Resonance Scaling of Circle Maps

by

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Declaration

This thesis has been composed by myself, and has not been submitted for any other degree or professional qualification. Unless otherwise attributed, the work is my own.

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Abstract

Throughout this thesis we deal with the scaling properties of resonance (or *Arnol'd*) tongues of circle maps. The motivation comes from the work of A.M. Davie, which we summarise as necessary.

We deal initially with the *Sine Circle Map* in the form

$$f_{k,\Omega}(x) = x + \Omega + k\sin^2(x) \mod \pi,$$

where $0 \leq \Omega < \pi$ and $0 \leq k < 1$, and consider rotation numbers of the form $\alpha = \frac{1}{n}$. Here we investigate the scaling behaviour of the intervals $I_{\frac{1}{n}}(k)$, where, for $\Omega \in I_{\frac{1}{n}}$, $f_{k,\Omega}$ has rotation number $\frac{1}{n}$, as $n \to \infty$. We know that $|I_{\frac{1}{n}}(k)| \sim \gamma_k n^{-3}$. Our concern is with estimating the behaviour of γ_k as $k \to 0$, to which end we improve on the estimates made in the original work by Davie, and consider the effect of errors neglected in the first order approximations.

In chapter 3 we deal with the same map, but now considering rotation numbers of the form $\alpha = (n + \frac{p}{q})^{-1}$, where $n, p, q \in \mathbf{N}$, with p, q co-prime. We investigate the widths of the intervals I_α. Specifically, we consider the asymptotic behaviour of $|I_\alpha|$ as $n \to \infty$ and $k \to 0$ in any manner. This behaviour is related to a polynomial in the first q Fourier coefficients of a particular period 1 map, derived from a transformation on the circle map. We derive the appropriate polynomials for values of $q \leq 20$.

In the final chapter we consider circle maps derived from the *Dissipative Standard Map*,

$$f_{k,\Omega}(x,\theta) = \left(Jx - k\sin(2\pi\theta),\ \ \theta + \Omega + Jx - k\sin(2\pi\theta)\right),$$

where $0 < J < 1$. The map may be regarded as a map of the cylinder $\mathbf{R} \times \mathbf{S}^1$ into itself. The theory of normal hyperbolicity shows that if k is small enough there exists a circle, V, homotopic to $\{0\} \times \mathbf{S}^1$, which is invariant under the action of $f_{k,\Omega}$, and on which is induced an orientation preserving circle homeomorphism. We obtain an approximation to the circle and the associated homeomorphism when $\Omega = \Omega^*$, such that $f_{k,\Omega^*}|_V$ has a fixed point, proceeding to investigate the scaling of $|I_{\frac{1}{n}}|$ from this basis. Finally, we numerically estimate the quantity derived in the analytical part of this chapter.

Contents

Chapter 1

Introduction

Resonance, or the possibility of resonance, arises often in physical systems - for example, where two or more oscillators are coupled and a rational relationship exists between the respective periods of oscillation. For this reason it is often known as mode, or phase locking. In such a situation we might represent the motion of the system as a curve on an n–torus, which, when the system is resonant, becomes a closed loop. We may then consider the behaviour of the system with attention restricted to the loop.

One way to study the phenomenon of resonance is to consider the time–one map of such a system. It is therefore natural, from a mathematical point of view, to abstract ideas from this situation and consider maps of the form

$$F(x) = x + g(x),$$

where g is continuous and periodic with period $\tau \in \mathbf{R}$. We therefore have $F(x+\tau) = F(x) + \tau$, and F naturally gives rise to the map

$$f(x) = F(x) \mod \tau,$$

which maps the circle $\tau\mathbf{S}^1 = \mathbf{R}/\tau\mathbf{Z}$ continuously onto itself. For obvious reasons we call f a *circle map*, although the notation is somewhat loose and may be applied to F. F is properly known as a *lift* of f. Note also that whilst it is convenient for this present discussion to distinguish between circle maps and lifts by the use of lower and upper case letters, the meaning is in general clear from the context and we will drop the practice after this chapter.

One of the simplest maps of this type (other than a rigid rotation,

$x \mapsto x + \alpha \mod \tau$, which is dynamically uninteresting) is the map

$$f(x) = x + \Omega + k \sin 2\pi x \mod 1,$$

where $0 \leq \Omega < 1$, $0 < k < 1$. f is often known as the *standard (circle) map.* Much of this thesis is concerned with the study of a related map,

$$f_{k,\Omega}(x) = x + \Omega + k \sin^2 x \mod \pi,$$

with $0 \leq \Omega < \pi$, $0 < k < 1$.

In the context of a circle map, resonance is associated with the existence of periodic points – that is, points x_0 such that

$$F^q(x_0) = x_0 + p,$$

where $p, q \in \mathbf{N}$, with $(p, q) = 1$. At this point, we observe that throughout this thesis the notation f^n, where f is a mapping and $n \in \mathbf{N}$, will denote the n fold composition of f with itself. If f is bijective we allow $n \in \mathbf{Z}$ with the obvious interpretation. We have also the following:

Definition 1.1 *For any map $f : X \to X$ and any point $x \in X$ we define the* **orbit** *of x to be the set $\{y \in X : y = f^n(x) \text{ or } x = f^n(y) \text{ for some } n \in \mathbf{N} \cup \{0\}\}$. If f is bijective we can simplify this definition to $\{y \in X : y = f^n(x), n \in \mathbf{Z}\}$. Typically we will write $x_n = f^n(x_0)$, so that the orbit of a point x_0 is the set $\{x_n : n \in \mathbf{Z}\}$, or just $\{x_n\}$.*

The most important tool for the study of resonance in circle maps is the *Rotation Number*, due to Poincaré (see [Po]).

Definition 1.2 *Let F be a lift of a circle homeomorphism, f. We define the* **rotation number** *ρ of f by*

$$\rho(f) = \lim_{n \to \infty} \frac{F^n(x)}{n} \mod 1$$

It can be shown that this limit always exists, and is independent both of x and of the particular lift chosen (see, for instance, [Ni]).

We may speak of the rotation number of F, rather than f, with the obvious interpretation.

The importance of the rotation number lies in the following simple result (see, for instance, [AP] for proof):

Theorem 1.3 *Let f be a circle homeomorphism, and F a lift. Let $p, q \in \mathbf{N}$ be such that $(p, q) = 1$, that is p and q are co-prime. Then f has rotation number $\frac{p}{q}$ iff $F^q(x) = x + p$ for some $x \in \mathbf{R}$.*

Now, it is easy to see from the definition that $\rho(f_{k,\Omega})$ is increasing, as a function of Ω. Poincaré stated, without proof (see [Ar1]), that the rotation number depends continuously on the map, and so we see that the set $I_{\frac{p}{q}} = \{\Omega : \rho(f_{k,\Omega}) = \frac{p}{q}\}$ is a closed interval.

We come now to the idea of the scaling of the intervals $I_{\frac{p}{q}}$. If the small parameter, k, is allowed to vary, then the question of how $|I_{\frac{p}{q}}|$ depends on k arises.

Arnol'd, [Ar1], studied the cosine map,

$$f_{k,\Omega}(x) = x + \Omega + k \cos x \mod 2\pi,$$

and found that $|I_{\frac{p}{q}}| = O(k^q)$ as $k \to 0$, also conjecturing the existence of the more general result (see [Ar2]). Thus the set $\{(\Omega, k) : \rho(f_{k,\Omega}) = \frac{p}{q}\}$ appears as a narrowing 'tongue' approaching the $k = 0$ axis, and hence the term *'Arnol'd tongues'*.

Further general scaling properties also came to light, primarily regarding the scaling of $|I_{\frac{p}{q}}|$ for fixed values of the non-linearity parameter, as $q \to \infty$. Ecke, Farmer and Umberger, [EFU], numerically observe the scaling behaviour $|I_{\frac{p}{q}}| = O(q^{-3})$ for fixed $k < 1$ in the case of the sine map. More recently, Jonker [Jo] (and see also [Da2]) has shown that this law holds for diffeomorphisms and differentiable homeomorphisms of the circle in general.

It is this latter form of scaling behaviour that we will be primarily concerned with in the chapters that follow.

Chapter 2

Scaling of $\frac{1}{n}$–tongues of the sine circle map

In this chapter we consider the asymptotics of the resonance (or *Arnol'd*) tongues of the *sine circle map*. More particularly, we consider the map in the form

$$f_{k,\Omega}(x) = x + \Omega + k\sin^2(x), \tag{2.1}$$

where $0 \leq \Omega < \pi$, $0 < k < 1$, and $x \in \mathbf{R}$. The map is much studied, and in particular we review here the work of Davie, [Da1], which is essential to the material that forms the main part of this chapter.

2.1 Resonance scaling for the sine circle map.

We first define $I_n(k) = \{\Omega : \rho(f_{k,\Omega}) = \frac{1}{n}\}$, where $\rho(\cdot)$ denotes rotation number. Then I_n is a closed interval, $[\alpha_n(k), \beta_n(k)]$.

The paper establishes the following:

Theorem 2.1 *For each k there are numbers $\alpha(k), \beta(k)$ such that*

$$\alpha_n(k) = \frac{\pi^2}{kn^2} + \frac{\alpha(k)}{n^3} + o(n^{-3})$$

and

$$\beta_n(k) = \frac{\pi^2}{kn^2} + \frac{\beta(k)}{n^3} + o(n^{-3}),$$

as $n \to \infty$. Thus $|I_n(k)| = \frac{\gamma(k)}{n^3} + o(n^{-3})$, where $\gamma(k) = \beta(k) - \alpha(k)$.

Also, as $k \to 0$, $\gamma(k) = k^{-1}e^{-\frac{2\pi}{k}}[A + o(1)]$, where A is a constant (numerical results show $A \simeq 650.0$).

We present now the main details of the proof.

Consider first of all the map $f_k(x) = x + k\sin^2(x)$, that is, $f_{k,\Omega}$ for $\Omega = 0$. Now, f_k is a diffeomorphism, and for $x_0 \in (0,\pi)$ the orbit $\{x_n\}$ is such that $x_n \to \pi$ as $n \to +\infty$, and $x_n \to 0$ as $n \to -\infty$. We find $\varphi : \mathbf{R} \to \mathbf{R}$ such that $\varphi(x_{n+1}) - \varphi(x_n) = 1 + O(x_n^2)$. Using φ we define increasing functions $g_k : (0,\pi) \to \mathbf{R}$ and $h_k : (0,\pi) \to \mathbf{R}$ by

$$g_k(x_0) = \lim_{n\to-\infty} \varphi(x_n) - n, \tag{2.2}$$

and

$$h_k(x_0) = \lim_{n\to\infty} \varphi(x_n) - n, \tag{2.3}$$

for $x_0 \in (0,\pi)$, noting that $g_k(f_k(x)) = g_k(x) + 1$, and similarly for h_k. We may now define the mapping $\tau_k(x) = h_k(x) - g_k(x)$, and, since g_k maps $(0,\pi)$ *onto* $\mathbf{R}$, we can write τ_k as $\tau_k(x) = \sigma_k(g_k(x))$. So then σ_k is a periodic function on $\mathbf{R}$ with period 1.

We now fix k and consider orbits of $f_{k,\Omega}$ for small $\Omega > 0$. These orbits now extend from $-\infty$ to $+\infty$, since $f_{k,\Omega}(x) \geq x + \Omega$. As before we find a function $\psi : \mathbf{R} \to \mathbf{R}$ such that $\psi(x_{n+1}) - \psi(x_n) \simeq 1$. By consideration of the orbits of a point x_0 under $f_{k,\Omega}$ and f_k, and of the mapping ψ, the following result is obtained.

For $\Omega \in I_N$,

$$\alpha_N(k) = \frac{\pi^2}{kN^2}\left(1 - \frac{2}{N}\Theta_k\right) + o(N^{-3}),$$

and

$$\beta_N(k) = \frac{\pi^2}{kN^2}\left(1 - \frac{2}{N}\theta_k\right) + o(N^{-3}),$$

where

$$\theta_k = \min_{\xi\in\mathbf{R}} \sigma_k(\xi),$$

and

$$\Theta_k = \max_{\xi\in\mathbf{R}} \sigma_k(\xi).$$

The first part of the result follows.

We now consider the estimation of γ as $k \to 0$. From the above, we have

$$\gamma(k) = -\frac{2\pi^2}{k}(\Theta_k - \theta_k).$$

Since σ_k is periodic with period 1 we can write it as a Fourier series,

$$\sigma_k(\xi) = \sum_{r\in\mathbf{Z}} c_r(k)e^{-2\pi i r\xi},$$

where, for any $\delta \in \mathbf{R}$,

$$\begin{aligned} c_r(k) &= \int_\delta^{\delta+1} \sigma_k(\xi) e^{2\pi i r\xi}\, d\xi \\ &= \int_{x_0}^{x_1} [h_k(x) - g_k(x)] e^{2\pi i r g_k(x)} g_k'(x)\, dx. \end{aligned} \tag{2.4}$$

Clearly, in employing 2.4, x_0 may be chosen arbitrarily. In order to estimate the integral, we extend f_k, h_k and g_k into the complex plane. Thus we consider orbits of

$$f_k(z) = z + k\sin^2(z), \qquad z \in \mathbf{C}.$$

We now fix a constant $C > 0$ sufficiently large so that when $z \in V$, with

$$V = \left\{ z \in \mathbf{C} : \tfrac{3}{8}\pi < \Re z < \tfrac{5}{8}\pi, 0 \leq \Im z \leq \tfrac{1}{2}\log\tfrac{1}{k} - C \right\},$$

we have $\left|k\sin^2(z)\right| \leq e^{-2C}$, and z has a unique orbit with $z_n \to 0$ as $n \to -\infty$ and $z_n \to \pi$ as $n \to \infty$.

By consideration of the first order non-linear O.D.E.

$$\dot{z} = k\sin^2 z$$

we obtain a mapping,

$$\Phi_k(z) = -k^{-1}\cot(z) + \log\sin(z),$$

such that

$$\Phi_k(z_{n+1}) - \Phi_k(z_n) - 1 = O\left(k^2 \left|\sin^2(z_n)\right| e^{2\Im z_n}\right).$$

We may now define complex analytic mappings $\tilde{g}_k : V \to \mathbf{C}$ and $\tilde{h}_k : V \to \mathbf{C}$ analogous to those obtained earlier on the real line, and we have

$$\left|-k^{-1}\cot(z_n) + \log\sin(z_n) - n - \tilde{g}_k(z_0)\right| = O\left(ke^{2\Im z_n}\right)$$

as $n \to -\infty$, and

$$\left|-k^{-1}\cot(z_n) + \log\sin(z_n) - n - \tilde{h}_k(z_0)\right| = O\left(ke^{2\Im z_n}\right)$$

as $n \to \infty$. When z is real, $\tilde{g}_k$ and $\tilde{h}_k$ coincide with g and h respectively, and for the sake of notational simplicity we shall drop the $\tilde{}$.

Now, we choose $x_0, z_0 \in V$, with x_0 real. By Cauchy's Theorem applied to a suitable contour we find that

$$c_r(k) = \int_{z_0}^{z_1} [h_k(z) - g_k(z)] e^{2\pi i r g_k(z)} g_k'(z)\, dz.$$

In order to estimate this, we make a k-dependent change of variable to transform f_k into a mapping which is independent of k, to a first approximation. We write $z = ib + w$, where $b = \frac{1}{2}\log\frac{1}{k}$, and consider the map

$$f(w) = w - \tfrac{1}{4} e^{-2iw},$$

for $w \in W$ with

$$W = \left\{ w \in \mathbf{C} : \tfrac{3}{8}\pi < \Re z < \tfrac{5}{8}\pi, \Im z < -C \right\}.$$

We now find that orbits of f satisfy $\Im w_n \to -\infty$ as $n \to \pm\infty$, and $\Re w_n \to \frac{\pi}{4}$ or $\frac{3\pi}{4}$ as $n \to -\infty$ or $+\infty$ respectively. In addition, we find that the limits

$$\lim_{n \to -\infty} i\left(2e^{2iw_n} - w_n\right) - n \tag{2.5}$$

and

$$\lim_{n \to +\infty} i\left(2e^{2iw_n} - w_n\right) - n \tag{2.6}$$

exist, and define analytic functions, G and H, on W. Fixing $w \in W$, and n large, negative, we find that, as $k \to 0$,

$$g_k(ib + w) - ik^{-1} - b \to G(w) + \tfrac{i\pi}{2} - \log 2, \tag{2.7}$$

with a similar result for h_k, H.

Thus we find that

$$e^{\frac{2\pi r}{k}} |c_r(k)| \to e^{-r\pi^2} \left| \int_{w_0}^{w_1} [H(w) - G(w)]\, e^{2\pi i r G(w)} G'(w)\, dw \right|,$$

as $k \to 0$. Clearly when k is small, $|c_1(k)|$ dominates $|c_r(k)|$, $r \geq 2$, and so

$$\Theta_k - \theta_k \sim 4\, |c_1(k)|\,,$$

giving

$$\gamma(k) = k^{-1} e^{-\frac{2\pi}{k}} (A + o(1)),$$

where

$$A = 8\pi^2 e^{-\pi^2} \left| \int_{w_0}^{w_1} [H(w) - G(w)]\, e^{2\pi i G(w)} G'(w)\, dw \right|,$$

which is then easily estimated numerically. □

2.2 Asymptotic behaviour of γ_k

Throughout the remainder of this chapter we will be concerned with the more precise estimation of $\gamma(k)$. In particular, we establish the following result:

Theorem 2.2 *For any $r \in \mathbf{N}$,*

$$\gamma(k) = k^{-1}e^{-\frac{2\pi}{k}}\left[A_0 + A_1 k + \cdots + A_r k^r + O(k^{r+1})\right],$$

where $A_0 = A$.

We also show how, in principle, the A_i can be found, giving an expression for A_1, and investigate the possibilities of numerical estimation.

The proof of Theorem 2.2 follows broadly the same lines as that of Theorem 2.1, but with some significant additions. We show how 2.2 and 2.3 may be generalised to give higher order approximations, and similarly 2.5 and 2.6; we consider the error involved in relating the orbits of the map $f_k(z) = z + k\sin^2(z)$ to those of $f(w) = w - \frac{1}{4}e^{-2iw}$; and finally we consider the convergence of higher order terms in 2.7, from which the generalised result follows.

Firstly, then, we look for a change of co-ordinates, Ψ, that will transform the map, f_k, approximately satisfying the relationship

$$\Psi \circ f_k \circ \Psi^{-1}(\xi) = \xi + 1.$$

Lemma 2.3 *Let*

$$V = \left\{z \in \mathbf{C} : \tfrac{3}{8}\pi < \Re z < \tfrac{5}{8}\pi, 0 \leq \Im z \leq \tfrac{1}{2}\log\tfrac{1}{k} - C\right\},$$

and let U be the set

$$U = \bigcup_{i=-\infty}^{\infty} f_k^i(V).$$

Then, given $N \in \mathbf{N}$, there exists a mapping, $\Psi_k : U \to \mathbf{C}$, such that

$$\Psi_k(f_k(z)) - \Psi_k(z) = 1 + O\left(k^{N+2}\sin^2 z \; e^{2(N+1)\Im z}\right). \tag{2.8}$$

Proof. We show that there exists a mapping of the form

$$\Psi_k(z) = \sum_{r=-1}^{m} k^r \psi_r(z), \tag{2.9}$$

which satisfies the requirement.

Let us first begin by assuming that this is the case. For such a Ψ_k, analytic on some neighbourhood of U, we form the Taylor series,

$$\begin{aligned} \Psi_k(f_k(z)) &= \Psi_k(z + k\sin^2(z)) \\ &= \Psi_k(z) + (k\sin^2 z)\Psi_k'(z) + \frac{(k\sin^2 z)^2}{2!}\Psi_k''(z) + \cdots \end{aligned} \tag{2.10}$$

Now, substituting 2.9 into 2.10 and comparing coefficients, we obtain

$$\psi_{-1}(z) = -\cot z$$

and

$$\psi_0(z) = \log\sin z$$

as in section 2.1. In general, for $r \geq 1$ we have the following relationship.

$$\begin{aligned} \psi_r'(z) &= -\frac{1}{2!}\sin^2 z\, \psi_{r-1}'' - \frac{1}{3!}(\sin^2 z)^2\, \psi_{r-2}'''(z) - \cdots \\ &\quad -\frac{1}{(r+2)!}(\sin^2 z)^{r+1}\, \psi_{-1}^{(r+2)}(z). \end{aligned} \tag{2.11}$$

From this it is possible to calculate recursively as many coefficients, ψ_r, as may be required, although this is a tedious and error prone operation by hand, and it is greatly simplified by the use of computer packages such as *Maple* or *Mathematica*. However, for present purposes we require general information about arbitrarily many coefficients. We pause in the proof of Lemma 2.3 to consider the following:

Definition 2.4 *The* **degree** *of the trigonometric monomial* $\sin^m z\cos^n z$ *is* $(m+n)$. *The degree of a trigonometric polynomial in* $\sin z$ *and* $\cos z$ *is the degree of the term of highest degree.*

Proposition 2.5 *Let* $r \geq 1$. *Then* $\psi_r(z)$ *has the form*

$$\psi_r(z) = \alpha_r + \beta_r z + T(z),$$

where $\alpha_r, \beta_r \in \mathbf{C}$, *and* $T(z)$ *is a trigonometric polynomial in* $\sin(z)$ *and* $\cos(z)$ *with degree* $2r$.

Proof. We first note that if $T(z)$ is a trigonometric monomial of degree n, then so also is $T'(z)$. Now, let $T_1(z)$ be a trigonometric polynomial of degree n, and let

$$\Theta(z) = \frac{T_1(z)}{\sin^{2m} z}.$$

Then

$$\begin{aligned} \Theta'(z) &= \frac{\sin^{2m} z T_1'(z) - 2mT_1(z)\sin^{2m-1} z \cos z}{\sin^{4m} z} \\ &= \frac{T_2(z)}{\sin^{2(m+1)} z}, \end{aligned}$$

where $T_2(z)$ is a trigonometric polynomial of degree $n+2$.

Now, since $\psi_{-1}'''(z)$ has the form

$$\frac{T_1(z)}{\sin^4 z},$$

and similarly

$$\psi_0''(z) = \frac{T_2(z)}{\sin^2 z},$$

where T_1, T_2 are trigonometric polynomials in $\sin z$ and $\cos z$ with degree 2 and 0 respectively, it follows trivially by induction that $\psi_r'(z)$ is a trigonometric polynomial in $\sin z$ and $\cos z$ with degree $2r$, and the result follows, with α_r arbitrary. □

Now, returning to the proof of the lemma , we choose all the $\psi_r(z)$ as above, and write

$$\Psi_{N,k}(z) = \sum_{r=-1}^{N} k^r \psi_r(z).$$

We now need to show that this $\Psi_{N,k}$ satisfies 2.8. Now, in general, for fixed z and ε small, we can write

$$\Psi_{N,k}(z+\varepsilon) = \Psi_{N,k}(z) + \varepsilon\Psi_{N,k}'(z) + \cdots + \frac{\varepsilon^{N+2}}{(N+2)!}\Psi_{N,k}^{(N+2)}(z) + R_{N+3}(z+\varepsilon),$$

say, where

$$R_{N+3}(z+\varepsilon) = \frac{\varepsilon^{N+3}}{(N+3)!}\Psi_{N,k}^{(N+3)}(z) + \frac{\varepsilon^{N+4}}{(N+4)!}\Psi_{N,k}^{(N+4)}(z) + \cdots .$$

Now

$$\Psi_{N,k}^{(N+2)}(z+\varepsilon) = \Psi_{N,k}^{(N+2)}(z) + \varepsilon\Psi_{N,k}^{(N+3)}(z) + \cdots,$$

so that

$$
\begin{aligned}
|R_{N+3}(z+\varepsilon)| &\leq \left|\varepsilon^{N+2}\left(\Psi_{N,k}^{(N+2)}(z+\varepsilon)-\Psi_{N,k}^{(N+2)}(z)\right)\right| \\
&= \left|\varepsilon^{N+3}\int_0^1 \Psi_{N,k}^{(N+3)}(z+\theta\varepsilon)\ d\theta\ \right| \\
&\leq |\varepsilon|^{N+3}\max_{\theta\in[0,1]}\left|\Psi_{N,k}^{(N+3)}(z+\theta\varepsilon)\right|,
\end{aligned}
$$

and given the form of $\psi_j, j=-1,0,1,\ldots,N$, we easily obtain

$$
\left|R_{N+3}(z+k\sin^2 z)\right| \ \leq\ D\left|k^{N+2}\sin^2 z\ e^{2(N+1)\Im z}\right|,
$$

for some constant, D.

We have chosen the ψ_r in order to equate coefficients in 2.10, and so we now have

$$
\begin{aligned}
\Psi_{N,k}(z+k\sin^2 z) &= \sum_{r=-1}^{N} k^r\psi_r(z) + k\sin^2 z\sum_{r=-1}^{N} k^r\psi_r'(z) \\
&\quad +\frac{1}{2!}(k\sin^2 z)^2\sum_{r=-1}^{N} k^r\psi_r''(z)+\cdots \\
&\quad +\frac{1}{(N+2)!}(k\sin^2 z)^{N+2}\sum_{r=-1}^{N} k^r\psi_r^{(N+2)}(z) \\
&\quad +O\left(k^{N+2}\sin^2 z\ \ e^{2(N+1)\Im z}\right) \\
&= \sum_{r=-1}^{N} k^r\psi_r(z)+1+0+\cdots+0 \\
&\quad +\frac{1}{2!}(k\sin^2 z)^2 k^N\psi_N''(z) \\
&\quad +\frac{1}{3!}(k\sin^2 z)^3\left[k^N\psi_N'''(z)+k^{N-1}\psi_{N-1}'''(z)\right]+\cdots \\
&\quad +\frac{1}{(N+2)!}(k\sin^2 z)^{N+2}\left[k^{N+2}\psi_N^{(N+2)}(z)+k^{N-1}\psi_{N-1}^{(N+2)}(z)\right. \\
&\quad \left.+\cdots+k^0\psi_0^{(N+2)}(z)\right] \\
&\quad +O\left(k^{N+2}\sin^2 z\ \ e^{2(N+1)\Im z}\right) \\
&= \Psi_{N,k}(z)+1+O\left(k^{N+2}\sin^2 z\ \ e^{2(N+1)\Im z}\right), \qquad (2.12)
\end{aligned}
$$

as required. □

We now turn our attention to the question of the error involved in relating the orbits of f_k to those of f. Recall that if we make the transformation $z = w + ib$, with $b = \frac{1}{2}\log\frac{1}{k}$, then

$$\begin{aligned} f_k(z) &= f_k(w+ib) \\ &= (w+ib) + k\sin^2(w+ib) \\ &= w + ib - \frac{1}{4}e^{-2iw} + O(k). \end{aligned} \tag{2.13}$$

Thus we obtain the 'limiting mapping', $f(w) = w - \frac{1}{4}e^{-2iw}$. Now, with V and W as in section 2.1, let $z_0 \in V$. We choose $w_0 \in W$ such that $w_0 = z_0 - ib$, and let $\{z_n\}$ and $\{w_n\}$ be the orbits of z_0 and w_0 under f_k and f respectively. We write

$$\zeta_n = z_n - (w_n + ib).$$

Consideration of this error was unnecessary in the previous section, but now we need more precision. The following Lemma gives the form of ζ_n for fixed n.

Lemma 2.6 *ζ_n is analytic in k, and we write*

$$\zeta_n = kb_{n,1} + k^2 b_{n,2} + \cdots \ .$$

Also, for $s = 1, 2, \ldots,$

$$b_{n,s} = O\left(n^s\right).$$

Proof. Fix $n \geq 1$ (the argument is similar for $n < 0$). From 2.13 it is clear that $\zeta_n \to 0$ as $k \to 0$. Also, $\zeta_0 = 0$. Now, since $\zeta_{n+1} = z_{n+1} - w_{n+1} - ib$, we obtain

$$\zeta_{n+1} = \zeta_n + \frac{k}{2} - \frac{k^2}{4}e^{2iw_n}e^{2i\zeta_n} - \frac{1}{4}e^{-2iw_n}\left(e^{-2i\zeta_n} - 1\right). \tag{2.14}$$

Thus ζ_{n+1} is analytic as a function of k and of ζ_n. The first part of the Lemma follows by induction.

Now, from 2.14, we can see that

$$\zeta_{n+1} = \frac{k}{2} + \zeta_n\left[1 + \frac{i}{2}e^{-2iw_n} + O(\zeta_n)\right] + O(k^2) \tag{2.15}$$

$$= \frac{k}{2}\left\{1 + \sum_{t=1}^{n}\left[\prod_{s=1}^{t}\left(1 + \frac{i}{2}e^{-2iw_{n+1-s}} + O(\zeta_{n+1-s})\right)\right]\right\} + O(k^2). \tag{2.16}$$

So,

$$\frac{\zeta_n}{k} \to \frac{1}{2}\left\{1+\sum_{t=1}^{n-1}\left[\prod_{s=1}^{t}\left(1+\frac{i}{2}e^{-2iw_{n-s}}\right)\right]\right\} = b_{n,1}, \tag{2.17}$$

as $k \to 0$. Now, recall that

$$2ie^{2iw_n} - iw_n - n \to G(w_0)$$

as $n \to \infty$. Hence we deduce that, for n large,

$$-\frac{i}{2}e^{-2iw_n} = \frac{1}{n} + \delta_n,$$

with $|\delta_n| < \frac{1}{n}$. So then $\left|1+\frac{i}{2}e^{-2iw_n}\right| < 1$. Now, given such an n, we choose a constant, $B > 1$, such that $|b_{n,1}| < Bn$, and suppose that for some $t \geq n$ we have $|b_{t,1}| < Bt$. Then, by 2.15,

$$b_{t+1,1} = \frac{1}{2} + b_{t,1}\left(1+\frac{i}{2}e^{-2iw_t}\right).$$

So

$$|b_{t+1,1}| \leq \frac{1}{2} + Bt \leq B(t+1).$$

Hence, by induction, $b_{n,1} = O(n)$.

Now let s be such that $1 \leq s < r$, and suppose that for $1 \leq l < s$ we have $b_{n,l} = O(n^l)$. Then from 2.14 we have

$$\begin{aligned}
b_{n+1,s} = {} & \\
& b_{n,s} - \frac{1}{4}e^{2iw_n}\left[2ib_{n,s-2} + \frac{(2i)^2}{2!}\left(b_{n,1}b_{n,s-3}+\cdots+b_{n,s-3}b_{n,1}\right)+\cdots+\frac{(2i)^{s-2}}{(s-2)!}b_{n,1}^{s-2}\right] \\
& -\frac{1}{4}e^{-2iw_n}\left[-2ib_{n,s} + \frac{(-2i)^2}{2!}\left(b_{n,1}b_{n,s-1}+\cdots+b_{n,s-1}b_{n,1}\right)+\cdots+\frac{(-2i)^s}{s!}b_{n,1}^{s}\right] \\
= {} & b_{n,s}\left(1+\frac{i}{2}e^{-2iw_n}\right) + O\left(n^{s-1}\right) \\
= {} & O\left(n^{s-1}\left|1+\sum_{t=1}^{n-1}\left[\prod_{u=1}^{t}\left(1+\frac{i}{2}e^{-2iw_{n-u}}\right)\right]\right|\right) \\
= {} & O\left(n^{s-1}|2b_{n,1}|\right) \\
= {} & O\left(n^s\right).
\end{aligned}$$

The result follows, by induction. □

Recall that in defining Ψ_k earlier, the constants, α_r, were arbitrary. We now impose the requirement that $|\psi_r(z_n)| \to 0$ as $n \to \pm\infty$. In fact, this means that we need two different $\Psi_{N,k}$ functions, which we will denote $\Psi^+_{N,k}$ and $\Psi^-_{N,k}$ respectively. Because of the form of ψ_r and the fact that $z_n \to \pi$ or 0 as $n \to \pm\infty$ respectively, the constants, α_r, are uniquely determined for each $\Psi_{N,k}$ by the requirement.

Lemma 2.7 *Let $N \geq 0$. There exist mappings, $g_k : V \to \mathbf{C}$ and $h_k : V \to \mathbf{C}$, such that*

$$\left|g_k(z_0) - \left[\Psi^-_{N,k}(z_n) - n\right]\right| = O\left(\frac{1}{n^{N+1}}\right),$$

as $n \to -\infty$, and

$$\left|h_k(z_0) - \left[\Psi^+_{N,k}(z_n) - n\right]\right| = O\left(\frac{1}{n^{N+1}}\right),$$

as $n \to \infty$.

Proof. We give the proof for g_k, that for h_k being similar. We have

$$\left|\Psi^-_{N,k}(f_k(z)) - \Psi^-_{N,k}(z) - 1\right| = O\left(k^{N+2} \sin^2 z \;\; e^{2(N+1)\Im z}\right),$$

from Lemma 2.3. Thus

$$\Psi^-_{N,k}(z_n) - n = \Psi^-_{N,k}(z_0) + O\left(\sum_{u=1}^{n} \left(k \sin^2 z_{-u}\right) \left(ke^{2\Im z_{-u}}\right)^{N+1}\right).$$

We need to show now that the error term converges, and to that end we consider the tail of the infinite sum. Recall that for $z_0 \in V$, we have $\Im z_n \to 0$ as $n \to \pm\infty$. Thus we can say

$$\begin{aligned}
0 &\leq \left|\sum_{u=n+1}^{\infty} \left(k \sin^2 z_{-u}\right) \left(ke^{2\Im z_{-u}}\right)^{N+1}\right| \\
&\leq \left(ke^{2\Im z_{-n}}\right)^{N+1} \left|\sum_{u=n+1}^{\infty} \left(k \sin^2 z_{-u}\right)\right| \\
&\leq K\left|\sum_{u=n+1}^{\infty} \left(k \sin^2 z_{-u}\right)\right|, && \text{for some } K > 0, \\
&\to 0, && \text{as } n \to \infty.
\end{aligned}$$

Thus there exists a g_k such that

$$\left|g_k(z_0) - \left[\Psi^-_{N,k}(z_n) - n\right]\right| = O\left(\left(ke^{2\Im z_n}\right)^{N+1}\right),$$

as $k \to 0$ and $n \to -\infty$.

Now

$$\begin{aligned} \Im z_n &= \Im(w_n + ib + \zeta_n) \\ &= \Im w_n + \frac{1}{2}\log\frac{1}{k} + O(kn), \end{aligned}$$

by Lemma 2.6. Hence

$$\begin{aligned} ke^{2\Im z_n} &= e^{2\Im w_n}(1 + O(kn)) \\ &= e^{2\Im w_n}(1 + o(1)), \end{aligned}$$

as $n \to -\infty$, provided $k = O\left(\frac{1}{n^\mu}\right)$, for some $\mu > 1$.

Now we also know that

$$2ie^{2iw_n} - iw_n - n \to G(w_0),$$

as $n \to -\infty$, and since

$$|w_{n+1} - w_n| = \left|\frac{1}{4}e^{-2iw_n}\right| < 1,$$

we see that $w_n = O(n)$. Thus $e^{2iw_n} = O(n)$, which gives $e^{2\Im w_n} = O\left(\frac{1}{n}\right)$, and the result follows. □

So, we now have functions, g_k and h_k, analytic on V, defined by

$$g_k(z_0) = \lim_{n\to-\infty} \Psi^-_{N,k}(z_n) - n,$$

and

$$h_k(z_0) = \lim_{n\to\infty} \Psi^+_{N,k}(z_n) - n.$$

Also, g_k satisfies

$$g_k(f(z)) = g_k(z) + 1,$$

and similarly for h_k. It follows from the definition, and from the fact that $\psi^\pm_r(z_n) \to 0$ as $n \to \pm\infty$ respectively, that when z_0 is real, these functions agree with the ones defined in section 2.1. As before we use the complex functions, g_k and h_k, to find the maximum and minimum of the σ_k function. Recall,

$$\sigma_k(\xi) = \sum_{r\in\mathbf{Z}} c_r(k)e^{-2\pi ir\xi},$$

with

$$c_r(k) = \int_{z_0}^{z_1} [h_k(z) - g_k(z)]e^{2\pi irg_k(z)}g_k'(z)\,dz. \qquad (2.18)$$

Lemma 2.8 *Let $w \in W$. Given $N \geq 1$, there exists an $M \in \mathbf{N}$ such that*

$$g_k(w+ib) = k^{-1}i + bP_M(k) + G_0(w) + kG_1(w) + \cdots + k^N G_N(w) + O\left(k^{N+1}\right),$$

and

$$h_k(w+ib) = k^{-1}i + bP_M(k) + H_0(w) + kH_1(w) + \cdots + k^N H_N(w) + O\left(k^{N+1}\right),$$

where P_M is a polynomial, and the $G_j, H_j, j = 0,\ldots,N$ are analytic on W.

Proof. We shall give the proof for h_k. Again, the proof for g_k is similar. Firstly, let $M_0 \in \mathbf{N}$. Now, recall that

$$\begin{aligned}
\Psi^+_{M_0,k}(z_n) - n &= \sum_{r=-1}^{M_0} k^r\psi_r^+(z_n) \\
&= -k^{-1}\cot z_n + \log\sin z_n + k\psi_1^+(z_n) + \cdots + k^{M_0}\psi^+_{M_0}(z_n) - n.
\end{aligned}$$

Now, since $z_n = w_n + ib + kb_{n,1} + k^2 b_{n,2} + \cdots$, we can expand the ψ_s^+ and rearrange, collecting terms in k^j. Thus,

$$\begin{aligned}
-k^{-1}\cot z_n &= -k^{-1}i\left[\frac{e^{iz_n} + e^{-iz_n}}{e^{iz_n} - e^{-iz_n}}\right] \\
&= k^{-1}i\left[\frac{1+e^{2iz_n}}{1-e^{2iz_n}}\right] \\
&= k^{-1}i\left(1 + 2e^{2iz_n} + 2e^{4iz_n} + \cdots\right) \\
&= k^{-1}i + 2ie^{2iw_n}e^{2i(kb_{n,1}+\cdots)} + 2ike^{4iw_n}e^{4i(kb_{n,1}+\cdots)} + \cdots
\end{aligned}$$

$$= \quad k^{-1}i + 2ie^{2iw_n}\left(1 + 2i(kb_{n,1} + \cdots) - 2(kb_{n,1} + \cdots)^2 - \cdots\right)$$

$$+2ike^{4iw_n}\left(1 + 4i(kb_{n,1} + \cdots) - 8(kb_{n,1} + \cdots)^2 - \cdots\right) + \cdots$$

$$= \quad k^{-1}i + 2ie^{2iw_n} + k\left(2ib_{n,1} + 2ie^{4iw_n}\right) + \cdots$$

Proceeding similarly with $\psi_0^+, \psi_1^+, \ldots, \psi_{M_0}^+$, we write

$$\Psi_{M_0,k}^+(z_n) - n = k^{-1}i + bP_{M_0}(k) + c_{n,0} + kc_{n,1} + k^2c_{n,2} + \cdots + k^N c_{n,N} + R_{k,n}.$$

Now, the contribution to $c_{n,s}$ from $-k^{-1}\cot z_n$ is

$$2i\left\{e^{2i(s+1)w_n} + e^{2isw_n}\, 2isb_{n,1} + e^{2i(s-1)w_n}\left(\frac{-[2(s-1)]^2}{2!}\right)b_{n,1}^2\right.$$

$$\left. +e^{2i(s-1)w_n}\, 2(s-1)ib_{n,2} + \cdots + e^{2iw_n}\frac{1}{s!}(2i)^s b_{n,1}^s\right\}$$

$$= \quad O\left(n^{s+1}\right), \qquad \text{as } n \to \infty.$$

A similar analysis for $\log\sin z_n$ gives its contribution as $O\left(n^s\right)$. From Proposition 2.5 we can write, for $1 \le r \le M_0$,

$$\psi_r^+(z_n) = \beta_{r,r}e^{2irz_n} + \beta_{r,r-1}e^{2i(r-1)z_n} + \cdots + \beta_{r,-r}e^{-2irz_n} + \beta_r z_n + \alpha_r.$$

Thus, for the contribution to $c_{n,s}$ from $\psi_r^+(z_n)$, we require the terms of order k^{s-r} from

$$\psi_r^+(z_n) = \sum_{j=-r}^{r} \beta_{r,j}k^j e^{2ijw_n}\left(1 + 2ij\zeta_n + \frac{1}{2!}(2ij\zeta_n)^2 + \cdots\right) + \beta_r(w_n + ib + \zeta_n) + \alpha_n.$$

The order k^{s-r} terms are

$$\sum_{\substack{j=-r \\ s-r-j\ge 0}}^{r} \beta_{r,j}k^{s-r}e^{2ijw_n}Q_{s-r-j}(b_{n,1}, \ldots, b_{n,s-r-j}) + \begin{cases} \beta_r b_{n,s-r} & \text{if } s \ge r \\ 0 & \text{otherwise} \end{cases}$$

$$+\begin{cases} \alpha_r & \text{if } s = r \\ 0 & \text{otherwise} \end{cases},$$

where

$$Q_{s-r-j}(b_{n,1}, \ldots, b_{n,s-r-j}) = \sum_{\substack{\Sigma x_i = s-r-j \\ x_i \ge 0}} B_{\mathbf{x}} b_{n,1}^{x_1} \cdots b_{n,s-r-j}^{x_{s-r-j}},$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_{s-r-j})$, and $B_{\mathbf{x}}$ is a constant.

Since $e^{2iw_n} = O(n)$, and $b_{n,j} = O(n^j)$, it is clear that the contribution to $c_{n,s}$ from $\psi_r^+(z_n)$ is $O(n^{s-r})$. Hence we see that

$$c_{n,s} = O\left(n^{s+1}\right). \tag{2.19}$$

Similarly, we obtain

$$R_{k,n} = O\left(k^{N+1} n^{N+2}\right). \tag{2.20}$$

Note that the polynomial, P_{M_0}, is derived from the $\beta_r z_n$ term in $\psi_r^+(z_n)$, where $z_n = w_n + ib + \zeta_n$. It is important also to realise that we derive the *same* polynomial for $\Psi_{M_0,k}^-(z_n)$. Also, it is clear that each $c_{n,s}$ defines an analytic function of $w_0 \in W$ which is independent of k. To prove the Lemma, however, we need to show that they converge to analytic functions on W. Now the set V depends on k, and it will be helpful, at this point, to make that clear in the notation. We write

$$V_k = \left\{z \in \mathbf{C} : \frac{3}{8}\pi < \Re z < \frac{5}{8}\pi, 0 \le \Im z \le \frac{1}{2}\log\frac{1}{k} - C\right\},$$

and we also introduce the set W_k,

$$W_k = \left\{w \in \mathbf{C} : \frac{3}{8}\pi < \Re w < \frac{5}{8}\pi, -\frac{1}{2}\log\frac{1}{k} \le \Im w \le -C\right\}.$$

W_k is thus the translation of V_k by $\frac{1}{2}\log\frac{1}{k}$ in the negative imaginary direction.

We show that the $c_{n,s}$ converge uniformly on every compact subset of W, and therefore define analytic functions on W.

Let $W^* \subseteq W$ be compact. Then there exists a k^* such that $W^* \subseteq W_{k^*}$. Now by Lemma 2.7, for $z_0 \in V_{k^*}$ and $k < k^*$, we have

$$\left|h_k(z_0) - \left[\Psi_{M_0,k}^+(z_n) - n\right]\right| = O\left(\frac{1}{n^{M_0+1}}\right).$$

So, if $m > n$,

$$|(c_{n,0} - c_{m,0}) + k(c_{n,1} - c_{m,1}) + \cdots + (R_{k,n} - R_{k,m})| = O\left(\frac{1}{n^{M_0+1}}\right).$$

Now, from 2.19 and 2.20, provided $km < 1$, we have

$$\left|k(c_{n,1} - c_{m,1}) + k^2(c_{n,2} - c_{m,2}) + \cdots + (R_{k,n} - R_{k,m})\right| = O\left(km^2\right).$$

Thus we obtain

$$|c_{n,0} - c_{m,0}| = O\left(\frac{1}{n^{M_0+1}} + km^2\right).$$

Now, suppose that $k \sim \frac{1}{m^\mu}$, $\mu > 1$. Then we have

$$|c_{n,0} - c_{m,0}| = O\left(\frac{1}{n^{M_0+1}} + m^{2-\mu}\right).$$

So if μ is large enough, then we may conclude that

$$|c_{n,0} - c_{m,0}| = O\left(\frac{1}{n^{M_0+1}}\right).$$

Thus $(c_{n,0})_{n=1}^\infty$ is a Cauchy sequence, and converges. Also, the convergence is uniform for $z_0 \in V_{k^*}$, and hence for $w_0 \in W_{k^*}$. Continuing in like manner, we obtain

$$|k(c_{n,1} - c_{m,1})| = O\left(\frac{1}{n^{M_0+1}} + k^2m^3\right).$$

Now, suppose that $n < m < 2n$, and let $k \sim \frac{1}{m^\mu}$, as before. Then we have

$$|c_{n,1} - c_{m,1}| = O\left(\frac{1}{n^{M_0+1-\mu}} + n^{3-\mu}\right).$$

Thus, if $4 \le \mu \le M_0$,

$$|c_{n,1} - c_{m,1}| = O\left(\frac{1}{n}\right).$$

We now remove the restriction $m < 2n$. There is a $q \in \{0, 1, 2, \ldots\}$ such that

$$2^q n < m \le 2^{q+1} n.$$

Thus

$$\begin{aligned} c_{n,1} - c_{m,1} &= (c_{n,1} - c_{2n,1}) + (c_{2n,1} - c_{4n,1}) + \cdots + (c_{2^q n,1} - c_{m,1}) \\ &= O\left(\frac{1}{n} + \frac{1}{2n} + \cdots + \frac{1}{2^q n}\right) \\ &= O\left(\frac{1}{n}\right). \end{aligned}$$

Thus $(c_{n,1})_{n=1}^\infty$ is Cauchy, and converges as required.

Now, since we can choose M as large as we like, it is clear that we can also make μ large enough so that

$$|c_{n,1} - c_{m,1}| = O\left(\frac{1}{n^\delta}\right),$$

with δ large. In fact, choosing

$$\mu = \frac{M+N+3}{N+2}$$

gives

$$\delta = \frac{M-3(N+1)}{N+2} > M_0 + 1,$$

provided M is large enough. Now, the contributions from $\psi^+_{M_0+1}, \ldots, \psi^+_M$ to $c_{n,1}$ are of order $O(n^{1-(M_0+1)}) = O(n^{-M_0})$. Thus, when $M = M_0$,

$$k\,|c_{n,1} - c_{m,1}| = O\left(\frac{1}{n^{M_0+\mu}}\right),$$

and so we obtain

$$k^2\,|c_{n,2} - c_{m,2}| = O\left(\frac{1}{n^{M_0+1}} + k^3 m^4 + \frac{1}{n^{M_0+\mu}}\right),$$

and hence,

$$\begin{aligned} |c_{n,2} - c_{m,2}| &= O\left(\frac{1}{n^{M_0+1-2\mu}} + \frac{1}{n^{\mu-4}} + \frac{1}{n^{M_0-\mu}}\right) \\ &= O\left(\frac{1}{n^{M_0+1-2\mu}} + \frac{1}{n^{\mu-4}}\right), \end{aligned}$$

since $M_0 - \mu > M_0 + 1 - 2\mu$. We may now repeat the process, obtaining, in general,

$$\begin{aligned} |c_{n,s} - c_{m,s}| &= O\left(\frac{1}{n^{M_0+1-s\mu}} + \frac{1}{n^{\mu-(s+2)}} + \frac{1}{n^{M_0+2-s-\mu}}\right) \\ &= O\left(\frac{1}{n^{M_0+1-s\mu}} + \frac{1}{n^{\mu-(s+2)}}\right), \end{aligned}$$

for $1 \le s \le N$, provided μ can be chosen to satisfy the following conditions:

(i) $M_0 + 1 - s\mu > 1$ for $1 \le s \le N+1$;

(ii) $\mu - (s+2) > 1$ for $1 \le s \le N$.

It is evident that the choice

$$\mu = \frac{M_0+N+3}{N+2}$$

will suffice, provided M_0 is large enough. In this way, we obtain

$$k^s\,|c_{n,s} - c_{m,s}| = O\left(\frac{1}{n^{\delta_s}}\right),$$

with

$$\delta_s = M_0 + 1 - s + s\mu \geq M_0 + 1,$$

for $0 \leq s \leq N$. Finally, then, we have

$$|R_{k,n} - R_{k,m}| = O\left(\frac{1}{n^{M_0+1}}\right),$$

from which we readily see that $\frac{R_{k,n}}{k^{N+1}}$ converges, so that

$$R_{k,n} = O\left(k^{N+1}\right),$$

as $k \to 0$ and $n \to \infty$.

So, then, we see that

$$\left|\left(\Psi^+_{M_0,k}(z_n) - n\right) - \left(k^{-1}i + bP_{M_0}(k) + G_0(w_0) + \cdots + k^N G_N(w_0)\right)\right|$$
$$= O\left(\frac{1}{n^{M_0+1}} + k^{N+1}\right),$$

as $k \to 0$ and $n \to \infty$, and so finally,

$$\begin{aligned}\left|h_k(w_0 + ib) - \left(k^{-1}i + bP_{M_0}(k) + G_0(w_0) + \cdots + k^N G_N(w_0)\right)\right| & \\ &= O\left(\frac{1}{n^{M_0+1}} + k^{N+1}\right) \\ &= O\left(k^{\frac{M_0+1}{\mu}} + k^{N+1}\right) \\ &= O\left(k^{N+1}\right),\end{aligned}$$

provided M_0 is large enough. □

The theorem now follows quite easily; substituting for $g_k(z), h_k(z)$ in 2.18, we obtain

$$\begin{aligned}|c_1(k)| &= \left|\int_{w_0}^{w_1} \left\{[H_0(w) - G_0(w)] + \cdots + k^N [H_N(w) - G_N(w)] + O(k^{N+1})\right\}\right. \\ &\quad \times\ e^{2\pi i\left\{k^{-1}i + bP_M(k) + \frac{i\pi}{2} + G_0(w) + \cdots + k^N G_N(w) + O(k^{N+1})\right\}} \\ &\quad \times \left.\left[G_0'(w) + \cdots + k^N G_N'(w) + O(k^{N+1})\right]\ dw\right|,\end{aligned} \tag{2.21}$$

and the result follows.

2.3 Numerical estimation of A_1

We turn our attention now to the numerical estimation of A_1. We use a Simpson type approximation for the integral 2.21. First of all though, we need to calculate sufficiently many of the ψ_r to obtain the required convergence. In terms of the notation of section 2.2, we have $N = 1$. Consideration of the conditions (i) and (ii) on μ suggests that $M > 8$ will suffice. Of course, this is conservatively large, and we will use $M = 5$, which, by experimentation, is adequate.

Now we have already,

$$\begin{aligned} \psi_{-1}(z) &= -\cot z; \\ \psi_0(z) &= \log \sin z. \end{aligned}$$

Using the relationship 2.11, we obtain,

$$\begin{aligned}
\psi_1(z) &= -\frac{1}{3}\sin z\cos z - \frac{z}{6} + \alpha_1; \\
\psi_2(z) &= -\frac{1}{4}\cos^4 z - \frac{1}{6}\sin^2 z + \alpha_2; \\
\psi_3(z) &= \frac{38}{135}\cos^5 z\sin z - \frac{119}{540}\cos^3 z\sin z - \frac{1}{30}\cos z\sin z - \frac{1}{36}z + \alpha_3; \\
\psi_4(z) &= \frac{1}{3}\cos^8 z - \frac{74}{135}\cos^6 z + \frac{83}{720}\cos^4 z + \frac{29}{360}\cos^2 z + \alpha_4; \\
\psi_5(z) &= -\frac{164}{525}\cos^9 z\sin z + \frac{20197}{75600}\cos^7 z\sin z + \frac{190439}{453600}\cos^5 z\sin z \\
&\quad -\frac{29009}{72576}\cos^3 z\sin z + \frac{3001}{80640}\cos z\sin z - \frac{419}{34560}z + \alpha_5.
\end{aligned}$$

Also, we have $\alpha_1 = 0$, for $\psi_1^-(z)$, and $\frac{\pi}{6}$ for $\psi_1^+(z)$. Expanding the ψ_r and collecting the appropriate terms, we obtain the following approximations: for n large, positive,

$$\begin{aligned}
H_0(w_0) &\simeq \frac{i\pi}{2} - \log 2 + 2ie^{2iw_n} - iw_n - \frac{i}{12}e^{-2iw_n} - \frac{1}{64}e^{-4iw_n} \\
&\quad + \frac{19i}{4320}e^{-6iw_n} + \frac{1}{768}e^{-8iw_n} - \frac{41i}{134400}e^{-10iw_n}; \qquad (2.22)
\end{aligned}$$

$$H_1(w_0) \simeq 2ie^{4iw_n} - (4b_{n,1}+1)e^{2iw_n} - ib_{n,1} - \frac{w_n}{6} + \frac{\pi}{6} - \left(\frac{b_{n,1}}{6} + \frac{1}{48}\right)e^{-2iw_n}$$

$$-\left(\frac{ib_{n,1}}{16}-\frac{11i}{2880}\right)e^{-4iw_n}+\left(\frac{19b_{n,1}}{720}+\frac{1}{540}\right)e^{-6iw_n}$$

$$-\left(\frac{ib_{n,1}}{96}+\frac{5407i}{3870720}\right)e^{-8iw_n}-\frac{164b_{n,1}}{53760}e^{-10iw_n}, \tag{2.23}$$

with similar expressions for $G_0(w_0), G_1(w_0)$, with n large, negative. In addition, a similar calculation to that in the proof of Lemma 2.6 for $n < 0$ gives

$$b_{n,1}=-\frac{1}{2}\left\{\sum_{t=n+1}^{0}\left[\prod_{s=n+1}^{t}\left(1+\frac{i}{2}e^{-2iw_s}\right)^{-1}\right]\right\}.$$

For $G_0'(w_0)$ and $G_1'(w_0)$ we use a difference estimate:

$$G_i'(w_0) = \frac{G_i(w_0+\varepsilon)-G_i(w_0-\varepsilon)}{2\varepsilon}+O(\varepsilon^2).$$

Now, we have

$$\begin{aligned}
|c_1(k)| &= \left|\int_{w_0}^{w_1}\left\{[H_0(w)-G_0(w)]+k\,[H_1(w)-G_1(w)]+O(k^2)\right\}\right.\\
&\quad \left.\times e^{2\pi i\left\{k^{-1}i+\frac{kib}{6}+\frac{i\pi}{2}+G_0(w)+kG_1(w)+O(k^2)\right\}}\left[G_0'(w)+kG_1'(w)+O(k^2)\right]dw\right|\\
&= e^{\frac{-2\pi}{k}}e^{-\pi^2}\left\{\left|\int_{w_0}^{w_1}[H_0(w)-G_0(w)]\,e^{2\pi iG_0(w)}G_0'(w)dw\right|\right.\\
&\quad +k\left|\int_{w_0}^{w_1}[H_0(w)-G_0(w)]\left[e^{2\pi iG_0(w)}G_1'(w)+2\pi iG_1(w)e^{2\pi iG_0(w)}G_0'(w)\right]\right.\\
&\quad \left.\left.[H_1(w)-G_1(w)]\,e^{2\pi iG_0(w)}G_0'(w)dw\right|+O\left(k^2\right)\right\} \qquad (2.24)\\
&= e^{\frac{-2\pi}{k}}\frac{1}{8\pi^2}\left[A_0+kA_1+O\left(k^2\right)\right].
\end{aligned}$$

2.3.1 Results and programs

The approximations, 2.22 and 2.23, are sufficient to obtain an estimate of A_1 using the second integral in 2.24. However, we need to give consideration to the choice of the limits of integration, or, more particularly, w_0, since w_1 is determined by w_0. Theoretically we should obtain approximately the same estimate regardless of the choice of $w_0 \in W$. However, we find that as $\Im w_0 \to -\infty$, $H_0(w_0) - G_0(w_0) \to 0$ and $\Im G_0(w_0) \to -\infty$. Since the relative error involved in our estimate of $H_0(w_0) - G_0(w_0)$ increases rapidly as $\Im w_0 \to -\infty$, we find there is only a small interval in $\Im w_0$ near 0 where the integral can be reasonably well estimated. Evaluating the integral over a range of w_0 values we obtain the following data.

$\lvert\int\rvert$	$\Im w_0$
7.20662790502871e+007	-3.00000000000000e-001
4.61409302724904e+007	-2.90000000000000e-001
2.97942421415212e+007	-2.80000000000000e-001
1.93966684094697e+007	-2.70000000000000e-001
1.27315170695301e+007	-2.60000000000000e-001
8.42737499087931e+006	-2.50000000000000e-001
5.63022883993885e+006	-2.40000000000000e-001
3.80425189737047e+006	-2.30000000000000e-001
2.61043853201874e+006	-2.20000000000000e-001
1.83266049403588e+006	-2.10000000000000e-001
1.33153535818652e+006	-2.00000000000000e-001
1.01530819982830e+006	-1.90000000000000e-001
8.21254006798849e+005	-1.80000000000000e-001
7.05049348515789e+005	-1.70000000000000e-001
6.35903061365152e+005	-1.60000000000000e-001
5.93817767072854e+005	-1.50000000000000e-001
5.66886746730438e+005	-1.40000000000000e-001
5.48462625274533e+005	-1.30000000000000e-001
5.34987972377826e+005	-1.20000000000000e-001
5.24560137020016e+005	-1.10000000000000e-001
5.16140341096238e+005	-9.99999999999997e-002
5.09137974611902e+005	-8.99999999999997e-002
5.03196439246782e+005	-7.99999999999997e-002
4.98085071045551e+005	-6.99999999999997e-002
4.93643252921929e+005	-5.99999999999997e-002
4.89751900694620e+005	-4.99999999999997e-002

4.86320862910543e+005	-3.99999999999997e-002
4.83278205498934e+005	-2.99999999999997e-002
4.80566414538667e+005	-1.99999999999997e-002
4.78138673573294e+005	-9.99999999999969e-003
4.75956468143587e+005	3.08780778723872e-016
4.73988304361610e+005	1.00000000000003e-002
4.72207858927168e+005	2.00000000000003e-002
4.70593162239516e+005	3.00000000000003e-002
4.69125850448329e+005	4.00000000000003e-002
4.67790506744822e+005	5.00000000000003e-002
4.66573839500267e+005	6.00000000000003e-002
4.65464572974336e+005	7.00000000000003e-002
4.64452924753408e+005	8.00000000000003e-002
4.63530322476074e+005	9.00000000000003e-002
4.62689320782182e+005	1.00000000000000e-001
4.61923277694736e+005	1.10000000000000e-001
4.61226254288507e+005	1.20000000000000e-001
4.60592922137809e+005	1.30000000000000e-001
4.60018450386608e+005	1.40000000000000e-001
4.59498404654552e+005	1.50000000000000e-001
4.59028727113075e+005	1.60000000000000e-001
4.58605653573064e+005	1.70000000000000e-001
4.58225673688599e+005	1.80000000000000e-001
4.57885500588759e+005	1.90000000000000e-001
4.57582030999148e+005	2.00000000000000e-001
4.57312315774994e+005	2.10000000000000e-001
4.57073533984293e+005	2.20000000000000e-001
4.56862972577594e+005	2.30000000000000e-001
4.56677999747501e+005	2.40000000000000e-001
4.56516047657056e+005	2.50000000000000e-001
4.56374592277234e+005	2.60000000000000e-001
4.56251136897695e+005	2.70000000000000e-001
4.56143195318592e+005	2.80000000000000e-001

```
4.56048279480739e+005  2.90000000000000e-001
4.55963885553530e+005  3.00000000000000e-001
4.55887477358960e+005  3.10000000000000e-001
4.55816483753687e+005  3.20000000000000e-001
4.55749062458658e+005  3.30000000000000e-001
4.68627110835984e+005  3.40000000000000e-001
9.84336981363252e+006  3.50000000000000e-001
```

It is clear from the data that there is, as expected, a 'window' in $\Im w_0$ where the estimate of the integral is substantially constant, although close inspection reveals a systematic decrease with increasing $\Im w_0$ until $\Im w_0$ reaches about 0.33. This is probably due to rounding error, which is an ever present hazard when subtracting almost equal quantities as happens here. However, on the basis of the data here presented we tentatively suggest that A_1 is somewhere in the range 450000—460000.

For completeness we present also the program used to obtain the above data, which is written in the *Pascal* programming language.

```
program newint3(input,output);
const pi=3.14159265358979e+000;
type complex=array [1..2] of real;
     orbit= array[-30000..30000] of complex;
     contour= array[0..30000] of complex;
var commult,rthpower,cominv,comexp,wone,
    zetaplus,zetaminus,integral,intgr,comadd,comdiv,a,one,
    minusone,
    zero,wnought,invg,two,four,six,fwn,bwn,wn,H,G,H1,G1,modcka,
    comminus,zetaminuscurrent,comlog,bwnplusone,error,minusn,
    Ginverse,G1inverse,trial,zetanought,Ginvbwn,wnoughtplush,
    wnpluseps,wnminuseps,wnplusteps,wnminusteps,
    df,delta,intgr1,intgr2,gcurrent,hcurrent,g1current,h1current,
    answer,g1dashedcurrent,gdashed,int:complex;
    eps,moda,modexp,b,modint:real;
    ll,num,s,numberofbands,m,n:integer;
    worbit,otherworbit:orbit;
    gcont,hcont,g1cont,h1cont,g1peps,g1dashed,gamma,
    gpeps,gdashedcheck,gdashedcont,g1meps,
    gmeps,gpteps,gmteps,
    g1pteps,g1mteps:contour;
```

```
function modulus(z:complex):real;
begin
  modulus:=sqrt(sqr(z[1])+sqr(z[2]))
end;(*modulus*)

procedure initialise;
begin
  one[1]:=1;one[2]:=0;
  minusone[1]:=-1;minusone[2]:=0;
  zero[1]:=0;zero[2]:=0;
  six[1]:=6;six[2]:=0;
  two[1]:=2;two[2]:=0;
  four[1]:=4;four[2]:=0;
end;(*initialise*)

procedure compadd(z1,z2:complex);
begin
  comadd[1]:=z1[1]+z2[1];
  comadd[2]:=z1[2]+z2[2]
end;(*compadd*)

procedure compminus(z1,z2:complex);
begin
  comminus[1]:=z1[1]-z2[1];
  comminus[2]:=z1[2]-z2[2]
end;(*compminus*)

procedure compmult(z1,z2:complex);
begin
  commult[1]:=(z1[1]*z2[1])-(z1[2]*z2[2]);
  commult[2]:=(z1[1]*z2[2])+(z1[2]*z2[1])
end;(*compmult*)

procedure comppower(w1:complex;power:integer);
var w2:complex;count:integer;
begin
  if power=0 then rthpower:=one else
  begin
    w2:=w1;
    for count:= 1 to (power-1) do
    begin
      compmult(w1,w2);
      w2:=commult
    end;(*for*)
    rthpower:=w2
```

```
  end(*else*)
end;(*comppower*)

procedure compinv(w:complex);
begin
  cominv[1]:=w[1]/(sqr(w[1])+sqr(w[2]));
  cominv[2]:=-w[2]/(sqr(w[1])+sqr(w[2]))
end;(*compinv*)

procedure compdiv(z1,z2:complex);
begin
  compinv(z2);
  compmult(z1,cominv);
  comdiv:=commult
end;(*compdiv*)

procedure complog(w:complex);
begin
  comlog[1]:=ln(sqrt(sqr(w[1])+sqr(w[2])));
  comlog[2]:=arctan(w[2]/w[1]);
  if w[1]<0 then
comlog[2]:=comlog[2]+pi
end;(*complog*)

procedure compexp(z:complex);
begin
  comexp[1]:=exp(z[1])*cos(z[2]);
  comexp[2]:=exp(z[1])*sin(z[2])
end;(*compexp*)

procedure fofw(w:complex);
var fhold1,fhold2:complex;
begin
  fhold1[1]:=2*w[2];
  fhold1[2]:=-2*w[1];
  compexp(fhold1);
  wn[1]:=w[1]-comexp[1]/4;
  wn[2]:=w[2]-comexp[2]/4
end;(*fofw*)

procedure finverse(w:complex);
var finvhold1,finvhold2:complex;
    limit,j:integer;
begin
  if w[2]<-1 then limit:=15 else limit:=100;
  bwn:=w;
```

```
  for j:=1 to limit do
  begin
    finvhold1[1]:=2*bwn[2];
    finvhold1[2]:=-2*bwn[1];
    compexp(finvhold1);
    bwn[1]:=w[1]+comexp[1]/4;
    bwn[2]:=w[2]+comexp[2]/4
  end;(*for-j*)
end;(*finverse*)

procedure fdashed(w:complex);
var fdhold1,fdhold2:complex;
begin
  fdhold1[1]:=w[2]*2;
  fdhold1[2]:=-w[1]*2;
  compexp(fdhold1);
  df[1]:=1-comexp[2]/2;
  df[2]:=comexp[1]
end;(*fdashed*)

procedure getwn(z:complex;k:integer);
var j:integer;
begin
  wn:=z;
  bwn:=z;
  worbit[0]:=z;
  for j:=1 to k do
  begin
    fofw(wn);
    finverse(bwn);
    worbit[j]:=wn;
    worbit[-j]:=bwn
  end(*for-j*)
end;(*getwn*)

procedure getotherwn(z:complex;k:integer);
var j:integer;
begin
  bwn:=z;
  otherworbit[0]:=z;
  for j:=1 to k do
  begin
    finverse(bwn);
    otherworbit[-j]:=bwn
  end(*for-j*)
end;(*getotherwn*)
```

```
procedure getgamma;
var i:integer;
begin
  compminus(wone,wnought);
  delta:=comminus;
  delta[1]:=delta[1]/numberofbands;
  delta[2]:=delta[2]/numberofbands;
  gamma[0]:=wnought;
  for i:=1 to numberofbands do
  begin
    compadd(delta,gamma[i-1]);
    gamma[i]:=comadd
  end(*for*)
end;(*getgamma*)

procedure getH;
var Hterm1,Hterm2,Hterm3,Hterm4:complex;
begin
  Hterm4[1]:=-2*worbit[num][2];
  Hterm4[2]:=2*worbit[num][1];
  compexp(Hterm4);
  Hterm4:=comexp;
  Hterm1[1]:=-2*Hterm4[2];
  Hterm1[2]:=2*Hterm4[1];
  Hterm1[1]:=Hterm1[1]+worbit[num][2];
  Hterm1[2]:=Hterm1[2]-worbit[num][1];
  compinv(Hterm4);
  Hterm1[1]:=Hterm1[1]+cominv[2]/12;
  Hterm1[2]:=Hterm1[2]-cominv[1]/12;
  compmult(cominv,cominv);
  Hterm1[1]:=Hterm1[1]-commult[1]/64;
  Hterm1[2]:=Hterm1[2]-commult[2]/64;
  comppower(cominv,3);
  Hterm1[1]:=Hterm1[1]-rthpower[2]*19/4320;
  Hterm1[2]:=Hterm1[2]+rthpower[1]*19/4320;
  comppower(cominv,4);
  Hterm1[1]:=Hterm1[1]+rthpower[1]/768;
  Hterm1[2]:=Hterm1[2]+rthpower[2]/768;
  comppower(cominv,5);
  Hterm1[1]:=Hterm1[1]-rthpower[1]*164/537600;
  Hterm1[2]:=Hterm1[2]-rthpower[2]*164/537600;
  comppower(cominv,6);
  Hterm1[1]:=Hterm1[1]-rthpower[1]*37/6635520;
  Hterm1[2]:=Hterm1[2]-rthpower[2]*37/6635520;
  comppower(cominv,7);
```

```
  Hterm1[1]:=Hterm1[1]-rthpower[1]*4.51472545e-5;
  Hterm1[2]:=Hterm1[2]-rthpower[2]*4.51472545e-5;
  comppower(cominv,8);
  Hterm1[1]:=Hterm1[1]-rthpower[1]*2.145918589e-5;
  Hterm1[2]:=Hterm1[2]-rthpower[2]*2.145918589e-5;
  Hterm1[1]:=Hterm1[1]-num;
  H:=Hterm1
end;(*getH*)

procedure getG;
var Gterm1,Gterm2,Gterm3,Gterm4:complex;
begin
  Gterm4[1]:=-2*worbit[-num][2];
  Gterm4[2]:=2*worbit[-num][1];
  compexp(Gterm4);
  Gterm4:=comexp;
  Gterm1[1]:=-2*Gterm4[2];
  Gterm1[2]:=2*Gterm4[1];
  Gterm1[1]:=Gterm1[1]+worbit[-num][2];
  Gterm1[2]:=Gterm1[2]-worbit[-num][1];
  compinv(Gterm4);
  Gterm1[1]:=Gterm1[1]+cominv[2]/12;
  Gterm1[2]:=Gterm1[2]-cominv[1]/12;
  compmult(cominv,cominv);
  Gterm1[1]:=Gterm1[1]-commult[1]/64;
  Gterm1[2]:=Gterm1[2]-commult[2]/64;
  comppower(cominv,3);
  Gterm1[1]:=Gterm1[1]-rthpower[2]*19/4320;
  Gterm1[2]:=Gterm1[2]+rthpower[1]*19/4320;
  comppower(cominv,4);
  Gterm1[1]:=Gterm1[1]+rthpower[1]/768;
  Gterm1[2]:=Gterm1[2]+rthpower[2]/768;
  comppower(cominv,5);
  Gterm1[1]:=Gterm1[1]-rthpower[1]*164/537600;
  Gterm1[2]:=Gterm1[2]-rthpower[2]*164/537600;
  comppower(cominv,6);
  Gterm1[1]:=Gterm1[1]-rthpower[1]*37/6635520;
  Gterm1[2]:=Gterm1[2]-rthpower[2]*37/6635520;
  comppower(cominv,7);
  Gterm1[1]:=Gterm1[1]-rthpower[1]*4.51472545e-5;
  Gterm1[2]:=Gterm1[2]-rthpower[2]*4.51472545e-5;
  comppower(cominv,8);
  Gterm1[1]:=Gterm1[1]-rthpower[1]*2.145918589e-5;
  Gterm1[2]:=Gterm1[2]-rthpower[2]*2.145918589e-5;
  Gterm1[1]:=Gterm1[1]+num;
  G:=Gterm1
```

```
end;(*getG*)

procedure getzetaplus;
var i:integer;
    zetaplusterm1,zetaplusterm2,zetaplusterm3,
    zetaplusterm4:complex;
begin
  zetaplus:=zero;
  zetaplusterm3:=one;
  for i:=1 to (num-1) do
  begin
    zetaplusterm1[1]:=2*worbit[num-i][2];
    zetaplusterm1[2]:=-2*worbit[num-i][1];
    compexp(zetaplusterm1);
    zetaplusterm2[1]:=1-comexp[2]/2;
    zetaplusterm2[2]:=comexp[1]/2;
    compmult(zetaplusterm3,zetaplusterm2);
    zetaplusterm3:=commult;
    compadd(zetaplus,zetaplusterm3);
    zetaplus:=comadd
  end;(*for i*)
  zetaplus[1]:=(zetaplus[1]/2 +0.5);
  zetaplus[2]:=zetaplus[2]/2
end;(*getzetaplus*)

procedure getzetaminus;
var i:integer;
    zetaminusterm1,zetaminusterm2,zetaminusterm3,
    zetaminusterm4:complex;
begin
  zetaminus:=zero;
  zetaminusterm3:=one;
  for i:=1 to (num) do
  begin
    zetaminusterm1[1]:=2*worbit[-num-1+i][2];
    zetaminusterm1[2]:=-2*worbit[-num-1+i][1];
    compexp(zetaminusterm1);
    zetaminusterm2[1]:=1-comexp[2]/2;
    zetaminusterm2[2]:=comexp[1]/2;
    compinv(zetaminusterm2);
    compmult(zetaminusterm3,cominv);
    zetaminusterm3:=commult;
    compadd(zetaminus,zetaminusterm3);
    zetaminus:=comadd
  end;(*for i*)
  zetaminus[1]:=-zetaminus[1]/2;
```

```
  zetaminus[2]:=-zetaminus[2]/2
end;(*getzetaminus*)

procedure getH1;
var H1term1,H1term2,H1term3,H1term4:complex;
begin
  H1term4[1]:=-2*worbit[num][2];
  H1term4[2]:=2*worbit[num][1];
  compexp(H1term4);
  H1term4:=comexp;
  H1term1:=zero;
  compmult(zetaplus,H1term4);
  H1term1[1]:=H1term1[1]-4*commult[1];
  H1term1[2]:=H1term1[2]-4*commult[2];
  compmult(H1term4,H1term4);
  H1term1[1]:=H1term1[1]-2*commult[2]+zetaplus[2]-H1term4[1]
-worbit[num][1]/6;
  H1term1[2]:=H1term1[2]+2*commult[1]-zetaplus[1]-H1term4[2]
-worbit[num][2]/6;
  compinv(H1term4);
  compmult(cominv,zetaplus);
  H1term1[1]:=H1term1[1]-commult[1]/6-cominv[1]/48;
  H1term1[2]:=H1term1[2]-commult[2]/6-cominv[2]/48;
  compmult(cominv,cominv);
  H1term2:=commult;
  H1term1[1]:=H1term1[1]-commult[2]*11/2880;
  H1term1[2]:=H1term1[2]+commult[1]*11/2880;
  compmult(H1term2,zetaplus);
  H1term1[1]:=H1term1[1]+commult[2]/16;
  H1term1[2]:=H1term1[2]-commult[1]/16;
  comppower(cominv,3);
  compmult(rthpower,zetaplus);
  H1term1[1]:=H1term1[1]+commult[1]*228/8640 + 3.14159265358979/6;
  H1term1[2]:=H1term1[2]+commult[2]*228/8640;
  H1term1[1]:=H1term1[1]+rthpower[1]/540;
  H1term1[2]:=H1term1[2]+rthpower[2]/540;
  comppower(cominv,4);
  compmult(rthpower,zetaplus);
  H1term1[1]:=H1term1[1]+commult[2]/96+rthpower[2]*5407/3870720;
  H1term1[2]:=H1term1[2]-commult[1]/96-rthpower[1]*5407/3870720;
  comppower(cominv,5);
  compmult(rthpower,zetaplus);
  H1term1[1]:=H1term1[1]-commult[1]*164/53760
 -rthpower[1]*18937/19353600;
  H1term1[2]:=H1term1[2]-commult[2]*164/53760
 -rthpower[2]*18937/19353600;
```

```
  comppower(cominv,6);
  compmult(rthpower,zetaplus);
  H1term1[1]:=H1term1[1]-commult[2]*37/552960
-rthpower[2]*1.625340325e-3;
  H1term1[2]:=H1term1[2]-commult[1]*37/552960
-rthpower[1]*1.625340325e-3;
  comppower(cominv,7);
  compmult(rthpower,zetaplus);
  H1term1[1]:=H1term1[1]-commult[1]*6.320615631e-4
+rthpower[1]*8.575247229e-5;
  H1term1[2]:=H1term1[2]-commult[2]*6.320615631e-4
+rthpower[2]*8.575247229e-5;
  comppower(cominv,8);
  compmult(rthpower,zetaplus);
  H1term1[1]:=H1term1[1]-commult[2]*443/1290240;
  H1term1[2]:=H1term1[2]+commult[1]*443/1290240;
  H1:=H1term1
end;(*getH1*)

procedure getG1;
var G1term1,G1term2,G1term3,G1term4:complex;
begin
  G1term4[1]:=-2*worbit[-num][2];
  G1term4[2]:=2*worbit[-num][1];
  compexp(G1term4);
  G1term4:=comexp;
  G1term1:=zero;
  compmult(zetaminus,G1term4);
  G1term1[1]:=G1term1[1]-4*commult[1];
  G1term1[2]:=G1term1[2]-4*commult[2];
  compmult(G1term4,G1term4);
  G1term1[1]:=G1term1[1]-2*commult[2]+zetaminus[2]-G1term4[1]
-worbit[-num][1]/6;
  G1term1[2]:=G1term1[2]+2*commult[1]-zetaminus[1]-G1term4[2]
-worbit[-num][2]/6;
  compinv(G1term4);
  compmult(cominv,zetaminus);
  G1term1[1]:=G1term1[1]-commult[1]/6-cominv[1]/48;
  G1term1[2]:=G1term1[2]-commult[2]/6-cominv[2]/48;
  compmult(cominv,cominv);
  G1term2:=commult;
  G1term1[1]:=G1term1[1]-commult[2]*11/2880;
  G1term1[2]:=G1term1[2]+commult[1]*11/2880;
  compmult(G1term2,zetaminus);
  G1term1[1]:=G1term1[1]+commult[2]/16;
  G1term1[2]:=G1term1[2]-commult[1]/16;
```

```
  comppower(cominv,3);
  compmult(rthpower,zetaminus);
  G1term1[1]:=G1term1[1]+commult[1]*228/8640;
  G1term1[2]:=G1term1[2]+commult[2]*228/8640;
  G1term1[1]:=G1term1[1]+rthpower[1]/540;
  G1term1[2]:=G1term1[2]+rthpower[2]/540;
  comppower(cominv,4);
  compmult(rthpower,zetaminus);
  G1term1[1]:=G1term1[1]+commult[2]/96+rthpower[2]*5407/3870720;
  G1term1[2]:=G1term1[2]-commult[1]/96-rthpower[1]*5407/3870720;
  comppower(cominv,5);
  compmult(rthpower,zetaminus);
  G1term1[1]:=G1term1[1]-commult[1]*164/53760
-rthpower[1]*18937/19353600;
  G1term1[2]:=G1term1[2]-commult[2]*164/53760
-rthpower[2]*18937/19353600;
  comppower(cominv,6);
  compmult(rthpower,zetaminus);
  G1term1[1]:=G1term1[1]-commult[2]*37/552960
-rthpower[2]*1.625340325e-3;
  G1term1[2]:=G1term1[2]-commult[1]*37/552960
-rthpower[1]*1.625340325e-3;
  comppower(cominv,7);
  compmult(rthpower,zetaminus);
  G1term1[1]:=G1term1[1]-commult[1]*6.320615631e-4
+rthpower[1]*8.575247229e-5;
  G1term1[2]:=G1term1[2]-commult[2]*6.320615631e-4
+rthpower[2]*8.575247229e-5;
  comppower(cominv,8);
  compmult(rthpower,zetaminus);
  G1term1[1]:=G1term1[1]-commult[2]*443/1290240;
  G1term1[2]:=G1term1[2]+commult[1]*443/1290240;
  G1:=G1term1
end;(*getG1*)

procedure getgdashed;
var gdashedterm1,gdashedterm2,gdashedterm3:complex;
    i:integer;
begin
  gdashedterm1:=one;
  for i:=0 to (num-1) do
  begin
    fdashed(worbit[-i]);
    compmult(gdashedterm1,df);
    gdashedterm1:=commult
  end;(*for-i*)
```

```
    gdashedterm2[1]:=-2*worbit[-num][2];
    gdashedterm2[2]:=2*worbit[-num][1];
    compexp(gdashedterm2);
    gdashedterm2:=comexp;
    compmult(gdashedterm2,four);
    compminus(zero,commult);
    gdashedterm3:=comminus;
    compinv(gdashedterm2);
    gdashedterm3[1]:=gdashedterm3[1]-cominv[1]/6;
    gdashedterm3[2]:=gdashedterm3[2]-1-cominv[2]/6;
    compmult(cominv,cominv);
    gdashedterm3[1]:=gdashedterm3[1]+commult[2]/16;
    gdashedterm3[2]:=gdashedterm3[2]-commult[1]/16;
    compmult(gdashedterm3,gdashedterm1);
    gdashed:=commult
  end;(*getgdashed*)

  procedure gethgs;
  var i:integer;
  begin
    for i:=0 to numberofbands do
    begin
      getwn(gamma[i],num);
      getzetaplus;
      getzetaminus;
      getG;
      getH;
      getG1;
      getH1;
      gcont[i]:=G;
      hcont[i]:=H;
      g1cont[i]:=G1;
      h1cont[i]:=H1
    end;(*for-i*)
    for i:=0 to numberofbands do
    begin
      wnpluseps:=gamma[i];
      wnpluseps[1]:=wnpluseps[1]+eps;
      getwn(wnpluseps,num);
      getzetaminus;
      getG;
      getG1;
      gpeps[i]:=G;
      g1peps[i]:=G1;
      wnminuseps:=gamma[i];
      wnminuseps[1]:=wnminuseps[1]-eps;
```

```
    getwn(wnminuseps,num);
    getzetaminus;
    getG;
    getG1;
    gmeps[i]:=G;
    g1meps[i]:=G1
  end;(*for-i*)
  for i:=0 to numberofbands do
  begin
    wnplusteps:=gamma[i];
    wnplusteps[1]:=wnplusteps[1]+(2*eps);
    getwn(wnplusteps,num);
    getzetaminus;
    getG;
    getG1;
    gpteps[i]:=G;
    g1pteps[i]:=G1;
    wnminusteps:=gamma[i];
    wnminusteps[1]:=wnminusteps[1]-(2*eps);
    getwn(wnminusteps,num);
    getzetaminus;
    getG;
    getG1;
    gmteps[i]:=G;
    g1mteps[i]:=G1
  end(*for-i*)
end;(*gethgs*)

procedure getG1dashed;
var G1dashedterm1,G1dashedterm2:complex;
    i:integer;
begin
  for i:=0 to numberofbands do
  begin
    g1dashed[i][1]:=(8*g1peps[i][1]-8*g1meps[i][1]-
         g1pteps[i][1]+g1mteps[i][1])/(12*eps);
    g1dashed[i][2]:=(8*g1peps[i][2]-8*g1meps[i][2]-
         g1pteps[i][2]+g1mteps[i][2])/(12*eps);
    gdashedcont[i][1]:=(8*gpeps[i][1]-8*gmeps[i][1]-
         gpteps[i][1]+gmteps[i][1])/(12*eps);
    gdashedcont[i][2]:=(8*gpeps[i][2]-8*gmeps[i][2]-
         gpteps[i][2]+gmteps[i][2])/(12*eps)
  end(*for-i*)
end;(*getG1dashed*)

procedure integrand1(k:integer);
```

```
var intgr11,intgr12,intgr13,intgr14:complex;
begin
  compminus(hcont[k],gcont[k]);
  intgr11:=comminus;
  intgr12[1]:=-2*pi*gcont[k][2];
  intgr12[2]:=2*pi*gcont[k][1];
  compexp(intgr12);
  compmult(comexp,g1dashed[k]);
  intgr13:=commult;
  compmult(intgr11,intgr13);
  intgr1:=commult;
  int:=intgr1
end;(*integrand1*)

procedure integrand2(k:integer);
var intgr21,intgr22,intgr23,intgr24,intgr2:complex;
begin
  compminus(hcont[k],gcont[k]);
  intgr21:=comminus;
  compmult(intgr21,g1cont[k]);
  intgr22[1]:=commult[2]*(-2)*pi;
  intgr22[2]:=commult[1]*(2)*pi;
  compminus(h1cont[k],g1cont[k]);
  compadd(intgr22,comminus);
  intgr21:=comadd;
  intgr22[1]:=(-2)*pi*gcont[k][2];
  intgr22[2]:=(2)*pi*gcont[k][1];
  compexp(intgr22);
  compmult(intgr21,comexp);
  intgr23:=commult;
  compmult(intgr23,gdashedcont[k]);
  intgr2:=commult;
  int:=intgr2
end;(*integrand2*)

procedure integrand(kk:integer);
begin
  integrand1(kk);
  integrand2(kk);
  compadd(intgr1,intgr2);
  int:=comadd
end;(*integrand*)

procedure integrate1;
var subtotalint11,subtotalint12,subtotalint13,
    subtotalint14:complex;
```

```
    h:real;
    count:integer;
begin
  integrand1(0);
  subtotalint11:=int;
  integrand1(numberofbands);
  compadd(subtotalint11,int);
  subtotalint11:=comadd;
  count:=1;
  subtotalint12:=zero;
  repeat
  begin
    integrand1(count);
    compadd(subtotalint12,int);
    subtotalint12:=comadd;
    count:=count+2
  end(*repeat*)
  until count>numberofbands;
  compmult(subtotalint12,four);
  compadd(subtotalint11,commult);
  subtotalint11:=comadd;
  count:=2;
  subtotalint13:=zero;
  if count<numberofbands then
  repeat
  begin
    integrand1(count);
    compadd(subtotalint13,int);
    subtotalint13:=comadd;
    count:=count+2
  end(*repeat*)
  until count>=numberofbands;
  compmult(subtotalint13,two);
  compadd(subtotalint11,commult);
  subtotalint11:=comadd;
  compmult(subtotalint11,delta);
  subtotalint11[1]:=commult[1]/3;
  subtotalint11[2]:=commult[2]/3;
  integral:=subtotalint11;
  modint:=modulus(integral)
end;(*integrate1*)

procedure integrate2;
var subtotalint21,subtotalint22,subtotalint23,
    subtotalint24:complex;
    h:real;
```

```
    count:integer;
begin
  integrand2(0);
  subtotalint21:=int;
  integrand2(numberofbands);
  compadd(subtotalint21,int);
  subtotalint21:=comadd;
  count:=1;
  subtotalint22:=zero;
  repeat
  begin
    integrand2(count);
    compadd(subtotalint22,int);
    subtotalint22:=comadd;
    count:=count+2
  end(*repeat*)
  until count>numberofbands;
  compmult(subtotalint22,four);
  compadd(subtotalint21,commult);
  subtotalint21:=comadd;
  count:=2;
  subtotalint23:=zero;
  if count<numberofbands then
  repeat
  begin
    integrand2(count);
    compadd(subtotalint23,int);
    subtotalint23:=comadd;
    count:=count+2
  end(*repeat*)
  until count>=numberofbands;
  compmult(subtotalint23,two);
  compadd(subtotalint21,commult);
  subtotalint21:=comadd;
  compmult(subtotalint21,delta);
  subtotalint21[1]:=commult[1]/3;
  subtotalint21[2]:=commult[2]/3;
  answer[1]:=integral[1]+subtotalint21[1];
  answer[2]:=integral[2]+subtotalint21[2];
  writeln(modulus(answer),' ',wnought[2])
end;(*integrate2*)

begin(*body*)
  initialise;
  num:=100;
  numberofbands:=40;
```

```
  eps:=1e-4;
  wnought[1]:=1.57;
  wnought[2]:=-0.3;
  repeat
  begin
    wnought[2]:=wnought[2]+0.01;
    fofw(wnought);
    wone:=wn;
    getgamma;
    gethgs;
    getG1dashed;
    integrate1;
    integrate2
  end(*repeat*)
  until wnought[2]>0.35
end.
```

Chapter 3

Scaling of $(n + \frac{p}{q})^{-1}$–tongues of the sine circle map

In this chapter we shall be considering the *sine circle map* as studied in chapter 3. We shall now, however, be investigating the width of tongues with rotation number $\frac{1}{n+\frac{p}{q}}$. The motivation for this work is the paper [Da3], which looked at tongues with rotation number $\frac{1}{n}$ as $n \to \infty$ and $k \to 0$ in any manner. We summarise this paper below, since it is essential to the material that follows.

3.1 Width of Arnol'd tongues for the sine circle map

The following result is obtained:

Theorem 3.1 *Let $f_{k.\Omega}(x) = x + \Omega + k \sin^2 x$, for $0 < k < 1, 0 < \Omega < \pi$, and let*

$$l(n,k) = \frac{1}{n^2\sqrt{n^2k^2 + 4\pi^2}} \left(\sqrt{1 + \left(\frac{2\pi}{nk}\right)^2} - \frac{2\pi}{nk} \right)^n .$$

Then

$$\frac{|I_n(k)|}{l(n,k)} \to A_0$$

as $n \to \infty$ and $k \to 0$, where A_0 is the same constant obtained in section 2.1.

The proof is by estimation of the range of $f^n_{k,\Omega}$ for $\Omega \in I_n(k)$, whence we obtain the range of Ω for which $f^n_{k,\Omega}(x) - x - \pi$ has a fixed point. As in [Da1], $f_{k,\Omega}$ is extended into the complex plane. With an extra dimension available, z_0 is allowed to vary in such a way that $f_{k,0}$ is transformed to a 'limiting mapping' independent of the small parameter k.

We begin by defining a mapping which satisfies

$$H(f_{k,\Omega}(z)) - H(z) \simeq 1,$$

for $z \in \mathbf{C}$. Writing

$$g(z) = \Omega + k \sin^2 z,$$

and

$$G(z) = \frac{1}{\sqrt{\Omega(k+\Omega)}} \tan^{-1}\left(\sqrt{\frac{k+\Omega}{\Omega}} \tan z\right), \tag{3.1}$$

(so that $G' = \frac{1}{g}$), we define

$$H(z) = G(z) + \tfrac{1}{2} \log g(z),$$

the branches being chosen so that $G(0) = 0$ and $H(0) = \frac{1}{2} \log \Omega$.

Now, let

$$\lambda = \frac{1}{\sqrt{\Omega(k+\Omega)}} \tanh^{-1}\left(\sqrt{\frac{\Omega}{k+\Omega}}\right).$$

We find that G maps the domain $V = \{z : |\Im G(z)| < \lambda\}$ conformally onto the strip $\{w : |\Im w| < \lambda\}$. We then also find that if $W_A = \{z \in V : \Im H(z) < \lambda - A\}$, where $A > 0$, then provided A is large enough, H maps W_A conformally onto the set $\{w : |\Im w| < \lambda - A\}$.

This now facilitates the estimation of the error in the approximation $H(f_{k,\Omega}(z)) - H(z) \simeq 1$. We obtain

$$\left|H(f_{k,\Omega}(z)) - H(z) - 1\right| \leq C_1 k e^{2y} \left|g(z)\right|, \tag{3.2}$$

where $y = \Im z$, and $C_1 > 0$. We are now able to define a mapping,

$$\tilde{f} = H \circ f_{k,\Omega} \circ H^{-1},$$

on the rectangle

$$R_A = \left\{w : |\Re w| < \tfrac{1}{2}, |\Im w| < \lambda - A\right\},$$

so that if $H(z) = w \in R_A$, then

$$\begin{aligned} \left|\tilde{f}(w) - w - 1\right| &\leq C_1 k e^{2y} \left|g(z)\right| \\ &\leq C_2 k \Omega e^{2y}. \end{aligned}$$

We now introduce the following important result:

Proposition 3.2 *There exists $\delta > 0$ such that if $0 < \varepsilon < \delta$ and if F is analytic on the rectangle*

$$R = \left\{z = x + iy : |x| < \tfrac{1}{2}, |y| < a\right\},$$

for some $a < 2$, satisfying

$$\int\int_R |F(x+iy) - (x+iy) - 1|\, dx\, dy < \varepsilon,$$

then there exists φ, analytic on the rectangle

$$R_0 = \left\{z : -\tfrac{1}{4} < x < \tfrac{5}{4}, |y| < a - 1\right\},$$

such that

$$|\varphi(z) - z| < B\varepsilon$$

for $z \in R_0$, and

$$\varphi(F(z)) = \varphi(z) + 1$$

for $|x| < \frac{1}{4}$, $|y| < a - 1$. (B is an absolute constant).

Now $\tilde{f}$ satisfies the hypotheses of Proposition 3.2, and we then obtain φ, analytic on the rectangle

$$Q_{A+1} = \left\{w : -\tfrac{1}{4} < \Re w < \tfrac{5}{4}, |\Im w| < \lambda - (A+1)\right\}.$$

φ satisfies

$$\varphi(\tilde{f}(w)) = \varphi(w) + 1,$$

for $w \in R_A \cap Q_{A+1}$, and

$$|\varphi(w) - w| < C_3\Omega,$$

for $w \in Q_{A+1}$. Writing now, $\theta = \varphi \circ H$, we have

$$\theta(f(z)) = \theta(z) + 1, \tag{3.3}$$

for $z \in H^{-1}(R_A \cap Q_{A+1})$, and

$$|\theta(z) - H(z)| < C_3\Omega,$$

for $z \in H^{-1}(Q_{A+1})$.

We are now in a position to define the mapping which is the crucial ingredient in the proof: we write

$$\sigma(w) = \theta\left[f^m\left(\theta^{-1}(w)\right) - \pi\right] - w - m, \tag{3.4}$$

where m is chosen so that $f^m\left(\theta^{-1}(w)\right) - \pi \in Q_{A+1}$. It follows from 3.3 that if more than one such m exists, $\sigma(w)$ is independent of the choice. Also, if $w, w+1 \in Q_{Q+2}$, then $\sigma(w+1) = \sigma(w)$. We may thus extend σ to an analytic function of period 1 on the strip $\{w : |\Im w| < \lambda - (A+2)\}$, and write

$$\sigma(w) = \sum_{r=-\infty}^{\infty} \sigma_r e^{-2\pi i r w},$$

where

$$\sigma_r = \int_0^1 \sigma(w) e^{-2\pi i r w}\, dw.$$

We now consider the behaviour of σ_r as $k, \Omega \to 0$ by means of the 'limiting map', $f_0(v) = v - \frac{1}{4}e^{-2iv}$. This is the same map as we derived in section 2.1, and it plays the same rôle. We define similarly the map

$$H_0(v) = 2ie^{2iv} - iv,$$

and thence the limits

$$h_\pm(v_0) = \lim_{m\to\pm\infty} H_0(v_m) - m,$$

on the set

$$U = \left\{v : \tfrac{3\pi}{8} < \Re v < \tfrac{5\pi}{8}, \Im v < -A-3\right\}.$$

h_+ and h_- are the same as the mappings h and g in section 2.1, and they satisfy

$$h_\pm(f_0(v) = h_\pm(v) + 1.$$

We define, as before, an analytic, period 1 map,

$$\sigma^0(w) = h_+\left(h_-(w)\right) - w,$$

on the half plane $\{w : \Im w < B\}$, for B large enough. By considering an orbit of f_0, $\{v_m\}$, and in particular by considering $v_{\pm m}$, where m is chosen large enough so that $e^{2\Im v_{\pm m}} < \varepsilon$, for $\varepsilon > 0$, we find that

$$\sigma(w) = \sigma^0(w - \mu) + E + O(\varepsilon),$$

E a constant, for $w \in Q_{A+3}$ with $\Im w = \lambda - A - 4$, where

$$\mu = \frac{\pi}{2\sqrt{\Omega(k+\Omega)}} - \log 2 + i\left(\lambda + \tfrac{\pi}{2}\right),$$

and we therefore obtain

$$\sigma_r = e^{2\pi i r\mu} \int_{-\mu+i(\lambda-A-4)}^{-\mu+i(\lambda-A-4)+1} e^{2\pi i r w} \sigma^0(w)\, dw \; + O\left(e^{-2\pi r\lambda}\varepsilon\right). \tag{3.5}$$

Since $\mu - i(\lambda + \frac{\pi}{2}) \in \mathbf{R}$, we obtain

$$|\sigma_r| = e^{-2\pi r\lambda - r\pi^2} \left(\left|\sigma_r^0\right| + O(\varepsilon)\right), \tag{3.6}$$

where $\sigma^0(w) = \sum_{r=-\infty}^{\infty} \sigma_r^0 e^{-2\pi i r w}$. Thus, as $k, \Omega \to 0$,

$$\max_{w\in\mathbf{R}} \sigma(w) - \min_{w\in\mathbf{R}} \sigma(w) = 4e^{-2\pi\lambda-\pi^2} \left(\left|\sigma_1^0\right| + o(1)\right).$$

Now, if $\Omega \in I_n(k)$, there exists an orbit, $\{\xi_m\}$, such that

$$f^n(\xi_m) - \pi = \xi_m. \tag{3.7}$$

Since $f(0) = \Omega$, and since f preserves order, there must be $x_0 \in [0, \Omega]$ such that 3.7 holds for $\xi_m = x_0$. By 3.2,

$$|H(x_0 + \pi) - H(x_0) - n| \le C_4 k,$$

and so

$$|G(x_0 + \pi) - G(x_0) - n| \le C_4 k.$$

However, from 3.1, we see that

$$|G(x_0 + \pi) - G(x_0)| = \frac{\pi}{\sqrt{\Omega(k + \Omega)}} \quad ,$$

and so we have

$$\left| \; n - \frac{\pi}{\sqrt{\Omega(k+\Omega)}} \; \right| \le C_4 k.$$

This then gives

$$\begin{aligned} e^{-2\pi\lambda} &= \exp\left\{-2n \tanh^{-1}\left[\frac{-nk + \sqrt{n^2k^2 + 4\pi^2}}{2\pi}\right]\right\} \left(1 + o(1)\right) \\ &= \left[\sqrt{1 + \left(\frac{2\pi}{nk}\right)^2} - \frac{2\pi}{nk}\right]^n \left(1 + o(1)\right). \end{aligned}$$

Now also, $|\Omega\theta'(x) - 1| = o(1)$ for $x \in [-\Omega, 2\Omega]$, and so we obtain

$$\begin{aligned} &\max_{x\in[0,\Omega]} \left(f^n(x) - x - \pi\right) - \min_{x\in[0,\Omega]} \left(f^n(x) - x - \pi\right) \\ &\qquad = 4e^{-\pi^2} \left|\sigma_1^0\right| \left[\sqrt{1 + \left(\frac{2\pi}{nk}\right)^2} - \frac{2\pi}{nk}\right]^n \left(1 + o(1)\right). \end{aligned}$$

Finally, we require an estimate of $\frac{\partial f^n}{\partial \Omega}$ to complete the argument, and for this we obtain

$$\frac{\partial f^n}{\partial \Omega}(x) = \frac{n^3}{2\pi^2}\left[-k + \sqrt{k^2 + \frac{4\pi^2}{n^2}}\ \right]\sqrt{k^2 + \frac{4\pi^2}{n^2}}\ \big(1 + o(1)\big),$$

for $x \in [0, \Omega]$ and $\Omega \in I_n(k)$. $\square$

3.2 The extension to rotation numbers $(n+\frac{p}{q})^{-1}$.

In order to estimate the width of $(n+\frac{p}{q})^{-1}$–tongues we develop the idea of the map 3.4. If $\alpha = (n+\frac{p}{q})^{-1}$, and $\Omega \in I_\alpha$, with $(p,q) = 1$, then there exists $x \in [0,\pi)$ such that

$$f_{k,\Omega}^{qn+p}(x) = x + q\pi.$$

[We note that if $(p,q) = 1$, then for any $n \in \mathbf{N}$, $(q, qn+p) = 1$.] Recall now the mapping which was the centre piece of the proof of Theorem 3.1:

$$\sigma(w) = \theta\left[f_{k,\Omega}^{m}(\theta^{-1}(w)) - \pi\right] - m - w,$$

for $w \in Q_{A+2}$. In order to estimate the range of $f_{k,\Omega}^{qn+p}(x) - x - q\pi$ we define

$$\psi(w) = \theta\left[f_{k,\Omega}^{m}(\theta^{-1}(w)) - \pi\right] - m,$$

for $w \in Q_{A+2}$, and consider

$$\psi^q(w) = \theta\left[f_{k,\Omega}^{m'}(\theta^{-1}(w)) - q\pi\right] - m',$$

with m' chosen so that $f_{k,\Omega}^{m'}(\theta^{-1}(w)) - q\pi \in Q_{A+1}$. Of course, for any given w, $\Im w$ not too large, m' will be near to $qn+p$ when $\Omega \in I_\alpha$. Recall that $\theta(f_{k,\Omega}(z)) = \theta(z) + 1$, so that again we see there is no ambiguity regarding the choice of m', and the map is well defined.

We define now

$$\tilde{\sigma}(w) = \psi^q(w) - w,$$

for $w \in Q_{A+2}$. Again $\tilde{\sigma}$ has the property $\tilde{\sigma}(w+1) = \tilde{\sigma}(w)$, provided $w, w+1 \in Q_{A+2}$, and so it can be analytically extended to a function of period 1 on the strip $\{w : |\Im w| < \lambda - A - 2\}$. We thus write

$$\tilde{\sigma}(x) = \sum_{r=-\infty}^{\infty} \tilde{\sigma}_r(x) e^{2\pi r i x}.$$

Now, since for $\Omega \in I_\alpha$ there exists x_0 such that $f_{k,\Omega}^{qn+p}(x_0) = x_0 + q\pi$, we see that $\tilde{\sigma}(\theta(x_0)) = -qn - p$, and hence conclude that

$$\tilde{\sigma}_0 = -qn - p + O(\tilde{\sigma}_1).$$

(Recall that $\theta(f_{k,\Omega}(z)) = \theta(z) + 1$, for $z \in H^{-1}(R_A \cap Q_{A+1})$.)

The problem is thus now, as before, one of estimating the range

$$\max_{x\in[0,\pi)} \tilde{\sigma}(x) - \min_{x\in[0,\pi)} \tilde{\sigma}(x).$$

3.2.1 The case $\alpha = (n+\frac{1}{2})^{-1}$.

For q small enough we may, in theory, expand $\tilde{\sigma}$ in terms of the Fourier coefficients of σ. We illustrate this process by considering $\alpha = \frac{1}{n+\frac{1}{2}}$, so that $p = 1, q = 2$. We have then, first of all,

$$\psi(x) = x + \sum_{r=-\infty}^{\infty} \sigma_r e^{2\pi i r x},$$

for $x \in \mathbf{R}$. Thus

$$\begin{aligned} \psi^2(x) &= x + \sigma_0 + \sigma_1 e^{2\pi i x} + \bar{\sigma}_1 e^{-2\pi i x} + \sigma_2 e^{4\pi i x} + \bar{\sigma}_2 e^{-4\pi i x} + \cdots \\ &\quad + \sigma_0 + \sigma_1 \exp\left\{2\pi i \{x + \sigma_0 + \sigma_1 e^{2\pi i x} + \bar{\sigma}_1 e^{-2\pi i x} + \cdots\}\right\} \\ &\quad + \bar{\sigma}_1 \exp\left\{-2\pi i \{x + \sigma_0 + \sigma_1 e^{2\pi i x} + \bar{\sigma}_1 e^{-2\pi i x} + \cdots\}\right\} \\ &\quad + \cdots. \end{aligned} \tag{3.8}$$

Now, writing $\tilde{\sigma}(x) = \psi^2(x) - x$ as the Fourier series

$$\tilde{\sigma}(x) = \sum_{r=-\infty}^{\infty} \tilde{\sigma}_r(x) e^{2\pi r i x},$$

we see that the constant term, $\tilde{\sigma}_0$, is given by

$$\tilde{\sigma}_0 = 2\sigma_0 + O(\sigma_1^2).$$

[Recall that $\sigma_r = O(\sigma_r^0 e^{-2\pi r\lambda - r\pi^2})$.] Thus we obtain

$$\sigma_0 = -n - \frac{1}{2} + O\left(\sigma_1^2 + \tilde{\sigma}_1\right).$$

Continuing with the expansion of 3.8, then, we eventually find

$$\begin{aligned} \tilde{\sigma}_1 &= O\left(\sigma_1^3\right), \\ \tilde{\sigma}_2 &= 2\left(\sigma_2 - i\pi\sigma_1^2\right) + O\left(\sigma_1^3\right), \end{aligned}$$

and of course,

$$\tilde{\sigma}_{-i} = \bar{\tilde{\sigma}}_i \ .$$

We thus easily see that

$$\max_{x\in\mathbf{R}} \tilde{\sigma}(x) - \min_{x\in\mathbf{R}} \tilde{\sigma}(x) = 8\left|\sigma_2 - i\pi\sigma_1^2\right| + O\left(\sigma_1^3\right),$$

and by 3.6 we obtain

$$\left|\sigma_2 - i\pi\sigma_1^2\right| = e^{-4\pi\lambda - 2\pi^2}\left(\left|\sigma_2^0 - i\pi(\sigma_1^0)^2\right| + o(1)\right),$$

as $k \to 0$.

Now, from 3.2 we obtain

$$|G(x_0 + 2\pi) - G(x_0) - (2n+1)| \leq Dk,$$

where D is some positive constant. Also we see directly from the definition of G that

$$G(x_0 + 2\pi) - G(x_0) = \frac{2\pi}{\sqrt{\Omega(k+\Omega)}},$$

and hence

$$\left|(2n+1) - \frac{2\pi}{\sqrt{\Omega(k+\Omega)}}\right| \leq Dk.$$

We therefore now have

$$e^{-4\pi\lambda} = \left[\sqrt{1 + \left(\frac{4\pi}{(2n+1)k}\right)^2} - \frac{4\pi}{(2n+1)k}\right]^{2n+1}\left(1 + o(1)\right),$$

as $k \to 0$. Also, our estimate of $\frac{\partial f_{k,\Omega}^{2n+1}}{\partial \Omega}$ is

$$\frac{\partial f_{k,\Omega}^{2n+1}}{\partial \Omega} = \frac{\Omega(n+\frac{1}{2})^2}{\pi^2}\sqrt{\left(n+\frac{1}{2}\right)^2 k^2 + 4\pi^2}\left(1 + o(1)\right),$$

for $x \in [0, \Omega]$ and $\Omega \in I_\alpha$. We thus finally obtain

$$\left|I_\alpha(k)\right| = \frac{8\pi^2 e^{-2\pi^2}\left|\sigma_2^0 - i\pi(\sigma_1^0)^2\right|}{(n+\frac{1}{2})^2\sqrt{(n+\frac{1}{2})^2k^2 + 4\pi^2}}\left[\sqrt{1 + \left(\frac{2\pi}{(n+\frac{1}{2})k}\right)^2} - \frac{2\pi}{(n+\frac{1}{2})k}\right]^{2n+1} \times \left(1 + o(1)\right),$$

as $n \to \infty, k \to 0$.

3.2.2 The general case.

More generally we are interested in $\alpha = (n + \frac{p}{q})^{-1}$, for $(p,q) = 1$. The problem with generalising the above method for $\frac{p}{q} = \frac{1}{2}$ is simply that for larger q, calculation of ψ^q rapidly becomes impractical by direct means. We use a result from [Da2].

Theorem 3.3 *Let $f_\Omega(x) = x + \Omega + g(x)$, where*

$$g(x) = \sum_{r=1}^{\infty} 2\varepsilon^r \Re\left(a_r e^{2\pi i r x}\right),$$

the series converging for ε small enough. Then for this family,

$$|I_\alpha| = |\Psi_\alpha(a_1, \dots, a_q)|\, \varepsilon^q + O(\varepsilon^{q+1}),$$

for each rational $\alpha = \frac{p}{q}$, where Ψ_α is a polynomial with complex coefficients.

For the detailed proof of this result we refer the reader to [Da2], and in particular to Theorem 7.2 and Proposition 7.3 therein.

Now consider again the map $\psi(x)$. From 3.5 we obtain

$$\sigma_r = e^{2\pi i r \mu}\left(\sigma_r^0 + o(1)\right),$$

as $n \to \infty$. Thus

$$\psi(x) \sim \psi^0(x) = x + \Omega_0 + \sum_{r=1}^{\infty} e^{-2\pi r\lambda - r\pi^2} 2\Re\left(e^{2\pi i r \Re\mu}\sigma_r^0\right),$$

as $n \to \infty$. We apply the theorem to ψ^0, or to be more precise, to the map

$$x \mapsto x + (\Omega_0 \bmod 1) + \sum_{r=1}^{\infty} e^{-2\pi r\lambda - r\pi^2} 2\Re\left(e^{2\pi i r \Re\mu}\sigma_r^0\right),$$

looking for the interval $I_{\frac{p}{q}}$. As we shall see, for each term in the polynomial, $K\sigma_1^{r_1}\dots\sigma_q^{r_q}$, we have $\sum_{m=1}^{q} m r_m = q$, so that we obtain

$$|I_\alpha| = \left|\Psi_{\frac{p}{q}}\left(e^{2\pi i \Re\mu}\sigma_1^0, \dots, e^{2\pi i q \Re\mu}\sigma_q^0\right)\right| e^{-2\pi q\lambda - q\pi^2} + o\left(e^{-2\pi q\lambda}\right).$$

An algorithm obtained from the proof of Theorem 3.3 enables us to calculate the required polynomial $\Psi_{\frac{p}{q}}$, and we give details of this, as well as tabulation of $|\Psi_{\frac{p}{q}}|$ for values of q up to 20 in the following section.

We see then that

$$\max_{x\in\mathbf{R}}(\psi^q(x) - x) - \min_{x\in\mathbf{R}}(\psi^q(x) - x) = q|\Psi_{\frac{p}{q}}| e^{-2\pi q\lambda - q\pi^2} + O\left(e^{-2\pi(q+1)\lambda - (q+1)\pi^2}\right),$$

and we then easily obtain the following result:

Theorem 3.4 *Let $f_{k.\Omega}(x) = x + \Omega + k\sin^2 x$, for $0 < k < 1, 0 < \Omega < \pi$, and let $\alpha = (n + \frac{p}{q})^{-1}$, and let*

$$l(n,k,p,q) = \frac{1}{(n+\frac{p}{q})^2\sqrt{(n+\frac{p}{q})^2k^2+4\pi^2}}\left[\sqrt{1+\left(\frac{2\pi}{(n+\frac{p}{q})k}\right)^2} - \frac{2\pi}{(n+\frac{p}{q})k}\right]^{qn+p}.$$

Then

$$\frac{|I_\alpha(k)|}{l(n,k,p,q)} \to 2\pi^2 e^{-q\pi^2}|\Psi_{\frac{p}{q}}|,$$

as $n \to \infty, k \to 0$.

3.3 Numerical results

We now turn to the estimation of $|\Psi_{\frac{p}{q}}|$ for $q = 1, 2, ..., 20$. For full details of the following algorithm we refer the reader to [Da2] and to the proofs of the aforementioned results.

We begin by defining $\gamma = e^{2\pi i\alpha}$, and we set $b_1 = \sigma_1$. Now, proceeding recursively, for $1 \leq k \leq q$ we define

$$c_k = \frac{b_k}{(\gamma^k - 1)},$$

and

$$b_k = \sum_{j=1}^{k} \sigma_j \sum (2\pi i j)^{r_1 + \cdots + r_m} \frac{c_1^{r_1} \ldots c_m^{r_m}}{r_1! \ldots r_m!},$$

where the second summation is taken over all sequences $r_1, \ldots, r_m$ such that

(i) $\quad r_i \geq 0;$

(ii) $\quad r_1 + 2r_2 + \cdots + mr_m = k - j.$

We then have $\Psi_\alpha = b_q$. Furthermore, it is easy to verify inductively that, as mentioned in the previous section, every term in Ψ_α has the form $K\sigma_1^{r_1} \ldots \sigma_q^{r_q}$, where $\sum_{m=1}^{q} mr_m = q$.

3.3.1 The Fourier coefficients

We derive the Fourier coefficients from numerical work done by Stewart, [St]. In this work, Stewart considers the map $g(v) = ve^v$, and derives a period-1 map (called σ, but we shall denote it by ψ, to avoid confusion) analogous to the map σ^0 involved in this chapter. Recall that $\sigma^0(w) = h_+(h_-^{-1}(w)) - w$, where $h_+(z_0) = \lim_{n\to\infty} H_0(z_n) - n$ and $h_-(z_0) = \lim_{n\to-\infty} H_0(z_n) - n$, z_n being iterates of the map $f(z) = z - \frac{1}{4}e^{-2iz}$. Now, it is easily verified that the transformation $z = u(v) = \frac{i}{2}\log -2iv$ is such that $f \circ u = u \circ g$. Defining $\Theta = H_0 \circ u$, we obtain

$$\Theta(v) = -\frac{1}{v} + \frac{1}{2}\log v + C,$$

where C is a constant, and we find that

$$\Theta(g(v)) - \Theta(v) = 1 + O(v^2). \tag{3.9}$$

Θ can then be used to define the map ψ in a manner analogous to the definition of σ^0. We note, however, that Stewart calculates additional terms for the map we have called Θ, so that the error term in equation 3.9 is of higher order. The effect of this is that the limits

$$\varphi_{\pm}(v_0) = \lim_{n\to\pm\infty} \Theta(v_n) - n$$

will differ by a constant, depending on the number of terms calculated for Θ.

We have, then,

$$\begin{aligned}\psi(\tilde{w}) &= \varphi_+(v) - \varphi_-(v),\\ \sigma^0(w) &= h_+(z) - h_-(z),\end{aligned}$$

where $\tilde{w} = \varphi_-(v)$ and $w = h_-(z)$. Thus

$$\sigma^0(w) = h_+ \circ u(v) - h_- \circ u(v).$$

Now,

$$\begin{aligned}h_+ \circ u(v) &= \lim_{n\to\infty} H_0 \circ f^n(u(v)) - n\\ &= \lim_{n\to\infty} H_0 \circ u \circ g^n(v) - n\\ &= \lim_{n\to\infty} \Theta(v_n) - n\\ &= \varphi_+(v),\end{aligned}$$

and similarly for h_-, φ_-. Hence we see that

$$\sigma^0(w) = \psi(\tilde{w}) = \psi(w+\nu),$$

for some constant ν. We thus have, for the Fourier coefficients we require,

$$\begin{aligned}\sigma_r^0 &= \int_{w_0}^{w_0+1} \psi(w+\nu)e^{-2\pi i r w}dw\\ &= \int_{w_0+\nu}^{w_0+\nu+1} \psi(\xi)e^{-2\pi i r(\xi-\nu)}d\xi\\ &= e^{2\pi i r\nu}\psi_r.\end{aligned}$$

From **[St]** we have the following estimates for $\psi_1, \ldots, \psi_{20}$:

$$\begin{aligned}
\psi_1 &= 1.0968632 \times 10^3 - 3.2721079 \times 10^2 i \\
\psi_2 &= 3.0300529 \times 10^5 - 7.1904567 \times 10^4 i \\
\psi_3 &= 3.7588902 \times 10^7 - 1.9265715 \times 10^7 i \\
\psi_4 &= 4.2745511 \times 10^9 - 2.4590307 \times 10^9 i \\
\psi_5 &= 1.3774875 \times 10^{11} - 4.8994609 \times 10^{11} i \\
\psi_6 &= 2.6166535 \times 10^{13} + 3.9449673 \times 10^{12} i \\
\psi_7 &= -3.3864992 \times 10^{16} - 1.0748475 \times 10^{16} i \\
\psi_8 &= 2.1925635 \times 10^{17} + 1.7196424 \times 10^{18} i \\
\psi_9 &= -1.1493705 \times 10^{18} - 3.4009912 \times 10^{20} i \\
\psi_{10} &= -1.3183358 \times 10^{22} + 5.3164967 \times 10^{22} i \\
\psi_{11} &= 3.9565245 \times 10^{24} - 6.9086502 \times 10^{24} i \\
\psi_{12} &= -8.7036497 \times 10^{26} + 7.5179384 \times 10^{26} i \\
\psi_{13} &= 1.6676413 \times 10^{29} - 8.3835279 \times 10^{28} i \\
\psi_{14} &= -3.256251 \times 10^{31} + 1.5434638 \times 10^{31} i \\
\psi_{15} &= 6.9358351 \times 10^{33} - 3.6867026 \times 10^{33} i \\
\psi_{16} &= -1.4760744 \times 10^{36} + 6.7328611 \times 10^{35} i \\
\psi_{17} &= 2.718999 \times 10^{38} - 5.9558781 \times 10^{37} i \\
\psi_{18} &= -3.7648585 \times 10^{40} - 8.0754807 \times 10^{39} i \\
\psi_{19} &= 2.9151277 \times 10^{42} + 4.0989653 \times 10^{42} i \\
\psi_{20} &= 1.2735682 \times 10^{44} - 7.6963318 \times 10^{44} i
\end{aligned}$$

Now, from earlier numerical work we have the estimate $\sigma_1^0 \approx 134330 - 85357.4i$, and so we may obtain the estimate $e^{2\pi i\nu} \approx 133.777 - 37.9118i$ by direct calculation.

3.3.2 Computed values of $|\Psi_{\frac{p}{q}}|$.

The data we present here were obtained using the program given at the end of this chapter, and figures quoted are real and imaginary parts of $\Psi_{\frac{p}{q}}$, followed by the modulus. No more than 4 significant figures should be regarded as reliable.

```
p/q= 1/1
      1.34329918e+005 -8.53573361e+004  1.59155276e+005

p/q= 1/2
     -6.77854793e+010 -3.80562999e+010  7.77377203e+010

p/q= 1/3
     -2.27667430e+016  6.49245146e+016  6.88005609e+016
p/q= 2/3
      2.20486157e+016  6.24799206e+016  6.62561841e+016

p/q= 1/4
      7.83065247e+022  3.14375935e+022  8.43814796e+022
p/q= 3/4
      1.05836213e+022 -7.64794058e+022  7.72082415e+022

p/q= 1/5
      7.33573961e+028 -1.09537237e+029  1.31832143e+029
p/q= 2/5
     -6.67494686e+028 -5.42036371e+028  8.59856141e+028
p/q= 3/5
     -7.72161534e+028 -3.27202060e+028  8.38626628e+028
p/q= 4/5
     -4.77469116e+028  1.03580042e+029  1.14055218e+029

p/q= 1/6
     -1.51677161e+035 -2.00811455e+035  2.51656913e+035
p/q= 5/6
      1.13908078e+035 -1.71392840e+035  2.05792507e+035

p/q= 1/7
     -5.58003886e+041  1.22318024e+041  5.71253040e+041
p/q= 2/7
      1.52286366e+041  1.08265157e+041  1.86848819e+041
p/q= 3/7
      1.09255853e+041 -1.21615057e+041  1.63484138e+041
p/q= 4/7
      1.61292124e+041 -2.46164659e+040  1.63159798e+041
p/q= 5/7
     -9.96704046e+039 -1.65320669e+041  1.65620849e+041
p/q= 6/7
     -2.61706235e+041  3.56715005e+041  4.42420329e+041

p/q= 1/8
     -4.34493063e+047  1.44700229e+048  1.51082753e+048
p/q= 3/8
```

```
     -6.06619292e+046 -2.26100299e+047  2.34096593e+047
p/q= 5/8
     -2.13911433e+047  4.15974893e+046  2.17918453e+047
p/q= 7/8
      6.22730078e+047 -9.20721835e+047  1.11154012e+048

p/q= 1/9
      2.98805850e+054  3.47057472e+054  4.57967057e+054
p/q= 2/9
     -3.66608113e+053 -4.75439811e+053  6.00370321e+053
p/q= 4/9
      3.46250569e+053  2.23252941e+053  4.11984626e+053
p/q= 5/9
     -3.69503655e+053  2.04502436e+053  4.22320018e+053
p/q= 7/9
      2.90761079e+053 -3.78040196e+053  4.76923888e+053
p/q= 8/9
     -1.50970775e+054  2.83460312e+054  3.21157163e+054

p/q= 1/10
      1.56266385e+061 -1.46653066e+060  1.56953032e+061
p/q= 3/10
      5.68201687e+059  3.36241591e+059  6.60235991e+059
p/q= 7/10
      2.87997462e+058 -5.70569293e+059  5.71295671e+059
p/q= 9/10
      3.41478659e+060 -9.95731716e+060  1.05265822e+061

p/q= 1/11
      3.21251831e+067 -5.08202193e+067  6.01225589e+067
p/q= 2/11
      2.29794243e+065  2.69884007e+066  2.70860538e+066
p/q= 3/11
      1.12214115e+066 -2.84071157e+065  1.15753928e+066
p/q= 4/11
      5.57538253e+065 -7.68576592e+065  9.49504545e+065
p/q= 5/11
     -3.30878959e+065  1.25865873e+066  1.30142333e+066
p/q= 6/11
      1.03882429e+066 -8.95141742e+065  1.37128941e+066
p/q= 7/11
     -4.12031803e+065  7.31067910e+065  8.39184423e+065
p/q= 8/11
      2.05654740e+065  9.10335723e+065  9.33276487e+065
p/q= 9/11
      1.68278091e+066 -9.27635753e+065  1.92152535e+066
```

```
p/q= 10/11
     -4.92757310e+066  3.83720570e+067  3.86871521e+067

p/q= 1/12
     -7.93795754e+073 -2.42242293e+074  2.54916547e+074
p/q= 5/12
     -1.83954286e+070  1.30069617e+072  1.30082625e+072
p/q= 7/12
     -1.25795102e+071  1.25633888e+072  1.26262100e+072
p/q= 11/12
     -1.74622759e+073 -1.56865717e+074  1.57834674e+074

p/q= 1/13
     -1.09200543e+081 -4.63656281e+080  1.18636125e+081
p/q= 2/13
      1.06967334e+079 -1.26686819e+079  1.65805792e+079
p/q= 3/13
     -3.51383859e+078  2.77018703e+078  4.47448297e+078
p/q= 4/13
      3.06127503e+078  7.12692437e+077  3.14314100e+078
p/q= 5/13
      2.05231012e+078  3.90724898e+077  2.08917274e+078
p/q= 6/13
     -4.90862700e+078  8.80936838e+077  4.98705010e+078
p/q= 7/13
     -3.74451071e+078  3.88699588e+078  5.39723053e+078
p/q= 8/13
      8.46342795e+077 -1.70836486e+078  1.90651688e+078
p/q= 9/13
      9.77197153e+077 -2.47640020e+078  2.66223069e+078
p/q= 10/13
     -3.06568375e+078  1.17289663e+078  3.28239294e+078
p/q= 11/13
      9.96301894e+078 -3.40595092e+078  1.05291143e+079
p/q= 12/13
      2.62448777e+080  6.58272843e+080  7.08662470e+080

p/q= 1/14
     -5.47503294e+087  2.49486884e+087  6.01667318e+087
p/q= 3/14
     -2.55763944e+084  8.99089574e+084  9.34760534e+084
p/q= 5/14
      5.02690306e+084  6.83130836e+083  5.07310774e+084
p/q= 9/14
      2.21079114e+084  3.62868710e+084  4.24911374e+084
p/q= 11/14
```

```
        4.41120004e+084 -4.63794219e+084  6.40071821e+084
p/q= 13/14
       -2.16678823e+087 -2.71744680e+087  3.47555583e+087

p/q= 1/15
       -1.08408226e+094  3.12148350e+094  3.30437491e+094
p/q= 2/15
       -1.33404123e+092 -1.12988425e+091  1.33881754e+092
p/q= 4/15
        6.86734781e+090 -7.98866429e+090  1.05346677e+091
p/q= 7/15
       -1.46503813e+091 -1.72556665e+091  2.26360707e+091
p/q= 8/15
        1.73287133e+091 -1.81931091e+091  2.51251572e+091
p/q= 11/15
       -4.62094783e+090 -6.16042277e+090  7.70090693e+090
p/q= 13/15
        7.21143366e+091 -2.55569053e+091  7.65090384e+091
p/q= 14/15
        1.54483213e+094  1.01746490e+094  1.84979489e+094

p/q= 1/16
        8.62967406e+100  1.75364248e+101  1.95447555e+101
p/q= 3/16
        6.43193686e+096 -4.79910999e+097  4.84201971e+097
p/q= 5/16
        1.83734691e+097 -4.52998930e+096  1.89236670e+097
p/q= 7/16
       -1.83549605e+096 -1.11520297e+097  1.13020711e+097
p/q= 9/16
        7.83813462e+096  8.48628437e+096  1.15522022e+097
p/q= 11/16
        1.19064391e+097 -1.02328922e+097  1.56995342e+097
p/q= 13/16
        2.84634047e+097  9.24051130e+096  2.99257825e+097
p/q= 15/16
       -1.02964850e+101 -2.61449211e+100  1.06232373e+101

p/q= 1/17
        1.16930206e+108  4.09772951e+107  1.23902428e+108
p/q= 2/17
        5.40449570e+104  1.28177172e+105  1.39105157e+105
p/q= 3/17
       -4.33077318e+103 -1.10262680e+104  1.18462730e+104
p/q= 4/17
        3.56701822e+103  2.78568460e+103  4.52588750e+103
```

```
p/q= 5/17
      2.21244961e+103 -5.19420614e+102  2.27260446e+103
p/q= 6/17
      7.26352949e+102  3.30464791e+103  3.38353165e+103
p/q= 7/17
     -6.40956489e+102 -1.51013736e+103  1.64053043e+103
p/q= 8/17
      3.17244647e+103 -1.15113022e+104  1.19404562e+104
p/q= 9/17
     -9.95498005e+103  9.22127312e+103  1.35695802e+104
p/q= 10/17
     -1.54105999e+103 -6.59741156e+101  1.54247155e+103
p/q= 11/17
      2.64045062e+103 -4.87546998e+102  2.68508501e+103
p/q= 12/17
      3.18319833e+102  1.71129868e+103  1.74065237e+103
p/q= 13/17
      2.81890518e+103  1.22095508e+103  3.07196318e+103
p/q= 14/17
     -6.62388513e+103  1.77944588e+103  6.85873763e+103
p/q= 15/17
      6.32448655e+104 -3.44219622e+104  7.20054477e+104
p/q= 16/17
      6.51652466e+107 -6.61334681e+106  6.54999673e+107

p/q= 1/18
      7.63327222e+114 -3.46431322e+114  8.38261957e+114
p/q= 5/18
      6.56972314e+108 -4.26435198e+109  4.31466226e+109
p/q= 7/18
     -9.79228757e+108  2.63289397e+109  2.80909587e+109
p/q= 11/18
      4.56095668e+108  2.47884169e+109  2.52045222e+109
p/q= 13/18
     -1.63204748e+108 -3.09066162e+109  3.09496770e+109
p/q= 17/18
     -3.85829523e+114  1.93564264e+114  4.31661378e+114

p/q= 1/19
      2.31123774e+121 -5.56881081e+121  6.02938419e+121
p/q= 2/19
      1.29907462e+118 -1.27380772e+118  1.81939027e+118
p/q= 3/19
      2.40280987e+116  7.70200234e+116  8.06810605e+116
p/q= 4/19
      2.10594378e+116 -2.39889012e+115  2.11956268e+116
```

```
p/q= 5/19
        3.46960349e+115 -1.20739841e+116  1.25626128e+116
p/q= 6/19
        9.31220635e+115 -1.03281377e+116  1.39063876e+116
p/q= 7/19
       -1.16440832e+115  5.00535181e+115  5.13900705e+115
p/q= 8/19
        2.03452462e+115 -4.14598642e+115  4.61827824e+115
p/q= 9/19
        6.67583966e+116 -2.71107049e+116  7.20532708e+116
p/q= 10/19
        6.76073752e+116 -4.93329748e+116  8.36928885e+116
p/q= 11/19
        4.35064995e+115  1.11079067e+115  4.49021279e+115
p/q= 12/19
       -3.77700858e+115 -1.91721586e+115  4.23574202e+115
p/q= 13/19
        1.12348259e+116 -1.17268283e+115  1.12958620e+116
p/q= 14/19
        7.89664617e+115  2.60956922e+115  8.31666233e+115
p/q= 15/19
        7.19775932e+115 -1.01986680e+116  1.24828109e+116
p/q= 16/19
       -3.42048851e+116 -2.46116838e+116  4.21391640e+116
p/q= 17/19
        6.14472717e+117 -6.00028191e+117  8.58842564e+117
p/q= 18/19
        2.01872731e+121 -2.25778782e+121  3.02867393e+121

p/q= 1/20
       -1.56481881e+128 -4.32023221e+128  4.59489545e+128
p/q= 3/20
        1.49991466e+123  1.71530410e+123  2.27859873e+123
p/q= 7/20
       -2.44235853e+122  1.21120071e+122  2.72619191e+122
p/q= 9/20
        9.00623800e+121  1.10895779e+122  1.42860443e+122
p/q= 11/20
        1.47137151e+122  4.63430944e+121  1.54262840e+122
p/q= 13/20
        1.45760264e+121 -2.04575122e+122  2.05093737e+122
p/q= 17/20
        1.11827140e+123 -6.02722821e+121  1.11989450e+123
p/q= 19/20
       -7.56834396e+127  2.12328285e+128  2.25413583e+128
```

3.3.3 The program

Finally, we present the program used to calculate the above data. Most important are the procedures used in calculating partitions of certain integers. Recall that in the algorithm we required to perform a summation over all sequences $r_1, \ldots, r_m$ such that

$$\begin{array}{lrcl} \text{(i)} & r_i & \geq & 0; \\ \text{(ii)} & r_1 + 2r_2 + \cdots + mr_m & = & k - j. \end{array}$$

Condition (ii) means that we are seeking partitions of $k - j$, and these are generated recursively in the procedure *findpartitions.*

```
program tonguewidth(input,output);
type complex=array [1..2] of real;
var zero,one,compproduct,rthpower,lambda,partialsum,
    nu,sum:complex;
    p,q,k,j,m:integer;
    r,t:array [0..100] of integer;
    partition:array [1..100] of integer;
    psi,b,c:array[1..20] of complex;
    modbq:real;

procedure initialise;
var i,l:integer;
begin
  for i:=1 to 20 do
  begin
    c[i][1]:=1;
    c[i][2]:=0
  end;(*for*)
  one[1]:=1;one[2]:=0;
  zero[1]:=0;zero[2]:=0;
  psi[1][1]:=1.0968632E03;psi[1][2]:=-3.2721079E02;
  psi[2][1]:=3.0300529E05;psi[2][2]:=-7.1904567E04;
  psi[3][1]:=3.7588902E07;psi[3][2]:=-1.9265715E07;
  psi[4][1]:=4.2745511E09;psi[4][2]:=-2.4590307E09;
  psi[5][1]:=1.3774875E11;psi[5][2]:=-4.8994609E11;
  psi[6][1]:=2.6166535E13;psi[6][2]:=3.9449673E12;
  psi[7][1]:=-3.3864992E16;psi[7][2]:=-1.0748475E16;
  psi[8][1]:=2.1925635E17;psi[8][2]:=1.7196424E18;
  psi[9][1]:=-1.1493705E18;psi[9][2]:=-3.4009912E20;
  psi[10][1]:=-1.3183358E22;psi[10][2]:=5.3164967E22;
  psi[11][1]:=3.9565245E24;psi[11][2]:=-6.9086502E24;
```

```
  psi[12][1]:=-8.7036497E26;psi[12][2]:=7.5179384E26;
  psi[13][1]:=1.6676413E29;psi[13][2]:=-8.3835279E28;
  psi[14][1]:=-3.256251E31;psi[14][2]:=1.5434638E31;
  psi[15][1]:=6.9358351E33;psi[15][2]:=-3.6867026E33;
  psi[16][1]:=-1.4760744E36;psi[16][2]:=6.7328611E35;
  psi[17][1]:=2.718999E38;psi[17][2]:=-5.9558781E37;
  psi[18][1]:=-3.7648585E40;psi[18][2]:=-8.0754807E39;
  psi[19][1]:=2.9151277E42;psi[19][2]:=4.0989653E42;
  psi[20][1]:=1.2735682E44;psi[20][2]:=-7.6963318E44;
  nu[1]:=1.33777E2;nu[2]:=-3.79118E1;

  for i:=1 to q do
  begin
    for l:=1 to i do
    begin
      psi[i][1]:=psi[i][1]/10000;
      psi[i][2]:=psi[i][2]/10000
    end(*for,l*)
  end;(*for,i*)
  lambda[1]:=cos(2*3.1415926*p/q);
  lambda[2]:=sin(2*3.1415926*p/q)
end;(*initialise*)

procedure compmult(z1,z2:complex);
begin
  compproduct[1]:=(z1[1]*z2[1])-(z1[2]*z2[2]);
  compproduct[2]:=(z1[1]*z2[2])+(z1[2]*z2[1])
end;(*compmult*)

procedure comppower(w1:complex;power:integer);
var w2:complex;count:integer;
begin
  if power=0 then rthpower:=one else
  begin
    w2:=w1;
    for count:= 1 to (power-1) do
    begin
      compmult(w1,w2);
      w2:=compproduct
    end;(*for*)
    rthpower:=w2
  end(*else*)
end;(*comppower*)

function factorial(n:integer):integer;
var count:integer;product:integer;
```

```
begin
  product:=1;
  if n<>0 then
  begin
    for count:=1 to n do
    product:=count*product
  end(*if*);
  factorial:=product
end;(*factorial*)

procedure zeroelsewhere(h:integer);
var i:integer;
begin
  for i:= k-j+1 to 20 do partition[i]:=0;
  if h>1 then for i:= 1 to h-1 do partition[i]:=0
end;(*zeroelsewhere*)

procedure addinterm;
var a,rtotal,divisor:integer;
    term,twopiij,numerator:complex;
begin
  rtotal:=0;numerator:=one;divisor:=1;
  for a:=1 to (k-j+1) do
  begin
    rtotal:=rtotal+partition[a];
    divisor:=divisor*factorial(partition[a]);
    comppower(c[a],partition[a]);
    compmult(numerator,rthpower);
    numerator:=compproduct
  end;(*for*)
  twopiij[1]:=0;twopiij[2]:=2*3.1415926*j;
  comppower(twopiij,rtotal);
  term:=rthpower;
  compmult(term,numerator);
  term:=compproduct;
  term[1]:=term[1]/divisor;
  term[2]:=term[2]/divisor;
  partialsum[1]:=partialsum[1]+term[1];
  partialsum[2]:=partialsum[2]+term[2]
end;(*addinterm*)

procedure findpartitions;
begin(*findpartitions*)
  m:=m-1;r[m]:=-1;
  if m=1 then
  begin
```

```
    partition[m]:=t[m];
    zeroelsewhere(1);
    addinterm
  end(*then*) else
  while m*r[m]<=(t[m]-m) do
  begin(*while*)
    r[m]:=r[m]+1;
    partition[m]:=r[m];
    t[m-1]:=t[m]-(m*r[m]);
    if t[m-1]>0 then findpartitions else
    begin
      zeroelsewhere(m);
      addinterm
    end(*else*)
  end(*while*);
  m:=m+1
end(*findpartitions*);

procedure findc(s:integer);
var denominator,conjugate:complex;
    norm:real;
begin
  comppower(lambda,s);
  denominator:=rthpower;
  denominator[1]:=denominator[1]-1;
  conjugate:=denominator;
  conjugate[2]:=-conjugate[2];
  norm:=sqr(conjugate[1])+sqr(conjugate[2]);
  compmult(b[s],conjugate);
  c[s][1]:=compproduct[1]/norm;
  c[s][2]:=compproduct[2]/norm
end;(*findc*)

function gcd(n1,n2:integer):integer;
var  newn,last:integer;
begin
  repeat
    if n1>n2 then
    begin
      n1:=n1 mod n2;
      newn:=n1;
      last:=n2
    end(*then*)
    else
    begin
      n2:=n2 mod n1;
```

```
      newn:=n2;
      last:=n1
    end;(*else*)
  until newn=0;
  gcd:=last
end;(*gcd*)


begin(*main body*)
for q:= 1 to 20 do
begin
  writeln;
  for p:=1 to q do
  begin
    if gcd(p,q)=1 then
    begin
  initialise;
  b[1]:=psi[1];
  findc(1);
  for k:=2 to q do
  begin
    sum:=zero;
    for j:=1 to k do
    begin
      partialsum:=zero;
      if k-j=0 then
      begin
        zeroelsewhere(2);
        addinterm;
        compmult(psi[j],partialsum);
        sum[1]:=sum[1]+compproduct[1];
        sum[2]:=sum[2]+compproduct[2]
      end(*then*) else
      begin
        m:=k-j+1;t[k-j]:=k-j;
        findpartitions;
        compmult(psi[j],partialsum);
        sum[1]:=sum[1]+compproduct[1];
        sum[2]:=sum[2]+compproduct[2]
      end(*else*)
    end;
    b[k]:=sum;
    findc(k)
   end;
   comppower(nu,q);
   compmult(b[q],rthpower);
```

```
  b[q]:=compproduct;
  modbq:=sqrt(sqr(b[q][1])+sqr(b[q][2]));
  writeln('    p/q= ',p:1,'/',q:1);
  writeln('             ',b[q][1]*10**(4*q):16,'
',b[q][2]*10**(4*q):16,' ',modbq*10**(4*q):16);
  writeln
      end(*then*)
    end;(*for,p*)
    writeln
  end(*for,q*)
end.(*main body*)
```

Chapter 4

Resonance scaling on invariant circles of the Dissipative Standard Map

4.1 Introduction

Circle maps arise in the consideration of higher dimensional systems, rather than just directly as one dimensional systems. In particular, when invariant circles exist for maps from $\mathbf{R}^n$ to $\mathbf{R}^n$, we can restrict attention to the invariant set and consider the resulting 'induced' circle map. In view of Jonker's result, [Jo], the existence of the n^{-3} scaling law similar to that derived from one dimensional systems is a consequence of the smoothness of the invariant curve. Here we turn our attention to resonance scaling properties of the planar map known as the Dissipative Standard Map,

$$\begin{aligned} x_{n+1} &= Jx_n - \tfrac{k}{2\pi}\sin 2\pi\theta_n \\ \theta_{n+1} &= \theta_n + \Omega + Jx_n - \tfrac{k}{2\pi}\sin 2\pi\theta_n, \end{aligned}$$

where $x_n, \theta_n \in \mathbf{R}$, and $0 \le J, k, \Omega < 1$. The map is dissipative in the sense that its Jacobian matrix has determinant J, with $|J| < 1$, and it may also be described as *area contracting*. Writing $f_{k,\Omega}: \mathbf{R} \times \mathbf{S}^1 \to \mathbf{R} \times \mathbf{S}^1$,

$$f_{k,\Omega}(x,\theta) = \left(Jx - \tfrac{k}{2\pi}\sin 2\pi\theta \quad , \quad \theta + \Omega + Jx - \tfrac{k}{2\pi}\sin 2\pi\theta \mod 1\right),$$

we have a homeomorphism of the cylinder $\mathbf{R} \times \mathbf{S}^1$. When $k = 0$, it is clear that $V = \{0\} \times \mathbf{S}^1$ is an invariant set under the action of $f_{k,\Omega}$. The theory of normal hyperbolicity (see [HPS]) shows that, given $r \in \mathbf{N}$, if $k > 0$ is small enough then there exists a C^r manifold, V', which is C^r near to V and invariant under $f_{k,\Omega}$. The area contracting condition means that only one such invariant 'circle', homotopic to $\{0\} \times \mathbf{S^1}$, can exist.

We think of $f_{k,\Omega}$ as inducing a circle map on V', and we here investigate the scaling of its resonance tongues: given $\alpha \in \mathbf{R}$, there exists an interval, I_α, such that for $\Omega \in I_\alpha$, $f_{k,\Omega}$ has rotation number α, provided k is small enough. The purpose of this chapter is to consider the scaling of $|I_{\frac{1}{n}}(J,k)|$ for large n. We shall see, using the results in [Da2], that given $k > 0$,

$$\left|I_{\frac{1}{n}}(J,k)\right| = \frac{\pi(1-J)}{kn^3}|\beta_2 - \beta_1| + o(n^{-3}),$$

for some constants β_1, β_2, and in particular, we establish the following result;

Theorem 4.1

$$|\beta_2 - \beta_1| \sim e^{-\frac{2\pi(1-J)}{k}} A(J),$$

as $k \to 0$, where $A(J)$ is a constant depending only on J.

In addition, we derive numerical estimates of $A(J)$ for several values of J.

4.2 Invariant circles of $f_{k,\Omega}$

We begin by considering the rotation number $\rho = 0$, characteristic of the case where $f_{k,\Omega}$ has a fixed point. If $\Omega_n \in I_{\frac{1}{n}}$, we have $f_{k,\Omega_n} \longrightarrow f_{k,\Omega^*}$ as $n \to \infty$, where Ω^* is the right hand end point of the interval I_0, and in this context we will invoke the results in [Da2]. In the meantime, we look for the right hand end point of I_0.

Proposition 4.2 *The right hand end point of I_0 is*

$$\Omega^* = \frac{k}{2\pi(1-J)}.$$

Further, f_{k,Ω^*} has a (non-hyperbolic) fixed point at

$$(x^*, \theta^*) = \left(\frac{-k}{2\pi(1-J)}\ ,\ \frac{1}{4}\right).$$

Proof Since $\Omega = \Omega^*$ marks the point at which the fixed point of $f_{k,\Omega}$ disappears, we anticipate that the fixed point will be non-hyperbolic, and so the linearization of $f_{k,\Omega}$ at the fixed point has at least one eigen value with modulus unity. We therefore consider

$$Df_{k,\Omega} = \begin{bmatrix} J & -k\cos 2\pi\theta \\ J & 1 - k\cos 2\pi\theta \end{bmatrix}.$$

This has eigen–values given by $|Df_{k,\Omega} - \lambda I| = 0$, which gives

$$\begin{aligned} 0 &= (J-\lambda)(1 - k\cos 2\pi\theta - \lambda) + Jk\cos 2\pi\theta \\ &= \lambda^2 + \lambda(k\cos 2\pi\theta - J - 1) + J. \end{aligned}$$

Thus, if the eigen–values of $Df_{k,\Omega}$ are λ_1, λ_2, then we have

$$\lambda_1 + \lambda_2 = 1 + J - k\cos 2\pi\theta,$$

and

$$\lambda_1\lambda_2 = J.$$

Thus we obtain $\lambda_1 = 1, \lambda_2 = J$, when $\cos 2\pi\theta = 0$, that is, when $\theta = \frac{1}{4} + \frac{m}{2}$, for $m \in \mathbf{Z}$.

Regarding $f_{k,\Omega}$ as a map of the cylinder $\mathbf{R}\times\mathbf{S}^1$, we consider the values $\theta = \frac{1}{4}, \frac{3}{4}$. Now, for any fixed point , (x_0, θ_0), with $\Omega = \Omega_0$, we have

$$x_0 = Jx_0 - \frac{k}{2\pi}\sin 2\pi\theta_0, \tag{4.1}$$

$$\theta_0 = \theta_0 + \Omega_0 + Jx_0 - \frac{k}{2\pi}\sin 2\pi\theta_0. \tag{4.2}$$

4.1 gives

$$(1-J)x_0 = -\frac{k}{2\pi}\sin 2\pi\theta_0,$$

whilst from 4.2 we obtain

$$Jx_0 = \frac{k}{2\pi}\sin 2\pi\theta_0 - \Omega_0.$$

Thus we have

$$Jx_0 = -(1-J)x_0 - \Omega_0,$$

which gives

$$x_0 = -\Omega_0.$$

Now, by 4.1, $\theta = \frac{1}{4}$ gives

$$\begin{aligned} x_0 &= -\frac{k}{2\pi(1-J)}\sin 2\pi\theta_0 \\ &= -\frac{k}{2\pi(1-J)}, \end{aligned}$$

and so we obtain

$$\Omega_0 = \frac{k}{2\pi(1-J)}.$$

Similarly, we find that $\theta = \frac{3}{4}$ leads to a negative value for Ω, and so we choose $\theta_0 = \frac{1}{4}$, together with $x_0 = -\frac{k}{2\pi(1-J)}$, and $\Omega_0 = \frac{k}{2\pi(1-J)}$.

Finally, we show that Ω_0 *is* the right hand end point of I_0, Ω^*, and so also $(x^*, \theta^*) = (x_0, \theta_0)$.

Let $\varepsilon > 0$, and let $\Omega_1 = \Omega_0 + \varepsilon$. We suppose that f_{k,Ω_1} has a fixed point, (x_1, θ_1). If so, then as before we obtain

$$\begin{aligned} x_1 &= -\Omega_1 \\ &= -\Omega_0 - \varepsilon \\ &= -\frac{k}{2\pi(1-J)} - \varepsilon. \end{aligned}$$

However, we also have, from 4.1,

$$\begin{aligned} x_1 &= -\frac{k}{2\pi(1-J)}\sin 2\pi\theta_1 \\ &> -\frac{k}{2\pi(1-J)} - \varepsilon, \end{aligned}$$

and thus we have a contradiction. □

We now consider the map f_{k,Ω^*}. If k is small enough, f_{k,Ω^*} will have an invariant circle through (x^*, θ^*), given by

$$V^* = \{(x,\theta) : x = u(\theta),\ 0 \le \theta < 1\},$$

with $u:\mathbf{R} \to \mathbf{R}$ being periodic with period 1. We look for an expression for u of the form

$$u(\theta) = \sum_{i=0}^{r-1} a_i(\theta - \theta^*)^i + O\left((\theta-\theta^*)^r\right),$$

near to $\theta = \theta^*$. This is reasonable, since V^* is C^r. By substitution into the definition of f_{k,Ω^*}, we have

$$Ju(\theta) - \frac{k}{2\pi}\sin 2\pi\theta = u\left[Ju(\theta) + \theta + \Omega^* - \frac{k}{2\pi}\sin 2\pi\theta\right].$$

Writing $\theta = \theta^* + \varphi$, we obtain

$$\begin{aligned} &Ju(\theta^*+\varphi) - \frac{k}{2\pi}\sin 2\pi(\theta^*+\varphi) = \\ &\quad u\left[Ju(\theta^*+\varphi) + \varphi + \theta^* + \Omega^* - \frac{k}{2\pi}\sin 2\pi(\theta^*+\varphi)\right]. \end{aligned}$$

So, by considering the relationship

$$\begin{aligned} &J\sum_{i=0}^{r-1} a_i\varphi^i - \frac{k}{2\pi}\cos 2\pi\varphi = \\ &\quad u\left[J\sum_{i=0}^{r-1} a_i\varphi^i + \varphi + \theta^* + \Omega^* - \frac{k}{2\pi}\cos 2\pi\varphi\right] + O(\varphi^r), \end{aligned}$$

we may, in principle, determine as many a_i as we like, and obtain an approximation valid for k small enough. Now, since $a_0 = x_0$, and $x_0, \Omega^*, \frac{k}{2\pi}\cos 2\pi\varphi = O(k)$, we can further write

$$J\sum_{i=0}^{r-1} a_i\varphi^i - \frac{k}{2\pi}\cos 2\pi\varphi$$

$$= \sum_{i=0}^{r-1} a_i \left[J \sum_{i=0}^{r-1} a_i \varphi^i + \varphi + \Omega^* - \frac{k}{2\pi} \cos 2\pi\varphi \right]^i$$

$$= \sum_{i=0}^{r-1} a_i \left[J \sum_{i=1}^{r-1} a_i \varphi^i + \varphi - \frac{k}{2\pi} \sum_{s=1}^{\infty} (-1)^s \frac{(2\pi\varphi)^{2s}}{(2s)!} \right]^i,$$

since

$$\begin{aligned} Ja_0 + \Omega^* - \frac{k}{2\pi} &= -\frac{Jk}{2\pi(1-J)} + \frac{k}{2\pi(1-J)} - \frac{k}{2\pi} \\ &= 0. \end{aligned}$$

Collecting terms in φ^i we obtain

(i) for $i = 1$:

$$Ja_1 = a_1(1 + Ja_1),$$

giving $a_1 = 0$ or $\frac{J-1}{J}$. To determine which is the required value we consider the eigen–vectors of the linearisation of f_{k,Ω^*} at the fixed point. Now, Df_{k,Ω^*} has eigen–values $\lambda_1 = 1$ and $\lambda_2 = J$ at the fixed point, and from these we obtain the corresponding eigen–vectors,

$$\begin{aligned} \mathbf{e}_1 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ \mathbf{e}_2 &= \begin{pmatrix} 1 \\ \frac{-J}{1-J} \end{pmatrix}. \end{aligned}$$

Now, these define the eigen–spaces E_1 and E_2, which are respectively the centre and stable manifolds of the non–hyperbolic fixed point of Df_{k,Ω^*}. We may think of Df_{k,Ω^*} as being hyperbolic, and in particular, contracting, with respect to the centre manifold, E_1, and hence we expect $u(\theta)$ to be tangential to E_1 at (x^*, θ^*). Evidently, then, we require $a_1 = 0$.

(ii) $i = 2$:

$$Ja_2 + k\pi = a_2,$$

and hence

$$a_2 = \frac{k\pi}{1-J}.$$

(iii) $i = 3$:

$$Ja_3 = a_3 + 2a_2(Ja_2 + k\pi),$$

giving

$$a_3(J-1) = \frac{2k\pi}{1-J}\left(\frac{2k\pi}{1-J} + k\pi\right),$$

and so we obtain

$$a_3 = -\frac{2k^2\pi^2}{(1-J)^3}.$$

Thus we find,

$$u(\theta^* + \varphi) = -\frac{k}{2\pi(1-J)} + \frac{k\pi}{1-J}\varphi^2 - \frac{2k^2\pi^2}{(1-J)^3}\varphi^3 + O\left(\varphi^4\right),$$

provided k is small enough.

So, then, restricted to the invariant circle, V^*, the action of f_{k,Ω^*} is described by the relationship

$$\begin{aligned}(\varphi_{n+1} + \theta^*) = (\varphi_n + \theta^*) + \Omega^* - \frac{Jk}{2\pi(1-J)} \\ + \frac{Jk\pi}{1-J}\varphi_n^2 - \frac{2Jk^2\pi^2}{(1-J)^3}\varphi_n^3 - \frac{k}{2\pi}\sin 2\pi(\varphi_n + \theta^*) + O\left(\varphi_n^4\right),\end{aligned}$$

which gives

$$\varphi_{n+1} = \varphi_n + \frac{k\pi}{1-J}\varphi_n^2 - \frac{2Jk^2\pi^2}{(1-J)^3}\varphi_n^3 + O\left(\varphi_n^4\right).$$

We shall call this derived mapping, v_{k,Ω^*}.

In the usual manner, we now seek a transformation, Λ, such that

$$\Lambda\left(v_{k,\Omega^*}(\varphi)\right) - \Lambda(\varphi) \simeq 1.$$

Accordingly, we consider the flow

$$\dot{\varphi} = \frac{k\pi}{1-J}\varphi^2 - \frac{2Jk^2\pi^2}{(1-J)^3}\varphi^3.$$

Setting

$$A = \frac{k\pi}{1-J}, \qquad B = -\frac{2Jk^2\pi^2}{(1-J)^3},$$

we obtain

$$\begin{aligned} t &= \int \frac{d\varphi}{A\varphi^2 + B\varphi^3} \\ &= \frac{1}{A}\int \frac{1}{\varphi^2} - \frac{C}{\varphi} + \frac{C^2}{1+C\varphi}\; d\varphi \\ &= \frac{1}{A}\left[-\frac{1}{\varphi} - C\log\varphi + C\log(1+C\varphi)\right],\end{aligned}$$

where $C = \frac{B}{A}$.

Thus we consider

$$
\begin{aligned}
&\frac{1}{A}\left[-\frac{1}{\varphi} - C\log\varphi + C\log(1+C\varphi)\right]_{\varphi_n}^{\varphi_{n+1}} \\
&= \frac{1}{A}\left\{ -\frac{1}{\varphi_n}\left(1 + A\varphi_n + B\varphi_n^2 + O(\varphi_n^3)\right)^{-1} \right. \\
&\quad -C\log\left(\varphi_n + A\varphi_n^2 + B\varphi_n^3 + O(\varphi_n^4)\right) \\
&\quad +C\log\left[1 + C\left(\varphi_n + A\varphi_n^2 + B\varphi_n^3 + O(\varphi_n^4)\right)\right] \\
&\quad \left. +\frac{1}{\varphi_n} + C\log\varphi_n - C\log(1+C\varphi_n)\right\} \\
&= \cdots \\
&\doteq 1 - A\varphi_n + O(\varphi_n^2).
\end{aligned}
$$

We shall require something more accurate than this, and in fact we find that

$$\int_{\varphi_n}^{\varphi_{n+1}} \frac{1}{\varphi^2} + \frac{A-C}{\varphi}\ d\varphi = 1 + O(\varphi_n^2).$$

So, then, we define a mapping, $\Lambda : (-1,0) \cup (0,1) \to \mathbf{R}$, by

$$
\begin{aligned}
\Lambda(\varphi) &= \frac{a}{A}\int \frac{1}{\varphi^2} + \frac{A-C}{\varphi}\ d\varphi \\
&= -\frac{1}{A\varphi} + \frac{A-C}{A}\log|\varphi| \\
&= -\frac{(1-J)}{k\pi\varphi} + \left(\frac{1+J}{1-J}\right)\log|\varphi|,
\end{aligned}
$$

so that

$$\Lambda(\varphi_{n+1}) - \Lambda(\varphi_n) - 1 = O(\varphi_n^2).$$

Now, if n is large enough, we have

$$
\begin{aligned}
\varphi_n^2 &= \frac{1}{A}\left(\varphi_{n+1} - \varphi_n - B\varphi_n^3 + O(\varphi_n^4)\right) \\
&= O(\varphi_{n+1} - \varphi_n).
\end{aligned}
$$

Thus we can say that the sums $\sum_{n=0}^{\pm\infty} \varphi_n^2$ converge. Accordingly, we define mappings,

$$g_k(\varphi_0) = \lim_{n \to -\infty} \Lambda(\varphi_n) - n, \tag{4.3}$$

and

$$h_k(\varphi_0) = \lim_{n \to +\infty} \Lambda(\varphi_n) - n, \tag{4.4}$$

for $\varphi_0 \in (0,1)$ and $\varphi_0 \in (-1,0)$ respectively. By the periodicity of $f_{k,\Omega}$ we can extend the definition of h_k so that both maps are defined on $(0,1)$. We need now to define the map $\sigma_k(g_k(\varphi_0)) = h_k(\varphi_0) - g_k(\varphi_0)$. It is easy to see that g_k is increasing on $(0,1)$. We shall see later that there exist analytic extensions of g_k and h_k into a domain in $\mathbf{C}$ containing the interval $(0,1)$. Since g_k is non-constant we conclude that g_k is *strictly* increasing on that interval. In addition, it is clear that g_k maps $(0,1)$ *onto* the real line, so that we may indeed define the period 1 function, $\sigma_k : \mathbf{R} \to \mathbf{R}$, by

$$\sigma_k(u) = h_k \circ g_k^{-1}(u) - u.$$

Now, in order to be able to relate the intervals I_α to this map, σ_k, we need briefly to consider what happens when $\Omega = \Omega^* + \delta$, $\delta > 0$. The theory of normal hyperbolicity again shows that if δ is small enough, $f_{k,\Omega}$ still has a C^r invariant circle,

$$\tilde{V} = \{(x,\theta) : x = \tilde{u}(\theta), \;\; 0 \le \theta < 1\},$$

say, and further, that $\tilde{V}$ is C^r near to V^*. Thus we can write

$$\tilde{u}(\theta) = \sum_{i=0}^{r-1} \tilde{a}_i(\theta - \theta^*)^i + O\left((\theta - \theta^*)^r\right),$$

for θ near to θ^*, and we have for δ small enough,

$$i!|a_i - \tilde{a}_i| < \varepsilon,$$

for any $\varepsilon > 0$. As earlier, we write

$$\begin{aligned} &J\tilde{u}(\theta^* + \varphi) - \frac{k}{2\pi}\sin 2\pi(\theta^* + \varphi) \\ &= \;\; \tilde{u}\left(J\tilde{u}(\theta^* + \varphi) + \varphi + \theta^* + \Omega^* + \delta - \frac{k}{2\pi}\sin 2\pi(\theta^* + \varphi)\right), \end{aligned}$$

and we obtain

$$
\begin{aligned}
&J\sum_{i=0}^{r-1}\tilde{a}_i\varphi^i - \frac{k}{2\pi}\left(1-\frac{(2\pi\varphi)^2}{2!}+\cdots\right) \\
&= \sum_{i=0}^{r-1}\tilde{a}_i\left(J\sum_{i=0}^{r-1}\tilde{a}_i\varphi^i+\varphi+\Omega^*+\delta-\frac{k}{2\pi}\left(1-\frac{(2\pi\varphi)^2}{2!}+\cdots\right)\right)^i+O(\varphi^r).
\end{aligned}
$$

As earlier, we seek to expand this and collect terms in φ^i. The problem is not as straight forward as it was earlier, since we now have a non-zero constant term in the argument of the power series on the right hand side. Specifically, we have

$$J\tilde{a}_0+\Omega^*+\delta-\frac{k}{2\pi}=\Delta,$$

say. Thus we obtain

(i) for $i=0$:

$$J\tilde{a}_0-\frac{k}{2\pi}=\tilde{a}_0+\tilde{a}_1\Delta+O\left(\Delta^2\right),$$

giving

$$\tilde{a}_0=\frac{\frac{k}{2\pi}+\tilde{a}_1\Delta}{J-1}.$$

(ii) For $i=1$:

$$J\tilde{a}_1=\tilde{a}_1(1+J\tilde{a}_1)+O(\Delta),$$

from which we obtain

$$\tilde{a}_1=O(\Delta).$$

Returning to the determination of $\tilde{a}_0$, we now have

$$
\begin{aligned}
\tilde{a}_0 &= \frac{\frac{k}{2\pi}+O(\Delta^2)}{J-1} \\
&= -\frac{k}{2\pi(1-J)}+O(\Delta^2).
\end{aligned}
$$

Now,

$$
\begin{aligned}
\Delta &= J\tilde{a}_0+\Omega^*+\delta-\frac{k}{2\pi} \\
&= -\frac{k}{2\pi(1-J)}+\Omega^*-\frac{k}{2\pi}+\delta+O(\Delta^2) \\
&= \delta+O(\Delta^2)
\end{aligned}
$$

Thus, if δ is small enough, $\Delta = O(\delta)$. So we have

$$\begin{aligned}
\tilde{a}_0 &= -\frac{k}{2\pi(1-J)} + O(\delta^2), \\
\tilde{a}_1 &= O(\delta).
\end{aligned}$$

Continuing, we find

$$J\tilde{a}_2 + k\pi = J\tilde{a}_1^2 + k\pi\tilde{a}_1 + \tilde{a}_2\left(2\Delta(\tilde{a}_2 + k\pi) + \tilde{a}_1^2 + 1\right) + O(\Delta),$$

giving

$$\tilde{a}_2 = \frac{k\pi}{1-J} + O(\delta).$$

Now, $f_{k,\Omega}$ may be regarded as a mapping, $f_k : \mathbf{R}^2 \times \mathbf{S}^1 \to \mathbf{R}^2 \times \mathbf{S}^1$, defined by

$$(\Omega, x, \theta) \mapsto \left(\Omega\,, \quad Jx - \frac{k}{2\pi}\sin 2\pi\theta, \theta + \Omega + Jx - \frac{k}{2\pi}\sin 2\pi\theta \mod 1\right).$$

By consideration of the map in this sense, we see that the invariant circle of $f_{k,\Omega}$ is just the restriction, $\Omega = \mathit{constant}$, of an invariant manifold of f_k, which is C^r for any r. Thus we can write

$$\tilde{u}(\theta) = -\frac{k}{2\pi(1-J)} + \mu\delta\varphi + \frac{k\pi}{1-J}\varphi^2 + \nu\varphi^3 + O\left(\varphi^4 + \delta^2\right),$$

where μ, ν are constants, the precise values of which will not concern us. So, restricted to the invariant circle the action of $f_{k,\Omega}$ is described by

$$\begin{aligned}
\varphi_{n+1} &= \varphi_n + \Omega^* + \delta + J\tilde{u}(\theta^* + \varphi_n) - \frac{k}{2\pi}\cos 2\pi\varphi_n \\
&= \varphi_n + \delta + J\mu\delta\varphi_n + \frac{k\pi}{1-J}\varphi_n^2 + J\nu\varphi_n^3 + O\left(\varphi^4 + \delta^2\right).
\end{aligned}$$

This map is, then, $v_{k,\Omega}$ where $\Omega = \Omega^* + \delta$. The coefficients of δ and φ^2 are of particular relevance in the application of the following result, due to Davie, [Da2], which enables us to relate the length of $|I_{\frac{1}{n}}(k)|$ to the mapping σ_k, defined earlier. We note at this point that in the original paper the relevant results are much more general than the special case presented here for sake of brevity. For the complete picture the reader is referred to sections 1–5 in [Da2], and especially to Proposition 4.2 and Theorem 5.1.

Theorem 4.3 *The end points of the interval $I_{\frac{1}{n}}$ are given by*

$$\Omega^* + \frac{\pi^2}{cd}\left(\frac{1}{n^2} + \frac{\beta_i}{n^3}\right) + o(n^{-3}), \quad i = 1, 2,$$

where c, d are the coefficients of δ and φ^2, mentioned above, β_1, β_2 are constants, and the length of the interval $[\beta_1, \beta_2]$ is equal to twice the length of the interval in s such that $\sigma_k(u) - s$ has a zero.

Recall that σ_k is a periodic function with period 1. We therefore, as in previous chapters, wish to estimate the quantity

$$\max_{u\in[0,1]} \sigma_k(u) - \min_{u\in[0,1]} \sigma_k(u),$$

and to that end we consider the Fourier coefficients, $\sigma_r(k)$ of σ_k.

4.3 Estimation of $|\sigma_r(k)|$

It would, of course, be possible at this point to obtain numerical estimates for $|\sigma_r(k)|$ for various different values of k. However, as in Chapter 2 we can determine the form of $\sigma_r(k)$ more explicitly by extending $f_{k,\Omega}$ into a complex domain. The facility of an extra dimension to play with enables us to convert the problem into one where we are seeking to estimate the quantities $\tilde{\sigma}_r$, say, which are independent of k, and where we can approximately determine the relationship between $\sigma_r(k)$ and $\tilde{\sigma}_r$.

4.3.1 Continuation into $\mathbf{C}^2$

Firstly, $f_{k,\Omega}$ is extended into $\mathbf{C} \times \mathbf{C}/\mathbf{Z}$ in the obvious way: we have

$$f_{k,\Omega}(x,\theta) = \left(Jx - \tfrac{k}{2\pi}\sin 2\pi\theta \quad , \quad \theta + \Omega + Jx - \tfrac{k}{2\pi}\sin 2\pi\theta \mod 1\right),$$

where now $x, \theta \in \mathbf{C}$. As before, J, k, Ω are real constants. We will be concerned only with the case $\Omega = \Omega^*$.

When $k = 0$ we have a family of invariant circles,

$$\Psi_\mu = \{(0,\theta) : \Im\theta = \mu, \mu \in \mathbf{R}\},$$

say, with $U = \cup_{\mu\in\mathbf{R}}\Psi_\mu$ being the plane, $x = 0$.

Now to apply the Normal Hyperbolicity Theorem to f_{k,Ω^*} we must have f_{k,Ω^*} C^r near to f_{0,Ω^*}. Considering the derivatives of these mappings, we have

$$\begin{aligned} f'_{0,\Omega^*}(x,\theta) &= (J, J+1), \\ f^{(r)}_{0,\Omega^*}(x,\theta) &= (0,0), & \text{for } r > 1. \end{aligned}$$

Also,

$$\begin{aligned} f'_{k,\Omega^*}(x,\theta) &= (J - k\cos 2\pi\theta, J+1-k\cos 2\pi\theta), \\ f^{(r)}_{k,\Omega^*}(x,\theta) &= (\,(2\pi)^{r-1}kT_r(2\pi\theta),\ (2\pi)^{r-1}kT_r(2\pi\theta)\,), & \text{for } r > 1, \end{aligned}$$

where $T_r(2\pi\theta) \in \{\pm\sin 2\pi\theta, \pm\cos 2\pi\theta\}$. Thus we see that f_{k,Ω^*} is near to f_{0,Ω^*} in the C^r topology provided that $ke^{\pm 2\pi i\theta}$ is sufficiently small. This will be the case if $|\Im\theta| \le \frac{1}{2\pi}\log\frac{1}{k}$. In fact we will constrain $|\Im\theta|$ to be bounded below this quantity, for reasons which will become clear, and say that if r is given, and $k > 0$ is small enough, there exists an f_{k,Ω^*}–invariant C^r manifold, $M \subseteq \mathbf{C}^2$, which is C^r near to $\{(0,\theta) : |\Im\theta| < \frac{1}{2\pi}\log\frac{1}{k} - C\}$, where C is a positive constant.

We now suppose that $(x_0, \theta_0) \in M$ is given and let $\{x_n, \theta_n\}$ be the orbit of (x_0, θ_0) under f_{k,Ω^*}. We therefore have

$$\begin{aligned} x_n &= Jx_{n-1} - \frac{k}{2\pi}\sin 2\pi\theta_{n-1} \\ &= \theta_n - \theta_{n-1} - \Omega^*. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \theta_{n+1} &= \theta_n + \Omega^* + J(\theta_n - \theta_{n-1}) - \frac{k}{2\pi}\sin 2\pi\theta_n - J\Omega^* \\ &= (1+J)\theta_n - J\theta_{n-1} - \frac{k}{2\pi}\sin 2\pi\theta_n + \frac{k}{2\pi}, \end{aligned}$$

which gives us the second order difference equation governing the the orbit $\{\theta_n\}$,

$$(\theta_{n+1} - 2\theta_n + \theta_{n-1}) + (1-J)(\theta_n - \theta_{n-1}) + \frac{k}{2\pi}(\sin 2\pi\theta_n - 1) = 0. \tag{4.5}$$

We use this now to consider the action of f_{k,Ω^*} restricted to M. Rearranging 4.5 we have

$$(\theta_{n+1} - \theta_n) - J(\theta_n - \theta_{n-1}) = \frac{k}{2\pi}(1 - \sin 2\pi\theta_n).$$

Putting $\theta = \theta^* + \varphi$, this becomes

$$(\varphi_{n+1} - \varphi_n) - J(\varphi_n - \varphi_{n-1}) = \frac{k}{2\pi}(1 - \cos 2\pi\varphi_n). \tag{4.6}$$

We seek an expression for the mapping,

$$\bar{f}_{k,\Omega^*}(\varphi_n) = \varphi_{n+1}.$$

Now, let

$$D_k = \left\{\varphi : \left|\Re\varphi - \frac{1}{2}\right| < \gamma,\ 0 < \Im\varphi_0 < \frac{1}{2\pi}\log\frac{1}{k} - C\right\},$$

where γ is a small positive constant. M is C^r and may be expanded in powers of k, so given k_0, small, and $\theta_0 \in D_{k_0}$, for $k < k_0$ we assume a solution of the form

$$\varphi_{n+1} - \varphi_n = k\varrho_1(\varphi_n) + k^2\varrho_2(\varphi_n) + O\left(k^3\varrho_3(\varphi_n)\right).$$

As a first approximation, then, we have

$$\varphi_{n+1} - \varphi_n = \frac{k(1 - \cos 2\pi\varphi_n)}{2\pi(1-J)} + O\left(k^2 \varrho_2(\varphi_n)\right).$$

We could, of course, derive the same expression for $\varphi_n - \varphi_{n-1}$. From 4.6, we write

$$(1-J)(\varphi_{n+1} - \varphi_n) = \frac{k}{2\pi}(1 - \cos 2\pi\varphi_n) + e_{n,2},$$

say, where

$$e_{n,2} = -J\left[(\varphi_{n+1} - \varphi_n) - (\varphi_n - \varphi_{n-1})\right].$$

We therefore have

$$\begin{aligned}
k^2 \varrho_2(\varphi_n) &\approx \frac{e_{n,2}}{1-J} \\
&= \frac{-Jk}{2\pi(1-J)}\left[(1 - \cos 2\pi\varphi_n) - (1 - \cos 2\pi\varphi_{n-1})\right] \\
&\qquad + O\left(k^2(\varrho_2(\varphi_n) - \varrho_2(\varphi_{n-1}))\right) \\
&= \frac{Jk}{2\pi(1-J)}\left[\cos 2\pi\varphi_n - \cos 2\pi\varphi_{n-1}\right] + O\left(k^3\right) \\
&= \frac{-Jk}{2\pi(1-J)}\left[2\pi((\varphi_n - \varphi_{n-1}) \sin 2\pi\varphi_n\right] + O\left(k^3\right) \\
&= \frac{-Jk}{2\pi(1-J)}\left[\frac{k(1 - \cos 2\pi\varphi_n)}{1-J} \sin 2\pi\varphi_n\right] + O\left(k^3\right)
\end{aligned}$$

Thus we obtain

$$\varrho_2(\varphi_n) = -\frac{J}{2\pi(1-J)^3}(1 - \cos 2\pi\varphi_n) \sin 2\pi\varphi_n.$$

Finally, by considering $k^2[\varrho_2(\varphi_n) - \varrho_2(\varphi_{n-1})]$ we have

$$\begin{aligned}
k^3 \varrho_3(\varphi_n) &\approx \frac{-J}{1-J}\left[\frac{-k}{2\pi(1-J)}(1 - \cos 2\pi\varphi_n)\right] \\
&\quad \times \frac{-Jk^2 2\pi}{2\pi(1-J)^3}\left[(1 - \cos 2\pi\varphi_n)\cos 2\pi\varphi_n + \sin^2 2\pi\varphi_n\right] \\
&= \frac{-J^2k^3}{2\pi(1-J)^5}\left[(1 - \cos 2\pi\varphi_n)^2 \cos 2\pi\varphi_n + (1 - \cos 2\pi\varphi_n)\sin^2 2\pi\varphi_n\right].
\end{aligned}$$

We therefore have

$$\varphi_{n+1} - \varphi_n = \frac{k(1 - \cos 2\pi\varphi_n)}{2\pi(1-J)} - \frac{Jk^2}{2\pi(1-J)^3}(1 - \cos 2\pi\varphi_n) \sin 2\pi\varphi_n$$

$$+O\left(k^3\left((1-\cos 2\pi\varphi_n)^2\cos 2\pi\varphi_n + (1-\cos 2\pi\varphi_n)\sin^2 2\pi\varphi_n\right)\right).$$

We use this to see how the orbits behave for large n. Now, for k small we have

$$\varphi_{n+1}-\varphi_n \approx \frac{k}{2\pi(1-J)}(1-\cos 2\pi\varphi_n).$$

Thus, writing $\varphi_n = \xi_n + i\eta_n$, we obtain

$$\begin{aligned}\Re(\varphi_{n+1}-\varphi_n) &\approx \frac{k}{2\pi(1-J)}\left[1-\frac{1}{2}\left(e^{-2\pi\eta_n}\cos 2\pi\xi_n + e^{2\pi\eta_n}\cos 2\pi\xi_n\right)\right] \\ &= \frac{k}{2\pi(1-J)}(1-\cos 2\pi\xi_n \cosh 2\pi\eta_n), \end{aligned} \qquad (4.7)$$

and similarly

$$\Im(\varphi_{n+1}-\varphi_n) \approx \frac{k}{2\pi(1-J)}\sin 2\pi\xi_n \sinh 2\pi\eta_n. \qquad (4.8)$$

We consider first the forward orbit of a point $\varphi_n = \xi_n + i\eta_n$, with $\frac{1}{2} < \xi_n < 1$ and $0 < \eta_n < \frac{1}{2\pi}\log\frac{1}{k} - C$. From 4.8 it is immediately clear that, for k small enough, $\Im(\varphi_{n+1}-\varphi_n) < 0$. We also note that the upper half of the φ–plane is invariant under $\bar{f}_{k,\Omega^*}$. From 4.7, however, we see that if $\xi_n > \frac{3}{4}$ then $\Re(\varphi_{n+1}-\varphi_n)$ is positive provided η_n is small enough, whilst it is certainly positive if $\frac{1}{4} < \xi_n \le \frac{3}{4}$.

The above information is almost sufficient to show that $\varphi_n \to 1$ as $n \to \infty$, but we need to ensure that the orbit does not jump the line $\Re\varphi = 1$. This can be seen to be the case from the fact that $\varphi_{n+1}-\varphi_n = \frac{k\pi}{(1-J)}(1-\varphi_n)^2 + O(k^2(1-\varphi_n)^3)$, and so $\Re(\varphi_{n+1}-\varphi_n) < \Re(1-\varphi_n)$.

Now for the backward orbit, we note that we may as easily use 4.5 to estimate the inverse map, $\bar{f}^{-1}_{k,\Omega^*}$, and in fact a very similar expression is obtained. In the same way we can show that if $0 < \xi_n < \frac{1}{2}$ and $0 < \eta_n < \frac{1}{2\pi}\log\frac{1}{k} - C$ then $\varphi_n \to 0$ as $n \to -\infty$.

Finally, similar considerations of the action of the map when ξ_n is near to $\frac{1}{2}$ easily show that the following result holds:

Proposition 4.4 *Let $\varphi_0 \in D_k = \{\varphi :\ |\Re\varphi - \frac{1}{2}| < \gamma,\ 0 < \Im\varphi_0 < \frac{1}{2\pi}\log\frac{1}{k} - C\}$, where γ is a small positive constant, and let x_0 be such that $(x_0,\varphi_0) \in M$. Then if $C > 0$ is large enough, there exists $k_0 > 0$ such that for $0 < k < k_0$, the orbit of (x_0,φ_0) under f_{k,Ω^*}, $\{x_n,\varphi_n\}$, lies in M, and $\varphi_n \to 0$ as $n \to -\infty$, $\varphi_n \to 1$ as $n \to +\infty$.*

As with the real case we now seek a mapping, $\Lambda_1 : D_k \to \mathbf{C}$, such that

$$\Lambda_1(\varphi_{n+1}) - \Lambda_1(\varphi_n) \approx 1.$$

We therefore consider the flow

$$\dot{\varphi} = \frac{k}{2\pi(1-J)}(1 - \cos 2\pi\varphi).$$

Now

$$\begin{aligned}
&\frac{2\pi(1-J)}{k}\int_{\varphi_n}^{\varphi_{n+1}} \frac{d\varphi}{1-\cos 2\pi\varphi} \\
&= \frac{2\pi(1-J)}{k}\int_0^{\varphi_{n+1}-\varphi_n} \frac{d\varphi}{1-\cos 2\pi(\varphi_n+\varphi)} \\
&= \frac{2\pi(1-J)}{k}\int_0^{\varphi_{n+1}-\varphi_n} \left\{\frac{1}{1-\cos 2\pi\varphi} - \frac{2\pi \sin 2\pi\varphi_n}{(1-\cos 2\pi\varphi_n)^2}\varphi \right. \\
&\qquad \left. - \frac{2\pi^2 \cos 2\pi\varphi_n(1-\cos 2\pi\varphi_n) - 4\pi^2\sin^2 2\pi\varphi_n}{(1-\cos 2\pi\varphi_n)^3}\varphi^2 + \cdots\right\} d\varphi \\
&= 1 + 2\pi(1-J)k\left\{-\frac{J\sin 2\pi\varphi_n}{2\pi(1-J)^3} - \frac{\sin 2\pi\varphi_n}{4\pi(1-J)^2}\right\} \\
&\qquad + O\left(k^2\left[(1-\cos 2\pi\varphi_n)\cos 2\pi\varphi_n + \sin^2 2\pi\varphi_n\right]\right) \\
&= 1 - \frac{k(1+J)}{2(1-J)^2}\sin 2\pi\varphi_n + O\left(k^2\left[(1-\cos 2\pi\varphi_n)\cos 2\pi\varphi_n + \sin^2 2\pi\varphi_n\right]\right)
\end{aligned}$$

We thus find that

$$\begin{aligned}
&\frac{2\pi(1-J)}{k}\int_{\varphi_n}^{\varphi_{n+1}} \frac{1}{1-\cos 2\pi\varphi} + \frac{k(1-J)\sin 2\pi\varphi}{2(1-J)^2(1-\cos 2\pi\varphi)}\; d\varphi \\
&\qquad = 1 + O\left(k^2\left[(1-\cos 2\pi\varphi_n)\cos 2\pi\varphi_n + \sin^2 2\pi\varphi_n\right]\right).
\end{aligned}$$

We therefore define $\Lambda_1 : D_k \to \mathbf{C}$ by

$$\Lambda_1(\varphi) = -\frac{(1-J)\cot \pi\varphi}{k} + \frac{1}{2}\left(\frac{1+J}{1-J}\right)\log(1-\cos 2\pi\varphi) + \mathit{constant}, \tag{4.9}$$

so that

$$\Lambda_1(\varphi_{n+1}) - \Lambda_1(\varphi_n) = 1 + O\left(k^2\left[(1-\cos 2\pi\varphi_n)\cos 2\pi\varphi_n + \sin^2 2\pi\varphi_n\right]\right). \tag{4.10}$$

Writing

$$\mathcal{D}_k = \bigcup_{i=-\infty}^{\infty} \bar{f}^{\,i}_{k,\Omega^*}(D_k),$$

we are now in a position also to define the mappings $g_{1,k}, h_{1,k} : \mathcal{D}_k \to \mathbf{C}$, analogous to the real mappings 4.3 and 4.4, by

$$g_{1,k}(\varphi_0) = \lim_{n\to-\infty} \Lambda_1(\varphi_n) - n, \tag{4.11}$$

and

$$h_{1,k}(\varphi_0) = \lim_{n\to+\infty} \Lambda_1(\varphi_n) - n. \tag{4.12}$$

We consider the question of the analyticity of $g_{1,k}$ and $h_{1,k}$. We show that $g_{1,k}$ is analytic, the proof for $h_{1,k}$ being similar.

We consider, for the moment, a small subset of $\mathcal{D}_k$,

$$E_k = \mathcal{D}_k \bigcap \{\varphi \;:\; |\varphi| < \mu\},$$

where μ is a small real constant. We shall show that $g_{1,k}$ is analytic on E_k.

Recall that near to the fixed point the manifold is given by $x = u(\theta)$, where

$$u(\theta) = u(\theta^* + \varphi) = \sum_{i=0}^{r} a_i\varphi^i + O\left(\varphi^{r+1}\right).$$

Previously we considered this expression in the context of $x, \theta \in \mathbf{R}$, but it applies equally well to the complex case if φ is small. Now, writing

$$u_r(\theta^* + \varphi) = \sum_{i=0}^{r} a_i\varphi^i,$$

we have an approximation to the manifold near to $\varphi = 0$.

Now, for $\varphi_0 \in \mathcal{D}_k$, with n large, negative, we have

$$\Lambda_1(\varphi_n) - n \approx g_{1,k}(\varphi_0).$$

Thus, for $\zeta \in g_{1,k}(\mathcal{D}_k)$,

$$\varphi_n \approx \Lambda_1^{-1}(n + \zeta).$$

For k small,

$$\Lambda_1(\varphi) \approx -\frac{(1-J)}{k}\cot \pi\varphi,$$

so that

$$\varphi_n \approx -\frac{i}{2\pi}\log\left\{\frac{k(n+\zeta) + i(1-J)}{k(n+\zeta) - i(1-J)}\right\}.$$

Expanding this we obtain

$$\varphi_n \approx -\frac{i}{2\pi}\left[\frac{2i(1-J)}{k(n+\zeta)} - \frac{2i(1-J)^3}{3k^3(n+\zeta)^3} + O\left(\frac{1}{(kn)^4}\right)\right],$$

and thus

$$\varphi_n = O\left(\frac{1}{kn}\right),$$

as $n \to -\infty$.

Now, we choose k, small. Given $\varphi_0 \in \mathcal{D}_k$, and defining x_0 such that $(x_0, \varphi_0) \in M$, we consider the limit,

$$\lim_{n\to-\infty} f_{k,\Omega^*}^{-n}\left(u_r(\varphi_n), \tfrac{1}{4} + \varphi_n\right).$$

For n large we have

$$\begin{aligned}\begin{pmatrix} x_n \\ \varphi_n \end{pmatrix} &= \begin{pmatrix} u_r(\varphi_n) + O(\varphi_n^{r+1}) \\ \varphi_n \end{pmatrix} \\ &= \begin{pmatrix} u_r(\varphi_n) + \varepsilon_n \\ \varphi_n \end{pmatrix},\end{aligned}$$

say.

Now, for $m > n$ we define $\varepsilon_m(n), \delta_m(n)$ by

$$\begin{pmatrix} \varepsilon_m \\ \delta_m \end{pmatrix} = \begin{pmatrix} x_m \\ \varphi_m \end{pmatrix} - f_{k,\Omega^*}^{-n+m}\begin{pmatrix} u_r(\varphi_n) \\ \frac{1}{4} + \varphi_n \end{pmatrix}.$$

To see the behaviour of ε_m, δ_m, let ε, δ be small. Then

$$\begin{aligned} f_{k,\Omega^*}\begin{pmatrix} x+\varepsilon \\ \frac{1}{4}+\varphi+\delta \end{pmatrix} &= \begin{pmatrix} J(x+\varepsilon) - \frac{k}{2\pi}\cos 2\pi(\varphi+\delta) \\ \frac{1}{4}+\varphi+\delta+\Omega^*+J(x+\varepsilon) - \frac{k}{2\pi}\cos 2\pi(\varphi+\delta) \end{pmatrix} \\ &= f_{k,\Omega^*}\begin{pmatrix} x \\ \frac{1}{4}+\varphi \end{pmatrix} + \begin{pmatrix} J\varepsilon + k\delta\sin 2\pi\varphi + O(k\delta^2) \\ \delta + J\varepsilon + k\delta\sin 2\pi\varphi + O(k\delta^2) \end{pmatrix} \end{aligned}$$

We therefore have the system,

$$\begin{aligned} \varepsilon_{m+1} &= J\varepsilon_m + k\delta_m \sin 2\pi\varphi_m + O(k\delta_m^2) \\ \delta_{m+1} &= \delta_m + J\varepsilon_m + k\delta_m\sin 2\pi\varphi_m + O(k\delta_m^2) \\ &= \delta_m + \varepsilon_{m+1} \end{aligned} \tag{4.13}$$

Now, we first suppose that, for some m, ε_m and $\alpha > 0$,

$$|\varepsilon_m| \le \alpha k|\delta_m\varphi_m|, \tag{4.14}$$

and we choose $\alpha \geq \frac{8\pi}{1-J}$. Now, provided δ_m remains small compared with φ_m, we can say

$$\begin{aligned}
|\varepsilon_{m+1}| &\leq J|\varepsilon_m| + 4k|\delta_m \sin 2\pi\varphi_m| \\
&\leq J|\varepsilon_m| + 8k\pi|\delta_m\varphi_m| \\
&\leq (8\pi + J\alpha)k|\delta_m\varphi_m| \\
&\leq \alpha k|\delta_m\varphi_m|,
\end{aligned}$$

and we can certainly choose α large enough to satisfy 4.14 when $m = n+1$. Thus, from 4.13 we obtain

$$\begin{aligned}
|\delta_{m+1}| &\leq |\delta_m| + \alpha k|\delta_m\varphi_m| \\
&\leq |\delta_m|\left(1 + \alpha k|\varphi_m|\right), \\
&\leq |\delta_m|\left(1 + \frac{\alpha k B_1}{|m|}\right),
\end{aligned}$$

for some $B_1 > 0$, and $n+1 \leq m \leq -1$. We thus obtain

$$|\delta_0| \leq |\delta_{n+1}|\left(1 + \frac{\alpha k B_1}{|n+1|}\right)\left(1 + \frac{\alpha k B_1}{|n+2|}\right)\cdots\left(1 + \frac{\alpha k B_1}{|-1|}\right).$$

Now, if n is large,

$$\left(1 + \frac{\alpha k B_1}{|n+1|}\right)\left(1 + \frac{\alpha k B_1}{|n+2|}\right)\cdots\left(1 + \frac{\alpha k B_1}{|-1|}\right) \approx \frac{|n+1|^{\alpha k B_1}}{\Gamma(\alpha k B_1 + 1)},$$

and so we see that

$$\delta_0(n) = O\left(\varepsilon_n n^{B_2}\right),$$

as $n \to -\infty$, for some $B_2 > 0$. Hence we deduce that if $\varepsilon_n = O(n^{-B_3})$, for $B_3 > 0$ sufficiently large, then both $\varepsilon_0(n), \delta_0(n) \to 0$ as $n \to -\infty$. More particularly, we see that if r is large enough, and k small, then

$$\lim_{n\to-\infty} f_{k,\Omega^*}^{-n}\left(u_r(\varphi_n), \varphi_n\right) = \left(u(\varphi_0), \varphi_0\right),$$

with convergence being uniform on any compact subset of E_k. Thus u is analytic on E_k. Then iteration under f_{k,Ω^*} shows u to be analytic on $\mathcal{D}_k$. We then easily see that $g_{1,k}$ is analytic on $\mathcal{D}_k$.

We also see that if $\varphi_0 \in \mathbf{R}$ then with the appropriate choice of the constant in 4.9, $g_{1,k}$ and $h_{1,k}$ are simply analytic continuations of g_k and h_k. It is also easy to see that in fact $g_{1,k}$ and $h_{1,k}$ may be defined on a neighbourhood of the open interval $(0,1)$, justifying the claim we made earlier that g_k is strictly increasing. In view of the agreement with the real functions we will drop the suffix, 1, and identify the functions just defined as g_k and h_k. Now, from 4.10 we have, as $n \to -\infty$,

$$\begin{aligned}\Lambda_1(\varphi_{n+1}) - \Lambda_1(\varphi_n) &= 1 + O\left(k^2 \left|\varphi_n\right|^2\right) \\ &= 1 + O\left(k \left|\varphi_{n+1} - \varphi_n\right|\right).\end{aligned}$$

Thus we see that

$$|\Lambda_1(\varphi_n) - n - g_k(\varphi_0)| = O(k|\varphi_n|), \tag{4.15}$$

as $n \to -\infty$, and similarly,

$$|\Lambda_1(\varphi_n) - n - h_k(\varphi_0)| = O(k|1 - \varphi_n|),$$

as $n \to +\infty$.

4.3.2 The limiting map

Now, let $\theta = y + ib$, with $y \in \mathbf{C}$ and $b = \frac{1}{2\pi} \log \frac{1}{k}$. Then we have

$$\begin{aligned}\sin 2\pi\theta &= \frac{1}{2i}\left(e^{2\pi i(y+ib)} - e^{-2\pi i(y+ib)}\right) \\ &= \frac{i}{2k} e^{-2\pi i y} + O(k).\end{aligned}$$

Thus we have, for f_{k,Ω^*},

$$\begin{aligned}x_{n+1} &= J x_n - \frac{i}{4\pi} e^{-2\pi i y_n} + O\left(k^2\right), \\ y_{n+1} + ib &= y_n + ib + J x_n - \frac{i}{4\pi} e^{-2\pi i y_n} + O(k).\end{aligned}$$

Thus our limiting map is

$$\begin{aligned}x_{n+1} &= J x_n - \frac{i}{4\pi} e^{-2\pi i y_n} \\ y_{n+1} &= y_n + x_{n+1}.\end{aligned} \tag{4.16}$$

Restricting our attention as before to orbits $\{x_n, y_n+ib\}$ on the invariant manifold, we wish to determine y_{n+1} exclusively in terms of y_n. We thus consider the expansion of x_{n+1} in powers of $e^{-2\pi i y_n}$. In particular we look for a solution of the form

$$x_{n+1} = \alpha_1 e^{-2\pi i y_n} + \alpha_2 e^{-4\pi i y_n} + O\left(e^{-6\pi i y_n}\right).$$

As a first approximation we have

$$x_{n+1} \approx \frac{-i}{4\pi(1-J)} e^{-2\pi i y_n},$$

so that $\alpha_1 = \frac{-i}{4\pi(1-J)}$, and we have

$$y_n \approx y_{n-1} - \frac{i}{4\pi(1-J)} e^{-2\pi i y_{n-1}}.$$

We therefore obtain

$$\begin{aligned} e^{-2\pi i y_n} - e^{-2\pi i y_{n-1}} &= e^{-2\pi i\left\{y_{n-1} - \frac{i}{4\pi(1-J)} e^{-2\pi i y_{n-1}} + O\left(e^{-4\pi i y_{n-1}}\right)\right\}} - e^{-2\pi i y_{n-1}} \\ &= -\frac{e^{-4\pi i y_{n-1}}}{2(1-J)} + O\left(e^{-6\pi i y_{n-1}}\right). \end{aligned} \tag{4.17}$$

Now, we have, from 4.16

$$\sum_{j=1}^{2} \alpha_j e^{-2j\pi i y_n} + O\left(e^{-6\pi i y_n}\right) = J \sum_{j=1}^{2} \alpha_j e^{-2j\pi i y_{n-1}} - \frac{i}{4\pi} e^{-2\pi i y_{n-1}} + O\left(e^{-6\pi i y_{n-1}}\right),$$

and so, using 4.17, we find

$$\alpha_2 e^{-4\pi i y_n} + \frac{iJ}{8\pi(1-J)^2} e^{-4\pi i y_{n-1}} = J\alpha_2 e^{-4\pi i y_{n-1}} + O\left(e^{-6\pi i y_{n-1}}\right),$$

and deduce that

$$\alpha_2 = -\frac{iJ}{8\pi(1-J)^3}.$$

Now defining the domain

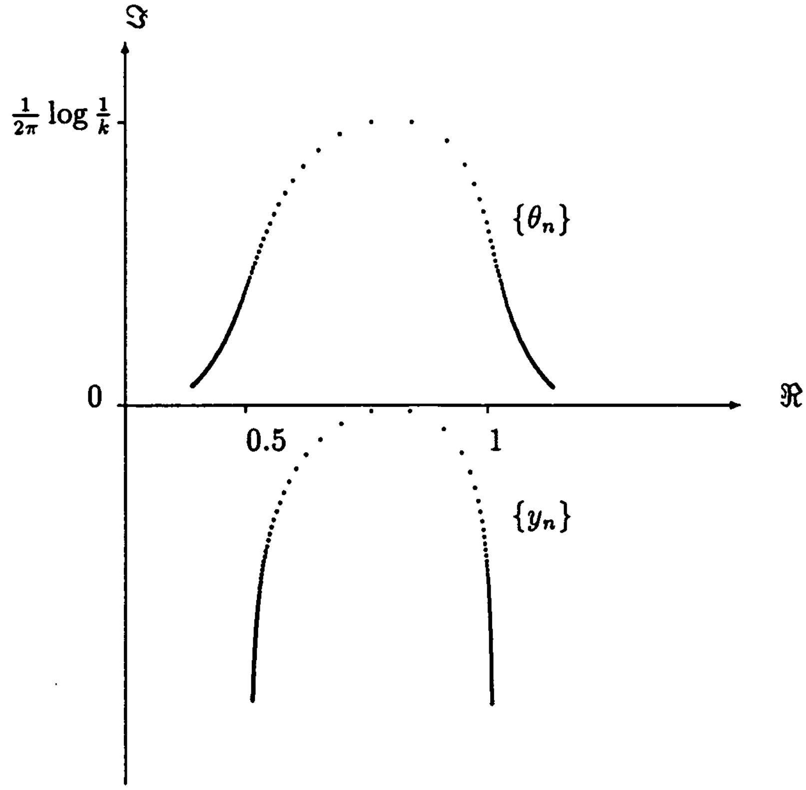
$$D_0 = \left\{y : \left|\Re y - \frac{3}{4}\right| < \gamma,\ \Im y < -C\right\},$$

we consider orbits of the map $f : \mathbf{C} \to \mathbf{C}$,

$$f(y) = y - \frac{i}{4\pi(1-J)} e^{-2\pi i y} - \frac{iJ}{8\pi(1-J)^3} e^{-4\pi i y},$$

for $y_0 \in D_0$. For such orbits, $\{y_n\}$, we find $\Re y_n \to \frac{1}{2}$ as $n \to -\infty$, and $\Re y_n \to 1$ as $n \to +\infty$, whilst $\Im y_n \to -\infty$ as $n \to \pm\infty$. We illustrate the relationship between the orbits $\{y_n\}$ and $\{\theta_n\}$ in figure 4.1.

Figure 4.1



As usual we wish to find a mapping, $\Psi : \mathbf{C} \to \mathbf{C}$, such that

$$\Psi(y_{n+1}) - \Psi(y_n) \approx 1.$$

We consider first

$$\begin{aligned}\int_{y_n}^{y_{n+1}} \frac{1}{\alpha_1} e^{2\pi i y}\, dy &= \left[\frac{1}{2\pi i \alpha_1} e^{2\pi i y}\right]_{y_n}^{y_n + \alpha_1 e^{-2\pi i y_n} + \alpha_2 e^{-4\pi i y_n}} \\ &= 1 + \left(\frac{\alpha_2}{\alpha_1} + \pi i \alpha_1\right) e^{-2\pi i y_n} + O\left(e^{-4\pi i y_n}\right),\end{aligned}$$

and we therefore obtain

$$\int_{y_n}^{y_{n+1}} \frac{1}{\alpha_1} e^{2\pi i y} - \left(\frac{\alpha_2}{\alpha_1^2} + \pi i\right) dy = 1 + O\left(e^{-4\pi i y_n}\right).$$

Accordingly we define

$$\begin{aligned}\Psi(y) &= \frac{1}{2\pi i \alpha_1} e^{2\pi i y} - \left(\frac{\alpha_2}{\alpha_1^2} + \pi i\right) y \\ &= 2(1-J) e^{2\pi i y} - \left(\frac{1+J}{1-J}\right) \pi i y,\end{aligned}$$

and see that the limits

$$\lim_{n\to-\infty} \Psi(y_n) - n$$

and

$$\lim_{n\to+\infty} \Psi(y_n) - n$$

exist, defining analytic functions on D_0, $G(y_0)$ and $H(y_0)$. We further see that

$$\left|(\Psi(y_n) - n) - G(y_0)\right| = O\left(e^{-2\pi i y_n}\right), \tag{4.18}$$

as $n \to -\infty$, and similarly,

$$\left|(\Psi(y_n) - n) - H(y_0)\right| = O\left(e^{-2\pi i y_n}\right),$$

as $n \to +\infty$.

We now investigate the relationship between these functions and 4.11 and 4.12. Firstly, let $y_0 \in D_0$ be chosen, and let $\theta_0(k) = y_0 + \frac{i}{2\pi}\log\frac{1}{k}$. Now, if k is small, we have $\varphi_n \approx \varphi_0 + z_n$, for some z_n not dependent on k. Thus, by 4.15, for n large negative, we obtain

$$\begin{aligned}
\left|\Lambda_1(\varphi_n) - n - g_k(\varphi_0)\right| &= O\left(k\left|\frac{i}{2\pi}\log\frac{1}{k} + z_n\right|\right) \\
&= O\left(k\log\frac{1}{k}\right),
\end{aligned}$$

with a similar result for n large positive, and $h_k(\varphi_0)$. Now, since $\varphi = \theta - \theta^* = \theta - \frac{1}{4}$, we have

$$\begin{aligned}
\cot\left(\pi\theta - \frac{\pi}{4}\right) &= \left(\frac{\cos\pi\theta + \sin\pi\theta}{\cos\pi\theta - \sin\pi\theta}\right) \\
&= \left[\frac{i\left(e^{i\pi y_0 - \frac{1}{2}\log\frac{1}{k}} + e^{-i\pi y_0 + \frac{1}{2}\log\frac{1}{k}}\right) + \left(e^{i\pi y_0 - \frac{1}{2}\log\frac{1}{k}} - e^{-i\pi y_0 + \frac{1}{2}\log\frac{1}{k}}\right)}{\left(e^{i\pi y_0 - \frac{1}{2}\log\frac{1}{k}} - e^{-i\pi y_0 + \frac{1}{2}\log\frac{1}{k}}\right) - i\left(e^{i\pi y_0 - \frac{1}{2}\log\frac{1}{k}} + e^{-i\pi y_0 + \frac{1}{2}\log\frac{1}{k}}\right)}\right] \\
&= \left[\frac{i\left(ke^{2i\pi y_0} + 1\right) + \left(ke^{2i\pi y_0} - 1\right)}{\left(ke^{2i\pi y_0} - 1\right) - i\left(ke^{2i\pi y_0} + 1\right)}\right] \\
&= \frac{ke^{2i\pi y_0} + i}{-ike^{2i\pi y_0} - 1} \\
&= \left(ke^{2i\pi y_0} + i\right)\left(-1 + ike^{2i\pi y_0} + O(k^2)\right) \\
&= -i - 2ke^{2i\pi y_0} + O(k^2).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\log\left(1-\cos\left(2\pi\theta-\frac{\pi}{2}\right)\right) &= \log(1-\sin 2\pi\theta) \\
&= \log\left[1-\frac{1}{2i}\left(ke^{2i\pi y_0}-\frac{1}{k}e^{-2i\pi y_0}\right)\right] \\
&= -\log\left(2ike^{2i\pi y_0}\right)+\log\left[1+2ike^{2i\pi y_0}-k^2e^{4i\pi y_0}\right] \\
&= -\log 2ik-2\pi i y_0+O(k).
\end{aligned}$$

Thus we have, for some $\beta \in \mathbf{C}$,

$$\begin{aligned}
\Lambda_1(\varphi_n)-n &= -\frac{(1-J)}{k}\left(-i-2ke^{2i\pi y_n}\right)+\frac{1}{2}\left(\frac{1+J}{1-J}\right)(-\log 2ik-2\pi i y_n)-n \\
&\quad +\beta \\
&= 2(1-J)e^{2i\pi y_n}-\left(\frac{1+J}{1-J}\right)\pi i y_n+\frac{(1-J)i}{k}-\frac{1}{2}\left(\frac{1+J}{1-J}\right)\log 2ik \\
&\quad +\beta+O(k) \\
&= G(y_0)+\frac{(1-J)i}{k}-\frac{1}{2}\left(\frac{1+J}{1-J}\right)\log 2ik+\beta+O\left(k+e^{-2\pi i y_n}\right),
\end{aligned}$$

by 4.18. A similar result also applies for n large, positive, and $H(y_0)$. Thus we have the following lemma:

Lemma 4.5 *Let $y_0 \in D_0$. Then we have*

$$g_k(\varphi_0) = G(y_0)+\frac{(1-J)i}{k}-\frac{1}{2}\left(\frac{1+J}{1-J}\right)\log 2ik+\beta+O\left(k\log\frac{1}{k}\right),$$

and

$$h_k(\varphi_0) = H(y_0)+\frac{(1-J)i}{k}-\frac{1}{2}\left(\frac{1+J}{1-J}\right)\log 2ik+\beta+O\left(k\log\frac{1}{k}\right).$$

We are now in a position to estimate $|\sigma_r(k)|$. We have

$$\begin{aligned}
\sigma_r(k) &= \int_{u_0}^{u_0+1}(\sigma_k(u)-u)\,e^{2\pi i r u}\,du \\
&= \int_{\varphi_0}^{\varphi_1}[h_k(\varphi)-g_k(\varphi)]\,e^{2\pi i r g_k(\varphi)}g_k'(\varphi)\,d\varphi
\end{aligned}$$

Using Lemma 4.5 we then obtain

$$|\sigma_r(k)| \exp\left\{\frac{2\pi r(1-J)}{k}\right\}$$

$$\longrightarrow \exp\left\{-\frac{3\pi^2}{2}\left(\frac{1+J}{1-J}\right)r\right\}\left|\int_{y_0}^{y_1}[H(y)-G(y)]\,e^{2\pi i r G(y)}G'(y)\,dy\right| \quad (4.19)$$

Evidently as $k \to 0$ we have

$$|\sigma_{r+1}(k)| = O\left(e^{-\frac{2\pi(1-J)}{k}}\sigma_r(k)\right),$$

so that

$$\max_{u\in[0,1]}\sigma_k(u) - \min_{u\in[0,1]}\sigma_k(u) \sim 4|\sigma_0(k)|,$$

and the theorem is proved. □

4.4 Numerical estimation of $|\sigma_0(k)|$

The numerical estimation of $|\sigma_0(k)|$ amounts to the estimation of the integral in 4.19 above. We do this in a similar way to the numerical work done in Chapter 3, although here it is altogether simpler. We do, however, have the additional feature that the value of the integral depends on J, and so we compute this for several different J. Again we find that the integrals can only be estimated well for $\Im y$ in a certain region, and we give a fuller listing of the data in the appendix.

In terms of the notation we have already,

$$\begin{aligned} A(J) &= 8\exp\left\{-\frac{3\pi^2}{2}\left(\frac{1+J}{1-J}\right)\right\}\left|\int_{y_0}^{y_1}[H(y)-G(y)]\,e^{2\pi i G(y)}G'(y)\,dy\right| \\ &= 8\exp\left\{-\frac{3\pi^2}{2}\left(\frac{1+J}{1-J}\right)\right\}B(J), \end{aligned}$$

say. We thus obtain the following estimates:

J	$B(J)$	$A(J)$
0.2	4.28×10^{11}	777
0.1	1.778×10^{9}	197.1
0.05	1.768×10^{8}	110.7
0.01	3.299×10^{7}	72.80
0.001	2.303×10^{7}	66.53
0.00001	2.214×10^{7}	65.87
0	2.213×10^{7}	65.86

Remark: When $J=0$ we have θ independent of x, and the map can be thought of as just

$$\theta_{n+1}=\theta_n+\Omega-\frac{k}{2\pi}\sin 2\pi\theta_n. \tag{4.20}$$

Recall the results of [Dal], where the *sine circle map* was studied, in the form

$$x_{n+1}=x_n+\Omega+k\sin^2 x. \tag{4.21}$$

Making the transformation $x\mapsto\pi\theta-\frac{\pi}{4}$ we obtain the map

$$\theta_{n+1}=\theta_n+\frac{1}{\pi}\left(\Omega+\frac{k}{2}\right)-\frac{k}{2\pi}\sin 2\pi\theta_n,$$

so that we would expect the width of $\pi \times I_{\frac{1}{n}}(k, 0)$ for 4.21 to be asymptotic to the width of $I_{\frac{1}{n}}(k)$ for 4.20 as $k \to 0$. In terms of our present notation, this means that we should have

$$\pi^2(1 - J)A(J) \approx 650.0,$$

when $J = 0$. In fact we do have $\pi^2 A(0) = 650.0$ as required.

4.4.1 Numerical data

The estimation of the constants, $A(J)$, is computationally similar to the the numerical work involved in chapter 2, and to avoid being repetitious we omit the program. However, we include the data produced from which we obtain the estimates given in the table in the previous section. Again, estimation of the integral is made using y_0 values with varying imaginary part, and for the same reasons discussed in section 2.3.1.

(i) $J = 0.2$

$\lvert\int\rvert$	$\Im y_0$
4.44850683987032e+012	-6.90000000000000e+000
2.27991897036061e+012	-6.80000000000000e+000
1.16116432361204e+012	-6.70000000000000e+000
6.20110135357472e+011	-6.60000000000000e+000
4.15867169105417e+011	-6.50000000000000e+000
3.79948997233665e+011	-6.40000000000000e+000
3.90846989467893e+011	-6.30000000000000e+000
4.04923795882818e+011	-6.20000000000000e+000
4.14600311951183e+011	-6.10000000000000e+000
4.20374212628581e+011	-6.00000000000000e+000
4.23645504849211e+011	-5.90000000000000e+000
4.25456952982522e+011	-5.80000000000000e+000
4.26448295313320e+011	-5.70000000000000e+000
4.26986664209537e+011	-5.60000000000000e+000
4.27276874563020e+011	-5.50000000000001e+000
4.27431692770530e+011	-5.40000000000001e+000
4.27512804722820e+011	-5.30000000000001e+000
4.27553841212394e+011	-5.20000000000001e+000
4.27573104744053e+011	-5.10000000000001e+000
4.27580545122283e+011	-5.00000000000001e+000
4.27581567664727e+011	-4.90000000000001e+000
4.27579107332272e+011	-4.80000000000001e+000
4.27574757162982e+011	-4.70000000000001e+000
4.27569382304183e+011	-4.60000000000001e+000
4.27563456369560e+011	-4.50000000000001e+000
4.27557255052201e+011	-4.40000000000001e+000
4.27551003495986e+011	-4.30000000000001e+000
4.27545127581173e+011	-4.20000000000001e+000
4.27544585966241e+011	-4.10000000000001e+000
4.30177522561754e+011	-4.00000000000001e+000

(ii) $J = 0.1$

$\lvert\int\rvert$	$\Im y_0$
4.36286239902700e+012	-6.90000000000000e+000
2.28961769568085e+012	-6.80000000000000e+000
1.20123746707083e+012	-6.70000000000000e+000
6.29937917323941e+011	-6.60000000000000e+000
3.30086683332036e+011	-6.50000000000000e+000
1.72719417518338e+011	-6.40000000000000e+000
9.01355009202116e+010	-6.30000000000000e+000
4.68003681608130e+010	-6.20000000000000e+000
2.40673832425675e+010	-6.10000000000000e+000
1.21585230076647e+010	-6.00000000000000e+000
5.96109316835625e+009	-5.90000000000000e+000
2.84280221706241e+009	-5.80000000000000e+000
1.54177838811173e+009	-5.70000000000000e+000
1.34151417960661e+009	-5.60000000000000e+000
1.47409415901882e+009	-5.50000000000001e+000
1.60033220253863e+009	-5.40000000000001e+000
1.67954838294728e+009	-5.30000000000001e+000
1.72464743318029e+009	-5.20000000000001e+000
1.74946083130564e+009	-5.10000000000001e+000
1.76291675733597e+009	-5.00000000000001e+000
1.77016200847370e+009	-4.90000000000001e+000
1.77404607100196e+009	-4.80000000000001e+000
1.77611985037929e+009	-4.70000000000001e+000
1.77722090152184e+009	-4.60000000000001e+000
1.77779986807665e+009	-4.50000000000001e+000
1.77809878030077e+009	-4.40000000000001e+000
1.77824751287840e+009	-4.30000000000001e+000
1.77831572718852e+009	-4.20000000000001e+000
1.77834078214754e+009	-4.10000000000001e+000
1.77834271631198e+009	-4.00000000000001e+000
1.77833228868075e+009	-3.90000000000001e+000
1.77831529799526e+009	-3.80000000000001e+000
1.77829491544252e+009	-3.70000000000001e+000
1.77827297493378e+009	-3.60000000000001e+000
1.77825076821781e+009	-3.50000000000001e+000
1.77822975817994e+009	-3.40000000000001e+000
1.77821279149599e+009	-3.30000000000001e+000
1.77823184564989e+009	-3.20000000000001e+000
1.88601868506875e+009	-3.10000000000001e+000

(iii) $J = 0.05$

$\lvert\int\rvert$	$\Im y_0$
4.04074828388899e+012	-6.90000000000000e+000
2.12000811077437e+012	-6.80000000000000e+000
1.11207470304354e+012	-6.70000000000000e+000
5.83232936420994e+011	-6.60000000000000e+000
3.05804148006132e+011	-6.50000000000000e+000
1.60287763314341e+011	-6.40000000000000e+000
8.39727963835968e+010	-6.30000000000000e+000
4.39553623257996e+010	-6.20000000000000e+000
2.29738559344384e+010	-6.10000000000000e+000
1.19741957601839e+010	-6.00000000000000e+000
6.20807334826132e+009	-5.90000000000000e+000
3.18576236940684e+009	-5.80000000000000e+000
1.60219408945339e+009	-5.70000000000000e+000
7.73925545988127e+008	-5.60000000000000e+000
3.44937050887698e+008	-5.50000000000001e+000
1.38637042968041e+008	-5.40000000000001e+000
9.38506450758156e+007	-5.30000000000001e+000
1.22273191098808e+008	-5.20000000000001e+000
1.46321116306629e+008	-5.10000000000001e+000
1.60354588605813e+008	-5.00000000000001e+000
1.68037491874735e+008	-4.90000000000001e+000
1.72163478819345e+008	-4.80000000000001e+000
1.74362623821716e+008	-4.70000000000001e+000
1.75530584125609e+008	-4.60000000000001e+000
1.76149496143557e+008	-4.50000000000001e+000
1.76476742514344e+008	-4.40000000000001e+000
1.76649217834461e+008	-4.30000000000001e+000
1.76739607817816e+008	-4.20000000000001e+000
1.76786474820132e+008	-4.10000000000001e+000
1.76810268774164e+008	-4.00000000000001e+000
1.76821832555922e+008	-3.90000000000001e+000
1.76826914825072e+008	-3.80000000000001e+000
1.76828563870523e+008	-3.70000000000001e+000
1.76828397335430e+008	-3.60000000000001e+000
1.76827276209031e+008	-3.50000000000001e+000
1.76825663655467e+008	-3.40000000000001e+000
1.76823818345960e+008	-3.30000000000001e+000
1.76821903377871e+008	-3.20000000000001e+000
1.76820057833569e+008	-3.10000000000001e+000
1.76818463353260e+008	-3.00000000000001e+000
1.76817366363812e+008	-2.90000000000001e+000
1.76788724182413e+008	-2.80000000000001e+000

(iv) $J = 0.01$

$\lvert f \rvert$	$\Im y_0$
5.78114030461431e+009	-5.90000000000000e+000
3.01275738735397e+009	-5.80000000000000e+000
1.56268211069374e+009	-5.70000000000000e+000
8.03218818378121e+008	-5.60000000000000e+000
4.05497212029778e+008	-5.50000000000000e+000
1.97256213627122e+008	-5.40000000000000e+000
8.83234325459802e+007	-5.30000000000000e+000
3.17419079114196e+007	-5.20000000000000e+000
7.53264447683457e+006	-5.10000000000000e+000
1.65867167853243e+007	-5.00000000000000e+000
2.42226281087278e+007	-4.90000000000000e+000
2.83630122878515e+007	-4.80000000000000e+000
3.05558684965107e+007	-4.70000000000000e+000
3.17111125627291e+007	-4.60000000000000e+000
3.23185254373228e+007	-4.50000000000001e+000
3.26375486094117e+007	-4.40000000000001e+000
3.28049408920690e+007	-4.30000000000001e+000
3.28926575084876e+007	-4.20000000000001e+000
3.29385241832895e+007	-4.10000000000001e+000
3.29624159153556e+007	-4.00000000000001e+000
3.29747721159960e+007	-3.90000000000001e+000
3.29810745794223e+007	-3.80000000000001e+000
3.29842010717848e+007	-3.70000000000001e+000
3.29856619666422e+007	-3.60000000000001e+000
3.29862498590276e+007	-3.50000000000001e+000
3.29863807713798e+007	-3.40000000000001e+000
3.29862733982216e+007	-3.30000000000001e+000
3.29860434100211e+007	-3.20000000000001e+000
3.29857533541073e+007	-3.10000000000001e+000
3.29854396076277e+007	-3.00000000000001e+000
3.29851279398191e+007	-2.90000000000001e+000
3.29848441679832e+007	-2.80000000000001e+000
3.29846234098240e+007	-2.70000000000001e+000
3.29845412506058e+007	-2.60000000000001e+000
3.30082143379126e+007	-2.50000000000001e+000

(v) $J = 0.001$

$\|J\|$	$\Im y_0$
5.65968883842204e+009	-5.90000000000000e+000
2.95306893801074e+009	-5.80000000000000e+000
1.53550499348245e+009	-5.70000000000000e+000
7.93167674921143e+008	-5.60000000000000e+000
4.04469646650674e+008	-5.50000000000000e+000
2.00962945253940e+008	-5.40000000000000e+000
9.44348865961544e+007	-5.30000000000000e+000
3.87252522043553e+007	-5.20000000000000e+000
9.94347819936175e+006	-5.10000000000000e+000
6.82215241267005e+006	-5.00000000000000e+000
1.43558601777961e+007	-4.90000000000000e+000
1.84665872115225e+007	-4.80000000000000e+000
2.06352959637762e+007	-4.70000000000000e+000
2.17744649542356e+007	-4.60000000000000e+000
2.23719986089379e+007	-4.50000000000001e+000
2.26851736003079e+007	-4.40000000000001e+000
2.28491866428799e+007	-4.30000000000001e+000
2.29349936580899e+007	-4.20000000000001e+000
2.29798120525631e+007	-4.10000000000001e+000
2.30031551415558e+007	-4.00000000000001e+000
2.30152504584547e+007	-3.90000000000001e+000
2.30214568015487e+007	-3.80000000000001e+000
2.30245810308245e+007	-3.70000000000001e+000
2.30260928521485e+007	-3.60000000000001e+000
2.30267615904839e+007	-3.50000000000001e+000
2.30269899119030e+007	-3.40000000000001e+000
2.30269886838271e+007	-3.30000000000001e+000
2.30268686596736e+007	-3.20000000000001e+000
2.30266886750471e+007	-3.10000000000001e+000
2.30264812508469e+007	-3.00000000000001e+000
2.30262666780711e+007	-2.90000000000001e+000
2.30260616000647e+007	-2.80000000000001e+000
2.30258854209065e+007	-2.70000000000001e+000
2.30257611666812e+007	-2.60000000000001e+000
2.30232213701251e+007	-2.50000000000001e+000
1.65356252661815e+007	-2.40000000000001e+000

(vi) $J = 0.00001$

$\lvert I \rvert$	$\Im y_0$
5.64596040429597e+009	-5.90000000000000e+000
2.94623163652679e+009	-5.80000000000000e+000
1.53229245312413e+009	-5.70000000000000e+000
7.91863324302108e+008	-5.60000000000000e+000
4.04171111444736e+008	-5.50000000000000e+000
2.01194049950564e+008	-5.40000000000000e+000
9.49420444085942e+007	-5.30000000000000e+000
3.93648740492561e+007	-5.20000000000000e+000
1.05520782570817e+007	-5.10000000000000e+000
5.96833030782586e+006	-5.00000000000000e+000
1.34844594667030e+007	-4.90000000000000e+000
1.75910302587404e+007	-4.80000000000000e+000
1.97563734609937e+007	-4.70000000000000e+000
2.08933788003747e+007	-4.60000000000000e+000
2.14896161574683e+007	-4.50000000000001e+000
2.18020387566927e+007	-4.40000000000001e+000
2.19656224861142e+007	-4.30000000000001e+000
2.20511885416765e+007	-4.20000000000001e+000
2.20958743536211e+007	-4.10000000000001e+000
2.21191468972663e+007	-4.00000000000001e+000
2.21312069674511e+007	-3.90000000000001e+000
2.21373980648459e+007	-3.80000000000001e+000
2.21405183431254e+007	-3.70000000000001e+000
2.21420325598285e+007	-3.60000000000001e+000
2.21427072509786e+007	-3.50000000000001e+000
2.21429434943104e+007	-3.40000000000001e+000
2.21429512467851e+007	-3.30000000000001e+000
2.21428407140386e+007	-3.20000000000001e+000
2.21426703572118e+007	-3.10000000000001e+000
2.21424723588256e+007	-3.00000000000001e+000
2.21422665816854e+007	-2.90000000000001e+000
2.21420689919325e+007	-2.80000000000001e+000
2.21418978502567e+007	-2.70000000000001e+000
2.21417756990259e+007	-2.60000000000001e+000
2.21399370653966e+007	-2.50000000000001e+000
1.93080418143582e+007	-2.40000000000001e+000

(vii) $J = 0$

$\lvert f \rvert$	$\Im y_0$
5.64582154428750e+009	-5.90000000000000e+000
2.94616208269964e+009	-5.80000000000000e+000
1.53225995725658e+009	-5.70000000000000e+000
7.91850158216538e+008	-5.60000000000000e+000
4.04168114870592e+008	-5.50000000000000e+000
2.01196242351556e+008	-5.40000000000000e+000
9.49470297859775e+007	-5.30000000000000e+000
3.93712101704299e+007	-5.20000000000000e+000
1.05582458901004e+007	-5.10000000000000e+000
5.95990766583480e+006	-5.00000000000000e+000
1.34758429056738e+007	-4.90000000000000e+000
1.75823705793220e+007	-4.80000000000000e+000
1.97476795415453e+007	-4.70000000000000e+000
2.08846630369789e+007	-4.60000000000000e+000
2.14808874853675e+007	-4.50000000000001e+000
2.17933022414832e+007	-4.40000000000001e+000
2.19568816584495e+007	-4.30000000000001e+000
2.20424452411815e+007	-4.20000000000001e+000
2.20871297065190e+007	-4.10000000000001e+000
2.21104015342082e+007	-4.00000000000001e+000
2.21224612402547e+007	-3.90000000000001e+000
2.21286521793399e+007	-3.80000000000001e+000
2.21317724122242e+007	-3.70000000000001e+000
2.21332866506719e+007	-3.60000000000001e+000
2.21339613999900e+007	-3.50000000000001e+000
2.21341977206581e+007	-3.40000000000001e+000
2.21342055612797e+007	-3.30000000000001e+000
2.21340951222235e+007	-3.20000000000001e+000
2.21339248604891e+007	-3.10000000000001e+000
2.21337269552464e+007	-3.00000000000001e+000
2.21335212652781e+007	-2.90000000000001e+000
2.21333237499804e+007	-2.80000000000001e+000
2.21331526590042e+007	-2.70000000000001e+000
2.21330305285923e+007	-2.60000000000001e+000
2.21311980004647e+007	-2.50000000000001e+000
1.93244001062085e+007	-2.40000000000001e+000

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