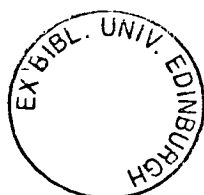# A Study of the Renormalization Group as a Model for Large-Eddy Simulations of Turbulence

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Abstract

This thesis presents a Renormalization Group approach for the modelling of homogeneous, isotropic and statistically stationary turbulence. The general problem is described and, following a discussion of various alternative approaches, it is outlined in general how the Renormalization Group may be used to reduce the number of degrees of freedom needed to accurately describe turbulence.

A critical discussion of the various Renormalization Group theories is then made before the new approach is introduced. This is based upon the two-field theory of McComb and Watt [Phys. Rev. A **46**, 4797 (1992)], and in particular the idea of a formal conditional average [W.D. McComb, W. Roberts & A.G. Watt, Phys. Rev A **45**, 3507 (1992)]. First, the formalism of the conditional average is redefined in terms of an ensemble of time-independent realizations. This resolves one problem, present in the two-field theory, regarding the order in which operations are performed. Second, a hypothesis which enables us to split conditional averages into low and high wavenumber velocity modes is introduced and discussed. It is then shown how the conditional average and hypothesis may be used together to eliminate from the system a finite band of high wavenumber modes, the effects of the eliminated modes being represented by an enhanced viscosity acting upon the remaining scales.

The mode elimination procedure is then used as the basis for a Renormalization Group calculation. This calculation is found to reach a fixed point, that is a point at which the equation of motion exhibits form-invariance under the Renormal-

ization Group transformation. Using the effective viscosity at this fixed point, a value for the Kolmogorov constant of $\alpha \approx 1.62$ is obtained.

A discussion and justification of the approximations used in the Renormalization Group calculation is then made. The basis for this is the introduction of a similarity solution for the velocity field, from which an expression for a local Reynolds number may be inferred. This local Reynolds number is shown to have a magnitude of less than 0.4. Using this approach, the approximations are all found to be equivalent to truncating at lowest non-trivial order an expansion in the local Reynolds number. Using the local Reynolds number, it is then shown that the fixed point of the Renormalization Group calculation corresponds to the onset of Kolmogorov scaling. As an aside, it is also shown how the similarity solution enables us to obtain an analytic model for the energy spectrum.

Finally, the fixed point effective viscosity was tested as the viscosity model in a 32^3 spectral large-eddy simulation. The results obtained are found to compare well with those obtained using an alternative analytic model, the test-field model of Kraichnan [J. Atmos. Sci **33**, 1521 (1976)], those obtained using an empirical viscosity model and those obtained in a 256^3 direct numerical simulation with identical parameters.

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I'd also like to thank all the members of the fluids group (I'd name you all individually, but I'd be bound to offend somebody by forgetting them!) for creating such a relaxed environment to work in and around.

Outside of the department, I'd like to thank my flatmates Esther, Tim and Owen, for always being there to talk to / moan at, and simply for being friends. Also, thanks to all the other friends who've supported me during the past few years, Dan, Dave, Jules, John, Owen, Leo, Simon H., Nick, Suzy, Terri, Rob, Simon V. and Sara. Finally, thanks to Liz simply for being there.

That leaves just my parents, who I'd like to thank for their continued support in everything I've chosen to do. I couldn't have done it without you.

Declaration

I declare that this thesis was composed by myself and that, except where explicitly stated otherwise in the text, the work contained therein is my own or was carried out in collaboration with Professor W.D. McComb. In addition to Professor McComb, the work in Chapter 7 was carried out in collaboration with Dr A.J. Young and A. Hunter.

Details of the work in Chapter 4 have been published in the Journal of Physics A: Mathematical and General [J. Phys. A: Math. Gen. **33**, L15 (2000)], and in the proceedings of the symposium 'Turbulence Structure and Vortex Dynamics', held at the Isaac Newton Institute, Cambridge, UK, 15-19 March 1999.

Aspects of the work in Chapters 6 and 7 were presented at the First International Symposium on Turbulence and Shear Flow Phenomena, Santa Barbara, USA, September 12-15, 1999, and are contained in the proceedings of this conference.

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Chapter 1

Introduction

1.1 The equations of fluid motion

The phenomenon of turbulence is perhaps the most easily visualized and yet least understood aspect of fluid dynamics, having both inspired and confounded generations of artists and scientists, from Leonardo da Vinci to Werner Heisenberg. Despite being of immense practical importance, and having been subject to intense study throughout the 20th century, there is relatively little consensus, and there exist a diverse variety of theoretical approaches, ranging from simple semi-empirical models, for instance the mixing-length model of Prandtl (see, for example, Hinze [1]) through to highly elaborate statistical models, for example the work of Canuto and Dubovikov [2,3].

The first scientific study of turbulence is generally taken to be that of Reynolds [4] in 1883, in which he studied the flow through long straight pipes of constant diameter and circular cross section by introducing coloured dye to the water. Using this method of 'colour bands', he found that for flow speeds below some critical value the flow was ordered (or *laminar*). However, once this critical speed was exceeded the flow then abruptly became turbulent, the dye being dispersed across the entire diameter of the pipe. The results of these experiments are illustrated in Figure 1.1, the pictures in which were obtained using Reynolds'

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original apparatus. Reynolds found that the criterion for this transition from laminar to turbulent flow could be simply expressed in terms of a dimensionless parameter, which we now refer to as the Reynolds number Re,

$$Re = \frac{Ud}{\nu}, \tag{1.1}$$

where U and d are representative velocity and length scales, in this instance the (bulk) mean velocity and the diameter of the pipe, whilst ν is the kinematic viscosity of the fluid flow. In the case of the pipe flow described, the transition from stable laminar to unstable turbulent flow occurs at a Reynolds number of approximately 2000.

This result has been amply verified. That is, for any fluid flow there is a critical Reynolds number above which the flow becomes turbulent. However, as with all empirical results, this knowledge does not address the fundamental questions of why the flow becomes turbulent, and what physics is occurring in a turbulent flow. The answers to these questions are almost certainly contained in the fundamental equations of fluid motion, the Navier-Stokes equation, which has been known since 1823 [6] and is essentially Newton's second law of motion (i.e. force $\propto$ rate of change of momentum), along with the continuity equation, which expresses the conservation of mass.

Throughout this thesis we shall consider only incompressible fluids, that is fluids in which the density ρ is constant. In this case the continuity equation takes the form

$$\frac{\partial U_\alpha(\boldsymbol{x}, t)}{\partial x_\alpha} = 0, \tag{1.2}$$

where $\boldsymbol{U}(\boldsymbol{x}, t)$ is the velocity field at position $\boldsymbol{x}$ and time t, whilst the conservation of momentum is expressed by

$$\frac{\partial U_\alpha(\boldsymbol{x}, t)}{\partial t} + \frac{\partial}{\partial x_\beta}(U_\alpha(\boldsymbol{x}, t) U_\beta(\boldsymbol{x}, t)) = -\frac{1}{\rho}\frac{\partial P(\boldsymbol{x}, t)}{\partial x_\alpha} + \frac{1}{\rho}\frac{\partial s_{\alpha\beta}(\boldsymbol{x}, t)}{\partial x_\beta}, \tag{1.3}$$

where $P(\boldsymbol{x}, t)$ is the pressure field and $s_{\alpha\beta}(\boldsymbol{x}, t)$ is the deviatoric stress tensor. Note that throughout the thesis well shall employ both Cartesian tensor notation
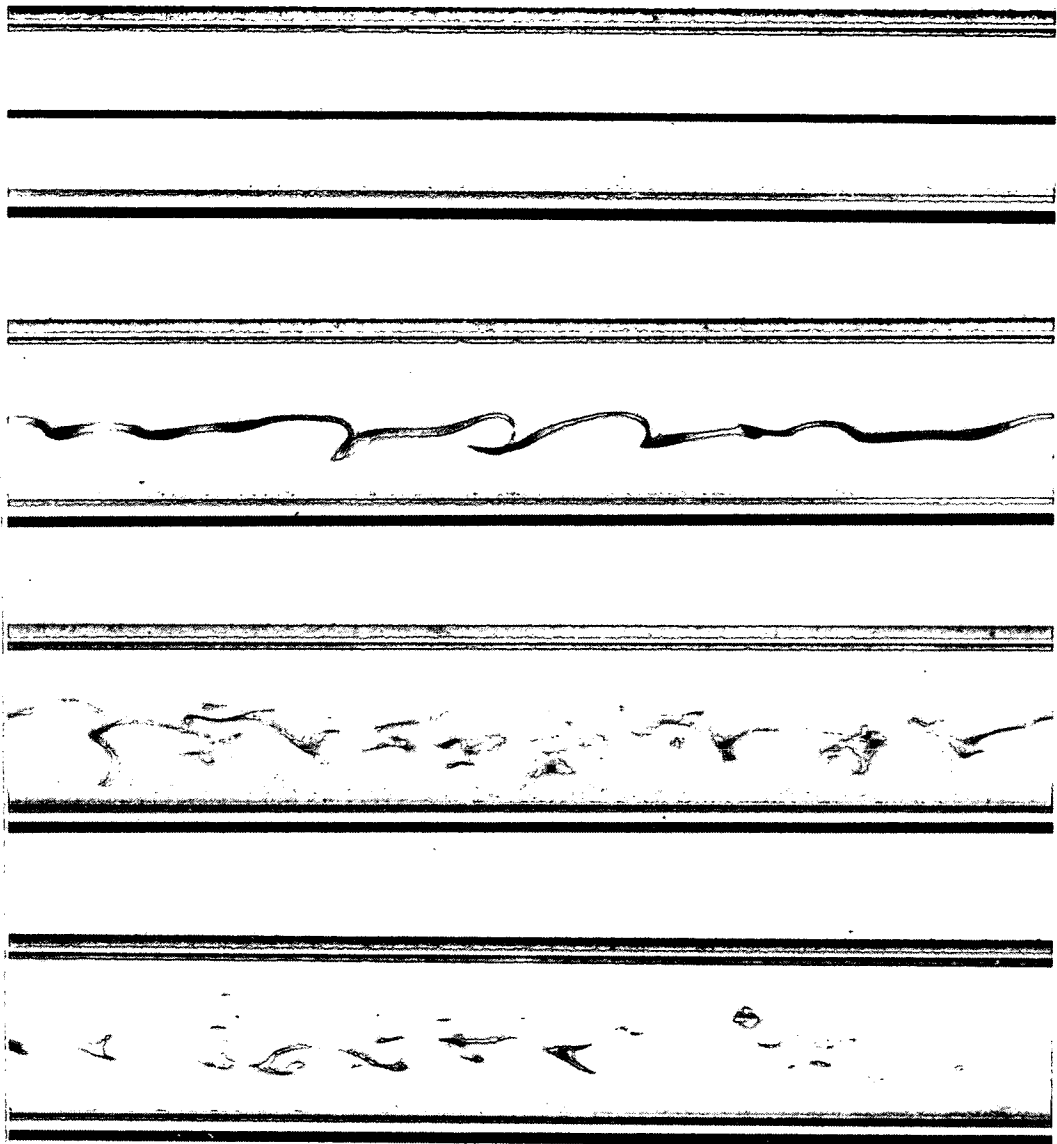
Figure 1.1: Illustration of Reynolds' dye experiment showing the onset of turbulence in the fluid as the Reynolds number is increased (Photographs taken from Van Dyke, *An Album of Fluid Motion* [5]).

and the Einstein summation convention that repeated indices are summed over. For a Newtonian fluid, $s_{\alpha\beta}(\boldsymbol{x}, t)$ is given by

$$s_{\alpha\beta}(\boldsymbol{x}, t) = \rho\nu \left(\frac{\partial U_\alpha(\boldsymbol{x}, t)}{\partial x_\beta} + \frac{\partial U_\beta(\boldsymbol{x}, t)}{\partial x_\alpha} \right), \qquad (1.4)$$

and hence by substituting (1.4), with the use of (1.2) equation (1.3) reduces to the Navier-Stokes equation (NSE),

$$\frac{\partial U_\alpha(\boldsymbol{x}, t)}{\partial t} + \frac{\partial}{\partial x_\beta}(U_\alpha(\boldsymbol{x}, t)U_\beta(\boldsymbol{x}, t)) = -\frac{1}{\rho}\frac{\partial P(\boldsymbol{x}, t)}{\partial x_\alpha} + \nu\frac{\partial^2 U_\alpha(\boldsymbol{x}, t)}{\partial x_\beta \partial x_\beta}. \qquad (1.5)$$

This equation enables us to gain an intuitive feel for the physical meaning of the Reynolds number. To see this we need only consider the ratio of the non-linear term, $\frac{\partial}{\partial x_\beta}(U_\alpha(\boldsymbol{x}, t)U_\beta(\boldsymbol{x}, t))$, to the viscous term, $\nu\frac{\partial^2 U_\alpha(\boldsymbol{x},t)}{\partial x_\beta \partial x_\beta}$. From a dimensional viewpoint, the non-linear term may be though of as the square of a velocity scale divided by a length scale, whilst the viscous term may be viewed as a viscosity multiplying a velocity scale divided by a length scale squared. Hence the ratio of these terms gives, using the notation of (1.1),

$$\frac{\text{non-linear term}}{\text{viscous term}} \sim \frac{U^2/d}{\nu U/d^2} = \frac{Ud}{\nu} = Re. \qquad (1.6)$$

Thus, the Reynolds number is simply a measure of the relative importance of the non-linear and viscous terms in the equation of motion. If Re is large then the non-linear term dominates, while if Re is small the viscous term is the more important.

The aim of this thesis is to devise an approach by which we may obtain a description of the statistical properties of turbulent flow, but given equation (1.5) the fundamental difficulty in achieving this is immediately apparent. We start by rearranging (1.5) to give

$$\left(\frac{\partial}{\partial t} - \nu\frac{\partial^2}{\partial x_\beta \partial x_\beta} \right) U_\alpha(\boldsymbol{x}, t) = -\frac{\partial}{\partial x_\beta}(U_\alpha(\boldsymbol{x}, t)U_\beta(\boldsymbol{x}, t)) - \frac{1}{\rho}\frac{\partial P(\boldsymbol{x}, t)}{\partial x_\alpha}, \qquad (1.7)$$

which may be written in a highly symbolic manner as

$$L_0 U = L_1 UU + L_2 P, \qquad (1.8)$$

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where L_0, L_1 and L_2 represent the respective (linear) differential operators. If we then denote the general operation of performing an average[1] by $\langle\cdot\rangle$, then averaging each term of (1.8) we find

$$L_0\langle U\rangle = L_1\langle UU\rangle + L_2\langle P\rangle. \tag{1.9}$$

As we shall shortly see, the pressure field P can be related to the velocity U using the continuity equation, (1.2), and hence this implies that a solution for $\langle U\rangle$ depends *in principle* only upon the second order moment $\langle UU\rangle$.

By multiplying each term of (1.8) by U and then averaging we readily obtain an equation for $\langle UU\rangle$

$$L_0\langle UU\rangle = L_1\langle UUU\rangle + L_2\langle UP\rangle, \tag{1.10}$$

and likewise, multiplying in turn by UU, UUU,..., we may generate the hierarchy of moment equations

$$L_0\langle UUU\rangle = L_1\langle UUUU\rangle + L_2\langle UUP\rangle, \tag{1.11}$$

$$L_0\langle UUUU\rangle = L_1\langle UUUUU\rangle + L_2\langle UUUP\rangle \tag{1.12}$$

and so on. Thus we have an open set of n equations for $n+1$ moments. The problem of closing this moment hierarchy is referred to as the 'closure problem' and is the underlying problem of turbulence theory.

1.2   The solenoidal Navier-Stokes equation

To see that the pressure field may be written in terms of the velocity field we simply take the divergence $(\partial/\partial x_\alpha)$ of equation (1.7). Doing this we obtain

$$\frac{1}{\rho}\frac{\partial^2}{\partial x_\alpha \partial x_\alpha}P(\boldsymbol{x},t) = -\frac{\partial^2}{\partial x_\alpha \partial x_\beta}U_\alpha(\boldsymbol{x},t)U_\beta(\boldsymbol{x},t), \tag{1.13}$$

---

[1] We shall give a more detailed discussion of what we mean by various types of averaging procedure in Chapter 3.

the terms linear in U vanishing according to (1.2). This is simply a form of Poisson's equation. Using (1.13) we can then obtain a form of the NSE in which there is no explicit dependence upon the pressure.

We start by considering the fluid to occupy a volume V bounded by a surface S, and apply the boundary condition

$$U_\alpha(\boldsymbol{x}, t) = 0 \text{ for } \boldsymbol{x} \text{ on } S. \tag{1.14}$$

Applying this boundary condition to the NSE, equation (1.5), we obtain

$$\frac{1}{\rho} \frac{\partial P(\boldsymbol{x}, t)}{\partial x_\alpha} = \nu \frac{\partial^2}{\partial x_\beta \partial x_\beta} U_\alpha(\boldsymbol{x}, t) \text{ for } \boldsymbol{x} \text{ on } S. \tag{1.15}$$

This can be re-expressed in terms of the normal derivatives

$$\frac{\partial}{\partial n} = n_\beta \frac{\partial}{\partial x_\beta} \text{ and } \frac{\partial^2}{\partial n^2} = n_\beta n_\gamma \frac{\partial^2}{\partial x_\beta \partial x_\gamma}, \tag{1.16}$$

where $n_\alpha(\boldsymbol{x})$ is the unit inward normal at $\boldsymbol{x}$ on S, to give

$$\frac{1}{\rho} \frac{\partial P(\boldsymbol{x}, t)}{\partial n} = \nu n_\beta \frac{\partial^2}{\partial n^2} U_\beta(\boldsymbol{x}, t) \text{ for } \boldsymbol{x} \text{ on } S. \tag{1.17}$$

Subject to the boundary condition (1.17), we can solve (1.13) for the pressure in terms of the Green function $G(\boldsymbol{x}, \boldsymbol{x}')$ which satisfies Laplace's equation in the form

$$\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}') = \delta(\boldsymbol{x} - \boldsymbol{x}'), \tag{1.18}$$

subject to the condition

$$\frac{\partial G(\boldsymbol{x}, \boldsymbol{x}')}{\partial n} = 0 \text{ for } \boldsymbol{x} \text{ on } S. \tag{1.19}$$

Doing this, the formal solution of (1.13) is found to be [7]

$$\begin{aligned} P(\boldsymbol{x}, t) = & -\rho \int_V \mathrm{d}^3 x' \, G(\boldsymbol{x}, \boldsymbol{x}') \frac{\partial^2 \{U_\beta(\boldsymbol{x}', t) U_\gamma(\boldsymbol{x}', t)\}}{\partial x'_\beta \partial x'_\gamma} \\ & + \rho \nu \int_S \mathrm{d}^2 x' \, G(\boldsymbol{x}, \boldsymbol{x}') n_\beta \frac{\partial^2 U_\beta(\boldsymbol{x}', t)}{\partial n^2}. \end{aligned} \tag{1.20}$$

Performing two partial integrations, using the boundary conditions and the symmetry of $G(\boldsymbol{x}, \boldsymbol{x}')$ under interchange of $\boldsymbol{x}$ and $\boldsymbol{x}'$ we obtain our final expression for the pressure

$$
\begin{aligned}
P(\boldsymbol{x}, t) &= -\rho \frac{\partial}{\partial x_\beta \partial x_\gamma} \int_V d^3x'\, G(\boldsymbol{x}, \boldsymbol{x}') U_\beta(\boldsymbol{x}', t) U_\gamma(\boldsymbol{x}', t) \\
&\quad + \rho \nu \int_S d^2x'\, G(\boldsymbol{x}, \boldsymbol{x}') n_\beta \frac{\partial^2 U_\beta(\boldsymbol{x}', t)}{\partial n^2}.
\end{aligned}
\tag{1.21}
$$

This is then substituted into the NSE (in the form given by equation (1.7)) to give

$$
\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x_\beta \partial x_\beta} \right) U_\alpha(\boldsymbol{x}, t) = -\frac{\partial}{\partial x_\beta} D_{\alpha\gamma}(\nabla) [U_\beta(\boldsymbol{x}, t) U_\gamma(\boldsymbol{x}, t)] - L_{\alpha\beta}(\nabla) [U_\beta(\boldsymbol{x}, t)],
\tag{1.22}
$$

where the operators $D_{\alpha\gamma}(\nabla)$ and $L_{\alpha\beta}(\nabla)$ are defined in terms of their effect on an arbitrary function $f(\boldsymbol{x})$ as

$$
D_{\alpha\gamma}(\nabla)[f(\boldsymbol{x})] = \delta_{\alpha\gamma} f(\boldsymbol{x}) - \frac{\partial^2}{\partial x_\alpha \partial x_\gamma} \int_V d^3x'\, G(\boldsymbol{x}, \boldsymbol{x}') f(\boldsymbol{x}')
\tag{1.23}
$$

and

$$
L_{\alpha\beta}(\nabla)[f(\boldsymbol{x})] = \nu \frac{\partial}{\partial x_\alpha} \int_S d^2x'\, G(\boldsymbol{x}, \boldsymbol{x}') n_\beta(\boldsymbol{x}') \frac{\partial^2 f(\boldsymbol{x}')}{\partial n^2}.
\tag{1.24}
$$

Equation (1.22) can be written in a more symmetric form by introducing the operator

$$
M_{\alpha\beta\gamma}(\nabla) = -\frac{1}{2} \left\{ \frac{\partial}{\partial x_\beta} D_{\alpha\gamma}(\nabla) + \frac{\partial}{\partial x_\gamma} D_{\alpha\beta}(\nabla) \right\},
\tag{1.25}
$$

which relies on the fact that the non-linear term in (1.22) must be unchanged under the interchange of the dummy indices β and γ. We can also extend the formulation to include flows subject to an external pressure gradient (i.e. a driving force). In this case, the external pressure would be such that

$$
\frac{\partial P_{\text{ext}}(\boldsymbol{x}, t)}{\partial x_\alpha} = \text{constant},
\tag{1.26}
$$

and hence it would satisfy Laplace's equation

$$
\frac{\partial^2 P_{\text{ext}}(\boldsymbol{x}, t)}{\partial x_\alpha \partial x_\alpha} = 0.
\tag{1.27}
$$

Referring back to equation (1.13) we can thus see that $P_{\text{ext}}(\boldsymbol{x}, t)$ may be simply added to $P(\boldsymbol{x}, t)$ as given by (1.21) without affecting the solution of Poisson's equation.

Making these two modifications leaves us with the final equation

$$\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x_\beta \partial x_\beta} \right) U_\alpha(\boldsymbol{x}, t) = M_{\alpha\beta\gamma}(\nabla)[U_\beta(\boldsymbol{x}, t)U_\gamma(\boldsymbol{x}, t)]$$

$$- L_{\alpha\beta}(\nabla)[U_\beta(\boldsymbol{x}, t)] + \frac{1}{\rho} \frac{\partial P_{\text{ext}}(\boldsymbol{x}, t)}{\partial x_\alpha}, \quad (1.28)$$

the solenoidal (or divergenceless) NSE.

1.3 The Navier-Stokes equation in Fourier space

In general, in theoretical approaches to studying turbulence we prefer to work in Fourier wavenumber (k) space. This has the dual benefits of converting differential operators into multipliers and giving us a comparatively simpler picture of the physics. To do this we must first Fourier transform (1.2) and (1.28) to obtain our equations of motion. However, before performing these Fourier transformations we shall first make some simplifying assumptions. We shall restrict ourselves to a system in which the surface S is at infinity, hence meaning there is no flow across S, which has zero mean velocity and in which there are no externally applied pressure gradients. We shall however introduce an arbitrary (divergenceless) forcing term $f_\alpha(\boldsymbol{x}, t)$ to the right hand side of equation (1.28), so that we may add energy to the system in order to counter viscous dissipation.

We also choose to use units in which the density ρ is unity, rewriting the viscosity as ν_0 to signify this change, and to rewrite (1.28) in terms of the fluctuation about its mean value. Following the procedure of Reynolds [8], the velocity field $U_\alpha(\boldsymbol{x}, t)$ may be decomposed as the sum of its mean value $\langle U_\alpha(\boldsymbol{x}, t) \rangle$ and the fluctuation from the mean $u_\alpha(\boldsymbol{x}, t)$, that is

$$U_\alpha(\boldsymbol{x}, t) = \langle U_\alpha(\boldsymbol{x}, t) \rangle + u_\alpha(\boldsymbol{x}, t). \quad (1.29)$$

From this, given our restriction to flows which have zero mean velocity it can be seen that the equations of motion may be rewritten in terms of the fluctuation simply by making the substitution $U_\alpha(\boldsymbol{x}, t) \to u_\alpha(\boldsymbol{x}, t)$. Applying these modifications and restrictions, equations (1.2) and (1.28) thus reduce to

$$\frac{\partial u_\alpha(\boldsymbol{x}, t)}{\partial x_\alpha} = 0 \tag{1.30}$$

and

$$\left(\frac{\partial}{\partial t} - \nu_0 \frac{\partial^2}{\partial x_\beta \partial x_\beta}\right) u_\alpha(\boldsymbol{x}, t) = f_\alpha(\boldsymbol{x}, t) + M_{\alpha\beta\gamma}(\nabla)[u_\beta(\boldsymbol{x}, t) u_\gamma(\boldsymbol{x}, t)]. \tag{1.31}$$

The Fourier space analogues of (1.30) and (1.31) may be found by introducing the transform pair

$$u_\alpha(\boldsymbol{x}, t) = \int \mathrm{d}^3 k \, u_\alpha(\boldsymbol{k}, t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{1.32}$$

and

$$u_\alpha(\boldsymbol{k}, t) = \left(\frac{1}{2\pi}\right)^3 \int \mathrm{d}^3 x \, u_\alpha(\boldsymbol{x}, t) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{1.33}$$

which relate $u_\alpha(\boldsymbol{x}, t)$ to its Fourier transform $u_\alpha(\boldsymbol{k}, t)$, and applying the results detailed in Appendix A.

From equation (1.32), equation (1.30) may be re-expressed as

$$\int \mathrm{d}^3 k \, (i k_\alpha) u_\alpha(\boldsymbol{k}, t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = 0, \tag{1.34}$$

and since this must hold for arbitrary $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$, the continuity equation becomes

$$k_\alpha u_\alpha(\boldsymbol{k}, t) = 0. \tag{1.35}$$

This indicates that $\boldsymbol{k}$ and $\boldsymbol{u}(\boldsymbol{k}, t)$ must be orthogonal to one another.

Obtaining the Fourier transform of equation (1.31) in not quite as simple a procedure. Equation (1.32) may again be used in re-expressing the left hand side of (1.31) as

$$\begin{aligned}
\text{LHS of (1.31)} &= \int \mathrm{d}^3 k \left(\frac{\partial}{\partial t} - \nu_0 (i k_\beta)^2\right) u_\alpha(\boldsymbol{k}, t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\
&= \int \mathrm{d}^3 k \left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_\alpha(\boldsymbol{k}, t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}},
\end{aligned} \tag{1.36}$$

and the Fourier transform of the forcing term $f_\alpha(\boldsymbol{k}, t)$ may be simply introduced using

$$f_\alpha(\boldsymbol{x}, t) = \int \mathrm{d}^3k\, f_\alpha(\boldsymbol{k}, t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{1.37}$$

but we still need to re-express the non-linear term.

From the form of equation (1.25) we can easily see that this reduces to the question of how can we re-express the term

$$N_\alpha(\boldsymbol{x}, t) \equiv \frac{\partial}{\partial x_\beta} D_{\alpha\gamma}(\nabla)[u_\beta(\boldsymbol{x}, t)u_\gamma(\boldsymbol{x}, t)]. \tag{1.38}$$

Using the definition of a Fourier transform and the convolution theorem, equation (A.10), we can rewrite this as

$$N_\alpha(\boldsymbol{x}, t) = \frac{\partial}{\partial x_\beta} D_{\alpha\gamma}(\nabla) \left[\int \mathrm{d}^3k \left\{ \int \mathrm{d}^3j\, u_\beta(\boldsymbol{j}, t)u_\gamma(\boldsymbol{k} - \boldsymbol{j}, t) \right\} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \right], \tag{1.39}$$

and substituting equation (1.23) we then have

$$N_\alpha(\boldsymbol{x}, t) = \int \mathrm{d}^3k\, \delta_{\alpha\gamma}(ik_\beta) \left\{ \int \mathrm{d}^3j\, u_\beta(\boldsymbol{j}, t)u_\gamma(\boldsymbol{k} - \boldsymbol{j}, t) \right\} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$
$$- \frac{\partial^3}{\partial x_\alpha \partial x_\beta \partial x_\gamma} \iint \mathrm{d}^3x' \mathrm{d}^3k\, G(\boldsymbol{x}, \boldsymbol{x}') \left\{ \int \mathrm{d}^3j\, u_\beta(\boldsymbol{j}, t)u_\gamma(\boldsymbol{k} - \boldsymbol{j}, t) \right\} e^{i\boldsymbol{k}\cdot\boldsymbol{x}'}. \tag{1.40}$$

Since $G(\boldsymbol{x}, \boldsymbol{x}')$ can depend only upon $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$ (see the discussion in Appendix A) we make this replacement for $\boldsymbol{x}'$ in the second term on the right hand side of (1.40), and making the further substitution

$$\int \mathrm{d}^3r\, G(r) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} = (2\pi)^3 G(k), \tag{1.41}$$

we obtain

$$N_\alpha(\boldsymbol{x}, t) = \int \mathrm{d}^3k\, \delta_{\alpha\gamma}(ik_\beta) \left\{ \int \mathrm{d}^3j\, u_\beta(\boldsymbol{j}, t)u_\gamma(\boldsymbol{k} - \boldsymbol{j}, t) \right\} e^{i\boldsymbol{k}\cdot\boldsymbol{j}}$$
$$- \frac{\partial^3}{\partial x_\alpha \partial x_\beta \partial x_\gamma} \int \mathrm{d}^3k\, (2\pi)^3 G(k) \left\{ \int \mathrm{d}^3j\, u_\beta(\boldsymbol{j}, t)u_\gamma(\boldsymbol{k} - \boldsymbol{j}, t) \right\} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{1.42}$$

From equation (A.19) we know

$$G(k) = -\left(\frac{1}{2\pi}\right)^3 \left(\frac{1}{k^2}\right), \tag{1.43}$$

and hence substituting for $G(k)$ and performing the remaining derivatives, we are left with the final expression for $N_\alpha(\boldsymbol{x}, t)$

$$N_\alpha(\boldsymbol{x}, t) = \int \mathrm{d}^3k \, (ik_\beta) \left(\delta_{\alpha\gamma} - \frac{k_\alpha k_\gamma}{k^2}\right) \left\{\int \mathrm{d}^3 j \, u_\beta(\boldsymbol{j}, t) u_\gamma(\boldsymbol{k} - \boldsymbol{j}, t)\right\} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{1.44}$$

This means we may re-express the non-linear term in (1.31) as

$$M_{\alpha\beta\gamma}(\nabla)[u_\beta(\boldsymbol{x}, t) u_\gamma(\boldsymbol{x}, t)] = \int \mathrm{d}^3k \, M_{\alpha\beta\gamma}(\boldsymbol{k}) \left\{\int \mathrm{d}^3 j \, u_\beta(\boldsymbol{j}, t) u_\gamma(\boldsymbol{k} - \boldsymbol{j}, t)\right\} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{1.45}$$

where

$$M_{\alpha\beta\gamma}(\boldsymbol{k}) = \frac{1}{2i}[k_\beta D_{\alpha\gamma}(\boldsymbol{k}) + k_\gamma D_{\alpha\beta}(\boldsymbol{k})] \tag{1.46}$$

and

$$D_{\alpha\beta}(\boldsymbol{k}) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}. \tag{1.47}$$

Re-expressing equation (1.31) in terms of equations (1.36), (1.37) and (1.45) and noting again that this expression must hold for arbitrary $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$, we are then left with the NSE in k-space

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_\alpha(\boldsymbol{k}, t) = f_\alpha(\boldsymbol{k}, t) + M_{\alpha\beta\gamma}(\boldsymbol{k}) \int \mathrm{d}^3 j \, u_\beta(\boldsymbol{j}, t) u_\gamma(\boldsymbol{k} - \boldsymbol{j}, t). \tag{1.48}$$

1.4 Homogeneous and isotropic turbulence

To further simplify the problem we are considering, we also choose to restrict our attention to flows which are both (spatially) homogeneous and isotropic. These are both statistical concepts and imply respectively that mean values of the flow do not change under either translation or rotation of the axes.

The implications of these restrictions are most easily seen in terms of the ($\boldsymbol{x}$-space) two point, two time velocity moment

$$Q_{\alpha\beta}(\boldsymbol{x}, \boldsymbol{x}'; t, t') = \langle u_\alpha(\boldsymbol{x}, t) u_\beta(\boldsymbol{x}', t')\rangle. \tag{1.49}$$

If we introduce

$$\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}', \tag{1.50}$$

then the assumption of homogeneity implies that $Q_{\alpha\beta}(\boldsymbol{x}, \boldsymbol{x}'; t, t')$ depends only upon the relative position $\boldsymbol{r}$, that is

$$\begin{aligned}
Q_{\alpha\beta}(\boldsymbol{x}, \boldsymbol{x}'; t, t') &= \langle u_\alpha(\boldsymbol{x}' + \boldsymbol{r}) u_\beta(\boldsymbol{x}', t') \rangle \\
&= \langle u_\alpha(\boldsymbol{0} + \boldsymbol{r}) u_\beta(\boldsymbol{0}, t') \rangle \\
&= Q_{\alpha\beta}(\boldsymbol{r}; t, t'),
\end{aligned} \tag{1.51}$$

and that it must be unaffected by interchange of $\boldsymbol{x}$ and $\boldsymbol{x}'$, meaning

$$Q_{\alpha\beta}(\boldsymbol{r}; t, t') = Q_{\alpha\beta}(-\boldsymbol{r}; t, t'). \tag{1.52}$$

The additional assumption of isotropy then implies the further symmetry requirement that

$$Q_{\alpha\beta}(\boldsymbol{r}; t, t') = Q_{\beta\alpha}(\boldsymbol{r}; t, t'). \tag{1.53}$$

Since we are choosing to work in Fourier space, we clearly need Fourier space forms for these relations. To obtain these results, we start by defining, in analogue to equations (1.32) and (1.33), the Fourier transform pair

$$Q_{\alpha\beta}(\boldsymbol{r}; t, t') = \int d^3r \, Q_{\alpha\beta}(\boldsymbol{k}; t, t') e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{1.54}$$

and

$$Q_{\alpha\beta}(\boldsymbol{k}; t, t') = \left(\frac{1}{2\pi}\right)^3 \int d^3k \, Q_{\alpha\beta}(\boldsymbol{r}; t, t') e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{1.55}$$

It is also useful to here recall equation (1.33),

$$u_\alpha(\boldsymbol{k}, t) = \left(\frac{1}{2\pi}\right)^3 \int d^3x \, u_\alpha(\boldsymbol{x}, t) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}.$$

Now since the velocity field in $\boldsymbol{x}$-space is real valued, this implies that

$$u_\alpha^*(\boldsymbol{x}, t) = u_\alpha(\boldsymbol{x}, t), \tag{1.56}$$

where the asterisk denotes the operation of complex conjugation. Hence from equation (1.33) we have

$$
\begin{aligned}
u_\alpha^*(\boldsymbol{k},t) &= \left(\frac{1}{2\pi}\right)^3 \int \mathrm{d}^3x \, u_\alpha(\boldsymbol{x},t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\
&= u_\alpha(-\boldsymbol{k},t).
\end{aligned}
\tag{1.57}
$$

Similarly, $Q_{\alpha\beta}(\boldsymbol{x};t,t')$ must also be real valued, and hence from equation (1.55)

$$
Q_{\alpha\beta}^*(\boldsymbol{k};t,t') = Q_{\alpha\beta}(-\boldsymbol{k};t,t').
\tag{1.58}
$$

This is the k-space analogue to equation (1.52).

The analogue to equation (1.53) is easily found, using equations (1.53) and (1.54), to be

$$
Q_{\alpha\beta}(\boldsymbol{k};t,t') = Q_{\beta\alpha}(\boldsymbol{k};t,t').
\tag{1.59}
$$

That is, the k-space symmetry requirement for isotropy is identical to that in x-space. There still remains, however, the question of how we express $Q_{\alpha\beta}(\boldsymbol{k};t,t')$ in terms of velocity modes.

We start by using equation (1.33) to write $\langle u_\alpha(\boldsymbol{k},t)u_\beta(\boldsymbol{k}',t')\rangle$ as

$$
\langle u_\alpha(\boldsymbol{k},t)u_\beta(\boldsymbol{k}',t')\rangle = \left(\frac{1}{2\pi}\right)^6 \iint \mathrm{d}^3x \, \mathrm{d}^3x' \, \langle u_\alpha(\boldsymbol{x},t)u_\beta(\boldsymbol{x}',t')\rangle e^{-i\boldsymbol{k}\cdot\boldsymbol{x}-i\boldsymbol{k}'\cdot\boldsymbol{x}'}.
\tag{1.60}
$$

Making the replacement $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{r}$ and invoking homogeneity in the form of equation (1.51), this may be rewritten as

$$
\langle u_\alpha(\boldsymbol{k},t)u_\beta(\boldsymbol{k}',t')\rangle = \left(\frac{1}{2\pi}\right)^6 \iint \mathrm{d}^3x \, \mathrm{d}^3r \, Q_{\alpha\beta}(\boldsymbol{r};t,t') e^{-i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}} e^{i\boldsymbol{k}'\cdot\boldsymbol{r}},
\tag{1.61}
$$

and using equation (A.5) to perform the integral with respect to $\boldsymbol{x}$, we are left with

$$
\langle u_\alpha(\boldsymbol{k},t)u_\beta(\boldsymbol{k}',t')\rangle = \left(\frac{1}{2\pi}\right)^3 \int \mathrm{d}^3r \, Q_{\alpha\beta}(\boldsymbol{r};t,t') e^{i\boldsymbol{k}'\cdot\boldsymbol{r}} \delta(\boldsymbol{k}+\boldsymbol{k}').
\tag{1.62}
$$

Comparison with equation (1.55) then implies

$$
\langle u_\alpha(\boldsymbol{k},t)u_\beta(\boldsymbol{k}',t')\rangle = Q_{\alpha\beta}(\boldsymbol{k}';t,t')\delta(\boldsymbol{k}+\boldsymbol{k}'),
\tag{1.63}
$$

and integrating this over $\boldsymbol{k}'$ gives the final expression

$$
Q_{\alpha\beta}(\boldsymbol{k};t,t') = \int \mathrm{d}^3k' \, \langle u_\alpha(\boldsymbol{k},t)u_\beta(\boldsymbol{k}',t')\rangle.
\tag{1.64}
$$

1.5 The isotropic spectrum tensor

The restriction to isotropic turbulence also allows us to express the correlation tensor $Q_{\alpha\beta}(\boldsymbol{k}; t, t')$ in a simpler manner than for the general case. In general, this tensor is described by nine scalar functions (in the 3-dimensional case), but following the approach of Robertson [9] we may reduce this number to one. The method for doing this is based on the idea that an isotropic tensor can be expressed in terms of the invariants of the rotation group. Using these arguments we find

$$Q_{\alpha\beta}(\boldsymbol{k}; t, t') = Q(k; t, t')\delta_{\alpha\beta} + A(k; t, t')k_\alpha k_\beta, \tag{1.65}$$

where $Q(k; t, t')$ and $A(k; t, t')$ are arbitrary even functions of $k = |\boldsymbol{k}|$.

One of these scalar functions may be eliminated using the continuity equation (1.35). If we multiply both sides of equation (1.65) by k_α and sum the repeated indices, then it follows that

$$k_\alpha Q_{\alpha\beta}(\boldsymbol{k}; t, t') = 0 = Q(k; t, t')k_\beta + A(k; t, t')k^2 k_\beta, \tag{1.66}$$

and as this must hold for arbitrary k_β, we have the relationship

$$Q(k; t, t') = -k^2 A(k; t, t'). \tag{1.67}$$

Using this to substitute for $A(k; t, t')$ in equation (1.65), we thus have

$$Q_{\alpha\beta}(\boldsymbol{k}; t, t') = Q(k; t, t')\delta_{\alpha\beta} - Q(k; t, t')\frac{k_\alpha k_\beta}{k^2} = D_{\alpha\beta}(\boldsymbol{k})Q(k; t, t'), \tag{1.68}$$

where $D_{\alpha\beta}(\boldsymbol{k})$ is the projection operator defined in equation (1.47).

The scalar function $Q(k; t, t')$ may also be related to the spectrum of the energy contained within the system. In real space, the total energy of the system is given by

$$
\begin{aligned}
E(t) &= \frac{1}{2}\langle u_\alpha(\boldsymbol{x}, t)u_\alpha(\boldsymbol{x}, t)\rangle \\
&= \frac{1}{2}\mathrm{tr}\, Q_{\alpha\beta}(\boldsymbol{r} = \boldsymbol{0}; t, t)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\mathrm{tr}\int \mathrm{d}^3k\, Q_{\alpha\beta}(\boldsymbol{k};t,t) \\
&= \frac{1}{2}\mathrm{tr}\int \mathrm{d}^3k\, D_{\alpha\beta}(\boldsymbol{k})Q(k,t),
\end{aligned}
\tag{1.69}
$$

where the penultimate line comes from equation (1.54), and where we have introduced the shorthand that $Q(k;t,t) = Q(k,t)$. Noting that

$$
\mathrm{tr}D_{\alpha\beta}(\boldsymbol{k}) = \delta_{\alpha\alpha} - \frac{k_\alpha k_\alpha}{k^2} = 3 - 1 = 2,
\tag{1.70}
$$

and performing the integral over $\boldsymbol{k}$ in spherical polar co-ordinates, this then reduces to

$$
\begin{aligned}
E(t) &= \int_0^\infty \mathrm{d}k\, 4\pi k^2 Q(k,t) \\
&= \int_0^\infty \mathrm{d}k\, E(k,t)
\end{aligned}
\tag{1.71}
$$

where the energy spectrum is given by

$$
E(k,t) = 4\pi k^2 Q(k,t).
\tag{1.72}
$$

$Q(k,t)$ is thus interpreted as the spectral energy density.

1.6   The energy balance equation

Using the NSE, equation (1.48), we may obtain an expression which describes the transport of turbulent energy within wavenumber space. We start by forming a dynamical equation for $\langle u_\alpha(\boldsymbol{k},t)u_\delta(\boldsymbol{\ell},t)\rangle$ as follows:

1. Multiply both sides of equation (1.48) by $u_\delta(\boldsymbol{\ell},t)$.

2. Use equation (1.48) to write a dynamic equation for $u_\delta(\boldsymbol{\ell},t)$ and then multiply both sides of this by $u_\alpha(\boldsymbol{k},t)$.

3. Add the equations formed by steps 1 and 2 and then take the average.

Doing this gives the result

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2 + \nu_0 l^2\right) \langle u_\alpha(\boldsymbol{k},t)u_\delta(\boldsymbol{\ell},t)\rangle = \langle f_\alpha(\boldsymbol{k},t)u_\delta(\boldsymbol{\ell},t)\rangle + \langle u_\alpha(\boldsymbol{k},t)f_\delta(\boldsymbol{\ell},t)\rangle$$
$$+ M_{\alpha\beta\gamma}(\boldsymbol{k}) \int \mathrm{d}^3 j \langle u_\beta(\boldsymbol{j},t)u_\gamma(\boldsymbol{k}-\boldsymbol{j},t)u_\delta(\boldsymbol{\ell},t)\rangle$$
$$+ M_{\delta\beta\gamma}(\boldsymbol{l}) \int \mathrm{d}^3 j \langle u_\beta(\boldsymbol{j},t)u_\gamma(\boldsymbol{l}-\boldsymbol{j},t)u_\alpha(\boldsymbol{k},t)\rangle.$$

$$(1.73)$$

Using similar arguments to those by which we obtain equation (1.63), we may obtain both the analogous result for the third order moment

$$\langle u_\alpha(\boldsymbol{k},t)u_\beta(\boldsymbol{j},t')u_\gamma(\boldsymbol{l},t'')\rangle = Q_{\alpha\beta\gamma}(\boldsymbol{j},\boldsymbol{l};t,t',t'')\delta(\boldsymbol{k}+\boldsymbol{j}+\boldsymbol{l}), \tag{1.74}$$

and also an expression for $\langle f_\alpha(\boldsymbol{k},t)u_\beta(\boldsymbol{j},t')\rangle$,

$$\langle f_\alpha(\boldsymbol{k},t)u_\beta(\boldsymbol{j},t')\rangle = W_{\alpha\beta}(\boldsymbol{j};t,t')\delta(\boldsymbol{k}+\boldsymbol{j}), \tag{1.75}$$

where $W_{\alpha\beta}(\boldsymbol{k};t,t')$ is the Fourier transform of

$$W_{\alpha\beta}(\boldsymbol{r};t,t') = \langle f_\alpha(\boldsymbol{x}+\boldsymbol{r},t)u_\beta(\boldsymbol{x},t')\rangle$$
$$= \langle f_\alpha(\boldsymbol{0}+\boldsymbol{r},t)u_\beta(\boldsymbol{0},t')\rangle, \tag{1.76}$$

and is subject to the same symmetry requirements as $Q_{\alpha\beta}(\boldsymbol{k};t,t')$.

Substituting equations (1.63), (1.74) and (1.75) into equation (1.73) and integrating with respect to $\boldsymbol{l}$ we are thus left with

$$\left(\frac{\partial}{\partial t} + 2\nu_0 k^2\right) Q_{\alpha\delta}(-\boldsymbol{k},t) = W_{\alpha\delta}(-\boldsymbol{k},t) + W_{\delta\alpha}(\boldsymbol{k},t)$$
$$+ M_{\alpha\beta\gamma}(\boldsymbol{k}) \int \mathrm{d}^3 j\, Q_{\beta\gamma\delta}(\boldsymbol{k}-\boldsymbol{j},-\boldsymbol{k},t) + M_{\delta\beta\gamma}(-\boldsymbol{k}) \int \mathrm{d}^3 j\, Q_{\beta\gamma\alpha}(-\boldsymbol{k}-\boldsymbol{j},\boldsymbol{k},t).$$

$$(1.77)$$

Using the same argument as used in Section 1.5 to obtain equation (1.68) we find

$$W_{\alpha\beta}(\boldsymbol{k};t,t') = D_{\alpha\beta}(\boldsymbol{k})\overline{W}(k;t,t'), \tag{1.78}$$

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where $\overline{W}(k; t, t')$ is a scalar function. Hence, recalling that $\mathrm{tr} D_{\alpha\beta}(\boldsymbol{k}) = 2$, if we take the trace of equation (1.77) and multiply the result by $2\pi k^2$ we obtain

$$\left(\frac{\partial}{\partial t} + 2\nu_0 k^2\right) E(k, t) = W(k, t) + T(k, t), \tag{1.79}$$

where $E(k, t)$ is as defined in equation (1.72),

$$W(k, t) = 8\pi k^2 \overline{W}(k; t, t), \tag{1.80}$$

and

$$T(k, t) = 2\pi k^2 M_{\alpha\beta\gamma}(\boldsymbol{k}) \int \mathrm{d}^3 j \left\{Q_{\beta\gamma\alpha}(\boldsymbol{k} - \boldsymbol{j}, -\boldsymbol{k}, t) - Q_{\beta\gamma\alpha}(-\boldsymbol{k} - \boldsymbol{j}, \boldsymbol{k}, t)\right\}. \tag{1.81}$$

Equation (1.79) is known as the energy balance equation.

The physical interpretation of each term in the energy balance equation is relatively obvious, and can be seen if we integrate each term over k. We start by considering the terms on the left hand side. Clearly,

$$\int_0^\infty \mathrm{d}k \, \frac{\partial E(k, t)}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty \mathrm{d}k \, E(k, t) = \frac{\mathrm{d}E(t)}{\mathrm{d}t} \tag{1.82}$$

represents the rate of change of energy contained within the system, while

$$\int_0^\infty \mathrm{d}k \, 2\nu_0 k^2 E(k, t) = \varepsilon_d(t) \tag{1.83}$$

represents the rate at which energy is lost from the system due to viscous dissipation. Similarly,

$$\int_0^\infty \mathrm{d}k \, W(k, t) = \varepsilon_w(t) \tag{1.84}$$

describes the rate at which energy is input by forcing. This leaves just the integral of $T(k, t)$ to be considered. We start by rewriting the one-dimensional integral over k as a three-dimensional integral over $\boldsymbol{k}$

$$\begin{aligned} \int_0^\infty \mathrm{d}k \, 2T(k, t) &= \int_0^\infty \mathrm{d}k \, 4\pi k^2 M_{\alpha\beta\gamma}(\boldsymbol{k}) \int \mathrm{d}^3 j \left\{Q_{\beta\gamma\alpha}(\boldsymbol{k} - \boldsymbol{j}, -\boldsymbol{k}, t) \right. \\ &\quad \left. - Q_{\beta\gamma\alpha}(-\boldsymbol{k} - \boldsymbol{j}, \boldsymbol{k}, t)\right\} \\ &= \iint \mathrm{d}^3 k \, \mathrm{d}^3 j \, M_{\alpha\beta\gamma}(\boldsymbol{k}) \left\{Q_{\beta\gamma\alpha}(\boldsymbol{k} - \boldsymbol{j}, -\boldsymbol{k}, t) \right. \\ &\quad \left. - Q_{\beta\gamma\alpha}(-\boldsymbol{k} - \boldsymbol{j}, \boldsymbol{k}, t)\right\}. \end{aligned} \tag{1.85}$$

Next, we note that from the definition of $D_{\alpha\beta}(\boldsymbol{k})$, equation (1.47), and the continuity equation we have

$$D_{\alpha\beta}(\boldsymbol{k})u_{\alpha}(\boldsymbol{k},t) = u_{\beta}(\boldsymbol{k},t), \tag{1.86}$$

which further implies that

$$D_{\alpha\beta}(\boldsymbol{k})Q_{\gamma\beta\alpha}(\boldsymbol{j},\boldsymbol{k};t,t',t'') = Q_{\gamma\beta\beta}(\boldsymbol{j},\boldsymbol{k};t,t',t''). \tag{1.87}$$

From equation (1.46), we also note, since $M_{\alpha\beta\gamma}(\boldsymbol{k})$ is symmetric under the interchange $\beta \leftrightarrow \gamma$, that we may rewrite $M_{\alpha\beta\gamma}(\boldsymbol{k})$ in the non-symmetric form

$$M_{\alpha\beta\gamma}(\boldsymbol{k}) = -ik_{\gamma}D_{\alpha\beta}(\boldsymbol{k}). \tag{1.88}$$

Using these two results, we are hence able to rewrite equation (1.85) as

$$\int_{0}^{\infty} dk\, 2T(k,t) = \iint d^{3}k\, d^{3}j(-i)\left\{k_{\gamma}Q_{\beta\gamma\beta}(\boldsymbol{k}-\boldsymbol{j},-\boldsymbol{k},t) - k_{\gamma}Q_{\beta\gamma\beta}(-\boldsymbol{k}-\boldsymbol{j},\boldsymbol{k},t)\right\}. \tag{1.89}$$

Again using the continuity equation we may see that

$$(k_{\gamma} - j_{\gamma})u_{\gamma}(\boldsymbol{k}-\boldsymbol{j}) = 0, \tag{1.90}$$

meaning

$$k_{\gamma}u_{\gamma}(\boldsymbol{k}-\boldsymbol{j}) = j_{\gamma}u_{\gamma}(\boldsymbol{k}-\boldsymbol{j}). \tag{1.91}$$

This enables us to replace the first k_{γ} on the right hand side of equation (1.89) by j_{γ}, to give

$$\int_{0}^{\infty} dk\, 2T(k,t) = \iint d^{3}k\, d^{3}j(-i)\left\{j_{\gamma}Q_{\beta\gamma\beta}(\boldsymbol{k}-\boldsymbol{j},-\boldsymbol{k},t) - k_{\gamma}Q_{\beta\gamma\beta}(-\boldsymbol{k}-\boldsymbol{j},\boldsymbol{k},t)\right\}. \tag{1.92}$$

Since each triple moment is symmetric under the interchange of $\boldsymbol{k}$ and $\boldsymbol{j}$ it hence follows that the integrand is antisymmetric under the same interchange, and therefore it vanishes when integrated over all space with respect to these variables. Thus we may conclude that

$$\int_{0}^{\infty} dk\, T(k,t) = 0. \tag{1.93}$$

That is, $T(k, t)$ neither adds nor removes energy from the system. Instead, it redistributes energy between the modes, which leads its description as the *energy transfer spectrum*. This means we have the intuitive result for the integrated energy balance equation that the rate of change of energy in the system equals the input rate minus the dissipation rate,

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = \varepsilon_w(t) - \varepsilon_d(t). \tag{1.94}$$

To simplify our future calculations, from this point onwards we shall also restrict ourselves to considering only stationary turbulence, that is turbulence in which the total energy contained within the system is constant. As can be seen from equation (1.94), this implies that $\varepsilon_w(t) = \varepsilon_d(t) \equiv \varepsilon$.

1.7 The Richardson cascade and Kolmogorov's 1941 theory

The usual interpretation of the energy balance equation is that energy is input to the system at large scales (small k) by the $W(k, t)$ term, transferred to small scales (high k) by the non-linear $T(k, t)$ term, and then dissipated at these small scales by the $2\nu_0 k^2 E(k, t)$ term. A characteristic wavenumber, representative of the scales at which dissipation occurs, may be introduced if we note that the only relevant physical parameters available are the viscosity ν_0 and the dissipation rate $\varepsilon_d \equiv \varepsilon$. On dimensional grounds we can then introduce the *Kolmogorov wavenumber*

$$k_d = \left(\frac{\varepsilon}{\nu_0^3}\right)^{1/4}. \tag{1.95}$$

If we then similarly define an associated velocity scale

$$v = (\nu_0 \varepsilon)^{1/4} \tag{1.96}$$

and form a local (in wavenumber) Reynolds number based on these scales, we find

$$R(k_d) = \frac{v k_d^{-1}}{\nu_0} = 1, \tag{1.97}$$

which implies that for $k \sim k_d$, dissipation processes are indeed dominant.

This description of turbulence as a process in which energy is input at the large scales and then cascades through intermediate scales to the smallest scales where it is dissipated, has long been viewed in experiments [10] and owes its origins to Richardson [11]. For this reason the process is often referred to as the *Richardson cascade*, or alternatively simply as the *energy cascade*. The experiments also show that the regions where energy is input and dissipated do not overlap with one another, even at relatively low Reynolds number, and further that their separation increases with increasing Reynolds number. Thus it follows that the region in which energy is transferred, the so-called *inertial range*, can be made to dominate over as large a range of wavenumbers as we like, simply by increasing the Reynolds number.

An understanding of physical processes occurring in the inertial range is not easily gained, since such processes are described by the non-linear term in the NSE. A simple and plausible description, which illustrates all the important points, has however been given by Frisch [12] and is illustrated in Figure 1.2. In this picture we assume that energy is input to the system at an upper scale L, which is representative of the physical size of the system, and then cascades down through successive generations of eddies with scales $\ell_n = Lr^n$ (n=1,2,3,...), where $0 < r < 1$, until it reaches the scale of the smallest eddies ($\sim k_d^{-1}$). If we then further assume that the number of eddies per unit volume grows with n as r^{-3n}, which ensures that small eddies are as space filling as large eddies, we find that the cascade displays *scale invariance* within the inertial range. Secondly, we also find that interactions within the cascade are predominantly *local* in scale[2]. The physical argument behind this statement follows from the decomposition of the interaction between two eddies into (a) the convection of one by the other, and (b) the shearing of one by the other. In the first of these interactions there is no

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[2]In turbulence, 'local' is conventionally taken to refer to localness of scale, or, more particularly, localness in k-space. Unless explicitly stated otherwise, we shall follow this convention.
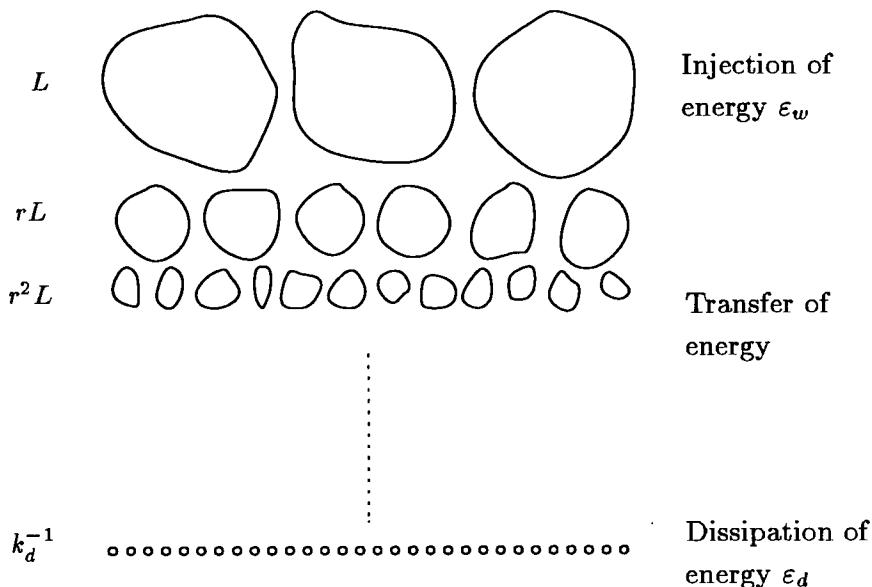
Figure 1.2: A schematic illustration of the transfer of energy within the Richardson cascade. Note that at each step the eddies are space filling.

energy transfer between the eddies, merely a phase change in the related Fourier coefficients, but in the second the internal distortion of the eddies will transfer energy to a smaller scale of disturbance. For eddies which differ vastly in size it would seem reasonable that interaction (a) is by far the most likely, meaning that the energy transfers are to some degree local in wavenumber space.

The assumptions of scale invariance within the inertial range and localness of energy transfer are encapsulated in arguably the most important theory regarding turbulence, that put forward by Kolmogorov in 1941 [13,14]. The two hypotheses he suggested are essentially similarity principles for the energy spectrum and can be expressed in k-space as follows. Firstly, if we assume that all energy is input to the system at a wavenumbers $\sim k_f$, where $k_f = L^{-1}$ and L is the physical size of the system, then it is argued that for $k \gg k_f$ the spectrum can only depend upon the viscosity and dissipation rate. On dimensional grounds this implies that we may write the energy spectrum as

$$E(k) = \nu_0^{5/4} \varepsilon^{1/4} f(k/k_d),$$ (1.98)

where f is an unknown function of universal form.

The second similarity hypothesis is that $E(k)$ should become independent of the viscosity as the Reynolds number tends to infinity. This implies that f must take the form

$$f(k/k_d) = \alpha(k/k_d)^{-5/3} \tag{1.99}$$

where α is the *Kolmogorov constant*. Substituting this back into equation (1.98) we thus obtain the energy spectrum

$$E(k) = \alpha\varepsilon^{2/3}k^{-5/3}, \tag{1.100}$$

a result which is referred to as the *Kolmogorov spectrum*.

In reality an infinite Reynolds number is, of course, unobtainable and it is hence of interest to find a similar result for large but finite Reynolds numbers. To obtain such a result we adapt the above arguments as follows. First we postulate that for sufficiently large Reynolds numbers there exists an inertial range of wavenumbers such that

$$k_f \ll k \ll k_d, \tag{1.101}$$

within which the energy spectrum is independent of the viscosity. Equation (1.99) is then modified to take the form

$$f(k/k_d) = \alpha(k/k_d)^{-5/3}F(k/k_d) \tag{1.102}$$

where F is another universal function, which satisfies $F(0) = 1$. Substituting this into equation (1.98) we then obtain the energy spectrum

$$E(k) = \alpha\varepsilon^{2/3}k^{-5/3}F(k/k_d) \tag{1.103}$$

for $k \gg k_f$, a result which tends asymptotically to the Kolmogorov spectrum within the inertial range.

1.8   Numerical simulation of turbulence

The NSE, equation (1.48), also enables us to view the turbulence problem in an alternative manner — namely as a mode coupling problem involving many degrees of freedom, the integral on the right hand side implying that in principle each mode depends upon every other mode. This approach to the problem is of particular relevance in the context of numerically simulating turbulent flows.

In a so-called *direct numerical simulation* (DNS) we attempt to simulate a turbulent flow by numerically evolving forward a discretized form of the NSE. If we view each discrete Fourier mode as a degree of freedom, then the difficulties inherent in this type of approach rapidly become apparent. For such simulations we clearly need to include all scales up to and including those of order k_d, and if we also note that we must resolve scales down to k_f then we may estimate the number of degrees of freedom of the system as

$$N \approx \left(\frac{k_d}{k_f}\right)^3. \tag{1.104}$$

Noting then that k_d increases with increasing Reynolds number, and that for any numerical calculation we are restricted by the available computer memory, there is hence an upper limit on the Reynolds number achievable in any given simulation. Indeed, it may be shown [15] that the number of degrees of freedom in a simulation may be related to the Reynolds number via

$$N \approx R^{9/4}. \tag{1.105}$$

At the present time, this restricts such numerical simulations to a maximum Reynolds number[3] of around 250 [16]. At such Reynolds numbers the inertial range is restricted to a very small range of wavenumbers, although despite this

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[3]Note that in numerical simulations, the Reynolds number referred to is usually taken to be the so-called Taylor-Reynolds number, whose definition is based upon the Taylor length scale (see [7]). This enables us to make a systematic comparison between different simulations. The Taylor-Reynolds number will be discussed further in Chapter 7.
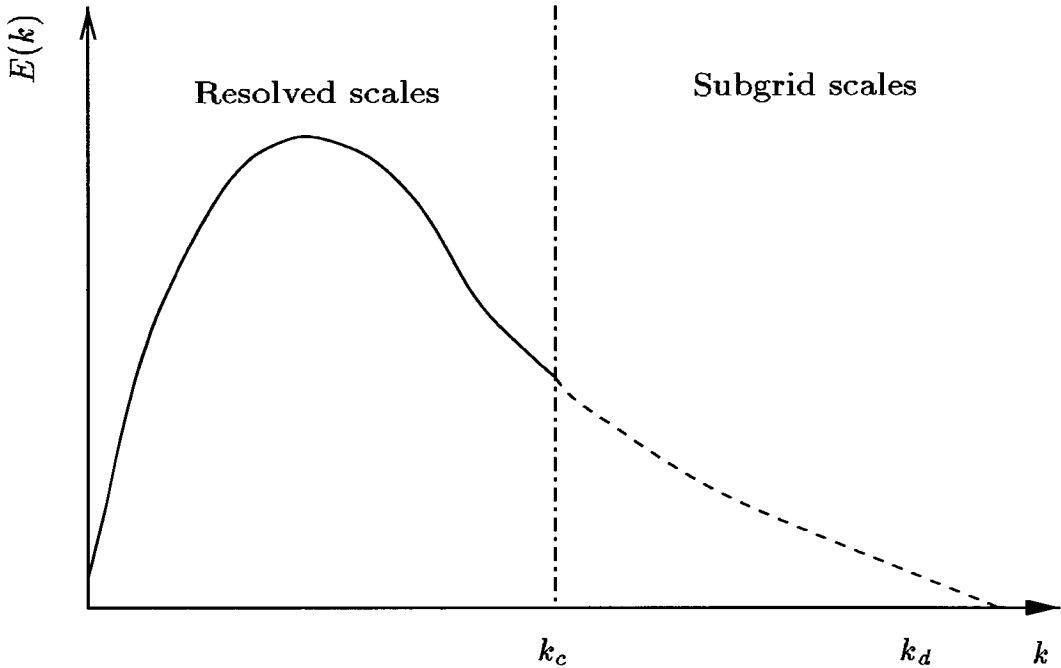
Figure 1.3: Wavenumber space divided up into resolved scales ($k < k_c$) and sub-grid scales ($k > k_c$) for the purposes of a large-eddy simulation.

the simulations obtain a value for the Kolmogorov constant of around 1.62, which is within the accepted experimental range of 1.6 to 2.5 [17].

Clearly however, a simulation with higher Reynolds number is desirable. Such an end by be achieved by either of two routes. Firstly we may simply wait until we have a more powerful computer at our disposal. Alternatively we may attempt to systematically reduce the number of degrees of freedom included in the simulation. One way in which this may be achieved is by performing a *large-eddy simulation* (LES). The basic idea of an LES is illustrated in Figure 1.3. Rather than simulate all the modes up to those of order k_d, as we would do in a DNS, we instead simulate only those wavenumbers below an arbitrarily chosen cutoff wavenumber k_c ($k_c \ll k_d$). Following the ideas of Boussinesq [18] and Heisenberg [19,20], the effect of the discarded, or *sub-grid*, modes upon the remaining *resolved* modes is then represented by an increased, wavenumber dependent, viscosity. That is, we assume that the dynamic properties of the resolved scales may be described by a

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truncated NSE in which we have made the substitution

$$\nu_0 \to \nu_0 + \delta\nu(k|k_c), \qquad (1.106)$$

where the notation $\delta\nu(k|k_c)$ indicates that this is the wavenumber dependent increment to the viscosity *given* the cutoff wavenumber k_c. Of course obtaining $\delta\nu(k|k_c)$ is not a simple procedure, but as we shall see in future chapters, our work provides one approach to obtaining such an *eddy-viscosity*. It would however also appear that the eddy-viscosity thus obtained is alone unlikely to be sufficient to describe the dynamics of the system as a whole.

1.9   Overview of the thesis

In the remainder of this thesis we shall put forward and test a new model for turbulent fluids. We shall begin in Chapter 2 by giving a critical overview of the general class of Renormalization Group approaches to the turbulence problem. In Chapters 3, 4 and 5 we shall then develop and carry out our RG calculation, the results we obtain being considered at the end of Chapter 5. In Chapter 6 we shall then consider two of the approximations made in our RG calculation, their justification being deferred until this point. Finally, in Chapter 7 we shall use the eddy-viscosity found in our calculation as the basis for a large-eddy simulation, comparing the results obtained to those of alternative models and a DNS, before bringing together our conclusions in Chapter 8.

Chapter 2

Modelling turbulence using the Renormalization Group

2.1 Theoretical approaches to turbulence

Over the years there have been many different attempts to theoretically describe turbulence. These start with the previously mentioned effective viscosity theory of Heisenberg [19, 20] and the quasi-normality hypothesis [21, 22] in which the moment hierarchy is closed by assuming that the fourth order moments may be related to second order moments as if for a normal distribution. However, these simple approaches have some fundamental problems, the Heisenberg theory giving rise to the incorrect spectrum in the dissipation region [23] (k^{-7} as opposed to exponential decay), whilst when solved numerically and integrated forward in time [24–26] the quasi-normality hypothesis gives an energy spectrum which becomes negative for certain values of k. Thus, it is immediately apparent that in order to accurately describe turbulence we will need more sophisticated models.

The earliest such models were based upon a perturbation expansion of the NSE in which we introduce the book-keeping parameter λ ($\lambda = 1$) as a factor in the non-linear term. This gives

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_\alpha(\boldsymbol{k}, t) = f_\alpha(\boldsymbol{k}, t) + \lambda M_{\alpha\beta\gamma}(\boldsymbol{k}) \int \mathrm{d}^3 j \, u_\beta(\boldsymbol{j}, t) u_\gamma(\boldsymbol{k} - \boldsymbol{j}, t), \quad (2.1)$$

along with the perturbation series

$$u_\alpha(\boldsymbol{k}, t) = u_\alpha^{(0)}(\boldsymbol{k}, t) + \lambda u_\alpha^{(1)}(\boldsymbol{k}, t) + \lambda^2 u_\alpha^{(2)}(\boldsymbol{k}, t) + \ldots . \tag{2.2}$$

Substituting (2.2) into (2.1) and equating coefficients of λ we then obtain the hierarchy of equations

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_\alpha^{(0)}(\boldsymbol{k}, t) = f_\alpha(\boldsymbol{k}, t), \tag{2.3}$$

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_\alpha^{(1)}(\boldsymbol{k}, t) = M_{\alpha\beta\gamma}(\boldsymbol{k}) \int \mathrm{d}^3 j \, u_\beta^{(0)}(\boldsymbol{j}, t) u_\gamma^{(0}(\boldsymbol{k} - \boldsymbol{j}, t), \tag{2.4}$$

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_\alpha^{(2)}(\boldsymbol{k}, t) = M_{\alpha\beta\gamma}(\boldsymbol{k}) \int \mathrm{d}^3 j \, 2 u_\beta^{(0)}(\boldsymbol{j}, t) u_\gamma^{(1)}(\boldsymbol{k} - \boldsymbol{j}, t), \tag{2.5}$$

$$\vdots \qquad \vdots$$

The formal solutions of these equations may then be substituted back into the RHS of (2.2) to give an exact expression for the velocity field in terms of the zero order field. If λ were small we could then truncate this expression at some appropriate order, as in a conventional perturbation theory. However since in fact $\lambda = 1$ the expansion is highly divergent, meaning we can only view the expansion as being in orders of complexity of the mode coupling. Hence there can be no simple justification for truncating the expansion at low order. To deal with this fact we need to introduce a renormalization scheme.

Such renormalized perturbation theories (RPT) fall into two distinct classes; those which obtain the Kolmogorov energy spectrum and those which do not. Into the first of these groups fall the well known direct-interaction approximation (DIA) of Kraichnan [27], the Edwards-Fokker-Planck (EFP) theory [28] and the self consistent field (SCF) theory of Herring [29, 30]. Although being based upon different principles, the DIA, say, using the idea that we may introduce an infinitesimal response tensor $G_{\alpha\beta}(k; t, t')$ such that a fluctuation in the velocity field $\delta u_\alpha(\boldsymbol{k}, t)$ may be related to a fluctuation in the forcing $\delta f_\alpha(\boldsymbol{k}, t)$ by

$$\delta u_\alpha(\boldsymbol{k}, t) = \int_{-\infty}^{t} \mathrm{d}t' \, G_{\alpha\beta}(\boldsymbol{k}; t, t') \delta f_\beta(\boldsymbol{k}, t'), \tag{2.6}$$

while the EFP theory is based upon an adaptation of Brownian motion theory and uses the probability distribution of fluctuations, connections between these theories can be made [31]. Indeed the SCF theory, which is in many ways similar to that of Edwards, yields the DIA equations in its later time-dependent formulation. However all these theories have a major flaw in that they predict an energy spectrum which differs significantly from that of Kolmogorov. In order to consider how we may rectify this problem we have to study the theories from the second group.

The first such theory, Kraichnan's Lagrangian history theory [32] is essentially just a re-formulation of the DIA theory, but in terms of quasi-Lagrangian co-ordinates rather than the Eulerian coordinates used in the earlier theory. This however leads to a theory which is considerably more complex than the previous approach, although it does now yield the Kolmogorov spectrum. Based upon this theory Kraichnan [33] reported the first theoretical prediction for the Kolmogorov constant, which at a value of $\alpha = 1.77$ lies within the current range of experimental values.

Similarly, attempts have also been made to modify the EFP theory in such a way that the Kolmogorov spectrum is obtained. The first of these attempts was made by Edwards and McComb [34] who used the idea of maximizing the turbulent entropy. This was later followed by the local energy transfer (LET) theory [35–38], the basic ansatz of which is that the velocity field is connected to itself at later times by an exact propagator.

The first of these approaches predicted a somewhat high value for the Kolmogorov constant of $\alpha = 3.6$, but this theory involves a potentially large uncontrolled approximation, where in order to facilitate a practical calculation Edwards and McComb simply drop a term from their equations without any real justification. In contrast the LET theory gives a value of $\alpha = 2.5$ which is at the upper end of the experimental range. This theory is still the subject of ongoing research

[39], as are several of the approaches just discussed. In the remainder of this thesis however we shall consider a different class of approaches to the turbulence problem, those which utilize the Renormalization Group.

2.2   The Renormalization Group

The origins of the Renormalization Group (RG) method lie within quantum field theory, where it was originally developed to investigate the uniqueness of renormalization procedures. As is well known, when applying perturbation theory to problems within quantum field theory we obtain results displaying divergences, which in *renormalizable* theories appear as an infinite correction to the bare parameters of the problem, for instance the mass or charge. The idea behind renormalization is that the physically observable value of any such parameter may be represented as a sum of the bare parameter plus a field correction calculated using perturbation theory, the only constraint being that this sum is finite. Individual quantities can however be infinite, and it is here where problems potentially lie since the operation of subtracting one infinity from another is not unique. To eliminate this non-uniqueness the normalization conditions must also be given and subjected to the requirement of *renormalization invariance.*

Within the perturbation theory, renormalization may be viewed as replacing the bare parameters in the equations for the unperturbed system by the renormalized parameters and adding counterterms to the perturbing part to compensate for these changes. Given this viewpoint, the normalization condition then consists of the requirement that at a given normalization point — namely for prescribed values of the coordinates, times, wavenumbers etc. — the field theoretic corrections exactly compensate the counterterm. Renormalization invariance then requires that the solution of the complete equation, as described by the infinite perturbation series, must not depend upon the method used for partitioning the equation into the perturbed and unperturbed parts. Since the choice of the normalization

point is usually associated with the introduction of a new scale, for instance the cutoff wavenumber in diverging integrals, the condition of renormalization invariance reduces to the requirement that there be no dependence upon the choice of this scale.

However, although renormalization invariance must apply to the perturbation series as a whole, individual terms within the series will vary with changes in the normalization conditions. These changes will occur according to laws determined by renormalization invariance, and it is found that the transformations describing the transition from one normalization point to another obey a group composition law. According to this, two successive transformations are also a transformation corresponding to a change in the normalization conditions and hence the set of these transformations forms a continuous group, the *renormalization group*.

In the RG transformations, the simplest case requires us to deal with scale transformations of the form $r \to r' = r/\lambda$, with a corresponding transformation of the numerical parameters $g \to g' = \tilde{g}(\lambda, g)$. From the group composition law, it follows that the transformation

$$g \to g'' = \tilde{g}(\lambda\lambda', g) \tag{2.7}$$

is identical to the two consecutive transformations

$$
\begin{aligned}
g &\to g' = \tilde{g}(\lambda, g) \\
g' &\to g'' = \tilde{g}(\lambda', g'),
\end{aligned}
\tag{2.8}
$$

and hence we have the result

$$\tilde{g}(x, g) = \tilde{g}(x/\lambda, \tilde{g}(\lambda, g)). \tag{2.9}$$

Differentiating this expression with respect to λ and then setting $\lambda = 1$ we obtain, noting the identity condition $\tilde{g}(1, g) = g$, the differential equation of the renormalization group

$$\left\{ -x\frac{\partial}{\partial x} + \beta(g)\frac{\partial}{\partial g} \right\} \tilde{g}(x, g) = 0, \tag{2.10}$$

where the RG function $\beta(g) = \partial\tilde{g}(x,g)/\partial x|_{x=1}$ is determined by the behaviour of a physical quantity near to the normalization point. Having calculated the RG function using perturbation theory, the solution of the resulting differential equation will then satisfy renormalization invariance in its entire range and reproduce the results of the lowest order perturbation approximations near the normalization point.

This field theoretic approach to RG is not however that which we shall apply within the context of this thesis. Instead, we shall follow a somewhat different, but equivalent, approach due to Wilson, for which he was subsequently awarded the 1982 Nobel prize.

2.3 Wilson's formulation of the Renormalization Group

In Wilson's approach [40–42], RG provides an approach for investigating problems involving many, equally important, length or time scales, and was originally applied to problems in critical phenomena, for instance the Kondo problem [41]. Relying upon the physical idea that interactions between modes are predominantly local, the essential idea of Wilson's approach is to deal with the problem in steps, each step representing a particular length scale. This enables us to systematically reduce the number of modes involved in the problem. To obtain this reduction we start by averaging over a narrow band of small scale modes (that is, high wavenumber modes), the average effect of these small scale modes being retained in the new equations for the remaining larger scales. These new equations are then rescaled so that they are defined on the original interval and the entire process, with the elimination of the small scales followed by rescaling, is performed repeatedly over narrow bands of increasing scale. In doing this the average effect of the small scale modes upon the large scales reduces to a change (or renormalization) of the transport parameters in the problem.

This procedure is iterated upon until the so-called *fixed point* of the RG transformation is reached. If we describe the RG transformation, that is the averaging out of a range of modes plus the accompanied rescaling, with the notation

$$g' = \mu[g], \tag{2.11}$$

where g is a general transport parameter, g' is its value after the RG transformation, and μ is a general functional representing the transformation, then the fixed point of the RG transformation is defined to be the particular set g^* which is invariant under the transformation, that is, which satisfies the expression

$$g^* = \mu[g^*]. \tag{2.12}$$

The physical explanation of the fixed point is relatively simple. At the fixed point, the system has become independent of the details at the largest scales and displays universal behaviour. This enables an explanation of the universality of various types of critical behaviour and allows us to obtain the critical exponents involved in scaling laws. As has been shown many times since, RG provides arguably the best technique for obtaining such values in a range of scaling problems.

2.4 Description of Turbulence using the Renormalization Group

The obvious analogies between the assumptions of universality and localness of the turbulent cascade, and the assumptions made in Wilson's theory immediately suggest that RG could be used to describe fully developed turbulence. This has given rise to several different theories, all of which fall under the banner 'renormalization group theories of turbulence'.

The first attempt to study a problem involving fluid motion using RG was that of Forster, Nelson and Stephen [43, 44], which although not strictly describing turbulence is of relevance. Indeed, it was arguably extended to describe turbulence

in the later work of Yakhot and Orszag [45, 46]. In addition to these theories there is also the work of McComb *et al.* [47–51], which has analogues with Rose's earlier work on passive scalar convection [52] and which pre-dates the work of Yakhot and Orszag, Zhou and Vahala [53–55] and Eyink [56]. We shall shortly discuss in chronological order, save for the theory of Yakhot and Orszag which is implicitly connected to that of Forster, Nelson and Stephen, the specific details and criticisms of each of these theories (for greater detail see the reviews contained in [7, 57–59]), but all such attempts to apply RG to the NSE follow the same basic algorithm.

We start by introducing an upper (ultra-violet) cutoff wavenumber k_0, and then decompose the velocity field at a wavenumber k_1 ($k_1 < k_0$) such that

$$u_\alpha(\boldsymbol{k}, t) = \begin{cases} u_\alpha^-(\boldsymbol{k}, t) \text{ for } 0 < k < k_1 \\ u_\alpha^+(\boldsymbol{k}, t) \text{ for } k_1 < k < k_0. \end{cases} \tag{2.13}$$

We further assume that the force term may also be decomposed in an identical manner, and given this decomposition the NSE may be re-written as two coupled equations

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_\alpha^-(\boldsymbol{k}, t) = f_\alpha^-(\boldsymbol{k}, t) + M_{\alpha\beta\gamma}^-(\boldsymbol{k}) \int \mathrm{d}^3 j \left\{ u_\beta^-(\boldsymbol{j}, t) u_\gamma^-(\boldsymbol{k} - \boldsymbol{j}, t) \right. $$
$$\left. + 2 u_\beta^-(\boldsymbol{j}, t) u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t) + u_\beta^+(\boldsymbol{j}, t) u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t) \right\} \tag{2.14}$$

and

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_\alpha^+(\boldsymbol{k}, t) = f_\alpha^+(\boldsymbol{k}, t) + M_{\alpha\beta\gamma}^+(\boldsymbol{k}) \int \mathrm{d}^3 j \left\{ u_\beta^-(\boldsymbol{j}, t) u_\gamma^-(\boldsymbol{k} - \boldsymbol{j}, t) \right. $$
$$\left. + 2 u_\beta^-(\boldsymbol{j}, t) u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t) + u_\beta^+(\boldsymbol{j}, t) u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t) \right\} \tag{2.15}$$

Given equations (2.14) and (2.15), the RG procedure then involves two stages

1. Solve equation (2.15) on $k_1 < k < k_0$ and then substitute this solution for the mean effect of the high-k modes into equation (2.14). We find that this results in an increment to the viscosity $\nu_0 \rightarrow \nu_1 = \nu_0 + \delta \nu_0$.

2. Rescale the basic variables so that the new NSE on $0 < k < k_1$ looks like the original NSE, equation (1.48), on $0 < k < k_0$.

These steps are then repeated until a fixed point is obtained. The differences between the various RG theories lie in how we go about performing these steps.

2.4.1   Forster, Nelson and Stephen

We consider first the theory of Forster, Nelson and Stephen (FNS) [43,44]. In this theory we start by restricting our system to the range $k \leq \Lambda$, where $\Lambda \ll k_d$, and defining the stirring forces via their autocorrelation (in d dimensions)[1]

$$\langle f_\alpha(\boldsymbol{k}, \omega) f_\beta(\boldsymbol{k}', \omega') \rangle = 2W(k)(2\pi)^{d+1} D_{\alpha\beta}(\boldsymbol{k}) \delta(\boldsymbol{k} + \boldsymbol{k}') \delta(\omega + \omega'). \tag{2.16}$$

The system is then further simplified by assuming $W(k)$ takes the form of a power law,

$$W(k) = W_0 k^{-y}, \tag{2.17}$$

and assuming that the stirring forces are multivariate normal with zero mean.

FNS then define $k_0 \equiv \Lambda$ and relate this value to k_1 using the definition, the choice of which simplifies later results, $k_1 = \Lambda \exp(-\ell)$, with ℓ being such that $0 < \exp(-\ell) < 1$. u^+ may then be eliminated from (2.14) by use of the perturbation series

$$u_\alpha^+(\boldsymbol{k}, \omega) = u_\alpha^{+(0)}(\boldsymbol{k}, \omega) + \lambda u_\alpha^{+(1)}(\boldsymbol{k}, \omega) + \lambda^2 u_\alpha^{+(2)} + \dots, \tag{2.18}$$

where the strength parameter $\lambda(= 1)$ indicates the order of the expansion.

In this approach it is assumed that the non-linear term gives rise to a perturbation about the solution which would be obtained if only the forcing term was present on the right hand side of the NSE. Hence, we introduce λ as a factor in the non-linear term. Having done this, by substituting the perturbation series into (2.15)

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[1]Note also that in this theory we work with a form of the NSE in which the time variable has been Fourier transformed along with the spatial variables. This gives rise to the ω variables in the subsequent equations.

and equating coefficients of λ, FNS obtain the hierarchy of equations

$$u_\alpha^{+(0)}(\boldsymbol{k},\omega) = G_0(\boldsymbol{k},\omega)f_\alpha^+(\boldsymbol{k},\omega), \tag{2.19}$$

$$
\begin{aligned}
u_\alpha^{+(1)}(\boldsymbol{k},\omega) = G_0(\boldsymbol{k},\omega)M_{\alpha\beta\gamma}^+(\boldsymbol{k})\int_{j\leq\Lambda}\mathrm{d}^3j\int\mathrm{d}\Omega\,\Big\{&u_\beta^-(\boldsymbol{j},\Omega)u_\gamma^-(\boldsymbol{k}-\boldsymbol{j},\omega-\Omega)\\
&+2u_\beta^-(\boldsymbol{j},\Omega)u_\gamma^{+(0)}(\boldsymbol{k}-\boldsymbol{j},\omega-\Omega)\\
&+u_\beta^{+(0)}(\boldsymbol{j},\Omega)u_\gamma^{+(0)}(\boldsymbol{k}-\boldsymbol{j},\omega-\Omega)\Big\},
\end{aligned} \tag{2.20}
$$

$$
\begin{aligned}
u_\alpha^{+(2)}(\boldsymbol{k},\omega) = G_0(\boldsymbol{k},\omega)M_{\alpha\beta\gamma}^+(\boldsymbol{k})\int_{j\leq\Lambda}\mathrm{d}^3j\int\mathrm{d}\Omega\,\Big\{&2u_\beta^-(\boldsymbol{j},\Omega)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},\omega-\Omega)\\
&+2u_\beta^{+(0)}(\boldsymbol{j},\Omega)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},\omega-\Omega)\Big\}
\end{aligned} \tag{2.21}
$$

and so on, where $G_0(\boldsymbol{k},\omega) = [i\omega + \nu_0 k^2]^{-1}$. Clearly all higher order terms in the perturbation series may be expressed in terms of $u_\alpha^{+(0)}(\boldsymbol{k},\omega)$, and since G_0 is statistically sharp we may thus average out the effects of the high frequencies using our earlier defined statistics of the forcing term. These imply, amongst others, the properties

$$\langle u_\alpha^{+(0)}(\boldsymbol{k},\omega)\rangle = 0, \qquad M_{\alpha\beta\gamma}^-(\boldsymbol{k})\langle u_\beta^{+(0)}(\boldsymbol{j},\Omega)u_\gamma^{+(0)}(\boldsymbol{k}-\boldsymbol{j},\omega-\Omega)\rangle = 0. \tag{2.22}$$

Substituting (2.18) into the version of (2.14) in which the time variable has also been Fourier transformed, FNS then obtained

$$
\begin{aligned}
\left(i\omega + \nu_0 k^2\right)u_\alpha^-(\boldsymbol{k},\omega) = f_\alpha^-(\boldsymbol{k},\omega) + \lambda M_{\alpha\beta\gamma}^-(\boldsymbol{k})\int_{j\leq\Lambda}\mathrm{d}^3j\int\mathrm{d}\Omega\,\times&\\
\times\Big\{u_\beta^-(\boldsymbol{j},\Omega)u_\gamma^-(\boldsymbol{k}-\boldsymbol{j},\omega-\Omega)&\\
+2u_\beta^-(\boldsymbol{j},\Omega)u_\gamma^{+(0)}(\boldsymbol{k}-\boldsymbol{j},\omega-\Omega)&\\
+u_\beta^{+(0)}(\boldsymbol{j},\Omega)u_\gamma^{+(0)}(\boldsymbol{k}-\boldsymbol{j},\omega-\Omega)\Big\}&\\
+\lambda^2 M_{\alpha\beta\gamma}^-(\boldsymbol{k})\int_{j\leq\Lambda}\mathrm{d}^3j\int\mathrm{d}\Omega\,\Big\{2u_\beta^-(\boldsymbol{j},\Omega)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},\omega-\Omega)&\\
+2u_\beta^{+(0)}(\boldsymbol{j},\Omega)u_\gamma^{+(1)}(\boldsymbol{k}-\boldsymbol{j},\omega-\Omega)\Big\} + \mathcal{O}\left(\lambda^3\right),&
\end{aligned} \tag{2.23}
$$

where we may substitute for $u^{+(1)}$ from (2.20).

The next step is to average out the effects of the high-k modes upon the low wavenumber modes. However this averaging cannot be achieved using the usual

ensemble average since, for example, $\langle u_\alpha^-(\boldsymbol{k},\omega)\rangle = 0$ and hence taking the ensemble average of (2.23) gives a trivial result. To get around this problem FNS instead apply a filtered ensemble average $\langle \cdot \rangle_f$, in which $\langle f^-\rangle_f = f^-$ and $\langle u^-\rangle_f = u^-$ whilst the filtered ensemble average of a term involving solely high-k modes is identical to the ordinary ensemble average. Substituting from (2.20) and applying such an average to (2.23), FNS then obtain the result, in very abbreviated notation[2],

$$
\begin{aligned}
(i\omega + \nu_0 k^2)u_k^- &= f_k^- + \lambda M_k^- \left\{ u_j^- u_{k-j}^- \right\} - \lambda^2 \Delta\nu_0 k^2 u_k^- \\
&\quad + 2\lambda^2 M_k^- M_{k-j}^+ G_0(k-j)u_j^- u_p^- u_{k-j-p}^-,
\end{aligned}
\tag{2.24}
$$

with the viscosity increment $\Delta\nu_0$ being given by

$$
\Delta\nu_0 = \frac{\lambda^2 W_0}{\nu_0^2 \Lambda^\epsilon} \frac{A(d)S_d}{(2\pi)^d} \frac{(1 - \exp(\epsilon\ell))}{\epsilon},
\tag{2.25}
$$

where

$$
\epsilon = 4 + y - d,
\tag{2.26}
$$

$$
A(d) = \frac{d^2 - d - \epsilon}{2d(d+2)},
\tag{2.27}
$$

$$
S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)},
\tag{2.28}
$$

$\Gamma(x)$ being the usual Gamma function.

In the limit $k \to 0$ FNS show that the last term on the right hand side of (2.24) (i.e. the triple non-linearity in u^-) is, as the RG operation proceeds, an irrelevant variable and so this leaves us with the final equation

$$
\left(i\omega + (\nu_0 + \Delta\nu_0)k^2\right) = f_\alpha^-(\boldsymbol{k},\omega) + \lambda M_{\alpha\beta\gamma}^-(\boldsymbol{k}) \int_{j\leq\Lambda} d^3j \int d\Omega\, u_\beta^-(\boldsymbol{j},\Omega)u_\gamma^-(\boldsymbol{k}-\boldsymbol{j},\omega-\Omega).
\tag{2.29}
$$

Thus FNS obtain an expression which is of identical form to that with which they originally start, save for the fact that it has an increased viscosity $\nu_1 = \nu_0 + \Delta\nu_0$, but which is defined on the range $0 < k < \Lambda\exp(-\ell)$.

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[2]This abbreviated notation, in which we drop all integrals, tensor labels and time variables shall be used throughout the remaining chapters. It is hoped that the actual form of the individual terms should be readily apparent.

As is required by the RG procedure, FNS then rescale this result onto the original range and subsequently iterate upon the complete operation until they reach the fixed point for a renormalized form of the strength parameter λ, an infinitesimal wavenumber band being eliminated at each step. Finally, they extended their theory to obtain the energy spectrum

$$E(k) \sim k^{-5/3+2(d-y)/3}. \tag{2.30}$$

As it stands, the FNS theory provides a relatively rigorous approach for applying RG to the NSE, but due to the requirements that Λ is below the inertial range and $k \to 0$ it does *not* actually describe turbulence. Instead, as acknowledged by the authors themselves, it is a theory which describes the long wavelength properties of stirred hydrodynamics. The theory is however open to some criticisms. Firstly, as pointed out by McComb [7], there is no consideration of the fact that the non-linear term will still transfer energy to values of $k > \Lambda$ despite the fact that the theory only includes smaller wavenumbers. This energy transfer will have an effect on the dissipation wavenumber, which is determined only by the rate of energy transfer and the viscosity, and consequently it is not immediately obvious that the condition $\Lambda \ll k_d$ will be satisfied in practice. There is also a more serious criticism due to Eyink [56], who in discussing the FNS theory stated

> "The equation for u^+ is solved perturbatively in the non-linearity in terms of u^- and f^+. This solution is then used to eliminate u^+ everywhere in the equation for u^- and subsequently this equation is averaged over the known statistics of f^+, **assuming independence from** u^-, [Eyink's emphasis] to give the effective dynamics of the variables u^-. However, this is an uncontrolled approximation, since the u^- variables get a statistical dependence on the forces f^+ through their coupling to the u^+ variables and a conditional average over the f^+ forces with u^- fixed will change the distribution of the forces f^+ in an unknown way."

This criticism has, however, been addressed by Hunter [60] and the other points are addressed in the theory of Yakhot and Orszag, which extends the FNS approach to actual turbulence.

2.4.2 Yakhot and Orszag

The theory of Yakhot and Orszag (YO) [45, 46] is essentially an extension of that of FNS, in its initial stages using virtually identical assumptions and mathematics as the earlier theory. YO differ however from FNS in that they claim their theory describes inertial range turbulence. This claim is based upon their so-called *correspondence principle*, according to which they consider the FNS forcing terms to be equivalent, in a statistical sense, to the boundary and initial conditions of a freely cascading turbulent flow, provided only that the forcing is chosen correctly.

Beyond introducing the correspondence principle, YO make two other additional assumptions. First they assume that the upper cutoff Λ is defined such that $\Lambda = \mathcal{O}\left(k_d\right)$, hence meaning that the system they consider contains the inertial range. Second, it is assumed, with no mathematical justification, that the triple u^- moment, which FNS show to be irrelevant in the limit $k \to 0$, is also irrelevant for scales within the inertial range. As a consequence of making such assumptions, YO obtain identical expression to FNS for quantities such as the energy spectrum, equation (2.30), but it is now claimed that they are also applicable to the inertial range. Using these results YO then proceed to extend the theory, in particular obtaining a value for the Kolmogorov constant.

From equation (2.30), we can see that when $y = d$, $E(k) \sim k^{-5/3}$, a result which provides their starting point. In this case however the triple non-linearity in u^- *doesn't* tend exponentially to zero as the RG iteration proceeds, but instead gives rise to logarithmic corrections to the spectrum. YO circumvent this problem simply by assuming that such contributions are small, although there is very little evidence to justify this assumption. Assuming however that this approach is legitimate, the Kolmogorov constant is obtained by first modifying the RG approach to eliminate a finite band of modes in the range $\Lambda(\ell) = \Lambda e^{-\ell} < k < \Lambda$.

Doing this they obtain a renormalized viscosity $\nu(\ell)$,

$$\nu(\ell) = \nu_0 \left[1 + \frac{3}{4} \frac{A(d) S_d}{(2\pi)^d} \lambda^2 (e^{4\ell} - 1) \right]^{1/3}, \qquad (2.31)$$

where $A(d)$ and S_d are as defined in equations (2.27) and (2.28) respectively, and, for modes with $k < \Lambda(\ell)$, the wavenumber dependent eddy-viscosity

$$\nu(k) = \left(\frac{3}{4} A_d W_0 \right)^{1/2} k^{-4/3}. \qquad (2.32)$$

In order to obtain a value for the Kolmogorov constant we next need to relate W_0 to the mean energy dissipation ε. YO achieved this by introducing an equation for the energy balance obtained from renormalized perturbation theory [61]. Substituting the Kolmogorov spectrum along with equation (2.32) into this expression then gives

$$\alpha = 1.496 \left(\frac{2 W_0 S_d}{(2\pi)^d \varepsilon} \right)^{1/6}, \qquad (2.33)$$

from which they proceed to obtain the final expression for the energy spectrum

$$E(k) = 1.617 \varepsilon^{2/3} k^{-5/3}. \qquad (2.34)$$

That is, the Kolmogorov spectrum with constant $\alpha = 1.617$.

Despite the fact that the approach of YO is in many respects identical to that of FNS, due to the additional assumptions needed to allow its extension to the inertial range the theory is open to considerably more criticism. Indeed the main justification for many of the assumptions seems simply to be that the results thus obtained are in apparently good agreement with experimental values.

One of the first and most major criticisms the YO theory is usually subjected to regards an arithmetical inconsistency in their calculation of α. This is the fact that they use two different values for ϵ, $\epsilon = 0$ and $\epsilon = 4$, at different points in the same calculation. However it has relatively recently been shown by Wang and Wu [62] that the need to do this arises from an error made by YO, when in changing an integration variable the region of integration was mistakenly left

unchanged[3]. Once this error is corrected it is found that the final result is the same as that given by YO, but a term which YO drop by setting $\epsilon = 0$ now disappears naturally. This means we can use a value of $\epsilon = 4$ consistently throughout the calculation. There do however remain several criticisms of the theory.

The correspondence principle, by which YO assume that the turbulent cascade can be described by carefully choosing the random forcing, is a particularly bold assumption, and allied to this is the fact that in order to have a Kolmogorov type $k^{-5/3}$ spectrum we require $\epsilon = 4$. Eyink argues [56] that there is no RG fixed point for $\epsilon > 3$ (although he also admits that this may not be the case if the theory is reformulated in terms of Lagrangian histories) and if true this casts doubt upon whether or not the RG theory can be extended to the necessary value of ϵ. Further to this, Eyink also shows that the triple non-linearity in u^-, which YO hope is irrelevant, remains finite as the iteration proceeds, only becoming irrelevant in the $k \to 0$ limit as used by FNS. This clearly brings into question the assumption that this term is negligible. Similarly there is also the question of whether the logarithmic corrections arising from setting $y = d$ are negligible. Again there is no justification for this, YO merely hope that they are!

Finally there is the question of whether use of a result from renormalized perturbation theory is legitimate. As discussed by McComb [7], use of the energy balance equation obtained from RPT requires, using conservation of energy, that the stirring forces are confined to a band such that $k_{\mathrm{max}}/k_{\mathrm{min}} = 1.007$. This could clearly give problems with application of the correspondence principle. The fact that the result is also reliant upon a different class of theory (with unknown convergence properties) also raises doubts about whether the results can even be described as arising from an RG theory. Indeed as a final comment it is worth noting that Kraichnan [63] has obtained essentially the same results as YO merely

---

[3]The same error was also made by FNS, but in their case it did not lead to any arithmetical inconsistency in their later calculations. Consequently, we should also make the same correction to their theory.

by using a simple perturbation model and making the same assumptions.

2.4.3 McComb *et al.*

An alternative to the RG approaches discussed thus far is provided by the work of McComb *et al.* [48,50,51,64]. In particular, rather than eliminate infinitesimal bands of wavenumbers they instead eliminate finite blocks of modes.

As in the previous theories, McComb *et al.* start from the forced NSE and perform the usual decomposition into u^- and u^+ modes. However, rather than prescribe the statistics of the forcing they instead assume that the forcing is purely to maintain stationarity and deal with the fact that the u^- and u^+ terms are not statistically independent of one another by introducing a so-called *conditional average* [65]. We shall discuss this conditional average (CA) in far greater detail in Chapter 3, but the basic idea is relatively simple to understand. Essentially, from the complete ensemble of turbulent realizations we select a *biased*, or conditional, subensemble, the members of which are selected to have their low-k modes differing from u^- by less than a small amount ξ. The CA is then defined to be the (sub)ensemble average over this subset of realizations.

The slight uncertainty in the low-k modes is required since if we chose the members of the biased subensemble to have identical low wavenumber modes to u^- then the deterministic nature of the NSE means that all the members of the subensemble would be identical. However given our ideas about the chaotic nature of the NSE, see for instance [66], and the localness of energy transfer in the turbulent cascade, it would seem reasonable to assume that outside the constrained low-k region the members of the biased subensemble are, to some extent at least, unconstrained. These ideas are illustrated schematically in Figure 2.1, which shows how members of the biased subensemble are selected from the full ensemble along with the results we would expect to obtain after averaging over both sets of realizations. As we show here, only those realizations which lie within

Full Ensemble   Conditional Subensemble
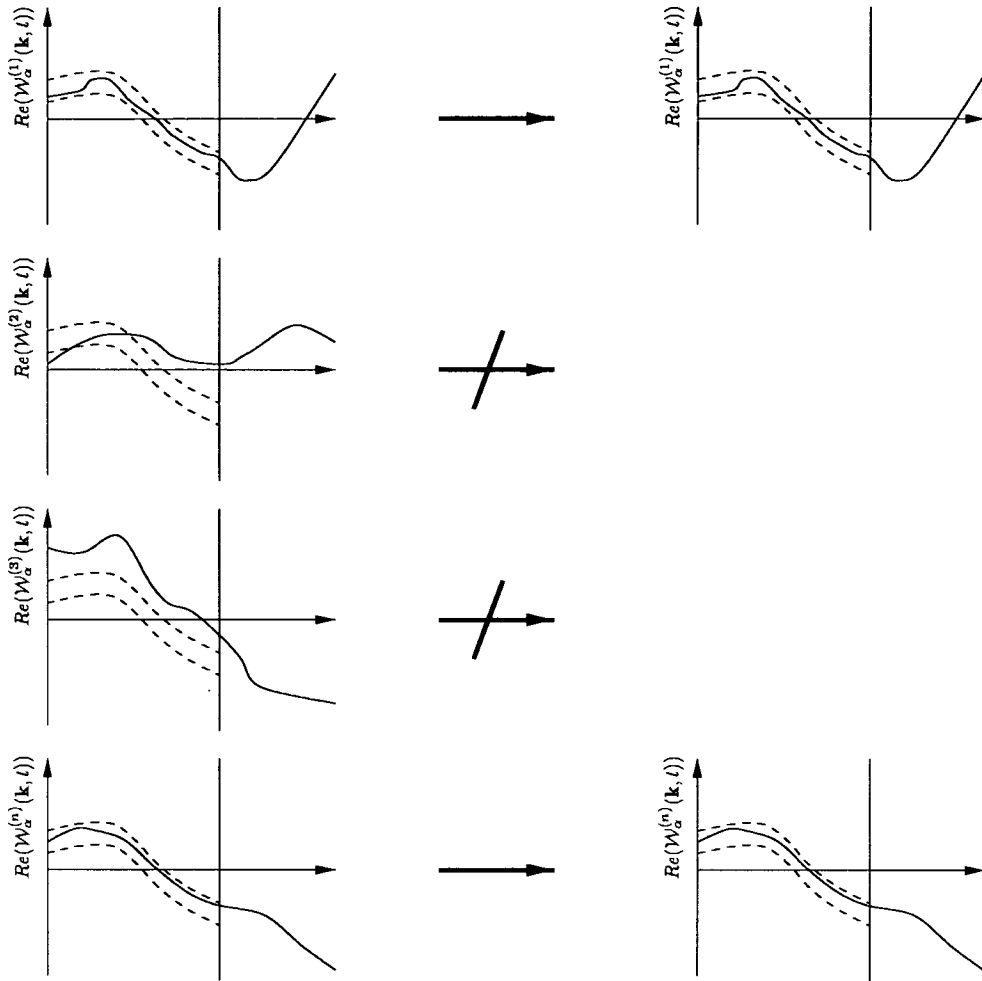
Ensemble Averages

Figure 2.1: Schematic illustration of the way in which we select members of the biased subensemble, and the results we would expect if we perform an ensemble average over each of these sets of realizations. The dotted lines indicate the limits of the range $u^- \pm \xi$. Note that since $\langle u \rangle = 0$, the average lies along the ordinate in the case of the full ensemble.

the dotted lines, that is within $\pm\xi$ of u^-, for the entire low-k region are included in the biased subensemble. Given this selection it would seem reasonable that the CA of u^-, $\langle u^-\rangle_c$ will satisfy

$$\langle u_\alpha^-(\boldsymbol{k},t)\rangle_c \approx u_\alpha^-(\boldsymbol{k},t). \tag{2.35}$$

As illustrated in the figure, we cannot however be certain as to what the CA of $u_\alpha^+(\boldsymbol{k},t)$ is, although it seems realistic to assume, provided k_0 and k_1 are well separated, that

$$\lim_{|\boldsymbol{k}|\to k_0} \langle u_\alpha^+(\boldsymbol{k},t)\rangle_c = \langle u_\alpha^+(\boldsymbol{k_0},t)\rangle, \tag{2.36}$$

where $\boldsymbol{k_0} \equiv \hat{\boldsymbol{k}}k_0$ and $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$. We shall refer to this condition as *asymptotic freedom*.

In practice each member of the biased subensemble, $Y^{(m)}(\boldsymbol{k},t)$, may be written in terms of u^- as

$$u_\alpha^-(\boldsymbol{k},t) + \Phi_\alpha^{-(m)}(\boldsymbol{k},t), \tag{2.37}$$

where the label m indicates each particular member of the subensemble. Given this selection McComb *et al.* further require the members of the subensemble to satisfy

$$\langle \Phi_\alpha(\boldsymbol{k},t)\rangle_c = 0, \tag{2.38}$$

so that

$$\langle u_\alpha^-(\boldsymbol{k},t)\rangle_c = u_\alpha^-(\boldsymbol{k},t). \tag{2.39}$$

The subensemble is also assumed to satisfy

$$\langle u_\alpha^-(\boldsymbol{k},t)u_\beta^-(\boldsymbol{j},t)\rangle_c = u_\alpha^-(\boldsymbol{k},t)u_\beta^-(\boldsymbol{j},t), \tag{2.40}$$

a result which can only hold as an approximation, requiring

$$\langle \phi_\alpha^-(\boldsymbol{k},t)\phi_\beta^-(\boldsymbol{j},t)\rangle_c \approx 0, \tag{2.41}$$

and given these properties the only remaining question is how to relate a CA involving u^+ to the full ensemble average.

McComb *et al.* considered the coupling of the u^- and u^+ modes by introducing the two field decomposition

$$u_\alpha^+(\boldsymbol{k}, t) = v_\alpha^+(\boldsymbol{k}, t) + \Delta_\alpha^+(\boldsymbol{k}, t), \tag{2.42}$$

where v^+ is any other realization of the turbulent ensemble, that is it has the same statistical properties as u^+ but no phase relationship to u^-, and Δ^+ is the phase difference between the two realizations u^+ and v^+. It can then be shown [65] that

$$
\begin{aligned}
\langle u_\alpha^+(\boldsymbol{k}, t) \rangle_c &= \langle v_\alpha^+(\boldsymbol{k}, t) \rangle + \langle \Delta_\alpha^+(\boldsymbol{k}, t) \rangle_c \\
&= \langle \Delta_\alpha^+(\boldsymbol{k}, t) \rangle_c.
\end{aligned} \tag{2.43}
$$

By introducing the ansatz, based upon the idea of local energy transfer, that v^+ and u^+ are related by

$$v_\alpha^+(\boldsymbol{k}, t) = u_\alpha^+(\boldsymbol{k_0}, t) + (\boldsymbol{k} - \boldsymbol{k_0}) \cdot \boldsymbol{\nabla}_k u_\alpha^+(\boldsymbol{k}, t)\big|_{k=k_0}, \tag{2.44}$$

equation (2.43) then reduces to

$$\langle u_\alpha^+(\boldsymbol{k}, t) \rangle_c = \langle \Delta_\alpha^+(\boldsymbol{k}, t) \rangle_c = \mathcal{O}\left(\eta^2\right), \tag{2.45}$$

where η is the bandwidth of the u^+ region. Hence, providing η is sufficiently small, terms involving this average may be deemed negligible.

In a similar way, the CA $\langle u^-(\boldsymbol{k}, t) u^+(\boldsymbol{j}, t) \rangle_c$ may also be deemed negligible, and given these results McComb *et al.* are then able to proceed to an RG calculation. Applying the CA to equations (2.14) and (2.15) they obtain, dropping the negligible terms,

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_\alpha^-(\boldsymbol{k}, t) &= M_{\alpha\beta\gamma}^-(\boldsymbol{k}) \int \mathrm{d}^3 j \left\{ u_\beta^-(\boldsymbol{j}, t) u_\gamma^-(\boldsymbol{k} - \boldsymbol{j}, t) + \right. \\
&\quad \left. + \langle u_\beta^+(\boldsymbol{j}, t) u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t) \rangle_c \right\}
\end{aligned} \tag{2.46}
$$

and

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_\alpha^+(\boldsymbol{k}, t) &= M_{\alpha\beta\gamma}^+(\boldsymbol{k}) \int \mathrm{d}^3 j \left\{ 2 u_\beta^-(\boldsymbol{j}, t) u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t) + \right. \\
&\quad \left. + u_\beta^+(\boldsymbol{j}, t) u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t) \right\} + H_\alpha(\boldsymbol{k}, t),
\end{aligned} \tag{2.47}
$$

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where $H_\alpha(\boldsymbol{k}, t)$ is an error term given by

$$H_\alpha(\boldsymbol{k}, t) = \left(\frac{\partial}{\partial t} + \nu_0 k^2\right) \langle u_\alpha^+(\boldsymbol{k}, t)\rangle_c - M_{\alpha\beta\gamma}^+(\boldsymbol{k}) \left\{\langle u_\beta^+(\boldsymbol{j}, t) u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t)\rangle_c\right\} \quad (2.48)$$

and, for simplicity, the forcing terms have been dropped from the equations since McComb *et al.* assume that its purpose is merely to maintain stationarity. Equation (2.47) is then used to obtain a dynamical equation for the CA in equation (2.46), the terms in this expression involving $H_\alpha(\boldsymbol{k}, t)$ being negligible.

By making the further *boundary layer* type approximations that (i) velocity components in the high-k band are much smaller than those in the retained modes, which allows them to neglect a $u^+ u^+ u^+$ term in comparison to a $u^- u^+ u^+$ term, and (ii) the velocity components in the retained modes evolve very slowly on the time scales of the u^+ modes, McComb *et al.* then find that the CA in (2.46) can be re-expressed in terms of an expression linear in u^-. Hence it may be interpreted as a viscosity increment $\delta\nu_0(k)$, leaving an equation, defined on $0 < k < k_1$, which is of identical form to the original NSE but which has viscosity $\nu_1(k) = \nu_0 + \delta\nu_0(k)$.

This approach can then be extended iteratively to eliminate further shells of wavenumbers. For the nth shell, $k_n = (1 - \eta)^n k_0$, this gives a viscosity recursion relation

$$\nu_{n+1}(k) = \nu_n(k) + \delta\nu_n(k) \quad (2.49)$$

which relates the viscosity on subsequent iterations, the value for the viscosity increment being given by

$$\delta\nu_n(k) = \frac{1}{k} \int d^3 j \, \frac{L(\boldsymbol{k}, \boldsymbol{j}) \left[Q(l)|_{l=k_n} + (l - k_n) \left.\frac{\partial Q(l)}{\partial l}\right|_{l=k_n}\right]}{\nu_n(j) j^2 + \nu_n(|\boldsymbol{k} - \boldsymbol{j}|)|\boldsymbol{k} - \boldsymbol{j}|^2}, \quad (2.50)$$

where $L(\boldsymbol{k}, \boldsymbol{j}) = -2M_{\rho\beta\gamma}(\boldsymbol{k}) M_{\beta\rho\delta}(\boldsymbol{j}) D_{\delta\gamma}(\boldsymbol{k} - \boldsymbol{j})$. By rewriting these expressions in terms of dimensionless variables, an RG calculation, which reached a fixed point, could then be performed. As with all the alternative RG approaches, the work of McComb *et al.* is however open to criticism.

One concern lies with the fact that in principle we should get the same result when we evaluate the $\langle u^+ u^+\rangle_c$ term in equation (2.46), regardless of whether we

substitute directly for each u^+ or else form a dynamical equation for u^+u^+, as is actually done. It appears that this is not the case as McComb *et al.* obtain uncontrolled expansions if they use direct substitution, but the only real argument for using one approach over the other is that forming a dynamical equation gives rise to a viscosity increment involving two inverse lifetimes in the denominator, that is terms of the form $\nu_0 k^2$, whereas direct substitution gives rise to an expression with just one. As was pointed out by Edwards [67], who drew an analogy with the Peierls-Boltzmann equation for phonon transport in solids, a full perturbation solution of the NSE would involve 'cross-sections' with three lifetimes. For the kind of mode elimination we are considering here, one of these lifetimes is associated with the explicit scales and so cannot appear in the expression for the eliminated modes, but this still means that we would expect two inverse lifetimes. This requirement appears to be the main justification for the approach taken. Allied to this, there is also an ambiguity in that if the two-field decomposition is made at an earlier stage, that is in the low-k equation rather than the high-k equation, then the term which gives rise to the viscosity increment appears to be of a lower order and hence, for consistency, should be neglected.

Thirdly, criticism can also be levelled regarding the way in which the v^+ field is related to u^+ by means of a first order Taylor series expansion about $k = k_0$. As discussed by Yang [64], this type of procedure, in which a Taylor series is used in connection with a chaotic system, has been criticized in other areas of mathematical physics. Most important however is the question of whether or not the conditional averaging procedure is valid. Recent work [68,69], although somewhat preliminary in nature, would appear to indicate that the approach is not without foundation, but as we shall discuss in the following chapter there is a question regarding the time-dependent nature of members of the turbulent ensemble, in particular regarding the way in which we select members of the biased subensemble.

Finally, it is also worth noting that an attempt to extend the theory presented here has been recently made by Yang [64], who, by the introduction of a model field with carefully prescribed characteristics, enabled a two-field type approach which also included a perturbation expansion. The model field has however been subject to question by Young [70], whose numerical simulations imply that some of the assumptions may be inconsistent with one another. Since the theory is of little bearing to the later chapters we shall not discuss this approach further. Indeed this alternative theory is still subject to most of the same criticisms as that detailed. We shall attempt to resolve these criticisms in the remaining chapters of the thesis, but first we complete our discussion of the application of RG to turbulence by considering the theory of Zhou and Vahala and that of Eyink.

2.4.4   Zhou and Vahala

The work of Zhou and Vahala [53–55] falls into the same category of approaches as that of McComb *et al.* in that it eliminates finite bands of wavenumbers. However it differs in several important aspects, in particular with the fact that they perform averages in a similar manner to FNS.

Again, Zhou and Vahala (ZV) start from the forced NSE and perform the usual decomposition into u^- and u^+ regions. Having done this they then proceed to consider both forced and freely decaying (i.e. $f_\alpha(\mathbf{k}, t) = 0$) turbulence, these two situations requiring different approaches. In the case of forced turbulence they proceed in the same way as FNS and YO by expanding u^+ as a perturbation series, where the zeroth order term is given by

$$u_\alpha^{+(0)}(\mathbf{k}, t) = \int d\tau\, G_0(k; t, \tau) f_\alpha^+(\mathbf{k}, t), \qquad (2.51)$$

and all higher order terms can be expressed in terms of $u^{+(0)}$. The random forcing is then defined such that $\langle f_\alpha^- \rangle = f_\alpha^-$ and $\langle f_\alpha^+ \rangle = 0$, and hence using (2.51) we can easily obtain the properties of the velocity field under the same average.

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In contrast, for freely decaying turbulence ZV claim that it is no longer appropriate to expand u^+ in a perturbation series. Instead they decompose it according to

$$u_\alpha^+(\boldsymbol{k}, t) = u_\alpha^{+(b)}(\boldsymbol{k}, t) + u_\alpha^{+(c)}(\boldsymbol{k}, t), \qquad (2.52)$$

where $u^{+(b)}$ corresponds to the "base" subgrid turbulence, which is described by the dynamical equation for u^+ with the terms involving u^- set equal to zero, that is

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_\alpha^{+(b)}(\boldsymbol{k}, t) = \lambda M_{\alpha\beta\gamma}(\boldsymbol{k}) \int \mathrm{d}^3 j \, u_\beta^{+(b)}(\boldsymbol{j}, t) u_\gamma^{+(b)}(\boldsymbol{k} - \boldsymbol{j}, t), \qquad (2.53)$$

and $u^{+(c)}$ describes the effects of the large scale field on the base subgrid turbulence. Equation (2.53) is simple to solve in a formal manner and from the solution ZV are subsequently able to obtain the results for the base subgrid field that $\langle u_\alpha^{+(b)} \rangle = 0$ and $\langle u_\alpha^{+(b)} u_\beta^{+(b)} \rangle = 0$, along with an expression for the correction $u^{+(c)}$,

$$u_\alpha^{+(c)} \simeq \lambda \iint \mathrm{d}^3 j \, \mathrm{d}\tau \, G_0(k; t, \tau) M_{\alpha\beta\gamma}(\boldsymbol{k}) \times$$
$$\times \left\{ u_\beta^-(\boldsymbol{j}, \tau) u_\gamma^-(\boldsymbol{k} - \boldsymbol{j}, \tau) + 2 u_\beta^{+(b)}(\boldsymbol{j}, \tau) u_\gamma^-(\boldsymbol{k} - \boldsymbol{j}, \tau) \right\}. (2.54)$$

For both forced and freely decaying turbulence, performing the average over the u^+ field then gives us, to order λ, results of the form

$$\langle u_\alpha^+(\boldsymbol{k}, t) \rangle = \lambda M_{\alpha\beta\gamma}(\boldsymbol{k}) \iint \mathrm{d}^3 j \, \mathrm{d}\tau \, G_0(k; t, \tau) u_\beta^-(\boldsymbol{j}, \tau) u_\gamma^-(\boldsymbol{k} - \boldsymbol{j}, \tau), \qquad (2.55)$$

and

$$\langle u_\alpha^+(\boldsymbol{j}, t) u_\beta^+(\boldsymbol{k} - \boldsymbol{j}, t) \rangle = 4\lambda M_{\alpha\delta\epsilon}(\boldsymbol{j}) \iint \mathrm{d}^3 p \, \mathrm{d}\tau \, G_0(k; t, \tau) \times$$
$$\times \langle u_\delta^{+(0)}(\boldsymbol{p}, \tau) u_\beta^{+(0)}(\boldsymbol{k} - \boldsymbol{j}, \tau) \rangle u_\epsilon^-(\boldsymbol{j} - \boldsymbol{p}, \tau).(2.56)$$

The only difference between the equations given above and those in the freely decaying case is that in equation (2.56) we replace the factor 4 by 2 and make the replacement $u^{+(0)} \rightarrow u^{+(b)}$. These results can then be substituted into the

averaged dynamical equation for the u^- modes to give

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_\alpha^-(\boldsymbol{k},t) + \int_{-\infty}^t \mathrm{d}\tau \sum_{m=0}^n \eta_0(k;t,\tau) u_\alpha^-(\boldsymbol{k},\tau) = f_\alpha^-(\boldsymbol{k},t) + \lambda M_{\alpha\beta\gamma}^-(\boldsymbol{k}) \times$$

$$\left[\int \mathrm{d}^3 j\, u_\beta^-(\boldsymbol{j},t) u_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t) + 2\lambda \iiint \mathrm{d}^3 j\, \mathrm{d}^3 p\, \mathrm{d}\tau\; M_{\beta\delta\epsilon}(\boldsymbol{j}) G_0(j;t,\tau) \times\right.$$

$$\left. \times u_\delta^-(\boldsymbol{p},\tau) u_\epsilon^-(\boldsymbol{j}-\boldsymbol{p},\tau) u_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t)\right],$$

$$(2.57)$$

where η_0 denotes the eddy damping due to the elimination of the subgrid scales. It should be noted here that the triple non-linearity violates form-invariance of the original NSE. Form-invariance with respect to (2.57) is however obtained after the second and subsequent iterations, ZV arguing that the triple non-linearity describes the strong coupling between the u^- and u^+ modes near to the crossover between these regimes.

From this calculation, ZV obtain an identical eddy-viscosity recursion relation to McComb *et al.* (i.e. equation (2.49)), but in this case the value for the viscosity increment $\delta\nu_n(k)$ is found to differ between the forced and decaying instances, being

$$\delta\nu_n(k) = 2\sum_{i=0}^n \int \mathrm{d}^3 j \frac{L(\boldsymbol{k},\boldsymbol{j}) Q(|\boldsymbol{k}-\boldsymbol{j}|)}{\nu_i(j) j^2 k^2} \qquad (2.58)$$

for freely decaying turbulence and

$$\delta\nu_n(k) = \frac{W_0}{k^2} \sum_{i=0}^n \int \mathrm{d}^3 j \frac{L(\boldsymbol{k},\boldsymbol{j}) |\boldsymbol{k}-\boldsymbol{j}|^{-3}}{\nu_i(j) j^2 \nu_n(\boldsymbol{k}-\boldsymbol{j}) |\boldsymbol{k}-\boldsymbol{j}|^2} \qquad (2.59)$$

for forced turbulence where the forcing spectrum has $y = 3$. By rewriting the viscosity and wavenumbers in terms of dimensionless variables, ZV were then able to obtain a renormalized recursion relation and an eddy-viscosity which reached a fixed point under the RG iteration.

A possible criticism of the ZV approach is that the dynamical equation for the retained modes is only form-invariant after the second and subsequent iterations, the resulting equation containing an additional triple non-linearity $u^- u^- u^-$ which breaks form invariance when compared to the original NSE. However the fact that

the final equation, including the triple non-linearity, is form invariant and reaches a fixed point would appear to indicate that the approach is legitimate.

It is also worth commenting on the fact that the expression for the freely decaying viscosity increment involves only one inverse lifetime in the denominator. As discussed in the previous section we would expect two inverse lifetimes, and this could bring into question the results for the freely decaying situation. In addition to this, it has also been found by Carati [71] that in the case of external power law forcing the ZV theory gives rise to the unphysical result of a negative viscosity for small bandwidths ($k_1/k_0 > 0.8$).

More important than any of these points however, a major criticism can be levelled regarding the manner in which the averages are performed. In taking averages in the manner of ZV, it has to be implicitly assumed that the u^- and u^+ modes are statistically independent of one another. This is required in order that we may have, for instance, $\langle u^+ u^- \rangle = \langle u^+ \rangle u^-$ and $\langle u^- \rangle = u^-$. As discussed in the Eyink criticism of FNS, in reality this cannot be the case since the NSE is in principle deterministic and the u^- and u^+ are part of the same velocity field. Hence, some sort of conditional averaging procedure is needed.

2.4.5 Eyink

In the same paper as he put forward his criticisms of the FNS and YO theories [56], Eyink also set out an alternative approach to applying RG to turbulence. The Eyink RG formulation differs significantly from the approaches of FNS and YO in being based upon the more general field theory formulation for stochastic mechanics due to Martin, Siggia and Rose (MSR) [72] and also in working in configuration space. In this approach, a randomly forced NSE, in which the forcing has zero mean and is defined by its covariance, is again used. Here however the dynamics of the subgrid modes are described in terms of a path integral representation for the probability generating functionals, using a so-called MSR

action. This takes the form

$$\mathcal{Z}[\boldsymbol{\eta}, \hat{\boldsymbol{\eta}}] = \int \mathcal{D}\boldsymbol{u}\mathcal{D}\hat{\boldsymbol{u}} \, e^{S[\boldsymbol{u},\hat{\boldsymbol{u}}]+i\langle\boldsymbol{\eta},\boldsymbol{u}\rangle+i\langle\hat{\boldsymbol{\eta}},\hat{\boldsymbol{u}}\rangle}, \tag{2.60}$$

the action $S[\boldsymbol{u}, \hat{\boldsymbol{u}}]$ being given by

$$
\begin{aligned}
S[\boldsymbol{u}, \hat{\boldsymbol{u}}] = {}& -i \int \mathrm{d}t \int \mathrm{d}^d\boldsymbol{r} \, \hat{\boldsymbol{u}} \cdot \left[\frac{\partial}{\partial t}\boldsymbol{u} - \nu_0 \nabla^2 \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} \right] \\
& - \int \mathrm{d}t \int \mathrm{d}^d\boldsymbol{r} \int \mathrm{d}^d\boldsymbol{r}' \hat{\boldsymbol{u}}(\boldsymbol{r}, t) F(\boldsymbol{r} - \boldsymbol{r}') \hat{\boldsymbol{u}}(\boldsymbol{r}', t)
\end{aligned} \tag{2.61}
$$

which is well defined if the fields are Fourier truncated at a wavenumber Λ and the time integrals are approximated by a discretization. It should also be noted that we obtain the statistical correlation and response functions by functional differentiation of (2.60) with respect to $\boldsymbol{\eta}, \hat{\boldsymbol{\eta}}$.

Having decomposed the velocity field $\boldsymbol{u}(\boldsymbol{r}, t)$ into u^- and u^+ terms (the real space equivalents to the u^- and u^+ modes in Fourier space) the effects of the high wavenumber modes are then simply dealt with, merely by integrating out the u^+ modes in (2.60). This gives an equation for the low wavenumber modes of identical form to (2.60) but with the substitutions $\boldsymbol{u} \to \boldsymbol{u}^-$, $\hat{\boldsymbol{u}} \to \hat{\boldsymbol{u}}^-$, $\boldsymbol{\eta} \to \boldsymbol{\eta}^-$, $\hat{\boldsymbol{\eta}} \to \hat{\boldsymbol{\eta}}^-$ and $S[\boldsymbol{u}, \hat{\boldsymbol{u}}] \to S^{\mathrm{eff}}[\boldsymbol{u}^-, \hat{\boldsymbol{u}}^-]$, where the only major difference is that effective action S^{eff} now contains infinitely many higher order non-linear terms, as opposed to the original cubic non-linearity. This result can then be rescaled onto the original interval and the process iterated upon as an RG calculation.

This approach has an advantage over both FNS and YO in that it is formally *exact* and *not* restricted to a weak coupling regime, as is required for the perturbation expansions of the earlier theories. As such it provides an impressively rigorous approach to the problem. However it is still open to some criticisms if we consider its practical application, as in order to perform any practical calculation we have to introduce approximations to deal with the infinite number of terms in S^{eff}. It should therefore be questioned whether in fact this approach is different in application to the earlier approaches of FNS and YO. That is, are the approximations needed to actually calculate anything any different from the

arbitrary truncation of the perturbation series for u^+ used in the earlier theories? In principle, the perturbation series has an infinite number of terms, all of which could be evaluated to give an exact result, so choosing to truncate at a certain power of λ is surely no different than truncating the expansion of S^{eff}. Indeed, as Eyink shows, if we perform the mode elimination procedure perturbatively to second order, we obtain essentially the same results as with the FNS and YO approaches.

Chapter 3

Formulation of the conditional average

3.1   The conditional average and its problems

The first problem we shall consider is that arising from the time-dependent nature of members of the turbulent ensemble. To do this we need first to have a thorough understanding of the nature of the averaging procedures used and the current theory. Hence we start by reviewing in detail the current formulation.

The current form of the CA is described in the paper of McComb, Roberts and Watt [65] (MRW) and as is normal in theoretical work on turbulence relies upon the assumption of ergodicity, namely that time averages, as would normally be measured in a experiment, and ensemble averages are equivalent to one another. Hence they work with an ensemble of realizations, which they define formally as the set $\mathcal{W}$, where

$$\mathcal{W} = \left\{ W_\alpha^{(n)}(\boldsymbol{k}, t) | \alpha = 1, 2, 3; 0 < k < k_0; n = 1, \ldots, N \right\}, \tag{3.1}$$

and each $W_\alpha^{(n)}(\boldsymbol{k}, t)$ is a particular solution of the NSE. Since each member of the set would be identical if they were subject to the same initial conditions, a so-called *deterministic ensemble*, the initial conditions also have to be defined carefully in order that they vary randomly from one realization to another, so

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generating a *chaotic ensemble*. This is done by setting, at some starting time t_0, say,

$$W_\alpha^{(n)}(\boldsymbol{k}, t_0) = w_\alpha^{(n)}(\boldsymbol{k}), \tag{3.2}$$

where the initial fields are all defined to have the same total kinetic energy. This ensures that although each member of the ensemble has the same overall excitation there is no dynamical connection between the individual members.

It thus follows that the only difference between any two members of the ensemble, denoted by superscripts p and q, is a phase difference $\Delta_\alpha^{(p,q)}(\boldsymbol{k}, t)$, which is defined as

$$W_\alpha^{(p)}(\boldsymbol{k}, t) - W_\alpha^{(q)}(\boldsymbol{k}, t) = \Delta_\alpha^{(p,q)}(\boldsymbol{k}, t), \tag{3.3}$$

such that $\Delta_\alpha^{(p,p)}(\boldsymbol{k}, t) = 0$ and

$$\langle \Delta_\alpha^{(p,q)}(\boldsymbol{k}, t) \rangle = 0. \tag{3.4}$$

It also follows that different realizations are uncorrelated, thus in analogue to equation (1.63) we have

$$\langle W_\alpha^{(p)}(\boldsymbol{k}, t) W_\beta^{(q)}(\boldsymbol{k}', t') \rangle = \delta_{pq} Q_{\alpha\beta}(\boldsymbol{k}'; t, t') \delta(\boldsymbol{k} + \boldsymbol{k}'), \tag{3.5}$$

where δ_{pq} is the Kronecker delta.

Having defined the ensemble in this manner, a formal ensemble average may then be defined. This is done by considering an arbitrary functional of the velocity field, $F(\boldsymbol{u}(\boldsymbol{k}, t))$. For any such functional, its ensemble average is just

$$\langle F(\boldsymbol{u}(\boldsymbol{k}, t)) \rangle = \frac{1}{N} \sum_{n=1}^{N} F(\boldsymbol{W}^{(n)}(\boldsymbol{k}, t)), \tag{3.6}$$

although there is clearly a fundamental requirement that N is sufficiently large for the average given by (3.6) to be independent of N.

In addition to the complete ensemble $\mathcal{W}$, we also need to introduce the concept of a *representative subensemble*. We start by defining a general subensemble $\mathcal{X} \subset \mathcal{W}$,

which is a subset of the whole ensemble with M (where $M < N$) members. In analogue to (3.1) we can formally define this subensemble as

$$\mathcal{X} = \left\{ X_\alpha^{(m)}(\boldsymbol{k}, t) | \alpha = 1, 2, 3; 0 < k < k_0; m = 1, \ldots, M \right\}, \qquad (3.7)$$

where for each m, $1 \leq m \leq M$, there exists some n, $1 \leq n \leq N$, such that $X_\alpha^{(m)}(\boldsymbol{k}, t) = W_\alpha^{(n)}(\boldsymbol{k}, t)$, and we may further define a subensemble average in the obvious manner as

$$\langle F(\boldsymbol{u}(\boldsymbol{k}, t)) \rangle_X = \frac{1}{M} \sum_{m=1}^{M} F(\boldsymbol{X}^{(m)}(\boldsymbol{k}, t)). \qquad (3.8)$$

If, for sufficiently large M, this subensemble average is indistinguishable from the average of the full ensemble, as given by equation (3.6), then the subensemble is *representative*. We shall subsequently assume all subensembles labelled by $X^{(m)}$ to be representative, and for clarity of our later arguments it is also worth noting that for any particular full ensemble there may be more than one representative subensemble, each of which consists of different members of $\mathcal{W}$. There may however be some overlap between the members included in two different subensembles.

The next step is to define the *biased subensemble* and hence the conditional average. As described in the previous chapter, the CA $\langle \cdot \rangle_c$ is defined as the subensemble average over the biased subensemble and has the ideal properties

$$\langle u_\alpha^-(\boldsymbol{k}, t) \rangle_c = u_\alpha^-(\boldsymbol{k}, t) \qquad (3.9)$$

and

$$\langle u_\alpha^-(\boldsymbol{k}, t) u_\beta^-(\boldsymbol{k}', t') \rangle_c = u_\alpha^-(\boldsymbol{k}, t) u_\beta^-(\boldsymbol{k}', t'), \qquad (3.10)$$

although, as we have already noted, the second of these properties can only hold as an approximation. As we also alluded to, this biased subensemble, $\mathcal{Y}$ say, is chosen by selecting from the complete ensemble $\mathcal{W}$ all those members whose low-k modes lie very close to a particular member which we have chosen as our reference. To put these ideas into more formal terms, we start by choosing our reference

field $u_\alpha(\boldsymbol{k},t)$, where clearly $u_\alpha(\boldsymbol{k},t) \in \mathcal{W}$, and then selecting the members of the biased ensemble $\mathcal{Y} \subset \mathcal{W}$, which is a subset of $\mathcal{W}$ with M ($M < N$) members, such that each member $Y_\alpha^{(m)}(\boldsymbol{k},t)$ satisfies the criterion

$$\max \left| \theta^-(k) Y_\alpha^{(m)}(\boldsymbol{k},t) - u_\alpha^-(\boldsymbol{k},t) \right| \leq \xi, \tag{3.11}$$

where ξ is some bounding value and $\theta^-(k)$ is a step function such that

$$\theta^-(k) = \begin{cases} 1 & \text{for } 0 \leq k \leq k_1 \\ 0 & \text{for } k_1 < k \leq k_0. \end{cases} \tag{3.12}$$

For later convenience, we also define another step function $\theta^+(k)$, where

$$\theta^+(k) = \begin{cases} 0 & \text{for } 0 \leq k \leq k_1 \\ 1 & \text{for } k_1 < k \leq k_0. \end{cases} \tag{3.13}$$

Given equation (3.11), the points made in Figure 2.1 should become immediately apparent, namely that the dotted lines represent $u_\alpha^-(\boldsymbol{k},t) + \xi$ and $u_\alpha^-(\boldsymbol{k},t) - \xi$ respectively, any member of $\mathcal{W}$ which lies within these bounds for the entire low-k region being selected as a member of $\mathcal{Y}$. This gives us the biased subensemble

$$\mathcal{Y} = \left\{ Y_\alpha^{(m)}(\boldsymbol{k},t) | \alpha = 1,2,3; 0 \leq k \leq k_0; m = 1, \ldots, M \right\}, \tag{3.14}$$

in which each member satisfies (3.11) and where for each m, $1 \leq m \leq M$, there exists some n, $1 \leq n \leq N$, such that $Y_\alpha^{(m)}(\boldsymbol{k},t) = W_\alpha^{(n)}(\boldsymbol{k},t)$. It is also worth re-iterating the fact that we cannot simply choose the subensemble to consist of those members of $\mathcal{W}$ which satisfy $\theta^-(k)W_\alpha^{(n)}(\boldsymbol{k},t) = u_\alpha^-(\boldsymbol{k},t)$, that is those members whose low-k modes are identical to $u_\alpha(\boldsymbol{k},t)$. This would give us a deterministic ensemble for the low wavenumber modes, and the determinism of the NSE would then ensure that the subensemble was deterministic for the high wavenumber modes, thus leading to the unhelpful result $\langle u_\alpha^+(\boldsymbol{k},t) \rangle_c = u_\alpha^+(\boldsymbol{k},t)$.

As with the members of the full ensemble, the members of the subensemble can be related to one another in terms of their phase differences. In particular we may relate each member to the reference field. If we index this preferred realization $u_\alpha(\boldsymbol{k},t)$ by the label $\aleph$, that is $u_\alpha(\boldsymbol{k},t) \equiv W_\alpha^{(\aleph)}(\boldsymbol{k},t)$, then from (3.3) we have

$$\theta^-(k) Y_\alpha^{(m)}(\boldsymbol{k},t) - u_\alpha^-(\boldsymbol{k},t) = \Delta_\alpha^{(m,\aleph)}(\boldsymbol{k},t). \tag{3.15}$$

The specification of the subensemble $\mathcal{Y}$ is then completed by requiring its members to be such that the conditional average of the phase differences vanishes in the low-k region, that is

$$\langle \Delta_\alpha^{(m,\aleph)}(\boldsymbol{k},t) \rangle_c = 0 \text{ for } 0 < k < k_1, \tag{3.16}$$

and by an obvious extension of (3.8), the conditional average of an arbitrary functional is then defined as

$$\langle F(\boldsymbol{u}(\boldsymbol{k},t)) \rangle_c \equiv \langle F(\boldsymbol{u}(\boldsymbol{k},t)) \rangle_Y = \frac{1}{M} \sum_{m=1}^{M} F(\boldsymbol{Y}^{(m)}(\boldsymbol{k},t)). \tag{3.17}$$

Following MRW we introduce

$$\Delta_\alpha^{(m,\aleph)}(\boldsymbol{k},t) = \phi_\alpha^{(m)}(\boldsymbol{k},t), \tag{3.18}$$

and seeing that we may then write any member of the biased subensemble as

$$\theta^-(k) Y_\alpha^{(m)}(\boldsymbol{k},t) = u_\alpha^-(\boldsymbol{k},t) + \theta^-(k)\phi_\alpha^{(m)}(\boldsymbol{k},t), \tag{3.19}$$

we are now in a position to consider the properties of conditional averages involving low-k and high-k modes. Using equations (3.17) and (3.19), for the terms relevant in our calculations these properties may be found to be

$$
\begin{aligned}
\langle u_\alpha^-(\boldsymbol{k},t) \rangle_c &= \frac{1}{M} \sum_{m=1}^{M} \theta^-(k) Y_\alpha^{(m)}(\boldsymbol{k},t) \\
&= u_\alpha^-(\boldsymbol{k},t) + \frac{1}{M} \sum_{m=1}^{M} \theta^-(k)\phi_\alpha^{(m)}(\boldsymbol{k},t) \\
&= u_\alpha^-(\boldsymbol{k},t),
\end{aligned} \tag{3.20}
$$

$$\langle u_\alpha^+(\boldsymbol{k},t) \rangle_c = \frac{1}{M} \sum_{m=1}^{M} \theta^+(k) Y_\alpha^{(m)}(\boldsymbol{k},t), \tag{3.21}$$

$$
\begin{aligned}
\langle u_\alpha^-(\boldsymbol{k},t) u_\beta^-(\boldsymbol{k}',t') \rangle_c &= \frac{1}{M} \sum_{m=1}^{M} \theta^-(k) Y_\alpha^{(m)}(\boldsymbol{k},t) \theta^-(k') Y_\beta^{(m)}(\boldsymbol{k}',t') \\
&= u_\alpha^-(\boldsymbol{k},t) u_\beta^-(\boldsymbol{k}',t') + 2u_\alpha^-(\boldsymbol{k},t) \frac{1}{M} \sum_{m=1}^{M} \theta^-(k')\phi_\beta^{(m)}(\boldsymbol{k}',t')
\end{aligned}
$$

$$+ \frac{1}{M} \sum_{m=1}^{M} \theta^-(k) \phi_\alpha^{(m)}(\boldsymbol{k}, t) \theta^-(k') \phi_\beta^{(m)}(\boldsymbol{k}', t')$$

$$= u_\alpha^-(\boldsymbol{k}, t) u_\beta^-(\boldsymbol{k}', t') + \langle \phi_\alpha^-(\boldsymbol{k}, t) \phi_\beta^-(\boldsymbol{k}', t') \rangle_c, \qquad (3.22)$$

$$\langle u_\alpha^-(\boldsymbol{k}, t) u_\beta^+(\boldsymbol{k}', t') \rangle_c = \frac{1}{M} \sum_{m=1}^{M} \theta^-(k) Y_\alpha^{(m)}(\boldsymbol{k}, t) \theta^+(k') Y_\beta^{(m)}(\boldsymbol{k}', t')$$

$$= u_\alpha^-(\boldsymbol{k}, t) \frac{1}{M} \sum_{m=1}^{M} \theta^+(k') Y_\beta^{(m)}(\boldsymbol{k}', t')$$

$$+ \frac{1}{M} \sum_{m=1}^{M} \theta^-(k) \phi^{(m)}(\boldsymbol{k}, t) \theta^+(k') Y^{(m)}(\boldsymbol{k}', t')$$

$$= u_\alpha^-(\boldsymbol{k}, t) \langle u_\beta^+(\boldsymbol{k}', t') \rangle_c + \langle \phi_\alpha^-(\boldsymbol{k}, t) u_\beta^+(\boldsymbol{k}', t') \rangle_c, \qquad (3.23)$$

and

$$\langle u_\alpha^+(\boldsymbol{k}, t) u_\beta^+(\boldsymbol{k}', t') \rangle_c = \frac{1}{M} \sum_{m=1}^{M} \theta^+(k) Y_\alpha^{(m)}(\boldsymbol{k}, t) \theta^+(k') Y_\beta^{(m)}(\boldsymbol{k}', t'), \qquad (3.24)$$

where results (3.20) and (3.22) follow from the fact that equations (3.16) and (3.18) imply

$$\frac{1}{M} \sum_{m=1}^{M} \theta^-(k) \phi_\alpha^{(m)}(\boldsymbol{k}, t) = \langle \phi_\alpha^-(\boldsymbol{k}, t) \rangle_c = 0. \qquad (3.25)$$
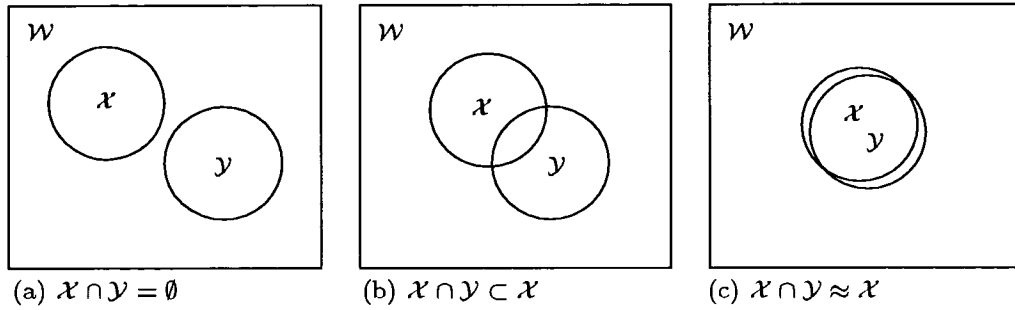
As an approximation, in order to satisfy (3.10) it is also assumed that the CA in (3.22) may be treated as a negligible error term.

As noted in the previous chapter, MRW then relate conditional averages involving u^+ to the full ensemble average by introducing a two field decomposition. To do this they write the high-k modes of each member of $\mathcal{Y}$ as the sum of the high-k modes of a corresponding member of a representative subensemble $\mathcal{X}$ plus a correction term. That is,

$$\theta^+(k) Y_\alpha^{(m)}(\boldsymbol{k}, t) = \theta^+(k) X_\alpha^{(m)}(\boldsymbol{k}, t) + \theta^+(k) \Delta_\alpha^{(m)}(\boldsymbol{k}, t), \qquad (3.26)$$

where $\Delta^{(m)}$ is shorthand for the phase difference between $Y^{(m)}$ and $X^{(m)}$. From equation (3.21) we can then write

$$\langle u_\alpha^+(\boldsymbol{k}, t) \rangle_c = \frac{1}{M} \sum_{m=1}^{M} \theta^+(k) X_\alpha^{(m)}(\boldsymbol{k}, t) + \frac{1}{M} \sum_{m=1}^{M} \theta^+(k) \Delta_\alpha^{(m)}(\boldsymbol{k}, t)$$

(a) $\mathcal{X} \cap \mathcal{Y} = \emptyset$ (b) $\mathcal{X} \cap \mathcal{Y} \subset \mathcal{X}$ (c) $\mathcal{X} \cap \mathcal{Y} \approx \mathcal{X}$

Figure 3.1: A schematic illustration of the relationship in the high-k region between the sets $\mathcal{W}$, $\mathcal{X}$ and $\mathcal{Y}$. In principle, having selected the subensembles $\mathcal{X}$ and $\mathcal{Y}$ their relationship can be as illustrated as being either (a) or (b), that is there can be either no intersection between these sets, or else there may be some members common to both sets. If we choose $\mathcal{X}$ to be the representative subensemble for which any such overlap is greatest, that is if we select from all possible representative subensembles that which shares the greatest number of members with $\mathcal{Y}$, then (c) illustrates our fundamental assumption that the members of $\mathcal{Y}$ form an ensemble approximately equivalent to a representative subensemble.

$$= \langle u_\alpha^+(\boldsymbol{k}, t)\rangle_X + \langle \Delta_\alpha^+(\boldsymbol{k}, t)\rangle_c$$

$$= \langle u_\alpha^+(\boldsymbol{k}, t)\rangle + \langle \Delta_\alpha^+(\boldsymbol{k}, t)\rangle_c$$

$$= \langle \Delta_\alpha^+(\boldsymbol{k}, t)\rangle_c, \tag{3.27}$$

where the third equality comes from the fact that the set $\mathcal{X}$ is representative, and the fourth comes from the velocity field having zero mean. To proceed from here we then require a set of conditions such the conditionally averaged error term $\langle \Delta_\alpha^+(\boldsymbol{k}, t)\rangle_c$ may be deemed small. This requires us to introduce our fundamental assumption regarding the nature of the conditional average, namely that although the low-k modes of the members of the biased subensemble are constrained, it is assumed that the chaotic nature of the NSE is such that the $Y_\alpha^{(m)}(\boldsymbol{k}, t)$ will approximate to a representative subensemble in the high-k region. This point is illustrated schematically and expanded upon in Figure 3.1, but it also worth noting the vitally important point that if we cannot make this assumption then it is *impossible* to define any non-trivial conditional averaging procedure.

An estimate for the magnitude of the error term can however be calculated, as was done by MRW. To do this they assume that in the high-k region each member

of $\mathcal{Y}$ can be written in terms of a Taylor series expansion about $\boldsymbol{k_0}$. This gives

$$Y_\alpha^{(m)}(\boldsymbol{k}, t) = Y_\alpha^{(m)}(\boldsymbol{k_0}, t) + (\boldsymbol{k} - \boldsymbol{k_0}) \cdot \boldsymbol{\nabla}_k Y_\alpha^{(m)}(\boldsymbol{k}, t)\Big|_{k=k_0} + \mathcal{O}\left(\eta^2\right), \qquad (3.28)$$

where η is the width of the high-k region, defined by

$$k_1 = (1 - \eta)k_0, \qquad (3.29)$$

the error term being of order η^2 since the maximum value of $|\boldsymbol{k} - \boldsymbol{k_0}|$ is ηk_0. This result can then be substituted into (3.21) in order to evaluate $\langle u_\alpha^+(\boldsymbol{k}, t)\rangle_c$, giving

$$\langle u_\alpha^+(\boldsymbol{k}, t)\rangle_c = \langle u_\alpha^+(\boldsymbol{k_0}, t)\rangle_c + (\boldsymbol{k} - \boldsymbol{k_0}) \cdot \langle \boldsymbol{\nabla}_k u_\alpha^+(\boldsymbol{k}, t)\Big|_{k=k_0}\rangle_c + \mathcal{O}\left(\eta^2\right). \qquad (3.30)$$

Assuming that (i) the condition of asymptotic freedom holds, that is

$$\lim_{|\boldsymbol{k}| \to k_0} \langle u_\alpha^+(\boldsymbol{k}, t)\rangle_c = \langle u_\alpha^+(\boldsymbol{k_0}, t)\rangle, \qquad (3.31)$$

holds, and (ii) that the operations of taking the divergence and the conditional average commute with one another, equation (3.30) then reduces to

$$\langle u_\alpha^+(\boldsymbol{k}, t)\rangle_c = \mathcal{O}\left(\eta^2\right), \qquad (3.32)$$

and hence by comparison with (3.27) we may see that

$$\langle \Delta_\alpha^+(\boldsymbol{k}, t)\rangle_c = \mathcal{O}\left(\eta^2\right). \qquad (3.33)$$

Now the condition of asymptotic freedom requires that the bandwidth is sufficiently large that the high-k modes of the members of $\mathcal{Y}$ are effectively unconstrained at $\boldsymbol{k_0}$, but result (3.32) requires that η is sufficiently small that terms of order η^2 may be neglected. Hence we have both upper and lower bounds on the allowed range of η. Given that η lies in this range, the CA as defined was then sufficient to allow MRW to carry out an RG calculation. However, as we shall now discuss, there is a potential difficulty with the formalism described thus far.

The difficulty is most clearly seen in the equation which defines the biased subensemble, equation (3.11),

$$\max\left|\theta^-(k)Y_\alpha^{(m)}(\boldsymbol{k}, t) - u_\alpha^-(\boldsymbol{k}, t)\right| \leq \xi.$$

This defining equation states that the biased subensemble consists of all the members of the full ensemble whose low-k modes are very close to those of our reference solution of the NSE, $u_\alpha^-(\boldsymbol{k}, t)$, for *all times*. However, the NSE displays chaotic behaviour [66], one of the defining principles being that two solutions which are virtually identical at a given time will rapidly diverge from one another as the system evolves. Hence it is extremely unlikely that any member of the ensemble will satisfy equation (3.11) for all times. This means that the biased subensemble will be either a very sparse or, most likely, an empty set. If this is the case then all the subsequent analysis is invalid.

On a more practical side, there has also been recent work concerned with studying RG using the results of numerical simulations [69,70]. If we wish to investigate the theory in this manner then we are restricted by the limitations of these numerical calculations. In particular, we need to recognize the fact that time restrictions mean we cannot perform the huge number of runs necessary to generate an ensemble of solutions as used in the theory. Instead we can only generate an ensemble of time-independent velocity fields, obtained by performing a single run for a large number of time steps and then sampling periodically in time.

Coupled together, these two points provide us with the motivation to re-formulate the conditional average in terms of time-independent realizations, mainly in order to resolve the theoretical difficulty described above, but also in order to make more transparent the way in which the resulting theory may be tested using DNS.

3.2   Definition of the turbulence ensemble

In order to reformulate the CA in terms of time-independent realizations, we first need to consider how we obtain the turbulence ensemble. We start from a particular solution of the NSE, $u_\alpha(\boldsymbol{k}, t)$, defined for all time and wavevectors. Unlike MRW, who define their ensemble as a set of such time-dependent solutions,

we have to time sample this particular solution in order to obtain the N members, $W_\alpha^{(n)}(\boldsymbol{k})$, of our ensemble[1] $\mathcal{W}$. In principle we could sample $u_\alpha(\boldsymbol{k}, t)$ using a Dirac delta function, according to

$$W_\alpha^{(n)}(\boldsymbol{k}) = \int_{-\infty}^{\infty} dt' \, u_\alpha(\boldsymbol{k}, t') \delta(t_n - t'), \qquad (3.34)$$

where each t_n denotes a different time. However, this instantaneous sample in time leads to some difficulty when we later need to consider the time derivative of functionals involving members of the ensemble. Clearly such an operation is meaningless for a function which has no time dependence and hence we need to find some way to circumvent this problem. To do this we instead define $W_\alpha^{(n)}(\boldsymbol{k})$ to be such that it has a slight dependence on time. This is done by using a function which is sharply peaked about a desired time and which tends to a Dirac delta function if we take a particular limit. We start by introducing a filtering function $P(t)$, which in our calculation we choose to take the form

$$P(t) = \left\{ \begin{array}{ll} \frac{1}{2\sigma} & \text{for} \quad |t| \leq \sigma \\ 0 & \text{otherwise} \end{array} \right., \qquad (3.35)$$

with a *small* value for σ. Clearly with this definition we have the result that

$$\int_{-\infty}^{\infty} dt \, P(t) = 1, \qquad (3.36)$$

and if we consider taking the limit $\sigma \to 0$ we see that

$$\lim_{\sigma \to 0} P(t) = \left\{ \begin{array}{ll} \infty & t = 0 \\ 0 & t \neq 0 \end{array} \right.. \qquad (3.37)$$

Hence, in the limit $\sigma \to 0$ the function $P(t)$ provides a delta function as desired. We then obtain $W_\alpha^{(n)}(\boldsymbol{k})$ using the definition

$$W_\alpha^{(n)}(\boldsymbol{k}) = \lim_{\sigma \to 0} \int_{-\infty}^{\infty} dt' \, u_\alpha(\boldsymbol{k}, t') P(t_n - t'). \qquad (3.38)$$

With this expression we are also able to obtain the time derivative of a member of the ensemble, provided only that we perform this operation prior to taking the limit on σ.

---

[1]From now, we shall assume all realizations to be time-independent.

3.3 Definition of the biased subensemble and conditional average

Having defined our ensemble $\mathcal{W}$, we next have to introduce a new criterion for the biased subensemble in order that we may define the conditional average. However, in introducing the new biased subensemble we also need to recall that we are now working with time independent realizations. Thus if we define two biased subensembles with respect our reference solution $u_\alpha(\boldsymbol{k}, t)$ at two different times, t_0 and t_1 say, then these two subensembles will contain different members of the full ensemble $\mathcal{W}$, although there may be some overlap in their members. This implies that we may only perform a CA involving products of $u_\alpha(\boldsymbol{k}, t)$ when each velocity field is defined at the same instant of time. That is, using this formalism we *cannot* perform a CA involving multiple time moments. The method by which we select such a subensemble is however identical, regardless of our choice of reference time, and is simply a modification of the approach used by MRW.

Given the reference solution of the NSE, $u_\alpha(\boldsymbol{k}, t)$, we choose the members of the biased subensemble $\mathcal{Y}$ by the criterion

$$\left| \theta^-(k) Y_\alpha^{(m)}(\boldsymbol{k}) - u_\alpha^-(\boldsymbol{k}, T) \right| \leq \xi, \tag{3.39}$$

where ξ is a small bounding value and $u_\alpha(\boldsymbol{k}, T)$ is the velocity field (time snapshot of $u_\alpha(\boldsymbol{k}, t)$ at $t = T$) we define our subensemble with respect to. Clearly this selection of the members of $\mathcal{Y}$ with reference to a time-independent *velocity field*, rather than to a time-dependent *solution of the NSE*, removes the problem of the MRW theory regarding the fact that the members of their subensemble need to be close for all times.

We also modify the theory of MRW by writing, in direct analogue to (3.19), the low-k modes of each member of $\mathcal{Y}$ in terms of $u_\alpha(\boldsymbol{k}, T)$, to give

$$\theta^-(k) Y_\alpha^{(m)}(\boldsymbol{k}) = u_\alpha^-(\boldsymbol{k}, T) + \theta^-(k) \phi_\alpha^{(m)}(\boldsymbol{k}), \tag{3.40}$$

where $\phi_\alpha^{(m)}(\boldsymbol{k})$ represents the difference between the velocity field of the mth member of $\mathcal{Y}$ and the reference velocity field. Again in analogue to the work of MRW, the CA of an arbitrary functional of $u_\alpha(\boldsymbol{k},t)$ *evaluated at time* $t = T$ may then be defined as

$$\langle F(\boldsymbol{u}(\boldsymbol{k},t))\rangle_c \big|_{t=T} = \frac{1}{M} \sum_{m=1}^{M} F(\boldsymbol{Y}^{(m)}(\boldsymbol{k})). \tag{3.41}$$

3.4 Evaluation of simple conditional averages

As was the case in the theory of MRW, we complete our definition of the biased subensemble by requiring it to be such that

$$\left\langle \phi_\alpha^-(\boldsymbol{k})\right\rangle_c \Big|_{t=T} = \frac{1}{M} \sum_{m=1}^{M} \theta^-(k)\phi_\alpha^{(m)}(\boldsymbol{k}) = 0. \tag{3.42}$$

Using this along with equations (3.40) and (3.41) we may then obtain the analogues to equations (3.20) and (3.22):

$$\begin{aligned}
\left\langle u_\alpha^-(\boldsymbol{k},t)\right\rangle_c \Big|_{t=T} &= \frac{1}{M} \sum_{m=1}^{M} \theta^-(k)Y_\alpha^{(m)}(\boldsymbol{k}) \\
&= u_\alpha^-(\boldsymbol{k},T) + \frac{1}{M} \sum_{m=1}^{M} \theta^-(k)\phi_\alpha^{(m)}(\boldsymbol{k}) \\
&= u_\alpha^-(\boldsymbol{k},T), \tag{3.43}
\end{aligned}$$

and

$$\begin{aligned}
\left\langle u_\alpha^-(\boldsymbol{k},t)u_\beta^-(\boldsymbol{j},t)\right\rangle_c \Big|_{t=T} &= \frac{1}{M} \sum_{m=1}^{M} \theta^-(k)Y_\alpha^{(m)}(\boldsymbol{k})\theta^-(j)Y_\beta^{(m)}(\boldsymbol{j}) \\
&= u_\alpha^-(\boldsymbol{k},T)u_\beta^-(\boldsymbol{j},T) + 2u_\alpha^-(\boldsymbol{k},T)\frac{1}{M} \sum_{m=1}^{M} \theta^-(j)\phi_\beta^{(m)}(\boldsymbol{j}) \\
&\quad + \frac{1}{M} \sum_{m=1}^{M} \theta^-(k)\phi_\alpha^{(m)}(\boldsymbol{k})\theta^-(j)\phi_\beta^{(m)}(\boldsymbol{j}) \\
&= u_\alpha^-(\boldsymbol{k},T)u_\beta^-(\boldsymbol{j},T) + \mathcal{O}\left(\left\langle \phi^-\phi^-\right\rangle_c \Big|_{t=T}\right), \tag{3.44}
\end{aligned}$$

where it is again assumed that $\left\langle \phi_\alpha^-(\boldsymbol{k},t)\phi_\beta^-(\boldsymbol{j},t)\right\rangle_c \Big|_{t=T}$ may be deemed negligible. Conditional averages involving high-k modes are also evaluated by following MRW

and writing the members of $\mathcal{Y}$ as a member of a representative ensemble $\mathcal{X}$ plus an correction term to give

$$\theta^+(k)Y_\alpha^{(m)}(\boldsymbol{k}) = \theta^+(k)X_\alpha^{(m)}(\boldsymbol{k}) + \theta^+(k)\Delta_\alpha^+(\boldsymbol{k}), \qquad (3.45)$$

where there is a one-to-one correspondence between the members of $\mathcal{X}$ and $\mathcal{Y}$. From equation (3.41) it then follows that

$$
\begin{aligned}
\left\langle u_\alpha^+(\boldsymbol{k},t)\right\rangle_c\Big|_{t=T} &= \frac{1}{M}\sum_{m=1}^{M}\theta^+(k)Y_\alpha^{(m)}(\boldsymbol{k})\\
&= \frac{1}{M}\sum_{m=1}^{M}\theta^+(k)X_\alpha^{(m)}(\boldsymbol{k}) + \frac{1}{M}\sum_{m=1}^{M}\theta^+(k)\Delta_\alpha^{(m)}(\boldsymbol{k})\\
&= \left\langle u_\alpha^+(\boldsymbol{k},t)\right\rangle + \frac{1}{M}\sum_{m=1}^{M}\theta^+(k)\Delta_\alpha^{(m)}(\boldsymbol{k})\\
&= \frac{1}{M}\sum_{m=1}^{M}\theta^+(k)\Delta_\alpha^{(m)}(\boldsymbol{k})\\
&= \left\langle \Delta_\alpha^+(\boldsymbol{k})\right\rangle_c\Big|_{t=T}, \qquad (3.46)
\end{aligned}
$$

and given the same assumptions as used by MRW, namely that each member of $\mathcal{Y}$ may be written as a Taylor series about $\boldsymbol{k}_0$ and the condition of asymptotic freedom holds, we will again find that

$$\left\langle u_\alpha^+(\boldsymbol{k},t)\right\rangle_c\Big|_{t=T} = \mathcal{O}\left(\eta^2\right), \qquad (3.47)$$

and hence may be deemed negligible. However, performing a CA involving time derivatives requires a little more thought.

3.5 Evaluation of conditional averages involving time derivatives

The difficulties in evaluating conditional averages which involve time derivatives arise from the fact that we are now dealing with an ensemble of time-independent (in the limit $\sigma \to 0$) velocity fields. However, provided we perform the operation of differentiation prior to taking the limit $\sigma \to 0$ in (3.38) we may calculate such a
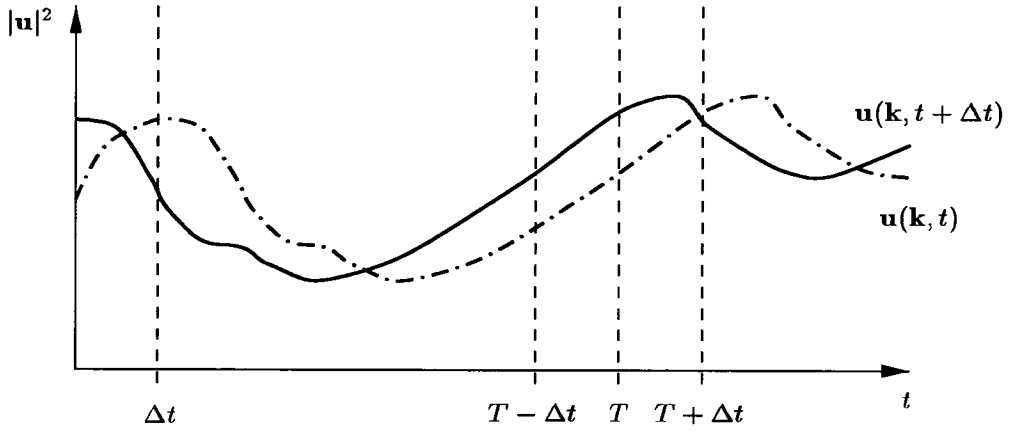
Figure 3.2: A schematic illustration of the relationship between $u(k, t)$ and $u(k, t+\Delta t)$, as needed in obtaining a conditional average involving a time derivative.

CA. To do this we first need to recall the formal definition for the time derivative of a general functional $F(u(k, t))$:

$$\frac{\partial F(u(k, t))}{\partial t} = \lim_{\Delta t \to 0} \frac{F(u(k, t + \Delta t)) - F(u(k, t))}{\Delta t}. \tag{3.48}$$

If we take the conditional average of this we find, assuming that the operations $\lim_{\Delta t \to 0}$ and $\langle \cdot \rangle_c |_{t=T}$ commute,

$$\left\langle \frac{\partial F(u(k, t))}{\partial t} \right\rangle_c \Bigg|_{t=T} = \lim_{\Delta t \to 0} \left\langle \frac{F(u(k, t + \Delta t)) - F(u(k, t))}{\Delta t} \right\rangle_c \Bigg|_{t=T}$$

$$= \lim_{\Delta t \to 0} \frac{\langle F(u(k, t + \Delta t)) \rangle_c |_{t=T} - \langle F(u(k, t)) \rangle_c |_{t=T}}{\Delta t}, \tag{3.49}$$

where the second step comes from the fact that the conditional average is a linear operator. The second CA on the right hand side of (3.49) is easily evaluated since this is just the usual CA, leaving only the question of how do we evaluate the first CA, $\langle F(u(k, t + \Delta t)) \rangle_c |_{t=T}$? To perform this conditional average, we first consider the relationship between $u(k, t)$ and $u(k, t + \Delta t)$, as illustrated schematically in Figure 3.2. As can be easily seen from this figure, at any given time T, say, the two fields are related to one another in such a way that

$$u(k, t)|_{t=T} = u(k, t + \Delta t)|_{t=T-\Delta t}$$

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$$= \left. \boldsymbol{u}(\boldsymbol{k}, t + \Delta t) \right|_{t + \Delta t = T}, \tag{3.50}$$

and consequently we can see that the members of a biased subensemble chosen with reference to $u_\alpha(\boldsymbol{k}, t)$ at time $t = T$ will be identical to the members of a biased subensemble chosen with reference to $u_\alpha(\boldsymbol{k}, t + \Delta t)$ at time $t = T - \Delta t$. Hence,

$$\begin{aligned}
\left. \langle F(\boldsymbol{u}(\boldsymbol{k}, t + \Delta t))) \rangle_c \right|_{t=T} &= \left. \langle F(\boldsymbol{u}(\boldsymbol{k}, t + \Delta t))) \rangle_c \right|_{t + \Delta t = T + \Delta t} \\
&= \left. \langle F(\boldsymbol{u}(\boldsymbol{k}, t))) \rangle_c \right|_{t = T + \Delta t},
\end{aligned} \tag{3.51}$$

meaning that we may rewrite equation (3.49) as

$$\begin{aligned}
\left\langle \frac{\partial F(\boldsymbol{u}(\boldsymbol{k}, t))}{\partial t} \right\rangle_c \Bigg|_{t=T} &= \lim_{\Delta t \to 0} \frac{\left. \langle F(\boldsymbol{u}(\boldsymbol{k}, t))) \rangle_c \right|_{t = T + \Delta t} - \left. \langle F(\boldsymbol{u}(\boldsymbol{k}, t))) \rangle_c \right|_{t=T}}{\Delta t} \\
&= \frac{\partial \left. \langle F(\boldsymbol{u}(\boldsymbol{k}, t))) \rangle_c \right|_{t=T}}{\partial T}.
\end{aligned} \tag{3.52}$$

Thus we see that the operations of performing a time derivative and conditional averaging commute with one another, although it should also be noted that this is only the case if Δt tends to zero at least as fast as the σ in the definition of members of the ensemble. That is, we require $\sigma \geq \Delta t$ as we perform the two necessary limits. Given however that this is the case, the fact that these two operators commute makes performing conditional averages on time derivatives a simple procedure, requiring only knowledge of our earlier results, equations (3.43), (3.44) and (3.47). For example, using (3.52) along with (3.43) and (3.47) we find

$$\begin{aligned}
\left\langle \frac{\partial u_\alpha^-(\boldsymbol{k}, t)}{\partial t} \right\rangle_c \Bigg|_{t=T} &= \frac{\partial \left. \langle u_\alpha^-(\boldsymbol{k}, t) \rangle_c \right|_{t=T}}{\partial T} \\
&= \frac{\partial u_\alpha^-(\boldsymbol{k}, T)}{\partial T}
\end{aligned} \tag{3.53}$$

and

$$\begin{aligned}
\left\langle \frac{\partial u_\alpha^+(\boldsymbol{k}, t)}{\partial t} \right\rangle_c \Bigg|_{t=T} &= \frac{\partial \left. \langle u_\alpha^+(\boldsymbol{k}, t) \rangle_c \right|_{t=T}}{\partial T} \\
&= \frac{\partial \left. \langle \Delta_\alpha^+(\boldsymbol{k}, t) \rangle_c \right|_{t=T}}{\partial T} \\
&= \mathcal{O}\left(\eta^2 \right).
\end{aligned} \tag{3.54}$$

3.6   A simplified notation for the conditional average

In practice, the notation we have used to indicate the CA evaluated with respect to a particular reference time, namely $\langle \cdot \rangle_c |_{t=T}$ proves to be somewhat unwieldy. Hence we desire a simpler notation. This is found by noting that we may obtain a final expression dependent upon the label t by the following steps:

1. Start by taking the CA of a general functional of $\boldsymbol{u}(\boldsymbol{k}, t)$ at a reference time T, say, that is $\langle F(\boldsymbol{u}(\boldsymbol{k}, t)) \rangle_c |_{t=T}$.

2. Next note, from equations (3.43), (3.44), (3.47), (3.53) and (3.54), that after performing this CA the result will always be either a functional of $\boldsymbol{u}(\boldsymbol{k}, T)$ or else negligible.

3. Since the choice of T is arbitrary, it is effectively a dummy variable, meaning it may be re-labelled $T \rightarrow t$.

Consequently, we shall subsequently revert to our earlier notation for the CA of $\langle \cdot \rangle_c$, but with the added implicit requirement that the above 3 steps are carried out, meaning that the biased subensemble is chosen with respect to $\boldsymbol{u}(\boldsymbol{k}, t)$ at some arbitrary instant of time.

3.7   Evaluation of the $\langle u_\beta^+(\boldsymbol{j}, t) u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t) \rangle_c$ term in the McComb *et al.* theory

One immediate benefit of our new formalism for the conditional average is that it resolves the criticism of the McComb *et al.* theory regarding the fact that we obtain differing results depending upon whether we substitute directly for each u^+ or else form a dynamical equation for $u^+ u^+$. To see this we start by rewriting the right hand side of equation (2.47) as the functional $\Theta_\alpha(\boldsymbol{u}(\boldsymbol{k}, t))$, to obtain the

simplified expression

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_\alpha^+(\boldsymbol{k}, t) = \Theta_\alpha(\boldsymbol{u}(\boldsymbol{k}, t)). \tag{3.55}$$

We first consider forming a dynamical equation for $\langle u^+ u^+ \rangle_c$. We obtain this by following the 3 steps:

1. Rewrite (3.55) in terms of $u_\beta^+(\boldsymbol{j}, t)$ and then multiply this expression by $u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t)$.

2. Rewrite (3.55) in terms of $u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t)$ and then multiply this expression by $u_\beta^+(\boldsymbol{j}, t)$.

3. Add the expressions from steps 1 and 2 and then take the CA.

Doing this we find

$$\left(\frac{\partial}{\partial t} + \nu_0 j^2 + \nu_0 |\boldsymbol{k} - \boldsymbol{j}|^2\right) \langle u_\beta^+(\boldsymbol{j}, t) u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t)\rangle_c = 2\langle\Theta_\beta(\boldsymbol{u}(\boldsymbol{j}, t)) u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t)\rangle_c. \tag{3.56}$$

By introducing the integrating factor $e^{(\nu_0 j^2 + \nu_0 |\boldsymbol{k} - \boldsymbol{j}|^2)t}$, this may then be formally solved as a first order differential equation to give

$$\langle u_\beta^+(\boldsymbol{j}, t) u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t)\rangle_c = 2\int_{-\infty}^{t} dt' \, e^{-(\nu_0 j^2 + \nu_0 |\boldsymbol{k} - \boldsymbol{j}|^2)(t - t')}\langle\Theta_\beta(\boldsymbol{u}(\boldsymbol{j}, t')) u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t')\rangle_c, \tag{3.57}$$

and as may be easily seen from this expression the conditional averages on the RHS involve only one time. This however is not the case if we substitute directly for u^+.

If we formally solve equation (3.55) we obtain

$$u_\beta^+(\boldsymbol{j}, t) = \int_{-\infty}^{t} dt' \, e^{-\nu_0 j^2 (t - t')}\Theta_\beta(\boldsymbol{u}(\boldsymbol{j}, t')), \tag{3.58}$$

and hence multiplying by $u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t)$ and taking the CA we have

$$\langle u_\beta^+(\boldsymbol{j}, t) u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t)\rangle_c = \int_{\infty}^{t} dt' \, e^{-\nu_0 j^2 (t - t')}\langle\Theta_\beta(\boldsymbol{u}(\boldsymbol{j}, t')) u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t)\rangle_c. \tag{3.59}$$

However, as we note in Section 3.3, the time independent nature of our realizations means that we can only perform a conditional average for products of velocity modes in which *all* the modes are defined at the *same* instant of time. Hence we cannot perform the conditional average on the right hand side of (3.59). Thus we are left with no ambiguity as to why we should choose one approach to finding $\langle u^+ u^+ \rangle_c$, namely that of forming a dynamical equation, over the other, and can verify that the approach taken by MRW is indeed the correct one. This fact is also likely to have implications for the work of Rose [52] and ZV [53, 54], which if reformulated in terms of our new ensemble would be subject to the same constraint, but we shall not here consider any such implications.

Chapter 4

Elimination of turbulent modes using a conditional average with asymptotic freedom

4.1   The hypothesis of local chaos

Having defined the CA we are now almost in a position to be able to eliminate a band of wavenumber modes. However, before we can do this we need first to consider how we may take the conditional average of mixed moments, that is moments which involve products of both u^- and u^+ modes. In Chapter 3 we obtained the general results that

$$\langle u_\alpha^-(\boldsymbol{k}, t)\rangle_c \;=\; u_\alpha^-(\boldsymbol{k}, t), \tag{4.1}$$

$$\langle \phi_\alpha^-(\boldsymbol{k}, t)\rangle_c \;=\; 0 \tag{4.2}$$

and

$$\langle u_\alpha^-(\boldsymbol{k}, t) u_\beta^-(\boldsymbol{j}, t)\rangle_c = u_\alpha^-(\boldsymbol{k}, t) u_\beta^-(\boldsymbol{j}, t) + \langle \phi_\alpha^-(\boldsymbol{k}, t) \phi_\beta^-(\boldsymbol{j}, t)\rangle_c, \tag{4.3}$$

where $\langle \phi^- \phi^-\rangle_c$ is assumed negligible, but as is easily seen from equation (2.14), if we take the conditional average of the low-pass filtered NSE then in addition to these results we also need to know $\langle u^- u^+\rangle_c$ and $\langle u^+ u^+\rangle_c$. In fact, using the high-pass filtered NSE, equation (2.15), we can always write a product of u^+ modes

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in terms of a higher order mixed mode product. Hence we need only consider the general manner in which we may perform the conditional average of a mixed moment. Based upon this fact and the idea of asymptotic freedom, we introduce the *Hypothesis of Local Chaos*. This forms the basic ansatz for all the following theory and is stated as follows:

For sufficiently large Reynolds number and corresponding k_0, there exists a cut-off wavenumber $k_1 < k_0$ such that a mixed conditional moment involving p low wavenumber and r high wavenumber modes takes the limiting form

$$\lim_{\xi \to 0} \langle u_\alpha^-(\mathbf{k_1}, t) u_\beta^-(\mathbf{k_2}, t) \ldots u_\gamma^-(\mathbf{k_p}, t) u_\delta^+(\mathbf{k_{p+1}}, t) u_\epsilon^+(\mathbf{k_{p+2}}, t) \ldots u_\sigma^+(\mathbf{k_{p+r}}, t) \rangle_c \to$$

$$u_\alpha^-(\mathbf{k_1}, t) u_\beta^-(\mathbf{k_2}, t) \ldots u_\gamma^-(\mathbf{k_p}, t) \lim_{\{\cdot\} \to k_0} \langle u_\delta^+(\mathbf{k_{p+1}}, t) u_\epsilon^+(\mathbf{k_{p+2}}, t) \ldots u_\sigma^+(\mathbf{k_{p+r}}, t) \rangle,$$

$$(4.4)$$

where $\lim_{\{\cdot\} \to k_0}$ means take the limit for all wavevector arguments of the u^+ modes within the average, with the condition that $\langle \phi^- \phi^- \rangle_c \approx 0$ satisfied as a corollary.

Given this hypothesis we are then able to evaluate all conditional averages involving a mixed product of u^- and u^+ modes. For instance

$$\lim_{\xi \to 0} \langle u_\alpha^-(\mathbf{j}, t) u_\beta^+(\mathbf{k} - \mathbf{j}, t) \rangle_c = u_\alpha^-(\mathbf{j}, t) \lim_{|k-j| \to k_0} \langle u_\beta^+(\mathbf{k} - \mathbf{j}, t) \rangle = 0, \qquad (4.5)$$

since $\langle u_\alpha(\mathbf{k}, t) \rangle = 0$ by definition, and similarly

$$\lim_{\xi \to 0} \langle u_\alpha^-(\mathbf{p}, t) u_\beta^-(\mathbf{j} - \mathbf{p}, t) u_\gamma^+(\mathbf{k} - \mathbf{j}, t) \rangle_c =$$

$$= 2 u_\alpha^-(\mathbf{p}, t) u_\beta^-(\mathbf{j} - \mathbf{p}, t) \lim_{|k-j| \to k_0} \langle u_\gamma^+(\mathbf{k} - \mathbf{j}, t) \rangle$$

$$= 0. \qquad (4.6)$$

It should however also be noted that the hypothesis as defined is considerably more general that absolutely necessary, since we shall only need to consider products containing at most two u^- modes.

4.2   The conditional projector in function space

The reasoning behind the hypothesis of local chaos, can be clarified if we consider the Hilbert space projection of products of the u^+ modes onto the u^- modes. We start by introducing the exact probability functional $P[u(k, t)]$ such that the expectation value of any well behaved functional F is given by

$$\langle F[u(k, t)] \rangle = \int \mathcal{D}u(k, t) P[u(k, t)] F[u(k, t)], \qquad (4.7)$$

where the functional integration is indicated symbolically. This operation is unaffected by our filtering the modes, and hence

$$\langle F[u^\pm(k, t)] \rangle = \int \mathcal{D}u(k, t) P[u(k, t)] F[u^\pm(k, t)]. \qquad (4.8)$$

In order to extract a conditional projection on the u^- modes, we then construct a projection operator $\mathcal{P}_c^-$ [73], such that its action on an arbitrary functional is given by

$$\mathcal{P}_c^- F[u(k, t)] = \int \mathrm{d}^3 j \int \mathrm{d}s\, u^-(j, s) \int \mathcal{D}u(k, t) P[u(k, t)] \frac{\delta F[u(k, t)]}{\delta u^-(j, s)}, \qquad (4.9)$$

where $\delta/\delta u^-$ denotes functional differentiation. If we assume the proper normalization

$$\int \mathcal{D}u(k, t) P[u(k, t)] = 1, \qquad (4.10)$$

and take $F[u(k, t)] = u^-(k, t)$ then we can easily see that

$$\mathcal{P}_c^- u^-(k, t) = \int \mathrm{d}^3 j \int \mathrm{d}s\, u^-(j, s) \delta(k - j) \delta(t - s) = u^-(k, t), \qquad (4.11)$$

as required.

In general, of course, we wish to project out products of u^- with functionals of the u^+ modes. Hence, we require, for example

$$\mathcal{P}_c^- u^-(k, t) f[u^+(k, t)] = \int \mathrm{d}^3 j \int \mathrm{d}s\, u^-(j, s) \int \mathcal{D}u(k, t) P[u(k, t)]$$
$$\times \frac{\delta u^-(k, t) f[u^+(k, t)]}{\delta u^-(j, s)}$$

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$$= \int \mathrm{d}^3j \int \mathrm{d}s\, \boldsymbol{u}^-(\boldsymbol{j},s) \int \mathcal{D}\boldsymbol{u}(\boldsymbol{k},t) P[\boldsymbol{u}(\boldsymbol{k},t)]$$

$$\times \left\{ \delta(\boldsymbol{k}-\boldsymbol{j})\delta(t-s)f[\boldsymbol{u}^+(\boldsymbol{k},t)] + \boldsymbol{u}^-(\boldsymbol{k},t)\frac{\delta f[\boldsymbol{u}^+(\boldsymbol{k},t)]}{\delta \boldsymbol{u}^-(\boldsymbol{j},s)} \right\}. \qquad (4.12)$$

The second term in the curly brackets is intractable, and is simply another way of expressing the turbulence problem, but if we are able to find a limit in which u^+ becomes independent of u^-, then this term vanishes and we are left with

$$\mathcal{P}_c^- \boldsymbol{u}^-(\boldsymbol{k},t) f[\boldsymbol{u}^+(\boldsymbol{k},t)] = \boldsymbol{u}^-(\boldsymbol{k},t)\langle f[\boldsymbol{u}^+(\boldsymbol{k},t)]\rangle. \qquad (4.13)$$

In making the hypothesis of local chaos, we postulate that such a limit exists under the defined conditions.

4.3 Elimination of a band of high wavenumber modes

4.3.1 The equations of motion for the system

Given the hypothesis of local chaos, we may now begin our mode elimination calculation. We start by introducing the assumption that all forcing is at very low wavenumbers and is applied purely to maintain stationarity. To indicate this fact we introduce the notation for the forcing term $\mathcal{F}_\alpha^{<<}(\boldsymbol{k},t)$, where the forcing is now assumed to be a Dirac delta function located at the origin. It is also assumed that

$$\langle \mathcal{F}_\alpha^{<<}(\boldsymbol{k},t)\rangle_c = \mathcal{F}_\alpha^{<<}(\boldsymbol{k},t), \qquad (4.14)$$

and given this definition for the forcing, we may rewrite equations (2.14) and (2.15) as

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_\alpha^-(\boldsymbol{k},t) = \mathcal{F}_\alpha^{<<}(\boldsymbol{k},t) + M_{\alpha\beta\gamma}^-(\boldsymbol{k}) \int \mathrm{d}^3j \left\{ u_\beta^-(\boldsymbol{j},t)u_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t) \right.$$

$$\left. + 2u_\beta^-(\boldsymbol{j},t)u_\gamma^+(\boldsymbol{k}-\boldsymbol{j},t) + u_\beta^+(\boldsymbol{j},t)u_\gamma^+(\boldsymbol{k}-\boldsymbol{j},t) \right\} \quad (4.15)$$

and

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_\alpha^+(\boldsymbol{k},t) = M_{\alpha\beta\gamma}^+(\boldsymbol{k}) \int \mathrm{d}^3j \left\{ u_\beta^-(\boldsymbol{j},t)u_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t) \right.$$

$$+ 2u_{\bar{\beta}}^-(\boldsymbol{j},t)u_{\gamma}^+(\boldsymbol{k}-\boldsymbol{j},t) + u_{\beta}^+(\boldsymbol{j},t)u_{\gamma}^+(\boldsymbol{k}-\boldsymbol{j},t)\big\} \quad (4.16)$$

respectively.

Taking the CA of the low-pass filtered NSE, equation (4.15), we obtain

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_{\alpha}^-(\boldsymbol{k},t) = \mathcal{F}_{\alpha}^{<<}(\boldsymbol{k},t) + M_{\alpha\beta\gamma}^-(\boldsymbol{k}) \int \mathrm{d}^3 j \, \big\{\langle u_{\bar{\beta}}^-(\boldsymbol{j},t)u_{\gamma}^-(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c$$

$$+ 2\langle u_{\bar{\beta}}^-(\boldsymbol{j},t)u_{\gamma}^+(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c + \langle u_{\beta}^+(\boldsymbol{j},t)u_{\gamma}^+(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c\big\}, (4.17)$$

where the conditional averages of u^- and the forcing are evaluated using equations (4.1), (4.2) and (4.14). This expression may then be rewritten as

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_{\alpha}^-(\boldsymbol{k},t) = \mathcal{F}_{\alpha}^{<<}(\boldsymbol{k},t) + M_{\alpha\beta\gamma}^-(\boldsymbol{k}) \int \mathrm{d}^3 j \, u_{\bar{\beta}}^-(\boldsymbol{j},t)u_{\gamma}^-(\boldsymbol{k}-\boldsymbol{j},t)$$

$$+ S_{\alpha}^-(\boldsymbol{k}|k_1) + M_{\alpha\beta\gamma}^-(\boldsymbol{k}) \int \mathrm{d}^3 j \, \lim_{\xi\to 0}\langle u_{\beta}^+(\boldsymbol{j},t)u_{\gamma}^+(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c,$$

$$(4.18)$$

where

$$S_{\alpha}^-(\boldsymbol{k}|k_1) = M_{\alpha\beta\gamma}^-(\boldsymbol{k}) \int \mathrm{d}^3 j \, \big\{\langle \phi_{\bar{\beta}}^-(\boldsymbol{j},t)\phi_{\gamma}^-(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c + 2\langle u_{\bar{\beta}}^-(\boldsymbol{j},t)u_{\gamma}^+(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c$$

$$+ \langle u_{\beta}^+(\boldsymbol{j},t)u_{\gamma}^+(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c - \lim_{\xi\to 0}\langle u_{\beta}^+(\boldsymbol{j},t)u_{\gamma}^+(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c\big\} \quad (4.19)$$

is viewed as an error term, the CA on the right hand side of (4.18) being written in terms of the limit $\xi \to 0$ in order to make it tractable using the hypothesis. However the hypothesis does not explicitly tell us how to evaluate the CA in (4.18). Instead we must first use the high-pass filtered NSE, equation (4.16), in order to form a governing equation for this quantity.

4.3.2   Iterative solution for the high wavenumber modes

To obtain an expression for $\lim_{\xi\to 0}\langle u^+ u^+\rangle_c$, we first use (4.16) to write dynamical equations for $u_{\beta}^+(\boldsymbol{j},t)$ and $u_{\gamma}^+(\boldsymbol{k}-\boldsymbol{j},t)$. We then multiply these equations by $u_{\gamma}^+(\boldsymbol{k}-\boldsymbol{j},t)$ and $u_{\beta}^+(\boldsymbol{j},t)$ respectively, add the resulting equations together, and

take the CA. After some re-arrangement of dummy variables, this gives us

$$\left(\frac{\partial}{\partial t} + \nu_0 j^2 + \nu_0 |\mathbf{k} - \mathbf{j}|^2\right) \langle u_\beta^+(\mathbf{j}, t) u_\gamma^+(\mathbf{k} - \mathbf{j}, t)\rangle_c =$$

$$= 2 M_{\beta\delta\epsilon}^+(\mathbf{j}) \int \mathrm{d}^3 p \left\{ \langle u_\delta^-(\mathbf{p}, t) u_\epsilon^-(\mathbf{j} - \mathbf{p}, t) u_\gamma^+(\mathbf{k} - \mathbf{j}, t)\rangle_c \right.$$

$$+ 2\langle u_\delta^-(\mathbf{p}, t) u_\epsilon^+(\mathbf{j} - \mathbf{p}, t) u_\gamma^+(\mathbf{k} - \mathbf{j}, t)\rangle_c$$

$$\left. + \langle u_\delta^+(\mathbf{p}, t) u_\epsilon^+(\mathbf{j} - \mathbf{p}, t) u_\gamma^+(\mathbf{k} - \mathbf{j}, t)\rangle_c \right\}. \tag{4.20}$$

Taking the limit $\xi \to 0$ and applying the hypothesis of local chaos, that is equation (4.4), it is easily seen that the first term on the right hand side of this expression is zero, since in the limit it involves the ensemble average of $u_\alpha^+(\mathbf{k}, t)$, and that the second term gives rise to a term linear in u^-. This leaves just the third term, which we may evaluate by iterating the above procedure to form a dynamical equation for $\langle u_\delta^+(\mathbf{p}, t) u_\epsilon^+(\mathbf{j} - \mathbf{p}, t) u_\gamma^+(\mathbf{k} - \mathbf{j}, t)\rangle_c$.

Doing this yields, in the abbreviated notation introduced in Chapter 2,

$$\left(\frac{\partial}{\partial t} + \nu_0 p^2 + \nu_0 |\mathbf{j} - \mathbf{p}|^2 + \nu_0 |\mathbf{k} - \mathbf{j}|^2\right) \langle u_p^+ u_{j-p}^+ u_{k-j}^+\rangle_c =$$

$$= 2 M_p^+ \left\{ \langle u_q^- u_{p-q}^- u_{j-p}^+ u_{k-j}^+\rangle_c + 2\langle u_q^- u_{p-q}^+ u_{j-p}^+ u_{k-j}^+\rangle_c + \langle u_q^+ u_{p-q}^+ u_{j-p}^+ u_{k-j}^+\rangle_c \right\}. \tag{4.21}$$

Again applying the hypothesis of local chaos, the first CA on the right hand side of this expression is also found to be zero. This can be seen if we note that (4.4) implies that in evaluating this term we will need to calculate the ensemble average $\langle u_\epsilon^+(\mathbf{j} - \mathbf{p}, t) u_\gamma^+(\mathbf{k} - \mathbf{j}, t)\rangle$. From equation (1.63) this is given by

$$\langle u_\epsilon^+(\mathbf{j} - \mathbf{p}, t) u_\gamma^+(\mathbf{k} - \mathbf{j}, t)\rangle = Q_{\epsilon\gamma}(\mathbf{k} - \mathbf{j}, t)\delta(\mathbf{k} - \mathbf{p}) \tag{4.22}$$

and hence can only give a contribution if $\mathbf{k} = \mathbf{p}$. However by definition $\mathbf{k}$ lies in the low wavenumber range, whilst from (4.21) $\mathbf{p}$ lies in the high wavenumber region. Thus this term gives no contribution, meaning we are again left with a term which is linear in u^- and a term involving only u^+ modes for which we can obtain a yet higher order moment expansion.

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In general we find that a similar pattern occurs for all higher order moments involving only products of u^+. That is, each such moment gives rise to a term involving a moment of two u^+ modes, which has to be zero for consistency in its wavevector arguments, a term linear in u^- and a moment involving only u^+ modes of next higher order. Hence by inverting the differential operators in equations (4.20), (4.21) and in the expressions for the higher order moments we are ultimately able to obtain a general expression for the CA on the right hand side of (4.18).

We invert the differential operators by following the same approach as used in Section 3.7, namely we introduce an integrating factor and then formally solve the equation as a first order differential equation. Doing this gives us the sequence of equations, for successively higher order moments,

$$
\langle u_\beta^+(\boldsymbol{j},t)u_\gamma^+(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c = \int_{-\infty}^t \mathrm{d}s\, e^{-(\nu_0 j^2 + \nu_0 |k-j|^2)(t-s)} 2M_{\beta\delta\epsilon}^+(\boldsymbol{j}) \times
$$
$$
\times \int \mathrm{d}^3p\, \Big\{ 2\langle u_\delta^-(\boldsymbol{p},s)u_\epsilon^+(\boldsymbol{j}-\boldsymbol{p},s)u_\gamma^+(\boldsymbol{k}-\boldsymbol{j},s)\rangle_c
$$
$$
+ \langle u_\delta^+(\boldsymbol{p},s)u_\epsilon^+(\boldsymbol{j}-\boldsymbol{p},s)u_\gamma^+(\boldsymbol{k}-\boldsymbol{j},s)\rangle_c \Big\}, \quad (4.23)
$$

$$
\langle u_\delta^+(\boldsymbol{p},t)u_\epsilon^+(\boldsymbol{j}-\boldsymbol{p},t)u_\gamma^+(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c = \int_{-\infty}^t \mathrm{d}s'\, e^{-(\nu_0 p^2 + \nu_0 |j-p|^2 + \nu_0 |k-j|^2)(t-s')} \times
$$
$$
\times 3M_{\delta\rho\sigma}^+(\boldsymbol{p}) \int \mathrm{d}^3q\, \Big\{ 2\langle u_\rho^-(\boldsymbol{q},s')u_\sigma^+(\boldsymbol{p}-\boldsymbol{q},s')u_\epsilon^+(\boldsymbol{j}-\boldsymbol{p},s')u_\gamma^+(\boldsymbol{k}-\boldsymbol{j},s')\rangle_c
$$
$$
+ \langle u_\rho^+(\boldsymbol{q},s')u_\sigma^+(\boldsymbol{p}-\boldsymbol{q},s')u_\epsilon^+(\boldsymbol{j}-\boldsymbol{p},s')u_\gamma^+(\boldsymbol{k}-\boldsymbol{j},s')\rangle_c \Big\}, \quad (4.24)
$$

and so on. Using this hierarchy of moments, we then find that the CA in equation (4.18) may be written as the moment expansion

$$
\lim_{\xi\to 0}\langle u_\beta^+(\boldsymbol{j},t)u_\gamma^+(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c =
$$
$$
= \int_{-\infty}^t \mathrm{d}s\, e^{-(\nu_0 j^2 + \nu_0 |k-j|^2)(t-s)} M_{\beta\delta\epsilon}^+(\boldsymbol{j}) \int \mathrm{d}^3p \times
$$
$$
\times \Big\{ 4u_\delta^-(\boldsymbol{p},s) \lim_{|k-j|\to k_0} D_{\epsilon\gamma}^+(\boldsymbol{k}-\boldsymbol{j})Q(|\boldsymbol{k}-\boldsymbol{j}|)\delta(\boldsymbol{k}-\boldsymbol{p})
$$
$$
+ \int_{-\infty}^s \mathrm{d}s'\, e^{-(\nu_0 p^2 + \nu_0 |j-p|^2 + \nu_0 |k-j|^2)(s-s')} 12M_{\delta\rho\sigma}^+(\boldsymbol{p}) \times
$$

$$\times \int d^3q\, u_\rho^-(\boldsymbol{q}, s') \lim_{\{\cdot\} \to k_0} Q_{\delta\epsilon\gamma}^+(\boldsymbol{j} - \boldsymbol{p}, \boldsymbol{k} - \boldsymbol{j})\delta(\boldsymbol{k} - \boldsymbol{j}) + \dots \Big\},$$

$$(4.25)$$

where the first and second terms in the curly brackets follow from equations (1.63) and (1.74) respectively.

4.3.3 Two approximations

In principle, (4.25) can be simply substituted into the right hand side of (4.18) to give our final result. However, because of the time integrations and the fact that the moment expansion is infinite, this leaves us with an expression which is of little use in practical calculations. To obtain a usable expression we introduce two approximations, leaving their justification until Chapter 6.

First we truncate the moment expansion at lowest non-trivial order. This leaves

$$M_{\alpha\beta\gamma}^-(\boldsymbol{k}) \int d^3j\, \lim_{\xi \to 0}\langle u_\beta^+(\boldsymbol{j}, t)u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t)\rangle_c =$$

$$= \int d^3j\, 4M_{\alpha\beta\gamma}^-(\boldsymbol{k})M_{\beta\delta\epsilon}^+(\boldsymbol{j}) \lim_{\boldsymbol{\ell} \to k_0} D_{\epsilon\gamma}^+(\boldsymbol{\ell})Q^+(\boldsymbol{\ell}) \int_{-\infty}^t ds\, e^{-\omega_2(j,\ell)(t-s)}u_\delta^-(\boldsymbol{k}, s),$$

$$(4.26)$$

where $\omega_2(k, \ell) = \nu_0 j^2 + \nu_0 \ell^2$ and $\boldsymbol{\ell} = \boldsymbol{k} - \boldsymbol{j}$, and where we have also performed the integral over $\boldsymbol{p}$. Next we have to perform the time integral. We start by rewriting the right hand side as

$$\text{RHS} = \int_{-\infty}^t ds\, e^{-\omega_2(j,\ell)(t-s)} B_{\alpha\delta}(\boldsymbol{k})u_\delta^-(\boldsymbol{k}, s),$$

$$(4.27)$$

where

$$B_{\alpha\delta}(\boldsymbol{k}) = 4 \int d^3j\, M_{\alpha\beta\gamma}^-(\boldsymbol{k})M_{\beta\delta\epsilon}^+(\boldsymbol{j}) \lim_{\boldsymbol{\ell} \to k_0} D_{\epsilon\gamma}^+(\boldsymbol{\ell})Q^+(\boldsymbol{\ell}),$$

$$(4.28)$$

and if we then change the integration variable to $\tau = t - s$ this becomes

$$\text{RHS} = \int_0^\infty d\tau\, e^{-\omega_2(j,\ell)\tau} B_{\alpha\delta}(\boldsymbol{k})u_\delta^-(\boldsymbol{k}, t - \tau).$$

$$(4.29)$$

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For an isotropic field we find [74] that

$$B_{\alpha\delta}(\boldsymbol{k})u_{\delta}^{-}(\boldsymbol{k}, t - \tau) = B(k)D_{\alpha\delta}(\boldsymbol{k})u_{\delta}(\boldsymbol{k}, t - \tau) \qquad (4.30)$$

$$= B(k)u_{\alpha}^{-}(\boldsymbol{k}, t - \tau), \qquad (4.31)$$

where

$$B(k) = \frac{1}{d-1}\mathrm{Tr}(B_{\alpha\delta}(\boldsymbol{k})) \qquad (4.32)$$

for a d-dimensional system, and hence in 3 dimensions we have

$$B_{\alpha\delta}(\boldsymbol{k})u_{\delta}^{-}(\boldsymbol{k}, t - \tau) = \frac{1}{2}B_{\delta\delta}(\boldsymbol{k})u_{\alpha}^{-}(\boldsymbol{k}, t - \tau). \qquad (4.33)$$

Thus we find

$$M_{\alpha\beta\gamma}^{-}(\boldsymbol{k})\int \mathrm{d}^{3}j \lim_{\xi\to 0}\langle u_{\beta}^{+}(\boldsymbol{j}, t)u_{\gamma}^{+}(\boldsymbol{k} - \boldsymbol{j}, t)\rangle_{c} =$$

$$= \int \mathrm{d}^{3}j \lim_{\ell\to k_{0}}\int_{0}^{\infty}\mathrm{d}\tau\, e^{-\omega_{2}(j,\ell)\tau}\frac{1}{2}4M_{\delta\beta\gamma}^{-}(\boldsymbol{k})M_{\beta\delta\epsilon}^{+}(\boldsymbol{j})D_{\epsilon\gamma}^{+}(\boldsymbol{\ell})Q^{+}(\ell)u_{\alpha}^{-}(\boldsymbol{k}, t - \tau)$$

$$= -\int \mathrm{d}^{3}j \lim_{\ell\to k_{0}}\int_{0}^{\infty}\mathrm{d}\tau\, e^{-\omega_{2}(j,\ell)\tau}L(\boldsymbol{k}, \boldsymbol{j})Q^{+}(\ell)u_{\alpha}^{-}(\boldsymbol{k}, t - \tau), \qquad (4.34)$$

where

$$L(\boldsymbol{k}, \boldsymbol{j}) = -2M_{\delta\beta\gamma}^{-}(\boldsymbol{k})M_{\beta\delta\epsilon}^{+}(\boldsymbol{j})D_{\epsilon\gamma}^{+}(\boldsymbol{k} - \boldsymbol{j}), \qquad (4.35)$$

which we rewrite as

$$M_{\alpha\beta\gamma}^{-}(\boldsymbol{k})\int \mathrm{d}^{3}j \lim_{\xi\to 0}\langle u_{\beta}^{+}(\boldsymbol{j}, t)u_{\gamma}^{+}(\boldsymbol{k} - \boldsymbol{j}, t)\rangle_{c} =$$

$$= -\int \mathrm{d}^{3}j \lim_{\ell\to k_{0}}L(\boldsymbol{k}, \boldsymbol{j})Q^{+}(\ell)\int_{0}^{\infty}\mathrm{d}\tau\, e^{-\omega_{2}(j,\ell)\tau}u_{\alpha}^{-}(\boldsymbol{k}, t - \tau). \qquad (4.36)$$

This still however leaves us with the question of how to perform the actual time integral.

In the work of McComb *et al.* [47–51, 65], the equivalent of this time integral was performed using a Markovian type approximation based upon the physical argument that the u^{+} modes evolve upon much faster timescales than the u^{-}. Using this argument, $u_{\alpha}^{-}(\boldsymbol{k}, t - \tau)$ was expanded as a Taylor series about $\tau = 0$ and then truncated at lowest order, leaving simply $u_{\alpha}^{-}(\boldsymbol{k}, t)$. Given this approximation

it is then a simple matter to perform the integral. Although still using the Taylor series expansion we shall not perform this truncation at zero order but shall instead introduce an alternative approximation. This approximation is somewhat more consistent than simply dropping terms in an ad-hoc manner. Again we shall leave its justification until Chapter 6.

Assuming the approach is applicable, writing $u_\alpha^-(\boldsymbol{k}, t - \tau)$ as a Taylor series about $\tau = 0$ we have

$$u_\alpha^-(\boldsymbol{k}, t - \tau) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A_n(\boldsymbol{k}, t)\tau^n, \tag{4.37}$$

where $A_n(\boldsymbol{k}, t)$ is defined as

$$A_n(\boldsymbol{k}, t) = \left.\frac{\partial^n u_\alpha^-(\boldsymbol{k}, s)}{\partial s^n}\right|_{s=t}. \tag{4.38}$$

If we then define

$$I(\boldsymbol{k}, t) = \int_0^\infty \mathrm{d}\tau \, e^{-\omega_2(j,\ell)\tau} u_\alpha^-(\boldsymbol{k}, t - \tau), \tag{4.39}$$

then by comparison to (4.37) we see that it may be re-written as

$$I(\boldsymbol{k}, t) = \sum_{n=0}^{\infty} I_n(\boldsymbol{k}, t), \tag{4.40}$$

with

$$I_n(\boldsymbol{k}, t) = \frac{(-1)^n}{n!} A_n(\boldsymbol{k}, t) \int_0^\infty \mathrm{d}\tau \, \tau^n e^{-\omega_2(j,\ell)\tau}. \tag{4.41}$$

Now a general result is that

$$\int \mathrm{d}s \, s^n e^{-Bs} = (-1)^n \frac{\partial^n}{\partial B^n} \int \mathrm{d}s \, e^{-Bs}, \tag{4.42}$$

and hence (4.41) may be re-expressed as

$$\begin{aligned} I_n(\boldsymbol{k}, t) &= \frac{A_n(\boldsymbol{k}, t)}{n!} \frac{\partial^n}{\partial \omega_2^n} \int_0^\infty \mathrm{d}\tau e^{-\omega_2(j,\ell)\tau} &\tag{4.43} \\ &= \frac{A_n(\boldsymbol{k}, t)}{n!} \frac{\partial^n}{\partial \omega_2^n} \left[\omega_2(j,\ell)^{-1}\right] &\tag{4.44} \\ &= (-1)^n A_n(\boldsymbol{k}, t) \, \omega_2(j,\ell)^{-(n+1)}, &\tag{4.45} \end{aligned}$$

where the final line follows from the further general result

$$\frac{\partial^n}{\partial x^n}(x^{-1}) = (-1)^n n! \, x^{-(n+1)}. \tag{4.46}$$

Hence we have

$$I(\boldsymbol{k}, t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\omega_2(j, \ell)^{n+1}} \left. \frac{\partial^n u_\alpha^-(\boldsymbol{k}, s)}{\partial s^n} \right|_{s=t}. \qquad (4.47)$$

To proceed from here we introduce our second approximation, first noting that the time derivative in (4.47) may be rewritten using the NSE as

$$\left. \frac{\partial u_\alpha^-(\boldsymbol{k}, s)}{\partial s} \right|_{s=t} = -\nu_0 k^2 u_\alpha^-(\boldsymbol{k}, t) + \text{NLT}, \qquad (4.48)$$

where 'NLT' refers to the non-linear term in the NSE and where we have neglected the forcing term in (4.15) on the grounds that, with the Dirac delta form we are assuming, it is only defined at the origin. Our second approximation is then simply that we may neglect the non-linear term in (4.48). Differentiating the resulting equation with respect to time we find, assuming all higher order derivatives exist, that

$$\left. \frac{\partial^n u_\alpha^-(\boldsymbol{k}, x)}{\partial s^n} \right|_{s=t} = (-1)^n (\nu_0 k^2)^n u_\alpha^-(\boldsymbol{k}, t), \qquad (4.49)$$

and hence substituting into (4.47) we have

$$I(\boldsymbol{k}, t) = \frac{u_\alpha^-(\boldsymbol{k}, t)}{\omega_2(j, \ell)} \sum_{n=0}^{\infty} \left[\frac{\nu_0 k^2}{\omega_2(j, \ell)} \right]^n. \qquad (4.50)$$

Now provided $|x| < 1$, we have the general result [75] that

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}, \qquad (4.51)$$

and if we note that

$$\frac{\nu_0 k^2}{\omega_2(j, \ell)} = \frac{\nu_0 k^2}{\nu_0 j^2 + \nu_0 |\boldsymbol{k} - \boldsymbol{j}|^2} = \frac{k^2}{j^2 + |\boldsymbol{k} - \boldsymbol{j}|^2} < \frac{1}{2}, \qquad (4.52)$$

since $0 < k < k_1$ whilst $k_1 < j, |\boldsymbol{k} - \boldsymbol{j}| < k_0$, we can then easily see that

$$\begin{aligned} I(\boldsymbol{k}, t) &= \frac{u_\alpha^-(\boldsymbol{k}, t)}{\nu_0 j^2 + \nu_0 |\boldsymbol{k} - \boldsymbol{j}|^2} \left[\frac{1}{1 - k^2/(j^2 + |\boldsymbol{k} - \boldsymbol{j}|^2)} \right] \\ &= \frac{u_\alpha^-(\boldsymbol{k}, t)}{\nu_0 j^2 + \nu_0 |\boldsymbol{k} - \boldsymbol{j}|^2} \left[\frac{(k^2/2) + j^2 - kj\mu}{j^2 - kj\mu} \right], \end{aligned} \qquad (4.53)$$

where μ is the cosine of the angle between the $\boldsymbol{k}$ and $\boldsymbol{j}$ wavevectors. Consequently we have

$$M_{\alpha\beta\gamma}(\boldsymbol{k}) \int \mathrm{d}^3j \lim_{\xi\to 0} \langle u_\beta^+(\boldsymbol{j},t) u_\gamma^+(\boldsymbol{k}-\boldsymbol{j},t) \rangle_c =$$

$$= -\int \mathrm{d}^3j \lim_{|k-j|\to k_0} \frac{L(\boldsymbol{k},\boldsymbol{j}) Q^+(|\boldsymbol{k}-\boldsymbol{j}|)}{\nu_0 j^2 + \nu_0 |\boldsymbol{k}-\boldsymbol{j}|^2} \left[\frac{(k^2/2) + j^2 - kj\mu}{j^2 - kj\mu} \right] u_\alpha^-(\boldsymbol{k},t).$$

$$(4.54)$$

Substituting (4.54) back into (4.18) we are then left with the final result

$$\left(\frac{\partial}{\partial t} + \nu_1(k)k^2 \right) u_\alpha^-(\boldsymbol{k},t) = \mathcal{F}_\alpha^{<<}(\boldsymbol{k},t) + S_\alpha^-(\boldsymbol{k}|k_1)$$

$$+ M_{\alpha\beta\gamma}^-(\boldsymbol{k}) \int \mathrm{d}^3j\, u_\beta^-(\boldsymbol{j},t) u_\gamma^-(\boldsymbol{k}-\boldsymbol{j},t), (4.55)$$

for $0 < k < k_1$, where

$$\nu_1(k) = \nu_0 + \delta\nu_0(k) \qquad (4.56)$$

and

$$\delta\nu_0(k) = \frac{1}{k^2} \int \mathrm{d}^3j \lim_{|k-j|\to k_0} \frac{L(\boldsymbol{k},\boldsymbol{j}) Q^+(|\boldsymbol{k}-\boldsymbol{j}|)}{\nu_0 j^2 + \nu_0 |\boldsymbol{k}-\boldsymbol{j}|^2} \left[\frac{(k^2/2) + j^2 - kj\mu}{j^2 - kj\mu} \right]. \qquad (4.57)$$

Hence we have achieved our first aim, namely that of eliminating a band of high wavenumber modes. Next we need to iterate upon this procedure to perform the RG calculation.

Chapter 5

The Renormalization Group calculation

5.1 Importance of the error term

5.1.1 The explicit scales equation of motion

In principle, extending the procedure of the previous chapter in order to form an RG calculation is relatively simple, requiring us to carry out the following steps:

1. Re-label $u_\alpha^-(\boldsymbol{k}, t) \to u_\alpha(\boldsymbol{k}, t)$ in equation (4.55), so that we now have a new NSE with effective viscosity $\nu_1(k)$ for modes on the interval $0 < k < k_1$.

2. Rescale the variables in this new equation so that it becomes defined on the original interval $0 < k < k_0$.

3. Decompose again into explicit modes and modes to be eliminated.

4. Repeat the procedures used in Chapter 4 to eliminate the high wavenumber modes.

These steps may then be repeated, eliminating successive wavenumber shells until the fixed point of the RG calculation is reached. However before we can do this we

need first to consider the error term $S_\alpha^-(\boldsymbol{k}|k_1)$ in equation (4.55). This apparently breaks the required form-invariance when compared to the original NSE.

If we consider the explicit form of this term, as given by equation (4.19), then it is immediately apparent that we cannot simply neglect this term in equation (4.18), nor at any stage shall we make such a supposition. Indeed it is likely that $S_\alpha^-(\boldsymbol{k}|k_1)$ *cannot* be neglected in equation (4.18), which, it should be noted, is an *exact* expression governing the dynamics of the low wavenumber modes. As has been shown by Young [70], a point which is given further support by Gong *et al.* [76], near the cutoff between the low and high wavenumber regions the $u^- u^+$ term in (4.15) has a significant effect upon the dynamics of the system. Thus any neglect of this term would be at best an uncontrolled approximation. However in obtaining an expression for the energy spectrum, and the values of the terms in this expression, we need to consider the renormalized energy balance equation obtained from (4.55), that is the analogue of equation (1.79), rather than equation (4.55) itself. In this instance it would appear to be a valid approximation to neglect the terms which arise from $S_\alpha^-(\boldsymbol{k}|k_1)$.

5.1.2   The explicit scales energy balance equation

To obtain the explicit scales energy balance equation from (4.55) we need to follow essentially the same procedure as in Section 1.6. That is we multiply (4.55) by $u_\delta^-(\boldsymbol{\ell}, t)$, then use (4.55) to write a dynamic equation for $u_\delta^-(\boldsymbol{\ell}, t)$ and multiply this by $u_\alpha^-(\boldsymbol{k}, t)$. We then add these two expressions together, average over the full turbulent ensemble, and integrate with respect to $\boldsymbol{\ell}$. This leaves us with

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + \nu_1(k)k^2 + \nu_1(\ell)\ell^2 \right) &\langle u_\alpha^-(\boldsymbol{k}, t)u_\delta^-(\boldsymbol{\ell}, t) \rangle = \\
&\langle \mathcal{F}_\alpha^{<<}(\boldsymbol{k}, t)u_\delta^-(\boldsymbol{\ell}, t) \rangle + \langle u_\alpha^-(\boldsymbol{k}, t)\mathcal{F}_\delta^{<<}(\boldsymbol{\ell}, t) \rangle \\
&+ M_{\alpha\beta\gamma}^-(\boldsymbol{k}) \int \mathrm{d}^3 j \, \langle u_\beta^-(\boldsymbol{j}, t)u_\gamma^-(\boldsymbol{k} - \boldsymbol{j}, t)u_\delta^-(\boldsymbol{\ell}, t) \rangle \\
&+ M_{\delta\beta\gamma}^-(\boldsymbol{\ell}) \int \mathrm{d}^3 j \, \langle u_\beta^-(\boldsymbol{j}, t)u_\gamma^-(\boldsymbol{\ell} - \boldsymbol{j}, t)u_\alpha^-(\boldsymbol{k}, t) \rangle
\end{aligned}
$$

$$+ \langle S_\alpha^-(\boldsymbol{k}|k_1)u_\delta^-(\boldsymbol{\ell},t)\rangle + \langle u_\alpha^-(\boldsymbol{k},t)S_\delta^-(\boldsymbol{\ell}|k_1)\rangle. \tag{5.1}$$

Applying the same approach by which we obtain equations (1.63) and (1.68), we next write

$$\langle S_\alpha^-(\boldsymbol{k}|k_1)u_\delta^-(\boldsymbol{\ell},t)\rangle = D_{\alpha\delta}^-(\boldsymbol{\ell})Z^-(\boldsymbol{\ell},t)\delta(\boldsymbol{k}+\boldsymbol{\ell}), \tag{5.2}$$

which defines $Z^-(\boldsymbol{\ell},t)$. If we then integrate equation (5.1) with respect to $\boldsymbol{\ell}$ we find, using equations (1.63), (1.74) and (1.75) along with (5.2),

$$\left(\frac{\partial}{\partial t} + 2\nu_1(k)k^2\right)Q_{\alpha\delta}^-(-\boldsymbol{k},t) = W_{\alpha\delta}^-(-\boldsymbol{k},t) + W_{\delta\alpha}^-(\boldsymbol{k},t)$$

$$+ M_{\alpha\beta\gamma}^-(\boldsymbol{k})\int \mathrm{d}^3 j\, Q_{\beta\gamma\delta}^{--}(\boldsymbol{k}-\boldsymbol{j},-\boldsymbol{k},t) + M_{\delta\beta\gamma}^-(-\boldsymbol{k})\int \mathrm{d}^3 j\, Q_{\beta\gamma\alpha}^{--}(-\boldsymbol{k}-\boldsymbol{j},\boldsymbol{k},t)$$

$$+ D_{\alpha\delta}^-(-\boldsymbol{k})Z^-(\boldsymbol{k},t) + D_{\delta\alpha}^-(\boldsymbol{k})Z^-(\boldsymbol{k},t), \tag{5.3}$$

where the superscript '$--$' on the triple moments indicates that both wavevectors which are explicitly written lie within the low wavenumber region. Finally we take the trace of (5.3) and multiply by $2\pi k^2$ to obtain our analogue to (1.79),

$$\left(\frac{\partial}{\partial t} + 2\nu_1(k)k^2\right)E^-(k,t) = W^-(k,t) + T^{--}(k,t) + 8\pi k^2 Z^-(k,t), \tag{5.4}$$

and if we further note that

$$\frac{\partial E^-(k,t)}{\partial t} = 0, \tag{5.5}$$

since the system is stationary, we are left with

$$2\nu_1(k)k^2 E^-(k,t) = W^-(k,t) + T^{--}(k,t) + 8\pi k^2 Z^-(k,t). \tag{5.6}$$

As was done with equation (1.79), we may integrate each term in (5.6) with respect to k, but here we shall integrate over the range 0 to k_1 rather than up to infinity. If we do this we have

$$\int_0^{k_1} \mathrm{d}k\, W^-(k,t) = \varepsilon(t), \tag{5.7}$$

due to our having defined the forcing to only act at very low wavenumbers, and we also find, using the same symmetry property as applied in Section 1.6, that the

transfer term $T^{--}(k,t)$ disappears when integrated over this range. This leaves us with

$$\varepsilon(t) = \int_0^{k_1} \mathrm{d}k \, 2\nu_1(k)k^2 E^-(k) - \int_0^{k_1} \mathrm{d}k \, 8\pi k^2 Z^-(k). \qquad (5.8)$$

This expression may be interpreted as stating that there are two contributions to energy dissipation in the system, one due to the RG eddy viscosity (which in turn incorporates the molecular viscosity), i.e. $\nu_1(k)$, and one arising from the error term. The fact that the error term should act in a dissipative manner is relatively obvious. If we re-examine equation (4.19) it is apparent that one effect of this term is to transfer energy to high wavenumbers, that is remove energy from the low wavenumber region, and hence it may be viewed as dissipating energy from our low wavenumber system. Indeed, we may rewrite the integrand of the second term on the right hand side of (5.8) as

$$8\pi k^2 Z^-(k) = -2\nu_S(k)k^2 E^-(k), \qquad (5.9)$$

where $\nu_S(k)$ denotes a contribution to the sub-grid viscosity, additional to that described by $\nu_1(k)$, arising from the transfer of energy to modes outside the low wavenumber region by the terms contained in $S^-(\boldsymbol{k}|k_1)$. Re-arranging (5.9), this may be found as

$$\nu_S(k) = 4\pi \frac{Z^-(k)}{E^-(k)}. \qquad (5.10)$$

Making such a substitution would have parallels with the work of Zhou and Vahala [54], who introduce a 'drain-eddy viscosity' to represent the loss of energy from the low wavenumber region caused by the triple non-linearity in their theory. This function is then added to their RG eddy viscosity in order to give a total eddy viscosity. It should however be noted that the expression from which our additional dissipative term arises is not the same as the triple non-linearity of Zhou and Vahala, and, as we shall now discuss, we feel that in our case the $Z^-(k)$ term (and hence the $\nu_S(k)$ term) is likely to be negligible in comparison to the RG eddy viscosity when considering energy, as opposed to momentum, transfers.

5.1.3   Neglect of the $Z^-(k)$ term

If we consider the ensemble average from which the $Z^-(k)$ term arises, that is

$$\langle S_\alpha^-(\boldsymbol{k}|k_1) u_\alpha^-(\boldsymbol{\ell}, t)\rangle,$$

then from the form of $S_\alpha^-(\boldsymbol{k}|k_1)$, as given in equation (4.19), we can easily see that each of the terms which make up $Z^-(\boldsymbol{k})$ involve a conditional average multiplied by $u_\alpha^-(\boldsymbol{\ell}, t)$. In evaluating the ensemble average of such a product we need to perform a double summation. That is, there is one summation due to the ensemble average and one due to the conditional average. To see the form of this double summation we need only consider the ensemble average of the more general functional

$$H[\boldsymbol{u}(\boldsymbol{k}, t)] = \langle h[\boldsymbol{u}(\boldsymbol{k}, t)]\rangle_c \boldsymbol{u}(\boldsymbol{k}, t), \tag{5.11}$$

where $h[\boldsymbol{u}(\boldsymbol{k}, t)]$ denotes a functional of which we take the conditional average.

Using the formal definition for the ensemble average, equation (3.6), the average of (5.11) is given by

$$
\begin{aligned}
\langle H[\boldsymbol{u}(\boldsymbol{k}, t)]\rangle &= \frac{1}{N} \sum_{n=1}^{N} H[\boldsymbol{W}^{(n)}(\boldsymbol{k}, t)] \\
&= \frac{1}{N} \sum_{n=1}^{N} \langle h[\boldsymbol{W}^{(n)}(\boldsymbol{k}, t)]\rangle_c \boldsymbol{W}^{(n)}(\boldsymbol{k}, t) \\
&= \frac{1}{N} \sum_{n=1}^{N} \left(\frac{1}{M} \sum_{m=1}^{M} h[\boldsymbol{Y}^{(m,n)}(\boldsymbol{k})] \right) \boldsymbol{W}^{(n)}(\boldsymbol{k}, t), \tag{5.12}
\end{aligned}
$$

where the notation $\boldsymbol{Y}^{(m,n)}(\boldsymbol{k})$ is used for the biased subensemble to indicate that for each value of n used in calculating the ensemble average we need to define a new biased subensemble with reference to $\boldsymbol{W}^{(n)}(\boldsymbol{k}, t)$. That is, the members of $\boldsymbol{Y}^{(m,n)}(\boldsymbol{k})$ are defined by the criterion (c.f. equation (3.39))

$$\left| \theta^-(k) Y_\alpha^{(m,n)}(\boldsymbol{k}) - \theta^-(k) W_\alpha^{(n)}(\boldsymbol{k}, t) \right| \leq \xi, \tag{5.13}$$

and hence in performing the double summation we first sum over all members of the full ensemble with low wavenumber modes close to a particular member
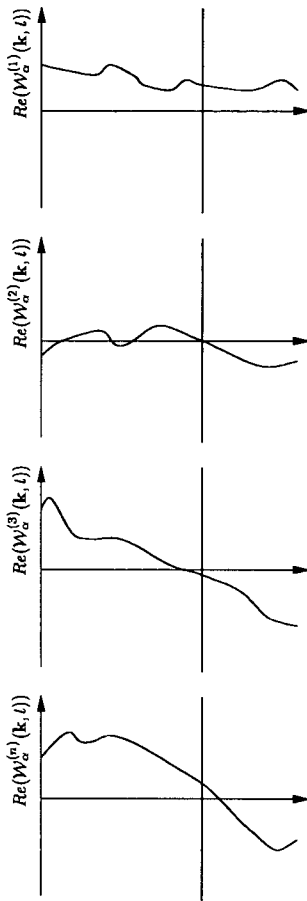
of the ensemble and then repeat this summation for every member of the entire ensemble. This summation is illustrated schematically in Figure 5.1. From this we may see that the effect of performing the requisite double summation could alternatively be viewed as performing a single summation over a set consisting of all the members of our turbulent ensemble $\mathcal{W}$ but in which some of the members of $\mathcal{W}$ are counted more than once. That is, we average over a new larger set, which contains all the members of $\mathcal{W}$, but in which some members have multiple entries. Now the initial turbulent ensemble was constructed according to the principle of equal *a priori* probabilities, but with the composite ensemble we are now considering this is no longer true. If it were true, then all the terms in this average involving $S_\alpha^-(\boldsymbol{k}|k_1)$ would vanish identically for all k_1. That is, the operation of performing an ordinary ensemble average would effectively lift the constraint imposed by the conditional averaging procedure. However, even though this cannot be strictly true, it would seem reasonable to assume, to a good approximation, that such a relaxation of the constraint does occur. For the remainder of the thesis we shall assume this to be the case.

## 5.2   Inductive treatment of the nth shell

Given the assumption that the constraint imposed by the conditional averaging procedure may be relaxed by taking the ensemble average, we can simplify our calculation by dropping the $S_\alpha^-(\boldsymbol{k}|k_1)$ term from equation (4.55), on the basis that the terms it ultimately gives rise to are negligible when calculating the values involved in the energy spectrum, as will be those arising from the error terms generated upon subsequent iterations. Making this simplification, (4.55) reduces to

$$\left(\frac{\partial}{\partial t} + \nu_1(k)k^2\right) u_\alpha^-(\boldsymbol{k}, t) = \mathcal{F}_\alpha^{<<}(\boldsymbol{k}, t) + M_{\alpha\beta\gamma}^-(\boldsymbol{k}) \int \mathrm{d}^3 j \, u_\beta^-(\boldsymbol{j}, t) u_\gamma^-(\boldsymbol{k} - \boldsymbol{j}, t),$$
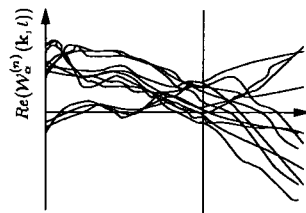
$$(5.14)$$

Sum over n Sum over m
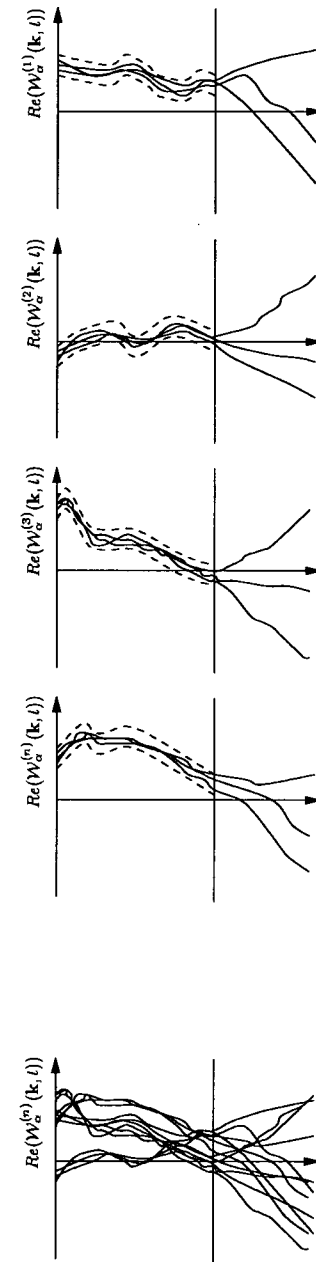
Double sum over m and n $\equiv$

Figure 5.1: Schematic illustration of the double summation in equation (5.12). The diagrams on the left hand side illustrate the summation over n which is performed in evaluating the full ensemble average, whilst the diagrams on the right hand side illustrate the m conditionally sampled members corresponding to each n. The double sum itself is the sum over all the individual diagrams on the right hand side, as indicated in the bottom illustration.

and it is then a simple matter to extend the mode elimination procedure to further shells and hence form an RG calculation. We do this as follows:

1. Set $u_\alpha^-(\boldsymbol{k}, t) = u_\alpha(\boldsymbol{k}, t)$ in the equation for the explicit modes, so that we have a new NSE with effective viscosity $\nu_1(k)$, but which is now defined on the interval $0 < k < k_1$.

2. Make the decomposition into u^- and u^+ modes, but this time at $k = k_2$, where $k_2 = (1 - \eta)k_1 = (1 - \eta)^2 k_0$, so that $u_\alpha^+(\boldsymbol{k}, t)$ is now defined in the range $k_2 < k < k_1$.

3. Repeat the mode elimination procedure to remove the new u^+ modes.

4. Repeat steps 1–3 for successive shells, where the nth shell in the procedure is defined by

$$k_n = (1 - \eta)^n k_0 \text{ for } 0 < \eta < 1. \tag{5.15}$$

By induction, equation (4.56) for the effective viscosity then generalizes to

$$\nu_{n+1}(k) = \nu_n(k) + \delta\nu_n(k), \tag{5.16}$$

and similarly (4.57) becomes

$$\delta\nu_n(k) = \frac{1}{k^2} \int d^3 j \lim_{|k-j| \to k_n} \frac{L(\boldsymbol{k}, \boldsymbol{j}) Q^+(|\boldsymbol{k} - \boldsymbol{j}|)}{\nu_n(j) j^2 + \nu_n(|\boldsymbol{k} - \boldsymbol{j}|)|\boldsymbol{k} - \boldsymbol{j}|^2} \left[\frac{(k^2/2) + j^2 - kj\mu}{j^2 - kj\mu} \right], \tag{5.17}$$

where, of course, the superscript on $Q^+(|\boldsymbol{k} - \boldsymbol{j}|)$ now means $|\boldsymbol{k} - \boldsymbol{j}|$ lies in the range $k_{n+1} < |\boldsymbol{k} - \boldsymbol{j}| < k_n$.

5.2.1 Energy equation for the retained modes

Likewise, equation (5.14) has the iterated form

$$\left(\frac{\partial}{\partial t} + \nu_n(k)k^2 \right) u_\alpha^-(\boldsymbol{k}, t) = \mathcal{F}_\alpha^{<<}(\boldsymbol{k}, t) + M_{\alpha\beta\gamma}^-(\boldsymbol{k}) \int d^3 j \, u_\beta^-(\boldsymbol{j}, t) u_\gamma^-(\boldsymbol{k} - \boldsymbol{j}, t), \tag{5.18}$$

for wavenumbers $0 < k, j, |\boldsymbol{k} - \boldsymbol{j}| < k_n$, and if we follow the calculation of Section 5.1.2 using this expression rather than equation (4.55), we obtain the analogue to (5.6)

$$2\nu_n(k)k^2 E(k) = W^-(k) + T^{--}(k), \tag{5.19}$$

defined on $0 < k < k_n$, where we have again assumed that the flow is stationary. Integrating (5.19) with respect to k from 0 to k_n we then find

$$\varepsilon_w = \int_0^{k_n} \mathrm{d}k\, 2\,\nu_n(k)k^2 E(k), \tag{5.20}$$

where the $T^{--}(k)$ term again disappears due to symmetry. This is just the renormalized version of the usual dissipation integral, equation (1.83), the increased effective viscosity compensating for the reduced upper limit on the region of integration.

5.2.2   Rescaling the equations

In order to satisfy the RG algorithm, we also need to rescale the expression for the nth cycle increment, the NSE and the recursion relation, so that we are considering expressions defined on the same interval. To do this we start by introducing the assumption that the energy spectrum in the high wavenumber band is described by a power law of the form

$$E(k) = \alpha\varepsilon^{2/3}k^{-5/3}F(k/k_d), \tag{5.21}$$

that is by equation (1.103), along with the scaling transformation

$$k = k_n k', \tag{5.22}$$

where k' is non-dimensional. To simplify the calculation, we also introduce the definition

$$h = (1 - \eta), \tag{5.23}$$

and hence have the additional expression

$$k_{n+1} = h k_n. \tag{5.24}$$

Rewriting (5.17) in terms of (5.22) we find

$$\delta\nu_n(k) = \frac{\alpha\varepsilon^{2/3}k_n^{-8/3}}{4\pi k'^2} \int \mathrm{d}^3 j' \lim_{\ell'\to 1} \frac{L(\boldsymbol{k}',\boldsymbol{j}')\ell'^{-11/3}F(k_n\ell'/k_d)}{\nu_n(k_n j')j'^2 + \nu_n(k_n\ell')\ell'^2} \left[\frac{k'^2/2 + j'^2 - k'j'\mu}{j'^2 - k'j'\mu}\right],$$
(5.25)

where $\boldsymbol{\ell} = \boldsymbol{k} - \boldsymbol{j}$, and if we impose the obvious consistency requirement that $\nu_n(k)$ and $\delta\nu_n(k)$ must scale in the same way, then from this expression and (5.16) we have

$$\nu_n(k_n k') = \alpha^{1/2}\varepsilon^{1/3}k_n^{-4/3}\tilde{\nu}_n(k'),$$
(5.26)

where $\tilde{\nu}_n(k')$ is a dimensionless function. With this expression we can scale all the relevant equations. Firstly, equation (5.26) can be extended to the $(n+1)^{\text{th}}$ iteration as

$$\nu_{n+1}(k_{n+1} k') = \alpha^{1/2}\varepsilon^{1/3}k_{n+1}^{-4/3}\tilde{\nu}_{n+1}(k'),$$
(5.27)

and in making this statement we implicitly rescale the space variable. That is, in equation (5.26) we have $k = k_n k'$, whereas in (5.27) we have $k = k_{n+1}k'$, which is equivalent to the rescaling $k' \to hk'$. In terms of this rescaled variable, equation (5.26) may be re-expressed as

$$\nu_n(k) = \alpha^{1/2}\varepsilon^{1/3}k_n^{-4/3}\tilde{\nu}_n(hk'),$$
(5.28)

and if we rewrite (5.25) in terms of $k = k_{n+1}k'$, rather than (5.22), we find

$$\delta\nu_n(k) = \frac{\alpha^{1/2}\varepsilon^{1/3}}{4\pi k'^2} \frac{k_{n+1}^{-8/3}}{k_n^{-4/3}} \int \mathrm{d}^3 j' \lim_{\ell'\to h^{-1}} \frac{L(\boldsymbol{k}',\boldsymbol{j}')Q'}{\tilde{\nu}_n(hj')j'^2 + \tilde{\nu}_n(h\ell')\ell'^2} \left[\frac{k'^2/2 + j'^2 - k'j'\mu}{j'^2 - k'j'\mu}\right],$$
(5.29)

where we have substituted from (5.28) for $\nu_n(k)$, and

$$Q' = \ell'^{-11/3}F(\ell/k_d).$$
(5.30)

Substituting equations (5.27), (5.28) and (5.29) into (5.17), we then find

$$\alpha^{1/2}\varepsilon^{1/3}k_{n+1}^{-4/3}\tilde{\nu}_{n+1}(k') = \alpha^{1/2}\varepsilon^{1/3}k_n^{-4/3}\tilde{\nu}_n(hk') + \alpha^{1/2}\varepsilon^{1/3}k_n^{-4/3}h^{-8/3}\delta\tilde{\nu}_n(k'), \quad (5.31)$$

where

$$\delta\tilde{\nu}_n(k') = \frac{1}{4\pi k'^2} \int \mathrm{d}^3 j' \lim_{\ell'\to h^{-1}} \frac{L(\boldsymbol{k}',\boldsymbol{j}')Q'}{\tilde{\nu}_n(hj')j'^2 + \tilde{\nu}_n(h\ell')\ell'^2} \left[\frac{k'^2/2 + j'^2 - k'j'\mu}{j'^2 - k'j'\mu}\right] \quad (5.32)$$

gives the explicit form for the scaled viscosity increment, the system being defined for the wavenumber bands $0 < k' < 1; 1 < j', \ell' < h^{-1}$. From equations (5.24) and (5.31) we can then obtain the scaled recursion relation

$$\tilde{\nu}_{n+1}(k') = h^{4/3}\tilde{\nu}_n(hk') + h^{-4/3}\delta\tilde{\nu}_n(k'). \tag{5.33}$$

5.3   The Renormalization Group calculation

Equations (5.32) and (5.33), which, it must be noted, are of identical form to the expressions obtained by McComb and Watt for the same quantities [51], describe the essential RG calculation, that is the elimination of a range of modes followed by a rescaling of the system. Thus we may now perform such a calculation, iterating until we reach the fixed point, which we define by the condition

$$\tilde{\nu}_{n+1}(k') = \tilde{\nu}_n(k') \equiv \tilde{\nu}_N(k'), \tag{5.34}$$

where N denotes the iteration upon which the fixed point is reached.

Following McComb and Watt [51], the Kolmogorov constant may then be obtained using the renormalized dissipation integral. At the fixed point equation (5.20) becomes

$$\varepsilon = \int_0^{k_N} dk\, 2\nu_N(k)k^2 E(k). \tag{5.35}$$

If we then note that the fixed point should indicate that the RG calculation has reached the high wavenumber end of the inertial range, that is we expect the energy spectrum below k_N to have Kolmogorov form, we may substitute equation (1.100) along with equations (5.22) and (5.27), in which we have set $n \equiv N$ and $n + 1 \equiv N$ respectively, to obtain

$$1 = 2\alpha^{3/2} \int_0^1 dk'\, \tilde{\nu}_N(k')k'^{1/3}. \tag{5.36}$$

Rearranging this, we are then left with the final expression for the Kolmogorov constant

$$\alpha = \left\{ 2 \int_0^1 dk'\, \tilde{\nu}_N(k')k'^{1/3} \right\}^{-2/3}. \tag{5.37}$$

Given these expressions, the RG calculation can then be carried out numerically to obtain values for both the eddy viscosities and the Kolmogorov constant for a wide range of bandwidths η. In fact these calculations are performed using

$$\omega_n(k) = \nu_n(k)k^2, \tag{5.38}$$

rather than $\nu_n(k)$, in order to counter problems caused by the factor $1/k^2$ in equation (5.17). Making this definition, we simply modify equations (5.28), (5.32) and (5.33) to obtain the expressions actually used, that is the scaling relation

$$\omega_n(k) = \alpha^{1/2}\varepsilon^{1/3}k_n^{2/3}\tilde{\omega}_n(hk'), \tag{5.39}$$

the recursion relation

$$\tilde{\omega}_{n+1}(k') = h^{-2/3}\tilde{\omega}_n(hk') + h^{2/3}\delta\tilde{\omega}_n(k') \tag{5.40}$$

and the scaled increment

$$\delta\tilde{\omega}_n(k') = \frac{1}{4\pi}\int d^3j' \lim_{\ell'\to h^{-1}} \frac{L(\boldsymbol{k}',\boldsymbol{j}')Q'}{\tilde{\omega}_n(hj') + \tilde{\omega}_n(h\ell')}\left[\frac{k'^2/2 + j'^2 - k'j'\mu}{j'^2 - k'j'\mu}\right], \tag{5.41}$$

which can be rewritten in terms of the spherical polar coordinates actually used in the calculation, where the vector $\boldsymbol{k}'$ is chosen to be coincident with the z axis, as

$$\delta\tilde{\omega}_n(k') = \frac{1}{2}\int dj' \int d\mu \lim_{\ell'\to h^{-1}} \frac{j'^2 L(k',j',\mu)Q'}{\tilde{\omega}_n(hj') + \tilde{\omega}_n(h\ell')}\left[\frac{k'^2/2 + j'^2 - k'j'\mu}{j'^2 - k'j'\mu}\right]. \tag{5.42}$$

In addition we also need to model the form of Q' and take the limit on ℓ'. To do this we first note that once the inertial range is reached, that is once we have Kolmogorov scaling, we would expect Q' to take the form $Q' = \ell'^{-11/3}$, which in the limit gives $Q' = h^{11/3}$. To obtain an approximation for the rest of the band, we extrapolate back to lower wavenumbers using a Taylor series expansion about $\ell' = h^{-1}$, assuming that the spectral energy density $Q(k)$ has Kolmogorov form at this point, to obtain

$$Q' = h^{11/3} - \frac{11}{3}h^{14/3}(\ell' - h^{-1}). \tag{5.43}$$

Although on the first few iterations of the RG calculation such an approximation is likely to be rather poor, as the calculation progresses its quality should improve, giving an accurate description once the fixed point is reached.

In order to perform the actual numerical calculation, the k', j' and μ ranges are first discretized, and then given an input scaled viscosity, $\tilde{\nu}_0(k')$, which is easily converted to $\tilde{\omega}_0(k')$ using (5.38). The viscosity increment is calculated by quadrature, using Simpson's rule to perform each of the integrations, repeatedly halving the width of the intervals until a specified fractional accuracy with respect to the value calculated at the previous width, in our case 10^{-6}, is achieved. The recursion relation is then used to iterate the calculation until the fixed point is reached, this being defined to be the iteration upon which the value of $\tilde{\omega}_{n+1}(k')$ differs from the value of $\tilde{\omega}_n(k')$ by less than 0.1% for each mesh point. The interpolation to values between our discrete mesh points, necessary for the rescaling, is achieved using a cubic spline fit. The form of the scaled effective viscosity is obtained on each step of this calculation, to ensure that the code is working properly and enable comparison with the earlier work of McComb and Watt [51]. Having reached the fixed point, this is then used to calculate a value for the Kolmogorov constant by substituting into (5.37).

5.4   Results and Discussion

The first results we shall present are those illustrated in Figures 5.2 and 5.3. These figures clearly illustrate that for given values of k' and η the RG calculation reaches the same fixed point regardless of the initial choice for $\tilde{\nu}_0(k')$. The same result is also found for alternative choices of k' and η, provided only that η lies within the so-called 'plateau' range of bandwidths, in which the calculated value of the Kolmogorov constant is insensitive to the choice of η. We shall shortly see that the illustrated bandwidth of $\eta = 0.4$ lies within this plateau region. As discussed by McComb and Watt [51], this result illustrates the principle of univer-
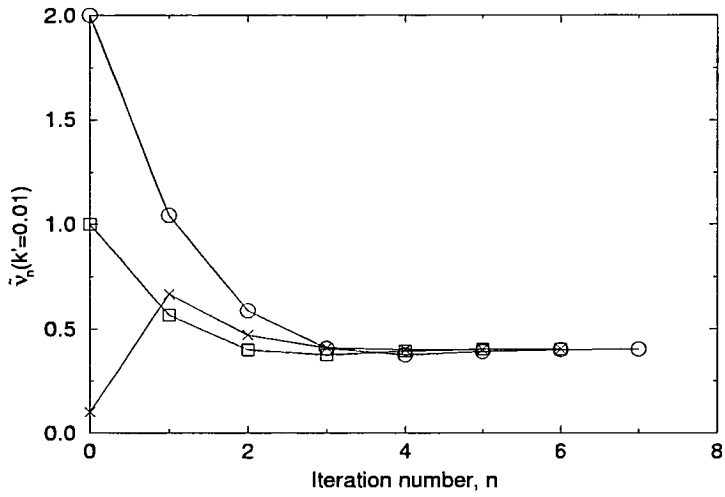
Figure 5.2: Convergence of the scaled effective viscosity to the fixed point for several values of initial viscosity. Values are plotted for $\tilde{\nu}_n(k')$ at $k' = 0.01$ and a bandwidth of $\eta = 0.4$.
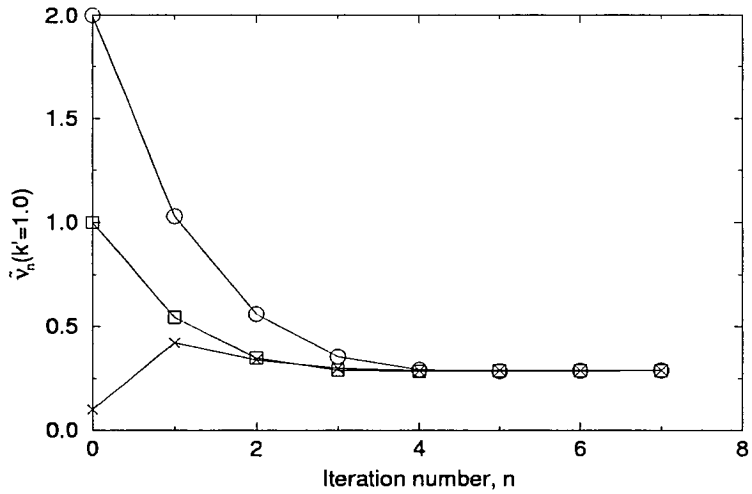

Figure 5.3: Convergence of the scaled effective viscosity to the fixed point for several values of initial viscosity. Values are plotted for $\tilde{\nu}_n(k')$ at $k' = 1.0$ and a bandwidth of $\eta = 0.4$.
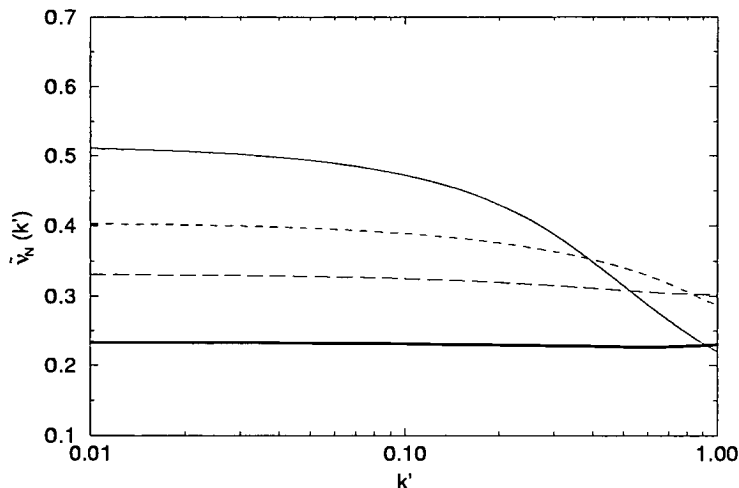
Figure 5.4: Dependence of the fixed point scaled eddy viscosity upon wavenumber k' for various bandwidths, $\eta = 0.2$ (——), $\eta = 0.4$ (-----), $\eta = 0.6$ (---) and $\eta = 0.8$ (——). $\tilde{\nu}_0(k') = 1.0$ for all of the plotted bandwidths.

sality, whereby the values obtained are independent of the details of the system, depending only upon the dynamics of the inertial range. The eddy viscosity thus obtained is valid for any Reynolds number which is sufficiently high for there to be an inertial range.

Next we consider the effect of varying the bandwidth upon the fixed point scaled eddy viscosity $\tilde{\nu}_N(k')$. This is illustrated in Figure 5.4. Clearly there is a significant dependence upon the choice of bandwidth, with the narrow bandwidths showing a much greater dependence upon k' than the wider band. That this should be so is relatively easily explained, the dependence upon wavenumber being large for narrow bands but small for wide bands due to the conditional average becoming more deterministic for small choices of η, greater scale separation also being possible with increasing bandwidth. We shall consider both these points further when discussing our results for the Kolmogorov constant.

The form of the eddy viscosity is also considered in Figure 5.5, in which we illustrate the evolution of the unscaled eddy viscosity as the RG calculation progresses through its iterations. This shows the expected result that the eddy viscosity rises
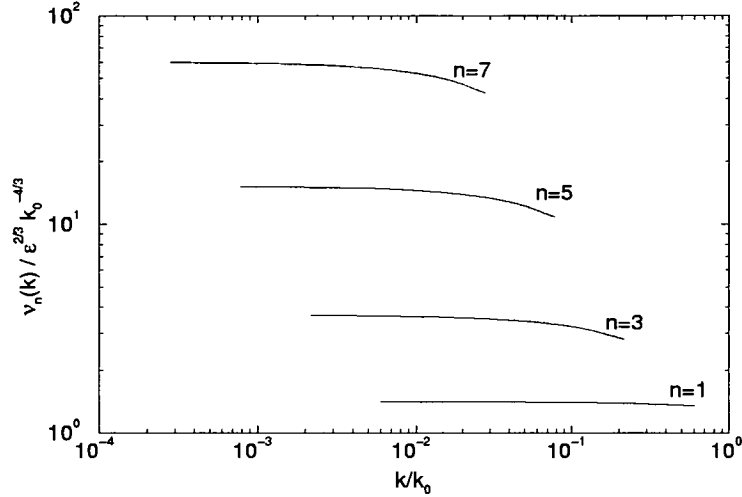
Figure 5.5: Evolution of the unscaled (dimensional) eddy viscosity during the RG calculation. Evaluated for $\eta = 0.4$. $n = 7$ corresponds to the fixed point.

as more modes are eliminated, this increase being necessary to compensate for the dissipation that would normally occur at the scales of the eliminated modes. Also clear is the characteristic result, also seen in Figure 5.4, that the effective viscosity shows an asymptotic trend to a constant value as k becomes small compared to the cutoff wavenumber, a result which reflects the increasing validity of the concept of an eddy viscosity as the retained and eliminated modes become more widely separated. In fact, that this must be so can be seen if we note that our equation (5.32) is essentially the same as that used by McComb and Watt, save for the extra factor

$$\frac{k'^2/2 + j'^2 - k'j'\mu}{j'^2 - k'j'\mu}$$

in the integrand. As was shown by Storkey [77], the expression of McComb and Watt must analytically tend to a constant as $k' \to 0$, and since in this limit we have

$$\lim_{k' \to 0} \frac{k'^2/2 + j'^2 - k'j'\mu}{j'^2 - k'j'\mu} = \frac{j'^2}{j'^2} = 1, \tag{5.44}$$

given that j' is restricted to being non-zero, our eddy viscosity must also tend to the same constant value, the value of the integrand in the limit $k' \to 0$ being the same for both approaches.

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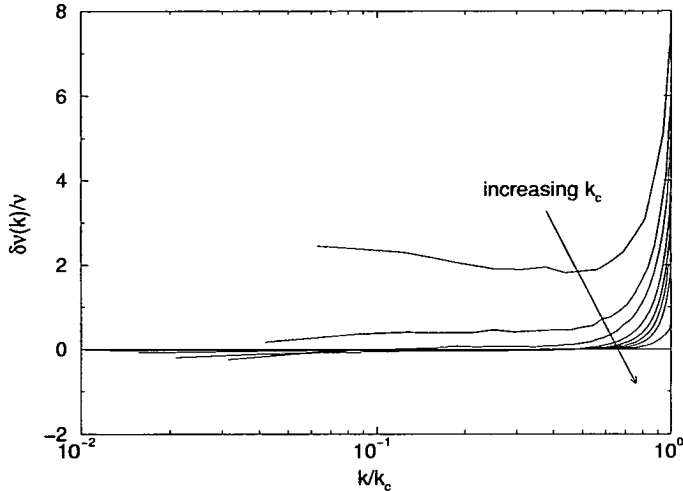
Figure 5.6: Eddy viscosities computed from DNS data, scaled on the molecular viscosity, for $k_c = 16, 24, 32, 48, 64, 80, 96, 112$. (*Figure reproduced from Young* [70].)

There is however a question regarding the form of the fixed point eddy viscosity, that is the $n = 7$ case, illustrated in Figure 5.5, in particular regarding the question of whether the eddy viscosity may be used in a spectral LES. We shall consider this question in more detail in Chapter 7, but it is worth noting here that the observed downturn as k increases is at odds with the results of, for example, Young [70], who used a DNS to calculate the effect of the subgrid modes upon the explicit scales and hence obtained an eddy viscosity which exactly represents the effects of the subgrid modes. These results are reproduced in Figure 5.6 (his Figure 6.4) and clearly show an upturn at the largest wavenumbers, regardless of the choice of cutoff k_c. That these results differ from our eddy viscosity should, perhaps, be no surprise since the $S^-(k|k_c)$ term, which we have neglected in our calculation of the eddy viscosity, will give a contribution. This contribution will be greatest near the cutoff between the resolved and subgrid modes, that is in the region where the form of our eddy viscosity differs from that of Young, and it would thus seem reasonable to suppose that if the $S^-(k|k_c)$ term was accounted for in our calculation, then this could give rise to an upturn in the eddy viscosity. As is discussed earlier, we would however expect the absence of the $S^-(k|k_c)$ term
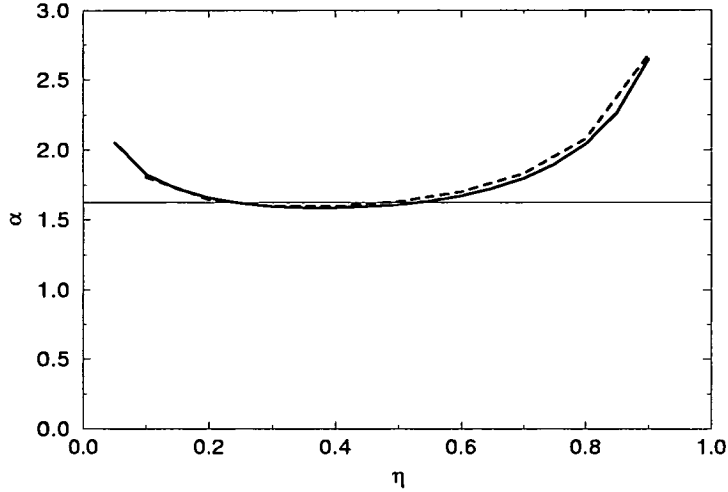
Figure 5.7: The variation with bandwidth η of the Kolmogorov constant α obtained from the RG calculation (——) in comparison to the Kolmogorov constant obtained from the RG calculation of McComb and Watt [51] (- - - -). The value for the Kolmogorov constant obtained in the DNS of Young [70], $\alpha = 1.624$, (——) is also plotted.

to have no effect upon the calculation of the Kolmogorov constant.

The calculation of the Kolmogorov constant for various bandwidths is illustrated in Figure 5.7. Immediately apparent from this figure is that our calculation gives good results, an approximate plateau region giving a value of α roughly equal to 1.6, in good agreement with both numerical simulations and experiment. The results also compare well to those obtained using the equations of McComb and Watt, our new theory giving essentially very similar predictions, but with an increased plateau region, although admittedly the increase in range is relatively small. We also note a good comparison with the DNS of Young [70], the result from which is explicitly plotted. At the ends of the range of bandwidths, our results move significantly away from their plateau values, but a good estimate for the value of α can be found by restricting ourselves to the range $0.2 < \eta < 0.6$, in which instance the Kolmogorov constant lies in the range $\alpha = 1.62 \pm 0.05$.

We should also comment on the increase in the predicted value of α as η tends to both 0 and 1. As we alluded to when considering the dependence of $\tilde{\nu}_N(k')$ with
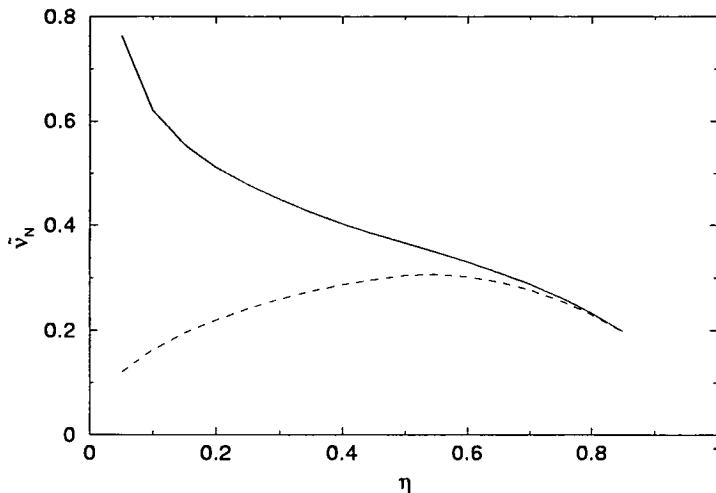
Figure 5.8: Variation of the scaled fixed point eddy-viscosity with bandwidth η for wavenumbers near the origin, $k' = 0.01$ (——), and near the cutoff, $k' = 1.00$ (-----).

η, such a breakdown in the theory as we approach these two limits is exactly what we would expect. As $\eta \to 0$, the conditional average becomes more deterministic, with the effect of the $S^-(k|k_1)$ term becoming significant, hence meaning that our assumptions are no longer valid. Likewise, as $\eta \to 1$ the bandwidth will become so large that the Taylor series expansion used to approximate Q' will break down, as will our assumption that a result defined in the limit $k \to k_n$ can be used, as an approximation, to describe the entire band. This point is given further support by the results plotted in Figure 5.8, which show that for a wide band ($\eta = 0.8$, say) the scaled fixed point eddy viscosity has relatively little dependence upon k' when compared to a narrow band ($\eta = 0.2$, say), for which there is a large variation depending upon whether or not we are near the cutoff. With a narrow band, in which all wavenumbers are comparatively close to the cutoff, the errors introduced by the neglect of the $S^-(k|k_1)$ term will clearly be far more significant than in the case of a wide band, where the errors are only likely to be significant for a relatively small range of the included wavenumbers. The large variation in $\tilde{\nu}_N(k')$ for the narrow band would tend to support this point. The fact that there is a reasonably wide plateau in the plot for the Kolmogorov constant would

however tend to suggest that, regarding the energy equation, the approximation of dropping the $S^-(k|k_1)$ term from our calculation is valid, in a heuristic sense, within the range $0.2 < \eta < 0.6$.

Finally we should note that even in the plateau region the value of α is *not* totally independent of η. As discussed by McComb and Watt [51] this is likely to be due to the fact that the graph is the result of a numerical calculation, in which the fixed point is only calculated to a finite accuracy. The uncertainty in α reflects this.

Chapter 6

Formal justification of the approximations

6.1 The dimensionless Navier-Stokes equation

In the previous two chapters we have shown both how we may eliminate a band of high wavenumber modes and then extend this procedure to provide the basis for a renormalization group calculation. However, as noted in Section 4.3.3, we have left until now the justification of two essential approximations, the truncation of the moment expansion at lowest order and the neglect of the non-linear term in performing the time integral. Both of these approximations may be justified by rescaling the equations of motion prior to the mode elimination calculation.

Our first step is to rewrite the NSE in its well known dimensionless form, defined on the interval 0 to 1. To do this we introduce the wavenumber transformation

$$\hat{\boldsymbol{k}} = \boldsymbol{k}/k_0, \tag{6.1}$$

along with corresponding time and velocity transformations

$$\hat{t} = t/\tau(k_0) \tag{6.2}$$

and

$$u_\alpha(\boldsymbol{k}, t) = V(k_0)\hat{u}_\alpha(\hat{\boldsymbol{k}}, \hat{t}), \tag{6.3}$$

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where $\tau(k_0)$ is an, as yet, undetermined timescale and $V(k_0)$ is the r.m.s. value of a velocity mode with $|\boldsymbol{k}| = k_0$, defined for any k by

$$V^2(k) = \frac{1}{k^3} \int d^3 j \, \langle u_\alpha(\boldsymbol{k}, t) u_\alpha(\boldsymbol{j}, t) \rangle, \tag{6.4}$$

where the factor of $1/k^3$ ensures the correct dimensions (recall that both $d^3 j$ and k^3 have dimensions L^{-3}).

Substituting (6.1), (6.2) and (6.3) into the NSE, equation (1.48), and multiplying through by $\tau(k_0)/V(k_0)$ we obtain[1]

$$\left(\frac{\partial}{\partial \hat{t}} + \nu_0(k)\tau(k_0)k_0^2 \hat{k}^2 \right) \hat{u}_\alpha(\hat{\boldsymbol{k}}, \hat{t}) = \tau(k_0)V(k_0)k_0^4 M_{\alpha\beta\gamma}(\hat{\boldsymbol{k}}) \int d^3 \hat{j} \, \hat{u}_\beta(\hat{\boldsymbol{j}}, \hat{t}) \hat{u}_\gamma(\hat{\boldsymbol{k}} - \hat{\boldsymbol{j}}, \hat{t}), \tag{6.5}$$

defined on $0 < \hat{k} < 1$, where we have pre-supposed the wavenumber dependence of the eddy-viscosity on later iterations by writing $\nu_0 = \nu_0(k) = \nu_0(k_0 \hat{k})$. Equation (6.5) can then be rewritten as

$$\left(\frac{\partial}{\partial \hat{t}} + \nu_0^\dagger(\hat{k})\hat{k}^2 \right) \hat{u}_\alpha(\hat{\boldsymbol{k}}, \hat{t}) = R_0(k_0) M_{\alpha\beta\gamma}(\hat{\boldsymbol{k}}) \int d^3 \hat{j} \, \hat{u}_\beta(\hat{\boldsymbol{j}}, \hat{t}) \hat{u}_\gamma(\hat{\boldsymbol{k}} - \hat{\boldsymbol{j}}, \hat{t}), \tag{6.6}$$

on $0 < \hat{k} < 1$, where

$$R_0(k_0) = \tau(k_0)V(k_0)k_0^4 \tag{6.7}$$

and

$$\nu_0^\dagger(\hat{k}) = \tau(k_0)k_0^2 \nu_0(k_0 \hat{k}). \tag{6.8}$$

Equation (6.6) is a dimensionless form of the NSE. If we consider the dimensions of the terms in (6.7), it can also be easily seen that $R_0(k_0)$ takes the form of a Reynolds number defined at the scale of k_0. That is, it may be viewed as the *local* Reynolds number at k_1. As we shall subsequently show, this parameter helps to provide the basis for justifying our approximations, the RG calculation in this chapter being based upon (6.6) rather than (1.48).

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[1]Note that, for clarity, in this chapter we shall neglect the forcing term present in equation (1.48). If desired, this term may be simply included in our calculation.

6.2   The Renormalization Group rescaling

As we discussed previously, the formal RG approach involves implementing the algorithm:

1. Start with the dynamic equation defined on the interval $0 < k < k_{\mathrm{max}}$.

2. Decompose the system into low and high wavenumber modes at a cutoff defined by $k_{\mathrm{cut}} = (1 - \eta)k_{\mathrm{max}} = h k_{\mathrm{max}}$, where ηk_{max} is the width of the high wavenumber band.

3. Average out the effects of the high wavenumber modes to obtain a dynamic equation for the interval $0 < k < k_{\mathrm{cut}}$.

4. Rescale this expression so that the system on $0 < k < k_{\mathrm{cut}}$ becomes redefined on the original interval.

5. Repeat steps 2 to 4 until the fixed point of the RG calculation is achieved.

It is this algorithm which we followed in Chapters 4 and 5. However there is no reason to suppose that we cannot interchange the order of steps 3 and 4, and in this chapter we propose to make such an interchange. Doing this enables us to formally justify our approximations.

In order to perform the rescaling prior to the mode elimination, all we need to ensure is that after both steps the system we are left with is defined on the original interval. Considering (6.6), it can be easily seen that this condition is achieved if we start by rescaling the original system onto the interval 0 to h^{-1}, the cutoff for the mode elimination in this instance being at $h.h^{-1} = 1$, as required. This rescaling is described by the transformation

$$k' = \hat{k}/h, \tag{6.9}$$

along with the transformations of the dimensionless time and velocity variables,

$$t' = \hat{t}/\hat{\tau}(h) \tag{6.10}$$

and

$$\psi_\alpha(\boldsymbol{k}', t') = \hat{u}_\alpha(\hat{\boldsymbol{k}}, \hat{t})/\hat{V}(h). \tag{6.11}$$

Substituting these transformations into (6.6) and multiplying through by $\hat{\tau}/\hat{V}$, we are then left with

$$\left(\frac{\partial}{\partial t'} + \hat{\nu}_0(k')k'^2\right)\psi_\alpha(\boldsymbol{k}', t') = \lambda_1(h, k_0)M_{\alpha\beta\gamma}(\boldsymbol{k}')\int \mathrm{d}^3 j'\, \psi_\beta(\boldsymbol{j}', t')\psi_\gamma(\boldsymbol{k}' - \boldsymbol{j}', t'), \tag{6.12}$$

on $0 < k' < h^{-1}$, where

$$\lambda_1(h, k_0) = \hat{\tau}(h)\hat{V}(h)h^4 R_0(k_0) \tag{6.13}$$

and

$$\hat{\nu}_0(k') = \hat{\tau}(h)h^2\nu_0^\dagger(hk'). \tag{6.14}$$

However if we consider equations (6.1) and (6.9) together, we find

$$\boldsymbol{k}' = \hat{\boldsymbol{k}}/h = \boldsymbol{k}/hk_0 = \boldsymbol{k}/k_1, \tag{6.15}$$

and hence it can be easily seen that scaling the dimensionless NSE by a factor h must give an identical result to scaling the original NSE by a factor k_1. Similarly, it must also follow that

$$t' = \hat{t}/\hat{\tau}(h) = t/\hat{\tau}(h)\tau(k_0) = t/\tau(k_1) \tag{6.16}$$

and

$$\psi_\alpha(\boldsymbol{k}', t') = \hat{u}_\alpha(\hat{\boldsymbol{k}}, \hat{t})/\hat{V}(h) = u_\alpha(\boldsymbol{k}, t)/\hat{V}(h)V(k_0) = u_\alpha(\boldsymbol{k}, t)/V(k_1), \tag{6.17}$$

which may be viewed as a similarity solution for the velocity field about k_1, the last equality in each of these expressions coming from scaling the NSE on k_1. Hence,

$$\hat{\tau}(h)\tau(k_0) = \tau(k_1) \tag{6.18}$$

and

$$\hat{V}(h)V(k_0) = V(k_1). \tag{6.19}$$

Substituting (6.7) and (6.8) for $R_0(k_0)$ and $\nu_0^\dagger(hk')$ respectively, we can then use these relations to re-express (6.13) and (6.14) as

$$\lambda_1(h,k_0) = \lambda_1(k_1) = \tau(k_1)V(k_1)k_1^4 \tag{6.20}$$

and

$$\hat{\nu}_0(k') = \tau(k_1)k_1^2\nu_0(k_1k'). \tag{6.21}$$

We should also comment here on the fact that we have now chosen to use $\lambda_1(k_1)$ to represent the term equivalent to $R_0(k_0)$ in (6.6). Again this term is simply the local Reynolds number (at k_1), but as we shall next see it is also in effect the expansion parameter which describes the order of terms in our mode elimination calculation. We make the replacement $R_1(k_1) \to \lambda_1(k_1)$ in order to emphasise this fact.

6.3   Mode elimination

Given that the form of (6.12) is essentially the same as the original NSE, we may thus go about eliminating a band of (dimensionless) high wavenumber modes in much the same way as before. Our first step is to divide up the system into low and high wavenumber regions. Making this decomposition, (6.12) reduces to two coupled equations

$$\left(\frac{\partial}{\partial t'} + \hat{\nu}_0(k')k'^2\right)\psi_\alpha^-(\boldsymbol{k}',t') = \lambda_1(k_1)M_{\alpha\beta\gamma}^-(\boldsymbol{k}')\int \mathrm{d}^3 j'\left\{\psi_\beta^-(\boldsymbol{j}',t')\psi_\gamma^-(\boldsymbol{k}'-\boldsymbol{j}',t')+\right.$$
$$\left. + 2\psi_\beta^-(\boldsymbol{j}',t')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}',t') + \psi_\beta^+(\boldsymbol{j}',t')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}',t')\right\}, \tag{6.22}$$

for $0 < k' < 1$, and

$$\left(\frac{\partial}{\partial t'} + \hat{\nu}_0(k')k'^2\right)\psi_\alpha^+(\boldsymbol{k}',t') = \lambda_1(k_1)M_{\alpha\beta\gamma}^+(\boldsymbol{k}')\int \mathrm{d}^3 j'\left\{\psi_\beta^-(\boldsymbol{j}',t')\psi_\gamma^-(\boldsymbol{k}'-\boldsymbol{j}',t')+\right.$$
$$\left. + 2\psi_\beta^-(\boldsymbol{j}',t')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}',t') + \psi_\beta^+(\boldsymbol{j}',t')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}',t')\right\}, \tag{6.23}$$

for $1 < k' < h^{-1}$.

Following the procedure of Section 4.3.1, if we re-define the conditional average in terms of an ensemble of $\psi_\alpha(\mathbf{k'}, t')$ fields, take the CA of (6.22) and rearrange, we obtain

$$\left(\frac{\partial}{\partial t'} + \hat{\nu}_0(k')k'^2 \right) \psi_\alpha^-(\mathbf{k'}, t') = \lambda_1(k_1) M_{\alpha\beta\gamma}^-(\mathbf{k'}) \int \mathrm{d}^3 j' \psi_\beta^-(\mathbf{j'}, t')\psi_\gamma^-(\mathbf{k'} - \mathbf{j'}, t') +$$
$$+ S_\alpha'^-(\mathbf{k'}|1) + \lambda_1(k_1) M_{\alpha\beta\gamma}^-(\mathbf{k'}) \int \mathrm{d}^3 j' \lim_{\xi \to 0} \langle \psi_\beta^+(\mathbf{j'}, t')\psi_\gamma^+(\mathbf{k'} - \mathbf{j'}, t') \rangle_c,$$

$$(6.24)$$

where, in analogue to (4.19),

$$S_\alpha'^-(\mathbf{k'}|1) = \lambda_1(k_1) M_{\alpha\beta\gamma}^-(\mathbf{k'}) \int \mathrm{d}^3 j' \left\{ \langle \phi_\beta'^-(\mathbf{j'}, t')\phi_\gamma'^-(\mathbf{k'} - \mathbf{j'}, t') \rangle_c + \right.$$
$$+ 2\langle \psi_\beta^-(\mathbf{j'}, t')\psi_\gamma^+(\mathbf{k'} - \mathbf{j'}, t') \rangle_c + \langle \psi_\beta^+(\mathbf{j'}, t')\psi_\gamma^+(\mathbf{k'} - \mathbf{j'}, t') \rangle_c -$$
$$\left. - \lim_{\xi \to 0} \langle \psi_\beta^+(\mathbf{j'}, t')\psi_\gamma^+(\mathbf{k'} - \mathbf{j'}, t') \rangle_c \right\}, \quad (6.25)$$

the ϕ' being analogous to the ϕ in equation (3.40).

As before, in order to evaluate the CA in (6.24) we need first to form a dynamical equation for $\langle \psi^+ \psi^+ \rangle_c$ using (6.23). Doing this we obtain

$$\left(\frac{\partial}{\partial t'} + \hat{\nu}_0(j')j'^2 + \hat{\nu}_0(|\mathbf{k'} - \mathbf{j'}|)|\mathbf{k'} - \mathbf{j'}|^2 \right) \langle \psi_\beta^+(\mathbf{j'}, t')\psi_\gamma^+(\mathbf{k'} - \mathbf{j'}, t') \rangle_c =$$
$$= 2\lambda_1(k_1) M_{\beta\delta\epsilon}^+(\mathbf{j'}) \int \mathrm{d}^3 p' \left\{ \langle \psi_\delta^-(\mathbf{p'}, t')\psi_\epsilon^-(\mathbf{j'} - \mathbf{p'}, t)\psi_\gamma^+(\mathbf{k'} - \mathbf{j'}, t') \rangle_c \right.$$
$$+ 2\langle \psi_\delta^-(\mathbf{p'}, t')\psi_\epsilon^+(\mathbf{j'} - \mathbf{p'}, t')\psi_\gamma^+(\mathbf{k'} - \mathbf{j'}, t') \rangle_c$$
$$\left. + \langle \psi_\delta^+(\mathbf{p'}, t')\psi_\epsilon^+(\mathbf{j'} - \mathbf{p'}, t')\psi_\gamma^+(\mathbf{k'} - \mathbf{j'}, t') \rangle_c \right\}. \quad (6.26)$$

Using identical arguments to those given in Section 4.3.2, if we take the limit $\xi \to 0$ and apply the hypothesis of local chaos then the first term on the right hand side is zero, the second term is linear in ψ^- and the third may be evaluated by iterating the above procedure to form a dynamical equation for $\langle \psi^+ \psi^+ \psi^+ \rangle_c$. As was the case with the velocity field (see Section 4.3.2), performing this iterative procedure leaves us with the sequence of equations for successively higher order

moments[2]

$$\langle \psi_\beta^+(\boldsymbol{j}',t')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}',t')\rangle_c = \int_{-\infty}^{t'} \mathrm{d}s' e^{-(\hat{\nu}_0 j'^2 + \hat{\nu}_0|\boldsymbol{k}'-\boldsymbol{j}'|^2)(t'-s')} \, 2\lambda_1(k_1) M_{\beta\delta\epsilon}^+(\boldsymbol{j}') \times$$

$$\times \int \mathrm{d}^3 p' \left\{ 2\langle \psi_\delta^-(\boldsymbol{p}',s')\psi_\epsilon^+(\boldsymbol{j}'-\boldsymbol{p}',s')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}',s')\rangle_c \right.$$

$$\left. + \langle \psi_\delta^+(\boldsymbol{p}',s')\psi_\epsilon^+(\boldsymbol{j}'-\boldsymbol{p}',s')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}',s')\rangle_c \right\}, \quad (6.27)$$

$$\langle \psi_\delta^+(\boldsymbol{p}',t')\psi_\epsilon^+(\boldsymbol{j}'-\boldsymbol{p}',t')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}',t')\rangle_c =$$

$$= \int_{-\infty}^{t'} \mathrm{d}s' e^{-(\hat{\nu}_0 p'^2 + \hat{\nu}_0|\boldsymbol{j}'-\boldsymbol{p}'|^2 + \hat{\nu}_0|\boldsymbol{k}'-\boldsymbol{j}'|^2)(t'-s')} \, 3\lambda_1(k_1) M_{\delta\rho\sigma}^+(\boldsymbol{p}') \times$$

$$\times \int \mathrm{d}^3 q' \left\{ 2\langle \psi_\rho^-(\boldsymbol{q}',s')\psi_\sigma^+(\boldsymbol{p}'-\boldsymbol{q}',s')\psi_\epsilon^+(\boldsymbol{j}'-\boldsymbol{p}',s')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}',s')\rangle_c \right.$$

$$\left. + \langle \psi_\rho^+(\boldsymbol{q}',s')\psi_\sigma^+(\boldsymbol{p}'-\boldsymbol{q}',s')\psi_\epsilon^+(\boldsymbol{j}'-\boldsymbol{p}',s')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}',s')\rangle_c \right\}, \quad (6.28)$$

and so on.

If we then define, in direct analogue to equations (1.63) and (1.74),

$$\langle \psi_\alpha(\boldsymbol{k}',t')\psi_\beta(\boldsymbol{j}',t')\rangle = D_{\alpha\beta}(\boldsymbol{k}')\hat{Q}(k')\delta(\boldsymbol{k}'+\boldsymbol{j}'), \quad (6.29)$$

and

$$\langle \psi_\alpha(\boldsymbol{k}',t')\psi_\beta(\boldsymbol{j}',t')\psi_\gamma(\boldsymbol{\ell}',t')\rangle = \hat{Q}_{\alpha\beta\gamma}(\boldsymbol{j}',\boldsymbol{\ell}')\delta(\boldsymbol{k}'+\boldsymbol{j}'+\boldsymbol{\ell}'), \quad (6.30)$$

where we have assumed to field to be stationary, we find that the CA in (6.24) can be written as the moment expansion

$$\lim_{\xi\to 0}\langle \psi_\beta^+(\boldsymbol{j}',t')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}',t')\rangle_c =$$

$$= \int_{-\infty}^{t'} \mathrm{d}s' \, e^{-(\hat{\nu}_0 j'^2 + \hat{\nu}_0|\boldsymbol{k}'-\boldsymbol{j}'|^2)(t'-s')} \lambda_1(k_1) M_{\beta\delta\epsilon}^+(\boldsymbol{j}') \int \mathrm{d}^3 p' \times$$

$$\times \left\{ 4\psi_\delta^-(\boldsymbol{p}',s') \lim_{|\boldsymbol{k}'-\boldsymbol{j}'|\to h^{-1}} D_{\epsilon\gamma}^+(\boldsymbol{k}'-\boldsymbol{j}')\hat{Q}^+(|\boldsymbol{k}'-\boldsymbol{j}'|)\delta(\boldsymbol{k}'-\boldsymbol{p}') \right.$$

$$+ \int_{-\infty}^{s'} \mathrm{d}r' e^{-(\hat{\nu}_0 p'^2 + \hat{\nu}_0|\boldsymbol{j}'-\boldsymbol{p}'|^2 + \hat{\nu}_0|\boldsymbol{k}'-\boldsymbol{j}'|^2)(s'-r')} \, 12\lambda_1^2(k_1) M_{\delta\rho\sigma}^+(\boldsymbol{p}') \times$$

$$\left. \times \int \mathrm{d}^3 q' \, \psi_\rho^-(\boldsymbol{q}',r') \lim_{\{\cdot\}\to h^{-1}} \hat{Q}_{\delta\epsilon\gamma}^+(\boldsymbol{j}'-\boldsymbol{p}',\boldsymbol{k}'-\boldsymbol{j}')\delta(\boldsymbol{k}'-\boldsymbol{j}') + \ldots \right\}.$$

$$(6.31)$$

---

[2] For clarity, in the following equations we drop the explicit dependence of $\hat{\nu}_0$ upon k' when $\hat{\nu}_0$ occurs as an exponent. The dependence upon (dimensionless) wavenumber is however implicitly assumed.

6.4 The approximations

6.4.1 Truncation of the moment expansion

Given equation (6.31), we are now in a position to start to justify the approximations we introduced in Section 4.3.3. As can be seen from this expression, the first of these approximations, namely the truncation of the moment expansion, is equivalent to neglecting the terms of order $\lambda_1^2(k_1)$ and greater in this expansion. Clearly the validity of this approximation depends upon the magnitude of $\lambda_1(k_1)$ being less than unity, but as we shall subsequently show, for our calculation to be self consistent this must be the case. In addition, given the definition of ψ_α in (6.3) it can also be seen that we must have

$$|\psi_\alpha^+(\mathbf{k}',t')|_{\text{rms}} < 1, \tag{6.32}$$

since the average magnitude of the velocity modes decreases with increasing wavenumber. Thus in a formal perturbation expansion of the ψ^+ modes, the high order terms would indeed be of lesser effect when compared to the lower orders. This gives further support to the approximation, which we shall hence assume to be valid. Next we need to consider the second approximation, the neglect of the non-linear term in performing the time integral.

6.4.2 Neglect of the non-linear term in performing the time integral

Given that we may justify truncating the moment expansion in (6.31), this leaves us with the expression for the CA term in (6.24)

$$\lambda_1(k_1)M_{\alpha\beta\gamma}^-(\mathbf{k}')\int \mathrm{d}^3 j' \lim_{\xi\to 0}\langle\psi_\beta^+(\mathbf{j}',t')\psi_\gamma^+(\mathbf{k}'-\mathbf{j}',t')\rangle_c =$$
$$= \int \mathrm{d}^3 j'\, 4\lambda_1^2(k_1)M_{\alpha\beta\gamma}^-(\mathbf{k}')M_{\beta\delta\epsilon}^+(\mathbf{j}')\lim_{\ell'\to h^{-1}}D_{\epsilon\gamma}(\ell')\hat{Q}^+(\ell') \times$$
$$\times \int_{-\infty}^{t'}\mathrm{d}s'\, e^{-\hat{\omega}_2(j',\ell')(t'-s')}\psi_\delta^-(\mathbf{k}',s'), \tag{6.33}$$

where $\boldsymbol{\ell'} = \boldsymbol{k'} - \boldsymbol{j'}$, $\hat{\omega}_2(j',\ell') = \hat{\nu}_0(k')j'^2 + \hat{\nu}_0(k')\ell'^2$, and where we have also performed the integral over $\boldsymbol{p'}$. Following the approach of Section 4.3.3, the right hand side of this expression may be rewritten as

$$\text{RHS} = \int_{-\infty}^{t'} \mathrm{d}s' e^{-\hat{\omega}_2(j',\ell')(t'-s')} B_{\alpha\delta}(\boldsymbol{k'})\psi_\delta^-(\boldsymbol{k'},s'), \qquad (6.34)$$

where

$$B_{\alpha\delta}(\boldsymbol{k'}) = 4 \int \mathrm{d}^3 j' M_{\alpha\beta\gamma}^-(\boldsymbol{k'})M_{\beta\delta\epsilon}^+(\boldsymbol{j'}) \lim_{\ell'\to h^{-1}} D_{\epsilon\gamma}^+(\boldsymbol{\ell'})\hat{Q}^+(\ell). \qquad (6.35)$$

We may then change the variable of integration to $\tau' = t' - s'$, and, invoking isotropy, rewrite

$$B_{\alpha\delta}(\boldsymbol{k'})\psi_\delta^-(\boldsymbol{k'},t'-\tau') = \frac{1}{2}B_{\delta\delta}(\boldsymbol{k'})\psi_\alpha^-(\boldsymbol{k'},t'-\tau'), \qquad (6.36)$$

to leave us with

$$\lambda_1(k_1)M_{\alpha\beta\gamma}^-(\boldsymbol{k'}) \int \mathrm{d}^3 j' \lim_{\xi\to 0}\langle\psi_\beta^+(\boldsymbol{j'},t')\psi_\gamma^+(\boldsymbol{k'}-\boldsymbol{j'},t')\rangle_c =$$
$$-\lambda_1^2(k_1) \int \mathrm{d}^3 j' \lim_{\ell'\to h^{-1}} L(\boldsymbol{k'},\boldsymbol{j'})\hat{Q}^+(\ell) \int_0^\infty \mathrm{d}\tau' \, e^{-\hat{\omega}_2(j',\ell')\tau'}\psi_\alpha^-(\boldsymbol{k'},t'-\tau'), \qquad (6.37)$$

where

$$L(\boldsymbol{k'},\boldsymbol{j'}) = -2M_{\delta\beta\gamma}^-(\boldsymbol{k'})M_{\beta\delta\epsilon}^+(\boldsymbol{j'})D_{\epsilon\gamma}^+(\boldsymbol{k'}-\boldsymbol{j'}). \qquad (6.38)$$

As with (4.36), we are however still left with the question of how to perform the time integral

$$I(\boldsymbol{k'},t') = \int_0^\infty \mathrm{d}\tau' \, e^{-\hat{\omega}_2(j',\ell')\tau'}\psi_\alpha^-(\boldsymbol{k'},t'-\tau'). \qquad (6.39)$$

If we again follow the procedure of Section 4.3.3 this reduces to

$$I(\boldsymbol{k'},t') = \sum_{n=0}^\infty \frac{(-1)^n}{\hat{\omega}_2(j',\ell')^{n+1}} \left.\frac{\partial^n\psi_\alpha^-(\boldsymbol{k'},s')}{\partial s'^n}\right|_{s'=t'}, \qquad (6.40)$$

where the derivative with respect to s' may be rewritten using (6.22) as

$$\left.\frac{\partial\psi_\alpha^-(\boldsymbol{k'},s')}{\partial s'}\right|_{s'=t'} = -\hat{\nu}_0(k')k'^2\psi_\alpha^-(\boldsymbol{k'},t') + \text{NLT}, \qquad (6.41)$$

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where 'NLT' refers to the non-linear term in this expression. The second approximation we made in Chapter 4 was that the non-linear term in this expression could be neglected. If we again make this assumption, we find

$$I(\mathbf{k}',t') = \frac{\psi_\alpha^-(\mathbf{k}',t')}{\hat\nu_0(j')j'^2 + \hat\nu_0(\ell')\ell'^2}\left[\frac{(k'^2/2) + j'^2 - k'j'\mu}{j'^2 - k'j'\mu}\right], \qquad (6.42)$$

where μ is the cosine of the angle between $\mathbf{k}'$ and $\mathbf{j}'$, but we are now able to justify this approximation. To see this we first need to note that the non-linear term on the right hand side of (6.41) is of order $\lambda_1(k_1)$, thus meaning the term we have dropped from (6.42) is also of order $\lambda_1(k_1)$. If we don't neglect the non-linear term, but instead retain it as a term of order $\lambda_1(k_1)$, then substituting for $I(\mathbf{k}',t')$ in (6.37) we have

$$\lambda_1(k_1)M_{\alpha\beta\gamma}^-(\mathbf{k}')\int \mathrm{d}^3j' \lim_{\xi\to 0}\langle\psi_\beta^+(\mathbf{j}',t')\psi_\gamma^+(\mathbf{k}'-\mathbf{j}',t')\rangle_c =$$

$$-\lambda_1^2(k_1)\int \mathrm{d}^3j' \lim_{\ell'\to h^{-1}}\frac{L(\mathbf{k}',\mathbf{j}')\hat{Q}^+(\ell')}{\hat\nu_0(j')j'^2 + \hat\nu_0(\ell')\ell'^2}\left[\frac{(k'^2/2) + j'^2 - k'j'\mu}{j'^2 - k'j'\mu}\right]\psi_\alpha^-(\mathbf{k}',t')$$

$$+\,\mathcal{O}\left(\lambda_1^3(k_1)\right), \qquad (6.43)$$

where the order λ_1^3 term arises from the non-linear term in (6.41). However in our previous truncation of the moment expansion we have dropped the terms that would be of order λ_1^3 in (6.33), and thus for consistency we must do the same again. Hence, substituting (6.43) into (6.24) and neglecting terms of order λ_1^3 and greater, we are left with

$$\left(\frac{\partial}{\partial t'} + \hat\nu_1(k')k'^2\right)\psi_\alpha^-(\mathbf{k}',t') = \lambda_1(k_1)M_{\alpha\beta\gamma}^-(\mathbf{k}')\int \mathrm{d}^3j'\,\psi_\beta^-(\mathbf{j}',t')\psi_\gamma^-(\mathbf{k}'-\mathbf{j}',t')$$

$$+\,S_\alpha'^-(\mathbf{k}'|1), \qquad (6.44)$$

for $0 < k' < 1$, where

$$\hat\nu_1(k') = \hat\nu_0(k') + \delta\hat\nu_0(k'), \qquad (6.45)$$

and

$$\delta\hat\nu_0(k') = \lambda_1^2(k_1)\frac{1}{k'^2}\int \mathrm{d}^3j' \lim_{\ell'\to h^{-1}}\frac{L(\mathbf{k}',\mathbf{j}')\hat{Q}^+(\ell')}{\hat\nu_0(j')j'^2 + \hat\nu_0(\ell')\ell'^2}\left[\frac{(k'^2/2) + j'^2 - k'j'\mu}{j'^2 - k'j'\mu}\right]$$

$$+\,\mathcal{O}\left(\lambda_1^3(k_1)\right), \qquad (6.46)$$

for $1 < k',\ell' < h^{-1}$.

6.5 The recursion relation

Given these expressions it can be easily shown that this approach gives an identical result to our earlier RG calculation. To see this we first note that from (6.29) and (B.8) we have

$$\hat{Q}(k') = \frac{Q(k)}{k_1^3 V^2(k_1)}. \tag{6.47}$$

If we then take the energy spectrum to be described by equation (1.103), that is

$$E(k) = \alpha \varepsilon^{2/3} k^{-5/3} F(k/k_d),$$

this may be rewritten as

$$\hat{Q}(\ell') = \frac{1}{k_1^3 V^2(k_1)} \frac{\alpha \varepsilon^{2/3}}{4\pi} k_1^{-11/3} \ell'^{-11/3} F(k_1 \ell'/k_d) \tag{6.48}$$

$$= \frac{1}{V^2(k_1)} \frac{\alpha \varepsilon^{2/3}}{4\pi} k_1^{-20/3} Q', \tag{6.49}$$

where $Q' = \ell'^{-11/3} F(k_1 \ell'/k_d)$, and substituting this into (6.46) along with (6.20) we thus find

$$\delta \hat{\nu}_0(k') = \frac{\tau^2 \alpha \varepsilon^{2/3} k_1^{4/3}}{4\pi k'^2} \int d^3 j' \lim_{\ell' \to h^{-1}} \frac{L(\boldsymbol{k'}, \boldsymbol{j'}) \hat{Q}^+(\ell')}{\hat{\nu}_0(j') j'^2 + \hat{\nu}_0(\ell') \ell'^2} \left[\frac{(k'^2/2) + j'^2 - k'j'\mu}{j'^2 - k'j'\mu} \right], \tag{6.50}$$

where for simplicity we now drop the terms of order λ_1^3 and greater from our expression. Substituting from (6.45) we then have

$$\hat{\nu}_1(k') = \hat{\nu}_0(k') + \\ + \frac{\tau^2 \alpha \varepsilon^{2/3} k_1^{4/3}}{4\pi k'^2} \int d^3 j' \lim_{\ell' \to h^{-1}} \frac{L(\boldsymbol{k'}, \boldsymbol{j'}) \hat{Q}^+(\ell')}{\hat{\nu}_0(j') j'^2 + \hat{\nu}_0(\ell') \ell'^2} \left[\frac{(k'^2/2) + j'^2 - k'j'\mu}{j'^2 - k'j'\mu} \right], \tag{6.51}$$

and if we further note that both terms on the right hand side of this expression must scale in the same manner since $\hat{\nu}_0$ is a dimensionless function, this implies

$$\tau(k_1) = \frac{1}{\alpha^{1/2} \varepsilon^{1/3} k_1^{2/3}}. \tag{6.52}$$

From (6.21) we then find

$$\nu_0(k_1 k') = \alpha^{1/2} \varepsilon^{1/3} k_1^{-4/3} \hat{\nu}_0(k'). \tag{6.53}$$

However, in this expression $\hat{\nu}_0(k')$ is defined on the interval $0 < k' < h^{-1}$ whereas $\hat{\nu}_1(k')$ in (6.51) is defined on $0 < k' < 1$. In order that we have a recursion relation in which both sides are defined on the same interval we would prefer to rewrite the right hand side of (6.53) in terms of a new variable defined on $0 < k' < 1$, and in addition we would prefer the expression to depend on k_0 rather than k_1 as this would make our labelling more consistent. If we note that $k_1 = hk_0$, this is achieved by introducing a new variable $\tilde{\nu}_0$ defined by

$$\tilde{\nu}_0(\tilde{k}) = h^{-4/3} \hat{\nu}_0(\tilde{k}/h), \tag{6.54}$$

where $\tilde{k}$ is a dummy variable defined on $0 < \tilde{k} < 1$. Re-expressing (6.53), we then have

$$\nu_0(k_0 \tilde{k}) = \alpha^{1/2} \varepsilon^{1/3} k_0^{-4/3} \tilde{\nu}_0(\tilde{k}) \tag{6.55}$$

By an obvious analogue we also have

$$\nu_1(k_1 k') = \alpha^{1/2} \varepsilon^{1/3} k_1^{-4/3} \tilde{\nu}_1(k'), \tag{6.56}$$

where it should be noted that $\tilde{\nu}_1$ and $\hat{\nu}_1$ are identical since they are defined on the same interval. Thus from (6.51), (6.52) and (6.55) we have the final recursion relation

$$\tilde{\nu}_1(k') = h^{4/3} \tilde{\nu}_0(hk') + h^{-4/3} \delta\tilde{\nu}_0(k'), \tag{6.57}$$

where $\delta\tilde{\nu}_0(k')$ is as defined in (5.32) for the case $n = 0$. As can be easily seen, we have thus regained the same results as our earlier calculation.

6.6 Magnitude of the local Reynolds number

As we noted in Section 6.4.1, for our two approximations to be valid we require the magnitude of $\lambda_1(k_1)$ to be less than unity. Thus far we have simply assumed

this to be the case, but given (6.52) we are now in a position to show this to be true. To see this, we start by substituting (6.52) into (6.20) to obtain

$$\lambda_1(k_1) = \frac{k_1^{10/3}}{\alpha^{1/2}\varepsilon^{1/3}}V(k_1). \tag{6.58}$$

If we then note that from (B.23) we have

$$E(k_1) = 2\pi k_1^5 V^2(k_1), \tag{6.59}$$

this implies that

$$\lambda_1(k_1) = \frac{k_1^{5/6}E(k_1)^{1/2}}{(2\pi\alpha)^{1/2}\varepsilon^{1/3}}. \tag{6.60}$$

An initial estimate for the magnitude of $\lambda_1(k_1)$ can be found if we take the Kolmogorov spectrum as providing an upper bound on the energy spectrum for all wavenumbers. Doing this we find

$$\lambda_1(k_1) \leq \left(\frac{1}{2\pi}\right)^{1/2} \approx 0.4, \tag{6.61}$$

regardless of the value of k_1, and since this value is less than unity it would thus appear that our neglect of higher order terms in the moment expansion is valid. In the worst case scenario, that is if we have the Kolmogorov spectrum for all k, we are neglecting terms of order 0.16 with respect to terms of order 0.4. However in reality the value of energy spectrum at k_1 is likely to be far smaller than that suggested by the Kolmogorov spectrum. This point is illustrated in Figure 6.1, which was obtained by substituting into (6.60) the energy spectrum obtained in the DNS of Young [70], the model spectra of Pao [78] and Qian [79], and the model spectrum obtained in Appendix B. In interpreting this figure it has to be remembered that k_1 is in fact a dependent variable, its value depending upon both k_0 and the choice of bandwidth η, but as can be readily seen here, for any k_1 in the vicinity of k_d the local Reynolds number is far less than 0.4. We amplify this point in Table 6.1, where explicit values for $\lambda_1(k_1)$ are given at various k_1 for each of the energy spectrum models.
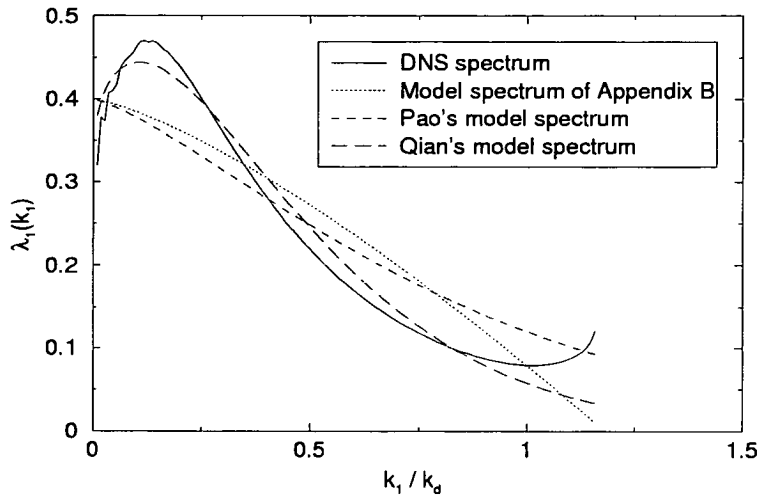
Figure 6.1: Variation of the local Reynolds number with the value of k_1, calculated for various model energy spectra: The DNS spectrum of Young [70], the model spectrum derived in Appendix B, and the model spectra of Pao [78] and Qian [79]. It should be noted that the upturn in the result obtained from the DNS spectrum is due to numerical errors brought about by truncating the system and should not be viewed as having any physical significance. Similarly the low wavenumber peaks in the values obtained using both DNS and Qian's model, which take a value greater than 0.4 and hence indicate a value of $E(k)$ greater than that predicted by the Kolmogorov spectrum, are most likely due to the numerical forcing, Qian's model being an empirical fit to DNS data.

		k_1/k_d				
		0.01	0.25	0.50	0.75	1.00
Model	DNS	0.32	0.39	0.22	0.12	0.08
	Appendix B model	0.40	0.35	0.27	0.18	0.08
	Pao's model	0.40	0.33	0.25	0.18	0.12
	Qian's model	0.38	0.39	0.24	0.13	0.06

Table 6.1: The value of the local Reynolds number at various values of k_1 for the energy spectrum obtained in the DNS of Young, and for the model spectra of Pao, Qian and Appendix B. For example, if the choices of η and k_0 are such that $k_1/k_d = 0.5$, then Qian's model gives $\lambda_1(k_1) = 0.24$.

Given that we expect our RG calculation to fail in both the narrow ($\eta \to 0$) and wide ($\eta \to 1$) bandwidth limits, if we follow Young and assume $k_0 \approx 1.2k_d$, then the lowest value of k_1 we would encounter in our calculations would be roughly $0.5k_d$, corresponding to a bandwidth of $\eta \approx 0.6$. At this wavenumber, $\lambda_1(k_1)$ will take a value of approximately 0.25, and hence in making our approximation we will be neglecting terms of order 0.06. This situation will improve yet further as the bandwidth is decreased, that is as k_1 becomes larger. Hence, in truncating the moment expansion at lowest order, we are neglecting terms approximately one order of magnitude smaller than those which we include. Given this fact, it does indeed seem reasonable to conclude that our truncation is legitimate.

6.7 Evolution of the local Reynolds number as the Renormalization Group calculation proceeds

Following the same approach as we have already carried out, it is a simple task to extend our calculation to rescale and eliminate further shells, and hence form an RG calculation. In doing this we will obtain identical expressions to those found in Chapter 5 for both the dimensionless recursion relation and the viscosity increment, namely equations (5.32) and (5.33), and also an identical expression for the Kolmogorov constant, Equation (5.37). Hence, we will also be able to regain all of Chapter 5's figures. However in addition we also find a generalisation of (6.60) for the local Reynolds number, that is the expansion parameter, on the nth iteration:

$$\lambda_n(k_n) = \frac{k_n^{5/6} E(k_n)^{1/2}}{(2\pi\alpha)^{1/2}\varepsilon^{1/3}}. \tag{6.62}$$

If we numerically calculate the value of the local Reynolds number on each iteration using the model of Pao [78] discussed in Appendix B,[3] we find the results

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[3]It was decided to use Pao's model for the energy spectrum since this is the better known of the two analytical spectrum models we consider in this thesis, the DNS and Qian's model
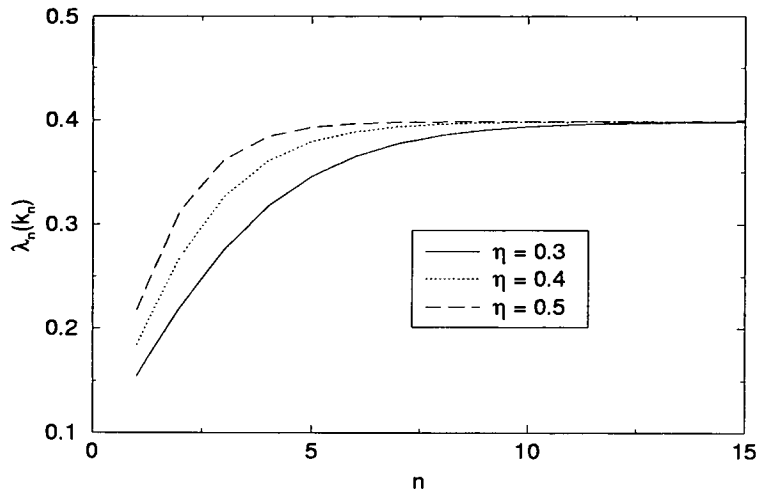
Figure 6.2: Evolution for various bandwidths of the local Reynolds number during the RG calculation as calculated using Pao's model [78] to represent the energy spectrum. Values were calculated assuming that $\alpha = 1.6$ and $k_0 = 1.2 k_d$.

illustrated in Figure 6.2. Regarding the actual RG calculation, with a bandwidth of $\eta = 0.3$ the fixed point is reached after 11 iterations, while $\eta = 0.4$ and $\eta = 0.5$ reach the fixed point after 7 and 6 iterations respectively. As can be seen in the figure, after these number of iterations the local Reynolds number reaches a fixed point (within a tolerance of $\pm 1\%$) regardless of the choice of bandwidth. Thus at the fixed point of the RG calculation we have

$$\lambda_n(k_n) = \lambda_{n+1}(k_{n+1}) = \lambda_N(k_N), \tag{6.63}$$

and hence from (6.62) it follows that

$$k_{n+1}^{5/3} E(k_{n+1}) = k_n^{5/3} E(k_n), \tag{6.64}$$

meaning that at the fixed point we have

$$E(k_N) \propto k_N^{-5/3}, \tag{6.65}$$

that is the same dependence on wavenumber as the Kolmogorov spectrum. If we further note that the magnitude of $\lambda_N(k_N)$ is approximately 0.4, then from

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not being used due to the likely influence of the artificial forcing at the low wavenumbers we achieve after several iterations of the RG algorithm.

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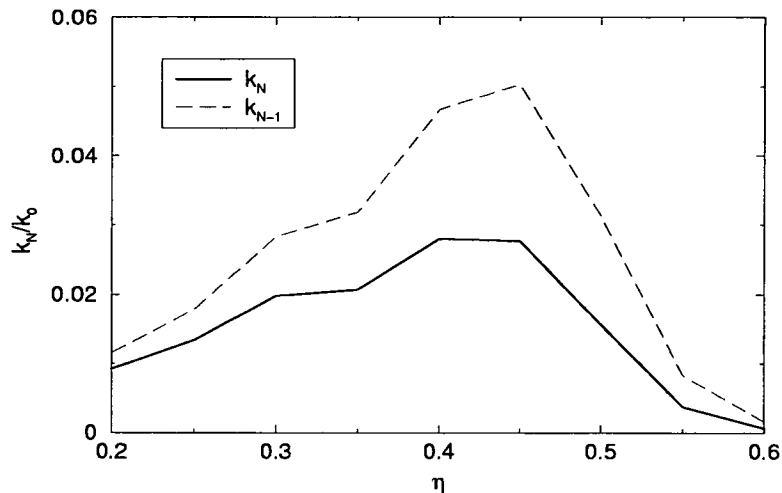
Figure 6.3: The variation of the fixed point wavenumber $k_N = (1 - \eta)^N k_0$ with bandwidth η, along with the value of the cutoff wavenumber on the previous iteration k_{N-1}.

(6.61) it follows that at the fixed point we have indeed obtained the Kolmogorov spectrum.

A further check regarding the consistency of this statement can be made if we consider the results illustrated in Figure 6.3, which were calculated using the number of iterations taken to reach the fixed point for each bandwidth η. The values for k_{N-1} are also plotted since we would expect the top of the inertial range to lie somewhere between k_{N-1} and k_N, k_N merely denoting the first cutoff wavenumber that actually lies within this scaling region, and given this figure it would seem reasonable that the top of the inertial range occurs at a wavenumber of around $0.02k_0$. If we then assume that $k_0 \approx 1.2k_d$, this implies a wavenumber of approximately $0.03k_d$. This value compares well to that found in the DNS of Young [70, Section 5.3.2], where the inertial range was found to lie about a wavenumber of approximately $0.04k_d$, and would thus appear to give further support to our conclusion that the fixed point corresponds to the onset of Kolmogorov scaling.

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Chapter 7

Large-eddy simulation using the Renormalization Group sub-grid model

7.1 Introduction

As we first alluded to in Section 1.8, the RG calculation we have carried out provides us with one approach to obtaining the eddy-viscosity required in a large eddy simulation (LES). In this chapter we shall consider how we actually perform such a simulation and the results of some preliminary calculations, comparing the results obtained using our RG eddy-viscosity with those obtained using alternative eddy-viscosity models, that found using the test-field model (TFM) of Kraichnan [80], which is a modified version of the direct interaction approximation (DIA), and that generated from a 256^3 DNS velocity field, along with the results obtained in a 256^3 DNS with identical parameters.

7.2 Large-eddy simulations

The idea of a large-eddy simulation was first put forward by Smagorinsky [81], who modelled the general circulation of the atmosphere on a finite-difference grid

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(in configuration space) and represented the drain of energy to scales smaller than the grid spacing using a subgrid model based upon a Heisenberg-type effective viscosity. The term 'large-eddy simulation' itself was, however, first coined by Leonard [82], who also introduced the idea of filtering as a formal convolution operation.

Given a (real space) simulation taking place on a grid with mesh spacing Δx, we first define the subgrid scales to be those scales with wavelength less than Δx. We then define the large (resolved) scales by the general filtering operation

$$
\begin{aligned}
\tilde{u}_\alpha(\boldsymbol{x},t) &= \int \mathrm{d}^3y\, G(\boldsymbol{x}-\boldsymbol{y})u_\alpha(\boldsymbol{y},t) \\
&= \int \mathrm{d}^3y\, G(\boldsymbol{y})u_\alpha(\boldsymbol{x}-\boldsymbol{y},t),
\end{aligned} \tag{7.1}
$$

which we write in contracted form as

$$
\tilde{u}_\alpha(\boldsymbol{x},t) = (G * u_\alpha)(\boldsymbol{x},t) \tag{7.2}
$$

The subgrid velocity field $u'_\alpha(\boldsymbol{x},t)$ can then be defined to satisfy

$$
u_\alpha(\boldsymbol{x},t) = \tilde{u}_\alpha(\boldsymbol{x},t) + u'_\alpha(\boldsymbol{x},t). \tag{7.3}
$$

Using integration by parts it can be shown [82] , provided only that u_α vanishes on the boundaries, that the filtering operation defined in equation (7.1) commutes with both spatial and temporal derivatives. Accordingly, if we apply the filtering to the continuity equation, equation (1.30), we find

$$
G * \frac{\partial u_\alpha}{\partial x_\alpha} = \frac{\partial \{G * u_\alpha\}}{\partial x_\alpha} = 0, \tag{7.4}
$$

and hence we have

$$
\frac{\partial \tilde{u}_\alpha(\boldsymbol{x},t)}{\partial x_\alpha} = 0. \tag{7.5}
$$

Similarly, if we apply the filter to the NSE we find

$$
\frac{\partial \tilde{u}_\alpha(\boldsymbol{x},t)}{\partial t} + \frac{\partial A_{\alpha\beta}(\boldsymbol{x},t)}{\partial x_\beta} = -\frac{\partial \tilde{p}(\boldsymbol{x},t)}{\partial x_\alpha} + \nu_0 \nabla^2 \tilde{u}_\alpha(\boldsymbol{x},t), \tag{7.6}
$$

where

$$A_{\alpha\beta}(\boldsymbol{x},t) = (G * \{u_\alpha u_\beta\})(\boldsymbol{x},t) \tag{7.7}$$

denotes the effect of the filter upon the non-linear term. Using (7.3), this can be rewritten as

$$
\begin{aligned}
A_{\alpha\beta}(\boldsymbol{x},t) &= (G * \{\tilde{u}_\alpha \tilde{u}_\beta + u'_\alpha \tilde{u}_\beta + \tilde{u}_\alpha u'_\beta + u'_\alpha u'_\beta\})(\boldsymbol{x},t) \\
&= (G * \{\tilde{u}_\alpha \tilde{u}_\beta\})(\boldsymbol{x},t) + T_{\alpha\beta}(\boldsymbol{x},t),
\end{aligned} \tag{7.8}
$$

where the *subgrid stress tensor*

$$T_{\alpha\beta}(\boldsymbol{x},t) = (G * \{u'_\alpha \tilde{u}_\beta + \tilde{u}_\alpha u'_\beta + u'_\alpha u'_\beta\})(\boldsymbol{x},t) \tag{7.9}$$

contains all the effects of the subgrid scales. Since only the terms involving $\tilde{u}_\alpha(\boldsymbol{x},t)$ alone are explicitly simulated, it is this latter term which we must model in a large-eddy simulation. There does however also remain the technical question of how to compute $G * \{\tilde{u}_\alpha \tilde{u}_\beta\}$. To do this, Leonard re-wrote the expression as

$$(G * \{\tilde{u}_\alpha \tilde{u}_\beta\})(\boldsymbol{x},t) = \tilde{u}_\alpha(\boldsymbol{x},t)\tilde{u}_\beta(\boldsymbol{x},t) + L_{\alpha\beta}(\boldsymbol{x},t), \tag{7.10}$$

where

$$L_{\alpha\beta}(\boldsymbol{x},t) = (G * \{\tilde{u}_\alpha \tilde{u}_\beta\})(\boldsymbol{x},t) - \tilde{u}_\alpha(\boldsymbol{x},t)\tilde{u}_\beta(\boldsymbol{x},t) \tag{7.11}$$

is referred to as the *Leonard stress tensor*.

Substituting (7.8) and (7.10) into (7.6) we are then left with

$$\frac{\partial \tilde{u}_\alpha}{\partial t} + \frac{\partial(\tilde{u}_\alpha \tilde{u}_\beta)}{\partial x_\beta} = -\frac{\partial \tilde{p}}{\partial x_\alpha} - \frac{\partial L_{\alpha\beta}}{\partial x_\beta} - \frac{\partial T_{\alpha\beta}}{\partial x_\beta} + \nu_0 \nabla^2 \tilde{u}_\alpha. \tag{7.12}$$

The only question now remaining is how to deal with the two stress terms in a practical simulation.

7.2.1 The Smagorinsky model for the subgrid stress

One way in which we may deal with the subgrid stress in real space (and the only approach which we shall consider here) is to introduce the model of Smagorinsky

[81]. This relies upon the traditional analogy between turbulence effects and molecular properties. Accordingly, by analogy with equation (1.4) we write the subgrid stress tensor in terms of the explicit scales as

$$T_{\alpha\beta}(\boldsymbol{x}, t) = -\nu_s \left(\frac{\partial \tilde{u}_\alpha(\boldsymbol{x}, t)}{\partial x_\beta} + \frac{\partial \tilde{u}_\beta(\boldsymbol{x}, t)}{\partial x_\alpha} \right), \tag{7.13}$$

where, on dimensional grounds, the subgrid effective viscosity takes the form

$$\nu_s = (c\Delta x)^2 \tilde{S}^{1/2}, \tag{7.14}$$

where c is a constant and

$$\tilde{S} = \frac{\partial \tilde{u}_\alpha}{\partial x_\beta} \left(\frac{\partial \tilde{u}_\alpha}{\partial x_\beta} + \frac{\partial \tilde{u}_\beta}{\partial x_\alpha} \right). \tag{7.15}$$

As was shown by Lilly [83], an approximate value for the constant may then be obtained by assuming that $k \sim 1/\Delta x$ lies in a region with Kolmogorov scaling for the energy spectrum (i.e. $E(k) \propto k^{-5/3}$). If this is the case, then if we adjust c so that the subgrid dissipation rate is equal to ε, we find

$$c \approx \frac{1}{\pi} \left(\frac{2}{3\alpha} \right)^{3/4}, \tag{7.16}$$

where α is the Kolmogorov constant.

7.2.2   The Leonard stress

The Leonard stress is somewhat easier to consider than the subgrid stress, since this only depends upon the choice of filter function. In particular, $L_{\alpha\beta}$ can vanish identically if we make the appropriate choice of G. This requires (c.f. (7.11))

$$(G * \{\tilde{u}_\alpha \tilde{u}_\beta\})(\boldsymbol{x}, t) = \tilde{u}_\alpha(\boldsymbol{x}, t)\tilde{u}_\beta(\boldsymbol{x}, t), \tag{7.17}$$

which may be alternatively written as

$$G * \tilde{F}_{\alpha\beta} = G * \{G * F_{\alpha\beta}\} = \tilde{F}_{\alpha\beta} \tag{7.18}$$
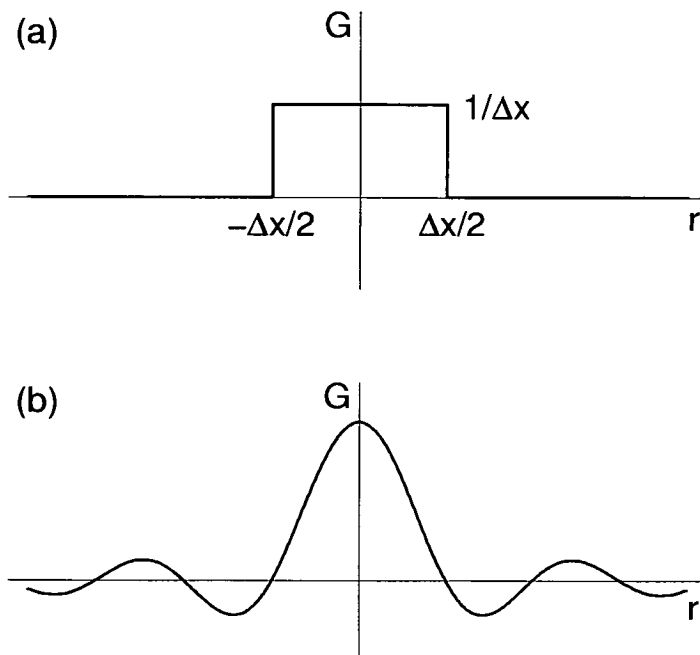
Figure 7.1: Two filter functions used to define the resolved scales in real space large-eddy simulations: (a) the 'top-hat' function, and (b) the sinc function $G = 2(\pi r)^{-1} \sin(\pi r/\Delta x)$.

if we make the replacement $\tilde{u}_\alpha(\boldsymbol{x}, t)\tilde{u}_\beta(\boldsymbol{x}, t) = \tilde{F}_{\alpha\beta}(\boldsymbol{x}, t)$. To obtain a suitable form for G, we then Fourier transform G into $g(k)$ and F into $f(k)$. From the convolution theorem (see Appendix A) we then have

$$g\tilde{f}_{\alpha\beta} = g^2 f_{\alpha\beta} = \tilde{f}_{\alpha\beta}, \qquad (7.19)$$

and hence we require G to be such that $g^2 = g = 0$ or 1.

We can satisfy this condition using one of the two filter functions illustrated in Figure 7.1. If we chose to use the first of these filters, Figure 7.1(a), and let Δx tend to infinity, then g (that is the Fourier transform of G) becomes a Dirac delta function and the condition is satisfied, a result which is unsurprising since in this instance (7.1) becomes a spatial average. If however we use the filter in Figure 7.1(b), we obtain a far more interesting result since the Fourier transform of this function (i.e. $g(k)$) is the unit top-hat function (in k-space rather than x-space), which clearly must satisfy the condition $g(k) = 0$ or 1. It is this 'spectrally sharp

filter' which we have been using in our earlier decompositions of the (k-space) NSE into low and high wavenumber regions, and it hence this which we use in our (spectral) large-eddy simulation.

7.3   The large-eddy simulation code

Our LES is based on the earlier DNS of Young [70]. The original DNS code was written to run on the Cray T3D supercomputer administered by the Edinburgh Parallel Computing Centre and has been well validated [70], yielding results in good agreement with alternative simulations, for instance those of Vincent and Meneguzzi [84], Yeung and Zhou [16] and Sreenivasan [85]. Given this, we shall not consider the technical aspects of the program, save to note that the original DNS is a standard pseudospectral simulation and uses a second order Runge-Kutta scheme to integrate the non-linear term, whilst the viscous term is treated analytically using an integrating factor. Partial dealiasing is achieved through the use of a random shifting method [86]. For further technical details, we refer the reader to the thesis of Young [70].

The modifications required to turn enable this code to be used in an LES were minimal, we simply needed to add an additional subprogram to account for the fact that the viscosity in an LES is a non-constant function of wavenumber, that is a subprogram which returned a value for $\nu_N(k)$ given k. Calling this subprogram wherever the DNS uses the molecular viscosity, we then have a functioning (spectral) LES program. Preliminary calculations were run for the three eddy-viscosity models mentioned in Section 7.1, that obtained using our RG calculation, that obtained by Kraichnan using the TFM [80] and that generated empirically from a DNS. These calculations were performed on a 32^3 grid using the same parameters as would be used in a 256^3 DNS (i.e. $\varepsilon = 0.149$ and $\nu_0 = 10^{-3}$) and hence the results should be comparable to those of the DNS.

7.4 Comparison of the eddy-viscosity models

Prior to running the LES code, the eddy-viscosities first had to be generated. This was simplest in the case of the RG eddy-viscosity, which was generated using equation (5.28) along with a scaled eddy-viscosity and corresponding value of α. Both of these are output by the RG program. The only question regarding the generation of this eddy-viscosity was what bandwidth the scaled eddy-viscosity should correspond to. Since it was found to lie well within the plateau region of Figure 5.7, we chose to use a value of $\eta = 0.4$.

The TFM eddy-viscosity was found by numerically integrating the expression [80]

$$\nu(k|k_c) = \left(\frac{\alpha\varepsilon^{1/3}}{4\pi\beta}\right)\frac{1}{k^2}\int_{j\geq k_c}\mathrm{d}^3j\ L(\boldsymbol{k},\boldsymbol{j})\frac{k^{11/3}}{|\boldsymbol{k}-\boldsymbol{j}|^{11/3}}\left[\frac{k^{-11/3}-j^{-11/3}}{k^{2/3}+j^{2/3}+|\boldsymbol{k}-\boldsymbol{j}|^{2/3}}\right],$$
$$(7.20)$$

for $k \leq k_c$, where $\beta = 0.19\alpha^2$. In this case we assumed a value for the Kolmogorov constant of $\alpha = 1.6$, and it should also be noted that to obtain (7.20) we need to assume a Kolmogorov spectrum for all wavenumbers.

In contrast to these analytic approaches, the final eddy-viscosity, which was computed on our behalf by Alistair Young, was obtained by truncating a 256^3 DNS velocity field and then calculating the eddy viscosity which modelled the effect of the subgrid terms (for details see [70], Section 6.3.1.). To obtain a smooth function for the eddy-viscosity, this numerical data was then fitted to the curve, which is a slightly modified form of that given by Lesieur and Rogallo [87],

$$\nu(k|k_c) = a_0 + a_1\exp\left(-a_2\left\{\frac{k_c}{k}\right\}\right), \qquad (7.21)$$

the parameters a_0, a_1 and a_2 being found using a least-squares method. This empirical function was used as the eddy-viscosity model.

The eddy-viscosities are illustrated in Figure 7.2, which, as in Section 5.4, shows that the value of the RG eddy-viscosity decreases with increasing wavenumber, unlike the alternative two models. As we discussed in Section 5.4, this downturn
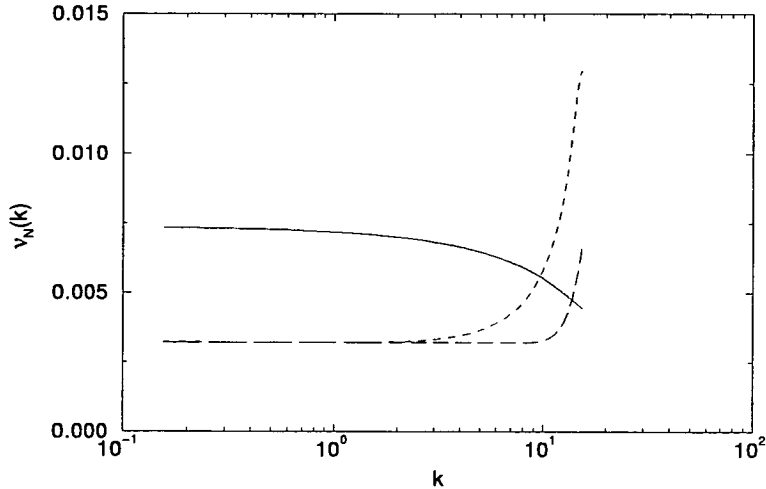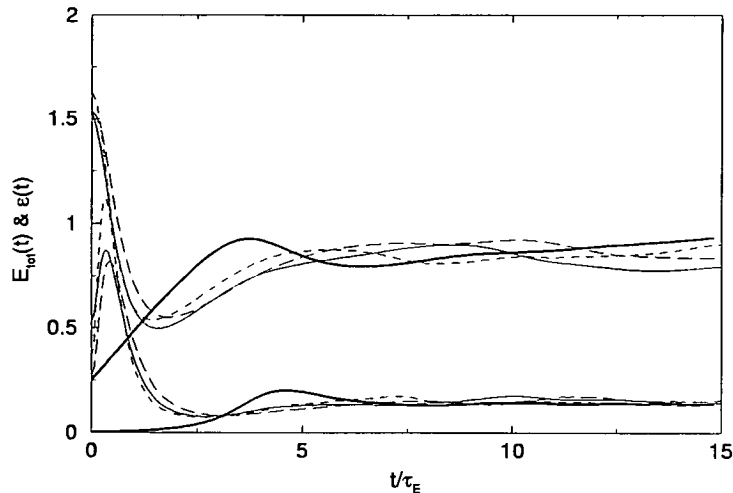
Figure 7.2: The eddy-viscosities used in our 32^3 LES: (a) The RG model (———), (b) The Kraichnan TFM model (- - - - -), and (c) The eddy-viscosity generated from a 256^3 DNS velocity field (— — —).

near the cutoff is almost certainly due to our neglect of the $S^-(k|k_c)$ term. If included this would most likely lead to us obtaining an eddy-viscosity with the same cusp as displayed by the alternatives. Here we are instead interested in how well the RG eddy-viscosity compares, as it stands, when used in our LES to obtain, for instance, the energy spectrum.

7.5 Results and Discussion

The results we obtained from our large-eddy simulations are illustrated in Figures 7.3 to 7.8. The first set of results, Figure 7.3, illustrates the time evolution of both the total energy in the system $E_{tot}(t)$ and the dissipation rate $\varepsilon(t)$. This clearly illustrates that all four simulations reach a stationary state after around 5-10 eddy-turnover times[1], with both the total energy and dissipation rate tending to mean values about which they only fluctuate slightly. It is also found, as

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[1]Note that in all the following figures we scale time axes in terms of the *eddy-turnover time* τ_E. This value, which we shall discuss further later in this section, is a measure of the typical time taken for a large scale structure in the system to undergo significant distortion due to the relative motion of its components [12]

Figure 7.3: Time evolution of the total energy $E_{tot}(t)$ (top group of lines) and dissipation rate $\varepsilon(t)$ (bottom group of lines) in our numerical simulations. Results are plotted for the 32^3 large-eddy simulations: (a) The RG model (———), (b) The Kraichnan TFM model (- - - - -), and (c) The eddy-viscosity generated from a 256^3 DNS velocity field (– – –), along with (d) The results from the 256^3 DNS (———). All these simulations used $\varepsilon = 0.149$ and $\nu_0 = 10^{-3}$.

expected, that the dissipation rate takes a value approximately identical to the rate at which is input to the system, that is $\varepsilon \approx 0.149$, for all four simulations. These results thus indicate that all the simulations are working correctly, giving us, after an initial transition period, a statistically stationary system in which energy is both put in and dissipated at the same rate, the amount of energy in the system being constant. We should also note that the evolved values for both the DNS and all the LES models are essentially the same. This most likely indicates that the system being modelled by the LES is identical to that modelled by the DNS, as we hoped. It is however worth commenting briefly on the different initial behaviour when the LES are compared to the DNS.

The differences between the simulations are entirely due to differences in the starting fields and have no physical significance. Instead, they are purely artifacts of the way the velocity fields are initially generated (for details of how this is done see Young [70]). The initially increasing value of, say, the total energy in the DNS
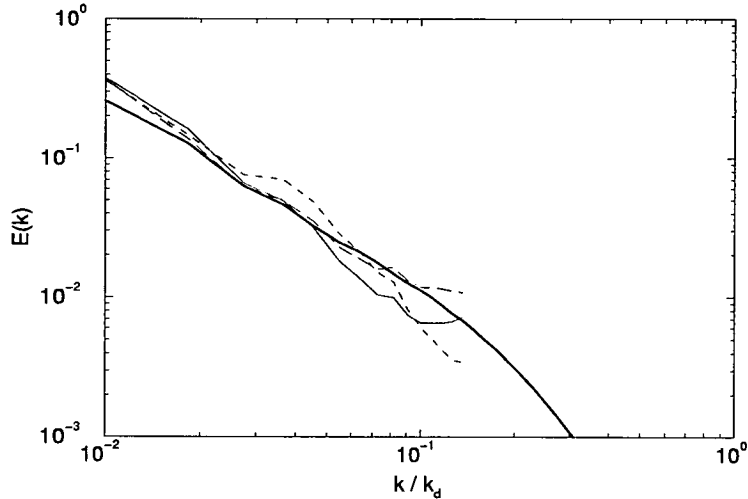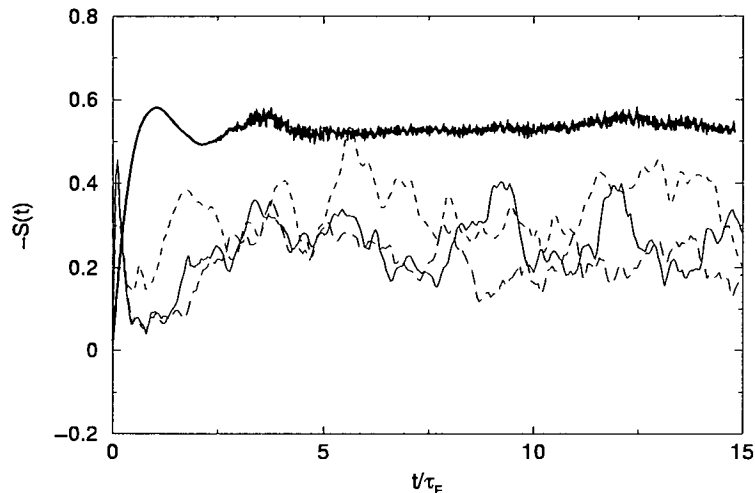
Figure 7.4: Time averaged energy spectra obtained from our simulations. Results are plotted for the 32^3 large-eddy simulations: (a) The RG model (———), (b) The Kraichnan TFM model (-----), and (c) The eddy-viscosity generated from a 256^3 DNS velocity field (— — —), along with (d) The results from the 256^3 DNS (———).

as opposed to the initial decrease exhibited by the LES, indicates only that the initial DNS velocity field had too low an energy for stationarity, whereas the initial LES velocity field had too high an energy. It is only once a stationary state has been reached that the results are physically meaningful. The same is true of all our following figures.

Our next figure, Figure 7.4, shows the time averaged energy spectra[2] obtained from the various simulations. Here it can be seen that all the LES models yield an energy spectrum which corresponds reasonably well with that of the 256^3 DNS. Again this indicates that the LES provide a reasonable model for the system simulated in the DNS. When compared to the other two LES models, it will however be noticed that there is a turn-up in the RG-model spectrum at the high wavenumber end. This is almost certainly due to the fact that the RG eddy-viscosity dissipates a smaller percentage of the total energy at high wavenumbers than is the case with the alternative models. Inspection of the single-time spectra

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[2]Note that the initial transitional period of the simulation is not contained in this or any subsequent time averages.
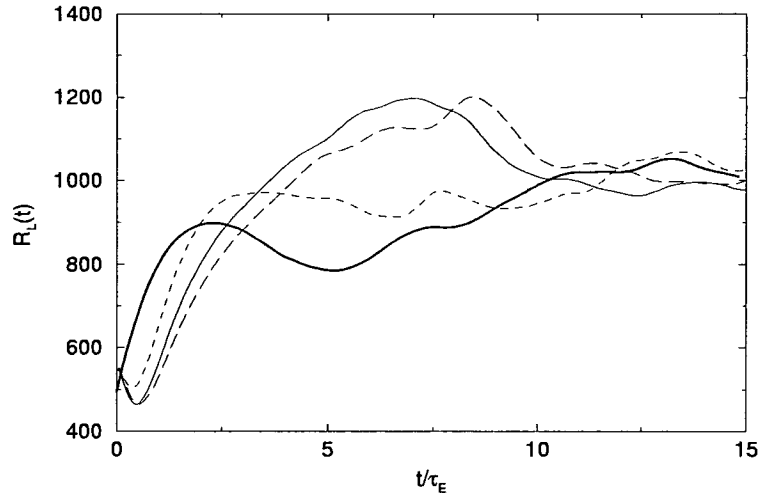
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Figure 7.5: Time evolution of the velocity derivative skewness $S(t)$ in our numerical simulations. Results are plotted for the 32^3 large-eddy simulations: (a) The RG model (———), (b) The Kraichnan TFM model (- - - - -), and (c) The eddy-viscosity generated from a 256^3 DNS velocity field (– – –), along with (d) The results from the 256^3 DNS (———).

for the RG model does, however, show no evidence that the energy in this region accumulates over time.

Figure 7.5 shows the time evolution of the (longitudinal) velocity derivative skewness S. This is defined in real space by

$$S(t) = \frac{\langle (\partial u_1(\boldsymbol{x}, t)/\partial x_1)^3 \rangle}{\langle (\partial u_1(\boldsymbol{x}, t)/\partial x_1)^2 \rangle^{3/2}}, \tag{7.22}$$

and is regarded as being one of the most sensitive turbulence parameters [7]. Here it is immediately apparent that the behaviour of the DNS differs significantly from that of the LES, the LES results (regardless of the model) both fluctuating more and having a smaller magnitude. The increased fluctuation of the LES models is probably a reflection of the sensitivity of skewness as a measure, the DNS being more stable due to the increased number of data points included in the averaging. Indeed, it has been found that a 128^3 LES using the same programs yields a skewness with a far smoother time evolution [88]. Likewise, the fact that the LES models all give a skewness of around -0.2 to -0.3, as opposed to the value of -0.5 given by this and other (see, for example, [89]) direct numerical

Figure 7.6: Time evolution of the integral Reynolds number $R_L(t)$ in our numerical simulations. Results are plotted for the 32^3 large-eddy simulations: (a) The RG model (——), (b) The Kraichnan TFM model (- - - -), and (c) The eddy-viscosity generated from a 256^3 DNS velocity field (— — —), along with (d) The results from the 256^3 DNS (——).

simulations, is easily accounted for. This is purely due to the fact that in an LES one loses the detail of the smallest scales, that is the subgrid scales. These scales give rise to a large proportion of the skewness [90] and thus it is only to be expected that an LES will return a skewness with lower magnitude than will a DNS in which they are included.

In Figures 7.6 and 7.7 we show the time evolution of the integral and microscale Reynolds numbers, $R_L(t)$ and $R_\lambda(t)$. These Reynolds numbers are based, respectively, upon the integral length scale
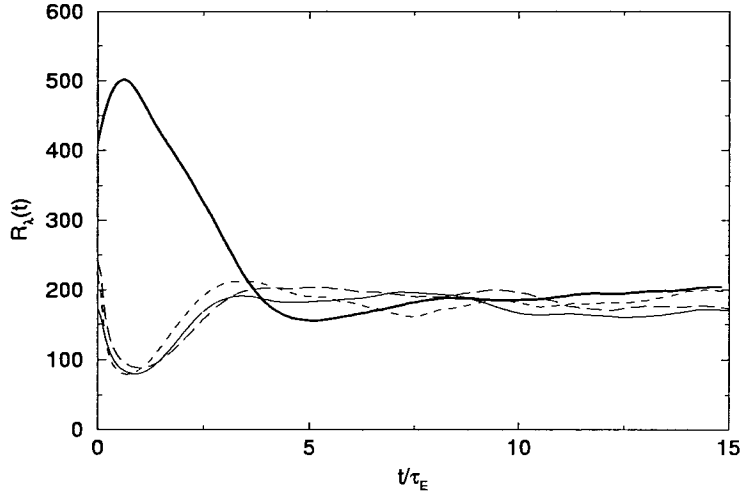
$$L(t) = \left[\frac{3\pi}{4} \int_0^\infty \mathrm{d}k\, k^{-1} E(k,t)\right] / E(k), \qquad (7.23)$$

which is taken to be representative of the large scales, and the Taylor microscale

$$\lambda(t) = \left[5E(t)/\int_0^\infty \mathrm{d}k\, k^2 E(k,t)\right]^{1/2}, \qquad (7.24)$$

which is taken to be intermediate between the energy input and dissipation scales (that is, representative of the inertial range) [91]. If we additionally introduce an

Figure 7.7: Time evolution of the microscale Reynolds number $R_\lambda(t)$ in our numerical simulations. Results are plotted for the 32^3 large-eddy simulations: (a) The RG model (———), (b) The Kraichnan TFM model (-----), and (c) The eddy-viscosity generated from a 256^3 DNS velocity field (———), along with (d) The results from the 256^3 DNS (———).

r.m.s. velocity, $u(t)$ defined by

$$u(t) = \sqrt{\frac{2}{3}E(t)}, \qquad (7.25)$$

then the integral and microscale Reynolds numbers are defined as

$$R_L(t) = \frac{L(t)u(t)}{\nu_0} \qquad (7.26)$$

and

$$R_\lambda(t) = \frac{\lambda(t)u(t)}{\nu_0}. \qquad (7.27)$$

In addition, we are now also in a position to define the previously mentioned eddy-turnover time τ_E, which is a measure of the typical time for a structure of size $L(t)$ to undergo significant distortion due to the relative motion of its components. This is defined by

$$\tau_E(t) = \frac{L(t)}{u(t)}. \qquad (7.28)$$

As can be seen in Figures 7.6 and 7.7, once the system reaches stationarity the values given by the large-eddy simulations again compare well with those given

by the DNS, with $R_L \approx 1000$ and $R_\lambda \approx 200$. It is however worth commenting on the fact that the integral Reynolds number appears to take a significantly longer time to reach its stationary state than does the microscale Reynolds number, taking around $10\tau_E$ as opposed to the $5\tau_E$ taken by R_λ. This is most likely due to the fact [7] that the small (physical) scales evolve on much faster timescales than the large scales. Since R_L depends upon larger scales than does R_λ, we should expect it to take a longer period to reach its stationary state.

Finally, on a more qualitative level, in Figure 7.8 we plot vorticity iso-surfaces for both the RG and TFM LES models in comparison to that of a 256^3 DNS truncated to 32^3. This truncation is made in order to remove the small scale structure that would otherwise be present, and hence provide a better comparison to the large-eddy simulations. As can be seen here, there is little qualitative difference between the simulations, both large-eddy simulations giving rise to similar amounts and types of structures as seen in the truncated DNS.

Aside from the figures we have just discussed, we may also obtain a more quantitative idea of how well the various LES models perform by considering the mean values of these quantities along with an estimate of the error. If we assume the error to be twice the standard deviation of the averaged data set, we obtain the results in Table 7.1. Save for the skewness values, all the LES models yield results which lie within the error bands of the DNS results, which would again confirm our belief that all the tested LES models perform well in comparison to the DNS.

In order to attempt to distinguish between the quality of the LES models, we finally compared their energy spectra to both the Kolmogorov spectrum and the time-averaged DNS spectrum using the χ^2 statistic for goodness of fit. This simply involves calculating the sum
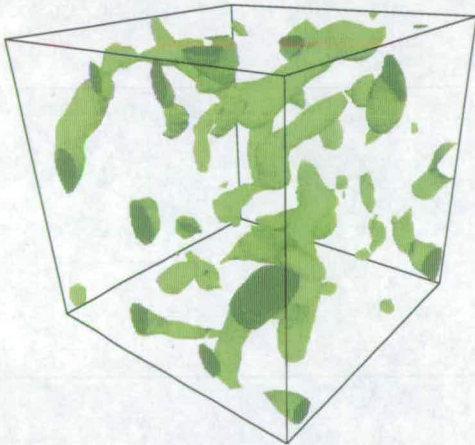
$$\chi^2 = \sum_i \frac{\left(y_i - y_i^{\text{model}}\right)^2}{\sigma_i^2}, \tag{7.29}$$

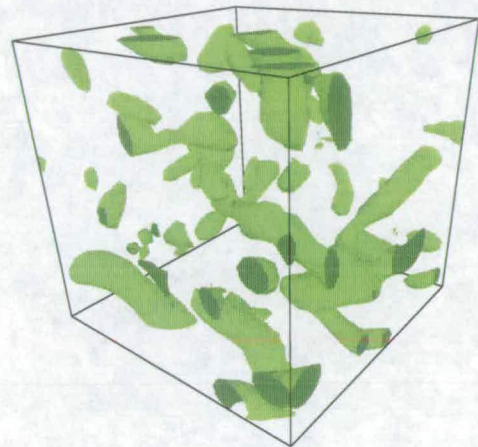where y_i is the observed value ($E(k)$ from our LES), y_i^{model} is the predicted value
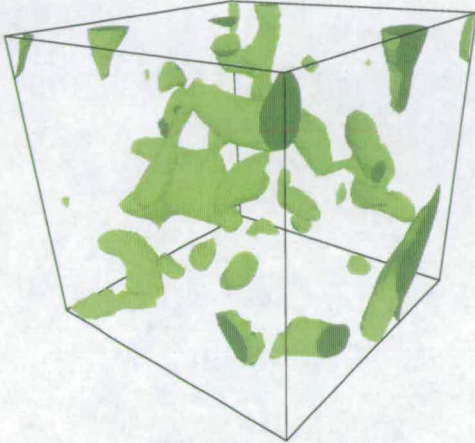
(a) The RG eddy-viscosity

(c) The truncated DNS

(b) The TFM eddy-viscosity

Figure 7.8: Vorticity iso-surfaces obtained from numerical simulations for (a) the RG eddy-viscosity, (b) the TFM eddy-viscosity and (c) the 256^3 DNS truncated onto a 32^3 grid. The plotted iso-surfaces are for a value of 55% of the maximum vorticity.

Model	RG model	TFM model	DNS model	256^3 DNS
$\bar{\varepsilon}$	0.147 ± 0.027	0.147 ± 0.034	0.150 ± 0.029	0.149 ± 0.039
$\bar{E}_{\text{tot}}$	0.832 ± 0.072	0.867 ± 0.100	0.854 ± 0.085	0.865 ± 0.083
$-\bar{S}$	0.241 ± 0.128	0.346 ± 0.137	0.215 ± 0.088	0.533 ± 0.025
$\bar{R}_L$	1060.6 ± 149.3	1031.7 ± 197.5	1023.1 ± 165.3	934.4 ± 185.5
$\bar{R}_\lambda$	177.8 ± 23.8	185.7 ± 31.9	180.8 ± 24.6	184.1 ± 27.8

Table 7.1: Time averaged values for the dissipation rate $\bar{\varepsilon}$, total energy $\bar{E}_{\text{tot}}$, skewness $\bar{S}$, integral length scale Reynolds number $\bar{R}_L$ and microscale Reynolds number $\bar{R}_\lambda$. Values are given for 32^3 large-eddy simulations using the RG eddy-viscosity, Kraichnan's TFM and the eddy-viscosity generated from a 256^3 DNS, along with results obtained from the 256^3 DNS of Young [70]. The quoted errors are for two standard deviations about the mean value.

Model	χ^2
RG model	48.6
TFM model	77.3
DNS model	88.21

Table 7.2: Goodness of fit (computed using the χ^2 statistic) of the LES models when compared to the Kolmogorov spectrum. A Kolmogorov constant of $\alpha = 1.6$ was assumed, whilst ε was taken to be the same as that used in the simulations, that is $\varepsilon = 0.149$.

we are comparing y_i to ($E(k)$ from the Kolmogorov spectrum or 256^3 DNS), and σ_i is the error on y_i. The smaller the value of χ^2 the better the data set fits the model[3].

Performing this calculation we find the results in Table 7.2, for the comparison to the Kolmogorov spectrum, and Table 7.3 for the comparison to the DNS energy spectrum. The third column in Table 7.3 is obtained from a modified form of (7.29), in which the square of the error on the DNS value is also added to the denominator in order to account for the fact that this value is also subject to uncertainty.

---

[3]Note that this statement assumes that the same number of data points are used in each calculation, as clearly an increase in this number will potentially lead to a corresponding increase in χ^2. All our calculations used 15 pairs (i.e. k and $E(k)$) of data points.

Model	χ^2	Modified χ^2
RG model	205.9	144.9
TFM model	276.3	215.2
DNS model	35.9	32.5

Table 7.3: Goodness of fit (computed using the χ^2 statistic) of the LES models when compared to the spectrum from the 256^3 DNS.

Since the TFM eddy-viscosity is calculated assuming the Kolmogorov spectrum, whilst the fixed point of the RG calculation corresponds to the onset of Kolmogorov scaling, we would expect both these models to compare favourably to the Kolmogorov spectrum. This is borne out by the results in Table 7.2, which show that both models provide a closer relation than the eddy-viscosity derived from the DNS, the RG model providing a slightly better fit. Similarly, we would expect the eddy-viscosity derived from the DNS to compare better to the DNS spectrum than either of the two analytic models, as indeed is the case. Of the analytic models, it would again appear that the RG eddy-viscosity compares better to the DNS spectrum than does the TFM model. The differences in the χ^2 values are however too insignificant to infer any superiority of one model over the other. We leave this question open for future study. We can however conclude that the RG eddy-viscosity we have obtained does give acceptable results when used in an LES.

Chapter 8

Conclusions

In this thesis we have developed and tested a Renormalization Group method for modelling turbulence in the context of homogeneous, isotropic and statistically stationary flows in incompressible fluids. This approach is based upon the earlier work of McComb *et al.* [50,51], and in particular upon the conditional averaging idea first introduced by McComb, Roberts and Watt in 1992 [65], the aim being to both improve upon this earlier theory and address the criticisms made of it.

Following an overview of the general theory of fluid turbulence, Chapter 1, in Chapter 2 we review more sophisticated theories of turbulence, starting with renormalized perturbation theories before moving onto the class of theories in which our work falls, those based upon use of the Renormalization Group. Here we critically discuss the relative merits of each of the alternative theories prior to the start of our own calculation.

We start in Chapter 3 by redefining the conditional average in terms of time-*independent* realizations as opposed to the time-*dependent* realizations used in the original theory. As discussed in the chapter itself, this redefinition is made since the ensemble as defined in the original reference [65] is likely to be either very sparse or else empty. Redefining in terms of time-independent realizations resolves this problem and has an additional immediate benefit in that it further resolves

a criticism of the earlier theory. As we discuss in Section 3.7, this redefinition of the ensemble means that the $\langle u_\beta^+(\boldsymbol{j}, t) u_\gamma^+(\boldsymbol{k} - \boldsymbol{j}, t)\rangle_c$ term, which in the McComb *et al.* theory can be considered in two ways, each of which gives a different result, can instead only be evaluated in one, well defined, manner.

Given the basic definition of the conditional average, in Chapter 4 we extend the idea in order to deal with conditional averages involving mixed products of low and high wavenumber modes. This is done by introducing the hypothesis of local chaos, which, under prescribed conditions, allows us to split such mixed moments into a product of low wavenumber modes multiplying the ordinary ensemble average of a product of high wavenumber modes. Using this hypothesis, we then proceeded to eliminate a band of high wavenumber modes, before showing in Chapter 5 how this procedure may be extended to form an RG calculation.

In performing this RG calculation, we do however have to make two approximations, one regarding the neglect of higher order terms in the moment expansion and one regarding the manner in which we perform a time integral. Both of these are unsubstantiated, save for simple physical arguments, in the theory of McComb *et al.* In Chapter 6 we justify these assumptions by introducing a similarity solution for $u_\alpha(\boldsymbol{k}, t)$. This allows us to rewrite the NSE in a dimensionless form and justify both assumptions as being equivalent to the neglect of higher order terms in an expansion based upon the local Reynolds number $\lambda_1(k_1)$. Performing the calculation in this manner leads us to a slightly different result for the viscosity increment than that obtained by McComb *et al.* We show that $\lambda_1(k_1)$ must take a value less than 0.4, and hence this truncation would seem reasonable. As an aside, in Appendix B we also show that the introduced similarity solution enables us to obtain analytically a model energy spectrum which, if subject to the same forcing and assumptions, is identical to the result derived in the RG theory of Forster, Nelson and Stephen [44].

As is discussed in Section 5.4, this RG calculation gives good results, in particular

giving a value for the Kolmogorov constant of $\alpha \approx 1.62$ over a wide range of bandwidths, in good agreement with both experiment and numerical simulations. From the RG calculation we also obtain an eddy-viscosity, which in Chapter 7 we use in a 32^3 large-eddy simulation, comparing the results obtained with this eddy-viscosity to those obtained using two alternative models, the TFM of Kraichnan and that derived from a 256^3 DNS. As can be seen in this chapter, each of these models gives results comparable to one another and to the results of a 256^3 DNS. Indeed with the simulations we have carried out it is impossible to say that any of the models considered performs significantly better than the alternatives. The question of why these considerably different eddy-viscosities should all perform in a virtually identical manner is clearly worthy of future investigation.

Appendix A

Properties of Fourier transforms

A.1 General properties

The Fourier transform of a general function $f_\alpha(\boldsymbol{x}, t)$ is defined by introducing the Fourier transform pair

$$f_\alpha(\boldsymbol{x}, t) = \int \mathrm{d}^3k \, f_\alpha(\boldsymbol{k}, t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{A.1}$$

and

$$f_\alpha(\boldsymbol{k}, t) = \left(\frac{1}{2\pi}\right)^3 \int \mathrm{d}^3x \, f_\alpha(\boldsymbol{x}, t) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{A.2}$$

With this definition, $f_\alpha(\boldsymbol{k}, t)$ is the Fourier transform (FT) of $f_\alpha(\boldsymbol{x}, t)$ and vice versa. Given equations (A.1) and (A.2) we can simply obtain several important results.

First we relate equations (A.1) and (A.2) to the Dirac delta function. To do this we substitute (A.1) into (A.2) to obtain

$$f_\alpha(\boldsymbol{k}, t) = \left(\frac{1}{2\pi}\right)^3 \int \mathrm{d}^3x \left\{ \int \mathrm{d}^3k' \, f_\alpha(\boldsymbol{k}', t) e^{i\boldsymbol{k}'\cdot\boldsymbol{x}} \right\} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{A.3}$$

$$= \left(\frac{1}{2\pi}\right)^3 \iint \mathrm{d}^3x \, \mathrm{d}^3k' \, f_\alpha(\boldsymbol{k}', t) e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}}. \tag{A.4}$$

It can then be easily seen that this statement can only hold true for arbitrary values of $\boldsymbol{k}$ provided

$$\int \mathrm{d}^3x \, e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}} = (2\pi)^3 \delta(\boldsymbol{k} - \boldsymbol{k}'). \tag{A.5}$$

Given equation (A.5) we can then obtain an expression for the FT of the product defined by

$$p_{\alpha\beta}(\boldsymbol{x}, t) = f_\alpha(\boldsymbol{x}, t) g_\beta(\boldsymbol{x}, t). \tag{A.6}$$

From (A.2), the FT of the left hand side may be written as

$$p_{\alpha\beta}(\boldsymbol{k}, t) = \left(\frac{1}{2\pi}\right)^3 \int \mathrm{d}^3 x \, p_{\alpha\beta}(\boldsymbol{x}, t) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{A.7}$$

which may be rewritten using (A.1) and (A.6) as

$$
\begin{aligned}
p_{\alpha\beta}(\boldsymbol{k}, t) &= \left(\frac{1}{2\pi}\right)^3 \int \mathrm{d}^3 x \left\{ \int \mathrm{d}^3 k' \, f_\alpha(\boldsymbol{k}', t) e^{i\boldsymbol{k}'\cdot\boldsymbol{x}} \right\} \\
&\qquad \times \left\{ \int \mathrm{d}^3 k'' \, g_\beta(\boldsymbol{k}'', t) e^{i\boldsymbol{k}''\cdot\boldsymbol{x}} \right\} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{A.8} \\
&= \left(\frac{1}{2\pi}\right)^3 \iiint \mathrm{d}^3 x \, \mathrm{d}^3 k' \, \mathrm{d}^3 k'' \, f_\alpha(\boldsymbol{k}', t) g_\beta(\boldsymbol{k}'', t) \\
&\qquad\qquad\qquad \times e^{-i(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{k}'')\cdot\boldsymbol{x}}. \tag{A.9}
\end{aligned}
$$

If we then use equation (A.5) in order to perform the integrals with respect to $\boldsymbol{x}$ and $\boldsymbol{k}''$ we obtain the final result

$$p_{\alpha\beta}(\boldsymbol{k}, t) = \int \mathrm{d}^3 k' \, f_\alpha(\boldsymbol{k}', t) g_\beta(\boldsymbol{k} - \boldsymbol{k}', t), \tag{A.10}$$

which is known as the *convolution theorem*. That is, a product in $\boldsymbol{x}$-space becomes a convolution in $\boldsymbol{k}$-space.

Finally we consider the effect of taking a derivative of $f_\alpha(\boldsymbol{x}, t)$. From (A.1) it can be easily seen that

$$\frac{\partial f_\alpha(\boldsymbol{x}, t)}{\partial x_\beta} = \int \mathrm{d}^3 k \, (ik_\beta) f_\alpha(\boldsymbol{k}, t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{A.11}$$

and this can be extended to any order of derivative to give

$$\frac{\partial^m f_\alpha(\boldsymbol{x}, t)}{\partial x_\beta^m} = \int \mathrm{d}^3 k \, (ik_\beta)^m f_\alpha(\boldsymbol{k}, t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{A.12}$$

Further details regarding the general definition and properties of the Fourier transform may be found in either Lighthill [92] or Wong [93].

A.2 The Fourier transform of $G(x, x')$

In Section 1.3 we need to know the FT of the Green's function $G(x, x')$ which satisfies Poisson's equation

$$\nabla^2 G(x, x') = \delta(x - x'). \tag{A.13}$$

By definition of the Dirac delta function, if (A.13) is defined over the entire three dimensional space then the right hand side of it must satisfy

$$\int d^3 x' \, \delta(x - x') = 1. \tag{A.14}$$

Hence its solution must be spherically symmetric in

$$r = x - x', \tag{A.15}$$

since nothing will be changed by putting x' at the origin. Thus, $G(x, x')$ must be a function of r only.

From equation (A.1), $G(r)$ is related to its Fourier transform $G(k)$ by

$$G(r) = \int d^3 k \, G(k) e^{ik \cdot r}, \tag{A.16}$$

and hence substituting this into the left hand side of equation (A.13) and using (A.5) to re-express the delta function, we are left with

$$\frac{\partial^2}{\partial r_\alpha \partial r_\alpha} \int d^3 k \, G(k) e^{ik \cdot r} = \left(\frac{1}{2\pi}\right)^3 \int d^3 k \, e^{ik \cdot r}, \tag{A.17}$$

that is,

$$-\int d^3 k \, k^2 G(k) e^{ik \cdot r} = \left(\frac{1}{2\pi}\right)^3 \int d^3 k \, e^{ik \cdot r}. \tag{A.18}$$

This must hold for arbitrary $e^{ik \cdot r}$ and thus we find that the FT of $G(r)$ is

$$G(k) = -\left(\frac{1}{2\pi}\right)^3 \left(\frac{1}{k^2}\right). \tag{A.19}$$

Appendix B

Analytic calculation of the energy spectrum

B.1 An integral equation for the energy spectrum

In order to use the similarity solution of Chapter 6 to obtain an expression describing the energy spectrum, we start by recalling that for a homogeneous and stationary velocity field we have, from equations (1.63) and (1.74),

$$\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{j},t)\rangle = Q_{\alpha\beta}(\mathbf{k})\delta(\mathbf{k}+\mathbf{j}) \tag{B.1}$$

and

$$\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{j},t)u_\gamma(\boldsymbol{\ell},t)\rangle = Q_{\alpha\beta\gamma}(\mathbf{k},\mathbf{j})\delta(\mathbf{k}+\mathbf{j}+\boldsymbol{\ell}). \tag{B.2}$$

Under the further assumption of isotropy (B.1) reduces to (see equation (1.68))

$$\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{j},t)\rangle = Q(k)D_{\alpha\beta}(\mathbf{k})\delta(\mathbf{k}+\mathbf{j}). \tag{B.3}$$

In addition, we rewrite equations (6.15) and (6.17) in terms of an arbitrary wavenumber κ, say, as

$$\mathbf{k}' = \mathbf{k}/\kappa, \tag{B.4}$$

and

$$u_\alpha(\mathbf{k},t) = V(\kappa)\psi_\alpha(\mathbf{k}',t'), \tag{B.5}$$

in order to emphasise that the calculation in this appendix is independent of the choice of this wavenumber. Using all these expressions along with (6.4), we can then obtain a relationship between $Q(k)$ and $V(k)$ thus:

1. Substitute equations (B.4) and (B.5) into equation (6.4) to obtain

$$V^2(k) = \frac{V^2(\kappa)}{k'^3} \int d^3j' \, \langle \psi_\alpha(\mathbf{k}', t') \psi_\alpha(\mathbf{j}', t') \rangle. \tag{B.6}$$

2. Substitute equations (B.4) and (B.5) into equation (B.3)

$$\kappa^3 V^2(\kappa) \langle \psi_\alpha(\mathbf{k}', t') \psi_\beta(\mathbf{j}', t') \rangle = Q(k) D_{\alpha\beta}(\mathbf{k}') \delta(\mathbf{k}' + \mathbf{j}'). \tag{B.7}$$

3. Take the trace of equation (B.7) and rearrange

$$\langle \psi_\alpha(\mathbf{k}', t') \psi_\alpha(\mathbf{j}', t') \rangle = \frac{2Q(k)}{\kappa^3 V^2(\kappa)} \delta(\mathbf{k}' + \mathbf{j}'). \tag{B.8}$$

4. Substitute this into equation (B.6) to obtain

$$V^2(k) = \frac{2Q(k)}{k'^3 \kappa^3} = \frac{2Q(k)}{k^3}. \tag{B.9}$$

As desired this gives us a relationship between $V(k)$ and $Q(k)$, namely

$$Q(k) = \frac{1}{2} k^3 V^2(k). \tag{B.10}$$

To proceed further, we first need to recall the energy balance equation, (1.79),

$$\left(\frac{\partial}{\partial t} + 2\nu_0 k^2 \right) E(k, t) = W(k, t) + T(k, t), \tag{B.11}$$

where the transfer spectrum $T(k)$ is given by

$$T(k) = 2\pi k^2 M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \, \{ Q_{\beta\gamma\alpha}(\mathbf{j}, \mathbf{k} - \mathbf{j}, -\mathbf{k}) - Q_{\beta\gamma\alpha}(\mathbf{j}, -\mathbf{k} - \mathbf{j}, \mathbf{k}) \}. \tag{B.12}$$

If we then return to (B.2) we may substitute (B.5) for $u_\alpha(\mathbf{k}, t)$ to obtain

$$Q_{\alpha\beta\gamma}(\mathbf{k}, \mathbf{j}) \delta(\mathbf{k} + \mathbf{j} + \mathbf{l}) = V^3(\kappa) \langle \psi_\alpha(\mathbf{k}', t') \psi_\beta(\mathbf{j}', t') \psi_\gamma(\mathbf{l}', t') \rangle, \tag{B.13}$$

and if we integrate this with respect to l we find

$$Q_{\alpha\beta\gamma}(\boldsymbol{k}, \boldsymbol{j}, -\boldsymbol{k} - \boldsymbol{j}) = V^3(\kappa)\kappa^3 \int d^3l' \, \langle \psi_\alpha(\boldsymbol{k}', t')\psi_\beta(\boldsymbol{j}', t')\psi_\gamma(\boldsymbol{l}', t') \rangle \quad (B.14)$$

$$= V^3(\kappa)\kappa^3 H_{\alpha\beta\gamma}(\boldsymbol{k}', \boldsymbol{j}'), \quad (B.15)$$

where

$$H_{\alpha\beta\gamma}(\boldsymbol{k}', \boldsymbol{j}') = \int d^3l' \, \langle \psi_\alpha(\boldsymbol{k}', t')\psi_\beta(\boldsymbol{j}', t')\psi_\gamma(\boldsymbol{l}', t') \rangle. \quad (B.16)$$

Along with (B.4), equation (B.15) can then be substituted into (B.12) to obtain

$$T(k) = 2\pi\kappa^9 V^3(\kappa)k'^2 M_{\alpha\beta\gamma}(\boldsymbol{k}') \int d^3j' \, \{H_{\beta\gamma\alpha}(\boldsymbol{j}', \boldsymbol{k}' - \boldsymbol{j}') - H_{\beta\gamma\alpha}(\boldsymbol{j}', -\boldsymbol{k}' - \boldsymbol{j}')\}. \quad (B.17)$$

If we integrate the energy balance equation with respect to k from 0 to κ, we have

$$\int_0^\kappa dk \, 2\nu_0 k^2 E(k) = \int_0^\kappa dk \, W(k) + \int_0^\kappa dk \, T(k), \quad (B.18)$$

where it has been assumed that the energy spectrum is stationary, and we may obtain the integral of the transfer spectrum using equation (B.17):

$$\int_0^\kappa dk \, T(k) = 2\pi\kappa^{10}V^3(\kappa) \int_0^1 dk' \, k'^2 M_{\alpha\beta\gamma}(\boldsymbol{k}') \int d^3j' \, \{\cdot\} \quad (B.19)$$

$$= \frac{1}{2}\kappa^{10}V^3(\kappa) \int_{|k'|\leq 1} d^3k' \, M_{\alpha\beta\gamma}(\boldsymbol{k}') \int d^3j' \, \{\cdot\} \quad (B.20)$$

$$= -A\kappa^{10}V^3(\kappa), \quad (B.21)$$

where $\{\cdot\}$ represents the terms in curly brackets in (B.17), the second line follows from isotropy and the last line, where A is a constant, comes from the fact that the integrals will be the same regardless of the choice of κ, since they only depend upon the dimensionless variable $\boldsymbol{k}'$. If we then substitute equation (B.21) into (B.18) and rearrange, we have a final expression for $V(\kappa)$:

$$V(\kappa) = A^{-1/3}\varepsilon^{1/3}\kappa^{-10/3} \left(\frac{1}{\varepsilon} \int_0^\kappa dj \, W(j) - \frac{1}{\varepsilon} \int_0^\kappa dj \, 2\nu_0 j^2 E(j) \right)^{1/3}. \quad (B.22)$$

Recalling equations (1.72) and (B.10) we find

$$E(\kappa) = 2\pi\kappa^5 V^2(\kappa), \quad (B.23)$$

and hence substituting from (B.22) we have our analytic integral equation for the energy spectrum, as desired:

$$E(\kappa) = \alpha \varepsilon^{2/3} \kappa^{-5/3} \left(\frac{1}{\varepsilon} \int_0^\kappa \mathrm{d}j\, W(j) - \frac{1}{\varepsilon} \int_0^\kappa \mathrm{d}j\, 2\nu_0 j^2 E(j) \right)^{2/3}, \qquad (B.24)$$

where $\alpha = 2\pi A^{-2/3}$. Clearly, if we restrict ourselves to wavenumbers above those at which the energy is input and below those at which energy dissipation occurs, that is the inertial range, this result reduces to the Kolmogorov spectrum $E(\kappa) = \alpha \varepsilon^{2/3} \kappa^{-5/3}$, where α may now be identified as the Kolmogorov constant.

B.2 Forster, Nelson and Stephen type forcing in the $k \to 0$ limit

To proceed from equation (B.24) we shall follow our earlier assumptions in considering all the forcing to occur at very low wavenumbers and restricting our attention to wavenumbers above this region. However, before doing this it is of interest to first consider the effect of substituting into (B.24) forcing of the type used by FNS and YO. Such a forcing has an input spectrum of power law form, see Section 2.4.1,

$$W_{\text{fns}}(k) = W_0 k^{-y}, \qquad (B.25)$$

Our definition of $W(k)$ does however differ slightly from that used by FNS, and in our case the equivalent statement to (B.25) is

$$\begin{aligned} W(k) &= 4\pi k^2 W_{\text{fns}}(k) \\ &= 4\pi W_0 k^{2-y}. \end{aligned} \qquad (B.26)$$

If we then proceed to substitute (B.26) into (B.24) we find

$$\begin{aligned} E(k) &= \alpha \varepsilon^{2/3} k^{-5/3} \left(\frac{1}{\varepsilon} \int_0^k \mathrm{d}j\, 4\pi W_0 j^{2-y} - \frac{1}{\varepsilon} \int_0^k \mathrm{d}j\, 2\nu_0 j^2 E(j) \right)^{2/3} \\ &= \alpha \varepsilon^{2/3} k^{-5/3} \left(\frac{4\pi W_0}{\varepsilon} \frac{k^{3-y}}{3-y} - \frac{1}{\varepsilon} \int_0^k \mathrm{d}j\, 2\nu_0 j^2 E(j) \right)^{2/3}, \qquad (B.27) \end{aligned}$$

where we have re-labelled $\kappa \to k$ and assume that $y \neq 3$.

In their RG calculation, FNS restrict themselves to the limit $k \to 0$. If we also make this restriction we may assume that the viscous term in (B.27) is negligible, enabling us to rewrite this expression as

$$\lim_{k \to 0} E(k) = \frac{\alpha (4\pi W_0)^{2/3}}{(3-y)^{2/3}} k^{-5/3+2(3-y)/3}. \tag{B.28}$$

That is, we obtain an energy spectrum which has an identical dependence upon wavenumber as that obtained by FNS and YO.

Similarly, if we consider the situation $y = 3$, assume that no forcing occurs below a lower limit $k_{\min}$ and apply the same approximations as previously, we obtain the result

$$E(k) = \alpha (4\pi)^{2/3} W_0^{2/3} k^{-5/3} \ln \left(\frac{k}{k_{\min}} \right), \tag{B.29}$$

which displays the expected logarithmic divergence. The results of YO have also be re-obtained by Kraichnan [63] using a perturbative approximation (the so-called 'distant-interaction algorithm'), but the question of why our simple, non-perturbative, calculation also yields the same expressions for the energy spectrum would clearly be worth pursuing.

B.3   Solution of the integral equation

Rather than assume a form for the forcing, in this section we shall instead assume all forcing to be constrained to very low wavenumbers and restrict our attention to wavenumbers above this region. Assuming the upper limit of this forcing region to be defined by k_f, say, then from equation (1.84) we have

$$\varepsilon = \int_0^k \mathrm{d}j \, W(j) \tag{B.30}$$

for any choice of $k \geq k_f$. Given that this is the case we may then rewrite (B.24) as

$$E(k) = \alpha \varepsilon^{2/3} k^{-5/3} \left(1 - \frac{1}{\varepsilon} \int_0^k \mathrm{d}j \, 2\nu_0 j^2 E(j) \right)^{2/3}, \tag{B.31}$$

where we have again re-labelled $\kappa \to k$.

We can obtain a simplified, non integral, form of equation (B.31) as follows:

1. Raise equation (B.31) to the power 3/2 and multiply through by $\alpha^{-3/2} k^{5/2}$ to obtain

    $$\alpha^{-3/2} k^{5/2} E^{3/2}(k) = \varepsilon - \int_0^k \mathrm{d}j \, 2\nu_0 j^2 E(j) \qquad (B.32)$$

2. Differentiate with respect to k

    $$\frac{3}{2}\alpha^{-3/2} k^{5/2} E^{1/2}(k)\frac{dE(k)}{dk} + \frac{5}{2}\alpha^{-3/2} k^{3/2} E^{3/2}(k) = -2\nu_0 k^2 E(k). \qquad (B.33)$$

3. Rearranging this we have

    $$\frac{dE(k)}{dk} + \frac{5}{3}k^{-1}E(k) = -\frac{4}{3}\alpha^{3/2}\nu_0 k^{-1/2}E^{1/2}(k). \qquad (B.34)$$

In order to solve this non-linear ODE we note that it has the form of a Bernoulli ODE [75]

$$\frac{dy(x)}{dx} + p(x)y(x) = g(x)y(x)^a, \qquad (B.35)$$

with $a = 1/2$. Such an equation can by transformed to a linear first order ODE by making the substitution

$$u(x) = y(x)^{1-a}, \qquad (B.36)$$

and so we make the replacement

$$u(k) = E^{1/2}(k) \qquad (B.37)$$

which implies

$$\frac{dE(k)}{dk} = 2u(k)\frac{du(k)}{dk}. \qquad (B.38)$$

Hence, substituting into equation (B.34) and rearranging we have

$$\frac{du(k)}{dk} + \frac{5}{6}k^{-1}u(k) = -\frac{2}{3}\alpha^{3/2}\nu_0 k^{-1/2}. \qquad (B.39)$$

This can be solved by multiplying through by the integrating factor $k^{5/6}$ to obtain

$$\frac{d}{dk}\left(k^{5/6}E^{1/2}(k)\right) = -\frac{2}{3}\alpha^{3/2}\nu_0 k^{1/3}, \qquad (B.40)$$

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and then integrating with respect to k which gives

$$k^{5/6}u(k) = C - \frac{1}{2}\alpha^{3/2}\nu_0 k^{4/3}, \tag{B.41}$$

where C is the constant of integration. Re-introducing $E(k)$ and rearranging, this gives the final result

$$E(k) = \alpha\varepsilon^{2/3}k^{-5/3}\left(\frac{C}{\alpha^{1/2}\varepsilon^{1/3}} - \frac{1}{2}\alpha\left(\frac{k}{k_d}\right)^{4/3}\right)^2, \tag{B.42}$$

where k_d is the Kolmogorov dissipation wavenumber.

If we then apply the approximate boundary condition that at $k = k_f$ we have the Kolmogorov spectrum, that is $E(k_f) = \alpha\varepsilon^{2/3}k_f^{-5/3}$ we find that C is given by

$$C = \alpha^{1/2}\varepsilon^{1/3}\left(1 + \frac{1}{2}\alpha\left(\frac{k_f}{k_d}\right)^{4/3}\right), \tag{B.43}$$

and hence

$$E(k) = \alpha\varepsilon^{2/3}k^{-5/3}\left(1 - \frac{1}{2}\alpha\frac{(k^{4/3} - k_f^{4/3})}{k_d^{4/3}}\right)^2. \tag{B.44}$$

B.4 Validity of the obtained energy spectrum

Prior to assessing the validity of the spectrum described in (B.44) we first note that it can be simplified slightly if we assume that $k_f \to 0$. In this instance it is reasonable to neglect the k_f term, meaning we obtain the simplified result

$$E(k) = \alpha\varepsilon^{2/3}k^{-5/3}\left(1 - \frac{1}{2}\alpha\left(\frac{k}{k_d}\right)^{4/3}\right)^2. \tag{B.45}$$

This expression is compared to the DNS spectrum of Young and the spectra obtained using the models of Pao [78] and Qian [79] in Figure B.1. Pao's model spectrum is obtained by assuming that the rate at which energy is transferred through wavenumber space has the same dependence on viscosity as the energy spectrum, and has the form

$$E(k) = \alpha\varepsilon^{2/3}k^{-5/3}\exp\left(-\left(\frac{3\alpha}{2}\right)\left(\frac{k}{k_d}\right)^{4/3}\right), \tag{B.46}$$
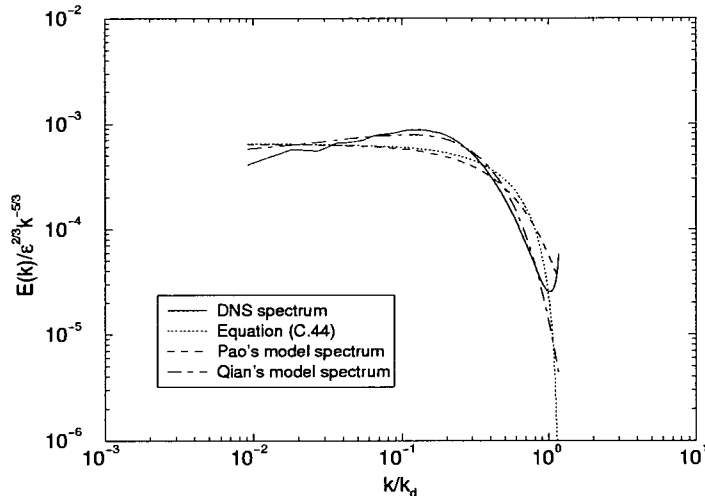
Figure B.1: Comparison of the compensated energy spectra given by equation (B.45), the models of Pao [78] and Qian [79], and that obtained in the DNS of Young [70]. All the model spectra were calculated using the same values for α, ε and ν_0 as used in the DNS ($\alpha = 1.624$, $\varepsilon = 0.149$ and $\nu_0 = 10^{-3}$).

whilst Qian's model is obtained by fitting the results of numerical experiments to an assumed spectrum with adjustable parameters, and has the form

$$E(k) = 1.19\varepsilon^{2/3}k^{-5/3}\left(1 + 5.3\left(\frac{k}{k_d}\right)^{2/3}\right)\exp\left(-5.4\left(\frac{k}{k_d}\right)^{4/3}\right). \qquad (B.47)$$

As can be seen in the figure, the spectrum described by (B.45) fits the results of the DNS better than the spectrum of Pao's model, and although the spectrum of Qian does give a better fit that either of the alternatives it has to be remembered that this result is purely empirical, with values chosen exactly so that it does match the results of numerical simulations. It would thus seem reasonable to conclude that the model spectrum we have obtained provides at least as good an approximation as the analytic theory of Pao, whilst using fewer restrictive assumptions. However, we should point out that the spectrum described by (B.45) has a significant flaw in that it predicts the spectrum will equal zero at $k = k_{\text{root}} = (2/\alpha)^{3/4}k_d \approx 1.18k_d$, assuming $\alpha = 1.6$, and then increase again as the wavenumber increases. We believe that this behaviour occurs as a result of

the approximations we make in solving the integral equation, in particular the boundary condition used, as equation (B.31) indicates the expected behaviour of $E(k)$ decreasing for all k and tending to zero only as $k \to \infty$. We do not however believe that this problem should invalidate the use of (B.45) as a model in most situations of interest. Indeed, it should be noted that if we include the forcing term, that is use (B.44) instead of (B.45), we find that the zero in the energy spectrum occurs at

$$k_{\text{root}} = \left(\frac{2}{\alpha}\right)^{3/4} k_d \left(1 + \frac{\alpha}{2}\frac{k_f}{k_d}\right)^{3/4}. \tag{B.48}$$

That is, including forcing up to a defined wavenumber has the effect of increasing the value of k_{root}, the value of which could be used to provide an estimate of the maximum relevant wavenumber in our system.

Appendix C

Published papers

Reproduced on the following pages is the paper:

Elimination of turbulence modes using a conditional average with asymptotic freedom,

W. D. McComb and C. Johnston, J. Phys. A: Math. Gen. **33**, L15 (2000).

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LETTER TO THE EDITOR

Elimination of turbulence modes using a conditional average with asymptotic freedom

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Abstract. The part of the nonlinear term in the Navier–Stokes equation which represents coupling to the small-scale modes may be averaged out by introducing a weak conditional average with asymptotic freedom in wavenumber. A residual deterministic part, while important for individual realizations, makes a negligible contribution to the renormalization of the dissipation rate. This is because the full ensemble average, needed to establish the energy balance, relaxes the constraint on the conditional average.

The application of renormalization group methods to dynamical problems in microscopic physics requires an average over small scales in which large scales are held fixed [1]. Unfortunately, the corresponding procedure for classical nonlinear systems, such as Navier–Stokes turbulence, is impossible, *in principle*, because of the deterministic nature of such systems. Recently, it has been proposed that the chaotic nature of turbulence may justify the use of an approximate conditional average [2]. In this paper we argue that the conditional elimination of a band of high-wavenumber modes may be accomplished in terms of a deterministic part, which has a coherent phase relation with the retained modes, and a random part, which is asymptotically free and may be averaged out with the introduction of an effective viscosity. The reduction of the number of modes takes place at a constant rate of energy dissipation, and it is further argued that the renormalization of this quantity can be adequately represented by the incoherent part only. This is because the full ensemble average, needed for the spectral energy balance, tends to 'lift' the constraint on the conditional average.

We consider incompressible fluid turbulence, as governed by the solenoidal Navier–Stokes equation (NSE)

$$(\partial_t + \nu_0 k^2)u_\alpha(k, t) = M_{\alpha\beta\gamma}(k) \int d^3 j \, u_\beta(j, t)u_\gamma(k - j, t) \tag{1}$$

where ν_0 is the kinematic viscosity of the fluid,

$$M_{\alpha\beta\gamma}(k) = (2i)^{-1}[k_\beta D_{\alpha\gamma}(k) + k_\gamma D_{\alpha\beta}(k)] \tag{2}$$

and the projector $D_{\alpha\beta}(k)$ is expressed in terms of the Kronecker delta $\delta_{\alpha\beta}$ as

$$D_{\alpha\beta}(k) = \delta_{\alpha\beta} - k_\alpha k_\beta |k|^{-2}. \tag{3}$$

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In order to pose a specific problem, we restrict our attention to stationary, isotropic, homogeneous turbulence, with dissipation rate ε and zero mean velocity. We also introduce an upper cutoff wavenumber $K_{\max}$, which is defined through the dissipation integral

$$\varepsilon = \int_0^\infty 2\nu_0 k^2 E(k)\,\mathrm{d}k \simeq \int_0^{K_{\max}} 2\nu_0 k^2 E(k)\,\mathrm{d}k \tag{4}$$

where $E(k)$ is the energy spectrum, so ensuring that $K_{\max}$ is of the same order of magnitude as the Kolmogorov dissipation wavenumber.

We then filter the velocity field at $|k| \equiv k = K_c$, where $0 < K_c < K_{\max}$, according to

$$u_\alpha(k, t) = \begin{cases} u_\alpha^-(k, t) & \text{for} \quad 0 < k < K_c \\ u_\alpha^+(k, t) & \text{for} \quad K_c < k < K_{\max}. \end{cases} \tag{5}$$

The NSE may be decomposed using (5), to give

$$(\partial_t + \nu_0 k^2)u_k^- = M_k^-(u_j^- u_{k-j}^- + 2u_j^- u_{k-j}^+ + u_j^+ u_{k-j}^+) \tag{6}$$

$$(\partial_t + \nu_0 k^2)u_k^+ = M_k^+(u_j^- u_{k-j}^- + 2u_j^- u_{k-j}^+ + u_j^+ u_{k-j}^+) \tag{7}$$

where, for simplicity, all vector indices and independent variables are contracted into a single subscript.

In order to obtain an expression for the average effect of the high-wavenumber modes upon a particular low-wavenumber mode, we need to average out the u^+ whilst holding the u^- constant. This requires a *conditional average* $\langle \cdot \rangle_c$, such that

$$\langle u_\alpha^-(k, t) \rangle_c = u_\alpha^-(k, t). \tag{8}$$

This is the *only* rigorous property we can attribute to the conditional average, and it should also be noted that it is vital to distinguish between this operation and that of a filtered ensemble average.

To establish the statistical properties of $u_\alpha(k, t)$ we consider an ensemble $\mathcal{W}$ consisting of the set of M time-independent realizations $\{W_\alpha^{(i)}(k)\}$, each realization† being labelled by an integer i. Subject to certain weak conditions, the ensemble average is

$$\langle u_\alpha(k, t) \rangle = \lim_{M \to \infty} \frac{1}{M} \sum_{i=1}^M W_\alpha^{(i)}(k) = \bar{U}_\alpha(k) \tag{9}$$

where $\bar{U}_\alpha(k)$ is the time average of $u_\alpha(k, t)$. This procedure can then be extended to any well behaved functional, $F[u_\alpha(k, t)]$, thus:

$$\langle F[u_\alpha(k, t)] \rangle = \lim_{M \to \infty} \frac{1}{M} \sum_{i=1}^M F[W_\alpha^{(i)}(k)]. \tag{10}$$

Now we consider how to perform a *conditional* average. To do this, we first select a subensemble, $\mathcal{Y} \equiv \{Y_\alpha^{(i)}(k)\} \subset \mathcal{W}$, and choose the members of this *biased* subensemble to be those N $(N \leqslant M)$ members of $\mathcal{W}$ satisfying the criterion

$$\lim_{\delta \to 0}(\max |\theta^-(k)W_\alpha^{(i)}(k) - u_\alpha^-(k, t_1)| \leqslant \delta) \tag{11}$$

where t_1 is some fixed time and $\theta^-(k) = 1$ for $0 < k < K_c$, and zero otherwise. The conditional average is then obtained by generalizing (9) and (10) to the biased subensemble, namely,

$$\langle u_\alpha(k, t) \rangle_c = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N Y_\alpha^{(i)}(k) \tag{12}$$

† Note that this differs from the formulation in [2], where each realization was time-dependent.

and

$$\langle F[u_\alpha(k, t)]\rangle_c = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} F[Y_\alpha^{(i)}(k)]. \tag{13}$$

It follows by construction that (8) holds, since from (11) and (12)

$$\langle u_\alpha^-(k, t)\rangle_c = \lim_{N\to\infty} \frac{1}{N}[Nu_\alpha^-(k, t)] = u_\alpha^-(k, t). \tag{14}$$

The difficulty now facing us lies in the nature of the subensemble, which is an example of deterministic chaos. This can be seen if we consider two extreme scenarios for the behaviour of u^+ under the conditional average. Firstly, if we assume that the subensemble is *strictly* deterministic, then in this instance u_k^+ is fully determined by prescribing u_k^-. Accordingly, equation (8) implies that $\langle u_j^- u_{k-j}^-\rangle_c = u_j^- u_{k-j}^-$, $\langle u_j^- u_{k-j}^+\rangle_c = u_j^- u_{k-j}^+$ and $\langle u_j^+ u_{k-j}^-\rangle_c = u_j^+ u_{k-j}^-$. Thus, the low-pass filtered NSE, equation (6), reduces back to itself under the conditional average. Secondly, if we assume that the subensemble is purely random, it follows that in this case, u_k^+ is independent of u_k^-. Hence, applying the conditional average to the low-pass filtered NSE, we find

$$(\partial_t + \nu_0 k^2)u_k^- = M_k^- u_j^- u_{k-j}^-$$

the $u_j^- u_{k-j}^+$ term being zero since the ensemble average of u^+ is zero, whilst the $u_j^+ u_{k-j}^-$ term is zero due to homogeneity. Thus in this scenario it appears that there is *no* effect of nonlinear coupling.

In reality we are faced with a situation somewhere between these two extremes, and so we replace our criterion for members of the biased subensemble, equation (11), which is equivalent to the first of these situations if $\delta = 0$, by the less precise criterion

$$\max |\theta^-(k)W_\alpha^{(i)}(k) - u_\alpha^-(k, t_1)| \leqslant \xi \tag{15}$$

where, in general, ξ is of the order of the turbulent velocities involved.

To obtain a non-trivial conditional average we must now identify those circumstances in which ξ may be neglected as being, in some sense, small. A measure of the 'smallness of ξ' can be identified by constructing the subensemble as

$$W_\alpha^{(i)}(k) = u_\alpha^-(k, t_1) + \phi_\alpha^{(i)}(k, t_1) \tag{16}$$

where i is any label satisfying (15). If we then further restrict the subensemble to be such that the set $\{\phi_\alpha^{(i)}(k, t_1)\}$ satisfies (8), we find that

$$\langle u_j^- u_{k-j}^-\rangle_c = u_j^- u_{k-j}^- + \langle \phi_j \phi_{k-j}\rangle_c. \tag{17}$$

Thus in order to maintain form invariance of the NSE under conditional averaging, we require

$$\langle \phi_j \phi_{k-j}\rangle_c \to 0 \tag{18}$$

in some limit. This is our criterion for the smallness of ξ.

If we further suppose that chaos and unpredictability are local characteristics of turbulence, and there is support for such a view [3,4], then if K_c and K_{max} are sufficiently far apart we might expect, due to the development of unpredictability as k is increased above K_c, that the effect of the constraint given in equation (15) would die away, such that

$$\lim_{k\to K_{max}} \langle u_\alpha^+(k, t)\rangle_c \to \langle u_\alpha^+(K_{max}, t)\rangle. \tag{19}$$

We refer to this property as *asymptotic freedom*. In order to extend this concept to higher-order moments, we introduce the following *hypothesis of local chaos*:

'For sufficiently large Reynolds' number and corresponding K_{max}, there exists a cut-off wavenumber $K_c < K_{max}$, such that a mixed conditional moment involving p low-wavenumber and r high-wavenumber modes takes the limiting form:

$$\lim_{\xi \to 0} \langle u_\alpha^-(k_1, t) u_\beta^-(k_2, t) \dots u_\gamma^-(k_p, t) u_\delta^+(k_{p+1}, t) u_\epsilon^+(k_{p+2}, t) \dots u_\sigma^+(k_{p+r}, t) \rangle_c$$

$$\to u_\alpha^-(k_1, t) u_\beta^-(k_2, t) \dots u_\gamma^-(k_p, t)$$

$$\times \lim_{\{\cdot\} \to K_{max}} \langle u_\delta^+(k_{p+1}, t) u_\epsilon^+(k_{p+2}, t) \dots u_\sigma^+(k_{p+r}, t) \rangle \quad (20)$$

where $\lim_{\{\cdot\} \to K_{max}}$ means take the limit for all wavevector arguments of the u^+ modes, with the condition of equation (18) satisfied as a corollary'.

This provides our definition of an asymptotic conditional average and we should emphasize that the numerical simulations of Machiels [4] provide independent verification of this behaviour. It may be used to evaluate all terms involving mixed products of u^- with u^+. For example,

$$\lim_{\xi \to 0} \langle u_j^- u_{k-j}^+ \rangle_c = u_j^- \lim_{\{\cdot\} \to K_{max}} \langle u_{k-j}^+ \rangle = 0 \quad (21)$$

since $\langle u_\alpha^+(k, t) \rangle = 0$. Note also, that the hypothesis as stated is more general than is necessary, since we shall only need to consider products containing at most two u^- modes.

If we then take the conditional average of the low-pass filtered NSE, equation (6), we obtain

$$(\partial_t + \nu_0 k^2) u_k^- = M_k^- \{ \langle u_j^- u_{k-j}^- \rangle_c + 2 \langle u_j^- u_{k-j}^+ \rangle_c + \langle u_j^+ u_{k-j}^+ \rangle_c \} \quad (22)$$

where the conditional average of u_k^- on the left-hand side has been evaluated using (8). This equation may be further rewritten as

$$(\partial_t + \nu_0 k^2) u_k^- = M_k^- u_j^- u_{k-j}^- + S^-(k|K_c) + M_k^- \lim_{\xi \to 0} \langle u_j^+ u_{k-j}^+ \rangle_c \quad (23)$$

where

$$S^-(k|K_c) = M_k^- \left\{ \langle \phi_j^- \phi_{k-j}^- \rangle_c + 2 \langle u_j^- u_{k-j}^+ \rangle_c + \langle u_j^+ u_{k-j}^+ \rangle_c - \lim_{\xi \to 0} \langle u_j^+ u_{k-j}^+ \rangle_c \right\}. \quad (24)$$

It should also be noted that the hypothesis *must* hold for $K_c \to 0$, as in this instance equation (22) reduces to the Reynolds equation, with $u_\alpha(k, t) \to \bar{U}_\alpha(k)$ as given by (9).

Our hypothesis does not explicitly tell us how evaluate the conditional average in (23), which involves a non-trivial projection of a product of u^+ modes in the Hilbert space of the u^- modes, but we may use the high-pass filtered NSE, equation (7), to obtain a governing equation for this quantity. To do this, we use (7) to write equations for u_j^+ and u_{k-j}^+, multiply these equations by u_{k-j}^+ and u_j^+, respectively, add the resulting equations together, and then take the conditional average. After some rearrangement of dummy variables, this gives

$$\lim_{\xi \to 0} (\partial_t + \nu_0 j^2 + \nu_0 |k - j|^2) \langle u_j^+ u_{k-j}^+ \rangle_c = \lim_{\xi \to 0} 2 M_j^+$$

$$\times \{ \langle u_p^- u_{j-p}^+ u_{k-j}^+ \rangle_c + 2 \langle u_p^- u_{j-p}^+ u_{k-j}^+ \rangle_c + \langle u_p^+ u_{j-p}^+ u_{k-j}^+ \rangle_c \}. \quad (25)$$

Applying the hypothesis as given by equation (20), it is easily seen that the first term on the right-hand side of (25) is zero, since in the limit it involves the ensemble average of u_k^+, while the second term gives rise to a term linear in u_k^-. The third term may be evaluated by iterating the above procedure to form a dynamical equation for $\langle u_p^+ u_{j-p}^+ u_{k-j}^+ \rangle_c$, which in turn gives rise to higher-order moments.

In general, we can show that a similar pattern occurs for all higher-order moments involving only products of u^+. That is, each such moment gives rise to a moment involving two u^- modes, which in general, has to be zero for consistency in its wavevector arguments, a term linear in

u_k^-, and a moment involving only u^+ modes of next higher order. Hence we may write the general result

$$M_{\alpha\beta\gamma}^-(\mathbf{k}) \int \mathrm{d}^3 j \lim_{\xi\to 0} \langle u_\beta^+(\mathbf{j}, t) u_\gamma^+(\mathbf{k}-\mathbf{j}, t)\rangle_c = \int_{-\infty}^t \mathrm{d}s\, A(\mathbf{k}, t-s) u_\alpha^-(\mathbf{k}, s) \qquad (26)$$

where $A(\mathbf{k}, t-s)$ has the form

$$A(\mathbf{k}, t-s) = \int \mathrm{d}^3 j \exp[-(\nu_0 j^2 + \nu_0|\mathbf{k}-\mathbf{j}|^2)(t-s)]$$

$$\times \left\{ 4 M_k^- M_j^+ \lim_{\{\cdot\}\to K_{\max}} \langle u_{j-p}^+ u_{k-j}^+\rangle \right.$$

$$\left. + 24 M_k^- M_j^+ L_{03}^{-1} M_p^+ \lim_{\{\cdot\}\to K_{\max}} \langle u_{p-q}^+ u_{j-p}^+ u_{k-j}^+\rangle + \cdots \right\} \qquad (27)$$

$L_{03} \equiv \partial_t + \nu_0 p^2 + \nu_0|\mathbf{j}-\mathbf{p}|^2 + \nu_0|\mathbf{k}-\mathbf{j}|^2$, and where higher-order terms are easily found by induction. Thus, in all, equation (23) for the low-wavenumber modes may be written as

$$(\partial_t + \nu_0 k^2) u_k^- - \int_{-\infty}^t \mathrm{d}s\, A(\mathbf{k}, t-s) u_\alpha^-(\mathbf{k}, s) = M_k^- u_j^- u_{k-j}^- + S(\mathbf{k}|K_c). \qquad (28)$$

In order to test the hypothesis, we make two approximations. First, we truncate the expansion of $A(\mathbf{k}, t)$ at lowest non-trivial order. This can be justified by the introduction of a *local* Reynolds number based on a length scale K_c^{-1}, the moment expansion being re-expressed as a power series in this parameter. Making the truncation in (26) and (27) leaves us with the expression

$$\lim_{\xi\to 0} M_{\alpha\beta\gamma}^-(\mathbf{k}) \quad \langle u_\beta^+(\mathbf{j}, t) u_\gamma^+(\mathbf{k}-\mathbf{j}, t)\rangle_c = \int_{-\infty}^t \mathrm{d}s \exp[-(\nu_0 j^2 + \nu_0|\mathbf{k}-\mathbf{j}|^2)(t-s)]$$

$$\times 4 M_{\alpha\beta\gamma}^-(\mathbf{k}) M_{\beta\delta\epsilon}^+(\mathbf{j}) \int \mathrm{d}^3 p \lim_{\{\cdot\}\to K_{\max}} \langle u_\epsilon^+(\mathbf{j}-\mathbf{p}, s) u_\gamma^+(\mathbf{k}-\mathbf{j}, s)\rangle u_\delta^-(\mathbf{k}, s). \qquad (29)$$

For stationary, homogeneous, and isotropic turbulence we may write

$$\langle u_\epsilon^+(\mathbf{j}-\mathbf{p}, s) u_\gamma^+(\mathbf{k}-\mathbf{j}, s)\rangle = Q(|\mathbf{k}-\mathbf{j}|) D_{\epsilon\gamma}(\mathbf{k}-\mathbf{j}) \delta(\mathbf{k}-\mathbf{p}) \qquad (30)$$

where $Q(k)$ is the spectral density and δ is the Dirac delta function. This leaves the question of how to perform the time integral

$$\int_{-\infty}^t \mathrm{d}s \exp[-(\nu_0 j^2 + \nu_0|\mathbf{k}-\mathbf{j}|^2)(t-s)] u_\delta^-(\mathbf{k}, s). \qquad (31)$$

To do this we change the variable of integration from s to $T = t-s$, expand the resultant $u_\delta^-(\mathbf{k}, t-T)$ as a Taylor series about $T = 0$, and then truncate the expansion at zero order, this approach being based upon the physical idea that the u^- modes are slowly evolving on timescales defined by the inverse of $\nu_0 j^2 + \nu_0|\mathbf{k}-\mathbf{j}|^2$.

We have investigated the validity of these two approximations using results from direct numerical simulations performed on a 256^3 grid, with Taylor–Reynolds number $R_\lambda = 190$. At this resolution the simulations have a very limited inertial range (see [5,6]), but nevertheless they indicate that there is a range of K_c ($K_c \gtrsim 0.5 K_{\max}$) where both approximations give rise to error terms of less than unity, and that the magnitude of these errors will decrease as we increase R_λ to the large values where we may reasonably expect our hypothesis to hold.

With these approximations, the right-hand side of (26) is simple to evaluate, and we are left with the final expression for the conditional average on the right-hand side of (23):

$$M_{\alpha\beta\gamma}^-(\mathbf{k}) \lim_{\xi\to 0} \langle u_\beta^+(\mathbf{j}, t) u_\gamma^+(\mathbf{k}-\mathbf{j}, t)\rangle_c$$

$$= 4 M_{\alpha\beta\gamma}^-(\mathbf{k}) M_{\beta\delta\epsilon}^+(\mathbf{j}) \lim_{|\mathbf{k}-\mathbf{j}|\to K_{\max}} \frac{Q(|\mathbf{k}-\mathbf{j}|) D_{\epsilon\gamma}(\mathbf{k}-\mathbf{j})}{\nu_0 j^2 + \nu_0|\mathbf{k}-\mathbf{j}|^2} u_\delta^-(\mathbf{k}, t) \qquad (32)$$

which is linear in u_k^-, meaning that it may be interpreted in terms of an increment to the viscosity.

In order to evaluate the limit, we make a first-order truncation of a Taylor series expansion in wavenumber of Q^+ about K_{max}. In this way, we re-obtain the results previously obtained using the two-field theory of McComb and Watt [7]. As they showed, a renormalization group calculation based upon these equations gives a prediction for the Kolmogorov constant of 1.60 ± 0.01, in good agreement with experiment, for $0.55 K_{max} \leqslant K_c \leqslant 0.75 K_{max}$. This calculation obtained the Kolmogorov exponent and pre-factor by assuming that the effective viscosity and its increment scale in the same way (which is true at the fixed point) and that the rate of energy transfer is renormalized. This latter assumption amounted, in our present terminology, to the neglect of $S(k|K_c)$ in equation (23).

A new justification of this step can now be offered as follows. The equation for the energy spectrum is obtained by multiplying the dynamical equation for $u_\alpha^-(k, t)$ by $u_\alpha^-(-k, t)$ and then performing an average over the full ensemble. Thus the effect of $S(k|K_c)$ is just

$$\langle S(k|K_c) u_\alpha^-(-k, t) \rangle.$$

If we consider the form of $S(k|K_c)$ we see that each of the terms in the above expression involves a conditional average. In evaluating such terms we perform a double summation, firstly summing over all members with low-wavenumber modes close to a particular member of the ensemble, and then repeating this summation for every member of the ensemble. Now, the initial ensemble was constructed according to the principle of equal *a priori* probabilities but this is no longer necessarily true of the composite ensemble which we are now considering. If it were true, then the terms making up $S(k|K_c)$ would vanish identically for all K_c. However, in view of the results of the renormalization group calculations [7], it seems likely that the contribution from $S(k|K_c)$ is small for K_c in the range $0.55 K_{max} \leqslant K_c \leqslant 0.75 K_{max}$. Thus, for this range of cut-off wavenumbers, it would appear that the renormalization group calculation of the effective viscosity [7] is valid in a heuristic sense.

Finally, it should be noted that this work does not suggest that $S(k|K_c)$ can be neglected in equation (23), which is the governing equation for a single realization. However, it does suggest that, having averaged out the chaotic part to yield an effective viscosity, one should consider modelling the relationship of $S(k|K_c)$ to the u_k^- modes as predominantly deterministic. Work along these lines will be the subject of a separate communication.

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