a basis for $\mathcal{O}_q(Sk_n)$. The fact that such monomials span $\mathcal{O}_q(Sk_n)$ is clear from the defining relations. The key step is to prove that these monomials are linearly independent. We give an alternative to Strickland's proof of this fact. Instead of her argument we use the *Diamond Lemma* (see, for example, [4, I.11.6]). We note that there is no restriction on q needed in our argument.

Proposition 3.2.2. *The ordered monomials form a basis for $\mathcal{O}_q(Sk_n)$.*

Proof. It remains to show that the ordered monomials are linearly independent. We are going to apply the *Diamond Lemma* so our first task is to define an appropriate total order, which we will denote by $\leq_w$, among the words on the letters a_{ij} $(1 \leq i < j \leq n)$. In [32, Theorem 1.4] the authors use the *Diamond Lemma* to show that the ordered monomials form a basis for $\mathcal{O}_q(M_n)$. We will later use this result to reduce the work that needs to be done in our case. Therefore, it is necessary for us to take the same total order (this also saves us the task of showing that our total order satisfies the requirements for application of the *Diamond Lemma*). To a word, $a_{i_1j_1} \cdots a_{i_dj_d}$, we associate a matrix $B = (b_{ij}) \in M_n(\mathbb{N})$, where b_{ij} is the number of times a_{ij} occurs in the word. For example, to the word $a_{11}a_{12}a_{21}^3a_{11}$ we associate the matrix B with $b_{11} = 2, b_{12} = 1, b_{21} = 3$ and all other entries zero. To such a matrix, B, we associate a sequence

$$(\sum_{i,j} b_{ij}, b_{11}, b_{12}, ..., b_{21}, b_{22}, ..., b_{nn}) \in \mathbb{N}^{1+n^2}.$$

We well-order $M_n(\mathbb{N})$ via the lexicographic ordering of such sequences. The ordering on words is then defined by first ordering via this well-order on the associated matrices, B, and then by the lexicographic ordering of the sequences $(i_1, j_1, i_2, j_2, ..., i_d, j_d) \in \mathbb{N}^{2d}$. So, for example,

$$1 <_w a_{nn} <_w a_{12} <_w a_{11} <_w a_{12}a_{34} <_w a_{34}a_{12} <_w a_{11}^2 <_w a_{22}a_{33}a_{44}.$$

From the relations (3.1.1)-(3.1.4) we get the following *reduction system*:

$$
\begin{aligned}
&a_{it}a_{ij} \mapsto q^{-1}a_{ij}a_{it} && \text{for } i<j<t\\
&a_{jt}a_{ij} \mapsto q^{-1}a_{ij}a_{jt} && \text{for } i<j<t\\
&a_{sj}a_{ij} \mapsto q^{-1}a_{ij}a_{sj} && \text{for } i<s<j\\
&a_{st}a_{ij} \mapsto a_{ij}a_{st} && \text{for } i<s<t<j\\
&a_{st}a_{ij} \mapsto a_{ij}a_{st} - \hat{q}a_{it}a_{sj} && \text{for } i<s<j<t\\
&a_{st}a_{ij} \mapsto a_{ij}a_{st} - \hat{q}a_{is}a_{jt} + q\hat{q}a_{it}a_{js} && \text{for } i<j<s<t.
\end{aligned}
$$

We note that in all cases the words on the RHS are lower with respect to our ordering than the one on the LHS. Applying the *Diamond Lemma*, our proof will be complete if we can show that all *ambiguities* are *resolvable* (for the exact definitions of these terms we refer the reader to [4, I.11], but it will be clear enough from the work we do below). We must show that a word, $a_{ij}a_{st}a_{kl}$, consisting of three distinct letters can be unambiguously resolved (i.e. put into ordered form using our reduction system). Consideration of the reduction system defined above yields that ambiguities arise when,

$$a_{ij} > a_{st} > a_{kl}$$

where we denote by, $<$, the ordering induced by the lexicographic ordering in the subscripts. Suppose $|\{i,j,s,t,k,l\}| = 4$. Then one can easily show that

$$a_{ij}, a_{st}, a_{kl} \in K\langle a_{uv} : u, v \in \{i,j,s,t,k,l\}\rangle \cong \mathcal{O}_q(Sk_4).$$

So we may as well assume that $a_{ij}, a_{st}, a_{kl} \in \mathcal{O}_q(Sk_4)$. Now from the relations (3.1.1), (3.1.3), (3.1.4), and (3.1.5), it can be readily seen that

$$K\langle a_{13}, a_{14}, a_{23}, a_{24}\rangle \cong \mathcal{O}_q(M_2).$$

So if $a_{ij}, a_{st}, a_{kl} \in \{a_{13}, a_{14}, a_{23}, a_{24}\}$ then all ambiguities are resolvable by the proof of [32, Theorem 1.4]. We are left with the cases:

$$
\begin{aligned}
(i)&\quad z > y > a_{12}\\
(ii)&\quad a_{34} > y > a_{12}\\
(iii)&\quad a_{34} > y > x,
\end{aligned}
$$

where $x, y, z \in \{a_{13}, a_{14}, a_{23}, a_{24}\}$. Firstly, let us note that a_{12} and a_{34} both q-commute with a_{13}, a_{14}, a_{23}, and a_{24}. Secondly, we note that

$$zy = \sum_i \lambda_i y_i z_i,$$

for some $y_i, z_i \in \{a_{13}, a_{14}, a_{23}, a_{24}\}$, $\lambda_i \in K$. Now, using our reduction system,

$$zya_{12}: \begin{cases} z(q^{-1}a_{12}y) \to q^{-1}(q^{-1}a_{12}z)y \to q^{-2}a_{12}(\sum_i \lambda_i y_i z_i), \\ (\sum_i \lambda_i y_i z_i)a_{12} \to \sum_i \lambda_i y_i (q^{-1}a_{12}z_i) \to q^{-2}a_{12}(\sum_i \lambda_i y_i z_i). \end{cases}$$

So ambiguity (i) is resolvable. Similarly (iii) is resolvable. We turn to (ii):

$$a_{34}ya_{12}: \begin{cases} a_{34}(q^{-1}a_{12}y) \to q^{-1}(a_{12}a_{34} - \hat{q}a_{13}a_{24} + q\hat{q}a_{14}a_{23})y, & (1) \\ (q^{-1}ya_{34})a_{12} \to q^{-1}y(a_{12}a_{34} - \hat{q}a_{13}a_{24} + q\hat{q}a_{14}a_{23}). & (2) \end{cases}$$

Further reductions depend on y. Suppose $y = a_{13}$. Then (1) reduces as follows:

$$\begin{aligned} &q^{-1}(a_{12}a_{34} - \hat{q}a_{13}a_{24} + q\hat{q}a_{14}a_{23})a_{13} \\ &\qquad \to q^{-1}\big(a_{12}(q^{-1}a_{13}a_{34}) - \hat{q}a_{13}(a_{13}a_{24} - \hat{q}a_{14}a_{23}) + \hat{q}a_{14}(q^{-1}a_{13}a_{23})\big) \\ &\qquad \to q^{-1}\big(q^{-1}a_{12}a_{13}a_{34} - \hat{q}a_{13}^2a_{24} - \hat{q}^2a_{13}a_{14}a_{23} + \hat{q}q^{-1}(q^{-1}a_{13}a_{14})a_{23}\big) \\ &\qquad = q^{-2}a_{12}a_{13}a_{34} - q^{-1}\hat{q}a_{13}^2a_{24} + (\hat{q}^2q^{-1} + \hat{q}q^{-2})a_{13}a_{14}a_{23} \\ &\qquad = q^{-2}a_{12}a_{13}a_{34} - q^{-1}\hat{q}a_{13}^2a_{24} + \hat{q}a_{13}a_{14}a_{23}, \end{aligned}$$

where the last equality holds since $\hat{q}q^{-1} + q^{-2} = 1$. We now show that we obtain the same result when we reduce (2):

$$\begin{aligned} &q^{-1}a_{13}(a_{12}a_{34} - \hat{q}a_{13}a_{24} + q\hat{q}a_{14}a_{23}) \\ &\qquad \to q^{-1}\big((q^{-1}a_{12}a_{13})a_{34} - \hat{q}a_{13}^2a_{24} + q\hat{q}a_{13}a_{14}a_{23}\big) \\ &\qquad = q^{-2}a_{12}a_{13}a_{34} - q^{-1}\hat{q}a_{13}^2a_{24} + \hat{q}a_{13}a_{14}a_{23}. \end{aligned}$$

If $y = a_{24}$ then the reductions are similar to those above (except this time (2) takes more steps and (1) less). If $y = a_{14}, a_{23}$ then y commutes or q-commutes with everything and the equivalence of the reductions is easily checked.

Now we suppose that $|\{i, j, s, t, k, l\}| \neq 4$. We will first argue that if i, j, s, t, k, and l are such that relation (3.1.6) does not apply to a_{ij}, a_{st}, and a_{kl}, then all ambiguities are equivalent to ones that are resolvable by the proof of [32, Theorem 1.4]. Suppose (3.1.6) does not apply to a_{ij}, a_{st}, and a_{kl}. Then, when we try to put the monomial $a_{ij}a_{st}a_{kl}$ into ordered form using our reduction system the reductions that we use only involve commutation, q-commutation, and relation (3.1.5). Apart from the q-commutation due to (3.1.2), these are all particular instances of the relations that define $\mathcal{O}_q(M_n)$. Immediately we may deduce from [32, Theorem 1.4] that if (3.1.2) does not apply then all ambiguities are resolvable. If (3.1.2) does apply then the resolution of the ambiguities is entirely similar to cases covered by [32, Theorem 1.4].

We now consider cases to which (3.1.6) applies. There are many such cases and we do not explicitly go through all of them here, instead we show that the ambiguity is resolvable in one of the more complicated examples. Consider the monomial $a_{45}a_{13}a_{12}$. Then one possible reduction is:

$$\begin{aligned}
a_{45}a_{13}a_{12} &\to a_{45}(q^{-1}a_{12}a_{13}) \\
&\to q^{-1}(a_{12}a_{45} - \hat{q}a_{14}a_{25} + q\hat{q}a_{15}a_{24})a_{13} \\
&\to q^{-1}a_{12}(a_{13}a_{45} - \hat{q}a_{14}a_{35} + q\hat{q}a_{15}a_{34}) \\
&\qquad - q^{-1}\hat{q}a_{14}(a_{13}a_{l5} - \hat{q}a_{15}a_{23}) + \hat{q}a_{15}(a_{13}a_{24} - \hat{q}a_{14}a_{23}) \\
&\to q^{-1}a_{12}a_{13}a_{45} - q^{-1}\hat{q}a_{12}a_{14}a_{35} + \hat{q}a_{12}a_{15}a_{34} - q^{-1}\hat{q}(q^{-1}a_{13}a_{14})a_{25} \\
&\qquad + q^{-1}\hat{q}^2a_{14}a_{15}a_{23} + \hat{q}(q^{-1}a_{13}a_{15})a_{24} - \hat{q}^2(q^{-1}a_{14}a_{15})a_{23} \\
&= q^{-1}a_{12}a_{13}a_{45} - q^{-1}\hat{q}a_{12}a_{14}a_{35} + \hat{q}a_{12}a_{15}a_{34} - q^{-2}\hat{q}a_{13}a_{14}a_{25} \\
&\qquad + \hat{q}q^{-1}a_{13}a_{15}a_{24}.
\end{aligned}$$

Reducing the monomial in the other order gives:

$$\begin{aligned}
a_{45}a_{13}a_{12} &\to (a_{13}a_{45} - \hat{q}a_{14}a_{35} + q\hat{q}a_{15}a_{34})a_{12} \\
&\to a_{13}(a_{12}a_{45} - \hat{q}a_{14}a_{25} + q\hat{q}a_{15}a_{24}) \\
&\qquad - \hat{q}a_{14}(a_{12}a_{35} - \hat{q}a_{13}a_{25} + q\hat{q}a_{15}a_{23}) \\
&\qquad\qquad + q\hat{q}a_{15}(a_{12}a_{34} - \hat{q}a_{13}a_{24} + q\hat{q}a_{14}a_{23}) \\
&\to (q^{-1}a_{12}a_{13})a_{45} - \hat{q}a_{13}a_{14}a_{25} + q\hat{q}a_{13}a_{15}a_{24} \\
&\qquad - \hat{q}(q^{-1}a_{12}a_{14})a_{35} + \hat{q}^2(q^{-1}a_{13}a_{14})a_{25} - q\hat{q}^2a_{14}a_{15}a_{23} \\
&\qquad + q\hat{q}(q^{-1}a_{12}a_{15})a_{34} - q\hat{q}^2(q^{-1}a_{13}a_{15})a_{24} + q^2\hat{q}^2(q^{-1}a_{14}a_{15})a_{23} \\
&= q^{-1}a_{12}a_{13}a_{45} - \hat{q}a_{13}a_{14}a_{25} + q\hat{q}a_{13}a_{15}a_{24} \\
&\qquad - \hat{q}q^{-1}a_{12}a_{14}a_{35} + \hat{q}^2q^{-1}a_{13}a_{14}a_{25} - q\hat{q}^2a_{14}a_{15}a_{23} \\
&\qquad\qquad + \hat{q}a_{12}a_{15}a_{34} - \hat{q}^2a_{13}a_{15}a_{24} + q\hat{q}^2a_{14}a_{15}a_{23} \\
&= q^{-1}a_{12}a_{13}a_{45} - (\hat{q} - \hat{q}^2q^{-1})a_{13}a_{14}a_{25} + (q\hat{q} - \hat{q}^2)a_{13}a_{15}a_{24} \\
&\qquad - \hat{q}q^{-1}a_{12}a_{14}a_{35} + \hat{q}a_{12}a_{15}a_{34} \\
&= q^{-1}a_{12}a_{13}a_{45} - \hat{q}q^{-2}a_{13}a_{14}a_{25} + q^{-1}\hat{q}a_{13}a_{15}a_{24} \\
&\qquad - \hat{q}q^{-1}a_{12}a_{14}a_{35} + \hat{q}a_{12}a_{15}a_{34},
\end{aligned}$$

and so the two reductions give the same result. This exact same argument will of course work for all monomials $a_{ij}a_{st}a_{kl}$ with $s = k < l < t < i < j$. In a similar manner one can show that all ambiguities are resolvable. □

In [38] $\mathcal{O}_q(Sk_n)$ is shown to be a domain. The proof, however, relies on q being transcendental. We will now show that this holds more generally.

Proposition 3.2.3. *$\mathcal{O}_q(Sk_n)$ is a domain.*

Proof. For this proof we adapt an idea taken from [4, Proposition I.8.17]. Order the generators of $\mathcal{O}_q(Sk_n)$ according to the lexicographic ordering of subscripts,

$$a_{12} < a_{13} < \cdots < a_{1n} < a_{23} < \cdots < a_{nn},$$

and label them, in order, $u_1 = a_{12}, u_2 = a_{13}, ..., u_{n-1} = a_{1n}, u_n = a_{23}, ..., u_{n(n-1)/2} = a_{nn}$. Then from (3.1.1)-(3.1.6) we may see that,

$$u_i u_j = q_{ij} u_j u_i + \sum_{\substack{s,t=j+1 \\ s<t}}^{i-1} \alpha_{st}^{ij} u_s u_t, \tag{3.2.1}$$

for $1 \le j < i \le n(n-1)/2$, where $q_{ij} \in K^\times$ and $\alpha_{st}^{ij} \in K$ (in-fact, $q_{ij} = 1$ or q^{-1} and $\alpha_{st}^{ij} = 0, \hat{q}$, or $-q\hat{q}$, depending on the values of i, j, s, t). We now set u_i to have degree $d_i = 2^i$. Let $\mathcal{O}_q(Sk_n)_0 = K \cdot 1$, and for $d > 0$, let $\mathcal{O}_q(Sk_n)_d$ be the K-subspace spanned by

$$\{u_{i_1} u_{i_2} \cdots u_{i_r} : d_{i_1} + \cdots + d_{i_r} \le d\}.$$

This clearly defines a filtration of $\mathcal{O}_q(Sk_n)$. It is also clear that the associated graded ring, $\mathrm{gr}(\mathcal{O}_q(Sk_n))$, is generated by the elements y_i, where

$$y_i := u_i + \mathcal{O}_q(Sk_n)_{d_i - 1} \in (\mathrm{gr}(\mathcal{O}_q(Sk_n)))_{d_i}.$$

For $i > j$ we know that $u_i u_j - q_{ij} u_j u_i$ is a linear combination of products $u_s u_t$ with $j < s < t < i$ (where this makes sense, and zero otherwise). Now,

$$d_s + d_t = 2^s + 2^t \le 2^{i-2} + 2^{i-1} < 2^{i-1} + 2^{i-1} = 2(2^{i-1}) = 2^i < 2^i + 2^j = d_i + d_j.$$

So we see that in $\mathrm{gr}(\mathcal{O}_q(Sk_n))$, $y_i y_j = q_{ij} y_j y_i$ for $i > j$. It follows that we have an onto algebra morphism

$$\gamma : \mathcal{O}_{\mathbf{q}}(K^{n(n-1)/2}) \longrightarrow \mathrm{gr}(\mathcal{O}_q(Sk_n)),$$

where $\mathbf{q}$ is the obvious multiplicatively antisymmetric matrix of scalars. Using GKdim we will argue in a similar manner to the proof of Proposition 2.6.1 that this is in-fact an isomorphism. Suppose, for a contradiction, that γ is not an isomorphism. Now since $\mathcal{O}_{\mathbf{q}}(K^{n(n-1)/2})$ is a domain [4, Theorem I.2.7], $Ker\gamma$ is a nonempty ideal containing a regular element, and so by [26, Proposition 3.15],

$$\mathrm{GKdim}(\mathcal{O}_{\mathbf{q}}(K^{n(n-1)/2})/Ker\gamma) < \mathrm{GKdim}(\mathcal{O}_{\mathbf{q}}(K^{n(n-1)/2}),$$

and hence

$$\mathrm{GKdim}(\mathrm{gr}(\mathcal{O}_q(Sk_n))) < \mathrm{GKdim}(\mathcal{O}_{\mathbf{q}}(K^{n(n-1)/2}).$$

Now $\text{GKdim}(\mathcal{O}_{\mathbf{q}}(K^{n(n-1)/2}) = n(n-1)/2$ [4, Proposition II.9.9], so if we can show that $\text{GKdim}(\text{gr}(\mathcal{O}_q(Sk_n))) = n(n-1)/2$ then we will have our contradiction. Since $\mathcal{O}_q(Sk_n)$ is a K-algebra with finite filtration and $\text{gr}(\mathcal{O}_q(Sk_n))$ is finitely generated, we may deduce that

$$\text{GKdim}(\text{gr}(\mathcal{O}_q(Sk_n))) = \text{GKdim}(\mathcal{O}_q(Sk_n))$$

by [26, Proposition 6.6]. An easy consequence of the previous Proposition, showing that the ordered monomials form a basis for $\mathcal{O}_q(Sk_n)$, is that each graded component of $\mathcal{O}_q(Sk_n)$ (with the canonical grading) is in bijection with the corresponding graded component of the commutative polynomial ring in $n(n-1)/2$ variables. It follows from the the definition of GKdim (Definition 1.4.1) that,

$$\text{GKdim}(\mathcal{O}_q(Sk_n)) = \text{GKdim}(K[x_1, \ldots, x_{n(n-1)/2}]) = n(n-1)/2.$$

Hence γ must be an isomorphism. Therefore, since $\mathcal{O}_{\mathbf{q}}(K^{n(n-1)/2})$ is a domain, $\text{gr}(\mathcal{O}_q(Sk_n))$ is a domain, and we may deduce that $\mathcal{O}_q(Sk_n)$ is a domain by [4, Lemma I.12.12]. □

Before we proceed to the main result of this section, we recall the following lemma taken from [4, Exercise I.1.L] which we will need in its proof:

Lemma 3.2.4. *Let $R \subseteq S$ be rings, and suppose that there is a regular element $d \in S$ such that $dR + R = Rd + R$ and $dR \cap R = 0 = Rd \cap R$. Then there exist unique maps $\tau, \delta : R \to R$ such that τ is an automorphism of R and δ is a τ-derivation on R.*

Proposition 3.2.5. *$\mathcal{O}_q(Sk_n)$ is an iterated skew polynomial ring.*

Proof. Let us define an ordered series of subalgebras,

$$B_{12}, B_{13}, ..., B_{1n}, B_{23}, ..., B_{n-1,n}$$

of $\mathcal{O}_q(Sk_n)$ as follows,

$$B_{ij} := K\langle a_{st} : a_{st} \leq a_{ij}\rangle,$$

where $\leq$ denotes the ordering of the generators, a_{ij} ($i < j$), induced by the lexicographic ordering of subscripts. Since the ordered monomials form a basis for $\mathcal{O}_q(Sk_n)$ we may deduce that that each subalgebra, B_{ij}, is a free module over the preceding subalgebra with basis $\{a_{ij}^m : m \geq 0\}$. Now $B_{12} = K[a_{12}]$. We will show that in general B_{ij} is a skew polynomial extension of the previous subalgebra, B_{rs}. Examining the relations (3.1.1)-(3.1.6), it is not hard to see that $a_{ij}B_{rs} + B_{rs} = B_{rs}a_{ij} + B_{rs}$. Now clearly since the ordered monomials form a basis for $\mathcal{O}_q(Sk_n)$ it follows that B_{rs} has a basis consisting of ordered monomials in $\{a_{st} : a_{st} \leq a_{rs}\}$. The fact that, $a_{ij} \notin \{a_{st} : a_{st} \leq a_{rs}\}$, together with the relations (3.1.1)-(3.1.6), implies $a_{ij}B_{rs} \cap B_{rs} = B_{rs}a_{ij} \cap B_{rs} = 0$. We have already shown that $\mathcal{O}_q(Sk_n)$ is a domain and so certainly a_{ij} is regular in B_{ij}. Therefore we may apply Lemma 3.2.4 to get $B_{ij} = B_{rs}[a_{ij}; \sigma, \tau]$. Hence $\mathcal{O}_q(Sk_n) = B_{n-1,n}$ is an iterated skew polynomial ring over K. □

We can now deduce, using Theorem 1.2.6, the following:

Corollary 3.2.6. *$\mathcal{O}_q(Sk_n)$ is noetherian.*

Before leaving this section we note that the work we have just done allows us to apply two results from [4] to our algebra $\mathcal{O}_q(Sk_n)$. These show that $\mathcal{O}_q(Sk_n)$ satisfies certain "nice" properties of noncommutative algebras.

Firstly we recall [4, Theorem II.7.19]:

Theorem 3.2.7. *Let A be a K-algebra with a $\mathbb{Z}_{\geq 0}$-filtration $(A_d)_{d\geq 0}$ such that $A_0 = K \cdot 1$ and all A_d are finite dimensional over K. Assume that grA can be generated by homogeneous elements $y_1, \ldots, y_m$ satisfying relations $y_i y_j = q_{ij} y_j y_i$ for $i > j$, for some $q_{ij} \in K^{\times}$. Then A is noetherian and satisfies the Nullstellensatz.*

Corollary 3.2.8. *$\mathcal{O}_q(Sk_n)$ satisfies the Nullstellensatz.*

Proof. Looking at the proof of Proposition 3.2.3 we immediately see that $\mathcal{O}_q(Sk_n)$ satisfies all the conditions of Theorem 3.2.7. □

Secondly we recall [4, Lemma II.9.10]:

Lemma 3.2.9. *Let $A = K[x_1][x_2; \tau_2, \delta_2] \cdots [x_n; \tau_n, \delta_n]$ be an iterated skew polynomial algebra over K. Then A is Auslander-regular. Now suppose that for $1 \leq j < i \leq n$, we have $\tau_i(x_j) \in K^{\times} x_j$ and $\delta_i(x_j) \in \sum_{s,t<i} K x_s x_t$. Then A is also Cohen-Macaulay.*

Corollary 3.2.10. *$\mathcal{O}_q(Sk_n)$ is Auslander-regular and Cohen-Macaulay.*

Proof. We have shown that $\mathcal{O}_q(Sk_n)$ is an iterated skew polynomial algebra over K. It remains to show that at each stage the automorphism and derivation of the extension are as required. However this is an immediate consequence of (3.2.1). □

3.3 A Torus Action

When studying an algebra one of the central areas of interest is its ideal theory. In particular the composition and structure of both the prime and primitive spectrums. In [4, Part II] the ideal theory of many quantized coordinate rings are investigated and numerous results are proved for certain iterated skew polynomial algebras. In this section we quickly show that $\mathcal{O}_q(Sk_n)$ satisfies (under certain conditions) the setup of [4, II.5.1] which allows us to deduce many results.

[4, II.5.1]:

(a) Let $A = K[x_1][x_2; \tau_2, \delta_2] \cdots [x_n; \tau_m, \delta_m]$ be an iterated skew polynomial algebra over K.

(b) Let H be a group acting on A by K-algebra automorphisms.

(c) Assume that $x_1, \ldots, x_m$ are H-eigenvectors.

(d) Assume that there exists $h_1, \ldots, h_m \in H$ such that $h_i(x_j) = \tau_i(x_j)$ for $j < i$ and such that the h_i-eigenvalue of x_i is not a root of unity for any i.

The reason a group acting by automorphisms on our algebra is of interest is because the study of the prime spectrum can be aided by considering its "H-stratification". To explain what we mean by this we give some definitions and results from [4, II.1 and II.2]:

Definitions and Results 3.3.1. *Let H be a group acting by automorphisms on a ring R. An H-*ideal *of R is an ideal of R such that $h(I) = I$ for all $h \in H$. An H-*prime *ideal of R is any proper H-ideal P of R such that in R/P any product of non-zero H-ideals of R/P is nonzero. The largest H-ideal contained in an ideal I in R is written*

$$(I : H) = \bigcap_{h \in H} h(I).$$

If P is a prime ideal in R then $(P : H)$ is H-prime. The "H-stratum" of the prime spectrum (specR) corresponding to an H-prime ideal J is

$$spec_J R = \{P \in specR : (P : H) = J\},$$

and we have an H-stratification of specR,

$$specR = \bigsqcup_{J \in H-specR} spec_J R.$$

Knowledge of the H-spectrum and the structure of the H-strata therefore leads to knowledge of the prime spectrum.

We now show that $\mathcal{O}_q(Sk_n)$ satisfies [4, II.5.1] (when q is not a root of unity) and then give some of the results that this allows us to deduce.

Lemma 3.3.2. *When q is not a root of unity $\mathcal{O}_q(Sk_n)$ satisfies [4, II.5.1].*

Proof. Condition (a) is satisfied by Proposition 3.2.5. Now instead of $x_1, \ldots, x_m$ we label the indeterminates by a_{ij} with $1 \leq i < j \leq n$ according to the lexicographic ordering of subscripts as in the proof of Proposition 3.2.3. Let H be the algebraic torus, $(K^\times)^n$ and let H act on $\mathcal{O}_q(Sk_n)$ by K-algebra automorphisms as follows,

$$(\alpha_1, \ldots, \alpha_n) \cdot a_{ij} = \alpha_i \alpha_j a_{ij},$$

for any $(\alpha_1, \ldots, \alpha_n) \in H$. Then clearly conditions (b) and (c) are satisfied. Now for (d) we label the distinguished elements of H by h_{ij} with $1 \leq i < j \leq n$ and define h_{ij} to be the row vector with all entries 1 except for q^{-1}'s in the i-th and j-th place, so for example,

$$\begin{aligned} h_{12} &= (q^{-1}, q^{-1}, 1, \ldots, 1) \\ h_{24} &= (1, q^{-1}, 1, q^{-1}, 1, \ldots, 1). \end{aligned}$$

So $h_{ij} \cdot a_{ij} = q^{-2} a_{ij}$. Since we are assuming q is not a root of unity then clearly the h_{ij}-eigenvalue of a_{ij} is not a root of unity for any (i, j). We note that if $(r, s) < (i, j)$, then $|\{r, s\} \cap \{i, j\}| \leq 1$ and so for $(r, s) < (i, j)$ we have,

$$h_{ij} \cdot a_{rs} = \begin{cases} q^{-1} a_{rs}, & \text{if } |\{r, s\} \cap \{i, j\}| = 1; \\ a_{rs}, & \text{if } |\{r, s\} \cap \{i, j\}| = 0. \end{cases}$$

Inspection of the relations (3.1.1)-(3.1.6) yields that condition (d) is satisfied. □

Theorem 3.3.3. [4, Theorem II.5.12] *Let A and H be as in (II.5.1). Then all H-prime ideals of A are completely prime, and there are at most 2^n of them.*

Corollary 3.3.4. *When q is not a root of unity all H-prime ideals of $\mathcal{O}_q(Sk_n)$ are completely prime and there are at most 2^n of them.*

Proof. By Lemma 3.3.2 we may apply [4, Theorem II.5.12]. □

To deduce further results we require the following lemma.

Lemma 3.3.5. *The H-action on $\mathcal{O}_q(Sk_n)$ is rational.*

Proof. By [4, Definition II.2.6 and Exercise II.2.F], since our H is a torus it suffices to show that the H-action on $\mathcal{O}_q(Sk_n)$ is semisimple, that is $\mathcal{O}_q(Sk_n)$ is a direct sum of its H-eigenspaces, and that the H-eigenvalues induce rational characters. These are easy consequences of the definition of H and Proposition 3.2.2. □

Theorem 3.3.6. [4, Theorem II.6.9] *Let A and H be as in (II.5.1). Assume also that H is a K-torus and that the H-action on A is rational. There are scalars $\lambda_{ij} \in K^\times$ such that $\tau_i(x_j) = \lambda_{ij}x_j$ for all $i > j$. If the subgroup $\langle \lambda_{ij} \rangle \subseteq K^\times$ is torsionfree, then all prime ideals of A are completely prime.*

Corollary 3.3.7. *Suppose q is not a root of unity. Then all prime ideals of $\mathcal{O}_q(Sk_n)$ are completely prime.*

Proof. By Lemma 3.3.2 and Lemma 3.3.5 the conditions on H are satisfied. Now in our case $\langle \lambda_{ij} \rangle \subseteq \langle q \rangle$ so assuming q is not a root of unity $\langle \lambda_{ij} \rangle$ is torsionfree. So we may apply [4, Theorem II.6.9]. □

Theorem 3.3.8. [4, Theorem II.8.4] *Let A be a noetherian K-algebra, with K infinite, and let H be a K-torus acting rationally on A by K-algebra automorphisms. Assume that H-spec is finite, and that A satisfies the Nullstellensatz over K. Then*

$$\begin{aligned} primA &= \{\textit{locally closed prime ideals}\} \\ &= \{\textit{rational prime ideals}\} \\ &= \bigsqcup_{J \in H-specA} \{\textit{maximal elements of } spec_J A\}. \end{aligned}$$

Corollary 3.3.9. *Suppose that K is infinite and q is not a root of unity. Then*

$$\begin{aligned} prim\mathcal{O}_q(Sk_n) &= \{\textit{locally closed prime ideals}\} \\ &= \{\textit{rational prime ideals}\} \\ &= \bigsqcup_{J \in H-spec\mathcal{O}_q(Sk_n)} \{\textit{maximal elements of } spec_J \mathcal{O}_q(Sk_n)\}. \end{aligned}$$

Proof. Again by Lemma 3.3.2, Lemma 3.3.5, and Corollary 3.3.4 the conditions on H and H-spec are satisfied. By Corollary 3.2.6 and Corollary 3.2.8 $\mathcal{O}_q(Sk_n)$ is noetherian and satisfies the Nullstellsatz over K. So we may apply [4, Theorem II.8.4]. □

3.4 A Laplace Expansion of Pf_q

Recall the expansion of the classical Pfaffian given in Theorem 1.5.5. We now prove that an analogous Laplace expansion holds in the quantum case. However before we do we make the following remark.

Remark 3.4.1. *We note here a key point that will be used implicitly in the proof of many results to come. The presentation of $U_q(\mathfrak{gl}_n)$ that we gave in Section 3.1 required q to be a non-root of unity. It follows that whenever we use the $U_q(\mathfrak{gl}_n)$-action in our proofs we should also impose this condition. However, this will not always be necessary. As in the proof of [23, Corollary 1.1], when the relations we establish using the $U_q(\mathfrak{gl}_n)$-action have coefficients in $\mathbb{Z}[q, q^{-1}]$ we will be able to drop the restriction on q. Our proofs will show that the relations hold in $\mathcal{O}_q(Sk_n)$ over $\mathbb{Z}[q, q^{-1}]$ and there is a natural homomorphism from $\mathcal{O}_q(Sk_n)$ over $\mathbb{Z}[q, q^{-1}]$ to $\mathcal{O}_q(Sk_n)$ over K which preserves the established relations.*

Proposition 3.4.2. *For fixed $i, k = 1, ..., n$ we have,*

$$\delta_{ik} Pf_q = \sum_{j=1}^{n} (-q)^{\mu_{kj}} a_{ij} [1...\check{k}...\check{j}...n]$$

where

$$\mu_{kj} = \begin{cases} j-k-1 & \text{if } j > k \\ j-k & \text{if } j < k. \end{cases}$$

We note that, for simplicity, we are including some redundant terms in our sum (odd-length q-Pfaffians and elements of the form a_{ss} both being identically zero).

Proof. Fixing k, we proceed by induction on i. We first prove the case $i = 1$. Now if $k = i = 1$ then the proposition reduces to the definition of Pf_q since $\mu_{1j} = j - 2$ for $j > 1$. So we must prove that for $k > 1$,

$$\sum_{j=1}^{n} (-q)^{\mu_{kj}} a_{1j} [1...\check{k}...\check{j}...n] = 0.$$

Now, using Definition 3.1.2, with $(i_1, ..., i_{n-2}) = (1, .., \check{k}, .., \check{j}, .., n)$ gives,

$$\sum_{j=1}^{n} (-q)^{\mu_{kj}} a_{1j} [1...\check{k}...\check{j}...n] = \sum_{j=2}^{n} (-q)^{\mu_{kj}} a_{1j} \left(\sum_{r=2}^{n-2} (-q)^{r-2} a_{1i_r} [i_2...\check{i_r}...i_{n-2}] \right),$$

since q-Pfaffians of odd length are identically zero we can rewrite this as,

$$\sum_{j=1}^{n} (-q)^{\mu_{kj}} a_{1j} [1...\check{k}...\check{j}...n] = \sum_{\substack{j=2 \\ j \neq k}}^{n} \sum_{r=2}^{n-2} (-q)^{\mu_{kj}+r-2} a_{1j} a_{1i_r} [2...\check{k}...\check{j}...\check{i_r}...n].$$

An application of relation (3.1.1) gives,

$$\sum_{j=1}^{n} (-q)^{\mu_{kj}} a_{1j} [1...\check{k}...\check{j}...n]$$

$$= \sum_{\substack{j=2 \\ j \neq k}}^{n} \sum_{r=2}^{n-2} (-q)^{\mu_{kj}+r-2} q^{\beta_{jr}} a_{1min(j,i_r)} a_{1max(j,i_r)} [2...\check{k}...\check{j}...\check{i_r}...n]$$

where $\beta_{jr} = \begin{cases} 0 & \text{if } j < i_r \\ -1 & \text{if } j > i_r \end{cases}$. The sum on the RHS consists of an even number of terms of the form $a_{1s}a_{1t}[2...\check{k}...\check{s}...\check{t}...n]$ where $s < t$. For a given $s < t$, terms of the above form appear twice, since $(s,t) = (min(j,i_r), max(j,i_r))$ yields the two possibilities, $(s,t) = (j,i_r)$ and $(s,t) = (i_r,j)$. We prove that the sum is zero by showing that, for a fixed pair $s < t$, the coefficients of $a_{1s}a_{1t}[2...\check{k}...\check{s}...\check{t}...n]$ that occur add up to zero.

There are three general cases to consider depending on the relative ordering of k, s, t. First suppose $k < s < t$. We need to calculate the value of $(-q)^{\mu_{kj}+r-2}q^{\beta_{jr}}$ in the two cases $(s,t) = (j,i_r)$ and $(s,t) = (i_r,j)$. If $(s,t) = (j,i_r)$ then we have $k < j < i_r$. Given that $(i_1,...,i_{n-2}) = (1,..,\check{k},..,\check{j},..,n)$ we may see that in this case $r = t-2$. This fact, together with the definitions of μ_{ij} and β_{ij}, gives

$$\begin{aligned}(-q)^{\mu_{kj}+r-2}q^{\beta_{jr}} &= (-q)^{(s-k-1)+(t-2)-2}q^0 \\ &= (-q)^{s-k+t-5}.\end{aligned}$$

If $(s,t) = (i_r,j)$ then we have $k < i_r < j$ and so $r = s-1$. Hence,

$$\begin{aligned}(-q)^{\mu_{kj}+r-2}q^{\beta_{jr}} &= (-q)^{(t-k-1)+(s-1)-2}q^{-1} \\ &= -(-q)^{s-k+t-5}.\end{aligned}$$

So the two coefficients do indeed sum to zero.

Next suppose $s < k < t$. If $(s,t) = (j,i_r)$ then we have $j < k < i_r$ and so $r = t-2$. Hence,

$$\begin{aligned}(-q)^{\mu_{kj}+r-2}q^{\beta_{jr}} &= (-q)^{(s-k)+(t-2)-2}q^0 \\ &= (-q)^{s-k+t-4}.\end{aligned}$$

If $(s,t) = (i_r,j)$ then we have $i_r < k < j$ and so $r = s$. Hence,

$$\begin{aligned}(-q)^{\mu_{kj}+r-2}q^{\beta_{jr}} &= (-q)^{(t-k-1)+(s)-2}q^{-1} \\ &= -(-q)^{s-k+t-4}.\end{aligned}$$

So again the two coefficients sum to zero.

Finally, suppose $s < t < k$. If $(s,t) = (j,i_r)$ then we have $j < i_r < k$ and so $r = t-1$. Hence,

$$\begin{aligned}(-q)^{\mu_{kj}+r-2}q^{\beta_{jr}} &= (-q)^{(s-k)+(t-1)-2}q^0 \\ &= (-q)^{s-k+t-3}.\end{aligned}$$

If $(s,t) = (i_r, j)$ then we have $i_r < j < k$ and so $r = s$. Hence,

$$\begin{aligned}(-q)^{\mu_{kj}+r-2}q^{\beta_{jr}} &= (-q)^{(t-k)+(s)-2}q^{-1}\\ &= -(-q)^{s-k+t-3}.\end{aligned}$$

So the two coefficients sum to zero in all cases. Thus we have proved the base case in our induction. Next we turn to the inductive step. Assume that we know,

$$\sum_{j=1}^{n}(-q)^{\mu_{kj}}a_{cj}[1...\check{k}...\check{j}...n] = \delta_{ck}\mathrm{Pf}_q. \tag{3.4.1}$$

We wish to show that the same statement holds for $c+1$. Throughout the following argument we will use Lemma 3.1.5 and Remark 3.1.6. We recall (3.1.13) and act on both sides of (3.4.1) by F_c, yielding,

$$\begin{aligned}0 &= F_c\left(\sum_{j=1}^{n}(-q)^{\mu_{kj}}a_{cj}[1...\check{k}...\check{j}...n]\right)\\ &= \sum_{j=1}^{n}(-q)^{\mu_{kj}}\Big(F_c(a_{cj})L_c^{-1}L_{c+1}\left([1..\check{k}..\check{j}..n]\right) + a_{cj}F_c\left([1..\check{k}..\check{j}..n]\right)\Big). \end{aligned}\tag{3.4.2}$$

At this point we must split our argument into three cases, $k = c$, $k = c+1$ and $k \neq c, c+1$. First suppose $k = c$. Then we have,

$$\begin{aligned}0 &= \sum_{j=1}^{n}(-q)^{\mu_{cj}}\Big(F_c(a_{cj})L_c^{-1}L_{c+1}\left([1..\check{c}..\check{j}..n]\right) + a_{cj}F_c\left([1..\check{c}..\check{j}..n]\right)\Big)\\ &= \sum_{\substack{j=1\\ j\neq c,c+1}}^{n} (-q)^{\mu_{cj}}a_{c+1,j}\left(q[1..\check{c}..\check{j}..n]\right),\end{aligned}$$

dividing through by q, and noting that $a_{c+1,c+1} = 0$ and $[1..\check{c}..n] = 0$, allows us to deduce the required identity,

$$\sum_{j=1}^{n}(-q)^{\mu_{cj}}a_{c+1,j}[1..\check{c}..\check{j}..n] = 0. \tag{3.4.3}$$

Now suppose $k \neq c, c+1$. Then using Lemma 3.1.5 and Remark 3.1.6 we obtain the following from (3.4.2),

$$0 = \left(\sum_{\substack{j=1\\ j\neq c}}^{n}(-q)^{\mu_{kj}}a_{c+1,j}[1..\check{k}..\check{j}..n]\right) + (-q)^{\mu_{k,c+1}}a_{c,c+1}[1..\check{k}..\check{c}..n]$$

where the last term on the RHS comes from the fact that, when $k \neq c, c+1$, $F_c\left([1..\check{k}..\check{j}..n]\right) = [1..\check{k}..\check{c}..n]$ if $j = c+1$ and $F_c\left([1..\check{k}..\check{j}..n]\right) = 0$ otherwise. We rewrite $a_{c,c+1}$ and deduce that,

$$0 = \left(\sum_{\substack{j=1 \\ j \neq c}}^{n} (-q)^{\mu_{kj}} a_{c+1,j} [1..\check{k}..\check{j}..n] \right) + (-q)^{\mu_{k,c+1}} (-q)^{-1} a_{c+1,c} [1..\check{k}..\check{c}..n],$$

and since $\mu_{k,c+1} - 1 = \mu_{kc}$ this gives,

$$\sum_{j=1}^{n} (-q)^{\mu_{kj}} a_{c+1,j} [1..\check{k}..\check{j}..n] = 0 \quad \text{for } k \neq c, c+1. \tag{3.4.4}$$

Finally suppose $k = c+1$. Then (3.4.2) says,

$$0 = \sum_{j=1}^{n} (-q)^{\mu_{c+1,j}} \Big(F_c(a_{cj}) L_c^{-1} L_{c+1} \left([1.., c \check{+} 1, ..\check{j}..n] \right) + a_{cj} F_c \left([1.., c \check{+} 1, ..\check{j}..n] \right) \Big),$$

we note that $a_{cc} = 0 = [1..., c \check{+} 1, ..n]$ and apply Lemma 3.1.5 and Remark 3.1.6,

$$0 = \sum_{\substack{j=1 \\ j \neq c, c+1}}^{n} (-q)^{\mu_{c+1,j}} \Big(a_{c+1,j} \left(q^{-1} [1.., c \check{+} 1, ..\check{j}..n] \right) + a_{cj} [1..\check{c}..\check{j}..n] \Big),$$

multiplying through by q and rearranging gives,

$$\begin{aligned} (-q) \sum_{\substack{j=1 \\ j \neq c, c+1}}^{n} (-q)^{\mu_{c+1,j}} a_{cj} [1..\check{c}..\check{j}..n] &= \sum_{\substack{j=1 \\ j \neq c, c+1}}^{n} (-q)^{\mu_{c+1,j}} a_{c+1,j} [1.., c \check{+} 1, ..\check{j}..n] \\ \sum_{\substack{j=1 \\ j \neq c, c+1}}^{n} (-q)^{\mu_{c+1,j}+1} a_{cj} [1..\check{c}..\check{j}..n] &= \sum_{\substack{j=1 \\ j \neq c, c+1}}^{n} (-q)^{\mu_{c+1,j}} a_{c+1,j} [1.., c \check{+} 1, ..\check{j}..n]. \end{aligned}$$

Now if $j \neq c+1$, $\mu_{c+1,j} + 1 = \mu_{cj}$, so

$$\sum_{\substack{j=1 \\ j \neq c+1}}^{n} (-q)^{\mu_{c,j}} a_{cj} [1..\check{c}..\check{j}..n] = \sum_{\substack{j=1 \\ j \neq c}}^{n} (-q)^{\mu_{c+1,j}} a_{c+1,j} [1.., c \check{+} 1, ..\check{j}..n]. \tag{3.4.5}$$

We now observe that since, $a_{c,c+1} = (-q)^{-1} a_{c+1,c}$, it follows that,

$$\begin{aligned} a_{c,c+1} [1..\check{c}, c \check{+} 1, ..n] &= (-q)^{-1} a_{c+1,c} [1..\check{c}, c \check{+} 1, ..n] \\ (-q)^{\mu_{c,c+1}} a_{c,c+1} [1..\check{c}, c \check{+} 1, ..n] &= (-q)^{\mu_{c,c+1}-1} a_{c+1,c} [1..\check{c}, c \check{+} 1, ..n], \end{aligned}$$

and since $\mu_{c,c+1} - 1 = 0 - 1 = -1 = c - (c+1) = \mu_{c+1,c}$ we have,

$$(-q)^{\mu_{c,c+1}} a_{c,c+1} [1..\check{c}, c \check{+} 1, ..n] = (-q)^{\mu_{c+1,c}} a_{c+1,c} [1..\check{c}, c \check{+} 1, ..n]. \tag{3.4.6}$$

Adding (3.4.5) and (3.4.6) gives us,

$$\sum_{j=1}^{n} (-q)^{\mu_{c+1,j}} a_{c+1,j} [1..(c \check{+} 1)..\check{j}..n] = \sum_{j=1}^{n} (-q)^{\mu_{c,j}} a_{cj} [1..\check{c}..\check{j}..n].$$

By our inductive hypothesis, (3.4.2), this says,

$$\sum_{j=1}^{n}(-q)^{\mu_{c+1,j}}a_{c+1,j}[1..(c\check{+}1)..\check{j}..n] = \mathrm{Pf}_q. \tag{3.4.7}$$

We are now finished, for (3.4.7), (3.4.4), and (3.4.3) together give,

$$\sum_{j=1}^{n}(-q)^{\mu_{kj}}a_{c+1,j}[1..\check{k}..\check{j}..n] = \delta_{k,c+1}\mathrm{Pf}_q$$

thus completing the inductive step. Keeping in mind Remark 3.4.1 we see that our proof is complete. □

The proof of the above proposition only relies upon the relative ordering of the numbers $1, ..., n$. So we have in-fact proved the following more general result:

Corollary 3.4.3. *Let $1 \le i_1 < i_2 < \cdots < i_{2h} \le n$ for some $1 \le h \le \lfloor n/2 \rfloor$. For fixed $r, t = 1, ..., 2h$,*

$$\delta_{rt}[i_1 i_2 ... i_{2h}] = \sum_{s=1}^{2h}(-q)^{\mu_{ts}}a_{i_r i_s}[i_1..\check{i_t}..\check{i_s}..i_{2h}].$$

3.5 Pf_q is Central

From now on we will use the notations a_{ij} and $[ij]$ for the generators of $\mathcal{O}_q(Sk_n)$ interchangeably. We thus also view the generators as length-2 q-Pfaffians. However we should make clear that although we will allow ourselves to write $[ji]$ when $j > i$, for q-Pfaffians of length greater than 2, writing $[i_1 \cdots i_{2h}]$ will always imply that $i_1 < \cdots < i_{2h}$.

Recall that in the case of standard quantum matrices, $\mathcal{O}_q(M_n)$, we have a distinguished element, namely the quantum determinant, det_q, that is well-known to be central. Since, in the world of skew-symmetric matrices, we can think of Pfaffians "playing a similar role" as the one determinants do in the world of general matrices, it is natural to expect that we would also have a centrality result for q-Pfaffians. We shall now proceed to prove that Pf_q is indeed central in $\mathcal{O}_q(Sk_n)$, a particularly nice property for an element of a noncommutative algebra to have.

Before we prove the main result of this section we require two Lemmas giving some specific q-Pfaffian commutation relations.

Lemma 3.5.1. *For $2 < l_1 < ... < l_{2m} \le n$,*

$$[12][l_1...l_{2m}] = [l_1...l_{2m}][12] + \hat{q}\sum_{r=1}^{2m}(-q)^{r-1}[1l_r][2l_1...\check{l_r}...l_{2m}].$$

Proof. We proceed by induction on m, the base case being the relation (3.1.6) of the algebra. Now, expanding $[l_1...l_{2m}]$ using Definition 3.1.2 gives,

$$[12][l_1...l_{2m}] = \sum_{r=2}^{2m}(-q)^{r-2}[12][l_1l_r][l_2..\check{l_r}..l_{2m}],$$

applying relation (3.1.6) we have,

$$[12][l_1...l_{2m}] = \sum_{r=2}^{2m}(-q)^{r-2}\left([l_1l_r][12] + \hat{q}[1l_1][2l_r] - q\hat{q}[1l_r][2l_1]\right)[l_2..\check{l_r}..l_{2m}],$$

setting $(u_1, ..., u_{2m-2}) = (l_2, .., \check{l_r}, .., l_{2m})$ and applying the inductive hypothesis to $[12][u_1...u_{2m}]$ gives,

$$\begin{aligned}&[12][l_1...l_{2m}] = \\ &\sum_{r=2}^{2m}(-q)^{r-2}\Bigg([l_1l_r]\Big([l_2..\check{l_r}..l_{2m}][12] + \hat{q}\sum_{k=1}^{2m-2}(-q)^{k-1}[1u_k][2u_1..\check{u_k}..u_{2m-2}]\Big) \\ &\qquad + \hat{q}[1l_1][2l_r][l_2..\check{l_r}..l_{2m}] - q\hat{q}[1l_r][2l_1][l_2..\check{l_r}..l_{2m}]\Bigg),\end{aligned}$$

and so,

$$\begin{aligned}[12][l_1...l_{2m}] = &\left(\sum_{r=2}^{2m}(-q)^{r-2}[l_1l_r][l_2..\check{l_r}..l_{2m}]\right)[12] \\ &+ \hat{q}\sum_{r=2}^{2m}\sum_{k=1}^{2m-2}(-q)^{r+k-3}[l_1l_r][1u_k][2u_1..\check{u_k}..u_{2m-2}] \\ &+ \hat{q}[1l_1]\left(\sum_{r=2}^{2m}(-q)^{r-2}[2l_r][l_2..\check{l_r}..l_{2m}]\right) + \hat{q}\sum_{r=2}^{2m}(-q)^{r-1}[1l_r][2l_1][l_2..\check{l_r}..l_{2m}],\end{aligned}$$

two applications of Definition 3.1.2 gives,

$$\begin{aligned}[12][l_1...l_{2m}] = [l_1...l_m][12] &+ \hat{q}\sum_{r=2}^{2m}\sum_{k=1}^{2m-2}(-q)^{r+k-3}[l_1l_r][1u_k][2u_1..\check{u_k}..u_{2m-2}] \\ &\hat{q}[1l_1][2l_2...l_{2m}] + \hat{q}\sum_{r=2}^{2m}(-q)^{r-1}[1l_r][2l_1][l_2..\check{l_r}..l_m].\end{aligned}$$

So, for our proof to be complete, it remains to be shown that,

$$\begin{aligned}\sum_{r=2}^{2m}\sum_{k=1}^{2m-2}(-q)^{r+k-3}[l_1l_r][1u_k][2u_1..\check{u_k}..u_{2m-2}] + \sum_{r=2}^{2m}(-q)^{r-1}[1l_r][2l_1][l_2..\check{l_r}..l_m] \\ = \sum_{r=2}^{2m}(-q)^{r-1}[1l_r][2l_1..\check{l_r}..l_{2m}]. \quad (3.5.1)\end{aligned}$$

We consider the RHS of this equation. Writing $(t_1, ..., t_{2m}) = (2, l_1, ...\check{l_r}, .., l_{2m})$ and using Definition 3.1.2 enables us to write the RHS as follows,

$$\sum_{r=2}^{2m}(-q)^{r-1}[1l_r][2l_1..\check{l_r}..l_{2m}] = \sum_{r=2}^{2m}(-q)^{r-1}[1l_r]\left(\sum_{k=2}^{2m}(-q)^{k-2}[2t_k][t_2..\check{t_k}..t_{2m}]\right)$$
$$= \sum_{r=2}^{2m}\sum_{k=2}^{2m}(-q)^{r+k-3}[1l_r][2t_k][t_2..\check{t_k}..t_{2m}],$$

noting that $t_2 = l_1$, we may rewrite this as,

$$\sum_{r=2}^{2m}(-q)^{r-1}[1l_r][2l_1..\check{l_r}..l_{2m}] = \sum_{r=2}^{2m}(-q)^{r-1}[1l_r][2l_1][l_2..\check{l_r}..l_{2m}]$$
$$+\sum_{r=2}^{2m}\sum_{k=3}^{2m}(-q)^{r+k-3}[1l_r][2t_k][t_2..\check{t_k}..t_{2m}].$$

So to prove (3.5.1) it suffices to show,

$$\sum_{r=2}^{2m}\sum_{k=1}^{2m-2}(-q)^{r+k-3}[l_1l_r][1u_k][2u_1..\check{u_k}..u_{2m-2}] = \sum_{r=2}^{2m}\sum_{k=3}^{2m}(-q)^{r+k-3}[1l_r][2t_k][t_2..\check{t_k}..t_{2m}].$$

Expanding the $(2m-2)$-length q-Pfaffians on both sides, using Definition 3.1.2, gives us, as our target,

$$\sum_{r=2}^{2m}\sum_{k=1}^{2m-2}\sum_{s=2}^{2m-2}(-q)^{r+k+s-5}[l_1l_r][1u_k][2w_s][w_2..\check{w_s}..w_{2m-2}]$$
$$= \sum_{r=2}^{2m}\sum_{k=3}^{2m}\sum_{s=2}^{2m-2}(-q)^{r+k+s-5}[1l_r][2t_k][l_1v_s][v_2..\check{v_s}..v_{2m-2}] \quad (3.5.2)$$

where $(w_1, ..., w_{2m-2}) = (2, u_1, .., \check{u_k}, .., u_{2m-2})$ and $(v_1, ..., v_{2m-2}) = (t_2, .., \check{t_k}, ..t_{2m})$. Now the RHS of this equation is composed of terms of the form,

$$[1l_a][2l_b][l_1l_c][l_2..\check{l_a}..\check{l_b}..\check{l_c}..l_{2m}] \quad (3.5.3)$$

with the numbers a, b, c running through all the possible triples of distinct elements from the set $\{2, ..., 2m\}$. We will show that this is also true of the LHS of (3.5.2). To prove the equality we will then show that, for a given triple (a, b, c), the coefficients of the respective terms appearing on either side of the equation are the same.

Our first step is to rearrange $[l_1l_r][1u_k][2w_s]$. This depends on the relative ordering of l_r, u_k, w_s so we must split this into the six appropriate cases:

(i) $\underline{u_k < w_s < l_r :}$
By (3.1.5),and then an application of (3.1.5) and (3.1.4) we have,

$$\begin{aligned}[l_1l_r][1u_k][2w_s] &= \left([1u_k][l_1l_r] - \hat{q}[1l_r][l_1u_k]\right)[2w_s] \\ &= [1u_k][2w_s][l_1l_r] - \hat{q}[1u_k][2l_r][l_1w_s] - \hat{q}[1l_r][2w_s][l_1u_k].\end{aligned}$$

(ii) $\underline{w_s < u_k < l_r :}$
By (3.1.5), and then (3.1.5) applied twice we have,

$$\begin{aligned}[l_1l_r][1u_k][2w_s] &= \left([1u_k][l_1l_r] - \hat{q}[1l_r][l_1u_k]\right)[2w_s] \\ &= [1u_k][2w_s][l_1l_r] - \hat{q}[1u_k][2l_r][l_1w_s] \\ &\qquad - \hat{q}[1l_r][2w_s][l_1u_k] + \hat{q}^2[1l_r][2u_k][l_1w_s].\end{aligned}$$

(iii) $\underline{u_k < l_r < w_s :}$
By (3.1.5), and then (3.1.4) applied twice we have,

$$\begin{aligned}[l_1l_r][1u_k][2w_s] &= \left([1u_k][l_1l_r] - \hat{q}[1l_r][l_1u_k]\right)[2w_s] \\ &= [1u_k][2w_s][l_1l_r] - \hat{q}[1l_r][2w_s][l_1u_k].\end{aligned}$$

(iv) $\underline{l_r < w_s < u_k :}$
Two applications of (3.1.4) give,

$$\begin{aligned}[l_1l_r][1u_k][2w_s] &= [1u_k][l_1l_r][2w_s] \\ &= [1u_k][2w_s][l_1l_r].\end{aligned}$$

(v) $\underline{w_s < l_r < u_k :}$
By (3.1.4) and then (3.1.5) we have,

$$\begin{aligned}[l_1l_r][1u_k][2w_s] &= [1u_k][l_1l_r][2w_s] \\ &= [1u_k][2w_s][l_1l_r] - \hat{q}[1u_k][2l_r][l_1w_s].\end{aligned}$$

(vi) $\underline{l_r < u_k < w_s :}$
Applying (3.1.4) twice gives,

$$\begin{aligned}[l_1l_r][1u_k][2w_s] &= [1u_k][l_1l_r][2w_s] \\ &= [1u_k][2w_s][l_1l_r].\end{aligned}$$

Now for $r = 2, ..., 2m$, $k = 1, ..., 2m-2$, and $s = 2, ..., 2m-2$, (l_r, u_k, w_s) runs through all possible triples of distinct numbers from the set $\{l_2, ..., l_{2m}\}$. So from $(i) - (vi)$ it follows that the LHS of (3.5.2) is composed of terms of the required type. It remains to show that, for a given triple (a, b, c), the coefficients of terms of the form (3.5.3) are the same on either side of the equation. We begin by calculating the coefficients of terms on the LHS, by examining the cases $(i)-(vi)$ above and seeing when a nonzero term of the required form arises. Again this depends on the relative ordering of l_a, l_b, l_c so we split this up into six cases:

(1) $\underline{l_a < l_b < l_c}$

The only nonzero term is in (i) when $(l_a, l_b, l_c) = (u_k, w_s, l_r)$. We wish to calculate the value of the coefficient which is, in this case, $(-q)^{r+k+s-5}$. So we must express r, k, s in terms of a, b, c. Let us recall how u_k, w_s, l_r are defined. We start with the ordered list $(l_2, ..., l_{2m})$. We remove l_r and relabel the new list $(u_1, ..., u_{2m-2})$. We then remove u_k and relabel the new list $(w_2, ..., w_{2m-2})$. And finally we remove w_s. So the values of k and s that give $u_k = l_a$ and $w_s = l_b$ will depend on whether the elements removed from the lists before the respective relabelings came before or after u_k (resp. w_s) in the ordering. In the case we are in the relative ordering of l_r, u_k, w_s is $u_k < w_s < l_r$ giving $(r, k, s) = (c, a-1, b-1)$. So the coefficient of the term in question is

$$(-q)^{r+k+s-5} = (-q)^{a+b+c-7}.$$

(2) $\underline{l_b < l_a < l_c}$

The only nonzero term is in (ii) when $(l_a, l_b, l_c) = (u_k, w_s, l_r)$. The relative ordering of l_r, u_k, w_s is $w_s < u_k < l_r$ giving $(r, k, s) = (c, a-1, b)$. So the coefficient is

$$(-q)^{r+k+s-5} = (-q)^{a+b+c-6}.$$

(3) $\underline{l_b < l_c < l_a}$

In this case there are two nonzero terms. The first occurs in (ii) when $(l_a, l_b, l_c) = (l_r, w_s, u_k)$ with coefficient $-\hat{q}(-q)^{r+k+s-5}$; the relative ordering in this case is $w_s < u_k < l_r$ giving $(r, k, s) = (a, c-1, b)$. The second occurs in (v) when $(l_a, l_b, l_c) = (u_k, w_s, l_r)$ with coefficient $(-q)^{r+k+s-5}$; the relative ordering in this case is $w_s < l_r < u_k$ giving $(r, k, s) = (c, a-2, b)$. So the combined coefficient is

$$(-q)^{a+b+c-7}(1 + q\hat{q}) = (-q)^{a+b+c-5}.$$

(4) $\underline{l_a < l_c < l_b}$

Again in this case there are two nonzero terms. The first occurs in (i) when $(l_a, l_b, l_c) = (u_k, l_r, w_s)$ with coefficient $-\hat{q}(-q)^{r+k+s-5}$; the relative ordering in this case is $u_k < w_s < l_r$ giving $(r, k, s) = (b, a-1, c-1)$. The second occurs in (iii) when $(l_a, l_b, l_c) = (u_k, w_s, l_r)$ with coefficient $(-q)^{r+k+s-5}$; the relative ordering in this case is $u_k < l_r < w_s$ giving $(r, k, s) = (c, a-1, b-2)$. So the combined coefficient is

$$(-q)^{a+b+c-8}(q\hat{q} + 1) = (-q)^{a+b+c-6}.$$

(5) $\underline{l_c < l_a < l_b}$

In this case there are three nonzero terms. The first occurs in (ii) when $(l_a, l_b, l_c) = (u_k, l_r, w_s)$ with coefficient $-\hat{q}(-q)^{r+k+s-5}$; the relative ordering in this case is $w_s < u_k < l_r$ giving $(r, k, s) = (b, a-1, c)$. The second occurs in (iii) when $(l_a, l_b, l_c) = (l_r, w_s, u_k)$ with coefficient $-\hat{q}(-q)^{r+k+s-5}$; the relative ordering in this case is $u_k < l_r < w_s$ giving $(r, k, s) = (a, c-1, b-2)$. The third occurs in (vi) when $(l_a, l_b, l_c) = (u_k, w_s, l_r)$ with coefficient $(-q)^{r+k+s-5}$; the relative ordering in this case is $l_r < u_k < w_s$ giving $(r, k, s) = (c, a-2, b-2)$. So the combined coefficient is

$$(-q)^{a+b+c-9}(q^3\hat{q} + q\hat{q} + 1) = (-q)^{a+b+c-5}.$$

(6) $\underline{l_c < l_b < l_a}$

In this last case there are four nonzero terms. The first occurs in (i) when $(l_a, l_b, l_c) = (l_r, w_s, u_k)$ with coefficient $-\hat{q}(-q)^{r+k+s-5}$; the relative ordering in this case is $u_k < w_s < l_r$ giving $(r, k, s) = (a, c-1, b-1)$. The second occurs in (ii) when $(l_a, l_b, l_c) = (l_r, u_k, w_s)$ with coefficient $\hat{q}^2(-q)^{r+k+s-5}$; the relative ordering in this case is $w_s < u_k < l_r$ giving $(r, k, s) = (a, b-1, c)$. The third occurs in (iv) when $(l_a, l_b, l_c) = (u_k, w_s, l_r)$ with coefficient $(-q)^{r+k+s-5}$; the relative ordering in this case is $l_r < w_s < u_k$ giving $(r, k, s) = (c, a-2, b-1)$. The last occurs in (v) when $(l_a, l_b, l_c) = (u_k, l_r, w_s)$ with coefficient $-\hat{q}(-q)^{r+k+s-5}$; the relative ordering in this case is $w_s < l_r < u_k$ giving $(r, k, s) = (b, a-2, c)$. So the combined coefficient is

$$(-q)^{a+b+c-8}(q\hat{q} + q^2\hat{q}^2 + 1 + q\hat{q}) = (-q)^{a+b+c-4}.$$

Our final task is to show that in cases (1) – (6) the coefficients that we have calculated on the LHS match those on the RHS of (3.5.2). Recall that the RHS is

$$\sum_{r=2}^{2m}\sum_{k=3}^{2m}\sum_{s=2}^{2m-2}(-q)^{r+k+s-5}[1l_r][2t_k][l_1v_s][v_2..\check{v_s}..v_{2m-2}]$$

where $(t_3, ..., t_{2m}) = (l_2, ...\check{l_r}, .., l_{2m})$ and $(v_2, ..., v_{2m-2}) = (t_3, .., \check{t_k}, ..t_{2m})$. For a given a, b, c the coefficient of the term of the form (3.5.3) is $(-q)^{r+k+s-5}$ where $(l_r, t_k, v_s) = (l_a, l_b, l_c)$. In case (1) the relative ordering of l_r, t_k, v_s is $l_r < t_k < v_s$ so $(r, k, s) = (a, b, c-2)$ and so the coefficient is $(-q)^{a+b+c-7}$ as required. In case (2) the relative ordering is $t_k < l_r < v_s$ so $(r, k, s) = (a, b+1, c-2)$ and so the coefficient is $(-q)^{a+b+c-6}$ as required. In case (3) the relative ordering is $t_k < v_s < l_r$ so $(r, k, s) = (a, b+1, c-1)$ and so the coefficient is $(-q)^{a+b+c-5}$ as required. In case (4) the relative ordering is $l_r < v_s < t_k$ so $(r, k, s) = (a, b, c-1)$ and so the coefficient is $(-q)^{a+b+c-6}$ as required. In case (5) the relative ordering is

$v_s < l_r < t_k$ so $(r,k,s) = (a,b,c)$ and so the coefficient is $(-q)^{a+b+c-5}$ as required. Finally, in case (6) the relative ordering is $v_s < t_k < l_r$ so $(r,k,s) = (a,b+1,c)$ and so the coefficient is $(-q)^{a+b+c-4}$ as required. □

Lemma 3.5.2. *For $2 < l_1 < \ldots < l_{2m-1} \leq n$,*

$$[12][2l_1 \ldots l_{2m-1}] = q[2l_1 \ldots l_{2m-1}][12].$$

Proof. Now, using Definition 3.1.2 and then relation (3.1.2) we have,

$$\begin{aligned}[12][2l_1 \ldots l_{2m-1}] &= [12]\left(\sum_{k=1}^{2m-1}(-q)^{k-1}[2l_k][l_1..\check{l_k}..l_{2m-1}]\right)\\ &= \sum_{k=1}^{2m-1}(-q)^{k-1}q[2l_k][12][l_1..\check{l_k}..l_{2m-1}].\end{aligned}$$

Applying Lemma 3.5.1 while setting $(j_1, \ldots, j_{2m-2}) = (l_1, .., \check{l_k}, .., l_{2m-1})$ gives,

$$[12][2l_1 \ldots l_{2m-1}] = \\ \sum_{k=1}^{2m-1}(-q)^{k-1}q[2l_k]\left([l_1..\check{l_k}..l_{2m-1}][12] + \hat{q}\sum_{r=1}^{2m-2}(-q)^{r-1}[1j_r][2j_1..\check{j_r}..j_{2m-2}]\right),$$

and by Definition 3.1.2 we can deduce

$$[12][2l_1 \ldots l_{2m-1}] = q[2l_1 \ldots l_{2m-1}][12] + q\hat{q}\sum_{k=1}^{2m-1}\sum_{r=1}^{2m-2}(-q)^{k+r-2}[2l_k][1j_r][2j_1..\check{j_r}..j_{2m-2}].$$

So it suffices to show that

$$\sum_{k=1}^{2m-1}\sum_{r=1}^{2m-2}(-q)^{k+r}[2l_k][1j_r][2j_1..\check{j_r}..j_{2m-2}] = 0. \tag{3.5.4}$$

Now, by Definition 3.1.2,

$$\begin{aligned}&\sum_{k=1}^{2m-1}\sum_{r=1}^{2m-2}(-q)^{k+r}[2l_k][1j_r][2j_1..\check{j_r}..j_{2m-2}]\\ &\qquad = \sum_{k=1}^{2m-1}\sum_{r=1}^{2m-2}\sum_{t=2}^{2m-2}(-q)^{k+r+t-2}[2l_k][1j_r][2w_t][w_2..\check{w_t}..w_{2m-2}]\end{aligned} \tag{3.5.5}$$

where $(w_1, \ldots, w_{2m-2}) = (2, j_1, .., \check{j_r}, .., j_{2m-2})$. We now express $[2l_k][1j_r][2w_t]$ in terms of ordered monomials. This depends on the relative ordering of l_k, j_r, w_t so we split this into the appropriate cases:

(*a*) $\underline{l_k < j_r < w_t:}$
By (3.1.4),

$$[2l_k][1j_r][2w_t] = [1j_r][2l_k][2w_t].$$

(b) $\underline{j_r < l_k < w_t:}$
By (3.1.5),

$$\begin{aligned}[2l_k][1j_r][2w_t] &= ([1j_r][2l_k] - \hat{q}[1l_k][2j_r])\,[2w_t] \\ &= [1j_r][2l_k][2w_t] - \hat{q}[1l_k][2j_r][2w_t].\end{aligned}$$

(c) $\underline{j_r < w_t < l_k:}$
By (3.1.5), and then (3.1.1) we have,

$$\begin{aligned}[2l_k][1j_r][2w_t] &= ([1j_r][2l_k] - \hat{q}[1l_k][2j_r])[2w_t] \\ &= q^{-1}[1j_r][2w_t][2l_k] - \hat{q}[1l_k][2j_r][2w_t].\end{aligned}$$

(d) $\underline{l_k < w_t < j_r:}$
By (3.1.4),

$$[2l_k][1j_r][2w_t] = [1j_r][2l_k][2w_t].$$

(e) $\underline{w_t < l_k < j_r:}$
By (3.1.4), and then (3.1.1) we have,

$$\begin{aligned}[2l_k][1j_r][2w_t] &= [1j_r][2l_k][2w_t] \\ &= q^{-1}[1j_r][2w_t][2l_k].\end{aligned}$$

(f) $\underline{w_t < j_r < l_k:}$
By (3.1.5), and then (3.1.1) applied twice we have,

$$\begin{aligned}[2l_k][1j_r][2w_t] &= ([1j_r][2l_k] - \hat{q}[1l_k][2j_r])[2w_t] \\ &= q^{-1}[1j_r][2w_t][2l_k] - q^{-1}\hat{q}[1l_k][2w_t][2j_r].\end{aligned}$$

So we may express the RHS of (3.5.5) as a sum of terms of the form

$$[1l_a][2l_b][2l_c][l_1..\check{l_a}..\check{l_b}..\check{l_c}..l_{2m-1}] \qquad \text{with } l_b < l_c.$$

We shall show that, for a given a, b, c, the coefficients of these terms are in-fact zero, thus proving (3.5.4). We proceed in a manner similar to the proof of (3.5.2) in Lemma 3.5.1 and split this into cases:

$\underline{l_a < l_b < l_c}$
From $(a)-(f)$ we have a contributing term in case (b) with coefficient $(-q)^{k+r+t-2}$, when $(l_a, l_b, l_c) = (j_r, l_k, w_t)$. Noting that j_r is chosen from $(j_1, ..., j_{2m-2}) = (l_1, .., \check{l_k}, .., l_{2m-1})$ and w_t is chosen from $(w_2, ..., w_{2m-2}) = (j_1, .., \check{j_r}, .., j_{2m-2})$ it follows that $(k, r, t) = (b, a, c-1)$. There is also a contributing term in case

(c) with coefficient $q^{-1}(-q)^{k+r+t-2}$, when $(l_a, l_b, l_c) = (j_r, w_t, l_k)$. In this case $(k, r, t) = (c, a, b)$. So the combined coefficient is,

$$(-q)^{a+b+c-3} + q^{-1}(-q)^{a+b+c-2} = 0.$$

$\underline{l_b < l_a < l_c}$

We have a contributing term in case (a) with coefficient $(-q)^{k+r+t-2}$, when $(l_a, l_b, l_c) = (j_r, l_k, w_t)$, and so $(k, r, t) = (b, a-1, c-1)$. There is a contributing term in case (b) with coefficient $-\hat{q}(-q)^{k+r+t-2}$, with $(l_a, l_b, l_c) = (l_k, j_r, w_t)$, and hence $(k, r, t) = (a, b, c-1)$. Also there is a contributing term in case (f) with coefficient $q^{-1}(-q)^{k+r+t-2}$, with $(l_a, l_b, l_c) = (j_r, w_t, l_k)$, and hence $(k, r, t) = (c, a, b+1)$. The combined coefficient is,

$$(-q)^{a+b+c-4}(1 + q\hat{q} - q^2) = 0.$$

$\underline{l_b < l_c < l_a}$

We have a contributing term in case (c) with coefficient $-\hat{q}(-q)^{k+r+t-2}$, when $(l_a, l_b, l_c) = (l_k, j_r, w_t)$, and so $(k, r, t) = (a, b, c)$. There is a contributing term in case (d) with coefficient $(-q)^{k+r+t-2}$, with $(l_a, l_b, l_c) = (j_r, l_k, w_t)$, and hence $(k, r, t) = (b, a-1, c)$. Also there is a contributing term in case (e) with coefficient $q^{-1}(-q)^{k+r+t-2}$, with $(l_a, l_b, l_c) = (j_r, w_t, l_k)$, and hence $(k, r, t) = (c, a-1, b+1)$. Finally, there is a contributing term in case (f) with coefficient $-q^{-1}\hat{q}(-q)^{k+r+t-2}$, with $(l_a, l_b, l_c) = (l_k, w_t, j_r)$, and so $(k, r, t) = (a, c, b+1)$. The combined coefficient is,

$$(-q)^{a+b+c-3}(q\hat{q} + 1 - 1 - q\hat{q}) = 0.$$

□

We are now in a position to prove the main result of this section,

Proposition 3.5.3. *Pf_q is central in $\mathcal{O}_q(Sk_n)$.*

Proof. Now, by definition,

$$\begin{aligned} a_{12}\mathrm{Pf}_q &= \sum_{r=2}^{n} (-q)^{r-2} a_{12} a_{1r} [2..\check{r}..n] \\ &= a_{12} a_{12} [3...n] + \sum_{r=3}^{n} (-q)^{r-2} a_{12} a_{1r} [2..\check{r}..n], \end{aligned}$$

applying Lemma 3.5.1 gives,

$$a_{12}\mathrm{Pf}_q = a_{12} \left([3...n] a_{12} + \hat{q} \sum_{r=3}^{n} (-q)^{r-3} a_{1r} [2..\check{r}..n] \right) + \sum_{r=3}^{n} (-q)^{r-2} a_{12} a_{1r} [2..\check{r}..n],$$

and hence,

$$\begin{aligned}a_{12}\mathrm{Pf}_q &= a_{12}[3...n]a_{12}+\\ &\qquad (q-q^{-1})\sum_{r=3}^{n}(-q)^{r-3}a_{12}a_{1r}[2..\check{r}..n]+\sum_{r=3}^{n}(-q)^{r-2}a_{12}a_{1r}[2..\check{r}..n]\\ &= a_{12}[3...n]a_{12}+(-q^{-1})\sum_{r=3}^{n}(-q)^{r-3}a_{12}a_{1r}[2..\check{r}..n],\end{aligned}$$

using (3.1.1) we have,

$$a_{12}\mathrm{Pf}_q = a_{12}[3...n]a_{12}+(-q^{-1})\sum_{r=3}^{n}(-q)^{r-3}qa_{1r}a_{12}[2..\check{r}..n].$$

Applying Lemma 3.5.2 to $a_{12}[2..\check{r}..n]$ gives,

$$a_{12}\mathrm{Pf}_q = a_{12}[3...n]a_{12}+(-q^{-1})\sum_{r=3}^{n}(-q)^{r-3}q^2a_{1r}[2..\check{r}..n]a_{12},$$

and so,

$$\begin{aligned}a_{12}\mathrm{Pf}_q &= \left(a_{12}[3...n]+\sum_{r=3}^{n}(-q)^{r-2}a_{1r}[2..\check{r}..n]\right)a_{12}\\ &= \mathrm{Pf}_q a_{12}.\end{aligned}$$

Now by Lemma 3.1.5, $F_s(\mathrm{Pf}_q)=0$, and since $\Delta(F_s)=F_s\otimes L_s^{-1}L_{s+1}+1\otimes F_s$, it follows that,

$$\begin{aligned}F_s(a_{ij}\mathrm{Pf}_q) &= F_s(a_{ij})L_s^{-1}L_{s+1}(\mathrm{Pf}_q)\\ &= F_s(a_{ij})\mathrm{Pf}_q.\end{aligned}$$

Similarly, $F_s(\mathrm{Pf}_q a_{ij})=\mathrm{Pf}_q F_s(a_{ij})$, so,

$$F_s(a_{ij}\mathrm{Pf}_q-\mathrm{Pf}_q a_{ij})=F_s(a_{ij})\mathrm{Pf}_q-\mathrm{Pf}_q F_s(a_{ij}).$$

We have shown, $a_{12}\mathrm{Pf}_q-\mathrm{Pf}_q a_{12}=0$. Hence,

$$\begin{aligned}F_2(a_{12}\mathrm{Pf}_q-\mathrm{Pf}_q a_{12})&=0\\ F_2(a_{12})\mathrm{Pf}_q-\mathrm{Pf}_q F_2(a_{12})&=0\\ a_{13}\mathrm{Pf}_q-\mathrm{Pf}_q a_{13}&=0.\end{aligned}$$

A trivial induction using the action of the F_k gives,

$$a_{1j}\mathrm{Pf}_q-\mathrm{Pf}_q a_{1j}=0,\quad \text{for } j=2,...,n.$$

For fixed j we may then proceed in a similar manner to prove

$$a_{ij}\mathrm{Pf}_q-\mathrm{Pf}_q a_{ij}=0,\quad \text{for } 1\le i<j\le n.$$

Keeping in mind Remark 3.4.1, the result follows. □

We end this section with the observation that the centrality of the q-Pfaffian enables us to employ another result of [4].

Recall [4, Lemma II.9.11]:

Lemma 3.5.4. *Let A be a noetherian, Auslander-regular, Cohen-Macaulay K-algebra.*
(a) If $z \in A$ is a central regular element, then $A/\langle z \rangle$ is Auslander-Gorenstein and Cohen-Macaulay.
(b) Assume that $A = \bigoplus_{i=0}^{\infty} A_i$ is a connected graded K-algebra. If $c \in A$ is a regular normal element such that $cA_i = A_i c$ for all i, then $A[c^{-1}]$ is Auslander-regular and Cohen-Macaulay.

Corollary 3.5.5. *$\mathcal{O}_q(Sk_n)/\langle Pf_q \rangle$ is Auslander-Gorenstein and Cohen-Macaulay and $\mathcal{O}_q(Sk_n)[Pf_q^{-1}]$ is Auslander-regular and Cohen-Macaulay.*

Proof. We saw at the end of Section 3.2 that $\mathcal{O}_q(Sk_n)$ is noetherian, Auslander-regular, and Cohen-Macaulay. Given that Pf_q is central and that $\mathcal{O}_q(Sk_n)$ is a domain (Proposition 3.2.3), the conditions of the previous lemma are all trivially met in our two cases. □

3.6 A Link with $G_q(2, n)$

We now show that there is a link between $\mathcal{O}_q(Sk_n)$ and $G_q(2, n)$. More specifically, we will show that if we factor out the ideal generated by the length-4 q-Pfaffians from $\mathcal{O}_q(Sk_n)$ then the resulting algebra is isomorphic to $G_q(2, n)$. To do this we first recall the presentation of $G_q(2, n)$, in terms of generators and relations, given in [12, Example (5.7)] (except we reverse the roles of q and q^{-1}):

$G_q(2, n)$ is the K-algebra generated by $\{b_{ij} : 1 \le i < j \le n\}$ subject to the following relations:

$$b_{ij}b_{it} = qb_{it}b_{ij}, \qquad \text{for } i < j < t, \qquad (3.6.1)$$
$$b_{ij}b_{jt} = qb_{jt}b_{ij}, \qquad \text{for } i < j < t, \qquad (3.6.2)$$
$$b_{ij}b_{sj} = qb_{sj}b_{ij}, \qquad \text{for } i < s < j, \qquad (3.6.3)$$
$$b_{ij}b_{st} = q^2 b_{st}b_{ij} - \hat{q}b_{it}b_{sj} + q^{-1}\hat{q}b_{is}b_{tj}, \qquad \text{for } i < s < t < j, \qquad (3.6.4)$$
$$b_{ij}b_{st} = q^2 b_{st}b_{ij} - \hat{q}b_{is}b_{jt}, \qquad \text{for } i < s < j < t, \qquad (3.6.5)$$
$$b_{ij}b_{st} = q^2 b_{st}b_{ij}, \qquad \text{for } i < j < s < t, \qquad (3.6.6)$$
$$b_{ij}b_{st} - qb_{is}b_{jt} + q^2 b_{it}b_{js} = 0, \qquad \text{for} i < j < s < t. \qquad (3.6.7)$$

Proposition 3.6.1. *Let I_4 be the ideal of $\mathcal{O}_q(Sk_n)$ generated by $\{[ijst] : 1 \le i < j < s < t \le n\}$. Then,*

$$\frac{\mathcal{O}_q(Sk_n)}{I_4} \cong G_q(2,n).$$

Proof. We define a map $G_q(2,n) \longrightarrow \mathcal{O}_q(Sk_n)/I_4$ by $b_{ij} \mapsto a_{ij}$, where we abuse terminology and think of the a_{ij} as generating $\mathcal{O}_q(Sk_n)/I_4$. This map is clearly onto. We show that it is an algebra morphism. It suffices to show that the a_{ij} satisfy the relations (3.6.1)-(3.6.7). Relations (3.6.1), (3.6.2), and (3.6.3) are the same as (3.1.1), (3.1.2), and (3.1.3). Since we are working in the algebra with I_4 factored out we have "set the length-4 q-Pfaffians to equal zero", and so (3.6.7) trivially holds for the a_{ij}. It remains to show that (3.6.4), (3.6.5), and (3.6.6) hold.

(3.6.4):
Let $i < s < t < j$. Then by (3.1.4) we have,

$$a_{ij}a_{st} = a_{st}a_{ij},$$

multiplying both sides by q^2 gives,

$$q^2 a_{ij}a_{st} = q^2 a_{st}a_{ij}.$$

Since $[istj] = 0$ we may add a multiple of it to the RHS and deduce,

$$\begin{aligned} q^2 a_{ij}a_{st} &= q^2 a_{st}a_{ij} + q^{-1}\hat{q}[istj] \\ &= q^2 a_{st}a_{ij} + q^{-1}\hat{q}(a_{is}a_{tj} - q a_{it}a_{sj} + q^2 a_{ij}a_{st}) \\ &= q^2 a_{st}a_{ij} + q^{-1}\hat{q}a_{is}a_{tj} - \hat{q}a_{it}a_{sj} + q\hat{q}a_{ij}a_{st} \\ (q^2 - q\hat{q})a_{ij}a_{st} &= q^2 a_{st}a_{ij} + q^{-1}\hat{q}a_{is}a_{tj} - \hat{q}a_{it}a_{sj} \\ a_{ij}a_{st} &= q^2 a_{st}a_{ij} + q^{-1}\hat{q}a_{is}a_{tj} - \hat{q}a_{it}a_{sj}, \end{aligned}$$

since $q^2 - q\hat{q} = 1$. This is relation (3.6.4) as required.

(3.6.5):
Let $i < s < j < t$. Then by (3.1.5) we have,

$$a_{ij}a_{st} = a_{st}a_{ij} + \hat{q}a_{it}a_{sj}.$$

Since $[isjt] = 0$ it follows that,

$$\begin{aligned} a_{ij}a_{st} &= a_{st}a_{ij} + \hat{q}(-q^{-2}a_{is}a_{jt} + q^{-1}a_{ij}a_{st}) \\ (1 - \hat{q}q^{-1})a_{ij}a_{st} &= a_{st}a_{ij} - \hat{q}q^{-2}a_{is}a_{jt} \\ q^{-2}a_{ij}a_{st} &= a_{st}a_{ij} - \hat{q}q^{-2}a_{is}a_{jt} \\ a_{ij}a_{st} &= q^2 a_{st}a_{ij} - \hat{q}a_{is}a_{jt}, \end{aligned}$$

and this is relation (3.6.5).

(3.6.6):
Let $i < j < s < t$. Then by (3.1.6) we have,

$$\begin{aligned} a_{ij}a_{st} &= a_{st}a_{ij} + \hat{q}a_{is}a_{jt} - q\hat{q}a_{it}a_{js} \\ &= a_{st}a_{ij} + \hat{q}(a_{is}a_{jt} - qa_{it}a_{js}). \end{aligned}$$

Since $[ijst] = 0$ it follows that,

$$\begin{aligned} a_{ij}a_{st} &= a_{st}a_{ij} + \hat{q}(q^{-1}a_{ij}a_{st}) \\ (1 - \hat{q}q^{-1})a_{ij}a_{st} &= a_{st}a_{ij} \\ q^{-2}a_{ij}a_{st} &= a_{st}a_{ij} \\ a_{ij}a_{st} &= q^{2}a_{st}a_{ij} \end{aligned}$$

and this is relation (3.6.6). So the map sending b_{ij} to a_{ij} is an onto algebra morphism. We can show in a similar manner that the map sending a_{ij} to b_{ij} is an onto algebra morphism, and the result follows. □

Corollary 3.6.2. *I_4 is completely prime.*

Proof. We know $G_q(2, n)$ is a domain [23, Theorem 1.4], so by the previous proposition $\mathcal{O}_q(Sk_n)/I_4$ is a domain. Hence I_4 is a completely prime ideal. □

Remark 3.6.3. *Let I_{2m} $(1 \leq 2m \leq n)$ be the ideal of $\mathcal{O}_q(Sk_n)$ generated by the length-$2m$ q-Pfaffians. In [38] Strickland shows that the previous result holds for all I_{2m}. However, her proof relies on q being transcendental and does not hold in our more general setting. It is natural to ask whether this result is true when there is no restriction on $q \neq 0$.*

Conjecture 3.6.4. *I_{2m} is a completely prime ideal for all $2 \leq 2m \leq n$.*

Chapter 4

$\mathcal{O}_q(Sk_n)$, A Different Perspective

"Quantum Skew-symmetric Matrices" are also mentioned in a paper by Noumi [31]. The main objects of concern in the paper are certain 'quantum homogeneous spaces', that is quantum analogues of the coordinate ring of certain homogeneous spaces. One of these in particular is of interest to us, namely the quantum analogue of the homogeneous space GL_{2m}/Sp_{2m}. In the classical world SL_{2m}/Sp_{2m} can be realized as an orbit of skew-symmetric matrices. It is in exactly this context in which Noumi refers to "Quantum Skew-symmetric Matrices". This approach is not the same as that given in the previous chapter. We describe this different method in the following section, extracting the relevant definitions from Noumi's remarks.

4.1 Noumi's Approach

Let n be even, say $n = 2m$. Let $T = (t_{ij})_{1\le i,j\le n}$ be the matrix of the n^2 canonical generators of $\mathcal{O}_q(M_n)$. Let us define the following R-matrix in $End_K(K^n \otimes_K K^n)$,

$$R = \sum_{1\le i,j\le n} q^{\delta_{ij}} e_{ii} \otimes e_{jj} + \hat{q} \sum_{1\le i<j\le n} e_{ij} \otimes e_{ji}, \tag{4.1.1}$$

where $e_{ij} \in End_K(K^n)$, $1 \le i,j \le n$, are the matrix units with respect to the natural basis of K^n and $\otimes$ is used here to mean the Kronecker product of matrices. We think of $R = (R^{ik}_{jl})$ as an $n^2 \times n^2$ matrix whose rows and columns are indexed by pairs (i,k) and (j,l) respectively. For a general $n \times n$ matrix $A = (a_{ij})$ we make the following notational definitions, $A_1 := A \otimes I$ and $A_2 := I \otimes A$, where I is the $n \times n$ identity matrix. So A_1 and A_2 are both $n^2 \times n^2$ matrices and, if we think of them also as having rows and columns indexed by pairs, then we have that $(A_1)^{ik}_{jl} = a_{ij}\delta_{kl}$ and $(A_2)^{ik}_{jl} = \delta_{ij}a_{kl}$. With this notation in place we are free to observe that the commutation relations of the t_{ij} can be succinctly expressed as follows,

$$RT_2T_1 = T_1T_2R.$$

In [31, Section 4, pg 41] the following definition is given

Definition 4.1.1. *$\mathcal{O}_q(Sk_n)_{(No)}$ is defined to be the K-subalgebra of $\mathcal{O}_q(M_n)$ generated by the following quadratic elements,*

$$x_{ij} := \sum_{k=1}^{m}(t_{i,2k-1}t_{j,2k} - qt_{i,2k}t_{j,2k-1}).$$

This can also be expressed as,

$$X = (x_{ij})_{1\leq i,j\leq n}, \quad X = TJT^t$$

where $J \in End_K(K^n)$ is the matrix of a "quantized"-symplectic form defined, in terms of the $e_{ij} \in End_K(K^n)$, as follows,

$$J := \sum_{k=1}^{m}(e_{2k-1,2k} - qe_{2k,2k-1}).$$

The relations amongst the x_{ij} are given in [31, Proposition 4.4],

Proposition 4.1.2. *The x_{ij} satisfy the following relations*

$$x_{ii} = 0 \text{ for } i = 1,..,n, \qquad x_{ji} = -qx_{ij} \text{ for } 1 \leq i < j \leq n,$$

$$RX_2R^{t_2}X_1 = X_1R^{t_2}X_2R,$$

where R^{t_2} is an $n^2 \times n^2$ matrix with rows and columns indexed by pairs $(i,k),(j,l)$ respectively, with entries given by $(R^{t_2})^{ik}_{jl} := R^{il}_{jk}$.

Finally, Noumi also makes the following definition [31, Remark 4.12], which can be seen as a quantum analogue of Definition 1.5.2:

Definition 4.1.3. *Let $\Omega_n := \{\sigma \in S_n : \sigma(2i-1) < \sigma(2i) \text{ for } i = 1,...,m\}$.*

$$Pf_{q(No)}(n) := \frac{1}{[m]_{q^4}!}\sum_{w\in\Omega_n}(-q)^{l(w)}x_{w(1)w(2)}\cdots x_{w(2m-1)w(2m)}$$

where $l(w)$, the length of w, is the number of inversions in w,

$$l(w) = \#\{i < j : w(i) > w(j)\}$$

and

$$[k]_{q^4} := \frac{1-q^{4k}}{1-q^4} = 1 + ... + q^{4(k-1)}, \qquad [m]_{q^4}! := \prod_{k=1}^{m}[k]_{q^4}.$$

We note that this definition is valid only when $[m]_{q^4}! \neq 0$. If we assume q is not a root of unity then this is certainly the case.

Example 4.1.4. *We now see what this definition means in the case $n = 4$.*

$$Pf_{q(No)}(4) = \frac{1}{[2]_{q^4}!}(x_{12}x_{34} - qx_{13}x_{24} + q^2x_{14}x_{23} \\ - q^3x_{24}x_{13} + q^2x_{23}x_{14} + q^4x_{34}x_{12}).$$

Contrast this with the 4×4 q-Pfaffian given in Example 3.1.4 in the last chapter.

Remark 4.1.5. *In making the above definition Noumi observes that*

$$Pf_{q(No)}(n) = det_q(T).$$

Since $det_q(T)$ is central in $\mathcal{O}_q(M_n)$ and $\mathcal{O}_q(Sk_n)_{(No)}$ is a subalgebra of $\mathcal{O}_q(M_n)$ this yields the centrality of $Pf_{q(No)}$ in $\mathcal{O}_q(Sk_n)_{(No)}$. This does not invalidate our result in the previous chapter since $Pf_{q(No)}$ is only defined when q is a non-root of unity.

We now generalise Noumi's definition, in the obvious way, to sub-q-Pfaffians of any (even) length.

Definition 4.1.6. *Let $1 \leq i_1 < i_2 < \cdots < i_{2h} \leq n$ for some $1 \leq h \leq m$. Let $\Omega_{2h} := \{\sigma \in S_{2h} : \sigma(2i-1) < \sigma(2i) \text{ for } i = 1, ..., h\}$. For q not a root of unity,*

$$[i_1...i_{2h}]_{(No)} := \frac{1}{[h]_{q^4}!}\sum_{w \in \Omega_{2h}} (-q)^{l(w)} x_{i_{w(1)}i_{w(2)}} \cdots x_{i_{w(2h-1)}i_{w(2h)}}.$$

So from Noumi [31] we have the notions of the coordinate ring of "Quantum Skew-symmetric Matrices" and of q-Pfaffians that are apparently distinct from those of the previous chapter. The question arises as to how these rival concepts are related. This matter is addressed in the next section.

4.2 The Equivalence of Strickland and Noumi

In Section 3.1, $\mathcal{O}_q(Sk_n)$ was presented abstractly in terms of generators and relations. In the previous section $\mathcal{O}_q(Sk_n)_{(No)}$ was defined to be a specific subalgebra of $\mathcal{O}_q(M_n)$. We shall now show that these two objects are in-fact (under certain conditions on q) isomorphic. First of all we show that, with no restrictions on q,

Proposition 4.2.1. *There is an onto algebra morphism,*

$$\mathcal{O}_q(Sk_n) \twoheadrightarrow \mathcal{O}_q(Sk_n)_{(No)} \quad \text{given by} \quad a_{ij} \mapsto x_{ij}.$$

Proof. It suffices to show the $x_{ij} \in \mathcal{O}_q(Sk_n)_{(No)}$ obey the same relations as the $a_{ij} \in \mathcal{O}_q(Sk_n)$. Comparing the relations of the a_{ij} given in Section 3.1 and those of the x_{ij} given in Proposition 4.1.2 it is clear that if we can deduce relations (3.1.1)-(3.1.6) from

$$RX_2R^{t_2}X_1 = X_1R^{t_2}X_2R \tag{4.2.1}$$

then we are done. Now, for a given $(i,k),(j,l)$, (4.2.1) says,

$$(RX_2R^{t_2}X_1)^{ik}_{jl} = (X_1R^{t_2}X_2R)^{ik}_{jl}.$$

Writing out what this means in full gives,

$$\sum_{a,b,c,d,e,f} R^{ik}_{ab}(X_2)^{ab}_{cd}(R^{t_2})^{cd}_{ef}(X_1)^{ef}_{jl} = \sum_{a,b,c,d,e,f} (X_1)^{ik}_{ab}(R^{t_2})^{ab}_{cd}(X_2)^{cd}_{ef}R^{ef}_{jl},$$

and then,

$$\sum_{a,b,c,d,e,f} R^{ik}_{ab}R^{cf}_{ed}\delta_{ac}\delta_{fl}x_{bd}x_{ej} = \sum_{a,b,c,d,e,f} R^{ef}_{jl}R^{ad}_{cb}\delta_{kb}\delta_{ce}x_{ia}x_{df}.$$

Taking into account the Kronecker deltas, we deduce,

$$\sum_{a,b,d,e} R^{ik}_{ab}R^{al}_{ed}x_{bd}x_{ej} = \sum_{a,d,e,f} R^{ef}_{jl}R^{ad}_{ek}x_{ia}x_{df}. \tag{4.2.2}$$

We now look to eliminate terms in (4.2.2) which are always zero, no matter the value of $(i,k),(j,l)$. To do this we translate the definition of R given in (4.1.1) into a more usable form,

$$R^{ik}_{jl} = \begin{cases} q, & i=j=k=l; \\ 1, & i=j\neq k=l; \\ \hat{q}, & i=l<j=k; \\ 0, & \text{otherwise.} \end{cases} \tag{4.2.3}$$

We now examine the LHS of (4.2.2) taking into account (4.2.3). We see that R^{ik}_{ab} is nonzero when $(a,b)=(i,k),(k,i)$; note that these two possibilities are distinct only when $k\neq i$. Suppose $(a,b)=(i,k)$ then $R^{ik}_{ab}R^{al}_{ed} = R^{ik}_{ik}R^{il}_{ed}$ and this is nonzero when $(e,d)=(i,l),(l,i)$; once again note that these two cases represent two distinct terms only when $i\neq l$. We proceed in this manner, taking into account all the possible cases, to eliminate all the terms on the LHS and RHS of (4.2.2) which are identically zero. After some careful thought, we deduce from (4.2.2) that,

$$\begin{aligned} R^{ik}_{ik}R^{il}_{il}x_{kl}x_{ij} &+ (1-\delta_{il})R^{ik}_{ik}R^{il}_{li}x_{ki}x_{lj} + (1-\delta_{ik})R^{ik}_{ki}R^{kl}_{kl}x_{il}x_{kj} \\ &+ (1-\delta_{kl})(1-\delta_{ki})R^{ik}_{ki}R^{kl}_{lk}x_{ik}x_{lj} = R^{jk}_{jk}R^{jl}_{jl}x_{ij}x_{kl} + (1-\delta_{kj})R^{kj}_{jk}R^{jl}_{jl}x_{ik}x_{jl} \\ &+ (1-\delta_{lj})R^{lk}_{lk}R^{lj}_{jl}x_{il}x_{kj} + (1-\delta_{jl})(1-\delta_{kl})R^{kl}_{lk}R^{lj}_{jl}x_{ik}x_{lj}. \end{aligned} \tag{4.2.4}$$

Recall the remaining relations from Proposition 4.1.2,

$$x_{ii} = 0 \ \forall i, \quad x_{ji} = -q x_{ij} \quad \text{for } j > i. \tag{4.2.5}$$

We complete the proof by examining (4.2.4) under various restrictions placed upon the values of $(i,k),(j,l)$, using (4.2.5) and (4.2.3) to simplify the resulting equation.

$\underline{(i,k),(j,l) = (i,i),(j,t), \quad i < j < t:}$
(4.2.4) gives,

$$\begin{aligned} R^{ii}_{ii}R^{it}_{it}x_{it}x_{ij} + R^{ii}_{ii}R^{it}_{ti}x_{ii}x_{tj} &= R^{ji}_{ji}R^{jt}_{jt}x_{ij}x_{it} + R^{ij}_{ji}R^{jt}_{jt}x_{ii}x_{jt} \\ &\quad + R^{ti}_{ti}R^{tj}_{jt}x_{it}x_{ij} + R^{it}_{ti}R^{tj}_{jt}x_{ii}x_{tj} \\ q x_{it}x_{ij} &= x_{ij}x_{it}. \end{aligned}$$

This is relation (3.1.1).

$\underline{(i,k),(j,l) = (i,j),(j,t), \quad i < j < t:}$
(4.2.4) gives,

$$\begin{aligned} R^{ij}_{ij}R^{it}_{it}x_{jt}x_{ij} + R^{ij}_{ij}R^{it}_{ti}x_{ji}x_{tj} + R^{ij}_{ji}R^{jt}_{jt}x_{it}x_{jj} + R^{ij}_{ji}R^{jt}_{tj}x_{ij}x_{tj} & \\ = R^{jj}_{jj}R^{jt}_{jt}x_{ij}x_{jt} + R^{tj}_{tj}R^{tj}_{jt}x_{it}x_{jj} + R^{jt}_{tj}R^{tj}_{jt}x_{ij}x_{tj} & \\ x_{jt}x_{ij} + \hat{q}q^2 x_{ij}x_{jt} - \hat{q}^2 q x_{ij}x_{jt} &= q x_{ij}x_{jt} \\ x_{jt}x_{ij} &= (q + \hat{q}^2 q - \hat{q}q^2) x_{ij}x_{jt} \\ x_{ij}x_{jt} &= q x_{jt}x_{ij}. \end{aligned}$$

This is relation (3.1.2).

$\underline{(i,k),(j,l) = (i,s),(j,j), \quad i < s < j:}$
(4.2.4) gives,

$$\begin{aligned} R^{is}_{is}R^{ij}_{ij}x_{sj}x_{ij} + R^{is}_{is}R^{ij}_{ji}x_{si}x_{jj} + R^{is}_{si}R^{sj}_{sj}x_{ij}x_{sj} + R^{is}_{si}R^{sj}_{js}x_{is}x_{jj} & \\ = R^{js}_{js}R^{jj}_{jj}x_{ij}x_{sj} + R^{sj}_{js}R^{jj}_{jj}x_{is}x_{jj} & \\ x_{sj}x_{ij} + \hat{q}x_{ij}x_{sj} &= q x_{ij}x_{sj} \\ x_{sj}x_{ij} &= (q - \hat{q}) x_{ij}x_{sj} \\ x_{ij}x_{sj} &= q x_{sj}x_{ij}. \end{aligned}$$

This is relation (3.1.3).

$\underline{(i,k),(j,l) = (i,s),(j,t), \quad i < s < t < j:}$
(4.2.4) gives,

$$\begin{aligned}
R^{is}_{is}R^{it}_{it}x_{st}x_{ij} + R^{is}_{is}R^{it}_{ti}x_{si}x_{tj} + R^{is}_{si}R^{st}_{st}x_{it}x_{sj} + R^{is}_{si}R^{st}_{ts}x_{is}x_{tj} \\
= R^{js}_{js}R^{jt}_{jt}x_{ij}x_{st} + R^{sj}_{js}R^{jt}_{jt}x_{is}x_{jt} + R^{ts}_{ts}R^{tj}_{jt}x_{it}x_{sj} + R^{st}_{ts}R^{tj}_{jt}x_{is}x_{tj} \\
x_{st}x_{ij} - \hat{q}qx_{is}x_{tj} + \hat{q}x_{it}x_{sj} + \hat{q}^2x_{is}x_{tj} = x_{ij}x_{st} - \hat{q}qx_{is}x_{tj} + \hat{q}x_{it}x_{sj} + \hat{q}^2x_{is}x_{tj} \\
x_{st}x_{ij} = x_{ij}x_{st}.
\end{aligned}$$

This is relation (3.1.4).

$\underline{(i,k),(j,l) = (i,s),(j,t), \quad i < s < j < t:}$
(4.2.4) gives,

$$\begin{aligned}
R^{is}_{is}R^{it}_{it}x_{st}x_{ij} + R^{is}_{is}R^{it}_{ti}x_{si}x_{tj} + R^{is}_{si}R^{st}_{st}x_{it}x_{sj} + R^{is}_{si}R^{st}_{ts}x_{is}x_{tj} \\
= R^{js}_{js}R^{jt}_{jt}x_{ij}x_{st} + R^{sj}_{js}R^{jt}_{jt}x_{is}x_{jt} + R^{ts}_{ts}R^{tj}_{jt}x_{it}x_{sj} + R^{st}_{ts}R^{tj}_{jt}x_{is}x_{tj} \\
x_{st}x_{ij} + \hat{q}q^2x_{is}x_{jt} + \hat{q}x_{it}x_{sj} - \hat{q}^2qx_{is}x_{jt} = x_{ij}x_{st} + \hat{q}x_{is}x_{jt} \\
x_{ij}x_{st} = x_{st}x_{ij} + (\hat{q}q^2 - \hat{q}^2q - \hat{q})x_{is}x_{jt} + \hat{q}x_{it}x_{sj} \\
x_{ij}x_{st} = x_{st}x_{ij} + \hat{q}x_{it}x_{sj}.
\end{aligned}$$

This is relation (3.1.5).

$\underline{(i,k),(j,l) = (i,s),(j,t), \quad i < j < s < t:}$
(4.2.4) gives,

$$\begin{aligned}
R^{is}_{is}R^{it}_{it}x_{st}x_{ij} + R^{is}_{is}R^{it}_{ti}x_{si}x_{tj} + R^{is}_{si}R^{st}_{st}x_{it}x_{sj} + R^{is}_{si}R^{st}_{ts}x_{is}x_{tj} \\
= R^{js}_{js}R^{jt}_{jt}x_{ij}x_{st} + R^{sj}_{js}R^{jt}_{jt}x_{is}x_{jt} + R^{ts}_{ts}R^{tj}_{jt}x_{it}x_{sj} + R^{st}_{ts}R^{tj}_{jt}x_{is}x_{tj} \\
x_{st}x_{ij} + \hat{q}q^2x_{is}x_{jt} - \hat{q}qx_{it}x_{js} - \hat{q}^2qx_{is}x_{jt} = x_{ij}x_{st} \\
x_{st}x_{ij} + (\hat{q}q^2 - \hat{q}^2q)x_{is}x_{jt} - \hat{q}qx_{it}x_{js} = x_{ij}x_{st} \\
x_{st}x_{ij} + \hat{q}x_{is}x_{jt} - \hat{q}qx_{it}x_{js} = x_{ij}x_{st}
\end{aligned}$$

This is relation (3.1.6). □

Our proof of the existence of an algebra isomorphism between $\mathcal{O}_q(Sk_n)$ and $\mathcal{O}_q(Sk_n)_{(No)}$ relies upon the following well-known theorem ([31],[38],[32],[34],[30]) concerning the representation theory of $U_q(\mathfrak{gl}_n)$. Before we state the theorem we require some definitions. In what follows, unless stated otherwise, modules will be assumed to be left-modules.

Definition 4.2.2. *Let M be a $U_q(\mathfrak{gl}_n)$-module. Let $m \in M$. We say m is a* weight vector *with* weight $\lambda = (\lambda_1, ..., \lambda_n)$, *if, for $s = 1, ..., n$, $L_s m = q^{\lambda_s} m$ where $\lambda_s \in \mathbb{Z}$. Furthermore, we say m is a* highest weight vector *if we also have that $E_k m = 0$ for $k = 1, .., n-1$.*

Definition 4.2.3. *Let M be a $U_q(\mathfrak{gl}_n)$-module. We say M is a* weight-representation of $U_q(\mathfrak{gl}_n)$ *if it has a K-basis of weight vectors.*

Theorem 4.2.4. *Let q be a non-root of unity. The finite-dimensional irreducible weight-representations of $U_q(\mathfrak{gl}_n)$ have the same indexing and characters as in the classical case of GL_n. That is any finite-dimensional irreducible weight-representation of $U_q(\mathfrak{gl}_n)$ is isomorphic to some $V(\lambda)$ with $\lambda \in P^+$; where $V(\lambda)$ is the unique irreducible $U_q(\mathfrak{gl}_n)$-module generated by a highest weight vector with weight λ, P is the weight lattice for GL_n, that is the free $\mathbb{Z}$-module of rank n with canonical basis $(\epsilon_j)_{1 \le j \le n}$, and P^+ is the set of dominant integral weights in P, that is*

$$P^+ = \{\lambda = \sum_{s=1}^{n} \lambda_s \epsilon_s \in P : \lambda_1 \ge ... \ge \lambda_n\}.$$

We now require some results concerning the representation theoretic properties of $\mathcal{O}_q(Sk_n)$ and $\mathcal{O}_q(Sk_n)_{(No)}$. So far we have not defined a $U_q(\mathfrak{gl}_n)$-action on $\mathcal{O}_q(Sk_n)_{(No)}$. In [31] Noumi gives the $U_q(\mathfrak{gl}_n)$-bimodule structure of $\mathcal{O}_q(GL_n)$ in terms of, so called, L-operators. It can be checked that these right and left $U_q(\mathfrak{gl}_n)$-actions are exactly those that are specified explicitly in his earlier joint paper [32, (1.35.a),(1.35.b),(1.35.c)]

$$L_s t_{ij} = q^{\delta_{sj}} t_{ij}, \qquad t_{ij} L_s = q^{\delta_{si}} t_{ij}, \tag{4.2.6}$$

$$E_s t_{ij} = \delta_{s+1,j} t_{i,j-1}, \qquad t_{ij} E_s = \delta_{si} t_{i+1,j}, \tag{4.2.7}$$

$$F_s t_{ij} = \delta_{sj} t_{i,j+1}, \qquad t_{ij} F_s = \delta_{s+1,i} t_{i-1,j}. \tag{4.2.8}$$

where the L_s, E_s, and F_s are the generators of $U_q(\mathfrak{gl}_n)$ and the t_{ij} are the generators of $\mathcal{O}_q(M_n)$, thought of as a subalgebra of $\mathcal{O}_q(GL_n)$ (we shall implicitly regard $\mathcal{O}_q(M_n)$ as a subalgebra of $\mathcal{O}_q(GL_n)$ hereafter). With this $U_q(\mathfrak{gl}_n)$-action $\mathcal{O}_q(Sk_n)_{(No)}$ is easily seen to be a right $U_q(\mathfrak{gl}_n)$-module. However the result we shall use concerning the representation theory of $\mathcal{O}_q(Sk_n)$ is in the context of it being a left $U_q(\mathfrak{gl}_n)$-module. This problem can be overcome, for there is in-fact another subalgebra of $\mathcal{O}_q(GL_n)$ which is isomorphic, as an algebra, to $\mathcal{O}_q(Sk_n)_{(No)}$ [31, Remark 4.5]. We shall call this algebra $\mathcal{O}_q(Sk_n)'_{(No)}$. It is defined by Noumi to be the K-subalgebra of $\mathcal{O}_q(M_n)$ generated by the quadratic elements y_{ij} defined by

$$Y = (y_{ij})_{1 \le i,j \le n}, \quad Y = T^t J^{-1} T. \tag{4.2.9}$$

The algebra isomorphism is given by $x_{ij} \mapsto y_{ij}$. This different subalgebra is indeed a left $U_q(\mathfrak{gl}_n)$-module under the stated action. In [31] Noumi states the following representation theoretic result relating to $\mathcal{O}_q(Sk_n)'_{(No)}$.

Proposition 4.2.5. [31, Remark 4.5] *Let q be a non-root of unity.*
Let $\langle \mathcal{O}_q(Sk_n)'_{(No)}, det_q(T)^{-1}\rangle$ be the subalgebra of $\mathcal{O}_q(GL_n)$ generated by the y_{ij} and $det_q(T)^{-1}$. Then we have the following multiplicity free decomposition as a $U_q(\mathfrak{gl}_n)$-module

$$\langle \mathcal{O}_q(Sk_n)'_{(No)}, det_q(T)^{-1}\rangle \cong \bigoplus_{\lambda \in P_H^+} V(\lambda)$$

where

$$P_H^+ = \{\lambda \in P : \lambda = \sum_{r=1}^{m} b_r \Lambda_{2r},\ b_r \in \mathbb{N} \text{ for } r = 1, ..., m-1 \text{ and } b_m \in \mathbb{Z}\}$$

denoting the fundamental weights by $\Lambda_r = \sum_{k=1}^{r} \epsilon_k$.

We may use this result to prove the following decomposition of $\mathcal{O}_q(Sk_n)'_{(No)}$.

Lemma 4.2.6. *Let q be a non-root of unity. As a $U_q(\mathfrak{gl}_n)$-module $\mathcal{O}_q(Sk_n)'_{(No)}$ has the following decomposition*

$$\mathcal{O}_q(Sk_n)'_{(No)} \cong \bigoplus_{\lambda \in P_{SS}^+} V(\lambda)$$

where

$$P_{SS}^+ = \{\lambda \in P : \lambda = \sum_{r=1}^{m} b_r \Lambda_{2r},\ b_r \in \mathbb{N} \text{ for } r = 1, ..., m\}$$

and each irreducible component occurs with multiplicity one.

Proof. Now from (4.2.9) it follows that the y_{ij} are given explicitly by

$$y_{ij} = \sum_{k=1}^{m} (t_{2k,i} t_{2k-1,j} - q^{-1} t_{2k-1,i} t_{2k,j}). \tag{4.2.10}$$

From (4.2.6) it follows that all the weights on $\mathcal{O}_q(Sk_n)'_{(No)}$ are nonnegative. Furthermore it easily follows from (4.2.6) that all positive weights in P_H^+ are possible on $\mathcal{O}_q(Sk_n)'_{(No)}$. The result then follows from the previous proposition. □

Remark 4.2.7. *We note that this decomposition into irreducibles is exactly the same as in the classical case (i.e. the coordinate ring of skew-symmetric matrices decomposes as a GL_n-module into irreducibles indexed by P_{SS}^+ [18, Proposition 4.2]).*

We now turn to $\mathcal{O}_q(Sk_n)$. Strickland proves that it too decomposes analogously to the classical case.

Proposition 4.2.8. [38, Proposition 1.3] *Let q be a non-root of unity. As a $U_q(\mathfrak{gl}_n)$-module $\mathcal{O}_q(Sk_n)$ has the following decomposition*

$$\mathcal{O}_q(Sk_n) \cong \bigoplus_{\lambda \in P_{SS}^+} V(\lambda)$$

where each irreducible component occurs with multiplicity one.

Remark 4.2.9. *We note that this is not the phrasing of the proposition as stated in [38]. However it is exactly the same result translated into the notation we are using.*

We are now in a position to prove the following,

Proposition 4.2.10. *For q not a root of unity,*

$$\mathcal{O}_q(Sk_n) \cong \mathcal{O}_q(Sk_n)_{(No)}.$$

Proof. We have the following onto algebra morphism by our work above,

$$\Psi : \mathcal{O}_q(Sk_n) \twoheadrightarrow \mathcal{O}_q(Sk_n)'_{(No)} \quad \text{given by} \quad a_{ij} \mapsto y_{ij}.$$

One can check from (4.2.6)-(4.2.8) and (4.2.10) that $U_q(\mathfrak{gl}_n)$ acts on the y_{ij} in exactly the same way as it acts on the a_{ij} (as given in (3.1.10)-(3.1.12)). Hence Ψ is clearly a $U_q(\mathfrak{gl}_n)$-module morphism. It follows that KerΨ is a $U_q(\mathfrak{gl}_n)$-module. Suppose Ker$\Psi \neq 0$. Then by Proposition 4.2.8 there must exist a $\gamma \in P_{SS}^+$ such that $V(\gamma) \subseteq$ KerΨ, where $V(\gamma)$ is an irreducible $U_q(\mathfrak{gl}_n)$-module that occurs once in the decomposition of $\mathcal{O}_q(Sk_n)$. That is $\Psi(V(\gamma)) = 0$. But by Lemma 4.2.6 $\mathcal{O}_q(Sk_n)'_{(No)}$ has the same decomposition into irreducible $U_q(\mathfrak{gl}_n)$-modules as $\mathcal{O}_q(Sk_n)$, with each irreducible component also occurring once. So $\Psi(V(\gamma)) = 0$ contradicts the surjectivity of Ψ. Hence Ker$\Psi = 0$ and we have shown that Ψ is an isomorphism. □

Now that we can view the x_{ij} and a_{ij} interchangeably (for q not a root of unity) we may apply our original definition of a q-Pfaffian, Definition 3.1.2, to the x_{ij} and ask whether it is related to Noumi's q-Pfaffian. We first consider the case $n = 4$.

Lemma 4.2.11. *For q not a root of unity,*

$$[1234]_{(No)} = [1234].$$

Proof.

$$[1234]_{(No)} = \frac{1}{[2]_{q^4}!}(x_{12}x_{34} - qx_{13}x_{24} + q^2x_{14}x_{23}$$
$$- q^3x_{24}x_{13} + q^2x_{23}x_{14} + q^4x_{34}x_{12}).$$

We rewrite this using (3.1.5), (3.1.4), (3.1.6), and Definition 3.1.2,

$$\begin{aligned}[1234]_{(No)} &= \frac{1}{[2]_{q^4}!}\big([1234] - q^3(x_{13}x_{24} - \hat{q}x_{14}x_{23}) + q^2x_{14}x_{23} \\ &\qquad + q^4(x_{12}x_{34} - \hat{q}x_{13}x_{24} + q\hat{q}x_{14}x_{23})\big) \\ &= \frac{1}{[2]_{q^4}!}\big([1234] + q^4(x_{12}x_{34} - qx_{13}x_{24} + q^2x_{14}x_{23})\big) \\ &= \frac{1}{[2]_{q^4}!}(1+q^4)[1234] \\ &= [1234].\end{aligned}$$

□

We now prove that the two q-Pfaffians are equal in general.

Proposition 4.2.12. *Let $1 \le i_1 < i_2 < \cdots < i_{2h} \le n$ for some $1 \le h \le m$. For q not a root of unity,*

$$[i_1 \ldots i_{2h}]_{(No)} = [i_1 \ldots i_{2h}].$$

Proof. We proceed by induction on h, the base case $h = 1$ being trivial. We note that the proof of Lemma 4.2.11 translates directly into a proof of the case $h = 2$. So we may assume $h > 2$. With this assumption in place we now prove the general inductive step. By definition,

$$[i_1 \ldots i_{2h}]_{(No)} = \frac{1}{[h]_{q^4}!}\sum_{w\in\Omega_{2h}} (-q)^{l(w)} x_{i_{w(1)}i_{w(2)}} \cdots x_{i_{w(2m-1)}i_{w(2m)}}$$

Recalling that $\Omega_n := \{\sigma \in S_n : \sigma(2i-1) < \sigma(2i) \text{ for } i = 1, \ldots, m\}$ allows us to rewrite the above equation as,

$$[i_1 \ldots i_{2h}]_{(No)} = \frac{1}{[h]_{q^4}!}\sum_{s=1}^{2h-1}\sum_{j=s+1}^{2h} x_{i_s i_j}\left(\sum_{\substack{w\in\Omega_{2h}\\ w(1)=s\\ w(2)=j}} (-q)^{l(w)} x_{i_{w(3)}i_{w(4)}} \cdots x_{i_{w(2h-1)}i_{w(2h)}}\right). \tag{4.2.11}$$

We will attempt to rewrite the sum in brackets in such away as to allow us to apply the inductive hypothesis. Fix $w(1) = s < j = w(2)$. Let $(l_1, \ldots, l_{2h-2}) = (i_1, .., \check{i_s}, .., \check{i_j}, .., i_{2h})$, that is,

$$l_k = \begin{cases} i_k, & 1 \le k < s; \\ i_{k+1}, & s \le k < j-1; \\ i_{k+2}, & j-1 \le k \le 2h-2. \end{cases} \tag{4.2.12}$$

Define $w' \in \Omega_{2h-2}$ by

$$w'(k) = \begin{cases} w(k+2), & 1 \leq w(k+2) < s; \\ w(k+2)-1, & s+1 \leq w(k+2) < j; \\ w(k+2)-2, & j+1 \leq w(k+2) \leq 2h. \end{cases} \tag{4.2.13}$$

We will now express $l(w)$ in terms of $l(w')$. By definition,

$$l(w) = \#\{i < j : w(i) > w(j)\}.$$

This can be expressed equivalently as,

$$l(w) = \sum_{r=1}^{2h} \#\{w(t) < w(r) : r < t\},$$

so,

$$l(w) = \#\{w(t) < w(1) : 1 < t\} + \#\{w(t) < w(2) : 2 < t\} + \sum_{r=3}^{2h} \#\{w(t) < w(r) : r < t\}.$$

We are assuming that $w(1) = s < j = w(2)$, so we have,

$$l(w) = \#\{w(t) < s : 1 < t\} + \#\{w(t) < j : 2 < t\} + \sum_{r=1}^{2h-2} \#\{w(t) < w(r+2) : r+2 < t\}.$$

Now the relative ordering of $w(3), ..., w(2h)$ is "the same" as $w'(1), ..., w'(2h-2)$ by definition of w'. So

$$\sum_{r=1}^{2h-2} \#\{w(t) < w(r+2) : r+2 < t\} = \sum_{r=1}^{2h-2} \#\{w'(t) < w'(r) : r < t\}.$$

Hence,

$$l(w) = (s-1) + (j-2) + \sum_{r=1}^{2h-2} \#\{w'(t) < w'(r) : r < t\}.$$

But by definition of $l(w')$ this is just,

$$l(w) = l(w') + s + j - 3. \tag{4.2.14}$$

Using (4.2.12), (4.2.13), and (4.2.14), we may rewrite the sum in brackets on the RHS of (4.2.14) and deduce that,

$$[i_1 ... i_{2h}]_{(No)} = \frac{1}{[h]_{q^4}!} \sum_{s=1}^{2h-1} \sum_{j=s+1}^{2h} (-q)^{s+j-3} x_{i_s i_j} \sum_{w' \in \Omega_{2h-2}} (-q)^{l(w')} x_{l_{w'(1)} l_{w'(2)}} \cdots x_{l_{w'(2h-3)} l_{w'(2h-2)}}.$$

By the inductive hypothesis it follows that,

$$[i_1...i_{2h}]_{(No)} = \frac{1}{[h]_{q^4}!}\sum_{s=1}^{2h-1}\sum_{j=s+1}^{2h}(-q)^{s+j-3}x_{i_si_j}\left([h-1]_{q^4}![l_1...l_{2h-2}]\right),$$

so,

$$[i_1...i_{2h}]_{(No)} = \frac{[h-1]_{q^4}!}{[h]_{q^4}!}\sum_{s=1}^{2h-1}\sum_{j=s+1}^{2h}(-q)^{s+j-3}x_{i_si_j}[i_1..\check{i_s}..\check{i_j}..i_{2h}].$$

Writing the $s=1$ term separately we have,

$$\begin{aligned}&[i_1...i_{2h}]_{(No)}\\&= \frac{1}{[h]_{q^4}}\left(\sum_{j=2}^{2h}(-q)^{j-2}x_{i_1i_j}[i_2..\check{i_j}..i_{2h}] + \sum_{s=2}^{2h-1}\sum_{j=s+1}^{2h}(-q)^{s+j-3}x_{i_si_j}[i_1..\check{i_s}..\check{i_j}..i_{2h}]\right),\end{aligned}$$

and by Definition 3.1.2 this is just,

$$[i_1...i_{2h}]_{(No)} = \frac{1}{[h]_{q^4}}\left([i_1...i_{2h}] + \sum_{s=2}^{2h-1}\sum_{j=s+1}^{2h}(-q)^{s+j-3}x_{i_si_j}[i_1..\check{i_s}..\check{i_j}..i_{2h}]\right). \quad (4.2.15)$$

Suppose

$$\sum_{s=2}^{2h-1}\sum_{j=s+1}^{2h}(-q)^{s+j-3}x_{i_si_j}[i_1..\check{i_s}..\check{i_j}..i_{2h}] = q^4[h-1]_{q^4}[i_1...i_{2h}]. \quad (4.2.16)$$

Then (4.2.15) would give,

$$\begin{aligned}[i_1...i_{2h}]_{(No)} &= \frac{1}{[h]_{q^4}}\left([i_1...i_{2h}] + q^4[h-1]_{q^4}[i_1...i_{2h}]\right)\\&= \frac{1}{[h]_{q^4}}[i_1...i_{2h}]\left(1+q^4[h-1]_{q^4}\right)\\&= \frac{1}{[h]_{q^4}}[i_1...i_{2h}]\left(1+q^4(1+q^4+...+q^{4(h-2)})\right)\\&= \frac{1}{[h]_{q^4}}[i_1...i_{2h}]\left(1+q^4+...+q^{4(h-1)}\right)\\&= \frac{1}{[h]_{q^4}}[i_1...i_{2h}]\left([h]_{q^4}\right)\\&= [i_1...i_{2h}],\end{aligned}$$

and the proof is complete. So it suffices to prove (4.2.16).

Now, writing $(l_1,...,l_{2h-2}) = (i_1,..,\check{i_s},..,\check{i_j},..,i_{2h})$ allows us to express the LHS of (4.2.16) as follows,

$$\sum_{s=2}^{2h-1}\sum_{j=s+1}^{2h}(-q)^{s+j-3}x_{i_si_j}[i_1..\check{i_s}..\check{i_j}..i_{2h}] = \sum_{s=2}^{2h-1}\sum_{j=s+1}^{2h}(-q)^{s+j-3}x_{i_si_j}[l_1...l_{2h-2}].$$

Using Definition 3.1.2 this gives,

$$\sum_{s=2}^{2h-1}\sum_{j=s+1}^{2h}(-q)^{s+j-3}x_{i_si_j}[i_1..\check{i_s}..\check{i_j}..i_{2h}]$$
$$=\sum_{s=2}^{2h-1}\sum_{j=s+1}^{2h}\sum_{r=2}^{2h-2}(-q)^{s+j+r-5}x_{i_si_j}x_{l_1l_r}[l_2..\check{l_r}..l_{2h-2}]$$
$$=\sum_{s=2}^{2h-1}\sum_{j=s+1}^{2h}\sum_{r=2}^{2h-2}(-q)^{s+j+r-5}x_{i_si_j}x_{i_1l_r}[i_2..\check{i_s}..\check{i_j}..\check{l_r}..i_{2h}]. \tag{4.2.17}$$

We proceed by rewriting the RHS of (4.2.17) in terms of ordered monomials. Now $i_1 < i_s < i_j$ and $i_1 < l_r$, so when we rearrange $x_{i_si_j}x_{i_1l_r}$ there are three cases we must consider:
(*i*) $\underline{i_1 < l_r < i_s < i_j}$ by (3.1.6), $x_{i_si_j}x_{i_1l_r} = x_{i_1l_r}x_{i_si_j} - \hat{q}x_{i_1i_s}x_{l_ri_j} + q\hat{q}x_{i_1i_j}x_{l_ri_s}$.
(*ii*) $\underline{i_1 < i_s < l_r < i_j}$ by (3.1.5), $x_{i_si_j}x_{i_1l_r} = x_{i_1l_r}x_{i_si_j} - \hat{q}x_{i_1i_j}x_{i_sl_r}$.
(*iii*) $\underline{i_1 < i_s < i_j < l_r}$ by (3.1.4), $x_{i_si_j}x_{i_1l_r} = x_{i_1l_r}x_{i_si_j}$.
So we can see that when written in terms of ordered monomials the RHS of (4.2.17) is a sum of terms of the form

$$x_{i_1i_a}x_{i_bi_c}[i_2..\check{i_a}..\check{i_b}..\check{i_c}..i_{2h}]$$

with $a = 2, ..., 2h$, $b = 2, ..., 2h-1$, and $c = b+1, ..., 2h$. For a given (a, b, c) we now calculate the coefficient of such a term in each of the three appropriate cases.

$\underline{a < b < c:}$
Examining (*i*) – (*iii*) we see that the only term that arises occurs in (*i*) when $(i_a, i_b, i_c) = (l_r, i_s, i_j)$. In this case $r = a$ so the coefficient is

$$(-q)^{s+j+r-5} = (-q)^{a+b+c-5}.$$

$\underline{b < a < c:}$
There are two such terms that arise. One occurs in (*i*) when $(i_a, i_b, i_c) = (i_s, l_r, i_j)$ with coefficient $-\hat{q}(-q)^{s+j+r-5}$; in this case $r = b$. The other occurs in (*ii*) when $(i_a, i_b, i_c) = (l_r, i_s, i_j)$ with coefficient $(-q)^{s+j+r-5}$; in this case $i_s < l_r < i_j$ so $r = a-1$. So overall the coefficient is

$$(-q)^{a+b+c-6}(q\hat{q}+1) = (-q)^{a+b+c-4}.$$

$\underline{b < c < a:}$
Three terms of this form arise. One occurs in (*i*) when $(i_a, i_b, i_c) = (i_j, l_r, i_s)$ with coefficient $q\hat{q}(-q)^{s+j+r-5}$; in this case $r = b$. The second occurs in (*ii*) when

$(i_a, i_b, i_c) = (i_j, i_s, l_r)$ with coefficient $-\hat{q}(-q)^{s+j+r-5}$; in this case $i_s < l_r < i_s$ so $r = c-1$. The third occurs in *(iii)* when $(i_a, i_b, i_c) = (l_r, i_s, i_j)$ with coefficient $(-q)^{s+j+r-5}$; in this case $i_s < i_j < l_r$ so $r = a-2$. So overall the coefficient is

$$(-q)^{a+b+c-7}(q^3\hat{q} + q\hat{q} + 1) = (-q)^{a+b+c-3}.$$

So, defining $\theta_{abc} = \begin{cases} 5, & a < b < c; \\ 4, & b < a < c; \\ 3, & b < c < a. \end{cases}$, we may deduce from (4.2.17) and the work just done that,

$$\sum_{s=2}^{2h-1}\sum_{j=s+1}^{2h}(-q)^{s+j-3}x_{i_si_j}[i_1..\check{i_s}..\check{i_j}..i_{2h}]$$
$$= \sum_{a=2}^{2h}\sum_{b=2}^{2h-1}\sum_{c=b+1}^{2h}(-q)^{a+b+c-\theta_{abc}}x_{i_1i_a}x_{i_bi_c}[i_2..\check{i_a}..\check{i_b}..\check{i_c}..i_{2h}],$$

so,

$$\sum_{s=2}^{2h-1}\sum_{j=s+1}^{2h}(-q)^{s+j-3}x_{i_si_j}[i_1..\check{i_s}..\check{i_j}..i_{2h}]$$
$$= \sum_{a=2}^{2h}(-q)^{a+2}x_{i_1i_a}\sum_{b=2}^{2h-1}\sum_{c=b+1}^{2h}(-q)^{b+c-(\theta_{abc}+2)}x_{i_bi_c}[i_2..\check{i_a}..\check{i_b}..\check{i_c}..i_{2h}].$$

Writing $(u_1, ..., u_{2h-4}) = (i_2, .., \check{i_a}, .., \check{i_b}, .., \check{i_c}, .., i_{2h})$ and using the inductive hypothesis gives,

$$\sum_{s=2}^{2h-1}\sum_{j=s+1}^{2h}(-q)^{s+j-3}x_{i_si_j}[i_1..\check{i_s}..\check{i_j}..i_{2h}]$$
$$= \sum_{a=2}^{2h}(-q)^{a+2}x_{i_1i_a}\sum_{b=2}^{2h-1}\sum_{c=b+1}^{2h}[(-q)^{b+c-(\theta_{abc}+2)}x_{i_bi_c}\times$$
$$\times\frac{1}{[h-2]_{q^4}!}\sum_{\sigma\in\Omega_{2h-4}}(-q)^{l(\sigma)}x_{u_{\sigma(1)}u_{\sigma(2)}}\cdots x_{u_{\sigma(2h-5)}u_{\sigma(2h-4)}}]. \quad (4.2.18)$$

Now let $(v_1, ..., v_{2h-2}) = (i_2, .., \check{i_a}, .., i_{2h})$ and define $\sigma' \in \Omega_{2h-2}$ so that $v_{\sigma'(1)} = i_b$, $v_{\sigma'(2)} = i_c$, and $v_{\sigma'(r)} = u_{\sigma(r-2)}$ for $r > 2$. Then,

$$l(\sigma') = \sum_{r=1}^{2h-2}\#\{\sigma'(t) < \sigma'(r) : r < t\}$$
$$= \#\{\sigma'(t) < \sigma'(1) : 1 < t\} + \#\{\sigma'(t) < \sigma'(2) : 2 < t\}$$
$$+ \sum_{r=3}^{2h-2}\#\{\sigma'(t) < \sigma'(r) : r < t\},$$

and similarly to our earlier calculation of $l(w)$ we find that,

$$l(\sigma') = l(\sigma) + \#\{\sigma'(t) < \sigma'(1) : 1 < t\} + \#\{\sigma'(t) < \sigma'(2) : 2 < t\}.$$

Since $(v_1, ..., v_{2h-2}) = (i_2, .., \check{i_a}, .., i_{2h})$ and $v_{\sigma'(1)} = i_b$, $v_{\sigma'(2)} = i_c$ the value of $\#\{\sigma'(t) < \sigma'(1) : 1 < t\} + \#\{\sigma'(t) < \sigma'(2) : 2 < t\}$ will depend on the relative ordering of a, b, c.

$\underline{a < b < c:}$ $\quad l(\sigma') = l(\sigma) + (b-3) + (c-4)$.
$\underline{b < a < c:}$ $\quad l(\sigma') = l(\sigma) + (b-2) + (c-4)$.
$\underline{b < c < a:}$ $\quad l(\sigma') = l(\sigma) + (b-2) + (c-3)$.

So, $l(\sigma') = l(\sigma) + b + c - (\theta_{abc} + 2)$. Hence from (4.2.18) we may deduce,

$$\begin{aligned}\sum_{s=2}^{2h-1} \sum_{j=s+1}^{2h} &(-q)^{s+j-3} x_{i_s i_j} [i_1..\check{i_s}..\check{i_j}..i_{2h}] \\ &= \sum_{a=2}^{2h} (-q)^{a+2} x_{i_1 i_a} \frac{1}{[h-2]_{q^4}!} \sum_{b=2}^{2h-1} \sum_{c=b+1}^{2h} \Big[x_{i_b i_c} \times \\ &\qquad \times \sum_{\substack{\sigma' \in \Omega_{2h-2} \\ v_{\sigma'(1)} = i_b \\ v_{\sigma'(2)} = i_c}} (-q)^{l(\sigma')} x_{v_{\sigma'(3)} v_{\sigma'(4)}} \cdots x_{v_{\sigma'(2h-3)} v_{\sigma'(2h-2)}} \Big],\end{aligned}$$

and so,

$$\begin{aligned}&\sum_{s=2}^{2h-1} \sum_{j=s+1}^{2h} (-q)^{s+j-3} x_{i_s i_j} [i_1..\check{i_s}..\check{i_j}..i_{2h}] \\ &= \sum_{a=2}^{2h} (-q)^{a+2} x_{i_1 i_a} \frac{1}{[h-2]_{q^4}!} \left(\sum_{\sigma' \in \Omega_{2h-2}} (-q)^{l(\sigma')} x_{v_{\sigma'(1)} v_{\sigma'(2)}} \cdots x_{v_{\sigma'(2h-3)} v_{\sigma'(2h-2)}} \right).\end{aligned}$$

By the inductive hypothesis applied to the sum in the brackets we have,

$$\begin{aligned}\sum_{s=2}^{2h-1} \sum_{j=s+1}^{2h} (-q)^{s+j-3} x_{i_s i_j} [i_1..\check{i_s}..\check{i_j}..i_{2h}] &= \sum_{a=2}^{2h} (-q)^{a+2} x_{i_1 i_a} \frac{[h-1]_{q^4}!}{[h-2]_{q^4}!} [v_1 ... v_{2h-2}] \\ &= q^4 [h-1]_{q^4} \sum_{a=2}^{2h} (-q)^{a-2} x_{i_1 i_a} [i_2..\check{i_a}, ..i_{2h}] \\ &= q^4 [h-1]_{q^4} [i_1 ... i_{2h}].\end{aligned}$$

So (4.2.16) is proved and we are done. □

Corollary 4.2.13. *For q not a root of unity,*

$$Pf_q = Pf_{q(No)}.$$

Remark 4.2.14. *We end this section by noting that, when q is not a root of unity, we are now in a position to view $\mathcal{O}_q(Sk_n)$ as a subalgebra of $\mathcal{O}_q(M_n)$. It is natural to ask what properties of $\mathcal{O}_q(Sk_n)$ can be deduced from its relationship with $\mathcal{O}_q(M_n)$, a much-studied object. We give one answer to this question in the next section.*

4.3 Another Laplace-type Expansion

Laplace expansions for *quantum minors* in $\mathcal{O}_q(M_n)$ are well known [32]. We use these known expansions and our realisation of $\mathcal{O}_q(Sk_n)$ as a subalgebra of $\mathcal{O}_q(M_n)$ to produce a Laplace-type expansion for q-Pfaffians which mirrors a classical result.

Let us first fix some notation. For $I = \{i_1 < ... < i_r\}, J = \{j_1 < ... < j_r\} \subseteq \{1, ..., n\}$ let ξ_J^I denote the quantum minor of $\mathcal{O}_q(M_n)$ with rows I and columns J. That is,

$$\xi_J^I := \sum_{\sigma \in S_r} (-q)^{l(\sigma)} t_{i_1 j_{\sigma(1)}} \cdots t_{i_r j_{\sigma(r)}}.$$

Define the symbol $sgn_q(I; J)$ as follows,

$$sgn_q(I;J) = \begin{cases} 0, & \text{if } I \cap J \neq \emptyset; \\ (-q)^{l(I;J)}, & \text{if } I \cap J = \emptyset, \end{cases}$$

where

$$l(I;J) = \#\{(i,j) : i \in I, j \in J, i > j\}.$$

Finally, for $I = \{i_1 < \cdots < i_{2h}\} \subseteq \{1, ..., n\}$ we denote $[i_1 ... i_{2h}]$ by $[I]$. With all this notation in place we may begin to gather the necessary results for our proof.

Proposition 4.3.1. *Let q be a non-root of unity. For $I = \{i_1 < \cdots < i_{2r}\} \subseteq \{1, ..., n\}$.*

$$[I]_{(No)} = \sum_{1 \leq k_1 < \cdots < k_r \leq m} \xi^I_{2k_1-1, 2k_1, ..., 2k_r-1, 2k_r}.$$

Proof. From Noumi [31, (4.43) and (4.59)] we have that,

$$\sum_{l_1 < j_1; ...; l_r < j_r} (v_{l_1} \wedge v_{j_1} \wedge ... \wedge v_{l_r} \wedge v_{j_r} \otimes x_{l_1 j_1} ... x_{l_r j_r})$$

$$= [r]_{q^4}! \sum_{s_1 < \cdots < s_{2r}} \left(v_{s_1} \wedge ... \wedge v_{s_{2r}} \otimes \sum_{1 \leq k_1 < \cdots < k_r \leq m} \xi^{s_1 ... s_{2r}}_{2k_1-1, 2k_1, ..., 2k_r-1, 2k_r} \right) \quad (4.3.1)$$

where the v_i are the canonical generators of the quantum exterior algebra $\bigwedge_q(V)$ with relations

$$v_i \wedge v_i = 0 \ (1 \leq i \leq n) \ \text{ and } \ v_j \wedge v_i = -q v_i \wedge v_j \ (1 \leq i < j \leq n).$$

Rearranging the LHS of (4.3.1) into a sum of terms of the form $v_{s_1} \wedge ... \wedge v_{s_{2r}}$ with $s_1 < \cdots < s_{2r}$ and then equating the term $v_{i_1} \wedge ... \wedge v_{i_{2r}} \otimes *$ with the respective term on the RHS gives us the result. $\square$

The known Laplace expansion of $\mathcal{O}_q(M_n)$ that we shall use is taken from [32]. We note that in that paper the ground field is taken to be $\mathbb{C}$ however the argument used is valid over any field.

Proposition 4.3.2. [32, Proposition 1.1] *Let r, r_1, r_2 be positive integers with $1 \leq r \leq n$ and $r_1 + r_2 = r$. Let I and J be subsets of $\{1, ..., n\}$ with $\#I = \#J = r$. Let J_1, J_2 be subsets of J with $\#J_1 = r_1, \#J_2 = r_2$. Then,*

$$sgn_q(J_1; J_2)\xi_J^I = \sum_{I_1 \sqcup I_2 = I} (-q)^{l(I_1;I_2)} \xi_{J_1}^{I_1} \xi_{J_2}^{I_2},$$

where the summation ranges over all partitions $I_1 \sqcup I_2 = I$ such that $\#I_1 = r_1, \#I_2 = r_2$.

Finally, before we proceed, we state the following combinatorial result taken from [37, Proposition 1.3.17] which we will need,

Lemma 4.3.3. *Let $\mathfrak{S}_{m,d} := \{(a_1, ..., a_m) : d$ of the a_i are 0 and $m - d$ are $1\}$. For $\pi = (a_1, ..., a_m) \in \mathfrak{S}_{m,d}$ define $inv(\pi) := \#\{i < j : a_i > a_j\}$.*

$$\begin{bmatrix} m \\ d \end{bmatrix}_{q^4} = \sum_{\pi \in \mathfrak{S}_{m,d}} q^{4inv(\pi)}.$$

where $\begin{bmatrix} m \\ d \end{bmatrix}_{q^4} := \frac{[m]_{q^4}!}{[d]_{q^4}![m-d]_{q^4}!}$.

We now have all the tools necessary to prove the main result of this section.

Proposition 4.3.4. *Let q be a non-root of unity. Let $n = 2m$. Let $1 \leq d \leq m$.*

$$\begin{bmatrix} m \\ d \end{bmatrix}_{q^4} Pf_q(n) = \sum_{\substack{\sigma \in S_n \\ \sigma(1) < \cdots < \sigma(2d) \\ \sigma(2d+1) < \cdots < \sigma(n)}} (-q)^{l(\sigma)} [\sigma(1) ... \sigma(2d)][\sigma(2d+1) ... \sigma(n)].$$

Proof. Fix $1 \leq d \leq m$ and let $s = m - d$. The following summation will be restricted to disjoint subsets I, J of $\{1, ..., n\}$. Now,

$$\sum_{\substack{\#I = 2d \\ \#J = 2s}} (-q)^{l(I;J)} [I][J] = \sum_{\substack{\#I = 2d \\ \#J = 2s}} (-q)^{l(I;J)} [I]_{(No)} [J]_{(No)},$$

using Proposition 4.2.12. We now apply Proposition 4.3.1,

$$\sum_{\substack{\#I=2d\\ \#J=2s}} (-q)^{l(I;J)}[I][J]$$

$$= \sum_{\substack{\#I=2d\\ \#J=2s}} (-q)^{l(I;J)} \left(\sum_{1\le k_1<\dots<k_d\le m} \xi^{I}_{2k_1-1,\dots,2k_d} \right) \left(\sum_{1\le l_1<\dots<l_s\le m} \xi^{J}_{2l_1-1,\dots,2l_s} \right),$$

rearranging the RHS gives,

$$\sum_{\substack{\#I=2d\\ \#J=2s}} (-q)^{l(I;J)}[I][J] = \sum_{\substack{1\le k_1<\dots<k_d\le m\\ 1\le l_1<\dots<l_s\le m}} \left(\sum_{\substack{\#I=2d\\ \#J=2s}} (-q)^{l(I;J)} \xi^{I}_{2k_1-1,\dots,2k_d} \xi^{J}_{2l_1-1,\dots,2l_s} \right).$$

An application of Proposition 4.3.2 gives,

$$\sum_{\substack{\#I=2d\\ \#J=2s}} (-q)^{l(I;J)}[I][J] = \sum_{\substack{1\le k_1<\dots<k_d\le m\\ 1\le l_1<\dots<l_s\le m}} sgn_q(K;L)\xi^{1,\dots,n}_{1,\dots,n},$$

where $K = \{2k_1 - 1, 2k_1, ..., 2k_d - 1, 2k_d\}$ and $L = \{2l_1 - 1, 2l_1, ..., 2l_s - 1, 2l_s\}$. Now $\xi^{1,\dots,n}_{1,\dots,n} = det_q(T)$, so,

$$\sum_{\substack{\#I=2d\\ \#J=2s}} (-q)^{l(I;J)}[I][J] = \left(\sum_{\substack{1\le k_1<\dots<k_d\le m\\ 1\le l_1<\dots<l_s\le m\\ K\cap L=\emptyset}} (-q)^{l(K;L)} \right) det_q(T),$$

applying Remark 4.1.5 gives,

$$\sum_{\substack{\#I=2d\\ \#J=2s}} (-q)^{l(I;J)}[I][J] = \left(\sum_{\substack{1\le k_1<\dots<k_d\le m\\ 1\le l_1<\dots<l_s\le m\\ K\cap L=\emptyset}} (-q)^{l(K;L)} \right) \mathrm{Pf}_{q(No)}.$$

Using Proposition 4.2.12 we can deduce,

$$\sum_{\substack{\#I=2d\\ \#J=2s}} (-q)^{l(I;J)}[I][J] = \left(\sum_{\substack{1\le k_1<\dots<k_d\le m\\ 1\le l_1<\dots<l_s\le m\\ K\cap L=\emptyset}} (-q)^{l(K;L)} \right) \mathrm{Pf}_q. \tag{4.3.2}$$

Now, recalling that we are restricting our summation to disjoint subsets I, J of $\{1, ..., n\}$, it is not hard to see that,

$$\sum_{\substack{\#I=2d\\ \#J=2s}} (-q)^{l(I;J)}[I][J] = \sum_{\substack{\sigma\in S_n\\ \sigma(1)<\dots<\sigma(2d)\\ \sigma(2d+1)<\dots<\sigma(n)}} (-q)^{l(\sigma)}[\sigma(1)...\sigma(2d)][\sigma(2d+1)...\sigma(n)]. \tag{4.3.3}$$

Combining (4.3.2) and (4.3.3), all that remains of our proof is to show,

$$\sum_{\substack{1\le k_1<\cdots<k_d\le m\\ 1\le l_1<\cdots<l_s\le m\\ K\cap L=\emptyset}} (-q)^{l(K;L)} = \begin{bmatrix} m \\ d \end{bmatrix}_{q^4}.$$

We use Lemma 4.3.3 to reduce the problem to showing that,

$$\sum_{\substack{1\le k_1<\cdots<k_d\le m\\ 1\le l_1<\cdots<l_s\le m\\ K\cap L=\emptyset}} (-q)^{l(K;L)} = \sum_{\pi\in\mathfrak{S}_{m,d}} q^{4inv(\pi)}.$$

Now,

$$\sum_{\substack{1\le k_1<\cdots<k_d\le m\\ 1\le l_1<\cdots<l_s\le m\\ K\cap L=\emptyset}} (-q)^{l(K;L)} = \sum_{1\le k_1<\cdots<k_d\le m} (-q)^{l(K;\tilde{K})},$$

where $\tilde{K} = \{1,...,n\}\backslash K$. We can define a bijection between $\mathfrak{S}_{m,d}$ and $\{(k_1,...,k_d) : 1\le k_1<\cdots<k_d\le m\}$ in the following obvious way,

$$\begin{aligned}\{(k_1,...,k_d) : 1\le k_1<\cdots<k_d\le m\} &\longrightarrow \mathfrak{S}_{m,d}\\ (k_1,...,k_d) &\longmapsto \pi_{k_1,...,k_d}\end{aligned}$$

where

$$\pi_{k_1,...,k_d} = (a_1,...,a_m), \quad \text{with} \quad a_i = \begin{cases} 0, & \text{if } i = k_j \text{ for some } j;\\ 1, & \text{otherwise.}\end{cases}$$

So clearly all that remains to be shown is $l(K;\tilde{K}) = 4inv(\pi_{k_1,...,k_d})$. For clarity let us recall the following notation,

$$\begin{aligned} K &= \{2k_1-1, 2k_1, ..., 2k_d-1, 2k_d\},\\ \tilde{K} &= \{2l_1-1, 2l_1, ..., 2l_s-1, 2l_s\}.\end{aligned}$$

Therefore we may write $a_i = \begin{cases} 0, & \text{if } i = k_j \text{ for some } j;\\ 1, & \text{if } i = l_j \text{ for some } j,\end{cases}$ since $\{k_1,...,k_d\}$ and $\{l_1,...,l_s\}$ partition $\{1,...,m\}$. Hence,

$$\begin{aligned} inv(\pi_{k_1,...,k_d}) &= \#\{i<j : a_i > a_j\}\\ &= \sum_{r=1}^{m} \#\{j<r : a_j > a_r\}\\ &= \sum_{i=1}^{d} \#\{j : k_i > l_j\}. \end{aligned} \tag{4.3.4}$$

By definition,

$$l(K;\tilde{K}) = \#\{(k,l) : k\in K, l\in\tilde{K}, k>l\}.$$

The RHS of this equation, taking into account the definitions of K and $\tilde{K}$, may be expressed as follows,

$$l(K;\tilde{K}) = \sum_{i=1}^{d} \Big(\#\{j : 2k_i > 2l_j\} + \#\{j : 2k_i > 2l_j - 1\} \\ + \#\{j : 2k_i - 1 > 2l_j\} + \#\{j : 2k_i - 1 > 2l_j - 1\} \Big). \quad (4.3.5)$$

Now

$$k_i > l_j \Leftrightarrow 2k_i > 2l_j \Leftrightarrow 2k_i - 1 > 2l_j - 1$$

and since k_i and l_j are distinct integers we also have

$$k_i > l_j \Leftrightarrow k_i - 1/2 > l_j \Leftrightarrow 2k_i - 1 > 2l_j$$

and

$$k_i > l_j \Leftrightarrow k_i > l_j - 1/2 \Leftrightarrow 2k_i > 2l_j - 1.$$

So (4.3.5) may be rewritten as,

$$l(K;\tilde{K}) = 4 \sum_{i=1}^{d} \#\{j : k_i > l_j\},$$

and so by (4.3.4),

$$l(K;\tilde{K}) = 4inv(\pi_{k_1,\ldots,k_d}).$$

□

Remark 4.3.5. *We note that this result serves as a q-analogue for Proposition 1.5.6 which holds in the classical case.*

Chapter 5

Further Properties of $\mathcal{O}_q(Sk_n)$

In the classical case, relations amongst Pfaffians are central to the understanding of skew-symmetric matrices and so-called "Pfaffian varieties". In-fact, in papers such as [7, Section 6], [2], and [11], the use of such relations is key. In the quantum world much work has been done on understanding quantum matrices and the associated *quantum determinantal ideals* through the use of quantum minors, and here too establishing relations amongst the quantum minors, as done in [33] and [16], plays a central role. We note that in [16] an analogous result to Conjecture 3.6.4 is proved. It is only natural, therefore, to look for similar results for quantum skew-symmetric matrices. In this chapter we concentrate on investigating the structure of $\mathcal{O}_q(Sk_n)$ in terms of the relations amongst the set of q-Pfaffians.

5.1 Commutation Relations

We first establish commutation relations between the generators of $\mathcal{O}_q(Sk_n)$ (that is length-2 q-Pfaffians) and q-Pfaffians of arbitrary size. It will prove expedient, in-order to clearly phrase the forthcoming proof of the relations, to first define the following terminology.

Definition 5.1.1. *Let $I = \{b_1 < \cdots < b_m\} \subseteq \{1, ..., n\}$. We say there is a* gap in I at $s+1$ *if $b_{s+1} \neq b_s + 1$ for some $s < m$. The gap is of* length k *if $b_{s+1} = b_s + (k+1)$. The* initial string *of I is $b_1 < \cdots < b_s$ where $b_{j+1} = b_j + 1$ for $1 \leq j < s$ and either $s = m$ or there is a gap in I at $s+1$; in this case the initial string is of* length s. *If there is a gap in I at $s+1$ then* the string in I at $s+1$ *is $b_{s+1} < \cdots < b_t$ where $b_{j+1} = b_j + 1$ for $s+1 \leq j < t$ and either $t = m$ or there is a gap in I at $t+1$; in this case the string is of* length $(t-s)$. *For an integer $1 \leq j \leq n$ we will say that j* lies in the string in I at $s+1$ *if $j = b_r$ for some $s+1 \leq r \leq t$ where $b_{s+1} < \cdots < b_t$ is a string in I (with the obvious adaptation* *to the case when j lies in the initial string of I). Similarly we will say that j* lies in a gap in I at $s+1$ *if there is a gap in I at $s+1$ and $b_s < j < b_{s+1}$. We will drop the suffix "at $s+1$" when using all of the above terminology if it is unimportant or obvious from the context. Finally, we will use phrases such as "the* first *gap of I" or "the gap* after *the* second *string of I" where the implicit ordering of strings and gaps involved is the canonical ordering induced by the ordering of $\mathbb{N}$.*

With these definitions in place we may proceed to prove the following result.

Lemma 5.1.2. *Let $i, j \in \{1, ..., n\}$ with $i < j$ and $I = \{b_1 < \cdots < b_m\} \subseteq \{1, ..., n\}$, where $m \geq 2$ is even. Recall that we denote $[b_1...b_m]$ by $[I]$.*
(a) If $i, j \in I$, then

$$[ij][I] = [I][ij].$$

(b) If $i, j \notin I$, then

$$\begin{aligned}[ij][I] = [I][ij] &+ \hat{q}\sum_{\substack{k\in I\\ k>j}}(-q)^{|(j,k)\cap I|}[ik][I\cup\{j\}\setminus\{k\}]\\ &- \hat{q}\sum_{\substack{k\in I\\ k<i}}(-q)^{-|(k,i)\cap I|}[kj][I\cup\{i\}\setminus\{k\}]\\ &+ \hat{q}^2\sum_{\substack{r,s\in I\\ r<i,s>j}}(-q)^{|(j,s)\cap I|-|(r,i)\cap I|}[rs][I\cup\{i,j\}\setminus\{r,s\}].\end{aligned}$$

(c) If $i \notin I, j \in I$, then

$$[ij][I] = q[I][ij] - \hat{q}\sum_{\substack{k\in I\\ k<i}}(-q)^{-|(k,i)\cap I|}[kj][I\cup\{i\}\setminus\{k\}].$$

(d) If $i \in I, j \notin I$, then

$$[ij][I] = q^{-1}[I][ij] + \hat{q}\sum_{\substack{k\in I\\ k>j}}(-q)^{|(j,k)\cap I|}[ik][I\cup\{j\}\setminus\{k\}].$$

Before commencing with the proof we make the following remark, reminding ourselves of an important point.

Remark 5.1.3. *Throughout the proofs in this chapter we will be using the $U_q(\mathfrak{gl}_n)$-action on $\mathcal{O}_q(Sk_n)$ to form inductive arguments and so we should technically require q to be a non-root of unity. Recalling Remark 3.4.1 we see that we may drop this restriction on q.*

Proof. (a) There is a K-algebra isomorphism

$$\mathcal{O}_q(Sk_m) \cong K\langle a_{rc} : r, c \in I\rangle \subseteq \mathcal{O}_q(Sk_n),$$

sending $\mathrm{Pf}_q(m)$ to $[I]$. So by Proposition 3.5.3, in which the centrality of $\mathrm{Pf}_q(m)$ in $\mathcal{O}_q(Sk_m)$ is proven, part (a) follows.

(b),(c),(d) Our first remark is to note that for $m = 2$ these relations are just the defining relations of $\mathcal{O}_q(Sk_n)$ given in (3.1.1)-(3.1.6). To prove the more general case we start with the specific commutation relations established in Lemma 3.5.1 and Lemma 3.5.2. We note that the proofs of Lemma 3.5.1 and Lemma 3.5.2 only rely on the relative orderings of $1, 2$ and the l_i. So in-fact Lemma 3.5.1 is an instance of relation (b) in the case $i < j < b_1$ and Lemma 3.5.2 is an instance of relation (c) in the case $i < j = b_1$. Fix $[I]$. We know that $[ij]$ and $[I]$ satisfy the required commutation relations when $i < j < b_1$ and $i < j = b_1$. Given these established starting points we will use the action of $U_q(\mathfrak{gl}_n)$ on $\mathcal{O}_q(Sk_n)$ to form an inductive argument proving that $[ij]$ and $[I]$ satisfy the appropriate relation for all $1 \leq i < j \leq n$. We do this by first holding $i < b_1$ fixed and using the action of the F_k's (3.1.11) to 'move j along', that is from cases where we know that $[ij]$ and $[I]$ satisfy the appropriate relation we will use the F_k-action to deduce that $[i, j+1]$ and $[I]$ satisfy the correct relation. Once we know that the appropriate commutation relation holds between $[I]$ and $[ij]$ for all $1 \leq i < j \leq n$ with $i < b_1$ we will then hold j fixed, with either $j \in I$ or $j \notin I$, and 'move i along', so that we can deduce that $[I]$ satisfies the correct relation with $[ij]$ for all $1 \leq i < j \leq n$.

<u>Until stated otherwise $i < b_1$:</u>

For the sake of clarity let us just repeat the specifics of the situation. I is fixed. We also fix $i < b_1$. We want to show that $[I]$ and $[ij]$ satisfy the appropriate relation (that is either (b) or (c) since $i \notin I$) for all $i < j \leq n$. Given that we know $[I]$ and $[ij]$ satisfy the right relation when $i < j < b_1$ and $i < j = b_1$ (that is when j comes before I and when it lies in the first place of the initial string of I) we will use the F_k-action to 'move j along' and show that the right relations hold for all j. We note that (b) and (c) are simplified in this case since the fact that $i < b_1$ means that in both relations there are terms that vanish.

Our first step is to move j along the initial string of I. We will prove by induction on the length of the initial string of I, call this r, that if j lies in the initial string then $[ij]$ and $[I]$ satisfy relation (c). The case $r = 1$ is already done since then $j = b_1$. We turn to the inductive step. Assume that I is such that $r = k + 1$. Then inductively we know that $[ib_t]$ and $[I]$ obey relation (c) for

$t = 1, ..., k$, so in particular we have,

$$[ib_k][I] - q[I][ib_k] = 0.$$

Recalling the comultiplication defined in (3.1.9), we act on this relation by F_{b_k},

$$F_{b_k}([ib_k])\,L_{b_k}^{-1}L_{b_k+1}([I]) + [ib_k]F_{b_k}([I]) \\ - qF_{b_k}([I])\,L_{b_k}^{-1}L_{b_k+1}([ib_k]) - q[I]F_{b_k}([ib_k]) = 0.$$

Throughout this proof we will use (3.1.11), Lemma 3.1.5, and Remark 3.1.6 to work out the various F_s-actions. In this case, noting that $b_k + 1 = b_{k+1} \in I$, the equation simplifies to,

$$[ib_{k+1}][I] - q[I][ib_{k+1}] = 0,$$

which is relation (c), since $i < b_1$. So the inductive step is complete.

We turn next to moving j off of the initial string of I. Suppose I has initial string $b_1 < \cdots < b_r$. We show by induction on r that $[i, b_r + 1]$ and $[I]$ satisfy relation (b). From above we know,

$$[ib_r][I] - q[I][ib_r] = 0.$$

Acting on this relation by F_{b_r} gives,

$$F_{b_r}([ib_r])\,L_{b_r}^{-1}L_{b_r+1}([I]) + [ib_r]F_{b_r}([I]) \\ - qF_{b_r}([I])\,L_{b_r}^{-1}L_{b_r+1}([ib_r]) - q[I]F_{b_r}([ib_r]) = 0.$$

As before we use (3.1.11), Lemma 3.1.5, and Remark 3.1.6 to simplify. Since $b_r + 1 \notin I$ it follows that,

$$q^{-1}[i, b_r+1][I] + [ib_r][I \cup \{b_r+1\} \setminus \{b_r\}] - [I \cup \{b_r+1\} \setminus \{b_r\}][ib_r] - q[I][i, b_r+1] = 0.$$

Multiplying through by q and rearranging gives,

$$[i, b_r + 1][I] - q^2[I][i, b_r + 1] \\ = -q\left([ib_r][I \cup \{b_r + 1\} \setminus \{b_r\}] - [I \cup \{b_r + 1\} \setminus \{b_r\}][ib_r]\right). \quad (5.1.1)$$

Consider the base case $r = 1$. Then $I \cup \{b_r+1\} \setminus \{b_r\} = b_1+1 < b_2 < \cdots < b_m$. We are assuming in this section that $i < b_1$ so $i < b_1+1$ and we may use Lemma 3.5.1 (translating it into the appropriate form) to say,

$$[ib_1][I \cup \{b_1 + 1\} \setminus \{b_1\}] - [I \cup \{b_1 + 1\} \setminus \{b_1\}][ib_1] \\ = \hat{q} \sum_{\substack{k \in I \cup \{b_1+1\} \setminus \{b_1\} \\ k > b_1}} (-q)^{|(b_1,k) \cap (I \cup \{b_1+1\} \setminus \{b_1\})|}[ik][I \cup \{b_1 + 1\} \setminus \{k\}].$$

Substituting this into the RHS of (5.1.1) with $r = 1$ gives,

$$\begin{aligned}
&[i, b_1 + 1][I] - q^2[I][i, b_1 + 1] \\
&\quad = -q\hat{q} \sum_{\substack{k \in I \cup \{b_1+1\} \setminus \{b_1\} \\ k > b_1}} (-q)^{|(b_1,k) \cap (I \cup \{b_1+1\} \setminus \{b_1\})|} [ik][I \cup \{b_1 + 1\} \setminus \{k\}] \\
&\quad = -q\hat{q}[i, b_1 + 1][I] \\
&\qquad\quad - q\hat{q} \sum_{\substack{k \in I \cup \{b_1+1\} \setminus \{b_1\} \\ k > b_1+1}} (-q)^{|(b_1,k) \cap (I \cup \{b_1+1\} \setminus \{b_1\})|} [ik][I \cup \{b_1 + 1\} \setminus \{k\}].
\end{aligned}$$

Since $\{k \in I \cup \{b_1 + 1\} \setminus \{b_1\} : k > b_1 + 1\} = \{k \in I : k > b_1 + 1\}$ we can rewrite the index of the summation on the RHS; together with rearranging this gives,

$$\begin{aligned}
&q^2[i, b_1 + 1][I] - q^2[I][i, b_1 + 1,] \\
&\qquad = -q\hat{q} \sum_{\substack{k \in I \\ k > b_1+1}} (-q)^{|(b_1,k) \cap (I \cup \{b_1+1\} \setminus \{b_1\})|} [ik][I \cup \{b_1 + 1\} \setminus \{k\}] \\
&[i, b_1 + 1][I] - [I][i, b_1 + 1] \\
&\qquad = -q^{-1}\hat{q} \sum_{\substack{k \in I \\ k > b_1+1}} (-q)^{|(b_1,k) \cap (I \cup \{b_1+1\} \setminus \{b_1\})|} [ik][I \cup \{b_1 + 1\} \setminus \{k\}] \\
&\qquad = \hat{q} \sum_{\substack{k \in I \\ k > b_1+1}} (-q)^{|(b_1,k) \cap (I \cup \{b_1+1\} \setminus \{b_1\})|-1} [ik][I \cup \{b_1 + 1\} \setminus \{k\}] \\
&\qquad = \hat{q} \sum_{\substack{k \in I \\ k > b_1+1}} (-q)^{|(b_1+1,k) \cap I|} [ik][I \cup \{b_1 + 1\} \setminus \{k\}]
\end{aligned}$$

where the last equality holds since $|(b_1+1, k) \cap I| = |(b_1, k) \cap (I \cup \{b_1+1\} \setminus \{b_1\})| - 1$. The above equation is relation (b), as required. So the base case of our induction is done and we turn to the inductive step. Now $I \cup \{b_r + 1\} \setminus \{b_r\}$ has initial string of length $r - 1$ and $b_r = b_{r-1} + 1$, so by the inductive hypothesis we have,

$$\begin{aligned}
&[ib_r][I \cup \{b_r + 1\} \setminus \{b_r\}] - [I \cup \{b_r + 1\} \setminus \{b_r\}][ib_r] \\
&\qquad = \hat{q} \sum_{\substack{k \in I \cup \{b_r+1\} \setminus \{b_r\} \\ k > b_r}} (-q)^{|(b_r,k) \cap (I \cup \{b_r+1\} \setminus \{b_r\})|} [ik][I \cup \{b_r + 1\} \setminus \{k\}].
\end{aligned}$$

We can now substitute this into the RHS of (5.1.1) and the rest of the proof of the inductive step follows in an exactly similar manner to the $r = 1$ case. So we have shown that we can move j off of the initial string of I and the appropriate relation (in this case relation (b)) is satisfied.

Next we turn to moving j along the first gap in I. We have just shown that if j lies immediately after the initial string, $b_1 < \cdots < b_r$, of I then $[ij]$ and $[I]$

satisfy relation (b). With this as the base case, we will now show by a quick induction on k that relation (b) is also satisfied if $j = b_r + k$, for any $k \geq 1$ if $r = m$ or for any $1 \leq k \leq p$ if the first gap in I has length p. We now prove the inductive step. Suppose $j, j+1 \notin I$. For the inductive hypothesis we assume,

$$[ij][I] - [I][ij] = \hat{q} \sum_{\substack{k \in I \\ k > j}} (-q)^{|(j,k) \cap I|} [ik][I \cup \{j\} \setminus \{k\}]. \tag{5.1.2}$$

We proceed by acting on this by F_j. Let us once again remind ourselves of (3.1.9),

$$\Delta(F_j) = F_j \otimes L_j^{-1} L_{j+1} + 1 \otimes F_j.$$

Since $j, j+1 \notin I$ it follows that,

$$F_j([I]) = 0,$$
$$L_j^{-1} L_{j+1}([I]) = [I],$$
$$F_j([I \cup \{j\} \setminus \{k\}]) = [I \cup \{j+1\} \setminus \{k\}],$$

for k such that $k > j$ and $k \in I$. Hence acting on (5.1.2) by F_j gives,

$$[i, j+1][I] - [I][i, j+1] = \hat{q} \sum_{\substack{k \in I \\ k > j}} (-q)^{|(j,k) \cap I|} [ik][I \cup \{j+1\} \setminus \{k\}].$$

Since $j, j+1 \notin I$ it follows that $|(j,k) \cap I| = |(j+1,k) \cap I|$. So we may rewrite the above equation as,

$$[i, j+1][I] - [I][i, j+1] = \hat{q} \sum_{\substack{k \in I \\ k > j+1}} (-q)^{|(j+1,k) \cap I|} [ik][I \cup \{j+1\} \setminus \{k\}].$$

This is relation (b) (since we are in the case $i < b_1$).

If I contains more than one string, then for our proof of the case $i < b_1$ to be complete we must show that we may 'move j from a gap to a string'. We do this now. Suppose $j \notin I$ and $j+1 \in I$ and assume inductively that,

$$[ij][I] - [I][ij] = \hat{q} \sum_{\substack{k \in I \\ k > j}} (-q)^{|(j,k) \cap I|} [ik][I \cup \{j\} \setminus \{k\}].$$

As usual we act on this by F_j giving,

$$q[i, j+1][I] - [I][i, j+1] = \hat{q}[i, j+1][I]. \tag{5.1.3}$$

To see why this is so, the main point to note is that, since $j+1 \in I$, by Lemma 3.1.5 it follows that unless $k = j+1$,

$$F_j([I \cup \{j\} \setminus \{k\}]) = 0,$$

and if $k = j + 1$ then,

$$F_j([I \cup \{j\} \setminus \{k\}]) = [I].$$

Simplifying (5.1.3) results in,

$$[i, j+1][I] - q[I][i, j+1] = 0,$$

which is relation (c) as required.

We have now done enough to show that we may deduce that $[I]$ and $[i, j+1]$ satisfy the appropriate relation from the knowledge that $[I]$ and $[ij]$ do, for all the possible transitions from j to $j+1$. We should make clear that though technically we have only shown that we can move j off of the initial string of I, the argument used can easily be adapted to work for moving j off of any string. All that is required for the adapted argument to hold is that the appropriate relation already be established for j lying in the gap preceding the string in question. This will be so because of the way our proof is constructed, since we are 'moving j from low to high values'. So one can see that the work done above allows us to make the transition from j to $j+1$ no matter if this keeps us in a string, takes us from a string to a gap, keeps us in a gap, or takes us from a gap to a string. This is all that is required to finish the case $i < b_1$.

<u>We now turn to more general cases where we allow the possibility that $i \geq b_1$.</u>

We now know that the appropriate relation is satisfied by $[I]$ and $[ij]$ for any $1 \leq i < j \leq n$ with the condition that $i < b_1$. So we will now think of j as fixed and from the known cases, that is from the cases $i < b_1$, we shall 'move i along' to deduce the required relations for any $i < j$. The details of this process will of course depend on whether j lies in I or not.

<u>$j \notin I$</u>:

As noted at the beginning of the proof the cases when $j < b_1$ have been done in Lemma 3.5.1. So we shall assume that $j > b_1$. We will initially make the assumption that $b_1 > 1$ so that we may use the work done in the previous section to form the base case of an inductive argument. That is, so we have something to 'move i along' from.

Our first step is to show that we may 'move i onto the initial string of I'. The work done above shows that $[ij]$ and $[I]$ satisfy relation (b) for all $i < b_1$. Since

$b_1 > 1$ it follows that $1 \leq b_1 - 1 < b_1$ so we know,

$$[b_1 - 1, j][I] - [I][b_1 - 1, j] = \hat{q} \sum_{\substack{k \in I \\ k > j}} (-q)^{|(j,k) \cap I|} [b_1 - 1, k][I \cup \{j\} \setminus \{k\}].$$

Noting that $F_{b_1 - 1}$ kills $[I]$ and $[I \cup \{j\} \setminus \{k\}]$ where $k > j > b_1$, we act on the above equation by F_{b_1-1} giving,

$$q[b_1 j][I] - [I][b_1 j] = q\hat{q} \sum_{\substack{k \in I \\ k > j}} (-q)^{|(j,k) \cap I|} [b_1 k][I \cup \{j\} \setminus \{k\}].$$

Dividing by q gives,

$$[b_1 j][I] - q^{-1}[I][b_1 j] = \hat{q} \sum_{\substack{k \in I \\ k > j}} (-q)^{|(j,k) \cap I|} [b_1 k][I \cup \{j\} \setminus \{k\}]$$

which is relation (d) as required.

We next turn to moving i along the string. Suppose $i, i+1 \in I$ and that we know $[I]$ and $[ij]$ satisfy relation (d),

$$[ij][I] - q^{-1}[I][ij] = \hat{q} \sum_{\substack{k \in I \\ k > j}} (-q)^{|(j,k) \cap I|} [ik][I \cup \{j\} \setminus \{k\}].$$

Since $i, i+1 \in I$, F_i kills $[I]$. Since $j > b_1$ and $j \notin I$ it follows that $j > i+1$ and so F_i also kills $[I \cup \{j\} \setminus \{k\}]$ for $k > j$. So acting on the above relation by F_i gives,

$$[i+1, j][I] - q^{-1}[I][i+1, j] = \hat{q} \sum_{\substack{k \in I \\ k > j}} (-q)^{|(j,k) \cap I|} [i+1, k][I \cup \{j\} \setminus \{k\}],$$

which is relation (d) as required. So we may move i along the first string of I and the required relations have been shown to hold.

If j lies immediately after the initial string of I then we are done, so suppose it does not. Then we must show that we may move i from a string to a gap. Let r be the length of the initial string of I. We are assuming that $j > b_r + 1$. We have just proved that,

$$[b_r j][I] - q^{-1}[I][b_r j] = \hat{q} \sum_{\substack{k \in I \\ k > j}} (-q)^{|(j,k) \cap I|} [b_r k][I \cup \{j\} \setminus \{k\}].$$

We act on this by F_{b_r} giving,

$$\begin{aligned}q^{-1}[b_r+1,j][I] + [b_r j][I \cup \{b_r+1\} \setminus \{b_r\}] \\ - q^{-2}[I \cup \{b_r+1\} \setminus \{b_r\}][b_r j] - q^{-1}[I][b_r+1,j] \\ = q^{-1}\hat{q} \sum_{\substack{k \in I \\ k>j}} (-q)^{|(j,k)\cap I|}[b_r+1,k][I \cup \{j\} \setminus \{k\}] \\ + \hat{q} \sum_{\substack{k \in I \\ k>j}} (-q)^{|(j,k)\cap I|}[b_r k][I \cup \{b_r+1, j\} \setminus \{b_r, k\}],\end{aligned}$$

Multiplying by q and rearranging gives,

$$\begin{aligned}[b_r+1,j][I] = [I][b_r+1,j] \\ - q[b_r j][I \cup \{b_r+1\} \setminus \{b_r\}] + q^{-1}[I \cup \{b_r+1\} \setminus \{b_r\}][b_r j] \\ + \hat{q} \sum_{\substack{k \in I \\ k>j}} (-q)^{|(j,k)\cap I|}[b_r+1,k][I \cup \{j\} \setminus \{k\}] \\ + q\hat{q} \sum_{\substack{k \in I \\ k>j}} (-q)^{|(j,k)\cap I|}[b_r k][I \cup \{b_r+1, j\} \setminus \{b_r, k\}]. \quad (5.1.4)\end{aligned}$$

We will prove by induction on r that $[b_r+1, j]$ and $[I]$ satisfy relation (b). Consider the case $r = 1$. Then $I \cup \{b_r+1\} \setminus \{b_r\} = b_1+1 < b_2 < \cdots < b_m$ and $b_1 < b_1+1$ so we know from the work done in a previous section of the proof that,

$$\begin{aligned}[b_1 j][I \cup \{b_1+1\} \setminus \{b_1\}] = [I \cup \{b_1+1\} \setminus \{b_1\}][b_1 j] \\ + \hat{q} \sum_{\substack{k \in I \cup \{b_1+1\} \setminus \{b_1\} \\ k>j}} (-q)^{|(j,k)\cap(I \cup \{b_1+1\} \setminus \{b_1\})|}[b_1 k][I \cup \{b_1+1, j\} \setminus \{b_1, k\}].\end{aligned}$$

We use this to rewrite $q^{-1}[I \cup \{b_1+1\} \setminus \{b_1\}][b_1 j]$ in (5.1.4) in the $r = 1$ case,

$$\begin{aligned}[b_1+1,j][I] = [I][b_1+1,j] \\ - q[b_1 j][I \cup \{b_1+1\} \setminus \{b_1\}] + q^{-1}[b_1 j][I \cup \{b_1+1\} \setminus \{b_1\}] \\ - q^{-1}\hat{q} \sum_{\substack{k \in I \cup \{b_1+1\} \setminus \{b_1\} \\ k>j}} (-q)^{|(j,k)\cap(I \cup \{b_1+1\} \setminus \{b_1\})|}[b_1 k][I \cup \{b_1+1, j\} \setminus \{b_1, k\}] \\ + \hat{q} \sum_{\substack{k \in I \\ k>j}} (-q)^{|(j,k)\cap I|}[b_1+1,k][I \cup \{j\} \setminus \{k\}] \\ + q\hat{q} \sum_{\substack{k \in I \\ k>j}} (-q)^{|(j,k)\cap I|}[b_1 k][I \cup \{b_1+1, j\} \setminus \{b_1, k\}].\end{aligned}$$

We note that since $j > b_1+1$, $\{k \in I \cup \{b_1+1\} \setminus \{b_1\} : k > j\} = \{k \in I : k > j\}$ and $|(j,k) \cap (I \cup \{b_1+1\} \setminus \{b_1\})| = |(j,k) \cap I|$. So the fourth term on the RHS can be rewritten; together with grouping like terms, this gives,

$$\begin{aligned}[b_1+1,j][I] = [I][b_1+1,j] &- \hat{q}[b_1j][I \cup \{b_1+1\} \setminus \{b_1\}] \\ &- q^{-1}\hat{q}\sum_{\substack{k\in I\\ k>j}}(-q)^{|(j,k)\cap I|}[b_1k][I \cup \{b_1+1,j\} \setminus \{b_1,k\}] \\ &+ \hat{q}\sum_{\substack{k\in I\\ k>j}}(-q)^{|(j,k)\cap I|}[b_1+1,k][I \cup \{j\} \setminus \{k\}] \\ &+ q\hat{q}\sum_{\substack{k\in I\\ k>j}}(-q)^{|(j,k)\cap I|}[b_1k][I \cup \{b_1+1,j\} \setminus \{b_1,k\}].\end{aligned}$$

Combining the third and fifth terms of the RHS produces,

$$\begin{aligned}[b_1+1,j][I] = [I][b_1+1,j] &- \hat{q}[b_1j][I \cup \{b_1+1\} \setminus \{b_1\}] \\ &+ \hat{q}\sum_{\substack{k\in I\\ k>j}}(-q)^{|(j,k)\cap I|}[b_1+1,k][I \cup \{j\} \setminus \{k\}] \\ &+ \hat{q}^2\sum_{\substack{k\in I\\ k>j}}(-q)^{|(j,k)\cap I|}[b_1k][I \cup \{b_1+1,j\} \setminus \{b_1,k\}].\end{aligned}$$

We rewrite the second and fourth terms on the RHS so that they are of the required form, noting, for example, that $\{k \in I : k < b_1+1\} = \{b_1\}$ and $|(r, b_1+1) \cap I| = 0$,

$$\begin{aligned}[b_1+1,j][I] = [I][b_1+1,j] &- \hat{q}\sum_{\substack{k\in I\\ k<b_1+1}}(-q)^{-|(k,b_1+1)\cap I|}[kj][I \cup \{b_1+1\} \setminus \{k\}] \\ &+ \hat{q}\sum_{\substack{k\in I\\ k>j}}(-q)^{|(j,k)\cap I|}[b_1+1,k][I \cup \{j\} \setminus \{k\}] \\ &+ \hat{q}^2\sum_{\substack{r,k\in I\\ r<b_1+1,k>j}}(-q)^{|(j,k)\cap I|-|(r,b_1+1)\cap I|}[rk][I \cup \{b_1+1,j\} \setminus \{r,k\}].\end{aligned}$$

This is an instance of relation (b), so the case when $r=1$ is done.

To show that in general $[b_r+1,j]$ and $[I]$ satisfy relation (b), we must now do the inductive step. Now, if I has an initial string of length r then $I \cup \{b_r+1\} \setminus \{b_r\}$ has initial string of length $r-1$, so by the inductive hypothesis we know that $[b_rj]$ and $[I \cup \{b_r+1\} \setminus \{b_r\}]$ satisfy relation (b). Using this fact we rewrite the third term on the RHS of (5.1.4) to produce,

$$
\begin{aligned}
[b_r+1,j][I] = [I][b_r+1,j]& \\
- q[b_rj][I\cup\{b_r+1\}\setminus\{b_r\}] + \Bigg(q^{-1}[b_rj][I\cup\{b_r+1\}\setminus\{b_r\}]& \\
- q^{-1}\hat{q} \sum_{\substack{k\in I\cup\{b_r+1\}\setminus\{b_r\} \\ k>j}} (-q)^{|(j,k)\cap(I\cup\{b_r+1\}\setminus\{b_r\})|}[b_rk][I\cup\{b_r+1,j\}\setminus\{b_r,k\}]& \\
+ q^{-1}\hat{q} \sum_{\substack{k\in I\cup\{b_r+1\}\setminus\{b_r\} \\ k<b_r}} (-q)^{-|(k,b_r)\cap(I\cup\{b_r+1\}\setminus\{b_r\})|}[kj][I\cup\{b_r+1\}\setminus\{k\}]& \\
- q^{-1}\hat{q}^2 \sum_{\substack{t,s\in I\cup\{b_r+1\}\setminus\{b_r\} \\ t<b_r, s>j}} (-q)^{|(j,s)\cap(I\cup\{b_r+1\}\setminus\{b_r\})|-|(t,b_r)\cap(I\cup\{b_r+1\}\setminus\{b_r\})|}\times& \\
\times [ts][I\cup\{b_r+1,j\}\setminus\{t,s\}]\Bigg)& \\
+ \hat{q}\sum_{\substack{k\in I \\ k>j}}(-q)^{|(j,k)\cap I|}[b_r+1,k][I\cup\{j\}\setminus\{k\}]& \\
+ q\hat{q}\sum_{\substack{k\in I \\ k>j}}(-q)^{|(j,k)\cap I|}[b_rk][I\cup\{b_r+1,j\}\setminus\{b_r,k\}].&
\end{aligned}
$$

We group together the second and third terms on the RHS. The fourth, fifth, and sixth terms are rewritten as follows: the indexes of the summations and the indices of $(-q)$ are simplified. We note that $j > b_r + 1$, so, for example in the fourth term, we may observe that, $|(j,k)\cap(I\cup\{b_r+1\}\setminus\{b_r\})| = |(j,k)\cap I|$. All of this rewriting gives us,

$$
\begin{aligned}
[b_r+1,j][I] = [I][b_r+1,j]& \\
- \hat{q}[b_rj][I\cup\{b_r+1\}\setminus\{b_r\}]& \\
- q^{-1}\hat{q}\sum_{\substack{k\in I \\ k>j}}(-q)^{|(j,k)\cap I|}[b_rk][I\cup\{b_r+1,j\}\setminus\{b_r,k\}]& \\
+ q^{-1}\hat{q}\sum_{\substack{k\in I \\ k<b_r}}(-q)^{-|(k,b_r)\cap I|}[kj][I\cup\{b_r+1\}\setminus\{k\}]& \\
- q^{-1}\hat{q}^2 \sum_{\substack{t,s\in I \\ t<b_r, s>j}} (-q)^{|(j,s)\cap I|-|(t,b_r)\cap I|}[ts][I\cup\{b_r+1,j\}\setminus\{t,s\}]& \\
+ \hat{q}\sum_{\substack{k\in I \\ k>j}}(-q)^{|(j,k)\cap I|}[b_r+1,k][I\cup\{j\}\setminus\{k\}]& \\
+ q\hat{q}\sum_{\substack{k\in I \\ k>j}}(-q)^{|(j,k)\cap I|}[b_rk][I\cup\{b_r+1,j\}\setminus\{b_r,k\}].&
\end{aligned}
$$

amalgamating the third and seventh terms on the RHS gives,

$$
\begin{aligned}
[b_r+1,j][I] = [I][b_r+1,j] & \\
& - \hat{q}[b_r j][I \cup \{b_r+1\} \setminus \{b_r\}] \\
& + q^{-1}\hat{q} \sum_{\substack{k \in I \\ k < b_r}} (-q)^{-|(k,b_r)\cap I|}[kj][I \cup \{b_r+1\} \setminus \{k\}] \\
& - q^{-1}\hat{q}^2 \sum_{\substack{t,s \in I \\ t<b_r, s>j}} (-q)^{|(j,s)\cap I| - |(t,b_r)\cap I|}[ts][I \cup \{b_r+1, j\} \setminus \{t,s\}] \\
& + \hat{q} \sum_{\substack{k \in I \\ k>j}} (-q)^{|(j,k)\cap I|}[b_r+1,k][I \cup \{j\} \setminus \{k\}] \\
& + \hat{q}^2 \sum_{\substack{k \in I \\ k>j}} (-q)^{|(j,k)\cap I|}[b_r k][I \cup \{b_r+1, j\} \setminus \{b_r, k\}].
\end{aligned}
$$

Since $|(k,b_r)\cap I| + 1 = |(k,b_r+1)\cap I|$ and $|(b_r,b_r+1)\cap I| = 0$ we can rewrite the above as,

$$
\begin{aligned}
[b_r+1,j][I] = [I][b_r+1,j] & \\
& - \hat{q}[b_r j][I \cup \{b_r+1\} \setminus \{b_r\}] \\
& - \hat{q} \sum_{\substack{k \in I \\ k < b_r}} (-q)^{-|(k,b_r+1)\cap I|}[kj][I \cup \{b_r+1\} \setminus \{k\}] \\
& + \hat{q}^2 \sum_{\substack{t,s \in I \\ t<b_r, s>j}} (-q)^{|(j,s)\cap I| - |(t,b_r+1)\cap I|}[ts][I \cup \{b_r+1, j\} \setminus \{t,s\}] \\
& + \hat{q} \sum_{\substack{k \in I \\ k>j}} (-q)^{|(j,k)\cap I|}[b_r+1,k][I \cup \{j\} \setminus \{k\}] \\
& + \hat{q}^2 \sum_{\substack{k \in I \\ k>j}} (-q)^{|(j,k)\cap I| - |(b_r,b_r+1)\cap I|}[b_r k][I \cup \{b_r+1, j\} \setminus \{b_r, k\}].
\end{aligned}
$$

Finally, grouping together the second and third terms, and the fourth and sixth terms on the RHS gives,

$$
\begin{aligned}
[b_r+1,j][I] = [I][b_r+1,j] & \\
& - \hat{q} \sum_{\substack{k \in I \\ k < b_r+1}} (-q)^{-|(k,b_r+1)\cap I|}[kj][I \cup \{b_r+1\} \setminus \{k\}] \\
& + \hat{q}^2 \sum_{\substack{t,s \in I \\ t<b_r+1, s>j}} (-q)^{|(j,s)\cap I| - |(t,b_r+1)\cap I|}[ts][I \cup \{b_r+1, j\} \setminus \{t,s\}] \\
& + \hat{q} \sum_{\substack{k \in I \\ k>j}} (-q)^{|(j,k)\cap I|}[b_r+1,k][I \cup \{j\} \setminus \{k\}],
\end{aligned}
$$

which is relation (b) as required, completing the inductive step. So we have just shown that if i lies immediately after the initial string, $b_1 < \cdots < b_r$, of I then $[ij]$ and $[I]$ satisfy relation (b). It remains to be shown that we can move i along a gap in I and that we can move i from a gap onto a string.

Firstly let us show that we can move i along a gap. Suppose that $i, i+1 \notin I$ and assume that we know,

$$\begin{aligned}[ij][I] = [I][ij] &+ \hat{q}\sum_{\substack{k\in I\\ k>j}}(-q)^{|(j,k)\cap I|}[ik][I\cup\{j\}\setminus\{k\}]\\ &- \hat{q}\sum_{\substack{k\in I\\ k<i}}(-q)^{-|(k,i)\cap I|}[kj][I\cup\{i\}\setminus\{k\}]\\ &+ \hat{q}^2\sum_{\substack{r,s\in I\\ r<i,s>j}}(-q)^{|(j,s)\cap I|-|(r,i)\cap I|}[rs][I\cup\{i,j\}\setminus\{r,s\}]. \qquad (5.1.5)\end{aligned}$$

Keeping in mind that $i+1 \notin I$ we act on this by F_i yielding,

$$\begin{aligned}[i+1,j][I] = [I][i+1,j] &+ \hat{q}\sum_{\substack{k\in I\\ k>j}}(-q)^{|(j,k)\cap I|}[i+1,k][I\cup\{j\}\setminus\{k\}]\\ &- \hat{q}\sum_{\substack{k\in I\\ k<i}}(-q)^{-|(k,i)\cap I|}[kj][I\cup\{i+1\}\setminus\{k\}]\\ &+ \hat{q}^2\sum_{\substack{r,s\in I\\ r<i,s>j}}(-q)^{|(j,s)\cap I|-|(r,i)\cap I|}[rs][I\cup\{i+1,j\}\setminus\{r,s\}].\end{aligned}$$

Since both $i, i+1 \notin I$ we may rewrite this as,

$$\begin{aligned}[i+1,j][I] = [I][i+1,j] &+ \hat{q}\sum_{\substack{k\in I\\ k>j}}(-q)^{|(j,k)\cap I|}[i+1,k][I\cup\{j\}\setminus\{k\}]\\ &- \hat{q}\sum_{\substack{k\in I\\ k<i+1}}(-q)^{-|(k,i+1)\cap I|}[kj][I\cup\{i+1\}\setminus\{k\}]\\ &+ \hat{q}^2\sum_{\substack{r,s\in I\\ r<i+1,s>j}}(-q)^{|(j,s)\cap I|-|(r,i+1)\cap I|}[rs][I\cup\{i+1,j\}\setminus\{r,s\}],\end{aligned}$$

which is relation (b) as required.

Now we show that we can move i from a gap onto a string. Suppose that $i \notin I$ and $i+1 \in I$. We again assume that $[I]$ and $[ij]$ satisfy relation (b), that is (5.1.5) and we act on this relation by F_i. This time $i+1 \in I$, so the third and fourth terms are killed, since, for example, $F_i\left([I\cup\{i\}\setminus\{k\}]\right) = 0$. So acting by

F_i on (5.1.5) produces,

$$q[i+1,j][I] = [I][i+1,j] + q\hat{q}\sum_{\substack{k\in I\\k>j}}(-q)^{|(j,k)\cap I|}[i+1,k][I\cup\{j\}\setminus\{k\}].$$

This is relation (d) as required. We have now done enough to show that we can move i from a gap onto a string, along a string, off of a string into a gap, and along a gap, while proving that the appropriate relations between $[I]$ and $[ij]$ are satisfied after each transition. This is almost sufficient; however, one small task remains. At the beginning of this section of the proof we made the assumption that $1 < b_1$ in order that the previous section of the proof could serve as the base case of our "overall" induction. So we must now show that relations (b) and (d) remain valid even when $b_1 = 1$. To see why this is so we make the observation that there is an obvious K-algebra isomorphism

$$\mathcal{O}_q(Sk_n) \cong K\langle a_{ij} : i,j > 2\rangle \subseteq \mathcal{O}_q(Sk_{n+2}),$$

sending a_{ij} to $a_{i+2,j+2}$. To establish a relation involving $I = \{b_1 < \cdots < b_m\}$, with $b_1 = 1$, we can look to the appropriate relation involving $I' = \{b'_1 < \cdots < b'_m\}$ where $b'_1 > 1$ and I is sent to I' under the above isomorphism. Since $b'_1 > 1$, by the work done so far, we will know the relation involving I' and so we may transfer the known relation to the $b_1 = 1$ case. We have now finished the case $j \notin I$.

<u>$j \in I$</u>:
If $i \in I$ then we are in case (a) which has already been done. So we assume $i \notin I$. The cases when $i < b_1$ have also been done. We therefore concern ourselves with proving that $[ij]$ and $[I]$ satisfy relation (c) when $i \notin I$ and $b_1 < i$. Given that we know $[I]$ and $[ij]$ commute if i lies in a string in I, our method will be to 'move i off of strings in I and then along the gaps'.

We begin by proving that relation (c) is satisfied by $[I]$ and $[ij]$ for $i = t+1$ where t lies in the initial string of I and $t+1 \notin I$. That is we will prove we can 'move i off of the initial string'. Now we know that,

$$[tj][I] = [I][tj].$$

Acting on this by F_t gives us,

$$q^{-1}[t+1,j][I] + [tj][I\cup\{t+1\}\setminus\{t\}] = q^{-1}[I\cup\{t+1\}\setminus\{t\}][tj] + [I][t+1,j].$$

Multiplying through by q and rearranging gives,

$$[t+1,j][I] = q[I][t+1,j] + [I\cup\{t+1\}\setminus\{t\}][tj] - q[tj][I\cup\{t+1\}\setminus\{t\}]. \quad (5.1.6)$$

Now suppose I has initial string of length 1. Then t comes before the initial string of $I \cup \{t+1\} \setminus \{t\}$ and so by previous work done in this proof we know,

$$[tj][I \cup \{t+1\} \setminus \{t\}] = q[I \cup \{t+1\} \setminus \{t\}][tj].$$

We can use this to substitute for the second term on the RHS of (5.1.6) giving,

$$[t+1,j][I] = q[I][t+1,j] - \hat{q}[tj][I \cup \{t+1\} \setminus \{t\}]$$

Writing the second term on the RHS in the form of a sum (albeit a sum with only one term) gives,

$$[t+1,j][I] = q[I][t+1,j] - \hat{q} \sum_{\substack{k \in I \\ k<t+1}} (-q)^{-|(k,t+1) \cap I|}[kj][I \cup \{t+1\} \setminus \{k\}],$$

which we can see is relation (c). Now suppose inductively that I has initial string of length r, then $I \cup \{t+1\} \setminus \{t\}$ has initial string of length $r-1$ and t comes immediately after it, so by our inductive hypothesis,

$$\begin{aligned}[tj][I \cup \{t+1\} \setminus \{t\}] = q[I \cup \{t+1\} \setminus \{t\}][tj] \\ - \hat{q} \sum_{\substack{k \in I \cup \{t+1\} \setminus \{t\} \\ k<t}} (-q)^{-|(k,t) \cap I \cup \{t+1\} \setminus \{t\}|}[kj][I \cup \{t+1\} \setminus \{k\}].\end{aligned}$$

As before, we use this identity to substitute for the second term on the RHS of (5.1.6), yielding,

$$\begin{aligned}[t+1,j][I] = q[I][t+1,j] - \hat{q}[tj][I \cup \{t+1\} \setminus \{t\}] \\ + q^{-1}\hat{q} \sum_{\substack{k \in I \cup \{t+1\} \setminus \{t\} \\ k<t}} (-q)^{-|(k,t) \cap I \cup \{t+1\} \setminus \{t\}|}[kj][I \cup \{t+1\} \setminus \{k\}].\end{aligned}$$

Rewriting the the index of the summation in an equivalent way gives,

$$\begin{aligned}[t+1,j][I] = q[I][t+1,j] - \hat{q}[tj][I \cup \{t+1\} \setminus \{t\}] \\ - \hat{q} \sum_{\substack{k \in I \\ k<t}} (-q)^{-|(k,t) \cap I \cup \{t+1\} \setminus \{t\}|-1}[kj][I \cup \{t+1\} \setminus \{k\}],\end{aligned}$$

since $|(k,t) \cap I \cup \{t+1\} \setminus \{t\}| + 1 = |(k,t+1) \cap I|$ we have,

$$\begin{aligned}[t+1,j][I] = q[I][t+1,j] - \hat{q}[tj][I \cup \{t+1\} \setminus \{t\}] \\ - \hat{q} \sum_{\substack{k \in I \\ k<t}} (-q)^{-|(k,t+1) \cap I|}[kj][I \cup \{t+1\} \setminus \{k\}],\end{aligned}$$

grouping the second and third terms on the RHS gives,

$$[t+1,j][I] = q[I][t+1,j] - \hat{q}\sum_{\substack{k\in I\\ k<t+1}}(-q)^{-|(k,t+1)\cap I|}[kj][I\cup\{t+1\}\setminus\{k\}].$$

This is relation (c) as required. So we have shown we can move i off of the initial string of I. One can see that a similar argument will hold for moving i off of any string in I given that we know relation (c) holds for i in the gap preceding the string in question. For example, suppose I contains more than one string and we want to know that $[ij]$ and $[I]$ satisfy relation (c) when i lies immediately after the second string. Now $i-1$ will lie in I and so we know that $[i-1,j][I]=[I][i-1,j]$. We may act on this by F_{i-1} similarly to above and then use the fact that if the second string is of length 1 then $i-1$ lies in the first gap of $I\cup\{i\}\setminus\{i-1\}$ as the base case of an inductive argument on the length of the second string of I in an exactly similar manner to the inductive argument that we have just done.

To finish the proof, we will now show that relation (c) is satisfied if i lies anywhere in a gap in I (or if I does not contain any gaps then anywhere after the initial string of I). Suppose $i\notin I$ and $i+1\notin I$. Suppose we know,

$$[ij][I] = q[I][ij] - \hat{q}\sum_{\substack{k\in I\\ k<i}}(-q)^{-|(k,i)\cap I|}[kj][I\cup\{i\}\setminus\{k\}].$$

Acting on this by F_i produces the following,

$$[i+1,j][I] = q[I][i+1,j] - \hat{q}\sum_{\substack{k\in I\\ k<i}}(-q)^{-|(k,i)\cap I|}[kj][I\cup\{i+1\}\setminus\{k\}],$$

Since $i\notin I$ the summation on the RHS can be rewritten,

$$[i+1,j][I] = q[I][i+1,j] - \hat{q}\sum_{\substack{k\in I\\ k<i+1}}(-q)^{-|(k,i+1)\cap I|}[kj][I\cup\{i+1\}\setminus\{k\}],$$

which is again relation (c). At last, our proof is complete. □

We now observe that there are equivalent expressions for the relations (c) and (d) given in the previous lemma.

Corollary 5.1.4. *Let $i, j \in \{1, ..., n\}$ with $i < j$ and $I = \{b_1 < \cdots < b_m\} \subseteq \{1, ..., n\}$, where $m \geq 2$ is even.*

(c2) If $i \notin I, j \in I$, then

$$[ij][I] = q^{-1}[I][ij] + \hat{q}\sum_{\substack{k\in I\\ k>i}}(-q)^{|(i,k)\cap I|-1}[jk][I \cup \{i\} \setminus \{k\}].$$

(d2) If $i \in I, j \notin I$, then

$$[ij][I] = q[I][ij] - \hat{q}\sum_{\substack{k\in I\\ k<j}}(-q)^{-|(k,j)\cap I|}[ik][I \cup \{j\} \setminus \{k\}].$$

Proof. We begin by rewriting Corollary 3.4.3 in more convenient notation for our purposes. It can easily be seen that, for fixed $r, t \in J$ where $J = \{j_1 < \cdots < j_{2h}\}$ and $|J|$ is even, we have,

$$\delta_{rt}[J] = \sum_{\substack{s\in J\\ s<t}}(-q)^{-|(s,t)\cap J|-1}[rs][J \setminus \{s,t\}] + \sum_{\substack{s\in J\\ s>t}}(-q)^{|(t,s)\cap J|}[rs][J \setminus \{s,t\}]. \quad (5.1.7)$$

First let us consider (c2) and suppose $i \notin I, j \in I$. We use equation 5.1.7 with $r = j$ and $J = I \cup \{i, t\}$ where we choose some $t > i, j, b_m$ (we note this is always possible since if it happens to be the case that $b_m = n$ then we may always think of our $[ij]$ and $[I]$ as lying in some $\mathcal{O}_q(Sk_N)$ with $N > n$ due to the canonical embedding $\mathcal{O}_q(Sk_n) \hookrightarrow \mathcal{O}_q(Sk_N)$), giving us,

$$\begin{aligned}
0 &= \sum_{\substack{s\in J\\ s<t}}(-q)^{-|(s,t)\cap J|-1}[js][J \setminus \{s,t\}] \\
&= \sum_{s\in I\cup\{i\}}(-q)^{-|(s,t)\cap(I\cup\{i\})|-1}[js][I \cup \{i\} \setminus \{s\}] \\
&= \sum_{\substack{s\in I\\ s<i}}(-q)^{-|(s,t)\cap(I\cup\{i\})|-1}[js][I \cup \{i\} \setminus \{s\}] + (-q)^{-|(i,t)\cap(I\cup\{i\})|-1}[ji][I] \\
&\qquad + \sum_{\substack{s\in I\\ s>i}}(-q)^{-|(s,t)\cap(I\cup\{i\})|-1}[js][I \cup \{i\} \setminus \{s\}].
\end{aligned}$$

Now a little thought will yield that,

$$|(s,t) \cap (I \cup \{i\})| = \begin{cases} |(s,i) \cap I| + |[i,t) \cap (I \cup \{i\})|, & s < i; \\ |[i,t) \cap (I \cup \{i\})| - 1, & s = i; \\ -(|(i,s) \cap I| + 2) + |[i,t) \cap (I \cup \{i\})|, & s > i. \end{cases}$$

So we may divide through by $(-q)^{-|[i,t)\cap(I\cup\{i\})|-1}$ in the previous equation to obtain,

$$\begin{aligned}
0 &= \sum_{\substack{s\in I\\ s<i}}(-q)^{-|(s,i)\cap I|}[js][I \cup \{i\} \setminus \{s\}] + (-q)[ji][I] \\
&\qquad + \sum_{\substack{s\in I\\ s>i}}(-q)^{|(i,s)\cap I|+2}[js][I \cup \{i\} \setminus \{s\}].
\end{aligned}$$

We replace $[js]$ and $[ji]$ with $-q[sj]$ and $-q[ij]$ respectively; then divide through by $-q$, obtaining,

$$0 = \sum_{\substack{s\in I\\ s<i}} (-q)^{-|(s,i)\cap I|}[sj][I \cup \{i\} \setminus \{s\}] + (-q)[ij][I]$$
$$+ \sum_{\substack{s\in I\\ s>i}} (-q)^{|(i,s)\cap I|+1}[js][I \cup \{i\} \setminus \{s\}].$$

Substituting into relation (c) of Lemma 5.1.2 gives us,

$$[ij][I] = q[I][ij] - \hat{q}\left(q[ij][I] + q\sum_{\substack{k\in I\\ k>i}} (-q)^{|(i,k)\cap I|}[jk][I \cup \{i\} \setminus \{k\}] \right),$$

which simplifies to,

$$[ij][I] = q^{-1}[I][ij] + \hat{q}\sum_{\substack{k\in I\\ k>i}} (-q)^{|(i,k)\cap I|-1}[jk][I \cup \{i\} \setminus \{k\}],$$

as required. Relation (d2) is proved in a similar manner. □

5.2 Reflection in the Anti-diagonal

The relations established in the last section all involve sums that consist of terms of the form $[rs][J]$, where the length-2 q-Pfaffian appears on the left. Inspired by the quantum matrices case, we are led to also look for relations in which the sums involve terms of the form $[J][rs]$, where the length-2 q-Pfaffian appears on the right. Indeed we shall need both types in the proof of Proposition 5.3.3.

As in [33], where certain automorphisms of $\mathcal{O}_q(M_n)$ are used in the proof of relations amongst quantum minors, we now define a map that will enable us to generate new commutation relations from those we have already determined.

Definition 5.2.1. *Define a map*

$$\tau : \mathcal{O}_q(Sk_n) \longrightarrow \mathcal{O}_q(Sk_n) \quad by \quad \tau(a_{ij}) = a_{n+1-j,n+1-i}, \quad for\ i,j = 1, ..., n$$

Trivially τ is a bijection. In-fact if we arrange the generators a_{ij} in an $n \times n$ matrix then we may think of τ as reflection in the anti-diagonal.

Lemma 5.2.2. *τ extends to an algebra anti-automorphism on $\mathcal{O}_q(Sk_n)$.*

Proof. We extend τ to an algebra anti-morphism in the obvious way. That is we insist that τ is K-linear and make the definition $\tau(a_{ij}a_{st}) := \tau(a_{st})\tau(a_{ij})$. It suffices to show that τ, thus extended, respects the defining relations (3.1.1)-(3.1.6) of $\mathcal{O}_q(Sk_n)$.

τ respects (3.1.1):
Let $i < j < t$. Then,

$$\begin{aligned}\tau(a_{ij}a_{it}) &= \tau(a_{it})\tau(a_{ij}) \\ &= a_{n+1-t,n+1-i}a_{n+1-j,n+1-i} \\ &= qa_{n+1-j,n+1-i}a_{n+1-t,n+1-i} \\ &= q\tau(a_{ij})\tau(a_{it}) \\ &= q\tau(a_{it}a_{ij}) \\ &= \tau(qa_{it}a_{ij}),\end{aligned}$$

where the third equality holds by (3.1.3) since $n+1-t < n+1-j < n+1-i$.

τ respects (3.1.2):
Let $i < j < t$. Then,

$$\begin{aligned}\tau(a_{ij}a_{jt}) &= \tau(a_{jt})\tau(a_{ij}) \\ &= a_{n+1-t,n+1-j}a_{n+1-j,n+1-i} \\ &= qa_{n+1-j,n+1-i}a_{n+1-t,n+1-j} \\ &= q\tau(a_{ij})\tau(a_{jt}) \\ &= \tau(qa_{jt}a_{ij}),\end{aligned}$$

where the third equality holds by (3.1.2) since $n+1-t < n+1-j < n+1-i$.

τ respects (3.1.3):
Let $i < s < j$. Then,

$$\begin{aligned}\tau(a_{ij}a_{sj}) &= \tau(a_{sj})\tau(a_{ij}) \\ &= a_{n+1-j,n+1-s}a_{n+1-j,n+1-i} \\ &= qa_{n+1-j,n+1-i}a_{n+1-j,n+1-s} \\ &= q\tau(a_{ij})\tau(a_{sj}) \\ &= \tau(qa_{sj}a_{ij}),\end{aligned}$$

where the third equality holds by (3.1.1) since $n+1-j < n+1-s < n+1-i$.

τ respects (3.1.4):

Let $i < s < t < j$. Then,

$$\begin{aligned}\tau(a_{ij}a_{st}) &= \tau(a_{st})\tau(a_{ij})\\ &= a_{n+1-t,n+1-s}a_{n+1-j,n+1-i}\\ &= a_{n+1-j,n+1-i}a_{n+1-t,n+1-s}\\ &= \tau(a_{ij})\tau(a_{st})\\ &= \tau(a_{st}a_{ij}),\end{aligned}$$

where the third equality holds by (3.1.4) since $n+1-j < n+1-t < n+1-s < n+1-i$.

τ respects (3.1.5):

Let $i < s < j < t$. Then,

$$\begin{aligned}\tau(a_{ij}a_{st}) &= a_{n+1-t,n+1-s}a_{n+1-j,n+1-i}\\ &= a_{n+1-j,n+1-i}a_{n+1-t,n+1-s} + \hat{q}a_{n+1-t,n+1-i}a_{n+1-j,n+1-s}\\ &= \tau(a_{ij})\tau(a_{st}) + \hat{q}\tau(a_{it})\tau(a_{sj})\\ &= \tau(a_{st}a_{ij} + \hat{q}a_{sj}a_{it})\\ &= \tau(a_{st}a_{ij} + \hat{q}a_{it}a_{sj}),\end{aligned}$$

where the second equality holds by (3.1.5) since $n+1-t < n+1-j < n+1-s < n+1-i$, and the last equality holds by (3.1.4).

τ respects (3.1.6):

Let $i < j < s < t$. Then,

$$\begin{aligned}\tau(a_{ij}a_{st}) &= a_{n+1-t,n+1-s}a_{n+1-j,n+1-i}\\ &= a_{n+1-j,n+1-i}a_{n+1-t,n+1-s} + \hat{q}a_{n+1-t,n+1-j}a_{n+1-s,n+1-i}\\ &\qquad - q\hat{q}a_{n+1-t,n+1-i}a_{n+1-s,n+1-j}\\ &= \tau(a_{ij})\tau(a_{st}) + \hat{q}\tau(a_{jt})\tau(a_{is}) - q\hat{q}\tau(a_{it})\tau(a_{js})\\ &= \tau(a_{st}a_{ij} + \hat{q}a_{is}a_{jt} - q\hat{q}a_{js}a_{it})\\ &= \tau(a_{st}a_{ij} + \hat{q}a_{is}a_{jt} - q\hat{q}a_{it}a_{js}),\end{aligned}$$

where the second equality holds by (3.1.6) since $n+1-t < n+1-s < n+1-j < n+1-i$, and the last equality holds by (3.1.4). □

In-order to use τ to generate new commutation relations we must first show what effect τ has on the q-Pfaffians. In-fact τ behaves very nicely.

Lemma 5.2.3. *Let $1 \leq i_1 < \cdots < i_{2m} \leq n$. Then,*

$$\tau([i_1 \cdots i_{2m}]) = [n+1-i_{2m}, \cdots, n+1-i_1].$$

The proof of this result requires the knowledge of the expansion of a q-Pfaffian "on the right". We will restrict ourselves in the following lemma only to the particular case needed, thus greatly simplifying the proof.

Lemma 5.2.4. *Let $1 \leq i_1 < \cdots < i_{2m} \leq n$. Then,*

$$[i_1 ... i_{2m}] = \sum_{r=1}^{2m-1} (-q)^{2m-r-1} [i_1 ... \check{i_r} ... i_{2m-1}][i_r i_{2m}]$$

Proof. By Corollary 3.4.3,

$$\begin{aligned}
[i_1 ... i_{2m}] &= \sum_{r=1}^{2m-1} (-q)^{\mu(2m)r} [i_{2m} i_r][i_1 ... \check{i_r} ... i_{2m-1}] \\
&= \sum_{r=1}^{2m-1} (-q)^{\mu(2m)r+1} [i_r i_{2m}][i_1 ... \check{i_r} ... i_{2m-1}] \\
&= \sum_{r=1}^{2m-1} (-q)^{r-2m+1} [i_r i_{2m}][i_1 ... \check{i_r} ... i_{2m-1}]
\end{aligned}$$

where the last equality holds since, as we recall from Proposition 3.4.2, $\mu_{ij} = j - i$ when $j < i$. We now apply relation (b) from Lemma 5.1.2 to the last term in the series yielding,

$$\begin{aligned}
[i_1 ... i_{2m}] = &\left([i_1 ... i_{2m-2}][i_{2m-1} i_{2m}] - \hat{q} \sum_{r=1}^{2m-2} (-q)^{-(2m-r-2)} [i_r i_{2m}][i_1 ... \check{i_r} ... i_{2m-1}] \right) \\
&+ \sum_{r=1}^{2m-2} (-q)^{r-2m+1} [i_r i_{2m}][i_1 ... \check{i_r} ... i_{2m-1}],
\end{aligned}$$

combining the two sums on the RHS gives,

$$[i_1 ... i_{2m}] = [i_1 ... i_{2m-2}][i_{2m-1} i_{2m}] + \sum_{r=1}^{2m-2} (-q)^{r-2m+3} [i_r i_{2m}][i_1 ... \check{i_r} ... i_{2m-1}].$$

We again apply relation (b) from Lemma 5.1.2, this time to the last term in the rightmost series, giving us,

$$\begin{aligned}
[i_1 ... i_{2m}] &= [i_1 ... i_{2m-2}][i_{2m-1} i_{2m}] \\
&+ (-q) \left([i_1 ... \check{i_{2m-2}} ... i_{2m-1}][i_{2m-2} i_{2m}] - \hat{q} \sum_{r=1}^{2m-3} (-q)^{r-2m+3} [i_r i_{2m}][i_1 ... \check{i_r} ... i_{2m-1}] \right) \\
&+ \sum_{r=1}^{2m-3} (-q)^{r-2m+3} [i_r i_{2m}][i_1 ... \check{i_r} ... i_{2m-1}]
\end{aligned}$$

$$[i_1...i_{2m}] =$$

$$\sum_{r=2m-2}^{2m-1}(-q)^{2m-1-r}[i_1...\check{i_r}...i_{2m-1}][i_r i_{2m}] + \sum_{r=1}^{2m-3}(-q)^{r-2m+5}[i_r i_{2m}][i_1...\check{i_r}...i_{2m-1}]$$

where the last equality holds by writing out $\hat{q}$ in full, that is $\hat{q} = q - q^{-1}$, and then simplifying. It should now be clear that repeated application of relation (b) from Lemma 5.1.2, at each stage to the last term in the rightmost series, will gives us the required result. For example, at the k-th stage we would have,

$$\sum_{r=2m-k+1}^{2m-1}(-q)^{2m-1-r}[i_1...\check{i_r}...i_{2m-1}][i_r i_{2m}] + \sum_{r=1}^{2m-k}(-q)^{r-2m+2k-1}[i_r i_{2m}][i_1...\check{i_r}...i_{2m-1}],$$

with an application of relation (b) giving,

$$\sum_{r=2m-k+1}^{2m-1}(-q)^{2m-1-r}[i_1...\check{i_r}...i_{2m-1}][i_r i_{2m}] +$$
$$(-q)^{k-1}\left([i_1...\check{i_{2m-k}}...i_{2m-1}][i_{2m-k} i_{2m}] - \hat{q}\sum_{r=1}^{2m-(k+1)}(-q)^{r-2m+k+1}[i_r i_{2m}][i_1...\check{i_r}...i_{2m-1}]\right)$$
$$+ \sum_{r=1}^{2m-(k+1)}(-q)^{r-2m+2k-1}[i_r i_{2m}][i_1...\check{i_r}...i_{2m-1}].$$

Simplifying this yields the expected result,

$$\sum_{r=2m-(k+1)+1}^{2m-1}(-q)^{2m-1-r}[i_1...\check{i_r}...i_{2m-1}][i_r i_{2m}]$$
$$+\hat{q}\sum_{r=1}^{2m-(k+1)}(-q)^{r-2m+2(k+1)}[i_r i_{2m}][i_1...\check{i_r}...i_{2m-1}].$$

□

Proof of Lemma 5.2.3. We prove this by induction on m. The case $m = 1$ is Definition 5.2.1. We proceed to the inductive step. By Definition 3.1.2 we have,

$$\tau([i_1...i_{2m}]) = \tau\left(\sum_{r=2}^{2m}(-q)^{r-2}[i_1 i_r][i_2...\check{i_r}...i_{2m}]\right)$$
$$= \sum_{r=2}^{2m}(-q)^{r-2}\tau([i_2...\check{i_r}...i_{2m}])\,\tau([i_1 i_r])$$
$$= \sum_{r=2}^{2m}(-q)^{r-2}\tau([i_2...\check{i_r}...i_{2m}])\,[n+1-i_r, n+1-i_1]$$
$$= \sum_{r=2}^{2m}(-q)^{r-2}[n+1-i_{2m},...,n+\check{1}-i_r,...,n+1-i_2][n+1-i_r, n+1-i_1],$$

where the last equality holds by inductive hypothesis. Let us now relabel the $n+1-i_k$, say $j_k = n+1-i_{2m+1-k}$ for $k = 1, ..., 2m$. So $j_1 < \cdots < j_{2m}$ and we have,

$$\begin{aligned}\tau([i_1...i_{2m}]) &= \sum_{k=1}^{2m-1} (-q)^{2m-k-1}[j_1...\check{j}_k...j_{2m-1}][j_k j_{2m}] \\ &= [j_1...j_{2m}]\end{aligned}$$

where the last equality holds by Lemma 5.2.4. □

Recall that we only proved a particular instance of expanding a q-Pfaffian "on the right" in Lemma 5.2.4 in-order to keep the proof manageable. With Lemma 5.2.3 now proved this allows us to give a more general Laplace expansion "on the right".

Lemma 5.2.5. *For fixed $r, t \in J$ where $J = \{j_1 < \cdots < j_{2h}\}$ and $|J|$ is even, we have,*

$$\delta_{rt}[J] = \sum_{\substack{s\in J\\ s>t}} (-q)^{-|(t,s)\cap J|-1}[J\setminus\{s,t\}][sr] + \sum_{\substack{s\in J\\ s<t}} (-q)^{|(s,t)\cap J|}[J\setminus\{s,t\}][sr].$$

Proof. We use the formulation of Corollary 3.4.3 as given in (5.1.7) and to it we apply τ. This gives the result. □

Knowing the effect of τ on q-Pfaffians also puts us in a position to obtain new commutation relations by applying τ to those of Lemma 5.1.2 and Corollary 5.1.4.

Lemma 5.2.6. *Let $i, j \in \{1, ..., n\}$ with $i < j$ and $I = \{b_1 < \cdots < b_m\} \subseteq \{1, ..., n\}$, where $m \geq 2$ is even.*
(B) If $i, j \notin I$, then

$$\begin{aligned}[ij][I] = [I][ij] &+ \hat{q}\sum_{\substack{k\in I\\ k>j}} (-q)^{-|(j,k)\cap I|}[I\cup\{j\}\setminus\{k\}][ik] \\ &- \hat{q}\sum_{\substack{k\in I\\ k<i}} (-q)^{|(k,i)\cap I|}[I\cup\{i\}\setminus\{k\}][kj] \\ &- \hat{q}^2 \sum_{\substack{r,s\in I\\ r<i, s>j}} (-q)^{-|(j,s)\cap I|+|(r,i)\cap I|}[I\cup\{i,j\}\setminus\{r,s\}][rs].\end{aligned}$$

(C) If $i \notin I, j \in I$, then

$$[ij][I] = q[I][ij] - q\hat{q}\sum_{\substack{k\in I\\ k<i}} (-q)^{|(k,i)\cap I|}[I\cup\{i\}\setminus\{k\}][kj].$$

(C2) If $i \notin I, j \in I$, then

$$[ij][I] = q^{-1}[I][ij] + q^{-1}\hat{q}\sum_{\substack{k\in I\\ k>i}}(-q)^{-|(i,k)\cap I|}[I\cup\{i\}\setminus\{k\}][kj].$$

(D) If $i \in I, j \notin I$, then

$$[ij][I] = q^{-1}[I][ij] + q^{-1}\hat{q}\sum_{\substack{k\in I\\ k>j}}(-q)^{-|(j,k)\cap I|}[I\cup\{j\}\setminus\{k\}][ik].$$

(D2) If $i \in I, j \notin I$, then

$$[ij][I] = q[I][ij] + \hat{q}\sum_{\substack{k\in I\\ k<j}}(-q)^{|(k,j)\cap I|}[I\cup\{j\}\setminus\{k\}][ki].$$

Proof. This follows by applying τ to relations (b), (c), and (d) in Lemma 5.1.2 and, (c2) and (d2) in Corollary 5.1.4, together with Lemma 5.2.3. □

5.3 A Partial Ordering

We have so far concerned ourselves with establishing various commutation relations amongst the q-Pfaffians of $\mathcal{O}_q(Sk_n)$. We now observe that, as in the classical case (see for example [11]), the set of q-Pfaffians of $\mathcal{O}_q(Sk_n)$ possesses the structure of a partially ordered set. We will show that the relations we have established are *well-behaved*, in a way that we will soon make precise, with respect to this partial order.

Definition 5.3.1. *The set $\{[i_1\cdots i_{2h}] \in \mathcal{O}_q(Sk_n) : 1 \le 2h \le n\}$ is endowed with the following partial order:*

$$[i_1\cdots i_s] \le [j_1\cdots j_t] \quad \textit{iff } s \ge t \textit{ and } i_r \le j_r \textit{ for } r = 1, ..., t.$$

The figure on page 115 illustrates this poset in the case $n = 6$.

We shall now give a precise formulation of what we mean when we say that our commutation relations are *well-behaved* with regards to this partial order.

Definition 5.3.2. *For an algebra A and a poset $\Omega \subseteq A$ we shall say that an element $c \in \Omega$ is* normal modulo lower elements in Ω, *, if c is normal in the algebra $A/\langle\{d \in \Omega : d < c\}\rangle$. If it is obvious from the context we shall drop the reference to Ω and just talk of c being* normal modulo lower elements. *For a given $c \in \Omega$ we will also talk of relations involving c holding* modulo lower elements *or* modulo elements lower than c *and we will write the relations using $=_{<c}$, by this we will mean that the relations are true in $A/\langle\{d \in \Omega : d < c\}\rangle$.*

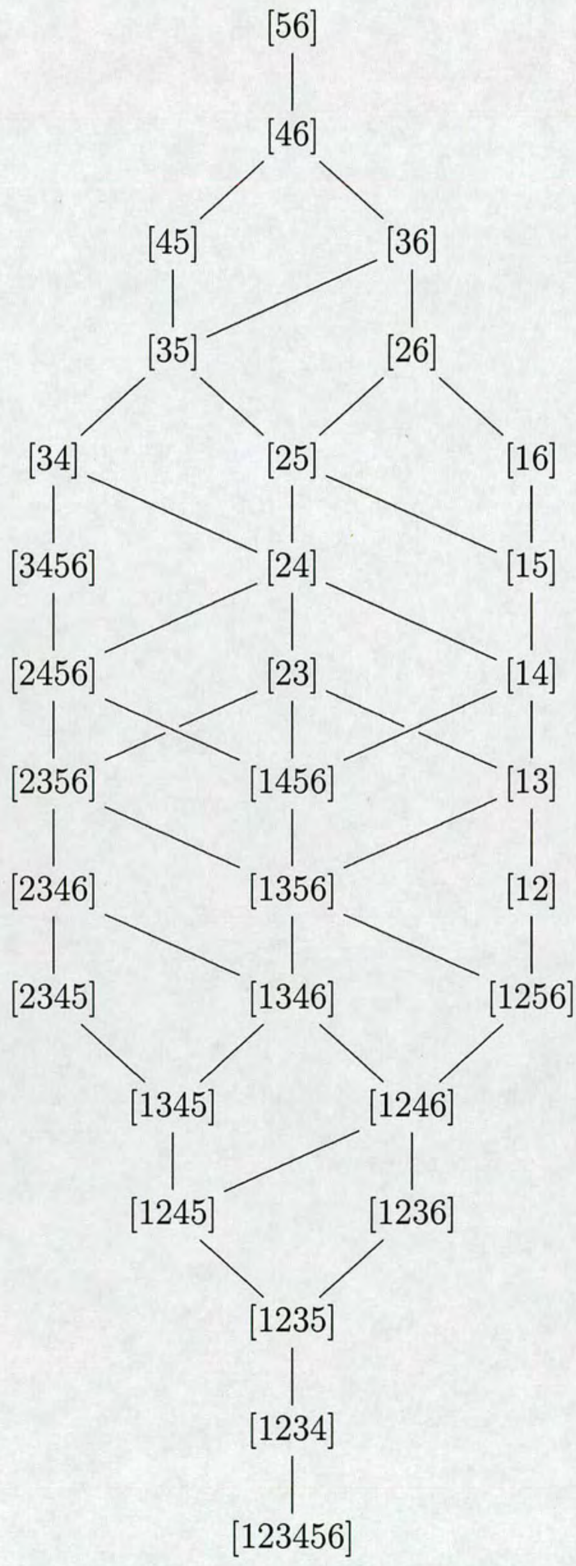


Figure 5.1: The 6 × 6 Poset

Proposition 5.3.3. *In $\mathcal{O}_q(Sk_n)$ the q-Pfaffians are normal modulo lower elements.*

Proof. It will suffice to show that, modulo lower elements, a given q-Pfaffian $[I]$ in $\mathcal{O}_q(Sk_n)$ commutes up to a scalar with the generators $[ij]$, $1 \leq i < j \leq n$. We shall proceed by using the commutation relations established in previous sections. Accordingly we must split up our argument into the usual cases.

$\underline{i, j \in I}$:

Trivial by relation (a) in Lemma 5.1.2.

$\underline{i \in I, j \notin I}$:

By relation (d) in Lemma 5.1.2 we have,

$$[ij][I] = q^{-1}[I][ij] + \hat{q} \sum_{\substack{k \in I \\ k > j}} (-q)^{|(j,k) \cap I|} [ik][I \cup \{j\} \setminus \{k\}].$$

For $k > j$ it is clear that $[I \cup \{j\} \setminus \{k\}] < [I]$. So, in the factor algebra $\mathcal{O}_q(Sk_n)/\langle\{[J] : [J] < [I]\}\rangle$, $[ij]$ and $[I]$ commute up to a scalar.

$\underline{i \notin I, j \in I}$:

By relation (c2) in Lemma 5.1.4 we have,

$$[ij][I] = q^{-1}[I][ij] + \hat{q} \sum_{\substack{k \in I \\ k > i}} (-q)^{|(i,k) \cap I| - 1} [jk][I \cup \{i\} \setminus \{k\}].$$

As in the previous case, for $k > i$, $[I \cup \{i\} \setminus \{k\}] < [I]$ and so we are done.

$\underline{i, j \notin I}$:

By relation (b) in Lemma 5.1.2 we have,

$$\begin{aligned}
[ij][I] = [I][ij] &+ \hat{q} \sum_{\substack{k \in I \\ k > j}} (-q)^{|(j,k) \cap I|} [ik][I \cup \{j\} \setminus \{k\}] \\
&- \hat{q} \sum_{\substack{k \in I \\ k < i}} (-q)^{-|(k,i) \cap I|} [kj][I \cup \{i\} \setminus \{k\}] \\
&+ \hat{q}^2 \sum_{\substack{r,s \in I \\ r < i, s > j}} (-q)^{|(j,s) \cap I| - |(r,i) \cap I|} [rs][I \cup \{i,j\} \setminus \{r,s\}]. \quad (5.3.1)
\end{aligned}$$

Now for $k > j$, $[I \cup \{j\} \setminus \{k\}] < [I]$ so, *modulo lower elements*, we may ignore the second term on the RHS. The final two terms are not apparently of the form which we may ignore. We may, however, rewrite them so that they are of the required form using the Laplace expansion of the q-Pfaffian given in (5.1.7). Now replacing J by $I \cup \{i,j\}$ in (5.1.7) gives,

$$[I \cup \{i,j\}] = \sum_{\substack{s \in I \cup \{i,j\} \\ s<r}} (-q)^{-|(s,r)\cap(I\cup\{i,j\})|-1}[rs][I \cup \{i,j\} \setminus \{s,r\}] + \sum_{\substack{s \in I \cup \{i,j\} \\ s>r}} (-q)^{|(r,s)\cap(I\cup\{i,j\})|}[rs][I \cup \{i,j\} \setminus \{s,r\}]. \quad (5.3.2)$$

Looking to rewrite the third term on the RHS of (5.3.1) we set $r = j$ in (5.3.2) giving us,

$$[I \cup \{i,j\}] = \sum_{\substack{s \in I \cup \{i,j\} \\ s<j}} (-q)^{-|(s,j)\cap(I\cup\{i,j\})|-1}[js][I \cup \{i\} \setminus \{s\}] + \sum_{\substack{s \in I \cup \{i,j\} \\ s>j}} (-q)^{|(j,s)\cap(I\cup\{i,j\})|}[js][I \cup \{i\} \setminus \{s\}].$$

Noting that $i < j$ we may rewrite the RHS of this expression as follows,

$$\begin{aligned}[I \cup \{i,j\}] = &\sum_{\substack{s \in I \cup \{i,j\} \\ s<i}} (-q)^{-|(s,j)\cap(I\cup\{i,j\})|-1}[js][I \cup \{i\} \setminus \{s\}] \\ &+ (-q)^{-|(i,j)\cap(I\cup\{i,j\})|-1}[ji][I] \\ &+ \sum_{\substack{s \in I \cup \{i,j\} \\ i<s<j}} (-q)^{-|(s,j)\cap(I\cup\{i,j\})|-1}[js][I \cup \{i\} \setminus \{s\}] \\ &+ \sum_{\substack{s \in I \cup \{i,j\} \\ s>j}} (-q)^{|(j,s)\cap(I\cup\{i,j\})|}[js][I \cup \{i\} \setminus \{s\}].\end{aligned}$$

We now simplify the RHS of this equation. For the first term, $s < i < j$, so

$$\begin{aligned}\{s \in I \cup \{i,j\} : s < i\} &= \{s \in I : s < i\}, \\ |(s,j) \cap (I \cup \{i,j\})| &= |(s,i) \cap I| + |(i,j) \cap I| + 1, \\ [js] &= -q[sj].\end{aligned}$$

For the second term, $|(i,j) \cap (I \cup \{i,j\})| = |(i,j) \cap I|$ and $[ji] = -q[ij]$. The last two terms are simplified in a similar manner leaving us with,

$$\begin{aligned}[I \cup \{i,j\}] = &\sum_{\substack{s \in I \\ s<i}} (-q)^{-|(s,i)\cap I|-|(i,j)\cap I|-1}[sj][I \cup \{i\} \setminus \{s\}] \\ &+ (-q)^{-|(i,j)\cap I|}[ij][I] \\ &+ \sum_{\substack{s \in I \\ i<s<j}} (-q)^{-|(s,j)\cap I|-1}[js][I \cup \{i\} \setminus \{s\}] \\ &+ \sum_{\substack{s \in I \\ s>j}} (-q)^{|(j,s)\cap I|}[js][I \cup \{i\} \setminus \{s\}].\end{aligned}$$

Multiplying through by $(-q)^{|(i,j)\cap I|+1}$ and rearranging gives,

$$\sum_{\substack{s\in I\\ s<i}}(-q)^{-|(s,i)\cap I|}[sj][I\cup\{i\}\setminus\{s\}] = q[ij][I]$$
$$+(-q)^{|(i,j)\cap I|+1}\left([I\cup\{i,j\}] - \sum_{\substack{s\in I\\ i<s<j}}(-q)^{-|(s,j)\cap I|-1}[js][I\cup\{i\}\setminus\{s\}]\right.$$
$$\left. - \sum_{\substack{s\in I\\ s>j}}(-q)^{|(j,s)\cap I|}[js][I\cup\{i\}\setminus\{s\}]\right). \quad (5.3.3)$$

Now $[I\cup\{i,j\}] < [I]$ and for both $i<s<j$ and $s>j>i$, $[I\cup\{i\}\setminus\{s\}] < [I]$. So *modulo elements lower than* $[I]$ (5.3.3) gives,

$$\sum_{\substack{s\in I\\ s<i}}(-q)^{-|(s,i)\cap I|}[sj][I\cup\{i\}\setminus\{s\}] =_{<[I]} q[ij][I].$$

Substituting this for the third term on the RHS of (5.3.1) and remembering that *modulo lower elements* we may ignore the second term on the RHS of (5.3.1), we have,

$$[ij][I] =_{<[I]} [I][ij] - \hat{q}q[ij][I]$$
$$+\hat{q}^2 \sum_{\substack{r,s\in I\\ r<i,s>j}}(-q)^{|(j,s)\cap I|-|(r,i)\cap I|}[rs][I\cup\{i,j\}\setminus\{r,s\}]. \quad (5.3.4)$$

So it remains to show that the last term on the RHS can be written in the required form. We go back to (5.3.2) and this time we fix $r>j>i$. This allows us to rewrite (5.3.2) as,

$$[I\cup\{i,j\}] = \sum_{\substack{s\in I\cup\{i,j\}\\ s<i}}(-q)^{-|(s,r)\cap(I\cup\{i,j\})|-1}[rs][I\cup\{i,j\}\setminus\{s,r\}]$$
$$+\sum_{\substack{s\in I\cup\{i,j\}\\ i\leq s<r}}(-q)^{-|(s,r)\cap(I\cup\{i,j\})|-1}[rs][I\cup\{i,j\}\setminus\{s,r\}]$$
$$+\sum_{\substack{s\in I\cup\{i,j\}\\ s>r}}(-q)^{|(r,s)\cap(I\cup\{i,j\})|}[rs][I\cup\{i,j\}\setminus\{s,r\}].$$

We rewrite the first term of the RHS of this equation as follows: we note that since $s<i<j<r$, we have,

$$\{s\in I\cup\{i,j\} : s<i\} = \{s\in I : s<i\},$$
$$|(s,r)\cap(I\cup\{i,j\})| = |(s,i)\cap I| + |[i,r)\cap(I\cup\{i,j\})|,$$
$$[rs] = -q[sr].$$

So we are able to deduce,

$$[I \cup \{i,j\}] = (-q)^{-|[i,r)\cap(I\cup\{i,j\})|} \sum_{\substack{s\in I\\ s<i}} (-q)^{-|(s,i)\cap I|}[sr][I\cup\{i,j\}\setminus\{s,r\}]$$
$$+ \sum_{\substack{s\in I\cup\{i,j\}\\ i\leq s<r}} (-q)^{-|(s,r)\cap(I\cup\{i,j\})|-1}[rs][I\cup\{i,j\}\setminus\{s,r\}]$$
$$+ \sum_{\substack{s\in I\cup\{i,j\}\\ s>r}} (-q)^{|(r,s)\cap(I\cup\{i,j\})|}[rs][I\cup\{i,j\}\setminus\{s,r\}].$$

Multiplying through by $(-q)^{|[i,r)\cap(I\cup\{i,j\})|}$ and rearranging gives,

$$\sum_{\substack{s\in I\\ s<i}} (-q)^{-|(s,i)\cap I|}[sr][I\cup\{i,j\}\setminus\{s,r\}] =$$
$$(-q)^{|[i,r)\cap(I\cup\{i,j\})|}\left([I\cup\{i,j\}] - \sum_{\substack{s\in I\cup\{i,j\}\\ i\leq s<r}} (-q)^{-|(s,r)\cap(I\cup\{i,j\})|-1}[rs][I\cup\{i,j\}\setminus\{s,r\}] \right.$$
$$\left. - \sum_{\substack{s\in I\cup\{i,j\}\\ s>r}} (-q)^{|(r,s)\cap(I\cup\{i,j\})|}[rs][I\cup\{i,j\}\setminus\{s,r\}] \right).$$

We can now substitute this into (5.3.4) (keeping in mind that the parameters r and s are interchanged in the two equations). From (5.3.4) we have,

$$[ij][I] =_{<[I]} [I][ij] - \hat{q}q[ij][I]$$
$$+ \hat{q}^2 \sum_{\substack{s\in I\\ s>j}} (-q)^{|(j,s)\cap I|} \left(\sum_{\substack{r\in I\\ r<i}} (-q)^{-|(r,i)\cap I|}[rs][I\cup\{i,j\}\setminus\{r,s\}] \right),$$

substituting for the expression in the brackets gives,

$$[ij][I] =_{<[I]} [I][ij] - \hat{q}q[ij][I]$$
$$+ \hat{q}^2 \sum_{\substack{s\in I\\ s>j}} (-q)^{|(j,s)\cap I|+|[i,s)\cap(I\cup\{i,j\})|} \left([I\cup\{i,j\}] \right.$$
$$- \sum_{\substack{r\in I\cup\{i,j\}\\ i\leq r<s}} (-q)^{-|(r,s)\cap(I\cup\{i,j\})|-1}[sr][I\cup\{i,j\}\setminus\{s,r\}]$$
$$\left. - \sum_{\substack{r\in I\cup\{i,j\}\\ r>s}} (-q)^{|(s,r)\cap(I\cup\{i,j\})|}[sr][I\cup\{i,j\}\setminus\{s,r\}] \right).$$

Since we are working *modulo elements lower than [I]* we may simplify this equation. It is true that $[I \cup \{i,j\}] < [I]$ and also, for $s > j > i$ and $r \geq i$, it is true that $[I \cup \{i,j\} \setminus \{s,r\}] < [I]$. So the above equation reduces to,

$$[ij][I] =_{<[I]} [I][ij] - \hat{q}q[ij][I].$$

Simplifying this gives the required result,

$$q^2[ij][I] =_{<[I]} [I][ij].$$

□

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Index