TOPICS IN THE THEORY OF NONSELF-ADJOINT OPERATOR ALGEBRAS

Stephen C. Power

Doctor of Science
University of Edinburgh
1987

University of
LANCASTER
Department of Mathematics

Lancaster, United Kingdom LA1 4YL
Telephone: (0524) 65201 Telex: 65111 Lancul G

6th January 1981

CONCERNING "TOPICS IN THE THEORY OF NON-SELF-ADJOINT OPERATOR ALGEBRA

1. The material of Chapter 8 will appear in "Tensor Products of non-self-adjoint operator algebras" in the Proceedings of the 7th Great Plains Operator Theory Seminar, special issue of Rocky Mountain Journal. This paper replaces [14] on page 8.

2. The results in papers 1, 2, 3, 4, 5, 6, 8, 11, 12 form a large part of the extensive research monograph of Davidson on Nest Algebras (to appear in the Pitman-Longman series).

3. The accompanying Errata sheet corrects some errors in the submission.

4. Remark My book on Hankel Operators (ref 25) is now a standard reference. (It is reviewed in Bull. Amer. Math Soc. (1983) vol 9. Perhaps this counterbalances the fact that much of the material of the submission is very recent!

Head of Department: Dr C. T. J. Dodson

ERRATA

page 7	3rd sentence	:	"It is not clear that each....................underlying spaces, but nevertheless we can often identify............"
page 157	paragraph 2	:	Delete the third sentence
page 168	paragraph 1	:	Delete the last sentence
page 187			Delete the last thirteen lines
page 189,190			Delete the second proof of part (i).

ACKNOWLEDGEMENTS

The work of William Arveson has been a great source of inspiration to many people working in nonself-adjoint operator algebras, and I am no exception to this rule. He has discovered the fundamental ideas for many of the sections of these notes. It is also a pleasure to thank my co-workers, Ken Davidson and Vern Paulsen, with whom many of the results below were obtained. Thanks are also due to Wendy Rush who typed up sections at breakneck speed, during a visit to the University of Waterloo.

PREFACE

In this thesis we describe new results and directions in the theory of nonself-adjoint operator algebras. The subject areas are detailed in the following list of contents and the first chapter presents a bird's eye view of the entire work. The mathematics is developed formally in the published papers and manuscripts that are bound in this volume, together with additional original text. A detailed breakdown of this assemblage is given at the end of Chapter 1.

LIST OF CONTENTS

ABSTRACT

Basic topics in the theory of nest algebras and nonself-adjoint operator algebras are developed, with particular reference to three connected theories: (i) Distance formulae and best approximation; (ii) Factorisation and decomposition theory; and (iii) Dilation theory and tensor products.

We begin with two approaches to the distance formula for nest algebras, together with applications to Hankel operators, to triangular trace class operators, and to quasitriangular algebras. A quasitriangular algebra is shown to be a subspace of best approximation, or proximinal subspace, and this serves as motivation for a general study of best approximation in a C*-algebra for spaces of type $S+I$. Here S is a closed nonself-adjoint subspace and I is a closed two-sided ideal. We obtain general Banach space generalisations by using the methods of M-ideals, and alternative constructive procedures in the C*-algebraic context. It is shown that many CSL algebras fail to be hyperreflexive, that is, they fail to possess a distance formula with constant, and, in particular, the infinite spatial tensor product of nontrivial nest algebras is not hyperreflexive.

A unified account is given for the factorisation of positive operators relative to outer operators in a nest algebra, and for the classical outer factorisation of positive matrix valued functions on the unit circle. The basic construction is an operator theoretic version of the Cholesky algorithm. This associates with a positive operator C, and a projection nest E, a positive operator-valued measure $C(\Delta)$ defined on the Borel σ-algebra of E. Generalizations are obtained of Arveson's inner-outer factorisation theory, and the Riesz-type factorisation of trace class operators in a nest algebra, and these generalisations extend to the context of II_∞ factors. The construction also provides a new approach to the extremal outer factorisation $f = hh^*+g$ (h outer, g positive and minimal) of a positive operator valued function on the unit circle, and gives new information on the relationship between h and f and between the prediction-error operator $h(0)h(0)^*$ and f.

Sz-Nagy's dilation theorem, and the Sz-Nagy-Foias commutant lifting theorem are key structure theorems for contraction operators which bear on model theory and the analysis of contractive representations of function algebras. We develop an analogous dilation theory for representations of finite dimensional nest algebras. The main dilation theorem is then established for σ-weakly contractive representations of a general nest algebra, and this requires an understanding of the subtle nonself-adjoint semidiscreteness structure of a nest algebra. Lifting theorems are obtained for commuting contractive representations, and for an operator in the commutant of a representation. These results are necessary for the analysis of complete operator cross norms on the algebraic tensor product of nonself-adjoint operator algebras. In particular we identify the maximal and minimal complete operator cross norms for the algebras $T(n) \otimes P(\mathbb{D})$, $T(n) \otimes T(m)$, and $T(n_1) \otimes T(n_2) \otimes T(n_3)$.

We also consider complementary topics, such as the infinite (minimal) tensor products $T(n_1) \otimes T(n_2) \otimes \ldots$, and the approximately finite nest algebras $\lim_k T(m_k)$.

CHAPTER 1 INTRODUCTION

The study of nonself-adjoint operator algebras is of considerable contemporary interest. The many recent conference proceedings, monographs, and published papers confirm this and reveal a deepening involvement with nearby areas of analysis, such as self-adjoint operator algebras, single operator theory, complex function theory and harmonic analysis.

Nest algebras were introduced by Ringrose in 1965 and have come to represent the most well understood class of weakly closed nonself-adjoint operator algebras being in many respects the most natural infinite dimensional analogues of the simplest noncommutative context, namely the algebra $T(n)$ of upper triangular $n \times n$ matrices. Also nest algebras provide important special cases in more general categories such as the commutative subspace lattice (CSL) algebras, subdiagonal algebras, and nonself-adjoint crossed products.

We shall present a systematic account of much of the structure theory of nest algebras and roughly speaking our topics fall into three broad themes:

(i) Distance formulae and best approximation (Chapter 2);

(ii) Factorisation and decomposition theory (Chapter 3);

(iii) Dilation theory and tensor products (Chapters 4,5,6,8).

In describing these areas below we confine our remarks to comments about the text and the topics theorein and make no detailed commentary on historical development or on recent relevant literature; such accounts can be found within the text.

Arveson's distance formula and its various proofs play an important part in the general theory and in Chapter 2 we discuss two proofs and associated ideas relating to trace class operators and the predual of the quotient space $L(H)/A$ when A is a nest algebra. As applications we obtain an analogue of Hardy's inequality for H^1 functions in the context of trace class triangular integral operators, and a proof of Nehari's theorem on Hankel operators. In fact Nehari's theorem can be thought of as an invariant form of the distance formula for the nest algebra $T(\mathbb{Z})$, and indeed there is a continuing parallel between a nest algebra and the Banach algebra H^∞ which becomes even more apparent within topics (ii) and (iii). Section (2.5) pursues the analogy between the quasitriangular algebra $A + K$ and the space $H^\infty + C$ and serves as an introduction to section (2.6) which contains the main body of material of Chapter 2. In this section quite general methods are developed in the context of nonself-adjoint subspaces of C*-algebras for the study of subspaces of best approximation (proximinal subspaces). However the main applications are in the context of nest algebras. In particular a formula is obtained for the distance $\mathrm{dist}(X, A+K)$ in terms of X and the underlying projection nest.

We remark that two more new proofs of Arveson's distance formula are obtained later as corollaries of the lifting theorems of Chapter 5 (see section (5.4)), and of the matrix completion theory of Chapter 6 (see Remark 2.6).

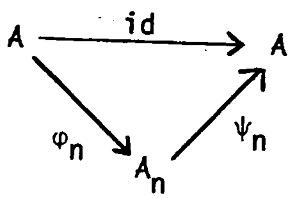
In the final section (2.7) it is shown how even in the context of a commutative subspace lattices L the operator norm distance $\mathrm{dist}(X, \mathrm{Alg}\, L)$ need not be comparable to the quantity $\beta(X) = \sup\{\|L^\perp X L\| : L \in L\}$. This settles a problem that had been open for some time and shows that CSL algebras need not be hyperreflexive. Also it is shown that an infinite tensor product of (nontrivial) nest algebras fails to be hyperreflexive.

In Chapter 3 we give a unified account of aspects of factorisation theory in the context of nest algebras. The fundamental construction here is an operator theoretic version of the Cholesky algorithm which associates a certain positive operator valued measure $C(\Delta)$ to a positive operator C and a projection nest $\mathcal{E}$. For trace class operators this leads to certain new integral representations and decompositions. As an easy application we obtain Lidskii's theorem on the equality of trace and spectral trace. We give a new approach and generalisation of Arveson's inner-outer factorisation theory, based on the derivation of the Cholesky factorisation $C = A^*A$, of a positive operator C, with A an outer operator in a nest algebra, through the analysis of $C(\Delta)$. We are also able to answer some questions of Shields and to generalise his Riesz factorisation theorem for trace class operators in $T(\mathbb{Z})$ to the case of a general well ordered nest algebra. Only a certain weak factorisation is available for trace class operators in a general nest algebra A, but this still leads to a Nehari-type theorem for bounded Hankel forms on the Hilbert space $A \cap C_2$. In fact this result extends to the context of nest subalgebras of II_∞ factors. An alternative analysis of these ideas is also available by means of the lifting theorems of Chapter 5 (section 5.4).

In the rest of the chapter we turn to the analysis of the extremal outer decomposition $f = hh^*+g$ of a positive operator valued function f on the unit circle, which arises through the analysis of $C(\Delta)$ in the context of a nest of uniform multiplicity and order type $\mathbb{Z}$. The explicit nature of the construction of $C(\Delta)$ leads to new information, and in particular to the solution of an old problem of Wiener on Masani on whether an explicit expression for the prediction-error operator $(h(0)h(0)^*)$ can be found in terms of the spectral density f. Even in the case where $g = 0$ and f admits the outer factorisation hh^* we can obtain new information on the relationship between h and f as well as new proofs of classical results. The approach here is based in part on the remarkable formula

$$H^*_{h^*}H_{h^*} = H^*_f \tilde{T}_f^{-1} H_f .$$

Semidiscreteness and approximately finite structure are well understood concepts in the theory of C*-algebras and von Neumann algebras, with many ramifications. In Chapter 4 we begin the analysis of semidiscreteness and related approximation properties in the context of nest algebras and certain other reflexive operator algebras (usually considered, for convenience only, on separable Hilbert spaces). In particular, by carefully examining the spectral type of a general projection nest E we can construct subalgebras of the nest algebra $\mathrm{Alg}\, E$ which are completely isometric copies of finite dimensional nest algebras $A_1, A_2, \ldots,$ with good approximation properties. More precisely we obtain the approximately commuting diagrams



which is to say that $\psi_n \circ \varphi_n$ converges pointwise to the identity map in the weak operator topology, with each φ_n σ-weakly continuous and completely contractive, and ψ_n a completely isometric embedding.

We do not know which CSL algebras are semidiscrete in the above sense, relative to finite dimensional CSL algebras. Nevertheless for the completely distributive CSL algebras we can obtain a good substitute property, which we call the complete CSL subalgebra approximation property, abbreviated CCAP (see section (4.3)).

The significance of the semidiscreteness of nest algebras arises partly from the fact that contractive representations of a finite dimensional nest algebra are completely contractive. We give a new simple direct proof of this fact in Chapter 4, by explicitly constructing $*$-dilations. From semidiscreteness it follows that a contractive σ-weakly continuous representations of a nest algebra is completely contractive. With the help of Arveson's dilation theory for completely contractive maps this leads to a general dilation theory for nest algebras. We remark that it seems to be an open problem whether <u>every</u> contractive representation of a nest algebra is completely contractive.

The Sz-Nagy Foias commutant lifting theorem and the closely related theorem of Ando on the existence of a commuting unitary dilation for a pair of commuting contractions, have played a prominent role in the dilation theory for contractions and in related areas of operator theory, such as interpolation problems. In Chapter 5 we develop analogous lifting theorems for contractive representations of a nest algebra. For example it is shown that if $\rho_1: T(n) \to L(H)$ and $\rho_2: T(m) \to L(H)$ are commuting contractive unital representations then there exists a Hilbert space $K \supset H$ and commuting unital $*$-representations $\pi_1: M_n \to L(K)$, $\pi_2: M_m \to L(H)$ such that

$$\rho_1(A_1)\rho_2(A_2) = P_H\pi_1(A_1)\pi_2(A_2)|_H$$

for all A_1 in $T(n)$ and A_2 in $T(m)$. The principal tool (Theorem 1.1 in Chapter 5) is essentially a structured form of the commutant lifting theorem for operators lying in certain spectral subspaces of a nilpotent automorphism. We also obtain a new proof of a lifting theorem of Ball and Gohberg for a contraction commuting with a contractive representation of a finite dimensional nest algebra, and this is also generalised to the context of general nest algebras. As an application we obtain a proof of the Nehari-type theorem for abstract Hankel operators.

In Chapter 6 we begin a study of dilation theory for contractive maps on certain subspaces of matrices defined by a sparcity pattern, and this includes the case of finite dimensional CSL algebras. The analysis here has considerable independant interest and is closely tied to completion problems for partially defined matrices. We obtain new proofs and generalisations of results of Dym-Gohberg, of Grone-Johnson-Sa-Wolkowicz, and a result of Haagerup on the completely bounded norm of a Schur product map on M_n.

Up to now our comments have been directed at either finite dimensional operator algebras, or at weakly closed operator algebras. Eventually there must be a closer harmony between the norm-closed and weakly closed contexts, as there is between C*-algebras and von Neumann algebras, but the study of norm closed nonself-adjoint operator algebras is, at the present time, fragmented. In Chapter 7 and in section (8.4) we discuss nest subalgebras of C*-algebras and infinite tensor products of upper triangular matrix algebras, namely, $T(n_1) \otimes T(n_2) \otimes \ldots$. We consider general structure and isomorphism theorems and pay particular attention to the analysis of all closed two-sided ideals. The reader can find further introductory remarks in the individual sections.

The theory of operator norms for the tensor product of nonself-adjoint operator algebras is complicated by the fact that even for very simple unital finite dimensional operator algebras, A_1 and A_2 say, the norm $\| \|_\rho$ induced by a faithful representation $\rho: A_1 \otimes A_2 \to L(H)$, of the algebraic tensor product, with the property that the restrictions $\rho|A_i$, $i = 1,2$ are completely isometric isomorphisms, need not be uniquely determined. The supremum of all such norms in fact gives what we call the maximal complete operator cross norm $\| \|_{max}$. Also it can be shown that each norm $\| \|_\rho$ dominates $\| \|_{spat}$, where $\| \|_{spat}$ is the norm induced by the natural representation coming from the Hilbert space tensor product of the underlying spaces, and so we can identify $\| \|_{spat}$ as the minimal complete operator cross norm. In the first three sections of Chapter 8 we develop these ideas and show that nevertheless $\| \|_{min} = \| \|_{max}$ for $T(n) \otimes P(\mathbb{D})$ and for $T(n) \otimes T(m)$. These results and certain generalisations depend on the lifting theorems of Chapter 5. It is an interesting point that an analysis of complete operator cross norms for nonself-adjoint operator algebras can hardly begin without essential involvement of the rather deep commutant lifting theorem of Sz-Nagy and Foias. This connection of ideas will undoubtedly be very significant in future studies for other CSL algebras and function algebras.

THE MANUSCRIPT

We now explain how the entire text has been assembled from the following published papers and unpublished manuscripts together with original text.

1. Analysis in nest algebras, Surveys of Recent Results in Operator Theory, Editor J. Conway, Pitman Research Notes in Mathematics, Longman, 1987 to appear.

2. (with K.R. Davidson) Best approximation in C*-algebras, J. fur der Reine und Angew. Math. 368 (1986), 43-62.

3. (with K.R. Davidson) Failure of the distance formula, Journal L.M.S. 32 (1985), 157-165.

4. Commutators with the triangular projection and Hankel forms on nest algebras, Journal L.M.S. 32 (1985), 272-282.

5. Nuclear operators in nest algebras, J. Operator Theory 10 (1983), 337-352.

6. Another proof of Liskii's theorem on the trace, Bull. London Math. Soc. 15 (1983), 146-148.

7. A Hardy-Littlewood-Fejer inequality for Volterra Integral operators, Indiana Univ. Math. J. 33 (1984), 667-671.

8. Factorisation in Analytic Operator Algebras, J. Funct. Anal. 67 (1986), 413-432.

9. Spectral Characterisation of the Wold-Zasuhin decomposition and prediction-error operator, to appear in J. of Functional Analysis.

10. (with C. Foias) Outer factorisation and Hankel operators, in preparation.

11. (with J. Ward and V.I. Paulsen) Semi-discreteness and dilation theory for nest algebras, to appear in the J. of Functional Analysis.

12. (with V.I. Paulsen) Lifting theorems for nest algebras, preprint, 1987.

13. (with V.I. Paulsen) Schur products and matrix completions, in preparation.

14. (with V.I. Paulsen) Tensor products and dilation theory for nonself-adjoint operator algebras, in preparation.

15. Infinite tensor products of upper triangular matrix algebras, preprint, 1987.

16. On ideals of nest subalgebras of C*-algebras, Proc. London Math. Soc. 50 (1985), 314-332.

Chapters 2 and 3 concern the material in the papers 1 to 10. The main results in Chapter 2 are the closing sections (2.6) and (2.7) which are the published papers 2, 3 (appendices 1 and 2). The sections (2.1) to (2.5) comprise original text which describes results in papers 1, 4, 7. Papers 4 and 7 appear as appendices 3 and 6. We have not included the survey paper 1, which is not yet published, but this is compensated for by the original text in both Chapters 1 and 2, which we have introduced to make a coherent and readable manuscript.

Chapter 3 describes results in the papers 5, 6, 8, 9 and 10. The published papers 5, 6, 8 are appendices 4, 5 and 7, and the unpublished papers 9 and 10 appear as sections (3.4) and (3.5).

In Chapter 4 section (4.1) is taken from paper 4, section (4.2) is paper 11, and section (4.3) is unpublished and forms part of the author's research with V.I. Paulsen on noncommutative dilation theory. Chapter 5 is paper 12. Chapter 6 is taken from a preliminary version of paper 13. Sections (8.1) to (8.3) of Chapter 8 is material that will appear in 14. Section (8.4) is the preprint 15.

Finally, Chapter 7 is paper 16, which appears as appendix 8, together with auxiliary text.

CHAPTER 2 Distance formulae and Best Approximation in C*-algebras

We start by introducing the basic concepts and notation. Let H be a complex Hilbert space. We refer to closed linear subspaces of H simply as subspaces. A subspace nest or nest in H is a family of subspaces which contains $\{0\}$ and H and which is totally ordered by inclusion. A complete nest is a nest that is closed under the formation of closed unions and arbitrary intersections. To each nest there is a unique minimal complete nest containing it called the completion. The nest algebra associated with a nest is the algebra of all bounded operators that have each element of the nest invariant.

Let Ω be a totally ordered set and suppose that H has an orthonormal basis indexed by Ω, namely $\{e_\omega: \omega \in \Omega\}$. Then the subspaces

$$N_\omega = \text{closed span}\{e_\sigma: \sigma \le \omega\} \quad \omega \in \Omega$$

together with $\{0\}$ and H form a nest. Let $T(\Omega)$ denote the nest algebra associated with Ω. If Ω has finite cardinality n, then $T(\Omega)$ is simply the algebra of upper triangular $n \times n$ matrices, which we write as $T(n)$. Of particular interest are the algebras $T(\mathbb{N})$, $T(\mathbb{Z})$ and $T(\mathbb{Q})$, for the natural numbers, integers, and rationals, respectively.

We prefer to talk of projection nests rather than subspace nests. A complete projection nest E is a totally ordered family of self-adjoint projections which contains 0 and I, and is closed in the strong operator topology. The nest algebra associated with E is denoted $\text{Alg}\, E$. Thus

$$\text{Alg}\, E = \{A: (I-E)AE = 0 \text{ for } E \in E\}$$

More generally we write Alg $\mathcal{L}$ for the operator algebra of operators which leave invariant each projection in $\mathcal{L}$, where $\mathcal{L}$ is a family of self-adjoint projections. Taking the dual viewpoint, if A is an algebra of operators then we write Lat A for the set of invariant self-adjoint projections L. That is

$$\text{Lat } A = \{L: L^2 = L = L^*, (I-L)AL = 0 \text{ for } A \in A\}$$

It is easily checked that Lat A is indeed a lattice relative to the usual ordering of self-adjoint projections. We say that A is a reflexive operator algebra if $A = \text{Alg Lat } A$, and that $A = \text{Alg } \mathcal{L}$ is a commutative subspace lattice, or CSL algebra, if $\mathcal{L}$ is family of commuting projections.

The canonical continuous projection nests are those associated with $L^2[0,1]$ and with $L^2(\mathbb{R})$. We say that the Volterra nest for $L^2[0,1]$ is the nest E of projection associated with the subspaces $L^2[0,t] \subset L^2[0,1]$ for $0 \leq t \leq 1$. The Volterra nest for $L^2(\mathbb{R})$ is defined similary in terms of the subspaces of functions supported on the intervals $(-\infty,t)$. Abusing earlier notation write $T([0,1])$ and $T(\mathbb{R})$ for the associated nest algebras.

The algebras $T(n)$, $T(\mathbb{N})$, $T(\mathbb{Z})$, $T(\mathbb{Q})$ and $T(\mathbb{R})$ have the property that $A \cap A^*$ is a maximal abelian operator algebra. These algebras are multiplicity free nest algebras, in the sense that the operator algebra $A \cap A^*$ is multiplicity free (or, equivalently, possess a cyclic vector).

If E is a projection nest on a finite dimensional Hilbert space then Alg E is called a finite dimensional nest algebra, and indeed every finite dimensional nest algebra is of this form, and is unitarily equivalent to an algebra of block upper triangular matrices.

Let L be a strongly closed commutative lattice of projections. An interval of L is any non zero projection of the form $F-E$ with E,F in L. An atom of L is a minimal interval. If $E \in L$, $E > 0$, define E_- in L by

$$E_- = \sup\{F: F \not\geq E\}.$$

If L is a (complete) projection nest then every atom has the form $Q = E - E_-$, and in this case E_- is called the immediate predecessor of E. A projection nest is purely atomic if it is generated by its atoms Q in the sense that $E = \sum_{Q \leq E} Q$ where the sum is taken over atoms Q and converges in the strong operator topology. In particular the nest of $T(\mathbb{Q})$ is purely atomic. If the projection nest E possesses no atoms then it is said to be continuous. In Chapter 4 we derive the spectral representation theorem for projection nests acting on a separable Hilbert space.

The rank one operators in a nest algebra Alg E form an important class. We write $e \otimes f$ for the rank one operator R such that $Rx = \langle x,f\rangle e$. It is easy to prove that $R \in \text{Alg } E$ if and only if there is a projection E in E such that $(I-E_-)f = f$ and $Ee = e$.

We write $C_p(H)$ for the von Neumann-Schatten classes of operators on the Hilbert space H, $1 \leq p \leq \infty$ and $K(H)$ for the compact operators.

(2.1) The Arveson distance formula

The following theorem of Arveson plays an important role in the general theory of nest algebras and quasitriangular algebras. We write $\text{dist}(X,A)$ for the operator norm distance.

(2.1.1) THEOREM. Let $\mathcal{A}$ be a nest algebra associated with the projection nest $\mathcal{E}$. Then for each operator X

$$\mathrm{dist}(X,\mathcal{A}) = \sup\{\|(I-E)XE\| : E \in \mathcal{E}\}$$

The original proof made use of analysis of the invariant subspaces of the inflation algebra $\mathbb{C}I \otimes \mathcal{A}$ on $\ell^2(\mathbb{N}) \otimes H$. We give two further proofs, each of which leads to further structure theory.

The first proof is an induction argument, the induction step of which is facilitated with the following fundamental lemma.

(2.1.2) LEMMA (Parrott). The minimum operator norm of the operator matrix $\begin{bmatrix} A & B \\ C & X \end{bmatrix}$, for variable X, is attained by an operator of the form $X_1 = C_1A^*B_1$. This minimum is equal to the maximum of the norm of the operators $\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$.

Proof. Without loss assume that the maximum norm of the last two operators is unity, so that $AA^* + BB^* \leq Q$ and $A^*A + C^*C \leq P$ where P (resp. Q) is the orthogonal projection onto the first summand in the decomposition of domain (resp. range) implied by the operator matrix presentation. Since $BB^* \leq Q - AA^*$ and $C^*C \leq P - A^*A$, by a well known factorization lemma of Douglas there exists contractions B_1, C_1 such that $B^* = B_1(Q-AA^*)^{\frac{1}{2}}$ and $C = C_1(P-A^*A)^{\frac{1}{2}}$. In particular let $X_0 = -C_1A^*B_1$ and we have

$$\begin{bmatrix} A & B \\ C & X_0 \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & C_1 \end{bmatrix} \begin{bmatrix} A & (Q-AA^*)^{1/2} \\ (P-A^*A)^{1/2} & -A^* \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & B_1 \end{bmatrix}$$

The middle matrix is a unitary operator, since $A(P-A^*A)^{1/2} = (Q-AA^*)^{1/2}A$, and so the operator norm at X_0 is unity. The stated maximum is a lower bound, and so the proof is complete.

(2.1.3) Remarks. The last lemma is closely related to a circle of important ideas related to the Sz-Nagy Foias commutant lifting theorem, (see Chapter 5) and embodies the idea of "one step extention". Here this is achieved by Douglas factorization and matrix construction. Similar constructions are used in the proof of the Sz-Nagy Foias theorem (Sz-Nagy Foias [35]). One corollary of such explicit constructions is that if the operator A is Lemma 2.1.2 lies in a particular ideal then the minimizing operator X_0 can be chosen from the same ideal. Parrott [20] showed in how the fundamental lemma leads to an immediate proof of the Nehari theorem for matricial Hankel operators. Prior proofs relied on the lifting theorem or on generalized Riesz factorization ideas that go back to Sarason's proof of an early version of the lifting theorem. In this version one step extension is achieved in a less explicit way by the Hahn Banach theorem. Parrott also showed how the fundamental lemma does indeed lead to a new proof of the lifting theorem. In Chapter 5 we return to these ideas. In fact we obtain yet another proof there of the Arveson distance formula as a corollary of a general lifting theorem for the commutant of a representation of a nest algebra. With the two proofs of this chapter, and with yet another proof in Chapter 6, based on Arveson's extension theorem for completely positive maps, we have, in fact, a total of 4 different proofs of the Arveson distance formula.

(2.1.4) LEMMA. The minimum operator norm, α say, of the operator matrices

$$\begin{bmatrix} X_{1,1} & X_{1,2} & \cdots & X_{1,n} \\ X_{2,1} & X_{2,2} & \cdots & X_{2,n} \\ \vdots & \vdots & & \vdots \\ Y_{n,1} & Y_{n,2} & \cdots & X_{n,n} \end{bmatrix}$$

where the upper triangular entries X_{ij} are variable and the Y_{ij} are fixed, is achieved and is equal to the maximum of the operator norms of the lower triangular block matrices. That is $\alpha = \beta$ where β is the maximum norm of the operator matrices

$$B_k = \begin{bmatrix} Y_{k,1} & \cdots & Y_{k,k-1} \\ \vdots & & \vdots \\ Y_{n,1} & \cdots & Y_{n,k-1} \end{bmatrix}, \quad 1 < k \leq n.$$

Proof. Define the operators $X_{i,j}$ that lie in the first row and the last column to be the zero operator (on the appropriate summand space). Choose $X_{2,2}$, using the last lemma, so that the operator norm of

$$\begin{bmatrix} Y_{2,1} & \cdots & X_{2,2} \\ \vdots & & \vdots \\ Y_{n,1} & & Y_{n,2} \end{bmatrix}$$

is no greater than β. Now, using the lemma again, choose $X_{3,3}$ in a similar way for the submatrix

$$\begin{bmatrix} Y_{31} & Y_{32} & X_{33} \\ Y_{41} & Y_{42} & Y_{43} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ Y_{n1} & Y_{n2} & Y_{n3} \end{bmatrix}$$

In this way construct $X_{22}, X_{33}, \ldots, X_{n-1,n-1}$. Similarly we can construct successive diagonals of X_{ij} until all are defined and the resulting operator has norm no greater than , and hence equal to, β.

Proof of Theorem 2.1.1.

$\mathcal{A}$ Is the intersection of the nest algebras Alg $\mathcal{F}$ taken over all finite subsets $\mathcal{F}$ of $\mathcal{E}$. Moreover dist$(X,\mathcal{F})$ = sup$\{\|(I-E)XE\|: E \in \mathcal{F}\}$ by the last lemma. It suffices to show that

$$\text{dist}(X,\mathcal{A}) = \sup\{\text{dist}(X,\text{Alg } \mathcal{F}): \mathcal{F} \subseteq \mathcal{E}, \text{ finite}\}.$$

Let σ denote this suprenum and let $\epsilon < 0$. Then the set

$$C_{\mathcal{F}} = \{A \in \text{Alg } \mathcal{F}: \|X-A\| \leq \sigma + \epsilon\}$$

is a nonempty set which is compact for the weak operator topology. The sets $C_{\mathcal{F}}$ have the finite intersection property, and so there is an operator A in the intersection, and hence in $\mathcal{A}$ with $\|X-A\| \leq \sigma + \epsilon$. Hence dist$(X,\mathcal{A}) \leq \sigma$. The reverse inequality is clear and so the proof is complete.

References. Arveson distance formula; Arveson [2], Lance [13], Power [24], [31], [29], [26], Parrott [20], Davis Kahan and Wienberger [6], Ball and Gohberg [4].

(2.2) Nehari's theorem for Hankel operators

Later we shall obtain some generalizations of the following theorem of Nehari on Hankel operators. We give a proof due to Parrott which in fact adapts easily to matricial Hankel operators. See also our remarks in 2.1.3.

Let $H^p \subseteq L^p$, $1 \leq p \leq \infty$, be the Hardy spaces for the unit circle, with norms determined by normalized Lebesgue measure and let $P: L^2 \to H^2$ be the Riesz projection. We write M_ϕ for the multiplication operator on L^2 determined by ϕ in L^∞, and we write $H_\phi = (I-P)M_\phi|H^2$ for the Hankel operator determined by ϕ, acting from H^2 to $(H^2)^\perp$. Clearly $H_\phi = 0$ if $\phi \in H^\infty$, and with respect to the orthonormal bases, $\{z^n: n \geq 0\}$, and $\{\bar{z}^{n+1} \cdot n \geq 0\}$, for H^2 and $(H^2)^\perp$ respectively, the operator H_ϕ has representing matrix $(a_{i+j})_{i,j=0}^\infty$ where $a_n = \hat{\phi}(-1-n)$ for $n = 0,1,\ldots,$ and where $\hat{\phi}(k)$ is the kth Fourier coefficient of ϕ, namely $\hat{\phi}(k) = \langle\phi, z^k\rangle$. So we see that H_ϕ depends only on the negative Fourier coefficients of ϕ. Moreover, $H_\phi = H_{\phi+h}$, for h in H^∞, and so $\|H_\phi\| \leq \|\phi+h\|_\infty$, and hence $\|H_\phi\| \leq \operatorname{dist}(\phi, H^\infty)$ where the distance is computed in L^∞.

(2.2.1) THEOREM. For ϕ in L^∞, $\|H_\phi\| = \operatorname{dist}(\phi, H^\infty)$.

Proof (Parrott). By Lemma 2.1.2 there is a complex number a_{-1} such that the operator determined by the matrix

$$\left[\begin{array}{c|ccc} a_{-1} & a_0 & a_1 & \cdots \\ \hline a_0 & a_1 & & \\ a_1 & & & \\ \vdots & & & \end{array}\right]$$

has norm equal to $\|A\|$ where A is the given bounded

Hankel operator $(a_{i+j})_{i,j=0}^{\infty}$. Repeating this argument with A replaced by A_{-1} we obtain a_{-2} and the Hankel operator A_{-2}, and, continuing in this way $A_{-3}, A_{-4}, \ldots$. It follows that the matrix $(a_{i+j})_{i,j\in\mathbb{Z}}$ determines a bounded operator B in $\ell^2(\mathbb{Z})$ which we identify with L^2, canonically, so that $PB|_{H^2} = A$. If F is the unitary operator on L^2 such that $Fz^n = z^{-n-1}$, for $n = 0,1,2,\ldots$ and $F^2 = I$, then it can be verified that FB commutes with the bilateral shift M_2, and so $FB = M_\psi$ for some multiplication operator, with $\|\psi\| = \|B\| = \|A\|$. Moreover, $\hat{\psi}(n) = a_n$ for $n = 0,1,2,\ldots$. In particular if A is identified with H_ϕ then $H_\phi = H_\psi$, and so $h = \phi - \psi$ belongs to H^∞ and $\|H_\phi\| = \|\phi - h\|$. In view of the remarks preceding the theorem, the proof is complete.

(2.2.2) Remarks. In many ways Nehari's theorem is the invariant form of the Arveson distance formula for $T(\mathbb{Z})$. It is useful to bear in mind these function theoretical connections since it may be that the analogous connections between distance problems for $T(\mathbb{Z}) \otimes T(\mathbb{Z})$ and bidisc function theory may shed some light on such problems. See section 2.7 for a discusssion of distance formulae in more general contexts.

The usual proof of Nehari's theorem makes use of the Riesz factorization of an H^2 function f as a product $f = f_1f_2$ with f_1, f_2 in H^2 and $\|f\|_1 = \|f_1\|_2\|f_2\|_2$. With this available the Hankel operator A can be used to define a bounded linear functional on H^1. This is extended, by the Hahn Banach theorem to a functional on L^1, with the same norm, from which we obtain a symbol function ψ for A (i.e. $A = H_\psi$) with $\|\psi\|_\infty = \|A\|$. We see in section (3.3) that there is an analogue of Riesz factorisation for trace class operators in certain nest algebras.

References. Nehari [17], Page [18], Bonsall and Power [5], Parrott [20], Power [23], [25], [31].

(2.3) Dual Space Methods

Recall that $C_1(H)$, the space of trace class operators on the separable Hilbert space H, is identified with the Banach space dual of $K(H)$ with the pairing $\langle K,T\rangle = \text{trace}(KT)$ for K in $K(H)$ and T in C_1. The following Lemma of Lance [13] provides a different approach to the distance formula and leads to decomposition theorems for trace class operators in a nest algebra. The proof depends on a linear decomposition of a positive operator. We investigate such decompositions in the next chapter where they form the basis of much factorization theorem.

(2.3.1) LEMMA. Let A be a trace-class operator and let E be an orthogonal projection such that $(I-E)AE = 0$. Then there exists a decomposition $A = A_1 + A_2$ such that

(i) $(I-E)A_1 = 0$, $A_2E = 0$

(ii) $\|A\|_1 = \|A_1\|_1 + \|A_2\|_1$

Proof. See Lance [13] or Power [26].

Let $\mathcal{E}$ be a complete nest of projections on H with nest algebra A. Let $A_1 = A \cap C_1$ and let $A_1^+ = \{A \in A_1 : QAQ = 0$ for all atoms Q of $\mathcal{E}\}$.

(2.3.2) LEMMA. (i) The extreme points of the unit ball of A_1 (resp.

A_1^+) are the rank one operators of unit norm in A_1 (resp. A_1^+).

(ii) For $\epsilon > 0$ and an operator A in A_1 (resp. A_1^+) there exist rank one operators $R_1, R_2, \ldots$ in A (resp. A_1^+) such that $A = R_1 + R_2 + \ldots$ and $\|R_1\|_1 + \|R_2\|_1 + \ldots \leq \|A\|_1 + \epsilon$.

Proof. Power [29].

In fact it follows easily from Lemma 2.3.1 that if A is an extreme point of the ball of A_1 then $A = EA(I-E_-)$ for some projection E in E. Now if $A = \Sigma A_k$ is any rank one Schmidt decomposition of A we can deduce that $A_k = EA_k(I-E_-)$ and hence $A_k \in A$ for all k, and hence since A is extreme, $A = A_k$ for some k. The assertion for A_1^+ is obtained similarly (with E_- above replaced by E).

For the proof of (ii) in the case of A_1 we let S denote the closed linear span in the operator norm of the rank one operators R such that $R = ER(I-E)$ for some projection E in E. It follows that A_1 is the annihilator of S, and hence that A_1 is the dual space of $K(H)/S$. In particular by the Krein Millman theorem the unit ball of A_1 is the closed convex hull of the extreme points, where the closure is taken in the weak star topology, which in this case corresponds to the weak operator topology. But if T_n is a sequence of finite rank operators such that $T_n \to T$ in the weak operator topology, and $\|T_n\|_1 \leq 1$, $\|T\|_1 = 1$, it follows that $\|T_n - T\|_1 \to 0$. The case of A_1^+ is proved similarly.

Second proof of Arveson distance formula

$L(H)/A$ is the Banach space dual of the preannihilator $A_\perp$ of the nest algebra A, where $A_\perp = \{T \in C_1: \mathrm{trace}(TA) = 0$ for all A in $A\}$. We claim that $A_\perp$ coincides with $\{T \in C_1: E^\perp TE_- = 0$ for all E in $E\} = A_1^+$. First note that $E_- XE^\perp \in A$ for all $X \in L(H)$, and so $E^\perp TE_- = 0$ for all T in $A_\perp$, and so $A_\perp \subset A_1^+$. On the other hand Lemma 2.3.2 (ii) shows that A_1^+ is the closed span of rank one operators R such that $ERE^\perp = R$. Such R lie in $A_\perp$, and so $A_\perp = A_1^+$. Now we compute, using Lemma 2.3.2 again,

$$\mathrm{dist}(X,A) = \|[x]\|_{L(H)/A}$$

$$= \sup\{|\mathrm{trace}(XT)|: T \in A_1^+, \|T\|_1 \leq 1\}$$

$$= \sup\{|\mathrm{trace}(XR)|: R = ERE^\perp, R \text{ rank one}, \|R\|_1 \leq 1, E \in E\}$$

$$= \sup\{|\mathrm{trace}(E^\perp XEY)|: Y \text{ rank}, \|Y\|_1 \leq 1; E \in E\}$$

$$= \sup\{\|E^\perp XE\|: E \in E\}.$$

<u>References</u>: Lance [13], Power [26], Power [31].

(2.4) <u>A Hardy-Littlewood-Fejér inequality for trace class integral operators</u>

We now describe an application of the decomposition theory of the previous section to integral operators.

Let μ denote a σ-finite Borel measure on the real Lie $\mathbb{R}$, and let $h(x,y)$, $k(x,y)$ denote measurable kernel functions which induce bounded integral operators Int h and Int k on $L^2(\mu)$ in the sense of Halmos and Sunder [11]. (Let dom k be the linear space of functions $f(y)$ in $L^2(\mu)$ such that $k(xky)f(y)$ is integral for almost every x and the function $(\mathrm{Int}\ k)f(x) = \int k(x,y)f(y)dy$ belongs to $L^2(\mu)$. If dom k is dense and $\|(\text{Int } k)f\|_2 \leq c\|f\|_2$ for all f is dom k, and some constant c, independent of f, then we say that k induces a bounded operator, namely the continuous extension of (Int k)(dom k.)

(2.4.1) THEOREM. If $h(x,y) = 0$ for all $x > y$, and if $k(x,y) \geq 0$ for $x \leq y$ then

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |h(x,y)|k(x,y)d\mu d\mu \leq \|\text{Int } k\|\|\text{Int } h\|_1$$

(2.4.2) COROLLARY. (A.L. Shields). Let $T = (t_{ij})$ be an operator in the nest algebra $T(\mathbb{N})$. Then

$$\sum_{j\geq i} \frac{|t_{ij}|}{1+j-i} \leq \pi\|T\|_1$$

(2.4.3) COROLLARY. Let $h(x,y)$ be a measurable kernel with respect to Lebesgue measure which induces a bounded integral operator Int h which belongs to $T(\mathbb{R})$. Then

$$\int\int_{y\leq x} \frac{|h(x,y)|}{y-x}\, dxdy \leq \pi\|\text{Int } h\|_1.$$

To obtain the first corollary let μ be counting measure on $\mathbb{N}$ and let $k(i,j) = (1+j-i)^{-1}$ for all i,j except the pairs $i,i+1$, for which $k(i,i+1) = 0$. This is essentially Hilberts second matrix which is known to have operator norm π. Similarly, for the second corollary notice that the kernel $k(x,y) = (y-x)^{-1}$ induces modulo a constant multiplier, the Hilbert transform on $L^2(\mathbb{R})$, as a singular integral operator, with norm π. Although Int k is not an integral operator in the sense above (since its domain is the zero function) the proof of the theorem is easily adapted.

The first corollary was obtained by A.L. Shields in an interesting paper emphasizing problems for upper triangular operators analogous to various problems in analytic function theory and harmonic analysis. His proof relied on a Riesz factorization theorem for upper triangular trace class operators (see Chapter 3). Both corollaries are analogue of the inequality $\sum_{n=0}^{\infty} |\hat{h}(n)|(n+1)^{-1} \le \pi\|h\|_1$ for the Fourier coefficients of a function in the Hardy class H^1.

The proof of Theorem 2.4.2 in Power [32] is different and is more analogous to that used in the atomic and molecular theory of analytic functions, where boundedness with respect to a "one norm" is first easily checked for special molecule functions and then shown to hold true in general by involving a decomposition theorem which expresses each analytic function in the space as a sum of molecules. The decomposition of Lemma 2.3.2 (ii) plays this role here. We leave it to the reader to verify Theorem 2.4.1 in the special case when Int h has rank one.

References: Shields [33], Power [32]

(2.5) Abstract Hankel operators and quasitriangular algebras

In this section we introduce some ideas encircling the quasitriangular algebra $QT(E) = T(E) + K$ associated with a projection nest E of order type $\mathbb{N}$, with finite dimensional atoms. We obtain a formula for $\operatorname{dist}(X, QT(E))$ by elementary means and explain why this distance is always achieved. In the next section we develop more general theorems and methods. Our framework here involves abstract Hankel operators and further analogues of theorems for classical Hankel operators and function theory on the circle.

Let E be a projection nest on the Hilbert space H consisting of 0 and I and finite rank projections $P_1, P_2, \ldots$ that increase to the identity. Regard $C_2(H)$ as a Hilbert space with inner product $\langle B_1, B_2\rangle = \mathrm{trace}(B_2^*B_1)$. Let P be the projection of triangular truncation from C_2 onto A_2, where $A_2 = T(E) \cap C_2$. For $X \in L(H)$ define the abstract Hankel operator H_X by $H_X = (I-P)L_XP$ where L_X is the operator on C_2 of left multiplication by X. Let Q_n denote the orthogonal projection of C_2 onto the subspace $C_2(P_n - P_{n-1})$, $n = 1,2,\ldots$ Then a simple calculation shows that with respect to the decomposition $C_2 = \sum_{n=1}^{\infty} \oplus C_2Q_n$ we have

$$H_X = \sum_{n=1}^{\infty} \oplus H_{P_n^{\perp} XP_n} | C_2Q_n \quad .$$

Moreover $\|H_{P_n^{\perp}XP_n} | C_2Q_n\| = \|P_n^{\perp}XP_n\|$ and so, by Arveson's theorem (2.1.1) $\|H_X\| = \mathrm{dist}(X, T(E))$, in direct analogy with Nehari's theorem (2.2.1).

The first part of the next theorem is a direct analogue of Hartman's theorem for Hankel operators $(\|H_\phi\| = \mathrm{dist}(\phi, H^\infty + C))$. The second part is analogous to the fact that the commutator $M_\phi P - PM_\phi$, associated with $\phi \in L^\infty$ and the Hardy space projection P, is a compact operator if and only if $\phi \in QC = (H^\infty + C) \cap \overline{(H^\infty + C)}$.

(2.5.1) THEOREM. Let X be a bounded operator. Then

(i) the Hankel operator H_X is a compact operator if and only if X belongs to the quasitriangular algebra $QT(E)$. Moreover

$$\mathrm{dist}(H_X, K(C_2)) = \mathrm{dist}(X, QT(E)) \; ;$$

(ii) the commutator $L_X P - P L_X$ determines a compact operator on C_2 if and only if X belongs to the C*-aglebra

$$QT(E) \cap QT(E)^*.$$

In the direct sum decomposition for H_X given above the summands $H_{P_n^{\perp}XP_n}|C_2Q_n$ have finite rank and norm $\|P_n^{\perp}XP_n\|$. From part (i) above it follows that $QT(E) = \{X: P_n^{\perp}XP_n \to 0 \text{ as } n \to \infty\}$, and that

$$\operatorname{dist}(X,QT(E)) = \lim_{n\to\infty} \|P_n^{\perp}XP_n\|.$$

The proof of Theorem 2.5.1 (Power [29]) is an argument analogous to the proof of Hartman's theorem, the key idea being the distance formula expressed in the form $\|H_X\| = \operatorname{dist}(X,T(E))$. The Hankel operator methodology is useful here in that it suggests that like $H^{\infty} + C$, the quasitriangular algebra $QT(E)$ is a space of best approximation, or proximinal space, in $L(H)$. That is, the distance of any operator X to $QT(E)$ is always attained. (The methods of the next section show the proximinality of $QT(E)$ for <u>any</u> nest E.) We sketch a proof here that uses a theorem of Axler, Berg, Jewell and Shields. (This theorem is also obtained in the following section.)

(2.5.2) <u>THEOREM.</u> Let E be a nest of finite rank projections. For each operator X there exists an operator Y such that $\|X-Y\| = \operatorname{dist}(X,QT(E))$.

<u>Proof.</u> Let $P_1,P_2,\ldots$ be the nontrivial projections of the nest, as before, and let $X_n = P_nXP_n$. Then $\|H_{X_n}\| \le \|H_X\|$ and the sequence H_{X_n} consists of finite rank operators converging to H_X in the strong

operator topology. It follows from the main result in Axler, Berg, Jewell, and Shields [ ], that there is a compact Hankel operator H_Y such that $\|H_X - H_Y\| = \text{dist}(H_X, K(C_2))$, which, by Theorem 2.5.1, aggres with $\text{dist}(X, QT(E))$. Moreover $Y \in QT(E)$. But $\|H_X - H_Y\| = \|H_{X-Y}\| =$ $\text{dist}(X-Y, T(E)) = \|X-Y-A\|$, for some operator A in $T(E)$ (since $T(E)$ is σ-weakly closed), and so $\text{dist}(X, QT(E)) = \|X-(A+Y)\|$ with $A+Y$ in $QT(E)$, as desired.

References. Hankel operators on the circle; Nehari [17], Hartman [12], Power [23], [25], Leuking [16]. Abstract Hankel operators; Power [29], Power [30], Paulsen and Power [21] (see Chapter 7). Quasitriangular algebras; see references of section 2.6 below.

CHAPTER 3 DECOMPOSITION THEORY AND FACTORISATION THEORY

In this chapter we consider an ordered decomposition associated with a positive operator C and a projection nest $\mathcal{E}$. In the matrix case this is a finite sum decomposition $C = C_1 + C_2 + \ldots + C_n$ associated with the Cholesky algorithm, but in general the decomposition is a positive operator valued measure $C(\Delta)$, defined on the Borel subsets Δ of $\mathcal{E}$, with the order topology, possessing certain minimality properties relative to the nest, with $C(\mathcal{E}) = C$. We call $C(\Delta)$ the Cholesky measure associated with C and $\mathcal{E}$. It will become clear that this construction plays a fundamental role in many aspects of the structure theory of operators in a nest algebra. In sections (3.1) and (3.2) we describe the Cholesky measure and its implications for the decomposition theory of trace class operators in a nest algebra. On the way we recover a classical theorem of Lidskii on the trace of a trace class operator. In section (3.3) we develop a new approach to and generalisations of the Arveson inner-outer factorisation theory for operators in a nest algebra. In particular we characterise nests such that every positive operator C admits an outer factorisation $C = A^*A$, with A an outer operator of the nest algebra. In subsequent sections we study the constructive Cholesky method in the outer factorisation of positive matrix valued functions on the unit circle. This new approach provides unity with the Arveson theory, and, being constructive, leads to new information, such as the description of the prediction-error operator in spectral terms and certain continuity properties for the outer factors.

(3.1) Construction of the Cholesky measure

We now outline the construction of $C(\Delta)$ given in Power [16]. As usual when we say an operator C is positive we mean more precisely that C is positive semidefinite.

(3.1.1) LEMMA. (E.C. Lance) Let C be a positive operator which has an operator matrix $\begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$ with respect to a given decomposition of H. Then $D_1 = \lim_{n\to\infty} B^*(A+n^{-1}I)B$ exists in the strong operator topology and the following hold.

(i) $D_1 \leq D$.

(ii) The operator $C_1 = \begin{bmatrix} A & B \\ B^* & D_1 \end{bmatrix}$ is positive.

(iii) If U is an operator on H and UC has the form $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$, then UC_1 and $U(C-C_1)$ have, respectively, the forms $\begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}$.

Let $C_2 = C - C_1$ so that $C = C_1 + C_2$. This is the Cholesky decomposition of C relative to the trivial nest $\{0,E,I\}$ associated with the projection E onto the first summand space. Note that from (i) and (ii) it follows that $C_1 \leq X$ for any positive operator X with $XE = CE$.

Let $[E,F)$ denote the Borel set in E of projections E_1 in E with $E \leq E_1 < F$. Define $C([0,E)) = C_1$ as above, define $C([E,F)) = C([0,F)) - C([0,E))$, and in general define $C(\Delta)$ for any Δ in the ring $R(E)$ generated by the semi-open intervals $[E,F)$. The minimality properties can be used to show that $C(\Delta)$ is a well defined additive operator valued measure on $R(E)$. Moreover it can be shown that on $R(E)$, $C(\Delta)$ is countably additive (left continuous), and so by standard operator measure theory, $C(\Delta)$ extends to a positive operator measure on the Borel sets of E. That is $C(\Delta)$ countably addiive relative to the weak operator topology.

In view of property (iii) it follows that if $A \in T(E)$ has polar decomposition $T = UC$ with U a partial isometry and C positive, then the operator $T([E,F)) = UC([E,F))$ has the following properties

$$T([E,F)) = FT([E,F))E_-^{\perp},$$
$$(F-E)T([E,F))(F-E) = (F-E)T(F-E).$$

The first equality shows that the operator valued measure $T(\Delta) = UC(\Delta)$ provides an upper triangular decomposition for T. In the case of trace class operators we can do better.

(3.2) <u>Integral representation for triangular trace class operators.</u>

Let $C_1 = C_1(H)$ denote the trace class. Recall that a C_1-valued function f on a σ-finite measure space (Ω,Σ,μ) is (weakly) measurable if $w \to \langle f(w)x,y\rangle$ is measurable for all pairs of vectors x,y in H, and is <u>integrable</u> if in addition the function $\|f(t)\|_1$ is integrable.

(3.2.1) <u>THEOREM.</u> Let E be a complete nest on a separable Hilbert space, and let T be a trace class operator in C_1. Then there exists a finite positive Borel measure τ on E, and an integrable C_1-valued function $E \to T_E$ on E, such that

(i) $T = \int_E T_E d\ (E)$

(ii) $\|T\|_1 = \int_E \|T_E\|_1 d\tau(E)$

(iii) $T_E = ET(I-E\ )$ almost everywhere.

The idea of the proof is to consider $T = UC$ and $C(\Delta)$ as in the previous section, and to note that for the scalar measure $\tau(\Delta) = \mathrm{trace}(C(\Delta))$, if, $\tau(\Delta) = 0$ then $C(\Delta) = 0$. Using the appropriate Radon Nikodyn theorem, (the trace class operators form a separable dual space), we can obtain an integral representation of $C(\Delta)$ and this leads to the desired integral representation.

The theorem above is the continuous version of Lemma 2.3.2. It is natural to ask whether the ϵ of that lemma can be removed, that is whether every trace class operator of $T(E)$ admits an <u>exact</u> sum decomposition $T = \sum_{n=1}^{N} R_n$ with $R_1, R_2, \ldots$ rank one operators of $T(E)$ such that $\|T\|_1 = \sum_{n=1}^{\infty} \|R_n\|_1$. This is not true in general. However the theorem and methods above can be used to obtain the following theorem.

(3.2.2) <u>THEOREM.</u> (i) Let E be a countable nest. Then every trace class operator T in $T(E)$ admits an exact rank one decomposition.

(ii) Let E be a general nest. If T is a trace class operator with positive imaginary part, then T admits an exact rank one decomposition.

We finish this section by outlining how Theorem 3.2.1 leads to a proof of the following theorem of Lidskii.

(3.2.3) <u>THEOREM.</u> (Lidskii). The trace of a trace class operator is the sum of its eigenvalues, counted with their algebraic multiplicities.

By the invariant subspace theorem for compact operators, together with Zorn's lemma we can construct a maximal projection nest E for a given trace class operator T so that $T \in T(E)$. By maximality the nonzero atoms $E - E_-$ are one dimensional. By an elementary argument, we can reduce to the case where $(E-E_-)T(E-E_-) = 0$ for all E in $\mathcal{E}$, so that T has no nonzero eigenvectors, and we are required to show that $\operatorname{trace} T = 0$. In this case we have, in the integral decomposition of Theorem 3.2.1, $(E-E_-)T_E(E-E_-) = 0$ and hence $\operatorname{trace}(T_E) = 0$, for all E. It follows that

$$\operatorname{trace}(T) = \int_{\mathcal{E}} \operatorname{trace}(T_E)\,d\tau(E) = 0,$$

as required.

References. Power [26],[27] , Lance [13], Erdos [8], Lidskii [15].

(3.3) The Arveson-Cholesky factorisation and related topics

We now give a new approach to Arveson's inner-outer factorisation theory for nest algebras, which leads to generalisations and further results. Let $A = \operatorname{Alg} \mathcal{E}$ be a nest algebra.

(3.3.1) DEFINITION. (Arveson) (i) An operator A in $\mathcal{A}$ is said to be outer if the range projection of A commutes with $\mathcal{E}$ and for every projection E in $\mathcal{E}$

$$(AE\mathcal{H})^- = (A\mathcal{H})^- \cap E\mathcal{H}.$$

(ii) An operator U in $\mathcal{A}$ is called inner if U is a partial isometry whose initial projection U^*U commutes with $\mathcal{E}$.

For certain projection nests $\mathcal{E}$ of discrete type the inner and outer operators play the role of inner and outer functions in the algebra H^∞ of bounded analytic functions on the unit disc.

We shall obtain analogues of the following factorisation results in function theory.

1. The Szego or outer factorisation of a positive function f:

   $f = h\bar{h}$ with h outer.

2. The inner-outer factorisation of an H^∞ function g:

   $g = uh$ with u inner and h outer.

3. The Riesz factorisation of H^1 functions:

   $h = h_1 h_2$ with $\|h\|_1 = \|h_1\|_2 \|h_2\|_2$.

The operator variants of 2 and 3 are Theorems 3.3.6 and 3.3.7, and these follow quickly from the Szego-type theorem 3.3.5. Our approach is unifying in that it also leads to the outer function factorisation $f = hh^*$ of a positive matrix valued function on the unit circle, when this factorisation is known to exist. More generally we can obtain the extremal outer decomposition $f = hh^* + g$ of <u>any</u> positive operator valued function. (The usual approach to these matters is through the Bearhny-Lax-Halmos theorem for shift invariant subspaces, and is accordingly less constructive.)

Note that if $A \in \mathcal{A}$ has the strict density property $\overline{AEH} = EH$ for E in $\mathcal{E}$ then A is outer. Also if A is invertible in $\mathcal{L}(H)$ then A is outer if and only if A is invertible in $\mathcal{A}$. On the other hand every operator in the diagonal algebra $\mathcal{A} \cap \mathcal{A}^*$ is outer. The next lemma characterizes the outer operators relative to a trivial three element projection nest. The precise nature of outerness for a well ordered nest can be understood in the proof of Theorem 3.3.5 (See Theorem 2.2 in Power [30]).

(3.3.2) DEFINITION. Let C be a positive operator and let E be a self-adjoint projection. Then C is said to be E-minimal if

$$E^{\perp}CE^{\perp} = \underset{t\to 0}{\text{s-lim}}\ E^{\perp}C(tE+ECE)^{-1}CE^{\perp}$$

where the inverse indicated is computed in $L(EH)$.

We have already observed the existence of the strong limit in the above definition, in Lemma 3.1.1.

Let us write R_X for the range projection of the operator X.

The proof of the next lemma is closely related to the constructions needed for the proof of Lemma 3.1.1.

(3.3.3) LEMMA. The following conditions are equivalent for an operator A with invariant self-adjoint projection E such that $R_AE = ER_A$.

(i) $(AEH)^{-} = (AH)^{-} \cap EH$.

(ii) $R_{AE} \geq R_{EA(I-E)}$.

(iii) A^*EA is E-minimal.

Moreover a positive operator C is E-minimal if and only if $C = A_1^*A_1$ where $A_1 = EA_1$ and A_1 satisfies condition (ii).

Proof. Since $(I-E)AE = 0$ the equivalence of (i) and (ii) is elementary.

Suppose now that (ii) holds and let

$$EA = \begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}$$

so that $R_{a_1} \geq R_{b_1}$. Then

$$\underset{t\to 0}{\text{s-lim}}\ b_1{}^*a_1(tE+a_1{}^*a_1)^{-1}a_1{}^*b_1$$

$$= \underset{t\to 0}{\text{s-lim}}\ b_1{}^*(tE+a_1a_1^*)^{-1}a_1a_1{}^*b_1$$

$$= b_1{}^*R_{a_1}b_1 = b_1{}^*b_1 = E^{\perp}A^*EAE^{\perp}.$$

and so (iii) holds. On the other hand, if (iii) holds then this computation shows that $b_1^*b_1 = b_1^*R_{a_1}b_1$ and so (ii) holds.

Now consider an E-minimal positive operator with operator matrix representation

$$C = \begin{bmatrix} a & b \\ b^* & c_1 \end{bmatrix}.$$

Let e_t denote the spectral projection for the operator a corresponding to the interval (t,∞). Then, for $t > 0$,

$$\begin{aligned} \|b^*a^{-1/2}e_t\|^2 &= \lim_{s\to 0} \|b^*(sE+a)^{-1/2}e_t(sE+a)^{-1/2}b\| \\ &\le \lim_{s\to 0} \|b^*(sE+a)^{-1}b\| \\ &\le \|c_1\|. \end{aligned}$$

It follows that the operator $d_t = b^*a^{-1/2}e_t$ converges strongly to an operator d as $t \to 0$. Since $c_1 \geq b^*a^{-1/2}e_ta^{-1/2}b$ it follows that $c_1 \geq dd^*$. On the other hand

$$\begin{bmatrix} a & b \\ b^* & dd^* \end{bmatrix} = \begin{bmatrix} a^{1/2} & 0 \\ d & d \end{bmatrix} \begin{bmatrix} a^{1/2} & d^* \\ 0 & 0 \end{bmatrix} \geq 0$$

and so, by minimality, $dd^* \geq c_1$. It is clear from the definition of d that the range projection of a (namely e_0) dominates the range projection of d^*. So C has the form required in the last part of the lemma. $\square$

Using the notation of the proof above we observe that the positive operator

$$C = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$$

factorises as $C = A^*A$ with

$$A = \begin{bmatrix} a^{1/2} & d^* \\ 0 & (e-c_1)^{1/2} \end{bmatrix}.$$

In view of the lemma A is an outer operator with respect to the nest $\{0,E,I\}$.

A consequence of the computations made in the above proof is the following algebraic feature of the outer factor:

$$X \in L(H) \quad XC = \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix} \quad XA^* = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}.$$

This is also a consequence of the following more general lemma which echoes the essential property of an H^∞ function h: If $\phi \in L^\infty$ and $\phi h \in H^\infty$ then $\phi \in H^\infty$.

(3.3.4) LEMMA (Arveson). Let $A \in \mathcal{A}$ be an outer operator and let X be an operator such that $XA \in \mathcal{A}$ and $X = 0$ on $(AH)^\perp$. Then X belongs to $\mathcal{A}$.

The next theorem generalises a result of Arveson.

(3.3.5) THEOREM. Let E be a well ordered nest of projections with nest algebra $\mathcal{A}$ and let C be a positive operator. Then there exists a factorisation $C = A^*A$ with A an outer operator in $\mathcal{A}$. Moreover the outer factor belongs to the von Neumann algebra generated by C and the nest.

Proof. See Power [30], [31].

It is well known and easily proven that the outer factor is unique up to a unitary diagonal factor, and in particular is uniquely determined if the diagonal part of A is a positive operator.

For universal factorisation it is necessary that the nest be well-ordered.

(3.3.5) THEOREM. Let $\mathcal{E}$ be a projection nest such that every positive operator C admits a factorisation $C = A^*A$ with A belonging to Alg $\mathcal{E}$. Then $\mathcal{E}$ is well ordered.

Proof. Power [30], [31].

It is curious that in the following generalisation of Arveson's inner-outer factorisation theorem we can drop the requirement that the projection 0 has a successor.

(3.3.6) THEOREM. Let $\mathcal{E}$ be a complete projection nest such that $E \neq E_+$ for all nonzero projections E, and let $T \in$ Alg $\mathcal{E}$. Then

(i) $T = UA$ where $U \in$ Alg $\mathcal{E}$ is an inner operator and $A \in$ Alg $\mathcal{E}$ is an outer operator.

(ii) If $T = UA = VB$ are two such factorisations then there is a partial isometry W in (Alg $\mathcal{E}$) $\cap$ (Alg $\mathcal{E}$)* such that $W^*W = R_A$, $WW^* = R_B$, $B = WA$ and $V = UW^*$.

(iii) If $T = UA$, as in (i), then U and A belong to the von Neumann algebra generated by T and $\mathcal{E}$.

Proof. Power [30], [31].

Let us introduce the following terminology to formulate the next theorem, which was obtained by Shields, by different methods, in the special case of the nest algebra $T(\mathbb{N})$. A projection lattice $\mathcal{L}$ is said to admit Riesz factorisation if for each T in (Alg $\mathcal{L}$) $\cap$ C_1

there exists operators A_1, A_2 in $(\text{Alg } L) \cap C_2$ such that

$$T = A_1A_2 \quad \text{and} \quad \|T\|_1 = \|A_1\|_2\|A_2\|_2$$

(3.3.7) THEOREM. Let E be a well ordered projection nest. Then E admits Riesz factorisation.

Proof. Power [30], [31].

It is an open problem exactly which projection lattices or nests admit Riesz factorisation. For nest it can be shown that the following condition is necessary: For all $0 < E < I$, $E_+ \neq E_-$.

For the Hardy Space H^1 associated with the ball or sphere in several complex dimensions it is known that Riesz factorisation fails, but that a good substitute is available, namely weak factorisation: each H^1 function f admits a decomposition $f = \Sigma g_k h_k$ with $\Sigma\|g_k\|_2\|h_k\|_2 \leq c\|f\|_1$ for some universal constant weak factorisation for nest algebras.

(3.3.8) THEOREM. Let A_1 be the trace class operators of a nest algebra A and let $\epsilon > 0$. Then for each operator T in A_1 there exist rank one operators $R_1, R_2, \ldots$ and $S_1, S_2, \ldots$ in A such that

(i) $T = \sum_{k=1}^{\infty} R_k S_k$

(ii) $\sum\|R_k\|_2\|S_k\|_2 \leq (1+\epsilon)\|T\|_1$

As in the function theory contexts, weak factorisation can be used to characterise the bounded Hankel forms on $A_2 = A \cap C_2$. A Hankel form on A_2 is a complex bilinear form $[\ ,\ ]$ such that

$$[A_1A_2,A_3] = [A_1,A_2A_3]$$

for all 3-tuples A_1,A_2,A_3 in A_2. The form is bounded if $[A_1,A_2]$ is bounded as A_1,A_2 range in the unit ball of A_2. We write $\|[\ ,\ ]\|$ for the least such bound.

(3.3.9) THEOREM. Let $[\ ,\ ]$ be a bounded Hankel form on A_2. Then there is an operator X such that $\|X\| = \|[\ ,\ ]\|$ and $[A_1,A_2] = \text{trace}(A_2XA_1)$ for all A_1 A_2 in A.

The proofs of (3.3.8) and (3.3.9) are given in Power [29], [30] and in [30] more general theorems are obtained in the context of finite factors, and their associated noncommutative L^p-spaces. Nevertheless the essential ideas already exist in the I context discussed here.

However, not all the results of this section have immediate natural counterparts in the context of finite factors. For example the literal translation of Theorem 3.3.5 is not valid since the methods of this section can be used to show that for a nest E, in a II_1 factor M, with order type $\mathbb{Z}$, every positive operator C in M admits a Cholesky factorisation relative to the nest subalgebra $M \cap \text{Alg } E$. A general outer factorisation theory for the II_1 context, even in the hyperfinite case, is not yet well understood. However the Gohberg Krein factorisation theory, which mainly concerns the LDU factorisation of invertible operators, can be carried out in the II_1 and II_∞ contexts. This has been done by Pitts [34]. Also, there are other approaches which we shall not go into here based on the boundedness of triangular truncation in the noncommutative L^2 space $L^2(M,\tau)$ associated with a semifinite factor M, with faithful normal semifinite trace τ.

References. Arveson [2] , Shields [33], Power [29],[30],[31],[32], Pitts [34].

(3.4) The outer factorisation of matrix and operator functions

Let f be an essentially bounded positive matrix valued function on the unit circle. If $f(e^{i\theta}) \geq \delta T$ almost everywhere for some $\delta > 0$, then it is well known that f admits a factorisation $f = hh^*$ where h is an analytic matrix valued function. The analysis of such factorisations formed the basis of Wiener and Masani's approach to the theoretical and computational aspects of the prediction theory of multi-variate stationary stochastic processes. The usual methods involve an analysis of the shift invariant subspaces of the multiple shift.

In the following two papers we develop an alternative approach, based on the more explicit methods of the Cholesky decomposition. Consequently we can obtain much more information on the relationship between the outer factor and the given function.

(3.4) ctd.

SPECTRAL CHARACTERISATION OF THE WOLD-ZASUHIN DECOMPOSITION AND PREDICTION-ERROR OPERATOR

S.C. Power,
Department of Mathematics,
University of Lancaster,
England, LA1 4YL.

1. Introduction

Nearly thirty years ago Wiener and Masani pointed out in the introduction of their celebrated paper [31] that for a general multivariate stationary stochastic process no relation had been given for the prediction-error matrix in terms of the spectrum of the process. In particular it was unknown how to characterise the rank of the process in spectral terms (cf.Masani [12, p369 Question 1]). Despite explicit progress in this connection with certain regular processes , such as the series representations in [32],[11],[22],[19] , or the iterative approach of [28],[29] , and despite progress in the structure theory of degenerate processes ([10],[14],[8],[26],[15]) ,a general relation or series expression has remained elusive.

In this paper we obtain spectral formulae ((2.2) and (2.5)) for the prediction-error matrix for a wide class of processes , namely those with essentially bounded spectral density. The characterisation is obtained in terms of Hilbert space operators. A new constructive approach is employed which is based on the linear decomposition of positive operators, rather than the traditional shift invariant subspace theory. We also obtain formulae for the outer factor that ensure the inheritance of smoothness and local properties in the case of a regular density,as well as a new characterisation of regularity (for essentially bounded spectral densities).

Let f(z) be an essentially bounded positive operator valued function on the unit circle, which is not identically zero, and consider the problem of obtaining a decomposition

$$f(z) = h(z)h(z)^* + g(z)$$

where g is also positive operator valued and where h is analytic, outer (in a sense specified below) and extremal in the sense that the function g is minimised. For scalar functions the Szego alternative provides the following solution. Subject to the normalisation $h(0) \geq 0$ there exists a unique maximal outer function h , and either $h = 0$ or $g = 0$. In the latter case we say that f admits outer factorisation, and a necessary and sufficient condition for this is the integrability of logf. For matrix valued functions the work of Wiener and Masani [31], Wiener and Akutowicz [30] and Helson and Lowdenslager [8] shows that the integrability of logdetf is a sufficient condition for outer factorisation, and this in turn was generalised to the setting of operator valued functions by Devinatz [3].See also [5]. If f(z) is the matricial spectral density of a multivariate stationary stochastic process then the process is purely non deterministic if and only if f admits outer factorisation. However the only known necessary and sufficient criterion for this event which is also valid for operator functions, seems to be that of Lowdenslager, namely

$$\bigcap_{n \geq 0} \{ z^n (\sqrt{f} H^2_{\mathbb{K}})^{-} \} = \{0\} .$$

Here z denotes the shift on a vectorial Hilbert space $L^2_{\mathbb{K}}$ with Hardy subspace $H^2_{\mathbb{K}}$.

Lowdenslager's condition is intimately connected with the usual approach to factorisation through the analysis of invariant subspaces and the Beurling-Lax-Halmos theorem, as exemplified in the books of Helson [7] and Sz-Nagy and Foias [27]. However, this approach does not reveal the dependence of the (essentially unique) outer term h on the original function f . Indeed in prediction theory there does not exist a spectral expression for the rank or the prediction error matrix $h(o)h(o)^*$ of a general stationary stochastic process. Also in the case of a regular (purely nondeterministic) process it is not clear how the outer factor is structurally related to the spectral density. Despite this, and despite the absence of an integral representation analogous to that for scalar outer functions, there are several such structure theorems in the literature ([22], [23] for example). We shall see how such results follow from a general inheritance principle based on the theory of Hankel operators and the remarkable formula

$$H_{h^*}^{*}H_{h^*} = H_f^{*}\tilde{T}_f^{-1}H_f \; . \tag{1.1}$$

where H_f is a Hankel operator and T_f is a Toeplitz operator associated with the regular spectral density f . The formula arises naturally in our constructive approach to the extremal decomposition of f. The method is based on an operator theoretic generalisation of the Cholesky factorisation of positive hermitian matrices and originates in the author's analysis [21] of the inner-outer factoration theory of Arveson [1] for operators in a nest algebra.

It would be desirable to write down a multiplicative integral formula for the prediction-error matrix or outer factor in terms of the spectral density(cf. [13]). An indication of the difficulty of this goal is expressed in (1.1) ; outer factorisation is closely tied to the inversion of matricial Toeplitz operators. On the other hand, perhaps the local inheritance properties for the outer factor (discussed in section 2) provide some evidence for the existance of such a formula.

The author is very grateful to G.Tunnicliffe Wilson and P. Masani for guidance in the literature of multivariate stochastic processes.

2. The main results

Our first purpose is to formulate the context, state the main results of the paper and to discuss some consequences.

Let $\mathcal{K}$ be a complex Hilbert space with Hilbert space tensor products

$$\mathcal{H} = l^2(\mathbb{Z}) \otimes \mathcal{K}\ ,\ \mathcal{H}_+ = l^2(\mathbb{Z}_+) \otimes \mathcal{K}$$

associated with $l^2(\mathbb{Z})$ and $l^2(\mathbb{Z}_+)$, the usual complex sequence Hilbert spaces for the integers and the non-negative integers, respectively. Regard $\mathcal{H}_+$ as the naturally embedded subspace of $\mathcal{H}$ with orthogonal projection P . When $\mathcal{K}$ is separable there are familiar identifications of $\mathcal{H}$ and $\mathcal{H}_+$ with the functional Hilbert spaces $L^2_{\mathcal{K}}$ and $H^2_{\mathcal{K}}$ respectively. Our development is independent of these realisations but nevertheless we shall retain some functional notation, even though $\mathcal{K}$ may be non-separable. Thus we write z for the bilateral shift on $\mathcal{H}$, we let L^∞ denote the commutant of this shift, and we write f, g, h, etc. for the operators in L^∞ . We also let H^∞ denote the subalgebra of L^∞ consisting of the operators that leave $\mathcal{H}_+$ invariant. An operator h in H^∞ is said to be *outer* if $(h\mathcal{H}_+)^- = (h\mathcal{H})^- \cap \mathcal{H}_+$. When h is nonzero and dim $\mathcal{K} = 1$ this notion coincides with the usual concept of an essentially bounded outer function, whilst if h has dense range then h is outer in the sense of Sz-Nagy and Foias [27]. We let Q denote the orthogonal projection of $\mathcal{H}$ onto $\mathbb{C}e_0 \otimes \mathcal{K}$ where e_0 is the central basis element of $l^2(\mathbb{Z})$. Notice that an operator f in L^∞ is uniquely determined by the operator Qf .

In the next section we construct an extremal decomposition $f = hh^* + g$ for each positive operator f in L^∞. When $\mathcal{K}$ has finite dimension and f is interpreted as a function on the unit circle representing the matricial spectral density of a multivariate stationary stochastic process then g is the spectral density for the deterministic part and h is the outer factor, or generating function, for the purely nondeterministic part. The extremal decomposition thus represents the spectral density decomposition associated with the Wold-Zasuhin decomposition of the process, and the operator $G(f) = (QhQ)(QhQ)^*$ 43
is the prediction-error matrix. These constructs are identified in the next theorem where we retain the prediction theoretic terminology even though $\mathcal{K}$ may

be a general complex Hilbert space.

We write

$$T_f = PfP, \; \tilde{T}_f = P^{\perp}fP^{\perp}, \; H_f = P^{\perp}fP,$$

for the Toeplitz operators and Hankel operator associated with f. We say that an operator X on $\mathcal{H}$ is *asymptotically vanishing* if the limit of the sequence $z^{-n}Xz^n$ exists in the weak operator topology and is the zero operator.

<u>THEOREM</u> Let f be a positive operator in L^∞. Then the limit

$$\lim_{t\to 0+} H_{f^*}^*(tP^{\perp} + \tilde{T}_f)^{-1}H_f \tag{2.1}$$

exists in the strong operator topology and determines a positive operator C_f.

(i) The prediction-error operator G(f) associated with the spectral density f is given by

$$G(f) = QfQ - QC_fQ\,. \tag{2.2}$$

(ii) The outer factor, or generating function, for the purely nondeterministic part of f is the outer operator in H^∞ given by the identity

$$Qh = G(f)^{-\frac{1}{2}}Q(T_f - C_f). \tag{2.3}$$

(iii) A purely nondeterministic process is determined by the spectral density f if and only if the operator C_f is asymptotically vanishing, and in this event we have the following relationship for the outer factor h :

$$H_{h^*}^*H_{h^*} = C_f. \tag{2.4}$$

The operator C_f is in fact determined as an increasing limit and it follows readily that under the normalisation $\|f\| \leq 1$ the prediction-error operator can be expressed as the following infinite series, convergent in the strong operator topology for $\mathcal{K}$:

$$\phi = 1-f$$

$$G(f) = Q - Q\phi Q - Q\phi P^{\perp}\phi Q - Q\phi P^{\perp}\phi P^{\perp}\phi Q - ... - Q\phi(P^{\perp}\phi P^{\perp})^{r}\phi Q - ... \quad . \tag{2.5}$$

Recall that the rank of the multivariate stationary stochastic process associated with a matricial spectral density f is defined to be the rank of G(f). Thus the formulae (2.2) and (2.5) provide a spectral determination of rank (cf [12]).

A rational spectral density gives rise to a purely nondeterministic process and a classical result of Rosanov [22] asserts that the generating function is also rational. This can be seen immediately from the third part of the theorem in view of the correspondence between finite rank Hankel operators and rational symbol functions. Similarly, if there exists a scalar function θ in H^{∞} with $f\theta$ in H^{∞} then $H_f T_\theta = H_{f\theta} = 0$, and so $H_{h^*\theta} = 0$ which means that $h^*\theta$ is also in H^{∞} (cf [23, Theorem 3.1]).

If f is an invertible operator in L^{∞} then the operator C_f is asymptotically vanishing because the operator T_f is invertible and $H_g z^n$ converges to zero in the strong operator topology for every symbol operator g. In this case formula (2.4), combined with the theory of Hankel and Toeplitz operators, leads to the very precise inheritance of structural properties. For example H_{h^*} belongs to a given von Neumann-Schatten class $C_p(H)$, or Schatten-Lorentz class, precisely when H_f does. In particular, by the results of Peller [17], h^* belongs to the vectorial Besov class $B_p^{1/p}(C_p(K))$ precisely when f does. Similarly there is inheritance for matricial function spaces that are defined in terms of the singular numbers of Hankel operators (or equivalently, in terms of rational approximation) such as the so called R-spaces ([18, Chapter 3]). Also, if the invertible matricial density function f is given by a matrix of functions of vanishing mean oscillation, then the same holds true for h since 45 such functions correspond to compact vectorial Hankel operators ([16], [24]). We also observe from standard local techniques that if the matrix entries for f are of vanishing mean oscillation on a given open arc, then h will inherit this property. Of course finer localisation methods, such as that expressed in [6], lead to finer inheritance.

In view of part (iii) of the theorem, a sufficient condition for regularity is that the operator C_f be compact,or lie in a given von Neumann-Schatten class $\mathcal{C}_p$. If f is invertible then $\mathcal{C}_p$-membership coincides with the notion of $\mathcal{C}_{2p}$-regularity, characterised by Peller and Hruscev [18]. But in general the condition expresses a weaker concept, and it is not clear to the author how this type of regularity may be otherwise characterised.

3. The proof of the theorem

We start with some general constructions for positive operator matrices. The first lemma embodies an important idea of E.C. Lance [9] (see also [2], [25]) and is the foundation stone of the approach. For the sake of completeness we give full details of all proofs.

LEMMA 1 Let H be a complex Hilbert space with orthogonal decomposition $H = H_1 \oplus H_2$ and let C be a positive operator on H. Then there exists a unique positive operator C_1 whose restriction operator $C_1|H_1$ agrees with $C|H_1$ and is minimal with respect to this property in the sense that

$$C_2 \geq 0,\ C_2|H_1 = C|H_1 \Rightarrow C_1 \leq C_2 .$$

Furthermore if C is represented by the operator matrix

$$\begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$$

then C_1 is represented by the operator matrix

$$\begin{bmatrix} a & b \\ b^* & c_1 \end{bmatrix}$$

where c_1 is the strong operator topology limit of the increasing sequence $b^*(a+n^{-1}I_1)^{-1}b$.

<u>Proof.</u> First recall that if a is an invertible operator on H_1 then the operator

$$C = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$$

is positive if and only if $c \geq b^*a^{-1}b$. Indeed the operator

$$A = \begin{bmatrix} a^{\frac{1}{2}} & a^{-\frac{1}{2}}b \\ 0 & I_2 \end{bmatrix}$$

is invertible and

$$C = A^* \begin{bmatrix} I_1 & 0 \\ 0 & c-b^*a^{-1}b \end{bmatrix} A \quad .$$

From this principle it follows that the increasing sequence $b^*(a+n^{-1}I_1)^{-1}b$ is dominated by the decreasing sequence $c + n^{-1}I_1$ and so converges in the strong operator topology to an operator $c_1 \leq c$. Thus the operator C_1 is positive and satisfies the required minimality condition.

We call the operator C_1 the *H_1-minimal part* of C and if $C = C_1$ we say that C is *H_1-minimal.* Note that the operator c_1 can be expressed as d^*d where d is the bounded operator $a^{-\frac{1}{2}}b$. Consequently

$$C = \begin{bmatrix} a^{\frac{1}{2}} & 0 \\ d^* & 0 \end{bmatrix} \begin{bmatrix} a^{\frac{1}{2}} & d \\ 0 & 0 \end{bmatrix} \quad .$$

Here d is an operator satisfying $d = Ed$ where E is the range projection of $a^{\frac{1}{2}}$. In fact if a 2×2 operator matrix admits such a factorisation then it is $\mathcal{H}_1$-minimal.

Now assume that the Hilbert space $\mathcal{H}$ has the finite orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots \oplus \mathcal{H}_n$$

an d recursively define the positive operators $C_1,\dots,C_n$. Let C_1 be the $\mathcal{H}_1$-minimal art of C, and, given $C_1,\dots,C_k$, where $1<k<n$, let C_{k+1} be the $\mathcal{M}_k$-minimal part of $C-(C_1+\dots+C_k)$, where $\mathcal{M}_k = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_k$. Also let $C_n = C-(C_1+\dots+C_{n-1})$ so that we arrive at the decomposition

$$C = C_1 + C_2 + \dots + C_n$$

which we call the *Cholesky decomposition* of C with respect to the given decomposition of $\mathcal{H}$. The next lemma expresses the convenient fact that this decomposition may be obtained through any reasonable recursive procedure.

<u>LEMMA 2.</u> For $1 \leq k < n$ the operator $C_1+\dots+C_k$ is the $\mathcal{M}_k$-minimal part of C.

<u>Proof.</u> Suppose first that $k=2$ and D is the $\mathcal{M}_2$-minimal part of the positive operator C. Let D_1 be the $\mathcal{H}_1$-minimal part of D and write $D = D_1 + D_2$. Since $C_1|\mathcal{H}_1 = D_1|\mathcal{H}_1$ it follows from Lemma 1 that $C_1 = D_1$. Also we have $D|\mathcal{M}_2 = (C_1 + C_2)|\mathcal{M}_2$ and so, by minimality, $D_1 + D_2 \leq C_1 + C_2$ and hence $D_2 \leq C_2$. But $D|\mathcal{M}_2 = (C_1 + C_2)|\mathcal{M}_2$ and so by the minimality of C_2 we have $C_2 \leq D_2$ and hence $C_2 = D_2$. The lemma is true for n=2 and the general case follows by induction with this special case.

<u>LEMMA 3</u> Let $\mathcal{H}$ be a complex Hilbert space with orthogonal decomposition

$$\mathcal{H} = \bigoplus_{-\infty<k<+\infty} \mathcal{H}_k$$

and let

$$\mathcal{M}_n = \bigoplus_{-\infty<k\leq n} \mathcal{H}_k \quad .$$

If C is a positive operator on $\mathcal{H}$ then there exists a unique representation

$$C = C_{-\infty} + \sum_{-\infty}^{+\infty} C_k$$

where $C_{-\infty}$ and C_k are positive operators, such that the series converges in the strong operator topology, and such that the operator

$$C_{-\infty} + \sum_{-\infty}^{n} C_k$$

is the $\mathcal{M}_n$-minimal part of C.

<u>Proof.</u> In view of Lemma 2 there is no notational ambiguity in writing

$$C = C^{(n)} + C_{-n} + C_{-n+1}+\ldots+C_{n-1} + C_n + R^{(n)}$$

for the Cholesky decomposition of the positive operator C with respect to the orthogonal decomposition

$$\mathcal{H} = \mathcal{M}_{-n-1} \oplus \mathcal{H}_{-n} \oplus \ldots \oplus \mathcal{H}_n \oplus \mathcal{N}_n \ .$$

Clearly the bounded sequence $R^{(n)}$ converges to zero in the strong operator topology. Also, by minimality, the sequence $C^{(n)}$ is decreasing and converges in the strong operator topology to a positive operator $C_{-\infty}$. The final assertions of the lemma follows from Lemma 2.

More general decompositions than that of Lemma 3 have been obtained in [20]. The part of this decomposition represented by the series is, in a sense, the factorisable part of C. Indeed we have, in view of our earlier remarks,

$$C_k = A_k^* A_k$$

where A_k has the form

$$A_k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_k^{\frac{1}{2}} & d_k \\ 0 & 0 & 0 \end{bmatrix}$$

with respect to $\mathcal{M}_{k-1} \oplus \mathcal{H}_k \oplus \mathcal{N}_k$, and so it follows that

$$\sum_{-\infty}^{\infty} C_k = \sum_{-\infty}^{\infty} A_k^* A_k$$

$$= \left[\sum_{-\infty}^{\infty} A_k\right]^* \left[\sum_{-\infty}^{\infty} A_k\right]$$

$$= A^* A$$

where A is the weak operator topology sum of the series ΣA_k and has upper triangular form with respect to the nest of subspaces $\mathcal{M}_k$, $k = 0, \pm 1, \ldots$.

We now return to the context and notation of the last section and apply this analysis to a positive operator $C = f$ in $\mathbb{L}^\infty$. This operator is represented by an infinite operator matrix with respect to the decomposition

$$\mathcal{H} = \bigoplus_{-\infty}^{\infty} \mathcal{K}$$

and possesses the Laurent form of constancy along diagonals. It follows that in the decomposition $f = C_{-\infty} + A^*A$ obtained above that A and $C_{-\infty}$ also have representing matrices of this form and we therefore write $h = A^*$, $g = C_{-\infty}$. Clearly h belongs to H^∞ and we obtain the important identities

$$T_{h^*}^{\;*} T_{h^*} = T_f - C_f \tag{3.1}$$

$$(QhQ)(Qh^*Q) = QfQ - QC_fQ\ . \tag{3.2}$$

The connection between the outerness of an operator h_1 in H^∞ and minimality lies in the following assertion. The operator h_1 is outer if and only if the operator $h_1(I-P)h^*{}_1$ is $\mathcal{H}_-$-minimal, where $\mathcal{H}_-$ denotes the range of I-P. In view of our earlier remarks this follows if we show that outerness is equivalent to the range projection of $P^\perp h_1{}^*P^\perp$ dominating that of $P^\perp h_1{}^*P$. But this is clear from the definition. We use this connection in the next lemma.

<u>LEMMA 4</u> The following conditions are equivalent for a positive operator f in L^∞.

(i) f admits a factorisation $f = hh^*$ with h an outer operator in H^∞.

(ii) $C_f = H_{h^*}{}^*H_{h^*}$ for some outer operator h in H^∞.

(iii) The operator C_f is asymptotically vanishing.

<u>Proof.</u> (i) => (ii). Note that $f = hh^* = PhPh^*P + h(I-P)h^*$. Since h is outer the operator $h(I-P)h^*$ is $\mathcal{H}_-$-minimal. From Lemma 1 it follows that $C_f = Ph(I-P)h^*P = H_{h^*}{}^*H_{h^*}$.

(ii) => (iii). Simply observe that $H_{h^*}z^n$ converges to zero in the strong operator topology.

(iii) => (i). Consider the decomposition $f = hh^* + g$ obtained above.

Then g is trivially zero if the $\mathcal{M}_n$-minimal part of f converges to zero in the strong operator topology as $n \to -\infty$. With respect to the decomposition

$$\mathcal{H} = \mathcal{M}_{n-1} \oplus \mathcal{M}_{n-1}^{\perp} = \left[\bigoplus_{-\infty}^{n-1} \mathcal{K}\right] \oplus \left[\bigoplus_{n}^{\infty} \mathcal{K}\right]$$

this minimal part has an operator matrix of the form

$$\begin{bmatrix} x_n & y_n \\ y_n^* & z^n C_f z^{-n} \end{bmatrix}$$

and so our hypothesis is equivalent to the condition $g = 0$. By the construction of h we see that $h(I-P)h^* = \sum_{-\infty}^{-1} C_k$ is $\mathcal{H}_-$-minimal, and hence h is outer.

<u>LEMMA 5.</u> Let f be a positive operator in L^∞ and let $f = hh^* + g$ be the decomposition obtained by the construction following Lemma 2. Then if h_1 is an outer operator in H^∞ such that $f \geq h_1 h_1^*$ then $hh^* \geq h_1 h_1^*$.

<u>Proof</u> Note that $g + h(I-P)h^*$ is the $\mathcal{H}_-$-minimal part of f, by our construction, and is thus dominated by $(f-h_1h_1^*) + h_1(I-P)h_1^*$. Since $z^{-n}h(I-P)h^*z^n$ converges to zero in the weak operator topology as $n\to\infty$ it follows that $g \leq f - h_1h_1^*$ as required.

The last lemma shows that the decomposition $f = hh^* + g$ is extremal. It remains only to show that this corresponds to the Wold-Zasuhin decomposition and that $(QhQ)(Qh^*Q)$ corresponds to the prediction-error matrix, in the case of finite dimensional $\mathcal{K}$.(For then the formulae (2.2) and (2.3) follow from (3.2) and (3.1) respectively). We do this by a well known argument with the Wold decompositions of shift invariant subspaces (see [27] and [26]).

Let $L^2{}_{\mathcal{K}}$ and $H^2{}_{\mathcal{K}}$ be the natural vector function space realisations of $\mathcal{H}$ 52 and H_+ respectively. Let f be a positive operator in L^∞ realised as a function f(z) on the circle $|z| = 1$ with values as operators on $\mathcal{K}$, ~~so that~~ and define the prediction-error matrix G(f) associated with the spectral density f(z) is the operator on $\mathcal{K}$ defined by

$$(G(f)a,a) = \inf_{p(0)=a} \int_0^{2\pi} p(z)^* f(z) p(z) \frac{d\theta}{2\pi} ,$$

where the infimum is taken over $\mathcal{K}$-valued analytic polynomials (with $\mathcal{K}$ realised as complex column vectors). The closed subspace $(\sqrt{f}H^2_{\mathcal{K}})^-$ is shift invariant and we have the Wold decompositions

$$\left[\sqrt{f}\, H^2_{\mathcal{K}}\right]^- = \left[\bigoplus_0^\infty z^n \mathcal{F}\right] \oplus \mathcal{N}, \quad \left[\sqrt{f}\, L^2_{\mathcal{K}}\right]^- = \left[\bigoplus_{-\infty}^\infty z^n \mathcal{F}\right] \oplus \mathcal{N},$$

where $\mathcal{N}$ is a reducing subspace for the shift and $\mathcal{F}$ is a wandering subspace with $\dim \mathcal{F} \leq \dim \mathcal{K}$. Let ϕ be a partial isometry that commutes with the shift and let G be a subspace of K so that dim G = dim F and ϕ canonically identifies $\bigoplus_{-\infty}^\infty z^n \mathcal{F}$ with $\bigoplus_{-\infty}^\infty z^n \mathcal{G}$. Let R_1 be the orthogonal projection onto $\bigoplus_{-\infty}^\infty z^n \mathcal{F}$ and let R_2 be the orthogonal projection onto $z\left[\sqrt{f}H^2_{\mathcal{K}}\right]^-$.

Observe that $h_1 = \phi R_1 \sqrt{f}$ is an outer operator in $\mathbf{H}^\infty$. Moreover, we have

$$(G(f)a,a) = \inf_{p(0)=a} \int_0^{2\pi} p(z)^* f(z) p(z) \frac{d\theta}{2\pi}$$

$$= \inf_{q(0)=0} \|\sqrt{f}(a-q)\|^2_{L^2_{\mathcal{K}}}$$

$$= \|(I-R_2)\sqrt{f}\, a\|^2_{L^2_{\mathcal{K}}}$$

$$= \|(I-R_2)R_1\sqrt{f}\, a\|^2_{L^2_{\mathcal{K}}}$$

$$= \|\Phi(I-R_2)R_1\sqrt{f}\, a\|^2_{L^2_{\mathcal{K}}}$$

$$= \|Qh_1 a\|^2$$

$$= (h_1(0)^* h_1(0)a,a)$$

$$= (G(h_1{}^* h_1)a,a).$$

The last equality follows from the outerness of h_1. Now a little argument shows that if g is the positive matrix function $f-h_1{}^*h_1$ then $G(g) = 0$ and g corresponds to a purely deterministic process. If h_2 is an outer operator in $\mathbf{H}^\infty$ such that $f \geq h_2{}^*h_2$ then $h_2 = X\sqrt{f}$, with X a contraction. Since h_2 belongs to $\mathbf{H}^\infty$ it follows that

$$X\mathcal{N} = X\bigcap_{n=0}^{\infty}[z^n(\sqrt{f}\,H^2_{\mathcal{K}})^-] \subset \bigcap_{n=0}^{\infty}[z^n(h_2H^2_{\mathcal{K}})^-] = \{0\}.$$

Hence $X = XR_1$, $h_2 = Yh_1$ with Y a contraction, and the decomposition $f=h^*{}_1h_1+g$ is extremal. Apply the construction above to $\tilde{f}$ where $\tilde{f}(z) = f(\bar{z})$ to obtain the extremal outer factorisation $\tilde{f} = h^*{}_1h_1 + g_1$ so that $f = \tilde{h}_1{}^*\tilde{h}_1 + \tilde{g}_1 = h_3h_3{}^* + \tilde{g}_1$ say, an extremal outer decomposition with $\tilde{g}_1$ a deterministic spectral density. By Lemma 5 this decomposition agrees with our construction (that is $g = \tilde{g}_1$) and since $G(f) = G(\tilde{h}_1{}^*\tilde{h}_1) = G(hh^*) = (QhQ)(QhQ)^*$, the proof is complete.

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(3.5) Outer factorisation and Hankel operators

Let K be a complex Hilbert space of dimension k $(1 \leq k \leq \infty)$ and let Φ be a positive operator in the commutant of the bilateral shift Z for the tensor product $H = \ell^2(\mathbb{Z}) \otimes K$. There has been much interest in the determination of when Φ admits an outer factorisation $\Phi = \Theta\Theta^*$, and in the connection between Φ and the essentially unique outer factor Θ. This interest stems from several sources; the classical origins in the Szego theorem that represents a positive function on the unit circle with integrable logarithm as the modulus of an appropriate analytic function, the factorisation of spectral density functions in multivariate prediction ([2], [5], [14], [15]), and in the connections with operator theory ([1], [9], [11], [13]). More generally it is known (Proposition 4.2 of [13]) that there always exists an essentially unique extremal outer decomposition $\Phi = \Theta\Theta^* + \Phi_d$ where $\Phi_d \geq 0$, Θ is an outer operator, and $\Theta\Theta^*$ is maximal with respect to the inequality $\Phi \geq \Theta\Theta^*$. This decomposition is of particular significance for the prediction theory of nonregular multivariate stationary stochastic processes ([13, p.224], [12], [10]).

A new constructive approach to the extremal outer decomposition was obtained recently by the second author ([9], [10]) through a study of minimal positivity and the linear decomposition of positive operators. Explicit limiting formulae were obtained for the outer factor Θ and the purely deterministic component Φ_d in terms of Hankel and Toeplitz operators associated with Φ. A feature of this characterisation is the factorisation of a positive operator as H^*H with H a Hankel operator.

In this note we characterise operators of the type $T + H^*H$, where T is a positive Toeplitz operator and H is a Hankel operator, and we relate the analysis to extremal outer decomposition. Moreover it is shown that the outer factor in the extremal decomposition of Φ is the limit as $t \to 0$, in the weak operator topology, of the outer factor Θ_t appearing in the factorisations

$$tI + \Phi = \Theta_t\Theta_t^*$$

of the invertible positive operator $tI + \Phi$ for $t > 0$. This result seems to be new, even in the context of finite dimensional K, and it enables the transference of known factorisation procedures for the regular case ($\Phi_d = 0$) to the general context.

1. Let V be a contraction and let W be a pure isometry of multiplicity k ($1 \leq k \leq \infty$) acting on the Hilbert spaces H_1, H_2 respectively. Let

$$C = T + H^*H \tag{1.1}$$

where T is a positive operator satisfying $V^*TV = T$ and H is an operator from H_1 to H_2 of Hankel type, satisfying $W^*H = HV$. The following conditions hold:

(i) $V^*CV \leq C$,

(ii) $\dim\{(C-V^*CV)H_2\} \leq k$.

THEOREM A. Let C be a positive operator on the Hilbert space H_1. Then C admits a factorisation of the form (1.1) if and only if the conditions (i) and (ii) hold. Moreover, C admits the factorisation

H^*H if and only if, in addition, $V^{*n}CV^n \to 0$ in the strong operator topology as $n \to \infty$.

Proof. Let C have the form (1.1) with $T = 0$. Then, since W is a unilateral shift the sequence $V^{*n}CV^n = H^*W^{*n}W^nH$ decreases to zero in the strong operator topology.

Assume now that C is a positive operator fulfilling conditions (i) and (ii), let T be the strong operator limit of the decreasing sequence $V^{*n}CV^n$, and let $C_1 = C - T$. Since $V^*TV = T$ the operator C_1 satisfies the conditions (i) and (ii) and $V^{*n}C_1V^n \to 0$ in the strong operator topology. Let $R = (C_1 - V^*C_1V)^{1/2}$. Then

$$\begin{aligned}\|C_1h\|^2 = (C_1h,h) &= \|Rh\|^2 + (C_1Vh,Vh)\\ &= \|Rh\|^2 + \|RVh\|^2 + (C_1V^2h,V^2h)\\ &= \sum_{n=0}^{k} \|RV^nh\|^2 + (C_1V^{k+1}h,V^{k+1}h)\\ &= \sum_{n=0}^{\infty} \|RV^nh\|^2 .\end{aligned}$$

Let W be the unilateral shift on $\ell^2(\mathbb{Z}_+) \otimes K$, where $K = (RH_1)^-$, and define the operator H from H_1 to $\ell^2(\mathbb{Z}_+) \otimes K$ by

$$Hh = (Rh, RVh, RV^2h, \ldots) \qquad (h \in H_1).$$

Then $\|Hh\|^2 = \|C_1^{1/2}h\|^2$, $C_1 = H^*H$, and $HV = W^*H$, as required.

Remarks 1. When $V = W^*$ the theorem provides a characterisation of the positive operators of the form TT^* where T commutes with the unilateral shift W (cf. [13], Proposition 5.1.).

2. Hruscev and Peller have asked ([3], page 94, problem 2) for

a characterisation of the positive operators that are unitarily equivalent to the modulus $(H^*H)^{1/2}$ of a scalar Hankel operator. Here $V = W$ and $k = 1$. Theorem A allows us to reformulate the problem; Determine the positive operators D for which there exists a scalar unilateral shift V such that $D^2 - V^*D^2V$ is a positive operator of rank 1 and $V^{*n}D^2V^n \to 0$ in the strong operator topolgy. (For compact operators the second condition always holds and this restricted problem has almost been resolved (cf. [7], section 2).)

2. Returning to the context of the introduction we say that the factorisation $\Phi = \Theta\Theta^*$ is the outer factorisation of the positive operator Φ if the following conditions are met; Θ is also an operator in the commutant of the shift Z and is analytic in the sense that Θ leaves invariant the subspace

$$H_+ = \ell^2(\mathbb{Z}_+) \otimes K,$$

Θ is outer in the sense that

$$\{\mathrm{ran}\ \Theta|H_+\}^- = \{\mathrm{ran}\ \Theta\}^- \cap H_+,$$

and, finally, $P_K\Theta|K \geq 0$, where P_K is the orthogonal projection onto the subspace $\mathbb{C}e_0 \otimes K$. Here e_0 is the central basis element in the standard basis for $\ell^2(\mathbb{Z})$.

The outer factor is unique when the outer factorisation exits ([13]) and we shall see that it can be understood in terms of the Hankel and Toeplitz operator entries of the representing operator matrix for Φ with respect to the orthogonal decomposition $H = H_- \oplus H_+$:

$$\Phi := \begin{bmatrix} \tilde{T}_\Phi & H_\Phi \\ H^*_{\bar{\Phi}} & T_\Phi \end{bmatrix}$$

Here we have $T_\Phi = P\Phi|H_+$, $\tilde{T}_\Phi = P^\perp\Phi|H_-$, and $H_\Phi = P^\perp\Phi|H_+$, where P is the orthogonal projection onto H_+. It is well known that the Toeplitz operators T_Ψ, for Ψ in the commutant of Z, are precisely the solutions T to the operator equation $T^*_Z T T_Z = T$, and that the Hankel operators H_Φ are characterised by the operator equation $\tilde{T}_Z H = HT_Z$ ([6], [8]).

In general, when a positive operator matrix is given, say

$$C := \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}, \tag{2.1}$$

with respect to some arbitrary nontrivial orthogonal decomposition, then the limit of the sequence $b^*(n^{-1}+a)^{-1}b$ exists in the strong operator topology and determines a positive operator c_1 with $c_1 \leq c$. (See Lance [4] and Power [9]). In fact $c_1 = b^*a^{-1}b$ where the inverse indicated is the pseudo-inverse: $a^{-1}y = x$ when x is the unique element orthogonal to the kernel of a satisfying $ax = y$. For Φ as above we write

$$C_\Phi := H^*_\Phi \tilde{T}^{-1}_\Phi H_\Phi := \lim_{t\to 0} H^*_\Phi(tP^\perp+\tilde{T}_\Phi)^{-1}H_\Phi.$$

<u>LEMMA 1.</u> $T^*_Z C_\Phi T_Z \leq C_\Phi$ and $\dim\{\mathrm{ran}(C_\Phi - T^*_Z C_\Phi T_Z)\}^- \leq k$.

<u>Proof.</u> Suppose first that Φ is an invertible positive operator commuting with Z. Then through the Beurling-Lax-Halmos theorem applied to the simply invariant subspace $\Phi^{1/2}H_+$ we obtain $\Phi = \Theta\Theta^*$

where Θ and Θ^{-1} are analytic operators. Observing that $\tilde{T}_\Phi = \tilde{T}_\theta \tilde{T}_{\theta^*}$ we compute that

$$\begin{aligned} C_\Phi &= H_\Phi^* \tilde{T}_\Phi^{-1} H_\Phi \\ &= P\Phi P^\perp (\tilde{T}_\Theta \tilde{T}_{\Theta^*})^{-1} P^\perp \Phi P \\ &= P\Theta P^\perp \Theta^* P^\perp (\tilde{T}_{\Theta^*}^{-1} \tilde{T}_\Theta^{-1}) P^\perp \Theta P^\perp \Theta^* P \\ &= H_{\Theta^*}^* H_{\Theta^*} \end{aligned}$$

Since $\tilde{T}_Z H_\Psi = H_\Psi T_Z$, for all operator symbols Ψ, we have

$$\begin{aligned} C_\Phi - T_Z^* C_\Phi T_Z &= H_{\Theta^*}^* (I - \tilde{T}_Z^* \tilde{T}_Z) H_{\Theta^*} \\ &= H_{\Theta^*}^* Q H_{\Theta^*} \end{aligned}$$

where Q is the orthogonal projection onto the subsapce $Z^*(\mathbb{C}e_0 \otimes K)$.

To deduce the lemma in the general case first observe that if $\Phi_t = tI + \Phi$, for $t > 0$, then Φ_t is invertible, $H_{\Phi_t} = H_\Phi$, and C_Φ is the increasing limit in the strong operator topology of the operators C_{Φ_t}, as $t \to 0$. The lemma now follows from the computations above, and the observation that if $T_n \to T$ in the strong operator topology then rank $T \leq \liminf_{n\to\infty}$ rank T_n.

<u>COROLLARY 2.</u> The operator C_Φ admits a decomposition

$$C_\Phi = T_{\Phi_d} + H^*H \tag{2.2}$$

where Φ_d is a positive operator in the commutant of the bilateral shift Z, and H is a Hankel operator satisfying $\tilde{T}_Z H = H T_Z$.

This corollary is an immediate consequence of Theorem A and the lemma. The operator Φ_d is uniquely determined and we have labelled this symbol operator as Φ_d because it coincides with the deterministic summand Φ_d in the extremal outer decomposition of Φ. Indeed, one of the main results in [10] is the fact that the outer factor Θ in the extremal decomposition $\Phi = \Theta\Theta^* + \Phi_d$ can be defined as the unique (outer) operator Θ with $P_K\Theta|K \geq 0$ satisfying

$$T_\Theta T_{\Theta^*} = T_\Phi - C_\Phi. \qquad (2.3)$$

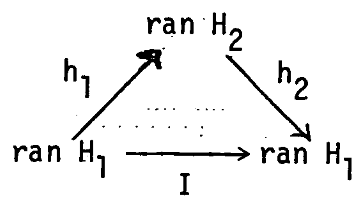
This shows that

$$\begin{aligned} C_\Phi &= T_{\Phi_d} + T_{\Theta\Theta^*} - T_\Theta T_{\Theta^*} \\ &= T_{\Phi_d} + H^*_{\Theta^*} H_{\Theta^*}, \end{aligned}$$

which justifies this notation Φ_d in Corollary 2, and indicates that the Hankel operator H, which is essentially uniquely determined in view of Proposition 3 below, is associated with an outer operator symbol. This fact indicates that the operator C_Φ has structural features in addition to the basic properties (i) and (ii) expressed in Lemma 1.

PROPOSITION 3. Let H_1, H_2 be Hankel operators satisfying $H_k T_Z = \tilde{T}_Z H_k$, $k = 1,2$. If $H_1^* H_1 = H_2^* H_2$ then $H_1 = XH_2$ where X is a partial isometry on $\ell^2(\mathbb{Z}) \otimes K$ of the form $I \otimes X_1$.

Proof. Assume that $H_1^* H_1 = H_2^* H_2$ so that there exists a contraction X, that is isometric on the range space ran H_1 of H_1, such that $H_2 = XH_1$. Observe that $\tilde{T}_Z X H_1 = \tilde{T}_Z H_2 = H_2 T_2 = XH_1 T_Z = X\tilde{T}_Z H_2$ so that $(\tilde{T}_Z X - X\tilde{T}_Z)|\text{ran } H_1 = 0$. By the Sz-Nagy Foias lifting theorem there exists an analytic operator h_1 in the commutant of Z which is contractive, such that $X|\text{ran } H_1 = T_{h_1}|\text{ran } H_1$. Similarly obtain a contractive analytic operator h_2 such that $H_1 = \tilde{T}_{h_2} H_2$. This means we have the following commutative diagram



However, ran H_1 and ran H_2 are invariant subspaces for the backward shift $\tilde{T}_z$ and hence have the forms $H_+^{\perp} \ominus v_1^* H_+^{\perp}$, $H_+^{\perp} \ominus v_2^* H_+^{\perp}$ respectively, for some inner operators (analytic partial isometries in the commutant of Z) v_1 and v_2. From the usual divisibility properties of inner operators we conclude that $h_1|\text{ran } H_1$ has the form $I \otimes X_1|\text{ran } H_1$ where X_1 is an operator on K, as desired.

3. <u>The convergence of outer factors</u>

<u>THEOREM C.</u> Let Φ be a positive operator in the commutant of the bilateral shift Z on the Hilbert space $\ell^2(\mathbb{Z}) \otimes K$, with the unique extremal outer decomposition $\Phi = \Phi_d + \Theta\Theta^*$. Let the operator $tI + \Phi$, for $t > 0$, have the unique outer factorisations $\Theta_t \Theta_t^*$. Then

$$\Theta = \underset{t \to 0}{\text{w-lim}}\, \Theta_t$$

where the limit is taken in the weak operator topology.

<u>Proof.</u> The proof rests on the essentially contructive formula for Θ given in formula (2.3). This formula shows that

$$T_\Theta T_{\Theta^*} = \underset{t\to 0}{\text{s-lim}}\,(T_{t+\Phi} - H_\Phi^* T_{t+\Phi}^{-1} H_\Phi)$$

$$= \underset{t\to 0}{\text{s-lim}}\,(T_{t+\Phi} - H_{t+\Phi}^* T_{t+\Phi}^{-1} H_{t+\Phi})$$

$$= \underset{t\to 0}{\text{s-lim}}\,(T_{\Theta_t} T_{\Theta_t^*}) \;.$$

With respect to the decomposition $H_+ = K \oplus K^\perp$ write

$$T_{\Theta^*} = \begin{bmatrix} A^{1/2} & A^{-1/2}B \\ 0 & * \end{bmatrix} \qquad T_{\Theta_t^*} = \begin{bmatrix} A_t^{1/2} & A_t^{-1/2}B_t \\ 0 & * \end{bmatrix}$$

(using the generalised inverse $A^{-1/2}$) and observe that $A_t \to A$ and $B_t \to B$ in the strong operator topology. Since $\{\Theta_t\colon 0 \le t \le 1\}$ is a norm bounded set and since

$$P_K \Theta_t^* = [0 \;\; A_t^{1/2} \;\; A_t^{-1/2}B_t],$$

with respect to $H_+^\perp \oplus K \oplus K^\perp$, the theorem will follow if it is shown that $A_t^{-1/2}B_t \to A^{-1/2}B$ in the weak operator topology as $t \to 0$.

To this end let L be a limit in the weak operator topology of some subnet $A_\alpha^{-1/2}B_\alpha$. For f in H_+ and g in K we have

$$\begin{aligned}(Lf, A^{1/2}g) &= \lim_\alpha (A_\alpha^{-1/2}B_\alpha f, A^{1/2}g)\\ &= \lim_\alpha (A_\alpha^{-1/2}B_\alpha f, A_\alpha^{1/2}g)\\ &= \lim_\alpha (B_\alpha f, g)\\ &= (Bf, g)\\ &= (A^{-1/2}Bf, A^{1/2}g).\end{aligned}$$

It follows that if P_A is the range projection of A, then all limit points of $\{P_A A_t^{-1/2} B_t : 0 \leq t \leq 1\}$, as $t \to 0$, coincide with $A^{-1/2}B$ and hence

$$A^{-1/2}B = \underset{t\to 0}{\text{w-lim}}\, P_A A_t^{-1/2} B_t.$$

In the special case when A is injective (and in particular in the multivariate context when A has full rank) the proof is now complete. But in general we need the following additional argument.

Using the identities $P_K = P - T_Z T_Z^*$ and $T_\Theta T_Z T_Z^* T_\Theta = T_Z T_\Theta T_\Theta^* T_Z$, observe that

$$T_\Theta P_K T_\Theta^* = \underset{t\to 0}{\text{s-lim}}\, T_{\Theta_t} P_K T_{\Theta_t}^*$$

In particular, examining the operator matrix entries we have

$$BA^{-1}B = \underset{t\to 0}{\text{s-lim}}\, B_t^* A_t^{-1} B_t.$$

Now introduce the notation $X_t = A_t^{-1/2} B_t$, $X = A^{-1/2}B$ so that $P_A X_t \to X$ (wot) and $|X_t| \to |X|$ (sot) as $t \to 0$. We now show that these conditions imply that $X_t \to X$ (wot) as $t \to 0$, as required.

Let $X_t = U_t|X_t|$ and $X = U|X|$ be the polar decompositions, and let P_u be the range projection of $|X|$. Then $P_A U_t |X_t| \to U|X|$, (wot), and so, since $|X_t| \to |X|$ (sot), we have $P_A U_t P_u |X_t| \to P_A U P_u |X|$ (wot) as $t \to 0$. Hence

$$\underset{t\to 0}{\text{w-lim}}\ P_A U_t P_u = P_A U P_u = U.$$

Let M be any limit point of the set $\{P_A^{\perp} U_t P_u : 0 \le t \le 1\}$ as $t \to 0$. Then, since $U + M$ is a contraction, $MP_u^{\perp} = 0$, and U is a partial isometry, it follows that $M = 0$. Hence $U_t P_u \to UP_u$ (wot), and $X_t \to X$ (wot) as $t \to 0$, completing the proof.

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CHAPTER 4 DENSITY, SEMIDISCRETENESS, AND DILATION THEORY

We now examine various density properties and finite dimensional structure in nest algebras, culminating in the dilation theorem for σ-weakly continuous contractive representations of a nest algebra. This may be viewed as a noncommutative analogue of Sz-Nagy's theorem that contractions possess unitary dilations. In the following chapter we will see a continuing analogy between the representation theory of a nest algebra and that for the complex polynomial algebra $P(\mathbb{D})$ for the disc.

In section 4.3 of this chapter we discuss analogous finite dimensional structure for various reflexive algebras with commutative subspace lattice.

(4.1) The Erdos density Theorem.

There is no natural analogue of the Kaplansky density theorem for nonself-adjoint operator algebras and so special arguments are often needed to show that the unit ball of a dense subalgebra is dense in the unit ball of the full algebra. The Erdos density theorem ((4.1.1) below) can be obtained in various ways. It is a consequence of the more general result Corollary 2.7 of section 2.6, it may be obtained by the direct construction of an approximate identity of finite rank operator, and it is a consequence of duality arguments combined with Lemma (2.3.2) (ii). We give the third proof below. In the next section we will obtain a refinement of the second approach to show that there exists certain 'good' subalgebras $A_n \subset A$, which are finite dimensional and consist of finite rank operators, such that the union of the unit balls of the subalgebras A_n is σ-weakly dense. This approach entails a close examination of the spectral representation of the projection nest E of the nest algebra A.

(4.1.1) THEOREM (Erdos). The finite rank operators in the unit ball of a nest algebra are dense in the σ-weak topology.

Proof. A typical rank one operator in the nest algebra $A = \mathrm{Alg}\,E$ has the form $EX(I-E_-)$ where E lies in E and X is a rank one operator. Thus a trace class operator A lies in the annihilator of the closed linear span, R say, of the rank one operators of A if and only if

$$0 = \mathrm{tr}(AEX(I-E_-)) = \mathrm{tr}((I-E_-)AEX)$$

for all rank one operators X and E in E. It follows that this annihilator is equal to A_1^+ where

$$A_1^+ = \{A \in C_1 : \ (I-E_-)AE = 0 \text{ for all } E \text{ in } E\}.$$

We now compute the annihilator of A_1^+ in $L(H)$. By Lemma 2.3.2 $(A_1^+)^\perp$ agrees with the annihilator of the collection of rank one operators in A_1^+. Each such operator has the form $EX(I-E)$, with $X \in L(H)$ of rank one and $E \in E$. Since $\mathrm{tr}(AEX(I-E)) = \mathrm{tr}(I-E)AEX)$ it follows that $(A_1^+)^\perp = A$.

Thus we have the following natural identification of the dual spaces of R and C_1/A_1^+.

$$R' = C_1/A_1^+ ,$$
$$(C_1/A_1^+)' = A.$$

Moreover the weak star topology on A coincides with the σ-weak topology. By Goldstine's theorem the unit ball of R is weak star dense in the unit ball of the second dual R'', and so we are done.

The density theorem has many uses. For example it provides a simple proof that the linear span $A + K(H)$, where $K(H)$ is the ideal of compact operators, is norm closed. It is also used in the characterization of the σ-weakly closed ideals of a nest algebra (cf. Erdos and Power [9]). However in the study of representation of a nest algebra we need the more refined density properties of semidiscreteness discussed in the following section.

<u>References</u>: Erdos [7], Power [29], [31].

In this note we show that a contractive σ-weakly continuous Hilbert space representation of a nest algebra admits a σ-weakly continuous dilation to the containing algebra of all operators. Our method is to establish first the complete contractivity of contractive representations through a semi-discreteness property for nest algebras relative to finite dimensional nest algebras (Theorem 2.1). This is obtained by an examination of the order type, spectral type and multiplicity of the nest, and by the construction of subalgebras that are completely isometric copies of finite dimensional nest algebras, with good approximation properties. With complete contractivity at hand, the desired dilation follows from Arveson's dilation theorem and auxiliary arguments.

We need to know that contractive representations of finite dimensional nest algebras are completely contractive, a fact first obtained by McAsey and Muhly [5]. We obtain this by the explicit construction of star dilations for contractive representations of finite dimensional nest algebras, and without recourse to Arveson's theorem.

An alternative approach to the dilation theorem can be found in Paulsen and Power [7], [8] based on the weaker notion of semi-discreteness relative to modules of M_n for the diagonal subalgebra, and on the dilation theory of contractive module representations. This alternative approach leads to generalizations of the results here to certain reflexive operator algebras with commutative invariant subspace lattice, and to the analysis of bounded representations. We also remark that the methods of this paper can be used in the dilation theory of commuting representations of nest algebras [8].

In the first section we constructively dilate contractive representations of finite dimensional nest algebras. In the second section we establish the semi-discreteness of nest algebras, and in the last section we obtain the dilation theorem.

Recall that a nest algebra is an algebra $\mathcal{A}$ of operators on a complex Hilbert space R such that each operator in $\mathcal{A}$ leaves invariant all subspaces in a preassigned nest of subspaces. We always assume R to be separable, and if R is finite dimensional we refer to $\mathcal{A}$ as a **finite dimensional nest algebra.** Such algebras are completely isometrically isomorphic to block upper triangular subalgebras of the complex matrix algebras M_n, $n = 1,2, \ldots$.

Let S be a subspace of $L(R)$, the algebra of all operators on R, and let $\rho : S \to L(H)$

be a linear representation of S as operators on the Hilbert space H. Write ρ_n for the induced map between the naturally normed spaces $M_n(S)$ and $M_n(L(H))$. We say that ρ is completely contractive (resp. completely positive, resp. completely bounded) if the maps ρ_n are contractive (resp. positive, resp. bounded) for n = 1,2, ...

The paper is self-contained except for the proof of Arveson's dilation theorem which we now state. General facts concerning completely bounded maps and dilations can be found in [6]. Basic properties of nest algebras are discussed in [9].

If $\mathcal{A}$ is a subalgebra of C^*-algebra $\mathcal{B}$ and if $\rho : \mathcal{A} \to L(H)$ is a representation then we say that $\pi : \mathcal{B} \to L(K)$ is a $\mathcal{B}$-dilation of ρ if π is a *-representation of $\mathcal{B}$ on a Hilbert space $K \supset H$ such that $\rho(A) = P_H \pi(A)|_H$ for all A in $\mathcal{A}$.

THEOREM (Arveson [1]). Let $\mathcal{A}$ be a subalgebra of the C^*-algebra $\mathcal{B}$ and let $\rho : \mathcal{A} \to L(H)$ be a unital homomorphism. Then the following conditions are equivalent:

(i) ρ has a $\mathcal{B}$-dilation,

(ii) ρ is completely contractive,

(iii) the induced map $\tilde{\rho} : \mathcal{A} + \mathcal{A}^* \to L(H)$, defined by $\tilde{\rho}(A_1 + A_2^*) = \rho(A_1) + \rho(A_2)^*$, is completely positive.

Recall that the dilation of the completely contractive representation ρ is achieved by first extending ρ to a completely contractive linear map from $\mathcal{B}$ to $L(H)$, and then dilating this map to a star homomorphism by means of Stinespring's dilation theorem. In particular, if $\mathcal{B}$ and H are separable then the dilation space K is separable.

1. Representation of finite dimensional nest algebras.

The contractive representations of a finite dimensional nest algebra have a simple and explicit characterization. The necessary and sufficient condition for contractivity is that the images of the matrix units are contractions. In fact we shall obtain an explicit dilation to a star representation of the enveloping matrix algebra from which it can be seen that contractive representations are completely contractive. We prove these facts and related observations in this section.

PROPOSITION 1.1. Let $\mathcal{A}$ be a finite dimensional nest algebra with enveloping matrix algebra $\mathcal{B}$, and let ρ be a representation of $\mathcal{A}$ on the Hilbert space H such that $\| \rho(e_{i,j}) \| \leq 1$ for each matrix unit $e_{i,j} \in \mathcal{A}$. Then there exists a Hilbert space K containing H as a subspace, and a star representation π of $\mathcal{B}$ on K, such that

$$\rho(A) = P_H \pi(A) \mid_H$$

for all A in $\mathcal{A}$.

Proof. Since $\rho(1)$ is an orthogonal projection we may assume, without loss of generality, that ρ is unital. Consider first the case of the $n \times n$ upper triangular matrix subalgebra $\mathcal{A}$ of the matrix algebra $\mathcal{B} = M_n$, so that $\mathcal{A}$ is spanned by the matrix units $e_{i,j}$, for $1 \leq i \leq j \leq n$. For each i the operator $\rho(e_{i,i})$ is a self-adjoint projection, E_i say, with range space H_i and $H = H_1 \oplus ... \oplus H_n$. Since ρ is a homomorphism the contraction $X_{ij} = \rho(e_{i,j})$ has range contained in H_i and kernel containing $(H_j)^{\perp}$, for $1 \leq i \leq j \leq n$. Let $T_{ij} = E_i X_{ij} \mid_{H_j}$, for $1 \leq i \leq j \leq n$, and we have $\rho((a_{ij})) = (a_{ij} T_{ij})$ as an operater matrix on $H_1 \oplus ... \oplus H_n$, for (a_{ij}) in $\mathcal{A}$, and $T_{ij} = T_{i,i+1} ... T_{j-1,j}$. Clearly the operators $T_i = T_{i,i+1}, i = 1, ..., n-1$ determine the representation. Conversely any family $\{T_i\} = \{T_1, ..., T_{n-1}\}$ of contractions $T_i : H_{i+1} \to H_i$ gives rise to a representation $\rho_{\{T_j\}}$ of $\mathcal{A}$, with $\| \rho(e_{ij}) \| \leq 1$.

We now construct a dilation $\rho_{\{V_i\}}$ for $\rho_{\{T_j\}}$ with $V_1, ..., V_{n-1}$ isometries. To simplify notation we restrict to the case where the dimension of H_i is constant and these subspaces are identified. If this does not already hold we can dilate ρ in a trivial way to a representation which does have this property. Let $K_i = R \oplus R \oplus ...$ with $R = H_i$ identified with the first summand. Let V_i be the operator on K_i which is the isometric dilation of T_i given by

$$V_i(r_1, r_2, ...) = (T_i r_1, D_i r_1, r_2, ...),$$

where $D_i = (I - T_i^* T_i)^{\frac{1}{2}}$. Observe that for $i < j$,

$$T_i T_{i+1} ... T_j = P_R (V_i V_{i+1} ... V_j) \mid_R .$$

Hence if $\rho_1 = \rho_{\{V_i\}}$ is the representation of $\mathcal{A}$ on $K = K_1 \oplus ... \oplus K_1$, n times, induced by $\{V_i\} = \{V_1, ..., V_{n-1}\}$, we have $\rho(A) = P_H \rho_1(A) \mid_H$, for A in $\mathcal{A}$.

Now consider the isometry $W = I \oplus V_1 \oplus V_1V_2 \oplus \ldots \oplus V_1 \ldots V_{n-1}$ on K, and the $*$-representation π of M_n on K given by $\pi((b_{ij})) = (b_{ij}I_{K_1})$. Observe that $\rho_1(A) = V^*\pi(A)V$ for A in $\mathcal{A}$. Thus after identifying H with VH, we have that $\rho(A) = P_H\pi(A)|_H$ for A in $\mathcal{A}$.

It remains to consider the case of a general finite dimensional nest algebra $\mathcal{A}$ associated with a subnest of the canonical projection nest in M_n. The proof above can be modified easily. On the other hand we can use the following useful general principle ([6, Proposition 2.12]).

Let M be a subspace of a unital C^*-algebra which contains the identity and let $\phi : M \to L(H)$ be a unital contraction. Then ϕ extends uniquely to a positive map $\tilde{\phi} : M + M^* \to L(H)$ with $\tilde{\phi}$ given by $\tilde{\phi}(a + b^*) = \phi(a) + \phi(b)^*$ for a,b in M.

In our context the representation ρ of $\mathcal{A}$ induces a representation ρ_u of the subalgebra, $\mathcal{A}_u$ of upper triangular $n \times n$ matrices. Moreover the representation $\rho_u(A) = P_H\pi(A)|_H$ leads to the positive extension map $\psi : M_n \to L(K)$ where $\psi(B) = P_H\pi(B)|_H$ for B in $M_n = \mathcal{A}_u + (\mathcal{A}_u)^*$. But, it must be that $\tilde{\rho} = \psi$ since they agree on $\mathcal{A}_u$. In particular $\rho(A) = P_H\pi(A)|_H$ for operators A in $\mathcal{A}$ as required. ■

Corollary 1.2. Let $\mathcal{A}$ be a finite dimensional nest algebra with enveloping matrix algebra M_n, and let ρ be a representation of $\mathcal{A}$ with $\| \rho(e_{ij}) \| \leq 1$ for each matrix unit e_{ij} in $\mathcal{A}$. Then ρ is completely contractive.

Remark 1.3. Let (π, K) be a unital star representation of the matrix algebra M_n on the Hilbert space K, and let M be a subspace of K which is semi-invariant for $\pi(\mathcal{A})$ where $\mathcal{A}$ is a finite dimensional nest algebra contained in M_n. Then the compression map $A \to P_M(\pi(A))|_M$ determines a representation (ρ, M) of $\mathcal{A}$. Such representations are called sub-star representations by Ball and Gohberg [2]. From Proposition 1.1 we see that every contractive representation is of this form.

1.4. The complete contractivity of representations of finite dimensional nest algebras can also be observed in the following way. Once more it will be enough to consider the algebra $\mathcal{A}$ of upper triangular $n \times n$ matrices and a unital contractive representation (ρ, H). Observe that the induced positive map $\tilde{\rho}$ of M_n is an inflated Schur product map in the following sense. There is an $n \times n$ operator matrix $T = (T_{ij})$ such that $\tilde{\rho}((x_{ij})) = \phi_T((x_{ij}))$

where $\phi_T((x_{ij})) = (x_{ij}T_{ij})$. Here if e_{ij}^* is in $\mathcal{A}$, we set $T_{ij} = T_{ij}^*$. We want to show that the map $\rho^{(k)} : M_k(\mathcal{A}) \to M_k(L(H))$ is contractive for every k. Equivalently we must show that $\tilde{\rho}^{(k)}$ is positive for every k. However $\tilde{\rho}^{(k)}$ is the inflated Schur product map on $M_{kn} = M_k(M_n)$ associated with the operator matrix $T^{(k)}$, the $k \times k$ matrix all of whose entries are T. Since $\tilde{\rho}$ is a positive map, T is a positive operator matrix and therefore so is $T^{(k)}$. It is sufficient then to see that a positive $r \times r$ operator matrix $S = (S_{ij})$, determines a positive mapping ϕ_S of M_r. Clearly $\phi_S(C) \geq 0$ if $C \geq 0$ and C has rank one. Since every positive operator in M_r is a positive linear combination of rank one operators we are done.

1.5. If ρ is a homomorphism from the upper triangular matrix algebra $\mathcal{A}$ of M_n into $L(H)$ then ρ is similar to a contractive representation. In fact we can first choose an invertible operator S_1 in $L(H)$ so that $\rho_1(\bullet) = S_1^{-1}\rho(\bullet)S_1$ determines a contractive (unital star) representation when restricted to the diagonal algebra $\mathcal{A} \cap \mathcal{A}^*$. A standard averaging argument achieves this (See [6, p. 127] for example). The representation ρ_1 is determined by the operators $X_i = \rho_1(e_{i,i+1})$. Let S_2 be the diagonal operator $\mathrm{diag}\{1, t, ..., t^{n-1}\}$ and we have $S_2^{-1}\rho_1(e_{i,i+1})S_2 = tX_i$. Thus $(S_1S_2)^{-1}\rho(\bullet)S_1S_2$ is a contractive representation if t is sufficiently small.

1.6. The methods of this section also apply directly to certain nest algebras associated with a projection nest which is of order ω. However to treat the general case we need to establish the semi-discreteness property in the next section.

2. Semi-discreteness of nest algebras.

Recall that a von Neumann algebra M is said to be semi-discrete if there exists nets of σ-weakly continuous completely positive maps $\phi_\lambda : M \to M_{n_\lambda}, \psi_\lambda : M_{n_\lambda} \to M$ such that $\psi_\lambda \circ \varphi_\lambda(X) \to X$ σ-weakly, for all X in M. The main result of this section is the following theorem which expresses an analogous property for nest algebras. In the case of a purely atomic nest E it is easy to obtain an elementary direct proof. However the general case requires an examination of the measure type and the spectral multiplicity of the projection nest.

THEOREM 2.1. Let $\mathcal{A}$ be the nest algebra associated with the nest of projections $\mathcal{E}$ acting on a separable Hilbert space H. Then there exists,

(i) a sequence $\mathcal{A}_n$ of finite dimensional nest algebras,

(ii) σ-weakly continuous completely contractive maps $\phi_n : \mathcal{A} \to \mathcal{A}_n$,

(iii) σ-weakly continuous completely isometric homomorphisms $\psi_n : \mathcal{A}_n \to \mathcal{A}$,

such that $\psi_n \circ \varphi_n(A) \to A$ σ-weakly for all A in $\mathcal{A}$.

We shall see from the proof below that ϕ_n and ψ_n are restrictions of completely positive mappings $\tilde{\phi}_n : L(H) \to \mathcal{B}_n, \tilde{\psi}_n : \mathcal{B}_n \to L(H)$ associated with the finite dimensional enveloping C^*-algebras $\mathcal{B}_n$ containing the algebras $\mathcal{A}_n$, and where $\tilde{\phi}_n, \tilde{\psi}_n$ have the properties required to show the semi-discreteness of $L(H)$. Thus, amongst the many pairs of sequences of maps which establish the semi-discreteness of $L(H)$, we find maps which respect upper triangularity.

Let $L^2(\mu)$ denote the Hilbert space of square integrable functions associated with a finite positive Borel measure μ on the unit interval [0,1]. For $0 \leq t \leq 1$ let M_t (respectively M_{t-}) be the operator of multiplication by the characteristic function of the interval [0,t] (respectively [0,t)). As usual we write $\mu_k >> \mu_{k+1}$ when the measure μ_{k+1} is absolutely continuous with respect to μ_k.

The following spectral theorem for projection nests acting on a separable Hilbert space is well known (See also [4]). For completeness we give a proof.

PROPOSITION 2.2. Let $\mathcal{E}$ be a complete projection nest on a separable Hilbert space. Then there exists a sequence $\mu_1 >> \mu_2 >> \ldots$ of regular Borel measures on [0,1] such that $\mathcal{E}$ is unitarily equivalent to the standard projection nest on $L^2(\mu_1) \oplus L^2(\mu_2) \oplus \ldots$ consisting of the projections $E_t = M_t \oplus M_t \oplus \ldots$ and $E_{t-} = M_{t-} \oplus M_{t-} \oplus \ldots$ for $0 \leq t \leq 1$.

Proof. Suppose first that x is a unit cyclic vector for the nest $\mathcal{E}$ on H, and let α be the left continuous function from $\mathcal{E}$, with the strong operator topology, to [0,1] given by $\alpha(E) = \| Ex \|^2$. Let $\sum_0$ be the algebra of sets generated by the necessarily non zero intervals $(\alpha(E), \alpha(F)]$ for $E < F$ in $\mathcal{E}$. The function $\mu((\alpha(E), \alpha(F)]) = \alpha(F) - \alpha(E)$ extends to a finitely additive set function on $\sum_0$, and using the extension theorem, μ extends to a measure on the σ-algebra $\sum$ generated by $\sum_0$, also denoted by μ. We can extend μ to a Borel measure in a natural way, so that $\mu((\alpha(E_-), \alpha(E)]) = \mu(\{\alpha(E)\})$ whenever $E_- < E$, where E_- is the strong operator limit of projections $F < E$ with

F in E. Now verify that if $I_k = (\alpha(E_k), \alpha(F_k)]$, $1 \le k \le n$ are disjoint intervals with characteristic function χ_{I_k}, then the linear mapping W defined by

$$W(\sum_{k=1}^{n} a_k(F_k - E_k)x) = \sum a_k \chi_{I_k}$$

extends to a unitary operator W from H onto $L^2(\mu)$. Moreover $W\mathcal{E}W^*$ is the standard projection nest on $L^2(\mu)$.

In general we may choose a sequence of orthogonal unit vector $x_1, x_2, \ldots$ so that $H = H_1 \oplus H_2 \oplus \ldots$ when H_k is the reducing subspace for $\mathcal{E}$ generated by $\mathcal{E}$ and x_k. Obtain the associated probability Borel measures $\eta_1, \eta_2, \ldots$ constructed as above, together with unitary operators $W_1, W_2, \ldots$, and we see that if $W = W_1 \oplus W_2 \oplus \ldots$ then $W\mathcal{E}W^*$ is the standard projection nest on $L^2(\eta_1) \oplus L^2(\eta_2) \oplus \ldots$. Finally we can observe that this standard projection nest is unitarily equivalent to the standard nest on $L^2(\mu_1) \oplus L^2(\mu_2) \oplus \ldots$, and that $\mu_1 >> \mu_2 \ldots$. ■

In view of the representation given above it will be enough to establish Theorem 2.1 for the special case of the standard projection nest on the Hilbert space $L^2(\mu_1) \oplus \ldots \oplus L^2(\mu_r)$ associated with the measures $\mu_1 >> \mu_2 >> \ldots >> \mu_r$. Indeed if we obtain the required maps $\phi_{n,r}$ and $\psi_{n,r}, n = 1, 2, \ldots$, in this case, and make natural subspace identifications, then the maps ϕ_{k_n}, ψ_{k_n} $n = 1, 2, \ldots$, have the required properties, for suitably large r.

To treat the special case we make a preliminary simplification. Let f_k be the Radon-Nikodym derivative $d\mu_k/d\mu_1$, for k=2, ..., r, and let $J_r = \{t : f_r(t) > 0\}$ so that $J_2 \supset J_3 \supset \ldots \supset J_r$, modulo sets of μ_1-measure zero. Then the standard projection nest on $L^2(\mu_1) \oplus \ldots \oplus L^2(\mu_r)$ is unitarily equivalent to the standard nest on $L^2(\mu_1) \oplus L^2(J_2, \mu_1) \oplus \ldots \oplus L^2(J_r, \mu_1)$. The implementing unitary operator is the operator $I \oplus X_2 \oplus \ldots \oplus X_r$ where X_k denotes multiplication by $f_k^{-1/2}$.

PROPOSITION 2.3. Let μ be a regular Borel measure on [0,1] with support J_1 and let $J_1 \supset J_2 \supset \ldots \supset J_r$ be Borel subsets of [0,1]. Then the nest algebra associated with the standard projection nest on $L^2(J_1, \mu) \oplus \ldots \oplus L^2(J_r, \mu)$ is semi-discrete in the sense of Theorem 2.1.

Proof. The main idea is to proceed directly with the construction of the subalgebras of $\mathcal{A}$ that are completely isometric copies of finite dimensional nest algebras. The subalgebras are associated with refining dissections of [0,1] in such a way that their union is dense in the ultra weak topology. Care must be taken to ensure that the matrix units taken to define these algebras do belong to $\mathcal{A}$, and in fact this is why we consider first the nest for $H = L^2(J_1, \mu) \oplus ... \oplus L^2(J_r, \mu)$.

Without loss of generality we may assume that $\mu(\{1\}) = 0$. Fix a natural number n and choose finite families of disjoint intervals $F_r \subseteq F_{r-1} \subseteq ... \subseteq F_1$ where each interval has the form [a,b), with (b-a)<1/n, and for each i the union U_i of the intervals in F_i satisfies $\mu(U_i \triangle J_i) < 1/n$. Enumerate the intervals in $F_1, I_k = [a_k, b_k), k = 1, ..., m$ such that if $k < \ell$ then $b_k \leq a_\ell$, and define $\Omega_j = \{k : I_k \epsilon F_j\}, j = 1, ..., r$.

We now construct "matrix units". For $k \epsilon \Omega_r$, let E^{ij}_{kk} be the canonical partial isometry on H with initial space $L^2(I_k \cap J_r, \mu) \subseteq L^2(J_j, \mu)$ and final space $L^2(I_k \cap J_r, \mu) \subseteq L^2(J_i, \mu), 1 \leq i, j \leq r$. If $k \epsilon \Omega_\ell \backslash \Omega_{\ell+1}$ then define E^{ij}_{kk} to be the canonical partial isometry on H with initial space $L^2(I_k \cap J_\ell, \mu) \subseteq L^2(J_j, \mu)$ and final space $L^2(I_k \cap J_\ell, \mu) \subseteq L^2(J_i, \mu)$ for $1 \leq i, j, \leq \ell$. Note that E^{ij}_{kk} has been defined for $1 \leq i, j \leq r_k$, where $r_k = max\{s : I_k \epsilon F_s\}$.

To construct the remaining matrix units, for $1 \leq i \leq r_k$, let e^i_k denote the characteristic function of the set $J_{r_k} \cap I_k$, normalized so that it has unit length and regarded as an element of $L^2(J_i, \mu)$. For $k < \ell$, we let $E^{ij}_{k,\ell} = e^i_k \otimes e^j_\ell$ denote the rank 1 operator with initial space contained in $L^2(J_j, \mu)$ and final space in $L^2(J_i, \mu)$ whose action is given by $e^i_k \otimes e^j_\ell(f) = < f, e^j_\ell > e^i_k$ for $f \epsilon L^2(J_j, \mu)$.

Now let $\{e_{k\ell}\}$ and $\{f_{ij}\}$ denote systems of matrix units for M_m and M_r, respectively. Let $\mathcal{A}_n \subseteq M_m \otimes M_r$ denote the subspace spanned by $\{e_{k\ell} \otimes f_{ij} : 1 \leq k \leq \ell \leq m, 1 \leq i \leq r_k, 1 \leq j \leq r_\ell\}$, i.e., for precisely those values of (k, ℓ, i, j) for which we have defined $E^{ij}_{k\ell}$. It is not difficult to see that $\mathcal{A}_n$ is a nest algebra and that the map $e_{k\ell} \otimes f_{ij} \to E^{ij}_{k\ell}$ defines a completely contractive homomorphism. Indeed, to see that this map is completely contractive by Proposition 1.1 it is sufficient to check that $\| E^{ij}_{k\ell} \| \leq 1$ and that $\{E^{ij}_{k\ell}\}$ multiply like matrix units. This defines the map $\psi_n : \mathcal{A}_n \to \mathcal{A}$.

To define a map $\phi_n : \mathcal{A} \to \mathcal{A}_n$ we simply set

$$\psi_n(A) = \sum < Ae_\ell^j, e_k^i > e_{k\ell} \otimes f_{ij},$$

which is essentially the compression of $\mathcal{A}$ to the span of $\{e_k^i\}$.

It is easy to check that $\phi_n \circ \psi_n$ is the identity map on $\mathcal{A}_n$ and hence ψ_n must be completely isometric. Also, for $X \in \psi(\mathcal{A}_n)$ we will have that $\phi_n \circ \psi_n(X) = X$.

Let $H_n = span\{e_k^i : 1 \leq i \leq r_k, 1 \leq k \leq m\}$. We claim that for every vector e in H, $\mathrm{dist}(e, H_n) \to 0$ as $n \to +\infty$. Using a simple approximation argument it is sufficient to show this for $e = \chi_I$ (the characteristic function of some interval I regarded as a vector in $L^2(J_i, \mu)$). But this follows readily from the fact that the intervals in F_i form an increasingly finer cover of J_i as $n \to +\infty$.

It remains to show that for each operator X in $\mathcal{A}$, $\psi_k \circ \phi_k(X) \to X$ in the σ-weak topology. Note that the sequence $X_k = \psi_k \circ \phi_k(X)$ is bounded so we need only check convergence in the weak operator topology. Let P_n denote this orthogonal projection onto H_n, so that $P_n \to I$ in the strong topology. A computation shows that $P_k X_k P_k = P_k X P_k$, for each k. Considering the identity

$$\begin{aligned} < Xf, g > - < X_k f, g > &= < Xf, g > - < P_k X P_k f, g > \\ &\quad - < X_k f, g > + < P_k X_k P_k f, g > \end{aligned}$$

we see that it suffices to check that $(X_k - X_k P_k)f \to 0$ for each vector f, and this is the case. ■

The proof of Theorem 2.1 is now complete. We can also modify the proof a little to obtain the following stronger density property.

COROLLARY 2.4. Let $\mathcal{A}$ be a nest algebra acting on a separable Hilbert space H. Then there exists subalgebras $\mathcal{C}_1, \mathcal{C}_2, \ldots$ which are completely isometrically isomorphic to finite dimensional nest algebras, and are such that $\mathrm{dist}(K, \mathcal{C}_n) \to 0$, as $n \to \infty$, for every compact operator K in $\mathcal{A}$.

Proof. Once again it will suffice to establish the corollary in the context of Proposition 2.3. Let the discrete component of the measure be supported on the countable or finite set D. Fix a natural number n and choose finite families of disjoint intervals,

$F_r \subseteq F_{r-1} \subseteq ... \subseteq F_1$, where each interval may be open, semi-open, closed, or a singleton, of length $< 1/n$. Arrange that the union of the singleton sets have μ measure greater than $\mu(D) - 1/n$, and that the union U_i of the intervals in F_i satisfies $\mu(U_i \triangle J_i) < \frac{1}{n}$. Enumerate the intervals in F_1 as $I_1, I_2, ..., I_m$, where the points, or point, in I_j lie to the left of points in I_{j+1}. Define $\Omega_j = \{k : I_k \epsilon F_j\}, j = 1, ..., r$, and $r_k = max\{S : I_k \epsilon F_s\}$.

Exactly as in the proof of Proposition 2.3 we can construct matrix units $E^{ij}_{k\ell}$, for $1 \leq i, j \leq r_k$, and $1 \leq k \leq \ell \leq m$, which determine a finite dimensional subalgebra, $\mathcal{C}_n$ say, which is completely isometrically isomorphic to a finite dimensional nest algebra. As before these algebras have the semi-discreteness density properties expressed in Theorem 2.1.

Each rank one operator R in $\mathcal{A}$ has the form $e \otimes f$ where for some projection E in the nest for $\mathcal{A}$, Ee = e and $(I - E_-)f = f$. Here E_- is the supremum of nest projections strictly less then E, and we observe that $E_- < E$ precisely when $E = M_t \oplus ... \oplus M_t$, and $\mu(\{t\}) > 0$. Our construction of the subalgebras $\mathcal{C}_n$ has the property that $(E - E_-)R(E - E_-)$ lies in $\mathcal{C}_n$ for all large enough n. We claim that the distance of the operators $ER(I - E)$ and $E_-R(I - E_-)$ from $\mathcal{C}_n$ tends to zero as $n \to \infty$. Since these operators have the form $Ee \otimes (I - E)f$ and $E_-e \otimes (I - E_-)f$, this is a consequence of a simple approximation argument using the fact that dist$(g, H_n) \to 0$ for every vector g in H. We have now shown that dist$(R, \mathcal{C}_n) \to 0$ for every rank one operator in the nest algebras. Since every compact operator in the nest algebra can be approximated by a linear span of such rank one operators (see [3] and [9]), the proof is complete. ■

3. Contractive representations of nest algebras.

We can now use the semi-discreteness properties of a nest algebra to extend the main results of section 1 for finite dimensional nest algebras to the general case. Notice however that the order is reversed; we first deduce the complete contractivity of σ-weakly continuous representations, and then use Arveson's dilation theorem to show that such representations admit star dilations to the enveloping algebra of all operators.

THEOREM 3.1. Let ρ be a contractive representation of a nest algebra acting on a separable Hilbert space, which is continuous for the σ-weakly topology. Then ρ is completely contractive.

Proof. Let $\mathcal{A}$ be the nest algebra and let (A_{ij}) be a matrix in $M_k(\mathcal{A})$. By Theorem 2.1 there exist finite dimensional nest algebras $\mathcal{A}_1, \mathcal{A}_2, \ldots$ and certain σ-weakly continuous maps $\phi_n : \mathcal{A} \to \mathcal{A}_n, \psi_n : \mathcal{A}_n \to \mathcal{A}$ such that $\psi_n \circ \phi_n$ (A) $\to$ A σ-weakly as $n \to \infty$ for all A in $\mathcal{A}$. Let $A_{ij}^n = \psi_n \circ \phi_n(A_{ij})$. Then $(A_{ij}^n) \to (A_{ij})$ σ-weakly, and so $(\rho(A_{ij}^n)) \to (\rho(A_{ij}))$ σ-weakly. Now $\| (\rho(A_{ij})) \| \leq \limsup \| (\rho(A_{ij}^n)) \| \leq \limsup \| (A_{ij}^n) \|$, by Corollary 1.2. Since $\psi_n \circ \phi_n$ is completely contractive we now obtain $\| (\rho(A_{ij})) \| \leq \| (A_{ij}) \|$, as required.

THEOREM 3.2. Let $\mathcal{A}$ be a nest algebra on a separable Hilbert space R, and let ρ be a unital contractive σ-weakly continuous representation of $\mathcal{A}$ on a separable Hilbert space H. Then there exists a separable Hilbert space K containing H as a subspace, and a σ-weakly continuous *-representation π of $L(R)$ on $L(K)$ such that

$$\rho(A) = P_H \pi(A) \mid_H \quad \text{for all A in } \mathcal{A}.$$

Proof. Let $\mathcal{B}_1$ denote the C^*-subalgebras of $L(R)$ generated by the identity and the compact operators, and let $\mathcal{A}_1 = \mathcal{A} \cap \mathcal{B}_1$. By Theorem 3.1, ρ is completely contractive on $\mathcal{A}$ and hence on $\mathcal{A}_1$, so by Arveson's theorem there exists $\pi_1 : \mathcal{B}_1 \to L(K)$ such that $\rho(A) = P_H \pi_1(A) \mid_H$ for all A in $\mathcal{A}_1$. By Strinesprings theorem, since $\mathcal{B}_1$ and H are separable, K is separable.

The representation π_1 of $\mathcal{B}_1$ decomposes as $\pi_1 = \pi \oplus \pi_0$ where π_0 is zero on the compacts and π is unitarily equivalent to an ampliation of the identity. By considering a sequence $\{K_n\}$ in $\mathcal{A}_1$ which converges σ-weakly to the identity we see that $\rho(1) = P_H \pi(1) \mid_H$. Hence H is orthogonal to the space on which π_0 acts, and consequently $\rho(A) = P_H \pi(A) \mid_H$ for all A in $\mathcal{A}_1$.

The representation π clearly extends to all of $L(R)$ since an ampliation of the identity is σ-weakly continuous. We still write π for this extension. But then $\rho(A) = P_H \pi(A) \mid_H$ holds for all A in $\mathcal{A}$, since both sides of this equation are σ-weakly continuous and $\mathcal{A}_1$ is σ-weakly dense in $\mathcal{A}$.

Corollary 3.3. Let $\mathcal{A}$ be a nest algebra on a separable Hilbert space R and let ρ be a σ-weakly continuous contractive representation of $\mathcal{A}$ on H. Then there exists a sequence of bounded operators $V_n : H \to R$ such that the series $\sum V_n^* A V_n$ converges *-strongly to $\rho(A)$ for every A in $\mathcal{A}$.

Proof. Let (π, K) be as in Theorem 3.2. In the proof of Theorem 3.2, we saw that K is unitarily equivalent to $R \oplus R \oplus \ldots$, and that π is unitarily equivalent to the map $A \to A \oplus A \oplus \ldots$. Since $H \subseteq K$ this unitary yields an isometry $V : H \to R \oplus R \oplus \ldots$, such that $P_H \pi(A) |_H = V^*(A \oplus A \oplus \ldots)V$. Letting V_n denote the projection of V onto the n-th copy of R yields the desired result. ■

Remark 3.4. Let $\mathcal{B}_1$ be the algebra of compact operators with identity, with subalgebra $\mathcal{A}_1 = \mathcal{A} \cap \mathcal{B}_1$, as in the proof of the last theorem. A $\mathcal{B}_1$-dilation π of a representation ρ of $\mathcal{A}_1$ on H is said to be minimal if the span of vectors Bh, with B in $\mathcal{B}_1$, h in H, is dense in the dilation space. Since $\mathcal{A}_1 + \mathcal{A}_1^*$ is norm dense in $\mathcal{B}_1$, standard elementary arguments show that every pair of minimal $\mathcal{B}_1$-dilations are unitarily equivalent, in the usual sense. From this follows the uniqueness up to unitary equivalence of minimal σ-weakly continuous $L(R)$-dilations of representations of nest algebras.

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(4.3) The complete approximation property for CSL algebras

In the last section it was shown how the semidiscreteness property of a nest algebra could be used to extend the finite dimensional dilation theorem to general nest algebras. We now examine related structural properties for more general reflexive operator algebras.

Semidiscreteness and complete approximation by subalgebras

A D_n-bimodule is a subspace of the complex matrix algebra M_n which is a bimodule for the diagonal algebra D_n of M_n. A unital D_n-bimodule is one that contains the identity, and hence contains D_n. The category of unital D_n-bimodules which are also subalgebras of M_n coincides (up to unitary equivalence) with the class of reflexive subalgebras of M_n with commutative subspace lattice. Such algebras are called finite dimensional CSL algebras.

(4.3.1) DEFINITION. Let A be a σ-weakly closed unital algebra of operators on a Hilbert space. Then A is said to be semidiscrete relative to finite dimensional CSL algebras, or CSL-semidiscrete, if there exists

(i) finite dimensional CSL algebras S_α indexed by a directed set,

(ii) σ-weakly continuous completely contractive maps $\phi_\alpha: A \to S_\alpha$,

(iii) completely isometric isomorphisms $\psi_\alpha: S_\alpha \to A$, such that $\psi_\alpha \circ \phi_\alpha(A) \to A$ σ-weakly for all A in A.

In a similar way we could define semidiscreteness relative to D_n-bimodules, but we shall not develop this here.

CSL-semidiscreteness is a strong property that implies hyperfiniteness in the category of CSL algebras. However additional structure is built in, from which it follows (as in the nest algebra case - see Theorem 3.1 in section 4.2) that if $\rho: A \to L(H)$ is a σ-weakly continuous map and $\rho|\psi_\alpha(S_\alpha)$ is completely contractive for every α, then ρ is also completely contractive. This conclusion follows from the fact that for the matricial algebra $M_n(\psi_\alpha(S_\alpha))$ (for fixed n), the union

$$\bigcup_\alpha \text{ball } M_n(\psi_\alpha(S_\alpha))$$

is σ-weakly dense in ball $M_n(A)$.

We formally identify this apparently weaker property in the next definition.

(4.3.2) DEFINITION. Let A be a σ-weakly closed unital algebra of operators on a Hilbert space. Then A is said to have the complete CSL algebra approximation property CCAP if there exist subalgebras $A_\alpha \subset A$ indexed by a directed set such that

(i) A_α is completely isometrically isomorphic to a finite dimensional CSL algebra

(ii) for $n = 1,2,\ldots$ and for every operator matrix A in $M_n(A)$ there exist operators A_α in $M_n(A_\alpha)$ such that $\|A_\alpha\| \leq \|A\|$ and $A_\alpha \to A$ σ-weakly.

We do not know that CCAP is strictly weaker than CSL semidiscreteness. Our main motivation for introducing this property is that we can show that certain CSL operator algebras have property CCAP, whilst it is not at all clear how to construct the maps ϕ_α required in the definition of CSL semidiscreteness.

On the other hand we remark that it is easy to show that a finite spatial tensor product of nest algebras is CSL-semidiscrete. It can also be shown that infinite tensor products of nest algebras are CSL-semidiscrete, but we will not develop these facts here.

Completely distributive CSL algebras

Let A be a CSL algebra (reflexive operator algebra with commutative subspace lattice) which enjoys the property that the linear span of the rank one operators in A is σ-weakly dense. By a result of Laurie and Longstaff [14] this occurs if and only if the projection lattice $L = \text{Lat } A$ is completely distributive, and for this reason we use the acronym of Gilfeather and Moore [10] and refer to A as a CDC algebra (completely distributive commutative lattice algebra). Nevertheless we only make use of the rank one density property of such algebras in the arguments below.

The next proposition shows that a reflexive operator algebra with commutative subspace lattice possess completely isometric copies of finite dimensional CSL algebras which uniformly approximate the rank one operators in the algebra, should such operators exist. From now on all operator algebras exist on a separable Hilbert space.

(4.3.3) PROPOSITION. Let A be a CSL algebra on a separable Hilbert space. Then there exist unital subalgebras $M_1, M_2, \ldots$ of A such that M_n is completely isometrically isomorphic to a unital finite dimensional CSL algebra, and $\text{dist}(R, M_n) \to 0$ as $n \to \infty$ for every rank one operator R in A.

Proof. Let $R = e \otimes f$ be the rank one operator $g \to \langle g,f \rangle e$, and suppose that R lies in A. Let L be the support projection of e, namely $L = \wedge\{E \in \mathcal{L}: Ee = e\}$, and let $L_- = \vee\{E \in \mathcal{L}: E \ngeq L\}$. Then $L_- f = 0$. Indeed, if $L_- f \neq 0$, then $Ef \neq 0$ for some projection E in $\mathcal{L}$ with $E \ngeq L$, and hence $E^\perp e \neq 0$. Thus $E^\perp(e \otimes f)E = E^\perp e \otimes E_t \neq 0$, contrary to our assumption. On the other hand if for some projection L we have $Le = e$ and $L_-^\perp f = f$, then a similar argument shows that $e \otimes f$ lies in A.

Since the underlying Hilbert space R say is separable we can choose a set of projections $L_1, L_2, \ldots$ in $\mathcal{L}$ which is dense in $\mathcal{L}$ relative to the strong operator topology. Recall that an atom of $\mathcal{L}$ is a minimal nonzero projection of the form $F - E$ with F, E in $\mathcal{L}$. We can assume that the sequence $L_1, L_2, \ldots$ is chosen so that if $\mathcal{L}_n$ is the finite sublattice generated by $L_1, \ldots, L_n$, then each atom Q of $\mathcal{L}$ appears as an atom of $\mathcal{L}_n$ for some n. Note that each projection L in $\mathcal{L}_n$ is of the form $Q_1 + \ldots + Q_n$ where Q_k is an atom of $\mathcal{L}_n$. Moreover the atoms of $\mathcal{L}_n$ are partially ordered by the relation $Q < Q'$ if and only if $QAQ' = Q\mathcal{L}(\mathcal{R})Q'$.

Choose nonzero vectors $x_1, x_2, \ldots$ in $\mathcal{R}$ so that the closed subspace $\mathcal{R}_n$ spanned by $\{Lx_n: L\}$ in $\mathcal{L}$ are pairwise orthogonal and have closed span $\mathcal{R}$. Let P_n be the orthogonal projection onto $\mathcal{R}_n$. Clearly QP_k belongs to A for $k = 1, 2, \ldots$ and each atom Q of $\mathcal{L}_n$.

Define $\mathcal{S}_n$ to be the algebra of operators spanned by the rank one operators

(i) $Q_1 x_k \otimes Q_2 x_\ell$; $1 \leq k \leq \ell \leq n$, Q_1, Q_2 atoms of $\mathcal{L}_n$ with $Q_1 < Q_2$.

(ii) $Qx_k \otimes Qx_k$; $1 \leq k \leq n$, Q an atom of $\mathcal{L}_n$.

By the orthogonality of the vectors Qx_k, for $1 \leq k \leq n$, and Q

an atom of L_n, it is clear that S_n is completely isometrically isomorphic to a D_m-module, where m is the number of these vectors which are not zero. By the transitivity of $<$ the space S_n is an algebra. Unfortunately the projections of type (ii) need not belong to A. (This will be the case however if Q is an atom of L and hence $Q < Q$.) Define M_n to be the subalgebra of A spanned by the rank one operators of type (i) as before, together with the operators

(ii)' QP_k, $1 \leq k \leq n$, Q an atom of L_n but not an atom of L.

Since x_k is cyclic in R_k for L it follows that $QP_k = 0$ if and only if $Qx_k \otimes Qx_k = 0$. Moreover,

$$QP_k(Qx_k \otimes Q_1x_\ell) = (Qx_k \otimes Q_1x_k)(Qx_k \otimes Q_1x_\ell)$$

when $Q < Q_1$, and so there is a natural algebra isomorphism $\alpha_n: M_n \to S_n$. This map is given by $X \to E_nXE_n$, where E_n is the orthogonal projection onto the span of the vectors Qx_k, for Q an atom of L_n and $1 \leq k \leq n$. In particular α_n is completely contractive. But in fact E_n commutes with S_n and so $X = E_nX \oplus E_n^\perp X$. Since $E_n^\perp X = E_n^\perp D$, where D is the diagonal part of X, it follows that $\|E_n^\perp X\| \leq \|X\|$ and hence $\|X\| = \|E_nX\|$. Similarly, α_n is completely isometric.

It remains to show that $\mathrm{dist}(R,M_n) \to 0$ for each rank one operator R in A. Suppose then that $R = e \otimes f$ with $Le = e$ and $L_-^\perp f = f$ for some projection L in L. Observe that if $LL_-^\perp \neq 0$ then the projection $Q = LL_-^\perp$ is an atom of L. (If Q' is a proper subinterval of Q then $L_- + Q' \not\geq L$ and $L_- + Q' \underset{\neq}{>} L_-$ contrary to the definition of L_-.) In this case then we see from our construction that $\mathrm{dist}(QRQ,M_n) \to 0$. On the other hand, from the density of $\{L_n\}$ in L, and the construction, it follows that $\text{dist}(QR(L_-^{\perp}-Q),M_n) \to 0$, and $\text{dist}((L-Q)RL_-^{\perp}M_n) \to 0$. (If Q_1,Q_2 are atoms in $\mathcal{L}_n$ with $Q_1 \leq L - Q$ and $Q_2 \leq L_-^{\perp}$ then $Q_1 < Q_2$, etc.). Hence $\text{dist}(R,M_n) \to 0$, completing the proof. □

In the case of a nest algebra on a separable Hilbert space there exists a sequence of finite rank contractions that converge to the identity in the σ-weak topology. We want this feature in the more general context of a CDC algebra.

(4.3.4) PROPOSITION. Let R_n be a sequence of finite rank operators which converges to the identity in the weak operator topology. Then there exists convex combinations S_n of $\{R_n\}$ such that $\|S_n\| \to 1$ and $S_n \to I$ in the σ-weak topology.

Proof. By the Banch Steinhaus theorem $\|R_n\|$ is bounded, and the proposition follows from a simple convexity argument. (Also see section (2.6), Lemma 4.3).

(4.3.5) THEOREM. Let A be a CDC algebra on a separable Hilbert space. Then A has the complete CSL algebra approximation property.

Proof. Using the last proposition we see that there is a sequence of contractive operators R_n, in the linear span of the rank one operators of A, which converges to the identity in the σ-weak topology. Proposition 4.3.2 shows that there exist subspace $M_1,M_2,\ldots$ satisfying condition (i) of Definition 4.3.2. such that $\text{dist}(R,M_n) \to 0$ as $n \to \infty$ for every rank one operator R. Let (A_{ij}) be a matrix in $M_r(A)$, and let $R_n^{(r)} = R_n \oplus \ldots \oplus R_n$, $n = 1,2,\ldots$, be the diagonal matrix in $M_r(A)$.

Set $(A^n_{ij}) = R^{(r)}_n(A_{ij})R^{(r)}_n$ and note that $(A^n_{ij}) \to (A_{ij})$ σ-weakly, and $\|(A^n_{ij})\| \leq \|(A_{ij})\|$. Since $\mathrm{dist}((A^n_{ij}), M_r(M_m)) \to 0$ as $m \to \infty$, for each n, it follows that condition (ii) of Definition 4.3.2 holds, completing the proof.

The last theorem is useful in the dilation theory of certain CSL algebras. We should remark however that at the time of writing (July 1987) the dilation theory for contractive representations of finite dimensional CSL algebras is incomplete. Also it is not known whether there exists a CSL algebra which fails to be CSL semidiscrete or fails to have the CCAP property.

References: The concepts and results of this section have not yet been published. They form part of the author's research with V.I. Paulsen on noncommutative non self-adjoint dilation theory (as do Chapters 4,5,6 and 8).

CHAPTER 5 LIFTING THEOREMS FOR NEST ALGEBRAS

The dilation and model theory for contractions on a Hilbert space begins with the Sz-Nagy dilation theorem which asserts that every contraction possesses a unitary dilation, or equivalently, that every contractive representation of the normed polynomial algebra $P(\mathbb{D})$ admits a *-dilation to $C(\mathbb{T})$. In the last chapter we obtained the dilation theorem for σ-weakly continuous contractive representations of nest algebras. For pairs of commuting contractions Sz-Nagy's theorem has two apparently different, but actually equivalent, generalisations, namely, Ando's dilation theorem and the Sz-Nagy-Foias commutant lifting theorem. In the present chapter (the text of which is taken from the preprint "Lifting theorems for nest algebras" by V.I. Paulsen and S.C. Power) we obtain analogues of these results for representations of nest algebras.

Recently Ball and Gohberg have studied the contractive representations of upper triangular matrix algebras which have *-dilations to the containing full matrix algebra, and in this context they obtain lifting theorem for a contraction commuting with the representation.

In section 1, we prove the analogue of Ando's theorem for a finite dimensional nest algebra and a commuting contraction, which yields a new proof of the Ball-Gohberg result. In section 3 we use the results of [9] to extend this result to arbitrary nest algebras on separable Hilbert spaces.

In section 2, we prove the analogue of Ando's theorem where both contractions are replaced by commuting contractive representations of finite dimensional nest algebras. We then extend this result in section 3 to arbitrary nest algebras on separable Hilbert spaces. In particular, we show that a pair of commuting σ-weakly continuous contractive representations of a pair of nest algebras admits a pair of commuting σ-weakly continuous *-dilations.

In section 4 we use a lifting theorem to characterise the operator norm of abstract Hankel operators H_X associated with a nest algebra $\mathcal{A}$. We find that

$$\|H_X\| = \mathrm{dist}(X,\mathcal{A}) = \sup_{E\in \mathrm{Lat}\,\mathcal{A}} \|(I-E)XE\|$$

which is analoguous to the Nehari theorem for classical Hankel operators, and which also includes the Arveson distance formula.

The lifting theorems have fundamental implications for tensor products of various non-selfadjoint operator algebras. We discuss this and related matters in another paper.

McAsey and Muhly have observed in [6] that contractive representations of upper triangular matrix algebras are completely contractive, and so, by Arveson's dilation theorem, admit *-dilations. This was obtained by direct construction in [9], and here we pursue similar techniques together with the Sz.-Nagy-Foias lifting theorem to obtain generalised lifting and dilation theorems in the finite dimensional case. The extension to σ-weakly continuous contractive representations of nest algebras is obtained by exploiting the semi-discreteness property obtained in [9]. This property says that for the given nest algebras A on a separable Hilbert space there are finite dimensional nest algebras A_n, completely contractive σ-weakly continuous maps $\varphi_n: A \to A_n$, and completely contractive homomorphisms $\psi_n: A_n \to A$, such that $\psi_n \circ \varphi_n(X)$ converges to X σ-weakly for each X in A.

This paper is self-contained with the exception of the proofs of semi-discreteness and the following two well-known results.

<u>The Sz.-Nagy-Foias lifting theorem</u>. Let T be a contraction on a Hilbert space H with isometric dilation V on a Hilbert space $K \supset H$, and let X be an operator with $XT = TX$. Then there exists an operator Y on K commuting with V such that $\|Y\| = \|X\|$ and $X = P_H Y|_H$, where P_H is the orthogonal projection from K to H.

We usually consider the isometric dilation V on $K = H \oplus H \oplus \ldots$ defined by $V(h_1, h_2, \ldots) = (Th_1, D_T h_1, h_2, \ldots)$, where $D_T = (I-T^*T)^{1/2}$. It is important to note that Y can be chosen in this case so that $Y^*H \subset H$, where H is identified with the first summand of K. (See [15].) In particular, we have $T^nX^m = P_H V^n Y^m|_H$, for $n,m = 0,1,2,\ldots$.

<u>The Arveson dilation theorem [2]</u>. Let A be a unital subalgebra of the C*-algebra B, and let $\rho: A \to L(H)$ be a contractive unital representation. Then the following conditions are equivalent:

(i) ρ is completely contractive;

(ii) there is a unital *-representation $\pi: B \to L(K)$ on a Hilbert space $K \supset H$ such that $\rho(A) = P_H \pi(A)|_H$ for all A in A.

Recall that a linear map φ from a space of operators S into $L(H)$ is said to be completely contractive if the induced maps φ_n between the normed operator matrix spaces $M_n(S)$ and $M_n(L(H))$ are contractive for $n = 1,2,\ldots$. The implication (ii) $\Rightarrow$ (i) is elementary, and the direction (i) $\Rightarrow$ (ii) is obtained in two stages. First the completely positive map $\tilde{\rho}$ defined on $A + A^*$ by $\tilde{\rho}(A_1+A_2^*) = \rho(A_1)+\rho(A_2)^*$, is extended to a completely positive map φ from B to $L(H)$, by an extension theorem of Arveson. Then φ is dilated to π

by means of Stinespring's theorem [14]. In particular, if $\mathcal{B}$ and $\mathcal{H}$ are separable, the dilation space $\mathcal{K}$ can be assumed separable. Further details may be found in [2] and [8].

A nest algebra $\mathcal{A}$ on a Hilbert space $\mathcal{R}$ is an algebra of operators which leaves invariant the subspaces in a pre-assigned nest of subspaces. We always take $\mathcal{R}$ to be separable, and if $\mathcal{R}$ is finite dimensional we call $\mathcal{A}$ a finite dimensional nest algebra. General facts about nest algebras, and the density of compact and finite rank operators, may be found in the lecture notes [13], or the forthcoming book of Davidson [5].

We write $C(\mathbb{T})$ for the C^*-algebra of continuous complex valued functions on the unit circle, and write $A(\mathbb{D})$ for the disc algebra regarded as a closed subalgebra of $C(\mathbb{T})$.

1. Lifting theorems for finite dimensional nest algebras.

The lifting theorem of Ball and Gohberg [4] asserts that if an operator X commutes with a contractive representation ρ of a finite dimensional nest algebra A then there is a norm preserving lifting Y commuting with the *-dilation π of ρ. Theorem 1.2 below is a generalisation of this which obtains a lifting with much more structure, and can be viewed as an analogue of Ando's theorem that commuting contractions admit commuting unitary dilations. Recall that Ando's theorem and the Sz.-Nagy-Foias lifting theorem are essentially equivalent. The deduction of the lifting theorem from Ando's theorem is elementary, whilst the other direction is obtained by a somewhat non trivial two-stage argument. For details, see the discussion in Parrott [7] and our remark 1.8 below.

The following result is a structured from of the Sz-Nagy Foias lifting theorem which will be used in the proofs of Theorems 1.2 and 2.1.

<u>THEOREM 1.1</u>. Let X_1, X_2 and T be contractions on the Hilbert space H such that $X_1 T = T X_2$, and such that with respect to the decomposition $H = H_1 \oplus \ldots \oplus H_1$ (m times), we have representing operator matrices

$$X_i = \begin{bmatrix} 0 & X_{i,1} & & & \\ & 0 & X_{i,2} & & \\ & & 0 & \ddots & \\ & & & \ddots & X_{i,m-1} \\ & & & & 0 \end{bmatrix},$$

$$T = \begin{bmatrix} T_1 & & & \\ & T_2 & & \\ & & \ddots & \\ & & & T_m \end{bmatrix},$$

for $i = 1,2,$ (where the unspecified entries are zero). Then there are isometric dilations $\tilde{X}_i$ on the Hilbert space $\tilde{H} = H \oplus H \oplus \ldots$ of the form

$$\tilde{X}_i = \begin{bmatrix} 0 & \tilde{X}_{i,1} & & & \\ & 0 & \tilde{X}_{i,2} & & \\ & & 0 & \ddots & \\ & & & \ddots & \tilde{X}_{i,m-1} \\ U & & & & 0 \end{bmatrix}, \quad i = 1,2,$$

with respect to $\tilde{H} = \tilde{H}_1 \oplus \ldots \oplus \tilde{H}_1$ (m times), where $\tilde{H}_1 = H_1 \oplus H_1 \oplus \ldots,$ where U is the unilateral shift on $\tilde{H}_1$, and there is a contraction $\tilde{T}$ on $\tilde{H}$ of the form $\tilde{T} = \tilde{T}_1 \oplus \ldots \oplus \tilde{T}_n$, such that $\tilde{X}_1\tilde{T} = \tilde{T}\tilde{X}_2$, and

$$\tilde{T}_i = \begin{bmatrix} T_i & 0 \\ * & * \end{bmatrix}$$

with respect to the decomposition $\tilde{H}_i = H_i \oplus (\tilde{H}_i \ominus H_i)$, $1 \le i \le m$.

<u>Proof</u>. Define $\tilde{X}_{ij}$ on $\tilde{H}_1$ by $\tilde{X}_{ij}(h_1,h_2,\ldots) = (X_{ij}h_1, D_{ij}h_1, h_2,\ldots)$, where $D_{ij} = (I - X_{ij}^* X_{ij})^{1/2}$, $1 \le i \le 2$, $1 \le j \le m-1$ and observe that the associated operator $\tilde{X}_i$ is an isometric dilation of X_i. By the Sz.-Nagy-Foias lifting theorem there is a contraction $\hat{T}$ on $\tilde{H}$ of the form

$$\hat{T} = \begin{bmatrix} T & 0 \\ * & * \end{bmatrix}$$

with respect to $\tilde{H} = H \oplus (H)^{\perp}$, such that $\tilde{X}_1\hat{T} = \hat{T}\tilde{X}_2$.

Let D be the diagonal operator $I \oplus wI \oplus \ldots \oplus w^{m-1}I$ on $H = H_1 \oplus \ldots \oplus H_1$ where w is the primitive m^{th} root of unity, and note that $D^*X_iD = wX_i$, $i = 1,2$, and $D^*TD = T$. Also D has a natural extension $\tilde{D}$ on $\tilde{H}$ such that $\tilde{D}^*\tilde{X}_i\tilde{D} = w\tilde{X}_i$. Observe that $(\tilde{D}^*)^j\hat{T}\tilde{D}^j$ has compression equal to T and also intertwines $\tilde{X}_1$ and $\tilde{X}_2$. It follows that the operator $\tilde{T} = m^{-1}\sum_{j=1}^{m}(\tilde{D}^*)^j\hat{T}\tilde{D}^j$ has the required properties. □.

The proof of the next theorem contains the basic construction used in [9] of *-dilations for contractive representations of finite dimensional nest algebras $\mathcal{A} \subseteq M_n$.

<u>THEOREM 1.2</u>. Let ρ be a contractive representation of a finite dimensional nest algebra $\mathcal{A} \subseteq M_n$ on the Hilbert space $\mathcal{H}$, and let X be a contraction that commutes with $\rho(A)$, for all A in $\mathcal{A}$. Then there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$, a *-homomorphism $\pi: M_n \to \mathcal{L}(\mathcal{H})$, and a unitary operator U on $\mathcal{K}$ which commutes with $\pi(B)$, for all B in $\mathcal{B}$, such that

$$X^n\rho(A) = P_{\mathcal{H}}U^n\pi(A)|_{\mathcal{H}},$$

for $n = 0,1,2,\ldots,$ and A in $\mathcal{A}$.

Proof. We may assume that ρ is unital. We first consider the case where $A = A_u$ is the upper triangular matrix algebra in M_n.

For each i the operator $E_i = \rho(e_{i,i})$ is a self-adjoint projection, with range space H_i and $H = H_1 \oplus \ldots \oplus H_n$. The contraction $\rho(e_{i,j})$ has range contained in H_i and kernel containing $(H_j)^{\perp}$, for $1 \le i \le j \le n$. Let $T_{ij} = E_i \rho(e_{ij}) E_j$ and we have $\rho((a_{ij})) = (a_{ij} T_{ij})$ as an operator matrix on $H_1 \oplus \ldots \oplus H_n$, for (a_{ij}) in A, and $T_{ij} = T_{i,i+1} \cdots T_{j-1,j}$, $1 \le i < j \le n$. The representation ρ is determined by the contractions $T_i = T_{i,i+1}$ and we write $\rho = \rho_{\{T_i\}}$ to indicate such a representation. Since X commutes with ρ we see that

$X = X_1 \oplus \ldots \oplus X_n$, a diagonal operator on $H_1 \oplus \ldots \oplus H_n$, and that $X_i T_i = T_i X_{i+1}$, for $1 \le i \le n-1$.

Without loss of generality we assume that $H_i = H_j$ for all $1 \le i,j \le n$. If this does not already hold then we can arrange it to be true for a trivial dilation of the pair ρ and X obtained by adding trivial summands.

Let $\hat{T}_i$ be the isometric dilation of the operator T_i acting on this space $\hat{H}_i = H_i \oplus H_i \oplus \ldots$, given by $\hat{T}_i(h_1,h_2,\ldots) = (T_i h_1, C_i h_1, h_2, \ldots)$ where $C_i = (I-T_i^* T_i)^{1/2}$, $1 \le i \le n-1$. By Theorem 1.1 (with reversed notation) there exist contractions $\hat{X}_i$ on $\hat{H}_i$ of the form

$$\begin{bmatrix} X_i & 0 \\ * & * \end{bmatrix}$$

with respect to the decomposition $\hat{H}_i = H_i \oplus (\hat{H}_i \ominus H_i)$, such that $\hat{X}_i \hat{T}_i = \hat{T}_i \hat{X}_{i+1}$, for $1 \le i \le n-1$.

These relations imply that $\hat{X}$ commutes with $\hat{\rho}(A) = \rho_{\{\hat{T}_i\}}(A)$ on $\hat{H}$ and that $X^n \rho(A) = P_H \hat{X}^n \hat{\rho}(A)|_H$ for all $n = 0,1,2,\ldots$, and A in A_u. Here we identify H_i with $H_i \oplus 0 \oplus 0 \ldots$ in $\hat{H}_i$.

Now define an isometry W on $\hat{H}$ by setting $W(\hat{h}_1,\ldots,\hat{h}_n) = (\hat{h}_1, \hat{T}_1\hat{h}_2, \ldots, \hat{T}_1 \ldots \hat{T}_{n-1}\hat{h}_n)$ and define a $*$-homomorphism $\pi_0 : M_n \to L(\hat{H})$ via $\pi_0(e_{ij}) = \hat{E}_{ij}$, where e_{ij} are the canonical matrix units in M_n and $\hat{E}_{ij}$ are the canonical matrix units for $\hat{H} = \hat{H}_1 \oplus \ldots \oplus \hat{H}_n$. (Recall that $H_i = H_j$ and so $\hat{H}_i = \hat{H}_j$). Let $Y = \hat{X}_1 \oplus \ldots \oplus \hat{X}_1$.

We claim that $W^*Y^n\pi_0(A)W = \hat{X}^n\pi_0(A)$ for all $n = 0,1,\dots,$ and A in A_u. To see this, note that for $2 \le i < j \le n$, $W^*Y^n\hat{E}_{ij}W$ is the operator matrix which is 0 except for the (i,j)-th entry which is,

$$\hat{T}^*_{i-1}\cdots\hat{T}^*_1\hat{X}^n_1\hat{T}_1\cdots\hat{T}_{j-1} = \hat{T}^*_{i-1}\cdots\hat{T}^*_1\hat{T}_1\cdots\hat{T}_{i-1}\hat{X}^n_i\hat{T}_i\cdots\hat{T}_{j-1} = \hat{X}^n_i\hat{T}_{i,j}.$$

This last quantity is clearly the (i,j)-th entry of $\hat{X}^n\hat{\rho}(E_{ij})$, which is also 0 in its remaining entries. The calculation for other E_{ij} in A_u follows similarly.

Thus, for any h, k in H, we have $\langle X^n\rho(A)h,k\rangle = \langle Y^n\pi_0(A)Wh,Wk\rangle$. If we identify H with $WH \subseteq \hat{H}$, then this last equation becomes $X^n\rho(A) = P_HY^n\pi_0(A)|_H$.

Finally, if we let U_1 be the unitary dilation of $\hat{X}_1$ on K_0, $\hat{H}_i \subseteq K_0$, set $U = U_1\oplus\dots\oplus U_1$ on $K = K_0\oplus\dots\oplus K_0$ (n copies), and let $\pi: M_n \to L(K)$ be the obvious representation, we then obtain the desired result, for the case that A is the algebra of upper triangular matrices. Note that H_i is contained in the i-th copy of K.

The case of a general nest subalgebra A of M_n is deduced by first restricting ρ to the upper triangulars A_u, applying the above result to obtain (π,U), and observing that the desired relations also hold for all A in A as well as just in A_u. To see this it will be sufficient to let $i < j$ such that $e_{ji} \in A$ and show that $X^n \rho(e_{ji}) = P_H U^n \pi(e_{ji})|_H$.

Let $W_i: H_i \to K_0$ be the isometric inclusion obtained above, so that $W: H \to K$ defined by $W(h_1,\ldots,h_n) = (W_1 h_1,\ldots,W_n h_n)$ satisfies $X^n \rho(A) = W^* U^n \pi(A) W$, for A in A_u. In terms of operator matrices this says that,

$$X_i^n T_{ij} = W_i^* U_1^n W_j,$$

for $n = 0,1,2,\ldots,$ and $1 \le i \le j \le n$, with $T_{ii} = I_{H_i}$. Since $\rho(E_{ij})\rho(E_{ji}) = \rho(E_{ii})$, we have that $T_{ij}T_{ji} = I_{H_i}$. Hence, $W_i^* W_j W_j^* W_i = W_i^* W_i$ and so $W_j W_j^* W_i = W_i$. Thus, $X_j^n T_{ji} = (W_j^* U_1^n W_j)(W_j^* W_i) = W_j^* U_1^n W_i$, and so the operator matrix $X^n \rho(E_{ji})$ is equal to $W^* U^n \pi(E_{ji}) W$. After again identifying H with WH, we obtain the desired result. □

What we have really shown in the above proof, is that the relations $X_i T_i = T_i X_{i+1}$, $i = 1,\ldots,n-1$, have a representation $(U_1, W_1,\ldots,W_n)$, where U_1 is unitary and the W_i are isometries, such that $X_i^n T_i = W_i^* U_1^n W_{i+1}$, and $W_i W_i^* W_{i+1} = W_{i+1}$. The initial relations determine a representation $\rho_{\{T_i\}}$ and commuting contraction X, while the latter clearly yield the dilation.

Let $\mathcal{A} \subseteq M_n$ be a nest algebra and let $M_n(C(\mathbb{T})) = M_nC(\mathbb{T})$ denote the algebra of $n \times n$ matrices with entries from $C(\mathbb{T})$. We identify $\mathcal{A} \otimes A(\mathbb{D})$ with the subalgebra of $M_n(C(\mathbb{T}))$ consisting of those matrices of functions (f_{ij}) such that f_{ij} belongs to $A(\mathbb{D})$ and $f_{ij} = 0$ if e_{ij} does not belong to $\mathcal{A}$. The next corollary is an immediate consequence of the last theorem and the complete contractivity of compression mappings and *-representations. By Arveson's dilation theorem it is in fact equivalent to Theorem 1.2.

<u>COROLLARY 1.3</u>. Let $\mathcal{A}$ be a finite dimensional nest algebra and let $\rho_1 : \mathcal{A} \to L(H)$ and $\rho_2 : A(\mathbb{D}) \to L(H)$ be commuting contractive representations. Then the representation $\rho_1 \otimes \rho_2$ of $\mathcal{A} \otimes A(\mathbb{D})$ defined by $\rho_1 \otimes \rho_2((f_{ij})) = \sum_{i,j} \rho_1(f_{ij})\rho_2(e_{ij})$ is completely contractive.

<u>COROLLARY 1.4</u>. (Ball and Gohberg) Let $\mathcal{A}$ be a finite dimensional nest algebra with enveloping matrix algebra M_n, let (ρ, H) be a representation of $\mathcal{A}$ with a contractive M_n-dilation (π, K), and let X be an operator on H such that $X\rho(A) = \rho(A)X$ for all operators A in $\mathcal{A}$. Then there exists an operator Y on K such that $\|Y\| = \|X\|$, $Y\pi(A) = \pi(A)Y$ for all A in M_n, and $X = P_H Y|_H$.

Proof: Let M be the minimal reducing subspaces for $\pi(M_n)$ which contains the subspace H. Then the associated restriction representation is a minimal M_n-dilation of (ρ,H), and is unique up to the usual notion of unitary equivalence of dilations.

Without loss, let X be a contraction, and let (π_1,K_1) and U in $L(K_1)$ be the commuting dilations of (ρ,H) and X provided by Theorem 1.1. If M_1 is the minimal reducing subspace for $\pi_1(M_n)$ containing H, then (π,M) and (π_1,M_1) are unitarily equivalent dilations, and so we may identify them. Define $Y_0 = P_M U|_M$ and note that Y_0 commutes with the operators $\pi(A)|_M$. Let $Y = Y_0 \oplus 0$ on $M \oplus M^{\perp} = K$ and we are finished. □

Remark 1.5. The intertwining version of the lifting theorem concerns an operator X satisfying $X\rho_1(A) = \rho_2(A)X$ for all A in A, where ρ_1 and ρ_2 are contractive representations of the nest algebra A. The existence of an intertwining extension for dilations π_1, π_2 of ρ_1, ρ_2 follow easily from the theorem above and the familiar observation that the contractive representation $\rho = \rho_2 \oplus \rho_1$ commutes with the operator

$$\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} .$$

1.6. Ball and Gohberg provide two proofs of the lifting theorem above both of which are quite different from ours. The most elaborate of these, which also yields information about all the commuting liftings, makes use of the Krein space approach to the analysis of invariant subspaces for representations of nest algebras ([3],[4]). The other argument uses a dual extremal formulation and a use of the Hahn-Banach theorem. This latter argument is analogous to Sarason's proof of his early version of the lifting theorem for contractions related to the unilateral shift.

1.7. A different proof of Theorem 1.2 can be given that is similar to arguments used to deduce Ando's theorem from the Sz.-Nagy-Foias lifting theorem as discussed by Parrott [7]. Here the lifting theorem is used to obtain a dilation $\tilde{T}$ of T commuting with the isometric dilation $\tilde{X}$ of the contraction X. At this point it must be observed that the pair $\tilde{T}, \tilde{X}$ provide a commuting power dilation of the commuting pair T, X. Next an extension $\hat{T}$ of $\tilde{T}$ is constructed using the unitary dilation $\hat{X}$ of $\tilde{X}$, so that $\hat{T}$ and $\hat{X}$ provide a commuting power dilation for the pair T, X. In fact $\hat{T}$ is essentially the strong limit of the sequence $(\hat{X}^*)^n\tilde{T}\hat{X}^n$. In this way the dilation problem is reduced to the case of a commuting pair where one of the contractions is unitary, and there are direct methods to treat this.

Suppose now that we have, as before, contractions on a common Hilbert space satisfying the relations $X_i T_i = T_i X_{i+1}$, $i = 1,\ldots,n-1$, and hence a representation ρ of the upper triangular $n \times n$ matrix algebra commuting with the contraction $X_1 \oplus \ldots \oplus X_n$. Let $\tilde{X}$ and $\hat{X}$ be the natural isometric and unitary dilations respectively for X, with summands on a common dilation space. Then, using the lifting theorem, we can obtain dilations $\tilde{T}_i$ of T_i, satisfying the dilated relations, and hence a representation $\tilde{\rho}$ of ρ such that $\tilde{\rho}$ and $\tilde{X}$ are a commuting dilating pair for ρ and X. As in the last paragraph we next construct the norm preserving extension $\hat{T}_i$ of $\tilde{T}_i$ at the strong limit of the sequence $(\hat{X}_i^*)^n \tilde{T}_i \hat{X}_{i+1}^n$, to obtain a representation $\hat{\rho}$ such that $\hat{\rho}$, X form a commuting dilating pair for $\hat{\rho}$, $\hat{X}$. Once more we have reduced to the case where X is a unitary contraction and various direct methods can be used for this case. One such method is indicated in the next remark.

<u>1.8</u>. For doubly commuting contractions Ando's theorem has a more elementary proof. Similarly, if both X and X^* commute with the representation ρ in the statement of Theorem 1.2, then we can provide more elementary arguments. A useful result in this context is the lifting theorem of Arveson for the commutant of the range of a completely

positive mapping (see [2] and [8, p.162]): if π is a unital *-representation of a C*-algebra B, on the Hilbert space K, and if $P: K \to H$ is an orthogonal projection, then there is a *-isomorphism from the commutant $\{P\pi(B)P\}'$ onto $\{\pi(B)\}' \cap \{P\}'$. Using this principle we can obtain a dilation π_1 of ρ commuting with X_1 and X_1^*, where X_1 is a dilation of X. Applying the principle again, for the C*-algebra generated by the unitary dilation U of X_1, we obtain a representation π commuting with U, with the required properties.

2. Commuting contractive representations of finite dimensional nest algebras.

We now turn to the proof of an Ando-type dilation theorem for a pair of commuting contractive representations of finite dimensional nest algebras.

THEOREM 2.1. Let ρ_1 and ρ_2 be contractive unital representations of the finite dimensional nest algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, on the common Hilbert space $\mathcal{H}$, such that $\rho_1(A_1)\rho_2(A_2) = \rho_2(A_2)\rho_1(A_1)$ for all A_i in $\mathcal{A}_i$, $i = 1,2$. Then there exist unital *-representations π_1, π_2 of the enveloping matrix algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, on a Hilbert space $\mathcal{K} \supset \mathcal{H}$, such that

(i) $\rho_1(A_1)\rho_2(A_2) = P_{\mathcal{H}}\pi_1(A_1)\pi_2(A_2)|_{\mathcal{H}}$,

(ii) $\pi_1(B_1)\pi_2(B_2) = \pi_2(B_2)\pi_1(B_1)$,

for all A_i in $\mathcal{A}_i$ and B_i in $\mathcal{B}_i$, $i = 1,2$.

<u>Proof</u>. Assume first that A_1 and A_2 are the algebras of upper triangular $n \times n$ and $m \times m$ matrices, respectively, spanned by the matrix units e_{ij}, $1 \le i \le j \le n$, and f_{ij}, $1 \le i \le j \le m$, respectively. Let $H_i = \rho_1(e_{ii})H$, $1 \le i \le n$, and let $H_{i,j} = \rho_2(f_{jj})H_i$ for $1 \le j \le m$. Without loss we may assume that $H_{i,j} = H_{1,1}$ for all i, j. With respect to the decomposition $H_1 \oplus \ldots \oplus H_n$ the operators $T = \rho_1(e_{1,2} + \ldots + e_{n-1,n})$ and $X = \rho_2(f_{1,2} + \ldots + f_{m-1,m})$ have representing operator matrices

$$T = \begin{bmatrix} 0 & T_1 & & & \\ & 0 & T_2 & & \\ & & \ddots & & \\ & & & 0 & T_{n-1} \\ & & & & 0 \end{bmatrix},$$

$$X = \begin{bmatrix} X_1 & & & \\ & X_2 & & \\ & & \ddots & \\ & & & X_n \end{bmatrix},$$

and with respect to $\mathcal{H}_i = \mathcal{H}_{i,1} \oplus \ldots \oplus \mathcal{H}_{i,m}$ we have

$$T_i = \begin{bmatrix} T_{i,1} & & & \\ & T_{1,2} & & \\ & & \ddots & \\ & & & T_{i,m} \end{bmatrix}$$

for $1 \le i \le n-1$, and,

$$X_i = \begin{bmatrix} 0 & X_{i,1} & & & \\ & 0 & X_{i,2} & & \\ & & \ddots & \ddots & \\ & & & \ddots & X_{i,m-1} \\ & & & & 0 \end{bmatrix}$$

for $1 \le i \le n$. Note that X commutes with T if and only if $X_{i,j}T_{i,j+1} = T_{i,j}X_{i+1,j}$, for $1 \le i \le n-1$ and $1 \le j \le m-1$. Conversely if we have operators satisfying these relations then the operators $T_1,\ldots,T_{n-1}$ determine a representation of $\mathcal{A}_1$ commuting with the representation

$\rho = \rho_2^1 \oplus \ldots \oplus \rho_2^n$ of A_2 on H, determined by the representations ρ_2^i of A_2 on H_i associated with the contractions $X_{i,1}, \ldots, X_{i,m-1}$, for $1 \le i \le n$.

By Theorem 1.1, and its proof, if $\tilde{X}_{i,j}$ is the usual isometric dilation of X_{ij} on $\tilde{H}_{i,j} = H_{i,j} \oplus H_{i,j} \oplus \ldots$ for $1 \le i \le n$, $1 \le j \le m-1$, then there are dilations $\tilde{T}_i = \tilde{T}_{i,1} \oplus \ldots \oplus \tilde{T}_{i,m}$ of T_i, $1 \le i \le n-1$, such that $\tilde{X}_{i,j}\tilde{T}_{i,j+1} = \tilde{T}_{i,j}\tilde{X}_{i+1,j}$. Hence we obtain associated commuting contractive representations $\tilde{\rho}_1$ and $\tilde{\rho}_2$. Moreover, in view of the special form of the operators $\tilde{T}_{i,j}$, products of the operator $\tilde{T}_{i,j}$ dilate the corresponding products of the operators T_{ij}, and hence $\rho_1(A_1)\rho_2(A_2) = P_H\tilde{\rho}_1(A_1)\tilde{\rho}_2(A_2)|_H$, for A_i in A_i, $i = 1,2$.

Exchanging the roles ρ_1 and ρ_2 in the argument above we may assume that $\tilde{\rho}_1$ is the dilation of ρ_1 obtained by the canonical isometric dilations $\hat{T}_1, \ldots, \hat{T}_{n-1}$ of $T_1, \ldots, T_{n-1}$, and that $\tilde{\rho}_2$ is a contractive commuting dilation such that this pair $\tilde{\rho}_1$, $\tilde{\rho}_2$ dilate the pair ρ_1, ρ_2. Write $\hat{X}_1 \oplus \ldots \oplus \hat{X}_n$ for the dilation $\tilde{\rho}_2(f_{1,2} + \ldots + f_{n-1,n})$ of X.

As in the proof of Theorem 1.2, define the isometry W on $\tilde{H}$ by $W(\hat{h}_1, \ldots, \hat{h}_n) = (\hat{h}_1, \hat{T}_1 h_2, \ldots, \hat{T}_1 \ldots \hat{T}_{n-1} h_n)$, and define the *-isomorphism $\sigma_1 : M_n \to L(\tilde{H})$ by $\sigma_1(e_{ij}) = \tilde{E}_{ij}$, where $\hat{E}_{ij}$ is the partial isometry identifying the j^{th} and i^{th} summands of $\tilde{H}$. Let $Y = \hat{X}_1 \oplus \ldots \oplus \hat{X}_1$ (n times), and observe that, as before, for A in A_1,

$$X^n \rho_1(A) = P_H Y^n \sigma_1(A)|_H$$

where we have identified H with $WH \subset \hat{H}$. Using Y we can construct a contractive unital representation $\tau = \tau_1 \oplus \ldots \oplus \tau_1$ (n times) of A_2, which commutes with σ_1, and satisfies

$$\rho_2(A_2)\rho_1(A_1) = P_H \tau(A_2)\sigma_1(A_1)|_H$$

for A_i in A_i, $i = 1,2$. We have now reduced to the case where one of the representations is an inflation and this can be dealt with in a very explicit way. Let $\pi_2^1 : M_m \to L(K_1)$, $K_1 \supset H_1$ be the canonical *-dilation of τ_1. Let $\pi_2 = \pi_2^1 \oplus \ldots \oplus \pi_2^1$ (n times) on $K = K_1 \oplus \ldots \oplus K_1$ and let $\pi_1 : M_n \to L(K)$ be the obvious representation, which dilates σ_1 and commutes with π_2. Then π_1 and π_2 give the desired dilation of ρ_1 and ρ_2.

The case of general finite dimensional nest algebras, A_1, A_2 is now derived by first restricting ρ_1 and ρ_2 to the upper triangular subalgebras $A_{1,u}$, $A_{2,u}$ respectively and obtaining the dilating commuting pair π_1, π_2 for the restrictions of ρ_1 and ρ_2. The argument in the final paragraph of the proof of Theorem 1.1 already shows that π_1 and π_2 necessarily have the dilation properties for A_1, A_2.

3. Dilation and lifting theorems.

We now generalise the results of the last two sections to general nest algebras acting on a separable Hilbert space $\mathcal{R}$. Our method is to use the semidiscretness of nest algebras to obtain the complete contractivity of a representation of a spatial tensor product algebra associated with the given representations.

It was shown in [9] that a nest algebra $\mathcal{A}$ on a separable Hilbert space is semidiscrete in the sense that there are finite dimensional nest algebras $\mathcal{A}_1, \mathcal{A}_2, \ldots,$ completely contractive σ-weakly continuous maps $\varphi_n: \mathcal{A} \to \mathcal{A}_n$, and completely isometric σ-weakly continuous homomorphisms $\psi_n: \mathcal{A}_n \to \mathcal{A}$, such that $\psi_n \circ \varphi_n(A) \to A$ σ-weakly for all A in $\mathcal{A}$. Moreover, we can arrange that $\operatorname{dist}(K, \psi_n(\mathcal{A}_n)) \to 0$ for each compact operator K in $\mathcal{A}$, and we shall need this extra detail in the proofs below.

THEOREM 3.1. Let $\mathcal{A}$ be a nest algebra on a separable Hilbert space $\mathcal{R}$, let ρ be a σ-weakly continuous contractive representation of $\mathcal{A}$ on $\mathcal{H}$, and let X be a contraction on $\mathcal{H}$ that commutes with $\rho(\mathcal{A})$. Then there is an inflation $\pi: \mathcal{L}(\mathcal{R}) \to \mathcal{L}(\mathcal{R} \oplus \mathcal{R} \oplus \ldots)$ given by $\pi(A) = A \oplus A \oplus \ldots$, with at most countably many copies, a unitary U that commutes with $\pi(\mathcal{A})$, and an isometry $V: \mathcal{H} \to \mathcal{R} \oplus \mathcal{R} \oplus \ldots$, such that

$$X^n \rho(A) = V^* U^n \pi(A) V$$

for all $n = 0,1,2,\ldots,$ and A in $\mathcal{A}$.

<u>Proof</u>. Let $\mathcal{B}_1$ denote the C*-algebra generated by the compact operators and the identity and let $\mathcal{A}_1 = \mathcal{A} \cap \mathcal{B}_1$. We regard $\mathcal{A}_1 \otimes A(\mathbb{D})$ as a subalgebra of the C*-algebra $\mathcal{B}_1 \otimes C(\mathbb{T})$.

Let $\mathcal{C}_1, \mathcal{C}_2, \ldots$ be subalgebras of $\mathcal{A}$ which are completely isometric images of finite dimensional nest algebras, and satisfy $\text{dist}(K, \mathcal{C}_n) \to 0$ for every compact operator K in $\mathcal{A}$. Clearly, $\text{dist}(A, \mathcal{C}_n) \to 0$ for every A in $\mathcal{A}_1$.

By Corollary 1.3, X and $\rho|\mathcal{C}_n$ gives rise to a completely contractive representation of $\mathcal{C}_n \otimes A(\mathbb{D})$. From this it follows that X and $\rho|\mathcal{A}_1$ gives rise to a completely contractive representation of the algebra $\mathcal{A}_1 \otimes A(\mathbb{D})$.

Hence, there exists a separable Hilbert space $\mathcal{K}$, a *-homomorphism $\pi: \mathcal{B}_1 \to \mathcal{L}(\mathcal{K})$, a unitary U on $\mathcal{K}$ which commutes with $\pi(\mathcal{B}_1)$, and an isometry $V: \mathcal{H} \to \mathcal{K}$ such that

$$X^n \rho(A) = V^* U^n \pi(A) V,$$

$n = 0,1,2,\ldots,$ and A in $\mathcal{A}_1$.

The $*$-homomorphism π decomposes as $\pi_1 \oplus \pi_0$ on $K = K_1 \oplus K_0$ with π_1 faithful on the compacts and π_0 zero on the compacts. Relative to this decomposition $U = U_1 \oplus U_0$ with U_i in the commutant of $\pi_i(B_1)$, $i = 1,2$.

Now using the σ-weak continuity of ρ, and choosing a sequence K_n of compacts in A_1 which converges σ-weakly to the identity (see [13] or [5]), we see that in fact, $VH \subseteq K_1$ and $X^n \rho(A) = V^* U_1^n \pi_1(A) V$ for A in A_1. Note that π_1 is, up to unitary equivalence, a countable direct sum of the identity representation. Hence, π_1 is σ-weakly continuous, and since A_1 is σ-weakly dense in A the remainder of the proof follows. □

The following corollary generalises the Ball-Gohberg theorem and is obtained easily from Theorem 3.1 and elementary arguments.

<u>COROLLARY 3.2</u>. Let A be a nest algebra on R, let ρ be a σ-weakly continuous contractive representation of A on H, with σ-weakly continuous $L(R)$-dilation π on $K \supset H$, and let X be an operator commuting with the range of ρ. Then there exists an operator Y on K which commutes with the range of π and satisfies $\|Y\| = \|X\|$, $X = P_H Y|_H$.

<u>THEOREM 3.3</u>. Let $\mathcal{A}_1, \mathcal{A}_2$ be nest algebras on separable Hilbert spaces $\mathcal{R}_1, \mathcal{R}_2$. Let ρ_1, ρ_2 be σ-weakly continuous representations of $\mathcal{A}_1$ and $\mathcal{A}_2$ on the separable Hilbert space $\mathcal{H}$, such that $\rho_1(A_1)\rho_2(A_2) = \rho_2(A_2)\rho_1(A_1)$ for all A_i in $\mathcal{A}_i$, $i = 1,2$. Then there exist σ-weakly continuous *-isomorphisms π_1, π_2 of $\mathcal{L}(\mathcal{R}_1)$ and $\mathcal{L}(\mathcal{R}_2)$ on a separable Hilbert space $\mathcal{K} \supseteq \mathcal{H}$, such that

(i) $\rho_1(A_1)\rho_2(A_2) = P_{\mathcal{H}}\pi_1(A_1)\pi_2(A_2)|_{\mathcal{H}}$,

(ii) $\pi_1(B_1)\pi_2(B_2) = \pi_2(B_2)\pi_1(B_1)$,

for all A_i in $\mathcal{A}_i$, B_i in $\mathcal{L}(\mathcal{R}_i)$, $i = 1,2$.

<u>Proof</u>. Let $\mathcal{C}_1^{(i)}, \mathcal{C}_2^{(i)}, \ldots,$ $i = 1,2$, be subalgebras of $\mathcal{A}_i$ which are completely isometric images of finite dimensional nest algebras, and which satisfy $\text{dist}(K_i, \mathcal{C}_n^{(i)}) \to 0$ for every compact operator K_i in $\mathcal{A}_i$. Let $\mathcal{A}_i^1$ be the C*-algebra generated by the compact operators in $\mathcal{A}_i$ together with the identity operator.

By Theorem 3.1 the representation $\rho_1 \otimes \rho_2$ restricted to $\mathcal{C}_n^{(1)} \otimes \mathcal{C}_n^{(2)}$ is completely contractive. From this it follows that $\rho_1 \otimes \rho_2$ is completely contractive on the operator algebra $\mathcal{A}_1^1 \otimes \mathcal{A}_2^1 \subset \mathcal{L}(\mathcal{R}_1 \otimes \mathcal{R}_2)$. Hence there exists a separable Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a *-isomorphism π of $\mathcal{B}_1 \otimes \mathcal{B}_2$ (where $\mathcal{B}_i$ is the C* algebra generated by the identity and compacts on R_i) which dilates $\rho_1 \otimes \rho_2$. As in the proof of Theorem 3.1 π decomposes as $\pi_1 \oplus \pi_0$ on $K_1 \oplus K_0$ with π_1 faithful on the compacts and π_0 zero on the compacts. Using the σ-weak continuity of $\rho_1 \otimes \rho_2$ and choosing sequences of compact operators K_n^i in A_i, which converge σ-weakly to the identity, we see that $H \subset K_1$, and that the restriction representations $\pi|B_1$ and $\pi|B_2$ provide the desired commuting dilations of ρ_1 and ρ_2. □

4. Generalised Hankel operators

It is well known that Nehari's theorem for Hankel operators on the Hardy space H^2 is a simple consequence of the Sz.-Nagy-Foias lifting theorem. Ball and Gohberg obtained an analogous Nehari theorem in the triangular matrix context, where triangular truncation replaces the Riesz projection. More general Nehari type theorems were also obtained independently in [11], [12], for general nest algebras and for nest subalgebras of semi-finite factors, the main tools there being generalised Riesz factorisation, and Arveson's distance formula. Here we note how such results and Arveson's distance formula follow from the lifting theorem, Theorem 3.1.

To prove these results, it will be useful to consider anti-representations, i.e., multiplication reversing representations. A general principle says that every dilation theorem about representations has a corresponding statement for anti-representations and we wish to point out why this is so. Let A be a subalgebra of the C*-algebra B and suppose that $\rho: A \to L(H)$ is a contractive anti-representation and we wish to know if ρ dilates to a *-anti-homomorphism $\pi: B \to L(K)$, $H \subseteq K$. We call this an anti-dilation. If we let B_{op} denote B with multiplication reversed then B_{op} is a C*-algebra and π is a *-homomorphism on B_{op}. Moreover, ρ is a representation of the subalgebra A_{op}. Thus, by Arveson's theorem it is enough to know that ρ is completely contractive on A_{op}. We must be careful though because the norms on $M_n(A_{op})$ are inherited from $M_n(B_{op})$. We use $\|(b_{ij})\|_{op}$ to denote the norm of (b_{ij}) in $M_n(B_{op})$. We leave it to the reader to check that $\|(b_{ij})\|_{op} = \|(b_{ij})^t\|$, where t denotes the transpose. Thus, to see that an anti-homomorphism has an anti-dilation one needs to verify that

$$\|(\rho(a_{ij}))\| \leq \|(a_{ij})\|_{op} = \|(a_{ij})^t\|.$$

Now if A is a nest algebra and $\rho: A \to L(H)$ is a contractive anti-representation, consider $\tilde{\rho}: A \to L(H)_{op}$, $\tilde{\rho}(a) = \rho(a)$. This is a contractive representation, and so completely contractive. Thus, $\|(a_{ij})\| \geq \|(\tilde{\rho}(a_{ij}))\| = \|(\rho(a_{ij}))\|_{op} = \|(\rho(a_{ij}))^t\|$ from which it follows that ρ has an anti-dilation. Hence, we have that every contractive anti-representation of a nest algebra has an anti-dilation.

Similar, arguments yield "anti" versions of our other theorems concerning nest algebras and we use these freely in what follows.

Let E be a complete nest of projections on a separable Hilbert space R, with nest algebra A. Let C_2 be the Hilbert space of Hilbert-Schmidt operators on R and let $H^2(E) = C_2 \cap A$ be the upper triangular subspace, with orthogonal projection $P: C_2 \to H^2(E)$. For X in $L(R)$ define the generalised multiplication operator L_X on $\mathcal{C}_2$ and the generalised Hankel operator $H_X: H^2(\mathcal{E}) \to (H^2(\mathcal{E}))^{\perp}$, by

$$L_X T = XT, \qquad T \in \mathcal{C}_2,$$

$$H_X A = P^{\perp} L_X |_{H^2(\mathcal{E})} .$$

THEOREM 4.1. Let $X \in \mathcal{L}(\mathcal{R})$. Then there exist an operator $Y \in \mathcal{L}(\mathcal{R})$ such that $H_X = H_Y$ and $\|Y\| = \|H_X\|$. Moreover,

$$\|H_X\| = \text{dist}(X, \mathcal{A}) = \sup_{E \in \mathcal{E}} \|(I-E)XE\| .$$

Proof. Let ρ_1, ρ_2 be the σ-weakly continuous contractive unital anti-representations of $\mathcal{A}$ on $H^2(\mathcal{E})$ and $(H^2(\mathcal{E}))^{\perp}$ given by

$$\rho_1(A) = R_A |_{H^2(\mathcal{E})} ,$$

$$\rho_2(A) = P^{\perp} R_A |_{(H^2(\mathcal{E}))^{\perp}} ,$$

where R_A is the right multiplication operator on $\mathcal{C}_2$, $R_A T = TA$. Then, for A_1 in $H^2(\mathcal{E})$ and A in $\mathcal{A}$ we have

$$\begin{aligned}\rho_2(A)H_X A_1 &= \rho_2(A)P^{\perp}(XA_1)\\ &= P^{\perp}((P^{\perp}(XA_1))A)\\ &= P^{\perp}((P^{\perp}(XA_1) + P(XA_1))A)\\ &= P^{\perp}((XA_1)A)\\ &= P^{\perp}(X(A_1A))\\ &= H_X\rho_1(A)A_1 \quad .\end{aligned}$$

By the intertwining version of the antirepresentation version of Theorem 3.1, there is a operator $\tilde{Y}$ on C_2 such that

(i) $\|\tilde{Y}\| = \|H_X\|$,

(ii) $\pi_2(B)\tilde{Y} = \tilde{Y}\pi_1(B)$, $B \in L(R)$,

(iii) $H_X = P^{\perp}\tilde{Y}|_{H^2(E)}$,

where π_1 and π_2 are the *-anti-isomorphisms of $L(R)$ on C_2 given by $\pi_i(B) = R_B$, and which are $R(L)$-dilations of ρ_1, ρ_2.

Condition (ii) implies that $\tilde{Y} = L_Y$ for some operator Y in $L(R)$ with $\|Y\| = \|\tilde{Y}\|$, and so the first part of the theorem follows. Note that if $H_X = H_Y$ then $A = X-Y$ belongs to A, and so $\text{dist}(X,A) \le \|Y\| = \|H_X\|$. The inequality $\|H_X\| \le \text{dist}(X,A)$ is elementary, and so the first equality holds. It remains only to show that

$$\|H_X\| = \sup_{E\in\mathcal{E}}\|(I-E)XE\|.$$

Note that if $Q = E-E_-$ is an atom of $\mathcal{E}$ then $\mathcal{C}_2Q$ is a reducing subspace for L_X and

$$H_X|H^2(\mathcal{E})Q = H_X|E\mathcal{C}_2Q = L_{E^\perp XE}|\mathcal{C}_2Q.$$

If $\mathcal{C}$ is purely atomic then $\mathcal{C}_2 = \oplus\mathcal{C}_2Q$, where the direct sum is taken over all atoms, and so $\|H_X\| = \sup\|L_{E^\perp XE}|\mathcal{C}_2Q\| = \sup\|E^\perp XE\|$, as desired.

In a general nest it is easy to see that if $F < E$ then $\|H_X\| \geq \|(I-E)XF\|$, by considering the subspace $F\mathcal{C}_2(E-F)$ of $H^2(\mathcal{E})$. Thus if $E_- = E$ we have $\|H_X\| \geq \|(I-E)XE\|$. Our earlier reasoning gives this inequality when E is an atom ($E \neq E_-$) and so it follows that we need only show that $\|H_X\|$ is dominated by $\sup_{E\in\mathcal{E}}\|(I-E)XE\|$. Choose $A \in H^2(\mathcal{E})$ and $B \in (H^2(\mathcal{E}))^\perp$ of unit norm so that $\langle XA,B\rangle \geq \|H_X\|-\epsilon$. There is a finite nest $\mathcal{E}_n \subset \mathcal{E}$ so that $\|P_n^\perp B-B\|_{\mathcal{C}_2} < \epsilon\|X\|$, where P_n is the trucation operator for $H^2(\mathcal{E}_n)$. Let $B_1 = P_n^\perp B$ and note that $H^2(\mathcal{E}_n) \supset H^2(\mathcal{E})$. Then, using the formula in the finite (purely atomic) case, we have

$$\max_{E \in \mathcal{E}_n} \|(I-E)XE\| = \|\mathcal{P}_n^{\perp} L_X \mathcal{P}_n\|$$

$$\geq |\langle XA, \mathcal{P}_n^{\perp} B\rangle|$$

$$\geq \||\langle XA, B\rangle| - \epsilon$$

$$\geq \|H_X\| - 2\epsilon,$$

and so

$$\sup_{E \in \mathcal{E}} \|(I-E)XE\| \geq \|H_X\|$$

as desired. □

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June, 1987

CHAPTER 6 SCHUR PRODUCTS, MATRIX COMPLETIONS, AND DILATION THEORY

We saw in Chapter 4 that a contractive Hilbert space representation of a finite dimensional nest algebra is completely contractive, and this result served as the cornerstone for a general dilation theory for nest algebras. To carry out a similar program for reflexive algebras with commutative subspace lattice, the so called CSL-algebra, requires an examination of Schur product maps and inflated Schur product maps on certain subspaces and subalgebras of M_n. Such a study has considerable independent interest and, as we shall see, is closely tied to completion problems for partially defined matrices.

In the next two sections we prove that a necessary and sufficient condition for a given partially positive matrix to have a positive completion is that a certain Schur product map defined on a certain subspace of matrices is a positive map. By analysing the positive elements of this subspace we obtain new proofs of the results of Dym-Gohberg [5] and Grone-Johnson-Sa-Wolcowitz [7]. We also observe that Arveson's distance formula is a consequence of this analysis. In the third and fourth sections we give some applications and some generalisations of these results to partially defined operator matrices. In the last section we discuss dilation theory and various open problems.

1. Introduction. An n x n complex matrix is partially defined if only some of its entries are specified with the unspecified entries treated as complex variables. A completion of a partially defined matrix is simply a specification of the unspecified entries. Matrix completion problems are concerned with determining whether or not a completion of a partially defined matrix exists which enjoys some property, e.g., contraction, positive, Toeplitz, Hankel. Generally, one knows that every fully specified submatrix already has this property.

Perhaps the best known result of this type is due to Dym-Gohberg [5]. They proved that if $T = (t_{ij})$ is a partially defined n x n matrix, $i,j = 1,\dots,n$, such that t_{ij} is defined only for $|i-j| \le k$, where $0 < k < n-1$, which has the property that all its fully defined k x k principal submatrices are positive semi-definite, then T can be completed to a positive semi-definite matrix. That is, if we are given complex numbers $\{t_{ij}\}$, $i,j = 1, \dots, n$, $|i-j| \le k$ such that each k x k matrix $T_l = (t_{l+i,\, l+j})$, $i,j = 1,\dots,k$ is positive semi-definite, $l=0, \dots, n-k$, then we may choose $\{t_{i,j}\}$, $|i-j| > k$ such that $T = (t_{i,j})$ is a positive semi-definite matrix. This result is usually summarized by saying that every partially positive banded matrix has a positive completion. Dym-Gohberg [5] also proved the analogous result for block-banded patterns.

The best result about positive completions is due to [7]. Before describing this result, it will be convenient to fix some notation.

A subset J of $\{1,\dots,n\} \times \{1,\dots,n\}$ will be called a pattern. A partially defined n x n matrix $T = (t_{ij})$ will be said to have pattern J if t_{ij} is specified if and only if $(i,j) \in J$. A pattern J will be called

<u>symmetric</u> if $(i,i) \in J$ for all i and if $(i,j) \in J$ then $(j,i) \in J$. A partially defined matrix T will be called <u>symmetric</u> provided that its pattern J is symmetric, that t_{ii} is real for all i, and that whenever t_{ij} is specified, then $t_{ji} = \bar{t}_{ij}$.

Let T be a partially defined n x n matrix with pattern J. By a <u>specified submatrix</u> of T we mean any K x L matrix of the form $B=(b_{k,l})$, where $b_{k,l} = t_{i_k,j_l}$ and $(i_k,j_l) \in J$ for $1 \le k \le K$, $1 \le l \le L$. A <u>principal specified submatrix</u> of T is a k x k specified submatrix $B = (b_{k,l})$ with $b_{k,l} = t_{i_k,i_l}$ where $(i_k,i_l) \in J$ for $1 \le k,l \le K$.

Throughout this paper, we shall use positive to mean positive semidefinite.

A partially defined matrix T is <u>partially positive</u> if it is symmetric and if every principal specified submatrix of T is positive.

Clearly, a necessary condition for a partially defined symmetric matrix to have a positive completion is that it is partially positive. However, not every partially positive matrix can be completed to a positive matrix, examples have been given in [7], and in section 3, we give a means of generating many new examples.

We give (Theorem 2.1) a necessary and sufficient condition for a given partially positive matrix to have a positive completion.

In [7], a characterization is given of those symmetric patterns J such that <u>every</u> partially positive matrix with pattern J has a positive completion. Their result implies the results of Dym-Gohberg cited above since the banded and block-banded patterns can be easily seen to meet this characterization. Not surprisingly, the characterization of these patterns in [7] is combinatorial. We describe this characterization in section 2.

To each pattern J we associate a subspace $\mathcal{S}_J$ of the n x n matrices, M_n, by setting,

$$\mathcal{S}_J = \{ (a_{ij}) \in M_n: a_{ij} = 0 \text{ if } (i,j) \notin J \}.$$

If $T = (t_{ij})$ is a partially defined matrix with pattern J, then T yields a well-defined linear map $\Phi_T: \mathcal{S}_J \to \mathcal{S}_J$ via $\Phi_T((a_{ij})) = (a_{ij}t_{ij})$

We shall refer to such maps as Schur product maps.

We prove in section 2, that a partially positive matrix T has a positive completion if and only if Φ_T is a positive map. That is, if and only if $\Phi_T(P)$ is positive for every positive P in $\mathcal{S}_J$. This result yields different proofs of theorems of Grone-Johnson-Sa-Wolkowitz [7], Dym-Gohberg [5], and Haagerup [8]. In section 3, we study generalizations of these results to partially defined matrices of operators.

There is another characterization of the above subspaces and maps which will be central. Let $\mathcal{D}_n \subseteq M_n$ be the subalgebra of M_n consisting of diagonal matrices. A $\mathcal{D}_n$-bimodule is a subspace of M_n which is invariant under left and right multiplication by elements of $\mathcal{D}_n$.

An operator system $\mathcal{S}$ is a subspace of a unital, C^*-algebra which contains the identity and has the property that if $S \in \mathcal{S}$ then $S^* \in \mathcal{S}$.

The following is immediate.

Proposition 1.1. Let $\mathcal{S} \subseteq M_n$ be a subspace, then $\mathcal{S}$ is a $\mathcal{D}_n$-bimodule if and only if $\mathcal{S} = S_J$ for some pattern J. Moreover $\mathcal{S}$ is also an operator system if and only if J is symmetric.

Let $\mathcal{S}_J$ be a $\mathcal{D}_n$-bimodule. A map $\Phi : \mathcal{S}_J \to M_n$ is a $\mathcal{D}_n$-<u>bimodule map</u> provided that $\Phi(D_1AD_2)=D_1\Phi(A)D_2$ for all $D_1,D_2 \in \mathcal{D}_n$ and $A \in \mathcal{S}_J$. It is not difficult to check that Φ is a $\mathcal{D}_n$-bimodule map if and only if there is a partially defined matrix T with pattern J such that $\Phi = \Phi_T$.

More generally, let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be Hilbert spaces, $\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n$, and let $\mathcal{L}(\mathcal{H})$ denote the bounded linear operators on $\mathcal{H}$. If for some pattern J we are given bounded linear operators $T_{ij}:\mathcal{H}_j \to \mathcal{H}_i$ for every $(i,j) \in J$, then we may define a linear map

$$\Phi_T: \mathcal{S}_J \to \mathcal{L}(\mathcal{H}) \text{ via } \Phi_T((a_{ij})) = (a_{ij}\, T_{ij}).$$

We shall refer to $T = (T_{ij})$ as a <u>partially defined operator matrix</u> and call Φ_T an <u>inflated Schur product map</u>. If we identify $D \in \mathcal{D}_n$ with the corresponding diagonal operator on $\mathcal{H}$, then we may regard $\mathcal{L}(\mathcal{H})$ as a $\mathcal{D}_n$-bimodule also. Clearly, a map $\Phi : \mathcal{S}_J \to \mathcal{L}(\mathcal{H})$ will be a $\mathcal{D}_n$-bimodule map if and only if it is the inflated Schur product map given by some partially defined operator matrix.

Our main technical tool will be a theorem of Arveson [1]. Let $\mathcal{A}$ be a C^*-algebra with 1, then there is a C^*-algebra consisting of n x n matrices with entries from $\mathcal{A}$, denoted $M_n(\mathcal{A})$. In the case of $\mathcal{L}(\mathcal{H})$ we can identify $M_n(\mathcal{L}(\mathcal{H}))$ with $\mathcal{L}(\mathcal{H} \oplus \ldots \oplus \mathcal{H})$ (n copies). If $\mathcal{S}$ is an operator system in $\mathcal{A}$, and $\Phi: \mathcal{S} \to \mathcal{L}(\mathcal{H})$ is a linear map, then we can define linear maps

$$\Phi^{(n)}: M_n(\mathcal{S}) \to M_n(\mathcal{L}(\mathcal{H})) \text{ via } \Phi^{(n)}((a_{ij})) = (\Phi(a_{ij})).$$

The map Φ is called <u>positive</u> if $\Phi(p)$ is positive for every positive p in $\mathcal{S}$, and <u>completely positive</u> if $\Phi^{(n)}$ is positive for every n.

<u>Arveson's [1] Extension Theorem 1.1</u>. Let $\mathcal{A}$ be a unital C^*-algebra, let $\mathcal{D}$ be a unital C^*-subalgebra of $\mathcal{A}$ and $\mathcal{L}(\mathcal{H})$, and let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system and $\mathcal{D}$-bimodule. Then every completely positive $\mathcal{D}$-bimodule map ϕ: $\mathcal{S} \to \mathcal{L}(\mathcal{H})$, can be extended to a completely positive $\mathcal{D}$-bimodule map on $\mathcal{A}$.

This theorem is proved, except for the $\mathcal{D}$-bimodule part, in [1]. The inclusion of the $\mathcal{D}$-bimodule action is standard and can be found in [3] or [10]. However, since the $\mathcal{D}$-bimodule version is not well-known, we indicate how it can be deduced from the usual version of Arveson's extension theorem for the special case of $\mathcal{D} = \mathcal{D}_n$.

Recall the Schwarz inequalities for completely positive maps [3]:

$$\phi(a)^*\phi(a) \le \|\phi(1)\|^2 \, \phi(a^*a),$$

$$\phi(a)\phi(a)^* \le \|\phi(1)\|^2 \, \phi(aa^*).$$

Now given a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n$, a subspace $\mathcal{S}_J \subseteq M_n$ and a $\mathcal{D}_n$-bimodule map $\phi = \phi_T : \mathcal{S}_J \to \mathcal{L}(\mathcal{H})$, let $\Phi: M_n \to \mathcal{L}(\mathcal{H})$ be any completely positive extension of ϕ. We argue that $\Phi = \Phi_{\tilde{T}}$ for some operator matrix $\tilde{T}$. To see this fix a matrix unit E_{ij}, so that $\Phi(E_{ij})$ has some operator matrix (B_{kl}). Applying the two Schwarz inequalities with $a = E_{ij}$, one finds that necessarily $B_{k,l} = 0$, except when $(k,l) = (i,j)$.

<u>Proposition 1.2</u>. Let $\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n$, let $T = (T_{ij}) \in \mathcal{L}(\mathcal{H})$ be an operator matrix, and let $\phi_T: M_n \to \mathcal{L}(\mathcal{H})$ be the inflated Schur product map associated with T. Then the following are equivalent:

i) ϕ_T is completely positive,

ii) ϕ_T is positive,

iii) T is positive.

Proof. Clearly, (i) implies (ii). Let P be the matrix of all 1's. Since P is positive, and $\Phi_T(P) = T$, we have that (ii) implies (iii).

Now assume that T is positive, let $h \in \mathcal{H}$, $h = h_1 \oplus \ldots \oplus h_n$, and let $A = (\bar{\alpha}_i \alpha_j)$ be a typical rank one positive in M_n. Then

$$\langle \Phi_T(A)h, h\rangle = \langle Th_\alpha, h_\alpha\rangle \geq 0, \text{ where } h_\alpha = (\alpha_1 h_1) \oplus \ldots \oplus (\alpha_n h_n).$$

Since every positive in M_n is a sum of rank 1 positives, we have that Φ_T is positive. Thus, (iii) implies (ii). But now notice that $\Phi_T^{(k)} = \Phi_{T^{(k)}}$ where $T^{(k)}$ is the operator matrix on $\mathcal{H} \oplus \ldots \oplus \mathcal{H}$ (k copies) which is T in every entry, i.e., $T^{(k)} = T \otimes P$ where P is the k x k matrix of 1's. Since $T^{(k)}$ is positive, $\Phi_T^{(k)}$ is positive and Φ_T is completely positive.

Corollary 1.3. Let J be a symmetric pattern, $\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n$ and let T be a partially defined operator matrix on $\mathcal{H}$ with pattern J. Then T has a positive completion if and only if the inflated Schur product map Φ_T is completely positive.

Proof. If T has a positive completion, $\tilde{T}$, then $\Phi_{\tilde{T}}$ is completely positive and hence so is $\Phi_T = \Phi_{\tilde{T}}|_{\mathcal{S}_J}$. Conversely, if Φ_T is completely positive, then by Arveson's extension theorem, it has a completely positive $\mathcal{D}_n$-bimodule extension $\tilde{\Phi}$ to M_n. But $\tilde{\Phi} = \Phi_{\tilde{T}}$ for some $\tilde{T}$ and clearly $\tilde{T}$ is a completion of T.

We also obtain a new proof of an old result of Choi's.

Corollary (Choi [4]) 1.4. Let $\psi: M_n \to \mathcal{L}(\mathcal{K})$, then ψ is completely positive if and only if $(\psi(E_{ij}))$ is positive, where E_{ij} are the standard matrix units.

Proof. If ψ is completely positive then $(\psi(E_{ij})) = \psi^{(n)}((E_{ij}))$ is positive. Conversely, if $T = (\psi(E_{ij}))$ is positive then $\Phi_T: M_n \to \mathcal{L}(\mathcal{K} \oplus \ldots \oplus \mathcal{K})$ (n copies) is completely positive. Also, the map $S: \mathcal{L}(\mathcal{K} \oplus \ldots \oplus \mathcal{K}) \to \mathcal{L}(\mathcal{K})$ defined by $S((B_{ij})) = \Sigma_{ij} B_{ij}$ can be easily seen to be completely positive. Hence, $\psi = S \circ \Phi_T$ is completely positive.

2. Matrix Completions. In Dym-Gohberg [5] and Grone-Johnson-Sa-Wolkowitz [7] conditions on a symmetric pattern J were studied that ensured that every partially positive matrix with pattern J has a positive completion. In this section, we derive a general condition that ensures that a given partially positive matrix will have a positive completion. We obtain some new information on the positive elements in the subspaces of the form $\mathcal{S}_J$ with J a symmetric pattern.

Every partially defined matrix also gives rise to a linear functional

$$\psi_T\colon \mathcal{S}_J \to \mathbb{C} \text{ via } \psi_T((a_{ij})) = \Sigma_{ij}\, a_{ij}t_{ij}.$$

Theorem 2.1. Let J be a symmetric pattern and let T be a partially defined matrix with pattern J. Then the following are equivalent:

i) T has a positive completion,

ii) $\Phi_T : \mathcal{S}_J \to M_n$ is positive,

iii) $\psi_T : \mathcal{S}_J \to \mathbb{C}$ is positive.

Proof. Let T be a positive completion of T. Note that for A in $\mathcal{S}_J$, $\Phi_T(A) = \Phi_{\tilde{T}}(A)$. Since Schur products of positive matrices are positive, if A is positive, then $\Phi_{\tilde{T}}(A)$ is positive. Thus, (i) implies (ii).

The map $\mathcal{S}\colon M_n \to \mathbb{C}$ defined by $\mathcal{S}((a_{ij})) = \Sigma_{ij}a_{ij}$ is positive and $\psi_T = \mathcal{S} \bullet \Phi_T$. Thus, (ii) implies (iii)

Finally, if ψ_T is positive, then by Krein's theorem (the 1-dimensional case of Arveson's theorem) ψ_T extends to a positive functional ψ on M_n. Set $\tilde{t}_{ij} = \psi(E_{ij})$, so that $\tilde{T} = (\tilde{t}_{ij})$ is a completion of T. If $\lambda_1, \ldots \lambda_n$ are complex numbers, then

$$\Sigma_{ij=1}^{n} \tilde{t}_{ij}\lambda_j\bar{\lambda}_i = \Sigma_{ij=1}^{n} \psi(\lambda_j\bar{\lambda}_i E_{ij}) = \psi((\bar{\lambda}_i\lambda_j)) \geq 0,$$

since $(\bar{\lambda}_i\lambda_j)$ is a positive matrix. Thus, $\tilde{T}$ is a positive completion of T.

Let I denote the n x n identity matrix, then an n x n matrix A is a contraction if and only if the 2n x 2n matrix

$$\begin{pmatrix} I & A \\ A^* & I \end{pmatrix}$$

is positive.

If $T = (t_{ij})$ is a partially defined matrix with pattern J, then $T^* = (\bar{t}_{ji})$ is the partially defined matrix with pattern $J^* = \{(j,i): (i,j) \in J\}$. If T is partially defined then $P = \begin{pmatrix} I & T \\ T^* & I \end{pmatrix}$ is a partially defined 2n x 2n matrix, with pattern J′ and

$$\mathcal{S}_{J'} = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} : A,D \in M_n, B,C^* \in \mathcal{S}_J\right\}.$$

<u>Corollary 2.2</u>. Let T be a partially defined matrix with pattern J and let $P = \begin{pmatrix} I & T \\ T^* & I \end{pmatrix}$. Then T can be completed to a contraction if and only if $\Phi_P: \mathcal{S}_{J'} \to M_{2n}$ is positive.

We now turn our attention to the result of [7]. We first need to introduce some notation from graph theory. Note that if J is a symmetric pattern, then we may associate a graph G_J with J. The graph G_J has vertices $\{v_1,\ldots,v_n\}$ with v_i and v_j adjacent if and only if $(i,j) \in J$.

A <u>k-cycle</u> in a graph G is a subset $\{w_1, \ldots, w_k\}$ of distinct vertices 138 of G, such that w_k and w_1 are adjacent and w_i and w_{i+1} are adjacent, $1 \le i \le k-1$. A graph G is <u>chordal</u> if every k-cycle in G contains three vertices which form a 3-cycle. A vertex v in G is <u>perfect</u> or <u>simplicial</u>, if any time v is adjacent to w and v is adjacent to w′, w and w′ are themselves adjacent. A graph G on n vertices has a <u>perfect vertex elimination scheme</u> if there is an enumeration of the vertices $\{w_1, \ldots, w_n\}$ such that w_i is a perfect vertex in the graph G_i generated by $\{w_i, \ldots, w_n\}$, $1 \le i \le n$.

The theorem of [7] states that for a fixed symmetric pattern J, every partially positive matrix with pattern J will have a positive completion if and only if G_J is a chordal graph. The results of Dym-Gohberg [5] follow from this result by observing that every block-banded pattern gives rise to a chordal graph.

<u>Lemma 2.3</u>. Let J be a symmetric pattern and let T be a partially defined matrix with pattern J. Then T is partially positive if and only if $\Phi_T(P)$ is positive for every rank 1 positive in $\mathcal{S}_J$.

<u>Proof</u>. Let $1 \le i_1 \le i_2 \le \ldots \le i_k \le n$ with (i_k, i_l) in J, for $1 \le k, l \le k$. Then the k x k principal submatrix (t_{i_k, i_l}) is positive if and only if $\Phi_T(P)$ is positive where P is the matrix with 1's in the (i_k, i_l) positions and 0's elsewhere. Note that P is rank 1 since $P = A^*A$ where A is the matrix which is 0 except for the first row, which has 1's in the i_k positions.

Conversely, assume T is partially positive. If $P = (\bar{\alpha}_i \alpha_j)$ is a rank 1 positive in $\mathcal{S}_J$, then since $\bar{\alpha}_i \alpha_j = 0$ for $(i,j) \notin J$, we have that there is some subset $1 \le i_1 \le \ldots \le i_k \le n$ such that $\alpha_i = 0$ unless $i = i_k$ for some

$1 \le k \le K$. But then $\Phi_T(P)$ is positive if and only if the k x k matrix $(\bar{\alpha}_{i_k} \alpha_{i_l} t_{i_1, i_l})$ is positive. This latter matrix is the Schur product of two positive K x K matrices and hence is positive.

<u>Theorem 2.4</u>. Let J be a symmetric pattern, then the following are equivalent:

i) there exists a permutation of the numbers {1,2, ..., n} such that with respect to this re-numbering every positive P in S_J factors as $P = A^*A$ with $A \in S_J$ and A upper triangular,

ii) every positive $P \in S_J$ is a sum of rank 1 positives in S_J,

iii) every partially positive matrix with pattern J has a positive completion,

iv) the graph G_J is chordal,

v) the graph G_J has a perfect vertex elimination scheme.

<u>Proof</u>. Assuming (ii), let T be partially positive. By Lemma 2.3, $\Phi_T(P)$ is positive for every rank 1 positive and hence for every positive P that can be expressed as a sum of rank 1 positives. Thus, Φ_T is a positive map and so by Theorem 2.1, T has a positive completion. By [7], (iii) and (iv) are equivalent. In fact, we only need the "easier" implication. Namely that (iii) implies (iv).

The proof that (iv) implies (v) can be found in [6, Theorem 4.1]. We remark that the converse is easy to see.

Now assume that G has a perfect vertex elimination scheme, and let $\{w_1, \ldots, w_n\}$ be the enumeration of the vertices so that w_i is perfect in the graph spanned by $\{w_i, \ldots, w_n\}$. Re-number so that $w_i = v_i$. We need to recall the Cholesky algorithm. If $P = (P_{ij})$ is a positive matrix, then $P_2 = P - P_{11}^{-1} (\bar{P}_{1i} P_{1j})$ is positive and is 0 in the first row and column. Let A_1 be the matrix which is 0 except for its first row which is $P_{11}^{-\frac{1}{2}} P_{1j}$, then $P_2 = P - A_1^*A_1$. Note that $A_1^*A_1 \in S_J$ if and only if $(i,j) \notin J$ implies that $\bar{P}_{1i}P_{1j} = 0$. But if $(i,j) \notin J$, then since v_1 is a perfect vertex either $(1,i) \notin J$ or $(1,j) \notin J$ and hence either P_{1i} or P_{1j} is 0. Thus, A_1, $A_1^*A_1$, and P_2 are all in S_J.

Repeating this step on P_2, we obtain a matrix A_2 which is 0 except for the 2nd row, which is an appropriately scaled version of the 2nd row of P_2, and, in particular, 0 in the (2,1)-entry. The fact that $A_2^*A_2 \in S_J$ follows from the fact that v_2 is perfect in the graph generated by $\{v_2, \ldots, v_n\}$.

Thus, by the Cholesky algorithm, we obtain matrices $A_1, \ldots, A_n$, in S_J with $A_i^*A_i \in S_J$, A_i supported on the i-th row, such that $A = A_1 + \ldots + A_n$ is upper triangular, in S_J, and $A^*A = A_1^*A_1 + \ldots + A_n^*A_n = P$. Thus, (v) implies both (i) and (ii).

To complete the proof it will be sufficient to prove that (i) implies (v). Assume that the renumbering has been made. We will show that v_1 is simplicial. Let (1,i) and (1,j) be in J. Consider the positive matrix $P = (p_{k,l})$, with $p_{11} = 2$, $p_{1i} = p_{i1} = p_{j1} = p_{1j} = p_{ii} = p_{jj} = 1$ and the remaining entries 0. If $P = A^*A$ with A upper triangular, then A is unique up to multiplication by a diagonal unitary. Computing the Cholesky factorization of P, we find that $a_{ij} \neq 0$. Since $A \in S_J$ we have that (i,j) is in J. 141
Thus, v_1 is a simplicial vertex.

The remainder of the proof that $\{v_1, \ldots, v_n\}$ forms a perfect vertex elimination scheme follows similarly.

Remark 2.5. The statement that every positive in S_J factors as A^*A with A in S_J is not equivalent to the above conditions. Let G_J be non-chordal and consider the 2n x 2n matrices,

$$S_{J'} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A \in S_J, B, C, D \in M_n \right\} .$$

It is not hard to show that every positive in $S_{J'}^+$ can be factored as X^*X, with X of the form $\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$, $B, C, D \in M_n$. However, J' is not chordal since J is not.

Remark 2.6. In [5], Dym-Gohberg observe that Arveson's distance formula [2] can be deduced from their completion result for partially positive banded matrices. From the above results we see that a proof of Dym-Gohberg's result can be derived, which uses Arveson's (Krein's) extension theorem. Thus, the distance formula can be deduced as a consequence of the extension theorem. Since this seems to have gone unnoticed before, we sketch in the key steps needed to deduce the distance formula from the extension theorem.

Arveson's distance formula says that a necessary and sufficient condition for a partially defined matrix T with only the lower triangular entries specified to be completable to a contraction is that it be a partial contraction, that is, only if each rectangular block below the main diagonal is a contraction. It is easily seen that T is a partial contraction if and only if the banded matrix $P = \begin{pmatrix} I & T \\ T^* & I \end{pmatrix}$ is partially positive. Thus, by Corollary 2.2, to prove Arveson's distance formula, it is enough to show that the map ϕ_P or ψ_P induced by this partially positive 142 banded matrix is positive. Note that when J is a banded pattern, or even block-banded, then we may apply the Cholesky algorithm directly, with no re-ordering, to decompose positive elements in $\mathcal{S}_J$ into sums of rank 1's in $\mathcal{S}_J$. Thus, by Lemma 2.3, if P is partially positive and J is block-banded, then Φ_P is positive. Thus, P has a positive completion. This last statement combined with Theorem 2.1 is the Dym-Gohberg theorem [5].

<u>3. Completely Bounded Maps</u>. In [8], Haagerup obtained a characterization of those matrices T for which the Schur product map $\phi_T\colon M_n \to M_n$ is a contraction, and proved additionally that $\|\phi_T\| = \|\phi_T\|_{cb}$, which we shall define in a moment. In this section we re-derive this result via matrix completions. In addition, we obtain a Hahn-Banach type extension theorem for Schur product maps defined on subspaces of M_n. We then extend these results to inflated Schur products.

If $\mathcal{A}$ and $\mathcal{B}$ are C*-algebras, $\mathcal{M} \subseteq \mathcal{A}$, $\mathcal{N} \subseteq \mathcal{B}$ subspaces, then we endow $M_n(\mathcal{M})$ and $M_n(\mathcal{N})$ with the norms they inherit as subspaces of $M_n(\mathcal{A})$ and $M_n(\mathcal{B})$, respectively. Given a map $\phi\colon \mathcal{M} \to \mathcal{N}$ we define maps $\phi^{(n)}\colon M_n(\mathcal{M}) \to M_n(\mathcal{N})$ via $\phi^n((a_{ij})) = (\phi(a_{ij}))$. It is not difficult to check that if ϕ is bounded, then $\phi^{(n)}$ is bounded. However, in general, $\sup_n \|\phi^{(n)}\|$ need not be finite. When it is, we say that ϕ is <u>completely bounded</u> and use $\|\phi\|_{cb}$ to denote this supremum.

Let Q_n denote the partially defined n x n matrix whose diagonal entries are 1, and whose remaining entries are unspecified. If T is a partially defined n x n matrix, then $T^{(m)}$ denotes the partially defined matrix in $M_{mn} = M_m(M_n)$ whose (k, l)-th block is T. In some sense $T^{(m)}$ is the tensor of T with the m x m matrix of all 1's. Note that if T has pattern J, then the map $\phi_T^{(m)}\colon M_m(\mathcal{S}_J) \to M_m(M_n)$ is given by the Schur product with $T^{(m)}$, i.e., $\phi_T^{(m)} = \phi_{T^{(m)}}$.

Finally, given a pattern J, let $\mathcal{S}_{\tilde{J}} = \left\{ \begin{pmatrix} D_1 & A \\ B^* & D_2 \end{pmatrix} : D_1, D_2 \in \mathcal{D}_n,\ A, B \in \mathcal{S}_J \right\}$. Note that there is a pattern $\tilde{J}$ so that, indeed $\mathcal{S}_{\tilde{J}} = \mathcal{S}_{\tilde{J}}$.

<u>Lemma 3.1</u>. Let T be a partially defined matrix with pattern J and let

$P = \begin{pmatrix} Q_n & T \\ T^* & Q_n \end{pmatrix}$. Then $\Phi_T\colon \mathcal{S}_J \to \mathcal{S}_J$ is a contraction if and only if $\Phi_P : \mathcal{S}_{\tilde{J}} \to \mathcal{S}_{\tilde{J}}$ is positive.

Proof. Assume that Φ_P is positive and let $A \in \mathcal{S}_J$ with $\|A\| \le 1$, then $R = \begin{pmatrix} I & A \\ A^* & I \end{pmatrix}$ is positive in $\mathcal{S}_{\tilde{J}}$ and so

$$\Phi_P(R) = \begin{pmatrix} I & \Phi_T(A) \\ \Phi_{T^*}(A) & I \end{pmatrix} = \begin{pmatrix} I & \Phi_T(A) \\ \Phi_T(A)^* & I \end{pmatrix}$$

is positive. Hence, $\|\Phi_T(A)\| \le 1$ and so Φ_T is a contraction.

Conversely, assume that Φ_T is a contraction, and let $\begin{pmatrix} D_1 & A \\ A^* & D_2 \end{pmatrix}$ be positive in $\mathcal{S}_{\tilde{J}}$, with D_1 and D_2 also invertible. Then

$$\Phi_P\left(\begin{pmatrix} D_1 & A \\ A^* & D_2 \end{pmatrix}\right) = \begin{pmatrix} D_1 & \Phi_T(A) \\ \Phi_T(A)^* & D_2 \end{pmatrix} =$$

$$\begin{pmatrix} D_1^{1/2} & 0 \\ 0 & D_2^{1/2} \end{pmatrix}\begin{pmatrix} I & D_1^{-1/2}\Phi_T(A)D^{-1/2} \\ D_2^{-1/2}\Phi_T(A)^*D_1^{-1/2} & I \end{pmatrix}\begin{pmatrix} D_1^{1/2} & 0 \\ 0 & D_2^{1/2} \end{pmatrix} =$$

$$\begin{pmatrix} D_1^{1/2} & 0 \\ 0 & D_2^{1/2} \end{pmatrix}\begin{pmatrix} I & \Phi_T(D_1^{-1/2}AD_2^{-1/2}) \\ \Phi_T(D_1^{-1/2}AD_2^{-1/2})^* & I \end{pmatrix}\begin{pmatrix} D_1^{1/2} & 0 \\ 0 & D_2^{1/2} \end{pmatrix}.$$

However, since $\begin{pmatrix} D_1 & A \\ A^* & D_2 \end{pmatrix}$ is positive, we have that $\| D_1^{-1/2}AD_2^{-1/2}\| \le 1$. Since Φ_T is a contraction, the middle term in the above product is positive. Hence, $\Phi_P\left(\begin{pmatrix} D_1 & A \\ A^* & D_2 \end{pmatrix}\right)$ is positive, when D_1 and D_2 are also assumed to be invertible. But since such the invertible positives are clearly dense in all the positives in $\mathcal{S}_{\tilde{J}}$, Φ_P is positive.

It is interesting to note that in the above calculation, we have directly used, for the first time, the fact that Φ is a $\mathcal{D}_n$-bimodule map.

<u>Theorem 3.2</u> Let $T = (t_{ij})$ be a partially defined matrix with pattern J. Then Φ_T: $\mathcal{S}_J \to \mathcal{S}_J$ is a contraction if and only if there exists vectors $v_1, \ldots, v_n, w_1, \ldots, w_n$ in $\mathbb{C}^n$ of norm less than or equal to 1, with $t_{ij} = \langle w_j, v_i \rangle$, whenever t_{ij} is specified.

<u>Proof</u>. If Φ_T is a contraction, then by Lemma 3.1 and Theorem 2.1, $P = \begin{pmatrix} Q_n & T \\ T^* & Q_n \end{pmatrix}$ posesses a positive completion $\tilde{P}$. Factor $\tilde{P} = A^*A$ with A upper triangular, so $A = \begin{pmatrix} V & W \\ 0 & X \end{pmatrix}$ and note that V^*W is a completion of T. Thus, if we let v_i denote the i-th column of V and w_j the j-th column of W, then $t_{ij} = \langle w_j, v_i \rangle$ wherever specified. The fact that the norms of these vectors is less than or equal to 1 follows from the fact that the diagonal entries of $\tilde{P}$ are 1's.

Conversely, assume that we are given such a representation of T. It will be sufficient to show that for $\tilde{T} = (\langle w_j, v_i \rangle)$, the map $\Phi_{\tilde{T}} : M_n \to M_n$ is a contraction. To this end let $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mu = (\mu_1, \ldots, \mu_n)$ be unit vectors in $\mathbb{C}^n$ and let $A = (a_{ij})$ be a contraction in M_n. Then

$$\langle \Phi_{\tilde{T}}(A)\lambda, \mu \rangle = \Sigma_{i,j}\, a_{ij}\lambda_j\bar{\mu}_i \langle w_j, v_i \rangle = \left\langle A \begin{bmatrix} \lambda_1 w_1 \\ \vdots \\ \lambda_n w_n \end{bmatrix}, \begin{bmatrix} \mu_1 v_1 \\ \vdots \\ \mu_n v_n \end{bmatrix} \right\rangle,$$

and this last inner product is less than one since A is a contraction and each of these vectors has norm less than or equal to 1.

<u>Corollary 3.3</u>. Let $T = (t_{ij})$ be a partially defined matrix with pattern J. Then T has a completion $\tilde{T}$ such that the extended map $\phi_{\tilde{T}}$ satisfies $\|\phi_T\| = \|\phi_{\tilde{T}}\|$. Moreover, $\|\phi_T\| = \|\phi_T\|_{cb} = \|\phi_{\tilde{T}}\|_{cb} = \|\phi_{\tilde{T}}\|_{cb}$.

<u>Proof</u>. We may assume $\|\phi_T\| = 1$. Set $\tilde{T} = (\langle w_j, v_i \rangle)$, then $\|\phi_T\| \le \|\phi_{\tilde{T}}\| \le 1$. Since $\phi_{\tilde{T}}^{(m)} = \phi_{\tilde{T}^{(m)}}$, to see that $\|\phi_T\|_{cb} \le \|\phi_{\tilde{T}}\|_{cb} \le 1$, it is sufficient to note that $\tilde{T}^{(m)}$ has the form required in Theorem 3.2. One needs only to repeat the v's and w's m times.

<u>Remark 3.4</u>. Corollary 3.3 shows that every $\mathcal{D}_n$-bimodule map ϕ_T into M_n defined on a $\mathcal{D}_n$-bimodule in M_n has a norm preserving extension $\phi_{\tilde{T}}$ to a $\mathcal{D}_n$-bimodule map on all of M_n.

Haagerup [8] obtains the representation of Theorem 3.2 for Schur product maps whose domain is all of M_n and the equality of the norm and cb-norm. It is interesting to note that his proof uses Grothendieck-type inequalities in a non-trivial fashion. Also, given the equality of the norm and cb-norm for ϕ we can deduce the existence of the extension from the cb-generalization of Arveson's theorem [11], [14].

<u>Remark 3.5</u>. Lemma 3.1 allows one to construct many examples of partially positive matrices with no positive completions. Notice that in the matrix P, the only fully defined principal submatrices are all of the form $\begin{pmatrix} 1 & t_{ij} \\ \bar{t}_{ij} & 1 \end{pmatrix}$, with t_{ij} specified.

Thus P will be partially positive as long as all the specified entries of T satisfy $|t_{ij}| \le 1$. However, P will have a positive completion only if $\|\phi_T\| \le 1$.

For an interesting example, let T be the n x n matrix whose upper triangular entries are 1's and whose lower triangular entries are 0's, so that $\Phi_T: M_n \to M_n$ is "triangular truncation". It is known [9] that $\|\Phi_T\|$ is of the order of *ln n*. Thus the corresponding P has no positive completion. In fact, for P to have a positive completion, its diagonal entries of 1 would need to be replaced by numbers on the order of *ln n*.

It is interesting to note that the graph associated with the pattern for P, when T is fully specified, is the bipartite graph on 2n vertices. This graph is in some senses not too far from chordal. Every cycle in this graph contains 4 vertices which lie on a 4-cycle.

4. Inflated Schur Products. In this section, we study the problem of when a partially defined operator matrix $T = (T_{ij})$ on $\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n$ can be completed to a positive operator. In particular, we will study whether or not the condition that the inflated Schur product map $\Phi_T\colon \mathcal{S}_J \to \mathcal{L}(\mathcal{H})$ is positive, is sufficient. By Arveson's extension theorem, if Φ_T is completely positive, then T can be completed to a positive operator. Thus, we are concerned with studying whether or not Φ_T positive, implies that Φ_T is completely positive.

When $\mathcal{S}_J = M_n$ then by Proposition 1.2 these two statements are equivalent. The condition that Φ_T is positive is equivalent to requiring that Φ_{T_x} is positive for all $x = (x_1, \ldots; x_n)$, where $T_x = (\langle T_{i,j}\, x_j, x_i \rangle)$ is a partially defined scalar matrix (Lemma 4.1). Thus, when Φ_T is positive, every T_x will have a positive completion. Hence, the question we are interested in studying is an interpolation type problem. Namely, if for every x, $T_x = (\langle T_{ij}\, x_j, x_i \rangle)$ has a positive completion, then can we choose operators such that $T = (T_{ij})$ has a positive completion?

We have been unable to obtain a definitive answer, but we obtain several positive results. We also relate this question to a problem concerning positive elements in $M_m(\mathcal{S}_J)$ for J symmetric.

We begin with some positive results.

Let $T = (T_{ij})$ be a partially defined operator matrix on $\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n$

with pattern J, and for each $x = x_1 \oplus \dots \oplus x_n$ in H let $T_x = (\langle T_{ij}x_j, x_i\rangle)$ be the partially defined matrix of scalars. We summarize the above observations in two lemmas.

<u>Lemma 4.1</u>. Let $T = (T_{ij})$ be a partially defined operator matrix on H. Then $\phi_T: S_J \to L(H)$ is positive if and only if $\phi_{T_x}: S_J \to S_J$ is positive, for every x in H.

<u>Lemma 4.2</u>. Let $T = (T_{ij})$ be a partially defined operator matrix on H. Then T is partially positive if and only if $\phi_T(P)$ is positive for every rank 1 positive P in S_J.

<u>Proof</u>. It is easy to see that T is partially positive if and only if T_x is partially positive for all s. But this implies that ϕ_T (P) is positive for every rank 1 positive P in S_J and every x, which yields the result.

<u>Theorem 4.3</u>. Let $T = (T_{ij})$ be a partially defined operator matrix on H with symmetric pattern J. If G_J is chordal, then every partially positive operator matrix has a positive completion.

<u>Proof</u>. We need to prove that $\phi_T: S_J \to L(H)$ is completely positive. Note that ϕ_T is positive, by Lemma 4.2 and the fact that every positive in S_J is a sum of rank 1 positives in S_J.

Now $\phi_T^{(m)} = \phi_{T^{(m)}}$ and since T is partially positive, $T^{(m)}$ is partially positive. The domain of $\phi_T^{(m)}$ is $M_m(S_J) = S_{J^{(m)}}$, where $J^{(m)}$ is a symmetric pattern on mn vertices. Thus, if we can prove that $G_{J^{(m)}}$ is chordal then by the above argument $\phi_T^{(m)}$ will be positive.

The graph $G_{J(m)}$ can be obtained from G_J as follows: Replace each vertex v_i in G_J by a complete graph G_i on m vertices. If v_i and v_j are adjacent, then every vertex in G_i is adjacent to every vertex in G_j.

It is easy to see that if G_J is chordal, then the graph obtained from G_J in this manner is also chordal. This completes the proof.

We finish this section by observing that a necessary and sufficient condition for the complete positivity of every positive map ϕ_T on S_J is that the positive cone of $M_r(S_J) = S_J \otimes M_r$ coincides with $(S_J)_+ \otimes (M_r)_+$, the cone generated by elementary tensors of positive elements.

THEOREM 4.4. Let J be a symmetric pattern. Then every positive map ϕ_T on S_J is completely positive if and only if $(S_J \otimes M_r)_+ = (S_J)_+ \otimes (M_r)_+$ for every r.

Proof. This theorem is a special case of a more general result for operator systems. See Corollary 5.7 of Paulsen [11], for example. □

5. Dilation theory

(5.1) DEFINITION. Let A be a finite dimensional CSL algebra, so that for some matrix algebra M_n, we have $\mathcal{D}_n \subseteq A \subseteq M_n$ where $\mathcal{D}_n$ is the diagonal algebra for M_n. Then A is said to be a chordal algebra if $A + A^* = S_J$ and G_J is a chordal graph.

Thus A is a chordal algebra if and only if its associated (undirected) graph G is a chordal graph.

(5.2) THEOREM. Let ρ be a contractive unital representation of a chordal algebra $A \subset M_n$. Then ρ is completely contractive.

Proof. Since ρ is contractive and unital the induced well defined mapping $\tilde{\rho}$ on $A + A^*$ is a positive map. See, for example, Proposition 2.4 of Paulsen [11]. Moreover, if $T = (T_{ij})$ is the partially defined operator matrix with pattern J given by $T_{ij} = \tilde{\rho}(e_{ij})$, for the matrix units e_{ij} in $A + A^*$, then $\tilde{\rho}(A) = \phi_T(A)$. By hypothesis J is chordal and so by Theorem 4.3 T has a positive completion. By Lemma 4.1 and Proposition 1.2 ϕ_T, and hence $\tilde{\rho}$ are completely positive unital maps. It follows that ρ is completely contractive, as required. □

The last theorem provides another proof that contractive representations of finite dimensional nest algebras are completely contractive. It is also easy to recognize other matrix algebras as being chordal algebras.

Example. Let $A \subset T(n)$ be spanned by $\mathcal{D}_n$ and the matrix units $e_{in}, e_{1,i}$, for $1 \leq i \leq n$. Then A is a chordal algebra.

Example. Let A be a finite dimensional CSL algebra such that the graph G for $A + A^*$ is a tree. Then, since G contains no cycles whatsoever, A is a chordal algebra.

Example. Let $1 < k < n$ and let $A \subset T(n)$ be the algebra spanned by the matrix units e_{ij} such that $1 \leq i \leq j \leq n$ and $i \leq k$. Then A is a chordal algebra

In view of the distinguished nature of chordal algebras it is profitable to consider CSL algebras that are semidiscrete relative to (finite dimensional) chordal CSL algebras, or have the property CCAP relative to chordal subalgebras. (See Chapter 4 section 4.3). Indeed, for such algebras every contractive σ-weakly continuous representation will be completely contractive, and hence admit *-dilations. However there are non chordal algebras for which every contractive representation is completely contractive (for example $T(n) \otimes T(m)$), and so, from the point of view of dilation theory the "chordally approximable" CSL algebra will form a very special class. A general dilation theory for CSL algebras must await a better understanding of the completely positive inflated Schur product maps on pattern subspaces of M_n.

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V. I. Paulsen
University of Houston
Houston, TX USA

S. C. Power
University of Lancaster
Lancaster, England

(7.1) Introduction

In this chapter we introduce some classes of nest subalgebras of C*-algebras and examine various structural and approximation properties particularly in connection with ideals. The analysis is summarised in the introduction of section (7.2) which appears in Appendix 8.

Of particular interest are the approximately finite nest algebras which are obtained as direct limits of directed systems

$$T(n_1) \to T(n_2) \to \dots .$$

where the embeddings are injective, unital, and are obtained by refinement. This means that the image of the canonical nest in $T(n_k)$ appears as a subnest of the canonical nest in $T(n_{k+1})$. Of necessity n_k divides n_{k+1} for all k, and we can regard the direct limit as a nest subalgebra of a UHF C*-algebra. A somewhat more general class is obtained by considering nest subalgebras of AF C*-algebras, which we call approximately finite nest subalgebras. (It should be noted that all nests are canonical in the sense that they are associated with a regular maximal abelian subalgebra of the AF C*-algebra). This class corresponds to direct systems of finite dimensional algebras of the form $T(r_1) \oplus \dots \oplus T(r_k)$.

Later, in (8.4), we consider infinite tensor products of the form $T(m_1) \otimes T(m_2) \otimes \dots$, which can be regarded as subalgebras of an associated approximately finite nest algebra.

(7.2) See Appendix 8

(7.3) Dilation theory

In section (4.2) it was shown that a contractive Hilbert space representation of the operator algebra $T(n)$ is completely contractive. This fact extends in a trivial way to approximately finite nest algebras, and more generally, to approximately finite nest subalgebras, and so we have a natural dilation theory for the contractive representations of this class of subalgebras of approximately finite C*-algebras. It is of considerable interest to pursue dilation theory in more general C*-algebraic contexts, and this general theme is sure to develop further in the near future. Apart from the intrinsic interest of such a study there are implications for the theory of tensor products of operator algebras, and we develop this in the next chapter. For example the result of Chapter 5 can be extended to the case of contractive representations of approximately finite nest subalgebras A_1 and A_2, and this leads to the equality of the maximal and minimal complete operator cross norms on the algebra $A_1 \otimes P(\mathbb{D})$ and $A_1 \otimes A_2$.

(8.1) The maximal complete operator cross norm.

Let A_1 and A_2 be algebras of operators on the complex Hilbert spaces H_1 and H_2, respectively, which contain the identity operator. In this section we study a maximum operator cross norm on the algebraic tensor product $A_1 \otimes A_2$. An operator norm on $A_1 \otimes A_2$ is a norm induced by a faithful unital representation on a Hilbert space. It is natural in our context to restrict attention to those operator norms for which the embeddings $A_i \to A_1 \otimes A_2$, $i = 1,2$, are complete isometrical isomorphisms. This is because we view an operator algebra A as carrying not only the given norm structure, but the induced operator norm structure on the matrix algebras $M_n(A)$. That is, operator algebras are matricially normed spaces, and we choose to restrict attention to operator norms on $A_1 \otimes A_2$ under which $A_1 \otimes \mathbb{C}$ (and $\mathbb{C} \otimes A_2$) can be identified with A_1 (and A_2) as matricially normed spaces. We call such a norm a complete operator cross norm on $A_1 \otimes A_2$.

The spatial norm $\| \|_{spat}$ on $A_1 \otimes A_2$ is the complete operator norm induced by the inclusion $A_1 \otimes A_2 \subset L(H_1 \otimes H_2)$. For C*-algebras it is well known that the spatial norm is the minimal C*-cross norm. Even in our wider generality, $\| \quad \|_{spat}$ coincides with the minimal complete operator cross norm on $A_1 \otimes A_2$. (We leave this as an exercise.)

Given commuting unital representations $\rho_i: A_i \to L(H)$, $i = 1,2,$, we write $\rho_1 \otimes \rho_2$ for the induced unital representation of $A_1 \otimes A_2$. We use the induced seminorms $\| \quad \|_{\rho_1 \otimes \rho_2}$ from such pairs to define the following maximal norm.

Throughout this section we write $A_1 \otimes A_2$ for the algebraic tensor product and $A_1 \otimes_{min} A_2$, $A_1 \otimes_{max} A_2$ when normed by the spatial and maximal operator cross norms. For convenience we also write $\rho_1 \otimes \rho_2$ for the representation of $A_1 \otimes A_2$ induced by commuting representations ρ_i of A_i, $i = 1,2$.

(8.1.1) <u>DEFINITION.</u> The maximal norm $\| \ \|_{max}$ on $A_1 \otimes A_2$ is the supremum of the seminorms $\| \ \|_{\rho_1 \otimes \rho_2}$ induced by all pairs ρ_1, ρ_2 of commuting completely contractive unital representations of A_1, A_2.

By taking direct sums over representations it can be seen that $\| \ \|_{max}$ is a complete operator cross norm with $\| \ \|_{max} \leq \| \ \|_{\gamma}$ for any other such $\| \ \|_{\gamma}$.

We now look at two illustrative examples where the maximal and spatial norm coincide, preceded by an elementary example where they differ.

(8.1.2) <u>Example.</u> Let $A \subset M_2$ be the two dimensional operator algebra spanned by the identity and the matrix unit $e_{1,2}$. Let $\rho_1 = \rho_2$ be the identity representation and note that the matrix $e_{1,2} \otimes I + I \otimes e_{1,2}$ has the form

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which has norm $\sqrt{2}$. On the other hand, the image of this operator under $\rho_1 \otimes \rho_2$ is the matrix,

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix},$$

which has norm 2. In particular $\| \ \|_{spat} \neq \| \ \|_{max}$.

(8.1.3) Example. Let $P(\mathbb{D})$ be the usual normed algebra of complex polynomials on the unit disc. Any pair of commuting completely contractive representations ρ_1, ρ_2 of $P(\mathbb{D})$ is determined by a pair of commuting contractions T_1, T_2. By Ando's theorem there are commuting unitaries U_1, U_2 which dilate T_1, T_2 in the sense that $T_1^n T_2^m$ is the compression of $U_1^n U_2^m$ for all $n,m = 0,1,2,\dots$. From this, and the contractive character of unital *-representations of $C(\Pi \times \Pi)$, we see that the induced representation $\rho_1 \otimes \rho_2$ of $P(\mathbb{D}) \otimes P(\mathbb{D})$, with the spatial norm, is contractive. It follows that $\| \; \|_{spat} = \| \; \|_{max}$.

(8.1.4) Example. Let $T(2) \subset M_2$ be the unital operator algebra of upper triangular 2×2 matrices and let ρ_1, ρ_2 be completely contractive commuting representations of $T(2)$, $P(\mathbb{D})$, respectively, on the Hilbert space H. Then there is a decomposition $H = H_1 \oplus H_2$ with respect to which ρ_1 and ρ_2 have the form

$$\rho_1 : \; (a_{ij}) \to \begin{bmatrix} a_{11} I_1 & a_{12} T \\ 0 & a_{22} I_2 \end{bmatrix}$$

$$\rho_2 : \; p(z) \to \begin{bmatrix} p(X_1) & 0 \\ 0 & p(X_2) \end{bmatrix}$$

where X_1, X_2, T are contractions with $X_1 T = T X_2$. By the Sz-Nagy-Foias lifting theorem there is a contraction $\tilde{T}$ and unitary dilations $\tilde{X}_1, \tilde{X}_2$ of X_1, X_2 acting on $K_1 \supset H_1$ and $K_2 \subset H_2$ respectively, such that $\tilde{X}_1 \tilde{T} = \tilde{T} \tilde{X}_2$ and $T = P_{H_1} \tilde{T}|_{H_2}$. The operators $\tilde{X}_1, \tilde{X}_2, \tilde{T}$ determine commuting representations $\tilde{\rho}_1, \tilde{\rho}_2$ of $T(2)$ and $P(\mathbb{D})$ on the Hilbert space $K = K_1 \oplus K_2$ such that $\rho_1 \otimes \rho_2 = P_H(\tilde{\rho}_1 \otimes \tilde{\rho}_2)|_H$. Since $\tilde{\rho}_2$ extends to a unital *-representation of $C(\partial\mathbb{D})$ it follows, by elementary arguments, that $\tilde{\rho}_1 \otimes \tilde{\rho}_2$ is contractive (as a map from the spatially normed tensor product). Thus $\| \cdot \|_{spat} = \| \|_{max}$ on $T(2) \otimes P(\mathbb{D})$.

The last two examples illustrate how the equality of $\| \cdot \|_{spat}$ and $\| \|_{max}$ is closely related to the possibility of lifting commuting representations $\rho_i: A_i \to L(H)$ to commuting dilations of containing C*-algebras. The following proposition is a consequence of Arveson's dilation theorem for completely contractive maps.

(8.1.5) <u>PROPOSITION</u>. Let B_1, B_2 be unital C*-algebras with $\| \cdot \|_{min} = \| \cdot \|_{max}$ on $M_n(B_1) \otimes B_2$ for $n = 1,2,\ldots,$ and let $A_i \subset B_i$, $i = 1,2$, be unital subalgebras. Then the following conditions are equivalent:

i) $\| \|_{min} = \| \cdot \|_{max}$ on $M_n(A_1) \otimes A_1$ for $n = 1,2,\ldots$

ii) For every pair of commuting completely contractive unital representations $\rho_1: A_1 \to L(H)$, $\rho_2: A_2 \to L(H)$, there is a Hilbert space $K \supset H$ and commuting unital *-representations $\pi_1: B_1 \to L(K)$, $\pi_2: B_2 \to L(K)$, such that $\rho_1(a_1)\rho_2(a_2) = P_H \pi_2(a_1)\pi_2(a_2)|_H$ for all a_1 in A_1, a_2 in A_2.

<u>Proof.</u> (i) $\Rightarrow$ (ii). Let ρ_1, ρ_2 be as in (ii). By (i), the induced representation $\rho_1 \otimes \rho_2$ of $A_1 \otimes_{min} A_2$ is contractive. Furthermore, the induced representation $(\rho_1 \otimes \rho_2)^{(n)}$ of $M_n(A_1 \otimes_{min} A_2)$ is contractive for each $n = 2,3,\ldots$ because $M_n(A_1 \otimes_{min} A_2)$ and $M_n(A_1) \otimes_{min} A_2$ are canonically isometrically isomorphic. Since $\rho_1 \otimes \rho_2$ is completely contractive and unital, there exists, by Arveson's theorem, a *-representation π of $B_1 \otimes_{min} B_2$ which dilates $\rho_1 \otimes \rho_2$,

and the restrictions of π to $B_1 \otimes_{min} \mathbb{C}$ and $\mathbb{C} \otimes_{min} B_2$ give the desired representations π_1 and π_2.

(ii) ⇒ (i) Let ρ_1 and ρ_2 be commuting completely contractive unital representations of A_1 and A_2 on H, and let $a \in M_n(A_1) \hat{\otimes} A_2$. Then, in view of the existence of π_1, π_2, as in (ii), we have

$$\begin{aligned} \|(\rho_1^{(n)} \otimes \rho_2)(a) &= \|(\pi_1^{(n)} \otimes \pi_2)(a)\| \\ &\leq \|a\|_{M_n(B_1) \otimes_{max} B_2} \\ &= \|a\|_{M_n(B_1) \otimes_{min} B_2} \\ &= \|a\|_{min}, \end{aligned}$$

and so (i) holds. □

References: Paulsen and Power [22].

(8.2) $T(n) \otimes P(\mathbb{D})$ and $T(n) \otimes T(m)$

In chapter 5 we obtained lifting theorems for commuting (completely) contractive representations for the pair $T(n)$, $P(\mathbb{D})$ and also for the pair $T(n)$, $T(m)$. Moreover it is well known that $\| \cdot \|_{min} = \| \ \|_{max}$ on $M_n \otimes C(\Pi)$ and on $M_n \otimes M_m$. Using these facts we otain the following theorem as a corollary to Proposition 8.1.5.

(8.2.1) THEOREM. For positive integers n,m the minimum and maximum complete operators cross norms agree on $P(\mathbb{D}) \otimes T(n)$ and on $T(n) \otimes T(m)$.

(8.2.2) REMARK. The last theorem extends to $P(\mathbb{D}) \otimes A_1$ and $A_1 \otimes A_2$ where A_1 and A_2 are approximately finite nest algebras (see Power [28]).

(8.3) $T(n_1) \otimes T(n_2) \otimes T(n_3)$

The next two propositions imply that there is no easy characterisation of the contractive representations or the completely contractive representations of the higher order tensor products of nest algebras and disc algebras. Although the second proposition immediately generalises the first, we include the proof of the former since it illustrates the close connection between multinest algebras and polydisc function algebras.

(8.3.1) PROPOSITION. There is a positive integer n_0 such that $\| \ \|_{min} \neq \| \ \|_{max}$ on $T(n) \otimes T(n) \otimes T(n)$ for all $n \geq n_0$.

(8.3.2) PROPOSITION. $\| \ \|_{min} \neq \| \ \|_{max}$ on $T(2) \otimes T(2) \otimes T(2)$.

Proofs. We first show that $\| \ \|_{min} \neq \| \ \|_{max}$ on $T(\mathbb{Z}) \otimes T(\mathbb{Z}) \otimes T(\mathbb{Z})$ by considering $P(\mathbb{D}^3)$ as a subalgebra of this operator algebra with the spatial norm, and exploiting a counterexample of Parrott.

Let R be the Hilbert space on which $T(\mathbb{Z})$ acts, and let G be another complex separable Hilbert space. Define

$$\rho_1 : T(\mathbb{Z}) \to T(\mathbb{Z}) \otimes_{min} T(\mathbb{Z}) \otimes_{min} T(\mathbb{Z}) \otimes_{min} L(H)$$

by $\rho_1(e_{ij}) = e_{ij} \otimes I \otimes I \otimes X_1^{j-i}$, for $j \geq i$, where X_1 is a contraction on G. Similarly, for contractions X_2, X_3 on G define ρ_2 and ρ_3 such that $\rho_2(e_{ij}) = I \otimes e_{ij} \otimes I \otimes X_2^{j-i}$ and $\rho_3(e_{ij}) = I \otimes I \otimes e_{ij} \otimes X_3^{j-i}$. These representations are well defined and completely contractive by the results of Chapter 4. We now show that the representation $\rho = \rho_1 \otimes \rho_2 \otimes \rho_3$ need not be contractive on $T(\mathbb{Z}) \otimes_{min} T(\mathbb{Z}) \otimes_{min} T(\mathbb{Z})$.

Let X_1, X_2, X_3 be a commuting triple of contractions for which there is a polynomial p in $P(\mathbb{D}^3)$ such that $\|p(X_1,X_2,X_3)\| > \|p\|_\infty$ (Parrott [19].)

Let W be the bilateral shift in $T(\mathbb{Z})$ and define $W_1 = W \otimes I \otimes I$, $W_2 = I \otimes W \otimes I$, $W_3 = I \otimes I \otimes W$ and $W_k = \tilde{W}_k \otimes X_k$, $k = 1,2,3$. We claim that $\|\rho(p(W_1,W_2,W_3))\| = \|p(\tilde{W}_1,\tilde{W}_2,\tilde{W}_3)\| > \|p\|_\infty = \|p(W_1,W_2,W_3)\|_{spat}$. The first and last equalities are clear. To see the inequality consider unit vectors x_n in H such that $\|Wx_n - x_n\| \to 0$ as $n \to \infty$. Let f,g be unit vectors in G and let $f_n = x_n \otimes x_n \otimes x_n \otimes f$, $g_n = x_n \otimes x_n \otimes x_n \otimes g$. Then, if p has the expansion $p(z_1 z_2 z_3) = \sum_\alpha a_\alpha z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}$, we compute

$$<p(\tilde{W}_1,\tilde{W}_2,\tilde{W}_3)f_n,g_n> = \Sigma_\alpha a_\alpha <W_1{}^1 x_n,x_n><W_2{}^2 x_n,x_n><W_3{}^3 x_n,x_n><X_1{}^1 X_2{}^2 X_3{}^3 f,g>$$

$$= \Sigma_\alpha a_\alpha <X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3} f,g> + o(n)$$

$$= <p(X_1,X_2,X_3)f,g> + o(n).$$

Choosing f,g appropriately, the claim follows, and so ρ is not contractive.

Let P_n be the diagonal projection in $T(\mathbb{Z})$ given by $P_n = e_{1,1} + \ldots + e_{n,n}$, so that $P_n T(\mathbb{Z}) P_n$ is naturally completely isometrically isomorphic to $T(n)$. Moreover, if $\rho_k^{(n)} = \rho_k | T(n)$ for $k = 1,2,3$, are the commuting completely contractive representations of $T(n)$ indued by ρ_k then $\rho_1^{(n)} \otimes \rho_2^{(n)} \otimes \rho_3^{(n)}(A) = \rho(A)$ for A in $T(n) \otimes T(n) \otimes T(n)$ (identified as a subalgebra of $T(\mathbb{Z}) \otimes T(\mathbb{Z}) \otimes T(\mathbb{Z})$). Since $\|P_n A P_n\| \to \|A\|$ the proposition follows from the noncontractivity of ρ.

We now turn to a direct proof that in fact even $T(2) \otimes T(2) \otimes T(2)$ does not have $\| \ \|_{min} = \| \ \|_{max}$.

Let U,V be unitary operators in M_2 and consider the operators

$$R = \begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$$

in M_4. Let ρ_R, ρ_S, ρ_T be the contractive representations of $T(2)$ into $M_{32} = M_2 \otimes M_2 \otimes M_2 \otimes M_4$ given by

$$\rho_R(e_{12}) = e_{12} \otimes I \otimes I \otimes R$$
$$\rho_R(e_{11}) = e_{11} \otimes I \otimes I \otimes I$$
$$\rho_R(e_{22}) = e_{22} \otimes I \otimes I \otimes I$$
$$\rho_S(e_{12}) = I \otimes e_{12} \otimes I \otimes S$$
$$\rho_S(e_{11}) = I \otimes e_{11} \otimes I \otimes I$$
$$\rho_S(e_{22}) = I \otimes e_{22} \otimes I \otimes I$$
$$\rho_T(e_{12}) = I \otimes I \otimes e_{12} \otimes T$$
$$\rho_T(e_{11}) = I \otimes I \otimes e_{11} \otimes I$$
$$\rho_T(e_{22}) = I \otimes I \otimes e_{22} \otimes I$$

Then ρ_R, ρ_S, ρ_T are contractive representations and are mutually commuting since all products of R, S, T are zero. Furthermore, $\rho_R \otimes \rho_S \otimes \rho_T$ can be interpreted as the mapping which transports the 8×8 matrix (a_{ij}) in $T(2) \otimes T(2) \otimes T(2)$ to the inflated Schur product

$$(a_{ij}I) \circ \begin{bmatrix} I & T & S & 0 & R & 0 & 0 & 0 \\ & I & & S & & R & & 0 \\ & & I & T & & & R & 0 \\ & & & I & & & & R \\ & & & & I & T & S & 0 \\ & & & & & I & & S \\ & & & & & & I & T \\ & & & & & & & I \end{bmatrix}$$

where I is the 4×4 identity matrix, and where undefined entries are also zero. Notice that the inflated Schur product map has norm dominating the norm of the submap

$$\begin{bmatrix} a & b & 0 \\ c & 0 & d \\ 0 & e & f \end{bmatrix} \to \begin{bmatrix} aS & bR & 0 \\ cT & 0 & dR \\ 0 & eT & tS \end{bmatrix}$$

Considering the special form of R, S, T this submap has norm agreeing with the norm of the inflated Schur map

$$\begin{bmatrix} a & b & 0 \\ c & 0 & d \\ 0 & e & f \end{bmatrix} \to \begin{bmatrix} aI & bU & 0 \\ cV & 0 & dU \\ 0 & eV & fI \end{bmatrix}$$

The norm of the image matrix agrees with the norm of

$$\begin{bmatrix} aI & bI & 0 \\ cI & 0 & dI \\ 0 & cI & fUV^*U^*V \end{bmatrix}$$

(Multiplying left and right by appropriate diagonal unitaries.) Now make the choice

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad U = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}$$

and note that

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} I & I & 0 \\ I & 0 & I \\ 0 & I & +I \end{bmatrix}$$

The first matrix has norm $\sqrt{2}$ while the latter has norm 2. Hence $\rho_R \otimes \rho_S \otimes \rho_T$ is not contractive.

References. Paulsen and Power [22]. (The simple argument above for $T(2) \otimes T(2) \otimes T(2)$ was obtained with Ken Davidson).

INFINITE TENSOR PRODUCTS OF UPPER TRIANGULAR MATRIX ALGEBRAS

Stephen Power*

Let $n \geq 2$ be an integer and let $T(n)$ be the algebra of $n \times n$ complex matrices which have zero entries below the main diagonal. Under the operator norm $T(n)$ is a Banach algebra, and for a sequence (n_k) of such integers there is a natural way to associate a unital Banach algebra

$$T((n_k)) = T(n_1) \otimes T(n_2) \times \ldots$$

which is an infinite tensor product in the sense of inductive limits.

In what follows we determine the group Aut $T((n_k))$ of Banach algebra automorphisms of $T((n_k))$. The quotient group Out $T((n_k))$, obtained from the normal subgroup of pointwise inner automorphisms, turns out to be the discrete group of permutations π such that $n_k = n_{\pi(k)}$, $k = 1,2,\ldots$. Thus, up to composition by pointwise inner automorphisms the set of outer automorphisms may be uncountable, countable, finite, or even trivial. In fact we describe all isomorphisms and epimorphisms between these Banach algebras.

We also determine the structure of the complete lattice Id $T))n_k))$ of all closed two-sided ideals of $T((n_k))$, with the natural lattice operations. The abstract framework needed concerns primary approximately finite lattices, and we develop a little general theory in this direction, inspired by Arveson's unique factorization theorem for primary complete distributive <u>metric</u> lattices. It turns out that the unordered set $\{n_1, n_2, \ldots\}$ is a complete lattice isomorphism invariant for the AF Lattice Id $T((n_k))$ and hence a complete Banach algebra isomorphism invariant for the algebras.

*This research was supported by a D.G. Bourgin visiting scholarship at the University of Houston, by a Fullbright travel grant, and by the Science and Engineering Research Council of Great Britain.

The algebras $T((n_k))$ can be regarded as the approximately finite versions of reflexive operator algebras associated with certain commutative subspace lattices defined on an infinite tensor product Hilbert space. Such algebras were introduced and studied by Arveson [1,Chapter 3]. He obtained complete similarity invariants for these algebras as a consequence of a unique factorization theorem mentioned above. We use a similar result in the class of approximately finite lattices and our proof derives directly from Arveson's arguments. However the arguments simplify considerably in our setting since the lattices under consideration are lattices of sets, under the usual set operations. Moreover we can also obtain the complete algebra isomorphism invariant purely from the factorization theory of finite primary lattices.

We can define $T((n_k))$ as a subalgebra of the well known Glimm algebra, or UHF C^*-algebra,

$$M((n_k)) = M(n_1) \otimes M(n_2) \otimes \dots .$$

Here $M(n)$ indicates the full $n \times n$ matrix complex algebra and the infinite tensor product is the C^*-algebra direct limit of the direct injective unital system $M(n_1) \to M(n_1n_2) \to \dots$, under natural embeddings. The isomorphism theory and automorphism groups of these algebras are well understood (see [4],[5],[7],[9], for example) and, being approximately finite C^*-algebras, K_0 theory is also available as a complete invariant. Thus $M((n_k))$ and $M((m_k))$ are isomorphic if and only if the sequences of partial products $(n_1,\dots,n_k)$ and $(m_1,\dots,m_k)$, satisfy the Glimm dinsibility criterion: each term from one sequence must divide some term of the other. In other language, (n_k) and (m_k) must determine the same supernatural number. It follows then that $T((n_k))$ and $T((m_k))$ may fail to be isomorphic even though their associated UHF algebras are isomorphic, just as with finite tensor products. We show that the K_0 group of $T((n_k))$ coincides with the K_0 group of the diagonal subalgebra, from which it follows that K-theory provides poor invariants for the algebras $T((n_k))$. However unlike the UHF algebras, which are simple, there is a rich ideal structure, and this structure can serve to study morphisms and the automorphism group. For example the automorphisms that fix the ideal lattice are precisely the pointwise inner automorphisms.

The results above and related matters are organized in the following way. In section one we define approximately finite lattices and note relevant examples and key properties such as complete distributivity and zero-one laws for factorizations. In section two we determine the ideal lattice of $T((n_k))$ as an AF lattice. Here we use standard approximation techniques associated with natural expectation mappings on the containing UHF algebra. We have used similar methods in [8] to study ideals in another class of non-self-adjoint subalgebras of AF algebras, namely in nest subalgebras associated with a maximal projection nest in the diagonal. Sections three and four use ideas of Arveson and develop the structure of prime elements in finite and approximately finite primary lattices, respectively. In section five we determine the nature of isomorphisms, epimorphisms and the automorphism group. In the final section we compute K_0.

For general lattice theory the reader may consult the standard reference Birkhoff [3], where ideal completions of lattices are discussed a little. Arveson's results are also described in his lecture notes [2].

It is a pleasure to record my thanks here for the warm hospitality that I received from the Department of Mathematics at the University of Houston, in the fall semester of 1986, when the research was completed. Vern Paulsen gets extra thanks for our endless mathematical conversations.

1. Approximately finite lattices

Let L_0 be a lattice with respect to meet and join operations $\vee$ and $\wedge$ respectively. An ideal of L_0 is a subset J which is closed under joins and is such that if $a \leq b$, with $a \in L_0$ and $b \in J$, then $a \in J$. In the lattice of all subsets of L_0 the collection of all ideals, including the empty set, forms a complete lattice $\hat{L}_0$ known as the ideal completion of L_0. The lattice L_0 is injectively embedded in $\hat{L}_0$ as the sublattice of principal ideals of L_0.

We say that a complete lattice L is approximately finite if there is a countable sublattice $L_0 \subset L$ such that L is isomorphic to $\hat{L}_0$ as a lattice. More precisely we require that the natural injection $L_0 \to \hat{L}_0$ extends to an isomorphism $L \to \hat{L}_0$.

Let $L_1 \subset L_2 \subset \ldots$ be a chain of finite sublattices of L_0 with union equal to L_0. Then there is a one to one correspondence between elements J of $\hat{L}_0$ and certain chains of ideals $J_1 \subset J_2 \subset \ldots$, where each J_k is an ideal in L_k. The correspondence is given by

$$J \to J \cap L_1, J \cap L_2, \ldots ,$$

(and so we require that the chain have the fullness property,

$$J_k = (\cup J_m) \cap L_k, \text{ for all } k).$$

Approximately finite lattices often arise naturally as the direct limit of a direct system of finite lattices. In fact the class of such limit lattices, which we shall define in terms of an ideal completion, coincides with the class of AF lattices, as we now indicate.

An injective direct system of finite lattices is a sequence of finite lattices $M_1, M_2, \ldots$ together with injective embeddings

$$M_1 \to M_2 \to \ldots .$$

The collection M_{00} of increasing sequences (m_k), with $m_k \in M_k$, and which are eventually constant, forms a lattice in a natural way. Identifying eventually equal sequences we obtain a countable lattice M_0 in which each lattice M_i is naturally and injectively embedded, say $M_i \to \alpha(M_i)$. Moreover M_0 is the union of the chain $\alpha(M_1) \subset \alpha(M_2) \subset \ldots$. We define the <u>direct</u> limit L of the original system to be the ideal completion of M_0, and we write $L = \lim_k M_k$.

We usually consider lattices which possess both a first and last element, denoted by 0 and 1 respectively, and refer to such as unital lattices. A morphism between unital lattices is said to be unital if it maps 0 to 0 and 1 to 1.

An element c of a lattice is <u>join-irreducible</u>, or <u>prime</u>, if $c = a \vee b$ implies that $a = c$ or $b = c$, and a unital lattice is primary if the unit 1 is prime. An element c is <u>meet-irreducible</u> if $c = a \wedge b$ implies $a = c$ or $b = c$. If the first element 0 of a unital lattice is meet-irreducible then we say that the lattice itself is meet-irreducible. There is an elementary duality between the theory of primary lattices and meet-irreducible lattices that arises through the converse lattice, $(L,<)$ say, of the lattice $(L,\leq)$; $a < b$ in $(L,<)$ if and only if $b \leq a$ in $(L,\leq)$, $a \wedge b$ in $(L,<)$ is the supremum $a \vee b$ in $(L,\leq)$ and $a \vee b$ in $(L,\wedge)$ is the infimum $a \wedge b$ in $(L,\leq)$. It is easy to check that $(L,<)$ is primary if and only if $(L,\leq)$ is meet-irreducible.

A finite lattice is primary if the supremum of all elements strictly less than 1 is also strictly less than 1, and is meet-irreducible if the infimum of all elements strictly greater than zero is also strictly greater than zero.

We now give some examples to illustrate the concepts above.

Examples 1. For $n = 2,3,\ldots$ write $L(n)$ for the totally ordered unital lattice $\{0,1,\ldots,n-1\}$. In particular $L(2)$ is the trivial unital lattice. These lattices are primary and meet-irreducible.

2. For $n,m = 2,3,\ldots$ let $L(n) \times L(m)$ be the product lattice of $L(n)$ and $L(m)$ with the product partial ordering. For $n,m > 2$ these lattices are neither primary nor meet-irreducible.

3. A subset A of the product set $\{1,\ldots,n-1\} \times \{1,\ldots,m-1\}$ for $n,m \geq 2$, is said to be increasing if (j_1,j_2) belongs to A whenever $j_1 \leq k_1$ and $j_2 \leq k_2$ for some element (k_1,k_2) in A. The totality of increasing sets, together with the empty set (which is also regarded as an increasing set), forms a lattice of sets (under the set operations) which we denote by $\mathrm{Inc}(n,m)$. Thus $\mathrm{Inc}(n,2)$ and $\mathrm{Inc}(2,n)$ are just copies of $L(n)$. Similarly we can define $\mathrm{Inc}(n_1,\ldots,n_r)$ for integers $n_1,\ldots,n_r$ that are greater than unity, and there are natural unital injections

$$\mathrm{Inc}(n_1,\ldots,n_r) \to \mathrm{Inc}(n_1,\ldots,n_s)$$

for $r < s$. Here the increasing set A gets mapped to the increasing set $A \times N_{r+1} \times \ldots \times N_s$, where $N_j = \{1,\ldots,n_j - 1\}$. Note that the lattice $\mathrm{Inc}(n_1,\ldots,n_r)$ is generated by r sublattices $L_1,\ldots,L_r$ where L_k is a copy of the nest lattice $L(n_k)$. These lattices are primary and meet-irreducible.

4. For a sequence (n_k), of integer $n_k \geq 2$, we can define the direct limit AF lattice associated with the system

$$\mathrm{Inc}(n_1,n_2) \to \mathrm{Inc}(n_1,n_2,n_3) \to \ldots .$$

We see later that such lattices are primary and meet-irreducible. The lattice can be thought of as the infinite tensor product of the nest lattices $L(n_1), L(n_2), \ldots$.

5. Let A be a partially ordered set with a last element a, and let L be a unital lattice. Then the collection, Inc(A,L) say, of increasing functions from A to L, forms a unital lattice. Thus f belongs to Inc(A,L) if $f: A \to L$ and $f(b) \leq f(c)$ if $b \leq c$. If L is a finite meet-irreducible lattice then Inc(A,L) is also meet-irreducible. For if 0^+ is the unique successor of 0 in L then the function f, such that $f(a) = 0^+$ and $f(b) = 0$ for all $b \neq a$, is the unique successor of the zero function.

For example, if L is a lattice then Inc(L,L(2)) is the lattice of increasing subsets of L.

The lattice structure that we will be concerned with in later sections is the lattice IdA of closed ideals of a unital Banach algebra A. Here the join operation is closed linear span and meet is intersection. Clearly IdA is a complete unital lattice. We shall look at a class of inductive limit Banach algebras where the ideal lattice IdA can be identified as a direct limit of explicit finite lattices. This identification is fairly standard analysis, but the analysis of the structure of IdA requires quite a bit of lattice theory. The payoff is that the structure of meet-irreducible elements can be made quite explicit (see Theorem 4.2) and this has considerable implications for the nature of isomorphisms and automorphisms of the algebra A.

We complete the present section by establishing complete distributivity, factorizations, and zero-one laws in the context of AF-lattices.

This information will be needed for the lattice theory in section 4.

<u>PROPOSITION</u> 1.1. Let L be an AF lattice and let $c_1, c_2, \ldots$ and b be elements of L. Then $\bigvee_{j=1}^{\infty} (b \wedge c_j) = b \wedge (\bigvee_{j=1}^{\infty} c_j)$.

<u>Proof</u>. This is immediate because L is a lattice of sets, and such lattices are completely distributive. ∎

<u>DEFINITION</u> 1.2. Sublattices L_1 and L_2 of a lattice are said to be independent of the following property holds: if $a \wedge b \leq a' \vee b'$, with a, a' in L_1 and b, b' in L_2, then $a \leq a'$ or $b \leq b'$.

<u>DEFINITION</u> 1.3 (Arveson [1]). Let L be a complete unital lattice. A <u>factorization</u> of L is a sequence of sublattices $L_1, L_2, \ldots$ such that

(i) $L = L_1 \vee L_2 \vee \ldots$

(ii) For every j the lattices L_j and $\bigvee_{k \neq j} L_k$ are independant.

(iii) $\bigcap_{n=1}^{\infty} (L_n \vee L_{n+1} \vee \ldots) = \{0,1\}$.

Similarly we shall say that $L_1, \ldots, L_n$ is a factorization of L if (i) and (ii) hold. Property (iii) is called the zero-one law for the sequence $L_1, L_2, \ldots$. The next proposition shows how zero-one laws arise naturally in certain direct limit AF lattices.

<u>PROPOSITION</u> 1.4. Let $L_1, L_2, \ldots$ be unital sublattices of a lattice L such that for each n the lattices $L_1, \ldots, L_n$ form a factorization of the lattice that they generate. If $L = \lim_n (L_1 \vee \ldots \vee L_n)$ then $L_1, L_2, \ldots$ is a factorization of L.

<u>Proof</u>. Let $M_k = L_1 \vee \ldots \vee L_k$ so that L is (isomorphic to) the AF lattice $\lim_k M_k$. This means that L is identified with the lattice of ideals of the

countable sublattice $L_0 = \bigcup_{k=1}^{\infty} M_k$. Moreover each such ideal β of L_0 is associated uniquely with the increasing sequence $\beta \cap M_1, \beta \cap M_2, \ldots$. In view of this correspondence we can establish properties of elements β in L by arguing locally with the finite lattice of ideals $\beta \cap M_k$ in M_k.

First we obtain property (ii) of Definition 1.3. Let $N_r = \bigvee_{j\neq r} L_j$, and note that N_r is simply the sublattice of ideals of $\bigcup_{k=1}^{\infty}(L_1 \vee \ldots \vee L_{r-1} \vee L_{r+1} \vee \ldots \vee L_k)$. Moreover for $n \geq r$ $N_r \cap M_n$ is the lattice of principal ideals determined by the sublattice $L_1 \vee \ldots \vee L_{r-1} \vee L_{r+1} \vee \ldots \vee L_n$. Suppose then that $\beta, \beta' \in N_r$ and $\alpha, \alpha' \in L_r$ (where all elements are ideals in L_0), and that $\alpha \wedge \beta \leq \alpha' \vee \beta'$, which means $\alpha \cap \beta \subset \alpha' \cup \beta'$, as sets. Then $\alpha \wedge \beta \cap M_n = (\alpha \cap M_n) \wedge (\beta \cap M_n)$ is contained in $(\alpha' \cap M_n) \cup (\beta' \cap M_n)$. From the given indepence of $L_1, \ldots, L_n$ it follows that $\alpha \cap M_n \subset \alpha' \cap M_n$ or $\beta \cap M_n \subset \beta' \cap M_n$. This alternative holds for all $n \geq r$, and so $\alpha \subset \alpha'$ or $\beta \subset \beta'$, as required.

Similarly it can be shown that if $\beta \in L_n \vee L_{n+1} \vee \ldots$, then for $n \leq m$ $\beta \cap M_m$ is an ideal in $L_n \vee \ldots \vee L_m$, and for $n > m$ $\beta \cap M_m = \{0\}$ or M_m. Hence for $\gamma \in \bigcap_{n=1}^{\infty}(L_n \vee L_{n+1} \vee \ldots)$ we have $\gamma \wedge M_m = \{0\}$ or M_m for all m, and so property (iii) holds. ■

<u>DEFINITION</u> 1.5. We say that the factorization $L_1, L_2, \ldots$ of the AF lattice L is a coherent factorization if L is isomorphic to the approximately finite lattice $\lim_n (L_1 \vee \ldots \vee L_n)$, as in the statement of Proposition 1.4.

<u>PROPOSITION</u> 1.6. Let $L_1, L_2, \ldots$ be a coherent factorization of the unital AF lattice L, and let $p_k \in L_k$ for $k = 1, 2, \ldots$. Then either $\wedge_k p_k$ is the zero element or $p_k = 1$ for all but a finite number of k.

<u>Proof</u>. Let $\beta = \wedge_k p_k$ which is identified with the ideal $\{x \in L_0 : x \leq p_k$ for all $k\}$, where L_0 is as in the proof of Proposition 1.4. Let $x \in \beta \cap M_r$,

where $M_r = L_1 \vee \ldots \vee L_r$, as before. Then $x \wedge 1 \leq 0 \vee p_k$ for all k, and so, by the independence of the lattices M_r and L_k for $k > r$, it follows that $x \leq 0$ or $1 \leq p_k$. Thus if $p_k \neq 1$ for an infinity of k, then $x = 0$. Hence $\beta = 0$. ∎

Our last proposition in this section is also an elementary consequence of local arguments. A similar assertion holds with primary replaced by meet-irreducible arguments.

PROPOSITION 1.7. Let $L = \lim_k L_k$ be the AF-lattice determined by finite primary unital lattices L_k. Then L is primary.

2. Id $T((n_k))$ as an AF lattice

The following notation will be useful. Let (n_k) be a sequence of integers, with $n_k \geq 2$ for all k, to avoid trivialities. Let $A = T((n_k)) = \bigotimes_{k=1}^{\infty} T(n_k)$, $B = M((n_k)) = \bigotimes_{k=1}^{\infty} M(n_k)$, $C = C((n_k)) = \bigotimes_{k=1}^{\infty} C(n_k)$, where $C(n_k)$ is the diagonal algebra $T(n_k) \cap T(n_k)^*$. Also, for r = 1,2,..., let us write A_r, B_r and C_r for the finite tensor product algebras associated with the r-tuple $n_1,\ldots,n_r$, regarded as the canonical subalgebras of A, B and C respectively.

We now define some important expectation maps on the algebra B. For $r < s$ let $U_{r,s}$ be the unitary group of the diagonal algebra $C(n_{r+1}) \otimes\ldots\otimes C(n_s) \subseteq C$, and let du denote Haar measure on $U_{r,s}$. The linear contractive map $\phi_{r,s}$ defined on B_s by

$$\phi_{r,s}(x) = \int_{U_{r,s}} u^* x u \, du, \qquad x \text{ in } B_s,$$

is a projection and has range equal to the subalgebra $M(n_1) \otimes \ldots \otimes M(n_r) \otimes C(n_{r+1}) \otimes \ldots \otimes C(n_s)$. Since $\phi_{r,t}$ extends $\phi_{r,s}$ when $s < t$, we can define ϕ_r on B as the pointwise limit

$$\phi_r(x) = \lim_{n\to\infty} \phi_{r,r+n}(x).$$

The map ϕ_r is a contractive projection onto the subalgebra $B_r \otimes C(n_{r+1}) \otimes \ldots$. In particular $\phi_r(x) \to x$ as $r \to \infty$ for every x in B.

PROPOSITION 2.1. Let J be a closed subspace of B that is a C-module. Then J is the closed union of the subspaces $J \cap B_n$, n = 1,2,... . In particular this holds true for ideals J of the subalgebra A.

Proof. Note that if x belongs to J then so do $\phi_{r,s}(x)$ and $\phi_r(x)$, for all $r < s$. However $\phi_{r,s}(x)$ lies in $B_s \cap J$, $\phi_{r,s}(x) \to \phi_r(x)$ as $s \to \infty$, and $\phi_r(x) \to x$, so the proposition follows. ∎

The synthesis property expressed in the last proposition to required to identify the ideal lattice of A. In fact the same feature holds for appropriate modules in general approximately finite C^+-algebras (see [8]).

Let us introduce a twisted partial ordering on the set of pairs

$$\delta(n) = \{(i,j): 1 \leq i \leq j \leq n\},$$

which reflects the ideal structure of T(n). We write $(i,j) \leq (k,\ell)$ when $i \geq k$ and $j \leq \ell$. If S is an increasing subset of $\delta(n)$ with respect to this ordering then the set J of matrices in M(n) supported by S is an ideal. Conversely every ideal arises in this way. More generally we have the following elementary proposition.

We write 2 for the trivial unital lattice L(2), and we use the notation of example 5 in section 1.

PROPOSITION 2.2. (i) The ideal lattice Id T(n) is isomorphic to $\mathrm{Inc}(\delta(n),2)$.

(ii) If A is any complex algebra then the ideal lattice $\mathrm{Id}(T(n) \otimes A)$ is isomorphic to the lattice $\mathrm{Inc}(\delta(n),\mathrm{Id}A)$.

In particular $T(n_2) \otimes T(n_2)$ has an ideal lattice which is isomorphic to $\mathrm{Inc}(\delta(n_1), \mathrm{Inc}(\delta(n_2),2))$, and we write this more simply as $\mathrm{Inc}(\delta(n_1),\delta(n_2),2)$. Similarly the r-fold tensor product $T(n_1) \otimes \dots \otimes T(n_r)$ has an ideal lattice denoted by $\mathrm{Inc}(\delta(n_1),\dots,\delta(n_r),2)$.

There are natural embeddings

$$\mathrm{Inc}(\delta(n_1),\ldots,\delta(n_r),2) \to \mathrm{Inc}(\delta(n_1),\ldots,\delta(n_s),2),$$

when $r \leq s$, which are most easily identified by checking first that $\mathrm{Inc}(\delta(n_1),\ldots,\delta(n_r),2)$ is isomorphic to $\mathrm{Inc}(\delta(n_1)\times\ldots\times\delta(n_r),2)$, the lattice of increasing subsets of the partially ordered product space $\delta(n_1)\times\ldots\times\delta(n_r)$. The embeddings above correspond precisely to the embedding $\mathrm{Id}\,A_r \to \mathrm{Id}\,A_{r+1}$ of the ideal lattice of $\mathrm{Id}\,A_r$. (Here an ideal J in $\mathrm{Id}\,A_r$ is identified with the ideal $\bar{J}$ in $\mathrm{Id}\,A_{r+1}$ that it generates.)

THEOREM 2.3. The ideal lattice of $T((n_k))$ is isomorphic to the approximately finite lattice $\lim_k \mathrm{Inc}(\delta(n_1)\times\ldots\times\delta(n_k),2)$.

Proof. We have observed that the limit lattice in the statement of the theorem is isomorphic to $\lim_k \mathrm{Id}A_k$ in a natural way, and so it remains only to show that $\mathrm{Id}A$ is isomorphic to $\lim_k \mathrm{Id}A_k$.

By Proposition 2.1 we can identify IdA with the set of sequences $J \cap A_1, J \cap A_2, \ldots$, for J in IdA. An increasing sequence $J_1, J_2, \ldots$ of ideals J_k of A_k, is such a sequence precisely when $J_r = A_r \cap (\cup_k J_k), r = 1,2,\ldots$. Let us call such a sequence an inductive sequence of ideals. Then, more precisely, Proposition 2.1 allows us to identify IdA with the lattice of increasing inductive sequences of ideals. From the definition of direct limits of lattices, we see that IdA is isomorphic to $\lim_k \mathrm{Id}A_k$. ∎

We have already observed that the limit lattice of a unital direct system of primary lattices is primary. Similar reasoning or direct arguments with Proposition 2.1 show that the ideal lattice $\mathrm{Id}T((n_k))$ is meet-irreducible.

Remark. Similar reasoning applies in the context of nest subalgebras of AF algebras considered in [8]. For example it is possible to define a natural upper triangular subalgebra, $TM((n_k))$ say, of $M((n_k))$, which is the inductive limit algebra $\lim_k T(n_1 \ldots n_k)$, with certain natural embeddings (by 'refinement'). For this algebra we can obtain the identification

$$\text{Id } MT((n_k)) = \lim_k \text{Inc}(\delta(n_1 \ldots n_k), 2).$$

3. Finite primary lattices

We now collect together some elementary facts concerning finite factorizations and finite primary lattices. The arguments here have been extracted from Arveson's paper [1].

PROPOSITION 3.1. Let M be a finite unital lattice with unital sublattices $L_1,\dots,L_n$ which form a factorization of M. If each factor L_k is primary then M is primary.

Proof. Let $e_1,\dots,e_n$ be the largest non-units in $L_1,\dots,L_n$ respectively. Suppose that $1 = a \vee b$ where

$$a = \bigvee_{k=1}^{\ell} a_k \quad , \quad b = \bigvee_{k=1}^{\ell} b_k$$

and where each element a_k or b_k is a finite meet of elements in the union of the lattices $L_1,\dots,L_n$. Define

$$\alpha_m = \vee\{a_k\colon a_k \le e_1 \vee \dots \vee e_m\},$$

$$\alpha_m' = \vee\{a_k\colon a_k \not\le e_1 \vee \dots \vee e_n\},$$

so that $a = \alpha_m \vee \alpha_m'$ for each $m = 1,\dots,n$. Note that α_m' belongs to $L_{m+1} \vee L_{m+2} \vee \dots \vee L_n$. Indeed if $a_k = x_1 \wedge \dots \wedge x_r$, and $a_k \not\le e_1 \vee \dots \vee e_n$, with each x_i lying in the union of $L_1,\dots,L_n$, then $x_i \not\le e_1 \vee \dots \vee e_n$ for all i. Thus, if $x_i \ne 1$ then x_i lies in the union of $L_{m+1},\dots,L_n$.

In a similar way construct the elements β_m, β_m' for b, and observe that

$$1 = a \vee b = (\alpha_m \vee \beta_m) \vee (\alpha_m' \vee \beta_m')$$

$$\le (e_1 \vee \dots \vee e_m) \vee (\alpha_m' \vee \beta_m')$$

and so, by independence,

$$1 \leq e_2 \vee \dots \vee e_m \vee \alpha_m' \vee \beta_m'.$$

Continuing in this way obtain $1 \leq \alpha_m' \vee \beta_m'$. Since α_m' and β_m' are decreasing sequences we conclude that either $\alpha_m' = 1$ for all m, or $\beta_n' = 1$ for all n. Hence a = 1 or b = 1 as required. ∎

In view of Proposition 1.7 we now deduce that if $L_1, L_2, \dots$ is a coherent factorization of the approximately finite lattice L, then L is primary if each factor L_k is primary.

COROLLARY 3.2. Let M be a finite unital lattice with unital sublattices $L_1, \dots, L_n$ which form a factorization of M. If p is a prime element of L_i for some i, and if M is primary, then p is a prime element of M.

Proof. Let p be a non-zero prime element of L_i and define $N = p \wedge M$, $N_k = p \wedge L_k$, for $k = 1, \dots, n$. We claim that $N_1, \dots, N_n$ is a factorization of N.

Clearly, $N_1, \dots, N_n$ generate N. Fix r and elements a, a' in L_r, b, b' in $\bigvee_{j \neq r} L_j$, and assume that

$$(p \wedge a) \wedge (p \wedge b) \leq (p \wedge a') \vee (p \wedge b').$$

If $r = i$ then $(p \wedge a) \wedge b = (p \wedge a) \wedge (p \wedge b) \leq ((p \wedge a') \vee b') \wedge p \leq (p \wedge a') \vee b'$. Hence $p \wedge a \leq p \wedge a'$ or $b \leq b'$. On the other hand if $r \neq i$ then $a \wedge (p \wedge b)$ $= (p \wedge a) \wedge (p \wedge b) \leq (p \wedge a') \vee (p \wedge b')$, and so $a \leq a'$ or $p \wedge b \leq p \wedge b'$. In both cases we have the desired alternative, $p \wedge a \leq p \wedge a'$ or $p \wedge b \leq p \wedge b'$.

We next show that each of the lattices N_k is primary, and the corollary will follow from Proposition 3.1.

Assume that $p = (p \wedge a) \vee (p \wedge b)$ with a and b in N_k. If $k = i$ then $p \wedge a = p$ or $p \wedge b = p$ because p is prime in L_i. On the other hand if $k \neq i$ then $p \wedge 1 = p = p \wedge (a \vee b) \leq a \vee b = 0 \vee (a \vee b)$ and so, by independence, $p \leq 0$ or $1 \leq a \vee b$. Hence $1 = a \vee b$ and $a = 1$ or $b = 1$ because M is primary. Hence $p \wedge a = p$ or $p \wedge b = b$ as required. ∎

<u>COROLLARY</u> 3.3. Let M be a unital primary lattice with unital primary sublattices $L_1, \ldots, L_n$ which form a factorization of M. Let p be an element of the form $p = \bigwedge_{r=1}^{\ell} p_r$ where each p_r is a prime in M_r. Then p is prime in M.

<u>Proof</u>. By Proposition 3.1 it suffices to show that each of the sublattices $p \wedge L_i$ is primary. Suppose then that a,b are elements of L_i such that $p = (p \wedge a) \vee (p \wedge b)$, and $p \neq 0$. Let $q_i = \bigwedge_{r \neq i} p_r$ so that $p_i \wedge q_i = p = ((p_i \wedge a) \vee (q_i \wedge b)) \wedge q_i$. Since the lattices L_i and $\bigvee_{j \neq i} L_j$ are independent it follows that $p_i = (p_i \wedge a) \vee (q_i \wedge b)$ and hence $p_i = p_i \wedge a$ or $p_i = p_i \wedge b$, since p_i is prime. Hence $p = p \wedge a$ or $p = p \wedge b$, and $p \wedge L_i$ is primary. ∎

The converse to the last corollary is also valid; every prime element p of the lattice M is of the form $p_1 \wedge \ldots \wedge p_n$ where each p_k is prime in L_k. We see this in the next section where we obtain an analogous representation for prime elements in certain approximately finite lattices admitting a factorization $L_1, L_2, \ldots$ by finite primary sublattices.

4. Prime elements and the unique factorization theorem

Our context in this section concerns approximately finite lattices L which arise as in the statement of Proposition 1.4, that is, L is isomorphic to the approximately finite lattice $\lim_n (L_1 \vee \ldots \vee L_n)$ associated with the sequence $L_1, L_2, \ldots$ which is a factorization of L by finite lattices. We refer to such a factorization as a coherent factorization. It was noted in the last section that if each of the lattices L_k is primary then L is primary.

A factorization $L_1, L_2, \ldots$ of L is said to be indecomposable when none of the sublattices L_k admits a nontrivial factorization. We shall obtain the following unique factorization theorem, which may be regarded as the approximately finite analogue of a theorem of Arveson for distributive metric lattices [1, Theorem 3.3.2].

THEOREM 4.1. Let $L_1, L_2, \ldots$ and $N_1, N_2, \ldots$ be two indecomposable coherent factorizations of the approximately finite unital primary lattice L. Then there is a permutation π of the natural numbers such that $N_k = L_{\pi(k)}$ for all k.

A key step in the proof of this result is the following theorem, which is the approximately finite version of Theorem 3.2.4 in [1], with a simpler proof. Note in particular that every prime p admits a finite representation $p = p_1 \wedge \ldots \wedge p_n$

THEOREM 4.2. Let $L_1, L_2 \ldots$ be a coherent factorization of the approximately finite primary unital lattice L. Let p_k be a prime of L_k for $k = 1, \ldots, m$. Then the element

$$p = p_1 \wedge p_2 \wedge \ldots \wedge p_m$$

is a prime in L. Moreover, every prime eelement p has this form, for some integer m depending on p.

<u>Proof</u>. We first show that for each prime $p \neq 0,1$ we have

$$p = \wedge\{a_k : a_k \geq p,\ a_k \in L_k\}.$$

(This is the AF version of Theorem 3.1.2 in [2]). Let p_1 denote the infimum and let $p_n = \wedge\{a : a \geq p, a \in L_1 \vee \ldots \vee L_n\}$. Then $p_1 \wedge p_n \geq p$, and in fact it will be enough to show that for each n, $p \geq p_1 \wedge p_n$. To see that this is enough, note that

$$p = p_1 \wedge p_n = \vee(p_1 \wedge p_n) = p_1 \wedge (\vee\ p_n) = p_1 \wedge 1.$$

The last two equalities here follow from infinite distributivity and the zero-one law, Propositions 1.1 and1.4 respectively.

Supppse then that $x \geq p$. We show that $x \geq p_1 \wedge p_n$. Let $\beta_1, \ldots, \beta_\ell$ be an enumeration of the elements of the form $x_1 \wedge \ldots \wedge x_{n-1}$ with x_i in L_i. Consider the collection N of elements of the form

$$\bigvee_{k=1}^{\ell} (\beta_k \wedge c_k)$$

with c_k in $L_n \vee L_{n+1} \vee \ldots$. Then N is a lattice and by Proposition 1.1, a complete lattice. Hence for some $c_1, \ldots, c_\ell$ we have

$$p \leq x = \bigvee_{k=1}^{\ell} (\beta_k \wedge c_k).$$

Since p is a prime element it follows that $p \leq \beta_k \wedge c_k$ for some k, and so $p \leq \beta_k$ and $p \leq c_k$. We have $p_1 \leq \beta_k$ and $p_n \leq c_k$ and so $p_1 \wedge p_n \leq \beta_k \wedge c_k \leq x$ as required.

We now obtain the last statement of the theorem. Let

$$p_k = \wedge\{a_k : a_k \geq p, a_k \in L_k, a_k \text{ is prime}\}.$$

Suppose that $p_k = a \vee b$ with a,b in L_k. Let $q_k = \bigwedge_{i \neq k} p_i$. Then $p = p_k \wedge q_k = (a \vee b) \wedge q_k = (a \wedge q_k) \vee (b \wedge q_n)$ and so $p = a \wedge q_k$ or $p = b \wedge q_k$. Suppose that $p = a \wedge q_k$. Then $p = p_k \wedge q_k = a \wedge q_k \leq a \vee 0$. By independence $p_k \leq a$ (since $q_k \neq 0$). Also $a \leq a \vee b = p_k$, and so $p_k = a$. The other case, namely $p = b \wedge q_n$ leads to $p_n = b$.

In view of Corollary 3.2 asnd Corollary 1.5 the proof is complete. ∎

The proof of Theorem 4.1 is completed exactly as in Arveson's paper. Thus from Proposition 4.1 the following refinement theorem is obtained in a straightforward way by using the sublattices $L_{m,n} = L_m \cap N_n$. (See Theorem 3.3.1 in [1]). Under the assumptions of the statement of Theorem 4.1 there is a double sequence $L_{m,n}$, $m,n \geq 1$ of finite sublattices of L such that

(i) For each m (resp. n) $L_{m,n}$ is the trivial sublattice {0,I} for all but finitely many values of n (resp. m), and

(ii) $L_{m1}, L_{m2}, \ldots$ and $L_{1n}, L_{2n}, \ldots$ are factorizations of L_m and L_n respectively.

In fact the above is obtained without using the assumption that the factorizations are indecomposable. With this assumption it follows that the doubly infinite matrix (L_{mn}) has exactly one nontrivial entry in every row and in every column. Let π be the permutation such that $L_{n,\pi^{-1}(n)}$ is the nontrivial entry in the n^{th} row. Then $L_n = L_{n1} \vee L_{n2} \vee \ldots = L_{n,\pi^{-1}(n)}$ and so $L_{\pi(n)} = L_{\pi(n),n} = \vee_j L_{jn} = N_n$, as required.

<u>Remark</u>. We have obtained the unique factorization theorem above without recourse to Arveson's factorization theorem for distributive metric lattices. It seems logical to make the elementary context independent of the topological one. However, it may well be possible to deduce our theorem from Arveson's by constructing normal valuations on AF lattices.

5. Isomorphisms and the automorphism group of $T((n_k))$

The following theorem characterizes the Banach algebra isomorphisms and epimorphisms between the algebras $T((n_k))$ and $T((m_k))$ where, as usual, (n_k) and (m_k) are sequences of positive integers greater than unity.

THEOREM 5.1. (i) $T((n_k))$ and $T((m_k))$ are isomorphic if and only if there is a permutation π such that $m_k = n_{\pi(k)}$, $k = 1,2,\ldots$.

(ii) There is an onto unital homomorphism from $T((n_k))$ to $T((m_k))$ if and only if there are finite sets of positive integers F_1, F_2 and a bijection $\pi: \mathbb{N} \to \mathbb{N}\backslash F_1$ such that

$$m_k = n_{\pi(k)} \quad , k \notin F_2$$

$$m_k \leq n_{\pi(k)} \quad , k \in F_2 .$$

We give two related proofs of the first part of this result. In one proof we focus on the structure of the countable lattice of invariant projections, Lat $T((n_k))$, and show that isomorphisms induce projection lattice isomorphisms. The set $\{n_1,n_2,\ldots\}$ is a complete lattice isomorphism invariant for Lat $T((n_k))$, and this fact depends only on the finite primary lattice factorization theory of section three. In the other proof, which we give first, we use the structure of the complete lattice of closed ideals, Id $T((n_k))$. Clearly Banach algebra isomorphisms induce ideal lattice isomorphisms, and once more the set $\{n_1,n_2,\ldots\}$ is a complete invariant for the lattice structure, although this is a consequence of the approximately finite primary lattice factorization theory of section four.

In some ways the ideal lattice approach seems more revealing, and is well adapted to the second part of the Theorem.

First Proof. Let $\mathcal{L}((n_k))$ be the approximately finite unital lattice $\lim_k \mathrm{Inc}(\delta(n_1),\ldots,\delta(n_k),2)$ so that by Proposition 2.3 $\mathcal{L}((n_k))$ and $\mathrm{Id}\,T((n_k))$ are isomorphic. By Proposition 1.7 $\mathcal{L}((n_k))$ is meet-irreducible. There are canonical identifications of the lattice $\mathcal{L}_j = \mathrm{Inc}(\delta(n_j),2)$ as a unital sublattice of $\mathcal{L}((n_k))$ and, by Proposition 1.4 $\mathcal{L}_1,\mathcal{L}_2,\ldots$ is a factorization of $\mathcal{L}((n_k))$. However the factorization is not indecomposable. Each sublattice $\mathcal{L}_j$ admits a factorization $\mathcal{L}_j^{\alpha} \vee \mathcal{L}_j^{\beta}$, where $\mathcal{L}_j^{\alpha}$ and $\mathcal{L}_j^{\beta}$ are copies of the nest lattice $L(n_j)$:

$$\mathcal{L}_j^{\alpha} = \{\phi_t \in \mathrm{Inc}(\delta(n_j),2):\ \phi_t((i,j)) = 1 \Longleftrightarrow 1 \le i \le t\}$$

$$\mathcal{L}_j^{\beta} = \{\psi_t \in \mathrm{Inc}(\delta(n_j),2):\ \psi_t((i,j)) = 0 \Longleftrightarrow t \le j \le 1\}.$$

With the converse order $\mathcal{L}((n_k))$ is a primary unital approximately finite lattice with coherent indecomposable factorization $\mathcal{L}_1^{\alpha},\mathcal{L}_1^{\beta},\mathcal{L}_2^{\alpha},\mathcal{L}_2^{\beta},\ldots$.

Suppose now that $T((n_k))$ and $T((m_k))$ are isomorphic as Banach algebras. Then $\mathcal{L}((n_k))$ and $\mathcal{L}((m_k))$ are isomorphic lattices. By Theorem 4.1 and the discussion above we obtain the desired permutation π for the first part of the theorem.

For the second part consider the ideal J that is the kernel of an onto unital homomorphism from $T((n_k))$ to $T((m_k))$. Since $\mathrm{Id}\,T((m_k))$ is meet-irreducible the zero ideal is not the intersection of two nonzero ideals. It follows that J is a meet-irreducible element of the ideal lattice $\mathrm{Id}\,T((n_k))$. (Equivalently, J is a prime element of the primary lattice with the converse order.) By Proposition 4.1 we conclude that J is the finite join $J_{k_1} \vee \ldots \vee J_{k_r}$ of nontrivial elementary ideals $J_{k_1},\ldots,J_{k_r}$. By an elementary ideal J_k we mean the meet-irreducible ideal generated by a meet-irreducible ideal $\hat{J}_k$ in one of the coordinate subalgebras $T(n_k)$. Thus, $T(n_k)/\hat{J}_k$ is isomorphic to $T(n_k')$ for some $1 \le n_k' \le n_k$, and

$$J_k = T(n_1) \otimes \ldots \otimes T(n_{k-1}) \otimes \hat{J}_k \otimes T((n_{k+1}) \otimes \ldots .$$

Note that we may have $n_k' = 1$. Indeed we set F_1 to be the finite set of k with $n_k' = 1$. In view of the first part of the theorem it remains to show that the quotient algebra $T((n_k))/J$ is isomorphic to $T(n_1') \otimes T(n_2') \otimes \ldots$, where we write $n_k' = n_k$ if $k \neq k_i$ for some $i = 1,\ldots,r$. However, there is a natural isomorphism

$$T(n_{k_1}) \otimes \ldots \otimes T(n_{k_r})/J_1 \vee \ldots \vee J_t \rightarrow T(n_{k_1}') \otimes \ldots \otimes T(n_{k_r}').$$

which is induced by a compression mapping. From this we obtain the required identification for the quotient $T((n_k))/J$. ■

Second proof of part (i). Write Lat $T((n_k))$ for the commutative lattice of self-adjoint projections p in $M((n_k))$ such that $(1-p)ap = 0$ for all a in $T((n_k))$. It follows, by standard arguments, that p lies in the diagonal algebra $C((n_k))$, and indeed, since distinct commuting projections cannot be close, p lies in the union of the projections in the finite dimensional aubalgebras $C(n_1) \otimes \ldots \otimes C(n_k)$, $k = 1,2,\ldots$. Thus Lat $T((n_k))$ is simply the union of the projections in the relative lattices $\mathrm{Lat}(T(n_1) \otimes \ldots \otimes T(n_k))$, computed in $M(n_1) \otimes \ldots \otimes M(n_k)$.

Suppose now that $\alpha: T((n_k)) \rightarrow T((n_k))$ is a Banach algebra isomorphism. Then for each projection p in Lat $T((n_k))$, $\alpha(p) = \hat{\alpha}(p) + s$ where $\hat{\alpha}(p)$ is a self-adjoint projection in $C((m_k))$ and r belongs to the Jacobson radical. The Jacobson radical coincides with the strictly upper triangular subalgebra of $T((n_k))$ and it is straightforward to obtain the direct sum decomposition $T((n_k)) = C((n_k)) + \mathrm{rad}\, T((n_k))$ (See [8]). There are elements x,y in $T((m_k))$ such that $x(1-\alpha(p)) = 1 - \hat{\alpha}(p)$ and$\alpha(p)y = \hat{\alpha}(p)$, and so we conclude that for a in $T((n_k))$, $(1-\hat{\alpha}(p))\alpha(a)\hat{\alpha}(p) = x(1-\alpha(p))\alpha(a)\alpha(p)y =$ $x\alpha((1-p)ap)y = 0$. Since α has an inverse isomorphism and since the projections commute we conclude that $\hat{\alpha}$ is a lattice isomorphism from Lat $T((n_k))$ onto Lat $T((n_k))$. The relative lattice $\text{Lat}(T(n_1)\otimes\ldots\otimes T(n_k))$ is isomorphic to the finite primary lattice $\text{Inc}(n_1,\ldots,n_k)$. It now follows easily from the structure of prime elements, given in section three, that $\hat{\alpha}$ induces the desired permutation π. ∎

The automorphism group. We can use the last theorem to obtain the following key lemma for the determination of the group Aut $T((n_k))$ of Banach algebra automorphisms of $T((n_k))$. In the subsequent two lemmas we determine that the automorphisms fixing the ideal lattice are precisely the pointwise inner automorphisms. We write α_π for the canonical permutation automorphism of $T((n_k))$ associated with a permutation π such that $n_k = n_{\pi(k)}$ for all k.

LEMMA 5.2. Let $\alpha \in \text{Aut}\, T((n_k))$. Then $\alpha = \beta\circ\alpha_\pi$ where α_π is a permutation automorphism and β is an automorphism with $\beta(J) = J$ for every two-sided ideal J.

Proof. We know from the proof of the last theorem that the meet irreducible ideals are precisely the ideals of the form $J_{k_1}\vee\ldots\vee J_{k_r}$, where each J_{k_i} is an elementary meet-irreducible ideal associated with the distinct coordinate algebra $T(n_i)$. Notice that the partially ordered set of meet-irreducible ideals of $T(n_i)$ is anti-isomorphic to $\delta(n_i)$, and therefore that the partially ordered set of all meet-irreducible ideals of $T((n_k))$ has a particularly transparent structure: given the converse order it coincides with the partially ordered set, $\delta(n_1)\times\delta(n_2)\times\ldots$ say, of finitely non-zero sequences $(t_1,t_2,\ldots)$, with t_i in $\delta(n_i)$, with the product partial ordering. The automorphism α induces an automorphism $\hat{\alpha}$ of this partially ordered set. But the automorphisms of $\delta(n_1)\times\delta(n_2)\times\ldots$ are compositions of a permutation automorphism $\hat{\alpha}_\pi$ and an automorphism $\hat{\beta}$

which acts locally. In fact each $\delta(n_i)$ supports a flip automorphism (exchanging coordinates in $\delta(n_i)$), and $\hat{\beta}$ must either fix or flip each coordinate. Since $\hat{\beta}$ derives from the <u>algebra</u> automorphism $\beta = \alpha \circ \alpha_\pi^{-1}$, it is easy to check that in fact $\hat{\beta}$ has no flip action, and hence that $\beta(J) = J$ for every elementary ideal, and hence for all ideals. ∎

The hypothesis in the next lemma cannot be relaxed too much as can be seen from the following example. Let A be the subalgebra of $T(4)$ spanned by the matrix units e_{ij} of $T(4)$ other than e_{12} and e_{34}. It can be seen that A admits automorphisms that preserve ideals but which are not inner. For example consider the automorphism α such that $\alpha(e_{14}) = -e_{14}$ and $\alpha(e_{ij}) = e_{ij}$ for all other matrix units in A. This fails to be inner because α fails to preserve the rank of some elements. (See [6] for related matters).

<u>LEMMA</u> 5.3. Let A be a subalgebra of the algebra $T(n)$ which contains the matrix units e_{ii}, e_{in}, for $1 \leq i \leq n$. If α is an automorphism of A such that $\alpha(J) = J$ for every two sided ideal J then α is an inner automorphism. Moreover the same holds true for ideal preserving automorphisms of the algebraic tensor product $A \otimes B$, where B is a commutative unital C^*-algebra.

<u>Proof</u>. By the ideal invariance of α we see that $\alpha(e_{11}) = e_{11} + \sum_{j=2}^{n} a_{1j}e_{1j}$.

Let a_{1r} be a nonzero coefficient with $r \geq 2$ and let $S_{1r}(\lambda) = I + \lambda e_{1r}$. Then $S_{1r}(\lambda)^{-1} = S_{1r}(-\lambda)$ and we see that $S_{1r}(\lambda)$ is an invertible element of A such that

$$(S_{1r}(\lambda)^{-1}\alpha(e_{11})S_{1r}(\lambda))_{1r} = a_{1r} - \lambda.$$

It follows that we may construct an invertible element S in A such that $S^{-1}\alpha(e_{11})S = e_{11}$.

Since α is an automorphism we observe that for $1 < i \leq j$

$$\begin{aligned}(S^{-1}\alpha(e_{ij})S)_{1r} &= (e_{11}S^{-1}\alpha(e_{ij})S)_{ir} \\ &= (S^{-1}\alpha(e_{11}e_{ij})S)_{1r} \\ &= 0.\end{aligned}$$

Thus α leaves invariant the subalgebra, A_1 say, spanned by $\{e_{ij}: e_{ij} \in A, 2 \leq i\}$. In particular, with respect to the associated decomposition $\mathbb{C}^n = \mathbb{C} \oplus \mathbb{C}^{n-1}$, α has the form

$$\alpha: \begin{pmatrix} a_{11} & \underline{a} \\ 0 & A_1 \end{pmatrix} \rightarrow \alpha(A) = \begin{pmatrix} a_{11} & \delta(\underline{a}) \\ 0 & \alpha_1(A_1) \end{pmatrix}$$

where α_1 is the restriction of α to A_1, and δ is a linear map on the linear space of row vectors $\underline{a}$.

We shall show that α is inner by induction on n. By the induction hypothesis α_1 is implemented by an invertible element T_1 of the algebra A_1. Conjugating by $T = e_{11} \oplus T_1$ obtain a new ideal preserving automorphism which is the identity map on A_1. Without loss then, we assume that α already has this form. In particular $\delta(\underline{a}\,A_1) = \delta(\underline{a})A_1$ for all operators A in A_1, from which it follows that $\delta(e_{1j}) = d_j e_{1j}$ for some scalars d_j (associated with indexes $j \geq 2$ for which e_{1j} is in A). Suppose e_{1j} lies in A. Then $d_n e_{1n} = \delta(e_{1n}) = \delta(e_{1j}e_{jn}) = \delta(e_{1j})\delta(e_{jn}) = d_j e_{1j}e_{jn} = d_j e_{1n}$. Thus all the d_j coincide with a single scalar d say. Thus $\alpha(\cdot) = D^{-1}\cdot D$ where D is the diagonal matrix with entries $1,d,d,\ldots,d$, and the first assertion is proven.

Note that $A \otimes B$ can be considered as the algebra of matrices from A whose entries are operators in B. Replacing the role of the scalar field by B in the argument above leads to an almost identical proof for the second assertion of the proposition. ∎

The next lemma characterizes the ideal fixing automorphisms as the pointwise inner automorphisms.

<u>LEMMA</u> 5.4. Let α be an automorphism of $T((n_k))$ such that $\alpha(J) = J$ for every closed two sided ideal J. Then there exist invertible operators S_r, with S_r and S_r^{-1} in $T((n_k))$, for $r = 1,2,\ldots$, such that $S_r^{-1}XS_r \to \alpha(X)$ as $r \to \infty$ for every element X of $T((n_k))$.

<u>Proof</u>. Let $A_r = \bigotimes_{k=1}^{r} T(n_k)$, $A^r = \bigotimes_{k=r+1}^{\infty} T(n_k)$, $C_r = \bigotimes_{k=1}^{r} C(n_k)$ $C^r = \bigotimes_{k=r+1}^{\infty} C(n_k)$, regarded as the usual subalgebras of $T((n_k))$. The Jacobson radical $\mathrm{rad}\, A^r$ of the subalgebra A^r is the strictly upper triangular part of A^r and we have $A^r = C^r + \mathrm{rad}\, A^r$. Moreover $J = A_r \otimes \mathrm{rad}\, A^r$ is an ideal such that the quotient $T((n_k))/J$ is canonically isomorphic to $A_r \otimes C^r$. To see this observe that $A_r \otimes \mathrm{rad}\, A^r$ is the kernel of the natural contractive homomorphism from $T((n_k))$ to $A_r \otimes C^r$). In particular, since J is invariant, α induces an automorphism α_r of $A_r \otimes C^r$, and moreover α_r leaves invariant the ideals of $A_r \otimes C^r$. The ascending subalgebras $A_r \otimes C^r$ have dense union in $T((n_k))$, and so it will be sufficient to show that each automorphism α_r is inner. This follows from the second part of Lemma 5.3, since the algebras A_r are subalgebras of $T(n_1 n_2 \ldots n_r)$ of the required form. ∎

The results above are summarized in the next theorem. We write $\mathrm{Out}(T((n_k)))$ for the quotient group determined by the normal subgroup of pointwise inner automorphisms.

<u>THEOREM</u> 5.5. Let $\Pi((n_k))$ be the discrete group of permutations π such that $n_k = n_{\pi(k)}, k = 1,2,\ldots$. Then each automorphism α in $\mathrm{Aut}\, T((n_k))$ admits a decomposition $\alpha = \beta \circ \alpha_\pi$ with β a pointwise inner automorphism and π in $\Pi((n_k))$. In particular $\mathrm{Out}\, T((n_k))$ is the discrete group $\Pi((n_k))$.

6. The K_0 group

Let A be the algebra $T((n_k))$ with diagonal subalgebra $C = C((n_k))$, associated as usual with integers $n_k \geq 2$, $k = 1,2,\ldots$. We show that $K_0(A) = K_0(C)$. In particular K_0 does not distinguish the isomorphism type.

Recall that A decomposes as a direct sum $A = C + \operatorname{rad} A$, where $\operatorname{rad} A$ is the Jacobson radical. Let $p = (p_{ij})$ be an idempotent in $M_n(A)$ and let $p_{ij} = c_{ij} + r_{ij}$ with c_{ij} in C and r_{ij} in $\operatorname{rad} A$, so that $c = (c_{ij})$ is an idempotent in $M_n(C)$. We show that there is continuous path of idempotents p^t, $0 \leq t \leq 1$, in $M_n(A)$, such that $p^2 = p$ and $p^0 = c$. From this it will follow that the natural map $K_0(A) \to K_0(C)$ induced by the quotient mapping, is an isomorphism.

Let $d_{t,k}$ be the invertible element of C given by

$$d_{t,k} = 1 \otimes \ldots \otimes 1 \otimes D_{t,k} \otimes 1 \otimes \ldots ,$$

$$D_{t,k} = \begin{bmatrix} 1 & & & & & \\ & t & & & & \\ & & t^2 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & t^{n_k-1} \end{bmatrix}, \quad 0 < t < 1.$$

Then the inner automorphism $\alpha_{t,k}: a \to d_{t,k}^{-1} a d_{t,k}$ is a contractive on A. It follows that we can define the pointwise inner homomorphism α_t by

$$\alpha_t(a) = \lim_{k\to\infty} \alpha_{t,1} \circ \alpha_{t,2} \circ \ldots \circ \alpha_{t,k}(a).$$

Indeed, this limit exists on a dense subspace, and the composed automorphisms are contractive. Note that α_t is a homomorphism and $\alpha_t(a), 0 < t \leq 1$, is a continuous path in A. A simple approximation argument shows that if $a = c + r$ with c in C and r in $\operatorname{rad} A$, then $\alpha_0(a) = \lim_{t\to 0} \alpha_t(a) = c$. Thus the idempotents

$p^t = (\alpha_t(p_{ij}))$ form a path with the desired properties.

In fact a similar argument works for any subalgebra A_1 of a nest subalgebra A of an AF algebra, as defined in [8], with the property that A_1 contains the diagonal algebra of A.

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APPENDIX 1

Best approximation in C^*-algebras

By *Kenneth R. Davidson**) at Waterloo and *Stephen C. Power* at Lancaster

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In this paper, methods are developed for obtaining best approximations to ideals of (generally non self-adjoint) subalgebras and subspaces of C^*-algebras. Suppose $\mathscr{J}$ is an ideal of a C^*-algebra $\mathfrak{A}$. Let $\mathscr{S}$ be a subspace of $\mathfrak{A}$ such that $\mathscr{S} \cap \mathscr{J}$ is $\mathscr{J}$-weakly dense in $\mathscr{S}$ (see section one). Then $\mathscr{S} \cap \mathscr{J}$ is proximinal in $\mathscr{S}$, and the natural map

$$\sigma : \mathscr{S}/\mathscr{S} \cap \mathscr{J} \longrightarrow \mathscr{S} + \mathscr{J}/\mathscr{J}$$

is isometric.

Our methods use the M-ideals introduced by Alfsen and Effros [2], and in fact yield a general Banach space theorem. The special topologies needed are introduced in section one, and the approximation theorem is proved in section 2. In section 3, a constructive proof is given based on the method of Axler, Berg, Jewell and Shields [4]. This section can be read independently on the first two sections. In fact, this was our original method of proof and was highlighted in a previous version of this paper. However, the hypotheses are apparently more stringent (although Corollary 2.7 shows that this is not really the case). In section 4, the usefulness of approximate identities for $\mathscr{J}$ in $\mathscr{S} \cap \mathscr{J}$ is pointed out.

Section 5 is devoted to applications to nest algebras. The most significant result in this section is a distance formula for an arbitrary operator T to the quasitriangular algebra $\mathscr{Q}\mathscr{T}(\mathscr{N})$ in terms of the function Φ_T taking $\mathscr{N}$ into $\mathscr{B}(\mathscr{H})$ given by

$$\Phi_T(N) = P(N)^{\perp}\, TP(N)$$

for N in $\mathscr{N}$. In [10], it is shown that T belongs to $\mathscr{Q}\mathscr{T}(\mathscr{N})$ if and only if Φ_T is norm continuous and compact valued. It is shown that the distance of Φ_T to the ideal of norm continuous compact valued functions is exactly the distance of T to $\mathscr{Q}\mathscr{T}(\mathscr{N})$.

In section 6, nest subalgebras of the compact operators are studied. It turns out that only in three simple cases can $\mathscr{T}(\mathscr{N}) \cap \mathscr{K}$ be proximinal in $\mathscr{K}$. The methods of this section mimic those of [17], and use the useful matricial arguments of [16], [7].

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*) Research partially supported by grants from NSERC (Canada) and SERC (Great Britain)

1. Topologies on C^*-algebras

The situation to be considered is the following: $\mathfrak{A}$ is a C^*-algebra with a (closed two-sided) ideal $\mathscr{J}$, and $\mathscr{S}$ is a (closed) subspace of $\mathfrak{A}$. If S is an element of $\mathscr{S}$, is there an element J in $\mathscr{S} \cap \mathscr{J}$ such that

$$\|S+J\| = \|S+\mathscr{J}\| ?$$

The existence of such best approximations in $\mathscr{S}$ itself can have many ramifications. (See section 5 for some applications.) Naturally, such approximations do not always exist. It is perhaps surprising then that if one stipulates that $\mathscr{S} \cap \mathscr{J}$ is "sufficiently rich", such an approximation is always possible.

We need some topologies on $\mathfrak{A}$ induced by $\mathscr{J}$ analogous to the weak operator topology, strong operator topology and strong* operator topology. A net A_α will be said to converge to A in the $\mathscr{J}$-weak topology ($A_\alpha \xrightarrow{\mathscr{J}w} A$) provided

$$\Phi(A_\alpha J) \longrightarrow \Phi(AJ)$$

for all J in $\mathscr{J}$ and Φ in $\mathscr{J}^*$. Similarly, the net A_α converges in the $\mathscr{J}$-strong topology ($A_\alpha \xrightarrow{\mathscr{J}s} A$) provided

$$A_\alpha J \longrightarrow AJ$$

for all J in $\mathscr{J}$. Lastly, A_α converges to A in the $\mathscr{J}$-strong* topology ($A_\alpha \xrightarrow{\mathscr{J}s^*} A$) provided

$$A_\alpha \xrightarrow{\mathscr{J}s} A \quad \text{and} \quad A_\alpha^* \xrightarrow{\mathscr{J}s} A^*.$$

This last topology is also known as the $\mathscr{J}$-strict topology, and was introduced by Busby [5] for the purpose of studying extensions of C^*-algebras.

There is a natural homomorphism taking $\mathfrak{A}$ into the multiplier algebra $\mathscr{M}(\mathscr{J})$ of $\mathscr{J}$. Since $\mathscr{M}(\mathscr{J})$ imbeds naturally into the bounded operators on $\mathscr{J}$, one sees readily that the $\mathscr{J}$-topologies (weak, strong, strong*) correspond with the topologies induced by the corresponding operator topologies on $\mathscr{B}(\mathscr{J})$. For example, if the C^*-algebra is $\mathscr{B}(\mathscr{H})$, the space of bounded operators on a Hilbert space $\mathscr{H}$, and the ideal is the ideal of compact operators $\mathscr{K}$, then the $\mathscr{K}$-weak topology is precisely the weak* (or ultra weak) topology on $\mathscr{B}(\mathscr{H})$. The $\mathscr{K}$-strong and $\mathscr{K}$-strong* topologies are the ultra-strong and ultra-strong* topologies.

The reader familiar with these topologies on $\mathscr{B}(\mathscr{H})$ will not be surprised by the following lemma.

Lemma 1.1. *The continuous linear functionals on $\mathfrak{A}$ with respect to the $\mathscr{J}$-weak, $\mathscr{J}$-strong and $\mathscr{J}$-strong* topologies coincide. In particular, they have the same closed convex sets.*

Proof. Identify $\mathfrak{A}$ with its image in $\mathscr{B}(\mathscr{J})$. By [8], Theorem VI. 1. 4, the weak operator topology and strong operator topology on $\mathscr{B}(\mathscr{X})$ have the same continuous linear functionals for any Banach space $\mathscr{X}$. For the $\mathscr{J}$-strong* topology, note that the dual has an adjoint operation

$$\Phi^*(A) = \overline{\Phi(A^*)}$$

which is continuous since adjoint is $\mathscr{J}$-strong* continuous. (The same applies to the $\mathscr{J}$-weak topology.) Thus, one may assume that $\Phi = \Phi^*$. In particular, Φ is real on the self adjoint part $\mathfrak{A}_{s.a.}$, and the $\mathscr{J}$-strong and $\mathscr{J}$-strong* topologies agree on $\mathfrak{A}_{s.a.}$. The real version of the above general theorem shows that Φ is $\mathscr{J}$-weak continuous on $\mathfrak{A}_{s.a.}$. Now linearly extending this to all of $\mathfrak{A}$ shows that Φ is $\mathscr{J}$-weak continuous as well. The other direction is trivial. □

The condition that $\mathscr{S} \cap \mathscr{J}$ is "sufficiently rich" can now be stated as the requirement that $\mathscr{S} \cap \mathscr{J}$ is $\mathscr{J}$-weakly dense in $\mathscr{S}$. Lemma 1. 1 shows that it is therefore $\mathscr{J}$-strong* dense. In Corollary 2. 7, it will be shown that, moreover, the unit ball of $\mathscr{S} \cap \mathscr{J}$ is $\mathscr{J}$-strong* dense in the unit ball of $\mathscr{S}$. This condition will be used to obtain more constructive methods in section 3.

A closed subspace $\mathscr{M}$ of a Banach space $\mathscr{X}$ is said to be an M-ideal [2] if there is a linear projection

$$\eta : \mathscr{X}^* \longrightarrow \mathscr{M}^\perp$$

from the dual space $\mathscr{X}^*$ onto the annihilator $\mathscr{M}^\perp$ of $\mathscr{M}$ in $\mathscr{X}^*$ such that for all Φ in $\mathscr{X}^*$,

$$\|\Phi\| = \|\eta\Phi\| + \|\Phi - \eta\Phi\|.$$

In this case, $\mathscr{M}^\perp$ is said to be an L-summand of $\mathscr{X}^*$ and η is called the L-projection onto $\mathscr{M}^\perp$. The fact that M-ideals are proximinal ([2], Corollary 5. 6 and [13], section 4 for an elementary proof) has been exploited by several authors (for example, [15] and [20]).

The M-ideals in a C^*-algebra are precisely the two sided ideals [20]. We indicate a proof that ideals are M-ideals which is convenient for our purposes. Recall that $\mathfrak{A}^{**}$ may be identified with the enveloping von Neumann algebra of $\mathfrak{A}$. Let P denote the central support projection for $\mathscr{J}$ in $\mathfrak{A}^{**}$. Then define the mapping η on $\mathfrak{A}^*$ by

$$(\eta\Phi)(A) = \Phi((I-P)A).$$

This is an L-projection onto $\mathfrak{A}^*(I-P) = \mathscr{J}^\perp$. So $\mathscr{J}$ is an M-ideal (see Takesaki [21], p. 171 for details about $\mathfrak{A}^{**}$).

Our approximation results can be put in a general Banach space setting. To state them, the analogue of the $\mathscr{J}$-weak topology is required. Let $\mathscr{M}$ be an M-ideal on a Banach space $\mathscr{X}$, and let η be the L-projection of $\mathscr{X}^*$ onto $\mathscr{M}^\perp$. One can identify $\mathscr{M}^*$ with the range of $1-\eta$. Indeed, this identification associates to any ϕ in $\mathscr{M}^*$ its unique Hahn-Banach extension $\tilde{\phi}$ in $\mathscr{X}^*$. The $\mathscr{M}^*$-topology on $\mathscr{X}$ is the weakest topology in which each $\tilde{\phi}$ is continuous. In particular, one has that a net M_α of elements of $\mathscr{M}$ converges $\mathscr{M}^*$ to X in $\mathscr{X}$ if and only if

$$\lim_\alpha \phi(M_\alpha) = \tilde{\phi}(X)$$

for all ϕ in $\mathscr{M}^*$.

It is frequently the case that $\mathscr{X}$ is an M-ideal in $\mathscr{X}^{**}$. For example, this is the case if $\mathscr{X} = \mathscr{K}(\ell^p)$, $1 < p < \infty$, the space of compact operators on ℓ^p [14]. In this case, the $\mathscr{X}^*$-topology is precisely the weak* topology on $\mathscr{X}^{**}$.

Returning to the setting of an ideal $\mathscr{J}$ in a C^*-algebra $\mathfrak{A}$ we have the following

Lemma 1.2. *The $\mathscr{J}^*$ topology and the $\mathscr{J}$ weak topology coincide.*

Proof. The $\mathscr{J}^*$ topology is determined by the unique Hahn-Banach extensions $\tilde{\phi}$ of functionals ϕ in $\mathscr{J}^*$. The $\mathscr{J}$-weak topology is determined by the functionals $\psi(.J)$ for ψ in $\mathscr{J}^*$ and J in $\mathscr{J}$. It will suffice then to show that each ϕ in $\mathscr{J}^*$ may be factored as $\phi = J\psi$ where $J\psi$ indicates the functional $\psi(.J)$ for some ψ in $\mathscr{J}^*$, J in $\mathscr{J}$. Observe that $\mathscr{J}^*$ is in fact a left Banach module for $\mathscr{J}$ under this multiplication. Moreover, if $\{E_\alpha\}$ is an approximate unit for $\mathscr{J}$ then it can be shown that $\{E_\alpha\}$ is an approximate unit for $\mathscr{J}^*$. Thus Cohen's factorisation theorem is applicable, and each ϕ in $\mathscr{J}^*$ admits the required factorisation. □

2. Proximinality of ideal perturbations

The main result of this paper can now be stated.

Theorem 2.1. *Let $\mathscr{M}$ be an M-ideal in a Banach space $\mathscr{X}$. Suppose that $\mathscr{S}$ is a subspace of $\mathscr{X}$ such that $\mathscr{S} \cap \mathscr{M}$ is $\mathscr{M}^*$-dense in $\mathscr{S}$. Then*

(i) *$\mathscr{S} \cap \mathscr{M}$ is an M-ideal in $\mathscr{S}$, and the quotient map*

$$\sigma : \mathscr{S}/\mathscr{S} \cap \mathscr{M} \longrightarrow \mathscr{S} + \mathscr{M}/\mathscr{M}$$

is isometric,

(ii) *$\mathscr{S} + \mathscr{M}/\mathscr{S}$ is an M-ideal in $\mathscr{X}/\mathscr{S}$,*

(iii) *if $\mathscr{S}$ is proximinal in $\mathscr{X}$, so is $\mathscr{S} + \mathscr{M}$.*

Corollary 2.2. *Let $\mathscr{J}$ be an ideal in a C^*-algebra $\mathfrak{A}$. Suppose that $\mathscr{S}$ is a subspace of $\mathfrak{A}$ such that $\mathscr{S} \cap \mathscr{J}$ is $\mathscr{J}$-weakly dense in $\mathscr{S}$. Then for each S in $\mathscr{S}$, there is an element J of $\mathscr{S} \cap \mathscr{J}$ such that*

$$\|S + J\| = \|S + \mathscr{J}\|.$$

Corollary 2.3. *Let $\mathscr{S}$ be a weak* closed subspace of $\mathscr{B}(\mathscr{H})$ such that $\mathscr{S} \cap \mathscr{K}$ is weak* dense in $\mathscr{S}$. Then the map*

$$\sigma : \mathscr{S}/\mathscr{S} \cap \mathscr{K} \longrightarrow \mathscr{S} + \mathscr{K}/\mathscr{K}$$

is isometric and $\mathscr{S} \cap \mathscr{K}$ is proximinal in $\mathscr{S}$. Furthermore, $\mathscr{S} + \mathscr{K}$ is proximinal in $\mathscr{B}(\mathscr{H})$.

Proof of Theorem 2.1. Let η be the L-projection of $\mathscr{X}^*$ onto $\mathscr{M}^\perp$. First, we show that $\eta\mathscr{S}^\perp$ is contained in $\mathscr{S}^\perp$. So let Φ belong to $\mathscr{S}^\perp$. For any S in $\mathscr{S}$, let S_α be a net in $\mathscr{S} \cap \mathscr{M}$ converging to S in the $\mathscr{M}^*$-topology. Then since $(1-\eta)\,\Phi$ belongs to $\mathscr{M}^*$,

$$\begin{aligned}\eta\Phi(S) &= \Phi(S) - (1-\eta)\,\Phi(S)\\ &= \lim_\alpha (1-\eta)\,\Phi(S_\alpha) = -\lim_\alpha \Phi(S_\alpha) = 0.\end{aligned}$$

Since η leaves $\mathscr{S}^\perp$ invariant, it induces a projection $\tilde{\eta}$ from $\mathscr{X}^*/\mathscr{S}^\perp$ onto $\mathscr{M}^\perp+\mathscr{S}^\perp/\mathscr{S}^\perp$. Now $\mathscr{X}^*/\mathscr{S}^\perp$ is isomorphic to $\mathscr{S}^*$, and $\mathscr{M}^\perp+\mathscr{S}^\perp/\mathscr{S}^\perp \cong (\mathscr{M}\cap\mathscr{S})^\perp/\mathscr{S}^\perp$ is identified in $\mathscr{S}^*$ with the annihilator of $\mathscr{M}\cap\mathscr{S}$. It is clear that $\tilde{\eta}$ is an L-projection, and thus $\mathscr{M}\cap\mathscr{S}$ is an M-ideal in $\mathscr{S}$. In particular, $\mathscr{S}\cap\mathscr{M}$ is proximinal in $\mathscr{S}$.

Furthermore, if ψ belongs to $\mathscr{M}^\perp+\mathscr{S}^\perp=(\mathscr{M}\cap\mathscr{S})^\perp$, then $\eta\psi$ belongs to $\mathscr{M}^\perp$ and $(1-\eta)\psi$ belongs to $(1-\eta)\mathscr{S}^\perp$ which is contained in $\mathscr{S}^\perp$. So if S belongs to $\mathscr{S}$, $\psi(S)=\eta\psi(S)$. Whence

$$\begin{aligned} d(S, \mathscr{S}\cap\mathscr{M}) &= \sup\{\psi(S): \|\psi\|\leqq 1,\ \psi\in\mathscr{M}^\perp+\mathscr{S}^\perp\} \\ &= \sup\{\eta\psi(S): \|\psi\|\leqq 1,\ \psi\in\mathscr{M}^\perp+\mathscr{S}^\perp\} \\ &= \sup\{\phi(S): \|\phi\|\leqq 1,\ \phi\in\mathscr{M}^\perp\} = d(S,\mathscr{M}). \end{aligned}$$

Hence the natural map

$$\sigma: \mathscr{S}/\mathscr{S}\cap\mathscr{M} \to \mathscr{S}+\mathscr{M}/\mathscr{M}$$

is isometric. Thus if S belongs to $\mathscr{S}$, there is an element M of $\mathscr{S}\cap\mathscr{M}$ such that

$$\|S-M\| = d(S,\mathscr{M}).$$

For assertion (ii), note that $(\mathscr{X}/\mathscr{S})^*$ is isometrically isomorphic to $\mathscr{S}^\perp$, and the annihilator of $\mathscr{M}+\mathscr{S}/\mathscr{S}$ is just $\mathscr{M}^\perp\cap\mathscr{S}^\perp$. Since η leaves $\mathscr{S}^\perp$ invariant, the restriction $\tilde{\eta}$ to $\mathscr{S}^\perp$ is the desired L-projection onto $(\mathscr{M}+\mathscr{S}/\mathscr{S})^\perp$.

To prove (iii), take any X in $\mathscr{X}$. By (ii), there is an element of M in $\mathscr{M}$ such that

$$d(X-M,\mathscr{S}) = d(X,\mathscr{S}+\mathscr{M}).$$

Since $\mathscr{S}$ is proximinal, there is an element S of $\mathscr{S}$ such that

$$\|X-(M+S)\| = d(X-M,\mathscr{S}) = d(X,\mathscr{S}+\mathscr{M}). \quad \square$$

Proof of Corollaries. Corollary 2. 2 is immediate from (i) and the equivalence of the $\mathscr{J}$-weak and $\mathscr{J}^*$-topologies. For Corallary 2. 3, note that weak* closed subspaces are always proximinal in $\mathscr{B}(\mathscr{H})$. □

Corollary 2. 4. *Suppose $\mathscr{X}$ is an M-ideal in $\mathscr{X}^{**}$, and that $\mathscr{S}$ is a weak* closed subspace of $\mathscr{X}^{**}$ such that $\mathscr{S}\cap\mathscr{X}$ is weak* dense in $\mathscr{S}$. Then $\mathscr{S}+\mathscr{X}$ is proximinal in $\mathscr{X}^{**}$, $\mathscr{S}\cap\mathscr{X}$ is proximinal in $\mathscr{S}$, and the map*

$$\sigma: \mathscr{S}/\mathscr{S}\cap\mathscr{X} \to \mathscr{S}+\mathscr{X}/\mathscr{X}$$

is isometric.

A special case of this is somewhat stronger than the main result of [4].

Corollary 2. 5. *Let $\mathscr{S}$ be a subspace of $\mathscr{B}(\ell^p)$, $1<p<\infty$, which is the weak* closure of $\mathscr{S}\cap\mathscr{K}(\ell^p)$. Then if S belongs to $\mathscr{S}$, there is a compact operator K in $\mathscr{S}$ such that*

$$\|S+K\| = \|S\|_e.$$

Furthermore, $\mathscr{S}+\mathscr{K}$ is proximinal in $\mathscr{B}(\ell^p)$.

In [10], a subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ is called local if $\mathcal{S}$ is the weak* closure of $\mathcal{S} \cap \mathcal{K}$. They show that the map

$$\tau : \mathcal{K}/\mathcal{S} \cap \mathcal{K} \to \mathcal{S} + \mathcal{K}/\mathcal{S}$$

is isometric. Thus they obtain that $\mathcal{S} + \mathcal{K}$ is closed. However, they do not obtain that the more natural map

$$\sigma : \mathcal{S}/\mathcal{S} \cap \mathcal{K} \to \mathcal{S} + \mathcal{K}/\mathcal{K}$$

is isometric. This now follows immediately from Corollary 2. 4.

The proof of Theorem 2. 1 allows us to deduce more from $\mathcal{M}^*$-density, namely that the unit ball is $\mathcal{M}^*$-dense in the ball as well.

Proposition 2. 6. *Let $\mathcal{M}$ be an M-ideal in a Banach space $\mathfrak{X}$. Suppose that $\mathcal{S} \cap \mathcal{M}$ is $\mathcal{M}^*$-dense in $\mathcal{S}$. Then the unit ball of $\mathcal{S} \cap \mathcal{M}$ is $\mathcal{M}^*$-dense in the ball of $\mathcal{S}$.*

Proof. There is a natural contractive linear map τ of $\mathcal{S}$ into $\mathcal{M}^{**}$ given by

$$\tau(S)(\phi) = \tilde{\phi}(S)$$

for ϕ in $\mathcal{M}^*$. The condition that $\mathcal{S} \cap \mathcal{M}$ is $\mathcal{M}^*$-dense is precisely that $\tau(\mathcal{S}) \cap \mathcal{M}$ be weak* dense in $\tau(\mathcal{S})$.

Since the L-projection η leaves $\mathcal{S}^\perp$ invariant, $\mathcal{S}^\perp$ splits as the L^1 direct sum

$$\mathcal{S}^\perp = (\eta \mathcal{S}^\perp) \oplus (1-\eta)\mathcal{S}^\perp .$$

And from the proof of Theorem 2. 1, one also has

$$(\mathcal{S} \cap \mathcal{M})^\perp = \mathcal{M}^\perp \oplus (1-\eta)\mathcal{S}^\perp .$$

So it is apparent that in $\mathcal{M}^*$ one has

$$(\tau(\mathcal{S}) \cap \mathcal{M})^\perp = (1-\eta)\mathcal{S}^\perp = \tau(\mathcal{S})_\perp .$$

Thus $\tau(\mathcal{S})$ is identified with (a subspace of) $(\tau(\mathcal{S}) \cap \mathcal{M})^{**}$.

A well known theorem in functional analysis states that the unit ball of any Banach space $\mathfrak{X}$ is weak* dense in the unit ball of its bidual $\mathfrak{X}^{**}$. Applying this to $\tau(\mathcal{S}) \cap \mathcal{M}$ yields that the ball of $\tau(\mathcal{S}) \cap \mathcal{M}$ is weak* dense in the ball of $\tau(\mathcal{S})$. Since τ is isometric on $\mathcal{S} \cap \mathcal{M}$, the ball of $\mathcal{S} \cap \mathcal{M}$ is $\mathcal{M}^*$-dense in the ball of $\mathcal{S}$. □

Corollary 2. 7. *Let $\mathcal{J}$ be an ideal in a C*-algebra $\mathfrak{A}$. Suppose that $\mathcal{S}$ is a subspace of $\mathfrak{A}$ and $\mathcal{S} \cap \mathcal{J}$ is $\mathcal{J}$-weakly dense in $\mathcal{S}$. Then the unit ball of $\mathcal{S} \cap \mathcal{J}$ is $\mathcal{J}$-strong* dense in the ball of $\mathcal{S}$.*

Proof. Apply Proposition 2. 6 and Lemma 1. 1. □

3. A constructive approach

The purpose of this section is to modify the technique of [4] to get a constructive method of obtaining best approximants. Corollary 2. 7 shows that $\mathcal{J}$-weak density implies the much stronger condition of bounded, $\mathcal{J}$-strong* density. The price to be paid here is that we assume, a priori, that such bounded nets are at hand.

The first lemma is an easy application of the functional calculus. A proof may be found in [1], Theorem 4.3.

Lemma 3.1. *Let $\mathcal{J}$ be a (closed two-sided) ideal of a C*-algebra $\mathfrak{A}$. For any A in $\mathfrak{A}$, there is an element J in $\mathcal{J}$ such that*

$$\|A+J\| = \|A+\mathcal{J}\|$$

and

$$\|J\| = \|A\| - \|A+\mathcal{J}\|.$$

Corollary 3.2. *Every ideal of a C*-algebra is proximinal.*

This corollary is immediate and elementary. It also follows from the M-ideal theory (see section 1 or [20]).

Next, we need another elementary result. This lemma is straight-forward in the commutative case, but is a bit more subtle in general.

Lemma 3.3. *Let A and B be positive elements of a C*-algebra. Then*

$$\|A+B\| \leqq \max\{\|A\|, \|B\|\} + \|AB\|^{\frac{1}{2}}.$$

Proof. Assume for convenience that A and B are operators. Suppose $A+B$ attains its norm, so that there is a unit vector x with

$$(A+B)x = \|A+B\|\, x.$$

Write $Ax = \alpha x + y$ and $Bx = \beta x - y$, where y is orthogonal to x. Let $\gamma = \|y\|$. To simplify computations, let us further normalize so that $1 = \alpha \geqq \beta$. The compression of A to span$\{x, y\}$ is positive, and is greater than

$$\begin{bmatrix} 1 & \gamma \\ \gamma & \gamma^2 \end{bmatrix}.$$

Thus $\|A\| \geqq 1 + \gamma^2$. Also

$$\|AB\| \geqq |(ABx, x)| = |(A\beta x, x) - (Ay, x)| = |\beta - \gamma^2|.$$

Hence

$$\|A\| + \|AB\|^{\frac{1}{2}} \geqq 1 + \gamma^2 + |\beta - \gamma^2|^{\frac{1}{2}} \geqq 1 + \beta$$

where the desired inequality follows from elementary calculus.

The general case is obtained by using approximate eigenvectors. □

The next lemma is the appropriate analogue of Theorem 2 of [4] for arbitrary ideals instead of the compacts. The interested reader should note that in the case of the compacts this proof can be simplified to some extent. To our minds, it provides a direct and more natural proof of the theorem in [4].

Lemma 3.4. *Let $\mathcal{J}$ be an ideal in a C*-algebra $\mathfrak{A}$. Suppose A belongs to $\mathfrak{A}$, and B_α is a net of elements such that $B_\alpha \xrightarrow{\mathcal{J}s^*} 0$. Then for each $\varepsilon > 0$, there is an α_0 so that for all $\alpha \geqq \alpha_0$,*

$$\|A + B_\alpha\| < \max\{\|A\|, \|A+\mathcal{J}\| + \|B_\alpha\|\} + \varepsilon.$$

Proof. By Lemma 3. 1, obtain J in $\mathscr{J}$ so that

$$\|A+J\| = \|A+\mathscr{J}\| \text{ and } \|J\| = \|A\| - \|A+\mathscr{J}\|.$$

Since $B_\alpha \xrightarrow{\mathscr{J}_{s^*}} 0$, one can choose α_0 so that for $\alpha \geqq \alpha_0$,

$$\|B_\alpha J^*\| < \frac{\varepsilon^2}{2} \quad \text{and} \quad \|J^* B_\alpha\| < \frac{\varepsilon^2}{2}.$$

Let $M_\alpha = \max\{\|J\|, \|B_\alpha\|\}$. Then

$$\begin{aligned}\|B_\alpha - J\|^2 &= \|(B_\alpha - J)^* (B_\alpha - J)\| \\ &\leqq \|B_\alpha^* B_\alpha + J^* J\| + \|B_\alpha^* J + J^* B_\alpha\| \\ &< \|B_\alpha^* B_\alpha + J^* J\| + \varepsilon^2.\end{aligned}$$

Since

$$\|B_\alpha^* B_\alpha J^* J\| \leqq \|B_\alpha^*\| \; \|B_\alpha J^*\| \; \|J\| < M_\alpha^2 \frac{\varepsilon^2}{2},$$

Lemma 3. 3 implies that

$$\|B_\alpha - J\|^2 < M_\alpha^2 + \left(M_\alpha^2 \frac{\varepsilon^2}{2}\right)^{\frac{1}{2}} + \varepsilon^2 < (M_\alpha + \varepsilon)^2.$$

It now follows immediately that

$$\begin{aligned}\|A + B_\alpha\| &\leqq \|A + J\| + \|B_\alpha - J\| \\ &\leqq \|A + \mathscr{J}\| + \max\{\|A\| - \|A + \mathscr{J}\|, \|B_\alpha\|\} + \varepsilon \\ &= \max\{\|A\|, \|A + \mathscr{J}\| + \|B_\alpha\|\} + \varepsilon. \quad \square\end{aligned}$$

From this, we deduce the analogue of the main theorem of [4]. Note that in the case that $\{J_n\}$ is a sequence, the boundedness condition is automatic by the Banach-Steinhaus Theorem.

Theorem 3. 5. *Let $\mathscr{J}$ be an ideal in a C^*-algebra $\mathfrak{A}$. Suppose that A in $\mathfrak{A}$ is not in $\mathscr{J}$, and J_α is a bounded net in $\mathscr{J}$ converging $\mathscr{J}$-strong* to A. Then there is an element J in the closed convex hull of $\{J_\alpha\}$ such that*

$$\|A - J\| = \|A + \mathscr{J}\|.$$

Proof. For convenience, normalize so that $\|A+\mathscr{J}\| = 1$. Let $B_\alpha = A - J_\alpha$ and $\beta = \sup \|B_\alpha\|$. Clearly, B_α tends to zero in the $\mathscr{J}$-strong* topology. Choose real numbers $t_k > 0$ so that $\sum_{k=1}^{\infty} t_k = 1$, and for all $m \geqq 1$,

$$\sum_{k \geqq m} t_k > \beta t_m.$$

(For example, take $C > \beta$ and $t_k = C^{-1}(1 - C^{-1})^{k-1}$.)

Now we will inductively choose α_k so that for all $m \geqq 0$,

$$\left\| \sum_{k=1}^{m} t_k B_{\alpha_k} \right\| < 1.$$

This is trivial for $m=0$, suppose it holds for $X_m=\sum_{k=1}^{m} t_k B_{\alpha_k}$. Apply Lemma 3.4 with

$$\varepsilon=\min\left\{1-\|X_m\|,\ 1-\sum_{k=1}^{m} t_k-\beta t_{m+1}\right\}.$$

Note that

$$\|X_m+\mathcal{J}\|=\left(\sum_{k=1}^{m} t_k\right)\|A+\mathcal{J}\|=\sum_{k=1}^{m} t_k .$$

Take $\alpha=\alpha_{m+1}$ so large that

$$\begin{aligned}\|X_m+t_{m+1}B_\alpha\| &< \varepsilon+\max\{\|X_m\|,\ \|X_m+\mathcal{J}\|+t_{m+1}\|B_\alpha\|\}\\ &< \varepsilon+\max\left\{\|X_m\|,\ \sum_{k=1}^{m} t_k+\beta t_{m+1}\right\}=1.\end{aligned}$$

It is now immediate that $B=\sum_{k=1}^{\infty} t_k B_{\alpha_k}$ converges in $\mathfrak{A}$ and satisfies $\|B\|\leqq 1$. It is also clear that

$$A-B=\sum_{k=1}^{\infty} t_k J_{\alpha_k}=J$$

belongs to $\mathcal{J}$. Thus

$$\|B\|=\|A+\mathcal{J}\|=1. \qquad \square$$

4. Approximate identities

It often occurs in our applications that $\mathcal{S}$ is in fact a subalgebra of $\mathfrak{A}$. In this case, a simple criterion for the $\mathcal{J}$-weak density of $\mathcal{S}\cap\mathcal{J}$ in $\mathcal{S}$ is the existence of an appropriate approximate identity.

Lemma 4.1. *Let $\mathcal{J}$ be an ideal in a C*-algebra $\mathfrak{A}$. Suppose that $\mathcal{S}$ is a subalgebra of $\mathfrak{A}$ such that $\mathcal{S}\cap\mathcal{J}$ contains an approximate identity $\{E_\alpha\}$ for $\mathcal{J}$. Then $\mathcal{S}\cap\mathcal{J}$ is $\mathcal{J}$-weakly dense in $\mathcal{S}$. If furthermore, $\{E_\alpha\}$ is bounded, then $E_\alpha S E_\alpha$ converges boundedly, $\mathcal{J}$-strong* to S.*

Proof. For S in $\mathcal{S}$, the net SE_α belongs to $\mathcal{S}\cap\mathcal{J}$. If J belongs to $\mathcal{J}$ and ϕ belongs to $\mathcal{J}^*$, then

$$|\phi(SJ)-\phi(SE_\alpha J)|\leqq\|\phi\|\ \|S\|\ \|J-E_\alpha J\|\to 0.$$

Hence SE_α converges $\mathcal{J}$-weakly to S.

If E_α is bounded, and J belongs to $\mathcal{J}$,

$$SJ-E_\alpha SE_\alpha J=(SJ-E_\alpha SJ)+E_\alpha S(J-E_\alpha J)$$

which converges to zero in norm. Similarly $J(S-E_\alpha SE_\alpha)$ tends to zero. Hence $E_\alpha SE_\alpha$ is a bounded net converging $\mathcal{J}$-strong* to S. $\square$

It happens that approximate identities with nice norm properties can be used to compute the distance to ideal perturbations of subalgebras. These will be of interest in the applications, so we develop the general framework in this section.

Lemma 4.2. *Let $\mathfrak{A}$ be a C^*-algebra with ideal $\mathscr{J}$. Let $\mathscr{S}$ be a subalgebra of $\mathfrak{A}$ such that $\mathscr{S} \cap \mathscr{J}$ contains a norm one approximate identity E_n for $\mathscr{J}$ satisfying*

$$\lim_{n \to \infty} \|I - E_n\| = 1.$$

Then for any A in $\mathfrak{A}$,

$$d(A, \mathscr{S} + \mathscr{J}) = \lim_{n \to \infty} d(A(I - E_n), \mathscr{S}).$$

Proof. Since $A(I - E_n)$ is a $\mathscr{J}$ perturbation of A, the right hand side dominates $d(A, \mathscr{S} + \mathscr{J})$. Conversely, if J is in $\mathscr{J}$ and S is in $\mathscr{S}$, then

$$(A - S - J)(I - E_n) = A(I - E_n) - S(I - E_n) - (J - JE_n).$$

Since $\|J - JE_n\|$ tends to zero, and $S(I - E_n)$ belongs to $\mathscr{S}$,

$$\lim_{n \to \infty} d(A(I - E_n), \mathscr{S}) \leqq \lim_{n \to \infty} \|(A - S - J)\| \, \|I - E_n\| = \|A - S - J\|.$$

Thus equality is assured. □

The next lemma shows that the desired approximate identities can be obtained from less well behaved ones.

Lemma 4.3. *Let $\mathfrak{A}$ be a separable C^*-algebra. Suppose $\{R_k\}$ is a bounded left approximate identity for $\mathfrak{A}$. Then there exist convex combinations E_n of $\{R_k\}$ such that*

$$\lim_{n \to \infty} \|E_n\| = \lim_{n \to \infty} \|I - E_n\| = 1$$

and E_n is a two sided approximate identity for $\mathfrak{A}$.

Proof. Let Q_j, $j \geqq 1$, be a fixed approximate unit for $\mathfrak{A}$ satisfying $0 \leqq Q_j = Q_j^* \leqq Q_{j+1} \leqq 1$. Let $C = \sup \|R_k\|$, and let N be a given integer. Choose an integer $M \geqq C^2 N^2$. Let $j_1 = N$, and alternately choose j_i and k_i, $1 \leqq i \leqq M$, such that

$$\|Q_{j_i} - R_{k_i} Q_{j_i}\| < \frac{1}{N},$$

$$\|R_{k_i} - R_{k_i} Q_{j_{i+1}}\| < \frac{1}{N}$$

and

$$\|Q_{j_i} - Q_{j_i} Q_{j_{i+1}}\| < \frac{1}{M}.$$

Then let

$$E_N = \frac{1}{M} \sum_{i=1}^{M} R_{k_i}$$

and

$$F_N = \frac{1}{M} \sum_{i=1}^{M} Q_{j_i}.$$

Now $Q_N \leqq F_N \leqq I$, so F_N is an approximate unit for $\mathfrak{A}$ satisfying the desired inequality. So compute

$$\begin{aligned}\|E_N - F_N\| &= \left\| \frac{1}{M} \sum_{i=1}^{M} R_{k_i} - Q_{j_i} \right\| \\ &= \left\| \frac{1}{M} \sum_{i=1}^{M} (R_{k_i} - R_{k_i} Q_{j_{i+1}}) + R_{k_i}(Q_{j_{i+1}} - Q_{j_i}) + (R_{k_i} Q_{j_1} - Q_{j_i}) \right\| \\ &\leqq \frac{2}{N} + \frac{1}{M} \left\| \sum_{i=1}^{M} R_{k_i} \Delta_i \right\|\end{aligned}$$

where $\Delta_i = Q_{j_{i+1}} - Q_{j_i}$. In the case in which Q_j are projections, this term is readily bounded by $CM^{-\frac{1}{2}} \leqq N^{-1}$. In general, note that for $|i-j| \geqq 2$, one has

$$\|\Delta_i \Delta_j\| < \frac{4}{M^2}.$$

Split the sum into the odd and even terms, and estimate them separately.

$$\begin{aligned}\left\| \frac{1}{M} \sum_{i\,\mathrm{even}} R_{k_i} \Delta_i \right\|^2 &= \frac{1}{M^2} \left\| \sum_{i=1}^{\frac{M}{2}} \sum_{j=1}^{\frac{M}{2}} R_{k_{2i}} \Delta_{2i} \Delta_{2j} R^*_{k_{2j}} \right\| \\ &\leqq \frac{1}{M^2} \left(\sum_{i=1}^{\frac{M}{2}} \|R_{k_{2i}} \Delta_{2i}^2 R^*_{k_{2i}}\| + \frac{M^2}{4} \cdot C^2 \cdot \frac{4}{M^2} \right) \\ &< \frac{1}{M^2} \left(\frac{M}{2} C^2 + C^2 \right) < \frac{1}{N^2}.\end{aligned}$$

The odd term is the same, so one obtains

$$\|E_N - F_N\| < \frac{4}{N}.$$

Thus

$$\lim_{n \to \infty} \|E_N\| = 1 = \lim_{n \to \infty} \|I - E_N\|. \qquad \square$$

5. Applications to nest algebras

A nest $\mathcal{N}$ is a totally ordered complete chain of subspaces in a Hilbert space $\mathscr{H}$. The associated nest algebra $\mathscr{T} = \mathscr{T}(\mathcal{N})$ consists of all operators leaving each element of the nest invariant. The quasitriangular algebra of $\mathcal{N}$ is the algebra $\mathscr{Q}\mathscr{T}(\mathcal{N}) = \mathscr{T}(\mathcal{N}) + \mathscr{K}$. It was the study of this algebra that led to the development of this paper.

It is a result of [10] that $\mathscr{Q}\mathscr{T}(\mathcal{N})$ is closed, but our results yield a much stronger result.

Theorem 5. 1. *Let $\mathcal{N}$ be a nest. Then the quasitriangular algebra $\mathcal{QT}$ is closed, and the map*

$$\sigma : \mathcal{T}/\mathcal{T} \cap \mathcal{K} \to \mathcal{QT}/\mathcal{K}$$

is isometric. Furthermore, $\mathcal{T} \cap \mathcal{K}$ is proximinal in $\mathcal{T}$ and $\mathcal{QT}$ is proximinal in $\mathcal{B}(\mathcal{H})$.

Proof. By [9], $\mathcal{T} \cap \mathcal{K}$ contains a bounded approximate identity for $\mathcal{K}$. Thus $\mathcal{T} \cap \mathcal{K}$ is $\mathcal{K}$-strong* dense in $\mathcal{T}$. The theorem is an immediate consequence of Corollary 2. 3. □

The fact that σ is isometric has been noticed (unpublished) by several people. The first author together with F. Gilfeather and D. Larson constructed a proof of this using the approach of Lemma 4. 2. However, their proof that such an approximate identity exists was much more difficult than the general technique used in Lemma 4. 3. The second author constructed a proof similar in flavour to [4] using the methods of [19]. We have also heard that N. T. Andersen had a third argument.

The proximinality of $\mathcal{QT}(\mathcal{N})$ in $\mathcal{B}(\mathcal{H})$ can also be approached by the methods of [19]. Also Timothy Feeman [11], [12] shows that for a discrete nest, $\mathcal{QT}(\mathcal{N})$ is proximinal in $\mathcal{B}(\mathcal{H})$. He proves this using both M-ideals and constructively as in [4].

In [10], the operators in $\mathcal{QT}(\mathcal{N})$ are characterized among all operators in $\mathcal{B}(\mathcal{H})$ in terms of their behaviour with respect to the nest (see below). It is natural to hope that a distance formula can be obtained along these lines. The methods of this paper will be used to obtain such a formula.

Let the nest $\mathcal{N}$ be endowed with the order topology (equivalent to the strong operator topology). Note that $\mathcal{N}$ is compact and Hausdorff. Let $C_{s*}(\mathcal{N}, \mathcal{B}(\mathcal{H}))$ denote the C^* algebra of all *-strongly continuous functions from $\mathcal{N}$ into $\mathcal{B}(\mathcal{H})$. Let $C_n(\mathcal{N}, \mathcal{K})$ denote the norm closed, two sided ideal of norm continuous functions from $\mathcal{N}$ into $\mathcal{K}$. Let π denote the quotient map

$$\pi : C_{s*}(\mathcal{N}, \mathcal{B}(\mathcal{H})) \to C_{s*}(\mathcal{N}, \mathcal{B}(\mathcal{H}))/C_n(\mathcal{N}, \mathcal{K}).$$

For F in $C_{s*}(\mathcal{N}, \mathcal{B}(\mathcal{H}))$, let $\|F\|_e$ denote $\|\pi F\|$.

Consider the map $\Phi : \mathcal{B}(\mathcal{H}) \to C_{s*}(\mathcal{N}, \mathcal{B}(\mathcal{H}))$ given by

$$\Phi(A)(P) = P^{\perp} A P \qquad (P \in \mathcal{N}).$$

It is clear that Φ is a concrete linear map with kernel $\mathcal{T}(\mathcal{N})$. Furthermore, it is an immediate consequence of the distance formula for nests [3] (see also [17]) that

$$\|\Phi(A)\| = \operatorname{dist}(A, \mathcal{T}(\mathcal{N})).$$

Thus Φ factors through the quotient map

$$\tau : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{T}(\mathcal{N}).$$

Let $\phi : \mathcal{B}(\mathcal{H})/\mathcal{T}(\mathcal{N}) \to C_{s*}(\mathcal{N}, \mathcal{B}(\mathcal{H}))$ be the induced isometric imbedding.

In [10], it is shown that an operator A in $\mathcal{B}(\mathcal{H})$ belongs to $\mathcal{QT}(\mathcal{N})$ if and only if $\Phi(A)$ is continuous and compact valued. That is, $\mathcal{QT}(\mathcal{N})$ coincides with the kernel of $\pi \circ \Phi$. So

$$\operatorname{Im}(\Phi|\mathcal{K}) = \Phi(\mathcal{QT}(\mathcal{N})) = (\operatorname{Im}\Phi) \cap C_n(\mathcal{N}, \mathcal{K}).$$

Since ϕ is isometric, it follows that

$$\mathcal{B}(\mathcal{H})/\mathcal{Q}\mathcal{T}(\mathcal{N}) = (\mathcal{B}(\mathcal{H})/\mathcal{T})/(\mathcal{Q}\mathcal{T}(\mathcal{N})/\mathcal{T})$$

is isometric to $\operatorname{Im}\Phi/\operatorname{Im}(\Phi|\mathcal{K})$, (say via $\tilde{\phi}$). Let

$$\tilde{\pi} : \mathcal{B}(\mathcal{H})/\mathcal{T}(\mathcal{N}) \to \mathcal{B}(\mathcal{H})/\mathcal{Q}\mathcal{T}(\mathcal{N})$$

and

$$\psi : \operatorname{Im}\Phi/\operatorname{Im}(\Phi|\mathcal{K}) \to \operatorname{Im}\Phi + C_n(\mathcal{N}, \mathcal{K})/C_n(\mathcal{N}, \mathcal{K})$$

be the canonical quotient maps. Both are contractive. We have the following diagram

$$\begin{array}{ccccc}
\mathcal{B}(\mathcal{H}) & \overset{\Phi}{\searrow} & & & \\
\downarrow \tau & & & & \\
\mathcal{B}(\mathcal{H})/\mathcal{T}(\mathcal{N}) & \xrightarrow{\ \phi\ } & C_{s^*}(\mathcal{N}, \mathcal{B}(\mathcal{H})) & \overset{\pi}{\searrow} & \\
\downarrow \tilde{\pi} & & & & \\
\mathcal{B}(\mathcal{H})/\mathcal{Q}\mathcal{T}(\mathcal{N}) & \xrightarrow{\ \tilde{\phi}\ } & \operatorname{Im}\Phi/\operatorname{Im}(\Phi|\mathcal{K}) & \xrightarrow{\ \psi\ } & C_{s^*}(\mathcal{N}, \mathcal{B}(\mathcal{H}))/C_n(\mathcal{N}, \mathcal{K})
\end{array}$$

Our aim is to show that ψ is isometric, which yields

$$d(A, \mathcal{Q}\mathcal{T}(\mathcal{N})) = \|\phi(A)\|_e.$$

Our methods yield the proximinality of $\mathcal{Q}\mathcal{T}(\mathcal{N})$ in $\mathcal{B}(\mathcal{H})$ as well.

Theorem 5. 2. *Let $\mathcal{N}$ be a nest on a Hilbert space $\mathcal{H}$. Then for every A in $\mathcal{B}(\mathcal{H})$, there is an operator T in $\mathcal{Q}\mathcal{T}(\mathcal{N})$ such that*

$$\|A - T\| = d(A, \mathcal{Q}\mathcal{T}(\mathcal{N})) = \|\phi(A)\|_e.$$

Proof. Let E_n be a bounded approximate identity for $\mathcal{T}(\mathcal{N}) \cap \mathcal{K}$. Then $E_n A E_n$ converges to A in the weak* topology by Lemma 4. 1. From the definition of Φ, it is apparent that Φ takes weak* converging sequences to functions which are uniformly weak* convergent. Since norm continuous functions in $C_n(\mathcal{N}, \mathcal{K})$ have compact range, it follows that Φ takes weak* converging sequences to $C_n(\mathcal{N}, \mathcal{K})$-weakly converging sequences. Thus

$$\Phi(E_n A E_n) \xrightarrow{C_n(\mathcal{N}, \mathcal{K}) - w} \Phi(A).$$

Hence by Theorem 3. 5 there is a compact operator K such that

$$\|\Phi(A - K)\| = \|\Phi(A)\|_e.$$

Thus the map ψ is isometric. Now $\mathcal{T}(\mathcal{N})$ is weakly closed and hence proximinal, so there is an operator T in $\mathcal{T}(\mathcal{N})$ such that

$$\begin{aligned}
\|A - K - T\| &= d(A - K, \mathcal{T}(\mathcal{N})) \\
&= \|\Phi(A - K)\| = \|\Phi(A)\|_e \\
&\leq d(A, \mathcal{Q}\mathcal{T}(\mathcal{N})).
\end{aligned}$$

Thus $\|A - (T + K)\| = \|\Phi(A)\|_e = d(A, \mathcal{Q}\mathcal{T}(\mathcal{N}))$ as desired. □

Remark 5.3. Take the special case of a nest $\mathscr{P} = \{P_n, n \geqq 1\}$ of finite rank projections increasing with supremum 1. Then $\mathscr{Q}\mathscr{T}(\mathscr{P})$ is the classical quasitriangular algebra. For A in $\mathscr{B}(\mathscr{H})$, the map Φ becomes

$$\Phi(A)(n) = \|P_n^\perp A P_n\|.$$

For $\Phi(A)$ to belong to $C_n(\mathscr{P}, \mathscr{K})$ merely means that

$$\lim_{n \to \infty} \|P_n^\perp A P_n\| = 0.$$

In this context, our formula yields a distance formula due to Arveson [3],

$$d(A, \mathscr{Q}\mathscr{T}(\mathscr{P})) = \limsup \|P_n^\perp A P_n\|.$$

However, this formula can be obtained much more simply by combining the distance formula for $\mathscr{T}(\mathscr{P})$ with the fact that P_n is a norm one approximate identity for $\mathscr{K}$ in $\mathscr{T}(\mathscr{P}) \cap \mathscr{K}$ such that $\|P_n^\perp\| = 1$ as in Lemma 4.2.

Indeed, it follows from Erdos's approximate identity of compacts in a nest algebra [9] and Lemma 4.3 that there is always an approximate identity E_n in $\mathscr{T}(\mathscr{N}) \cap \mathscr{K}$ such that

$$\lim_{n \to \infty} \|E_n\| = \lim_{n \to \infty} \|I - E_n\| = 1.$$

Thus, Lemma 4.2 yields the formula

$$\begin{aligned} d(A, \mathscr{Q}\mathscr{T}(\mathscr{N})) &= \lim_{n \to \infty} d(A(I - E_n), \mathscr{T}(\mathscr{N})) \\ &= \lim_{n \to \infty} \lim_{P \in \mathscr{N}} \|P^\perp A(I - E_n) P\|. \end{aligned}$$

Remark 5.4. In [18], the second author defined the notion of a nest subalgebra $\mathscr{A}$ of an AF algebra $\mathscr{B}$. If $\mathscr{J}$ is any ideal of $\mathscr{B}$, he showed that $\mathscr{A} + \mathscr{J}$ is closed and the map

$$\sigma : \mathscr{A}/\mathscr{A} \cap \mathscr{J} \to \mathscr{A} + \mathscr{J}/\mathscr{J}$$

is isometric. Observing that $\mathscr{A} \cap \mathscr{J}$ always contains an approximate identity for $\mathscr{J}$ yields this corollary from Lemma 4.1 and Theorem 2.1. □

Remark 5.5. Consider the crossed product C^*-algebra $L^\infty(\mathbb{R}) \times_\tau \mathbb{R}$ corresponding to the translation action of $\mathbb{R}$. Let $\mathscr{S}$ be the nest subalgebra of elements A for which $P_t^\perp A P_t = 0$ for all projections P_t in $L^\infty(\mathbb{R})$ corresponding to the intervals $(-\infty, t]$ for all t in $\mathbb{R}$. There is a natural, faithful semifinite trace on this crossed product that determines a closed ideal $\mathscr{J}$ generated by the positive finite trace elements. One can check that the directed set of finite projections in $L^\infty(\mathbb{R})$ provides a bounded approximate identity for $\mathscr{J}$ in $\mathscr{J} \cap \mathscr{S}$. Consequently, $\mathscr{S} + \mathscr{J}$ is proximinal.

6. Nest subalgebras of the compact operators

Let $\mathcal{N}$ be a nest. By Corollary 5.1, $\mathcal{T} \cap \mathcal{K}$ is always proximinal in $\mathcal{T}$. However, it turns out that $\mathcal{T} \cap \mathcal{K}$ is rarely proximinal in $\mathcal{K}$, as the following theorem shows.

Theorem 6.1. *$\mathcal{T}(\mathcal{N}) \cap \mathcal{K}$ is proximinal in $\mathcal{K}$ if and only if the order type of $\mathcal{N}$ is finite, $\mathbb{N} \cup \{\infty\}$, $\{-\infty\} \cup -\mathbb{N}$, or $\{-\infty\} \cup \mathbb{Z} \cup \{+\infty\}$.*

Lemma 6.2. *If A, B and C are operators in $\mathcal{B}(\mathcal{H}_1)$, $\mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ respectively, then there is an operator X on $\mathcal{H}_2$ such that*

$$\left\| \begin{bmatrix} A & B \\ C & X \end{bmatrix} \right\| = \max \left\{ \| [A \ \ B] \|, \left\| \begin{bmatrix} A \\ C \end{bmatrix} \right\| \right\}.$$

Furthermore, if A is compact, then X can be taken to be compact.

Proof. This lemma except for the last sentence is a result in [16], [7]. In [6], it is shown that X can be taken to be of the form KAL for certain operators K and L, thus X is compact if A is. □

Proof of theorem 6.1. First suppose that $\mathcal{N}$ is finite. Then elements of the nest algebra are upper triangular $n \times n$ matrices with operator entries. If K is compact, then K is an $n \times n$ matrix (K_{ij}) with compact entries. The distance formula for nest algebras gives

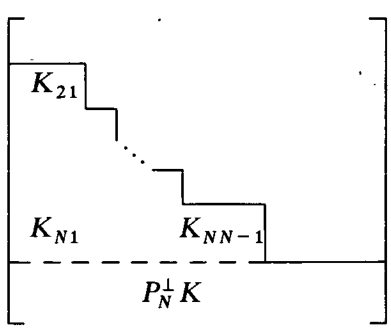
$$d(K, \mathcal{T}) = \max_{1 \leqq k < n} \|P_k^\perp K P_k\|$$

where P_k is the diagonal projection onto the first k blocks. Following the technique of [17], we start with the lower triangular entries of K and fill in the remaining blocks successively without increasing the norm of the blocks. Lemma 6.2 ensures that the new blocks are all compact, so a best compact approximant is obtained.

Next suppose that $\mathcal{N} = \{P_n, \ n \geqq 1\}$ and P_n increase to the identity. Given K compact but not triangular, one can find an integer N so large that

$$\|P_N^\perp K\| < d(K, \mathcal{T}).$$

Consider the lower triangular partial matrix

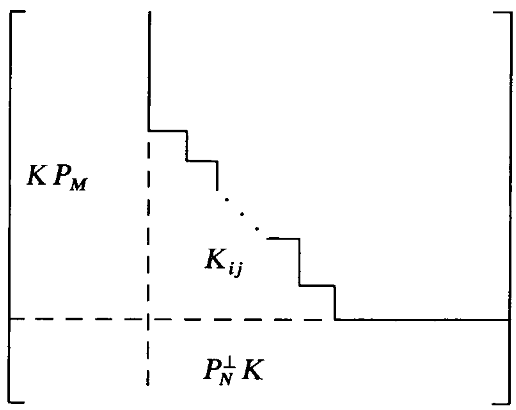


By the distance formula and the choice of N, all the complete rectangles have norm at most $d(K, \mathcal{T})$. So the matrix can be filled in as in the preceding paragraph.

The complementary nest $\{P_n^\perp, n \geqq 1\}$ is dealt with in the same way. Finally, if $\mathcal{N} = \{P_n, n \in \mathbb{Z}\}$ with $\inf P_n = 0$ and $\sup P_n = I$, proceed in a similar way. Given K compact but not triangular, choose N and M so that

$$\|P_N^\perp K\| < d(K, \mathcal{T}),$$
$$\|K P_M\| < d(K, \mathcal{T}).$$

Consider the partial matrix



$M < j < i \leqq N.$

This is filled in the same manner.

Now, suppose that $\mathcal{N}$ has some other order type ω. Then ω has a limit point other than 0 or I. That is, $\mathcal{N}$ contains a projection $P \neq \{0, I\}$ which is either of the form

$$P = \bigvee \{P' \in \mathcal{N} : P' < P\} \quad \text{or} \quad P = \bigwedge \{P' \in \mathcal{N} : P' > P\}.$$

For convenience, assume the former. Let x be a unit vector such that $x = Px$ but $x \neq P'x$ for any P' in $\mathcal{N}$ less than P. Let y be a unit vector such that $y = P^\perp y$. Set

$$K = (x + y) \otimes (x + y)^*.$$

So K is twice the rank one projection onto the span of $x + y$. For any projection Q,

$$\|Q^\perp K Q\| = \|Q^\perp(x + y)\| \; \|Q(x + y)\| \leqq \sup\{ab : a^2 + b^2 = 2\} = 1.$$

Hence

$$d(K, \mathcal{T}) = \sup\{\|Q^\perp K Q\| : Q \in \mathcal{N}\} = \|P^\perp K P\| = \|y\| \; \|x\| = 1.$$

Let T be any triangular operator such that $\|K - T\| = 1$. Since $Kx = x + y$ and $P^\perp(K - T)x = P^\perp K x = y$, it follows that $P(K - T)x = 0$. Therefore $Tx = x$. Let P_n be a strictly increasing sequence in $\mathcal{N}$ with $\sup P_n = P$, and let $Q_n = P - P_n$. Then

$$Q_n x = Q_n T x = Q_n(P_n^\perp T) P x = Q_n(P_n^\perp T P_n^\perp) P x = Q_n T(Q_n x).$$

Hence $\|Q_n T\| \geqq 1$. But Q_n tends to zero in the strong operator topology. Thus if T were compact, one would have

$$\lim_{n \to \infty} \|Q_n T\| = 0.$$

This shows that there is no best compact triangular approximant to K. □

Example 6. 3. It is interesting to make a more detailed analysis of a special case of the counterexamples produced in this proof. Let $\mathcal{N}=\{P_t, 0\leqq t\leqq 1\}$ be the nest on $L^2(0, 1)$, where $P_t\mathcal{H}$ is the set of functions supported on $[0, t]$. The operator K may be taken to be the projection $1\otimes 1$ where 1 is the constant function. Or, one might prefer to take K to be the Volterra operator V given by

$$Vf(y)=\int_0^y f(t)\,dt.$$

It is routine to verify that $1\otimes 1-V$ is a compact operator in the nest algebra $\mathcal{T}(\mathcal{N})$. As in the proof above,

$$d(V, \mathcal{T})=\|P_{\frac{1}{2}}^{\perp} V P_{\frac{1}{2}}\|=\frac{1}{2}$$

and V has no best compact triangular approximant.

Let D be the diagonal operator

$$Df(y)=\begin{cases} yf(y) & 0\leqq y\leqq\frac{1}{2},\\ (1-y)f(y) & \frac{1}{2}\leqq y\leqq 1.\end{cases}$$

It will be shown that $\|V-D\|=\frac{1}{2}$, so that

$$\|V-D\|=d(V, \mathcal{T})=d(V, \mathcal{D})$$

where $\mathcal{D}$ is the multiplication algebra on $L^2(0, 1)$ by $L^\infty(0, 1)$ functions.

Fix an integer N. Let

$$x_i=\sqrt{2N}\,\chi_{\left[\frac{i-1}{2N}, \frac{i}{2N}\right]}, \quad 1\leqq i\leqq 2N.$$

Let Q_N be the orthogonal projection onto span$\{x_i, 1\leqq i\leqq 2N\}$. An easy computation shows that

$$(Vx_i, x_j)=\begin{cases} 0 & i>j,\\ \dfrac{1}{4N} & i=j,\\ \dfrac{1}{2N} & i<j,\end{cases}$$

$$(Dx_i, x_j)=\begin{cases} 0 & i\neq j,\\ \dfrac{i-\frac{1}{2}}{2N} & i=j,\quad 1\leqq i\leqq N,\\ 1-\dfrac{i-\frac{1}{2}}{2N} & i=j,\quad N+1\leqq i\leqq 2N.\end{cases}$$

Thus $Q_N(V-D)\,Q_N|Q_N\mathscr{H}$ has the form

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & & & & & 0 \\ \frac{1}{2N} & \frac{-1}{2N} & 0 & \ddots & & & & & \vdots \\ \frac{1}{2N} & \frac{1}{2N} & \frac{-2}{2N} & \ddots & & & 0 & & \\ \vdots & & \ddots & \frac{-(N-1)}{2N} & \ddots & & & & \\ & & & \ddots & \frac{-(N-1)}{2N} & \ddots & & & \\ & & & & \ddots & \frac{-(N-2)}{2N} & \ddots & & \\ & & \frac{1}{2N} & & & \ddots & \ddots & 0 & 0 \\ & & & & & & \frac{1}{2N} & \frac{-1}{2N} & 0 \\ \frac{1}{2N} & \cdots & & & & & \frac{1}{2N} & \frac{1}{2N} & 0 \end{bmatrix}$$

Think of this as a 2×2 matrix with $N\times N$ entries

$$\begin{bmatrix} R_1 & 0 \\ \frac{1}{2}P & R_2 \end{bmatrix}$$

where P is the rank one projection onto $\operatorname{span}\{(1,\ldots,1)\}$. By inspection, one sees that each row of R_1 is orthogonal to every other row and to the range of P. Hence $R_1=R_1P^{\perp}$ and

$$\|R_1\| = \max_{1\le i\le N} \|i^{\text{th}}\text{-row of } R_1\|$$
$$= \left[(N-1)\left(\frac{1}{2N}\right)^2+\left(\frac{N-1}{2N}\right)^2\right]^{\frac{1}{2}} = \frac{1}{2}\left(\frac{N-1}{N}\right)^{\frac{1}{2}} < \frac{1}{2}.$$

It follows that

$$\left\| \begin{bmatrix} R_1 \\ \frac{1}{2} P \end{bmatrix} \right\| = \left\| \begin{bmatrix} R_1 P^{\perp} \\ \frac{1}{2} P \end{bmatrix} \right\| = \max \left\{ \|R_1\|, \left\| \frac{1}{2} P \right\| \right\} = \frac{1}{2}.$$

Similarly, $R_2 = P^{\perp} R_2$ and $\left\| \left[\frac{1}{2} P, R_2 \right] \right\| = \frac{1}{2}$. Thus

$$2 Q_N (V-D) Q_N \mathscr{H} = \begin{bmatrix} 2R_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P^{\perp} & 0 \\ P & P^{\perp} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 2R_2 \end{bmatrix}.$$

The centre factor on the right side is a partial isometry, so the product has norm (at most) one. Hence

$$\|Q_N (V-D) Q_N\| = \frac{1}{2}.$$

Since Q_N tends strongly to I, it follows that

$$\|V-D\| = \frac{1}{2}$$

as desired.

This best approximant is not unique, as $D + \alpha P_{\frac{1}{2}} V^* P_{\frac{1}{2}}^{\perp}$ is equally close for all $|\alpha| \leqq \frac{1}{2}$. We do not know if there are other best approximations. □

Finally, we mention another curious fact about the classical nest case.

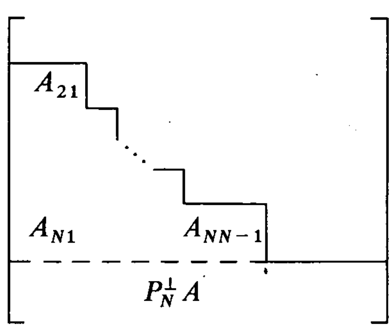
Theorem 6.4. *Let $\mathscr{N} = \{P_n;\ n \geqq 1\}$ be a nest of increasing finite rank projections P_n with $\sup P_n = I$. Then for all A in $\mathscr{B}(\mathscr{H})$,*

$$\begin{aligned} d(A, \mathscr{T} \cap \mathscr{K}) &= \max \{d(A, \mathscr{K}), d(A, \mathscr{T})\} \\ &= \max \left\{ \|A\|_e, \sup_{n \geqq 1} \|P_n^{\perp} A P_n\| \right\}. \end{aligned}$$

Proof. The proof follows what is by now a familiar line. Given $\varepsilon > 0$, choose N so large that

$$\|P_N^{\perp} A\| < \|A\|_e + \varepsilon.$$

Then consider the partial matrix



The rectangles filled in already have norm at most

$$\max\{\|P_N^\perp A\|,\ \|P_n^\perp A P_n\|,\quad 1 \leqq n \leqq N-1\}$$

which is less than $\max\{d(A, \mathscr{K}), d(A, \mathscr{T})\} + \varepsilon$. The "filling in" procedure produces a finite rank upper triangular approximation. □

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Pure Mathematics Department, University of Waterloo, Waterloo, Ontario, Canada N2L-3G1

Department of Mathematics, Cartmel College, University of Lancaster, Bailrigg, Lancaster, U.K. LA1-4YL

Eingegangen 31. Oktober 1984, in revidierter Form 18. Dezember 1984

APPENDIX 2

FAILURE OF THE DISTANCE FORMULA

KENNETH R. DAVIDSON AND STEPHEN C. POWER

Given any reflexive algebra $\mathcal{A}$ of operators on a Hilbert space $\mathcal{H}$, there is a convenient lower bound for the distance of an operator T in $\mathcal{B}(\mathcal{H})$ from $\mathcal{A}$ in terms of the lattice of invariant subspaces, Lat $\mathcal{A}$, of the algebra $\mathcal{A}$:

$$\inf_{A \in \mathcal{A}} \|T - A\| \geqslant \sup_{P \in \operatorname{Lat} \mathcal{A}} \|(I-P)TP\|.$$

Furthermore, it is easy to see that when the right-hand side vanishes, then T belongs to $\mathcal{A}$. None the less, it is not too surprising that these measures are not comparable in general [12]. However, in two important cases, they are comparable—when $\mathcal{A}$ is a nest algebra, they are equal (Arveson [2], see also [16, 13]), and when $\mathcal{A}$ is a type I von Neumann algebra, they agree within a factor of two (Christensen [6], see also [17]). It has been asked [1, 8, 10, 13, 14] if these measures are comparable for all algebras with commutative subspace lattices [3]. This has proved to be a rather elusive problem, and the purpose of this note is to provide a large class of counterexamples. For example, if $\mathcal{L}$ is the tensor product of infinitely many non-trivial nests, then Alg $\mathcal{L}$ fails to have a distance formula.

1. *The key example*

Let $A_0 = [1]$ be a 1×1 matrix. For $n \geqslant 0$, let A_{n+1} be the $3^{n+1} \times 3^{n+1}$ matrix given by

$$A_{n+1} = \begin{bmatrix} 0 & A_n & A_n \\ A_n & 0 & A_n \\ A_n & A_n & 0 \end{bmatrix}.$$

Let $\mathcal{S}_n$ denote the set of all $3^n \times 3^n$ matrices S such that the zero entries of S include all the non-zero entries of A_n. Let $\mathcal{D}_n$ denote the algebra of $3^n \times 3^n$ diagonal matrices. Then $\mathcal{S}_0 = \{[0]\}$, $\mathcal{S}_1 = \mathcal{D}_1$, and $\mathcal{S}_{n+1}$ consists of all matrices of the form

$$\begin{bmatrix} X_1 & S_{12} & S_{13} \\ S_{21} & X_2 & S_{23} \\ S_{31} & S_{32} & X_3 \end{bmatrix},$$

where X_i are arbitrary $3^n \times 3^n$ matrices, and S_{ij} belong to $\mathcal{S}_n$. Finally, define an algebra $\mathcal{A}_n$ consisting of all $2 \cdot 3^n \times 2 \cdot 3^n$ matrices of the form

$$\begin{bmatrix} D_1 & S \\ 0 & D_2 \end{bmatrix}$$

Received 16 July 1984.

1980 *Mathematics Subject Classification* 47D25.

The first author was partially supported by the SERC and NSERC.

J. London Math. Soc. (2) 32 (1985) 157–165

such that D_1 and D_2 belong to $\mathcal{D}_n$ and S belongs to $\mathcal{S}_n$. Note that $\mathcal{A}_n$ is reflexive, and that $\mathcal{L}_n = \operatorname{Lat} \mathcal{A}_n$ is a commutative subspace lattice consisting of all diagonal projections $P = P_1 \oplus P_2$ such that the range of $\mathcal{S}_n P_2$ is contained in the range of P_1.

Consider the matrix

$$T_n = \begin{bmatrix} 0 & A_n \\ 0 & 0 \end{bmatrix}.$$

Comparison of the two distance measures to $\mathcal{A}_n$ will show that the distance constant

$$\sup_T \frac{d(T, \mathcal{A}_n)}{\beta(T)}$$

is at least $(\frac{9}{8})^{\frac{1}{2}n}$

THEOREM 1.1. *With $\mathcal{A}_n$ and T_n as above,*

$$\beta(T_n) = \sup_{P \in \mathcal{L}_n} \| P^{\perp} T_n P \| = 2^{\frac{1}{2}n},$$

and

$$d(T_n, \mathcal{A}_n) = \inf_{A \in \mathcal{A}_n} \| T_n - A \| = (\tfrac{3}{2})^n.$$

LEMMA 1.2. *Given Y, X_1, X_2 and X_3 in $\mathcal{B}(\mathcal{H})$,*

$$\left\| \begin{bmatrix} X_1 & Y & Y \\ Y & X_2 & Y \\ Y & Y & X_3 \end{bmatrix} \right\| \geq \tfrac{3}{2} \| Y \|.$$

Equality is achieved by taking $X_1 = X_2 = X_3 = -\frac{1}{2} Y$.

Proof. It is well known that in the scalar case

$$\inf_{x_i \in \mathbb{C}} \left\| \begin{bmatrix} x_1 & 1 & 1 \\ 1 & x_2 & 1 \\ 1 & 1 & x_3 \end{bmatrix} \right\| = \tfrac{3}{2}$$

and the infimum is attained by taking $x_1 = x_2 = x_3 = -\frac{1}{2}$. Let x and y be unit vectors such that $\gamma = |(Yx, y)|$ is close to $\| Y \|$. Let $P_x = x \otimes x$ and $P_y = y \otimes y$ be the rank one projections with ranges $\mathbb{C}x$ and $\mathbb{C}y$, respectively. Then, setting $x_i = (X_i x, y)$, we obtain

$$\left\| \begin{bmatrix} X_1 & Y & Y \\ Y & X_2 & Y \\ Y & Y & X_3 \end{bmatrix} \right\| \geq \left\| \begin{bmatrix} P_y & 0 & 0 \\ 0 & P_y & 0 \\ 0 & 0 & P_y \end{bmatrix} \begin{bmatrix} X_1 & Y & Y \\ Y & X_2 & Y \\ Y & Y & X_3 \end{bmatrix} \begin{bmatrix} P_x & 0 & 0 \\ 0 & P_x & 0 \\ 0 & 0 & P_x \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} x_1 & \gamma & \gamma \\ \gamma & x_2 & \gamma \\ \gamma & \gamma & x_3 \end{bmatrix} \right\| \geq \tfrac{3}{2} |\gamma|.$$

Taking the supremum over pairs x, y results in the desired inequality.

With $X_1 = X_2 = X_3 = -\frac{1}{2} Y$, we have

$$\left\| \begin{bmatrix} -\frac{1}{2}Y & Y & Y \\ Y & -\frac{1}{2}Y & Y \\ Y & Y & -\frac{1}{2}Y \end{bmatrix} \right\| = \left\| \begin{bmatrix} -\frac{1}{2} & 1 & 1 \\ 1 & -\frac{1}{2} & 1 \\ 1 & 1 & -\frac{1}{2} \end{bmatrix} \otimes Y \right\| = \tfrac{3}{2} \| Y \|.$$

REMARK 1.3. An analogous result can be obtained for any fixed array of operators Y in a matrix. The constant obtained will be precisely the constant obtained in the scalar case.

Proof of Theorem 1.1. Since the non-zero entries of $\mathscr{S}_n$ coincide with the zeros of A_n, it is easy to see that $\beta(T_n)$ is the maximum norm of all rectangular arrays of ones occurring in A_n, namely $\max(kl)^{\frac{1}{2}}$ over all $k\times l$ arrays of ones. For A_1, this is seen to be $\sqrt{2}$ by inspection. As $A_{n+1}=A_1\otimes A_n$, it follows readily by induction that

$$\beta(T_{n+1})=2^{\frac{1}{2}}\beta(T_n)=2^{\frac{1}{2}(n+1)}.$$

Now suppose that $\tilde{B}_n=\begin{bmatrix}* & B_n\\ 0 & *\end{bmatrix}$ belongs to $\mathscr{A}_n$ and

$$\|T_n-\tilde{B}_n\|=d(T_n,\mathscr{A}_n).$$

Then

$$d(T_n,\mathscr{A}_n)=\|A_n-B_n\|=\inf\{\|A_n-S\|:S\in\mathscr{S}_n\}.$$

Think of A_n and B_n as

$$A_n=\begin{bmatrix}0 & A_{n-1} & A_{n-1}\\ A_{n-1} & 0 & A_{n-1}\\ A_{n-1} & A_{n-1} & 0\end{bmatrix},\qquad B_n=\begin{bmatrix}X_1 & S_{12} & S_{13}\\ S_{21} & X_2 & S_{23}\\ S_{31} & S_{32} & X_3\end{bmatrix}.$$

Given a permutation π on three elements, then π acts on a 3×3 matrix by simultaneously permuting the rows and columns. It is clear that this action preserves both A_n and $\mathscr{S}_n$. Each diagonal term is taken to each diagonal position twice, and the off-diagonal entries are cyclicly permuted as π runs through all of S_3. By averaging over S_3, we may assume that the nearest B_n in $\mathscr{S}_n$ to A_n has the form

$$B_n=\begin{bmatrix}X & S & S\\ S & X & S\\ S & S & X\end{bmatrix}.$$

Thus an application of Lemma 1.2 yields that

$$\|A_n-B_n\|=\left\|\begin{bmatrix}-X & A_{n-1}-S & A_{n-1}-S\\ A_{n-1}-S & -X & A_{n-1}-S\\ A_{n-1}-S & A_{n-1}-S & -X\end{bmatrix}\right\|$$

$$\geqslant \tfrac{3}{2}\inf\{\|A_{n-1}-S\|:S\in\mathscr{S}_{n-1}\}.$$

Furthermore, equality holds here, so the desired equality follows by induction.

2. *The general situation*

The result mentioned in the introduction will be obtained by imbedding the previous examples into our given algebras. Recall that if $\mathscr{L}$ is a commutative subspace lattice and $L_1, L_2\in\mathscr{L}$ are such that $L_1\supset L_2$, then the subspace $L_1\ominus L_2$ is called an interval. Minimal intervals are called atoms. For finite lattices, the atoms span the space. There is a partial order $\succ$ on atoms given by setting $F\prec E$ if $F\,\mathrm{Alg}\,\mathscr{L}E=F\mathscr{B}(\mathscr{H})E$ and $F\nprec E$ if $F\,\mathrm{Alg}\,\mathscr{L}E=\{0\}$. These two possibilities are mutually exclusive. In general, one extends $\prec$ to intervals by setting $F\prec E$ if $F\,\mathrm{Alg}\,\mathscr{L}E=F\mathscr{B}(\mathscr{H})E$, but naturally $F\nprec E$ is a weaker notion.

LEMMA 2.1. *Suppose that $\mathcal{L} = \mathcal{N}_1 \otimes \ldots \otimes \mathcal{N}_k$ is a tensor product of non-trivial nests. Let $A = (a_{ij})$ be a $k \times k$ matrix of zeros and ones. Then $\mathcal{L}$ contains intervals $G_1, \ldots, G_k$ and $H_1, \ldots, H_k$ such that $G_i \prec H_j$ if $a_{ij} = 0$ and $G_i(\mathrm{Alg}\,\mathcal{L})H_j = 0$ otherwise.*

Proof. Since $\mathcal{N}_i$ is a non-trivial nest, it can be split into intervals E_i^+, E_i^- and F_i^-, F_i^- such that $E_i^- \prec F_i^-$, $E_i^- + E_i^+ \prec F_i^-$, and

$$E_i^+ \,\mathrm{alg}\,\mathcal{N}_i F_i^- = \{0\}.$$

To see this, note that if $\mathcal{N}$ has two atoms $G \prec H$, then $E_i^- = F_i^- = G$ and $E_i^+ = F_i^+ = H$ will suffice. If not, then $\mathcal{N}_i$ has a continuous part order isomorphic to $[0, 1]$. Taking E_i^-, F_i^-, E_i^+ and F_i^+ corresponding to $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{3}{4}]$ and $[\frac{3}{4}, 1]$, respectively, meets the requirements.

Now for $1 \leqslant i \leqslant k$, define

$$G_i = E_1^- \otimes \ldots \otimes E_{i-1}^- \otimes E_i^+ \otimes E_{i+1}^- \otimes \ldots \otimes E_k^-.$$

Using the matrix (a_{ij}), define

$$H_j = F_1^{\varepsilon_1} \otimes \ldots \otimes F_k^{\varepsilon_k},$$

where $\varepsilon_i = +$ if $a_{ij} = 0$ and $\varepsilon_i = -$ if $a_{ij} = 1$. It is immediate that G_i and H_i have the desired properties.

THEOREM 2.2. *Suppose that the $\mathcal{N}_i$ are non-trivial nests for $1 \leqslant i \leqslant 3^n$, and $\mathcal{L}'$ is any commutative subspace lattice; let $\mathcal{L} = \mathcal{N}_1 \otimes \ldots \otimes \mathcal{N}_{3^n} \otimes \mathcal{L}'$. Then the distance constant for $\mathrm{Alg}\,\mathcal{L}$ is at least $(\frac{9}{8})^{\frac{1}{2}n}$.*

Proof. Let $A_n = (a_{ij})$ be the $3^n \times 3^n$ matrix defined in Section 1. Let G_i and H_j be the intervals of $\mathcal{L}_0 = \mathcal{N}_1 \otimes \ldots \otimes \mathcal{N}_{3^n}$ provided by Lemma 2.1. Let x_i and y_j be unit vectors in G_i and H_j, respectively. Let u_{ij} be the rank-one partial isometry taking y_j onto x_i. Let P_x and P_y be the projections onto the span of $\{x_i : 1 \leqslant i \leqslant 3^n\}$ and $\{y_j : 1 \leqslant j \leqslant 3^n\}$, respectively. Let $\mathcal{H}_0$ be the Hilbert space which supports $\mathcal{L}_0$. It is clear from this construction that $P_x \mathcal{B}(\mathcal{H}_0) P_y$ is linearly isometric to $\mathcal{M}_{3^n}$ in such a way that $P_x \,\mathrm{Alg}\,\mathcal{L}_0 P_y$ corresponds to $\mathcal{S}_n$. This correspondence sends

$$T = \sum_{i,j=1}^{3^n} a_{ij} u_{ij}$$

onto A_n.

Since the map taking X to $P_x X P_y$ is contractive, it follows that

$$d(T, \mathrm{Alg}\,\mathcal{L}_0) \geqslant d(A_n, \mathcal{S}_n) = (\tfrac{3}{2})^n;$$

indeed, this is easily seen to be an equality. On the other hand, suppose that P is a projection in $\mathcal{L}_0$. If $a_{ij} = 0$, then $G_i \mathcal{B}(\mathcal{H}) H_j$ is contained in $\mathrm{Alg}\,\mathcal{L}$. Thus if $PH_j \neq 0$, it follows that $P^\perp G_i = 0$. Let J be the set of j such that $PH_j \neq 0$. Then the set I of i such that $P^\perp G_i \neq 0$ is contained in the set I' of i such that the entries a_{1j} with (i,j) in $I' \times J$ consist entirely of ones. Hence

$$\| P^\perp T P \| = \| P^\perp (\sum_{i \in I} \sum_{j \in J} u_{ij}) P \| \leqslant (|I| \cdot |J|)^{\frac{1}{2}} \leqslant 2^{\frac{1}{2}n}.$$

This shows that the distance constant for $\mathcal{L}_0$ is at least $(\frac{9}{8})^{\frac{1}{2}n}$.

Finally, for $\mathcal{L} = \mathcal{L}_0 \otimes \mathcal{L}'$, take the operator $T \otimes I$. Now every operator in $\mathrm{Alg}\,\mathcal{L}$ is contained in $\mathrm{Alg}\,\mathcal{L}_0 \otimes \mathcal{B}(\mathcal{H})$. This latter algebra can be thought of as all infinite bounded matrices (A_{ij}) with entries from Alg $\mathcal{L}_0$. In this context, $T \otimes I$ is the matrix with entries T on the diagonal and zero elsewhere. Thus

$$\begin{aligned} d(T\otimes I, \operatorname{Alg}\mathcal{L}) &\geqslant d(T\otimes I, \operatorname{Alg}\mathcal{L}_0 \otimes \mathcal{B}(\mathcal{H})) \\ &\geqslant \inf\{\|T - A_{11}\| : A_{11} \in \operatorname{Alg}\mathcal{L}_0\} \\ &= d(T, \operatorname{Alg}\mathcal{L}_0). \end{aligned}$$

Now $\mathcal{L}'$ is a commutative subspace lattice, so it is contained in a σ-complete Boolean algebra of projections $\mathcal{E}$. The projections P of the form

$$P = \sum_{n=1}^{\infty} P_n \otimes E_n,$$

where the P_n belong to $\mathcal{L}_0$ and the E_n are pairwise orthogonal elements of $\mathcal{E}$, are strongly dense in $\mathcal{L}_0 \otimes \mathcal{E}$. Without loss of generality, we shall always suppose that $\sum_{n=1}^{\infty} E_n = I$, so that

$$P^{\perp} = \sum_{n=1}^{\infty} I \otimes E_n - \sum_{n=1}^{\infty} P_n \otimes E_n = \sum_{n=1}^{\infty} P_n^{\perp} \otimes E_n.$$

Hence

$$\begin{aligned} \sup_{P \in \mathcal{L}_0 \otimes \mathcal{L}'} \|P^{\perp}(T\otimes I)P\| &\leqslant \sup_{P \in \mathcal{P}} \|P^{\perp}(T\otimes I)P\| = \sup_{P \in \mathcal{P}} \left\| \sum_{n=1}^{\infty} (P_n^{\perp} T P_n \otimes E_n) \right\| \\ &= \sup_{P \in \mathcal{P}} \sup_{n} \|P_n^{\perp} T P_n\| = \sup_{P \in \mathcal{L}_0} \|P^{\perp} T P\|. \end{aligned}$$

Thus the distance constant for $\mathcal{L}$ is at least as great as that for $\mathcal{L}_0$.

COROLLARY 2.3. *If $\mathcal{L}$ is the infinite tensor product of non-trivial nests, then $\mathcal{L}$ fails to have a distance formula.*

REMARK 2.4. The key ingredient of this proof is Lemma 2.1 which says that arbitrary 0, 1 matrices can be 'imbedded' in the graph of the order for $\mathcal{L}$ (see [3]). It can be seen that this can be accomplished in many lattices of 'infinite width'. However, this does not hold for all lattices of infinite width, as the following example shows.

EXAMPLE 2.5. Let $\{e_n : n \geqslant 1\}$ be an orthogonal basis for $\mathcal{H}$. Let $\mathcal{D}$ be the diagonal algebra, and let $\mathcal{S}$ denote all operators with zero diagonal. Let $\mathcal{A}$ be the algebra of all operators on $\mathcal{H} \otimes \mathcal{H}$ of the form

$$\begin{bmatrix} D_1 & S \\ 0 & D_2 \end{bmatrix},$$

where D_i belong to $\mathcal{D}$ and S belongs to $\mathcal{S}$.

CLAIM. Lat *A has infinite width, and distance constant at most* 3.

Proof. Let $T = [T_{ij}]$ be a 2×2 operator acting on $\mathcal{H} \oplus \mathcal{H}$. Note that for any diagonal projection P, $P \oplus 0$ and $I \oplus P$ are invariant projections for $\mathcal{A}$. Hence $\beta(T) = \sup_{Q \in \operatorname{Lat}\mathcal{A}} \|Q^{\perp} T Q\|$ is at least

$$\max\{\sup_{P \in \mathcal{D}} \|P^{\perp} T_{11} P\|, \sup_{P \in \mathcal{D}} \|P^{\perp} T_{22} P\|, \|T_{21}\|, \|\delta(T_{12})\|\}.$$

where $\delta(T_{12})$ is the diagonal of T_{12}. By [6, 17], every type-I von Neumann algebra has distance constant 2. So there are diagonal operators D_1 and D_2 such that

$$\max\{\|T_{11}-D_1\|, \|T_{22}-D_2\|\} \leqslant 2\beta(T).$$

Since

$$A = \begin{bmatrix} D_1 & T_{12}-\delta(T_{12}) \\ 0 & D_2 \end{bmatrix}$$

belongs to $\mathscr{A}$, we obtain

$$d(T, \mathscr{A}) \leqslant \left\| \begin{bmatrix} T_{11}-D_1 & 0 \\ 0 & T_{22}-D_2 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 0 & \delta(T_{12}) \\ T_{21} & 0 \end{bmatrix} \right\| \leqslant 3\beta(T).$$

Hence $\mathscr{A}$ has distance constant at most 3.

Let $\{E_n : n \geqslant 1\}$ and $\{F_n : n \geqslant 1\}$ be the atoms of $\mathscr{D}\otimes 0$ and $0\otimes\mathscr{D}$, with the natural correspondence. Let $<$ be the partial order on the atoms of Lat $\mathscr{A}$. It is clear that

$$E_i < E_j \Leftrightarrow i = j,$$
$$F_i < F_j \Leftrightarrow i = j,$$
$$E_i < F_j \Leftrightarrow i \neq j,$$
$$F_j \nless E_i \quad \text{for all } i, j.$$

If Lat $\mathscr{A}$ had width n, there would be n linear orders $<_k$, $1 \leqslant k \leqslant n$, so that $E < F$ if and only if $E <_k F$ for $1 \leqslant k \leqslant n$. Consider the first $n+1$ atoms $F_1, \ldots, F_{n+1}$. For each k, pick j_k so that

$$F_{j_k} <_k F_j \qquad \text{for } 1 \leqslant j \leqslant n+1.$$

Let j_0 be chosen in $\{1, \ldots, n+1\}\setminus\{j_1, \ldots, j_n\}$. Then

$$E_{j_0} <_k F_{j_k} <_k F_{j_0}$$

for every k, $1 \leqslant k \leqslant n$. Hence $E_{j_0} < F_{j_0}$, which is absurd. Thus Lat $\mathscr{A}$ has infinite width.

3. *Lattice perturbations*

Two lattices are said to be close if there is a lattice isomorphism θ of one onto the other such that $\|\theta - \mathrm{id}\|$ is small. (The distance between two subspaces is taken to be the norm of the difference of the projections onto them.) Two algebras are said to be close if the Hausdorff distance between their unit balls is small. There are nice perturbation results for various classes of algebras giving the equivalence of close algebras, close lattices and similarity (or unitary equivalence) via an operator close to one [11, 5, 4]. In particular, it is shown in [4] that this situation holds for algebras close to finite-dimensional CSL algebras.

In [9], it is shown that if $\mathscr{A}$ is a CSL algebra and $\mathscr{B}$ is a norm-closed algebra close to $\mathscr{A}$, then Lat $\mathscr{B}$ is close to Lat $\mathscr{A}$. In this section, it will be shown that the failure of the distance constant gives rise to lattices which are similar and close, but for which any implementing similarity is necessarily far from the identity. This puts certain limits on the potential perturbation results for this class of algebras.

Let $\mathscr{L}$ be a commutative subspace lattice without a distance constant. Let $0 < \varepsilon < \frac{1}{6}$ be given, and let T be an operator such that

$$\|T\| = d(T, \mathrm{Alg}\,\mathscr{L}) = 1, \qquad \beta(T) = \sup_{P\in\mathscr{L}} \|P^\perp TP\| < \varepsilon.$$

Let $V = I+\frac{1}{2}T$. Then V is invertible, $\|V^{-1}\| \leqslant 2$, $d(V, \operatorname{Alg}\mathscr{L}) = \frac{1}{2}$ and $\beta(V) < \frac{1}{2}\varepsilon$. Let $\mathscr{M} = V\mathscr{L}$. Clearly, $\mathscr{M}$ is similar to $\mathscr{L}$ and $V(\operatorname{Alg}\mathscr{L})V^{-1} = \operatorname{Alg}\mathscr{M}$. The map $\theta:\mathscr{L}\to\mathscr{M}$ defined by

$$\theta(L) = VL$$

is obviously a lattice isomorphism.

First, it will be shown that $\|\theta-\mathrm{id}\| < 2\varepsilon$. To see this, fix L in $\mathscr{L}$ and $M = VL$. Let P_L and P_M be the orthogonal projections onto L and M, respectively; then

$$\|P_L-P_M\| = \|P_L P_M^{\perp} - P_L^{\perp} P_M\| = \max\{\|P_L P_M^{\perp}\|, \|P_L^{\perp} P_M\|\}.$$

Now if x is a unit vector in M, then $y = V^{-1}x$ belongs to L and $\|y\| \leqslant 2$. Thus

$$\|P_L^{\perp} P_M x\| = \|P_L^{\perp} V P_L y\| \leqslant \beta(V)\|y\| < \varepsilon.$$

Hence $\|P_L^{\perp} P_M\| < \varepsilon$.

Decomposing P_M relative to $L\oplus L$, we have

$$P_M = \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}$$

and $\|[Y^* Z]\| = \left\|\begin{bmatrix} Y \\ Z \end{bmatrix}\right\| < \varepsilon$. In particular,

$$\|X-X^2\| = \|YY^*\| < \varepsilon^2.$$

Since $0 \leqslant X \leqslant I$, it follows from the functional calculus that the spectrum of X is contained in $[0, 2\varepsilon^2] \cup [1-2\varepsilon^2, 1]$. So either $\|I-X\| < 2\varepsilon^2$ or $\|I-X\| \geqslant 1-2\varepsilon^2$. Thus

$$\|P_L P_M^{\perp}\| = \|[I-X, -Y]\| < 2\varepsilon^2+\varepsilon < 2\varepsilon,$$

or $\|P_L P_M\| \geqslant 1-2\varepsilon^2 > \frac{17}{18}$. This latter inequality, however, is impossible as, for every y in L,

$$\|P_M y\| \geqslant \frac{|(y, Vy)|}{\|Vy\|} \geqslant \frac{\|y\|^2-\frac{1}{2}|(y, Ty)|}{\frac{3}{2}\|y\|} \geqslant \frac{1}{3}\|y\|$$

and thus $\|P_M^{\perp} y\| \leqslant (2\sqrt{2}/3)\|y\|$. Hence

$$\|P_M^{\perp} P_L\| \leqslant 2\sqrt{2}/3 < \tfrac{17}{18}.$$

Thus $\|\theta-\mathrm{id}\| < 2\varepsilon$.

Now suppose that S is an invertible operator such that $S\mathscr{L} = \mathscr{M}$ and $\|S-I\| < \frac{1}{4}$. Then $S^{-1}V$ takes $\mathscr{L}$ onto $\mathscr{L}$, and the automorphism Ψ induced by $S^{-1}V$ satisfies

$$\|\Psi-\mathrm{id}\| \leqslant \|\theta-\mathrm{id}\| + \|S-I\| < 1.$$

But any two projections in $\mathscr{L}$ differ by 1 in norm, so that $\Psi = \mathrm{id}$. In particular, $A = S^{-1}V$ belongs to $\operatorname{Alg}\mathscr{L}$. Hence

$$\tfrac{1}{2} = d(V, \operatorname{Alg}\mathscr{L}) = d(A+(S-I)A, \operatorname{Alg}\mathscr{L}) < \|S-I\|\,\|S^{-1}\|\,\|V\| < \tfrac{1}{4}\cdot\tfrac{4}{3}\cdot\tfrac{3}{2} = \tfrac{1}{2},$$

a contradiction. Thus, any similarity S implementing θ is far from I.

4. *Further remarks*

1. It is known [1.14] that the existence of a distance formula for a reflexive operator algebra is equivalent to the following decomposability property of the preannihilator $\mathscr{A}_{\perp}$: there exists a constant $c > 0$ such that each trace class operator T in $\mathscr{A}_{\perp}$ admits a representation $T = \sum_k R_k$, where R_k are rank-one operators in $\mathscr{A}_{\perp}$ and $\sum_k \|R_k\|_1 \leqslant c\|T\|_1$. Of course, it makes sense to ask whether this decomposability

property holds for any closed space of trace class operators that is known to be the closed linear span of its rank-one members. Our key example and this duality provide many indecomposable spaces. For example, let $B_1 = [1]$ be a 1×1 matrix, and for $n \geq 1$ let B_{n+1} be the $3^n \times 3^n$ matrix given by

$$B_{n+1} = \begin{bmatrix} B_n & B_n & 0 \\ B_n & 0 & B_n \\ 0 & B_n & B_n \end{bmatrix};$$

let B_∞ be the infinite matrix whose upper left-hand blocks of order 3^n agree with B_{n+1} for $n = 0, 1, \ldots$. Then the zero entries of B_∞ specify a class of rank-one matrix units whose closed span, in the trace class, is indecomposable.

2. We indicate two function-theoretic connections that point to the importance and difficulty of establishing distance formulae for reflexive operator algebras. First, let $\{e_n : n \geq 1\}$ be an orthonormal basis for $\mathscr{H}$, and let $\mathscr{C}$ denote the set of those operators C whose matrix (c_{jk}) satisfies

$$\sum_{j+k=l} c_{jk} = 0$$

for $l = 2, 3, \ldots$. Let $\mathscr{A}$ be the algebra of all operators on $\mathscr{H} \oplus \mathscr{H}$ of the form

$$\begin{bmatrix} \lambda I & C \\ 0 & \mu I \end{bmatrix},$$

where λ, μ are complex numbers and C belongs to $\mathscr{C}$. The existence of a distance formula for $\mathscr{A}$ is thus equivalent to the decomposibility of the preannihilator $\mathscr{C}_\perp$. However, $\mathscr{C}_\perp$ is the space of trace class Hankel operators, and the proof that this is decomposable depends on the recently discovered decomposition properties of Bergman spaces obtained by Coifman and Rochberg [7] (see also [15]).

For the second connection let $\{e_n : n = 0, \pm 1, \ldots\}$ be an orthonormal basis for $\mathscr{H}$ and let $\mathscr{C}$ denote the set of those operators C such that $c_{jk} = 0$ whenever $k-j$ belongs to $\Lambda = \{1, 2, 4, 8, \ldots\}$. In this case a rectangular submatrix that is disjoint from the support of $\mathscr{C}$ must consist of a single row or a single column. Consequently a distance formula is valid for the associated (commutative subspace lattice) algebra $\mathscr{A}$, constructed as above, if and only if the distance

$$\inf_{C \in \mathscr{C}} \| T - C \|$$

is equivalent to the supremum of the Hilbert-space norm of certain lacunary subrows and subcolumns of T. If T is a multiplication operator corresponding to the L^∞ function ϕ this supremum is seen to be

$$\beta(\phi) = \left(\sum_{k=0}^{\infty} |\hat{\phi}(2^k)|^2 \right)^{\frac{1}{2}},$$

where $\hat{\phi}(n)$ denotes the n-th Fourier coefficient. Moreover, by a standard averaging argument the distance from T to $\mathscr{C}$ is achieved by a multiplication operator in the class $L^\infty_\Lambda = \{\psi \in L^\infty : \hat{\psi}(2^k) = 0, k = 0, 1, \ldots\}$. So a distance formula for $\mathscr{A}$ leads to the existence of a universal constant c such that

$$\beta(\phi) \leq \inf_{\psi \in L^\infty_\Lambda} \| \phi - \psi \|_\infty \leq c\beta(\phi),$$

for all ϕ in L^∞.

The existence of such a constant was shown to us by W. Rudin. The set L^{∞}_{Λ} is weak* closed, and its preannihilator in L^1 is

$$L^1_{\Lambda} = \{f \in L^1 : \hat{f}(n) = 0 \text{ for } n \notin \Lambda\}.$$

Since Λ is lacunary, there is a constant C such that

$$\| f \|_2 \leq C \| f \|_1 \tag{4.1}$$

for all f in L^1_{Λ} [18, Section 5.7.7]. Hence

$$\begin{aligned}\inf_{\psi \in L^{\infty}_{\Lambda}} \| \phi - \psi \| &= \sup\{|\langle \phi, f\rangle| : f \in L^1_{\Lambda}, \| f \|_1 \leq 1\} \\ &\leq \sup\{\| \phi|_{\Lambda} \|_2 \| f \|_2 : \| f \|_1 \leq 1\} \\ &\leq C\beta(\phi).\end{aligned}$$

Conversely, since the Fourier transforms of L^{∞} functions are dense in l^2, a reversal of this argument shows that the existence of a distance constant C implies that (4.1) holds for all f in L^1_{Λ}.

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Department of Mathematics
Cartmel College
University of Lancaster
Bailrigg
Lancaster LA1 4YL

APPENDIX 3

COMMUTATORS WITH THE TRIANGULAR PROJECTION AND HANKEL FORMS ON NEST ALGEBRAS

STEPHEN POWER

Let $\mathscr{B}_p$, $1 \leqslant p < \infty$, denote the von Neumann–Schatten classes and let $\mathscr{B}$ denote the bounded linear operators acting on a separable complex Hilbert space. Let $\mathscr{K}$ denote the compact operators. Associated with every totally ordered family, or nest, of self-adjoint projections in $\mathscr{B}$ there is a nest algebra $\mathscr{A}$ and a transformation $\mathscr{P}$ of lower triangular truncation. It is known that $\mathscr{P}$ possesses boundedness and weak type properties on the classes $\mathscr{B}_p$, $1 < p < \infty$, and on the Schatten–Lorentz classes, respectively, that are analogous to those of the Riesz projection (for functions on the unit circle). See [12, 13, 2] for example.

We take the parallel with the Riesz projection further. For certain triangular projections of discrete type it is shown that the commutator

$$\mathscr{P}B - B\mathscr{P}$$

determines a compact operator on $\mathscr{B}_2$ if and only if the operator B (acting as a left multiplier) belongs to the C*-algebra

$$(\mathscr{A} + \mathscr{K}) \cap (A + \mathscr{K})^*.$$

This algebra plays the role of the bounded functions on the circle of vanishing mean oscillation (the quasicontinuous functions). For function space contexts see [33, 35, 6]. The triangular conjugate $\tilde{X}$ of an operator X on l^2 is introduced to provide an alternative description of this C*-algebra. Moreover, a characterisation of $\mathscr{B} + \tilde{\mathscr{B}}$ is given that is analogous to Fefferman's description [11] of $L^\infty + \tilde{L}^\infty$ as the functions of bounded mean oscillation. The main idea involved is an 'atomic' decomposition property for the predual of $\mathscr{B} + \tilde{\mathscr{B}}$.

Our approach to commutators involves characterising the bounded bilinear forms [,] on the Hilbert–Schmidt subspace $\mathscr{A}_2 = \mathscr{B}_2 \cap \mathscr{A}$ that satisfy the identity

$$[A_1 A_2, A_3] = [A_1, A_2 A_3]$$

for all triples in $\mathscr{A}_2$. Such forms are known as Hankel forms. The characterisation is based on a weak factorisation property for the operators in $\mathscr{A}_1$, the triangular trace-class operators, together with the weak star density of the finite-rank operators of $\mathscr{A}$. These facts are related, and the latter, due to Erdos [8], is given a new proof. The factorisation property is linked closely to the atomic decomposition mentioned above, to the distance formula of Arveson [3, 4], and to related ideas discussed in [17, 27, 28, 18].

An operator X in $\mathscr{B}$ determines a Hankel operator H_X on $\mathscr{A}_2$ such that

$$H_X A = (I - \mathscr{P}) XA$$

Received 2 July 1984.

1980 *Mathematics Subject Classification* 47D25.

J. London Math. Soc. (2) 32 (1985) 272–282

for A in $\mathscr{A}_2$. In the case of a finitely ascending discrete nest of order type $\mathbb{N}$ the compactness of H_X is shown to correspond to the quasitriangularity of the symbol operator X. This connection is a useful one. We deduce that the difference of two truncation operators is compact precisely when their corresponding nests are asymptotic. Also the techniques of Axler, Berg, Jewell and Shields are applicable and we conclude that $\mathscr{A}+\mathscr{K}$ is proximal in this case; that is, every operator possesses a best quasitriangular approximant in the operator norm. More general results on the proximinality of perturbed spaces are obtained in [7].

1. *Weak factorisation and Hankel forms*

Throughout the paper we let $(\mathscr{B}_p, \| \|_p)$, $1 \leqslant p < \infty$, denote the von Neumann–Schatten classes of operators that act on a complex separable Hilbert space $\mathscr{H}$. The Banach space of compact operators is denoted by $\mathscr{K}$ and we identify the dual space with $\mathscr{B}_1$ by means of the pairing

$$\langle K, B \rangle = \operatorname{trace}(BK)$$

for B in $\mathscr{B}$ and K in $\mathscr{K}$. The dual space of $\mathscr{B}_1$ is identified with $\mathscr{B}$ in the same manner.

In this section we consider a complete nest $\mathscr{E}$ of self-adjoint projections E on $\mathscr{H}$. Thus $\mathscr{E}$ contains the projections 0 and I, $\mathscr{E}$ is closed in the strong operator topology, and any two projections are comparable with respect to the usual ordering; $F < E$ if and only if $E-F$ is a non-zero positive operator. If $E \in \mathscr{E}$ and $E > 0$ then $E_- = \sup\{F \in \mathscr{E} : F < E\}$. Similarly, if $E \in \mathscr{E}$ and $E < I$, we let $E_+ = \inf\{F \in \mathscr{E} : F > E\}$. If $F > E$ then the projection $F-E$ is called an interval of $\mathscr{E}$. The atoms of $\mathscr{E}$ are the irreducible intervals. The nest algebra $\mathscr{A}$ associated with $\mathscr{E}$ is the set $\{A \in \mathscr{B} : (I-E)AE = 0 \text{ for } E \in \mathscr{E}\}$. This consists of the operators that leave invariant all the subspaces $E\mathscr{H}$, and is often written as $\operatorname{Alg}\mathscr{E}$. We shall write $\mathscr{A}^+$ for the collection of those operators A in $\mathscr{A}$ for which $QAQ = 0$ for all atoms Q. We also let $\mathscr{A}_p = \mathscr{A} \cap \mathscr{B}_p$ and $\mathscr{A}_p^+ = \mathscr{A}_p \cap \mathscr{A}^+$, for $1 \leqslant p < \infty$.

We first obtain a decomposition for operators in $\mathscr{A}_1$ that has proved to be useful [29, 30]. We give a quick existential approach to this that is based on the Krein–Millman theorem rather than the constructive methods of [28]. Our starting point however is the same fundamental lemma of Lance [17].

LEMMA 1.1. *Let A be a trace-class operator that leaves invariant a proper closed subspace, and let E denote the orthogonal projection onto this subspace. Then there exists a trace-class decomposition $A = A_1 + A_2$ such that*

(i) $\| A \|_1 = \| A_1 \|_1 + \| A_2 \|_1$,

(ii) $A_1 = EA_1$ *and* $A_2 E = 0$.

LEMMA 1.2. *The extreme points of the closed unit ball of $\mathscr{A}_1$ are the rank-one operators in the unit sphere. Each such operator has the form $e \otimes f$ where $Ee = 0$ and $E_+ f = f$ for some E in $\mathscr{E}$, and where e and f are unit vectors.*

Proof. Let A be an extreme point. First we show that there is a projection E in $\mathscr{E}$ such that $A = E_+ A(I-E)$.

Suppose that $E \in \mathscr{E}$, $A \neq EA$ and $AE \neq 0$. Let $A = A_1 + A_2$ be the decomposition of Lemma 1.1 associated with E. Then $A_1 E = AE$, and so $A_1 \neq 0$. Also

$(I-E)A = (I-E)A_2$, and so $A_2 \neq 0$. Thus by Lemma 1.1 (i) A is not an extreme point. This contradiction shows that we have the alternative, $A = EA$ or $AE \neq 0$, for all $E \in \mathscr{E}$. Now let $E = \sup\{F: AF = 0\}$. It follows that $AE = 0$ and $A = GA$ for all $G > E$. Hence $A = E_+ A(I-E)$.

Let $A = \sum \lambda_k R_k$ be a Schmidt decomposition for A, with $\lambda_1, \lambda_2, \ldots$ the singular number sequence of A, and $R_1, R_2, \ldots$ rank-one operators of unit norm. Then

$$A = E_+ A(I-E) = \sum \lambda_k E_+ R_k(I-E).$$

Since $\| A \|_1 = \sum \lambda_k$ it follows that $\lambda_k = \lambda_k \| E_+ R_k(I-E) \|_1$ for all k. In particular $R_k = E_+ R_k(I-E)$ for all $\lambda_k \neq 0$. But this condition on R_k implies membership of $\mathscr{A}$. Since A is an extreme point it follows that $\lambda_2 = \lambda_3 = \ldots = 0$.

To complete the proof observe that a rank-one operator of unit norm is an extreme point in the unit ball of $\mathscr{B}_1$.

LEMMA 1.3. *Let $\varepsilon > 0$ and $A \in \mathscr{A}_1$. Then there exists a sequence $R_1, R_2, \ldots$ of rank-one operators in $\mathscr{A}$ such that*

(i) $A = R_1 + R_2 + \ldots$,
(ii) $\sum \| R_k \|_1 < \| A \|_1 + \varepsilon$.

Proof. In view of the Krein–Millman theorem, Lemma 1.2 and elementary functional analysis, it will be sufficient to show that $\mathscr{A}_1$ is a dual space. Let $\mathscr{S}$ denote the norm-closed linear span of the rank-one operators R such that $R = ER(I-E)$ for some E in $\mathscr{E}$. Then an operator A in $\mathscr{B}_1$ belongs to the annihilator of $\mathscr{S}$ if and only if

$$\operatorname{trace}(X(I-E)AE) = \operatorname{trace}(AEX(I-E)) = 0$$

for all E in $\mathscr{E}$, and all rank-one operators X. It follows that $\mathscr{A}_1$ is the annihilator of $\mathscr{S}$, and thus equal to the dual space of $\mathscr{K}/\mathscr{S}$ through standard duality.

COROLLARY 1.4 [8]. *The finite-rank operators in the operator norm unit ball of a nest algebra are dense in the ultraweak topology.*

Proof. The rank-one operators of $\mathscr{A}$ are described in Lemma 1.2 (this part of the lemma is a well-known and useful fact due to Ringrose [31]). Let $\mathscr{R}^-$ denote the closed linear span of these operators with respect to the operator norm. Then, as in the proof of Lemma 1.3, the annihilator of $\mathscr{R}^-$ in $\mathscr{B}_1$ is $\mathscr{A}_1^+$. Also the operators of $\mathscr{A}_1^+$ admit a decomposition into rank-one operators as in Lemma 1.3. (The proof follows the same pattern.) It is now clear that the annihilator of $\mathscr{A}_1^+$ in $\mathscr{B}$ is equal to the annihilator of the rank-one operators of $\mathscr{A}^+$. But this is the collection of operators A for which

$$\operatorname{trace}(X(I-E)AE) = \operatorname{trace}(EX(I-E)A) = 0$$

for all E in $\mathscr{E}$ and all rank-one operators X, and so coincides with $\mathscr{A}$.

We have shown that $\mathscr{A}$ is the second annihilator of $\mathscr{R}^-$ in the standard duality, and thus is naturally identified with the second dual of $\mathscr{R}^-$. Moreover the weak star topology corresponds to the relative weak star, or ultraweak topology on $\mathscr{A}$. A well-known Banach space principle (sometimes called Goldstine's theorem) now shows that the unit ball of $\mathscr{R}^-$ is weak star, and so ultraweakly, dense in the unit ball of $\mathscr{A}$. The corollary follows.

REMARKS. The original proof of Corollary 1.4 made use of the representation theory of nests and is quite different in character from the one above. A consequence of the density is the apparently weaker assertion that a nest algebra is local in the sense of [10]; that is, the finite-rank operators of $\mathscr{A}$ are ultraweakly dense in the algebra. In fact the Erdos density result may be obtained from localness by using the duality arguments of the proof of Corollary 1.4. However no real simplification arises through this approach since the core of the proof of localness in [10] requires Lidskii's theorem that the trace and the spectral trace of a trace-class operator agree. Indeed, it appears to be of more interest to obtain the surprisingly difficult theorem of Lidskii from triangular density properties, as in [9, 29]. These ideas seem to be strongly tied to the Hilbert space setting (see [16, 20]).

The finite-rank operators of $\mathscr{A}$ are operator norm dense in $\mathscr{A} \cap K$ [8]. This simple consequence of Corollary 1.4 is in fact not so deep and may be obtained by direct methods which are valid in wider Banach space contexts (such as the natural nests on $L^p(\mathbb{R}, \mu)$, $1 < p < \infty$) where decomposition theorems for triangular nuclear operators are not at hand.

One of the consequences of localness obtained in [10] is that the sum $\mathscr{A} + \mathscr{K}$ is closed. It is amusing to note that this may be obtained directly from Corollary 1.4 and the Banach space arguments of Rudin [32] for spaces of type $H^\infty + C$. Approximate identity arguments of this nature also appear in [19].

THEOREM 1.5. *Let $\varepsilon > 0$ and $A \in \mathscr{A}_1$. Then there exist rank-one operators $B_1, B_2, \ldots$ and $C_1, C_2, \ldots$ in $\mathscr{A}$ such that*

(i) $A = \sum_k B_k C_k$,

(ii) $\sum_k \| B_k \|_2 \| C_k \|_2 < \| A \|_1 + \varepsilon$.

Proof. Suppose first that $R \in \mathscr{A}_1$ is a non-zero rank-one operator and thus of the form $e \otimes f$ with $Ee = 0$ and $E_+ f = f$ for some $E \in \mathscr{E}$ (Lemma 1.2). If $E < E_+$, let g be a unit vector in the range of $E_+ - E$, so that the operators $B = g \otimes f$ and $C = e \otimes g$ belong to $\mathscr{A}$. Then $R = BC$ and $\| R \|_1 = \| B \|_2 \| C \|_2$. On the other hand, if $E = E_+$, choose $F > E$ so that $\| R - R(I-F) \|_1 < \varepsilon$. Let $R(I-F) = R_1 = e_1 \otimes f$ and choose a unit vector g_1 in the range of $F - E$ so that $B_1 = g_1 \otimes f$ and $C_1 = e_1 \otimes g_1$ belong to $\mathscr{A}$. Then $R_1 = B_1 C_1$ and $\| R_1 \|_1 = \| B_1 \|_2 \| C_1 \|_2$.

In conjunction with Lemma 1.3 the constructions above show that for $\varepsilon > 0$ and A in $\mathscr{A}_1$, there exist rank-one operators $B_1, B_2, \ldots$ and $C_1, C_2, \ldots$ in $\mathscr{A}$ such that

$$\textstyle\sum_k \| B_k \|_2 \| C_k \|_2 < \| A \|_1 + \varepsilon \quad \text{and} \quad \| A - \sum_k B_k C_k \|_1 < \varepsilon.$$

Iterative use of this principle completes the proof.

A bilinear form $[\,,\,]$ on a complex algebra is called a Hankel form if the identity $[A_1 A_2, A_3] = [A_1, A_2 A_3]$ holds for all triples. A bilinear form $[\,,\,]$ on a normed space is said to be bounded if $|[A_1, A_2]|$ is bounded for all couples A_1, A_2 in the unit ball. Characterisations of bounded Hankel forms on function spaces have been found by Nehari [22] for the complex polynomials with the H^2 norm, by Coifman, Rochberg and Weiss [6] for complex polynomials in several variables and the Hilbert space norms for the unit sphere and ball, and by Peetre [23] for other Bergman space norms. A key step in obtaining these results, as with our next theorem, is the use of weak factorisation ($A = \sum B_k C_k$).

THEOREM 1.6. *Let* $[\ ,\ ]$ *be a bounded Hankel form on* $\mathscr{A}_2$. *Then there exists a bounded operator* X *such that*

(i) $[A_1, A_2] = \operatorname{trace}(A_2 X A_1)$ *for all* A_1, A_2 *in* $\mathscr{A}_2$,

(ii) $\|X\| = \sup\{|[A_1, A_2]| : \|A_1\|_2 \leqslant 1, \|A_2\|_2 \leqslant 1\}$.

Proof. Using Corollary 1.4 fix a bounded sequence R_n of operators in $\mathscr{R}$ that converge to the identity in the ultraweak topology. Let $A = \sum B_k C_k$ be a weak factorisation of an operator A in $\mathscr{A}_1$, as given by Theorem 1.5. Since the series also converges in the Hilbert–Schmidt norm we have

$$\begin{aligned}\sum_k [B_k, C_k] &= \lim_n \sum_k [B_k, C_k R_n] \\ &= \lim_n \sum_k [B_k C_k, R_n] \\ &= \lim_n [\sum_k B_k C_k, R_n] \\ &= \lim_n [A, R_n].\end{aligned}$$

Let us denote this limit by $\Phi(A)$ and thereby define a linear functional on $\mathscr{A}_1$. Thus $\Phi(A) = [A_1, A_2]$ if $A = A_1 A_2$ with A_1, A_2 in $\mathscr{A}_2$. If α denotes the supremum in Theorem 1.6(ii) we have

$$|\Phi(A)| \leqslant \sum_k |[B_k, C_k]| \leqslant \alpha \sum_k \|B_k\|_2 \|C_k\|_2$$

and so it follows from Theorem 1.5 that the norm of Φ is no greater than α. Hence the norm is precisely α. Let X be an operator in $\mathscr{B}$ that implements any norm-preserving extension of Φ to a functional on $\mathscr{B}_1$. With this X the theorem follows.

REMARKS. Let $[\ ,\ ]_X$ denote the Hankel form determined by an operator X in $\mathscr{B}$ and the equation $[A_1, A_2]_X = \operatorname{trace}(A_2 X A_1)$. Then, by weak factorisation, the form is the zero form if and only if X is in the annihilator of $\mathscr{A}_1$, and therefore (as in the proof of Corollary 1.4), if and only if X is in $\mathscr{A}^+$. It now follows from Theorem 1.6 that

$$\sup\{|[A_1, A_2]_X| : \|A_i\|_2 \leqslant 1, i = 1, 2\} = \operatorname{dist}(X, \mathscr{A}^+).$$

The quantity on the left is called the norm of the form.

As mentioned earlier, the space $\mathscr{A}_1^+$ also admits a decomposition as in Lemma 1.3, and this leads to the characterisation of the Hankel forms on the product space $\mathscr{A}_2^+ \times \mathscr{A}_2$. In this case the norm of the form implemented by the operator X is $\operatorname{dist}(X, \mathscr{A})$.

The theorem suggests the attractive problem of characterising the bounded Hankel forms on reflexive algebras, both on Hilbert space and general Banach spaces. Because of the close connections with the existence of distance formulas progress will probably depend on new developments in this topic.

2. *Commutators and triangular conjugation*

In this section we specialise to a nest $\mathscr{E}$ that consists of 0, I and an increasing sequence of finite-rank projections $P_1, P_2, \ldots$ that converge in the strong operator topology to the identity. We regard $\mathscr{B}_2$ as a complex Hilbert space with an inner product given by $(B_1, B_2) = \text{trace}(B_2^* B_1)$. The triangular projection associated with $\mathscr{E}$ is the orthogonal projection $\mathscr{P}$ of $\mathscr{B}_2$ onto $\mathscr{A}_2$. We write $\mathscr{P}_- = I - \mathscr{P}$ for the complementary projection and $\mathscr{P}_+$ for the orthogonal projection with range $\mathscr{A}_2^+$. A complex linear unitary operator $\mathscr{C}$ is defined on $\mathscr{B}_2$ by the adjoint operation; $\mathscr{C}B = B^*$.

Consider the bounded Hankel form $[A_1, A_2] = \text{trace}(A_2 X A_1)$ that is induced by a bounded operator X. If $A_1 \in \mathscr{A}_2^+$ and $A_2 \in \mathscr{A}_2$ then

$$\begin{aligned}[A_2, A_1]_X &= \text{trace}(XA_2A_1)\\ &= (A_1, (XA_2)^*)\\ &= (\mathscr{P}_+A_1, \mathscr{C}(XA_2))\\ &= (A_1, \mathscr{P}_+\mathscr{C}(XA_2))\\ &= (A_1, \mathscr{C}H_X A_2),\end{aligned}$$

where H_X is the Hankel operator $(I-\mathscr{P})X\mathscr{P}$. The Hankel operator belongs to $\mathscr{B}(\mathscr{B}_2)$, and we can see from the above that its operator norm coincides with the norm of the Hankel form on $\mathscr{A}_2 \times \mathscr{A}_2^+$. Thus by our remarks in Section 1 we have

$$\| H_X \| = \text{dist}(X, \mathscr{A}).$$

THEOREM 2.1. *Let X be a bounded operator. Then*

(i) *the Hankel operator H_X is a compact operator if and only if X belongs to the quasitriangular algebra $\mathscr{A} + \mathscr{K}$. Moreover*

$$\text{dist}(H_X, \mathscr{K}(\mathscr{B}_2)) = \text{dist}(X, \mathscr{A} + K);$$

(ii) *the commutator $X\mathscr{P} - \mathscr{P}X$ determines a compact operator on $\mathscr{B}_2$ if and only if X belongs to the C*-algebra*

$$(\mathscr{A} + \mathscr{K}) \cap (\mathscr{A} + \mathscr{K})^*.$$

Proof. Note that $H_X = 0$ if $X \in \mathscr{A}$. Also, if $X = P_n X P_n$ for some n, then H_X has finite rank. Indeed, if $A \in \mathscr{A}_2$, then $XA(I-P_n) \in \mathscr{A}_2$, and so the range of H_X is contained in $(I-\mathscr{P})X\mathscr{A}_2 P_n = (I-\mathscr{P})XP_n\mathscr{A}_2 P_n$, which is finite dimensional. Since $\| H_X \| \leqslant \| X \|$ it follows that H_X is a compact operator when $X \in \mathscr{K}$, and so too when $X \in \mathscr{A} + \mathscr{K}$.

Suppose now that H_X is a compact operator. For $n = 1, 2, \ldots$ let $S_n = n^{-1}P_n + (I - P_n)$ so that S_n is a bounded sequence of invertible operators that converge to zero in the strong operator topology. The operator of left multiplication by S_n on $\mathscr{B}_2$ also converges to zero in the strong operator topology (of $\mathscr{B}(\mathscr{B}_2)$) and so, using the compactness of H_X, we see that $H_{XS_n} = H_X S_n$ converges to zero in operator norm. By the identity preceding Theorem 2.1 there exist operators $A_n \in \mathscr{A}$ such that the norm of $XS_n + A_n$ tends to zero as n tends to infinity. Let $\pi(T)$ denote the coset of T in the Calkin algebra $\mathscr{B}/\mathscr{K}$. Then

$$\| \pi(X + A_n S_n^{-1}) \| \leqslant \| \pi(XS_n + A_n) \| \, \| \pi(S_n^{-1}) \|$$

and $\| \pi(S_n^{-1}) \| \leqslant 1$. Since $A_n S_n^{-1} \in \mathscr{A} + \mathscr{K}$, it follows that $X \in \mathscr{A} + \mathscr{K}$.

Now let $K \in \mathscr{K}(\mathscr{B}_2)$, so that the operator KS_n converges to zero, where once again S_n is regarded as a left multiplier. For large enough n, we have

$$\begin{aligned}\| H_X + K \| &\geqslant \| H_{XS_n} + KS_n \| \geqslant \| H_{XS_n} \| - \varepsilon\\ &= \text{dist}(XS_n, \mathscr{A}) - \varepsilon \geqslant \text{dist}(X, \mathscr{A} + \mathscr{K}) - \varepsilon.\end{aligned}$$

On the other hand, by our opening comments, if $Y \in \mathscr{A}+\mathscr{K}$, we have

$$\operatorname{dist}(H_X, \mathscr{K}(\mathscr{B}_2)) = \operatorname{dist}(H_{X+Y}, \mathscr{K}(\mathscr{B}_2)) \leqslant \| X+Y \|,$$

and so the proof of (i) is complete.

(ii) Note that

$$X\mathscr{P}-\mathscr{P}X = (X\mathscr{P}-\mathscr{P}X\mathscr{P})-(\mathscr{P}X-\mathscr{P}X\mathscr{P}) = H_X-(H_{X^*})^*,$$

so that, by (i), the condition on X is sufficient for compactness. On the other hand, if the commutator is compact then so too are the operators $\mathscr{P}X\mathscr{P}-\mathscr{P}X$ and $X\mathscr{P}-\mathscr{P}X\mathscr{P}$. Thus, by (i) again, the condition is necessary.

It has been shown by Plastiras [25] that another finitely ascending nest, $Q_1, Q_2, \ldots$ say, determines the quasitriangular algebra $\mathscr{A}+\mathscr{K}$ if and only if $\{P_n\}$ and $\{Q_n\}$ are asymptotic. This means that P_n-Q_{n+k} converges to zero in norm, as n tends to infinity, for some fixed integer k. Ken Davidson noticed the following consequence.

COROLLARY 2.2. *Let $\mathscr{P}$ and $\mathscr{Q}$ be the projections of triangular truncation determined by the finitely ascending nests $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots$ respectively. Then $\mathscr{P}-\mathscr{Q}$ is a compact operator if and only if $\| P_n-Q_{n+k} \| \to 0$, as $n \to \infty$, for some integer k.*

Proof. Let $E_n = P_n-P_{n-1}$ and $F_n = Q_n-Q_{n-1}$. For an operator X we have

$$\begin{aligned}(\mathscr{P}-\mathscr{Q})X &= \sum_{k=1} P_k X E_{k+1} - \sum_{k=1} Q_k X F_{k+1} \\ &= \sum_{k=1} (P_k-Q_k) X E_{k+1} + \sum_{k=1} Q_k X(E_{k+1}-F_{k+1}) \\ &= \sum_{k=1} (P_k-Q_k) X E_{k+1} + \sum_{k=1} Q_k X(P_{k+1}-Q_{k+1}) - \sum_{k=1} Q_k X(P_k-Q_k).\end{aligned}$$

Since the projections E_n and Q_n have finite rank it follows that $\mathscr{P}-\mathscr{Q}$ is a compact operator if the nests are asymptotic.

On the other hand, if $\mathscr{P}$ is a compact perturbation of $\mathscr{Q}$ then the Hankel operator $(I-\mathscr{P})X\mathscr{P}$ is compact if and only if the Hankel operator $(I-\mathscr{Q})X\mathscr{Q}$ is compact. By Theorem 2.1 the quasitriangular algebras for $\mathscr{P}$ and for $\mathscr{Q}$ coincide, and so, by our earlier comments, $\{P_n\}$ and $\{Q_n\}$ are asymptotic.

REMARKS. (1) If X is a bounded operator and $X_n = P_n X P_n$, then the bounded sequence H_{X_n} of finite-rank Hankel operators converges to H_X in the strong topology. Because of this the results of [5] may be applied to show that there exists a compact Hankel operator H_Y such that

$$\| H_X - H_Y \| = \operatorname{dist}(H_X, \mathscr{K}(\mathscr{B}_2)).$$

In conjunction with the equality $\| H_X \| = \operatorname{dist}(X, \mathscr{A})$ this leads to the fact that $\mathscr{A}+\mathscr{K}$ is proximinal; that is, every operator possesses a best approximation in the operator norm from the quasitriangular algebra $\mathscr{A}+\mathscr{K}$. The proof follows that of [5] concerning the proximinality of $H^\infty+C$ in L^∞. However, there are rather more direct methods available, including the M-ideal techniques of Alfsen and Effros [1], and these are discussed in [7]. These methods cover the context of quasitriangular algebras with respect to an arbitrary nest, as well as certain Banach space contexts. The M-ideal approach was exploited by Leucking [21] to obtain the proximinality of $H^\infty+C$.

(2) The first part of the theorem yields another proof that the operators X that are quasitriangular with respect to $P_1, P_2, \ldots$ ($(I-P_n)XP_n \to 0$ as $n \to \infty$) are precisely the operators in the quasitriangular algebra $\mathscr{A}+\mathscr{K}$. Indeed, let Q_n denote the orthogonal projection of $\mathscr{B}_2$ onto the subspace $\mathscr{B}_2(P_n-P_{n-1})$. Then $H_X Q_n = Q_n H_{(I-P_n)XP_n} Q_n$. Consequently

$$H_X \oplus 0 = \bigoplus_{n=1}^{\infty} H_{(I-P_n)XP_n}.$$

Since the symbol operators for the summands are of finite rank, the summands themselves are compact operators, and so, by the quasitriangularity of X, it follows that H_X is compact.

(3) The equalities at the beginning of this section show that H_X is closely related to the Hankel operator S_X on $\mathscr{A}_2$ that is defined by

$$S_X : A \longrightarrow \mathscr{P}(XA)^*.$$

For example, S_X is compact if and only if the symbol operator X belongs to $\mathscr{A}^+ + \mathscr{K}$. This assertion does not hold for more general nests, such as the Volterra nest for $L^2[0, 1]$. In the case of the standard multiplicity-one nest of order type $\mathbb{N}$ or $\mathbb{Z}$ the following characterisations can be made:

(i) S_X is of finite rank if and only if the representing matrix of $\mathscr{P}X^*$ is finitely non-zero;

(ii) $S_X \geq 0$ if and only if X is a positive diagonal operator modulo $\mathscr{A}^+$;

(iii) S_X is a Hilbert-Schmidt operator if and only if

$$\sum_{j \geq k} (j-k+1)\,|x_{jk}|^2$$

is finite, where $X = (x_{jk})$.

More generally, it is possible to use the decomposition for H_X above to characterise when H_X and S_X belong to a given von Neumann–Schatten class. The corresponding characterisations for the classical Hankel operators are due to Peller [24].

$\mathscr{P}$-triangular conjugates

There is a strong formal similarity between Theorem 2.1 and certain function space settings involving classical Hankel operators, commutators, and the space $(H^\infty + C) \cap (\overline{H^\infty + C})$. See [14, 33, 26]. This mirroring can be taken further with the notion of the triangular conjugate of an operator.

Let $\mathscr{E}$ be a nest of multiplicity one and order type $\mathbb{N}$ or $\mathbb{Z}$, and let $\mathscr{F}$ denote the associated linear space of matrices with only a finite number of non-zero entries. The dual space of $\mathscr{F}$ of (conjugate) linear functionals is identified with the space $\mathscr{M}$ of all matrices under the pairing $(M, F) = \text{trace}\,(F^* M)$, for F in $\mathscr{F}$ and M in $\mathscr{M}$. The $\mathscr{P}$-triangular conjugate of a matrix M is defined to be the matrix

$$\tilde{M} = (-i\mathscr{P}_+ + i\mathscr{P}_-)\,M.$$

Note that for every matrix M the matrix $M + i\tilde{M}$ is upper triangular, and if M has zero diagonal, then $M + i\tilde{M} = 2\mathscr{P}_+ M$. If we let $\mathscr{D}$ denote the linear space of diagonal matrices, it follows easily from these facts that

$$(\mathscr{A}+\mathscr{K}) \cap (\mathscr{A}+\mathscr{K})^* = \mathscr{D} \cap \mathscr{B} + (\mathscr{K} + \tilde{\mathscr{K}}) \cap \mathscr{B}.$$

This provides an alternative description of the C*-algebra of Theorem 2.1. In fact we can give an analogue of Fefferman's characterisation [11] of $L^\infty + \tilde{L}^\infty$ as the functions of bounded mean oscillation. The identification of $\mathscr{B}+\tilde{\mathscr{B}}$ with the dual space of $\mathscr{F}_1$ follows elementary duality principles as in the function space setting. However, the realisation of $B+\tilde{B}$ as a certain space of matrices for which the 'oscillation' $\| ME-EM \|$ is bounded, for $E \in \mathscr{E}$, lies as deep as the decomposition result of Lemma 1.3. We see these facts in the next theorem.

For a matrix M in $\mathscr{B}+\tilde{\mathscr{B}}$ define the norm

$$\| M \|_* = \inf\{\max(\| F \|, \| G \|) : M = F+\tilde{G}, F, G \in \mathscr{B}\},$$

so that $(\mathscr{B}+\tilde{\mathscr{B}}, \| \|_*)$ is a Banach space. Let $\mathscr{F}_1$ denote the completion of $\mathscr{F}$ with respect to the norm

$$\| F \|_{\mathscr{F}_1} = \| F \|_1 + \| \tilde{F} \|_1.$$

THEOREM 2.3. *The following conditions are equivalent for a matrix M:*

(i) *M belongs to $\mathscr{B}+\tilde{\mathscr{B}}$;*

(ii) *M determines a continuous functional on the Banach space $\mathscr{F}_1$;*

(iii) *the diagonal of M is bounded and the set of commutators $ME-EM$, for E in $\mathscr{E}$, is uniformly bounded in operator norm.*

Also the Banach space $\mathscr{B}+\tilde{\mathscr{B}}$ is isometrically isomorphic to the dual of $\mathscr{F}_1$.

Proof. (i) ⇒ (ii) If $M = B_1+\tilde{B}_2$ with B_1, B_2 in $\mathscr{B}$, and if F is a matrix of $\mathscr{F}$, then

$$|(M, F)| = |(B_1+\tilde{B}_2, F)| = |(B_1, F)-(B_2, \tilde{F})| \leq \| B_1 \| \, \| F \|_1 + \| B_2 \| \, \| \tilde{F} \|_1.$$

Thus $|(M, F)| \leq \| M \|_* \| F \|_{\mathscr{F}_1}$.

(ii) ⇒ (i) Let Λ be the linear map between the Banach spaces $\mathscr{B} \oplus \mathscr{B}$ and $\mathscr{B}+\tilde{\mathscr{B}}$ such that $(B_1 \oplus B_2) \to B_1+\tilde{B}_2$. Then the induced mapping on the quotient space $(\mathscr{B} \ominus \mathscr{B})/\ker \Lambda$ is an isometrical isomorphism. It follows that the predual of $\mathscr{B}+\tilde{\mathscr{B}}$ is naturally identifiable with the annihilator of $\ker \Lambda$ in the predual $\mathscr{B}_1 \oplus \mathscr{B}_1$. However the operator $B_1 \oplus B_2$ belongs to this kernel if and only if $\tilde{B}_2 = -B_1$, which is the condition $B_2 = \tilde{B}_1$. Thus the annihilator consists of operators $C_1 \oplus C_2$ such that

$$\operatorname{trace}((C_1+\tilde{C}_2) B) = \operatorname{trace}(C_1 B + C_2 \tilde{B}) = 0$$

for all B in $\mathscr{B}$ with zero diagonal and with $\tilde{B}$ in $\mathscr{B}$. Hence the annihilator is identified with the subspace of elements of the form $-\tilde{C}_2 \oplus C_2$. Clearly this subspace is isomorphic to $\mathscr{F}_1$.

(ii) ⇒ (iii) Let the matrix M determine a $\| \|_{\mathscr{F}_1}$-continuous functional of norm $\alpha > 0$. Let $R = e \otimes f$, where e and f are unit vectors such that $Ee = 0$ and $E_+ f = f$, for some E in $\mathscr{E}$. Since R belongs to $\mathscr{A}$, it follows that $\| \tilde{R} \|_1 \leq 2$ and $\| R \|_{\mathscr{F}_1} \leq 3$. We have

$$(M, R) = \operatorname{trace}(R^* M) = (Me, f) = (E_+ M(I-E)e, f)$$

and so $\| E_+ M(I-E) \| \leq 3\alpha$. Since $\| F \|_{\mathscr{F}_1} = \| F^* \|_{\mathscr{F}_1}$ it follows that $\| E_+ M^*(I-E) \| \leq 3\alpha$. The boundedness of these norms is equivalent to property (iii).

(iii) ⇒ (ii) By the assertions above, if (iii) holds for a matrix M then for some $\beta > 0$ we have $|(M, R)| \leq \beta$ and $|(M, R^*)| \leq \beta$ for all R in $\mathscr{A}$ of rank one and unit operator norm. Let F be a self-adjoint operator in the unit ball of $\mathscr{F}_1$. Let $\sum R_k$ denote a decomposition of $F+i\tilde{F}$, as provided by Lemma 1.3, with $\Sigma \| R_k \| \leqslant 2$. Then $F = \Sigma A_k$, where $A_k = \frac{1}{2}(R_k + R_k^*)$, and so

$$|(M, F)| \leqslant \tfrac{1}{2}\Sigma(|(M, R_k)| + |(M, R_k^*)|) \leqslant \beta;$$

(ii) now follows.

REMARKS. The last part of the proof shows that elements of the Banach space $\mathscr{F}_1$ admit an atomic decomposition (in the sense of harmonic analysis) into sums of operators of bounded rank.

A constructive approach to the $(L^\infty + \tilde{L}^\infty)$-representation of a function of bounded mean oscillation has been given by Jones [15]. It would be interesting to discover an operator-theoretic variant of this process and thereby give a direct proof of the implication (iii) ⇒ (i), and possibly provide insight into how the Arveson distance formula is attained.

Like BMO, the space $\mathscr{B}+\tilde{\mathscr{B}}$ has the following monotonicity property. If $0 \leqslant |y_{ij}| \leqslant x_{ij}$ and the matrix $X = (x_{ij})$ belongs to $\mathscr{B}+\tilde{\mathscr{B}}$, then so does the matrix $Y = (y_{ij})$. This result follows from weak factorisation and is dual to the Hardy inequality of Shields [34] (see also [30]).

Note added in proof. T. G. Feeman has also obtained the proximality of quasi-triangular algebras associated with discrete nests (in a paper to appear in *Trans. Amer. Math. Soc.*). Theorem 1.6(i) has been generalised to semifinite factors in the author's preprint 'Factorisation in analytic operator algebras'.

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Department of Mathematics
Cartmel College
University of Lancaster
Bailrigg
Lancaster LA1 4YL

APPENDIX 4

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10(1983), 337–352

NUCLEAR OPERATORS IN NEST ALGEBRAS

S. C. POWER

1. INTRODUCTION

The main result shows that each nuclear operator T in a nest algebra Alg $\mathscr{E}$ admits a representation

$$T = \int_{\mathscr{E}} T_E \mathrm{d}\tau(E),$$

where τ is a finite positive Borel measure on the nest and $T \to T_E$ is a nuclear operator valued function on $\mathscr{E}$ such that $T_E = ET_E(I - E_-)$ almost everywhere. This representation leads to conditions under which T can be decomposed into an *exact* sum of rank one operators in Alg $\mathscr{E}$ in the following sense:

$$T = \sum_{i=1}^{\infty} R_i, \quad \|T\|_1 = \sum_{i=1}^{\infty} \|R_i\|_1$$

with $R_1, R_2, \ldots$ rank one operators in Alg $\mathscr{E}$. We call this property exact decomposability and it is shown, in particular, that T is exactly decomposable if $\mathscr{E}$ is countable or if T is dissipative.

A basic result required in the analysis is a construction of Lance, Lemmas 3.2, 3.3 of [11], which splits an upper triangular 2×2 operator matrix into a sum of two operators of the form $\begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}$. An indication of some of the power of this decomposition is given in the fact that it leads naturally to a useful result of Parrott [14]. In [11] it is used to derive Arveson's distance formula [1], to which Parrott's result is closely related [15].

In Section 3 we make inductive use of the lemma, and an inherent left continuity, in order to associate with each positive operator C and nest $\mathscr{E}$ a positive operator valued Borel measure $C(\Delta)$ on $\mathscr{E}$. If this construction is applied to the positive part C of an operator $T = UC$ in Alg $\mathscr{E}$ then the operator measure $UC(\Delta)$ on $\mathscr{E}$ provides the appropriate generalisation of Lance's construction, and in case $\mathscr{E}$

has three elements coincides with this construction. In Section 4 we give a Radon-Nikodym theorem for nuclear operator valued measures. For a nuclear operator T this allows us to form the derivative T_E of the measure $UC(\Delta)$ with respect to the scalar measure $\tau(\Delta) = \operatorname{trace} C(\Delta)$ and thereby obtain the main result. The relationship between C and $C(\Delta)$ seems to be worthy of further analysis.

In Section 5 we complete the proof of the main result and give various applications. A natural corollary, of wider interest, is discussed more fully in [16]. This is Lidskii's theorem that the trace of a nuclear operator is the sum of its eigenvalues (counted with their algebraic multiplicity).

NOTATION. We fix a separable complex Hilbert space H. The term *subspace* means *closed* linear subspace. We let $\mathscr{E}$ denote a complete *nest* of self-adjoint projections on H. Thus $\mathscr{E}$ is a totally ordered (under range inclusion) family which contains the projections 0 and I, and which is closed in the strong operator topology. If $E \in \mathscr{E}$ and $E \neq 0$ (resp. $E \neq I$) then we define E_- (resp. E_+) as the supremum (resp. infimum) of the collection of F in $\mathscr{E}$ with $F < E$ (resp. $F > E$). The algebra of all bounded linear operators on H is denoted by $B(H)$, and $B_1(H)$ denotes the class of nuclear operators (trace class operators). The nuclear operators form a Banach space under the norm

$$\|T\|_1 = \operatorname{tr}((T^*T)^{1/2})$$

where tr denotes the trace on $B_1(H)$.

The nest algebra $\operatorname{Alg} \mathscr{E}$ associated with a nest $\mathscr{E}$ is the algebra of all operators T such that $(I - E)TE = 0$ for all $E \in \mathscr{E}$. We denote the family of nuclear operators in $\operatorname{Alg} \mathscr{E}$ by $\operatorname{Alg}_1 \mathscr{E}$. The rank one operator $x \to (u, x)v$ is denoted $u \otimes v$.

2. A LEMMA OF E.C. LANCE

Our starting point is the following fundamental lemma of [11], reformulated in a manner appropriate for later induction.

LEMMA 2.1. *Let C be a positive operator which has an operator matrix* $\begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$ *with respect to a given decomposition of H. Then the limit, as $n \to \infty$, of the sequence $B^*(A + n^{-1}I)^{-1}B$ exists in the strong operator topology. If D_1 denotes this limit then the following hold.*

(i) $D_1 \leqslant D$.

(ii) *The operator* $C_1 = \begin{bmatrix} A & B \\ B^* & D_1 \end{bmatrix}$ *is positive.*

(iii) *If U is an operator on H and UC has the form* $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$, *then UC_1 and $U(C - C_1)$ have, respectively, the forms* $\begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$ *and* $\begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}$.

COROLLARY 2.2. *If* $T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix}$ *is a nuclear operator then there exist T_2' and T_2'' so that if* $R = \begin{bmatrix} T_1 & T_2' \\ 0 & 0 \end{bmatrix}$ *and* $S = \begin{bmatrix} 0 & T_2'' \\ 0 & T_3 \end{bmatrix}$ *then $T = R + S$ and* $\|T\|_1 = \|R\|_1 + \|S\|_1$.

Proof. The corollary follows immediately from an application of the lemma to the polar decomposition $T = UC$. Note that

$$\|T\|_1 = \text{tr}(C) = \text{tr}(C_1) + \text{tr}(C - C_1) = \|UC_1\|_1 + \|U(C - C_1)\|_1,$$

so we may take $R = UC_1$ and $S = U(C - C_1)$.

The corollary may be used now to obtain a useful result of Parrott (see [14] and its footnote for partial anticipations). The proof below makes free use of the $B_1(H)$, $B(H)$ duality and is closely related to the discussions of the distance formula in [11] and [12].

COROLLARY 2.3.

$$\inf_X \left\| \begin{bmatrix} X & A \\ C & B \end{bmatrix} \right\| = \max \left\{ \left\| \begin{bmatrix} 0 & A \\ 0 & B \end{bmatrix} \right\|, \left\| \begin{bmatrix} 0 & 0 \\ C & B \end{bmatrix} \right\| \right\}.$$

Proof. Let us suppose that the operator matrices are relative to an orthogonal decomposition $H = H_1 \oplus H_2$. If $Z \in B(H)$ then write Z_r for the functional on the annihilator of $B(H_1)$ which is induced by Z. That is, Z determines a functional on $B_1(H)$ and Z_r is the restriction of Z to the annihilator mentioned. This annihilator is simply the collection of nuclear operators whose first operator matrix entry is zero. If $Z = \begin{bmatrix} 0 & A \\ C & B \end{bmatrix}$

$$\|Z_r\| \leq \inf_X \left\| \begin{bmatrix} X & A \\ C & B \end{bmatrix} \right\|, \tag{2.1}$$

since operators X in $B(H_1)$ induce the zero functional on the annihilator. On the other hand, by the Hahn-Banach theorem, Z_r has a norm maintaining extension, and so equality occurs in (2.1). But, using Corollary 2.2, we see that

$$\|Z_r\| = \sup_{\left\|\begin{bmatrix} 0 & U \\ V & W \end{bmatrix}\right\|_1 = 1} \left| \operatorname{tr}\left(\begin{bmatrix} 0 & U \\ V & W \end{bmatrix} Z\right)\right| =$$

$$= \sup_{\left\|\begin{bmatrix} 0 & 0 \\ V & W_1 \end{bmatrix}\right\|_1 + \left\|\begin{bmatrix} 0 & U \\ 0 & W_2 \end{bmatrix}\right\|_1 = 1} \left| \operatorname{tr}\left(\begin{bmatrix} 0 & 0 \\ V & W_1 \end{bmatrix} Z\right) + \operatorname{tr}\left(\begin{bmatrix} 0 & U \\ 0 & W_2 \end{bmatrix} Z\right)\right| =$$

$$= \sup_{\left\|\begin{bmatrix} 0 & 0 \\ V & W_1 \end{bmatrix}\right\|_1 + \left\|\begin{bmatrix} 0 & U \\ 0 & W_2 \end{bmatrix}\right\|_1 = 1} \left| \operatorname{tr}\left(\begin{bmatrix} 0 & 0 \\ V & W_1 \end{bmatrix}\begin{bmatrix} 0 & A \\ 0 & B \end{bmatrix}\right) + \operatorname{tr}\left(\begin{bmatrix} 0 & U \\ 0 & W_2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ C & B \end{bmatrix}\right)\right| =$$

$$= \max\left\{\left\|\begin{bmatrix} 0 & A \\ 0 & B \end{bmatrix}\right\|, \left\|\begin{bmatrix} 0 & 0 \\ C & B \end{bmatrix}\right\|\right\}.$$

The last equality follows because the supremum of $\operatorname{tr}\left(\begin{bmatrix} 0 & 0 \\ V & W_1 \end{bmatrix}\begin{bmatrix} 0 & A \\ 0 & B \end{bmatrix}\right)$ as $\begin{bmatrix} 0 & 0 \\ V & W_1 \end{bmatrix}$ varies in the unit ball of $B_1(H)$, is the operator norm of $\begin{bmatrix} 0 & A \\ 0 & B \end{bmatrix}$. The corollary is now proven.

A well known result of Ringrose (see Erdos [5]) asserts that each operator T in Alg $\mathscr{E}$ with finite rank n may be written as a sum of n rank one operators in Alg $\mathscr{E}$. Lemma 2.1 provides an alternative proof of this with the strengthening of the conclusion to an *exact* sum, as we now show. Moreover the method provides a constructive rather than existential approach and so may be of added interest. Extensions of Ringrose's result have been made by various authors to reflexive algebras Alg $\mathscr{L}$ for certain commutative subspace lattices $\mathscr{L}$. We refer the reader to Hopenwasser and Moore [10] for a good discussion of this and for the following two results:

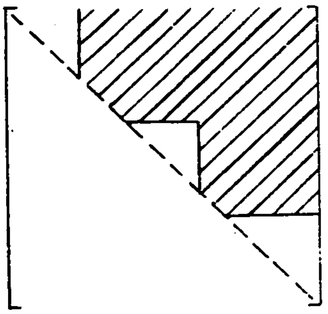
(i) decomposition into rank ones is possible if $\mathscr{L}$ has finite width (although the length of the sum may have to be greater than the rank),

(ii) there is a totally atomic $\mathscr{L}$ and a rank two operator in Alg $\mathscr{L}$ that cannot be written as a sum of rank one operators in Alg $\mathscr{L}$.

Before proceeding it is convenient to introduce the following concept.

DEFINITION 2.4. An operator T in Alg $\mathscr{E}$ is said to be *suspended* by a set $\mathscr{G} \subseteq \mathscr{E}$ if $(E - F)T(E - F) = 0$ whenever the interval $(F, E]$ is disjoint from $\mathscr{G}$.

If T is suspended by two disjoint intervals then T looks like this



One can verify that T is suspended by a singleton $E \neq 0$ if and only if $ET(I - E_-) = T$. Each rank one operator in Alg $\mathscr{E}$ is thus suspended by a singleton since, as is well known, it may be expressed as $e \otimes f$ with $f \in E$ and $e \in I - E_-$, for some $E \neq 0$. If $T \in \text{Alg}_1 \mathscr{E}$ is suspended by the singleton E then it is easy to obtain an exact decomposition of T. Let $C = \sum_{i=1}^{\infty} C_i$ be *any* decomposition of C into positive rank one operators where $T = UC$ is the polar decomposition of T. Then $T = \sum_{i=1}^{\infty} UC_i$ is an exact sum. Also

$$\sum_{i=1}^{\infty} UC_i = T = ET(I - E_-) = \sum_{i=1}^{\infty} EUC_i(I - E_-),$$

and so $\|EUC_i(I - E_-)\|_1 = \|UC_i\|_1$, $i = 1, 2, \ldots$, and hence each summand UC_i belongs to Alg $\mathscr{E}$ and is suspended by E.

It can be shown that every exact sum $X = \sum_{i=1}^{\infty} X_i$, with each X_i of rank one, must arise through a rank one positive decomposition of the positive part of the nuclear operator X. One often takes a spectral decomposition for the positive part, giving a Schmidt expansion for X, but in our context this takes no account of the invariant subspaces of X and need not correspond to the internal exact decomposition for Alg $\mathscr{E}$ obtained below.

COROLLARY 2.5. *Let $T \in \text{Alg}_1 \mathscr{E}$ be a finite rank operator of rank n. Then there are rank one operators $R_1, R_2, \ldots, R_n$ in* Alg $\mathscr{E}$ *with $T = R_1 + R_2 + \ldots + R_n$ and $\|T\|_1 = \|R_1\|_1 + \|R_2\|_1 + \ldots + \|R_n\|_1$.*

Proof. We use the notation of Lemma 2.1. Let $T = UC$ be the polar decomposition, let $E \in \mathscr{E}$, $E \neq 0, I$ and let C_1 be constructed from C, as in Lemma 2.1, relative to the decomposition induced by E. Let $C_2 = C - C_1$. We first show that $\text{rank}\, C_1 + \text{rank}\, C_2 = \text{rank}\, C$.

Let P denote the range projection of A. Then the positivity of C_1 shows that $B^*P = B^*$ (see Lance's proof). Thus

$$D_1 = \lim_n B^*(A + n^{-1})^{-1}B = \lim_n B^*P(A + n^{-1})^{-1}PB = B^*(PAP)^{-1}B$$

where $(PAP)^{-1}$ denotes, informally, the operator which is 0 on $(I - P)H$ and the inverse of PAP on PH. Let S be the invertible operator

$$S = \begin{bmatrix} I & 0 \\ B^*(PAP)^{-1} & I \end{bmatrix}.$$

Then since $B^*(PAP)^{-1}A = B^*P = B^*$, and $B^*(PAP)^{-1}B = D_1$ we have

(2.2) $$C_1 = S\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}.$$

Also

(2.3) $$C_2 = SC_2 = S\begin{bmatrix} 0 & 0 \\ 0 & D - D_1 \end{bmatrix}.$$

Since $B^*P = B^*$ we have

$$\ker \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}^* \supseteq \ker \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},$$

and thus

$$\operatorname{rank} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = \operatorname{rank} A.$$

Hence

$$\operatorname{rank} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} + \operatorname{rank} \begin{bmatrix} 0 & 0 \\ 0 & D - D_1 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A & B \\ 0 & D - D_1 \end{bmatrix}.$$

Now 'apply' S^{-1} to this last equation and use (2.2), (2.3) to see that $\operatorname{rank} C_1 + \operatorname{rank} C_2 = \operatorname{rank} C$, as desired.

To complete the proof the above is used inductively until we obtain $C = K_1 + K_2 + \ldots + K_m$ relative to $0 = E_0 < E_1 < \ldots < E_{k-1} < E_k = I$ with the following properties:

(i) $\operatorname{rank} K_i > 0$;

(ii) $\operatorname{rank} C = \sum_i \operatorname{rank} K_i$;

(iii) UK_i is suspended by $[E_{i-1}, E_i)$, $i = 1, 2, \ldots, k$;

(iv) K_i cannot be further decomposed with non zero summands relative to any projection in $[E_{i-1}, E_i)$.

Plainly, (iii) and (iv) show that UK_i is in fact suspended by a single projection. The proof is now completed.

REMARK. As observed in [11] there is a version of Corollary 2.2 for upper triangular operator matrices relative to decompositions of both domain space and range space. For example suppose P, Q are self-adjoint projections with $Q < P$ and that T has the form

$$T = Q \overset{P}{\left[\begin{array}{c|c} T_1 & T_2 \\ \hline 0 & T_3 \end{array}\right]}.$$

We construct an associated operator $\widetilde{T}$, so that $\widetilde{T}$ is upper triangular and

$$\widetilde{T} = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ T_1 & T_2 & 0 \\ \hline 0 & T_3 & 0 \end{array}\right].$$

It can be checked that the Lance decomposition of $\widetilde{T}$ provides an associated decomposition of T.

With the ideas above one can obtain a version of Corollary 2.5 for finite rank operators in a weakly closed operator module of Alg $\mathscr{E}$, and hence a strengthening of Lemma 2.1 of [8].

3. OPERATOR VALUED MEASURES

We now make inductive use of Lemma 2.1 to associate with each positive operator C in $B(H)$ a positive operator valued measure. This association will depend only upon the fixed nest $\mathscr{E}$. The construction of Lemma 2.1 has an inherent left continuity property with respect to the weak operator topology. This is expressed in Lemma 3.2 and provides just the continuity property required for extending finitely additive measures to measures.

Let $\mathscr{F}$ be the finite subnest $0 = E_0 < E_1 < \ldots < E_n = I$ of $\mathscr{E}$. Let C be a fixed positive operator on H and decompose C as in Lemma 2.1 with respect to E_1 to obtain $C = C_1 + C_2'$. Next decompose C_2' with respect to E_2 to obtain $C_2' = C_2 + C_3'$, and so on, until we have the following decomposition

$$C = C_1 + C_2 + \ldots + C_n \tag{3.1}$$

associated with $\mathcal{F}$. (Here C_1 has the form $\begin{bmatrix} A & B \\ B^* & D_1 \end{bmatrix}$ and $C_3' = C - C_1$, and so on.) We now define $C_{\mathcal{F}}[E_{i-1}, E_i) = C_i$, $i = 1, 2, \ldots, n$. The next lemma shows that $C_{\mathcal{F}}[E, F)$ is independent of the subnest $\mathcal{F}$, and so we shall denote the common value by $C[E, F)$.

Let $\mathcal{R}(\mathcal{E})$ be the ring of subsets of $\mathcal{E}$ generated by the collection of semi-intervals $[E, F)$ with $E, F \in \mathcal{E}$, $E < F$.

LEMMA 3.1 (i) *The operator $C_{\mathcal{F}}[E_{i-1}, E_i)$ is independent of $\mathcal{F}$, the finite nest containing E_{i-1} and E_i.*

(ii) *The correspondence $[E, F) \to C[E, F)$ extends to a finitely additive positive operator valued function on $\mathcal{R}(\mathcal{E})$.*

Proof. We first claim that the decomposition (3.1) arises independently of the *order* of successive applications of Lemma 2.1. More specifically consider a quadruple subnest $0 = E_0 < E_1 < E_2 < E_3 = I$. Use Lemma 2.1 to decompose C as $C' + C_3'$ relative to E_2. Next decompose C' relative to E_1 as $C' = C_1' + C_2'$. We show that, with the notation used earlier, $C_1' = C_1$, $C_2' = C_2$ and $C_3' = C_3$. That $C_1' = C_1$ should be clear. Since $C_1 + C_2$ is positive and $(C_1 + C_2)E_2 = C'E_2$ it follows, by the minimality property of Lemma 2.1(i), that $C' \leqslant C_1 + C_2$. Hence $C_1' + C_2' \leqslant C_1 + C_2$ and $C_2' \leqslant C_2$. But $C_2'E_2 = C_2E_2$, and so, by minimality again, $C_2 \leqslant C_2'$. Thus $C_2 = C_2'$ and $C_3 = C_3'$. Our original claim now follows easily by induction with the quadruple case.

The proof of (i) is now immediate, because if two finite subnests $\mathcal{F}_1$ and $\mathcal{F}_2$ determine $C_{\mathcal{F}_1}[E, F)$ and $C_{\mathcal{F}_2}[E, F)$ then, from the above, $C_{\mathcal{F}_1}[E, F) = C_{\mathcal{F}_1 \cup \mathcal{F}_2}[E, F) = C_{\mathcal{F}_2}[E, F)$.

To establish (ii) we need only verify that if $E < F < G$ belong to $\mathcal{E}$ then $C[E, G) = C[E, F) + C[F, G)$. This too is an immediate consequence of the claim and its proof.

LEMMA 3.2. *If $E \in \mathcal{E}$ and $E_- = E$ then $C[F, E)$ converges to zero in the weak operator topology as F increases to E with $F < E$.*

Proof. Note that, with respect to the Hilbert space decomposition induced by E, $C[0, E)$ has the form $\begin{bmatrix} A & B \\ B^* & D_1 \end{bmatrix}$, as in Lemma 2.1. Also with respect to the decomposition induced by F $(F < E)$, $C[0, F)$ has the form $\begin{bmatrix} A' & B' \\ B'^* & D_1' \end{bmatrix}$. Moreover, since $E_- = E$, we have $A' \to A$, $B' \to B$ in the weak operator topology as $F \to E$, $F < E$. Thus the monotone increasing net $C[0, F)$ converges in the weak operator topology to an operator $X \leqslant C[0, E)$ which has the form $\begin{bmatrix} A & B \\ B^* & Z \end{bmatrix}$ with respect to

E. But, by the minimality of D_1, $C[0, E) \leqslant X$. Hence $X = C[0, E)$ and the lemma follows.

From the last two lemmas and the basic theory of positive operator valued measures, [2, p. 15], there is a unique positive operator valued set function $C(\Delta)$ defined on the Borel subsets Δ of $\mathscr{E}$ ($\mathscr{E}$ is metrized by the strong operator topology), which coincides with $C[E, F)$ on $\mathscr{R}(\mathscr{E})$, and is such that

$$C(\Delta) = \sum_{i=1}^{\infty} C(\Delta_i) \tag{3.2}$$

whenever Δ is a disjoint union on Borel subsets Δ_i, and convergence is with respect to weak operator topology.

It follows from Lemma 2.1 (iii) and the constructions above that if $UC \in \operatorname{Alg}\mathscr{E}$ then $UC[E, F) \in \operatorname{Alg}\mathscr{E}$ and is suspended by $[E, F)$ for each $E, F \in \mathscr{E}$, $E < F$.

4. A RADON-NIKODYM THEOREM

We now establish some integration theory for nuclear operator valued functions sufficient for our application. No attempt is made at generality.

Let (Ω, Σ, μ) be a sigma finite measure space. A function $f: \Omega \to B_1(H)$ is said to be *measurable* if the function $t \to (f(t)x, y)$, $t \in \Omega$, is measurable for every pair of vectors x, y in H. In view of our separability assumption on H it would suffice here to require measurability for x, y in a dense subset. If f is such a measurable function then, again by separability, $t \to \|f(t)\|_1$ is measurable. The function f is said to be *integrable* if $t \to \|f(t)\|_1$ is integrable. Simple applications of Lebesgue's dominated convergence theorem reveal that for an integrable function f the sesquilinear form $[\,,\,]$ defined by

$$[x, y] = \int_{\Omega} (f(t)x, y)\,\mathrm{d}\mu(t)$$

satisfies

$$\sum_{n=1}^{\infty} |[x_n, y_n]| \leqslant \int_{\Omega} \|f(t)\|_1\,\mathrm{d}\mu(t)$$

for every pair of orthonormal sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$. Hence there exists a nuclear operator T such that $[x, y] = (Tx, y)$ for $x, y \in H$. The operator T is called the integral of f and we write $T = \int_{\Omega} f\,\mathrm{d}\mu$.

THEOREM 4.1. *Let (Ω, Σ, μ) be a sigma finite measure space and let $C(\Delta)$ be an operator valued measure on Σ such that $C(\Omega)$ is nuclear and $C(\Delta) = 0$ whenever $\Delta \in \Sigma$*

and $\mu(\Delta) = 0$. *Then there exists a positive integrable nuclear operator valued function* $D(t)$ *such that* $C(\Delta) = \int_\Delta D(t)\mathrm{d}\mu(t)$ *for all* $\Delta \in \Sigma$.

Proof. Let Q denote a subset of H consisting of all linear combinations, with coefficients in $\mathbf{Q} + i\mathbf{Q}$ of a fixed orthonormal basis $e_1, e_2, \ldots$. For x, y in H let $\mu_{x,y}$ denote the scalar complex measure on Σ defined by $\mu_{x,y}(\Delta) = (C(\Delta)x, y)$. By the Radon-Nikodym theorem there exists a measurable integrable function $D_{x,y}$ such that $\mu_{x,y}(\Delta) = \int_\Delta D_{x,y}(t)\mathrm{d}\mu(t)$. The derivative $D_{x,y}(t)$ is determined almost everywhere. Thus it is possible to choose a null set N so that for all $t \notin N$ the mapping $x, y \to D_{x,y}(t)$ is a finite and sesquilinear form, over $\mathbf{Q} + i\mathbf{Q}$ on the vector pairs x, y in Q. Also, by the monotone convergence theorem,

$$\int_\Omega \left(\sum_n D_{e_n, e_n}(t) \right) \mathrm{d}\mu(t) = \sum_n \int_\Omega D_{e_n, e_n}(t)\,\mathrm{d}\mu(t) = \tag{4.1}$$

$$= \sum_n \mu_{e_n, e_n}(\Omega) = \sum_n (C(\Omega)e_n, e_n) = \mathrm{tr}(C(\Omega)).$$

Hence we can arrange N so that $\sum_n D_{e_n, e_n}(t)$ is finite for all $t \notin N$. It follows by standard arguments that for each $t \notin N$ there exist a positive nuclear operator $D(t)$ such that $D_{x,y}(t) = (D(t)x, y)$ for all $x, y \in Q$. Set $D(t) = 0$ for $t \in N$. Since Q is dense it follows that $D(t)$ is measurable and, by (4.1), integrable. Since

$$(C(\Delta)x, y) = \mu_{x,y}(\Delta) = \int_\Delta D_{x,y}(t)\mathrm{d}\mu(t) =$$

$$= \int_\Delta (D(t)x, y)\mathrm{d}\mu(t) = \left(\int_\Delta D(t)\mathrm{d}\mu(t)x, y \right) \quad \text{for } x, y \in Q,$$

the theorem follows.

The integral of an integrable function has been defined in a weak sense and such a description could be used to integrate suitable $B(H)$ valued functions. For $B_1(H)$ valued functions however the integral exists in the following, much stronger, sense, and this will be useful.

THEOREM 4.2. *Let* (Ω, Σ, μ) *be a sigma finite measure space and let* $D(t)$ *be an integrable nuclear operator valued function on* Ω. *Then for each* $\varepsilon > 0$ *there exists a measurable partition* $\Delta_1, \Delta_2, \ldots, \Delta_r$ *of* Ω *and* $t_i \in \Delta_i$ *for* $i = 1, 2, \ldots r$ *such that*

$$\left\| \int_\Omega D(t)\mathrm{d}\mu(t) - \sum_{i=1}^r D(t_i)\mu(\Delta_i) \right\|_1 < \varepsilon.$$

Proof. We make the simplifying assumption that $\mu(\Omega) = 1$ and that $\|D(t)\|_1 \leqslant M$ almost everywhere since the theorem follows easily from this special case Let P_n, $n = 1, 2, \ldots$ be finite rank projections such that P_n tends strongly to the identity. If $X \in B_1(H)$ then $P_n X P_n \to X$ in $B_1(H)$. Thus $P_n D(t) P_n \to D(t)$ in $B_1(H)$ for almost every t. In particular there is a measurable set K with $\mu(K) < \frac{\varepsilon}{5M}$ and an integer N_0 such that $\|P_n D(t) P_n - D(t)\|_1 < \frac{\varepsilon}{5}$ for all $n > N_0$ and $t \notin K$. Also there exists an $N > N_0$ such that

$$\left\| \int P_N D(t) P_N \mathrm{d}\mu(t) - \int D(t) \mathrm{d}\mu \right\|_1 < \frac{\varepsilon}{5}.$$

Since $P_N D(t) P_N$ is an integrable operator valued function with values in $B(\mathbf{C}^n)$ it follows from the integration theory for scalar functions that there exists a partition $\Delta_1, \Delta_2, \ldots, \Delta_r$ of Ω such that

$$\left\| \int P_N D(t) P_N \mathrm{d}\mu(t) - \sum_{i=1}^{r} P_N D(t_i) P_N \mu(\Delta_i) \right\|_1 < \frac{\varepsilon}{5}$$

for almost every choice of $t_i \in \Delta_i$, $i = 1, 2, \ldots, r$. We can also assume that $K = \bigcup_{i=1}^{s} \Delta_i$ for some $s \leqslant r$. It follows that

$$\left\| \sum_{i=1}^{r} P_N D(t_i) P_N \mu(\Delta_i) - \sum_{i=s+1}^{r} P_N D(t_i) P_N \mu(\Delta_i) \right\|_1 \leqslant \frac{\varepsilon}{5},$$

$$\left\| \sum_{i=s+1}^{r} P_N D(t_i) P_N \mu(\Delta_i) - \sum_{i=s+1}^{r} D(t_i) \mu(\Delta_i) \right\|_1 \leqslant \frac{\varepsilon}{5}$$

and

$$\left\| \sum_{i=s+1}^{r} D(t_i) \mu(\Delta_i) - \sum_{i=1}^{r} D(t_i) \mu(\Delta_i) \right\|_1 \leqslant \frac{\varepsilon}{5}.$$

Combine the displayed inequalities above and the theorem follows.

5. MAIN RESULT AND APPLICATIONS

THEOREM 5.1. *Let $T \in \mathrm{Alg}_1 \mathscr{E}$. Then there exists a finite positive Borel measure τ on $\mathscr{E}$ and an integrable nuclear operator valued function $E \to T_E$ on $\mathscr{E}$ such that*

(i) $$T = \int_{\mathscr{E}} T_E \mathrm{d}\tau(E),$$

(ii) $$\|T\|_1 = \int_{\mathscr{E}} \|T_E\|_1 \mathrm{d}\tau(E),$$

(iii) $$T_E = ET(I - E_-) \text{ almost everywhere.}$$

Proof. Let $T = UC$ be a polar decomposition of T with U an isometry and C a positive operator. By the construction of Section 3 there is a nuclear operator valued measure $C(\Delta)$ defined on the Borel algebra of $\mathscr{E}$, such that $UC[E, F)$ is suspended by $[E, F)$ whenever $E, F \in \mathscr{E}$, $E < F$. Let τ be the scalar Borel measure on $\mathscr{E}$ defined by $\tau(\Delta) = \mathrm{tr}(C(\Delta))$. Plainly $C(\Delta)$ is absolutely continuous with respect to τ and so, by Theorem 4.1, there exists a positive integrable $B_1(H)$ valued derivative $E \to D_E$ such that $C(\Delta) = \int_{\Delta} D_E \,\mathrm{d}\tau(E)$. Define $T_E = UD_E$. Then $E \to T_E$ is integrable and (i) and (ii) follow.

Let $\mathscr{G}$ be a countable order dense subset of $\mathscr{E}$ and let $\mathscr{I}$ be the collection of intervals $\Delta = (F, G]$ whose endpoints belong to $\mathscr{G}$. To establish (iii) it will be sufficient, in view of the remarks following Definition 2.4, to show that for almost every E we have $\Delta T_E \Delta = 0$ for every projection $\Delta = G - F$ with $\Delta \in \mathscr{I}$ and $E \notin \Delta$. (The notational economy here should cause no confusion.)

Fix M, N in $\mathscr{G}$ with $M < N$ and consider a scalar step function $\varphi(E)$ on $[M, N)$ on the form $\varphi(E) = \sum_{k=1}^{n} a_k \chi_{\Delta_k}(E)$, where $\Delta_k = [E_{k-1}, E_k)$ and $M = E_0 < E_1 < \ldots < E_n = N$ is a finite measurable partition. Since $\int_{\Delta_k} T_E \,\mathrm{d}\tau = UC(\Delta_k)$ is suspended by Δ_k it follows that $\int_{[M,N)} \varphi(E) T_E \,\mathrm{d}\tau$ is suspended by $[M, N)$ and thus that

$$\int_{[M,N)} \varphi(E) \Delta T_E \Delta \,\mathrm{d}\tau = \Delta \int_{[M,N)} \varphi(E) T_E \,\mathrm{d}\tau \; \Delta = 0$$

for every $\Delta \in \mathscr{I}$ which is disjoint from $[M, N)$. Since φ is arbitrary it follows that there is a null set $\Delta_{M,N}$ such that $\Delta T_E \Delta = 0$ for all $E \in [M, N) \setminus \Delta_{M,N}$ and all Δ disjoint from $[M, N)$. Let Δ^* be the union of all the sets $\Delta_{M,N}$ with M, N in $\mathscr{G}$. Then it follows that if $E \notin \Delta^*$ then $\Delta T_E \Delta = 0$ for all $\Delta \in \mathscr{I}$ with $E \notin \Delta$. Thus (iii) is proven, since $\tau(\Delta^*) = 0$.

Recall that an operator $T \in \mathrm{Alg}_1 \mathscr{E}$ is said to be *exactly decomposable* if there exist rank one operators $R_1, R_2, \ldots$ in $\mathrm{Alg}\,\mathscr{E}$ such that $\|T\|_1 = \sum_{i=1}^{\infty} \|R_i\|_1$ and $T = \sum_{i=1}^{\infty} R_i$.

COROLLARY 5.2. (i) *If $\mathscr{E}$ is countable then each T in* $\mathrm{Alg}_1\,\mathscr{E}$ *is exactly decomposable.*

(ii) *Let* $T \in \mathrm{Alg}_1\,\mathscr{E}$ *and let* $\varepsilon > 0$. *Then there exist rank one operators* $R_1, R_2, \ldots$ *in* $\mathrm{Alg}\,\mathscr{E}$ *such that* $T = \sum_{i=1}^{\infty} R_i$ *and* $\sum_{i=1}^{\infty} \|R_i\|_1 < \|T\|_1 + \varepsilon$.

Proof. (i) Theorem 5.1 shows that $T = \sum_{E \in \mathscr{E}} \tau(\{E\})T_E$ and that this sum is exact. Since T_E is nuclear and suspended by a singleton, our remarks following Definition 2.4 show that each T_E is exactly decomposable. This proves (i).

(ii) Note first that if $S \in \mathrm{Alg}_1\,\mathscr{E}$ is suspended by a finite number of points then S is exactly decomposable. This is a consequence of Theorem 5.1 but follows from Corollary 2.2 more directly. Theorems 5.1 and 4.2 show that there is an approximating sum S_1, which is suspended by a finite number of points, such that $\|T - S_1\|_1 < \varepsilon/2$. Similarly obtain $S_2, S_3, \ldots$ each suspended by a finite number of points, such that

$$\|T - (S_1 + \ldots + S_n)\|_1 < \frac{\varepsilon}{2^n}, \qquad n = 1, 2, \ldots$$

$$\|S_n\|_1 < \frac{\varepsilon}{2^{n+1}}, \qquad n = 2, 3, \ldots .$$

Write each S_j as an exact decomposition $S_j = \sum_{i=1}^{\infty} R_i^{(j)}$. Then $T = \sum_{i,j} R_i^{(j)}$ and $\sum_{i,j} \|R_i^{(j)}\|_1 < \|T\|_1 + \varepsilon$.

REMARK. The second part of the corollary shows that every nuclear operator is approximately decomposable, and shows that in the unit ball of $\mathrm{Alg}_1\,\mathscr{E}$ the finite rank operators are dense. This could also be obtained as a consequence of Erdos' density theorem: In the unit ball of $\mathrm{Alg}\,\mathscr{E}$ the finite rank operators are dense in the weak operator topology [5]. This useful result (e.g. see [6], [8]) is usually applied in the equivalent form: there is a net F_α of finite rank operators in $\mathrm{Alg}\,\mathscr{E}$ with $\|F_\alpha\| \leqslant 1$ and $F_\alpha \to I$ in the weak topology. This looks like a bounded approximate identity for the weak operator topology, and in fact provides a (norm) bounded approximate identity for the Banach algebra $(\mathrm{Alg}\,\mathscr{E}) \cap \mathscr{K}$ with the operator norm ($\mathscr{K}$ = the compact operators). In particular factorisation is possible (by means of Cohen's factorisation theorem [3, p. 61]). This algebra is rather interesting, being radical if $\mathscr{E}$ is continuous. All closed ideals can be described by using the methods of [8]. Each closed ideal J of $(\mathrm{Alg}\,\mathscr{E}) \cap \mathscr{K}$ is of the form

$$J = \{X \in (\mathrm{Alg}\,\mathscr{E}) \cap \mathscr{K} \mid (I - \widetilde{E})XE = 0, \text{ all } E \in \mathscr{E}\}$$

where $E \to \widetilde{E}$ is a left continuous order homomorphism of $\mathscr{E}$, with $\widetilde{E} \leqslant E$ for all

E in $\mathscr{E}$. A similar description holds for the closed ideals of the Banach algebra $(\mathrm{Alg}_1\,\mathscr{E}, \|\cdot\|_1)$.

REMARK. It also follows from Corollary 5.2 (ii) that the upper triangular integral (in the usual sense [7]) of an operator $T \in \mathrm{Alg}_1\,\mathscr{E}$ converges to T in the nuclear norm. That is, if $\mathscr{U}_{\mathscr{F}}(T) = \sum E_i T(E_i - E_{i-1})$ is the upper triangular sum associated with a finite subset $\mathscr{F} = \{E_0 < E_1 < \dots < E_n\}$ then $\mathscr{U}_{\mathscr{F}}(T)$ converges $\|\ \|_1$ to the operator T as $\mathscr{F}$ runs through the directed set of all finite subsets.

This contrasts sharply with the well known fact that $\mathscr{U}_{\mathscr{F}}(X)$ need not converge $\|\ \|_1$ for $X \in B_1(H)$ (although it does converge $\|\ \|_p$, $1 < p < \infty$). Indeed the canonical projection from $B_1(H)$ to $\mathrm{Alg}_1\,\mathscr{E}$ is not $\|\ \|_1$ bounded if $\mathscr{E}$ is infinite. Let us digress a moment to indicate that $\mathrm{Alg}_1\,\mathscr{E}$ has *no* complement in $B_1(H)$. The proof is modeled on Newman's proof that H^1 has no complement in L^1 [9]. Specifically we show that if there is a continuous projection $\pi : B_1(H) \to \mathrm{Alg}_1\,\mathscr{E}$ then, by averaging, we can deduce the uniform boundedness of certain canonical projections on $\mathrm{Alg}\,\mathscr{F}$, $\mathscr{F}$ a finite subnest of $\mathscr{E}$, and thus obtain a contradiction. Indeed for a given finite subnest $\mathscr{F}$ let $G_{\mathscr{F}}$ denote the unitary group in $\mathscr{F}''$ (the double commutant) with Haar measure $\mathrm{d}U$. Define

$$\pi_{\mathscr{F}}(X) = \int_{G_{\mathscr{F}}} \int_{G_{\mathscr{F}}} U^* \pi(UXV^*) V \,\mathrm{d}U\,\mathrm{d}V.$$

This exists as a Riemann integral of $\|\ \|_1$ continuous $B_1(H)$ valued functions on $G_{\mathscr{F}} \times G_{\mathscr{F}}$. We have $\|\pi_{\mathscr{F}}\| \leqslant \|\pi\|$, for the operator norms of these mappings, and, since $G_{\mathscr{F}}\mathrm{Alg}_1\,\mathscr{E} = (\mathrm{Alg}_1\,\mathscr{E})G_{\mathscr{F}} = \mathrm{Alg}_1\,\mathscr{E}$ it follows that $\pi_{\mathscr{F}}$ is a projection. Since $\pi_{\mathscr{F}}(WXY) = W\pi_{\mathscr{F}}(X)Y$ for $W, Y \in G_{\mathscr{F}}$ it follows that $\pi_{\mathscr{F}}(SXT) = S\pi_{\mathscr{F}}(X)T$ for $S, T \in \mathscr{F}''$. In particular

$$(E_j - E_{j-1})\,\pi_{\mathscr{F}}(X)\,(E_k - E_{k-1}) = 0$$

for $j > k$. If $\tilde{\pi}_{\mathscr{F}}$ denotes the restriction to operators X with $0 = (E_j - E_{j-1})X(E_j - E_{j-1})$ then it follows that $\tilde{\pi}_{\mathscr{F}}$ is the canonical projection into $\mathrm{Alg}\,\mathscr{F}$. Now we have $\|\tilde{\pi}_{\mathscr{F}}\| \leqslant \|\pi\|$ for all $\mathscr{F}$, which is a contradiction.

THEOREM 5.3. *Let* $T \in \mathrm{Alg}_1\,\mathscr{E}$. *If* T *is dissipative then* T *is exactly decomposable.*

Proof. Recall that an operator is dissipative if $\mathrm{i}(T^* - T) \geqslant 0$. Let $T = \int_{\mathscr{E}} T_E \,\mathrm{d}\tau$ be the decomposition of Theorem 5.1. Since $\mathrm{tr}(T_E) = 0$ when $E_- = E$ we have,

$$\mathrm{tr}(\mathrm{i}(T^* - T)) = \int_{\mathscr{E}} \mathrm{tr}(\mathrm{i}(T_E^* - T_E))\,\mathrm{d}\tau =$$

$$= \int_{\mathscr{D}} \mathrm{i}\,\mathrm{tr}(T_E^* - T_E)\,\mathrm{d}\tau$$

where $\mathscr{D} = \{E : E_- < E\}$. This shows that $\mathscr{D}$ is non void if $T^* - T \neq 0$. Let H_0 be the closed span of $\{(E - E_-)H : E \in \mathscr{D}\}$. This is the subspace on which $\mathscr{E}$ is totally atomic. More precisely, if P is the orthogonal projection onto H_0 then P commutes with $\mathscr{E}$ and if $P \neq 0$ then $\mathscr{E}_0 = \{EP : E \in \mathscr{E}\}$ is a totally atomic nest on H_0. Moreover $\mathscr{E}_1 = \{E(I - P) : E \in \mathscr{E}\}$ is a continuous nest on $H_1 = (I - P)H$ if $P \neq I$. Let us write

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$$

relative to the decomposition $H_0 \oplus H_1$. Since T_4 is also dissipative and belongs to the continuous nest algebra $\operatorname{Alg} \mathscr{E}_1$, by our initial observation T_4 is self-adjoint. Hence $T_4 = 0$. But since T is dissipative this now implies that $T_2 = T_3^*$. Thus $T_2^* T_2 = T_3 T_2 = (I - P)TPT(I - P)$ is a compact self-adjoint operator in a continuous nest algebra, and so $T_2 = 0$. By Corollary 5.3(i) T_1 is exactly decomposable relative to $\mathscr{E}_0$, and this provides an exact decomposition relative to $\mathscr{E}$.

REMARK. The first part of this proof shows that a non zero dissipative nuclear operator cannot possess a continuous nest of invariant subspaces. In fact it is a theorem of Lidskii that the closed range of T is the closed linear span of the principal vectors of T. This is a simple consequence (see [17, p. 149]) of another well known theorem of his, namely that the trace of a nuclear operator is the sum of the eigenvalues counted with their algebraic multiplicity [13], [4, p. 1104], [17, p. 139], [18, Chapter 3], [6]. It is shown in [16] how the formula $\operatorname{tr}(T) = \int \operatorname{tr}(T_E)\, d\tau$ also leads to this result, thereby providing a triangularisation proof. (The triangularisation proof of [6] uses Erdos' density theorem.)

REMARK. If $T \in \operatorname{Alg} \mathscr{E}$ and $C(\Delta)$ is the operator measure for $C = |T|$ then it may happen that $\tau(\Delta) = \operatorname{tr}(C(\Delta))$ is a *locally finite* measure in the sense that $\tau((E,F)) < < +\infty$ for all $E > 0$ and $F < I$ and $0_+ = 0$ and $I_- = I$. In this case we could refer to T as a *locally nuclear* operator. Such an operator admits a representation $T = \int T_E\, d\tau$ which exists, for example, as a weak integral. One can obtain a mild generalisation of Lidskii's trace theorem: If T is locally nuclear with eigenvalues $\lambda_1(T), \lambda_2(T), \ldots$ counted with their algebraic multiplicity, such that $\sum_{i=1}^{\infty} |\lambda_i(T)| < +\infty$ then

$$\sum_{i=1}^{\infty} \lambda_i(T) = \lim \operatorname{tr}((F - E)T(F - E)) \quad \text{as } E \downarrow 0,\ F \uparrow I.$$

It may be of interest to obtain external characterisations of locally nuclear operators and of the *sigma nuclear* operators, where sigma nuclear means $\tau(\Delta)$ is sigma finite.

REMARK. We do not have an example of a nuclear operator T which is not exactly decomposable.

If the measure τ of Theorem 5.1 is discrete then, as in the proof of Corollary 5.2(i), T is exactly decomposable. However there are exactly decomposable operators for which τ is continuous.

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S. C. POWER
Department of Mathematics,
University of Lancaster, Lancaster,
England.

Received July 26, 1982.

APPENDIX 5

ANOTHER PROOF OF LIDSKII'S THEOREM ON THE TRACE

S. C. POWER

An important theorem of Lidskii [5, p. 101] shows that the trace of a nuclear (trace class) operator on a Hilbert space is the sum of the eigenvalues (repeated according to their algebraic multiplicities). Most proofs require developing properties of the determinant on the trace class and some complex function theory. See for example [2], [9] and [10]. An exception is in Erdos [4] where the result is obtained by a reduction to triangular form. The key to the proof is a rather delicate density property: the finite rank operators in the unit ball of a nest algebra are strongly dense [3].

We indicate here an alternative approach which uses the triangular form. The idea is to decompose a nuclear operator T as an integral "along the diagonal" of building block nuclear operators of the form

$$T_E = \left[\begin{array}{c|c} \ddots \; 0 & * \\ \hline 0 & 0 \; \ddots \end{array}\right].$$

It then follows that the trace of T is the integral of the scalar function $E \to \text{trace}(T_E)$. However, by choosing an appropriate basis, the trace of T_E is easily computed (it depends on how T_E meets the diagonal) and this leads to the theorem.

Preliminaries. We fix a separable complex Hilbert space H. We let $\mathscr{E}$ denote a (complete) *nest* of self-adjoint projections on H. Thus $\mathscr{E}$ is a totally ordered (under range inclusion) family which contains the projections 0 and I, and which is closed in the strong operator topology. If $E \in \mathscr{E}$ and $E \neq 0$ then we define E_- as the supremum of the collection of $F \in \mathscr{E}$ with $F < E$. A nest is said to be *simple* if rank $(E - E_-) \leqslant 1$ for all $E \in \mathscr{E}$. The *nest algebra* $\text{Alg}\,\mathscr{E}$ associated with the nest $\mathscr{E}$ is the collection of bounded operators X for which $(I-E)XE = 0$ for all $E \in \mathscr{E}$. We denote the collection of nuclear operators in $\text{Alg}\,\mathscr{E}$ by $\text{Alg}_1\,\mathscr{E}$, and we write $\|X\|_1$ for the nuclear norm.

By saying that a nuclear operator T is *triangularised* we mean that there is a simple nest $\mathscr{E}$ for which $T \in \text{Alg}_1\,\mathscr{E}$. That such a nest exists is a consequence of the invariant subspace theorem for compact operators and a little induction with Zorn's lemma. Such details can be found in [1], [2] or [9]. For each $E \in \mathscr{E}$ we can define the *diagonal coefficient* $\alpha_E(T) \in \mathbb{C}$ as follows. If $E = E_-$ then $\alpha_E(T) = 0$. If $E_- < E$ then $\alpha_E(T)$ is the unique scalar in the spectrum of the rank one operator $(E-E_-)T \mid (E-E_-)H$. The non zero diagonal coefficients are in fact eigenvalues of T. Indeed if $\alpha_E(T) \neq 0$ consider the restriction operator $R = (\alpha_E(T)I - T) \mid EH$. From the definition of the diagonal coefficient we see that R has proper range and so

Received 2 June, 1982.

Bull. London Math. Soc., 15 (1983) 146–148

is not invertible. Thus by Riesz–Schauder theory $\alpha_E(T)$ is an eigenvalue of $T \mid EH$ and therefore of T.

We can give $\mathscr{E}$ the strong operator topology (equivalent to the order topology here) and consider integrals with respect to Borel measures μ on $\mathscr{E}$ as follows. A nuclear operator valued function $E \to X_E$ is said to be *integrable* with respect to μ if $E \to (X_E f, g)$ is measurable for all $f, g \in H$ and if $E \to \|X_E\|_1$ is integrable with respect to μ. In this case $\int X_E d\mu$ exists as the unique nuclear operator implementing the sesquilinear from $f, g \to \int (X_E f, g) d\mu$.

Details of the next decomposition theorem and some applications are in [8]. At the bottom of the proof is a construction of Lance [6, Lemmas 3.2, 3.3] which asserts the theorem when $\mathscr{E}$ has three elements! The general version below is achieved by exploiting (i) induction, (ii) a continuity inherent in Lance's construction and (iii) a natural Radon–Nikodym theorem for nuclear operator valued measures.

THEOREM. *Let $T \in \text{Alg}_1 \mathscr{E}$. Then there exists a finite positive Borel measure τ on $\mathscr{E}$ and an integrable nuclear operator valued function $E \to T_E$ on $\mathscr{E}$ such that*

(i) $T = \int T_E d\tau$,

(ii) $\|T\|_1 = \int \|T_E\|_1 d\tau$,

(iii) $T_E = ET_E(I - E_-)$ *almost everywhere.*

COROLLARY (Lidskii [7]). *The trace of a nuclear operator is the sum of the eigenvalues, counted with their algebraic multiplicities.*

Proof. We first repeat a simple argument ([5, p. 103]) to reduce to the quasinilpotent case. Let $\mathscr{P}_T$ be the closed linear span of all principal vectors for the nuclear operator T which correspond to non zero eigenvalues. Thus $\mathscr{P}_T =$ closed span $\{x \in H \mid (\lambda I - T)^n x = 0 \text{ for some } n > 0, \lambda \neq 0\}$. In view of the Riesz–Schauder theory we can obtain an orthonormal basis $x_1, x_2, \ldots$ for $\mathscr{P}_T$, by successive orthogonalisation of principal vectors, such that

$$\text{trace}(T \mid \mathscr{P}_T) = \sum_{i=1}^{\infty} (Tx_i, x_i) = \sum_{i=1}^{\infty} \lambda_i(T),$$

where $\lambda_1(T), \lambda_2(T), \ldots$ are the eigenvalues of T counted with their algebraic multiplicities. Let P denote the orthogonal projection onto $\mathscr{P}_T$. By the invariance of $\mathscr{P}_T$ we have

$$T = TP + PT(I-P) + (I-P)T(I-P)$$

and so

$$\text{trace}(T) = \text{trace}(T \mid \mathscr{P}_T) + \text{trace}\big((I-P)T(I-P)\big).$$

Since the operator $(I-P)T(I-P)$ can have no non-zero eigenvalues (by Riesz–Schauder theory) it is now sufficient to establish the corollary in the case when T has no non-zero eigenvalues.

Let T be nuclear and quasinilpotent. We may assume, by our earlier comments, that $T \in \text{Alg}_1 \mathscr{E}$ with $\mathscr{E}$ a simple nest. The theorem applies and there exists a measurable function $E \to T_E$ and a Borel measure τ on $\mathscr{E}$ satisfying (i), (ii) and (iii).

Let $\mathscr{D}$ be the countable set $\{E \in \mathscr{E} \mid E_- < E\}$. Then, by (iii), trace$(T_E) = 0$ for almost every $E \in \mathscr{E} \setminus \mathscr{D}$. So by (i), and Lebesgue's dominated convergence theorem,

$$\operatorname{trace}(T) = \int_{\mathscr{E}} \operatorname{trace}(T_E)\,d\tau = \sum_{E \in \mathscr{D}} \operatorname{trace}(T_E)\tau(\{E\}). \tag{1}$$

Now if $F_- < F$ then by (i) and (iii)

$$(F-F_-)T(F-F_-) = \int_{\mathscr{E}} (F-F_-)T_E(F-F_-)\,d\tau = (F-F_-)T_F(F-F_-)\tau(\{F\}).$$

Hence

$$\begin{aligned}\operatorname{trace}(T_F)\tau(\{F\}) &= \operatorname{trace}\big((F-F_-)T_F(F-F_-)\big)\tau(\{F\}) \\ &= \operatorname{trace}\big((F-F_-)T(F-F_-)\big),\end{aligned} \tag{2}$$

which is the diagonal coefficient of T at F. Since T is quasinilpotent these coefficients are zero and so, by (1) and (2), trace$(T) = 0$, completing the proof.

REMARKS. (i) In fact (1) and (2) show directly that the trace is the sum of the diagonal coefficients.

(ii) Since the integrals above actually exist as $\| \ \|_1$ limits of approximating sums it can be deduced from the theorem that the finite rank operators are $\| \ \|_1$ dense in the unit ball of $\text{Alg}_1 \mathscr{E}$. This fact, which was used in [4], also follows from Erdos' density result.

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Cartmel College,
University of Lancaster,
Bailrigg, Lancaster,
LA1 4YL.

APPENDIX 6

A Hardy-Littlewood-Fejér Inequality for Volterra Integral Operators

S. C. POWER

For an operator T on the Hilbert space $\ell^2(\mathbf{N})$ that is upper triangular with respect to the standard basis, the following inequality holds,

$$\sum_{j \geq i} \frac{|t_{ij}|}{1+j-i} \leqq \pi \|T\|_1. \tag{1}$$

Here (t_{ij}) denotes the representing matrix of T (and so $t_{ij} = 0$ for $i > j$), and $\|T\|_1$ denotes the trace of $(T^*T)^{1/2}$. This result was obtained by Shields [9] as a natural analogue of the Hardy-Littlewood-Fejér inequality

$$\sum_{n=0}^{\infty} \frac{|\hat{h}(n)|}{n+1} \leqq \pi \|h\|_1, \tag{2}$$

for the Fourier coefficients $\hat{h}(n)$ of a function h in the usual Hardy space H^1 of the circle (see [5, page 70]). Thus the upper triangular operators in the Schatten class C_1, the space of operators T for which $\|T\|_1$ is finite, play the role of the space H^1. The space C_1 is referred to as the space of trace class, or nuclear, operators.

In Theorem 1 we give a version of (1) for an integral operator on $L^2(\mu)$ (where μ is a σ-finite Borel measure on the real line) whose kernel function is upper triangular in the obvious sense. Two special cases, where μ is counting measure for the integers, and where μ is Lebesgue measure on $\mathbf{R}$, resolve problems raised in [9]. Shields' account, which prompted this note, should be consulted for a full historical perspective on the ideas interlacing (1) and (2).

A crucial step [9, Lemma 3] used in obtaining (1) is the factorization $T = AB$, with A,B upper triangular Hilbert-Schmidt operators such that $\|T\|_1 = \|A\|_2\|B\|_2$, where $\|X\|_2$ denotes the Hilbert-Schmidt norm $(\mathrm{tr}(X^*X))^{1/2}$. After this the proof runs in perfect parallel with the proof of (2) that is based on the Riesz factorization $h = h_1 . h_2$, with h_1,h_2 functions in H^2 such that $\|h\|_1 = \|h_1\|_2\|h_2\|_2$. Our method is different and rests on a decomposition of an upper triangular integral operator of trace class into a sum of rank one upper triangular operators, with control of the $\| \ \|_1$ norms (Lemma 2). This approach resembles that used in the atomic and molecular theory of analytic functions [2]. In that theory the boundedness of an op-

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eration with respect to a "one norm" is first easily checked for special molecule functions and then shown to hold true in general by invoking a decomposition theorem which expresses each analytic function as a sum of molecules. It is the decomposition theorem that embraces the hard analysis, and that is the case here. The molecular (atomic?) analogues are the rank one summands.

The inequalities. Let μ denote a σ-finite Borel measure on the real line $\mathbf{R}$, and let $h(x,y)$, $k(x,y)$ denote kernel functions which induce bounded integral operators Int h, Int k on $L^2(\mu)$ in the sense of Halmos and Sunder [4, page 17].

Theorem 1. *If* $h(x,y) = 0$ *for all* $x > y$, *and if* $k(x,y) \geqq 0$ *for* $x \leqq y$, *then*

$$\int_{\mathbf{R}}\int_{\mathbf{R}} |h(x,y)|\,k(x,y)\,d\mu d\mu \leqq \|\text{Int } k\|\|\text{Int } h\|_1. \tag{3}$$

Remarks. The substance of the inequality (3) (and similarly for (1) or (2)) is that it is an assertion for $|h(x,y)|$. Moreover, (3) may fail if $h(x,y)$ is not upper triangular. This is a consequence of the unboundedness (when $L^2(\mu)$ has infinite dimension) of the mapping Int $k \to$ Int $|k|$ with respect to the trace class norm. This in turn is easily derived from the unboundedness of the upper triangular projection mapping with respect to the trace class norm. (On the other hand, if $k(x,y)$ is upper triangular one can drop the upper triangular assumption on h and (3) is valid.)

Notation. Let $\mathscr{E}$ denote the natural nest of distinct projections on $L^2(\mu)$ corresponding to (perhaps not all) intervals of the form $(-\infty,x)$ and $(-\infty,x]$, together with the projections O and I. Recall that the nest algebra Alg $\mathscr{E}$ is the family of operators which leave invariant each projection in $\mathscr{E}$. Thus the operator Int h of Theorem 1 belongs to Alg $\mathscr{E}$. A converse of this also holds [9, Proposition 1]. For a rank one integral operator this coincides with a special case of the characterization (Ringrose [8]) of rank one operators in a general nest algebra. Differently said, the following three assertions coincide: (a) The rank one operator $u \otimes v$ belongs to Alg $\mathscr{E}$. (b) There exists a projection E in $\mathscr{E}$ for which $Ev = v$ and $(I - E_-)u = u$, where E_- is the supremum of F in $\mathscr{E}$ with $F < E$. (c) The integral operator Int h, with $h(x,y) = v(x)u(y)$, is upper triangular.

The following lemma is the key to the proof of Theorem 1.

Lemma 2. *Let* Int h *be a trace class integral operator in* Alg $\mathscr{E}$, *and let* $\varepsilon > 0$. *Then there exist rank one operators* $T_1, T_2, \ldots$ *in* Alg $\mathscr{E}$ *such that*

(i) Int $h = \sum_{i=1}^{\infty} T_i$,

(ii) $\sum_{i=1}^{\infty} \|T_i\|_1 \leqq \|\text{Int } h\|_1 + \varepsilon$.

Remarks. 1. This lemma is a special case of Corollary 5.2(ii) of [7] which concerns nuclear operators in general nest algebras. The proof is rather involved and uses a Radon-Nikodym theorem for nuclear operator valued measures.

2. A different proof of Lemma 2 can be given, as we now indicate, by appealing to the Erdos density theorem [3]. This important (but nonelementary) result states that the finite rank operators in the unit ball of a nest algebra are dense in the strong operator topology. Consequently (exercise) the finite rank operators in the $\| \|_1$ unit ball of (Alg $\mathscr{E}$)$\cap$ C_1 are $\| \|_1$ dense. For this reason it is enough to establish the lemma for a finite rank operator. But in this case a strong form of the lemma holds in the sense that ε can be taken to be 0. For this fact and its proof see Corollary 2.5 of [7]. The proof rests on a decomposition lemma of Lance [6, Lemma 3.3] for 2 × 2 upper triangular trace class operator matrices.

3. There is a stronger version of Lemma 2 available for countable discrete nests in which we can take $\varepsilon = 0$ and assert equality in (ii). See Corollary 5.2(i) of [7].

Lemma 3. *Let h,k be as in Theorem 1 and suppose that* Int *h has rank one. Then inequality (3) is valid.*

Proof. We have Int $h = u \otimes v$, where u,v belong to $L^2(\mu)$, and $h(x,y)$ is the triangular kernel $v(x)u(y)$. Thus

$$\begin{aligned}\int_{\mathbf{R}}\int_{\mathbf{R}} |h(x,y)|k(x,y)\,d\mu d\mu &= \langle(\text{Int }k)|u|,|v|\rangle\\ &\leqq \|\text{Int }k\|\,\|u\|_2\,\|v\|_2\\ &= \|\text{Int }k\|\,\|\text{Int }h\|_1.\end{aligned}$$

The proof of Theorem 1 now follows. Let Int $h_i = T_i$, with T_i as in Lemma 2, so that $h(x,y) = \sum_{i=1}^{\infty} h_i(x,y)$ almost everywhere. Thus

$$\begin{aligned}\int_{\mathbf{R}}\int_{\mathbf{R}} |h(x,y)|k(x,y)\,d\mu d\mu &\leqq \sum_{i=1}^{\infty}\int_{\mathbf{R}}\int_{\mathbf{R}} |h_i(x,y)|k(x,y)\,d\mu\, d\mu\\ &\leqq \sum_{i=1}^{\infty} \|\text{Int }h_i\|_1\|\text{Int }k\|\\ &\leqq (\|\text{Int }h\|_1 + \varepsilon)\|\text{Int }k\|,\end{aligned}$$

and so (3) follows.

Remark. The constant $\|\text{Int }k\|$ is not necessarily the sharpest bound in (3) (for fixed k) because certain lower triangular perturbations of Int k do not affect the left-hand side. It can be seen from the proofs of Lemma 3 and the theorem that

$$\sup_{E\in\mathscr{E}} \|E(\text{Int }k)(I - E_-)\|$$

is the best possible replacement. Using Arveson's distance formula [1] we can also write this constant as

$$\text{dist}(\text{Int }k, (\text{Alg}_s\mathscr{E})^*).$$

This is the operator norm distance from Int k to $(\mathrm{Alg}_s\mathscr{E})^*$, where $\mathrm{Alg}_s\mathscr{E}$ are the strictly upper triangular operators (those operators X in Alg $\mathscr{E}$ satisfying $QXQ = 0$ for every atomic projection Q).

Shields' inequality (1) follows from Theorem 1 by letting μ be the counting measure on $\mathbf{N}$ and by taking $k(i,j) = (1 + j - i)^{-1}$ for all i,j except the pairs $i,i + 1$, for which $k(i,i + 1) = 0$. This is (essentially) Hilbert's second matrix which has operator norm π. Similarly a version of (1) holds for $\ell_2(\mathbf{Z})$. To obtain natural variants for the real line consider the kernel $k(x,y) = (y - x)^{-1}$ which induces (modulo a constant multiplier) the Hilbert transform on $L^2(\mathbf{R})$, as a singular integral operator, with norm π. Although Int k is not an integral operator in the sense used above, the next corollary follows from Theorem 1 and a little elementary approximation. The operators Int h of this corollary are Volterra integral operators.

Corollary 4. *Let $h(x,y) = 0$ for all $x > y$. Then*

$$\int_{y \geqq x}\!\!\int \frac{|h(x,y)|}{y - x}\, dxdy \leqq \pi\|\mathrm{Int}\, h\|_1. \tag{4}$$

Remark. The constant π is best possible in (4) because π is the operator norm of $EX(I - E)$, where X is the Hilbert transform (with kernel $(y - x)^{-1}$) and E is projection onto $L^2(-\infty,0)$. (A natural proof uses the Fourier-Plancherel transform.)

Shields asks whether the norm exact factorization $T = AB$ mentioned in the introduction holds for trace class operators T in an arbitrary nest algebra. From such a fact would follow alternative proofs of the above results. Let T be a rank one operator in a general nest algebra of the form $x \otimes y$, where $Ey = y$ and $(I - E_-)x = x$. Suppose moreover that $E_- < E$. Then the factorization is valid since one can take $A = e \otimes y$ and $B = x \otimes e$, where e is any unit vector in $E - E_-$. Consequently, in view of Remark 3 above, for (general) nests $\mathscr{E}$ of order type $\mathbf{N}$ or $\mathbf{Z}$ we have the following exact weak factorization for a trace class operator T in Alg $\mathscr{E}$;

$$T = \sum_{i=1}^{\infty} A_iB_i, \qquad \|T\|_1 = \sum_{i=1}^{\infty} \|A_i\|_2\|B_i\|_2, \tag{5}$$

where $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ are rank one operators in Alg $\mathscr{E}$.

Question. Is the exact decomposition (5) valid in an arbitrary nest algebra?

A weaker question still is to ask whether the ε in Lemma 2 can be dispensed with. Equivalently, in the terminology of [7], we ask the following.

Question. Is every trace class triangular integral operator exactly decomposable?

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UNIVERSITY OF LANCASTER—BAILRIGG, LANCASTER, LA1 4YL ENGLAND

Received February 22, 1983

APPENDIX 8

ON IDEALS OF NEST SUBALGEBRAS OF C*-ALGEBRAS

S. C. POWER

[Received 17 October 1983]

One of the attractions of non-self-adjoint operators and operator algebras is given by the connection and the parallels that exist with analytic function spaces and harmonic analysis. These parallels can serve as a source for interesting conjectures. See, for example, work by Arveson [**1, 2**] and Loebl and Muhly [**16**] for algebras and analyticity, and Shields [**23**] and Power [**19**] for operators and harmonic analysis. The non-self-adjoint context we consider here, namely nest subalgebras of C*-algebras, is a setting where analytic function theory and operator algebras combine quite strongly, especially when the ambient C*-algebra is infinite. In this paper we begin an analysis of the norm closed ideals of nest subalgebras of C*-algebras.

The theory of ideals of the algebra of upper triangular $n \times n$ matrices is easily understood. Each ideal I is described by an order homomorphism α from the finite lattice $\{0, 1, \ldots, n\}$ into itself, such that $\alpha(k) \leqslant k$. We write

$$I = I[\alpha] = \{(x_{ij}) \colon x_{ij} = 0 \text{ whenever } i > \alpha(j)\}.$$

This is the space of matrices which vanish below the boundary determined by α. A precise analogue of this result for the weakly closed ideals of a nest algebra was obtained by Erdos and Power [**9**]. Whilst the determination of various norm closed ideals of a nest algebra is of importance (see Ringrose [**20**], Lance [**15**], Erdos [**8**], and Hopenwasser [**14**], for example), the analysis of *all* such ideals for a non-self-adjoint algebra is more natural and tractable in the context of nest subalgebras of C*-algebras. These are the norm topology analogues of nest subalgebras of von Neumann algebras, and have received less attention than their weakly closed brothers. Witness the work of Gilfeather and Larson [**10, 11, 12**] and the literature cited therein.

Our analysis is arranged as follows. In the first section we consider approximately finite C*-algebras and nest subalgebras with respect to a maximal subnest of a (prescribed) diagonal algebra. (It is shown in Proposition 1.6 that such algebras do not depend on the choice of the projection nest). This setting lies closest to that of finite dimensionality. In § 2 we look at C*-algebras of operators on $L^2[0, 1]$ and their Volterra nest subalgebras. The boundary α of an ideal of such an algebra appears as a certain increasing function from $[0, 1]$ to $[0, 1]$. We observe that under fairly natural circumstances, involving simple C*-algebras, to each boundary function α there correspond a minimal ideal $I(\alpha)$ and a maximal ideal $I[\alpha]$. Using natural representations we see that these considerations apply to elementary crossed products, such as $C \otimes T$, where T is the rotation group and C is a commutative C*-algebra of functions on T, and to the Cuntz algebras O_n [**5**]. Section 3 is devoted to the C*-algebra O_2 and its Volterra nest subalgebra. Theorem 3.10 gives an alternative, representation free, description of this algebra.

A.M.S. (1980) subject classification: primary 47D; secondary 46L.
Proc. London Math. Soc. (3), 50 (1985), 314–332.

For many basic examples the quotient of a nest subalgebra by the Jacobson radical may be represented as a commutative function algebra. In the AF case this quotient is seen to be a copy of the diagonal, but in other settings, and in particular for O_2, analytic function algebras may appear, associated with certain maximal ideal points of the diagonal. In this way the ideal theory is tied to the ideal theory of function algebras. Thus the sum of two closed ideals need not be closed, because this phenomenon occurs in the disc algebra. On the other hand, for approximately finite nest subalgebras (as defined in § 1), a variant of Arveson's distance formula, and an inductivity property for ideals, show that such sums are automatically closed (Theorem 1.9). Further consequences for the nest subalgebra A of O_2, obtained by exploiting function theory of the disc algebra, are the following assertions. Ideals that contain the radical are principal ideals (Corollary 3.9). A spectral corona condition, namely,

$$|\hat{a}_1(x)|+\ldots+|\hat{a}_n(x)| \geqslant \delta,$$

for x in the ideal space of $A/\operatorname{rad} A$, provides a necessary and sufficient condition on the n-tuple $a_1, \ldots, a_n$ in A for the solution (in A) of the interpolation problem

$$b_1 a_1+\ldots+b_n a_n = 1.$$

The group of invertible elements of A is pathwise connected.

When the containing C*-algebra is simple it happens that the nest subalgebras we consider are 'ideal irreducible' (see Corollary 1.4 and Theorem 2.3), as in the $n \times n$ matrix case. That is, non-zero closed ideals have non-zero intersection (another algebraic parallel with analytic function spaces). This is probably true in a very wide generality.

This research was completed during a visit to Michigan State University. I would like to thank students and faculty, and in particular, Sheldon Axler, for their warm hospitality and their stimulation.

I am in debt to Geoffrey Price for suggesting Lemma 1.2 and to Alan Hopenwasser and Ken Davidson for some useful suggestions.

Notation. We write $M(n)$ for the C*-algebra of $n \times n$ complex matrices and $N(n)$ for the subalgebra of upper triangular matrices. More generally, if $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, we write $M(\mathbf{n})$ for the standard finite-dimensional C*-algebra $M(n_1) \oplus \ldots \oplus M(n_r)$, and $N(\mathbf{n})$ for its upper triangular subalgebra.

1. *AF nest subalgebras and AF nest algebras*

The results of this section concern a maximal nest subalgebra A of an approximately finite C*-algebra B, and the (closed) ideals of A. It is shown that A is a principal ideal algebra and, in the case when B is simple, that non-zero ideals have non-zero intersection. A variant of Arveson's distance formula leads to the automatic closure of the sum of two ideals. The key property required for all these assertions is the inductivity of the ideals of A.

A nest of projections in a unital C*-algebra is a totally ordered family of self-adjoint projections containing 0 and 1. If L is a nest of projections in a C*-algebra B then we let $\operatorname{Alg} L$ denote the *nest subalgebra*

$$\operatorname{Alg} L = \{b \in B : (1-p)bp = 0,\ p \in L\}$$

that is determined by L.

In this section we fix a unital AF algebra B with an ascending chain of finite-dimensional C*-algebras $B_1, B_2, \ldots$ whose closed union is B. In order to construct a nest L we chose a chain $C_1, C_2, \ldots$ of maximal abelian self-adjoint subalgebras of $B_1, B_2, \ldots$ respectively, and take L to be a *maximal* nest of projections in their union. In fact L is also a maximal subnest in C, the closed union of $C_1, C_2, \ldots$. Any two nest subalgebras constructed in this way are isometrically isomorphic (see Proposition 1.6 and the following discussion), and so we may speak of *the approximately finite nest subalgebra* associated with B and the family $B_1, B_2, \ldots$. In the hyperfinite case, where every B_n is a copy of a matrix algebra, it seems appropriate to refer to the algebra $\operatorname{Alg} L$ as an *approximately finite nest algebra.* Indeed, it follows from Proposition 1.6 that these algebras are the direct limits of directed systems

$$N(n_1) \to N(n_2) \to \ldots$$

of upper triangular matrix algebras. The embeddings indicated here are those that respect the standard nest L_n of projections in the diagonal of $N(n)$. (L_n consists of the projections $p_k = e_{11} + \ldots + e_{kk}$, together with 0, where $e_{11}, \ldots, e_{nn}$ are the diagonal matrix units.) That is, under the above embedding, we obtain a directed system of nests

$$L_{n_1} \to L_{n_2} \to \ldots.$$

(These embeddings are not the usual standard embeddings in the sense of Goodearl [13] or Effros [7, p. 9]. However, they are the natural embeddings that arise when $N(n)$ is represented as an operator algebra on $L^2[0,1]$ in the obvious way, using a partition of $[0,1]$ into n equal subintervals.)

Let us now fix C, L, and $A = \operatorname{Alg} L$ as above. If X is a subspace of B then we define an *X-module* to be a closed subspace of B that is closed under multiplication by elements of X. In particular, an A-module that is contained in A is a (two-sided closed) ideal of A. Since the analysis of A-modules is similar to that of ideals we shall consider this generality. Moreover, modules appear naturally as coefficient spaces for a nest subalgebra of O_2 (see (3.10)). The following terminology is convenient, and the concept is crucial.

DEFINITION 1.1. A closed subset I of B is said to be *inductive* if I is the closed union of the subsets $I \cap B_n$, for $n = 1, 2, \ldots$.

Thus C is an inductive m.a.s.a. An elementary C*-algebraic argument [25, p. 21], using the isometric nature of injective maps, gives the well-known result that closed ideals of AF algebras are inductive. We show that more is true. Any C-module (and therefore any ideal of A) is inductive.

In the proof of the next lemma we will use the fact that when B_n is a factor we have

$$B = \operatorname{span}\{vx \colon v \in B_n,\ x \in B_n^c\}. \tag{1.1}$$

Here X^c signifies the commutant of X in B. To see this note first that B_n^c is the closed span of $B_n^c \cap B_r$, for $r = n, n+1, \ldots$ (see [25, p. 11]). Since the span of (1.1) is closed it suffices to verify that

$$B_r = \operatorname{span}\{vx \colon v \in B_n,\ x \in B_n^c \cap B_r\},$$

for $r = n, n+1, \ldots$. By our hypothesis, and standard arguments, it is enough to consider the case where $B_n = M(n)$, appearing as a standard unital subalgebra of

$B_r = M(\mathbf{n})$. Thus $\mathbf{n} = (n_1, \ldots, n_s)$, $n_i = nk_i$ for some natural numbers $k_1, \ldots, k_s$, and $M(n)$ is embedded with multiplicity $k_1 + \ldots + k_s$ in the natural (standard) sense. For the case where $s = 1$ the required assertion is readily verified, and the general case follows naturally from this.

LEMMA 1.2. *Let $1 \leqslant n < r$, let $e_1, \ldots, e_p$ be the minimal projections of $B_n^c \cap B_r$, and define*

$$\varphi_{n,r}(x) = \sum_{i=1}^{p} e_i x e_i, \quad \text{for } x \in B.$$

Then $\varphi_n(x) = \lim_r \varphi_{n,r}(x)$ exists and may be written as $\varphi_n(x) = \sum_{i=1}^{l} v_i d_i$ where $v_1, \ldots, v_l$ are the matrix units of B_n, and d_i is in the closed span of $C_{n+1}, C_{n+2}, \ldots$.

Proof. Suppose first that B_n is a factor, with matrix units $v_1, \ldots, v_l$. Using (1.1) write x in B in the form $\sum_{j=1}^{l} v_j x_j$ with x_j in B_n^c. Then

$$\varphi_{n,r}(x) = \sum_{j=1}^{l} v_j \varphi_{n,r}(x_j).$$

As noted earlier, B_n^c is an approximately finite C*-algebra. Indeed, it is the closed union of $B_n^c \cap B_r$, for $r \geqslant n$. Thus, as $r \to \infty$, $\varphi_{n,r}(x_j)$ converges to an element of the diagonal of B_n^c.

In the general case B_n possesses minimal central projections $f_1, \ldots, f_t$. Each minimal projection of $B_n^c \cap B_r$ appears as a subprojection of one of these, and so it is clear that $\varphi_{n,r}$ may be decomposed as

$$\varphi_{n,r} = \varphi_{n,r}^{(1)} \oplus \ldots \oplus \varphi_{n,r}^{(t)},$$

where $\varphi_{n,r}^{(j)}$ is defined in terms of the subprojections of f_j, for $1 \leqslant j \leqslant t$. But the lemma has been established for the context of the factors $f_j B_n f_j$ and the corresponding mappings $\varphi_{n,r}^{(j)}$, and so the general case follows.

LEMMA 1.3. *A C-module is inductive.*

Proof. Let M be a C-module and x an element of M with unit norm. For $\varepsilon > 0$ choose n and y in B_n with $\| y - x \| < \varepsilon$. Notice that $y = \varphi_{n,r}(y) = \varphi_n(y)$ and so

$$\| \varphi_n(x) - x \| \leqslant \| \varphi_n(x) - y \| + \| y - x \| \leqslant 2\varepsilon.$$

We have $\varphi_n(x) = \sum_{i=1}^{l} v_i d_i$ as in Lemma 1.2. Since $\varphi_n(x)$ is in M, by the module property, so too is each $v_i d_i$. In fact if e_i, f_i are minimal projections in C_n such that $e_i v_i f_i = v_i$ then

$$v d_i = e_i v_i f_i d_i = \sum_{j=1}^{l} e_i v_j f_i d_j = e_i \varphi_n(x) f_i.$$

Now, for each i, consider the set $\{d \in C^{(n+1)} : v_i d \in M\}$ where $C^{(k)}$ is the closed span of $C_k, C_{k+1}, \ldots$. This is an ideal of $C^{(k)}$ and therefore inductive with respect to $C_k, C_{k+1}, \ldots$. Combining the above we see that $\varphi_n(x)$ is in the closed span of $B_n C_{n+r} \cap M$, for $r = 1, 2, \ldots$, and thus that M is inductive.

COROLLARY 1.4. *Let A be an approximately finite nest subalgebra of the AF algebra B. Then each closed ideal is a principal ideal. If B is simple then proper ideals have proper intersection.*

Proof. We may assume that matrix units have been chosen in all the algebras B_k so that each matrix unit in B_k is a sum of matrix units in B_{k+1} (see, for example, [25, p. 14]). Let I be a proper closed ideal of A. Consider the sequence of matrix units $v_1, v_2, \ldots$, which successively exhausts the matrix units of $I \cap B_k$, for $k = 1, 2, \ldots$. Form a subsequence $w_1, w_2, \ldots$ by successively striking out v_k that are 'subordinate' to a previous matrix unit. In this way we obtain a sequence $w_1, w_2, \ldots$ such that if $w_k = \sum_i v_{l,i}$ with $v_{l,i}$ matrix units in B_l then no $v_{l,i}$ appears in the sequence $w_{k+1}, w_{k+2}, \ldots$. The resulting sequence has the following two properties.

(i) The ideal generated by $w_1, w_2, \ldots$ coincides with I. In fact since ideals are inductive, by Lemma 1.3, we need only show that the ideal contains each v_k. But to each v_k there exists a w_l to which v_k is subordinate. That is $v_k = q_k w_l p_k$ where $q_k = v_k v_k^*$ and $p_k = v_k^* v_k$ are the final and initial projections of v_k. These projections belong to C and so the assertion (i) is justified.

(ii) Fix k and assume that $w_k = \sum_i v_{l,i}$ as above. Then, if $p_{l,i}$ and $q_{l,i}$ are the initial and final projections of $v_{l,i}$, we have, for each i,

$$q_{l,i} w_j p_{l,i} = 0 \quad \text{for } j < k.$$

This should be clear after a moment's thought. The raison d'être of the deletion process is that these equalities remain true for all w_j in $B_1, B_2, \ldots, B_l$, with the unique exception of w_k.

We claim that I is the principal ideal, $I(x)$ say, generated by the element

$$x = \sum_{k=1}^{\infty} \frac{w_k}{2^k}.$$

By (i) above it suffices to show that w_k is in $I(x)$. However, we see from (ii) that the norm of

$$\sum_i q_{l,i} x p_{l,i} - \frac{w_k}{2^k}$$

tends to zero as l tends to infinity, and now the claim follows.

We now show that non-zero ideals I,J of A have proper intersection when B is simple.

By the inductivity of ideals there exists an n such that $I \cap B_n$ and $J \cap B_n$ contain non-zero matrix units u and v respectively. Let $e_1, \ldots, e_r$ be the minimal projections of C_n arranged in the order determined by L. That is, e appears before f if and only if there exists p in $L \cap C_n$ such that $pf = 0$ and $pe = e$. In this circumstance we easily see that $eBf \subset A$. (Indeed, for any q in L we have the alternative $qf = 0$ or $qe = e$.) Let e denote the initial projection of u and let f denote the final projection of v, and suppose for the moment that e appears before f. Since B is simple, $\{0\} \neq eBf$ [25, Chapter 1], and so there is a non-zero element exf in A. Thus $uexfv$ is a non-zero element of $I \cap J$. If f appears before e, then the initial projection of v appears before the final projection of u and so the above argument, with u and v switched, is valid.

REMARK. In fact the proof above shows that C-modules are singly generated.

The uniqueness of A

We next show that the algebra A does not depend on the particular choice of maximal subnest L of the diagonal algebra C.

Elementary arguments show that $L = \bigcup L_n$ where $L_n = L \cap B_n$. In fact we claim that

$$A = (\mathrm{Alg}\, L) \cap B = \text{closed span}\{(\mathrm{Alg}\, L_n) \cap B_n\}. \tag{1.2}$$

Because A is itself inductive this amounts to the claim that

$$(\mathrm{Alg}\, L) \cap B_n \supset (\mathrm{Alg}\, L_n) \cap B_n,$$

the reverse inclusion being clear. To see this pick $q \in L \backslash L_n$ and *consecutive* projections p_1, p_2 of L_n, such that $p_1 < q < p_2$. Note that if $x \in B_n$ then

$$(1-q)xq = (1-q)(p_2 x p_1 + (1-p_2)xp_2)p.$$

Thus if x also belongs to $\mathrm{Alg}\, L_n$ then

$$(1-q)xq = (1-q)(1-p_1)(p_2 x p_1)q = (1-q)p_2(1-p_1)xp_1 q = 0.$$

Now fix L', another maximal subnest of C, so that (1.2) holds with L' and L'_n in place of L and L_n. The next elementary lemma is needed to construct isomorphisms θ_n between $(\mathrm{Alg}\, L_n) \cap B_n$ and $(\mathrm{Alg}\, L'_n) \cap B_n$ in such a way that θ_m extends θ_n for $m > n$. The procedure is analogous to fundamental C*-algebra arguments of Bratteli [3].

LEMMA 1.5. *Let P and Q be two maximal subnests of a finite-dimensional C*-algebra D and let R be a maximal subnest of a C*-algebra D_1 contained in D such that $R \subset P \cap Q$. Then there exists a unitary element u in D such that $u^* \mathrm{Alg}\, Pu = \mathrm{Alg}\, Q$ and $u^* x u = x$ for all x in $(\mathrm{Alg}\, R) \cap D_1$.*

Proof. Let $E = \{e_1, \ldots, e_v\}$ (respectively $F = \{f_1, \ldots, f_v\}$) be maximal families of minimal projections in P^{cc} (respectively Q^{cc}) with the ordering determined by P (respectively Q). Similarly, let $g_1, \ldots, g_\mu$ be a maximal set of minimal projections in R^{cc}. Then there exist numbers $1 = j(0) < j(1) < \ldots < j(\mu) = v$ such that

$$g_i = \sum_{k=j(i-1)}^{j(i)} p_k = \sum_{k=j(i-1)}^{j(i)} q_k, \quad \text{for } i = 1, \ldots, \mu.$$

Clearly there is a unitary element v_i in $g_i D g_i$ such that $v_i^* E_i v_i = F_i$, as unordered sets, where

$$E_i = \{e_{j(i-1)}, \ldots, e_{j(i)}\}, \quad F_i = \{f_{j(i-1)}, \ldots, f_{j(i)}\}, \quad \text{for } i = 1, \ldots, \mu.$$

Moreover, v_i can be chosen so that if e, e' are *equivalent* projections in E_i and e appears before e', then $v_i^* e v_i$ appears before $v_i^* e' v_i$ in the ordered set F_i. (Use the transposition unitaries which exchange such projections and leave the other elements of E_i fixed.) Set $v = v_1 \oplus \ldots \oplus v_\mu$. Then v is a unitary element and

$$E = \{e_1, \ldots, e_v\} = \{v^* f_1 v, \ldots, v^* f_v f\} = v^* F v$$

as unordered sets, and such that if E' is an ordered subset of equivalent projections in E, then E' appears as an ordered subset of the ordered set $v^* F v$. Let $p_1, \ldots, p_v$ and $q_1, \ldots, q_v$ be the non-zero projections in P and Q respectively.

Thus

$$p_l = \sum_{m=1}^{l} e_m, \quad q_l = \sum_{m=1}^{l} f_m, \quad \text{for } l = 1, \ldots, \mu.$$

By our construction of v, if z is a minimal central projection of D then

$$\{zp_l\}_{l=1}^{v} = \{zv^* q_l v\}_{l=1}^{v}.$$

(Indeed, if e, e' are in E and ze and ze' are non-zero, then e and e' are equivalent.) Thus $z \operatorname{Alg} P = z \operatorname{Alg}(v^*Qv) = zv^*(\operatorname{Alg} Q)v$, and hence $\operatorname{Alg} P = v^* \operatorname{Alg} Qv$, since both algebras contain the central projections. Now because $R \subset P \cap Q$ it follows that the mapping $x \to v^*xv$ defines an automorphism of $(\operatorname{Alg} R) \cap D_1$ which fixes the projections in R. This automorphism is implemented by a unitary element d in R^{cc} (finite-dimensional exercise). That is, $v^*xv = d^*xd$ for appropriate x. Set $u = vd^*$ and the lemma is proved.

PROPOSITION 1.6. *$\operatorname{Alg} L$ and $\operatorname{Alg} L'$ are isometrically isomorphic.*

Proof. Let $A_n = (\operatorname{Alg} L_n) \cap B_n$ and $A'_n = (\operatorname{Alg} L'_n) \cap B_n$. We need only show that there exist unitary operators u_n in B_n such that $u_n^* A_n u_n = A'_n$ and the automorphic action of u_{n+1} on A_{n+1} extends that of u_n on A_n. By Lemma 1.5, u_1 exists. Assume that $u_1, \ldots, u_n$ have been constructed. Let $A''_{n+1} = u_n^* A_{n+1} u_n$ and $L''_{n+1} = u_n^* L_{n+1} u_n$, so that $L''_{n+1} \cap L'_{n+1}$ contains L'_n. By Lemma 5.1 there exists a unitary element v_{n+1} in B_{n+1} such that $v_{n+1}^* A'_{n+1} v_{n+1} = A''_{n+1}$ and such that the automorphism for v_{n+1} fixes A'_n. Thus

$$v_{n+1} u_n^* A_{n+1} u_n v_{n+1}^* = A_{n+1} \quad \text{and} \quad v_{n+1} u_n^* A_n u_n v_{n+1}^* = u_n^* A_n u_n,$$

since $A'_n = u_n^* A_n u_n$. Set $u_{n+1} = u_n v_{n+1}^*$, and the induction step is complete.

To complete our original assertion, that approximately finite nest subalgebras depend only on the chain $B_1, B_2, \ldots$, we need to show that $\operatorname{Alg} L$ is isometrically isomorphic to $\operatorname{Alg} \tilde{L}$ when $\tilde{L}$ is a maximal subnest of the union of $\tilde{C}_1, \tilde{C}_2, \ldots$, another chain of maximal abelian subalgebras. This is now straightforward. There is an automorphism φ of B such that $\varphi(\tilde{C}_n) = C_n$. Since $\operatorname{Alg} \tilde{L}$ and $\operatorname{Alg} \varphi(\tilde{L})$ are isometrically isomorphic, and, by Proposition 1.6, $\operatorname{Alg} \varphi(\tilde{L})$ and $\operatorname{Alg} L$ are similarly isomorphic, we have finished. (Notice, however, that we have not shown that the isomorphism class of $\operatorname{Alg} L$ is independent of $B_1, B_2, \ldots$ although this is probably true.)

Sum of ideals and modules

Let $0 = p_0 < p_1 < \ldots < p_v = 1$ be the canonical subnest associated with the algebra $N(\mathbf{n})$. Furthermore, let α be a mapping from $\{0, 1, \ldots, v\}$ into itself with $\alpha(i) \leqslant \alpha(j)$ for $i \leqslant j$, and set

$$I[\alpha] = \{x \in M(\mathbf{n}) \colon (1 - p_{\alpha(i)}) x p_i = 0,\ i = 0, 1, \ldots, v\}.$$

We omit the elementary verifications that $I[\alpha] N(\mathbf{n}) \subset I[\alpha]$, $N(\mathbf{n}) I[\alpha] \subset I[\alpha]$, and that all $N(\mathbf{n})$-modules in $M(\mathbf{n})$ arise in this fashion. Note that $I[\alpha]$ is an ideal if $\alpha(i) \leqslant i$ for all i. The following lemma is a variation on a theme of Arveson [2]. The essentials of the proof can be found in [**18**].

LEMMA 1.7. *For x in $M(\mathbf{n})$ the following distance formula holds*

$$\operatorname{dist}(x, I[\alpha]) = \max\{\| (1 - p_{\alpha(i)}) x p_i \| \colon i = 1, \ldots, v\}.$$

LEMMA 1.8. *Let I_1, I_2 be two $N(\mathbf{n})$ modules and let $x_1 \in I_1$, $x_2 \in I_2$. Then*

$$\operatorname{dist}(x_1 + x_2, I_1 \cap I_2) = \max\{\operatorname{dist}(x_1, I_1 \cap I_2), \operatorname{dist}(x_2, I_1 \cap I_2)\}.$$

Proof. If α_1 and α_2 are the associated boundary maps for I_1, I_2 respectively, then $I_1 \cap I_2 = I[\alpha]$ where $\alpha(i) = \min\{\alpha_1(i), \alpha_2(i)\}$, for $i = 0, 1, \ldots, v$. Also if $k \in \{1, 2\}$ and

$\alpha(i) = \alpha_k(i)$ then $(1-p_{\alpha(i)})x_k p_i = 0$. Hence the set of numbers $\|(1-p_{\alpha(i)})(x_1+x_2)p_i\|$, with $i = 1, \ldots, \nu$, coincides with the numbers $\|(1-p_{\alpha(i)})x_k p_i\|$, for $k = 1, 2$ and $i = 1, \ldots, \nu$. The lemma now follows from Lemma 1.7.

THEOREM 1.9. *Let I_1 and I_2 be closed modules for an approximately finite nest subalgebra. Then I_1+I_2 is closed. Moreover, if $x_1 \in I_1$ and $x_2 \in I_2$ then*

$$\operatorname{dist}(x_1+x_2, I_1 \cap I_2) = \max\{\operatorname{dist}(x_1, I_1 \cap I_2), \operatorname{dist}(x_2, I_1 \cap I_2)\}.$$

Proof. We may assume, by Proposition 1.6, that for a given k, $B_k = M(\mathbf{n}_k)$ and $A_k = B_k \cap A = N(\mathbf{n}_k)$, so that the distance formula of Lemma 1.8 holds for the A_k-modules, $I_1 \cap B_k$ and $I_2 \cap B_k$. Since, by Lemma 1.3 the module $I_1 \cap I_2$ is inductive, this gives the required distance formula for I_1, I_2. This formula shows that $I_1/I_1 \cap I_2 + I_2/I_1 \cap I_2$ is a closed subspace of the quotient space $B/I_1 \cap I_2$, and so I_1+I_2 is norm closed, which completes the proof.

REMARKS. 1. An elementary consequence of the inductivity of ideals is that the radical and the commutator ideal of an approximately finite nest subalgebra A coincide with the closed union of $\operatorname{rad}(A \cap B_n)$, for $n = 1, 2, \ldots$. The elements of this ideal are characterized as those elements of A that satisfy a natural Ringrose-type criterion (see [20]) with respect to finite partitions induced by the nest. Also we have $A = C + \operatorname{rad} A$.

2. For general nest subalgebras of C*-algebras sums of ideals need not be closed, and $A/\operatorname{rad} A$ need not be isomorphic to the diagonal algebra C, even when this quotient is known to be commutative. We shall see this in § 3. However, the following natural example shows this, and is of independent interest. Verifications are left to the reader.

Let B be the operator algebra on $L^2(T)$ generated by the continuous functions $C(T)$, acting as multiplication operators, and the Hardy space projection p. These are the usual spaces and operators associated with the circle T. Let E be the discrete nest consisting of the projections 0, 1 and p_n, with $n \in \mathbb{Z}$, where p_n has range equal to the closed span of $\{z^k : k \leqslant n\}$. The algebra $A = B \cap \operatorname{Alg} E$ is the algebra of operators in B whose representing matrices are upper triangular. The commutator ideal of B is equal to K, the space of compact operators. (This, and other facts about B, can be found in [6], for example.) The radical of A, which is also the commutator ideal, is the algebra of *strictly* upper triangular compact operators. The quotient B/K is isomorphic to $C(T) \oplus C(T)$ under a map that sends the coset of multiplication by z to $z \oplus z$, and that of p to $0 \oplus 1$. The quotient $A/\operatorname{rad} A$ is isomorphic to a function algebra on

$$\bar{\mathbb{D}}_1 \cup \mathbb{Z} \cup \bar{\mathbb{D}}_2$$

where $\mathbb{D}_1, \mathbb{D}_2$ are open unit discs. The centres of the closed discs $\bar{\mathbb{D}}_1$ and $\bar{\mathbb{D}}_2$ are identified with the point $-\infty$ and $+\infty$ of the two-point compactification $\bar{\mathbb{Z}}$ of $\mathbb{Z}$, and the function algebra consists of the continuous functions on $\bar{\mathbb{D}}_1 \cup \mathbb{Z} \cup \bar{\mathbb{D}}_2$ that are analytic on the discs. (The topology is the natural one.)

The ideals I of A are specified by a boundary map α from $\bar{\mathbb{Z}}$ to $\bar{\mathbb{Z}}$, such that $\alpha(m) \leqslant \alpha(n)$ if $m \leqslant n$ and $\alpha(n) \leqslant n$, for all m, n in $\bar{\mathbb{Z}}$. If $\alpha(-\infty) = -\infty$ then we must additionally specify an ideal $I_{-\infty}$ of the disc algebra $A(\mathbb{D}_1)$. Similarly, if $\alpha(+\infty) = +\infty$ then we must specify an ideal $I_{+\infty}$. Each ideal is thus determined by a triple $(I_{-\infty}, \alpha, I_{+\infty})$.

Many facts about A may now be deduced from the corresponding facts for the disc algebra. For example,

(i) A is a principal ideal algebra (cf. [2]),

(ii) there are closed ideals in A whose sum is not closed.

2. *Volterra nest subalgebras*

In this section L denotes the Volterra nest of projections on $L^2[0,1]$. Thus L consists of the projections p_t, for $0 \leqslant t \leqslant 1$, where p_t is the orthogonal projection onto $L^2[0,t]$, viewed as a subspace of $L^2[0,1]$. For a fixed C*-algebra B of operators on $L^2[0,1]$ we define the *Volterra nest subalgebra* as the algebra

$$A = B \cap \operatorname{Alg} L = \{x \in P\colon (1-p_t)xp_t = 0,\ 0 \leqslant t \leqslant 1\}.$$

In contrast to the approximately finite nests, L is a complete lattice, and the definition of the boundary map of an A-module (within B) is a natural one.

DEFINITION 2.1. Let I be a closed subspace of B which is an A-module. The *boundary map* of I is the function $\alpha(t)$ from $[0,1]$ to $[0,1]$ defined by

$$\alpha(t) = \inf\{\alpha \in [0,1]\colon (1-p_\alpha)xp_t = 0, \text{ for all } x \text{ in } I\}.$$

PROPOSITION 2.2. *The boundary map α of an A-module satisfies the following:*

(i) $\alpha(0) = 0$;

(ii) *α is increasing;*

(iii) *α is left continuous.*

Proof. (i) and (ii) are clear. To see that α is left continuous at a point t in $(0,1]$ pick any value $\beta < \alpha(t)$. Then there exists an operator x in the module such that $(1-p_\beta)xp_t \neq 0$. Hence $(1-p_\beta)xp_s \neq 0$ for some $s < t$ (by weak operator topology continuity). Hence $\beta \leqslant \alpha(s)$ and (iii) follows.

Under a mild assumption, which we now impose, the boundary maps of modules are characterized by the properties of Proposition 2.2. We assume henceforth that $(p-q)B(p-q) \neq \{0\}$ for all p, q in L with $p > q$. For a function α, satisfying (i)–(iii) above, the following modules have α as a boundary map,

$$I[\alpha] = \{x \in B\colon (1-p_{\alpha(t)})xp_t = 0 \text{ for all } t \in [0,1]\}, \tag{2.1}$$

$$I(\alpha) = \text{closed span}\{x \in B\colon x = p_\beta x(1-p_t) \text{ for some } t \text{ and } \beta < \alpha(t)\}. \tag{2.2}$$

The strict inequality $\beta < \alpha(t)$ should be noted since replacement by $\beta \leqslant \alpha(t)$ may lead to an intermediate module.

If $\pi(t) = t$ denotes the position function on $[0,1]$ then $I(\pi) \subset \operatorname{rad} A$, the radical of A. This is because $I(\pi)$ is generated by operators x for which there exists a positive integer $n = n(x)$ such that $(ax)^n = 0$ for all a in A. If it can be shown that $A/I(\pi)$ is commutative then we have

$$\operatorname{rad} A \supset I(\pi) \supset \operatorname{com} A,$$

where $\operatorname{com} A$ denotes the commutator ideal of A. In the examples below this is the case, and often these ideals coincide (cf. §§ 1 and 3).

We now indicate that for a large class of C*-algebras the module $I(\alpha)$ is the *minimal* module with boundary map α.

Let Γ denote a dense subgroup of the unit circle, and for γ in Γ let u_γ denote the rotation unitary operator such that $(u_\gamma f)(x) = f(x+\gamma)$ for f in $L^2[0,1]$. Here, and later, we identify the circle with $[0,1]$ in the usual way and take addition modulo 1.

The following theorem, although somewhat specialized, applies to a wide class of crossed products and to the nest subalgebra in the next section.

THEOREM 2.3. *Let B be a simple C*-algebra of operators that contains the operators p_γ, u_γ for γ in Γ. If I is a closed module for the Volterra nest subalgebra, with boundary map α, then I contains $I(\alpha)$.*

REMARKS. 1. We omit the uninspiring proof of this theorem, since it follows closely the procedure for showing that $I = I[\alpha]$ when I is a module for $N(n)$. Thus the simplicity of B and operator matrix arguments are used to show that 'small superboundary compressions' of I are equal to the corresponding compressions of B. These compressions are then 'swept out', under the action of A, giving the generators of $I(\alpha)$.

2. If A is a Volterra nest subalgebra, as in Theorem 2.3, then the ideals $I(\alpha)$ have proper intersection. (Compare this with Corollary 1.5.) Just how general is this phenomenon?

3. If we drop the simplicity assumption then the conclusion can fail in various ways. Let B be the C*-algebra B_1+K, where K denotes the compact operators and B_1 denotes the operator algebra generated by $PC = C^*(\{L\})$ (piecewise continuous multiplications) and the full rotation group of unitary operators u_γ, for $\gamma \in T$. The algebra B_1 provides a faithful realization of the crossed product $PC \otimes T$ and is simple because PC has no proper rotation-invariant ideals. Each module I of the Volterra nest subalgebra $B_1 \cap \operatorname{Alg} L$ determines a boundary α and an essential boundary α_e with $\alpha_e \leqslant \alpha$. The function α_e is computed in the Calkin algebra in the obvious way. The appropriate analogue of the theorem is that each module I of B contains $(I(\alpha) \cap K) + I(\alpha_e)$.

On the other hand, let B be the highly non-simple C*-algebra $L^\infty(T) \otimes T$. Rudin [22] has shown the existence of a measurable subset E of the circle for which E and $T\setminus E$ are *permanently positive*. This concept, for a set E, means that the intersection of any finite number of translates of E has positive measure. It readily follows that the characteristic functions for E, and $T\setminus E$, generate different rotation-invariant ideals in $L^\infty(T)$. From this, and the elementary ideal theory for crossed products, we obtain distinct ideals of B and proper modules of the Volterra nest subalgebra without the property of the theorem.

It is natural at this point to mention the non-self-adjoint subalgebra $H^\infty \otimes T$ of $L^\infty(T) \otimes T$ and its Volterra nest subalgebra. The ideal theory here requires knowledge of all the rotation-invariant ideals of H^∞. The ideals $z^n H^\infty$ are the obvious ones. What others are there?

Crossed products

Let Γ be as above, a dense subgroup of the unit interval, and let C be a Γ-invariant C*-subalgebra of $L^\infty[0,1]$ which contains the nest $L_\Gamma = \{p_\gamma: \gamma \in \Gamma\}$. Moreover, suppose that C has no Γ-invariant ideals. Then the crossed product $C \otimes \Gamma$ is a simple C*-algebra isomorphic to the norm closed operator algebra B on $L^2[0,1]$ generated by u_γ, for $\gamma \in \Gamma$, and the multiplication operators associated with C. Theorem 2.3 applies to B and can be used to obtain a characterization of the ideals of the Volterra nest subalgebra $A = B \cap \operatorname{Alg} L = B \cap \operatorname{Alg} L_\Gamma$. Each closed ideal of A is specified by a boundary map α together with a prescription of how the operators of the ideal can 'vanish on the boundary'. A crucial step is to obtain the following coefficient characterization ((2.4) below) of C-modules. (Precisely this kind of characterization was needed by Muhly [17] in a different context concerning analytic crossed products.)

To each x in B can be associated a generalized Fourier series

$$x \sim \sum_{\gamma \in \Gamma} \varphi_\gamma u_\gamma, \quad \text{with } \varphi_\gamma \in C, \tag{2.3}$$

where $\varphi_\gamma = E(xu_\gamma^*)$ and E is the conditional expectation of B relative to the diagonal algebra C. This expectation may be defined by

$$E(x) = \lim_n \sum_{i=1}^{n} (p_i^{(n)} - p_{i-1}^{(n)}) x (p_i^{(n)} - p_{i-1}^{(n)}),$$

where the limit is taken as the size of the Γ-partition, $0 = p_0^{(n)} < p_1^{(n)} < \dots < p_n^{(n)} = 1$, tends to zero. A vital property of the series of (2.3) is that Bochner–Fejér approximation is valid. This means that x is a norm limit of finite sums

$$\sum_{\gamma \in \Omega_n} r_{\gamma,\Omega_n} \varphi_\gamma u_\gamma$$

where $\{r_{\gamma,\Omega_n} : \gamma \in \Omega_n\}$, for $n = 1, 2, \dots$, are finite sets of real numbers. (This can be obtained from the general theory of Banach space-valued almost periodic functions [4].) This kind of Cesaro sum approximation serves as an analogue of inductivity in the AF case. (In fact it may be used to establish inductivity for the C-modules of uniformly hyperfinite AF algebras through their realization as tensor product algebras.) Suppose now that $M(C)$ is the Gelfand space of C and $Z_\gamma \subset M(C)$, for $\gamma \in \Gamma$, is a family of compact subsets. Then

$$I = \{x \in B : x = \sum_\gamma \varphi_\gamma u_\gamma,\ \varphi_\gamma(z) = 0,\ z \in Z_\gamma\} \tag{2.4}$$

is clearly a C-module. The approximation property shows that all such modules arise this way.

3. *A nest subalgebra of O_2*

In [5] Cuntz has shown the importance of the class of C*-algebras O_n, for $n = 2, 3, \dots$, within the theory of infinite C*-algebras. In this section we consider a triangular, non-self-adjoint subalgebra A of O_2. This algebra may be specified by its generators, or as a Volterra nest subalgebra of a natural realization of O_2 on $L^2[0, 1]$. The equivalence of these descriptions is given by Theorem 3.10. The proof requires the inductivity of modules of uniformly hyperfinite nest algebras (cf. § 1) together with a fundamental Cesaro-sum convergence property for the generalized Fourier series of elements of O_2. This convergence property is Lemma 3.8 and, like inductivity, and the Bochner–Fejér summability of the series (2.3), plays a key role in the description of modules for the diagonal. Since most of the basic properties of A follow more readily from the generator specification of A, we shall introduce A in this way and postpone the connections with O_2 until later.

The algebra A and its radical

Let $\alpha, \beta, \gamma, \delta$, with $\alpha < \beta$ and $\gamma < \delta$, be four dyadic points in the unit interval $[0, 1]$ such that $\delta - \gamma = 2^n(\beta - \alpha)$ for some integer n. Then $v = v(\alpha, \beta, \gamma, \delta)$ denotes the natural partial isometry with initial space $L^2[\alpha, \beta]$ and final space $L^2[\gamma, \delta]$, and n is called the *index of dilation* of v. We refer to these operators as the *dyadic partial isometries*. We see that $v(\alpha, \beta, \gamma, \delta)$ belongs to the Volterra nest algebra if and only if one of the following conditions hold:

(i) $n = 0$ and $\gamma \leqslant \alpha$;
(ii) $n < 0$ and $\gamma \leqslant \alpha$;
(iii) $n > 0$ and $\delta \leqslant \beta$.

We define the operator algebra A as the norm closed linear span of the dyadic partial isometries that lie in the Volterra nest algebra $\operatorname{Alg} L$. We see later that the closed algebra B generated by all the dyadic partial isometries is a faithful realization of O_2 and that $A = B \cap \operatorname{Alg} L$.

First we obtain a representation of $A/\operatorname{rad} A$, where $\operatorname{rad} A$ denotes the Jacobson radical of A, as a commutative function algebra.

It was shown by Ringrose [20] that the radical of a full nest algebra of operators may be described as the intersection of certain diagonal ideals (not to be confused with ideals of the diagonal algebra $A \cap A^*$). We give a direct proof of the analogue of this result for A.

The *diagonal ideals* of A are the norm closed ideals I_0, I_1, I_{t+}, I_{t-}, for $0 < t < 1$, defined in terms of the Volterra nest $L = \{p_t\colon 0 \leqslant t \leqslant 1\}$ as follows:

$$I_0 = \{x \in A\colon p_\delta x p_\delta \to 0 \text{ as } \delta \to 0\};$$

$$I_1 = \{x \in A\colon (1-p_\delta)x(1-p_\delta) \to 0 \text{ as } \delta \to 1\};$$

$$I_{t+} = \{x \in A\colon (p_{t+\delta}-p_t)x(p_{t+\delta}-p_t) \to 0 \text{ as } \delta \to 0\};$$

$$I_{t-} = \{x \in A\colon (p_t-p_{t-\delta})x(p_t-p_{t-\delta}) \to 0 \text{ as } \delta \to 0\}.$$

Recall that π is the trivial boundary map $\pi(t) = t$, that $I(\pi)$ is given by (2.2), and that $\operatorname{com} A$ is the ideal generated by the commutators of A.

LEMMA 3.1. *The commutator ideal of A satisfies* $\operatorname{com} A = I(\pi) = \bigcap_r I_r$, *where the intersection is taken over all diagonal ideals I_r.*

Proof. Note first that for $x \in B$, $r \in [0, 1)$, and $s \in [0, 1)$ we have

$$(p_r - p_{r-\delta})x(p_{t+\delta}-p_t) \to 0 \quad \text{as } \delta \to 0. \tag{3.1}$$

Indeed this property follows at once for the dyadic partial isometries that generate B.

Suppose that the operator x belongs to the intersection of the diagonal ideals. Then, by a compactness argument, there exists a partition $1 = q_1 + \ldots + q_n$ by projections q_i of dyadic intervals such that $\| q_i x q_i \| < \varepsilon$ for $i = 1, \ldots, n$. Thus it will follow that $x \in I(\pi)$ if we show that $x - (q_1 x q_1 + \ldots + q_n x q_n)$ belongs to $I(\pi)$. This follows quickly from the definition of $I(\pi)$ and the property of (3.1). Since $I(\pi)$ is contained in each diagonal ideal, it follows that $I(\pi)$ coincides with their intersection.

To see that $A/I(\pi)$ is commutative we need only show that the cosets $v+I$ and $w+I$ commute when I is a diagonal ideal and v and w are dyadic partial isometries in $\operatorname{Alg} L$. Suppose that $I = I_{t+}$. Note that if $v = v(\alpha, \beta, \gamma, \delta)$ then $v+I \neq 0$ if and only if $\alpha = \gamma = t$. Since $v \in A$ it follows that $\delta \leqslant \beta$. Similarly, if $w = v(\alpha', \beta', \gamma', \delta')$ and $w+I \neq 0$, then $\alpha = \alpha'$ and $\alpha' \leqslant \beta'$. But in this case $vw = wv$, so in all cases $v+I$ and $w+I$ commute.

We have shown that $I(\pi) \supset \text{com}\, A$. For the reverse inclusion consider the operator matrix identity

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$$

and it becomes clear that the commutator ideal (in fact even just the linear span of the commutators) contains elements of the form $p_\gamma y(1-p_\gamma)$ where $y \in A$ and γ is dyadic. In view of (3.1) the linear span of such elements is precisely $I(\pi)$. Thus $\text{com}\, A = I(\pi)$ and the proof is complete.

In fact it follows from the proof of Lemma 3.1 that the intersection may be taken over the dyadic points only, and that the quotient norm in $A/\text{com}\, A$ may be computed as

$$\| x + \text{com}\, A \| = \lim_{k \to \infty} \max_j \| q_j x q_j \|, \tag{3.2}$$

where q_j are the projections associated with the intervals

$$[(j-1)2^k, j/2^k], \quad \text{for } j = 1, \ldots, 2^k.$$

It is possible to describe the quotient $A/\text{rad}\, A$ as a commutative function algebra. This algebra contains a copy of the diagonal algebra $A \cap A^*$ (which turns out to be the commutative C*-algebra C generated by dyadic (diagonal) projections) together with disc algebras that appear over dyadic maximal ideal points of C. (Compare with the example of Remark 2 following Theorem 1.9.) Thus all questions concerning ideals that contain the radical are reduced to questions about this function algebra.

Let $M(C)$ denote the maximal ideal space (character space) of C. As a set $M(C)$ is identified with the non-dyadic points $\alpha \in (0, 1)$ together with dyadic pairs $\alpha+, \alpha-$ that correspond to the right limit and left limit evaluation functionals of dyadic points $\alpha \in [0, 1]$. (Of course we do not have $0-$ or $1+$.) Let X be the subset of $M(C) \times \bar{\mathbb{D}}$ containing the points $(t, 0)$, for non-dyadic t, and the sets $\{t\} \times \bar{\mathbb{D}}$, for dyadic characters t, where $\mathbb{D}$ is the open unit disc, and $\bar{\mathbb{D}}$ its closure.

The diagonal algebra C can be realized as the subalgebra of $L^\infty[0, 1]$ generated by the characteristic functions χ of intervals $(\alpha, \beta]$ whose endpoints are dyadic. With no real confusion we let $\hat{\chi}_{(\alpha,\beta]}$ denote both the Gelfand transform, a continuous function on $M(C)$, and the function on X given by

$$\hat{\chi}_{(\alpha,\beta]}(t, z) = \hat{\chi}_{(\alpha,\beta]}(t), \quad \text{where } (t, z) \in X.$$

If t is a dyadic point in $M(C)$ and if f is a function in the disc algebra with $f(0) = 0$, then we define $f_{(t)}$ on X by

$$f_{(t)}((s, z)) = \begin{cases} f(z) & \text{if } s = t, \\ 0 & \text{if } s \neq t. \end{cases}$$

Define $\hat{A}$ to be the function algebra on X generated by the functions $f_{(t)}$ above, and the functions $\hat{\chi}_{(\alpha,\beta]}$.

THEOREM 3.2. (a) $\text{rad}\, A = \text{com}\, A = I(\pi)$.

(b) *$A/\text{rad}\, A$ is naturally isometrically isomorphic to the function algebra $\hat{A}$. This isomorphism associates (the cosets of) the dyadic partial isometries $v = v(\alpha, \beta, \gamma, \delta)$ of A with functions in $\hat{A}$ as follows:*

(i) *if $\alpha = \gamma$ and $\beta = \delta$ then v is mapped to $\hat{\chi}_{(\alpha,\beta]}$;*

(ii) *if* $\alpha = \gamma$ *and* $\delta < \beta$ *then* v *is mapped to* $z^k_{(\alpha+)}$ *where* $-k$ *is the index of dilation of* v;
(iii) *if* $\beta = \delta$ *and* $\gamma < \alpha$ *then* v *is mapped to* $z^k_{(\beta-)}$ *where* k *is the index of dilation of* v;
(iv) *in all other cases* v *is mapped to the zero function.*
(c) *The natural mapping* φ *such that*

$$\varphi\colon A/\operatorname{rad} A \to \bigoplus_r A/I_r,$$

and where the direct sum extends over all the diagonal ideals, is an isometric algebra monomorphism.

Proof. We first show that under the coset correspondence indicated in (i)–(iv) the quotient algebra $A/\operatorname{com} A$ is isometrically isomorphic to $\hat{A}$.

Let A_0 denote the unclosed algebra generated by the dyadic partial isometries v in A. Then $v+\operatorname{com} A$ is non-zero if and only if $v = v(\alpha, \beta, \gamma, \delta)$ with
(i) $\alpha = \gamma$ and $\beta = \delta$, or
(ii) $\alpha = \gamma$ and $\delta < \beta$ or
(iii) $\beta = \delta$ and $\gamma < \alpha$.
We distinguish these three classes by saying that
(i) v is a diagonal operator,
(ii) v is associated with $\alpha+$, and
(iii) v is associated with $\beta-$.
For each $x \in A$ the coset $x+\operatorname{com} A$ can be written almost uniquely in the reduced form

$$x+\operatorname{com} A = \left(d+\sum_{t\in\Omega}\sum_{i=1}^{n(t)} \lambda_{t,i}v_t^i\right)+\operatorname{com} A,$$

where d is an operator in C, Ω is a finite (dyadic) subset of $M(C)$, $\lambda_{t,i}$ are complex numbers, and v_t is a dyadic partial isometry associated with t whose index of dilation has modulus 1. There is some choice available for the v_t but d and $\lambda_{t,i}$ are uniquely determined. It was observed in the last proof that the coset of $v_t v_s$ is zero if $t \neq s$. It follows then that the map θ from $A_0+\operatorname{com} A$ to $\hat{A}$, defined by

$$\theta(x+\operatorname{com} A) = \hat{d}+\sum_{t\in\Omega}\sum_{i=1}^{n(t)} \lambda_{t,i}z^i_{(t)}, \tag{3.3}$$

is well defined and a homomorphism.

We now show that θ is isometric. Let $x \in A_0$ have a coset represented as above. Fix an integer k and let q_j be the projections of (3.2). Then

$$\begin{aligned}\| x+\operatorname{com} A \| &= \left\| \sum_{j=1}^{2^k} q_j x q_j+\operatorname{com} A \right\| \\ &= \max_j \| q_j x q_j+\operatorname{com} A \| \\ &= \max_j \left\| dq_j+\sum_{t\in\Omega}\sum_{i=1}^{n(t)} \lambda_{t,i}v_t^i q_j+\operatorname{com} A \right\|. \end{aligned} \tag{3.4}$$

Also, if $\hat{\chi}_j$ denotes the function on X, associated with q_j, then

$$\| \theta(x+\operatorname{com} A) \| = \max_j \left\| \hat{d}\hat{\chi}_j+\sum_{t\in\Omega}\sum_{i=1}^{n(t)} \lambda_{t,i}z^i_{(t)}\hat{\chi}_j \right\|. \tag{3.5}$$

Now if we take k large enough, so that the functions $\hat{\chi}_j$ separate the points of Ω, then the summations in (3.4) and (3.5) simplify. Thus, to see that θ is isometric on $A_0+\text{com}\,A$ we need only show that, with the χ_j so chosen, we have

$$\left\| dg_j+\sum_{i=1}^{n(t)} \lambda_{t,i}v_t^i q_j+\text{com}\,A \right\| = \max\left\{\| dg_j \|_\infty, \left\| \hat{d}(t)1+\sum_{i=1}^{n(t)} \lambda_{t,i}v_t^i \right\|\right\} \tag{3.6}$$

when t is associated with an endpoint of χ_j. Indeed, in this case, the quantity on the left-hand side of (3.6) is precisely the function norm of

$$\hat{d}\hat{\chi}_j+\sum_{i=1}^{n(t)} \lambda_{t,i}z_{(t)}^i\hat{\chi}_j.$$

The equality (3.6) follows from (3.2) and the observation that if $f(v_t)$ is any polynomial in 1 and v_t, with $t = \alpha+$ say, then for all $\delta > 0$,

$$\| f(v_t) \| = \| f(v_t)(p_{\alpha+\delta}-p_\alpha) \|.$$

A similar assertion holds when $t = \beta-$. Thus θ is isometric.

We know that rad $A \supset I(\pi)$ (see §2) and that $I(\pi) = \text{com}\,A$. The equality of these ideals will follow therefore if we show that com A is the intersection of the ideals of codimension 1. (Indeed the Jacobson radical of a unital Banach algebra coincides with the intersection of the maximal left ideals.) Let I be such an ideal; then $I \supset \text{com}\,A$. On the other hand, for each point w in X, the collection, J_w say, of all a in A such that $\theta(a)$ vanishes at w is a maximal ideal, and, in view of the first part of the proof, com A is precisely the intersection of these ideals.

It remains to prove (c). Since rad A is the intersection of the diagonal ideals I_r, φ is well defined. To see that φ is isometric it suffices to show that $\| \varphi(w) \| = \| w \|$ for w in A_0. This follows from elementary considerations, as in the first part of this proof.

Let us write $a \to \hat{a}$ for the homomorphism from A to $\hat{A}$ obtained from Theorem 3.7. The corollaries below follow from their analogues for the disc algebra.

Corollary 3.3. *The sum of two closed ideals of A need not be closed.*

Corollary 3.4. *A closed ideal of A that contains the radical is a principal ideal.*

Corollary 3.5 (Corona Theorem). *Let $a_1, \ldots, a_n$ belong to A. In order that there exist elements $b_1, \ldots, b_n$ in A satisfying*

$$b_1a_1+\ldots+b_na_n = 1$$

it is necessary and sufficient that there exist $\delta > 0$ such that

$$|\hat{a}_1(x)|+\ldots+|\hat{a}_n(x)| \geqslant \delta, \quad \text{for } x \in X.$$

Proofs. Corollary 3.3. The algebra A/I_0 is a copy of the disc algebra; so we may choose closed ideals J_1, J_2 for which J_1+J_2 is not closed. (See Stegenga **[24]**, for example.) Now I_0+J_1 and I_0+J_2 are closed ideals of A with non-closed sum.

Corollary 3.4. We first show that $\hat{A}$ is a principal ideal domain in the Banach algebra sense. Let I be an ideal of $\hat{A}$. Choose d in C so that $\hat{d}$ generates the ideal $\hat{I} \cap \hat{C}$ in $\hat{C}$. For each dyadic point t in $M(C)$ let $g^{(t)}$ be a function in the disc algebra that generates the ideal of functions $g(z)$ such that $g(z) = h(t, z)$ for some h in I and all z in the disc. This choice is possible because the disc algebra is a principal ideal domain (Rudin [21]). Now let $f = \hat{d}+\sum_t c_t z g_{(t)}^{(t)}$, where $c_t > 0$, $\sum_t c_t$ is finite, and summations extend over all dyadic points of $M(C)$. Then $f \in \hat{A}$ and f is a generator for the ideal $\hat{I}$.

Suppose now that I is an ideal of A which contains the radical. Choose $a \in A$ so that $\hat{a}$ generates $\hat{I}$ and choose r in rad A so that the ideal generated by r is rad A (this possibility follows from Theorem 2.3). It follows, by Theorem 2.3, that the ideal generated by $a+r$ is I.

Corollary 3.5. First recall the elementary corona theorem for the disc algebra. Given functions $f_1,\dots,f_n$ such that $|f_1(z)|+\dots+|f_n(z)| \geqslant \delta$ for $|z| \leqslant 1$, there exist functions $g_1,\dots,g_n$ in the disc algebra such that $f_1g_1+\dots+f_ng_n = 1$. Now fix a dyadic point t in $[0,1]$ and consider the quotient A/I_t^+, which is a copy of the disc algebra. It follows from the hypothesis on $a_1,\dots,a_n$ that there exist $b_1^t,\dots,b_n^t$ in A such that $b_1^t a_1+\dots+b_n^t a_n = 1$ modulo I_t^+. Thus for some $\delta = \delta(t) > 0$ and projection $q_t = p_{t+\delta}-p_t$ we have

$$\| q_t(b_1^t q_t a_1+\dots+b_n^t q_t a_n)q_t - q_t \| < \tfrac{1}{2}. \tag{3.7}$$

A simple compactness argument leads to a dyadic partition $1 = q_{t_1}+\dots+q_{t_m}$ such that (3.7) holds for each $q_t = q_{t_i}$. Let

$$c_j = \sum_{i=1}^{m} q_{t_i} b_j^t q_{t_i},$$

so that $\sum_j c_j(\sum_i q_{t_i} a_j q_{t_i}) = 1$. Since $a_j - \sum_i q_i a_j q_i$ belongs to the radical, it follows that $\sum_j c_j a_j \in 1+\text{rad}\,A$, and is therefore left invertible with left inverse c say. Set $b_j = cc_j$ and the proof is complete.

REMARKS. 1. The pathwise connectedness of the set of invertible elements of A is another consequence of Theorem 3.2.

2. The algebra A is *subdiagonal* in the sense that there is an expectation of B relative to C that is multiplicative on A (cf. [1]). Indeed the diagonal algebra C is complemented in A by the closed two-sided ideal of elements a such that $\hat{a}(t,0) = 0$ for all t in $M(C)$.

3. It seems most likely that A is a principal ideal algebra.

The algebra O_2

Let $s_1 = v(0,1,0,\frac{1}{2})$ and $s_2 = v(0,1,\frac{1}{2},1)$. These are the natural isometries that squeeze $L^2[0,1]$ into $L^2[0,\frac{1}{2}]$ and $L^2[\frac{1}{2},1]$ respectively, and satisfy the relation $s_1s_1^*+s_2s_2^* = 1$. Up to isomorphism, a unique C*-algebra is generated by *any* two isometries that satisfy this relation. This is a result of Cuntz [5] and the algebra is denoted O_2. Our presentation of O_2 as a certain operator algebra on $L^2[0,1]$ is one where it is natural to consider the Volterra nest subalgebra $O_2 \cap \text{Alg}\,L$. A little reflection is sufficient to see that $O_2 = B$. Indeed any dyadic partial isometry can be written as a word in the operators s_1, s_2, s_1^*, s_2^*. It does not seem clear however that $O_2 \cap \text{Alg}\,L = A$. Loosely put, this assertion states that the triangular subalgebra of B $(= O_2)$ is generated by those generators of B (the dyadic partial isometries) that are triangular. There is an obvious parallel here with continuous functions, trigonometric polynomials, and analyticity (triangularity). However, this parallel is dangerous because (unlike the disc algebra) A has a curious *non-Dirichlet* property (cf. [1]): $A+A^*$ is not dense in O_2. This follows from Theorem 3.10, the main result of this subsection.

We require four lemmas. For basic facts about O_2, including Lemma 3.7, we refer the reader to [5].

Let W_k denote the words of length k in the letters 1, 2. If $\mu = \mu_1 \dots \mu_k \in W_k$ then write $s_\mu = s_{\mu_1} \dots s_{\mu_k}$, $l(\mu) = k$, for the length of μ, and let

$$d(\mu) = \frac{\mu_1 - 1}{2} + \dots + \frac{\mu_k - 1}{2^k}.$$

Every word in the operators s_1, s_2 and their adjoints can be reduced to the form $s_\mu s_\nu^*$ for certain unique words μ, ν.

LEMMA 3.6. *Let $\mu, \nu \in W_k$. Then $s_\mu s_\nu^*$ is the canonical partial isometry with initial space $L^2[d(\nu), d(\nu)+2^{-k}]$ and final space $L^2[d(\mu), d(\mu)+2^{-k}]$.*

Thus $\{s_\mu s_\nu^* : \mu, \nu \in W_k\}$ is a set of matrix units for a finite-dimensional operator algebra, F_k say, isomorphic to $M(2^k)$. We write F for the closed union of these algebras. Thus F is a uniformly hyperfinite C*-algebra, embedded in O_2.

LEMMA 3.7 (Cuntz). *Each operator a in the star algebra generated by s_1 and s_2 has a unique representation*

$$a = \sum_{i=1}^{N} (s_1^*)^i a_{-i} + a_0 + \sum_{i=1}^{N} a_i s_1^i \tag{3.8}$$

with $a_i \in F$. Moreover, the maps $E_i(a) = a_i$ extend to continuous contractive linear maps from O_2 to F.

The extension of E_i to O_2 is also denoted by E_i. Thus each a in O_2 determines a coefficient sequence $a_i = E_i(a)$ and an associated generalized Fourier series, namely the infinite-sum version of (3.8). The next lemma expresses the convergence of the Cesaro sums of this series. We use the automorphisms ρ_λ, where $|\lambda| = 1$, of O_2, that are determined by the equations

$$\rho_\lambda(s_1) = \lambda s_1, \quad \rho_\lambda(s_2) = \lambda s_2.$$

LEMMA 3.8. *If $a \in O_2$ and $a_i = E_i(a)$, then a is the norm limit of the sequence*

$$a_0 + \sum_{i=1}^{N} \left(1 - \frac{i}{N}\right)((s_1^*)^i a_{-i} + a_i s_1^i). \tag{3.9}$$

In particular, $a = 0$ if and only if $a_i = 0$ for all i.

Proof. The function $\lambda \to \rho_\lambda(a)$ is a continuous O_2-valued function on the circle and is uniformly approximated by its Cesaro sums, $\sigma_N(\lambda)$ say. In particular, $\sigma_N(1)$ converges in norm to a. However, since ρ_λ fixes F, the Fourier coefficients of $\lambda \to \rho_\lambda(a)$ are just the terms of the generalized Fourier series for a. Thus $\sigma_N(1)$ is the limit of the sequence given by (3.2), and the proof is complete.

The following modules for the uniformly hyperfinite nest algebra $F \cap \operatorname{Alg} L$ turn out to be the Fourier coefficient spaces for the operators of A:

$$\begin{aligned} M_n &= \{x \in F : (1-p_t)xp_{t/2^n} = 0,\ 0 \leqslant t \leqslant 1\} \quad \text{for } n \geqslant 0; \\ M_{-n} &= \{x \in F : (1-p_{t/2^n})xp_t = 0,\ 0 \leqslant t \leqslant 1\} \quad \text{for } n > 0. \end{aligned} \tag{3.10}$$

LEMMA 3.9. *Let* $a \in O_2$. *Then* $a \in A$ *if and only if* $a_n \in M_n$ *for all integers* n.

Proof. We have $s_1 p_t = p_{t/2} s_1$ for $0 \leqslant t \leqslant 1$, and so

$$(1-p_t)a_n s_1^n p_t = (1-p_t)a_n(p_{t/2^n})s_1^n$$

for $n \geqslant 0$, and

$$(1-p_t)(s_1^*)^n a_{-n} p_t = (s_1^*)^n(1-p_{t/2^n})a_{-n}p_t$$

for $n > 0$. Use the second part of Lemma 3.8 for the operators $(1-p_t)ap_t$ and the lemma follows.

THEOREM 3.10. *The nest subalgebra* $O_2 \cap \operatorname{Alg} L$ *is generated by the words in* s_1, s_2, s_1^*, s_2^* *that are contained in* $O_2 \cap \operatorname{Alg} L$. *Moreover,*
(i) *if* $l(\mu) \geqslant l(v)$ *then* $s_\mu s_v^* \in O_2 \cap \operatorname{Alg} L$ *if and only if* $d(\mu) \leqslant d(v)$,
(ii) *if* $l(\mu) < l(v)$ *then* $s_\mu s_v^* \in O_2 \cap \operatorname{Alg} L$ *if and only if*

$$d(\mu)+2^{-l(\mu)} \leqslant d(v)+2^{-l(v)}.$$

Proof. By Lemma 1.3, the F-modules M_n are inductive. Therefore the first statement of the theorem follows from Lemmas 3.8 and 3.9.

Let $r = l(\mu)$ and $s = l(v)$. If $r = s$ then (i) follows immediately from Lemma 3.6. Now suppose that $r > s$ with $k = r-s$ and let $1_k v$ denote the word composed of v and the letter 1 appearing k times. Thus $s_\mu s_v^* = s_\mu s_v^* s_1^{*k} s_1^j = s_\mu s_{1_k v}^* s_1^k$, which belongs to A if and only if $s_\mu s_{1_k v}^* \in M_k$. By Lemma 3.6 this is the case if and only if $d(1_k v) \geqslant 2^{-k} d(\mu)$, which is precisely the condition $d(v) \geqslant d(\mu)$.

On the other hand, if $r < s$ let $k = s-r$. Then $s_\mu s_v^* \in A$ if and only if $s_{1_k\mu} s_v^* \in M_{-k}$. But $s_{1_k\mu} s_v^*$ is the canonical partial isometry from

$$L^2[d(v), d(v)+2^{-s}] \quad \text{to} \quad L^2[d(1_k\mu), d(1_k\mu)+2^{-r}].$$

Examining how M_{-k} is defined we see that this operator lies in M_{-k} if and only if

$$d(v)+2^{-s} \geqslant 2^k(d(1_k\mu)+2^{-s}),$$

which is the condition

$$d(v)+2^{-s} \geqslant d(\mu)+2^{-r}.$$

REMARKS AND PROBLEMS. 1. The asymmetry present in the assertions (i) and (ii) of Theorem 3.10 reflect the fact that A is *non-Dirichlet* in the sense that $A+A^*$ is not dense in O_2. In fact let $v = v(\alpha, \beta, \gamma, \delta)$ be a dyadic partial isometry with $\alpha < \gamma < \delta < \beta$. Then v belongs to $A+A^*$ only when the fixed point of the function from $[\alpha, \beta]$ to $[\gamma, \delta]$ that implements v is dyadic.

2. A similar analysis can be made of the infinite C*-algebra associated with unitaries on $L^2(\mathbb{R})$ that are induced by the homeomorphisms $x \to ax+b$, where $a, b \in \mathbb{R}$. Here analytic almost periodic functions appear. It is natural then to enquire: what kind of function algebra can be realized as the quotient A/I_0 of a Volterra nest subalgebra A? ($I_0 = \{x \in A\colon p_\delta x p_\delta \to 0 \text{ as } \delta \to 0\}$).

3. Let $H^\infty(s_1), H^\infty(s_2)$ denote the weakly closed operator algebras on $L^2[0,1]$ that are generated by s_1 and s_2 respectively. Define A^∞ to be the Volterra nest subalgebra of $C^*(H^\infty(s_1), H^\infty(s_2))$. A reasonable blind guess is that A^∞/I_0 is a copy of H^∞ and that 'A^∞ is to H^∞ as A is to the disc algebra'. Is this so?

4. Of course nest subalgebras with A/I_0 non-commutative have not been touched in this paper. In this context it would be interesting to construct a multiplicity-1 nest subalgebra A with A isomorphic to A/I_0.

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Department of Mathematics
University of Lancaster
Lancaster LA1 4YL

APPENDIX 7

D.Sc. 1988

Reprinted from JOURNAL OF FUNCTIONAL ANALYSIS

Vol. 67, No. 3, July 1986
Printed in Belgium

Factorization in Analytic Operator Algebras

S. C. POWER

University of Lancaster, Lancaster, LA1 4YL, England

Communicated by C. Foias

Received July 23, 1985; revised January 20, 1986

A constructive and unified approach is used to obtain the upper–lower factorization of positive operators and the outer function factorization of positive operator valued functions on the circle. For a projection nest $\mathscr{E}$ it is shown that every positive operator admits a canonical factorization $C = A^*A$, with A an outer operator, if and only if $\mathscr{E}$ is well ordered. With new methods we generalize the inner–outer factorizations obtained by Arveson, for nests of order type $\mathbb{Z}$, and the Riesz factorization, due to Shields, for trace class triangular operators. Weak factorization is obtained in noncommutative H^1 spaces associated with (general) nest subalgebras of a semifinite factor. Characterizations of a Nehari type are given for the associated Hankel forms and Hankel operators.

Contents.

1. INTRODUCTION

The lower–upper factorization of an operator has played a significant role in various areas of analysis, both in the solutions of specific problems in numerical analysis, integral equations, and prediction theory, for example, and in the general structure theory of Hilbert space operators. The factorization of a positive invertible finite matrix C as A^*A with A and its inverse in upper triangular form is known, especially to numerical analysts, as the Cholesky decomposition. Using an operator theoretic variant of the inner–outer factorization of Hardy space functions, Arveson [2] extended this to Hilbert space operators in the context of triangularity with respect to a fixed projection nest of order type $\mathbb{Z}$. Earlier, in work of significance to integral operators, Gohberg and Krein [9] obtained lower–upper factorizations with respect to arbitrary projection nests in the case of

0022-1236/86 $3.00

operators that differ from the identity by a sufficiently compact perturbation. Their methods were different and relied on the convergence of the triangular operator integral in symmetrically normed ideals. In the recent startling advances in the similarity theory of nests, initiated by Andersen [1] (see [4, 6] for different perspectives), Larson [13] has shown that there exist operators of the form identity plus compact that do not admit a lower–upper factorization with respect to a continuous nest. All these results are principally concerned with factorizations of invertible or essentially invertible operators.

Using a limiting argument, valid for nests of multiplicity one and order type $\mathbb{N}$, Shields [24] obtained Cholesky decompositions for *all* positive operators. This was shown to be significant for the associated noncommutative Hardy spaces, and variants of the Riesz factorization of functions and Hardy's inequality were obtained. The lack of a general Cholesky decomposition, even for a finite nest, impeded the extension of these results to more general nests. However, it was observed in Power [21, 22] that weak factorization and trace class decompositions could be used as a good substitute for Riesz factorization. This approach is reminiscent of the success of weak factorization [5, 18] and molecular decomposition [23] in higher dimensional Hardy spaces and Bergman spaces.

In this paper we give a new direct approach, that is essentially of a constructive nature, to obtain factorizations of Cholesky–Arveson type and which can be applied to *arbitrary* positive operators in the presence of a well-ordered nest. The well-ordered context is the appropriate framework for such universal factorization (see Corollary 2.5). In this way our viewpoint differs from that of Larson [13, Sect. 4] who has shown that the *countability* of the (complete) nest is the necessary and sufficient condition for the outer factorization $C = A^*A$ of every positive *invertible* operator. In contrast to Arveson's methods our constructions lead directly to the outer factor. From this main result we easily obtain generalizations and different proofs of the inner–outer factorization of operators and Shields' Riesz factorization mentioned above.

We also obtain weak factorization in noncommutative H^1 spaces associated with general nests in a semifinite factor. In this way we are able to characterize the associated Hankel forms and Hankel operators. For example, the celebrated theorem of Nehari [16] has its analog in the formula

$$\|H_x\| = \operatorname{dist}(x, H^\infty(M, \mathscr{E}, \tau)),$$

where H_x is the Hankel operator related to left multiplication by the operator x in the semifinite factor M (Theorem 8.1).

It is notable that in the context of positive operator valued functions ϕ on the circle, the construction also leads directly to the factorization $\phi = hh^*$ with h an outer operator valued function with $h(0)$ positive, when this factorization is known to exist. Such factorization is usually obtained indirectly through the Beurling–Lax–Halmos theorem (as in Helson's book [10], for example). Moreover in Theorem 3.1 we obtain a new condition for such factorization, namely

$$\lim_{n \to \infty} T_{\psi}^{-1/2} H_{\psi^*}^* \bar{z}^n = 0$$

where $\psi(z) = \phi(\bar{z})$ and where H_ψ and T_ψ are the associated Hankel and Toeplitz operators. The possibly unbounded operator T_ψ^{-1} must be appropriately interpreted, and the limit taken in the strong operator topology. Thus we have a new perspective on the rich ideas encircling outer factorization, prediction theory, and the Beurling–Lax–Halmos theorem.

The nest subalgebras $H^\infty(M, \mathscr{E}, \tau)$, defined below, are related to (but usually quite distinct from) the analytic operator algebras of McAsey, Muhly, and Saito [15], and, of course, to certain subdiagonal algebras introduced by Arveson [2]. Moreover, as nest subalgebras, they fall within the context studied by Gilfeather and Larson [8]. There are interesting connections with these studies but we do not pursue them here.

We use the following notation. Let M be a factor with faithful semifinite normal trace τ and let

$$L^p = L^p(M) = L^p(M, \tau), \qquad 1 \leqslant p \leqslant \infty,$$

be the usual noncommutative Lebesgue spaces. Let $\mathscr{E}$ be a complete nest of self-adjoint projections in M and define the noncommutative Hardy space

$$H^p = H^p(M, \mathscr{E}) = H^p(M, \mathscr{E}, \tau)$$

to be the closed subspace of L^p of elements x for which $(1-e)\,xe = 0$ for all e in $\mathscr{E}$. In particular $L^\infty = M$ and H^∞ is the nest subalgebra of M induced by $\mathscr{E}$. Also write

$$H_0^p = H_0^p(M, \mathscr{E}) = H_0^p(M, \mathscr{E}, \tau)$$

for the closed subspace of H^p of elements x for which $\tau(xa) = 0$ for all a in H^∞. The von Neumann algebra generated by $\mathscr{E}$ is called the *core* of $\mathscr{E}$ and the nest is said to be *compatible* with τ, or simply compatible, if the restriction of τ to the core is semifinite.

An *atom* of the nest $\mathscr{E}$ is a non-zero projection of the form $e_+ - e$, where $e_+ = \inf\{f: f > e, f \text{ in } \mathscr{E}\}$ is the immediate successor of e, and the nest is said to be *purely atomic* if the identity operator is the sum, in the strong operator topology, of these atoms. If no atoms exist then $\mathscr{E}$ is said to be a *continuous* nest. For any projection $e<I$ in any nest $\mathscr{E}$ we define e_+ as above, and similarly, if $e>0$, we let $e_- = \sup\{f: f<e, f \text{ in } \mathscr{E}\}$. A nest is *well ordered* if $e<e_+$ for all $e<I$. We write $\operatorname{Alg}\mathscr{E}$ for the nest algebra associated with $\mathscr{E}$, so that

$$H^\infty = L^\infty \cap \operatorname{Alg}\mathscr{E}.$$

For convenience we assume that all Hilbert spaces are complex and separable. We usually write $\mathscr{H}$ for the underlying Hilbert space, and $\mathscr{L}(\mathscr{H})$ for the associated algebra of bounded operators.

2. Arveson–Cholesky Factorizations

In finite dimensions the result of the next theorem is more easily obtained and, when used inductively, leads to a Cholesky type decomposition for an arbitrary positive operator. The proof of the general case below builds on an idea of Lance [12].

Theorem 2.1. *Let C be a positive operator on a Hilbert space with operator matrix*

$$\begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$$

with respect to a prescribed decomposition. Then the limit A of the sequence

$$A_n = \begin{bmatrix} (a+n^{-1})^{1/2} & (a+n^{-1})^{-1/2}\, b \\ 0 & (c-b^*(a+n^{-1})^{-1}\, b)^{1/2} \end{bmatrix}$$

exists in the weak operator topology. Moreover $C=A^*A$ and UA^* has upper triangular form if and only if UC has upper triangular form.

Proof. Recall that if a is an invertible positive operator then

$$\begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$$

is positive if and only if $c \geqslant b^*a^{-1}b$. Since $c+n^{-1}I \geqslant 0$ it follows that $b^*(a+n^{-1}I_1)^{-1}\, b \leqslant c+n^{-1}I_2$, where I, I_1, I_2 are the appropriate identity operators. The increasing sequence $b^*(a+n^{-1}I_1)\, b$ converges in the weak operator topology to an operator $c_1 \leqslant c$. Let e_t denote the spectral projection for the operator a corresponding to the interval (t, ∞). Then, for $t > 0$,

$$\begin{aligned}\|b^*a^{-1/2}e_t\|^2 &= \lim_{n\to\infty} \|b^*(a+n^{-1})^{-1/2}\, e_t(a+n^{-1})^{-1/2}\, b\| \\ &\leqslant \lim_{n\to\infty} \|b^*(a+n^{-1})^{-1}\, b\| \\ &\leqslant \|c_1\|.\end{aligned}$$

It follows that $d_t = b^*a^{-1/2}e_t$ converges to an operator d in the star strong topology as $t \to 0$. Moreover $c_1 = dd^*$. To see this note first that

$$\begin{bmatrix} a & b \\ b^* & dd^* \end{bmatrix} = \begin{bmatrix} a^{1/2} & 0 \\ d & 0 \end{bmatrix} \begin{bmatrix} a^{1/2} & d^* \\ 0 & 0 \end{bmatrix} \geqslant 0$$

and so, by our earlier argument, with dd^* replacing c, we have $c_1 \leqslant dd^*$. On the other hand,

$$b^*(a+n^{-1})^{-1}\, b \geqslant b^*(a+n^{-1})^{-1/2}\, e_t(a+n^{-1})^{-1/2}\, b$$

and so $c_1 \geqslant b^*a^{-1/2}e_t a^{-1/2}b$. Let $t \to 0$ and it follows that $c_1 \geqslant dd^*$. Now let

$$A = \begin{bmatrix} a^{1/2} & d^* \\ 0 & (c - dd^*)^{1/2} \end{bmatrix}$$

and it remains only to show that UA^* is upper triangular when UC is. But if

$$U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$$

and $u_3 a + u_4 b^* = 0$ then

$$u_3 a^{1/2} + u_4 d^* = \lim_{t\to 0} (u_3 a + u_4 b^*)\, a^{-1/2}e_t = 0,$$

completing the proof.

We now obtain a Cholesky factorization relative to a well-ordered nest. The case of a finite nest is particularly straightforward, but in general some care must be taken with the accumulation points.

THEOREM 2.2. *Let $\mathcal{E}$ be a well-ordered nest of projections and let C be a positive operator. Then there exists a factorization $C = A^*A$, with A in* Alg $\mathcal{E}$, *with the property that UA^* belongs to* Alg $\mathcal{E}$ *whenever U is an* *operator such that* UC *belongs to* $\operatorname{Alg}\mathscr{E}$. *Moreover* A *belongs to the von Neumann algebra generated by* C *and the nest.*

Proof. It has been shown in [4] how the constructions used in the proof of Theorem 2.1 lead to a positive operator valued measure $C(\Delta)$, defined on the Borel algebra of $\mathscr{E}$, with the order topology, that has the following properties. The total mass is $C(\mathscr{E})=C$, and if $Q=[E, F)$ is a half open interval of $\mathscr{E}$ then $C(Q)$ has the form

$$C(Q)=\lim_{n\to\infty}\begin{bmatrix}0 & 0 & 0\\ 0 & a & b\\ 0 & b^* & b^*(a+n^{-1})^{-1}b\end{bmatrix}\begin{matrix}E\mathscr{H}\\ (F-E)\mathscr{H}\\ (I-F)\mathscr{H}.\end{matrix}$$

(Indeed $C[0, F)$ is defined in this way, with $C[0, F)\,F=CF$, and $C[0, F)=C[0, E)+C[E, F)$. $C(\Delta)$ is constructed first on the ring generated by the semiintervals, and then after establishing the required continuity, extended to a positive operator valued measure, with convergence in the weak operator topology).

From the proof of Theorem 2.1 we may write $C(Q)=A_Q^*A_Q$ where A_Q has the form

$$A_Q=\lim_{t\to 0}\begin{bmatrix}0 & 0 & 0\\ 0 & a^{1/2} & e_t a^{-1/2}b\\ 0 & 0 & 0\end{bmatrix}\begin{matrix}E\mathscr{H}\\ (F-E)\mathscr{H}\\ (I-F)\mathscr{H},\end{matrix}$$

and where e_t is the spectral projection for the positive operator a for the interval (t, ∞), and convergence occurs in the star strong topology. Now let Q be a partition of $\mathscr{E}\backslash\{I\}$ by disjoint intervals Q of the above form. Then, since $C(\Delta)$ is a positive operator measure we have

$$C=\Sigma_Q C(Q)=\Sigma_Q A_Q^*A_Q$$

with convergence in the weak operator topology. If $\mathscr{F}$ is a finite subset of Q then

$$\Big(\sum_{Q\in\mathscr{F}}A_Q\Big)^*\Big(\sum_{Q\in\mathscr{F}}A_Q\Big)=\sum_{Q\in\mathscr{F}}A_Q^*A_Q\leqslant C.$$

In particular the finite sums of the series $\Sigma_Q A_Q$ are uniformly bounded in the operator norm. It is clear that the series

$$\Sigma_Q(A_Q x, y)$$

converges when the support of y is contained in a finite number of the intervals of Q. Since the collection of these vectors is dense, we conclude that the series $\Sigma_Q A_Q$ converges in the weak operator topology to an operator A such that $C = A^*A$.

We now use the hypothesis that $\mathscr{E}$ is well ordered. In this case the set $\{[E, E_+): E \text{ in } \mathscr{E}, E \neq I\}$ is a maximal partition of $\mathscr{E}$, and the associated operator A, constructed above, belongs to Alg $\mathscr{E}$. It follows from the proof of Theorem 2.1 that A belongs to the von Neumann algebra generated by C and $\mathscr{E}$, and has the desired property.

We refer to the specific factorization obtained in the proof of Theorem 2.2 as *the Cholesky factorization* of C associated with the well-ordered nest $\mathscr{E}$. The next two corollaries show that this decomposition generalizes results of Arveson obtained for invertible positive operators relative to nests of order type $\mathbb{N}$. Following Arveson we say that an operator A in Alg $\mathscr{E}$ is an *outer operator* if the range projection of A commutes with $\mathscr{E}$ and if $AE\mathscr{H}$ is dense in $A\mathscr{H} \cap E\mathscr{H}$ for every projection E in $\mathscr{E}$. In particular if A is invertible, with inverse in Alg $\mathscr{E}$, then A is outer.

COROLLARY 2.3. *Let $\mathscr{E}$ be a well-ordered nest and let $C = A^*A$ be the Cholesky factorization. Then A is an outer operator. Moreover if C is invertible then A is invertible with inverse in* Alg $\mathscr{E}$.

Proof. In view of the special form of the operators A_Q in the representation $A = \Sigma_Q A_Q$ it is possible to check that A is an outer operator. If C is an invertible operator then A will be seen to be invertible if we show that the range of A is dense. This in turn is a consequence of the fact that the operator a in the representation of A_Q is an invertible operator on $Q\mathscr{H}$, for every Q. To see this observe that the operator E_+CE_+ on $E_+\mathscr{H}$ is invertible and has the form

$$E_+CE_+ = \begin{bmatrix} ECE & B \\ B^* & B^*(ECE)^{-1}B + a \end{bmatrix} \begin{matrix} E\mathscr{H} \\ (E_+ - E)\mathscr{H} \end{matrix}.$$

Hence, noting that $B = EB$, we see that the operator

$$\begin{bmatrix} ECE & 0 \\ B^* & a \end{bmatrix} = \begin{bmatrix} ECE & B \\ B^* & B^*(ECE)^{-1}B + a \end{bmatrix} \begin{bmatrix} I & -(ECE)^{-1}B \\ 0 & I \end{bmatrix}$$

is invertible, and so a is invertible, as required.

COROLLARY 2.4. *Let $\mathscr{E}$ be a well-ordered nest of projections and let T be an operator in* Alg $\mathscr{E}$ *that is invertible. Then $T = UA$, where U, A belong to*

$\operatorname{Alg}\mathscr{E}$, U is an isometry, and A is invertible with inverse in $\operatorname{Alg}\mathscr{E}$. *Moreover U and A belong to the von Neumann algebra generated by T and $\mathscr{E}$.*

Proof. Let $T = VC$ be a polar decomposition of T, with C a positive invertible operator and V an isometry. Let $C^2 = A^*A$ be the Cholesky factorization of C^2 and define $U = VC^{-1}A^*$. Since $VC^{-1}C^2$ is in $\operatorname{Alg}\mathscr{E}$ it follows that U is also in $\operatorname{Alg}\mathscr{E}$. Also $U^*U = AC^{-2}A^* = I$. The remaining assertions follow from Corollary 2.3 and the constructive nature of the proof of Theorem 2.2.

If we relax the hypothesis that the nest is well ordered then there are operators that do not admit a Cholesky factorization.

COROLLARY 2.5. *Let $\mathscr{E}$ be a projection nest. Then every positive operator admits a Cholesky factorization with respect to $\mathscr{E}$ if and only if $\mathscr{E}$ is well ordered.*

Proof. We need only show that if E is a projection in the nest with $E = E_+$ $(E \neq I)$ then there is a non-factorizable positive operator. Let f be a unit vector such that $f = (I - E)f$ and $(F - E)f \neq 0$ for all $F > E$, and let $C = E + f \otimes f$. Suppose that $C = A^*A$ is a Cholesky factorization. Then $E = EA^*AE = EA^*EAE$ and EAE is an isometry on $E\mathscr{H}$. Since $\|A\| = \|C\| = 1$ it follows that the range of $EA(I - E)$ is orthogonal to the range of AE. But A is an outer operator and so this entails $EA(I - E) = 0$, and hence $f \otimes f = (E^\perp A E^\perp)^* \, (E^\perp A E^\perp) = A_1^* A_1$ say. Since A_1 is of rank one and $E = E_+$ it follows that $A_1(F - E) = 0$ for some projection $F > E$, and this now contradicts our hypothesis on the vector f.

Remarks 1. The inner and outer factors of Corollary 2.4 belong to the von Neumann algebra generated by the nest and the operator. It follows that this inner–outer factorization of invertible operators is valid in any nest subalgebra of a von Neumann algebra M associated with a well-ordered nest contained in M. In particular, since the positive operators of a von Neumann algebra constitute a spanning set, it follows from Corollary 2.3 that

$$L^\infty(M) = \operatorname{span}\{h^*h \colon h \text{ invertible in } H^\infty(M, \mathscr{E})\}$$

in the case of a well-ordered nest $\mathscr{E}$, in the semifinite factor M. In fact a weaker structural condition, with h unrestricted in $H^\infty(M, \mathscr{E})$, holds more generally. Indeed, using factorization in nests of order type $\mathbb{Z}$, Larson [13, Proposition 4.13] deduced that every invertible positive operator C admits a factorization A^*A with A leaving invariant *any* prescribed nest. However, A is not necessarily invertible or outer.

2. Corollary 2.4 is in fact a special case of a general inner–outer factorization theorem concerning arbitrary operators T in a nest algebra Alg $\mathscr{E}$ such that $\mathscr{E}$ has the property $E \neq E_+$ for all $E \neq 0$ (well ordered except, possibly, at 0).

3. There is clearly a strong formal analogy between the inner–outer factorization of operators and that of functions. However, the operator version in the case of the multiplicity one nest of order type $\mathbb{N}$ is weaker. In fact any operator T in Alg $\mathbb{N}$ with non-zero diagonal is an outer operator and T^*T is the Cholesky factorization of the positive operator T^*T. In particular, as Arveson has already observed in [3], the operator factorization of a coanalytic Toeplitz operator $T_{\bar{h}}$ is quite unrelated to the functional inner–outer factorization of h. However, we see in the next section that functional factorization is closely related to the Cholesky construction in the case of order type $\mathbb{Z}$.

3. Factorization of Positive Operator Functions

It is instructive to examine the Cholesky construction in the context of the multiplication operators M_ϕ on the Hilbert space $L^2(T)$, for the circle T, with respect to the nest $\mathscr{E}$ consisting of 0, the identity operator, and the projections E_n onto the subspaces $z^n\overline{H^2(T)}$, for integers n, where $H^2(T)$ is the Hardy subspace. Indeed a function f in $H^\infty(T)$ is an outer function if and only if the multiplication operator M_f^* is an outer operator with respect to this nest. The nest $\mathscr{E}$ is not well ordered. Nevertheless to each positive function ϕ in $L^\infty(T)$, and associated positive operator $C = M_\phi$, there is a uniquely determined positive operator valued measure $C(\Delta)$, as described in the proof of Theorem 2.2 and more fully in [20, Sect. 3]. In particular,

$$C([0, E_k)) = \begin{bmatrix} A_k & B_k \\ B_k^* & D_k \end{bmatrix} \begin{matrix} E_k\mathscr{H} \\ (I-E_k)\mathscr{H} \end{matrix}$$

where $A_k = E_k M_\phi E_k$, $B_k = E_k M_\phi (I - E_k)$, and $D_k = \lim_n B_k^*(n^{-1} + A_k)^{-1} B_k$. Also

$$C([E_k, I)) = \begin{bmatrix} 0 & 0 \\ 0 & F_k \end{bmatrix} \begin{matrix} E_k\mathscr{H} \\ (I-E_k)\mathscr{H} \end{matrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & b^* & c \end{bmatrix} \begin{matrix} E_k\mathscr{H} \\ (E_{k+1}-E_k)\mathscr{H} \\ (I-E_{k+1})\mathscr{H}, \end{matrix}$$

and so

$$C(\{E_k\}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & b^* & b^*a^{-1}b \end{bmatrix} \begin{matrix} E_k\mathscr{H} \\ (E_{k+1}-E_k)\mathscr{H} \\ (I-E_{k+1})\mathscr{H}, \end{matrix}$$

where a and b are defined as above in terms of the first row of $F_k = (I-E_k)M_\phi(I-E_k)-D_k$. The multiplication operator C has a Laurent matrix (constant on diagonals) and from this it follows that the operators $C_k = C(\{E_k\})$ are simply translates of each other, and that the operator $C((0, I)) = \Sigma_k C_k$ is a multiplication operator. As in the proof of Theorem 2.2 the operators C_k factors as $A_k^* A_k$ and $C((0, I)) = A^*A$, where $A = \sum_{k=-\infty}^{\infty} A_k$ is a coanalytic multiplication operator with Laurent representing matrix

$$A = \begin{bmatrix} \cdot & & & & & & \\ & \cdot & & & & & \\ \cdot & \cdot & \sqrt{a} & \sqrt{a^{-1}}\,b_1 & \sqrt{a^{-1}}\,b_2 & \cdot & \cdot \\ \cdot & \cdot & 0 & \sqrt{a} & \sqrt{a^{-1}}\,b_1 & \cdot & \cdot \\ \cdot & \cdot & 0 & 0 & \sqrt{a} & \cdot & \cdot \\ & & & & & \cdot & \\ & & & & & & \cdot \end{bmatrix}.$$

If $\phi = |h|^2$, with h an invertible outer function and $h(0)=1$, then $A = M_{\bar{h}}$. In fact one can verify directly that F_k reduces to the operator $(I-E_k)$ $M_h(I-E_k)M_h^*(I-E_k)$ by making use of the relations $E_kM_hM_{\bar{h}}E_k = E_kM_hE_kM_{\bar{h}}E_k$ and $(E_kM_hE_k)^{-1} = E_kM_h^{-1}E_k$. Thus $C = C((0, I))$ and $C(\{0\}) = 0$. On the other hand, since we always have $C = C(\{0\}) + C((0, I)) = C(\{0\}) + A^*A$ it follows that in general $\phi = \phi_0 + |h|^2$, where h is outer and ϕ_0 is positive. In particular if $\phi = 0$ on a set of positive measure then, since h cannot so vanish, $h = 0$, $\phi = \phi_0$, and $C = C(\{0\})$.

The moral to be drawn from the last remark is that in certain circumstances, for non-well-ordered nests, the measure $C(\Delta)$ may be concentrated at zero, or have mass at zero, and that the Cholesky factorization is not automatic. We can identify this circumstance precisely, even, as we now indicate, in the setting of infinite multiplicity, and this leads to a new operator theoretic perspective, and approach to, the circle of ideas surrounding the outer function factorization of a positive operator valued function, as investigated by Devinatz [7], Masani and Wiener [14], Helson and Lowdenslager [11], and many others since. First we need a little more notation. The context that follows is well known and developed, for example, in the books of Helson [10] and Sz-Nagy and Foias [25].

Here too can be found discussions of outer function factorization by means of the Beurling–Lax–Halmos theorem.

Let $\mathscr{K}$ be a separable Hilbert space, let $L^2_{\mathscr{K}} = L^2(T) \otimes \mathscr{K}$, with subspace $H^2_{\mathscr{K}} = H^2(T) \otimes \mathscr{K}$, and let P denote the orthogonal projection of $L^2_{\mathscr{K}}$ onto $H^2_{\mathscr{K}}$. Define $L^\infty_{\mathscr{L}(\mathscr{K})}$ as the algebra of bounded operators on $L^2_{\mathscr{K}}$ whose representing operator matrices, with entries in $\mathscr{L}(\mathscr{K})$, have the Laurent form of constancy along diagonals. In fact $L^\infty_{\mathscr{L}(\mathscr{K})}$ is the commutant of the bilateral shift $M_z \otimes I$ which we denote simply by z. Let $\mathscr{E}_{\mathscr{K}}$ be the nest containing 0, the identity operator, and the projections $\tilde{E}_n = E_n \otimes I$, for n in $\mathbb{Z}$, and write $H^\infty_{\mathscr{L}(\mathscr{K})}$ for the intersection of $L^\infty_{\mathscr{L}(\mathscr{K})}$ and $(\mathrm{Alg}\, \mathscr{E}_{\mathscr{K}})^*$. Finally, for ϕ in $L^\infty_{\mathscr{L}(\mathscr{K})}$ we let $T_\phi = P\phi P$ and $H_\phi = (I-P)\,\phi P$. These are the Toeplitz and Hankel operators associated with ϕ, defined in our context as operators on $L^2_{\mathscr{K}}$.

For a positive operator $\phi = C$ in $L^\infty_{\mathscr{L}(\mathscr{K})}$, the arguments above apply. It follows that there is a factorization $C = A^*A$ with A an outer operator relative to $\mathscr{E}_{\mathscr{K}}$, and moreover belonging to $(H^\infty_{\mathscr{L}(\mathscr{K})})^*$, if and only if $C(\{0\}) = 0$. (There is a natural dual formulation, with the dual nest, that leads to a factorization $C = B^*B$ with B an outer operator, relative to the dual nest, and belonging to $H^\infty_{\mathscr{L}(\mathscr{K})}$.) Our notion of outer operator here coincides precisely with the usual notion of outer for these model spaces, namely that the restriction of B to $H^2_{\mathscr{K}}$ should have dense range in $H^2_{\mathscr{K}} \cap \operatorname{ran} B$.

THEOREM 3.1. *Let ϕ be a positive operator in $L^\infty_{\mathscr{L}(\mathscr{K})}$ and let $\psi(z) = \varphi(\bar{z})$. Suppose that*

$$\lim_{n\to\infty} \lim_{m\to\infty} z^n H_\psi (T_\psi + m^{-1})^{-1} H^*_{\psi^*} z^{-n} = 0,$$

where the limit exists in this strong operator topology. Then there exists a factorization $\phi = hh^*$, with h an outer operator in $H^\infty_{\mathscr{L}(\mathscr{K})}$. In particular ϕ admits such a factorization if ϕ is invertible.

Proof. With $\phi = C$ we see from the definition of the operator measure $C(\Delta)$, as above, that

$$C([0, \tilde{E}_n)) = z^{n-1} \begin{bmatrix} P^\perp \phi P^\perp & P^\perp \phi P \\ P\phi P^\perp & X \end{bmatrix} z^{-n+1}$$

where $X = \lim_m P\phi P^\perp (P^\perp \phi P^\perp + m^{-1})^{-1} P^\perp \phi P$. Thus $C([0, \tilde{E}_n))$ decreases to zero in the weak star topology, as $n \to -\infty$, if and only if $z^n X z^{-n}$ converges to zero in the weak operator topology as $n \to -\infty$. This is equivalent to the stated condition, as can be seen by conjugation with the natural unitary operator that exchanges past and future. As we observed before, and the argument applies equally well in the present higher multiplicity setting, C admits the desired factorization if and only if $C(\{0\}) = 0$, and so the first part of the theorem is established.

If ϕ is invertible, as well as positive, then the Toeplitz operator T_ϕ is invertible. Moreover for a vector g in $L^2_{\mathscr{K}}$ we have

$$\begin{aligned}\lim_{n\to\infty} H_\psi^* z^{-n} g &= \lim_{n\to\infty} P\psi(I-P)\, z^{-n} g \\ &= \lim_{n\to\infty} P\psi z^{-n} g \\ &= 0\end{aligned}$$

from which it follows that $z^n H_\psi T_\psi^{-1} H_{\psi^*}^* z^{-n}$ converges to zero in the weak operator topology, completing the proof of the theorem.

Remarks. 1. In general $T_\psi^{-1/2}$ is an unbounded self-adjoint operator, but the proof of Theorem 2.1 shows that since ϕ is positive the operator $T_\psi^{-1/2} H_{\psi^*}^*$ is bounded. Thus the condition of the theorem coincides with that stated in the Introduction.

2. The theorem applies to positive matrix valued functions on the circle which may fail the non-degenerate requirement of prediction theory of the integrability of $\log \det \phi$. Indeed $\det \phi$ may be identically zero. It seems likely then that the Cholesky construction is significant for non-deterministic multivariate stationary stochastic processes, since factorization of the spectral density function is a key step in the analysis.

3. If the Hankel operator H_ψ has finite rank then the operator $T_\psi^{-1/2} H_\psi^*$ is well defined and also has finite rank, so the hypothesis of the theorem holds. Hence such ϕ admit outer function factorization. In particular if ϕ is a positive rational $n \times n$ matrix function then ϕ admits factorization. Such factorization is well known in prediction theory[1] but our particular viewpoint seems to be new.

4. Riesz Factorization and Weak Factorization

We introduce some terminology and show how weak factorization in an abstract, possibly noncommutative, Hardy space leads to the identification of the associated bounded Hankel forms.

Let H denote a complex algebra carrying norms $\| \ \|_1$, $\| \ \|_2$ such that $\|ab\|_1 \leq \|a\|_2 \|b\|_2$ for all a, b in H. We say that H has the *finite weak factorization* property if there exists a constant K_1 such that each element a in H admits a representation $a = b_1 c_1 + \cdots + b_n c_n$, with factors in H, such that

$$\|b_1\|_2 \|c_1\|_2 + \cdots + \|b_n\|_2 \|c_n\|_2 \leq K_1 \|a\|_1.$$

[1] P. Masani, Recent trends in multivariate prediction theory, *in* "Multivariate Analysis" (P. R. Krishnaiah, Ed.), pp. 351–382, Academic Press, New York, 1966.

The index n is unrestricted. If we can take K_1 equal to unity then we say that H admits *exact* finite weak factorization.

Denote the completions of H with respect to $\| \ \|_1$ and $\| \ \|_2$ by H^1 and H^2, respectively. A simple iterative argument shows that if H admits finite weak factorization with constant K_1 then H^1 admits *weak factorization* with constant K_2, in the following sense. Every element a in H^1 admits a representation $a = \sum_{k=1}^{\infty} b_k c_k$ with b_k, c_k in H and

$$\sum_{k=1}^{\infty} \|b_k\|_2 \, \|c_k\|_2 \leqslant K_2 \, \|a\|_1.$$

Moreover we can choose $K_2 > K_1$ to be arbitrary close to K_1. If $K_1 = 1$ we say that H^1 admits *almost exact* weak factorization. If in fact it is possible to take $K_2 = 1$ we say that H^1 admits *exact* weak factorization.

It is a simple consequence of the Riesz factorization of H^2 functions that the algebra of complex polynomials, endowed with the Hardy space norms, has the finite weak factorization property. In fact K_1 can be chosen arbitrarily greater than unity, and the length of the factorization can be restricted to two terms. Coifman, Rochberg, and Weiss [5] have shown that weak factorization is valid for the Hardy space of the sphere and ball in $\mathbb{C}^n$. It follows that the space of complex polynomials in n complex variables admits finite weak factorization.

A *bounded Hankel form* $[\ , \ ]$ on H is a bilinear form such that

$$[ab, c] = [a, bc]$$

for all a, b, c in H, and such that

$$|[a, c]| \leqslant K_3 \, \|a\|_2 \, \|c\|_2$$

for all a, c in H. Similarly we can define bounded Hankel forms on the completion H^2, where we take a, c in H^2 and b in H and require that H^2 be a two sided H-module.

A sequence r_n in H is said to be a $\| \ \|_2$-approximate identity if $\|ar_n - a\|_2 \to 0$ and $\|r_n a - a\|_2 \to 0$, as $n \to \infty$, for all a in H. The next lemma concerns Hankel forms on H, but clearly there is an analogous result for Hankel forms on H^2 when H^1 admits weak factorization. Trivial examples, with $H \cdot H = \{0\}$ for example, show that the approximate identity hypothesis cannot be dropped.

LEMMA 4.1. *Let H, $\| \ \|_1$, $\| \ \|_2$ be as above and suppose that H possesses the finite weak factorization property and a $\| \ \|_2$-approximate identity. Then for each bounded Hankel form $[\ , \ ]$ on H there exists a functional Φ in the dual space of H^1 such that $[a, b] = \Phi(ab)$ for a, b in H.*

Proof. Let r_n be the approximate identity. Define Φ on H by $\Phi(a) = [b_1, c_1] + \cdots + [b_m, c_m]$, where $a = b_1 c_1 + \cdots + b_m c_m$ is any factorization of a. Since

$$\begin{aligned}\sum_{k=1}^{m} [b_k, c_k] &= \lim_{n\to\infty} \sum_{k=1}^{m} [b_k, c_k r_n] \\ &= \lim_{n\to\infty} \sum_{k=1}^{m} [b_k c_k, r_n] \\ &= \lim_{n\to\infty} [a, r_n],\end{aligned}$$

the functional Φ is well defined. Moreover, by appropriate choice of factorization, we have

$$\begin{aligned}|\Phi(a)| &\leqslant \sum_{k=1}^{m} |[b_k, c_k]| \\ &\leqslant K_3 \sum_{k=1}^{m} \|b_k\|_2 \, \|c_k\|_1 \\ &\leqslant K_3 K_1 \, \|a\|_1.\end{aligned}$$

Thus Φ can be extended to a continuous linear functional on H^1 with norm no greater than $K_3 K_1$, where K_1 is the factorization constant, and K_3 is the norm of the form. This completes the proof.

It follows from the Hahn–Banach theorem that Φ, and therefore $[\ ,\ ]$, is implemented by an element of the dual of L^1, the natural enveloping Lebesgue space. For the contexts below this means that the Hankel form is implemented by an element x in $L^\infty(M, \tau)$, in the sense that $[a, b] = \tau(bxa)$. Moreover x can be chosen with $\|x\| = K_1 \, \|[\ ,\ ]\|$, where K_1 is the weak factorization constant and $\|[\ ,\ ]\|$ denotes the norm of the form, namely, the supremum of $|[a, b]|$ for a, b in the $\|\ \|_2$-unit ball of H^2.

The strongest form of weak factorization in H^1 is, of course, when every element h can be factored as $h_1 h_2$ with h_1, h_2 in H^1 and $\|h\|_1 = \|h_1\|_2 \, \|h_2\|_2$. We say that H^1 admits *Riesz factorization* in this case.

THEOREM 4.2. *Let τ be a faithful semifinite normal trace on $\mathcal{L}(\mathcal{H})$ and let $\mathcal{E}$ be a well ordered projection nest. Then $H^1(\mathcal{L}(\mathcal{H}), \mathcal{E}, \tau)$ admits Riesz factorization.*

Proof. Let h be an operator in H^1 with a polar decomposition $h = uc$. By Theorem 2.2 we may factor c as a^*a with a and ua^* leaving the nest invariant. Let $h_1 = ua^*$ and $h_2 = a$. Then $h = h_1 h_2$ is a Riesz factorization with respect to the von Neumann–Schatten norms as desired.

Similarly the space $H^1(M, \mathscr{E}, \tau)$ admits Riesz factorization when $\mathscr{E}$ is a well-ordered nest in the semifinite factor M. This may be seen by repeating the constructions of Theorems 2.1 and 2.2 in the context of $L^1(M, \tau)$, the details of which we leave to the reader. In the next three sections we obtain weak factorization in more general contexts. In Sections 5 and 6 we in fact only need Riesz factorization for finite nests (which does not require the construction of the measure $C(\Delta)$). In Section 7 we use completely different duality methods based on Arveson's distance formula.

5. Hyperfinite Nests and Purely Atomic Nests

We note two elementary settings wherein weak factorization and the characterization of Hankel forms is obtained easily by approximation through finite dimensional subalgebras.

Let M be the hyperfinite II_1 factor with a given sequence of nested matrix algebras $B_1 \subseteq B_2 \subseteq \cdots$ whose union is dense. Let $\mathscr{E}_n$ be a maximal projection nest in B_n such that $\mathscr{E}_1 \subseteq \mathscr{E}_2 \subseteq \cdots$. The weakly closed union $\mathscr{E}$ of these nests is a complete nest in M and determines a nest subalgebra $H^\infty(M, \mathscr{E})$. Moreover $H^\infty(M, \mathscr{E})$ is the weak operator topology closed union of the subalgebras $H^\infty(B_n, \mathscr{E}_n, \tau_n)$, where τ_n is the normalized trace on B_n. Similarly, writing τ for the normalized semifinite normal trace on M, $H^p(M, \mathscr{E}, \tau)$ is the $\| \ \|_p$-closed union of the isometrically embedded spaces $H^p(B_n, \mathscr{E}_n, \tau_n)$, for $1 \leqslant p \leqslant \infty$. We refer to the nest $\mathscr{E}$ as a *canonical nest* associated with M. Clearly it is maximal and continuous. The finite dimensional spaces $H^1(B_n, \mathscr{E}_n, \tau_n)$ admit Riesz factorization, by Theorem 2.2 and the proof of Theorem 4.2 (also see Shields [24]), and so $H^1(M, \mathscr{E}, \tau)$ admits almost exact weak factorization.

In a similar way, if $\mathscr{E}$ is a purely atomic nest, not necessarily compatible, in a semifinite factor M, then $H^1(M, \mathscr{E}, \tau)$ can be viewed as the closed union of a sequence of finite dimensional H^1 spaces and we obtain that $H^1(M, \mathscr{E}, \tau)$ admits almost exact weak factorization. The following two theorems now follow from the arguments of the last section.

THEOREM 5.1. *Let $\mathscr{E}$ be a canonical nest in the hyperfinite II_1 factor M and let τ denote the normalized trace. If* [,] *is a bounded Hankel form on $H^2(M, \mathscr{E}, \tau)$ then there exists an operator x in M such that $[a, b] = \tau(bxa)$ and $\|x\| = \|[\ ,\ ]\|$.*

THEOREM 5.2. *Let $\mathscr{E}$ be a purely atomic nest in the semifinite factor M with faithful semifinite normal trace τ. If* [,] *is a bounded Hankel form on $H^2(M, \mathscr{E}, \tau)$ then there exists an operator x in M such that $[a, b] = \tau(bxa)$ and $\|x\| = \|[\ ,\ ]\|$.*

6. Continuous Nests and Compatible Nests

We now characterize the Hankel forms on $H^2(M, \mathscr{E}, \tau)$ when M is a semifinite factor and $\mathscr{E}$ is any compatible nest. The weak factorization of $H^1(M, \mathscr{E}, \tau)$ is obtained through the decomposition $H^1 = L^1(D, \tau) + H_0^1$, where D is the diagonal algebra $H^\infty \cap (H^\infty)^*$, and the fact that H_0^1 admits almost exact weak factorization in case $\mathscr{E}$ is continuous. For this reason we only obtain the estimate $\|x\| \leqslant 3 \, \|[\ ,\ ]\|$ for the implementing operator. It may be that the constant 3 is just an artifact of our proof.

Proposition 6.1. *Let $\mathscr{E}$ be a continuous nest in a* II_1 *factor M and let*

$$H_0 = \operatorname{span}\{ex(1-f) \colon x \in M, e, f \in \mathscr{E}, e < f\}.$$

Then H_0 admits almost exact weak factorization.

Proof. Let x belong to H_0. Since $\mathscr{E}$ is continuous we may choose a sufficiently fine subnest $0 = e_0 < e_1 < \cdots < e_n = 1$ of $\mathscr{E}$ so that $\tau(e_j - e_{j-1}) = n^{-1}$ for $j = 1, \ldots, n$ and

$$x = e_1 x(1-e_4) + (e_2 - e_1)\, x(1-e_5) + \cdots + (e_{n-4} - e_{n-5})\, x(1-e_{n-1}).$$

Since M is a factor there is a partial isometry v with initial space equal to the range of $e_n - e_1$ and such that $ve_1 = 0$, $(e_j - e_{j-1})\, v = v(e_{j+1} - e_j)$ for $j = 1, \ldots, n-1$.

Let $w = v^{*2}$ and note that for $j = 4, \ldots, n-1$ we have

$$w(e_{j-3} - e_{j-4})\, x(1-e_j)\, w = (e_{j-1} - e_{j-2})\, wxw(1-e_{j-2})$$

and so wxw leaves the finite subnest invariant. By the proof of Theorem 4.2 there is a Riesz decomposition $wxw = rs$ with $\|wxw\|_1 = \|r\|_2 \, \|s\|_2$ and r, s operators in M that leave invariant the finite subnest. However $x = w^*wxwww^*$ and so $x = (w^*r)(sw^*)$ is a norm exact factorization. Since r and s leave invariant $e_1, \ldots, e_n$ it follows that w^*r and sw^* belong to H_0. To complete the proof we need only show that the subspace $H_0^1 = H_0^1(M, \mathscr{E}, \tau)$ defined in the Introduction, coincides with the $\|\ \|_1$ closure of H_0. This follows from the inequality $\|x\|_1 \leqslant \|x\|_2\, \tau(1)$ and elementary arguments (or from Theorem 7.1 below).

Theorem 6.2. *Let $\mathscr{E}$ be a compatible nest in the semifinite factor M with faithful normal semifinite trace τ. If $[\ ,\ ]$ is a bounded Hankel form on $H^2(M, \mathscr{E}, \tau)$ then there exists an operator x in M such that $[a, b] = \tau(bxa)$ and $\|x\| \leqslant 3 \, \|[\ ,\ ]\|$.*

Proof. Suppose first that M is a finite factor. To establish the theorem in this case it will be enough to show that H admits weak factorization with constant arbitrarily close to 3. There is a $\|\ \|_1$-continuous projection E_1 from $L^1(M, \tau)$ to $L^1(D, \tau)$, where $D = H^\infty \cap (H^\infty)^*$ and $L^1(D, \tau)$ is identified with the $\|\ \|_1$-closure of D in $L^1(M, \tau)$. In fact let E_Δ be the projection on B defined by a finite subnest Δ of $\mathscr{E}$, where $E_\Delta(x) = \sum qxq$, the sum being taken over the atoms q of Δ. Then $\lim_\Delta \|E_\Delta(x) - E(x)\|_2 = 0$, for x in M, where E is the normal expectation of M onto D. Hence $\lim_\Delta \|E_\Delta(x) - E(x)\|_1 = 0$, and so E_1 can be defined as the continuous extension of E, and $\|E_1\| = 1$. Since $\tau(x) = \tau(E(x))$ it follows that H_0^1 is the kernel of the restriction of E_1 to H^1 and that $H^1 = L^1(D, \tau) + H_0^1$. If $x = k + h$ with k in $L^1(D, \tau)$ and h in H_0^1 then $\|k\|_1 \leqslant \|x\|_1$ and $\|h\|_1 \leqslant 2\,\|x\|_1$. Since k can be exactly factored in terms of $L^2(D, \tau)$, which is contained in H^2, we will obtain the required factorization if we show that H_0^1 admits almost exact weak factorization with respect to H^2 (not H_0^2!). When $\mathscr{E}$ is continuous we have already observed this in Proposition 6.1. Since M has no minimal projections there exists a continuous nest $\mathscr{N}$ in M that contains $\mathscr{E}$. Observe that $H_0^1(M, \mathscr{E}, \tau)$ is contained in $H_0^1(M, \mathscr{N}, \tau)$ and that $H_0^2(M, \mathscr{N}, \tau)$ is contained in $H^2(M, \mathscr{E}, \tau)$. In view of Proposition 6.1, $H_0^1(M, \mathscr{N}, \tau)$ admits almost exact weak factorization relative to $H_0^2(M, \mathscr{N}, \tau)$ and so $H_0^1(M, \mathscr{E}, \tau)$ admits almost exact weak factorization relative to $H^2(M, \mathscr{E}, \tau)$, as desired.

To deduce the general case use the compatibility of $\mathscr{E}$ to obtain a sequence p_n of projections in the weak closure of $\mathscr{E}$ that converge strongly to the identity. Since $\mathscr{E}_n = p_n\mathscr{E}$ is a nest in the finite factor $M_n = p_n M p_n$ the theorem applies and the restriction of $[\ ,\ ]$ to $H^2(M_n, \mathscr{E}_n, \tau)$ is implemented by an operator x_n, of appropriate norm, in M_n. It follows that $[\ ,\ ]$ is implemented by any weak operator topology cluster point of $\{x_n\}$, and this completes the proof.

7. Duality Methods

Returning now to the context of an arbitrary nest $\mathscr{E}$ in a semifinite factor M we have the following variant of Arveson's distance formula,

$$\operatorname{dist}(x, H^\infty(M, \mathscr{E}, \tau)) = \sup_{e \in \mathscr{E}} \|(1 - e)\, xe\|.$$

This can be obtained from the proof given in [19] of Arveson's distance formula and which is based on constructive arguments of Parrott [17] for the 2×2 case. These constructions involve only the factors in the polar decompositions of compressions of x and so the distance from x to the full nest algebra associated with $\mathscr{E}$ is achieved by an element of M.

The Banach space $H_0^1(M, \mathscr{E}, \tau)$ is the preannihilator of H^∞, and so has a dual space that is naturally isometrically isomorphic to L^∞/H^∞. It follows from this duality and the distance formula that the unit ball of H_0^1 is the closed convex hull of elements of the form $h = ey(1-e)$, where e is in $\mathscr{E}$, y in L^1 and $\|y\|_1 \leqslant 1$. By an elementary approximation argument every element h in H_0 admits a decomposition $h = \sum_{k=1}^{\infty} h_k$, where $\sum_{k=1}^{\infty} \|h_k\|_1 \leqslant (1+\varepsilon)\|h\|_1$ and h_k has the special form $h_k = e_k h_k (1-e_k)$ with e_k in $\mathscr{E}$. We now factorize these elementary block operators to obtain an almost exact weak factorization for H_0^1 relative to H^2 and H_0^2.

THEOREM 7.1. *Let $\mathscr{E}$ be a projection nest in the semifinite factor M with faithful semifinite normal trace τ, let h belong to $H_0^1(M, \mathscr{E}, \tau)$ and let $\varepsilon > 0$. Then there exist elements $x_1, x_2, \ldots$ in $H_0^2(M, \mathscr{E}, \tau)$ and elements $y_1, y_2, \ldots$ in $H^2(M, \mathscr{E}, \tau)$ such that*

(i) $h = \sum_{k=1}^{\infty} x_k y_k$

(ii) $\sum_{k=1}^{\infty} \|x_k\|_2 \|y_k\|_2 < (1+\varepsilon)\|h\|_1$.

Proof. We may assume that $h = eh(1-e)$ for some e in $\mathscr{E}$. Write L_1^1 and L_1^2 for the unit balls of L^1 and L^2. Suppose first that $e_- < e$, where $e_- = \sup\{g \colon g < e, g \text{ in } \mathscr{E}\}$. Then $eL^2(e-e_-)$ is contained in H^2 and $(e-e_-)L^2(1-e)$ is contained in H_0^2. It will be sufficient then to show that L_1 is contained in the $\| \ \|_1$-closed convex hull of the set $F = \{x(e-e_-)y \colon x, y \in L_1^2\}$. Fix z in L_1^1 and let $z = z_1 z_2$ with z_1, z_2 in L_1^2. Let $\{q_n\}$ be an orthogonal family of self-adjoint projections each of which is equivalent to a subprojection of $e-e_-$, and such that $\sum q_n = 1$. Let $\alpha_n = \|z_1 q_n\|_2 \|q_n z_2\|_2$ and $w_n = \alpha_n^{-1} z_1 q_n z_2$ so that $\|w_n\|_1 \leqslant 1$ and $z = \sum \alpha_n w_n$. By the Cauchy–Schwarz inequality $\sum \alpha_n \leqslant 1$. Since $q_n = u_n(e-e_-)v_n$ for some partial isometries u_n, v_n in M it follows that w_n belongs to F, and that z lies in the closed convex hull of F, as desired.

If, on the other hand, $e = e_-$ then there is a projection f in $\mathscr{E}$ with $f < e$ and $\|(f-e)h\|_1 < \varepsilon/2$. Let $h_1 = h - (f-e)h$ so that $\|h - h_1\| < \varepsilon/2\ \|h\|$ and $h_1 = fh_1(1-e)$. Now $fL^2(e-f)$ and $(e-f)L^2(1-e)$ belong to H_0^2, and so we may obtain an almost exact weak factorization of h_1 relative to H_0^2 if we show that L_1^1 is contained in the $\| \ \|_1$-convex hull of $L_1^2(e-f)L_1^2$. This follows as above. A simple iterative argument completes the proof.

8. HANKEL OPERATORS

Let P and P_0 be the orthogonal projections from $L^2(M, \tau)$ onto $H^2(M, \mathscr{E}, \tau)$ and $H_0^1(M, \mathscr{E}, \tau)$, respectively. Define the Hankel operator $H_x = (I-P)L_x P$ on L^2, where L_x is the operator of left multiplication by the operator x in M. Let J be the conjugate linear isometry $y \to y^*$ on L^2

and note that $J(I-P)=P_0J$. Thus for h in H^∞, h_1 in H^2 and h_0 in H_0^2 we have

$$\begin{aligned}(JH_xh_1, hh_0) &= (P_0J(xh_1), hh_0)\\ &= (J(xh_1), hh_0)\\ &= (h^*J(xh_1), h_0)\\ &= (J(xh_1h), h_0)\\ &= (JH_x(h_1h), h_0).\end{aligned}$$

Define $[h_1, h_0]=(h_0, JH_xh_1)$ and we thereby establish a correspondence between bounded Hankel operators H_x and bounded Hankel forms $[\ ,\ ]$ on $H^2\times H_0^2$. Moreover $[h_1, h_0]=\tau(h_0xh_1)$. By Theorem 7.1 this form determines a bounded linear functional on H^1 whose norm is the operator norm $\|H_x\|$. By the Hahn–Banach theorem the functional is implemented by an operator y with $\|y\|=\|H_x\|$. Thus $H_x=H_y$ and so $x-y$ belongs to H^∞, the set of symbol operators that determine the zero Hankel operator. Thus we have the following Nehari type theorem in semifinite factors. The I_∞ case is also in [22].

THEOREM 8.1. *Let $\mathscr{E}$ be a projection nest in the semifinite factor M with faithful semifinite normal trace τ. Let x be an operator of M that determines the Hankel operator H_x on $L^2(M, \tau)$. Then*

$$\|H_x\| = \operatorname{dist}(x, H^\infty(M, \mathscr{E}, \tau)).$$

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Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium