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Foundations of algebraic surgery

Andrew Ranicki*

Department of Mathematics and Statistics
University of Edinburgh, Scotland, UK

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LNS

*aar@maths.ed.ac.uk

Abstract

The algebraic theory of surgery on chain complexes C with Poincaré duality

$$H^*(C) \;\cong\; H_{n-*}(C)$$

describes geometric surgeries on the chain level. The algebraic effect of a geometric surgery on an n-dimensional manifold M is an algebraic surgery on the n-dimensional symmetric Poincaré complex (C,ϕ) over $\mathbb{Z}[\pi_1(M)]$ with the homology of the universal cover $\widetilde{M}$

$$H_*(C) \;=\; H_*(\widetilde{M}) \; .$$

The algebraic effect of a geometric surgery on an n-dimensional normal map $(f,b):M\to X$ is an algebraic surgery on the kernel n-dimensional quadratic Poincaré complex (C,ψ) over $\mathbb{Z}[\pi_1(X)]$ with homology

$$H_*(C) \;=\; K_*(M) \;=\; \ker(f_*:H_*(\widetilde{M})\to H_*(\widetilde{X})) \; .$$

For $n>4$ and i-connected (f,b) with $2i\leqslant n$ there is a one-one correspondence between geometric surgeries on (f,b) killing elements $x\in K_i(M)$ and algebraic surgeries on (C,ψ) killing $x\in H_i(C)$. The Wall surgery obstruction of an n-dimensional normal map $(f,b):M\to X$

$$\sigma_*(f,b)\in L_n(\mathbb{Z}[\pi_1(X)])$$

was originally defined by first making (f,b) $[n/2]$-connected by geometric surgery below the middle dimension, using forms for even n and automorphisms of forms for odd n. The algebraic theory of surgery identifies $\sigma_*(f,b)$ with the cobordism class of the kernel quadratic Poincaré complex (C,ψ), so the algebraic surgery obstruction has the same formulation for odd and even n. The identification is used for $n=2i$ (resp. $2i+1$) to find a representative form (resp. automorphism) without preliminary geometric surgeries.

Keywords: surgery, quadratic form, chain complex, Poincaré duality

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Contents

1 Introduction

We compare the homology and chain level descriptions of surgery on a manifold, using a minimum of algebraic development.

Manifolds M are to be finite-dimensional, compact, and oriented (unless stated otherwise), with $C(M)$ denoting the cellular chain complex for some CW structure on M.

Cobordisms $(W; M, M')$ are to be oriented (unless stated otherwise) with $\partial W = M \cup -M'$, where $-M'$ denotes M' with the opposite orientation.

1.1 Background

The surgery method of classifying manifolds within a homotopy type was first applied by Kervaire and Milnor [2] to exotic spheres, using exact sequences to describe the homology effect of geometric surgery. Homology was quite adequate for the subsequent development of surgery theory on simply-connected manifolds (Browder [1], Novikov). Wall [10] used a combination of topology and homology to describe the effect of surgery on non-simply-connected manifolds. In general, the homology $\mathbb{Z}[\pi_1(M)]$-modules $H_*(\widetilde{M})$ of the universal cover $\widetilde{M}$ of a compact manifold M are not finitely generated, so a chain level approach is indicated. The algebraic theory of surgery of Ranicki [4],[5] provided a model for surgery using chain complexes with Poincaré duality.

Surgery was originally developed for differentiable manifolds, but has since been extended to PL and topological manifolds. The algebraic theory of surgery applies to all categories of manifolds.

1.2 The algebraic effect of a geometric surgery

Let M be an n-dimensional manifold. Surgery on $S^i \times D^{n-i} \subset M$ results in an n-dimensional manifold

$$M' = (M \backslash S^i \times D^{n-i}) \cup D^{i+1} \times S^{n-i-1} .$$

The trace of the surgery is the cobordism $(W; M, M')$ given by attaching a $(i+1)$-handle at $S^i \times D^{n-i} \subset M$

$$W = M \times I \cup D^{i+1} \times D^{n-i} .$$

The trace of the surgery on $D^{i+1} \times S^{n-i-1} \subset M'$ is the cobordism $(W'; M', M)$ with

$$W' = -W = M' \times I \cup D^{i+1} \times D^{n-i} .$$

In fact, every cobordism of manifolds is a union of the traces of surgeries.

In terms of homotopy theory the trace W is obtained from M by attaching an $(i+1)$-cell, and M' is then obtained from W by detaching an $(n-i)$-cell, with homotopy equivalences

$$W \simeq M \cup_x D^{i+1} \simeq M' \cup_{x'} D^{n-i} ,$$

with $x : S^i \to M$ the inclusion $S^i \times \{0\} \subset S^i \times D^{n-i} \subset M$, and similarly for $x' : S^{n-i-1} \to M'$. The immediate homology effect of the surgery is to kill $x \in H_i(M)$,

$$H_i(W) \;=\; H_i(M)/\langle x\rangle$$

with $\langle x\rangle \subseteq H_i(M)$ the subgroup generated by x. On the chain level

(i) $C(W)$ is chain equivalent to the algebraic mapping cone $\mathcal{C}(x)$ of a chain map $x : S^i\mathbb{Z} \to C(M)$ representing $x \in H_i(M)$, where

$$S^i\mathbb{Z} \;:\; \dots \to 0 \to \mathbb{Z} \to 0 \to \dots \quad \text{(concentrated in degree } i\text{)} ,$$

and similarly for $C(W) \simeq \mathcal{C}(x' : S^{n-i-1}\mathbb{Z} \to C(M'))$,

(ii) there is defined a commutative braid of chain homotopy exact sequences of chain complexes

$$\begin{tikzcd}
S^i\mathbb{Z} \arrow[dr] \arrow[rr, bend left, "x"] & & C(M) \arrow[dr] \arrow[rr, bend left, "x^*"] & & S^{n-i}\mathbb{Z} \\
& C(W, M \cup M')_{*+1} \arrow[ur] \arrow[dr] & & C(W) \arrow[ur, "y"] \arrow[dr, "y'"'] & \\
S^{n-i-1}\mathbb{Z} \arrow[ur] \arrow[rr, bend right, "x'"'] & & C(M') \arrow[ur] \arrow[rr, bend right, "x'^*"'] & & S^{i+1}\mathbb{Z}
\end{tikzcd}$$

with $x^* : C(M) \to S^{n-i}\mathbb{Z}$ a chain map representing the Poincaré dual $x^* \in H^{n-i}(M)$ of $x \in H_i(M)$, and similarly for x'^*,

(iii) $C(M')$ is chain equivalent to the dimension shifted algebraic mapping cone $\mathcal{C}(y)_{*+1}$ of a chain map $y : C(W) \to S^{n-i}\mathbb{Z}$ representing a cohomology class $y \in H^{n-i}(W)$ with image the Poincaré dual $x^* \in H^{n-i}(M)$ of $x \in H_i(M)$, and similarly for $C(M)$.

Algebraic surgery gives a precise algebraic model for a chain complex in the chain homotopy type of $C(M')$, which is obtained from $C(M)$ by attaching x and detaching y.

The homology groups $H_*(M), H_*(M'), H_*(W)$ are related by the long exact sequences

$$\dots \to H_r(M) \to H_r(W) \to H_r(W, M) \to H_{r-1}(M) \to \dots \;,$$
$$\dots \to H_r(M') \to H_r(W) \to H_r(W, M') \to H_{r-1}(M') \to \dots \;.$$

It now follows from the excision isomorphisms

$$H_r(W, M) \;=\; H_r(D^{i+1}, S^i) \;=\; \begin{cases} \mathbb{Z} & \text{for } r = i+1 \\ 0 & \text{otherwise ,} \end{cases}$$

$$H_r(W, M') \;=\; H_r(D^{n-i}, S^{n-i-1}) \;=\; \begin{cases} \mathbb{Z} & \text{for } r = n-i \\ 0 & \text{otherwise} \end{cases}$$

that

$$H_r(M) \ = \ H_r(W) \ = \ H_r(M') \text{ for } r \neq i, i+1, n-i-1, n-i \ .$$

The relationship between $H_r(M), H_r(M'), H_r(W)$ for $r = i, i+1, n-i-1, n-i$ is more complicated, especially in the middle dimensional cases $n = 2i, 2i+1$.

Here are some of the advantages of chain complexes over homology in describing the algebraic effects of surgery on manifolds. The chain complex method :

- (•) makes it easier to follow the passage from the embedding $S^i \times D^{n-i} \subset M$ to the homology $H_*(M')$ on the chain level;
- (•) provides a uniform description for all n, i ;
- (•) avoids the indeterminacies inherent in exact sequences;
- (•) works just as well in the non-simply connected case;
- (•) keeps track of the effect of successive surgeries.

Surgery on manifolds is described algebraically by surgery on chain complexes with symmetric Poincaré duality. The applications of surgery to the classification of manifolds involve a normal map $(f, b) : M \to X$, and only surgeries with an extension of (f, b) to a normal map on the trace

$$((g, c); (f, b), (f', b')) \ : \ (W; M, M') \to X \times ([0, 1]; \{0\}, \{1\})$$

are considered. Surgery on normal maps is described algebraically by surgery on chain complexes with quadratic Poincaré duality. The quadratic refinement corresponds to the additional information carried by the bundle map $b : \nu_M \to \nu_X$. The formulae for algebraic surgery on symmetric Poincaré complexes are entirely analogous to the formulae for quadratic Poincaré complexes.

1.3 The Principle of Algebraic Surgery

In its simplest form, the Principle states that for a cobordism of n-dimensional manifolds $(W; M, M')$ the chain homotopy type of $C(M')$ and the Poincaré duality chain equivalence

$$[M'] \cap - \ : \ C(M')^{n-*} \ \simeq \ C(M')$$

can be obtained from

- (i) the chain homotopy type of $C(M)$,
- (ii) the Poincaré duality chain equivalence

  $$\phi_0 \ = \ [M] \cap - \ : \ C(M)^{n-*} \ \simeq \ C(M)$$

  and the chain homotopy

  $$\phi_1 \ : \ (\phi_0)^* \ \simeq \ \phi_0 \ : \ C(M)^{n-*} \to C(M)$$

  determined up to higher chain homotopies by topology,

(iii) the chain homotopy class of the chain map $j : C(M) \to C(W, M')$ induced by the inclusion $M \subset W$,

(iv) the chain homotopy

$$\delta\phi_0 \;:\; j\phi_0 j^* \;\simeq\; 0 \;:\; C(W,M')^{n-*} \to C(W,M')$$

determined up to higher chain homotopies by topology.

The chain complex $C(M')$ is chain equivalent to the chain complex C' obtained from $C(M)$ by algebraic surgery, with

$$C'_r \;=\; C_r(M) \oplus C_{r+1}(W,M') \oplus C^{n-r-1}(W,M') \;.$$

See §3 for formulae for the differentials and Poincaré duality of C'.

In particular, if $(W; M, M')$ is the trace of a surgery on $S^i \times D^{n-i} \subset M$ as in §1.2 then $C(W,M')$ is chain equivalent to $S^{n-i}\mathbb{Z}$, and replacing $C(W,M')$ by $S^{n-i}\mathbb{Z}$ in the formula for C'_r gives a smaller chain complex (also denoted by C')

$$\begin{aligned} C' \;:\; C_n(M) \to \dots \to C_{n-i}(M) \xrightarrow{d\oplus y} C_{n-i-1}(M) \oplus \mathbb{Z} \xrightarrow{d\oplus 0} C_{n-i-2}(M) \to \dots \\ \to C_{i+2}(M) \xrightarrow{d\oplus 0} C_{i+1}(M) \oplus \mathbb{Z} \xrightarrow{d\oplus x} C_i(M) \to \dots \to C_0(M) \end{aligned}$$

chain equivalent to $C(M')$. The attaching chain map $x : S^i\mathbb{Z} \to C(M)$ and the chain map $j : C(M) \to C(W,M') \simeq S^{n-i}\mathbb{Z}$ in (iii) are determined by the homotopy class of the core embedding $S^i \times \{0\} \subset M$. The detaching chain map $y : \mathcal{C}(x) \to S^{n-i}\mathbb{Z}$ and the chain homotopy $\delta\phi_0$ in (iv) are determined by the framing of the core, and are much more subtle. (See the Examples below). In this case the algebraic surgery kills the homology class $x \in H_i(M)$. In the general algebraic context surgery kills entire subcomplexes rather than just individual homology classes.

Example. The effect of surgery on $S^0 \times D^1 \subset M = S^1$ is a double cover of S^1. There are two possibilities:
(i) If the two paths $S^0 \times D^1 \subset S^1$ move in opposite senses the effect of the surgery is the trivial double cover $M' = S^1 \cup S^1$ of S^1, and the trace $(W; M, M')$ is given by the orientable

$$W \;=\; \mathrm{cl.}(S^2 \backslash (D^2 \cup D^2 \cup D^2)) \;.$$

(ii) If the two paths $S^0 \times D^1 \subset S^1$ move in the same sense the effect of the surgery is the nontrivial double cover $M'' = S^1$ of S^1, and the trace $(W'; M, M'')$ is given by the nonorientable

$$W' \;=\; \mathrm{cl.}(\text{Möbius band}\backslash D^2) \;. \qquad \square$$

More generally:

Example. As usual, let $O(j)$ be the orthogonal group of $\mathbb{R}^j$. For any map $\omega : S^i \to O(j)$ write $n = i + j$, and define an embedding

$$e_\omega \;:\; S^i \times D^j \to S^n = S^i \times D^j \cup D^{i+1} \times S^{j-1} \;;\; (x,y) \mapsto (x, \omega(x)(y)) \;.$$

Surgery on $M = S^n$ killing e_ω has effect the $(j-1)$-sphere bundle over $S^{i+1} = D^{i+1} \cup D^{i+1}$

$$M' \ = \ S(\omega) \ = \ D^{i+1} \times S^{j-1} \cup_\omega D^{i+1} \times S^{j-1}$$

of the j-plane vector bundle over S^{i+1}

$$E(\omega) \ = \ D^{i+1} \times \mathbb{R}^j \cup_\omega D^{i+1} \times \mathbb{R}^j$$

classified by $\omega \in \pi_i(O(j)) = \pi_{i+1}(BO(j))$. The trace of the surgery is

$$W \ = \ \mathrm{cl.}(D(\omega)\backslash D^{n+1}) \ ,$$

with

$$D(\omega) \ = \ D^{i+1} \times D^j \cup_\omega D^{i+1} \times D^j$$

the j-disk bundle of ω, which fits into a fibre bundle

$$(D^j, S^{j-1}) \to (D(\omega), S(\omega)) \to S^{i+1} \ .$$

Exercise: work out the algebraic effect of the surgery! $\square$

2 Forms and formations

The quadratic L-groups $L_*(A)$ were originally defined by Wall, with $L_{2i}(A)$ a Witt-type group of stable isomorphism classes of nonsingular $(-)^i$-quadratic forms over a ring with involution A, and $L_{2i+1}(A)$ a Whitehead-type group of automorphisms of $(-)^i$-quadratic forms over A (now replaced by formations). The surgery obstruction of a normal map $(f,b) : M \to X$ from an n-dimensional manifold M to an n-dimensional Poincaré complex X

$$\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)])$$

was defined by first making (f,b) i-connected for $n = 2i$ (resp. $2i+1$) by surgery below the middle dimension. The surgery obstruction is such that $\sigma_*(f,b) = 0$ if (and for $n > 4$ only if) (f,b) is normal bordant to a homotopy equivalence.

Let A be an associative ring with 1, and with an involution $A \to A; a \mapsto \overline{a}$ satisfying

$$\overline{a+b} \ = \ \overline{a} + \overline{b} \ , \ \overline{ab} \ = \ \overline{b}\overline{a} \ , \ \overline{\overline{a}} \ = \ a \ , \ \overline{1} \ = \ 1 \ .$$

In the applications to topology $A = \mathbb{Z}[\pi]$ is a group ring with the involution

$$\mathbb{Z}[\pi] \to \mathbb{Z}[\pi] \ ; \ x \ = \ \sum_{g\in\pi} n_g g \mapsto \overline{x} \ = \ \sum_{g\in\pi} n_g g^{-1} \ .$$

The dual of a left A-module is the left A-module

$$K^* \ = \ \mathrm{Hom}_A(K,A) \ , \ A \times K^* \to K^* \ ; \ (a,f) \mapsto (x \mapsto f(x)\overline{a}) \ .$$

The dual of an A-module morphism $f : K \to L$ is the A-module morphism

$$f^* \ : \ L^* \to K^* \ ; \ g \mapsto (x \mapsto g(f(x))) \ .$$

For f.g. free K, L identify

$$f^{**} \ = \ f \ : \ K^{**} \ = \ K \to L^{**} \ = \ L \ ,$$

using the isomorphism $K \to K^{**}; x \mapsto (f \mapsto \overline{f(x)})$ to identify $K = K^{**}$, and similarly for L.

A $(-)^i$-*quadratic form* (K, λ, μ) is a f.g. free A-module K together with a $(-)^i$-symmetric form

$$\lambda \ = \ (-)^i \lambda^* \ : \ K \to K^*$$

and a function

$$\mu \ : \ K \to Q_{(-)^i}(A) \ = \ A/\{a - (-)^i \overline{a} \,|\, a \in A\}$$

such that

$$\lambda(x)(x) \ = \ \mu(x) + (-)^i \overline{\mu(x)} \ , \ \mu(ax) \ = \ a\mu(x)\overline{a} \ , \ \mu(x+y) \ = \ \mu(x) + \mu(y) + \lambda(x,y) \ .$$

The form is *nonsingular* if $\lambda : K \to K^*$ is an isomorphism.

A *lagrangian* for a nonsingular $(-)^i$-quadratic form (K, λ, μ) is a f.g. free direct summand $L \subset K$ such that $\lambda(L)(L) = 0$, $\mu(L) = 0$, and $L = L^\perp$, where

$$L^\perp \ = \ \{x \in K \,|\, \lambda(x)(L) = 0\} \ .$$

A nonsingular form admits a lagrangian if and only if it is isomorphic to the hyperbolic form

$$H_{(-)^i}(L) \ = \ (L \oplus L^*, \begin{pmatrix} 0 & 1 \\ (-)^i & 0 \end{pmatrix}, \mu)$$

with $\mu(x, f) = f(x)$.

The $2i$-*dimensional quadratic* L-*group* $L_{2i}(A)$ is the Witt group of stable isomorphism classes of nonsingular $(-)^i$-quadratic forms (K, λ, μ) over A, where stability is with respect to the hyperbolic forms.

As in Chapter 5 of Wall [10] the surgery obstruction of an i-connected $2i$-dimensional normal map $(f, b) : M \to X$ is the Witt class

$$\sigma_*(f, b) \ = \ (K_i(M), \lambda, \mu) \in L_{2i}(\mathbb{Z}[\pi_1(X)])$$

of the kernel nonsingular $(-)^i$-quadratic form $(K_i(M), \lambda, \mu)$ over $\mathbb{Z}[\pi_1(X)]$, with λ, μ defined by geometric intersection numbers. An algebraic surgery on (f, b) removing $S^j \times D^{2i-j} \subset M$ for $j = i-1$ (resp. i) correspond to the algebraic surgery of the addition (resp. subtraction) of the hyperbolic form $H_{(-)^i}(\mathbb{Z}[\pi_1(X)])$ to (resp. from) the kernel form.

A *nonsingular* $(-)^i$-*quadratic formation* $(K, \lambda, \mu; F, G)$ is a nonsingular $(-)^i$-quadratic form (K, λ, μ) together with an ordered pair of lagrangians F, G.

The $(2i+1)$-*dimensional quadratic* L-*group* $L_{2i+1}(A)$ is the group of stable isomorphism classes of nonsingular $(-)^i$-quadratic formations $(K, \lambda, \mu; F, G)$ over A, where stability is with respect to the formations such that either F, G are direct complements in K or share a common lagrangian complement in K.

The surgery obstruction of an i-connected $(2i+1)$-dimensional normal map $(f,b) : M \to X$

$$\sigma_*(f,b) \;=\; (K,\lambda,\mu;F,G) \in L_{2i+1}(\mathbb{Z}[\pi_1(X)])$$

is the Witt-type equivalence class of a kernel $(-)^i$-quadratic formation over $\mathbb{Z}[\pi_1(X)]$ with

$$F \cap G \;=\; K_{i+1}(M) \quad , \quad K/(F+G) \;=\; K_i(M) \; .$$

As in Chapter 6 of Wall [10] such a kernel formation $(K,\lambda,\mu;F,G)$ is obtained by realizing any finite set $\{x_1,x_2,\dots,x_k\} \subset K_i(M)$ of $\mathbb{Z}[\pi_1(X)]$-module generators by a high-dimensional Heegaard-type decomposition of (f,b) as a union of normal maps

$$(f,b) \;=\; (f_0,b_0) \cup (g,c) \;:\; M \;=\; M_0 \cup U \to X \;=\; X_0 \cup D^{2i+1}$$

with

$$\begin{aligned}
&(g,c) \;:\; (U,\partial U) \;=\; (\#_k S^i \times D^{i+1}, \#_k S^i \times S^i) \to (D^{2i+1}, S^{2i}) \; , \\
&F \;=\; \mathrm{im}(K_{i+1}(U,\partial U) \to K_i(\partial U)) \;=\; \mathbb{Z}[\pi_1(X)]^k \; , \\
&G \;=\; \mathrm{im}(K_{i+1}(M_0,\partial U) \to K_i(\partial U)) \;\cong\; \mathbb{Z}[\pi_1(X)]^k \; , \\
&K \;=\; K_i(\partial U) \;=\; F \oplus F^* \; , \; (\lambda,\mu) \;=\; \text{hyperbolic } (-)^i\text{-quadratic form} \; .
\end{aligned}$$

3 Surgery on symmetric Poincaré complexes

Symmetric Poincaré complexes are chain complexes with the Poincaré duality properties of manifolds. A manifold M determines a symmetric Poincaré complex (C,ϕ), such that a surgery on M determines an algebraic surgery on (C,ϕ). However, not every algebraic surgery on (C,ϕ) can be realized by a surgery on M.

Given a f.g. free A-module chain complex

$$C \;:\; \cdots \to C_{r+1} \xrightarrow{d} C_r \xrightarrow{d} C_{r-1} \to \dots$$

write the dual f.g. free A-modules as

$$C^r \;=\; (C_r)^* \; .$$

For any $n \geqslant 0$ let C^{n-*} be the f.g. free A-module chain complex with

$$d_{C^{n-*}} \;=\; (-)^r d_C^* \;:\; (C^{n-*})_r \;=\; C^{n-r} \to (C^{n-*})_{r-1} \;=\; C^{n-r+1} \; .$$

The duality isomorphisms

$$T \;:\; \mathrm{Hom}_A(C^p, C_q) \to \mathrm{Hom}_A(C^q, C_p) \; ; \; \phi \mapsto (-)^{pq}\phi^*$$

are involutions with the property that the dual of a chain map $f : C^{n-*} \to C$ is a chain map $Tf : C^{n-*} \to C$, with $T(Tf) = f$.

The algebraic mapping cone $\mathcal{C}(f)$ of a chain map $f : C \to D$ is the chain complex with

$$d_{C(f)} = \begin{pmatrix} d_D & (-1)^r f \\ 0 & d_C \end{pmatrix} : C(f)_r = D_r \oplus C_{r-1} \to C(f)_{r-1} = D_{r-1} \oplus C_{r-2} .$$

An *n-dimensional symmetric complex* (C, ϕ) over A is a f.g. free A-module chain complex

$$C : C_n \xrightarrow{d_C} C_{n-1} \to \cdots \to C_1 \xrightarrow{d_C} C_0$$

together with a collection of A-module morphisms

$$\phi = \{\phi_s : C^{n-r+s} \to C_r \mid s \geqslant 0\}$$

such that

$$d_C\phi_s + (-1)^r\phi_s d_C^* + (-1)^{n+s-1}(\phi_{s-1} + (-1)^s T\phi_{s-1}) = 0 : C^{n-r+s-1} \to C_r$$
$$(s \geqslant 0, \phi_{-1} = 0) .$$

Thus $\phi_0 : C^{n-*} \to C$ is a chain map, ϕ_1 is a chain homotopy $\phi_1 : \phi_0 \simeq T\phi_0 : C^{n-*} \to C$, and so on ... . More intrinsically, ϕ is an n-dimensional cycle in the $\mathbb{Z}$-module chain complex

$$\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_A(C^*, C))$$

with

$$W : \cdots \to \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2]$$

the free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$. The symmetric complex (C, ϕ) is *Poincaré* if the chain map $\phi_0 : C^{n-*} \to C$ is a chain equivalence.

Example. (Mishchenko [3]) An n-dimensional manifold M and a regular covering $\widetilde{M}$ with group of covering translations π determine an n-dimensional symmetric Poincaré complex over $\mathbb{Z}[\pi]$ $(C(\widetilde{M}), \phi)$ with

$$\phi_0 = [M] \cap - : C(\widetilde{M})^{n-*} \to C(\widetilde{M})$$

the Poincaré duality chain equivalence. The higher chain homotopies $\phi_1, \phi_2, \ldots$ are determined by an equivariant analogue of the construction of the Steenrod squares. □

An $(n+1)$-*dimensional symmetric pair* $(j : C \to D, (\delta\phi, \phi))$ is an n-dimensional symmetric complex (C, ϕ) together with a chain map $j : C \to D$ to an $(n+1)$-dimensional f.g. free A-module chain complex D and A-module morphisms $\delta\phi = \{\delta\phi_s : D^{n+1-r+s} \to D_r \mid s \geqslant 0\}$ such that

$$j\phi_s j^* = d_D\delta\phi_s + (-)^r\delta\phi_s d_D^* + (-)^{n+s+1}(\delta\phi_{s-1} + (-)^s T\delta\phi_{s-1}) : D^{n+1-r-s} \to D_r$$
$$(s \geqslant 0, \delta\phi_{-1} = 0) .$$

The pair is *Poincaré* if the chain map

$$\begin{pmatrix} \delta\phi_0 \\ \phi_0 j^* \end{pmatrix} : D^{n+1-*} \to \mathcal{C}(j)$$

is a chain equivalence, in which case (C,ϕ) is a n-dimensional symmetric Poincaré complex.

A *cobordism* of n-dimensional symmetric Poincaré complexes (C,ϕ), (C',ϕ') is an $(n+1)$-dimensional symmetric Poincaré pair of the type $(C\oplus C'\to D,(\delta\phi,\phi\oplus -\phi'))$. Symmetric complexes (C,ϕ), (C',ϕ') are *homotopy equivalent* if there exists a cobordism with $C\to D$, $C'\to D$ chain equivalences.

Example. A 0-dimensional symmetric complex (C,ϕ) is a f.g. free A-module C_0 together with a symmetric form ϕ_0 on C^0. The complex is Poincaré if and only if the form is nonsingular. Two 0-dimensional symmetric Poincaré complexes (C,ϕ), (C',ϕ') are cobordant if and only if the forms (C^0,ϕ_0), (C'^0,ϕ'_0) are Witt-equivalent, i.e. become isomorphic after stabilization with metabolic forms $(L\oplus L^*, \begin{pmatrix}\lambda & 1\\ 1 & 0\end{pmatrix})$. □

Example. An $(n+1)$-dimensional manifold with boundary $(W,\partial W)$ and cover $(\widetilde{W},\partial\widetilde{W})$ determines an $(n+1)$-dimensional symmetric Poincaré pair $(j: C(\partial\widetilde{W})\to C(\widetilde{W}),(\delta\phi,\phi))$ over $\mathbb{Z}[\pi]$ with

$$\begin{pmatrix}\delta\phi_0\\ \phi_0 j^*\end{pmatrix} = [W]\cap - \ :\ D^{n+1-*} = C(\widetilde{W})^{n-*}\to\mathcal{C}(j) = C(\widetilde{W},\partial\widetilde{W})$$

the Poincaré-Lefschetz duality chain equivalence. □

The *data* for *algebraic surgery* on an n-dimensional symmetric Poincaré complex (C,ϕ) is an $(n+1)$-dimensional symmetric pair $(j: C\to D,(\delta\phi,\phi))$. The *effect* of the algebraic surgery is the n-dimensional symmetric Poincaré complex (C',ϕ') with

$$\begin{aligned}
d_{C'} &= \begin{pmatrix} d_C & 0 & (-)^{n+1}\phi_0 j^*\\ (-)^r j & d_D & (-)^r\delta\phi_0\\ 0 & 0 & d_D^*\end{pmatrix}:\\
&\qquad C'_r = C_r\oplus D_{r+1}\oplus D^{n-r+1}\to C'_{r-1} = C_{r-1}\oplus D_r\oplus D^{n-r+2}\ ,\\
\phi'_0 &= \begin{pmatrix}\phi_0 & 0 & 0\\ (-)^{n-r}jT\phi_1 & (-)^{n-r}T\delta\phi_1 & 0\\ 0 & 1 & 0\end{pmatrix}:\\
&\qquad C'^{n-r} = C^{n-r}\oplus D^{n-r+1}\oplus D_{r+1}\to C'_r = C_r\oplus D_{r+1}\oplus D^{n-r+1}\ ,\\
\phi'_s &= \begin{pmatrix}\phi_s & 0 & 0\\ (-)^{n-r}jT\phi_{s+1} & (-)^{n-r+s}T\delta\phi_{s+1} & 0\\ 0 & 0 & 0\end{pmatrix}:\\
&\qquad C'^{n-r+s} = C^{n-r+s}\oplus D^{n-r+s+1}\oplus D_{r-s+1}\to C'_r = C_r\oplus D_{r+1}\oplus D^{n-r+1}\quad (s\geqslant 1)\ .
\end{aligned}$$

Remark. The appearance of the chain homotopy $\phi_1:\phi_0\simeq T\phi_0$ in the formula for the Poincaré duality chain equivalence ϕ'_0 is a reason for taking account of ϕ_1. The appearance of the higher chain homotopy $\phi_2:\phi_1\simeq T\phi_1$ in the formula for ϕ'_1 is a reason for taking account of ϕ_2. And so on .... □

The *trace* of an algebraic surgery is the $(n+1)$-dimensional symmetric Poincaré cobordism between (C,ϕ) and (C',ϕ')

$$((f\ \ f'): C\oplus C'\to D',(0,\phi\oplus -\phi'))$$

defined by

$$\begin{aligned}
d_{D'} &= \begin{pmatrix} d_C & (-)^{n+1}\phi_0 j^* \\ 0 & d_D^* \end{pmatrix} : D'_r = C_r \oplus D^{n-r+1} \to D'_{r-1} = C_{r-1} \oplus D^{n-r+2} , \\
f &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} : C_r \to D'_r = C_r \oplus D^{n-r+1} , \\
f' &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} : C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \to D'_r = C_r \oplus D^{n-r+1} .
\end{aligned}$$

Theorem. (Ranicki [4]) *The cobordism of symmetric Poincaré complexes is the equivalence relation generated by homotopy equivalence and algebraic surgery.*

Proof. Homotopy equivalent complexes are cobordant, by definition. Surgery equivalent complexes are cobordant by the trace construction.

Conversely, suppose given a cobordism of n-dimensional symmetric Poincaré complexes

$$\Gamma = ((f\ f') : C \oplus C' \to D, (\delta\phi, \phi \oplus -\phi')) .$$

Let

$$\overline{\Gamma} = ((\overline{f}\ \overline{f}') : C \oplus \overline{C}' \to \overline{D}, (0, \phi \oplus -\phi'))$$

be the trace of the algebraic surgery on (C,ϕ) with data $(j : C \to \mathcal{C}(f'), (\delta\phi/\phi', \phi))$ given by

$$\begin{aligned}
j &= \begin{pmatrix} f \\ 0 \end{pmatrix} : C_r \to \mathcal{C}(f')_r = D_r \oplus C'_{r-1} , \\
(\delta\phi/\phi')_s &= \begin{pmatrix} \delta\phi_s & (-)^s f'\phi'_s \\ 0 & (-)^{n-r+s} T\phi'_{s-1} \end{pmatrix} : \\
&\qquad \mathcal{C}(f')^{n-r+s+1} = D^{n-r+s+1} \oplus C'^{n-r+s} \to \mathcal{C}(f')_r = D_r \oplus C'_{r-1} \ \ (s \geqslant 0, \phi'_{-1} = 0) .
\end{aligned}$$

The A-module morphisms

$$\begin{aligned}
g &= (0\ \ 0\ \ 1\ \ 0\ \ 0) : \overline{C}'_r = C_r \oplus D_{r+1} \oplus C'_r \oplus D^{n-r+1} \oplus C'^{n-r} \to C'_r , \\
h &= (f\ \ \delta\phi_0\ \ f'\phi'_0) : \overline{D}_r = C_r \oplus D^{n-r+1} \oplus C'^{n-r} \to D_r
\end{aligned}$$

define a homotopy equivalence $(h, 1_C \oplus g) : \overline{\Gamma} \to \Gamma$. $\square$

Symmetric Surgery Principle. For any $(n+1)$-dimensional cobordism $(W; M, M')$ and regular cover $(\widetilde{W}; \widetilde{M}, \widetilde{M}')$ with group π the symmetric Poincaré complex $(C(\widetilde{M}'), \phi')$ is homotopy equivalent to the effect of algebraic surgery on $(C(\widetilde{M}), \phi)$ with data

$$(j : C(\widetilde{M}) \to C(\widetilde{W}, \widetilde{M}'), (\delta\phi', \phi)) .$$

Proof. The manifold cobordism determines a cobordism of n-dimensional symmetric Poincaré complexes over $\mathbb{Z}[\pi]$

$$\Gamma = (C(\widetilde{M}) \oplus C(\widetilde{M}') \to C(\widetilde{W}), (\delta\phi, \phi \oplus -\phi')) .$$

Now apply the Theorem to Γ, with $\delta\phi' = \delta\phi/\phi'$. $\square$

Example. If $(W; M, M')$ is the trace of a surgery on $S^i \times D^{n-i} \subset M$ then

$$C(\widetilde{W}, \widetilde{M}') \simeq S^{n-i}\mathbb{Z}[\pi]$$

is concentrated in dimension $(n-i)$, and the effect is to kill the spherical (co)homology class

$$j = [S^i] \in H^{n-i}(\widetilde{M}) \cong H_i(\widetilde{M}) .$$

The embedding $S^i \subset M$ determines $j : C(\widetilde{M}) \to S^{n-i}\mathbb{Z}[\pi]$, and the choice of extension to an embedding $S^i \times D^{n-i} \subset M$ determines $\delta\phi'$. □

The cobordism groups $L^n(A)$ $(n \geqslant 0)$ start with the symmetric Witt group $L^0(A)$. The symmetric signature map from manifold bordism to symmetric Poincaré bordism

$$\sigma^* : \Omega_n(X) \to L^n(\mathbb{Z}[\pi_1(X)]) ; \ M \mapsto (C(\widetilde{M}), \phi)$$

is a generalization of the signature map

$$\sigma^* : \Omega_{4k} \to L^{4k}(\mathbb{Z}) = \mathbb{Z} ; \ M \mapsto \text{signature}(H^{2k}(M), \text{intersection form}) .$$

Although the symmetric signature maps σ^* are not isomorphisms in general, they do provide many invariants. The symmetric and quadratic L-groups only differ in 8-torsion :

(i) the symmetrization maps

$$1 + T : L_n(A) \to L^n(A) ; \ (C, \psi) \mapsto (C, (1+T)\psi)$$

are isomorphisms modulo 8-torsion,

(ii) if $1/2 \in A$ the symmetrization maps are isomorphisms.

4 Surgery on quadratic Poincaré complexes

Quadratic Poincaré complexes are chain complexes with the Poincaré duality properties of kernels of normal maps. The quadratic Poincaré analogues of cobordism and surgery are defined by analogy with the symmetric case. Although there are many similarities between the quadratic and symmetric theories, there is one essential difference : the quadratic Poincaré cobordism groups are the Wall surgery obstruction groups $L_*(A)$, so for $A = \mathbb{Z}[\pi]$ every element is geometrically significant.

An *n-dimensional quadratic complex* (C, ψ) over A is a f.g. free A-module chain complex C together with a collection of A-module morphisms

$$\psi = \{\psi_s : C^{n-r-s} \to C_r \,|\, s \geqslant 0\}$$

such that

$$d_C\psi_s + (-1)^r\psi_s d_C^* + (-1)^{n-s-1}(\psi_{s+1} + (-1)^{s+1}T\psi_{s+1}) = 0 : C^{n-r-s-1} \to C_r \quad (s \geqslant 0) .$$

More intrinsically, ψ is an n-dimensional cycle in the $\mathbb{Z}$-module chain complex

$$W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_A(C^*, C)$$

with W the free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$ (as above). The quadratic complex (C, ψ) is *Poincaré* if the chain map $(1+T)\psi_0 : C^{n-*} \to C$ is a chain equivalence. A quadratic complex (C, ψ) determines the symmetric complex (C, ϕ) with $\phi_0 = (1+T)\psi_0$, $\phi_s = 0$ $(s \geqslant 1)$.

Example. ([5]) An n-dimensional normal map $(f, b) : M \to X$ and a regular covering $\widetilde{X}$ of X with group of covering translations π determine a *kernel* n-dimensional quadratic Poincaré complex (C, ψ) over $\mathbb{Z}[\pi]$ with $C = \mathcal{C}(f^!)$ the algebraic mapping cone of the Umkehr chain map

$$f^! \ : \ C(\widetilde{X}) \ \simeq \ C(\widetilde{X})^{n-*} \xrightarrow{f^*} C(\widetilde{M})^{n-*} \ \simeq \ C(\widetilde{M})$$

and $(1+T)\psi_0 = [M] \cap - : C^{n-*} \to C$ the Poincaré duality chain equivalence. It follows from $f_*[M] = [X] \in H_m(X)$ (f is degree 1) that there exists a chain homotopy $ff^! \simeq 1 : C(\widetilde{X}) \to C(\widetilde{X})$. The homology $\mathbb{Z}[\pi]$-modules of C are thus the kernels of f

$$H_*(C) \ = \ K_*(M) \ = \ \ker(f_* : H_*(\widetilde{M}) \to H_*(\widetilde{X})) \ ,$$

such that

$$H_*(\widetilde{M}) \ = \ K_*(M) \oplus H_*(\widetilde{X}) \ . \qquad \square$$

An $(n+1)$-*dimensional quadratic pair* $(j : C \to D, (\delta\psi, \psi))$ is an n-dimensional quadratic complex (C, ψ) together with a chain map $j : C \to D$ to an $(n+1)$-dimensional f.g. free A-module chain complex D and A-module morphisms

$$\delta\psi \ = \ \{\delta\psi_s : D^{n+1-r-s} \to D_r \mid s \geqslant 0\}$$

such that

$$j\psi_s j^* \ = \ d_D \delta\psi_s + (-)^r \delta\psi_s d_D^* + (-)^{n+s+1}(\delta\psi_{s+1} + (-)^s T\delta\psi_{s+1}) \ : \ D^{n+1-r-s} \to D_r \ \ (s \geqslant 0) \ .$$

The pair is *Poincaré* if the chain map

$$\begin{pmatrix} (1+T)\delta\psi_0 \\ (1+T)\psi_0 j^* \end{pmatrix} \ : \ D^{n+1-*} \to \mathcal{C}(j)$$

is a chain equivalence, in which case (C, ψ) is a n-dimensional quadratic Poincaré complex.

A *cobordism* of n-dimensional quadratic Poincaré complexes (C, ψ), (C', ψ') is an $(n+1)$-dimensional quadratic Poincaré pair of the type $(C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))$. Quadratic complexes (C, ψ), (C', ψ') are *homotopy equivalent* if there exists a cobordism with $C \to D$, $C' \to D$ chain equivalences.

Example. An $(n+1)$-dimensional normal map of pairs $(g, c) : (W, \partial W) \to (Y, \partial Y)$ determines a kernel $(n+1)$-dimensional quadratic Poincaré pair over $\mathbb{Z}[\pi]$ $(j : C(\partial g^!) \to C(g^!), (\delta\psi, \psi))$ with

$$\begin{pmatrix} (1+T)\delta\psi_0 j^* \\ (1+T)\psi_0 \end{pmatrix} \ = \ [W] \cap - \ : \ C(g^!)^{n+1-*} \to \mathcal{C}(j)$$

the Poincaré-Lefschetz duality chain equivalence. $\square$

The *data* for *algebraic surgery* on an n-dimensional quadratic Poincaré complex (C,ψ) is an $(n+1)$-dimensional quadratic pair $(j : C \to D, (\delta\psi,\psi))$. The *effect* of the algebraic surgery is the n-dimensional quadratic Poincaré complex (C',ψ') with

$$\begin{aligned}
d_{C'} &= \begin{pmatrix} d_C & 0 & (-)^{n+1}(1+T)\psi_0 j^* \\ (-)^r j & d_D & (-)^r(1+T)\delta\psi_0 \\ 0 & 0 & d_D^* \end{pmatrix} : \\
&\qquad C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \to C'_{r-1} = C_{r-1} \oplus D_r \oplus D^{n-r+2} , \\
\psi'_0 &= \begin{pmatrix} \psi_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : \\
&\qquad C'^{n-r} = C^{n-r} \oplus D^{n-r+1} \oplus D_{r+1} \to C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} , \\
\psi'_s &= \begin{pmatrix} \psi_s & (-)^{r+s}T\psi_{s-1}j^* & 0 \\ 0 & (-)^{n-r-s+1}T\delta\psi_{s-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} : \\
&\qquad C'^{n-r-s} = C^{n-r-s} \oplus D^{n-r-s+1} \oplus D_{r+s+1} \to C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \quad (s \geqslant 1) .
\end{aligned}$$

The trace of the algebraic surgery is an $(n+1)$-dimensional quadratic Poincaré cobordism $((f\ f') : C \oplus C' \to D', (\delta\psi', \psi \oplus -\psi'))$ defined by analogy with the symmetric case. As in the symmetric case :

Theorem. (Ranicki [4]) *The cobordism of quadratic Poincaré complexes is the equivalence relation generated by homotopy equivalence and algebraic surgery.* $\square$

Quadratic Surgery Principle. For a bordism of n-dimensional normal maps

$$((g,c);(f,b),(f',b')) \ : \ (W;M,M') \to X \times ([0,1];\{0\},\{1\})$$

the quadratic Poincaré complex $(C(f'^{!}),\psi')$ is homotopy equivalent to the effect of algebraic surgery on $(C(f^{!}),\psi)$ with data $(C(f^{!}) \to C(g^{!},f'^{!}),(\delta\psi,\psi))$. $\square$

Example. If $((g,c);(f,b),(f',b'))$ is the trace of a surgery on $S^i \times D^{n-i} \subset M$ then

$$C(g^{!},f'^{!}) \ \simeq \ S^{n-i}\mathbb{Z}[\pi]$$

is concentrated in dimension $(n-i)$. $\square$

A n-dimensional quadratic Poincaré complex (C,ψ) is *highly-connected* if it is homotopy equivalent to a complex (also denoted (C,ψ)) with

$$\begin{array}{lll} C & : & \dots \to 0 \to C_i \to 0 \to \dots & \text{if } n = 2i \\ C & : & \dots \to 0 \to C_{i+1} \to C_i \to 0 \to \dots & \text{if } n = 2i+1. \end{array}$$

Example. (i) The quadratic kernel (C,ψ) of an n-dimensional normal map $(f,b) : M \to X$ is highly-connected if and only if $f : M \to X$ is i-connected, that is $\pi_r(f) = 0$ for $r \leqslant i$.

(ii) The quadratic Poincaré kernel (C,ψ) of an i-connected $2i$-dimensional normal map $(f,b): M \to X$ is essentially the same as the geometric $(-)^i$-quadratic intersection form $(K_i(M),\lambda,\mu)$ of Wall [10], with

$$\begin{aligned}\lambda &= (1+T)\psi_0 \;:\; H^i(C) \;=\; K^i(M) \to H_i(C) \;=\; K_i(M) \;\cong\; H^i(C)^*\ ,\\ \mu(x) &= \psi_0(x)(x) \in Q_{(-)^i}(\mathbb{Z}[\pi_1(X)])\ .\end{aligned}$$

For $i \geqslant 3$ an element $x \in K_i(M)$ can be killed by a geometric surgery if and only if $\mu(x)=0$, if and only if there exists algebraic surgery data $(x: C \to S^i\mathbb{Z}[\pi_1(X)],(\delta\psi,\psi))$. The effect of the surgery is a normal map $(f',b'): M' \to X$ with quadratic Poincaré kernel (C',ψ') such that

$$C' \;:\; \cdots \to 0 \to \mathbb{Z}[\pi_1(X)] \xrightarrow{x} K_i(M) \xrightarrow{x^*\lambda} \mathbb{Z}[\pi_1(X)] \to 0 \to \dots\ . \qquad \square$$

Theorem. (Ranicki [4])
(i) *Every n-dimensional quadratic Poincaré complex (C,ψ) is cobordant to a highly-connected complex.*
(ii) *The cobordism group of n-dimensional quadratic complexes over A is isomorphic to $L_n(A)$, with the 4-periodicity isomorphisms given by*

$$L_n(A) \to L_{n+4}(A)\ ;\ (C,\psi) \mapsto (C_{*-2},\psi)\ .$$

Proof: (i) Let $n=2i$ or $2i+1$. Let D be the quotient complex of C with $D_r = C_r$ for $r > n-i$, and let $j: C \to D$ be the projection. The effect of algebraic surgery on (C,ψ) with data $(j: C \to D,(0,\psi))$ is homotopy equivalent to a highly-connected complex (C',ψ').
(ii) $(n=2i)$ A highly-connected $2i$-dimensional quadratic Poincaré complex (C,ψ) is essentially the same as a nonsingular $(-)^i$-quadratic form (C^0,ψ_0). The relative version of (i) shows that a null-cobordism of (C,ψ) is essentially the same as an isomorphism of forms

$$(C^0,\psi_0) \oplus \text{hyperbolic} \;\cong\; \text{hyperbolic}\ ,$$

which is precisely the condition for $(C^0,\psi_0) = 0 \in L_{2i}(A)$.
(ii) $(n=2i+1)$ A highly-connected $(2i+1)$-dimensional quadratic Poincaré complex (C,ψ) is essentially the same as a nonsingular $(-)^i$-quadratic formation. See [4] for further details. $\square$

Instant surgery obstruction for $n=2i$. A $2i$-dimensional quadratic Poincaré complex (C,ψ) is cobordant to the highly-connected complex (C',ψ') with

$$(C'^i,\psi'_0) \;=\; \left(\operatorname{coker}\left(\begin{pmatrix} d^* & 0 \\ (-)^{i+1}(1+T)\psi_0 & d\end{pmatrix} : C^{i-1}\oplus C_{i+2} \to C^i \oplus C_{i+1}\right), \begin{pmatrix}\psi_0 & d \\ 0 & 0\end{pmatrix}\right)\ .$$

Thus if (C,ψ) is the quadratic Poincaré kernel of a $2i$-dimensional normal map $(f,b): M \to X$ then (C'^i,ψ'_0) is a nonsingular $(-)^i$-quadratic form representing the surgery obstruction $\sigma_*(f,b) \in L_{2i}(\mathbb{Z}[\pi_1(X)])$ (without preliminary geometric surgeries below the middle dimension). $\square$

See §I.4 of [4] for the instant surgery obstruction formation in the case $n = 2i+1$.

5 The localization exact sequence

For any morphism of rings with involution $f : A \to B$ there is defined an exact sequence of L-groups

$$\dots \to L_n(A) \xrightarrow{f} L_n(B) \to L_n(f) \to L_{n-1}(A) \to \dots$$

with the relative L-group $L_n(f)$ the cobordism groups of pairs

$$\begin{aligned}&((n-1)\text{-dimensional quadratic Poincaré complex } (C,\psi) \text{ over } A,\\ &\qquad n\text{-dimensional quadratic Poincaré pair } (B \otimes_A C \to D, (\delta\psi, 1 \otimes \psi)) \text{ over } B)\ .\end{aligned}$$

Algebraic surgery provides a particularly useful expression for the relative L-groups $L_*(A \to S^{-1}A)$ of the localization map $A \to S^{-1}A$ inverting a multiplicatively closed subset $S \subset A$ of central non-zero divisors.

Localization exact sequence. (Ranicki [6]) The relative L-group $L_n(A \to S^{-1}A)$ is isomorphic to the cobordism group $L_n(A, S)$ of $(n-1)$-dimensional quadratic Poincaré complexes over A which are $S^{-1}A$-acyclic.

Proof Clearing denominators it is possible to lift every quadratic Poincaré pair over $S^{-1}A$ as above to an n-dimensional quadratic pair $(C \to D', (\delta\psi', \psi))$ over A. This is the data for algebraic surgery on (C, ψ) with effect a cobordant $(n-1)$-dimensional quadratic Poincaré complex (C', ψ') over A which is $S^{-1}A$-acyclic (i.e. $H_*(S^{-1}A \otimes_A C') = 0$). $\square$

The localization exact sequence

$$\dots \to L_n(A) \to L_n(S^{-1}A) \to L_n(A, S) \to L_{n-1}(A) \to \dots$$

and its extensions to noncommutative localization and to symmetric L-theory have many applications to the computation of L-groups, as well as to surgery on submanifolds (cf. Ranicki [7]).

Example. The localization of $A = \mathbb{Z}$ inverting $S = \mathbb{Z}\backslash\{0\} \subset A$ is $S^{-1}A = \mathbb{Q}$. See Chapter 4 of [6] for detailed accounts of the way in which the classification of quadratic forms over $\mathbb{Q}$ is combined with the localization exact sequence

$$\begin{aligned}&\dots \to L^n(\mathbb{Z}) \to L^n(\mathbb{Q}) \to L^n(\mathbb{Z}, S) \to L^{n-1}(\mathbb{Z}) \to \dots\ ,\\ &\dots \to L_n(\mathbb{Z}) \to L_n(\mathbb{Q}) \to L_n(\mathbb{Z}, S) \to L_{n-1}(\mathbb{Z}) \to \dots\end{aligned}$$

to give

$$L^n(\mathbb{Z}) \;=\; \begin{cases}\mathbb{Z} & \text{(signature)}\\ \mathbb{Z}_2 & \text{(deRham invariant)}\\ 0 & \\ 0 & \end{cases} \text{if } n \equiv \begin{cases}0\\1\\2\\3\end{cases} \pmod 4\ ,$$

$$L_n(\mathbb{Z}) \;=\; \begin{cases}\mathbb{Z} & \text{(signature)/8}\\ 0 & \\ \mathbb{Z}_2 & \text{(Arf invariant)}\\ 0 & \end{cases} \text{if } n \equiv \begin{cases}0\\1\\2\\3\end{cases} \pmod 4\ .$$

$\square$

References

[1] W. Browder, *Surgery on simply-connected manifolds*, Ergebnisse der Mathematik 65, Springer (1972)

[2] M. Kervaire and J. Milnor, *Groups of homotopy spheres I.*, Ann. of Maths. 77, 504–537 (1963)

[3] A.S. Mishchenko, *Homotopy invariants of non–simply connected manifolds, III. Higher signatures*, Izv. Akad. Nauk SSSR, ser. mat. 35, 1316–1355 (1971)

[4] A.A. Ranicki, *The algebraic theory of surgery I. Foundations*, Proc. Lond. Math. Soc. (3) 40, 87–192 (1980)

[5] ———, *The algebraic theory of surgery II. Applications to topology*, Proc. Lond. Math. Soc. (3) 40, 193–283 (1980)

[6] ———, *Exact sequences in the algebraic theory of surgery*, Mathematical Notes 26, Princeton (1982) http://www.maths.ed.ac.uk/~aar/books/exact.pdf

[7] ———, *High dimensional knot theory*, Springer Monograph, Springer (1998)

[8] ———, *An introduction to algebraic surgery*, in Surveys on Surgery Theory, Volume 2, Annals of Mathematics Studies 149, Princeton, 81–164 (2001)
e-print http://arXiv.org/abs/math.AT/0008071

[9] ———, *Algebraic Poincaré cobordism*, Proc. 1999 Conference for 60th birthday of J. Milgram , Stanford, Contemporary Mathematics 279, A.M.S., 213–255 (2001)
e-print http://arXiv.org/abs/math.AT/0008228

[10] C.T.C. Wall, *Surgery on compact manifolds*, 1st Edition, Academic Press (1970), 2nd Edition, Mathematical Surveys and Monographs 69, A.M.S. (1999)