

**On the Vassiliev invariants for knots and for
pure braids.**

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Abstract.

The study of Vassiliev knot invariants arose from Vassiliev's work on singularity theory and from the perturbative Chern-Simons theory of Witten. One reason for studying Vassiliev invariants is that they give topological ways of looking at "quantum" knot invariants — that is invariants which arise by generalizing the Jones polynomial.

This thesis contains various results on Vassiliev invariants: common themes running through include their polynomial nature, their functoriality, and the use of Gauß diagrams. The first chapter examines the functoriality of Vassiliev invariants and describes how they can be defined on different types of knotty objects such as knots, framed knots and braids, and how algebraic structure naturally arises. An explicit form of the relationship between the framed and unframed knot theory is given. Chapter 2 considers the important question of whether a Vassiliev invariant can be naïvely obtained from a combinatorial object called a weight system. A partial answer to this is given by showing how "half" of the steps in such a transition can be performed canonically and explicitly. In Chapter 3 the first two non-trivial invariants for knots, evaluated on prime knots up to twelve crossing are examined, and some surprising graphs are obtained by plotting them. A number of results for torus knots are proved, relating unknotting number and crossing number to the first two Vassiliev invariants.

The second half of the thesis is concerned primarily with Vassiliev invariants of pure braids and their connection with de Rham homotopy theory. In Chapter 4 a simple derivation is given showing the relationship between Vassiliev invariants and the lower central series of the pure braid groups. This is used to obtain closed formulæ for the actual number of invariants of each type. Chapter 5 is a digression on de Rham homotopy theory and explains the geometric connections between Chen's iterated integrals, higher order Albanese manifolds, and Sullivan's 1-minimal models. A method of Chen's for obtaining integral invariants of elements of the fundamental group from a 1-minimal model is given, and in Chapter 6 this is used to find Vassiliev invariants of pure braids at low order: this extends work of M. A. Berger. Finally, a similar method using currents is employed to obtain a combinatorial formula for a type two invariant which is independent of winding numbers.

Dedication.

To my parents.

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Introduction.

This introduction contains some comments on the history of Vassiliev invariants, a look at some themes present in the thesis and a synopsis of the contents.

Historical context.

Pre-Vassiliev. Gauß could arguably be described as the first mathematician to consider knots and links. As well as being interested in which patterns of over and undercrossings could occur, he also wrote down the so-called Gauß linking of a two component link which is now well known in the context of electromagnetic induction. In [28] he said; “A major task from the boundary of *Geometria Situs* and *Geometria Magnetudinus* would be to count the linking number of two closed or infinite curves.” There is also a fragment in the library at Göttingen, indicating that he had also considered braids.

British physicists, notably Kelvin [41] and Tait, hearing of the German work on knots through Helmholtz and knowing the topological properties of smoke rings, suggested that knotted vortices of æther could give a plausible atomic theory — different knot types corresponding to different elements. This led Tait [71] to try to classify knots of small complexity, i.e. up to ten crossings, in fact he managed to classify alternating knots up to that level. The atomic theory faded away and knot theory was left to the mathematicians.

In the twentieth century, Alexander considered the algebraic topology of knot complements and defined the Alexander polynomial. Conway [20] renormalized this so that it could be defined purely in terms of knot diagram combinatorics. Meanwhile, Milnor [55] had generalized the Gauß linking numbers to “higher order” invariants: whereas the Gauß invariants were related to the abelianization of a group, the Milnor invariants corresponded to higher nilpotent quotients.

The quantum and Vassiliev approaches. Following the excitement caused by the discovery of the Jones polynomial in 1984 and also the consternation caused by the fact that it was only understood in terms of the combinatorics of knot diagrams, Atiyah challenged Witten to place the Jones polynomial in some topological context.

This he duly did in the seminal paper [79] relating the Jones polynomial to Chern-Simons theory. Unfortunately for the mathematical community, the machinery involved includes the notorious Feynman path integral — this is based on physical intuition rather than mathematical rigour, and has so far withstood attempts at rigourization. However, the so-called perturbative approach to the Witten formalism can be put on a very sound mathematical footing — it is analogous to considering the Taylor expansion of a function rather than the function itself, or in the case of the Jones polynomial, substituting $q = e^x$ and expanding in powers of x .

Independently, Vassiliev discovered his invariants whilst considering the cohomology of the space of knots in \mathbb{R}^3 . He actually obtained a filtration on the space of knot invariants by looking at a spectral sequence coming from applying duality to the space of all embeddings of a circle in \mathbb{R}^3 [74]. Birman and Lin [13] showed that these invariants could be characterized as ‘finite-type’, i.e. without recourse to the spectral sequence.

It was whilst working on the perturbative Chern-Simons theory for his thesis under Witten that Bar-Natan noticed similarities between diagrammatic relations in this perturbative theory and the Vassiliev theory. Indeed [13] the coefficient of x^n in the expansion of the Jones polynomial mentioned above is a Vassiliev invariant of type n . This led to the synthesis which became [6]: the main theorem is a construction, due to Kontsevich [45], of a universal Vassiliev invariant which means that many questions about Vassiliev invariants can be reduced to combinatorial questions. For instance, this means that the dimension of the space of rational Vassiliev knot invariants of given type can be calculated by solving some (difficult) combinatorial problem; some low order values are tabulated in [6].

Since then Vassiliev invariants have been generalized to braids [67], string links [8], tangles (see e.g. [5]), and other such knotty objects. It can be proved that in a suitable sense, the Milnor invariants alluded to above are of finite type. Also Vassiliev invariants have been used [53, 10] to prove that the Alexander-Conway polynomial can be recovered from the *coloured* Jones polynomial which is a generalization of the usual Jones polynomial.

Some themes.

This thesis examines various aspects of the theory of Vassiliev invariants. Here are some of the recurrent themes.

Vassiliev invariants as polynomials. Vassiliev invariants can be most easily defined in terms of an analogy with polynomial functions: viz, a vanishing derivative

condition. For a knot invariant taking values in some abelian group (e.g. the rational numbers), a derivative can be defined which measures how the invariant alters under crossing switches. This derivative is a map from “knots with a double point” to the abelian group — the double point encoding where the crossing switch is taking place. Higher derivatives can be defined similarly. A Vassiliev invariant of type n is an invariant whose $(n + 1)$ th derivative vanishes identically. This is seen to be analogous to one way of characterizing polynomials. One theme of this thesis is that “Vassiliev invariants are like polynomials” is a useful paradigm (this is the theme of [76]).

Results in the literature demonstrating this usefulness include the following: the results of Dean [22] and Trapp [73] on twist sequences; of Alvarez and Labastida [2] on torus knots; and of Stanford [66] and Bar-Natan [7] on bounds and algorithms which are polynomial in crossing number.

In this thesis these are put into other contexts. For instance the twist sequence result is generalized in Corollary 9 which says that one gets similar results when forming sequences by satelliting a fixed pattern on a fixed companion and altering the framing. In Chapter 3 it is seen that when graphing the canonical type two knot invariant (x) against the canonical type three knot invariant (y) for knots with crossing number up to twelve, one sees something reminiscent of plotting $x^3 = y^2$, agreeing with the suggestion that x is like a quadratic and y is like a cubic. One other result is that the fact that the product of two Vassiliev invariants is Vassiliev can be proved via a Leibniz theorem analogy, viz the derivatives of the product can be expressed in terms of the derivatives of the factors.

If a degree n polynomial is differentiated n times a constant function is obtained: if a type n Vassiliev invariant is differentiated n times then a combinatorial object called a weight system is obtained. One would like to be able to find an explicit way of going from weight system to Vassiliev invariant. Chapter 2 proceeds by trying to do this “integration” naïvely. One problem with this is the “constants of integration” which come in at each integration step. By analogy with integrating odd and even polynomials, the natural operation of taking the mirror image of a knot can be utilized to perform half of the integration steps canonically and explicitly. (I can’t yet see how to complete this to a full integration from weight system to invariant.)

Functoriality. Vassiliev invariants can be defined for various classes of knotty objects — braids, string-links, links, framed knots, *etc.* (see [8, 65]) — in an obvious manner which is functorial with respect to many maps between the classes: “many maps” here being things like inclusions, wiring maps, satelliting, forgetful maps *etc.*

In [65], Stanford considers functoriality in terms of categories of diagrams and Reidemeister type moves, but the approach in Chapter 1 here is more in the skein theoretic vein of Przytycki [60]. It seems to be the right framework for the algebraic approach for pure braids — as used in Chapters 4–6. The approach is to consider the module (over some fixed ring) freely generated by the set of knotty objects under consideration, e.g. the free \mathbb{Z} module generated by the set of knots. A filtration is imposed on this module by considering knotty objects with double points as linear combinations of genuine knotty objects. Vassiliev invariants are then the invariants dual to this filtration. Functorial refers to maps respecting these filtrations.

In Chapter 1 the relationship between framed and unframed invariants is described in this manner, with the framing number being easily identifiable; and the functoriality of the mirror image map is a key to the half integration of weight systems in Chapter 2.

Gauß diagrams. Gauß diagrams are a way of encoding knot and braid diagrams. A Gauß diagram consists of disjoint circles/lines representing the strings of the knot/link/braid and of arrows connecting points on the strings which meet at crossings of the diagram. The orientation of an arrow indicates which part of the string is the over-crossing and the arrows are marked to indicate whether they represent positive or negative crossings. Gauß diagrams look quite similar to chord diagrams. A Gauß diagram formula for an invariant is one which calculates the invariant from the Gauß diagram of any diagram of the knot/link/braid. For instance the usual combinatorial formulæ for Gauß linking or for winding numbers can naturally be written as Gauß diagram formulæ. Polyak and Viro [59] gave Gauß diagram formulæ for the second and third order knot invariants and for one of the fourth order knot invariants. It is seen in Chapter 3 that the naïve approach of Chapter 2 leads to formulæ for the second and third order knot invariants.

Trying to replace one-forms with currents to obtain a combinatorial formula for the type two pure braid invariant naturally leads to a Gauß diagram formula in Chapter 6.

Use of pure braids. The Vassiliev theory of pure braids is very appealing in several respects. Pure braid groups have nice classifying spaces. The configuration space of k ordered, distinct points in the complex plane is a classifying space for the pure braid group on k strands, i.e. the only non-trivial homotopy group of such a configuration space is the fundamental group and this is isomorphic to a pure braid group. Studying the Vassiliev theory of pure braid groups is precisely the same as looking at the rational/de Rham homotopy theory of the classifying spaces (this was noted by Kohno [44]) — both are related to the nilpotent quotients of the pure braid groups. So the Vassiliev invariants for pure braids lie appealingly in the intersection of

geometry, topology and algebra. As well as this purely aesthetic reason for studying pure braids, two other aspects motivated my study. These are as follows.

The Kontsevich integral was defined in terms of iterated integrals, as a generalization of Chen's power series connection for the configuration spaces mentioned above; however, in this case, the points are signed ± 1 , and signed pairs are allowed to be spontaneously created, or else mutually annihilate one another. My intention was to improve my understanding of the Kontsevich integral by looking at the much simpler braid version of iterated integrals.

In an attempt to generalize winding number as a topological fluid invariant, Berger [12] was led to writing down an integral formula which I recognised as "the" type two invariant of pure braids. His derivation would be considered far from rigorous by pure mathematicians, and he also claimed to derive a combinatorial formula for this. I was unconvinced by his combinatorial formula and tried to derive one myself.

Synopsis.

Chapter 1 starts by recalling definitions of Hopf algebras. It then goes on to introduce the Vassiliev filtration associated to some class of knotty objects: this is not the most intuitive way of defining Vassiliev invariants, but it allows the algebraic structure to be made explicit from the start. Section 3 looks at the well known case of unframed knots, and introduces chord diagrams and Kontsevich's Theorem. Section 4 goes into the case of framed knots and shows that the only difference can be polynomials in the framing number. This leads to some results on satelliting.

Chapter 2 came from an attempt at proving the Kontsevich Theorem from a naïve point of view — namely to go from the combinatorial data of a weight system to an actual knot invariant. The Vassiliev knot-space point of view is reviewed, as is Stanford's criterion for integrability of an invariant of knots with double points. Then the mirror image map is used to obtain a combinatorial formula for half of the integration steps.

The first two non-trivial Vassiliev knot invariants are examined in Chapter 3. The ideas of Chapter 2 are used to obtain simple bounds on them and used to derive Gauß diagram formulæ for them, like those of Polyak and Viro. In Section 3 the values of these invariants on the prime knots up to twelve crossings are plotted, and this reveals some remarkable patterns. The formulæ of Alvarez and Labastida for the values of these invariants on torus knots are used to prove some results suggested by the graphs for the torus knot case, and these two invariants for torus knots are related to the crossing and unknotting numbers.

Chapter 4 introduces pure braids together with useful group theoretic and topological properties. Vassiliev invariants of pure braids are seen as generalizing winding numbers in the sense that as winding numbers are dual to the abelianization of pure braid groups, so Vassiliev invariants are dual to higher nilpotent quotients of the pure braid groups (this is a cleaned up proof of observations of Kohno and Stanford). Finally these ideas are used to find explicit formulæ for numbers of “independent” invariants.

De Rham homotopy theory is the subject of Chapter 5. Iterated integrals are introduced and the geometric connection with Sullivan’s 1-minimal models is shown in terms of “higher order Albaneses”. (This material seems to be known to the experts [35] but appears not to be explicitly in the literature.) This is used to show how generators of 1-minimal models give rise to functionals on the fundamental group.

Chapter 6 specializes the theory of Chapter 5 to the case of classifying spaces of the pure braid groups to answer a question of M. A. Berger [12] on generalizing winding numbers. Thus a method is given for obtaining integral formulæ for generating sets of Vassiliev invariants. Low order invariants are tabulated. A combinatorial formula is obtained for the type two invariant, by replacing smooth 1-forms by currents supported on hyperplanes.

Appendix A contains some of the calculations from Chapter 6 and Appendix B lists some some questions and further problems.

CHAPTER 1

Abstract Vassiliev theory.

*And should I then presume?
And how should I begin?*

— T. S. ELIOT, *The Love Song of J. Alfred Prufrock*.

This chapter presents the abstract structure of Vassiliev theory in a manner dual to the approach of later chapters. This approach has the advantage of making the Hopf algebraic structure very explicit. It also provides a clear framework for two results from my Part III Essay [77].

Section 1 recalls some useful notions concerning bialgebras and Hopf algebras. Section 2 defines the Vassiliev bialgebra filtration for any monoid of knotty objects. Section 3 looks specifically at the case of knots, and Section 4 considers the extra structure of framing on knots.

1. Bialgebras and Hopf algebras.

First recall the definitions of a bialgebra. A bialgebra H over a commutative unital ring \mathcal{R} is an associative algebra over \mathcal{R} with a unit, $\epsilon: \mathcal{R} \rightarrow H$, and an augmentation $\eta: H \rightarrow \mathcal{R}$, and with a coassociative comultiplication $\Delta: H \rightarrow H \otimes H$ which is also a map of algebras and for which η is a counit.

Typically the comultiplication will be the adjoint of some naturally defined associative product on the linear dual, $H^\vee := \text{Hom}(H, \mathcal{R})$. A standard example of a bialgebra is the monoid algebra $\mathcal{R}\mathcal{M}$, of a monoid \mathcal{M} : the comultiplication comes from the natural pointwise product on the dual (which comes from the product on the ring \mathcal{R}) and this comultiplication is given explicitly by the linear map defined on the elements of \mathcal{M} by $m \mapsto m \otimes m$.

A Hopf algebra is a bialgebra equipped with an antipode $S: H \rightarrow H$ which is an anti-automorphism, so $S(ab) = S(b)S(a)$, and satisfies $m(S \otimes \text{id})\Delta = \epsilon\eta = m(\text{id} \otimes S)\Delta$. The group algebra $\mathcal{R}G$ of a group G is a Hopf algebra as the linear extension of the inverse of the group is an antipode.

Two important subsets of a bialgebra are the following: the *group-like* elements $\mathcal{G}(H) = \{g \in H: \Delta g = g \otimes g\}$ — these form a monoid, and a group if H is a Hopf

algebra; and the *primitive* elements $\mathcal{P}(H) = \{x \in H: \Delta x = 1 \otimes x + x \otimes 1\}$ — these form a Lie algebra under the commutator product: $[x, y] = xy - yx$.

There are notions of graded and of filtered bialgebras in which the underlying modules are \mathbb{Z} graded/filtered and all of the maps are of degree zero. A connected graded bialgebra is one which is trivial in negative gradings and in which the degree zero part is isomorphic to \mathcal{R} as a ring via the augmentation and unit maps.

For modules A, B the twist map $\tau: A \otimes B \rightarrow B \otimes A$ is defined via $a \otimes b \mapsto (-1)^{\deg(a)\deg(b)} b \otimes a$ where if A and B are not graded then take them to be concentrated in degree zero. A bialgebra is commutative if the multiplication m satisfies $m \circ \tau = m$ and cocommutative if $\tau \circ \Delta = \Delta$.

The structure of graded bialgebras was studied by Milnor and Moore in [54] and relevant results include the following:

THEOREM 1 ([54]). *Let H be a connected graded bialgebra.*

- (i) *For \mathcal{R} a characteristic zero field, H is generated by its primitive elements if and only if it is cocommutative.*
- (ii) *For \mathcal{R} a characteristic zero field, H is isomorphic to the polynomial algebra generated by its primitive elements if and only if it is cocommutative and commutative.*
- (iii) *H has an antipode (and so is a Hopf algebra). \square*

In the appendix to [62] Quillen studies the structure of a certain class of filtered bialgebras called *complete Hopf algebras*, however in the following result his hypotheses can be weakened. One requires the definition of the exponential map, $\exp x = \sum_{i=0}^{\infty} x^i/i!$, on the augmentation ideal, $\ker \eta$, so characteristic zero and some completeness are necessary.

THEOREM 2 ([62]). *The map \exp induces a bijection of sets $\mathcal{P}(H) \xrightarrow{\cong} \mathcal{G}(H)$ and the product on $\mathcal{G}(H)$ pulls back to a product on $\mathcal{P}(H)$ via the Baker-Campbell-Hausdorff formula. \square*

2. Vassiliev invariants and Hopf algebra structure.

Defining the Vassiliev invariants in the following manner makes the algebraic structure very explicit but is rather abstract. Consider some monoid \mathcal{K} of oriented knotty objects — e.g. knots with connected sum, links with distant union, braids on a fixed number of strands with braid composition. Fix a commutative ring \mathcal{R} with unity and division by two.¹ Vassiliev theory is concerned with a topologically defined bialgebra filtration on the monoid algebra $\mathcal{R}\mathcal{K}$.

¹Not strictly necessary, perhaps.

One approach to this filtration is via “knotty objects with double points”. A knotty object in \mathcal{K} will be something like an isotopy class of embeddings of a 1-manifold. In a knotty object with double points, the image of a map of this 1-manifold is allowed a finite number of transversal self-intersections — these are the double points. Such an object with double points is to be considered as an element of \mathcal{RK} via the formal resolution of the double points (also known as the Vassiliev skein relation), viz:

$$\times := \times - \times \in \mathcal{RK}.$$

This is supposed to be an equation on all sets of three diagrams which are identical except inside some ball where they look as shown. So a knotty object with n double points is considered as an alternating sum of 2^n “proper” knotty objects. This is well-defined if the knotty objects with double points are considered up to rigid vertex isotopy. The filtration \mathcal{F} is defined on \mathcal{RK} via

$$\mathcal{F}_n \mathcal{RK} = \{\mathcal{R}\text{-linear combinations of knotty objects with at least } n \text{ double points}\}.$$

For a reasonable product on \mathcal{K} (e.g. those mentioned above) the product of a knotty object with n double points and a knotty object with m double points is a knotty object with $(n + m)$ double points, thus the product respects the filtration. Indeed so does the coproduct. This can be demonstrated in terms of the following Leibniz Theorem, which I proved in the dual setting in [77].

If a knotty object has n double points, order them and denote it by $K(\times \dots \times)$. Let \times represent $1/2(\times + \times)$. Then for $I \in C_n := \{\times, \times\}^n$ let $K(I)$ be the linear combination of knots obtained from K by replacing the double points by the corresponding elements of I . Further, let $\hat{I} \in C_n$ be obtained by replacing each \times in I by a \times and *vice versa*. Then

THEOREM 3. *The coproduct respects the filtration, as*

$$\Delta\left(K(\underbrace{\times \dots \times}_n)\right) = \sum_{I \in C_n} K(I) \otimes K(\hat{I}),$$

and this is in $\mathcal{F}_n(\mathcal{RK} \otimes \mathcal{RK})$.

PROOF. This is proved by induction on the number of double points in K . The case $n = 0$ is trivial and the inductive step is as follows:

$$\begin{aligned} \Delta\left(K(\underbrace{\times \dots \times}_n)\right) &= \Delta\left(K(\underbrace{\times \dots \times}_{n-1} \times)\right) - \Delta\left(K(\underbrace{\times \dots \times}_{n-1} \times)\right) \\ &\stackrel{\text{ind}}{\text{hyp}} \sum_{J \in C_{n-1}} \left[K(J \times) \otimes K(\hat{J} \times) - K(J \times) \otimes K(\hat{J} \times) \right] \end{aligned}$$

$$\begin{aligned}
&= \sum_{J \in \mathcal{C}_{n-1}} \frac{1}{2} \left[(\mathbb{K}(J \times) - \mathbb{K}(J \times)) \otimes (\mathbb{K}(\hat{J} \times) + \mathbb{K}(\hat{J} \times)) \right. \\
&\quad \left. + (\mathbb{K}(J \times) + \mathbb{K}(J \times)) \otimes (\mathbb{K}(\hat{J} \times) - \mathbb{K}(\hat{J} \times)) \right] \\
&= \sum_{J \in \mathcal{C}_{n-1}} \left[\mathbb{K}(J \times) \otimes \mathbb{K}(\hat{J} \times) + \mathbb{K}(J \times) \otimes \mathbb{K}(\hat{J} \times) \right] \\
&= \sum_{I \in \mathcal{C}_n} \mathbb{K}(I) \otimes \mathbb{K}(\hat{I}).
\end{aligned}$$

□

The Vassiliev invariants are then defined as follows. An \mathcal{R} -valued invariant of \mathcal{K} is said to be Vassiliev of type n if its linear extension to \mathcal{RK} vanishes on all knotty objects with more than n double points. Denote the set of type n invariants by $\mathcal{V}^n \mathcal{RK}$ and the set of all the Vassiliev invariants, $\bigcup \mathcal{V}^n \mathcal{RK}$, by \mathcal{VRK} , then from the definitions,

$$\mathcal{V}^n \mathcal{RK} \cong (\mathcal{RK} / \mathcal{F}_{n+1} \mathcal{RK})^\vee.$$

Consider the graded object associated to the filtration:

$$\text{gr}_\bullet \mathcal{RK} := \bigoplus_{\hat{i}} \mathcal{F}_i \mathcal{RK} / \mathcal{F}_{i+1} \mathcal{RK},$$

the hat on the sum meaning completion with respect to the grading. This inherits a bialgebra structure. For the cases of \mathcal{K} above with $\mathcal{R} = \mathbb{Q}$ (or $\mathcal{R} \supset \mathbb{Q}$), Kontsevich's theorem [6, 45] gives a map $\tilde{Z}: \mathbb{Q}\mathcal{K} \rightarrow \mathcal{A}_\bullet \mathbb{Q}\mathcal{K}$, where $\mathcal{A}_\bullet \mathbb{Q}\mathcal{K} = \bigoplus_{\hat{i} \geq 0} \mathcal{A}_i \mathbb{Q}\mathcal{K}$ is a combinatorially defined object — a space of chord diagrams — and \tilde{Z} induces an isomorphism

$$\text{gr}_\bullet \mathbb{Q}\mathcal{K} \cong \mathcal{A}_\bullet \mathbb{Q}\mathcal{K}.$$

Dually this means that $\mathcal{V}^n \mathbb{Q}\mathcal{K} / \mathcal{V}^{n-1} \mathbb{Q}\mathcal{K} \cong (\mathcal{A}_n \mathbb{Q}\mathcal{K})^\vee$ and that $\mathcal{V}\mathbb{Q}\mathcal{K} \cong \bigoplus (\mathcal{A}_i \mathbb{Q}\mathcal{K})^\vee = (\mathcal{A}_\bullet \mathbb{Q}\mathcal{K})^\vee$, i.e. the filtration splits.²

Standard problems include the following:

- Prove Kontsevich's fundamental theorem from an elementary point of view (see [11] for various proofs and moral objections to them). This is part of the *raison d'être* of Chapter 2.
- Understand the structure of the \mathcal{A}_n , e.g. their dimensions, and further algebraic structure. This is considered for pure braids in Chapter 4.

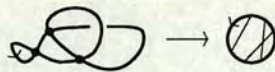
²However, the splitting is not, *a priori*, canonical.

- Find a way of calculating combinatorial formulæ for the invariants which generate the algebra \mathcal{V} . (This is related to both the above problems.) Chapters 2, 3, 5 and 6 look at this.
- Do Vassiliev invariants distinguish the knotty objects of \mathcal{K} ?
- How do Vassiliev invariants relate to classical invariants? To other topological constructions? These are considered in Chapters 3, 4, and 5.

3. The case of knots.

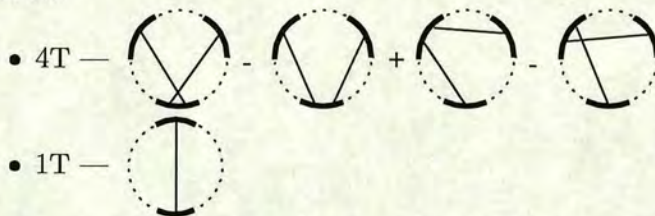
Let **Knot** be the monoid of oriented knots with connected sum, and let \mathcal{A}_\bullet be shorthand for the bialgebra $\mathcal{A}_\bullet \mathbb{Q}\mathbf{Knot}$, defined combinatorially in terms of chord diagrams as follows. A chord diagram of degree n is an oriented circle with n chords with distinct endpoints marked on it, considered up to orientation preserving diffeomorphisms of the circle (conventionally these are drawn with the orientation being anti-clockwise). An element of $\mathcal{F}_n \mathbb{Q}\mathbf{Knot} / \mathcal{F}_{n+1} \mathbb{Q}\mathbf{Knot}$ can be represented by a linear combination of knots with n double points. The underlying chord diagram of a knot with n double points is the chord diagram which indicates the points on the source circle which are identified at the double points.

E.g.



It is shown in [13] that two knots with n double points have the same underlying chord diagram if and only if they are related by a sequence of crossing changes; so if two knots with n double points have the same underlying chord diagram then as elements of $\mathcal{F}_n \mathbb{Q}\mathbf{Knot}$ they differ by an element of $\mathcal{F}_{n+1} \mathbb{Q}\mathbf{Knot}$.

Consider the subspaces $4T$ and $1T$ of the vector space of linear combinations of degree n chord diagrams which are generated respectively by the following kinds of elements:



where in the $4T$ the four diagrams an identical set of $n - 2$ chords are not marked, their endpoints lying on the dotted arcs; and the $1T$ generators have $n - 1$ other chords which do not intersect the pictured chord (called the isolated chord).

If a knot with n double points, K , has underlying chord diagram with an isolated chord then as an element of $\mathbb{Q}\mathbf{Knot} / \mathcal{F}_{n+1} \mathbb{Q}\mathbf{Knot}$ it can be represented by a knot with

double points which has a *nugatory* double point — this is precisely what is shown in the diagram:

$$[K] = \left[\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \right] = \left[\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right] - \left[\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \right] = 0,$$

as the resolution of the nugatory double point gives two isotopic knots with double points (give one half of one of them a full twist). This is not the case when considering framed knots, as will be mentioned again below.

An explanation of 4T is given in Chapter 2.

Define

$$\mathcal{A}_n := \langle \text{degree } n \text{ chord diagrams} \rangle_{\mathbb{Q}} / 4T, 1T.$$

Then Kontsevich's Theorem (see [6, 45]) gives a map $\tilde{Z}: \mathbb{Q}\mathbf{Knot} \rightarrow \mathcal{A}_\bullet$ inducing isomorphisms

$$\mathcal{F}_n \mathbb{Q}\mathbf{Knot} / \mathcal{F}_{n+1} \mathbb{Q}\mathbf{Knot} \xrightarrow{\cong} \mathcal{A}_n$$

which are given by the “underlying chord diagram” maps.

Thus \mathcal{A}_\bullet inherits the bialgebra structure of $\mathbb{Q}\mathbf{Knot}$, and it is immediate that these are as given in [6]. For example, the product is given by “connected sum”, as illustrated by the following:

$$\bigotimes \cdot \bigotimes = \text{---} \bigcirc \text{---} \bigcirc \text{---} = \bigotimes.$$

Although this seems to be dependent on where the chord diagrams are cut, it is in fact well-defined modulo the 4T relation [6]. Further, \mathcal{A}_\bullet is connected graded, so by Theorem 1 it is actually a Hopf algebra. Perhaps this is related to the fact that $\mathbf{Knot} / \mathcal{F}_n \mathbb{Q}\mathbf{Knot}$ is actually a group (rather than just a monoid) — see [30, 57].

In fact, in the normalization considered here, the Kontsevich integral, $\tilde{Z}: \mathbb{Q}\mathbf{Knot} \rightarrow \mathcal{A}_\bullet$, is a map of *filtered* bialgebras (this being stronger than just descending to a bialgebra map on the associated graded objects). This means, for instance that \tilde{Z} is multiplicative under the connected sum of knots, i.e. $\tilde{Z}(K \# K') = \tilde{Z}(K) \cdot \tilde{Z}(K')$, and that knots get mapped to group-like elements in \mathcal{A}_\bullet .

As a monoid \mathbf{Knot} does not have a lot of structure, it is just a commutative monoid on a countably infinite set of generators (the prime knots), but the topological information in the filtration is very rich; this should be compared with the case of pure braids on k strands, where the algebraic structure is a lot richer, but the Vassiliev filtration is very easy to understand (see Chapter 4 and on).

$\mathbb{Q}\mathbf{Knot}$, and hence also \mathcal{A}_\bullet , is commutative and cocommutative so by Theorem 1, \mathcal{A}_\bullet is generated as an algebra by its primitive elements. Further as it is complete, the exponential function defines a bijection between the primitive elements and the

group-like elements. Thus

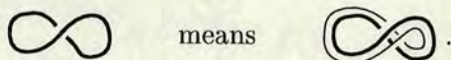
$$\log \circ \tilde{Z}: \mathbf{Knot} \rightarrow \mathcal{P}(\mathcal{A}_\bullet)$$

is well defined and additive under connected sum.

Some structure of \mathbf{Knot} can be read in \mathcal{A}_\bullet , for instance the mirror image map, $\mu: \mathbf{Knot} \rightarrow \mathbf{Knot}$. Defining $J: \mathcal{A}_\bullet \rightarrow \mathcal{A}_\bullet$ on homogeneous elements by $x \mapsto (-1)^{\deg x} x$ then μ and J intertwine \tilde{Z} ; further, as J is an algebra map, μ and J also intertwine $\log \circ \tilde{Z}$. This symmetry is utilized in Chapter 2.

4. The case of framed knots.

A framed knot is a knot equipped with a homotopy class of a non-vanishing section of its normal bundle: this can be thought of as being a “parallel” copy of the knot arbitrarily close, or else a framed knot can be thought of as an embedded orientable ribbon rather than an embedded string. Framed knots will be drawn with the “blackboard framing”, i.e. the parallel copy is pushed off the knot in the plane of the paper, so



For knots in \mathbb{R}^3 or S^3 (which are what is being considered here), the framing can be described by the *framing number* — this is the Gauß linking number of the knot and a parallel copy “pushed-off” along the framing.

4.1. The framed and unframed Kontsevich integrals. The set of oriented knots, \mathbf{Knot} , can be thought of as sitting inside the set of framed oriented knots, ${}^f\mathbf{Knot}$ — in three-space every knot has a canonical framing, the zero framing. Connected sum can be defined on framed knots, the underlying unframed knots are connect summed and the framing numbers are added together. Thus the “assign zero framing” map, $\xi: \mathbf{Knot} \rightarrow {}^f\mathbf{Knot}$, is a monoid map and ${}^f\mathbf{Knot}$ is just a trivial extension of \mathbf{Knot} by \mathbb{Z} . The extra generators for ${}^f\mathbf{Knot}$ are the ± 1 framed unknots:

$${}^f\mathbf{Knot} \cong \mathbf{Knot}[\infty, \infty] \cong \mathbf{Knot} \times \mathbb{Z}.$$

By [65], both the zero framing map, ξ , and the forgetful map $\mathcal{U}: {}^f\mathbf{Knot} \rightarrow \mathbf{Knot}$ respect the Vassiliev filtrations. In the Vassiliev theory for framed knots the 1T relation does not hold — for instance, the resolution of a nugatory double point for a knot with one double point gives two framed knots with a difference of two in framing number. So let ${}^f\mathcal{A}_\bullet := \mathcal{A}_\bullet \cdot \mathbb{Q} \cdot {}^f\mathbf{Knot} := \langle \text{chord diagrams} \rangle_{\mathbb{Q}} / 4T$. There is the natural “quotient by 1T” map $\mathcal{U}_{\mathcal{A}}: {}^f\mathcal{A}_\bullet \rightarrow \mathcal{A}_\bullet$.

Le and Murakami [47] define a framed version of the Kontsevich integral, which is a bialgebra map ${}^f\tilde{Z}: \mathbb{Q} \cdot {}^f\mathbf{Knot} \rightarrow {}^f\mathcal{A}_\bullet$ (although they really consider a slightly different

normalization). They show the following behaviour under change of framing:

$${}^f\tilde{Z}(K\# \infty) = {}^f\tilde{Z}(K) \cdot \exp(\Theta/2),$$

and also that $\mathbb{U}_{\mathcal{A}} \circ {}^f\tilde{Z} = \tilde{Z} \circ \mathbb{U}$. Furthermore, as expected (\tilde{Z} maps the unit to the unit) the unframed unknot maps to the empty chord diagram.

DEFINITION 1. Define $\xi_{\mathcal{A}}: \mathcal{A}_{\bullet} \rightarrow {}^f\mathcal{A}_{\bullet}$ as follows. For D a chord diagram and S a subset of its chords, let $D_{\setminus S}$ be the diagram obtained from D by removing the chords in S . Define

$$\xi_{\mathcal{A}}(D) = \sum_{\text{chord subsets } S} (-\Theta)^{|S|} \cdot D_{\setminus S},$$

and extend linearly.

E.g.

$$\begin{aligned} \xi_{\mathcal{A}}(\text{⊗}) &= \text{⊗} + 3(-\ominus) \cdot \text{⊗} + 3(-\ominus)^2 \cdot \text{⊗} + (-\ominus)^3 \cdot \text{⊗} \\ &= \text{⊗} - 3\text{⊗} + 2\text{⊗} \end{aligned}$$

In [77] I proved the following.

THEOREM 4. $\xi_{\mathcal{A}}$ is a well defined map of Hopf algebras and satisfies $\mathbb{U}_{\mathcal{A}} \circ \xi_{\mathcal{A}} = id$. □

COROLLARY 5. There is a Hopf algebra isomorphism ${}^f\mathcal{A}_{\bullet} \cong \mathcal{A}_{\bullet}[\Theta]$, and a vector space isomorphism $\mathcal{P}({}^f\mathcal{A}_{\bullet}) \cong \mathcal{P}(\mathcal{A}_{\bullet}) \oplus \langle \Theta \rangle_{\mathbb{Q}}$, with $\xi_{\mathcal{A}}|_{\mathcal{P}}$ being the natural inclusion. □

Define the map $w: {}^f\mathbf{Knot} \rightarrow \mathbb{Z}$ to be the framing number, and the map $w_{\mathcal{A}}: {}^f\mathcal{A}_{\bullet} \rightarrow \mathbb{Q}$ by taking twice the coefficient of Θ .

The precise formalism of the relationship between the framed and unframed cases given in this next theorem is interestingly absent from the literature — perhaps because of the penchant for a slightly different normalization which behaves well on links.

THEOREM 6. The following diagram of monoids commutes and the vertical lines are short exact:

$$\begin{array}{ccccc} \mathbb{Z} & \hookrightarrow & \mathbb{Q} & = & \mathbb{Q} \\ w \uparrow & & w_{\mathcal{A}} \uparrow & & w_{\mathcal{A}} \uparrow \\ {}^f\mathbf{Knot} & \xrightarrow{{}^f\tilde{Z}} & \mathcal{G}({}^f\mathcal{A}_{\bullet}) & \xrightarrow{\log} & \mathcal{P}({}^f\mathcal{A}_{\bullet}) \\ \xi \uparrow & & \xi_{\mathcal{A}} \uparrow & & \xi_{\mathcal{A}} \uparrow \\ \mathbf{Knot} & \xrightarrow{\tilde{Z}} & \mathcal{G}(\mathcal{A}_{\bullet}) & \xrightarrow{\log} & \mathcal{P}(\mathcal{A}_{\bullet}). \end{array}$$

PROOF. The fact that $\xi_{\mathcal{A}}$ is a coalgebra maps means that it restricts to a map of primitive elements and to a map of group-like elements. The fact that it is an algebra map means that it commutes with \log and \exp .

The commutativity of the top right square is due to the fact that a group-like element of ${}^f\mathcal{A}_\bullet$ is of the form

$$\bigcirc + \frac{w}{2} \ominus + \text{terms of degree 2 and greater.}$$

(Remember a group-like element is in the image of \exp , and that there is only one chord diagram of degree zero and one of degree one.)

The exactness of the left and right columns is immediate. The exactness of the centre column follows from the exactness of the right hand column, the commutativity of the right-hand side, and the fact that \log is an isomorphism.

Consider the map $w_{\mathcal{A}} \circ {}^f\tilde{Z} \circ \xi$, by the usual yoga this must be a type one invariant of unframed knots, and the only such are constant invariants. Looking at the value of ${}^f\tilde{Z}$ on the zero framed unknot above, it is seen that $w_{\mathcal{A}} \circ {}^f\tilde{Z} \circ \xi$ must be the zero map. Further, under the behaviour of connect summing with a nonzero framed unknot as above, it is seen that $w_{\mathcal{A}} \circ {}^f\tilde{Z}$ is precisely the framing number, w .

Finally it is necessary to show that the bottom left square commutes. From the last paragraph, ${}^f\tilde{Z} \circ \xi$ maps into the kernel of $w_{\mathcal{A}}|_{\mathcal{G}}$, but $\xi_{\mathcal{A}}|_{\mathcal{G}}$ is an isomorphism onto the kernel of $w_{\mathcal{A}}|_{\mathcal{G}}$ with inverse $U_{\mathcal{A}}$, and also U is a left inverse of ξ , so $U_{\mathcal{A}} \circ {}^f\tilde{Z} = \tilde{Z} \circ U$ from above gives ${}^f\tilde{Z} \circ \xi = \xi_{\mathcal{A}} \circ \tilde{Z}$. \square

4.2. Relationship with Vassiliev invariant. The map ${}^f\tilde{Z}$ is *not* a universal Vassiliev invariant for framed knots; this fact was observed by Kassel and Turaev [39]. The problem is essentially that the crossing change operation preserves the *parity* of framing number.³ Thus, for instance, there is a non-constant type zero invariant which evaluates to one on knots with odd framing number and evaluates to zero on knots with even framing number. However, a universal Vassiliev invariant can be defined in the following way.

Let $\mathbb{Q}(\mathbb{Z}/2)$ be the group (Hopf) algebra of $\mathbb{Z}/2$, considered graded of degree zero. Write $\mathbb{Z}/2$ multiplicatively with elements 1 and $\bar{1}$. For K a framed knot, let $r(K) \in \mathbb{Z}/2$ be the mod 2 reduction of the framing number of K (i.e. even framing maps to 1, odd framing maps to $\bar{1}$). Then define ${}^{ff}\tilde{Z}: \mathbb{Q}^f\mathbf{Knot} \rightarrow \mathbb{Q}(\mathbb{Z}/2) \otimes_{\mathbb{Q}} {}^f\mathcal{A}_\bullet$ by $K \mapsto r(K) \otimes {}^f\tilde{Z}(K)$. It is easy to see that ${}^{ff}\tilde{Z}$ is still a bialgebra map. It is also a universal Vassiliev invariant

³In the sense of the next chapter, the space of singular framed knots has two path components.

in the sense that it induces isomorphisms

$$\mathcal{F}_n \mathbb{Q}^f \mathbf{Knot} / \mathcal{F}_{n+1} \mathbb{Q}^f \mathbf{Knot} \xrightarrow{\cong} \mathbb{Q}(\mathbb{Z}/2) \otimes {}^f \mathcal{A}_n.$$

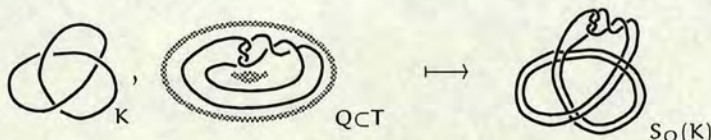
This means that a Vassiliev invariant of framed knots is the sum of an invariant of knots with even framing number and an invariant of knots with odd framing number, each of these invariants factoring through ${}^f \tilde{\mathbb{Z}}$. Combining this with Theorem 6 one obtains the following:

PROPOSITION 7. *A finite type invariant of framed knots is of the form $p(K) + (-1)^{w(K)} q(K)$ where $p(K)$ and $q(K)$ are polynomials in finite type invariants of unframed knots and the framing number, w .*

PROOF. By the Hopf algebra structure theory, an element of ${}^f \mathcal{A}_\bullet$ is a polynomial in Θ and elements of \mathcal{A}_\bullet . So by the above, the invariant restricted to even framed knots is some polynomial, $x(K)$, in finite type invariants of unframed knots and the framing number, w ; the restriction to odd framed knots is also such a polynomial, $y(K)$. One can then just take $p(K) = \frac{1}{2}(x(K) + y(K))$ and $q(K) = \frac{1}{2}(x(K) - y(K))$. \square

4.3. Satelliting operations. Framed knots have the natural operations of satelliting on them. These are well behaved with respect to the Vassiliev theory. First recall the definition of satelliting. For Q a knot in a solid torus T and K a framed knot, define $S_Q(K)$ to be the (unframed) knot formed by replacing a tubular neighbourhood of K by the solid torus containing T , the longitude of T being glued in as prescribed by the framing of K . In this case $S_Q(K)$ is called a satellite knot, Q is called the pattern and K is the companion.

E.g.



For a fixed pattern Q , the satelliting map $S_Q: {}^f \mathbf{Knot} \rightarrow \mathbf{Knot}$ preserves the Vassiliev filtration [65, 60]. Dualizing to the invariants, this means that

PROPOSITION 8 ([65, 60]). *If v is a type n invariant of unframed knots and Q is some fixed pattern then $v \circ S_Q$ is a type n invariant of framed knots.* \square

Combining this with Proposition 7 one obtains the following corollary:

COROLLARY 9. *Let K_i denote the knot K equipped with its i th framing, let Q be some fixed pattern and let v be an invariant of unframed knots of type n . As a function of i , $v \circ S_Q(K_i)$ is of the form $\bar{p}(i) + (-1)^i \bar{q}(i)$ where \bar{p} and \bar{q} are polynomials of degree at most n .*

Note that I don't know if \bar{q} is ever non-zero.

PROOF. By Propositions 7 and 8, $v \circ S_Q(K_i)$ is of the form $p(K_i) + (-1)^i q(K_i)$ where p and q are polynomials in finite type invariants of unframed knots and the framing number, but the underlying unframed knot is constant and the framing number is i , thus $v \circ S_Q(K_i)$ is of the required form. \square

Let W_h be the Whitehead pattern, drawn as



The following I believe was observed by X.-S. Lin and was communicated to me by Matt Greenwood at Knots 95 in Warsaw. The proof is my own.

THEOREM 10. *The Whitehead satelliting map S_{W_h} in general increases the Vassiliev filtration by one, i.e. induces maps $\mathcal{F}_n \mathbb{Q}^f \mathbf{Knot} \rightarrow \mathcal{F}_{n+1} \mathbb{Q} \mathbf{Knot}$ for $n \geq 1$. Dualizing, this means that if v is a type $n \geq 1$ invariant of unframed knots then $v \circ S_{W_h}$ is a type $n - 1$ invariant.*

PROOF. Consider the case of a framed knot with just one double point, the general case is the same but messier to write down. Concentrate on a neighbourhood of the double point and the place to where the "clasp" gets mapped:

$$\begin{aligned}
 S_{W_h}(\text{X } \curvearrowright) &= S_{W_h}(\text{X } \curvearrowright) - S_{W_h}(\text{X } \curvearrowright) \\
 &= (\text{X } \curvearrowright) - (\text{X } \curvearrowright) \\
 &= (\text{X } \curvearrowright) + (\text{X } \curvearrowright) \\
 &\quad - (\text{X } \curvearrowright) - (\text{X } \curvearrowright).
 \end{aligned}$$

But the two with unlinked clasps are unknots, and hence cancel. Thus

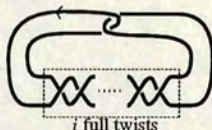
$$S_{W_h}(\text{X } \curvearrowright) = (\text{X } \curvearrowright) - (\text{X } \curvearrowright)$$

$$\begin{aligned}
&= \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) \left(\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right) - \left(\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \right) \left(\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right) \\
&\quad + \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) \left(\begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array} \right) - \left(\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \right) \left(\begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array} \right) \\
&\in \mathcal{F}_2\mathbb{Q}\mathbf{Knot}.
\end{aligned}$$

In general, a framed knot with n double points will map to an alternating sum of 4^n knots with $n + 1$ double points — one at the clasp and one at each of the sites of the original double points.

Note that when dualizing, the case $n = 1$ has to be considered separately, but that type one invariants of unframed knots are just constants, so induce a constant invariant on framed knots. \square

The i th twisted double of the unknot, $\text{Wh}(i)$, is the Whitehead satellite of the i -framed unknot. It can be pictured as



where i full twists means i positive full twists if i is positive and $-i$ negative full twists if i is negative.

COROLLARY 11. *If ν is a type n invariant of unframed knots, then, as a function of i , $\nu(\text{Wh}(i))$ is a polynomial of degree at most $n - 1$.*

PROOF. By Theorem 10 and the argument of Corollary 9, $\nu(\text{Wh}(i))$ is of the form $\bar{p}(i) + (-1)^i \bar{q}(i)$, where \bar{p} and \bar{q} are polynomials of degree at most $n - 1$. But by the result of Dean [22] and Trapp [73] on twist sequences, $\nu(\text{Wh}(i))$ is a polynomial in i of degree at most n , thus \bar{q} is the zero polynomial. \square

This will be revisited in Chapter 3.

CHAPTER 2

Half integration for knots.

Why, it's a looking glass book of course! And if I hold it up to a glass, the words will all go the right way round again.

— ALICE, *Through the Looking Glass*.

This chapter is concerned with attempting to prove Kontsevich's Theorem from a naïve point of view. Section 1 introduces the knot space approach to Vassiliev invariants, more akin to Vassiliev's own approach than that of Chapter 1. Section 1.2 presents Stanford's results on necessary and sufficient conditions for extending an invariant of knots with $i + 1$ double points to knots with i double points. In Section 2 it is shown that this can be achieved in a canonical fashion when a certain symmetry is present — in the integration of a weight system this will happen automatically in at least half the steps.

1. Introduction to the knot-space view.

1.1. Vassiliev knot-space view. Vassiliev's original spectral sequence approach was rather different to the approach of the Chapter 1, and was axiomatized by Birman and Lin in [13]; it can be described in the following manner. Consider the suitably topologised space of smooth maps $S^1 \rightarrow \mathbb{R}^3$ with at most a finite number of transversal self-intersections — double points. This space is stratified, the i th stratum consisting of knots with i double points. So the top stratum is an open dense set consisting of “proper” knots, the connected components corresponding to the distinct knot types. These “chambers” are separated by “walls” consisting of knots with a single double point. A path in one of these chambers corresponds to a continuous deformation of a knot, and a path which passes transversally through a wall corresponds to pushing one piece of the knot through another — a so-called crossing change.

As the embeddings are oriented, the walls are co-oriented, a transverse path intersects a wall positively if it corresponds to a negative crossing (\times) becoming a positive crossing (\times).

A knot invariant, v , taking values in some abelian group G , corresponds to an assignment of an element of the group to each connected component of the top stratum: this can be extended to knots with a single double point by considering how the

function jumps on a path through knot space passing transversely through the wall corresponding to the knot with a double point. Symbolically this is written as

$$v(\times) = v(\nearrow) - v(\searrow).$$

Inductively this can be continued to the higher codimension strata, i.e. v can be extended to knots with an arbitrary number of double points. Note that this process leads to invariants of rigid vertex isotopy.

A knot invariant is said to be Vassiliev if it vanishes on all strata of sufficiently large codimension, and specifically is said to be of type n if it vanishes on all strata of codimension greater than n (i.e. on knots with more than n double points). This is easily seen to be the same as the definition given in Chapter 1. It should be noted that these knots with double points are not the same objects as considered in Chapter 1. There they were considered as elements of \mathcal{RKnot} and so had various relations imposed on them such as $\infty = 0$. Here they are considered as objects with no relations between them (but the invariants defined on them will turn out to satisfy certain relations).

This framework has certain conceptual advantages over the abstract definition given earlier. The idea of measuring “jumps” in the knot invariant as it passes through a wall can be thought of as being analogous to differentiating the invariant at that double point. This leads to the Vassiliev condition being analogous to some vanishing derivatives condition, i.e. “Vassiliev invariants are like polynomials”. In fact this is a reasonably fruitful paradigm as has been seen in Chapter 1, and as will be seen below and in Chapter 3.

1.2. Integration and Stanford’s Theorem. Some of the notions of the above paragraph can be generalized in the following useful definitions.

- DEFINITION 2. • A singular isotopy, $\Phi(t)$, of knots with n double points is a path in the union of the n th and $(n+1)$ th strata such that the path only intersects the $(n+1)$ th stratum transversally and a finite number of times (see Figure 2.1). The intersections $\{\Phi_s: 1 \leq s \leq r\}$ of the path with the $(n+1)$ th strata are called the *singularities* of the singular isotopy and the indices $\{\sigma_s = \pm 1: 1 \leq s \leq r\}$ give the signs of the corresponding intersection.
- An invariant of knots with i double points (considered up to rigid vertex isotopy) will be called an *i-invariant*.
 - If P and Q are respectively an i -invariant and an $(i-1)$ -invariant which satisfy

$$P(\times) = Q(\nearrow) - Q(\searrow),$$

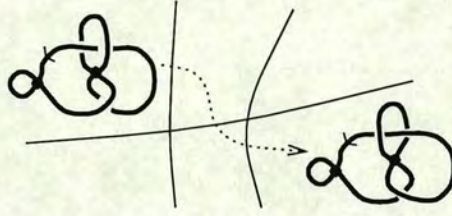


Figure 2.1. A singular isotopy of knots with two double points, the passage through the walls corresponding to crossing changes.

then say that P is the derivative of Q , Q is an integral of P and write $Q' = P$. (So P gives the “jumps” of Q .)

Using this terminology, if Q is an i -invariant for which a derivative exists (a necessary and sufficient condition is given below) and $\Phi(t)$ is a singular isotopy of knots with i double points from k_0 to k_1 , then

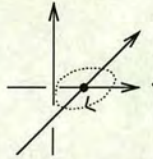
$$(*) \quad Q(k_1) - Q(k_0) = \sum_{s=1}^r \sigma_s Q'(\Phi_s).$$

Specifically, if the singular isotopy is closed (i.e. $k_0 = k_1$) then both sides vanish.

One wants to know which i -invariants come from genuine knot invariants, and similarly which i -invariants are derivatives of $(i - 1)$ -invariants.

Suppose that P is an i -invariant which is the derivative of some $(i - 1)$ -invariant, then it must satisfy the following relations.

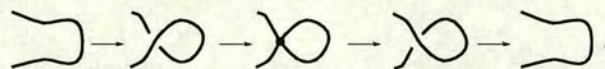
- (i) *Topological four term relation (T4T)*: Consider the closed singular isotopy of knots with $i - 1$ double points given by passing the vertical strand shown below through the others in the indicated manner:



Then $(*)$ gives the T4T relation:

$$P\left(\begin{array}{c} \uparrow \\ \diagdown \\ \diagup \\ \downarrow \end{array}\right) - P\left(\begin{array}{c} \uparrow \\ \diagup \\ \diagdown \\ \downarrow \end{array}\right) - P\left(\begin{array}{c} \uparrow \\ \downarrow \\ \diagdown \\ \diagup \end{array}\right) + P\left(\begin{array}{c} \uparrow \\ \downarrow \\ \diagup \\ \diagdown \end{array}\right) = 0.$$

- (ii) *Topological one term relation (T1T)*: Similarly consider the closed singular isotopy given by



Apply (*) to obtain:

$$P(\bigcirc) = 0.$$

(iii) *Differentiability*: Here the closed singular isotopy

$$\times \times \rightarrow \times \times \rightarrow \times \times \rightarrow \times \times \rightarrow \times \times$$

gives (after reordering)

$$P(\times \times) - P(\times \times) = P(\times \times) - P(\times \times).$$

REMARK 1. • It can be seen that (iii) is the necessary and sufficient condition alluded to above for P to have a derivative — it is saying that it does not matter which one of the $i + 1$ double points is resolved.

- The above three closed singular isotopies can be thought of as loops around respectively the following ‘higher order singularities’: a knot with $i - 1$ double points and one triple point; a knot with i double points and one cusp singularity; and a knot with $i + 1$ double points.

In his thesis (see [68]) Stanford proved the following.

THEOREM 12. *The above three conditions are necessary and sufficient for P to be the derivative of some $(i - 1)$ -invariant.*

In view of this, say that an i -invariant is *integrable* if it satisfies the above three conditions.

Unfortunately Stanford’s Theorem is inadequate in the following ways:

- it is non-constructive;
- it says nothing about whether the integral of an integrable invariant is itself integrable;
- if an invariant P does integrate to an invariant Q , then Q is in general not unique, if W is a differentiable $(i - 1)$ -invariant with $W' = 0$ then $Q + W$ is also an integral of P : i.e. one can add on constants of integration.

Below it will be seen that these problems can be overcome in cases displaying a reasonable symmetry.

1.3. Weight systems. A degree i weight system is an integrable i -invariant which has zero derivative. In particular it follows that the n th derivative of a type n knot invariant is a degree n weight system. Kontsevich’s Theorem implies that, over \mathbb{Q} , every weight system comes from a knot invariant in this way. The goal is to try to show this combinatorially, and possibly prove it for rings other than \mathbb{Q} , by integrating from a weight system to a knot invariant.

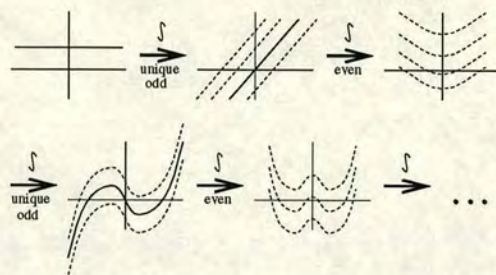


Figure 2.2. Semi-systematic integration of even and odd polynomials.

A key property of weight systems is that they are invariant under crossing changes. This is immediate as the derivative measures the jump under crossing changes and here the derivative vanishes.

2. Half integration.

In view of the problem of integration and the analogy with polynomials, consider a slight digression:

2.1. Integrating odd and even polynomials. If one wants to integrate from a constant function on the real line to a degree n polynomial, then one must perform n integrations and so pick up n constants of integration. It is natural to want to do this canonically. One way of doing this is to consider odd and even functions, i.e. the ± 1 eigenspaces for the involution adjoint to $x \mapsto -x$ on the real line (which is the range).

Note that the constants of integration are the constant functions and that these are even. By a simple direct sum decomposition argument this means that if a function has an even integral then all of its integrals are even and if a function has an odd integral then this is the unique odd integral.

Getting back to the integration of a constant function, one proceeds by taking the unique linear odd function which is its integral. This can then be integrated to a (non-unique) even function, then to a unique (given the previous choice) odd function. This continues until n integrations have been performed, $\lceil \frac{n}{2} \rceil^1$ of which will have been canonical. See Figure 2.2.

2.2. Back to finite type invariants. To apply the above argument it transpires that the involution on the space of knots with i double points that is relevant is that of the mirror image: $k \mapsto \bar{k}$. This decomposes the space of i -invariants into a $+1$

¹The ceiling function $\lceil x \rceil$ returns the least integer greater than or equal to x .

eigenspace — the even² invariants, and a -1 eigenspace — the odd invariants. The following should make apparent the above discussion of polynomials:

FACT 1. (i) [74, Theorem V.5.2] *If v is a type n knot invariant then one can find some invariant, v_1 , of lower degree so that $v + v_1$ is odd/even as n is respectively odd/even.*

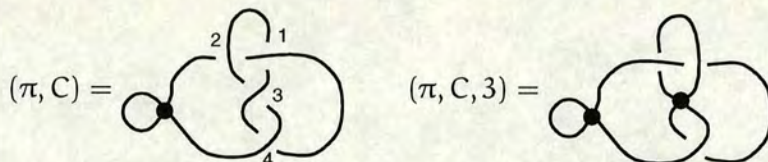
(ii) [23, implicit] *A weight system is even.*

The first was noted by Vassiliev when he considered the effect of orientation reversal of \mathbb{R}^3 on his spectral sequence and the second is clear as there is a singular isotopy from a projection of a knot with n double points to its mirror image obtained by flipping each of the crossings in the projection in turn.

In view of the latter, make the following definition which identifies the singularities in the singular isotopy from a knot projection by switching the crossings in some given order:

DEFINITION 3. If π is a regular projection of a knot with i double points, $C = \{a_1, \dots, a_r\}$ is an ordering of the crossings in π , and $1 \leq s \leq r$ then define (π, C, s) to be the knot with $i + 1$ double points obtained by switching the first $s - 1$ crossings of π and replacing the crossing a_s with a double point.

E.g.



Note that the index, σ_s , of the s th singularity, (π, C, s) , in the singular isotopy, is *minus* the sign, $\epsilon(a_s)$, of the s th crossing, a_s .

Domergue and Donato [23] gave a simple formula for integrating a type n weight system to an $(n - 1)$ -invariant; this can be generalized considerably in the following manner:

THEOREM 13. (i) *If P is an integrable even i -invariant, then it has a unique odd integral Q . Further, Q is itself integrable and is given on a projection π of a knot with $i - 1$ double points, k , by*

$$Q(k) = \frac{1}{2} \sum_{s=1}^r \epsilon(a_s) P((\pi, C, s)),$$

for any ordering C .

²Not to be confused with Bar-Natan's *inversion* even invariants [6].

(ii) If Q is an integrable odd i -invariant, then all integrals of Q are even.


PROOF. (i) Suppose that P is an integrable even i -invariant then by Stanford's Theorem there exists an $(i - 1)$ -invariant, \hat{Q} , which is an integral of P . Let Q be the odd part of \hat{Q} , so $Q(k) = \frac{1}{2}(\hat{Q}(k) - \hat{Q}(\bar{k}))$. Note that, using the overline notation, $\overline{\times}$ means the selected crossing is positive *before* the mirror image is taken. Q is differentiable and:

$$\begin{aligned} Q'(\times) &= Q(\times) - Q(\overline{\times}) \\ &= \frac{1}{2}(\hat{Q}(\times) - \hat{Q}(\overline{\times}) - \hat{Q}(\times) + \hat{Q}(\overline{\overline{\times}})) \\ &= \frac{1}{2}(\hat{Q}(\times) - \hat{Q}(\times) + \hat{Q}(\overline{\times}) - \hat{Q}(\overline{\overline{\times}})) \\ &= \frac{1}{2}(P(\times) + P(\overline{\overline{\times}})) \\ &= P(\times), \end{aligned}$$

as P is even. Thus Q is an odd integral of P . Equivalently, it could be shown (as in [75]) that the even part of \hat{Q} is just a weight system.

To see for a projection, π , of a knot with double points, k , that Q is given by the formula, observe first that as Q is odd it satisfies $Q(k) = 1/2(Q(k) - Q(\bar{k}))$ and the ordering of the crossings gives a singular isotopy from k to \bar{k} , then apply (*).

Finally it is necessary to show that Q satisfies the T4T and T1T relations.

For T4T consider the term in the T4T expression which looks locally like , and suppose that C is an ordering of the crossings which has the pictured crossing as the final crossing. Then

$$Q\left(\begin{array}{c} \uparrow \\ \times \\ \downarrow \end{array}\right) = \left[\frac{1}{2} \sum_{s=1}^{r-1} \epsilon(a_s) P\left(\left(\begin{array}{c} \uparrow \\ \times \\ \downarrow \end{array}, C, s\right)\right) \right] + \frac{1}{2} P\left(\overline{\begin{array}{c} \uparrow \\ \times \\ \downarrow \end{array}}\right).$$

Taking corresponding orderings of the crossings in the other three terms in the T4T gives similar expressions. On summing them together, the four last terms cancel in pairs and the four sums together cancel out because P satisfies the T4T.

T1T is straightforward:

$$Q\left(\begin{array}{c} \times \\ \times \end{array}\right) = \frac{1}{2} \sum_s \epsilon(a_s) P\left(\left(\begin{array}{c} \times \\ \times \end{array}, C, s\right)\right) = 0,$$

as P satisfies T1T.

(ii) Suppose that R is an integral of the odd i -invariant Q , that k is a knot with $i - 1$ double points, and that Φ is a singular isotopy from k to \bar{k} . Then $\overline{\Phi}$ is a singular

isotopy from \bar{k} to k ; $\overline{(\Phi)}_s = \overline{(\Phi_s)}$ are the singularities of $\overline{\Phi}$; and $\bar{\sigma}_s = -\sigma_s$ are the indices of the singularities. So by (*),

$$\begin{aligned} R(\bar{k}) - R(k) &= \sum_s \sigma_s Q(\Phi_s) = \sum_s -\sigma_s Q(\overline{(\Phi_s)}) \\ &= \sum_s \bar{\sigma}_s Q(\overline{(\Phi)}_s) \\ &= R(k) - R(\bar{k}), \end{aligned}$$

whence $R(\bar{k}) = R(k)$. □

REMARK 2. In [75], as well as a sketch of the above proof, a proof was given which doesn't require Stanford's Theorem, and instead utilizes the folk Reidemeister Theorem for knots with double points which is stated in [40] but has not been proved. Interestingly, Stanford [66] has since given a proof of his theorem based on the folk Reidemeister Theorem.

The hope is now to find some method of integrating integrable odd invariants, though no progress has yet been made, but this would give a proof of a Kontsevich-type theorem. The best so far is the following:

COROLLARY 14. *If the transition from a degree n weight system to a type n knot invariant is thought of as n steps of integration (e.g. the completion of an actuality table — see [13]), then at least half of the steps can be performed canonically.*

PROOF. A weight system is an even integrable n -invariant, this integrates to a unique integrable, odd $(n - 1)$ invariant. This integrates, by whatever process, to an even $(n - 2)$ invariant. The procedure continues until an odd or even knot invariant is obtained. □

CHAPTER 3

On v_2 and v_3 .

'First the fish must be caught.'
That is easy: a baby, I think, could have caught it.
— THE RED QUEEN, *Through the Looking Glass*.

This chapter is a *mélange* of bits and pieces on the first two non-trivial Vassiliev knot invariants. In Sections 1 and 2 bounds and formulæ for them are determined. Section 3 is concerned with looking at the actual values of v_2 and v_3 for the knots up to twelve crossings, from which some interesting graphs are obtained and in Section 4 formulæ of Alvarez and Labastida for torus knots are considered. Some of the investigations here were inspired by a weekend reading of [1].

1. Bounds on v_2 and v_3 .

As mentioned at the end of the last chapter, there is some work to be done before there is a naïve canonical way of going from weight system to knot invariant. However, at low order, certain tricks and lack of ambiguity can be utilized to obtain genuine canonical knot invariants from the degree two and degree three weight systems. Recall that there is one weight system (up to multiplication by a scalar) of degree zero, none of degree one, one of degree two and one of degree three. This can be translated to one (up to scaling) even degree two invariant vanishing on the unknot and one odd degree three invariant. Normalize these so that they take value plus one on the positive trefoil and denote them respectively v_2 and v_3 . Note that these are additive with respect to connected sum.

In degree four the space of primitive weight systems is two dimensional: this means that the space of even, additive, type four invariants is three dimensional, v_2 is one element in this space, but how to pick two other vectors to form a canonical basis is an interesting problem. Perhaps considering eigenvectors of cabling operations will be of some use.

The invariant v_2 has appeared in various guises previously in knot theory: it is the coefficient of z^2 in the Conway polynomial, is minus one sixth of the second derivative of the Jones polynomial evaluated at 1 and its reduction modulo two is the Arf invariant.

Also v_3 can be identified with a linear combination of the second and third derivatives of the Jones polynomial evaluated at 1.

Lin and Wang examined v_2 (with a different normalization) from a topological Chern-Simons point of view, and they proved the following:

THEOREM 15 ([48]). *If a knot K has a projection with c crossings then there is the bound*

$$|v_2(K)| \leq \frac{1}{4}c(c-1).$$

□

Following this Bar-Natan [7] proved that if a knot K has crossing number c and v is a type n invariant then $v(K) = O(c^n)$ (see also [66]) but the proof is not conceptually very enlightening, and gives no estimates on bounds. One can obtain the following bound:

THEOREM 16. *For a knot K with crossing number c*

$$|v_3(K)| \leq \frac{1}{4}c(c-1)(c-2).$$

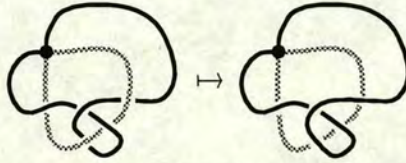
PROOF. Every knot projection can be turned into a projection of the unknot by a sequence of crossing changes. One way to do this is to pick a base point on the projection, away from a crossing, and travel around the knot ensuring that the first time a particular crossing is encountered it is via the underpass (otherwise switch the crossing). The result is called an ascending diagram and can readily be seen to represent the unknot. An unknot can be achieved by a sequence which consists of at most $c/2$ crossing changes — otherwise switch the complementary set of crossings instead. So there is a singular isotopy from K to the unknot with at most $c/2$ singularities, hence

$$v_3(K) = v_3(K) - v_3(\text{unknot}) = \sum_{s=1}^l \epsilon_s v_3(K'_s) \quad l \leq c/2,$$

where the K'_s are the singularities of the crossing switch sequence.

Similarly, by a sequence of crossing changes, every projection of a knot with one double point, K' , which has $c-1$ crossings can be turned into a projection in which the double point is nugatory, i.e. in which the two resolutions of the double point give the same knot: the double point decomposes K' into two “lobes” and crossing changes can be made so that one lobe lies entirely over the other lobe.

E.g.



Again this can be done with a sequence involving at most $(c - 1)/2$ crossing changes. So, as any invariant vanishes on a knot with a nugatory double point (see Chapter 1 Section 3),

$$v_3(K') = \sum_{r=1}^m \epsilon_r v_3(K_r'') \quad m \leq (c - 1)/2,$$

where again the K_r'' are the singularities.

To evaluate v_3 on a projection, π'' , of a knot with two double points one can use the formula of [23], which is a specialization of Theorem 13: pick some ordering of the crossings $C = (a_1, \dots, a_{c-2})$ and switch each of them in turn.

$$v_3(K'') = \frac{1}{2} \sum_{t=1}^{c-2} \epsilon(a_t) v_3((\pi'', C, t)).$$

Then recall that v_3 evaluated on a knot with three double points depends only on the underlying chord diagram, with $v_3 \left(\begin{array}{c} \bigcirc \\ \diagup \quad \diagdown \\ \bigcirc \end{array} \right) = 1$, $v_3 \left(\begin{array}{c} \bigcirc \\ \diagup \quad \diagup \\ \bigcirc \end{array} \right) = 2$ and v_3 vanishing on all other chord diagrams. This gives from the previous equation $|v_3(K'')| \leq c - 2$; and working backwards, the required inequality is obtained. \square

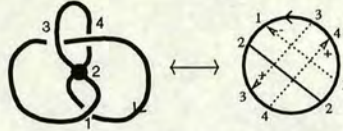
2. Gauß diagram formulæ.

Of course the above proof actually gives an algorithm for calculating v_2 and v_3 for a given knot projection — express the knot as a sum of knots with one double point, then as a sum of knots with two double points. Then either evaluate v_2 on these (i.e. see if the double points are in a crossed configuration) or evaluate v_3 on them. In fact this algorithm can be performed on the Gauß diagram of the knot projection. A slightly extended definition of Gauß diagram can be given as:

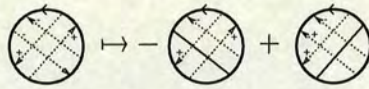
DEFINITION 4. Let π be a regular projection of a knot with double points. The Gauß diagram, $G(\pi)$, of π consists of an oriented circle representing the preimage of the knot and the circle is equipped with two types of chords indicating the points of the circle that are identified at the double points and crossings of the projection: dotted, oriented chords marked with a + or a - correspond to crossings, with the marking indicating the sign of the crossing and the orientation pointing towards that point on the underpass of the crossing; and solid chords correspond to double points.

Gauß diagrams can also be based, i.e. have a base point corresponding to a base point on the knot diagram: the base point will be indicated by the orientation arrow.

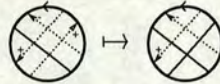
E.g.



To obtain an algorithm on the Gauß diagrams just run through the algorithm as described above. Start with a based knot diagram and the corresponding Gauß diagram, begin at the base point and switch crossings to achieve an ascending knot diagram. On the Gauß diagram level this means switching the oriented chords so that they are first encountered at the arrow head.



Next switch the crossings on the knots with one double point so that the lobe with the base point lies over the other lobe. Do this on the Gauß diagram level by starting at the end of the double point (after the base point) and switching the chords so that the arrow heads of the chords crossing the solid chord are all on the side not containing the base point.



It is then possible to evaluate v_2 on these. To evaluate v_3 on these proceed as in Theorem 13, start at the base point and traverse the circle, switching the chords the first time that they are met. Take half of the resulting sum.

This algorithm can be used to obtain formulæ for v_2 and v_3 . First the notion of an arrow diagram is required.

DEFINITION 5. Define an *arrow diagram* to be an oriented circle equipped with oriented chords with distinct endpoints. A based arrow diagram is an arrow diagram equipped with a base point on the circle which is distinct from the endpoints of the chords. (Again, base points will be indicated by the orientation arrow.)

Suppose G is a Gauß diagram and A is an arrow diagram. Let an *instance* of A in G be an orientation preserving map $A \rightarrow G$ which is a diffeomorphism of the circles, and maps chords to chords; if the arrow diagram is based, then it should also map base point to base point. (In other words an instance is a sub-diagram of G which looks like A .) Let the *multiplicity* of an instance be the product of the markings of the chords in

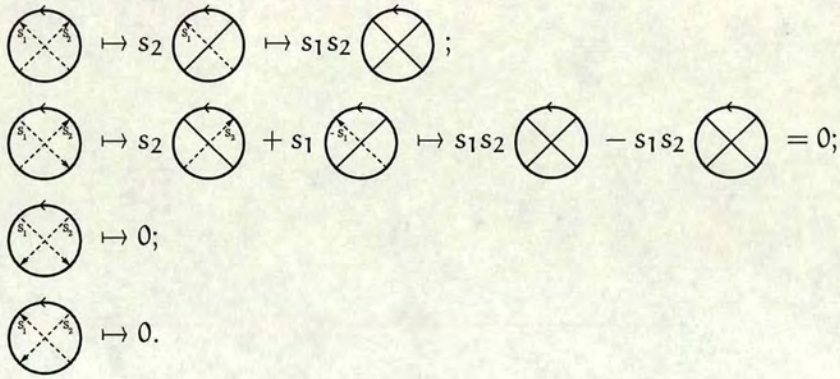
the image. Define $\langle A, G \rangle$ to be the number of instances (counted with multiplicity) of A in G . Extend linearly to linear combinations of arrow diagrams.

The following formula was announced, but not proved, in [59] (it can also be deduced from the Chern-Simons approach to v_2 of [48]).

THEOREM 17. *If K is a knot with projection π which has Gauß diagram G , then*

$$v_2(K) = \left\langle \bigcirc \otimes, G \right\rangle.$$

PROOF. Think of the possible sub-diagrams of G which would give a chord diagram of the form $\bigcirc \otimes$ after the algorithm. The possibilities are as below (s_i denotes the marking of the chords):



□

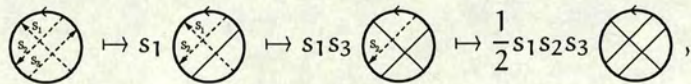
Similarly there is:

THEOREM 18. *For a knot K with a projection which has based Gauß diagram G ,*

$$v_3(K) = \left\langle \frac{1}{2} \left(\bigcirc \otimes + \bigcirc \otimes + \bigcirc \otimes + \bigcirc \otimes + \bigcirc \otimes - \bigcirc \otimes + \bigcirc \otimes - \bigcirc \otimes + 2 \bigcirc \otimes \right) + \bigcirc \otimes + \bigcirc \otimes, G \right\rangle.$$

PROOF. One proceeds as in the proof of the previous theorem, and laboriously checks each plausible diagram,

e.g.



recalling the values of v_3 on chord diagrams given at the end of the proof of Theorem 16.

□

Crossing number	3	4	5	6	7	8	9	10	11	12
Maximum $ v_2 $	1	1	3	2	6	5	10	9	15	14
Bound on $ v_2 $	1.5	2	5	7.5	11.5	14	18	22.5	27.5	33
Maximum $ v_3 $	1	0	5	1	14	10	30	25	55	49
Bound on $ v_3 $	1.5	6	15	30	57.5	84	126	180	247.5	330

Table 3.1. Comparing actual maxima and minima of $|v_2|$ and $|v_3|$ with the bounds of Section 1.

Polyak and Viro claimed (but did not prove) a simpler formula for v_3 in terms of unbased diagrams, viz:

$$v_3(K) = \left\langle \frac{1}{2} \left(\text{Diagram 1} + \text{Diagram 2} \right), G \right\rangle.$$

I cannot yet prove that this is the same as the above.

Gauß diagram formulæ will return in the pure braids sections.

3. Comparison with actual data.

How sharp are the bounds of Section 1? Stanford has calculated Vassiliev invariants up to order six for the prime knots up to ten crossings, the programs and data files of which are available as [64]; and Thistlethwaite has calculated Jones polynomials for knots up to twelve crossings, these are available from [72]. Here the data for v_2 and v_3 will be examined more closely (similar analysis of higher order invariants might have to wait until a canonical splitting of higher order invariants is sorted out). Table 3.1 lists the bounds from Section 1 together with the actual maxima and minima for $|v_2|$ and $|v_3|$.

Looking at the data, one notes that for odd crossing number the maxima are achieved precisely by the $(2, 2b+1)$ torus knots, and these dominate the $2b+2$ crossing knots as well. Alvarez and Labastida [2] (see Section 4 below) explicitly give, for crossing number $c = 2b + 1$,

$$\begin{aligned} v_2(T(2, c)) &= (c^2 - 1)/8, \\ v_3(T(2, c)) &= c(c^2 - 1)/24. \end{aligned}$$

One could conjecture that these give bounds on v_2 and v_3 . It might be noted that the $(2, c)$ -torus knots have maximal unknotting number for knots whose minimal unknotting projection is a minimal crossing projection.¹

¹The $(2, c)$ -torus knot has crossing number c (see Section 4.4) and unknotting number $(c - 1)/2$ (see Section 4.5).

i	-3	-2	-1	0	1	2	3	4
$\text{Wh}(i)$	8_1	6_1	4_1	0_1	3_1	5_2	7_2	9_2
$v_2(\text{Wh}(i))$	-3	-2	-1	0	1	2	3	4
$v_3(\text{Wh}(i))$	3	1	0	0	1	3	6	10

Table 3.2. The values of v_2 and v_3 on the twisted Whitehead doubles of the unknot. The knot notation, e.g 3_1 , refers to Alexander-Briggs notation (see [16]).

It is interesting and surprising to plot v_2 against v_3 for knots up to twelve crossings, this is done in Figure 3.1. The symmetry in the v_2 -axis is expected, as this is just the effect of taking the mirror image of a knot. The fish shape is not expected. This shape suggests some bound of the form

$$\text{cubic in } v_2(K) \leq (v_3(K))^2 \leq \text{another cubic in } v_2(K).$$

Such bounds, independent of crossing number do in fact exist for torus knots, as will be seen below. Although it might be possible to find such cubic bounds depending on the crossing number, this cannot be the case in general: two reasons for this are as follows.

Firstly, examine Table 3.2 which gives the values of v_2 and v_3 on the twisted Whitehead doubles of the unknot, $\text{Wh}(i)$ (see Chapter 1, Section 4). It is not difficult to prove from the method of Section 1 that $v_2(\text{Wh}(i)) = i$, alternatively this can be immediately deduced from Corollary 11, and this corollary can be used to deduce that $v_3(\text{Wh}(i)) = \frac{1}{2}i(i+1)$. These then form a sequence of unknotting number one knots (excepting the unknot), which map into the (v_2, v_3) -plane as a nice quadratic. This contradicts any bounds of the above form.

Secondly, for any $(a, b) \in \mathbb{Z}^2$ one can obtain a prime (alternating) knot with (v_2, v_3) equal to (a, b) in the following manner: connect sum suitably many positive and negative trefoil knots (with $(v_2, v_3) = (1, \pm 1)$) and figure eight knots (with $(v_2, v_3) = (-1, 0)$), to obtain a composite knot with $(v_2, v_3) = (a, b)$, then Stanford [67] gives a method for constructing a prime knot with the same v_2 and v_3 .

There does appear to be a qualitative difference between the pictures for odd and even crossing numbers in Figure 3.1. The even crossing number ones seem to be more concentrated in the ‘body’ of the ‘fish’ and the odd ones more in the ‘tail’. Note that for each odd crossing number, c , there is the $(2, c)$ -torus knot and the Whitehead double $\text{Wh}((c-1)/2)$ with a (v_2, v_3) of $((c-1)/2, (c^2-1)/8)$; and for even crossing number, c , there is the Whitehead double $\text{Wh}(1-c/2)$ with a (v_2, v_3) of $(1-c/2, (c-2)c/8)$. Also for up to twelve crossings (and it is conjectured for all knots), the amphicheiral

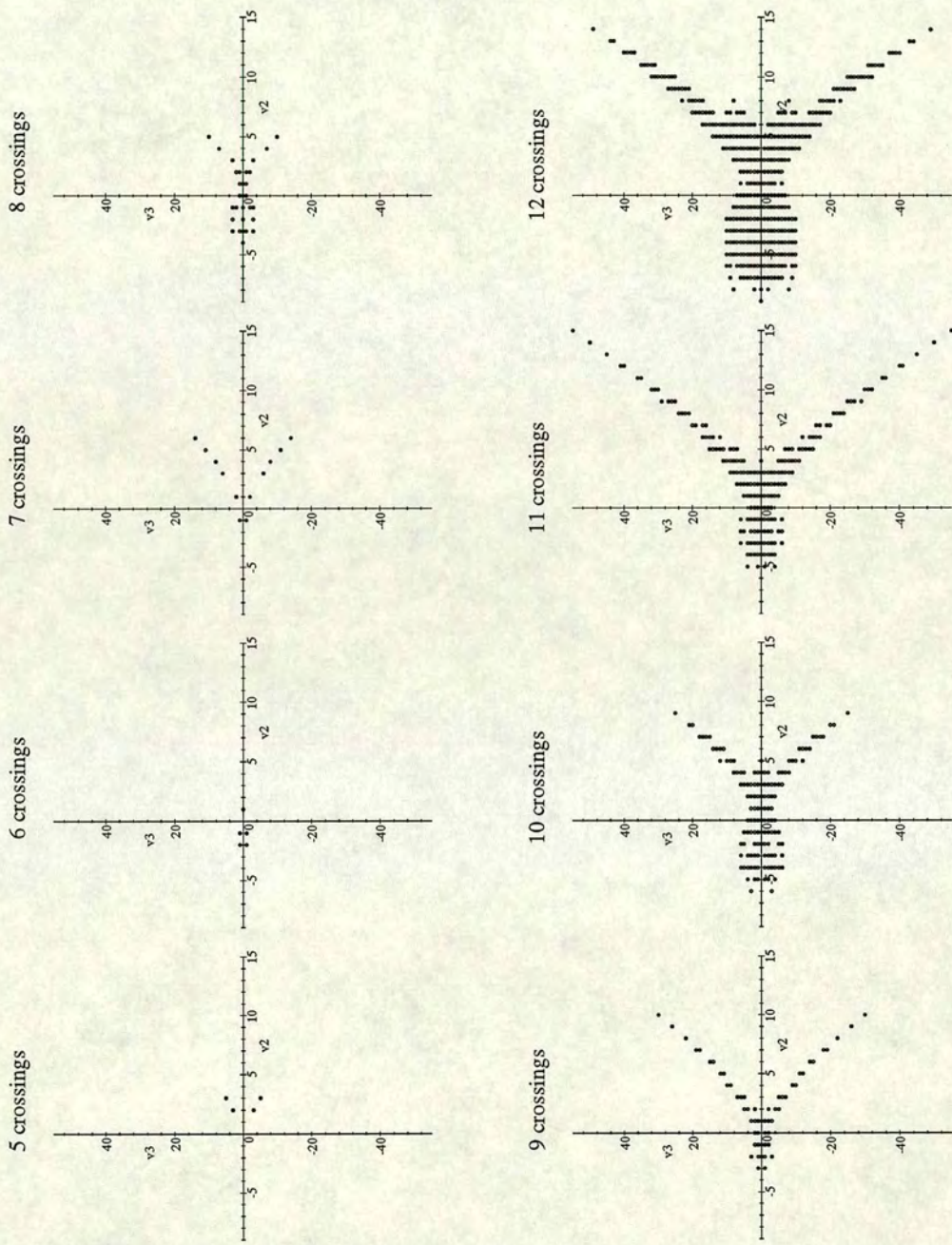


Figure 3.1. Plots by crossing number of v_2 and v_3 . The data for up to ten crossing knots were taken from Stanford [64] and for 11 and 12 crossing knots were calculated from Thistlethwaite's tables of Jones polynomials [72].

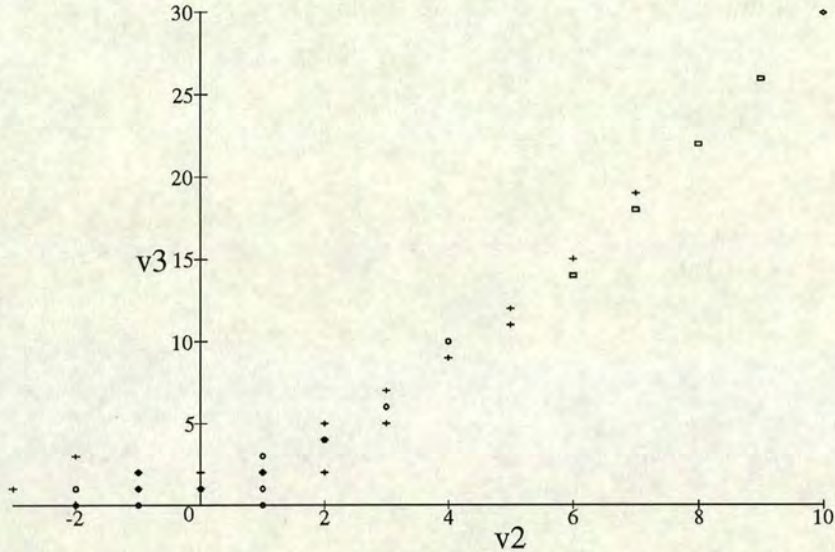


Figure 3.2. Plot of v_2 against v_3 for nine crossing knots (with positive v_3 half only), showing the unknotting number [42]: unknotting number one = circle; two = cross; three = square; four = diamond.

knots — that is those equivalent to their mirror image, and hence with $v_3 = 0$ — all have even crossing number.

In Figure 3.2 the unknotting numbers have been appended to the graph for knots with nine crossings. There does appear to be some correlation, but two things to bear in mind are the fact that minimal unknotting projections are coinciding with minimal crossing projections at these low crossing numbers, and that there are a lot more knots out there. Bounds between the unknotting number and v_2 are shown below for torus knots, but for instance the Whitehead doubles example above shows that there are unknotting number one knots with arbitrarily large v_2 and v_3 .

4. Torus knots.

4.1. Introduction. Some of the properties of v_2 and v_3 suggested above can be proved for the case of torus knots. Recall that a torus knot is a knot that can be isotoped so that it sits on a standardly embedded 2-torus (see Figure 3.3). For p and q coprime integers, the (p, q) -torus knot, $T(p, q)$ is the knot which has a representative which embeds in the torus so that it wraps p times around longitudinally and q times around meridionally (with the signs of p and q giving the orientation). $T(p, q)$ is the unknot if and only if p or q is ± 1 , and for $T(p, q)$ nontrivial, $T(p, q)$ is the same knot as $T(p', q')$ if and only if (p', q') equals one of the following (p, q) , (q, p) , $(-p, -q)$, or $(-q, -p)$. Further $T(p, -q)$ is the mirror image of $T(p, q)$. See [16].

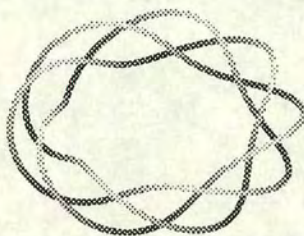


Figure 3.3. A $(3,8)$ -torus knot.

Alvarez and Labastida [2] give formulæ for v_2 and v_3 evaluated on torus knots as follows:

$$v_2(T(p, q)) = \frac{1}{24}(p^2 - 1)(q^2 - 1),$$

$$v_3(T(p, q)) = \frac{1}{144}pq(p^2 - 1)(q^2 - 1).$$

Note that these have the required properties under the symmetries of p and q mentioned above, and that these are integer valued on torus knots (i.e. when p and q are coprime). Also $T \mapsto (v_2(T), v_3(T))$ is injective for torus knots, that is to say torus knots are determined by their (v_2, v_3) .

Alvarez and Labastida prove that if v is a Vassiliev invariant coming from one of the standard polynomials (Jones, HOMFLY, *etc.*) then $v(T(p, q))$ is a polynomial in p and q .²

Remember that most knots are not torus knots.

4.2. Cubic bounds. With the above formulæ it is straightforward to prove bounds, for torus knots, of the form suggested in the last section.

PROPOSITION 19. *If T is a torus knot then*

$$\frac{2}{3}v_2(T)^3 + \frac{1}{3}v_2(T)^2 \leq v_3(T)^2 \leq \frac{8}{9}v_2(T)^3 + \frac{1}{9}v_2(T)^2.$$

Further, the right hand bound is tight in the sense that there exist torus knots with arbitrarily large v_2 and v_3 such that equality holds.

²This ought to be true for any Vassiliev invariant but I do not yet know how to prove it.

PROOF. Suppose that T is a (p, q) -torus knot then

$$\begin{aligned} v_3(T)^2 - \frac{2}{3}v_2(T)^3 &= \left(\frac{1}{144}pq(p^2 - 1)(q^2 - 1) \right)^2 - \frac{2}{3} \left(\frac{1}{24}(p^2 - 1)(q^2 - 1) \right)^3 \\ &= \frac{1}{12^4}(p^2 - 1)^2(q^2 - 1)^2[p^2 + q^2 - 1] \\ &\geq \frac{1}{12^3}(p^2 - 1)^2(q^2 - 1)^2 \quad \text{as } p^2 + q^2 \geq 13 \\ &= \frac{1}{3}v_2(T)^2, \end{aligned}$$

hence the first inequality (with equality only in the case of torus knots).

For the second,

$$\begin{aligned} \frac{8}{9}v_2(T)^3 - v_3(T)^2 &= \frac{8}{9} \left(\frac{1}{24}(p^2 - 1)(q^2 - 1) \right)^3 - \left(\frac{1}{144}pq(p^2 - 1)(q^2 - 1) \right)^2 \\ &= \frac{1}{4.27} \left[\frac{1}{24}(p^2 - 1)(q^2 - 1) \right]^2 \{4(p^2 - 1)(q^2 - 1) - 3p^2q^2\} \\ &= \frac{1}{4.27}v_2(T)^2 \{(p^2 - 4)(q^2 - 4) - 12\} \\ &\geq \frac{1}{4.27}v_2(T)^2\{-12\} = -\frac{1}{9}v_2(T)^2, \end{aligned}$$

and note that equality occurs precisely when T is a $(2, q)$ -torus knot. \square

Although the left hand bound has the correct asymptotic behaviour, for a tight bound a different form of cubic is required.

PROPOSITION 20. *For a torus knot T ,*

$$\frac{2}{3}v_2(T)^3 + \frac{1}{3}v_2(T)v_3(T) \leq v_3(T)^2,$$

and this bound is tight in the sense of the previous proposition.

PROOF. Using the notation of the previous proof,

$$\begin{aligned} v_3(T)^2 - \frac{2}{3}v_2(T)^3 - \frac{1}{3}v_2(T)v_3(T) &= \frac{1}{36.24^2}(p^2 - 1)^2(q^2 - 1)^2 \left((p - q)^2 - 1 \right) \\ &\geq 0, \end{aligned}$$

with equality if and only if T is a $(p, p + 1)$ torus knot. \square

Given that half the torus knots (those with positive v_3) can be thought of as lying in the region $q > p > 0$ in the (p, q) -plane, these bounds are not surprising. Graphically this can be seen in Figure 3.4.

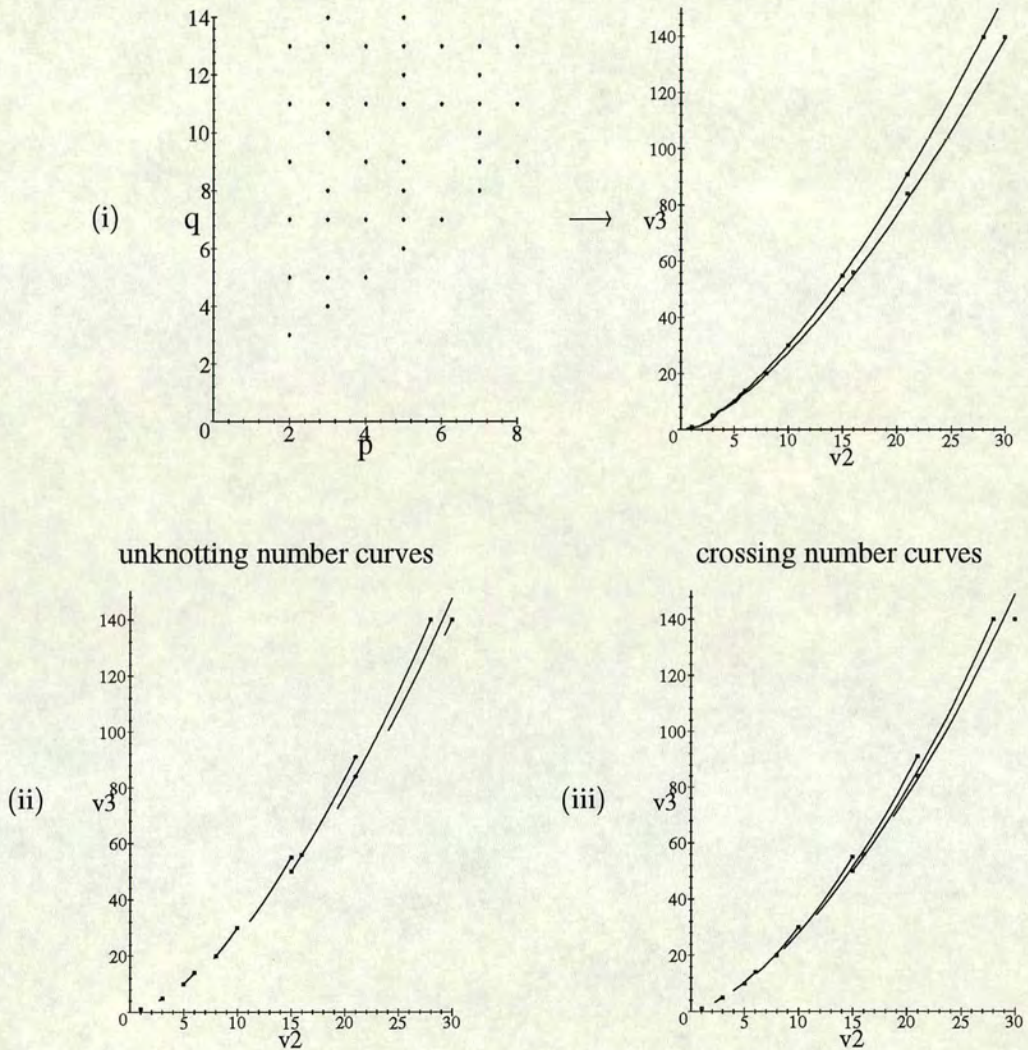


Figure 3.4. Torus knots in the (v_2, v_3) -plane: (i) mapping torus knots from the (p, q) -plane into the region of the (v_2, v_3) -plane given by Propositions 19 and 20; (ii) torus unknotting number curves for $u = 1, \dots, 9$ (see Section 4.3); (iii) torus crossing number curves for $c = 3, 5, \dots, 17$ (see Section 4.4).

4.3. Torus knots and unknotting number. By Kronheimer and Mrowka's [46] positive solution to the Milnor conjecture the following formula is known for the unknotting number, u , of torus knots:

$$u(T(p, q)) = \frac{1}{2} (|p| - 1) (|q| - 1).$$

As a consequence, the following easily verifiable relationship is obtained:

PROPOSITION 21. *For a torus knot T ,*

$$v_2(T)^2 + \frac{1}{6}u(T)(u(T) - 1)v_2(T) = u(T)|v_3(T)|,$$

and given $v_2(T)$ and $v_3(T)$ then $u(T)$ is the smaller of the two roots. \square

So for a fixed unknotting number, the torus knots lie on a quadratic in the (v_2, v_3) -plane (c.f. the Whitehead knots in Section 4.1). This is pictured in Figure 3.4. The segments of curves shown were chosen by the following proposition.

PROPOSITION 22. *For a torus knot T ,*

$$\frac{1}{2}u(T)(u(T) + 1) \geq v_2(T) \geq \frac{1}{6}u(T) \left(u(T) + \sqrt{8u(T) + 1} + 2 \right),$$

and both bounds are tight.

PROOF. If T is a (p, q) -torus knot, then a minimal amount of manipulation gives

$$\begin{aligned} \frac{1}{2}u(T)(u(T) + 1) - v_2(T) &= \frac{1}{12}(|p| - 1)(|q| - 1)(|p| - 2)(|q| - 2) \\ &\geq 0, \end{aligned}$$

with equality if and only if T is a $(2, q)$ -torus knot.

For the right hand bound, firstly, let a and b be distinct positive integers, then $(a - b)^2 \geq 1$, so $(a + b)^2 \geq 4ab + 1$ and thus $a + b \geq \sqrt{4ab + 1}$, with equality precisely when a and b differ by one.

Now for T a (p, q) -torus knot,

$$\begin{aligned} v_2(T) - \frac{1}{6}u(T) \left(u(T) + \sqrt{8u(T) + 1} + 2 \right) \\ = \frac{1}{12}(|p| - 1)(|q| - 1) \left\{ |p| + |q| - 2 - \sqrt{4(|p| - 1)(|q| - 1) + 1} \right\} \\ \geq 0, \end{aligned}$$

by putting $a = |p| - 1$, $b = |q| - 1$ in the above paragraph. Note that equality occurs precisely when T is a $(p, p + 1)$ -torus knot. \square

Weakening the right hand bound to $v_2 \geq \frac{1}{6}u(T)(u(T) + 5)$ and inverting the inequalities reveals the following corollary.

COROLLARY 23. *For a torus knot T ,*

$$\sqrt{1 + 8v_2(T)} - 1 \leq 2u(T) \leq \sqrt{24v_2(T) + 25} - 5,$$

and the left hand bound is tight (in the sense of Section 4.2). \square

4.4. Torus knots and crossing number. By the work of Murasugi [56], a similar formula is known for the crossing number, c , of torus knots:

$$c(T(p, q)) = |q|(|p| - 1), \quad \text{when } |p| < |q|.$$

This leads to the following relation;

PROPOSITION 24. *If T is a torus knot, and $\rho(T) = \left\lfloor \frac{6v_3(T)}{v_2(T)} \right\rfloor$ then*

$$24v_2(T)(c(T) - \rho(T))^2 = c(T) \left((c(T) - \rho(T))^2 - 1 \right) (2\rho(T) - c(T)),$$

and

$$c(T) = \rho(T) - \frac{1}{2} \left(\sqrt{(\rho(T) - 1)^2 - 24v_2(T)} + \sqrt{(\rho(T) + 1)^2 - 24v_2(T)} \right).$$

PROOF. This is easily verified; note that if T is a (p, q) -torus knot then $\rho(T) = |pq|$ and $c(T) - \rho(T) = |q|$. \square

This isn't as nice a relationship as with the unknotting number: for a fixed crossing number the relationship is a not particularly nice quartic between v_2 and v_3 . However, the crossing number curves can still be graphed, as in Figure 3.4 — the length of arc segments plotted there being determined by the following proposition.

PROPOSITION 25. *For a torus knot T ,*

$$\frac{1}{8} (c(T)^2 - 1) \geq v_2(T) \geq \frac{1}{24} c(T) (c(T) + 1 + 2\sqrt{c(T) + 1}),$$

and these bounds are tight (in the sense of Section 4.2).

PROOF. Suppose that T is a (p, q) -torus knot with $q > p > 0$ — this just avoids excessive modulus signs in the calculation — then for the left hand bound,

$$\begin{aligned} \frac{1}{8} (c(T)^2 - 1) - v_2(T) &= \frac{1}{24} \left\{ 3 \left([q(p-1)]^2 - 1 \right) - (p^2 - 1)(q^2 - 1) \right\} \\ &= \frac{1}{24} \left\{ 2q^2p^2 - 6q^2p + 4q^2 + p^2 - 4 \right\} \\ &= \frac{1}{24} (p-2) \left\{ (2q^2 + 1)(p-1) + 3 \right\} \\ &\geq 0, \end{aligned}$$

and equality occurs precisely when T is a $(2, q)$ -torus knot.

For the right hand bound,

$$\begin{aligned} 24v_2(T) - c(T) & \left(c(T) + 1 + 2\sqrt{c(T) + 1} \right) \\ & = (p^2 - 1)(q^2 - 1) - q(p - 1) \left(q(p - 1) + 1 + 2\sqrt{q(p - 1) + 1} \right) \\ & = (p - 1) \left\{ 2q^2 - q - 1 - p - 2q\sqrt{qp - q + 1} \right\}, \end{aligned}$$

and claim that this is non-negative and is zero precisely when $q = p + 1$.

To prove the claim, note

$$(q - 1)^2 = q(q - 1) - q - 1 \geq qp - q - 1$$

as $q - p - 1 \geq 0$, and so also

$$(q - 1)^2 + \frac{2(q - 1)(q - p - 1)}{2q} + \left[\frac{q - p - 1}{2q} \right]^2 \geq qp - q - 1 > 0,$$

thus, by taking square roots,

$$(q - 1) + \frac{q - p - 1}{2q} \geq \sqrt{qp - q - 1},$$

from which the claim follows on multiplying through by $2q$. □

Weakening the right hand bound to $v_2 \geq \frac{1}{24}c(c + 5)$ and inverting, gives

COROLLARY 26. *For a torus knot T*

$$\frac{1}{24} \left(\sqrt{25 + 96v_2(T)} - 5 \right) \geq c(T) \geq 2\sqrt{8v_2(T) + 1},$$

and the right hand bound is tight in the previous sense. □

CHAPTER 4

Vassiliev invariants for pure braids.

*Rapunzel, Rapunzel,
Let down your golden hair.*

— TRAD., *Fairy Tale*.

In this chapter the Vassiliev invariants of pure braids are related to the algebraic properties of the pure braid groups; specifically, to the lower central series of the pure braid groups. This was initially observed by Stanford, but here is used to obtain dimensions of the spaces of independent finite type invariants.

Section 1 introduces pure braid groups and requisite notions from algebra. Section 2 contains the details of how Vassiliev invariants can be characterized algebraically and Section 3 uses this algebra to give explicit formulæ for the number of invariants.

1. Preliminaries on pure braid groups.

Begin with some definitions and group theoretic properties of pure braids.

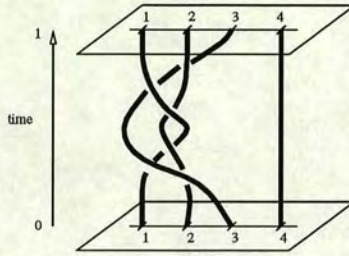
1.1. Pure braids. A convenient way to define the pure braid groups is as follows:

DEFINITION 6. (i) The configuration space, $\mathbb{C}[k]$, of k *ordered* distinct points in the complex plane, is the space $\{(z_1, \dots, z_k) \in \mathbb{C}^k \mid z_i \neq z_j \forall i \neq j\}$; it will be taken to have base point $(1, 2, \dots, k)$.

(ii) A *pure braid on k strands* is a homotopy class of piecewise-smooth based loops in the configuration space $\mathbb{C}[k]$. (Where a distinction is required, a *geometric braid* will mean a specific representative, $\gamma: S^1 \rightarrow \mathbb{C}[k]$, and a *topological braid* will mean an equivalence class: the distinction will not generally be made.) These form a group, P_k , under the usual operation of loop concatenation.

The usual notion of pure braid is recovered when these loops are thought of as living in $\mathbb{R}^3 \cong \mathbb{C}_z \times \mathbb{R}_t$, the time axis lying vertical. For instance, the following is a

pure braid on four strands:



Note that multiplication of braids $\beta\gamma$ will mean β followed by γ , i.e. γ on top of β .

- REMARK 3. (i) The pure braid group, P_k , is the subgroup of the full braid group, B_k , given as the kernel of the map to the symmetric group which associates to a braid the permutation induced on the strands.
- (ii) Artin [4] gave several equivalent definitions of braid groups, but this one was introduced by Fox and Neuwirth [27].

Fadell and Neuwirth [24] presented the notion that the configuration space $C[k]$ fibres over $C[k - 1]$ by “forgetting the last co-ordinate”:

$$C \setminus \{1, 2, \dots, k - 1\} \rightarrow C[k] \rightarrow C[k - 1].$$

This fibration admits a section:

$$s: C[k - 1] \rightarrow C[k]; \quad (z_1, \dots, z_{k-1}) \mapsto (z_1, \dots, z_{k-1}, \max_{i=1, \dots, k-1} (\operatorname{Re} z_i) + 1).$$

Inductively, $C[k]$ sits atop an iterated fibration which has punctured surfaces (i.e. Eilenberg-MacLane spaces) as fibres, so the homotopy exact sequence implies that $C[k]$ is an Eilenberg-MacLane space, that is to say, a $K(P_k, 1)$.

Applying the fundamental group functor and writing F_k for the free group on k generators, one obtains the split exact sequence of groups:

$$1 \rightarrow F_{k-1} \hookrightarrow P_k \twoheadrightarrow P_{k-1} \rightarrow 1.$$

Recall the concept of a semi-direct product of groups. Let A be a group which acts (on the right) on the group B . Define $A \ltimes B$ to be the group which has underlying set $A \times B$ and multiplication given by

$$(a_1, b_1)(a_2, b_2) := (a_1 a_2, b_1^{a_2} b_2).$$

A group G is isomorphic to $A \ltimes B$ precisely when there is a split short exact sequence

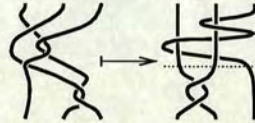
$$1 \rightarrow B \hookrightarrow G \twoheadrightarrow_s A \rightarrow 1,$$

with the action of A on B given by conjugation: $b^a = s(a^{-1})bs(a)$ (thinking of B as a normal subgroup of G).

One can then conclude that P_k is a semi-direct product:

$$(\dagger) \quad P_k = P_{k-1} \ltimes F_{k-1}.$$

Geometrically, this is Artin's "combing of the braid" [4], writing the braid as a product of a braid in which the last strand remains straight and uninvolved in the braiding (i.e. an element of P_{k-1}), and one in which the other $(k - 1)$ strands remain straight while the final strand winds around them (which can be thought of as an element of F_{k-1}).



Of course this can be repeated so that the pure braid group P_k is an iterated semi-direct product of free groups: $P_k = (\dots (F_1 \dots) \ltimes F_{k-2}) \ltimes F_{k-1}$. This is very useful for deducing algebraic properties of the pure braid groups.

1.2. Useful algebraic properties of the pure braid groups.

1.2.1. *Residual nilpotence.* Recall the definition of the lower central series of a group.

DEFINITION 7. If G is a group then the lower central series,

$$\Gamma^1 G \triangleright \Gamma^2 G \triangleright \dots \triangleright \Gamma^n G \triangleright \dots,$$

is defined inductively via

$$\Gamma^1 G = G, \quad \Gamma^{n+1} G = [\Gamma^n G, G];$$

where $[A, B]$ denotes the group generated by the commutators $[a, b] = aba^{-1}b^{-1}$ with $a \in A, b \in B$.

The lower central series is the smallest series of subgroups of G which satisfies $\Gamma^1 G = G, [\Gamma^i G, \Gamma^j G] \subset \Gamma^{i+j} G$ and $\Gamma^n G / \Gamma^{n+1} G$ is abelian, in the sense that if $\{\Xi^i G\}$ is another such series (called a *descending central series*), then $\Xi^i G \triangleright \Gamma^i G$ for all i . Also, the groups of the lower central series are actually *characteristic* subgroups, i.e. they are invariant under *all* automorphisms of the original group, not just the inner automorphisms.

Nilpotence will play an important role in what follows, so again, recall the definitions:

DEFINITION 8. A group G is said to be n -stage nilpotent if $\Gamma^n G = \{1\}$, i.e. all commutators of length n are trivial; and G is said to be residually nilpotent if $\bigcap_{n=1}^{\infty} \Gamma^n G = \{1\}$.

The free group F_k is residually nilpotent (see [49]) and Falk and Randell [26] used (†) to prove

THEOREM 27. P_k is residually nilpotent. □

1.2.2. *Structure of abelian quotients.* The abelian groups $\Gamma^n P_k / \Gamma^{n+1} P_k$ will be of interest so the following notation will be adopted:

NOTATION 1. For a group G and $n \in \{1, 2, \dots\}$, let $G\{n\}$ denote the abelian group $\Gamma^n G / \Gamma^{n+1} G$, so, e.g. $G\{1\}$ is the abelianization of G .

Falk and Randell proved the following:

THEOREM 28. [25] *If $G \cong A \times B$ and A acts trivially on the abelianization of B then for all $n \geq 1$, $G\{n\} \cong A\{n\} \oplus B\{n\}$.* □

It is straight-forward to see that P_{k-1} acts trivially on the abelianization of F_{k-1} because each generator of the free group is mapped to a conjugate element by the pure braid action. It is known [49] that $F_k\{n\}$ is free abelian of rank $\frac{1}{n} \sum_{m|n} \mu\left(\frac{n}{m}\right) k^m$, where μ is the Möbius function of number theory (see [36]). Thus Theorem 28 and an induction gives the structure of the lower central series factor groups of the pure braid groups:

COROLLARY 29. $P_k\{n\}$ is a free abelian group of rank φ_n^k , where

$$\varphi_n^k = \frac{1}{n} \sum_{m|n} \mu\left(\frac{n}{m}\right) \sum_{i=1}^{k-1} i^m.$$

□

2. Finite type invariants as generalized winding numbers.

Now switch attention to finite type invariants of pure braids, these can be considered as generalizations of winding numbers.

2.1. Winding numbers. For a pure braid β , one of the first kind of invariants that might come to mind is the winding numbers: for $i \neq j$ how many times does the i th strand wind around the j th strand? This is the (i, j) th winding number $w_{ij}(\beta)$. Of course this is analogous to Gauß linking of links. As with winding numbers of curves

in the complex plane, these winding numbers can be calculated from an integral as follows:

$$w_{ij}(\beta) = \frac{1}{2\pi i} \int_{\beta} d \ln(z_i - z_j) = \frac{1}{2\pi i} \int_{\beta} \frac{dz_i - dz_j}{z_i - z_j}.$$

Note that $w_{ij} = w_{ji}$.

For example

$$\beta = \text{[Diagram of a braid with three strands: the top strand crosses over the middle, which crosses over the bottom];} \quad \begin{aligned} w_{12}(\beta) &= -1 \\ w_{23}(\beta) &= 0 \\ w_{13}(\beta) &= 1 \end{aligned}$$

It is well known that these can be calculated from a braid diagram by counting, with sign, half the number of crossings between the i th and j th strands. This corresponds to replacing the 1-form $\frac{1}{2\pi i} d \ln(z_i - z_j)$ with a cohomologous current which is supported on the hyperplane in $\mathbb{C}[k]$ given by $\{z \in \mathbb{C}[k] \mid \text{Re } z_i = \text{Re } z_j\}$. This idea of combinatorial calculation will be returned to in Chapter 6.

In the case of Gauß linking, Maxwell observed in [52] that the so-called Whitehead link is a non-split two component link with zero linking numbers. Also here: winding numbers do not serve as complete invariants of pure braids. Similarly winding numbers do not serve as complete invariants of braids. Observe the pigtail braid:



All three winding numbers vanish but the braid is non-trivial. Next, finite type invariants will be shown to generalize winding numbers.

It should be noted here that the cohomology classes of the winding forms $\omega_{ij} := \frac{1}{2\pi i} d \ln(z_i - z_j)$ for $i \neq j$ generate the (integral) cohomology of $\mathbb{C}[k]$. The Arnold algebra \mathcal{A}_k^\bullet is defined to be the subalgebra of the de Rham complex, $\Lambda^\bullet(\mathbb{C}[k])$, of the configuration space $\mathbb{C}[k]$, generated by these winding forms, and it is a theorem of Arnold [3] that the inclusion $\mathcal{A}_k^\bullet \rightarrow \Lambda^\bullet(\mathbb{C}[k])$ induces an isomorphism on cohomology and also that the Arnold algebra can be described as the exterior algebra on the winding forms modulo the Arnold relations: $\omega_{ij} \wedge \omega_{jl} + \omega_{jl} \wedge \omega_{li} + \omega_{li} \wedge \omega_{ij} = 0$ for i, j , and l distinct.

2.2. Algebraicization of the finite type condition. The following algebraicization is implicit in the work of Stanford [67, 65].

Consider finite type invariants for braids as defined in Chapter 1. Recall that a pure braid invariant $v : P_k \rightarrow \mathcal{R}$, taking values in the ring \mathcal{R} , can be extended to braids

with double points via the Vassiliev skein relation:

$$v(\text{X}) = v(\text{X}) - v(\text{X}).$$

and that an invariant is of type $n \in \mathbb{N}$ if it vanishes on pure braids with more than n double points.

DEFINITION 9. Let $\mathcal{V}_k^n = \{\text{invariants of } P_k \text{ of type } \leq n\}$, and let $\overline{\mathcal{V}}_k^n = \{v \in \mathcal{V}_k^n \mid v(1) = 0\}$, where 1 is the trivial braid.

It is straightforward to show that \mathcal{V}_k^0 is the set of constant invariants, that $\overline{\mathcal{V}}_k^n \cong \mathcal{V}_k^n / \mathcal{V}_k^0$, and that $v \in \overline{\mathcal{V}}_k^1$ if and only if v is a linear combination of winding numbers.

Singular braids can be formally considered as elements of the group algebra \mathcal{RP}_k via the relation

$$\text{X} = \text{X} - \text{X} \in \mathcal{RP}_k.$$

Consider the augmentation $\epsilon: \mathcal{RP}_k \rightarrow \mathcal{R}$ defined by the linear extension of $\epsilon(\beta) = 1$ for all pure braids β , and let $J_{\mathcal{R}}$ be the augmentation ideal, i.e.

$$J_{\mathcal{R}} := \ker \epsilon = \langle p - 1 \mid p \in P_k \rangle \triangleleft \mathcal{RP}_k.$$

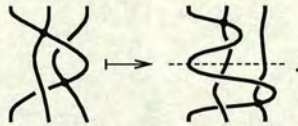
FACT 2. Any pure braid can be transformed to the trivial braid by a sequence of crossing changes. □

Thus

$$\begin{aligned} J_{\mathcal{R}} &= \langle (p_1 - p_2) + (p_2 - p_3) + \dots + (p_j - 1) \\ &\quad \mid p_i, p_{i+1} \in P_k \text{ differ by a crossing change (with } p_{j+1} = 1) \rangle \\ &= \langle (p - q) \mid p, q \text{ differ by a crossing change} \rangle \\ &= \langle \text{singular braids} \rangle. \end{aligned}$$

FACT 3. Any singular pure braid with n double points can be written as the product of n singular pure braids, each with one double point. □

For example,



REMARK 4. This is a fundamental point where the theory diverges from the case of knots: the statement analogous to Fact 3 for knots is not true.

Fact 3 means that $J_{\mathcal{R}}^n = \langle \text{braids with } n \text{ double points} \rangle$ and this gives a fundamental identification:

THEOREM 30. *There is a canonical isomorphism*

$$\mathcal{V}_k^n \cong (\mathcal{R}P_k/J_{\mathcal{R}}^{n+1})^\vee.$$

The topological filtration of pure braid invariants has therefore been rewritten as a purely algebraic filtration. This is the key to the methods that follow. Some more algebra must be considered.

2.3. Dimension subgroups. For a group G and \mathcal{R} a commutative ring with unity, a series of characteristic subgroups can be defined in the following manner.

DEFINITION 10. The augmentation ideal, $J_{\mathcal{R}}$, of the group ring $\mathcal{R}G$ is defined, as above, as the kernel of the augmentation map $\epsilon: \mathcal{R}G \rightarrow G$, and the n th dimension subgroup $\Delta_{\mathcal{R}}^n G$ is defined as the subgroup associated to the n th power of the augmentation ideal, viz

$$\Delta_{\mathcal{R}}^n G := G \cap (1 + J_{\mathcal{R}}^n).$$

Then

$$G = \Delta_{\mathcal{R}}^1 G \triangleright \Delta_{\mathcal{R}}^2 G \triangleright \dots \triangleright \Delta_{\mathcal{R}}^n G \triangleright \dots$$

This is a descending central series and so $\Gamma^n G \triangleleft \Delta_{\mathcal{R}}^n G$. Two important cases are $\mathcal{R} = \mathbb{Z}$ and \mathcal{R} is a characteristic zero field. The canonical unital ring map $\mathbb{Z} \rightarrow \mathcal{R}$ induces an inclusion

$$\Delta_{\mathbb{Z}}^n G \hookrightarrow \Delta_{\mathcal{R}}^n G.$$

A group is said to have the dimension subgroup property over \mathcal{R} if the dimension subgroups over \mathcal{R} are precisely the lower central subgroups. For instance free groups have the dimension subgroup property over the integers, but this does not hold for groups in general (see [37]).

If \mathcal{R} is a characteristic zero field, denote $\Delta_{\mathcal{R}}^n G$ by $\Delta_0^n G$, this is independent of the field because of the following result going back to Malcev:

$$\Delta_0^n G = \overline{\Gamma^n G} := \{x \in G \mid x^a \in \Gamma^n G \text{ for some } a \geq 1\},$$

the *isolator* (or *rational closure*) of $\Gamma^n G$.

From this it is easy to deduce

THEOREM 31. *If the lower central factor groups, $G\{n\} = \Gamma^n G / \Gamma^{n+1} G$, of G are torsion free then G has the dimension subgroup property over the integers and over any characteristic zero field.*

PROOF. In view of the sequence

$$\Gamma^n G \triangleleft \Delta_{\mathbb{Z}}^n G \hookrightarrow \Delta_0^n G = \overline{\Gamma^n G},$$

it suffices to show $\Gamma^n G = \overline{\Gamma^n G}$ for all n . Suppose not, then there is an $x \in G$ and $n, a \geq 2$ such that $x^a \in \Gamma^n G$ but $x \notin \Gamma^n G$. However, for some $m < n$, $x \in \Gamma^m G$ but $x \notin \Gamma^{m+1} G$ and $x^a \in \Gamma^n G \triangleleft \Gamma^{m+1} G$. So $[x] \neq 1 \in G\{m\}$ but $[x]^a = [x^a] = 1 \in G\{m\}$ thus contradicting the hypothesis. \square

Applying this theorem with the Corollary 29, the dimension subgroup property can be deduced for the pure braid groups:

COROLLARY 32. For all n , $\Delta_{\mathbb{Z}}^n P_k = \Delta_0^n P_k = \Gamma^n P_k$. \square

REMARK 5. Alternatively one could use the result of Sandling [63] which says that a group which is the semi-direct product of groups having the dimension subgroup property, itself has the dimension subgroup property. This is applied by the usual induction on (\dagger) .

See [57] for a way of expressing an element of $\Gamma^n P_k$ as a linear combination of braids with $(n-1)$ double points modulo braids with n double points.

An immediate consequence, via Theorem 30, is

COROLLARY 33. For β a k strand pure braid, $\beta \in \Gamma^{n+1} P_k$ if and only if $v(\beta) = 0$ for all $v \in \overline{V}_k^n$, i.e. if and only if β is indistinguishable from the trivial braid by type n invariants. \square

REMARK 6. In [67] Stanford obtained the “if” part of the above statement, and in [65] he noted that P_k has the dimension subgroup property, but failed to put the two together to obtain the “only if”. In [44] Kohno derived the “if and only if” by expressing finite type pure braid invariants in terms of de Rham homotopy theory (see the next chapter) and using the dimension subgroup property. I was able to synthesize these two approaches into the above simple argument.

3. Counting numbers of invariants.

The purpose of this section is to obtain the number of invariants of each type for each pure braid group. Because of the algebraic structure of the invariants, it is necessary only to obtain the invariants which are not expressible as sums of products of lower order invariants — these are the *indecomposable* invariants. Further, it is sensible to look at the number of “new” invariants of each type, i.e. the dimension of $\mathcal{V}_k^n / \mathcal{V}_k^{n+1}$. Then it is possible to consider those which are not induced from pure braid groups on fewer strands.

$k \setminus n$	1	2	3	4	5	6	7	8	9
2	1	1	1	1	1	1	1	1	1
3	3	7	15	31	63	127	255	511	1023
4	6	25	90	301	966	3025	9330	28501	86526
5	10	65	350	1701	7770	34105	145750	611501	2532530
6	15	140	1050	6951	42525	246730	1379400	7508501	40075035
7	21	266	2646	22827	179487	1323652	9321312	63436373	420693273
8	28	462	5880	63987	627396	5715424	49329280	408741333	3281882604
9	36	750	11880	159027	1899612	20912320	216627840	2141764053	20415995028
10	45	1155	22275	359502	5135130	67128490	820784250	9528822303	106175395755

Table 4.1. The dimensions of the spaces of type n invariants modulo type $(n - 1)$ invariants of the pure braid groups, P_k (included for comparison).

3.1. Numbers of indecomposable invariants. A result of Quillen [61] says (using the notation of the last section) that $\bigoplus J_{\mathbb{F}}^n/J_{\mathbb{F}}^{n+1}$ is the universal enveloping algebra of the Lie algebra $\bigoplus(\Gamma^n G/\Gamma^{n+1}G \otimes \mathbb{F})$. Translating that into the language of pure braids gives (unsurprisingly in view of Corollary 33) the following theorem:

THEOREM 34. *The dimension of the space of indecomposable type n invariants modulo type $n - 1$ invariants is equal to the rank, φ_n^k , of the free abelian group $\Gamma^n(P_k)/\Gamma^{n+1}(P_k)$. \square*

Recall that these ranks were given in Corollary 29.

REMARK 7. There are several ways to derive the above theorem. Kohno [43] wrote down the following formula which is analogous to the Witt formula for free groups [49]:

$$\prod_{n=1}^{\infty} (1 - t^n)^{\varphi_n^k} = \prod_{j=1}^{k-1} (1 - jt).$$

The left hand side can be interpreted as the reciprocal of the Poincaré series of a polynomial algebra with φ_n^k generators in degree n , and the right hand side can be interpreted as the reciprocal of the Poincaré series of the space of non-decreasing chord diagrams on k strands — non-decreasing chord diagrams were defined in [9] and shown to give a basis for the space of all chord diagrams modulo $4T$. The formulæ for φ_n^k given above can then be recovered from Kohno’s Witt-like formula (as in [49] for free groups) by taking logarithms, differentiating with respect to t , expanding in powers of t , comparing coefficients of t , and then using Möbius inversion.

3.2. Reducing by invariants from lower pure braid groups. There are $\binom{k}{l}$ maps from the k -strand pure braid group to the l -strand pure braid group obtained by picking l strands and “forgetting” the rest, thus each invariant of l -strand braids induces $\binom{k}{l}$ invariants of P_k . For instance, all type 1 invariants are linear combinations of winding numbers, and so are induced from P_2 .

$k \setminus n$	1	2	3	4	5	6	7	8	9	10
2	1	0	0	0	0	0	0	0	0	0
3	3	1	2	3	6	9	18	30	56	99
4	6	4	10	21	54	125	330	840	2240	5979
5	10	10	30	81	258	795	2670	9000	31360	110733
6	15	20	70	231	882	3375	13830	57750	248360	1086981
7	21	35	140	546	2436	11110	53820	267540	1368080	7132818
8	28	56	252	1134	5796	30654	171468	987840	5851776	35378658
9	36	84	420	2142	12348	74250	471060	3084480	20764800	142749558
10	45	120	660	3762	24156	162690	1154340	8464500	63811440	491422086

Table 4.2. The dimensions, φ_n^k of the spaces of indecomposable type n invariants modulo type $(n - 1)$ invariants of the pure braid groups P_k — see Theorem 34 and Corollary 29.

$k \setminus n$	1	2	3	4	5	6	7	8	9	10
2	1	0	0	0	0	0	0	0	0	0
3	0	1	2	3	6	9	18	30	56	99
4	0	0	2	9	30	89	258	720	2016	5583
5	0	0	0	6	48	260	1200	5100	20720	81828
6	0	0	0	0	24	300	2400	15750	92680	510288
7	0	0	0	0	0	120	2160	23940	211680	1643544
8	0	0	0	0	0	0	720	17640	258720	2963520
9	0	0	0	0	0	0	0	5040	161280	3024000
10	0	0	0	0	0	0	0	0	40320	1632960

Table 4.3. The dimensions, ψ_n^k , of the spaces of reduced type n indecomposable invariants modulo type $(n - 1)$ invariants of the pure braid groups P_k — see Theorem 35.

To see how many genuinely new invariants come from the P_k , one can calculate the dimension, ψ_n^k , of the space of type n indecomposable invariants of P_k modulo the type n invariants induced from lower braid groups. Then these dimensions satisfy

$$\varphi_n^l = \sum_{k=1}^l \binom{l}{k} \psi_n^k.$$

Define $\text{sur}(m, k)$ for $m > 0$, to be the number of surjections from an m element set to a k element set, with the convention that $\text{sur}(m, 0) = 0$.

THEOREM 35. *The reduced dimensions of type n invariants of P_k are given by*

$$\psi_n^k = \frac{1}{n} \sum_{m|n} \mu\left(\frac{n}{m}\right) \text{sur}(m, k - 1).$$

REMARK 8. For instance, if $k > 2$ and either $n < 2(k - 1)$ or n is prime, then $\psi_n^k = \text{sur}(n, k - 1)/n$.

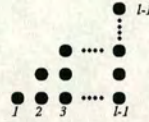


The key point is the following combinatorial identity:

LEMMA 36. For $m > 0$,

$$\sum_{i=1}^{l-1} i^m = \sum_{j=1}^l \binom{l}{j} \text{sur}(m, j - 1).$$

PROOF (OF LEMMA). The left hand side can be seen as the number of maps from an m element set to the following set, such that the image is ‘vertical’.



The proof proceeds by counting these maps in a different way. So,

$$\begin{aligned} \sum_{i=1}^{l-1} i^m &= \sum_{r=1}^{l-1} (\text{number of maps of image size } r) = \sum_{r=1}^{l-1} \sum_{k=1}^{l-1} \binom{k}{r} \text{sur}(m, r) \\ &= \sum_{r=1}^{l-1} \binom{l}{r+1} \text{sur}(m, r) && \text{by a simple identity} \\ &= \sum_{j=1}^l \binom{l}{j} \text{sur}(m, j - 1) && \text{by relabelling and } \text{sur}(m, 0) = 0. \end{aligned}$$

□

PROOF (OF THEOREM).

$$\begin{aligned} \sum_{k=1}^l \binom{l}{k} \psi_n^k &= \varphi_n^l = \frac{1}{n} \sum_{m|n} \mu\left(\frac{n}{m}\right) \sum_{i=1}^{l-1} i^m \\ &= \frac{1}{n} \sum_{m|n} \mu\left(\frac{n}{m}\right) \sum_{j=1}^l \binom{l}{j} \text{sur}(m, j - 1) \\ &= \sum_{j=1}^l \binom{l}{j} \left[\frac{1}{n} \sum_{m|n} \mu\left(\frac{n}{m}\right) \text{sur}(m, j - 1) \right]. \end{aligned}$$

The theorem then follows as the matrix $\left(\binom{i}{j}\right)_{1 \leq i, j \leq l}$ is invertible — it has inverse $\left((-1)^{i-j} \binom{j}{i}\right)_{1 \leq i, j \leq l}$. □

REMARK 9. If the same sort of reduction is performed with the ranks of the quotients of the lower central series of the free group F_k , then the number obtained is

$\frac{1}{n} \sum_{m|n} \mu\left(\frac{n}{m}\right) \text{sur}(m, k)$, reflecting the philosophy that modulo braiding on fewer than k strands, P_k looks like F_{k-1} .

These dimensions are tabulated in Table 4.3. Note that, by [8], the entries along the leading diagonal of the table correspond to Milnor invariants.

CHAPTER 5

Iterated integrals and minimal models for $\pi_1(X) \otimes \mathbb{R}$.

*And there's a mighty judgment coming, but I may be wrong
You see, you hear these funny voices
In the Tower of Song*

— LEONARD COHEN, *Tower of Song*.

This chapter presents some of the theory of de Rham homotopy theory of the fundamental group. The theme is that of generalizing abelianization via towers of nilpotent objects.

Section 1 introduces Chen's iterated integrals. Section 2 demonstrates the use of currents with these. Section 3 discusses tensoring a finitely generated group with the reals. Section 4 describes the algebraic idea of Sullivan's 1-minimal models. Section 5 gives a geometric relation between 1-minimal models and iterated integrals. Section 6 tells how to obtain functions on nilpotent quotients of the fundamental group by using minimal models. This will be applied to the case of pure braids in the next chapter.

1. Chen's iterated integrals.

Chen's method of iterated integrals generalizes the notion of integrating forms over cycles, and allows access to deeper homotopy information than just the abelianization of the fundamental group. The filtration induced on the fundamental group in the case of the configuration space $\mathbb{C}[k]$ is precisely the Vassiliev filtration on the pure braid group, thus iterated integrals give explicit formulæ for finite type pure braid invariants. Material for this section was harvested from [18, 34, 33].

1.1. Iterated integrals. Let (X, x_0) be a pointed smooth manifold, and let LX be the space of piecewise-smooth based loops on X . For θ a one-form on X , consider the functional on this loop space given by:

$$\int \theta: LX \rightarrow \mathbb{R}; \quad \gamma \mapsto \int_{\gamma} \theta.$$

This is a homotopy functional (that is, it depends only on the class of γ in the fundamental group of X) if and only if θ is exact, i.e. $d\theta = 0$. In fact, it is well known that if this is so, then $\int_{\gamma} \theta$ depends only on the class of γ in the abelianization of the

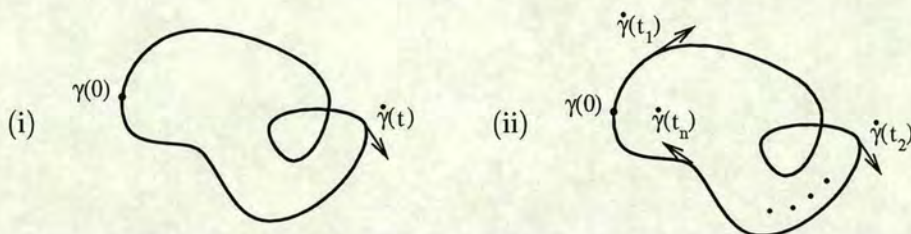


Figure 5.1. (i) To evaluate $\int_{\gamma} \theta$ the 1-form θ is evaluated on the tangent vector, $\dot{\gamma}(t)$, to the curve at each point, $\gamma(t)$, around the curve. (ii) To evaluate a length n iterated integral, n ordered tangent vectors around the curve are considered simultaneously.

fundamental group, and $\int \theta$ depends only on the class of θ in the de Rham cohomology group $H^1(\wedge^{\bullet} X, d)$. This corresponds to the composition of the de Rham isomorphism with the Hurewicz homomorphism:

$$H^1(\wedge^{\bullet} X, d) \cong H^1(X, \mathbb{R}) \cong \left(\pi_1(X) / \Gamma^2 \right)^{\vee}.$$

For n one-forms $\theta_1, \dots, \theta_n$, Chen defined $\int \theta_1 \dots \theta_n$, a functional on the loop space, by

$$\int_{\gamma} \theta_1 \dots \theta_n := \int \dots \int_{0 \leq t_1 < \dots < t_n \leq 1} \gamma^* \theta_1(t_1) \wedge \dots \wedge \gamma^* \theta_n(t_n).$$

Call such a functional a basic iterated integral of length n . See Figure 5.1 for a clue to what is being calculated; see also the discussion of currents below.

An iterated integral will mean an \mathbb{R} -linear combination of basic iterated integrals and constant functionals on LX . The length of an iterated integral is the largest of the lengths of its summands — constant functionals have zero length.

Chen defined a notion of differentiable space and its associated de Rham complex. In this setting one can take the exterior derivative of an iterated integral and obtain a one form on the loop space LX . The exterior derivative is the linear extension of the following function on basic iterated integrals:

$$d \int \theta_1 \dots \theta_n = - \sum_{j=1}^n \int \theta_1 \dots d\theta_j \dots \theta_n - \sum_{j=1}^{n-1} \int \theta_1 \dots (\theta_j \wedge \theta_{j+1}) \dots \theta_n.$$

Indeed, an iterated integral, I , is locally constant on the loop space of X (i.e. is a homotopy functional) precisely when $dI = 0$. Let $\mathcal{J}_n(X)$ be the space of homotopy invariant iterated integrals of length n or less (thought of as a subspace of the space of functionals on $\pi_1(X)$). The following is Chen's π_1 Theorem:

THEOREM 37. (i) For $J \triangleleft \mathbb{R}\pi_1(X)$ the augmentation ideal of $\pi_1(X)$ defined in Chapter 4.2, there is a natural isomorphism

$$J_n(X) \cong (\mathbb{R}\pi_1(X)/J^{n+1})^\vee.$$

(ii) If $A^\bullet \subset \Lambda^\bullet X$ is a sub-differential graded algebra, such that the inclusion map induces an isomorphism on cohomology, then all elements in $J_n(X)$ can be written as iterated integrals in forms from A^\bullet .

This can be applied immediately to the case of pure braids, $X = \mathbb{C}[k]$, with Theorem 30, to obtain

COROLLARY 38. An invariant of P_k is of type n if and only if it can be written as a homotopy invariant iterated integral of length less than or equal to n , in the ω_{ij} , i.e. $J_n(\mathbb{C}[k]) = \mathcal{V}_k^n$. \square

One can then look for finite type invariants by trying to write down homotopy invariant iterated integrals on the configuration space $\mathbb{C}[k]$. Recall the winding forms ω_{ij} defined in Section 2.2 of the previous chapter. A few homotopy invariant iterated integrals present themselves immediately:

$$\int \omega_{ij}, \quad \int (\omega_{ij}\omega_{lm} + \omega_{lm}\omega_{ij}), \quad \int (\omega_{ij}\omega_{jl} + \omega_{jl}\omega_{il} + \omega_{il}\omega_{ij}).$$

These are homotopy invariant because of, respectively, the exactness of the ω_{ij} , the anti-symmetry of \wedge , and the Arnold identity. The first is a winding number, the second is the product of two winding numbers (see below) and the third is the Milnor triple invariant (modulo products of invariants of type one).¹ What should come after these is not obvious. What is required is a method for generating homotopy invariant iterated integrals which are indecomposable — that is, not products of lower order invariants. Minimal models will give a recipe for just this. First, the Hopf algebraic structure of the space of iterated integrals should be mentioned.

1.2. Hopf algebraic structure and iterated integrals. Explicit formulæ exist for the operations in the Hopf algebra. The product is given by

$$\int_{\gamma} \theta_1 \dots \theta_n \cdot \int_{\gamma} \theta_{n+1} \dots \theta_{n+m} = \sum_{(n,m)\text{-shuffles } \sigma} \int_{\gamma} \theta_{\sigma(1)} \dots \theta_{\sigma(n+m)}$$

where an (n, m) -shuffle, σ , is a permutation of $\{1, \dots, n+m\}$ such that if $1 \leq a < b \leq n$ and $n+1 \leq c < d \leq n+m$ then $\sigma(a) < \sigma(b)$ and $\sigma(c) < \sigma(d)$.

¹This follows from later calculations, and the fact that the only type two invariant modulo products of type one invariants is the Milnor triple invariant — this follows from the dimension counts in the previous chapter and in [8].

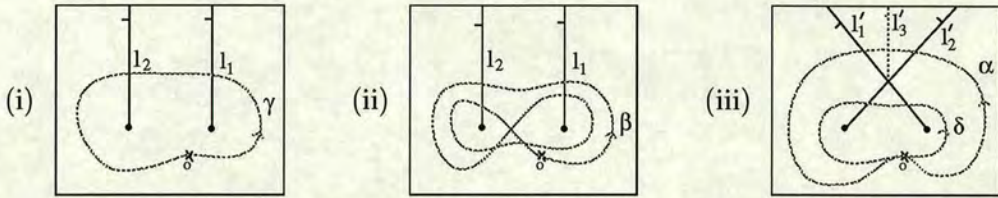


Figure 5.2. Paths and lines paths in the twice punctured plane. The co-orientation of each line is represented by the mark on the line which indicates the positive side of the line.

The coproduct is described via

$$\int_{\gamma\beta} \theta_1 \dots \theta_n = \sum_{i=0}^n \int_{\gamma} \theta_1 \dots \theta_i \cdot \int_{\beta} \theta_{i+1} \dots \theta_n.$$

Finally the antipode comes from

$$\int_{\gamma^{-1}} \theta_1 \dots \theta_n = (-1)^n \int_{\gamma} \theta_n \dots \theta_1.$$

REMARK 10. From the product formula comes $\int \theta_1 \cdot \int \theta_2 = \int (\theta_1 \theta_2 + \theta_2 \theta_1)$, explaining the comment about product of winding numbers above.

2. Currents.

An approach to iterated integrals which has both a more intuitive feel and an aspect of calculability was presented by Hain in [31]. It uses the idea of currents: the word current will here used to mean a distribution valued form supported on a sub-manifold (see also [21]) — this can be thought of as the limit of forms supported in regions arbitrarily close to the sub-manifold (see also Appendix A Section 4). From this approach the notion of a non-abelian intersection theory can be developed. These notions are used later to derive a combinatorial type two invariant of pure braids which is independent of the winding numbers.

2.1. Hain's example. One example given by Hain in [31] is the following. Consider the twice punctured plane, with two disjoint, co-oriented half-lines l_1 and l_2 as in Figure 5.2(i). Let w_1 and w_2 be the currents supported respectively on l_1 and l_2 , each representing the Poincaré dual of the respective line. Suppose γ is a generic based loop (i.e. intersects l_1 and l_2 transversely) then the signed intersection number of l_1 and γ can be written as $\int_{\gamma} w_1$. Indeed if $\{a_1, a_2\}$ is the basis of the first homology of the twice punctured plane with a_1 (respectively a_2) represented by a loop circling the right (respectively left) puncture, then the homology class of γ is $(\int_{\gamma} w_1) a_1 + (\int_{\gamma} w_2) a_2$. Both w_1 and w_2 are closed, and so both $\int_{\gamma} w_1$ and $\int_{\gamma} w_2$ are homotopy invariants.

If a curve, β , represents a commutator element in the fundamental group, such as in Figure 5.2(ii), then both $\int_{\beta} \omega_1$ and $\int_{\beta} \omega_2$ vanish, as β represents zero in homology. However, one can consider, for instance, the iterated integral $\int \omega_1 \omega_2$. This is a homotopy invariant as $d \int \omega_1 \omega_2 = - \int d\omega_1 - \int d\omega_2 - \int \omega_1 \wedge \omega_2 = -0 - 0 - 0$. If the definition of $\int_{\beta} \omega_1 \omega_2$ is unpacked, then it is seen that $\int_{\beta} \omega_1 \omega_2$ counts, with sign, the number of pairs (t_1, t_2) with $0 < t_1 < t_2 < 1$ such that $\beta(t_1) \in l_1$ and $\beta(t_2) \in l_2$. For the curve β in Figure 5.2(ii) there are three such pairs, but one is counted negative, so $\int_{\beta} \omega_1 \omega_2 = +1$.

Hain describes this as a non-abelian intersection theory; he also describes how the Hopf algebraic formulæ of the last section can be easily seen to hold if the forms are all currents supported on disjoint closed hypersurfaces.

2.2. The case of intersecting lines. To look at perturbing Hain's example so that the lines intersect, it is necessary to think a little about co-orientation. A co-orientation of a sub-manifold is an orientation of the normal bundle, i.e. a choice of non-vanishing section of the top exterior power of the co-normal bundle, this section considered up to multiplication by a positive function. Of course for a hyper-surface, a co-orientation is the same as a choice of one side of the hyper-surface being positive and the other negative.

For K a co-oriented sub-manifold, denote its co-orientation by $\text{co}(K)$. Adopt the following conventions. If K is a sub-manifold with boundary, then denote by $\text{in}(\partial K)$ the class (up to scaling by a positive function) of covectors on the boundary which are positive when evaluated on vectors pointing into K . Let the induced orientation on the boundary of K be given by $\text{co}(\partial K) = \text{in}(\partial K) \wedge \text{co}(K)$. Also [14, pp. 66–69] if K and L are transverse, co-oriented sub-manifolds then let the induced orientation on the ordered intersection, $(K \cap L)$, be given by $\text{co}(K \cap L) = \text{co}(K) \wedge \text{co}(L)$. With this convention, if K and L are also closed, with currents ω_K , ω_L , and $\omega_{K \cap L}$ having the obvious supports and representing the obvious Poincaré duals, then $\omega_{K \cap L} = \omega_K \wedge \omega_L$.

Let l'_1 and l'_2 be the intersecting closed half-lines as in Figure 5.2(iii). For $i = 1, 2$, let ω'_i be the current² supported on l'_i which represents the Poincaré dual of l'_i . Note that $\int \omega'_1 \omega'_2$ is *not* a homotopy invariant — for instance the two curves, α and δ , pictured in Figure 5.2(iii) are homotopic, but $\int_{\alpha} \omega'_1 \omega'_2 = 0$ and $\int_{\delta} \omega'_1 \omega'_2 = 1$ — this is because $d \int \omega'_1 \omega'_2 = \int \omega'_1 \wedge \omega'_2 \neq 0$. However, all is not lost. Now $\omega'_1 \wedge \omega'_2$ is concentrated at $l'_1 \cap l'_2$, and the co-oriented half-line l'_2 has as its boundary $l'_1 \cap l'_2$ with

²For the reader concerned about the use of currents, in Appendix A Section 4 it is shown how to construct C^∞ forms supported in ϵ -neighbourhoods of l'_1 , l'_2 , l'_3 , and $\partial l'_3$ which do (ϵ -approximately) the same job as the above currents.

opposite co-orientation. Let w'_3 be the current supported on l'_3 such that w'_3 restricted to the twice punctured plane minus the boundary of l'_3 , represents the Poincaré dual of l'_3 in this thrice punctured plane. Then dw'_3 is the 2-current supported on $\partial l'_3$ which represents its Poincaré dual, i.e. it equals $-w'_1 \wedge w'_2$.

This means that $d(\int w'_1 w'_2 + \int w'_3) = 0$, hence $\int w'_1 w'_2 + \int w'_3$ is a homotopy invariant, and, as one can probably see from Figure 5.2(iii), it is equal to $\int w_1 w_2$ in Hain's example above.

This example is very similar to that used to derive a combinatorial pure braid invariant in Chapter 6.

3. Nilpotent towers and Malcev groups.

This section contains some more algebra necessary for understanding de Rham homotopy theory. I have used material from [38, 58, 51, 50, 17, 37].

3.1. Malcev groups. Let G be a finitely generated group. Recall from Chapter 5 that $\{\Delta_0^i G\}_{i \in \mathbb{N}}$ are the characteristic zero dimension subgroups and that they are completely characterized by being the rational closure of the corresponding subgroups of the lower central series $\{\Gamma^i G\}_{i \in \mathbb{N}}$. The quotients G/Δ_0^i are torsion-free nilpotent and the factor groups $\Delta_0^i G/\Delta_0^{i+1} G$ are free abelian. Thus the following is a tower of torsion-free nilpotent groups built by iterated central extensions.

$$\begin{array}{ccc}
 & & \vdots \\
 & & \downarrow \\
 \Delta_0^3/\Delta_0^4 & \longrightarrow & G/\Delta_0^4 \\
 & & \downarrow \\
 \Delta_0^2/\Delta_0^3 & \longrightarrow & G/\Delta_0^3 \\
 & & \downarrow \\
 G/\Delta_0^2 & \longrightarrow & G/\Delta_0^2 \\
 & & \downarrow \\
 & & \{1\}.
 \end{array}$$

In the case that G is torsion-free nilpotent, $\Delta_0^i G = \{1\}$ for i sufficiently large, so G sits on top of a finite such tower. If G is just nilpotent, then $\Delta_0^i G$ eventually stabilizes to the subgroup of G which consists of all torsion elements.

Each stage in the tower can be embedded co-compactly in a simply-connected nilpotent Lie group. Of course \mathbb{Z}^n embeds co-compactly as the obvious lattice in \mathbb{R}^n .

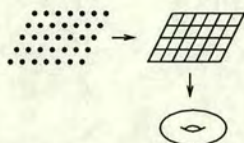


Figure 5.3. The 2-torus as the quotient of \mathbb{R}^2 by \mathbb{Z}^2 .

If a group H embeds, then a central extension of the form

$$\begin{array}{ccccc} \mathbb{Z}^r & \hookrightarrow & G & \longrightarrow & H \\ \downarrow & & & & \downarrow \\ \mathbb{R}^r & \dashrightarrow & G \otimes \mathbb{R} & \dashrightarrow & H \otimes \mathbb{R} \end{array}$$

can be embedded co-compactly. This process can be carried out iteratively up the tower. Thus a finitely generated torsion-free nilpotent group H embeds co-compactly in a simply connected nilpotent Lie group $H \otimes \mathbb{R}$. In fact this is well defined up to isomorphism. For a general finitely generated group G , the Malcev completion $G \rightarrow G \otimes \mathbb{R}$ is defined³ as the limit of the maps $G \rightarrow G/\Delta_0^i \hookrightarrow (G/\Delta_0^i) \otimes \mathbb{R}$.

For a simply connected nilpotent Lie group the exponential map is a diffeomorphism from the Lie algebra to the Lie group and the group multiplication pulls back to a well defined product (the Baker-Campbell-Hausdorff formula gives a finite sum as the Lie algebra is nilpotent). The Lie algebra is similarly built up from a tower of central extensions corresponding to the Lie group tower. Taking the limit of the tower one obtains the Malcev Lie algebra, $\mathcal{L}(G \otimes \mathbb{R})$.

The following maps can thus be constructed:

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & \downarrow & & \downarrow & & \downarrow & \\ \vdots & G/\Delta_0^3 & \hookrightarrow & (G/\Delta_0^3) \otimes \mathbb{R} & \xrightarrow{\cong} & \mathcal{L}((G/\Delta_0^3) \otimes \mathbb{R}) & \\ \nearrow & \downarrow & & \downarrow & & \downarrow & \\ G & \longrightarrow & G/\Delta_0^2 & \hookrightarrow & (G/\Delta_0^2) \otimes \mathbb{R} & \xrightarrow{\cong} & \mathcal{L}((G/\Delta_0^2) \otimes \mathbb{R}). \end{array}$$

3.2. Lie algebra cohomology. The purpose here is, in the case of a torsion-free nilpotent Lie algebra, to relate the Lie algebra cohomology to that of the its group and of a certain compact manifold.

The cohomology of a group H can be defined as the cohomology of any $K(H, 1)$ (see [15] for group cohomology). For H a finitely generated, torsion free, nilpotent

³This can actually be done functorially [62].

group, a compact $K(H, 1)$ which is a manifold can be constructed as follows:⁴ embed H co-compactly into the simply connected Lie group $H \otimes \mathbb{R}$; the quotient space, $H \otimes \mathbb{R}/H$, is the required Eilenberg-MacLane space.

To define Lie algebra cohomology consider the right invariant forms, $\Lambda^\bullet(H \otimes \mathbb{R})^{H \otimes \mathbb{R}}$, on $H \otimes \mathbb{R}$. By the usual identification of the Lie algebra \mathfrak{h} with the right invariant vector fields, the right invariant forms can be identified with the differential graded algebra, $\Lambda^\bullet(\mathfrak{h}^\vee)$, freely generated by the dual of the Lie algebra and with the differential given by dualizing the bracket.⁵ The real cohomology of \mathfrak{h} is defined to be the cohomology of this differential graded algebra, i.e.

$$H^\bullet(\mathfrak{h}, \mathbb{R}) := H^\bullet\left(\Lambda^\bullet(\mathfrak{h}^\vee)\right).$$

There is a canonical isomorphism — the push-forward — between the forms, $\Lambda^\bullet(H \otimes \mathbb{R})^H$, on $H \otimes \mathbb{R}$ invariant under the action of H (which is embedded in $H \otimes \mathbb{R}$) and the de Rham complex, $\Lambda^\bullet(H \otimes \mathbb{R}/H)$, of the quotient space.

An important fact [58] is that the inclusion of $(H \otimes \mathbb{R})$ -invariant forms on $H \otimes \mathbb{R}$ into the H -invariant forms on $H \otimes \mathbb{R}$,

$$\Lambda^\bullet(\mathfrak{h}^\vee) \cong \Lambda^\bullet(H \otimes \mathbb{R})^{H \otimes \mathbb{R}} \hookrightarrow \Lambda^\bullet(H \otimes \mathbb{R})^H \cong \Lambda^\bullet(H \otimes \mathbb{R}/H),$$

induces an isomorphism on cohomology, i.e.

$$H^\bullet(\mathfrak{h}, \mathbb{R}) \cong H^\bullet(H, \mathbb{R}).$$

This will be used below.

4. Sullivan's 1-minimal models.

Another approach to real π_1 theory is via Sullivan's 1-minimal models, which will be used to calculate pure braid invariants in the next chapter. See [29, 70] for some further material.

4.1. Definitions. If A^\bullet is a differential graded algebra then a 1-minimal model of A^\bullet is a differential graded algebra, \mathcal{M}^\bullet , freely generated in degree one, with a map of differential graded algebras

$$\rho: \mathcal{M}^\bullet \rightarrow A^\bullet,$$

⁴On first reading, the reader is advised to picture H as the abelian group \mathbb{Z}^2 , $H \otimes \mathbb{R}$ as \mathbb{R}^2 (with \mathbb{Z}^2 embedded as the obvious lattice) and $H \otimes \mathbb{R}/H$ as a standard 2-torus (see Figure 5.3); after that perhaps try to picture the Heisenberg group.

⁵This notation is potentially confusing, as the exterior algebra, denoted $\Lambda^\bullet(V)$, on the vector space V , is covariant in V , whereas the de Rham complex, denoted $\Lambda^\bullet(X)$, on the manifold X , is contravariant in X .

which induces an isomorphism, $H^1(\mathcal{M}^\bullet) \cong H^1(A^\bullet)$, on the first cohomology groups and induces an injection on the second cohomology groups. Note that the indecomposable elements of \mathcal{M}^\bullet can be identified as the homogeneous elements of degree one, \mathcal{M}^1 .

Such a 1-minimal model has a natural structure as an iterated sequence of elementary extensions, $\mathcal{M}^\bullet(i) = \mathcal{M}^\bullet(i-1) \otimes \Lambda^\bullet V(i)$, with $V(i)$ being homogeneous of degree one and $dV(i) \subset \mathcal{M}^1(i-1) \wedge \mathcal{M}^1(i-1)$. So $\mathcal{M}^\bullet(1) = \Lambda^\bullet(\ker d^1 : \mathcal{M}^1 \rightarrow \mathcal{M}^2)$. Hence there is a tower:

$$\begin{array}{ccc}
 & & \mathcal{M}^\bullet \\
 & & \uparrow \\
 & & \vdots \\
 & & \uparrow \\
 \Lambda^\bullet V(3) & \longleftarrow & \mathcal{M}^\bullet(3) \\
 & & \uparrow \\
 \Lambda^\bullet V(2) & \longleftarrow & \mathcal{M}^\bullet(2) \\
 & & \uparrow \\
 & & \mathcal{M}^\bullet(1).
 \end{array}$$

$\mathcal{M}^\bullet(i)$ is called the i th stage of the 1-minimal model.

4.2. Existence and construction. If A^\bullet is an augmented differential graded algebra then a 1-minimal model of A^\bullet exists, and the space is unique up to isomorphism with the map being unique up to a suitable notion of homotopy (see [29]). A minimal model can be constructed as follows.⁶

Set $\mathcal{M}^\bullet(1) = \{\Lambda^\bullet(H^1(A^\bullet)) | d = 0\}$, i.e. the free differential graded algebra on the first cohomology group of A^\bullet equipped with trivial differential, then define $\rho_1 : \mathcal{M}^\bullet(1) \rightarrow A^\bullet$ by choosing a set of representatives for $H^1(A^\bullet)$ and extend to an algebra map.

Continuing inductively, suppose that $\mathcal{M}^\bullet(i)$ and $\rho_i : \mathcal{M}^\bullet(i) \rightarrow A^\bullet$ have been defined. Let $V(i+1) := \ker\{H^2\rho_i : H^2\mathcal{M}^\bullet(i) \rightarrow H^2(A^\bullet)\}$ then choose a linear map

$$V(i+1) \rightarrow \mathcal{M}^2(i) \times A^1; \quad v \mapsto (m_v, a_v)$$

where m_v is a 2-cocycle representing v , i.e. $v = [m_v]$ and where $\rho_i(m_v) = da_v$. Define $dv = m_v$ and $\rho_{i+1}v = a_v$ then extend to $\mathcal{M}^\bullet(i+1) := \mathcal{M}^\bullet(i) \otimes \Lambda^\bullet V(i+1)$.

Take $\rho : \mathcal{M}^\bullet \rightarrow A^\bullet$ to be the direct limit of the $\rho_i : \mathcal{M}^\bullet(i) \rightarrow A^\bullet$. By construction it will have the required cohomological properties.

4.3. 1-minimal model of a manifold. If X is a smooth manifold, then a 1-minimal model of X means a 1-minimal model of the de Rham complex on X . Of

⁶This construction is used in Appendix A but can be skipped on first reading.

course if $A^\bullet \subset \Lambda^\bullet X$ is a sub-differential graded algebra of the de Rham complex, the inclusion inducing an isomorphism on cohomology, then a 1-minimal model of A^\bullet will give a 1-minimal model of X .

Sullivan's π_1 theorem [70] is the following:

THEOREM 39. *If $\mathcal{M}^\bullet \rightarrow \Lambda^\bullet X$ is a 1-minimal model of X then are natural Lie algebra isomorphisms*

$$\left(\mathcal{M}^1(i)\right)^\vee \cong \mathcal{L}\left(\pi_1(X)/\Delta_0^i \otimes \mathbb{R}\right),$$

the left hand side having a bracket by dualizing the differential from $\mathcal{M}^\bullet(i)$. \square

Dually this is that $\mathcal{M}^\bullet(i)$ is isomorphic to $\Lambda^\bullet(\mathcal{L}(\pi_1(X)/\Delta_0^i \otimes \mathbb{R})^\vee)$ as a differential graded algebra, the latter having the differential obtained by dualizing the bracket. Following Section 3 above, this should be suggestive of a geometric interpretation in terms of invariant differential forms — this is the subject of the next section.

5. Higher order Albaneses.

The idea of this section is to give a geometric connection between the 1-minimal model and iterated integrals approaches, and was inspired by [69, 17].

5.1. Classical real analytic Albanese. Let X be a smooth manifold and let ΠX be the fundamental groupoid of X — that is the groupoid of homotopy⁷ classes of piecewise-smooth paths on X . Picking⁸ a set, $\{\theta_1, \dots, \theta_r\}$, of representative one-forms for a basis of $H^1(X, \mathbb{R})$, integration on X induces a groupoid homomorphism:

$$I: \Pi X \rightarrow H^1(X, \mathbb{R})^\vee.$$

With respect to the dual basis of $H^1(X, \mathbb{R})^\vee$, the map I is given by

$$I: [\beta] \mapsto \left(\int_\beta \theta_1, \dots, \int_\beta \theta_r \right).$$

If x_0 is a base point of X , then the fundamental group, $\pi_1(X, x_0)$, is a subgroupoid of the fundamental groupoid, and its image in $H^1(X, \mathbb{R})^\vee$ is a lattice, \mathcal{P} , called the period lattice.⁹ A map

$$\text{Alb}: X \rightarrow H^1(X, \mathbb{R})^\vee / \mathcal{P},$$

⁷Homotopy relative to the endpoints.

⁸In the context of Hodge theory, this can be done canonically by choosing the harmonic forms.

⁹It is interesting to note that the map I depends on the choice of $\{\theta_1, \dots, \theta_r\}$, but the period lattice does not.

called the Albanese, from X to an r -dimensional torus, is then obtained and, with respect to the above basis, is given by choosing a path β from x_0 to x and setting

$$\text{Alb}(x) = \left(\int_{\beta} \theta_1, \dots, \int_{\beta} \theta_r \right) \mathcal{P}.$$

In the case of a complex curve this is a real analytic version of the Jacobian.

The map Alb induces an isomorphism

$$\pi_1(X, x_0)/\Delta_0^2 \cong \pi_1(H^1(X, \mathbb{R})^\vee/\mathcal{P}).$$

Note that for a group G , G/Δ_0^2 is the abelianization modulo torsion of G .

5.2. Generalizing to higher nilpotency. The classical Albanese above only detects the torsion-free part of the abelianization of the fundamental group. This can be generalized, using iterated integrals, to obtain higher order, torsion-free, nilpotent quotients of the fundamental group.

The iterated integrals defined in Section 1 actually give functionals on the space of all piecewise-smooth paths on X , not just the loops on X , and an iterated integral gives a functional on the fundamental groupoid (i.e. is homotopy invariant) precisely when $dI = 0$, as in the loop case.

DEFINITION 11. Define $\mathcal{G}(i)$ to be the simply connected $(i - 1)$ -stage nilpotent Lie group $(\pi_1(X, x_0)/\Delta_0^i) \otimes \mathbb{R}$, i.e. the Lie group in which the i^{th} nilpotent quotient of the fundamental group embeds.

One can then define groupoid maps to the tower of simply connected nilpotent Lie groups by using homotopy invariant iterated integrals:

$$\begin{array}{ccc} & \vdots & \\ & \downarrow & \\ & \vdots & \mathcal{G}(3) \\ \nearrow & & \downarrow \\ \Pi X & \longrightarrow & \mathcal{G}(2). \end{array}$$

Chen [17] does this by the Lie transport map and essentially it looks like

$$I_i: \Pi X \rightarrow \mathcal{G}(i)$$

$$[\beta] \mapsto \left(\int_{\beta} \theta_1, \dots, \int_{\beta} \theta_r, I_{2,1}(\beta), \dots, I_{2,r_2}(\beta), I_{3,1}(\beta), \dots, I_{i,r_i}(\beta) \right).$$

where each $I_{n,s}$ is a homotopy invariant iterated integral of length n . The multiplication for the groupoid structure on the right hand side is the one defined in Section 1.2.

The image of the fundamental group, $\pi_1(X, x_0)$ forms a lattice $\mathcal{P}_i \subset \mathcal{G}(i)$ for each i . Quotienting out by these lattices gives a tower of compact nil-manifolds¹⁰ $N(i) := \mathcal{G}(i)/\mathcal{P}_i$, and well defined maps:

$$\begin{array}{ccc} & \vdots & \\ & \downarrow & \\ & \vdots & N(3) \\ \nearrow & & \downarrow \\ X & \longrightarrow & N(2). \end{array}$$

The tower induces isomorphisms $\pi_1(X, x_0)/\Delta_0^i \cong \pi_1(N(i))$. The maps $\text{Alb}_i: X \rightarrow N(i)$ are the higher order Albaneses.

5.3. A geometric interpretation of the 1-minimal model. Given the tower of Albaneses above, there are the maps

$$\begin{array}{ccc} & \mathcal{G}(i) & \\ & \downarrow q_i & \\ X & \xrightarrow{\text{Alb}_i} & N(i), \end{array}$$

with q_i being the quotient by the period lattice \mathcal{P}_i . Again, one can look at the right-invariant forms on the Lie group $\mathcal{G}(i)$, these push forward to forms on the nil-manifold $N(i)$ and pull back to forms on the original space X . Recalling that right-invariant forms can be identified with the free differential graded algebra on the the dual, $\mathfrak{g}(i)^\vee$, of the Lie algebra, $\mathfrak{g}(i)$, of $\mathcal{G}(i)$, one obtains a map

$$\text{Alb}_i^* q_{i*}: \Lambda^\bullet(\mathfrak{g}(i)^\vee) \rightarrow \Lambda^\bullet(X).$$

Note that the induced map on H^1 is an isomorphism (by the discussion above on Lie algebra cohomology), and $\Lambda^\bullet(\mathfrak{g}(i)^\vee)$ is, by definition, freely generated in degree one. It then follows from the nilpotency of $\mathfrak{g}(i)$, that the map $\text{Alb}_i^* q_{i*}$ is the i th stage of a 1-minimal model of X and gives a geometric interpretation of Sullivan's π_1 theorem (Theorem 39).¹¹

6. Iterated integrals from minimal models.

As a partial converse to the geometric construction of a minimal model from iterated integrals, Chen [19] (following Sullivan [69]) describes how every indecomposable in

¹⁰A nil-manifold is a manifold with nilpotent fundamental group and no other non-trivial homotopy group.

¹¹In fact, [35], one can prove Sullivan's π_1 theorem by this approach.

the minimal model can be used to detect the relevant homotopy classes via iterated integrals.

6.1. Geometric version. First consider the geometric minimal model constructed above. Suppose that $\alpha \in \mathfrak{g}(\mathfrak{n})^\vee$ then one can form the following composition:

$$\pi_1(X, x_0) \rightarrow \pi_1(X, x_0)/\Delta_0^n \hookrightarrow \mathcal{G}(\mathfrak{n}) \xrightarrow{\log} \mathfrak{g}(\mathfrak{n}) \xrightarrow{\alpha} \mathbb{R}.$$

The map $\pi_1(X, x_0) \rightarrow \mathcal{G}(\mathfrak{n})$ is a vector of iterated integrals of length \mathfrak{n} . As $\mathcal{G}(\mathfrak{n})$ is \mathfrak{n} -stage nilpotent, \log is just a polynomial of degree \mathfrak{n} . The composition is just going to be an iterated integral of length \mathfrak{n} , and this will depend linearly on the α chosen, with the different choices of α detecting all elements of $\pi_1(X, x_0)/\Delta_0^n$.

The idea is to be able to associate homotopy invariant iterated integrals to arbitrary 1-minimal models.

6.2. General case. First recall that $J_n(X)$ is the space of functionals on $\pi_1(X)$ spanned by homotopy invariant iterated integrals of length \mathfrak{n} or less, and let $J(X)$ be the space spanned by all homotopy invariant iterated integrals. The following theorem can be extricated from [19]:

THEOREM 40 (Chen). *If $\rho: \mathcal{M}^\bullet \rightarrow \Lambda^\bullet X$ is a 1-minimal model for X , and $\overline{\mathcal{M}^\bullet}$ is the algebra obtained by placing all the generators of \mathcal{M}^\bullet in degree zero, then one can construct an isomorphism of algebras*

$$\varphi: \overline{\mathcal{M}^\bullet} \rightarrow J(X). \quad \square$$

In the terminology of finite type pure braid invariants, this means that every weight system can be *explicitly* integrated to a finite type invariant.

The idea is to construct a map from the generators of the \mathfrak{n} th stage:

$$\varphi_n: \mathcal{M}^1(\mathfrak{n}) \rightarrow J(X).$$

To do this it is necessary to know something about the bar construction on a differential graded algebra. The bar construction, $B^\bullet(A^\bullet)$, on a differential graded algebra A^\bullet is a differential graded Hopf algebra; the underlying vector space is that underlying the tensor algebra of A^\bullet , but with a different grading. A typical basis element, $a_1 \otimes \cdots \otimes a_r$, of $B^\bullet(A^\bullet)$ is written as

$$\mathbf{a} = [a_1|a_2|\dots|a_r] \in B^\bullet(A^\bullet).$$

Such an element has grading

$$\deg \mathbf{a} = \sum_{i=1}^r (\deg a_i - 1).$$

Define the map $J : A^\bullet \rightarrow A^\bullet$ by $a \mapsto (-1)^{\deg a} a$ and then the differential of the bar construction is determined by

$$\begin{aligned} d\mathbf{a} &= \sum_{1 \leq i \leq r} (-1)^i [J\mathbf{a}_1 | \dots | J\mathbf{a}_{i-1} | d\mathbf{a}_i | \mathbf{a}_{i+1} | \dots | \mathbf{a}_r] \\ &\quad - \sum_{1 \leq i < r} (-1)^i [J\mathbf{a}_1 | \dots | J\mathbf{a}_{i-1} | J\mathbf{a}_i \wedge \mathbf{a}_{i+1} | \mathbf{a}_{i+2} | \dots | \mathbf{a}_r]. \end{aligned}$$

For the case of the 1-minimal model $\rho : \mathcal{M}^\bullet \rightarrow \Lambda^\bullet(X)$, there is a map from elements of the bar construction which consist of sums of strings of elements of \mathcal{M}^1 to the space of iterated integrals, given by

$$[\mathbf{m}_1 | \dots | \mathbf{m}_r] \mapsto \int \rho(\mathbf{m}_1) \dots \rho(\mathbf{m}_r).$$

Note that this commutes with the differentials. Thus what is required is a map from \mathcal{M}^1 to the cocycles in the bar construction. Chen [19] proves that if $m \in \mathcal{M}^1(n)$ then $[m]$ can be completed to a cocycle in the bar construction by adding elements of the form $[\mathbf{m}_1 | \dots | \mathbf{m}_r]$ with $r \geq 2$ and $m_i \in \mathcal{M}^1(n-1)$. This then gives a suitable map $\varphi : \mathcal{M}^1(n) \rightarrow ZB^\bullet(\mathcal{M}^\bullet(n)) \rightarrow \mathcal{J}(X)$.

The methodology described here is used explicitly in Appendix A to find pure braid invariants of low order, and the results are collated in the next chapter.

CHAPTER 6

Obtaining explicit pure braid invariants.

*It's monstrous, horrid, shocking,
Beyond the power of thinking,
Not to know, interlocking
Is no mere form of linking.*

— JAMES CLERK MAXWELL, (*Cats*) *Cradle Song*.

In this chapter the abstract de Rham π_1 theory of the previous chapter is applied to the braid theory of Chapter 4 to obtain some concrete formulæ for pure braid invariants. This is done by building low order minimal models. This solves a problem of M. A. Berger.

In Section 1 a dictionary translating from the language of the previous chapter to the language of finite type pure braid invariants is given, and Kohno's description of the chord diagram algebra of pure braids is given. Calculations from Appendix A are summarized in Section 2, giving low order minimal models for P_3 and P_4 , and integral formulæ for corresponding invariants. Section 3 presents a combinatorial formula for "the" second order invariant independent of winding numbers.

1. Translating to pure braids.

1.1. Dictionary. Firstly note the dictionary of Table 6.1 which helps translate the abstract ideas of Chapter 5 to the concrete notions of Vassiliev invariants.

1.2. Kohno's description of \mathfrak{p}_k . If the Malcev Lie algebra of the pure braid group of k strands is denoted by $P_k \rightarrow \mathfrak{p}_k$, then Kohno [43] gave a description of the Lie algebra \mathfrak{p}_k . Generators of \mathfrak{p}_k are $\{Y_{ij}\}_{1 \leq i < j \leq k}$ although it is simpler here to take generators $\{Y_{ij}\}_{1 \leq i \neq j \leq k}$ with the extra relations $Y_{ij} = Y_{ji}$. The other relations are:

$$\begin{aligned} [Y_{ij}, Y_{rs}] &= 0, & \text{if } i, j, r, s \text{ are distinct;} \\ [Y_{ij}, Y_{ir} + Y_{jr}] &= 0, & \text{if } i, j, r \text{ are distinct.} \end{aligned}$$

The universal enveloping algebra of \mathfrak{p}_k is the so-called algebra of chord diagrams, and in this the generator Y_{ij} is written as a chord diagram on k vertical strands with

Manifold X	Configuration space $\mathbb{C}[k]$
Fundamental group $\pi_1(X)$	Pure braid group P_k
Augmentation ideal $J \triangleleft \mathbb{R}\pi_1(X)$	Linear combination of braids with at least one double point.
$(\mathbb{R}\pi_1(X)/J^{n+1})^\vee$	Type n invariants \mathcal{V}_k^n
Space of functionals spanned by homotopy invariant iterated integrals, $\mathcal{J}(X)$	Space of finite type invariants
Model $\mathcal{A}^\bullet \subset \Lambda^\bullet(X)$	Arnold algebra generated by winding forms $\{\omega_{ij}\}$
Malcev Lie algebra $\pi_1(X) \rightarrow \mathfrak{g}$	Universal indecomposable invariant $P_k \rightarrow \mathfrak{p}_k$
i th Malcev Lie algebra, $\mathfrak{g}(i)$	i th nilpotent quotient of Lie algebra of chord diagrams, $\mathfrak{p}_k(i)$
Malcev Lie group $\pi_1(X) \rightarrow \mathcal{G}$	Universal finite type invariant, (a.k.a. Kontsevich integral) $P_k \rightarrow \mathcal{U}(\mathfrak{p}_k)$
Generator, $\alpha \in \mathcal{M}^1(\mathfrak{n})$, of minimal model	Indecomposable weight system of degree \mathfrak{n}

Table 6.1. A dictionary to translate from the abstract ideas of the previous chapter to the concrete braid theory. Note that the translation is not precise.

a chord joining the i th and j th strands:



Multiplication is by placing diagrams on top of one another as with the braid multiplication. The relations coming from \mathfrak{p}_k are known as the commutativity and 4T relations.

So in fact the universal Vassiliev invariant is $P_k \rightarrow \mathcal{U}(\mathfrak{p}_k)$ and elements of the dual space to $\mathcal{U}(\mathfrak{p}_k)$ are weight systems. A standard Hopf algebra result is that the elements dual to the primitives are the indecomposable in the dual space, so the dual of \mathfrak{p}_k are indecomposable weight systems. Thus to find a set of generators for the indecomposable weight systems then, one could dualize \mathfrak{p}_k and find bases for the nilpotent quotients (the n th nilpotent quotient being the space of weight systems of degree \mathfrak{n} or less). To obtain generators for the invariants one should then use these in conjunction with

n		d	ρ	φ
1	α^1		ω_{23}	$\int \omega_{23}$
	α^2		ω_{31}	$\int \omega_{31}$
	α^3		ω_{12}	$\int \omega_{12}$
2	μ	$\chi^1 \wedge \chi^2$	0	$\int \Upsilon^2 \Upsilon^1$
3	ι_1	$\mu \wedge \chi^1$	0	$-\int \Upsilon^2 \Upsilon^1 \Upsilon^1$
	ι_2	$\mu \wedge \chi^2$	0	$\int \Upsilon^1 \Upsilon^2 \Upsilon^2$
4	κ_1	$\iota_1 \wedge \chi^1$	0	$\int \Upsilon^2 \Upsilon^1 \Upsilon^1 \Upsilon^1$
	κ_2	$\iota_2 \wedge \chi^2$	0	$-\int \Upsilon^1 \Upsilon^2 \Upsilon^2 \Upsilon^2$
	κ_3	$\iota_2 \wedge \chi^1 + \iota_1 \wedge \chi^2$	0	$-\int \Upsilon^1 \Upsilon^2 \Upsilon^2 \Upsilon^1 - \int \Upsilon^1 \Upsilon^2 \Upsilon^1 \Upsilon^2 - \int \Upsilon^1 \Upsilon^1 \Upsilon^2 \Upsilon^2$
5	λ_1	$\kappa_1 \wedge \chi^1$	0	$-\int \Upsilon^2 \Upsilon^1 \Upsilon^1 \Upsilon^1 \Upsilon^1$
	λ_2	$\kappa_2 \wedge \chi^2$	0	$\int \Upsilon^1 \Upsilon^2 \Upsilon^2 \Upsilon^2 \Upsilon^2$
	λ'_1	$\kappa_1 \wedge \chi^2 + \kappa_3 \wedge \chi^1$	0	$-\int \Upsilon^2 \Upsilon^1 \Upsilon^1 \Upsilon^1 \Upsilon^2 - \int \Upsilon^2 \Upsilon^1 \Upsilon^1 \Upsilon^2 \Upsilon^1$ $-\int \Upsilon^2 \Upsilon^1 \Upsilon^2 \Upsilon^1 \Upsilon^1 - \int \Upsilon^2 \Upsilon^2 \Upsilon^1 \Upsilon^1 \Upsilon^1$
	λ'_2	$\kappa_2 \wedge \chi^1 + \kappa_3 \wedge \chi^2$	0	$\int \Upsilon^1 \Upsilon^2 \Upsilon^2 \Upsilon^2 \Upsilon^1 + \int \Upsilon^1 \Upsilon^2 \Upsilon^2 \Upsilon^1 \Upsilon^2$ $+ \int \Upsilon^1 \Upsilon^2 \Upsilon^1 \Upsilon^2 \Upsilon^2 + \int \Upsilon^1 \Upsilon^1 \Upsilon^2 \Upsilon^2 \Upsilon^2$
	λ''_1	$\kappa_1 \wedge \chi^2 + \iota_1 \wedge \mu$	0	$-\int \Upsilon^1 \Upsilon^1 \Upsilon^2 \Upsilon^1 \Upsilon^2 - 2 \int \Upsilon^2 \Upsilon^1 \Upsilon^1 \Upsilon^2 \Upsilon^2$
	λ''_2	$\kappa_2 \wedge \chi^1 - \iota_2 \wedge \mu$	0	$\int \Upsilon^2 \Upsilon^2 \Upsilon^1 \Upsilon^2 \Upsilon^1 + 2 \int \Upsilon^1 \Upsilon^2 \Upsilon^2 \Upsilon^1 \Upsilon^1$

Table 6.2. A 1-minimal model to stage five with associated invariants, for P_3 . Note the simplifying definitions $\chi^1 := \alpha^1 - \alpha^2$, $\chi^2 := \alpha^1 - \alpha^3$, $\Upsilon^1 := \omega_{23} - \omega_{31}$ and $\Upsilon^2 := \omega_{23} - \omega_{12}$.

Chen's Lie connection $P_k \rightarrow \mathfrak{p}_k$. The approach taken here, however, is to build up minimal models directly for $\mathbb{C}[k]$ and use the theory of Chapter 5 to then construct invariants. An attempt is then made to do this on the level of currents in the hope of constructing combinatorial invariants.

2. Minimal models and integral formulæ for invariants of low order.

2.1. Three strand braids. The calculations in Appendix A lead to the summary in Table 6.2 which gives a description of a 1-minimal model for $\mathbb{C}[3]$ (or more precisely, for the Arnold algebra of winding forms) up to the fifth stage. The elements χ^1 and χ^2 are defined respectively to be $\alpha^1 - \alpha^2$ and $\alpha^1 - \alpha^3$ — this is so that the Arnold relation can be expressed in terms of a simple product: $\chi^1 \wedge \chi^2 = 0$.

So, for instance, one can read off that

$$\mathcal{M}^\bullet(2) = \left\{ \Lambda^\bullet(\alpha^1, \alpha^2, \alpha^3, \mu) \mid d\alpha^i = 0, d\mu = \chi^1 \wedge \chi^2 \right\}$$

with the map given as

$$\rho: \mathcal{M}^\bullet(3) \rightarrow \Lambda^\bullet(\mathbb{C}[3]); \alpha^1 \mapsto \omega_{23}, \alpha^2 \mapsto \omega_{31}, \alpha^3 \mapsto \omega_{12}, \mu \mapsto 0.$$

The one forms $\Upsilon^1 = \omega_{23} - \omega_{31}$ and $\Upsilon^2 = \omega_{23} - \omega_{21}$ are the images of χ^1 and χ^2 . The braid invariants associated to the generators of the minimal model are then

$$\alpha^1 \mapsto \int \omega_{23}, \alpha^2 \mapsto \int \omega_{31}, \alpha^3 \mapsto \int \omega_{12}, \mu \mapsto \int \Upsilon^2 \Upsilon^1.$$

These braid invariants are uniquely determined modulo products of lower order.

Note that the numbers of invariants match those in Table 4.3.

REMARK 11. The generators given here are far from canonical and are chosen to simplify the calculations. It can be asked if Hain's canonical idempotent on the bar construction [32] will reduce the choices.

2.2. Four strand braids. As shown in Appendix A and Table 4.3, the first two stages of the minimal model for P_4 are generated by elements which are induced by the “forgetting strings” maps to the two lower braid groups, P_2 and P_3 . Two new forms of generator occur at stage three however. In Appendix A suitable invariants corresponding to these generators are given as

$$\begin{aligned} & \int \omega_{42}(\omega_{41} - \omega_{12})(\omega_{23} - \omega_{34}) + \int \omega_{42}(\omega_{23} - \omega_{34})(\omega_{41} - \omega_{12}) \\ & + \int \omega_{13}(\omega_{34} - \omega_{41})(\omega_{12} - \omega_{23}) + \int \omega_{13}(\omega_{12} - \omega_{23})(\omega_{34} - \omega_{41}) \\ & - \int (\omega_{23} + \omega_{41})\omega_{12}\omega_{34} - \int (\omega_{23} + \omega_{41})\omega_{34}\omega_{12} \\ & - \int (\omega_{12} + \omega_{34})\omega_{41}\omega_{23} - \int (\omega_{12} + \omega_{34})\omega_{23}\omega_{41}, \end{aligned}$$

and

$$\begin{aligned} & \int \omega_{23}(\omega_{34} - \omega_{24})(\omega_{31} - \omega_{12}) + \int \omega_{23}(\omega_{31} - \omega_{12})(\omega_{34} - \omega_{24}) \\ & + \int \omega_{41}(\omega_{12} - \omega_{24})(\omega_{31} - \omega_{34}) + \int \omega_{41}(\omega_{31} - \omega_{34})(\omega_{12} - \omega_{24}) \\ & + \int (\omega_{24} + \omega_{13})\omega_{34}\omega_{12} + \int (\omega_{24} + \omega_{13})\omega_{12}\omega_{34} \\ & + \int (\omega_{12} + \omega_{34})\omega_{24}\omega_{13} + \int (\omega_{12} + \omega_{34})\omega_{13}\omega_{24}. \end{aligned}$$

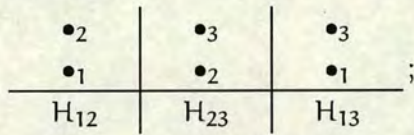
3. Towards combinatorial formulæ.

The goal is to get combinatorial expressions for a set of generating invariants. The idea is to try to map the minimal model generators to currents supported on submanifolds of $\mathbb{C}[k]$, so that evaluating iterated integrals based on these currents becomes a combinatorial operation, resulting in Gauß diagram type formulæ. This is achieved here in the limited case of invariants up to type two — the case of type one invariants is just the classical combinatorial way of calculating the winding numbers.

3.1. Beginning with $\mathbb{C}[3]$. To build a 1-minimal model mapping into currents, first a set of current representatives for a basis of $H^1(\mathbb{C}[3])$ must be found. A suitable set is supported on the following hyperplanes:

$$\begin{aligned} H_{12} &= \{z \in \mathbb{C}[3] \mid \operatorname{Re} z_1 = \operatorname{Re} z_2, \operatorname{Im} z_1 < \operatorname{Im} z_2\}; \\ H_{23} &= \{z \in \mathbb{C}[3] \mid \operatorname{Re} z_2 = \operatorname{Re} z_3, \operatorname{Im} z_2 < \operatorname{Im} z_3\}; \\ H_{13} &= \{z \in \mathbb{C}[3] \mid \operatorname{Re} z_1 = \operatorname{Re} z_3, \operatorname{Im} z_1 < \operatorname{Im} z_3\}. \end{aligned}$$

These are oriented so that the Poincaré dual of H_{ij} is the cohomology class of the winding form ω_{ij} . Then pick the current h_{ij} supported on H_{ij} representing the Poincaré dual. The hyperplanes can be pictured as the following configurations (real axis across the page, imaginary axis up the page):



the third point being allowed to be anywhere in the complex plane. So one can define a map, ρ' , from \mathcal{M}^1 into the 1-currents on $\mathbb{C}[3]$ by:

$$\rho'(\alpha^1) = h_{23}, \quad \rho'(\alpha^2) = h_{31}, \quad \rho'(\alpha^3) = (h_{12}).$$

The corresponding pure braid invariants are the winding numbers $w_{ij}(\beta) = \int_{\beta} h_{ij}$. These hyperplanes were chosen so that this integral can be done combinatorially: as h_{ij} represents the Poincaré dual of the hyperplane H_{ij} , $\int_{\beta} h_{ij}$ is just the signed intersection number of the braid β (as a loop in $\mathbb{C}[3]$) with H_{ij} — from a braid diagram of β , this is just the number of times (counted with sign) that strand i passes in front of strand j . This translates immediately to the Gauß diagram formalism of Chapter 3. Recall that the Gauß diagram encodes the crossing information in a diagram: the analogous definition is made here for braids. Then by the above discussion, e.g.

$$w_{12}(\beta) = \left\langle \begin{array}{c} \uparrow \uparrow \uparrow \\ \text{H} \end{array}, G_{\beta} \right\rangle$$

for G_{β} the Gauß diagram of the projection of the braid β , and the notation means count (with sign) the number of instances of the arrow diagram (on the left) in the Gauß diagram (on the right).

For example,

$$w_{12} \left(\begin{array}{c} \text{Y} \\ | \end{array} \right) = \left\langle \begin{array}{c} \uparrow \uparrow \uparrow \\ \text{H} \end{array}, \begin{array}{c} \uparrow \uparrow \uparrow \\ \text{H} \\ \uparrow \uparrow \uparrow \end{array} \right\rangle = +1.$$

To get the next level of invariants it is necessary to go to the second stage of the 1-minimal model and this requires examining the kernel of the wedge product. For the Arnold algebra this is contained in the Arnold relation:

$$\omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{31} + \omega_{31} \wedge \omega_{12} = 0.$$

As the h_{ij} are cohomologous to the ω_{ij} , on the cohomology level there is the relation:

$$[h_{12}] \wedge [h_{23}] + [h_{23}] \wedge [h_{13}] + [h_{13}] \wedge [h_{12}] = 0,$$

i.e. on the form level there is a 1-form k_{123} such that

$$h_{12} \wedge h_{23} + h_{23} \wedge h_{13} + h_{13} \wedge h_{12} = dk_{123}.$$

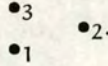
Translating that into the language of sub-manifolds, one would hope for a hypersurface $K_{123} \in C[3]$ such that

$$(H_{12} \cap H_{23}) \cup (H_{23} \cap H_{13}) \cup (H_{13} \cap H_{12}) = \partial K_{123}.$$

The calculation in Appendix A shows that setting

$$K_{123} = \{z \in C[3] \mid \operatorname{Re} z_1 = \operatorname{Re} z_3 \leq \operatorname{Re} z_2, \operatorname{Im} z_1 < \operatorname{Im} z_3\}$$

with the orientation as for H_{13} , gives the correct boundary. K_{123} can be pictured as:



This can be used to construct the second stage of the 1 minimal model, $\rho'_2: \mathcal{M}^\bullet(2) \rightarrow \Lambda^\bullet(X)$, so

$$\rho'(\mu) = k_{123}.$$

A suitable invariant is then given by

$$w_{123}(\beta) := \int_\beta (h_{12}h_{23} + h_{23}h_{13} + h_{13}h_{12} - k_{123})$$

To translate $\int k_{123}$ into a Gauß diagram formula it is necessary to encode some additional information in the Gauß diagram, namely whether the second strand is on the right when the first strand passes over the third. This can be indicated by placing an 'r' on the level of any such arrow on the Gauß diagram. So

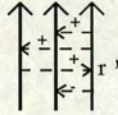
$$w_{123}(\beta) = \left\langle \begin{array}{c} \uparrow \uparrow \uparrow \\ \text{H} \text{H} \text{H} \\ \uparrow \uparrow \uparrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ \text{H} \text{H} \text{H} \\ \uparrow \uparrow \uparrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ \text{H} \text{H} \text{H} \\ \uparrow \uparrow \uparrow \end{array} - \begin{array}{c} \uparrow \uparrow \uparrow \\ \text{H} \text{H} \text{H} \\ \uparrow \uparrow \uparrow \end{array} \text{r}, G_\beta \right\rangle.$$

Here are two examples. first look at the pigtail braid:

$$w_{123} \left(\begin{array}{c} \text{pigtail braid} \end{array} \right) = \left\langle \begin{array}{c} \uparrow \uparrow \uparrow \\ \leftarrow \rightarrow \rightarrow \\ \uparrow \uparrow \uparrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ \leftarrow \rightarrow \rightarrow \\ \uparrow \uparrow \uparrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ \leftarrow \rightarrow \rightarrow \\ \uparrow \uparrow \uparrow \end{array} - \begin{array}{c} \uparrow \uparrow \uparrow \\ \leftarrow \rightarrow \rightarrow \\ \uparrow \uparrow \uparrow \end{array}, \begin{array}{c} \uparrow \uparrow \uparrow \\ \leftarrow \rightarrow \rightarrow \\ \uparrow \uparrow \uparrow \\ \leftarrow \rightarrow \rightarrow \\ \uparrow \uparrow \uparrow \\ \leftarrow \rightarrow \rightarrow \\ \uparrow \uparrow \uparrow \end{array} \right\rangle$$

$$= (+1 - 1 + 1) + 0 + 0 - 0 = 1.$$

Whereas the next braid has Gauß diagram



hence

$$w_{123} \left(\begin{array}{c} \text{Gauß diagram} \end{array} \right) = 0 + 0 + 0 - 1 = -1.$$

APPENDIX A

Calculations.

*Hi ho! Hi ho!
It's off to work we go.*

— DISNEY, *Snow White and the Seven Dwarfs*.

This appendix mostly contains calculations for Chapter 6, but also contains a section on approximating currents with C^∞ -forms pertinent to Chapter 5 Section 2. In Section 1 a minimal model for the Arnold algebra is constructed up to degree five for the three strand braid group and up to degree three for the four strand braid group: the former is 15 dimensional and the latter 21 dimensional. In Section 2 integral are constructed which give pure braid invariants corresponding to the generators of the minimal models of Section 1. In Section 3 the co-orientations of some intersections and boundaries of sub-manifolds of the configuration space $\mathbb{C}[3]$ are calculated to prove that the right choices are made in Chapter 6 Section 3. Finally, in Section 4 the example of Chapter 5 Section 2.2 is done with C^∞ forms supported arbitrarily close to the sub-manifolds, rather than with currents.

1. 1-minimal model for the Arnold algebra.

Recall from Section 4 of Chapter 5 that the 1-minimal model of a manifold X is constructed inductively, stage by stage, and at stage n consists of a differential graded algebra, $\mathcal{M}^\bullet(n)$, freely generated in degree 1, which is equipped with a map of differential graded algebras $\rho_n: \mathcal{M}^\bullet(n) \rightarrow \wedge^\bullet X$.

1.1. Three strand pure braids. Begin with the Arnold algebra:

$$\mathcal{A}_3^\bullet = \wedge^\bullet(\omega_{12}, \omega_{23}, \omega_{31}) / (\omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{31} + \omega_{31} \wedge \omega_{12}),$$

this is equipped with the trivial differential and forms a model for the de Rham algebra of the configuration space $\mathbb{C}[3]$.

For the first stage, $\rho_1: \mathcal{M}^\bullet(1) \rightarrow \mathcal{A}_3^\bullet$, of the minimal model take

$$\mathcal{M}^\bullet(1) := \left\{ \wedge \left(\alpha^1, \alpha^2, \alpha^3 \right) \mid d = 0 \right\}; \quad \rho(\alpha^1) = \omega_{23}, \quad \rho(\alpha^2) = \omega_{31}, \quad \rho(\alpha^3) = \omega_{12}.$$

To extend this to a second stage, consider

$$\begin{aligned} V(1) &= \ker \left\{ H^2 \rho_1: H^2 \mathcal{M}^\bullet(1) \rightarrow H^2 \mathcal{A}_3^\bullet \right\} \\ &= \left\langle \alpha^1 \wedge \alpha^2 + \alpha^2 \wedge \alpha^3 + \alpha^3 \wedge \alpha^1 \right\rangle = \left\langle (\alpha^1 - \alpha^2) \wedge (\alpha^1 - \alpha^3) \right\rangle. \end{aligned}$$

Set

$$\chi^1 := \alpha^1 - \alpha^2; \quad \chi^2 := \alpha^1 - \alpha^3.$$

Now a single generator is added to $\mathcal{M}^\bullet(1)$ to obtain

$$\mathcal{M}^\bullet(2) := \left\{ \mathcal{M}^\bullet(1) \otimes \Lambda^\bullet(\mu) \mid d\mu = \chi^1 \wedge \chi^2 \right\}; \quad \rho_2(\mu) := 0.$$

Next to calculate $H^2 \mathcal{M}^\bullet(2)$ first calculate the space, $Z^2 \mathcal{M}^\bullet(2)$, of cocycles:¹

$$\begin{aligned} d(a_3 \alpha^1 \wedge \alpha^2 + a_1 \alpha^2 \wedge \alpha^3 + a_2 \alpha^3 \wedge \alpha^1 + b_1 \mu \wedge \alpha^1 + b_2 \mu \wedge \alpha^2 + b_3 \mu \wedge \alpha^3) \\ = b_1 \alpha^2 \wedge \alpha^3 \wedge \alpha^1 + b_2 \alpha^3 \wedge \alpha^1 \wedge \alpha^2 + b_3 \alpha^1 \wedge \alpha^2 \wedge \alpha^3 \\ = (b_1 + b_2 + b_3) \alpha^1 \wedge \alpha^2 \wedge \alpha^3. \end{aligned}$$

So

$$Z^2 \mathcal{M}^\bullet(2) = \left\{ \mu \wedge (b_1 \alpha^1 + b_2 \alpha^2 + b_3 \alpha^3) + a_3 \alpha^1 \wedge \alpha^2 + a_1 \alpha^2 \wedge \alpha^3 \mid \sum b_i = 0 \right\}.$$

The co-boundaries, $B^2 \mathcal{M}^\bullet(2)$, are calculated via

$$d(a_1 \alpha^1 + a_2 \alpha^2 + a_3 \alpha^3 + b\mu) = b\chi^1 \wedge \chi^2.$$

Thus

$$H^2 \mathcal{M}^\bullet(2) = \left\{ a_3 \alpha^1 \wedge \alpha^2 + a_1 \alpha^2 \wedge \alpha^3 + a_2 \alpha^3 \wedge \alpha^1 + \sum_i b_i \chi^i \wedge \mu \right\} / \left\langle \chi^1 \wedge \chi^2 \right\rangle,$$

but

$$\rho_2(a_3 \alpha^1 \wedge \alpha^2 + a_1 \alpha^2 \wedge \alpha^3 + a_2 \alpha^3 \wedge \alpha^1 + \sum_i b_i \chi^i \wedge \mu) = a_3 \alpha^1 \wedge \alpha^2 + a_1 \alpha^2 \wedge \alpha^3 + a_2 \alpha^3 \wedge \alpha^1.$$

So

$$V(2) := \ker(H^2 \rho_2) = \left\{ [b_1 \mu \wedge \chi^1 + b_2 \mu \wedge \chi^2] \right\}.$$

This can then be used to construct:

$$\mathcal{M}^\bullet(3) := \left\{ \mathcal{M}^\bullet(2) \otimes \Lambda^\bullet(\iota_1, \iota_2) \mid d\iota_i = \mu \wedge \chi^i \right\}; \quad \rho_3(\iota_i) := 0.$$

Further calculations show

$$V(3) := \ker(H^2 \rho_3) = \left\{ [a_1 \iota_1 \wedge \chi^1 + a_2 \iota_2 \wedge \chi^2 + b(\iota_2 \wedge \chi^1 + \iota_1 \wedge \chi^2)] \right\} \subset H^2 \mathcal{M}^\bullet(3).$$

¹Latin letters will refer to real numbers, Greek letters will be elements of algebras.

This leads to defining

$$\mathcal{M}^\bullet(4) := \{\mathcal{M}^\bullet(3) \otimes \Lambda^\bullet(\kappa_1, \kappa_2, \kappa_3) \mid d\kappa_1 = \iota_1 \wedge \chi^1, d\kappa_2 = \iota_2 \wedge \chi^2, d\kappa_3 = \iota_2 \wedge \chi^1 + \iota_1 \wedge \chi^2\};$$

$$\rho_4(\kappa_i) := 0.$$

Additional straightforward but tedious calculations give

$$\begin{aligned} V(4) &:= \ker(H^2 \rho_4) \\ &= \{[a_1 \kappa_1 \wedge \chi^1 + a_2 \kappa_2 \wedge \chi^2 + (b_1 \kappa_1 \wedge \chi^2 - b_2 \kappa_3 \wedge \chi^1 - b_3 \iota_1 \wedge \mu) \\ &\quad + (c_1 \kappa_2 \wedge \chi^1 - c_2 \kappa_3 \wedge \chi^2 + c_3 \iota_2 \wedge \mu)] \mid \sum b_i = \sum c_i = 0\} \\ &\subset H^2 \mathcal{M}^\bullet(4). \end{aligned}$$

This leads to making the definition:

$$\begin{aligned} \mathcal{M}^\bullet(5) &:= \{\mathcal{M}^\bullet(4) \otimes \Lambda^\bullet(\lambda_1, \lambda_2, \lambda'_1, \lambda'_2, \lambda''_1, \lambda''_2) \mid \\ &\quad d\lambda_1 = \kappa_1 \wedge \chi^1, \quad d\lambda_2 = \kappa_2 \wedge \chi^2, \\ &\quad d\lambda'_1 = \kappa_1 \wedge \chi^2 + \kappa_3 \wedge \chi^1, \quad d\lambda'_2 = \kappa_2 \wedge \chi^1 + \kappa_3 \wedge \chi^2, \\ &\quad d\lambda''_1 = \kappa_1 \wedge \chi^2 + \iota_1 \wedge \mu, \quad d\lambda''_2 = \kappa_2 \wedge \chi^1 - \iota_2 \wedge \mu\}; \\ \rho_5(\lambda_i) &:= 0. \end{aligned}$$

These results are summarized in Table 6.2.

1.2. Four strand pure braids. The first two stages of the minimal model for P_4 are just induced by the “forgetting strings” maps to the two lower braid groups, P_2 and P_3 , so take

$$\begin{aligned} \mathcal{M}^\bullet(2) &:= \left\{ \Lambda^\bullet \left(\alpha^{12}, \alpha^{13}, \alpha^{41}, \alpha^{23}, \alpha^{24}, \alpha^{34}, \mu^1, \mu^2, \mu^3, \mu^4 \right) \mid dx_{ij} = 0; \right. \\ &\quad d\mu^i = \alpha^{jl} \wedge \alpha^{lm} + \alpha^{lm} \wedge \alpha^{mj} + \alpha^{mj} \wedge \alpha^{jm} \text{ (where } (i, j, l, m) \text{ is} \\ &\quad \left. \text{the standard cyclic ordering of } (1, 2, 3, 4)) \right\}; \\ \rho_2(\alpha^{ij}) &:= \omega_{ij}, \quad \rho_2(\mu^i) := 0. \end{aligned}$$

In degree three there are similarly ι_1^i and ι_2^i (for $i = 1, 2, 3, 4$) induced in the same way, but there are two more generators required, as the elements in $\mathcal{M}^2(2)$ of the form

$$\begin{aligned} &a \left(\mu^1 \wedge \alpha^{12} - \mu^2 \wedge \alpha^{12} + \mu^3 \wedge \alpha^{34} - \mu^4 \wedge \alpha^{34} \right) \\ &+ b \left(\mu^1 \wedge \alpha^{13} - \mu^2 \wedge \alpha^{24} + \mu^3 \wedge \alpha^{13} - \mu^4 \wedge \alpha^{24} \right) \\ &+ c \left(\mu^1 \wedge \alpha^{14} - \mu^2 \wedge \alpha^{23} + \mu^3 \wedge \alpha^{23} - \mu^4 \wedge \alpha^{14} \right), \end{aligned}$$

where $a + b + c = 0$, are co-cycles. So take the two generators to be ν_1 and ν_2 with

$$\begin{aligned} d\nu_1 &:= \mu^1 \wedge (\alpha^{41} - \alpha^{12}) + \mu^2 \wedge (\alpha^{12} - \alpha^{23}) + \mu^3 \wedge (\alpha^{23} - \alpha^{34}) + \mu^4 \wedge (\alpha^{34} - \alpha^{41}), \\ d\nu_2 &:= \mu^1 \wedge (\alpha^{31} - \alpha^{12}) + \mu^2 \wedge (\alpha^{12} - \alpha^{24}) + \mu^3 \wedge (\alpha^{13} - \alpha^{34}) + \mu^4 \wedge (\alpha^{34} - \alpha^{42}), \\ \rho_3(\nu_1) &:= 0 \quad \rho_3(\nu_2) := 0. \end{aligned}$$

2. Integrals for invariants.

Here the method from Section 5 of Chapter 5 is used to obtain invariants from the generators of the minimal model. Recall that, by Chen, for α a generator in the n th stage of the 1-minimal model there exists a (not necessarily unique) 1-cocycle, $\hat{\varphi}(\alpha)$, in the bar construction of the form $[\alpha] + \sum_i [\theta_{i,1} | \dots | \theta_{i,r_i}]$ such that $r_i \geq 2$ for each i and each $\theta_{i,j}$ is in $\mathcal{M}^1(n-1)$. The image, $\varphi(\alpha) := \rho_n \circ \hat{\varphi}(\alpha)$, of this cocycle in the space of iterated integrals is the required invariant corresponding to α .

Recall that in the bar construction, for $\theta_1, \dots, \theta_i$ of degree one the differential of $[\theta_1 | \dots | \theta_i]$ is given by

$$d[\theta_1 | \dots | \theta_i] = - \sum_{r=1}^i [\theta_1 | \dots | d\theta_r | \dots | \theta_i] - \sum_{r=1}^{i-1} [\theta_1 | \dots | \theta_r \wedge \theta_{r+1} | \dots | \theta_i].$$

2.1. Three strands, stage two. The above formula means that

$$-d[\mu] = [d\mu] = [\chi^1 \wedge \chi^2] = -[\chi^2 \wedge \chi^1] = d[\chi^2 | \chi^1],$$

(as $d\chi^i = 0$). Thus setting

$$\hat{\varphi}(\mu) := [\mu] + [\chi^2 | \chi^1]$$

gives a cocycle in the bar construction, and hence

$$\varphi(\mu) = \int \gamma^2 \gamma^1$$

is a suitable homotopy invariant iterated integral.

2.2. Three strands, stage three. Working similarly, see that

$$\begin{aligned} -d[\iota_1] &= [d\iota_1] = [\mu \wedge \chi^1] = -d[\mu | \chi^1] - [d\mu | \chi^1] = -d[\mu | \chi^1] - [\chi^1 \wedge \chi^2 | \chi^1] \\ &= -d[\mu | \chi^1] + [\chi^2 \wedge \chi^1 | \chi^1] = -d[\mu | \chi^1] - d[\chi^2 | \chi^1 | \chi^1] \end{aligned}$$

and so take

$$\hat{\varphi}(\iota_1) := [\iota_1] - [\mu | \chi^1] - [\chi^2 | \chi^1 | \chi^1],$$

which gives

$$\varphi(\iota_1) = -\int \gamma^2 \gamma^1 \gamma^1.$$

An analogous calculation leads to setting

$$\varphi(\iota_2) = \int \gamma^1 \gamma^2 \gamma^2.$$

2.3. Three strands, stage four. One can verify

$$-d[\kappa_1] = d \left\{ -[\iota_1 | \chi^1] + [\mu | \chi^1 | \chi^1] + [\chi^2 | \chi^1 | \chi^1 | \chi^1] \right\},$$

so set

$$\hat{\varphi}(\kappa_1) := [\kappa_1] - [\iota_1 | \chi^1] + [\mu | \chi^1 | \chi^1] + [\chi^2 | \chi^1 | \chi^1 | \chi^1],$$

hence

$$\varphi(\kappa_1) = \int \gamma^2 \gamma^1 \gamma^1 \gamma^1.$$

Similarly one is lead to setting

$$\varphi(\kappa_2) := -\int \gamma^1 \gamma^2 \gamma^2 \gamma^2.$$

Also, as

$$\begin{aligned} -d[\kappa_3] = d \left\{ & -[\iota_2 | \chi^1] - [\iota_1 | \chi^2] + [\mu | \chi^2 | \chi^1] + [\mu | \chi^1 | \chi^2] \right. \\ & \left. - [\chi^1 | \chi^2 | \chi^2 | \chi^1] - [\chi^1 | \chi^2 | \chi^1 | \chi^2] - [\chi^1 | \chi^1 | \chi^2 | \chi^2] \right\}, \end{aligned}$$

set

$$\varphi(\kappa_3) := -\int \gamma^1 \gamma^2 \gamma^2 \gamma^1 - \int \gamma^1 \gamma^2 \gamma^1 \gamma^2 - \int \gamma^1 \gamma^1 \gamma^2 \gamma^2.$$

2.4. Three strands, stage five. By the same calculation as for κ_1 and κ_2 one can set

$$\varphi(\lambda_1) := -\int \gamma^2 \gamma^1 \gamma^1 \gamma^1 \gamma^1; \quad \varphi(\lambda_2) := \int \gamma^1 \gamma^2 \gamma^2 \gamma^2 \gamma^2.$$

Now,

$$\begin{aligned} -d[\lambda'_1] = d \left\{ & -[\kappa_1 | \chi^2] - [\kappa_3 | \chi^1] + [\iota_1 | \chi^1 | \chi^2] + [\iota_2 | \chi^1 | \chi^1] + [\iota_1 | \chi^2 | \chi^1] \right. \\ & - [\mu | \chi^1 | \chi^1 | \chi^2] - [\mu | \chi^2 | \chi^1 | \chi^1] - [\mu | \chi^1 | \chi^2 | \chi^1] \\ & \left. - [\chi^2 | \chi^1 | \chi^1 | \chi^1 | \chi^2] - [\chi^2 | \chi^1 | \chi^1 | \chi^2 | \chi^1] - [\chi^2 | \chi^1 | \chi^2 | \chi^1 | \chi^1] - [\chi^2 | \chi^2 | \chi^1 | \chi^1 | \chi^1] \right\}. \end{aligned}$$

So take

$$\varphi(\lambda'_1) := -\int \gamma^2 \gamma^1 \gamma^1 \gamma^1 \gamma^2 - \int \gamma^2 \gamma^1 \gamma^1 \gamma^2 \gamma^1 - \int \gamma^2 \gamma^1 \gamma^2 \gamma^1 \gamma^1 - \int \gamma^2 \gamma^2 \gamma^1 \gamma^1 \gamma^1,$$

and similarly

$$\varphi(\lambda'_2) := \int \gamma^1 \gamma^2 \gamma^2 \gamma^2 \gamma^1 + \int \gamma^1 \gamma^2 \gamma^2 \gamma^1 \gamma^2 + \int \gamma^1 \gamma^2 \gamma^1 \gamma^2 \gamma^2 + \int \gamma^1 \gamma^1 \gamma^2 \gamma^2 \gamma^2.$$

It can be verified that

$$-d[\lambda_1''] := d\left\{-[\kappa_1|x^2] - [\iota_1|\mu] + [\iota_1|x^1|x^2] - [x^1|\mu|\mu] + [x^1|\mu|x^1|x^2] + [x^1|x^1|x^2|\mu] + [x^1|x^1|\mu|x^2] - [x^1|x^1|x^2|x^1|x^2] - 2[x^1|x^1|x^1|x^2|x^2]\right\}.$$

So one sets

$$\varphi(\lambda_1'') := -\int \gamma^1 \gamma^1 \gamma^2 \gamma^1 \gamma^2 - 2 \int \gamma^1 \gamma^1 \gamma^1 \gamma^2 \gamma^2,$$

and likewise

$$\varphi(\lambda_2'') := \int \gamma^2 \gamma^2 \gamma^1 \gamma^2 \gamma^1 + 2 \int \gamma^2 \gamma^2 \gamma^2 \gamma^1 \gamma^1.$$

2.5. Four strands, stage three. An easy calculation gives that

$$\begin{aligned} -d[\nu_1] = d\left\{ & -[\mu^1|(\alpha^{41} - \alpha^{12})] - [\mu^2|(\alpha^{12} - \alpha^{23})] \right. \\ & - [\mu^3|(\alpha^{23} - \alpha^{34})] - [\mu^4|(\alpha^{34} - \alpha^{41})] \\ & + [\alpha^{42}|(\alpha^{41} - \alpha^{12})|(\alpha^{23} - \alpha^{34})] + [\alpha^{42}|(\alpha^{23} - \alpha^{34})|(\alpha^{41} - \alpha^{12})] \\ & + [\alpha^{13}|(\alpha^{34} - \alpha^{41})|(\alpha^{12} - \alpha^{23})] + [\alpha^{13}|(\alpha^{12} - \alpha^{23})|(\alpha^{34} - \alpha^{41})] \\ & - [(\alpha^{23} + \alpha^{41})|\alpha^{12}|\alpha^{34}] - [(\alpha^{23} + \alpha^{41})|\alpha^{34}|\alpha^{12}] \\ & \left. - [(\alpha^{12} + \alpha^{34})|\alpha^{41}|\alpha^{23}] - [(\alpha^{12} + \alpha^{34})|\alpha^{23}|\alpha^{41}]\right\}. \end{aligned}$$

So a suitable invariant is

$$\begin{aligned} \varphi(\nu_1) := & \int \omega_{42}(\omega_{41} - \omega_{12})(\omega_{23} - \omega_{34}) + \int \omega_{42}(\omega_{23} - \omega_{34})(\omega_{41} - \omega_{12}) \\ & + \int \omega_{13}(\omega_{34} - \omega_{41})(\omega_{12} - \omega_{23}) + \int \omega_{13}(\omega_{12} - \omega_{23})(\omega_{34} - \omega_{41}) \\ & - \int (\omega_{23} + \omega_{41})\omega_{12}\omega_{34} - \int (\omega_{23} + \omega_{41})\omega_{34}\omega_{12} \\ & - \int (\omega_{12} + \omega_{34})\omega_{41}\omega_{23} - \int (\omega_{12} + \omega_{34})\omega_{23}\omega_{41}. \end{aligned}$$

Similarly set

$$\begin{aligned} \varphi(\nu_2) := & \int \omega_{23}(\omega_{34} - \omega_{24})(\omega_{31} - \omega_{12}) + \int \omega_{23}(\omega_{31} - \omega_{12})(\omega_{34} - \omega_{24}) \\ & + \int \omega_{41}(\omega_{12} - \omega_{24})(\omega_{31} - \omega_{34}) + \int \omega_{41}(\omega_{31} - \omega_{34})(\omega_{12} - \omega_{24}) \\ & + \int (\omega_{24} + \omega_{13})\omega_{34}\omega_{12} + \int (\omega_{24} + \omega_{13})\omega_{12}\omega_{34} \\ & + \int (\omega_{12} + \omega_{34})\omega_{24}\omega_{13} + \int (\omega_{12} + \omega_{34})\omega_{13}\omega_{24}. \end{aligned}$$

3. Co-oriented intersections in $\mathbb{C}[3]$.

In this section, some co-oriented intersections and boundaries are calculated for certain sub-manifolds of the configuration space $\mathbb{C}[3]$. Consider the configuration space

$\mathbb{C}[3]$ to be a subspace of \mathbb{R}^6 via the obvious embedding

$$(z_1, z_2, z_3) \mapsto (x_1, y_1, x_2, y_2, x_3, y_3),$$

with $z_j = x_j + iy_j$. Equip \mathbb{R}^6 with its standard metric.

Take the tangent space at each point of \mathbb{R}^6 to have basis $\{\partial_{x_i}, \partial_{y_i} \mid i = 1, 2, 3\}$, and the cotangent space at each point to have dual basis $\{dx_i, dy_i \mid i = 1, 2, 3\}$.

The subspace $H_{ij} = \{z \in \mathbb{C}[3] \mid \operatorname{Re} z_1 = \operatorname{Re} z_2, \operatorname{Im} z_i < \operatorname{Im} z_j\}$ has tangent space at each point z given by

$$T_z H_{ij} = \left\{ \sum_{r=1}^3 (a_r \partial_{x_r} + b_r \partial_{y_r}) \mid a_i = a_j; a_r, b_r \in \mathbb{R} \right\}$$

(i.e. the i th and j th points move equal amounts in the real direction). With the standard metric on \mathbb{R}^6 , the normal space to H_{ij} can be identified with

$$N_z H_{ij} = \{a(\partial_{x_i} - \partial_{x_j}) \mid a \in \mathbb{R}\},$$

which may be pictured as

$$\begin{array}{c} \leftarrow \bullet_j \\ \bullet_i \rightarrow \end{array}$$

The co-normal space at each point $z \in H_{ij}$ can be identified with

$$N_z^* H_{ij} = \langle dx_i, dx_j \rangle / (dx_i + dx_j = 0).$$

This is one dimensional. To make a positive crossing correspond to a positive transversal intersection of H_{ij} , take the positive co-orientation of H_{ij} to be the class of $dx_i - dx_j$.

Now consider the space $H_{ijl} = \{z \in \mathbb{C}[3] \mid \operatorname{Re} z_1 = \operatorname{Re} z_2 = \operatorname{Re} z_3, \operatorname{Im} z_i < \operatorname{Im} z_j < \operatorname{Im} z_l\}$, pictured as

$$\begin{array}{c} \bullet_l \\ \bullet_j \\ \bullet_i. \end{array}$$

The tangent space at each point $z \in H_{ijl}$ is given by

$$T_z H_{ijl} = \left\{ \sum_{r=1}^3 (a_r \partial_{x_r} + b_r \partial_{y_r}) \mid a_1 = a_2 = a_3 \right\}$$

(so all three points move sideways by the same amount).

The normal space is

$$N_z H_{ijl} = \{a_1 \partial_{x_1} + a_2 \partial_{x_2} + a_3 \partial_{x_3} \mid a_1 + a_2 + a_3 = 0\},$$

so the co-normal space can be taken to be

$$N_z^* H_{ijl} = \langle dx_1, dx_2, dx_3 \rangle / (dx_1 + dx_2 + dx_3 = 0).$$

A co-orientation of H_{ijl} is a class of section of $\Lambda^2 NH_{ijl}$; for the sake of calculation, make the arbitrary choice that the positive co-orientation of H_{ijl} corresponds to the class of $dx_1 \wedge dx_2$.

All the machinery is now in place to examine $(H_{12} \cap H_{23}) \cup (H_{23} \cap H_{13}) \cup (H_{13} \cap H_{12})$. The manifold underlying $H_{12} \cap H_{23}$ is H_{123} — pictorially:

$$\begin{array}{ccc} \bullet_2 & \bullet_3 & \bullet_3 \\ \cap & & = \\ \bullet_1 & \bullet_2 & \bullet_2 \\ & & \bullet_1 \end{array}$$

To calculate the co-orientation on this:

$$\begin{aligned} \text{co}(H_{12} \cap H_{23}) &= \text{co}(H_{12}) \wedge \text{co}(H_{23}) = (dx_1 - dx_2) \wedge (dx_2 - dx_3) \\ &= (dx_1 - dx_2) \wedge (2dx_2 + dx_1) = 3dx_1 \wedge dx_2 \end{aligned}$$

(recalling that $-dx_3 = dx_1 + dx_2$ in $\Lambda^* NH_{ijl}$). This is a *positive* orientation.

For $H_{23} \cap H_{13}$, the underlying submanifold is seen via

$$H_{23} \cap H_{13} = \begin{array}{ccc} \bullet_3 & \bullet_3 & \bullet_3 \\ \cap & & = \\ \bullet_2 & \bullet_1 & \bullet_2 \cup \bullet_1 \\ & & \bullet_1 \quad \bullet_2 \end{array} = H_{123} \cup H_{213}.$$

Calculate the co-orientation:

$$\text{co}(H_{23} \cap H_{13}) = \text{co}(H_{23}) \wedge \text{co}(H_{13}) = (dx_2 - dx_3) \wedge (dx_1 - dx_3) = -3dx_1 \wedge dx_2,$$

i.e. the co-orientation is *negative*.

Similarly,

$$H_{13} \cap H_{12} = \begin{array}{ccc} \bullet_3 & \bullet_2 & \bullet_3 \\ \cap & & = \\ \bullet_1 & \bullet_1 & \bullet_2 \cup \bullet_3 \\ & & \bullet_1 \quad \bullet_1 \end{array} = H_{123} \cup H_{132},$$

and

$$\text{co}(H_{13} \cap H_{12}) = \text{co}(H_{13}) \wedge \text{co}(H_{12}) = (dx_1 - dx_3) \wedge (dx_1 - dx_2) = -3dx_1 \wedge dx_2,$$

so this is also negative.

Putting this all together gives

$$\begin{aligned} (H_{12} \cap H_{23}) \cup (H_{23} \cap H_{13}) \cup (H_{13} \cap H_{12}) &= +H_{123} \cup -(H_{123} \cup H_{213}) \cup -(H_{123} \cup H_{132}) \\ &= -(H_{123} \cup H_{213} \cup H_{132}) = \begin{array}{ccc} \bullet_3 & \bullet_3 & \bullet_2 \\ \cup & \cup & \cup \\ \bullet_2 & \bullet_1 & \bullet_3 \\ \bullet_1 & \bullet_2 & \bullet_1 \end{array} . \end{aligned}$$

This looks plausibly like it might be the boundary of

$$K_{123} = \begin{matrix} \bullet_3 \\ \bullet_1 \end{matrix} \bullet_2.$$

It is necessary to check the co-orientations of

$$\partial K_{123} = \begin{matrix} \bullet_2 & \bullet_3 & \bullet_3 \\ \bullet_3 & \cup & \bullet_2 & \cup & \bullet_1 \\ \bullet_1 & \bullet_1 & \bullet_2 \end{matrix}.$$

Recall the formula $\text{co}(\partial K) = \text{in}(\partial K) \wedge \text{co}(K)$, where $\text{in}(\partial K)$ corresponds to the inward pointing normal of the boundary. K_{123} is ‘half’ of H_{13} (the other half being the part with the second point of the left of the first and third points) and has the same tangent space and co-orientation, so

$$\text{co}(\partial K_{123}) = (dx_2) \wedge (dx_1 - dx_3) = -2dx_1 \wedge dx_2,$$

i.e. the co-orientation is *negative*. Thus it has been shown that

$$(H_{12} \cap H_{23}) \cup (H_{23} \cap H_{13}) \cup (H_{13} \cap H_{12}) = \partial K_{123}$$

as oriented submanifolds, as required.

4. Poincaré duals via C^∞ forms supported on ϵ -neighbourhoods.

The intention of this section is to show how the currents used in Section 2 of Chapter 5 can be represented by C^∞ forms supported within an ϵ -neighbourhood of the corresponding sub-manifolds.

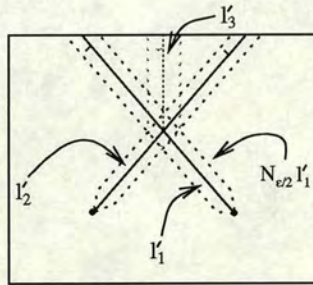


Figure A.1. Lines and ϵ -neighbourhoods in the twice punctured plane.

Consider Figure A.1. Let PP be the twice punctured plane, with the punctures at $(\pm 1, 0)$. Fix $0 < \epsilon \ll 1$. Let $N_{\epsilon/2} l'_1$ be a tubular neighbourhood of l'_1 which is mostly of radius $\epsilon/2$, but which tapers in to the puncture. There is the oriented ‘‘projection to the fibre’’ map $\pi: N_{\epsilon/2} l'_1 \mapsto I$, where I is the interval $(-1, 1)$. Let $\xi \in \Lambda^1(I, \partial I)$ be

a representative of the generator of $H^1(I, \partial I)$, define the closed one-form $\eta_1 \in \Lambda^1(PP)$ to be $\pi^*\xi$ on $N_{\epsilon/2}l'_1$ and zero elsewhere. This represents the Thom class of the normal bundle and hence the Poincaré dual of l'_1 . Similarly define η_2 for l'_2 . Then $\eta_1 \wedge \eta_2$ represents the Poincaré dual of $l'_1 \cap l'_2$ (with the induced orientation), and it is supported within an ϵ -neighbourhood of $l'_1 \cap l'_2$.

Define $\eta_3 \in \Lambda^1(PP)$ by

$$\eta_3(x, y) := - \int_{y'=0}^y \eta_1 \wedge \eta_2(x, y').$$

Then η_3 is supported in an ϵ -neighbourhood of l'_3 . Also $d\eta_3 = -\eta_1 \wedge \eta_2$ and thus $d(\int \eta_1 \eta_2 + \int \eta_3) = -\int \eta_1 \wedge \eta_2 - \int d\eta_3 = 0$. So the iterated integral $\int \eta_1 \eta_2 + \int \eta_3$ is a homotopy invariant. Further, if N is an ϵ -neighbourhood of $l'_1 \cap l'_2$ then $\eta_3|_{(PP \setminus N)}$ represents the Poincaré dual of $l'_3 \setminus N$ in $PP \setminus N$, and it is supported in an ϵ -neighbourhood of $l'_3 \setminus N$, and $d\eta_3$ is Poincaré dual to $\partial l'_3$.

APPENDIX B

Problems and further questions.

*Listen pal, you can't just waltz in here, use my toaster
and spout universal truths without qualification!*

— HAL HARTLEY, *Surviving Desire*.

Here, some possible questions are listed chapter-by-chapter.

1. Abstract Vassiliev theory.

PROBLEM 1.1. Is there any torsion? In other words, is all of the information in the rational invariants?

2. Half integration for knots.

PROBLEM 2.1. Is division by two necessary in the half integration, or can it be done over the integers?

PROBLEM 2.2. Does a similar integration method exist for the even steps? Can it be seen in terms of some symmetry or other preservation of functoriality such as cabling or the Hopf algebra structure?

3. On v_2 and v_3 .

PROBLEM 3.1. Can one find e.g. a canonical basis for the space of type four, even, additive invariants? Can canonical bases be found for higher order invariants? The spaces of weight systems/chord diagrams split as direct sums of eigenspaces of cabling operations; can this be done similarly for the invariants?

PROBLEM 3.2. Are the bounds for v_2 and v_3 of Section 1 tight? Data suggests that the bounds given for the $(2, c)$ -torus knots are best — are they?

PROBLEM 3.3. Does the fish pattern persist in the graphs of knots with higher crossing number?

PROBLEM 3.4. Is there some qualitative distinction between knots with odd and even crossing number which explains the perceived difference in the fish?

PROBLEM 3.5. Is there any relationship with unknotting number, e.g. are there knots of arbitrary unknotting number with $(v_2, v_3) = (0, 0)$?

PROBLEM 3.6. If v is a type n invariant, is $v(T(p, q))$ a polynomial in p and q of degree n in each? Alvarez and Labastida proved this only up to degree six.

PROBLEM 3.7. Can the cubic bounds between v_2 and v_3 for torus knots be extended to positive knots and negative knots? It is easy, for example, to get the linear bound $2v_3 \geq v_2 \geq 1$ for non-trivial positive knots using the Gauß diagram formulæ.

PROBLEM 3.8. For a knot K with $(6|v_3(K)| - |v_2(K)|)^2 \geq 24v_2(K)^3$, let $\rho(K) = 6|v_3(K)|/v_2(K)$ and then define the *pseudo-unknotting number*, $\tilde{u}(K)$, and the *pseudo-uncrossing number*, $\tilde{c}(K)$, by

$$\begin{aligned}\tilde{u}(K) &:= \frac{1}{2} \left(1 + \rho(K) - \sqrt{(1 + \rho(K))^2 - 24v_2(K)} \right); \\ \tilde{c}(K) &:= \rho - \frac{1}{2} \left(\sqrt{(1 + \rho(K))^2 - 24v_2(K)} + \sqrt{(1 - \rho(K))^2 - 24v_2(K)} \right).\end{aligned}$$

For torus knots, the pseudo-unknotting and pseudo-crossing numbers coincide with the usual unknotting and crossing numbers. Do they have any meaning for other knots? Does the necessary bound for K have any topological interpretation?

As an example, consider the Whitehead knots $Wh(i)$, for $i > 0$ these all have unknotting number equal to one; in this case $\tilde{u}(Wh(1)) = 1$, and $\tilde{u}(Wh(i)) \rightarrow 2$ as $i \rightarrow \infty$.

4. Vassiliev invariants for pure braids.

PROBLEM 4.1. Can Theorem 34 (which describes the numbers of invariants) be proved by considering chord diagrams?

PROBLEM 4.2. The symmetric group acts on the space of chord diagrams; can this be seen on the braid level?

6. Obtaining explicit pure braid invariants.

PROBLEM 6.1. Write a computer program to generate the n th stage of the minimal model of P_k , and to produce a corresponding set of generating invariants.

PROBLEM 6.2. Find higher combinatorial formulæ.

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