RADICALS AND RESIDUALS IN WREATH PRODUCTS OF GROUPS

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Abstract

The wreath product $A \text{ wr}^{\Lambda} B$ of a group A with a group B acting as a group of permutations on a set Λ is studied. The $\underline{\underline{X}}$ - radical of $A \text{ wr}^{\Lambda} B$ for general group classes $\underline{\underline{X}}$ closed under certain operations is characterised, and necessary and sufficient conditions are given for $A \text{ wr}^{\Lambda} B$ to be an $\underline{\underline{X}}$ - group for some special classes $\underline{\underline{X}}$.

Subsequently the $\underline{\underline{X}}$ - residual of $A \text{ wr}^{\Lambda} B$ for general classes $\underline{\underline{X}}$, closed under some closure operations, is characterised, and a partial characterisation of the nilpotent residual of $A \text{ wr}^{\Lambda} B$ is obtained.

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Chapter 1 Introduction and preliminary definitions

Section 1.1 Introduction

Let A be a group, and let B be a group acting as a group of permutations on a set Λ. The wreath product $W = A \text{ wr}^{\Lambda} B$ of A and B is the split extension of the direct power A^{Λ} by B, acting as follows:

$$f^b(\lambda) = f(\lambda b^{-1}) \qquad \text{for all } f \in A^{\Lambda},\ b \in B, \text{ and } \lambda \in \Lambda$$

Let $\underline{\underline{X}}$ be a class of groups; then the $\underline{\underline{X}}$ - radical of any group G is defined as

$$\rho(G : \underline{\underline{X}}) = \langle N : N \triangleleft G,\ N \in \underline{\underline{X}} \rangle$$

and the $\underline{\underline{X}}$ - residual of G is defined as

$$\rho_*(G : \underline{\underline{X}}) = \cap\{ N : N \triangleleft G,\ G/N \in \underline{\underline{X}} \}$$

In this thesis we study $\underline{\underline{X}}$ - radicals and $\underline{\underline{X}}$ - residuals of wreath products for various classes $\underline{\underline{X}}$.

We use P. Hall's language of closure operations; see page 4 for a list of special classes and closure operations used in this thesis.

In Chapter 2 we characterise the $\underline{\underline{X}}$ - radical of $A \text{ wr}^{\Lambda} B$, where $\underline{\underline{X}} = \langle Q, S_n, D_p, N_o \rangle \underline{\underline{X}}$. ($\underline{\underline{X}} = D_p \underline{\underline{X}}$ if every direct power of an $\underline{\underline{X}}$ - group is an $\underline{\underline{X}}$ - group.)

If $A \notin \underline{\underline{X}}$, the radical is given in terms of $\rho(A : \underline{\underline{X}})$ and $\rho(B : \underline{\underline{X}})$; if $A \in \underline{\underline{X}}$, the radical depends on when a wreath product is an $\underline{\underline{X}}$ - group. We suggest the general form of a theorem specifying when $A \text{ wr}^{\Lambda} B$ is an $\underline{\underline{X}}$ - group, and extend this to $L\underline{\underline{X}}$.

In Chapter 3 we give conditions for $A \text{ wr}^{\Lambda} B$ to be nilpotent, locally nilpotent, a Baer group, a Gruenberg group, and a ZA-group. We then characterise $\rho(A \text{ wr}^{\Lambda} B : \underline{\underline{X}})$ completely for these classes $\underline{\underline{X}}$.

In Chapter 4 we study the $\underline{\underline{X}}$ - residual of $A \text{ wr}^{\Lambda} B$ for various classes $\underline{\underline{X}}$, and characterise the residual for $\underline{\underline{X}} = \langle S, Q, R_o, W_o \rangle \underline{\underline{X}}$. ($\underline{\underline{X}} = W_o \underline{\underline{X}}$ if and only if whenever A and B are $\underline{\underline{X}}$ - groups, then $A \text{ wr}^{\Lambda} B$ is an $\underline{\underline{X}}$ - group.) We also show how the general case can be reduced to more specific cases.

In Chapter 5 we study the nilpotent residual of $A \text{ wr}^{\Lambda} B$. Let $C_B(\Lambda) = \{ b \in B : \lambda b = \lambda \quad \forall \lambda \in \Lambda \}$. We reduce the general case to two special cases, viz. $B/C_B(\Lambda) \in \underline{\underline{F}}_p$ for some prime p, and $B/C_B(\Lambda) \notin \underline{\underline{F}}_p$ for any prime p. We obtain lower bounds for $\rho_*(A \text{ wr}^{\Lambda} B : \underline{\underline{N}})$, and lower and upper bounds for the residual of the standard wreath product A wr B (i.e. $A \text{ wr}^{\Lambda} B$ with $\Lambda = B$ and B acting in the right regular representation). We characterise $\rho_*(A \text{ wr}^{\Lambda} B : \underline{\underline{N}})$ for the cases $B/C_B(\Lambda) \in \underline{\underline{F}}_p$ and B a perfect group, and characterise $\rho_*(A \text{ wr } B : \underline{\underline{N}})$ when A/A' and $B/\rho_*(B : \underline{\underline{N}})$ are periodic.

Section 1.2 Preliminary definitions and notation

Let A be a group and let Λ be a set. Then the <u>Cartesian power $\text{Cr } A^{\Lambda}$</u> of A is the group of all functions from Λ to A with componentwise multiplication, i.e.

$$fg(\lambda) = f(\lambda)g(\lambda) \qquad \text{for all } f,g \in \text{Cr } A^{\Lambda}, \text{ and all } \lambda \in \Lambda$$

For $f \in \text{Cr } A^{\Lambda}$, define the <u>support $\sigma(f)$</u> of f by

$$\sigma(f) = \{ \lambda \in \Lambda : f(\lambda) \neq 1 \}$$

Then the <u>direct power $\text{Dr } A^{\Lambda}$</u> of A is the subgroup of $\text{Cr } A^{\Lambda}$

consisting of all functions from Λ to A with finite support.

We will usually write A^{Λ} for Dr A^{Λ} where no confusion will arise.

For $\lambda \in \Lambda$, define $A_{\lambda} \subseteq \mathrm{Cr}\ A^{\Lambda}$ by

$$A_{\lambda} = \{ f \in \mathrm{Cr}\ A^{\Lambda} : \sigma(f) \subseteq \{\lambda\} \}$$

Then A_{λ} is isomorphic to A for all $\lambda \in \Lambda$, and Dr A^{Λ} is the direct product of its subgroups A_{λ} for $\lambda \in \Lambda$.

For $f \in \mathrm{Dr}\ A^{\Lambda}$ and $\mu \in \sigma(f)$, define $f_{\mu} \in A_{\mu}$ by $f_{\mu}(\mu) = f(\mu)$.

Then

$$f = \prod_{\mu \in \sigma(f)} f_{\mu}$$

this product being well defined since the A_{μ} commute elementwise.

Let $a \in A$ and $\lambda \in \Lambda$. Then define $a_{\lambda} \in A_{\lambda}$ by $a_{\lambda}(\lambda) = a$.

Let Λ be a set and let B be a group. Call (Λ , B) a <u>pair</u> if B acts as a group of permutations on Λ, i.e. if there exists a map $(\lambda, b) \mapsto \lambda b$ from $\Lambda \times B \to \Lambda$ such that for each $b \in B$, the map $\lambda \mapsto \lambda b$ for $\lambda \in \Lambda$ is a permutation of Λ, and moreover for all $\lambda \in \Lambda$ and $b_1, b_2 \in B$, $\lambda(b_1 b_2) = (\lambda b_1) b_2$. Define $C_B(\Lambda) \trianglelefteq B$ by

$$C_B(\Lambda) = \{ b \in B : \lambda b = \lambda \quad \forall \lambda \in \Lambda \}$$

Then say (Λ , B) is <u>trivial</u> if $B = C_B(\Lambda)$, and <u>faithful</u> if $C_B(\Lambda) = \{1\}$. (Λ , B) is <u>transitive</u> if B acts transitively on Λ.

Let A be a group, and let (Λ , B) be a pair. Then the <u>unrestricted wreath product $A \mathrm{Wr}^{\Lambda} B$</u> is defined as follows. The <u>base group</u> of $A \mathrm{Wr}^{\Lambda} B$ is Cr A^{Λ}. Define B as a group of automorphisms of Cr A^{Λ} by defining f^b for all $f \in \mathrm{Cr}\ A^{\Lambda}$ and all $b \in B$ by

$$f^{b}(\lambda) = f(\lambda b^{-1}) \qquad \forall \lambda \in \Lambda$$

Then $A \mathrm{Wr}^{\Lambda} B$ is the split extension of Cr A^{Λ} by B acting in this fashion. The elements of $A \, Wr^{\Lambda} \, B$ are thus formal products bf, where $b \in B$ and $f \in Cr \, A^{\Lambda}$, with multiplication

$$(bf)(cg) = (bc)(f^{c}g) \qquad \text{for all } b,c \in B \text{ and all } f,g \in Cr \, A^{\Lambda}$$

We identify $b \in B$ with $b1$ and $f \in Cr \, A^{\Lambda}$ with $1f$, making these into actual products; and $f^{b} = b^{-1}fb$ as usual. $Cr \, A^{\Lambda}$ and B are now subgroups of $A \, Wr^{\Lambda} \, B$, with $Cr \, A^{\Lambda} \triangle A \, Wr^{\Lambda} \, B$, $\{A \, Wr^{\Lambda} \, B\}/Cr \, A^{\Lambda} \cong B$, and $Cr \, A^{\Lambda} \cap B = \{1\}$.

The restricted wreath product $A \, wr^{\Lambda} \, B$ is defined as above, with $Dr \, A^{\Lambda}$ replacing $Cr \, A^{\Lambda}$.

The standard unrestricted wreath product $A \, Wr \, B$ of two abstract groups A and B is defined as $A \, Wr^{B} \, B$ where $(B \, , \, B)$ is the right regular representation of B. The standard restricted wreath product $A \, wr \, B$ is $A \, wr^{B} \, B$ with $(B \, , \, B)$ the right regular representation of B.

$\underline{\underline{X}}$ will denote a (group theoretic) class of groups, i.e. a class of groups such that $\{1\} \in \underline{\underline{X}}$ and $H \cong G \in \underline{\underline{X}} \Rightarrow H \in \underline{\underline{X}}$. We will often write "H is an $\underline{\underline{X}}$ - group" for $H \in \underline{\underline{X}}$.

We will use P. Hall's language of closure operations (see e.g. [17] Volume 1 Chapter 1).

Some special operations are given by:

$\underline{\underline{X}} = C\underline{\underline{X}}$	if the Cartesian product of any collection of $\underline{\underline{X}}$ - groups is an $\underline{\underline{X}}$ - group
$\underline{\underline{X}} = C_{p} \, \underline{\underline{X}}$	if any Cartesian power of any $\underline{\underline{X}}$ - group is an $\underline{\underline{X}}$ - group
$\underline{\underline{X}} = D(D_{o}) \, \underline{\underline{X}}$	if the direct product of any collection (any pair) of $\underline{\underline{X}}$ - groups is an $\underline{\underline{X}}$ - group

$\mathfrak{X} = D_p \mathfrak{X}$ if any direct power of any $\mathfrak{X}$ - group is an $\mathfrak{X}$ - group

$\mathfrak{X} = L\mathfrak{X}$ if G is an $\mathfrak{X}$ - group whenever every finite subset of G is contained in some $\mathfrak{X}$ - subgroup of G

$\mathfrak{X} = N(N_o)\mathfrak{X}$ if the product of any collection (any pair) of normal $\mathfrak{X}$ - subgroups is an $\mathfrak{X}$ - group

Note that $G \in N\mathfrak{X} \iff$ G is generated by its subnormal $\mathfrak{X}$ - subgroups (see e.g. [17] Volume 1 Lemma 1.31).

$\mathfrak{X} = \acute{N}\mathfrak{X}$ if every group generated by its ascendant $\mathfrak{X}$ - subgroups is an $\mathfrak{X}$ - group

$\mathfrak{X} = P\mathfrak{X}$ if for any group G whenever $N \triangle G$, $N \in \mathfrak{X}$, and $G/N \in \mathfrak{X}$ then $G \in \mathfrak{X}$

$\mathfrak{X} = \acute{P}\mathfrak{X}$ if every group having an ascending series all of whose factors are $\mathfrak{X}$ - groups is an $\mathfrak{X}$ - group

$\mathfrak{X} = Q\mathfrak{X}$ if every homomorphic image of an $\mathfrak{X}$ - group is an $\mathfrak{X}$ - group

$\mathfrak{X} = R(R_o)\mathfrak{X}$ if given a set (finite set) η of normal subgroups of G with $G/N \in \mathfrak{X}$ for $N \in \eta$, then $G/\cap\eta \in \mathfrak{X}$, for any group G

$\mathfrak{X} = S\mathfrak{X}$ if every subgroup of an $\mathfrak{X}$ - group is an $\mathfrak{X}$ - group

$\mathfrak{X} = S_n\mathfrak{X}$ if every normal subgroup of an $\mathfrak{X}$ - group is an $\mathfrak{X}$ - group

$\mathfrak{X} = W_o\mathfrak{X}$ if whenever A is a group, (Λ, B) is a pair, and A and B are $\mathfrak{X}$ - groups, then $A \text{ wr}^{\Lambda} B$ is an $\mathfrak{X}$ - group

Some special classes are:

$\mathfrak{A}$ abelian groups

$\underline{\underline{F}}_p$	finite p-groups
$\underline{\underline{G}}$	finitely generated groups
$\underline{\underline{N}}$	nilpotent groups
$\overline{\underline{\underline{N}}}_p$	nilpotent p-groups of finite exponent
$\underline{\underline{S}}$	soluble groups
$\overline{\underline{\underline{S}}}_p$	soluble p-groups of finite exponent
$\underline{\underline{T}}$	trivial groups
$\acute{\underline{\underline{Z}}}$	ZA - groups

$\underline{\underline{NA}}$ and $\acute{\underline{\underline{NA}}}$ are the classes of Baer groups and Gruenberg groups respectively.

If $\underline{\underline{X}}$ is a class of groups and G is a group, then the <u>$\underline{\underline{X}}$ - radical $\rho(G : \underline{\underline{X}})$</u> of G is defined as

$$\rho(G : \underline{\underline{X}}) = \langle N : N \triangleleft G, N \in \underline{\underline{X}} \rangle$$

and the <u>$\underline{\underline{X}}$ - residual $\rho_*(G : \underline{\underline{X}})$</u> of G is defined as

$$\rho_*(G : \underline{\underline{X}}) = \cap\{ N : N \triangleleft G, G/N \in \underline{\underline{X}} \}$$

Z^+ will denote the set $\{1,2,3,....\}$

N will denote the set $\{0,1,2,....\}$

P will denote the set of primes in Z^+

We will denote the direct product of groups G_1 and G_2 by $G_1 \times G_2$

Chapter 2 Radicals of wreath products

Section 2.1

We prove

Theorem 2.1.1 Let A be a group, (Λ , B) a pair, and $\underline{\underline{X}}$ a class of groups such that $\underline{\underline{X}} = \langle Q, S_n, D_p, N_o \rangle \underline{\underline{X}}$. Let $W = A \text{ wr}^{\Lambda} B$.
Then

(a) If $A \notin \underline{\underline{X}}$,

$$\rho(W : \underline{\underline{X}}) = \{ \rho(B : \underline{\underline{X}}) \cap C_B(\Lambda) \} \rho(A^{\Lambda} : \underline{\underline{X}})$$
$$= \{ \rho(B : \underline{\underline{X}}) \cap C_B(\Lambda) \} \rho(A : \underline{\underline{X}})^{\Lambda}$$

(b) If $A \in \underline{\underline{X}}$,

$$\rho(W : \underline{\underline{X}}) = \langle B_1 : B_1 \triangle B, A \text{ wr}^{\Lambda} B_1 \in \underline{\underline{X}} \rangle A^{\Lambda}$$

The proof is accomplished with the aid of several lemmas.

Definition 2.1.2 Let A be a group and let Λ be a set. Let $\lambda \in \Lambda$. Define the projection $\omega_{\lambda} : \text{Cr } A^{\Lambda} \to A$ by

$$f\omega_{\lambda} = f(\lambda) \qquad \text{for all } f \in \text{Cr } A^{\Lambda}$$

ω_{λ} is a homomorphism onto A. We will usually write the restriction of ω_{λ} to $\text{Dr } A^{\Lambda}$ as ω_{λ} also.

Lemma 2.1.3 Let A be a group, (Λ , B) be a pair, and let $W = A \text{ wr}^{\Lambda} B$. Suppose N is a normal subgroup of W such that $N \not\leq C_B(\Lambda)A^{\Lambda}$. Then $\exists\ \lambda \in \Lambda$ such that $(N \cap A^{\Lambda})\omega_{\lambda} = A$.

Proof: $N \not\leq C_B(\Lambda)A^{\Lambda} \Rightarrow \exists\ b \in B$ and $g \in A^{\Lambda}$ such that $bg \in N$ and $b \notin C_B(\Lambda)$.

Hence $\exists\ \lambda \in \Lambda$ such that $\lambda b \neq \lambda$. Let $a \in A$. Then since $N \triangleleft W$,

$$(bg)^{a_{\lambda}} \in N \Rightarrow (bg)^{-1}(bg)^{a_{\lambda}} \in N \Rightarrow g^{-1}a_{\lambda}^{-b}ga_{\lambda} \in N \cap A^{\Lambda}$$

Now $(g^{-1}a_\lambda^{-b}ga_\lambda)\omega_\lambda = g(\lambda)^{-1}a_\lambda(\lambda b^{-1})^{-1}g(\lambda)a_\lambda(\lambda) = a_\lambda(\lambda) = a$

and hence $a \in (N \cap A^\Lambda)\omega_\lambda$

Hence $A \leqslant (N \cap A^\Lambda)\omega_\lambda$. By definition, $(N \cap A^\Lambda)\omega_\lambda \leqslant A$; and so $(N \cap A^\Lambda)\omega_\lambda = A$.

<u>Lemma 2.1.4</u> Let A be a group, (Λ , B) be a pair, and let $W = A \text{ wr}^\Lambda B$. Let $\underline{\underline{X}}$ be a class of groups such that $X = \langle Q, S_n \rangle \underline{\underline{X}}$; suppose that $A \notin \underline{\underline{X}}$ and that N is a normal $\underline{\underline{X}}$ - subgroup of W. Then $N \leqslant C_B(\Lambda)A^\Lambda = C_B(\Lambda) \times A^\Lambda$.

<u>Proof:</u> Suppose $N \nleqslant C_B(\Lambda)A^\Lambda$; then by Lemma 2.1.3, $\exists \lambda \in \Lambda$ such that $A = (N \cap A^\Lambda)\omega_\lambda$, and so $A \in \langle Q, S_n \rangle \underline{\underline{X}} = \underline{\underline{X}}$, which is a contradiction. Hence $N \leqslant C_B(\Lambda)A^\Lambda$.

$C_B(\Lambda)A^\Lambda = C_B(\Lambda)\times A^\Lambda$ is immediate, since $C_B(\Lambda) \cap A^\Lambda = \{1\}$, and if $f \in A^\Lambda$, $b \in C_B(\Lambda)$ then

$$f^b(\lambda) = f(\lambda) \quad \forall \lambda \in \Lambda$$

i.e. $$f^b = f .$$

This gives us

<u>Lemma 2.1.5</u> Let $\underline{\underline{X}}$ be a class of groups such that $\underline{\underline{X}} = \langle Q, S_n \rangle \underline{\underline{X}}$, and suppose that A is a group such that $A \notin \underline{\underline{X}}$. Let (Λ , B) be a pair and let $W = A \text{ wr}^\Lambda B$. Let N be a normal $\underline{\underline{X}}$ - subgroup of W. Then

$$N \leqslant \langle B_1 : B_1 \trianglelefteq B,\ B_1 \leqslant C_B(\Lambda),\ B_1 \in \underline{\underline{X}} \rangle \rho(A^\Lambda : \underline{\underline{X}})$$

<u>Proof:</u> By Lemma 2.1.4, $N \leqslant C_B(\Lambda) \times A^\Lambda$; let N_C and N_Λ be the projections of N into $C_B(\Lambda)$ and A^Λ respectively. Then N_C and N_Λ are subgroups of W, and since N is normal in W, $N_\Lambda \trianglelefteq A^\Lambda$ and $N_C \trianglelefteq B$.

$N \in \underline{\underline{X}} \Rightarrow N_\Lambda$ and $N_C \in Q\underline{\underline{X}} = \underline{\underline{X}}$.

Hence $N_\Lambda \leqslant \rho(A^\Lambda : \underline{\underline{X}})$ and $N_C \leqslant \langle B_1 : B_1 \vartriangleleft B,\ B_1 \leqslant C_B(\Lambda),\ B_1 \in \underline{\underline{X}} \rangle$

Hence $N \leqslant N_C N_\Lambda \leqslant \langle B_1 : B_1 \vartriangleleft B,\ B_1 \leqslant C_B(\Lambda),\ B_1 \in \underline{\underline{X}} \rangle \rho(A^\Lambda : \underline{\underline{X}})$

As an immediate corollary we have

Corollary 2.1.6 Let A be a group, (Λ , B) be a pair, and let $W = A \text{ wr}^\Lambda B$. Let $\underline{\underline{X}}$ be a class of groups such that $\underline{\underline{X}} = \langle Q, S_n \rangle \underline{\underline{X}}$, and suppose $A \notin \underline{\underline{X}}$. Then

$$\rho(W : \underline{\underline{X}}) \leqslant \langle B_1 : B_1 \vartriangleleft B,\ B_1 \leqslant C_B(\Lambda),\ B_1 \in \underline{\underline{X}} \rangle \rho(A^\Lambda : \underline{\underline{X}})$$

To prove ~~the~~ a partial converse of Corollary 2.1.6, we will need the following general results about radicals of direct products of groups.

Lemma 2.1.7 Let I be a set and let $\{ G_i : i \in I \}$ be a family of groups. Let $\underline{\underline{X}}$ be a class of groups such that $\underline{\underline{X}} = Q\underline{\underline{X}}$. Then

$$\rho(\operatorname*{Dr}_{i\in I} G_i : \underline{\underline{X}}) \leqslant \operatorname*{Dr}_{i\in I} \rho(G_i : \underline{\underline{X}}) \quad \text{and}$$

$$\rho(\operatorname*{Cr}_{i\in I} G_i : \underline{\underline{X}}) \leqslant \operatorname*{Cr}_{i\in I} \rho(G_i : \underline{\underline{X}})$$

Proof: We prove the result for $\operatorname*{Dr}_{i\in I} G_i$; the proof for $\operatorname*{Cr}_{i\in I} G_i$ is the same.

Let H be a normal $\underline{\underline{X}}$ - subgroup of $\operatorname*{Dr}_{i\in I} G_i$. Then $H\omega_i \vartriangleleft G_i \ \forall\, i \in I$, and $H\omega_i \in Q\underline{\underline{X}} = \underline{\underline{X}} \ \forall\, i \in I$.

Hence $H\omega_i \leqslant \rho(G_i : X) \ \forall\, i \in I$, and so

$$H \leqslant \operatorname*{Dr}_{i\in I} H\omega_i \leqslant \operatorname*{Dr}_{i\in I} \rho(G_i : \underline{\underline{X}})$$

H was any normal $\underline{\underline{X}}$ - subgroup of $\operatorname*{Dr}_{i\in I} G_i$; therefore

$$\rho(\underset{i\in I}{\mathrm{Dr}}\, G_i : \underline{\underline{X}}) \leqslant \underset{i\in I}{\mathrm{Dr}}\, \rho(G_i : \underline{\underline{X}})$$

Lemma 2.1.8 Let $\{ G_i : i \in I \}$ be a family of groups, and let $\underline{\underline{X}}$ be a class of groups. Then

$$\underset{i\in I}{\mathrm{Dr}}\, \rho(G_i : \underline{\underline{X}}) \leqslant \rho(\underset{i\in I}{\mathrm{Dr}}\, G_i : \underline{\underline{X}})$$

Proof: We use internal direct products. Let $i \in I$, and let H be a normal $\underline{\underline{X}}$ - subgroup of G_i. Then since G_i commutes with G_j for all $j \in I\setminus\{i\}$, $H \trianglelefteq \underset{i\in I}{\mathrm{Dr}}\, G_i$; hence $H \leqslant \rho(\underset{i\in I}{\mathrm{Dr}}\, G_i : \underline{\underline{X}})$.

Hence $\rho(G_i : \underline{\underline{X}}) \leqslant \rho(\underset{i\in I}{\mathrm{Dr}}\, G_i : \underline{\underline{X}})$ for all $i \in I$, and so

$$\underset{i\in I}{\mathrm{Dr}}\, \rho(G_i : \underline{\underline{X}}) \leqslant \rho(\underset{i\in I}{\mathrm{Dr}}\, G_i : \underline{\underline{X}})$$

We have as an immediate corollary

Corollary 2.1.9 Let $\{ G_i : i \in I \}$ be a family of groups. Let $\underline{\underline{X}}$ be a class of groups such that $\underline{\underline{X}} = Q\underline{\underline{X}}$. Then

$$\rho(\underset{i\in I}{\mathrm{Dr}}\, G_i : \underline{\underline{X}}) = \underset{i\in I}{\mathrm{Dr}}\, \rho(G_i : \underline{\underline{X}})$$

Lemma 2.1.10 Let A be a group and let Λ be a set. Let $\underline{\underline{X}}$ be a class of groups. Then

$$\rho(A : \underline{\underline{X}})^{\Lambda} = \langle N^{\Lambda} : N \trianglelefteq A,\ N \in \underline{\underline{X}} \rangle$$

Proof: Let N be a normal $\underline{\underline{X}}$ - subgroup of A. Then $N \leqslant \rho(A : \underline{\underline{X}})$ and so $N^{\Lambda} \leqslant \rho(A : \underline{\underline{X}})^{\Lambda}$; hence

$$\langle N^{\Lambda} : N \trianglelefteq A,\ N \in \underline{\underline{X}} \rangle \leqslant \rho(A : \underline{\underline{X}})^{\Lambda}$$

Let $f \in \rho(A : \underline{\underline{X}})^{\Lambda}$ and let $\sigma(f) = \{ \lambda_1, \lambda_2, \ldots \lambda_n \}$. Then $f(\lambda_i) \in \rho(A : \underline{\underline{X}})$ for $1 \leqslant i \leqslant n$; hence for each $i \in \{1,2,\ldots,n\}$ $\exists$ integer k_i , normal $\underline{\underline{X}}$ - subgroups N_{ij} of A for $1 \leqslant j \leqslant k_i$, and elements n_{ij} of N_{ij} for $1 \leqslant j \leqslant k_i$, such that $f(\lambda_i) = n_{i1}n_{i2}\ldots\ldots n_{ik_i}$, For $1 \leq j \leq k_i$, $1 \leq i \leq n$, define $f_{ij} : \Lambda \to N_{ij}$ by

$$f_{ij}(\lambda_i) = n_{ij}$$

$$f_{ij}(\lambda) = 1 \qquad \forall \lambda \neq \lambda_i$$

Then $f_{ij} \in N_{ij}^{\Lambda}$ and

$$f = f_{11}f_{12}\ldots f_{1k_1}f_{21}\ldots f_{2k_2}\ldots\ldots f_{n1}\ldots f_{nk_n}$$

$$\in \langle N^{\Lambda} : N \trianglelefteq A,\ N \in \underline{\underline{X}} \rangle$$

and so $\quad \rho(A : \underline{\underline{X}}) = \langle N^{\Lambda} : N \trianglelefteq A,\ N \in \underline{\underline{X}} \rangle$

<u>Lemma 2.1.11</u> Let A be a group, (Λ , B) be a pair, and let $W = A \operatorname{wr}^{\Lambda} B$. Let $\underline{\underline{X}}$ be a class of groups such that $\underline{\underline{X}} = \langle Q, S_n, D_p \rangle \underline{\underline{X}}$. Then

$$\rho(A^{\Lambda} : \underline{\underline{X}}) = \rho(A : \underline{\underline{X}})^{\Lambda} \leq \rho(W : \underline{\underline{X}})$$

and $\quad \langle B_1 : B_1 \trianglelefteq B,\ B_1 \leq C_B(\Lambda),\ B_1 \in \underline{\underline{X}} \rangle \leq \rho(W : \underline{\underline{X}})$

<u>Proof:</u> By Corollary 2.1.9 and Lemma 2.1.10

$$\rho(A^{\Lambda} : \underline{\underline{X}}) = \rho(A : \underline{\underline{X}})^{\Lambda} = \langle N^{\Lambda} : N \trianglelefteq A,\ N \in \underline{\underline{X}} \rangle$$

$N \trianglelefteq A$ and $N \in \underline{\underline{X}} \Rightarrow N^{\Lambda} \trianglelefteq W$ and $N^{\Lambda} \in D_p \underline{\underline{X}} = \underline{\underline{X}}$

$$\Rightarrow N^{\Lambda} \leq \rho(W : \underline{\underline{X}})$$

Hence $\langle N^{\Lambda} : N \trianglelefteq A,\ N \in \underline{\underline{X}} \rangle \leq \rho(W : \underline{\underline{X}})$ and so

$$\rho(A^{\Lambda} : X) = \rho(A : X)^{\Lambda} \leq \rho(W : X)$$

Let $B_1 \trianglelefteq B$, $B_1 \leq C_B(\Lambda)$. Then $B_1 \trianglelefteq W$; for B_1 is normalised by B, and commutes with A^{Λ} (see proof of Lemma 2.1.4).

Hence $\quad \langle B_1 : B_1 \trianglelefteq B,\ B_1 \leq C_B(\Lambda),\ B_1 \in \underline{\underline{X}} \rangle \leq \rho(W : \underline{\underline{X}})$

Thus we have

<u>Proposition 2.1.12</u> Let A be a group, let (Λ , B) be a pair, and let $W = A \operatorname{wr}^{\Lambda} B$. Let $\underline{\underline{X}}$ be a class of groups such that $\underline{\underline{X}} = \langle Q, S_n, D_p \rangle \underline{\underline{X}}$, and suppose that $A \notin \underline{\underline{X}}$. Then

$$\rho(W : X) = \langle B_1 : B_1 \vartriangleleft B,\ B_1 \leqslant C_B(\Lambda),\ B_1 \in \underline{\underline{X}} \rangle \rho(A^\Lambda : \underline{\underline{X}})$$
$$= \langle B_1 : B_1 \vartriangleleft B,\ B_1 \leqslant C_B(\Lambda),\ B_1 \in \underline{\underline{X}} \rangle \rho(A : \underline{\underline{X}})^\Lambda$$

If further $\underline{\underline{X}} = N_0 \underline{\underline{X}}$, then

$$\rho(W : \underline{\underline{X}}) = \{ \rho(B : \underline{\underline{X}}) \cap C_B(\Lambda) \} \rho(A^\Lambda : \underline{\underline{X}})$$
$$= \{ \rho(B : \underline{\underline{X}}) \cap C_B(\Lambda) \} \rho(A : \underline{\underline{X}})^\Lambda$$

<u>Proof:</u> The first part of the proposition is immediate from Lemma 2.1.11 and Corollary 2.1.6.

We show that if further $\underline{\underline{X}} = N_0 \underline{\underline{X}}$, then

$$\langle B_1 : B_1 \vartriangleleft B,\ B_1 \leqslant C_B(\Lambda),\ B_1 \in \underline{\underline{X}} \rangle = \rho(B : X) \cap C_B(\Lambda)$$

For brevity we will write $\underline{B} = \langle B_1 : B_1 \vartriangleleft B,\ B_1 \leqslant C_B(\Lambda),\ B_1 \in \underline{\underline{X}} \rangle$

The inclusion $\underline{B} \leqslant \rho(B : \underline{\underline{X}}) \cap C_B(\Lambda)$ is clear.

Let $b \in \rho(B : \underline{\underline{X}}) \cap C_B(\Lambda)$. Since $\underline{\underline{X}} = N_0 \underline{\underline{X}}$, $\exists\, N \vartriangleleft B$ such that $N \in \underline{\underline{X}}$ and $b \in N$. Then $b \in N \cap C_B(\Lambda)$, and $N \cap C_B(\Lambda) \in S_n \underline{\underline{X}} = \underline{\underline{X}}$; $N \cap C_B(\Lambda) \vartriangleleft B$ since $N \vartriangleleft B$ and $C_B(\Lambda) \vartriangleleft B$. Hence $b \in \underline{B}$, and hence $\underline{B} = \rho(B : \underline{\underline{X}}) \cap C_B(\Lambda)$.

Thus we have Theorem 2.1.1 part (a).

None of the closure operations Q, S_n, and D_p may be dropped from the hypotheses of Proposition 2.1.12.

<u>D_p - closure:</u> Let $\underline{\underline{X}}$ be the class of groups of order 1 or 2 ; then $\underline{\underline{X}} = \langle Q, S_n \rangle \underline{\underline{X}}$ but $\underline{\underline{X}} \neq D_p \underline{\underline{X}}$. Let A be the cyclic group of order 4, generated by a, say; let B be the cyclic group of order 2, generated by b. Then $A \notin \underline{\underline{X}}$. Let W = A wr B. We will write $f \in A^B$ as $(f(1), f(b))$. Then

$$\rho(A^B : \underline{\underline{X}}) = \{ 1,\ (1,a^2),\ (a^2,1),\ (a^2,a^2) \} \quad \text{and}$$
$$\rho(A : \underline{\underline{X}}) = \{ 1,\ a^2 \}$$

so that $\rho(A^B : \underline{\underline{X}}) = \rho(A : \underline{\underline{X}})^B$; but $\rho(W : \underline{\underline{X}}) = \{ 1,\ (a^2,a^2) \}$

So $\rho(A^B : \underline{\underline{X}}) \not\leq \rho(W : \underline{\underline{X}})$.

<u>S_n - closure:</u> Let D_3 be the dihedral group of order 6, i.e.

$$D_3 = \langle a,b : a^3 = b^2 = 1,\ abab = 1 \rangle$$
$$= \{ 1, a, a^2, b, ab, ba \}$$

Then $\langle a \rangle = C_3$, the cyclic group of order 3, is normal in D_3. Note that $D_3 = \langle b, ab \rangle$, since $a = (ab)b$ and $ba = b(ab)b$.

Let $\underline{\underline{X}} = QD_p(D_3)$. We show that $\underline{\underline{X}} \neq S_n \underline{\underline{X}}$ by showing that $C_3 \notin \underline{\underline{X}}$.

Let Λ be any set and suppose $\exists\ N \triangleleft D_3^{\Lambda}$ such that $D_3^{\Lambda}/N \cong C_3$. Then $f^3 \in N$ for all $f \in D_3^{\Lambda}$; in particular,

$$b_\lambda = (b_\lambda)^3 \in N \quad \text{and} \quad (ab)_\lambda = (ab)_\lambda^3 \in N$$

Thus $\quad D_3^{\Lambda} = \langle b, ab \rangle^{\Lambda} \leq N$, which implies that $C_3 = \{1\}$, a contradiction.

Hence $C_3 \notin \underline{\underline{X}}$, and $\underline{\underline{X}} \neq S_n \underline{\underline{X}}$.

Let $A = C_3 = \langle a \rangle$,say, so that $A \notin \underline{\underline{X}}$, and let B be the cyclic group of order 2, generated by b, say. Let $W = A \text{ wr } B$. Then $N = \langle (a, a^{-1}), b \rangle \cong D_3 \in \underline{\underline{X}}$, and $N \triangleleft W$; but $N \not\leq A^B$.

<u>Q - closure:</u> Let $\underline{\underline{X}} = \underline{\underline{F}}^{-S}$, the class of torsion-free groups. Then $\underline{\underline{X}} = \langle S_n, D_p \rangle \underline{\underline{X}}$, but $\underline{\underline{X}} \neq Q\underline{\underline{X}}$. Let A be the cyclic group of order 2, generated by a , say, so that $A \notin \underline{\underline{X}}$;let B be the infinite cyclic group, generated by b, say. Let $\Lambda = \{\lambda\}$, with $\lambda b = \lambda$. Let $W = A \text{ wr}^{\Lambda} B = A \times B$, since $B = C_B(\Lambda)$. Then $\rho(A : \underline{\underline{X}}) = \{1\}$ and $\langle B_1 : B_1 \triangleleft B,\ B_1 \leq C_B(\Lambda),\ B_1 \in \underline{\underline{X}} \rangle = B$.
Now $\langle ab \rangle \triangleleft W$ since $W \in \underline{\underline{A}}$, and $\langle ab \rangle \in \underline{\underline{F}}^{-S}$; hence $\langle ab \rangle \leq \rho(W : \underline{\underline{X}})$
But

$$\langle ab \rangle \not\leq B = \rho(A : \underline{\underline{X}}) \langle B_1 : B_1 \triangleleft B,\ B_1 \leq C_B(\Lambda),\ B_1 \in \underline{\underline{X}} \rangle$$

Now consider the case $A \in \underline{\underline{X}}$. We have

<u>Proposition 2.1.13</u> Let $\underline{\underline{X}}$ be a class of groups such that $\underline{\underline{X}} = \langle D_p, N_0 \rangle \underline{\underline{X}}$, and let A be an $\underline{\underline{X}}$ - group. Let (Λ, B) be a pair and let $W = A \text{ wr}^{\Lambda} B$. Then

$$\rho(W : \underline{\underline{X}}) = \langle B_1 : B_1 \trianglelefteq B, B_1 A^{\Lambda} \in \underline{\underline{X}} \rangle A^{\Lambda}$$

<u>Proof:</u> $\underline{\underline{X}} = \langle D_p, N_0 \rangle \underline{\underline{X}}$ implies that

$$\rho(W : \underline{\underline{X}}) = \langle N : N \trianglelefteq W, N \in \underline{\underline{X}}, N \geq A^{\Lambda} \rangle \text{ ; for}$$

$A \in \underline{\underline{X}} \Rightarrow A^{\Lambda} \in D_p \underline{\underline{X}} = \underline{\underline{X}}$, and hence if N is a normal $\underline{\underline{X}}$ - subgroup of W, $N.A^{\Lambda} \in N_0 \underline{\underline{X}} = \underline{\underline{X}}$, and $N.A^{\Lambda} \geq A^{\Lambda}$; so $\rho(W : \underline{\underline{X}}) \geq N.A^{\Lambda}$ for all such N, i.e.

$$\rho(W : \underline{\underline{X}}) = \langle N : A^{\Lambda} \leq N \trianglelefteq W, N \in \underline{\underline{X}} \rangle$$

Now $A^{\Lambda} \leq N \trianglelefteq W \iff N = (N \cap B)A^{\Lambda}$, where $N \cap B \trianglelefteq B$,

and so

$$\begin{aligned} \rho(W : \underline{\underline{X}}) &= \langle B_1 A^{\Lambda} : B_1 \trianglelefteq B, B_1 A^{\Lambda} \in \underline{\underline{X}} \rangle \\ &= \langle B_1 : B_1 \trianglelefteq B, B_1 A^{\Lambda} \in \underline{\underline{X}} \rangle A^{\Lambda} \end{aligned}$$

Now if $B_1 \leq B$, (Λ, B_1) is clearly a pair with the same action as (Λ, B) ; so since the action is unchanged,

$$A \text{ wr}^{\Lambda} B_1 = B_1 A^{\Lambda} \leq BA^{\Lambda} = A \text{ wr}^{\Lambda} B$$

Hence we have Theorem 2.1.1 (b) , and thus both parts of Theorem 2.1.1 are proved.

<u>Note:</u> <u>The standard case</u>

Theorem 2.1.1 (b) gives only

$$\begin{aligned} \rho(A \text{ wr } B : \underline{\underline{X}}) &= \langle B_1 : B_1 \trianglelefteq B, B_1 A^{B} \in \underline{\underline{X}} \rangle A^{B} \\ &= \langle B_1 : B_1 \trianglelefteq B, A \text{ wr}^{B} B_1 \quad \underline{\underline{X}} \rangle A^{B} \end{aligned}$$

where $A \text{ wr}^{B} B_1$ is no longer the standard wreath product. However, the groups $A \text{ wr}^{B} B_1$ in this expression are isomorphic to the standard wreath products $A^{T_1} \text{ wr } B_1$, where T_1 is a (left) transversal to B_1 in B. (See proof of Theorem 5.4 of [14].)

Unrestricted wreath products

Lemmas 2.1.3, 2.1.4, 2.1.5, and 2.1.6 also hold for unrestricted wreath products without change of proof, and so we have

Proposition 2.1.14 Let A be a group, let (Λ, B) be a pair, and let $W = A \, Wr^{\Lambda} \, B$. Let $\underline{\underline{X}}$ be a class of groups such that $\underline{\underline{X}} = \langle Q, S_n \rangle \underline{\underline{X}}$ and suppose that $A \notin \underline{\underline{X}}$. Then

$$\rho(W : \underline{\underline{X}}) \leqslant \langle B_1 : B_1 \triangle B, B_1 \leqslant C_B(\Lambda), B_1 \in \underline{\underline{X}} \rangle \rho(Cr \, A^{\Lambda} : \underline{\underline{X}})$$

Corollary 2.1.9 does not hold for Cartesian products; e.g. let $\underline{\underline{X}}$ be the class of abelian p-groups for some prime p. Let G be any abelian p-group with infinite exponent. Then $\rho(G : \underline{\underline{X}}) = G$, but $\rho(Cr \, G^N : \underline{\underline{X}}) < Cr \, G^N$; $Cr \, G^N$ contains elements of infinite order which cannot be contained in the join of any finite number of p-groups.

Clearly, to characterise the radical further, we need conditions for W to be an $\underline{\underline{X}}$ - group. We look at some conditions for general classes $\underline{\underline{X}}$ in Sections 2.2 - 2.4, and in Chapter 3 obtain conditions for some specific group classes.

Section 2.2 P - closed classes

We have immediately

Proposition 2.2.1 Let $\underline{\underline{X}}$ be a class of groups such that $\underline{\underline{X}} = \langle P, S, D_p \rangle \underline{\underline{X}}$. Let A be a group, let (Λ, B) be a pair, and let $W = A \operatorname{wr}^{\Lambda} B$. Then

$$W \in \underline{\underline{X}} \iff A \in \underline{\underline{X}} \text{ and } B \in \underline{\underline{X}}$$

Proof: $W \in \underline{\underline{X}} \Rightarrow A$ and $B \in S\underline{\underline{X}} = \underline{\underline{X}}$

Now suppose $A \in \underline{\underline{X}}$ and $B \in \underline{\underline{X}}$. Then $A^{\Lambda} \in D_p \underline{\underline{X}} = \underline{\underline{X}}$, and so $A^{\Lambda}B \in P\underline{\underline{X}} = \underline{\underline{X}}$. Hence we have the result.

In this case, if $A \in \underline{\underline{X}}$,

$$\langle B_1 : B_1 \triangleleft B,\ A \operatorname{wr}^{\Lambda} B_1 \in \underline{\underline{X}} \rangle = \langle B_1 : B_1 \triangleleft B,\ B_1 \in \underline{\underline{X}} \rangle = \rho(B : \underline{\underline{X}})$$

and so we have

Theorem 2.2.2 Let $\underline{\underline{X}}$ be a class of groups such that $\underline{\underline{X}} = \langle Q, S, D_p, P \rangle \underline{\underline{X}}$. Let A be a group, let (Λ , B) be a pair, and let $W = A \operatorname{wr}^{\Lambda} B$. Then

(a) If $A \notin \underline{\underline{X}}$,

$$\rho(W : \underline{\underline{X}}) = \{ \rho(B : \underline{\underline{X}}) \cap C_B(\Lambda) \} \rho(A : \underline{\underline{X}})^{\Lambda} = \{ \rho(B : \underline{\underline{X}}) \cap C_B(\Lambda) \} \rho(A^{\Lambda} : \underline{\underline{X}})$$

(b) If $A \in \underline{\underline{X}}$,

$$\rho(W : \underline{\underline{X}}) = \rho(B : \underline{\underline{X}}) A^{\Lambda}$$

Examples of such classes are soluble groups and π-groups for any set of primes π. In [15], D.B. Parker proves this result for standard wreath products with $\underline{\underline{X}}$ the class of finite π-groups for any set π of primes.

Section 2.3 Some conditions for W to be an $\underline{\underline{X}}$ - group

Let $\underline{\underline{X}}$ be a class of groups.

Note that if $\underline{\underline{X}} = \langle S_n, D_p, D_o \rangle \underline{\underline{X}}$, then

$$A \text{ wr}^{\Lambda} B \in \underline{\underline{X}} \iff A \in \underline{\underline{X}} \text{ and } B \in \underline{\underline{X}}$$

if A or (Λ, B) is trivial; for then $W = A^{\Lambda} \times B$.

Suppose we have a hypothesis of the following form

Hypothesis (*) Let $\underline{\underline{X}}$ be a class of groups partitioned into non-empty 'subclasses' $\underline{\underline{X}}_i$ for $i \in I$, where if $G \in \underline{\underline{X}}_i$ and $H \cong G$, then $H \in \underline{\underline{X}}_i$. Suppose also that for each $i \in I$, there exists a class of pairs $\mathfrak{X}_i$ such that if A is any group and (Λ, B) is any pair, then

$$A \text{ wr}^{\Lambda} B \in \underline{\underline{X}} \iff \exists\, i \in I \text{ such that } A \in \underline{\underline{X}}_i \text{ and } (\Lambda, B) \in \mathfrak{X}_i$$

If $\underline{\underline{X}} = S_n \underline{\underline{X}}$, then $A \text{ wr}^{\Lambda} B \in \underline{\underline{X}}$ implies that $A \in \underline{\underline{X}}$, and so (*) can always be satisfied by taking each $\underline{\underline{X}}_i$ to comprise all copies of a single group in $\underline{\underline{X}}$, and then choosing $\mathfrak{X}_i$ appropriately. The interest in (*) comes when there are relatively few elements in I . For example, for $\underline{\underline{X}} = \underline{\underline{N}}$ we show in Chapter 3 that (*) holds with the $\underline{\underline{X}}_i$ as $\underline{\underline{T}}$, $\overline{\underline{\underline{N}}}_p \setminus \underline{\underline{T}}$ for $p \in P$, and $\underline{\underline{N}} \setminus \bigcup_{p \in P} \overline{\underline{\underline{N}}}_p$, and suitable $\mathfrak{X}_i$.

Now consider the $\underline{\underline{X}}$ - radical of $A \text{ wr}^{\Lambda} B$.

Let $\underline{\underline{X}}$ be a class of groups such that $\underline{\underline{X}} = \langle Q, S_n, D_p, N_o \rangle \underline{\underline{X}}$. Let $A \in \underline{\underline{X}}$, let (Λ, B) be a pair, and let $W = A \text{ wr}^{\Lambda} B$.

Then we have from Theorem 2.1.1 (b)

$$\rho(W : \underline{\underline{X}}) = \langle B_1 : B_1 \vartriangleleft B,\ A \text{ wr}^{\Lambda} B_1 \in \underline{\underline{X}} \rangle A^{\Lambda}$$

Suppose (*) holds, and define

$$\rho((\Lambda, B) : \mathfrak{X}_i) = \langle M : M \vartriangleleft B,\ (\Lambda, M) \in \mathfrak{X}_i \rangle$$

for $i \in I$.

$A \in \underline{\underline{X}} \Rightarrow \exists\, i \in I$ such that $A \in \underline{\underline{X}}_i$; thus

$\langle B_1 : B_1 \vartriangleleft B,\ A \text{ wr}^{\Lambda} B_1 \in \underline{\underline{X}} \rangle A^{\Lambda}$

$= \langle B_1 : B_1 \vartriangleleft B,\ (\Lambda, B_1) \in \mathfrak{X}_i \rangle A^{\Lambda}$ by (*)

i.e. $\langle B_1 : B_1 \Delta B, A \text{ wr}^{\Lambda} B_1 \in \underline{\underline{X}} \rangle A^{\Lambda} = \rho((\Lambda , B) : \mathcal{X}_i) A^{\Lambda}$

Hence we have

Theorem 2.3.1 Let $\underline{\underline{X}}$ be a $\langle Q, S_n, N_o, D_p \rangle$ - closed class, and suppose (*) holds for $\underline{\underline{X}}$. Let A be a group, let (Λ , B) be a pair, and let $W = A \text{ wr}^{\Lambda} B$.

If $A \notin \underline{\underline{X}}$, then

$$\rho(W : \underline{\underline{X}}) = \{ \rho(B : \underline{\underline{X}}) \cap C_B(\Lambda) \} \rho(A^{\Lambda} : \underline{\underline{X}})$$

On the other hand, if $\exists\, i \in I$ such that $A \in \underline{\underline{X}}_i$, then

$$\rho(W : \underline{\underline{X}}) = \rho((\Lambda , B) : \mathcal{X}_i) A^{\Lambda}$$

Standard case The above result is valid for the standard case, but involves calculation of $\rho((B , B) : \mathcal{X})$. The following may in some cases be more straightforward.

Suppose we have

Hypothesis (*) (standard) Let $\underline{\underline{X}}$ be a class of groups partitioned into non-empty subclasses $\underline{\underline{X}}_{i1}$ for $i \in I$, where if $G \in \underline{\underline{X}}_{i1}$ and $H \cong G$, then $H \in \underline{\underline{X}}_{i1}$. Suppose also that for each $i \in I$, there exists a class of groups $\underline{\underline{X}}_{i2}$ such that if A and B are any groups, then

$A \text{ wr } B \in \underline{\underline{X}} \iff \exists\, i \in I$ such that $A \in \underline{\underline{X}}_{i1}$ and $B \in \underline{\underline{X}}_{i2}$

Note that if (*) holds, then so does (*)(standard), with $\underline{\underline{X}}_{i2} = \{ B : (B , B) \in \mathcal{X}_i \}$ for all $i \in I$.

Let $\underline{\underline{X}}$ be a class of groups such that (*)(standard) holds, and such that $\underline{\underline{X}} = \langle Q, S_n, D_p, N_o \rangle \underline{\underline{X}}$. Suppose further that any non-trivial direct power of an $\underline{\underline{X}}_{i1}$ - group is an $\underline{\underline{X}}_{i1}$ - group, for all $i \in I$. Let B be any group, and suppose $A \in \underline{\underline{X}}$, so that there exists $i \in I$ such that $A \in \underline{\underline{X}}_{i1}$.

Let $B_1 \triangleleft B$ and let T_1 be a left transversal to B_1 in B. Then

$$A^{T_1} \text{ wr } B_1 \in \underline{\underline{X}} \iff B_1 \in \underline{\underline{X}}_{i2}$$

since $A^{T_1} \in D_p \underline{\underline{X}}_{i1} = \underline{\underline{X}}_{i1}$; and hence by the remarks after Proposition 2.1.13,

$$\rho(A \text{ wr } B : \underline{\underline{X}}) = \langle B_1 : B_1 \triangleleft B,\ B_1 \in \underline{\underline{X}}_{i2} \rangle A^B$$
$$= \rho(B : \underline{\underline{X}}_{i2}) A^B$$

Thus we have

Theorem 2.3.2 Let $\underline{\underline{X}}$ be a $\langle Q, S_n, D_p, N_o \rangle$ - closed class of groups such that (*)(standard) holds for $\underline{\underline{X}}$. Suppose further that any non-trivial direct power of an $\underline{\underline{X}}_{i1}$ - group is an $\underline{\underline{X}}_{i1}$ - group, for all $i \in I$. Let A and B be groups. Let W = A wr B .

If $A \notin \underline{\underline{X}}$,

$$\rho(W : \underline{\underline{X}}) = \rho(A : \underline{\underline{X}})^B = \rho(A^B : \underline{\underline{X}})$$

If on the other hand, $\exists\, i \in I$ such that $A \in \underline{\underline{X}}_{i1}$,

$$\rho(W : \underline{\underline{X}}) = \rho(B : \underline{\underline{X}}_{i2}) A^B$$

For example, let $\underline{\underline{X}} = \underline{\underline{N}}$. Then we have

Theorem 2.3.3 [1] Let A and B be non-trivial groups. Then A wr B $\in \underline{\underline{X}} \iff \exists$ prime p such that $A \in \overline{\underline{\underline{N}}}_p$ and $B \in \underline{\underline{F}}_p$

Clearly A wr B $\in \underline{\underline{N}} \iff A \in \underline{\underline{N}}$ and $B \in \underline{\underline{N}}$, if A or B is trivial.

Thus we may take $I = P \cup \{0,1\}$, and

$\underline{\underline{X}}_{01} = \underline{\underline{T}}$, $\underline{\underline{X}}_{11} = \underline{\underline{N}} \setminus \bigcup_{p \in P} \overline{\underline{\underline{N}}}_p$, $\underline{\underline{X}}_{p1} = \overline{\underline{\underline{N}}}_p \setminus \underline{\underline{T}}$ for all $p \in P$

$\underline{\underline{X}}_{02} = \underline{\underline{N}}$, $\underline{\underline{X}}_{12} = \underline{\underline{T}}$, $\underline{\underline{X}}_{p2} = \underline{\underline{F}}_p$ for all $p \in P$

and (*)(standard) holds.

If $A \in \underline{\underline{T}}$, then $\rho(W : \underline{\underline{N}}) = \rho(B : \underline{\underline{N}}) A^B = \rho(B : \underline{\underline{N}})$, and if

$A \in \underline{\underline{N}} \setminus \bigcup_{p \in P} \overline{\underline{\underline{N}}}_p$, $\rho(W : \underline{\underline{N}}) = \rho(B : \underline{\underline{T}})A^B = A^B = \rho(A : \underline{\underline{N}})^B$

Therefore since $\underline{\underline{N}} = \langle Q, S_n, D_p, N_o \rangle \underline{\underline{N}}$, we have

<u>Theorem 2.3.4</u> Let A and B be groups, and let $W = A \text{ wr } B$.

If A is trivial,

$$\rho(W : \underline{\underline{N}}) = \rho(B : \underline{\underline{N}})$$

If $A \notin \bigcup_{p \in P} \overline{\underline{\underline{N}}}_p$,

$$\rho(W : \underline{\underline{N}}) = \rho(A : \underline{\underline{N}})^B = \rho(A^B : \underline{\underline{N}})$$

Finally, if $\exists\, p \in P$ such that $A \in \overline{\underline{\underline{N}}}_p$, then

$$\rho(W : \underline{\underline{N}}) = \rho(B : \underline{\underline{F}}_p)A^B$$

Section 2.4 Extension to $W \in L\underline{\underline{X}}$

Let $\underline{\underline{X}}$ be a class of groups for which we have a set I and classes $\underline{\underline{X}}_i$ and $\mathfrak{X}_i$ such that Hypothesis (*) holds. Given that $\underline{\underline{X}}$ and $L\underline{\underline{X}}$ are closed under certain operations, we obtain necessary and sufficient conditions for W to be a locally - $\underline{\underline{X}}$ - group.

Define (Λ_1 , B_1) to be a subpair of (Λ , B) if $\Lambda_1 \subseteq \Lambda$, $B_1 \leqslant B$, and (Λ_1 , B_1) is a pair with the same action as (Λ , B). Note that if $B_1 \leqslant B$, (Λ , B_1) is always a subpair. If $\mathfrak{X}$ is a class of pairs, then say $\mathfrak{X} = S\mathfrak{X}$ if every subpair of an $\mathfrak{X}$ - pair is an $\mathfrak{X}$ - pair.

Let $\mathfrak{X}$ be a class of pairs. Then say

$(\Lambda , B) \in L\mathfrak{X} \iff$ for all finite subsets Λ_1 of Λ and B_1 of B,

$\exists\, \Lambda_2$ and B_2 such that $\Lambda_1 \subseteq \Lambda_2$, $B_1 \subseteq B_2$ and

(Λ_2 , B_2) is an $\mathfrak{X}$ - subpair of (Λ , B)

Note that if $\underline{\underline{X}} = \{ B : (B , B) \in \mathfrak{X} \}$, and $\mathfrak{X} = S\mathfrak{X}$, then $B \in L\underline{\underline{X}} \iff (B , B) \in L\mathfrak{X}$.

If $\underline{\underline{X}}_i$ is a subclass of groups such that $\underline{\underline{T}} \not\subseteq \underline{\underline{X}}_i$, let $L\underline{\underline{X}}_i$

denote $[\, L(\underline{\underline{X}}_i \cup \underline{\underline{T}})\,] \setminus \underline{\underline{T}}$, i.e. all non-trivial groups A for which any finite subset lies in an $\underline{\underline{X}}_i$ - subgroup, and let $S\underline{\underline{X}}_i$ denote $[\, S(\underline{\underline{X}}_i \cup \underline{\underline{T}})\,] \setminus \underline{\underline{T}}$; $L\underline{\underline{T}} = \underline{\underline{T}} = S\underline{\underline{T}}$ as usual.

Then we have

Theorem 2.4.1 Let $\underline{\underline{X}}$ be a class of groups with Hypothesis (*) holding. Suppose also that $S(\underline{\underline{X}}_i \cup \underline{\underline{T}}) = \underline{\underline{X}}_i \cup \underline{\underline{T}}$ for each $i \in I$. Let A be a group and let (Λ , B) be a pair. Then
$A \text{ wr}^{\Lambda} B \in L\underline{\underline{X}} \iff \exists\, i \in I$ such that $A \in L\underline{\underline{X}}_i$ and $(\Lambda , B) \in L\, \mathfrak{X}_i$

Proof: Note first that $\underline{\underline{X}} = S\underline{\underline{X}}$, since $S(\underline{\underline{X}}_i \cup \underline{\underline{T}}) = \underline{\underline{X}}_i \cup \underline{\underline{T}}$ for all $i \in I$.

We prove first that the conditions are sufficient. Suppose $\exists\, i \in I$ such that $A \in L\underline{\underline{X}}_i$ and $(\Lambda , B) \in L\, \mathfrak{X}_i$.

Let $W = A \text{ wr}^{\Lambda} B$ and let H be any finite subset of W. Define

$$H_\Lambda = \{\, f \in A^\Lambda : \exists\, b \in B \text{ such that } bf \in H \,\}$$
$$H_B = \{\, b \in B : \exists\, f \in A^\Lambda \text{ such that } bf \in H \}$$
$$H_A = \{\, f(\lambda) : f \in H_\Lambda ,\ \lambda \in \Lambda \,\}$$
$$H_\sigma = \cup\{\, \sigma(f) : f \in H_\Lambda \,\}$$

Then since H is finite, so are H_Λ, H_B, H_A, and H_σ. Hence since $A \in L\underline{\underline{X}}_i$, there exists $A_1 \leq A$ such that $H_A \subseteq A_1 \in \underline{\underline{X}}_i$, and since $(\Lambda , B) \in L\, \mathfrak{X}_i$, $\exists\, \Lambda_1 \subseteq \Lambda$ and $B_1 \leq B$ such that $H_\sigma \subseteq \Lambda_1$, $H_B \subseteq B_1$, and (Λ_1 , B_1) is an $\mathfrak{X}_i$- subpair of (Λ , B). Identify A^{Λ_1} with $\{\, f : f \in A^\Lambda \text{ and } \sigma(f) \subseteq \Lambda_1 \,\}$; then we have

$$H \subseteq \langle H_\Lambda, H_B \rangle \leq A_1^{\Lambda_1} B_1 \leq W$$

But $A_1^{\Lambda_1} B_1 = A_1 \text{ wr}^{\Lambda_1} B_1 \in \underline{\underline{X}}$, by (*) ; so H is contained in an $\underline{\underline{X}}$ - subgroup of W. Therefore $W \in L\underline{\underline{X}}$.

Conversely, suppose $W \in L\underline{\underline{X}}$. Then $A \in SL\underline{\underline{X}} \leq LS\underline{\underline{X}} = L\underline{\underline{X}}$. If A_1 and A_2 are any finitely generated subgroups of A then

$\langle A_1 \not\!\wr A_2 \rangle$ is also finitely generated, and so since $\underline{\underline{X}} = S\underline{\underline{X}}$, $\langle A_1 \not\!\wr A_2 \rangle \in \underline{\underline{X}}_i$ for some $i \in I$. If $A = \{1\}$, then $A = \langle A_1 \not\!\wr A_2 \rangle \in \underline{\underline{X}}_i$, and so $A \in L\underline{\underline{X}}_i$. If $A \neq \{1\}$, and $A_1 \neq \{1\} \neq A_2$, then A_1 and A_2 are elements of $S(\underline{\underline{X}}_i \cup \underline{\underline{T}}) \setminus \underline{\underline{T}} \leq \underline{\underline{X}}_i$; i.e. any two non-trivial finitely generated subgroups of A both lie in $\underline{\underline{X}}_i$. Hence $A \in L\underline{\underline{X}}_i$.

Now let A_1 and B_1 be finitely generated subgroups of A and B respectively, and let Λ_1 be any finite subset of Λ. Then $A_1^{\Lambda_1 B_1} B_1$ is a finitely generated subgroup of W, and so, since $W \in L\underline{\underline{X}}$, $A_1 \text{ wr}^{\Lambda_1 B_1} B_1 = A_1^{\Lambda_1 B_1} B_1 \in \underline{\underline{X}}$. Hence since $A_1 \in \underline{\underline{X}}_i$, by (*) we have that $(\Lambda_1 B_1, B_1) \in \mathfrak{X}_i$

So $(\Lambda_1, B_1) \subseteq (\Lambda_1 B_1, B_1) \in \mathfrak{X}_i$ and $(\Lambda, B) \in L\mathfrak{X}_i$

Hence result.

Therefore we have, by Theorem 2.4.1 and Theorem 2.3.1,

Theorem 2.4.2 Let $\underline{\underline{X}}$ be a class of groups such that (*) holds, with $S(\underline{\underline{X}}_i \cup \underline{\underline{T}}) = \underline{\underline{X}}_i \cup \underline{\underline{T}}$ for all $i \in I$. Suppose further that $L\underline{\underline{X}} = \langle Q, S_n, D_p, N_o \rangle L\underline{\underline{X}}$. Let A be a group, let (Λ, B) be a pair, and let $W = A \text{ wr}^{\Lambda} B$.

If $A \notin L\underline{\underline{X}}$,

$$\rho(W : L\underline{\underline{X}}) = \{ \rho(B : L\underline{\underline{X}}) \cap C_B(\Lambda) \} \rho(A^{\Lambda} : L\underline{\underline{X}})$$

On the other hand, if $\exists\, i \in I$ such that $A \in L\underline{\underline{X}}_i$, then

$$\rho(W : LX) = \rho((\Lambda, B) : L\mathfrak{X}_i) A^{\Lambda}$$

For the standard case we have

Theorem 2.4.3 Let $\underline{\underline{X}}$ be a class of groups with (*)(standard) holding, and $S(\underline{\underline{X}}_{i1} \cup \underline{\underline{T}}) = \underline{\underline{X}}_{i1} \cup \underline{\underline{T}}$ for each $i \in I$. Let

A and B be groups. Then

$A \text{ wr } B \in L\underline{\underline{X}} \iff \exists\, i \in I$ such that $A \in L\underline{\underline{X}}_{i1}$ and $B \in L\underline{\underline{X}}_{i2}$

<u>Proof:</u> The proof is very similar to that of Theorem 2.4.1 ; we give an outline.

To prove that the conditions are sufficient, suppose there exists $i \in I$ such that $A \in L\underline{\underline{X}}_{i1}$ and $B \in L\underline{\underline{X}}_{i2}$ and let $H \subseteq A \text{ wr } B$ be a finite subset. Define

$$H_{A^B} = \{ f \in A^B \; ; \; \exists\, b \in B \text{ such that } bf \in H \}$$

Let H_B and H_A be as before. Let

$$H_\sigma = \cup\{ \sigma(f) \; : \; f \in H_{A^B} \} \cup H_B$$

Then $H \subseteq \langle H_A \rangle^{\langle H_\sigma \rangle} \langle H_\sigma \rangle = \langle H_A \rangle \text{ wr } \langle H_\sigma \rangle \in \underline{\underline{X}}$, by (*)(standard) and so $W \in L\underline{\underline{X}}$.

Now suppose $W \in L\underline{\underline{X}}$; then $\exists\, i \in I$ such that $A \in L\underline{\underline{X}}_{i1}$ as before. Let $A_1 \subseteq A$ and $B_1 \subseteq B$ be finite subsets; then $\langle A_1 \rangle^{\langle B_1 \rangle} \langle B_1 \rangle \in \underline{\underline{G}}$ and so $\langle A_1 \rangle^{\langle B_1 \rangle} \langle B_1 \rangle \in \underline{\underline{X}}$; hence $\langle A_1 \rangle \text{ wr } \langle B_1 \rangle \in \underline{\underline{X}}$, and so $B_1 \in \underline{\underline{X}}_{i2}$. Hence $B \in L\underline{\underline{X}}_{i2}$, and we have the result.

<u>Note:</u> This result is an immediate corollary to Theorem 2.4.1, if $\mathfrak{X}_i = s\mathfrak{X}_i$ & if $\underline{\underline{X}}_{i2} = \{ B : (B , B) \in \mathfrak{X}_i \}$; but it also holds if we have only (*)(standard) and not (*).

<u>Theorem 2.4.4</u> Let $\underline{\underline{X}}$ be a class of groups for which (*)(standard) holds, and suppose that $S(\underline{\underline{X}}_{i1} \cup \underline{\underline{T}}) = \underline{\underline{X}}_{i1} \cup \underline{\underline{T}} \;\; \forall\, i \in I$. Suppose further that $L\underline{\underline{X}} = \langle Q, S_n, D_p, N_o \rangle L\underline{\underline{X}}$. Let A and B be groups, and let $W = A \text{ wr } B$.

If $A \notin L\underline{\underline{X}}$,

$$\rho(W : L\underline{\underline{X}}) = \rho(A^B : L\underline{\underline{X}}) = \rho(A : L\underline{\underline{X}})^B$$

If there exists $i \in I$ such that $A \in L\underline{\underline{X}}_{i1}$,

$$\rho(W : L\underline{\underline{X}}) = \rho(B : L\underline{\underline{X}}_{i2})A^B$$

For example, let $\underline{\underline{X}} = \underline{\underline{N}}$. Then from Theorem 2.3.3, we have

Theorem 2.4.5 Let $I = \{0,1\} \cup P$, and let

$\underline{\underline{X}}_{01} = \underline{\underline{T}}$, $\underline{\underline{X}}_{11} = \underline{\underline{N}} \setminus \bigcup_{p \in P} \underline{\underline{\overline{N}}}_p$, $\underline{\underline{X}}_{p1} = \underline{\underline{\overline{N}}}_p \setminus \underline{\underline{T}}$ $\forall \; p \in P$ and

$\underline{\underline{X}}_{02} = \underline{\underline{N}}$, $\underline{\underline{X}}_{12} = \underline{\underline{T}}$, $\underline{\underline{X}}_{p2} = \underline{\underline{F}}_p$ for all $p \in P$

Let A and B be groups. Then

$$A \text{ wr } B \in L\underline{\underline{N}} \iff \exists \; i \in I \text{ such that } A \in L\underline{\underline{X}}_{i1} \text{ and } B \in L\underline{\underline{X}}_{i2}$$

and since $L\underline{\underline{N}} = \langle Q, S_n, N_o, D_p \rangle L\underline{\underline{N}}$ we have

Theorem 2.4.6 Let A and B be groups and let $W = A \text{ wr } B$.

If $A = \{1\}$,

$$\rho(W : L\underline{\underline{N}}) = \rho(B : L\underline{\underline{N}})$$

If $\exists \; p \in P$ such that $A \in L\underline{\underline{\overline{N}}}_p \setminus \underline{\underline{T}}$,

$$\rho(W : L\underline{\underline{N}}) = \rho(B : L\underline{\underline{F}}_p)A^B$$

Otherwise,

$$\rho(W : L\underline{\underline{N}}) = \rho(A^B : L\underline{\underline{N}}) = \rho(A : L\underline{\underline{N}})^B$$

Chapter 3 Necessary and sufficient conditions for W to be an $\underline{\underline{X}}$ - group for some special classes $\underline{\underline{X}}$

In this chapter we give conditions for $A \text{ wr}^{\Lambda} B$ to be a Baer group (Theorem 3.2.1), a nilpotent group (Theorem 3.3.1), a locally nilpotent group (Theorem 3.4.1), a Gruenberg group (Theorem 3.5.1), and a ZA - group (Theorem 3.6.1). We then use these results to obtain $\rho(W : \underline{\underline{X}})$ for these classes $\underline{\underline{X}}$.

Section 3.1 Preliminary results

Let (Λ , B) be a pair and let $\Lambda' \subseteq \Lambda$. Then define

$$C_B(\Lambda') = \{ b \in B : \lambda b = \lambda \quad \forall \quad \lambda \in \Lambda' \}$$

<u>Lemma 3.1.1</u> Let (Λ , B) be a pair. Let $\Lambda' \subseteq \Lambda$. Then $C_B(\Lambda')$ is a subgroup of B ; and if $\Lambda'B \subseteq \Lambda'$, then $C_B(\Lambda') \triangle B$.

<u>Proof:</u> The proof consists of straightforward checking.

<u>Lemma 3.1.2</u> Let (Λ , B) be a pair, and let $\Lambda' \subseteq \Lambda$, $\Lambda'B \subseteq \Lambda'$. Then

(a) $(\Lambda' , \{B/C_B(\Lambda')\})$ is a faithful pair, with action

$$\lambda b C_B(\Lambda') = \lambda b \quad \forall \quad \lambda \in \Lambda'$$

(b) Let the orbits of B on Λ be $\{ \Lambda_i : i \in I \}$. Then the orbits of $\{\bar{B}/C_B(\Lambda')\}$ on Λ' are $\{ \Lambda_j : j \in J\}$, where

$$j \in J \iff \Lambda_j \subseteq \Lambda'$$

<u>Proof:</u> (a) The action is well defined on Λ', for if $b, \beta \in B$, then

$$\begin{aligned} & bC_B(\Lambda') = \beta C_B(\Lambda') \\ \Rightarrow\quad & b\beta^{-1} \in C_B(\Lambda') \\ \Rightarrow\quad & \lambda b\beta^{-1} = \lambda \quad \forall \quad \lambda \in \Lambda' \\ \Rightarrow\quad & \lambda b = \lambda\beta \quad \forall \quad \lambda \in \Lambda' \\ \Rightarrow\quad & \lambda b C_B(\Lambda') = \lambda\beta C_B(\Lambda') \quad \forall \quad \lambda \in \Lambda' \end{aligned}$$

and $\lambda \in \Lambda' \Rightarrow \lambda b C_B(\Lambda') = \lambda b \in \Lambda' \quad \forall\ b \in B$.

$bC_B(\Lambda')$ is a permutation since b is.

(b) $\phi \neq J$ since $\phi \neq \Lambda'$, and $\Lambda' B \subseteq \Lambda'$ means that Λ' must contain at least one orbit Λ_j. Let Λ_j be an orbit of B on Λ such that $\Lambda_j \subseteq \Lambda'$. Then if $\lambda, \lambda' \in \Lambda_j$, there exists $b \in B$ such that $\lambda b = \lambda'$, i.e. $\lambda b C_B(\Lambda') = \lambda'$; so $\Lambda_j \subseteq \Lambda_j'$ for some orbit Λ_j' of $B/C_B(\Lambda')$ on Λ'.

Now let $\lambda' \in \Lambda_j'$; then if $\lambda \in \Lambda_j \subseteq \Lambda_j'$, there exists $\beta \in B$ such that

$$\lambda = \lambda' \beta C_B(\Lambda') = \lambda' \beta$$

$$\Rightarrow \quad \lambda' \in \Lambda_j . \qquad\qquad \text{Hence } \Lambda_j' \subseteq \Lambda_j .$$

Hence $\Lambda_j' = \Lambda_j$, and so the orbits of $B/C_B(\Lambda')$ on Λ' are precisely $\{ \Lambda_j : j \in J \}$.

Thus we have in fact that $B/C_B(\Lambda')$ acts precisely as B on Λ' but is faithful.

Lemma 3.1.3 Let (Λ, B) be a pair, and let Λ' be a subset of Λ such that $\Lambda' B \subseteq \Lambda'$. Let the orbits of $B/C_B(\Lambda')$ on Λ' be $\{ \Lambda_j : j \in J \}$. Then for all $j \in J$

$$B/C_B(\Lambda_j) \cong \{B/C_B(\Lambda')\}/\{C_B(\Lambda_j)/C_B(\Lambda')\} = \{B/C_B(\Lambda')\}/C_{\{B/C_B(\Lambda')\}}(\Lambda_j)$$

Proof: $\Lambda' B \subseteq \Lambda' \Rightarrow C_B(\Lambda') \triangleleft B$ and hence $C_B(\Lambda') \triangleleft C_B(\Lambda_j)$, and so the first part of the lemma is immediate from the 3rd Isomorphism Theorem.

$C_B(\Lambda_j)/C_B(\Lambda') = C_{\{B/C_B(\Lambda')\}}(\Lambda_j)$ is routine checking.

Lemma 3.1.4 Let A be a group, let (Λ , B) be a pair, and let $W = A \text{ wr}^{\Lambda} B$. Then

$$\{A \text{ wr}^{\Lambda} B\}/C_B(\Lambda) \cong A \text{ wr}^{\Lambda} \{B/C_B(\Lambda)\}$$

Proof: $C_B(\Lambda) \Delta W$ since $C_B(\Lambda)$ is normal in B, and is normalised by A^{Λ} (see proof of Lemma 2.1.4).

Define $\theta : W \to A \text{ wr}^{\Lambda} \{B/C_B(\Lambda)\}$ by $bf\theta = bC_B(\Lambda)f$ for $b \in B$ and $f \in A^{\Lambda}$

θ is clearly well defined.

θ is a homomorphism :

Let $b,c \in B$ and $f,g \in A^{\Lambda}$. Then since $\lambda c = \lambda cC_B(\Lambda)\ \forall\ \lambda \in \Lambda$,

$$\begin{aligned}(bfcg)\theta &= bcC_B(\Lambda)f^c g \\ &= bcC_B(\Lambda)f^{cC_B(\Lambda)}g \\ &= bC_B(\Lambda)f\ cC_B(\Lambda)g \\ &= (bf)\theta\ (cg)\theta \quad \text{as required.}\end{aligned}$$

θ is clearly an epimorphism, and

$bf\theta = 1 \iff bC_B(\Lambda) = 1$ and $f = 1 \iff bf \in C_B(\Lambda)$, so $\text{Ker } \theta = C_B(\Lambda)$

Hence

$$\{A \text{ wr}^{\Lambda} B\}/C_B(\Lambda) \cong A \text{ wr}^{\Lambda} \{B/C_B(\Lambda)\}$$

Proposition 3.1.5 Let A be a group, let (Λ , B) be a trivial pair, and let $W = A \text{ wr}^{\Lambda} B$. Let $\underline{\underline{X}}$ be a class of groups such that $\underline{\underline{X}} = \langle D_o, D_p, S_n \rangle \underline{\underline{X}}$. Then

$$W \in \underline{\underline{X}} \iff A \in \underline{\underline{X}} \text{ and } B \in \underline{\underline{X}}$$

Proof: $C_B(\Lambda) = B \Rightarrow W = A^{\Lambda} \times B$, from which the result is immediate.

Notation: Let $\underline{\underline{X}}$ be a class of groups and let p be a prime. Then $\underline{\underline{X}}_p$ will denote the class of $\underline{\underline{X}}$ - p - groups.

Section 3.2 $W \in \underline{\underline{NA}}$

Recall that for primes p, $\bar{\underline{\underline{S}}}_p$ is the class of soluble p-groups of finite exponent.

Define classes $\mathcal{B}_p$ of pairs for primes p by

$(\Lambda , B) \in \mathcal{B}_p \iff B \in \underline{\underline{NA}}$ and $B/C_B(\Lambda)$ is a p-group.

We prove

Theorem 3.2.1 Let A be a group, let (Λ , B) be a pair, and let $W = A \text{ wr}^{\Lambda} B$. Let $I = \{0,1\} \cup P$, and partition $\underline{\underline{NA}}$ into subclasses

$\underline{\underline{X}}_0 = \underline{\underline{T}}$, $\underline{\underline{X}}_p = \bar{\underline{\underline{S}}}_p \setminus \underline{\underline{T}}$ for all $p \in P$, $\underline{\underline{X}}_1 = \underline{\underline{NA}} \setminus \bigcup_{p \in P} \bar{\underline{\underline{S}}}_p$

Define classes of pairs

$\mathcal{X}_0 = \{ (\Lambda , B) : B \in \underline{\underline{NA}} \}$ $\qquad \mathcal{X}_p = \mathcal{B}_p$ for all $p \in P$

$\mathcal{X}_1 = \{ (\Lambda , B) : B \in \underline{\underline{NA}}$ and $C_B(\Lambda) = B \}$

Then

$$W \in \underline{\underline{NA}} \iff \exists \; i \in I \text{ such that } \underline{\underline{A}} \in \underline{\underline{X}}_i \text{ and } (\Lambda , B) \in \mathcal{X}_i$$

This result is valid for standard wreath products; however if we put $X_{i2} = \{ B : (B , B) \in \mathcal{X}_i \}$ for all $i \in I$, we have

Theorem 3.2.2 Let A and B be groups and let $W = A \text{ wr } B$. Let $I = \{0,1\} \cup P$, and let

$\underline{\underline{X}}_{01} = \underline{\underline{T}}$, $\underline{\underline{X}}_{11} = \underline{\underline{NA}} \setminus \bigcup_{p \in P} \bar{\underline{\underline{S}}}_p$, $\underline{\underline{X}}_{p1} = \bar{\underline{\underline{S}}}_p \setminus \underline{\underline{T}}$ for all $p \in P$

$\underline{\underline{X}}_{02} = \underline{\underline{NA}}$, $\underline{\underline{X}}_{12} = \underline{\underline{T}}$, $\underline{\underline{X}}_{p2} = (\underline{\underline{NA}})_p$ for all $p \in P$

Then

$$A \text{ wr } B \in \underline{\underline{NA}} \iff \exists \; i \in I \text{ such that } A \in \underline{\underline{X}}_{i1} \text{ and } B \in \underline{\underline{X}}_{i2}$$

To prove Theorem 3.2.1, we require the following theorem, and two lemmas of P. Hall.

Notation: For any group G, subgroups H, K,

$$[H,K] = \langle [h,k] : h \in H, k \in K \rangle \quad ;$$

commutators are left normed, i.e. $[h,k_1,k_2] = [[h,k_1],k_2]$ etc.
$G' = G^{(1)} = [G,G]$ and $G^{(n)} = [G^{(n-1)},G^{(n-1)}]$ for all $n \in Z^+$.
We will write H sn G to mean that H is a subnormal subgroup of G.

Theorem 3.2.3 [[17] Volume 2 Theorem 7.17]
Let G be a non-trivial soluble p-group of finite exponent and let ℓ be the minimum length of a normal series of G with elementary abelian factors. Then $[G, \underbrace{g, \ldots, g}_{r}] = \{1\}$ for all $g \in G$ where $r = 1 + p + p^2 + \cdots + p^{\ell-1}$. In particular, G is a Baer group and an r - Engel group.

Lemma 3.2.4 [[7] Lemma 4] Let A be a group and let (Λ, B) be a transitive pair. Let $W = A \text{ wr}^{\Lambda} B$. Suppose $N \trianglelefteq W$ and $B \cap N \not\leq C_B(\Lambda)$. Then $(A')^{\Lambda} \leq N$.

The lemma is stated and proved for the faithful case in [7]; we will require this more general form. The proof is essentially the same as in [7].

Corollary 3.2.5 Let A be a group and let (Λ, B) be a transitive pair. Let $W = A \text{ wr}^{\Lambda} B$. Suppose N is subnormal in W and $N \cap B \not\leq C_B(\Lambda)$. Then there exists $n \in Z^+$ and $\lambda \in \Lambda$ such that $A_{\lambda}^{(n)} \leq N$.

Proof: Since $N \cap B \not\leq C_B(\Lambda)$, $\exists\ \lambda \in \Lambda$ and $b \in N \cap B$ such that $\lambda b \neq \lambda$. We prove by induction on k that if $H \geq N$ is any subnormal subgroup of W of defect k , then $A_\lambda^{(k)} \leqslant H$.

Suppose $k = 1$, and let $H \geqslant N$ be any subgroup of defect 1 . Then $H \triangleleft W$; and $H \cap B \geqslant N \cap B$ implies that $H \cap B \not\leqslant C_B(\Lambda)$. Hence by Lemma 3.2.4 , $(A')^\Lambda \leqslant H$, and so $A'_\lambda \leqslant H$ certainly.

Suppose the result holds for some $k \geqslant 1$ and suppose $H \geqslant N$ is subnormal of defect $k + 1$. Then $\exists\, H_1 \leqslant W$ such that $H \triangleleft H_1$ and H_1 is subnormal of defect K. $N \leqslant H \leqslant H_1$, and so by the induction hypothesis, $A_\lambda^{(k)} \leqslant H_1$.

Let $a, \alpha \in A^{(k)}$. Then

$$[a_\lambda, \alpha_\lambda] = [a_\lambda a_\lambda^{-b}, \alpha_\lambda] \quad \text{since } \lambda b \neq \lambda$$

$$= [a_\lambda^{-1}, b, \alpha_\lambda] \in H \quad \text{since } b \in N \leqslant H \text{ and } H \triangleleft H_1$$

So $A_\lambda^{(k+1)} \leqslant H$; hence result by induction.

In particular, $A_\lambda^{(n)} \leqslant N$ where n is the defect of N .

<u>Lemma 3.2.6</u> [[7] Lemma 7] Let A be a group, let (Λ , B) be a non-trivial faithful pair, and let $W = A \operatorname{wr}^\Lambda B$. Let T be a subnormal subgroup of W such that $\{1\} \neq T \leqslant B$. Then A is of exponent p^r for some prime p and $r \geqslant 0$; and if $A \neq \{1\}$, then T is of exponent p^s for some $s > 0$.

This lemma is stated in [7] for (Λ , B) transitive, but is valid with the same proof for (Λ , B) intransitive.

<u>Proof of Theorem 3.2.1</u>

Note firstly that if $A = \{1\}$,

$$A \operatorname{wr}^\Lambda B \in \underline{\underline{NA}} \iff B \in \underline{\underline{NA}}$$

Thus we obtain classes $\underline{\underline{X}}_0 = \underline{\underline{T}}$ and $\mathcal{X}_0 = \{ (\Lambda , B) : B \in \underline{\underline{NA}} \}$.

Now suppose that neither A nor (Λ , B) is trivial. We show that

$$A \text{ wr}^{\Lambda} B \in N\underline{\underline{A}} \iff \exists \; p \in P \text{ such that } A \in \overline{\underline{\underline{S}}}_p \text{ and } (\Lambda , B) \in \mathcal{B}_p$$

Suppose firstly that $\exists \; p \in P$ such that $A \in \overline{\underline{\underline{S}}}_p$ and $(\Lambda , B) \in \mathcal{B}_p$. Then $A^{\Lambda} \in \overline{\underline{\underline{S}}}_p$ and so by Theorem 3.2.3, A^{Λ} is a Baer group. Hence for all $f \in A^{\Lambda}$, $\langle f \rangle$ sn $A^{\Lambda} \triangleleft W$, and so $\langle f \rangle$ sn W. Thus we require to show that for all $b \in B$, $\langle b \rangle$ sn W, for then W will be generated by subnormal Baer subgroups, and will therefore be a Baer group.

Let $b \in B$. Then $\langle b \rangle C_B(\Lambda)/C_B(\Lambda)$ is a cyclic p-group and hence

$$A^{\Lambda}\{\langle b \rangle C_B(\Lambda)/C_B(\Lambda)\} \in \overline{\underline{\underline{S}}}_p \leq N\underline{\underline{A}} \text{ by Theorem 3.2.3}$$

Hence $\langle b \rangle C_B(\Lambda)$ sn $A^{\Lambda}\langle b \rangle C_B(\Lambda)$.

Now $B \in N\underline{\underline{A}} \Rightarrow \langle b \rangle C_B(\Lambda) \in N\underline{\underline{A}}$ and $\langle b \rangle$ sn $\langle b \rangle C_B(\Lambda)$ sn B ; so $\langle b \rangle$ sn $\langle b \rangle C_B(\Lambda)$ sn $A^{\Lambda}\langle b \rangle C_B(\Lambda)$ sn $A^{\Lambda}B = W$. Therefore W is a Baer group, and these conditions are sufficient.

Conversely, let $W \in N\underline{\underline{A}}$. We show firstly that A is soluble. Since $B > C_B(\Lambda)$, there exists an orbit Θ such that $B > C_B(\Theta)$. Let $\underline{B} = B/C_B(\Theta)$, and let $\underline{W} = A \text{ wr}^{\Theta} \underline{B}$. Then $\underline{W} \in QS(W)$, and so $\underline{W} \in N\underline{\underline{A}}$. Let $1 \neq c \in \underline{B}$. Then $\langle c \rangle$ is subnormal in $\underline{W}$, of defect n, say; and so by Corollary 3.2.5, there exists $\theta \in \Theta$ such that $A_{\theta}^{(n)} \leq \langle c \rangle$. But $\langle c \rangle \cap A^{\Theta} = \{1\}$; so $A_{\theta}^{(n)} = \{1\}$, and so $A^{(n)} = \{1\}$, i.e. A is soluble.

Now consider $W_f = A \text{ wr}^{\Lambda} \{B/C_B(\Lambda)\}$. $W_f \cong W/C_B(\Lambda)$ and so $W_f \in Q(N\underline{\underline{A}}) = N\underline{\underline{A}}$. $C_B(\Lambda) < B$ and so there exists $b \in B \setminus C_B(\Lambda)$ such that $\langle b \rangle C_B(\Lambda)$ is subnormal in W_f. Then by Lemma 3.2.6 there exists a prime p such that A is a p-group of finite exponent and $\langle b \rangle C_B(\Lambda)$ has p-power order.

Hence $A \in \overline{\underline{\underline{S}}}_p$ and $B/C_B(\Lambda)$ is a p-group. B is clearly a Baer group since $N\underline{\underline{A}} = S(N\underline{\underline{A}})$. Hence $(\Lambda , B) \in \mathcal{B}_p$, and the conditions are necessary.

So now we have classes $\underline{\underline{X}}_p = \overline{\underline{\underline{S}}}_p \setminus \underline{\underline{T}}$, and $\mathfrak{X}_p = \mathcal{B}_p$ for all $p \in P$.

Suppose finally that (Λ , B) is trivial. Then $A \text{ wr}^{\Lambda} B \in N\underline{\underline{A}} \iff A \in N\underline{\underline{A}}$ and $B \in N\underline{\underline{A}}$, by Proposition 3.1.5; so we obtain classes $X_1 = N\underline{\underline{A}} \setminus \bigcup_{p \in P} \overline{\underline{\underline{S}}}_p$ and $\mathfrak{X}_1 = \{ (\Lambda , B) : B \in NA \text{ and } B = C_B(\Lambda) \}$.

Hence we have Theorem 3.2.1.

Section 3.3 $W \in \underline{\underline{N}}$

Define a class of pairs $\mathcal{N}$ by

$$(\Lambda , B) \in \mathcal{N} \iff B \in \underline{\underline{N}}$$

Let p be a prime and define classes of pairs $\mathcal{K}_p$, $\overline{\mathcal{K}}_p$, and $\overline{\mathcal{L}}_p$ as follows:

$(\Lambda , B) \in \mathcal{K}_p \iff B/C_B(\Theta)$ is a finite p-group for all orbits Θ

$(\Lambda , B) \in \overline{\mathcal{K}}_p \iff (\Lambda , B) \in \mathcal{K}_p$ and $\exists\, n \in N$ such that $|B/C_B(\Theta)| \leq p^n$ for all orbits Θ .

$(\Lambda , B) \in \overline{\mathcal{L}}_p \iff (\Lambda , B) \in \overline{\mathcal{K}}_p$ and $B \in \underline{\underline{N}}$

We prove

Theorem 3.3.1 Let $I = \{0,1\} \cup P$ and define subclasses of $\underline{\underline{N}}$ by $\underline{\underline{X}}_0 = \underline{\underline{T}}$, $\underline{\underline{X}}_1 = \underline{\underline{N}} \setminus \bigcup_{p \in P} \underline{\underline{N}}_p$, $\underline{\underline{X}}_p = \underline{\underline{N}}_p \setminus \underline{\underline{T}}$ for all $p \in P$ and classes of pairs

$\mathfrak{X}_0 = \mathcal{N}$, $\mathfrak{X}_1 = \{ (\Lambda , B) : (\Lambda , B) \in \mathcal{N} \text{ and } B = C_B(\Lambda) \}$ and

$\mathfrak{X}_p = \overline{\mathcal{L}_p}$ for all $p \in P$.

Let A be a group and let (Λ , B) be a pair. Then

$A \text{ wr}^{\Lambda} B \in \underline{\underline{N}} \iff \exists\ i \in I$ such that $A \in \underline{\underline{X}}_i$ and $(\Lambda , B) \in \mathfrak{X}_i$

Putting $\underline{\underline{X}}_{i2} = \{ B : (B , B) \in \mathfrak{X}_i \}$ for all $i \in I$, we have for the standard case

Theorem 3.3.2 Let A and B be groups. Let $I = \{0,1\} \cup P$, and let

$\underline{\underline{X}}_{01} = \underline{\underline{T}}$, $\underline{\underline{X}}_{11} = \underline{\underline{N}} \setminus \bigcup_{p \in P} \overline{\underline{\underline{N}}}_p$, $\underline{\underline{X}}_{p1} = \overline{\underline{\underline{N}}}_p \setminus \underline{\underline{T}}$ $\forall$ $p \in P$

$\underline{\underline{X}}_{02} = \underline{\underline{N}}$, $\underline{\underline{X}}_{12} = \underline{\underline{T}}$, $\underline{\underline{X}}_{p2} = \underline{\underline{F}}_p$ for all $p \in P$

Then

$$A \text{ wr } B \in \underline{\underline{N}} \iff \exists\ i \in I \text{ such that } A \in \underline{\underline{X}}_{i1} \text{ and } B \in \underline{\underline{X}}_{i2}$$

These extend results of G. Baumslag and J.D.P. Meldrum, viz

Theorem 3.3.3 [1] Let A and B be non-trivial groups and let $W = A \text{ wr } B$. Then

$W \in N \iff$ there exists prime p such that $A \in \overline{\underline{\underline{N}}}_p$ and $B \in \underline{\underline{F}}_p$

Theorem 3.3.4 [[13] (unpublished)] Let A be a group, and let (Λ , B) be a pair such that $B \in \underline{\underline{N}}$. Let $W = A \text{ wr}^{\Lambda} B$.

Then

$W \in \underline{\underline{N}} \iff A \text{ wr}^{\Theta} \{B/C_B(\Theta)\}$ is nilpotent of bounded class for each orbit Θ

This result is required for the proof of Theorem 3.3.1 ; we prove it using a similar proof (Proposition 3.3.10).

Theorem 3.3.5 [[13] (unpublished)] Let A be a non-trivial group and let (Λ , B) be a non-trivial faithful transitive pair.

Then

$W \in \underline{\underline{N}} \iff$ there exists prime p such that $A \in \overline{\underline{\underline{N}}}_p$ and $B \in \underline{\underline{F}}_p$

To prove Theorem 3.3.1 we will need several results about the nilpotency class of nilpotent wreath products.

Notation: Denote the lower central series of a group G by

$$\gamma_1(G) = G$$
$$\gamma_{\alpha+1}(G) = [\gamma_\alpha(G), G] \quad \text{for all ordinals } \alpha$$
$$\gamma_\lambda(G) = \bigcap_{\mu<\lambda} \gamma_\mu(G) \quad \text{for all limit ordinals } \lambda$$

Lemma 3.3.6 Let A be a group, let (Λ, B) be a pair, and let $W = A \operatorname{wr}^\Lambda B$. Let $f \in A^\Lambda$ and let $b_1, \ldots, b_k \in B$. Then

$$\sigma([f, b_1, \ldots, b_k]) \subseteq \sigma(f)B \quad \text{and}$$
$$|\sigma([f, b_1, \ldots, b_k])| \leqslant 2^k |\sigma(f)|$$

Proof: We prove by induction on i that for $1 \leqslant i \leqslant k$,

$$\sigma([f, b_1, \ldots, b_i]) \subseteq \sigma(f)B \quad \text{and}$$
$$|\sigma([f, b_1, \ldots, b_i])| \leqslant 2^i |\sigma(f)|$$

Suppose $i = 1$. Then $[f, b_1] = f^{-1} f^{b_1}$ and so

$$\sigma([f, b_1]) \subseteq \sigma(f^{-1}) \cup \sigma(f^{b_1})$$
$$= \sigma(f) \cup \sigma(f)b_1$$
$$\subseteq \sigma(f)B$$

and $|\sigma([f, b_1])| \leqslant 2|\sigma(f)|$

Suppose the result holds for some i, $1 \leqslant i \leqslant k-1$. Then $[f, b_1, \ldots, b_{i+1}] = [f, b_1, \ldots, b_i]^{-1} [f, b_1, \ldots, b_i]^{b_{i+1}}$ and so

$$\sigma([f, b_1, \ldots, b_{i+1}]) \subseteq \sigma([f, b_1, \ldots, b_i]) \cup \sigma([f, b_1, \ldots, b_i])b_1$$
$$\subseteq \sigma(f)B \quad \text{by the induction hypothesis}$$

and $\quad |\sigma([f,b_1,\ldots,b_{i+1}])| \leq 2|\sigma([f,b_1,\ldots,b_i])|$

$\leq 2^{i+1}|\sigma(f)|$ by the induction hypothesis

Hence we have the result.

Lemma 3.3.7 Let A be a non-trivial group, let (Λ, B) be a pair, and let $W = A \text{ wr}^{\Lambda} B$. Let Θ be an orbit of Λ, and suppose there exists non-negative integer n such that $2^n < |\Theta|$. Let $\theta \in \Theta$ and let $1 \neq f \in A_\theta$. Then there exists $b_1,\ldots,b_{n+1} \in B$ such that $[f,b_1,\ldots,b_{n+1}] \neq 1$

Proof: We prove by induction on k that $[f,b_1,\ldots,b_{k+1}] \neq 1$ for $0 \leq k \leq n$

Let $k = 0$. $1 < |\Theta|$, and so there exists $\theta_1 \in \Theta$ such that $\theta_1 \neq \theta$. Since B is transitive on Θ, there exists $b_1 \in B$ such that $\theta b_1 = \theta_1$. Then

$$[f,b_1](\theta_1) = f(\theta_1)^{-1}f(\theta_1)^{b_1} = f(\theta) \neq 1$$

Hence $\theta_1 \in \sigma([f,b_1])$ and $[f,b_1] \neq 1$.

Now suppose the result holds for some k, $0 \leq k < n$. Then by the induction hypothesis there exists $b_1,\ldots,b_{k+1} \in B$ such that $[f,b_1,\ldots,b_{k+1}] \neq 1$; hence there exists $\mu \in \sigma([f,b_1,\ldots,b_{k+1}]) \subseteq \sigma(f)B \subseteq \Theta$ by Lemma 3.3.6. Also by Lemma 3.3.6,

$|\sigma([f,b_1,\ldots,b_{k+1}])| \leq 2^{k+1}|\sigma(f)| = 2^{k+1} < |\Theta|$, and hence there exists $\lambda \in \Theta \setminus \sigma([f,b_1,\ldots,b_{k+1}])$. Then since B is transitive on Θ, there exists $b_{k+2} \in B$ such that $\mu b_{k+2} = \lambda$; and so

$$\begin{aligned}[f,b_1,\ldots,b_{k+1},b_{k+2}](\lambda) &= [f,b_1,\ldots,b_{k+1}]^{-1}(\lambda)[f,b_1,\ldots,b_{k+1}](\lambda b_{k+2}^{-1}) \\ &= [f,b_1,\ldots,b_{k+1}](\mu) \neq 1\end{aligned}$$

Hence result by induction. In particular, putting $k = n$ we have the result.

We have the following corollary

Corollary 3.3.8 Let A be a non-trivial group, let (Λ, B) be a pair, and let $W = A \text{ wr}^{\Lambda} B$. Suppose $W \in \underline{\underline{N}}$, and let $n(W)$ be the nilpotency class of W. Then each orbit of Λ is finite, and hence $B/C_B(\Theta)$ is finite for all orbits Θ; further $|B/C_B(\Theta)| \leq 2^{n(W)}!$ for all orbits Θ.

Proof: Let Θ be an orbit and suppose $2^{n(W)} < |\Theta|$. Then by Lemma 3.3.7, $\gamma_{n(W)+2} \neq \{1\}$, which is a contradiction. Hence $|\Theta| \leq 2^{n(W)}$ for all orbits Θ, and so

$$|B/C_B(\Theta)| \leq 2^{n(W)}! \quad \text{for all orbits } \Theta.$$

The following lemma is proved in rather more general form than is required here; it is needed for a later chapter.

Lemma 3.3.9 Let A be a group, let (Λ, B) be a pair, and let $W = A \text{ wr}^{\Lambda} B$. Then

(a) $\gamma_n(W) = [A^{\Lambda},_{n-1} W]\gamma_n(B) \qquad \text{for all } n \in Z^+$

(b) Let $W_f = A \text{ wr}^{\Lambda} \{B/C_B(\Lambda)\}$. Then

$$[A^{\Lambda},_n W] = [A^{\Lambda},_n W_f] \qquad \text{for all } n \in N$$

(c) Suppose $\{A_i : i \in I\}$ is a family of groups such that $A = \underset{i \in I}{\text{Dr}} A_i$. Then

$$[A^{\Lambda},_n W] = \underset{i \in I}{\text{Dr}} \ \underset{\Theta \text{ an orbit}}{\text{Dr}} [A_i^{\Theta},_n A_i^{\Theta} B] \quad \text{for all } n \in N$$

Proof: (a) [This is essentially proved in [13]]

The proof is by induction on n.

The case $n = 1$ is immediate:

$$\gamma_1(W) = W = A^{\Lambda}B = [A^{\Lambda},{}_0W]\gamma_1(B)$$

Suppose the result is true for some $n \geq 1$. Then

$$\begin{aligned}\gamma_{n+1}(W) &= [\gamma_n(W),W]\\ &= [\ [A^{\Lambda},{}_{n-1}W]\gamma_n(B),\ W] \text{ by the induction hypothesis}\\ &= \langle [\ [A^{\Lambda},{}_{n-1}W],\ W]^{\gamma_n(B)},\ [\gamma_n(B),W] \rangle\\ &= [A^{\Lambda},{}_nW][\gamma_n(B),A^{\Lambda}B] \quad \text{since } [A^{\Lambda},{}_nW] \vartriangleleft W\\ &= [A^{\Lambda},{}_nW]\langle [\gamma_n(B),B],\ [\gamma_n(B),A^{\Lambda}]^B \rangle\\ &= [A^{\Lambda},{}_nW][A^{\Lambda},\gamma_n(B)]\gamma_{n+1}(B) \quad \text{since } [A^{\Lambda},\gamma_n(B)] \text{ is normalised by } B\\ &\leq [A^{\Lambda},{}_nW]\gamma_{n+1}(B) \quad \text{since } [A^{\Lambda},\gamma_n(B)] \leq [A^{\Lambda},{}_nW]\end{aligned}$$

(See e.g. [6]. The result is a corollary to Theorem 10.3.6)

Clearly $[A^{\Lambda},{}_nW]\gamma_{n+1}(B) \leq \gamma_{n+1}(W)$; hence

$$[A^{\Lambda},{}_nW]\gamma_{n+1}(B) = \gamma_{n+1}(W)$$

Hence result by induction.

(b) [This is also proved in [13]]

Let $f \in A^{\Lambda}$, $b \in B$, $\beta \in C_B(\Lambda)$. Then

$$f^{b\beta}(\lambda) = f(\lambda\beta^{-1}b^{-1}) = f^b(\lambda) \quad \forall \ \lambda \in \Lambda$$

i.e. $f^{bC_B(\Lambda)} = f^b$

Hence $(A^{\Lambda})^B = (A^{\Lambda})^{\{B/C_B(\Lambda)\}}$, and $[A^{\Lambda},B] = [A^{\Lambda},\ B/C_B(\Lambda)]$.

The result then follows from this.

(c) Note that $A_i \text{ wr}^{\Theta} B = A_i^{\Theta}B \leq W$ for all $i \in I$ and all Θ.

We prove by induction on n that

$$[A^{\Lambda},{}_nW] = \underset{i\in I}{\mathrm{Dr}} \ \underset{\Theta \text{ an orbit}}{\mathrm{Dr}} [A_i^{\Theta},{}_nW] \quad \text{for all } n \in N$$

The case $n = 0$ is immediate:

$$[A^{\Lambda},{}_0W] = A^{\Lambda} = \underset{i\in I}{\mathrm{Dr}} \ \underset{\Theta \text{ an orbit}}{\mathrm{Dr}} \ A_i^{\Theta} = \underset{i\in I}{\mathrm{Dr}} \ \underset{\Theta \text{ an orbit}}{\mathrm{Dr}} \ [A_i^{\Theta},{}_0W]$$

Suppose the result holds for some $n \geq 0$. Then

$$[A^{\Lambda},{}_{n+1}W] = [\underset{i\in I}{\mathrm{Dr}} \ \underset{\Theta \text{ an orbit}}{\mathrm{Dr}} \ [A_i^{\Theta},{}_nW], W]$$

Since $A_i^{\Theta} \triangleleft W$, $[A_i^{\Theta},{}_nW] \leq A_i^{\Theta}$ for all $i \in I$ and all orbits Θ;

further $[A_i, A_j] = \{1\}$ if $i \neq j$ and $[A^{\Sigma},A^{\Psi}] = \{1\}$ if Σ and Ψ are distinct orbits. Hence

$$[\underset{i\in I}{\mathrm{Dr}} \ \underset{\Theta \text{ an orbit}}{\mathrm{Dr}} \ [A_i^{\Theta},{}_nW], W] = \langle [[A_i^{\Theta},{}_nW],W] : i \in I, \Theta \text{ an orbit} \rangle$$

$$= \underset{i\in I}{\mathrm{Dr}} \ \underset{\Theta \text{ an orbit}}{\mathrm{Dr}} \ [A_i^{\Theta},{}_{n+1}W]$$

Hence result by induction.

Again since $[A_i, A_j] = \{1\} = [A^{\Sigma},A^{\Psi}]$ if $i \neq j$ and $\Sigma \neq \Psi$,

$$\underset{i\in I}{\mathrm{Dr}} \ \underset{\Theta \text{ an orbit}}{\mathrm{Dr}} \ [A_i^{\Theta},{}_nW] = \underset{i\in I}{\mathrm{Dr}} \ \underset{\Theta \text{ an orbit}}{\mathrm{Dr}} \ [A_i^{\Theta},{}_n A_i^{\Theta}B] \quad \forall \ n \in N$$

Hence result.

<u>Proposition 3.3.10</u> [[13] (unpublished)] Let A be a group, let (Λ, B) be a pair, and let $W = A \text{ wr}^{\Lambda} B$. Then

$W \in \underline{\underline{N}} \iff B \in \underline{\underline{N}}$ and $A \text{ wr}^{\Theta} \{B/C_B(\Theta)\}$ is nilpotent of bounded nilpotency class for all orbits Θ of Λ

<u>Proof:</u> Denote $A \text{ wr}^{\Theta} B$ by W_{Θ} and $A \text{ wr}^{\Theta} \{B/C_B(\Theta)\}$ by $W_{\Theta f}$.

$W \in \underline{\underline{N}} \Rightarrow B \in S\underline{\underline{N}} = \underline{\underline{N}}$, and $W_{\Theta f} \in QS\underline{\underline{N}} = \underline{\underline{N}}$, with $n(W_{\Theta f}) \leq n(W)$ for all orbits Θ.

Now suppose $B \in \underline{\underline{N}}$ and $W_{\Theta f}$ is nilpotent for all orbits Θ, with $n(W_{\Theta f}) \leq n$, say. Let $m = \max(n , n(B))$. Then

$$\gamma_{m+1}(W) = [A^{\Lambda},{}_m W]\gamma_{m+1}(B) \qquad \text{by Lemma 3.3.9 (a)}$$

$$= \underset{\Theta \text{ an orbit}}{\mathrm{Dr}} [A^{\Theta},{}_m W_{\Theta}] \qquad \text{by Lemma 3.3.9 (c)}$$

$$= \underset{\Theta \text{ an orbit}}{\mathrm{Dr}} [A^{\Theta},{}_m W_{\Theta f}] \qquad \text{by Lemma 3.3.9 (b)}$$

$$= \{1\} \qquad \text{by hypothesis}$$

Hence W is nilpotent.

<u>Lemma 3.3.11</u> Let G be a group. Then every subgroup H of G induces a transitive permutational representation of G by assigning to any element a of G the permutation $Hr \mapsto Hra$ of the right cosets of H. All transitive permutational representations of G can be obtained in this way.
See e.g. [9] p120.

<u>Lemma 3.3.12</u> Let A be a group. Let (Λ_1, B) and (Λ_2, B) be two pairs, (Λ_1, B) being the representation induced by H, say, and (Λ_2, B) the representation induced by G, say, where $H \leq G \leq B$. Let $W_1 = A \text{ Wr}^{\Lambda_1} B$ and $W_2 = A \text{ Wr}^{\Lambda_2} B$. Then there exists a monomorphism from W_2 into W_1.

<u>Proof:</u> Let R be a right transversal to G in B and S be a right transversal to H in G. Then $SR = \{sr : s \in S, r \in R\}$ is a right transversal to H in B. We may write

$$\Lambda_1 = \{Hsr : s \in S, r \in R\}$$

$$\Lambda_2 = \{Gr : r \in R\}$$

with $(Hsr)b = Hsrb$ and $(Gr)b = Grb$ for all $b \in B$.

Define $\theta : W_2 \to W_1$ by

$$b\theta = b \quad \forall\ b \in B$$

and $$f\theta(Hsr) = f(Gr) \quad \text{for all } s \in S,\ r \in R, \text{ and } f \in Cr\ A^{\Lambda_1}$$

(Note that even if the support of f is finite, the support of $f\theta$ need not be.)

Then $f\theta$ is well defined:

Let $s, s_1 \in S$ and $r, r_1 \in R$ such that $Hsr = Hs_1r_1$.

Then $$srr_1^{-1}s_1^{-1} \in H \leqslant G$$

$$\Rightarrow \quad rr_1^{-1} \in G$$

$$\Rightarrow \quad Gr = Gr_1$$

$$\Rightarrow \quad f\theta(Hsr) = f\theta(Hs_1r_1)$$

and so θ is well defined.

θ is a homomorphism:

Let $b, c \in B$ and $f, g \in Cr\ A^{\Lambda_2}$. Let $s \in S$ and $r \in R$.

Then $$(f^c)\,\theta(Hsr) = f^c(Gr) = f(Grc^{-1})$$

and $$(f\theta)^c(Hsr) = f\theta(Hsrc^{-1})$$

Let $r_1 \in R$ and $s_1 \in S$ be such that $Hsrc^{-1} = Hs_1r_1$.

Then $$src^{-1}r_1^{-1}s_1^{-1} \in H \leqslant G$$

$$\Rightarrow \quad rc^{-1}r_1^{-1} \in G$$

$$\Rightarrow \quad Grc^{-1} = Gr_1$$

Hence $(f\theta)^c(Hsr) = f(Gr_1) = f(Grc^{-1}) = (f^c)\,\theta(Hsr)$.

So $(f\theta)^c = (f^c)\,\theta$.

Also, $$(f^c g)\,\theta(Hsr) = f^c g(Gr) = (f\theta)^c(Hsr)\ g\theta(Hsr) = (f\theta)^c g\theta(Hsr)$$

Hence $(bfcg)\theta = bc(f^c g)\theta = (bf)\theta(cg)\theta$ as required.

θ is a monomorphism:

Let $b, c \in B$ and $f, g \in \mathrm{Cr}\ A^{\Lambda_2}$, and suppose $bf\theta = cg\theta$.

Then $b = c$ and $f\theta(Hsr) = g\theta(Hsr)$ for all $s \in S$, $r \in R$

$\Rightarrow$ $b = c$ and $f(Gr) = g(Gr)$ for all $r \in R$

$\Rightarrow$ $b = c$ and $f = g$

Hence θ is a monomorphism.

Corollary 3.3.13 Let A be a group, let (Λ, B) be a transitive pair, and let $W = A\ \mathrm{Wr}^{\Lambda}\ B$. Then we may embed W in A Wr B.

Proof: Let (Λ, B) be the representation induced by H, say. The right regular representation (B, B) is induced by $\{1\} \leq H$; so by Lemma 3.3.12 we may embed $A\ \mathrm{Wr}^{\Lambda}\ B$ in A Wr B.

We will require the following theorem

Theorem 3.3.14 [[18] Theorem 4.7] Let p be a prime and let $A \in \overline{\underline{\underline{N}}}_p$ and $B \in \underline{\underline{F}}_p$. Let $W = A \text{ wr } B$. Let A have exponent p^k and B have order p^t. Then W is nilpotent of class $n(W)$ and

$$n(W) \leq \begin{cases} k\, n(A)\, p^{t-2}\, (2p-1) & \text{if B is not cyclic} \\ n(A)\, p^{t-1}\, (kp - k + 1) & \text{if B is cyclic} \end{cases}$$

Hence we have

Proposition 3.3.15 Let p be a prime, and let $A \in \overline{\underline{\underline{N}}}_p$ and $(\Lambda, B) \in \overline{\mathcal{K}}_p$. For orbits Θ of Λ let $W_{\Theta f} = A \text{ wr}^{\Theta} \{B/C_B(\Theta)\}$. Then $W_{\Theta f} \in \underline{\underline{N}}$ for all orbits Θ, and there exists $n \in Z^+$ such that $n(W_{\Theta f}) \leq n$ for all orbits Θ.

Proof: Let Θ be an orbit. $A \in \overline{\underline{\underline{N}}}_p$ and $B/C_B(\Theta) \in \underline{\underline{F}}_p$ implies that

$A \text{ wr } \{B/C_B(\Theta)\} \in \underline{\underline{N}}$ by Theorem 3.3.3.

Note that since $B/C_B(\Theta)$ is finite, $A \text{ Wr } \{B/C_B(\Theta)\} = A \text{ wr } \{B/C_B(\Theta)\}$ and $A \text{ Wr}^{\Theta}\{B/C_B(\Theta)\} = A \text{ wr}^{\Theta} \{B/C_B(\Theta)\}$.

Hence by Corollary 3.3.13, $W_{\Theta f}$ is nilpotent.

$(\Lambda , B) \in \overline{\mathcal{K}}_p \Rightarrow$ there exists $t \in N$ such that

$$|B/C_B(\Theta)| \leq p^t \text{ for all orbits } \Theta .$$

Hence by Theorem 3.3.14, if the exponent of A is p^k,

$$n(A \text{ wr } \{B/C_B(\Theta)\}) \leq \begin{cases} k\, n(A)\, p^{t-2} (2p - 1) & \text{if B is not cyclic} \\ n(A)\, p^{t-1} (kp - k + 1) & \text{if B is cyclic} \end{cases}$$

$$= n \quad \text{say}$$

and so by Corollary 3.3.13, $n(W_{\Theta f}) \leq n$.

Since Θ was any orbit, we have the result.

<u>Proof of Theorem 3.3.1</u>

If $A \in \underline{\underline{T}}$, then clearly $A \text{ wr}^{\Lambda} B \in \underline{\underline{N}} \iff B \in \underline{\underline{N}}$; thus we obtain classes $\underline{\underline{X}}_0$ and $\mathfrak{X}_0$.

We now show that if $A \notin \underline{\underline{T}}$ and (Λ , B) is non-trivial,

$A \text{ wr}^{\Lambda} B \in \underline{\underline{N}} \iff \exists\, p \in P$ such that $A \in \overline{\underline{\underline{N}}}_p$ and $(\Lambda , B) \in \overline{\mathcal{L}}_p$

We prove first that these conditions are necessary.

Let $W \in \underline{\underline{N}}$. Then W is a Baer group and so by Theorem 3.2.1 there exists prime p such that $A \in \overline{\underline{\underline{S}}}_p$ and $B/C_B(\Lambda)$ is a p-group. $A \leq W$ and so $A \in S\underline{\underline{N}} = \underline{\underline{N}}$; hence $A \in \overline{\underline{\underline{N}}}_p$.

Let Θ be an orbit of B on Λ . Then $C_B(\Lambda) \,\Delta\, C_B(\Theta)$ and so $B/C_B(\Theta) \cong \{B/C_B(\Lambda)\}/\{C_B(\Theta)/C_B(\Lambda)\}$; therefore $B/C_B(\Theta)$ is a p-group.

By Corollary 3.3.8, $B/C_B(\Theta)$ is finite of bounded order for all orbits Θ . Thus $(\Lambda , B) \in \overline{\mathcal{K}}_p$. B is a subgroup of W, and so $B \in \underline{\underline{SN}} = \underline{\underline{N}}$; so $(\Lambda , B) \in \bar{\mathcal{L}}_p$, and so the conditions are necessary.

Conversely, suppose there exists $p \in P$ such that $A \in \bar{\underline{\underline{N}}}_p$ and $(\Lambda , B) \in \bar{\mathcal{L}}_p$. Then by Proposition 3.3.15, $A \text{ wr}^{\Theta} \{B/C_B(\Theta)\}$ is nilpotent of bounded class for all orbits Θ. Then since $B \in \underline{\underline{N}}$, by Proposition 3.3.10 $W \in \underline{\underline{N}}$. Thus the conditions are sufficient, and we have the result.

Thus we obtain classes $\underline{\underline{X}}_p$ and $\mathcal{X}_p$ for all $p \in P$.

Finally, suppose (Λ , B) is trivial; then

$$A \text{ wr}^{\Lambda} B \in \underline{\underline{N}} \iff A \in \underline{\underline{N}} \text{ and } B \in \underline{\underline{N}}$$

and so we have the remaining subclass of $\underline{\underline{N}}$, $\underline{\underline{X}}_1 = \underline{\underline{N}} \setminus \bigcup_{p \in P} \bar{\underline{\underline{N}}}_p$ with $\mathcal{X}_1 = \{ (\Lambda , B) : B \in \underline{\underline{N}} \text{ and } B = C_B(\Lambda) \}$

Section 3.4 $W \in L\underline{\underline{N}}$

We have

Theorem 3.4.1 Let $I = \{0,1\} \cup P$, and let

$\underline{\underline{X}}_0 = \underline{\underline{T}}$, $\underline{\underline{X}}_1 = \underline{\underline{N}} \setminus \bigcup_{p \in P} \bar{\underline{\underline{N}}}_p$, $\underline{\underline{X}}_p = \bar{\underline{\underline{N}}}_p \setminus \underline{\underline{T}} \quad \forall \ p \in P$

$\mathcal{X}_0 = \mathcal{n}$, $\mathcal{X}_1 = \{ (\Lambda , B) : B \in \underline{\underline{N}} \text{ and } B = C_B(\Lambda) \}$

$\mathcal{X}_p = \bar{\mathcal{L}}_p$ for all $p \in P$

Let A be a group, let (Λ , B) be a pair, and let $W = A \text{ wr}^{\Lambda} B$. Then

$W \in L\underline{\underline{N}} \iff \exists \ i \in I$ such that $A \in L\underline{\underline{X}}_i$ and $(\Lambda , B) \in L\mathcal{X}_i$

Proof: This is immediate from Theorem 2.4.1 and Theorem 3.3.1, since $S(\underline{\underline{X}}_i \cup \underline{\underline{T}}) = \underline{\underline{X}}_i \cup \underline{\underline{T}}$

For the standard result we have

Theorem 3.4.2 Let A and B be groups. Let $I = \{0,1\} \cup P$ and let

$\underline{\underline{X}}_{01} = \underline{\underline{T}}$, $\underline{\underline{X}}_{11} = \underline{\underline{N}} \setminus \bigcup_{p \in P} \overline{\underline{\underline{N}}}_p$, $\underline{\underline{X}}_{p2} = \overline{\underline{\underline{N}}}_p \setminus \underline{\underline{T}}$ $\forall$ $p \in P$

$\underline{\underline{X}}_{02} = \underline{\underline{N}}$, $\underline{\underline{X}}_{12} = \underline{\underline{T}}$, $\underline{\underline{X}}_{p2} = \underline{\underline{F}}_p$ for all $p \in P$

Then

A wr B $\in L\underline{\underline{N}}$ <=> $\exists$ $i \in I$ such that $A \in L\underline{\underline{X}}_{i1}$ and $B \in L\underline{\underline{X}}_{i2}$

Proof: This is immediate from Theorem 2.4.3 and Theorem 3.3.2.

Section 3.5 W $\in \acute{N}\underline{\underline{A}}$

Let $\mathcal{J} = \{ (\Lambda , B) : B \in \acute{N}\underline{\underline{A}} \}$.

We prove

Theorem 3.5.1 Let $I = \{0,1\} \cup P$, and define

$\underline{\underline{X}}_0 = \underline{\underline{T}}$, $\underline{\underline{X}}_1 = \acute{N}\underline{\underline{A}} \setminus \bigcup_{p \in P} (\acute{N}\underline{\underline{A}})_p$, $\underline{\underline{X}}_p = (\acute{N}\underline{\underline{A}})_p \setminus \underline{\underline{T}}$ $\forall$ $p \in P$

$\mathcal{X}_0 = \mathcal{J}$, $\mathcal{X}_1 = \{ (\Lambda , B) : B \in \acute{N}\underline{\underline{A}}$ and $B = C_B(\Lambda) \}$

$\mathcal{X}_p = L \overline{\mathcal{K}}_p \cap \mathcal{J}$ for all $p \in P$

Let A be a group and let (Λ , B) be a pair. Then

$A \text{ wr}^{\Lambda} B \in \acute{N}\underline{\underline{A}}$ <=> $\exists$ $i \in I$ such that $A \in \underline{\underline{X}}_i$ and $(\Lambda , B) \in \mathcal{X}_i$

Putting $X_{i2} = \{ B : (B , B) \in \mathcal{X}_i \}$ for all $i \in I$, we have

Theorem 3.5.2 Let $I = \{0,1\} \cup P$, and let

$\underline{\underline{X}}_{01} = \underline{\underline{T}}$, $\underline{\underline{X}}_{11} = \acute{N}\underline{\underline{A}} \setminus \bigcup_{p \in P} (\acute{N}\underline{\underline{A}})_p$, $\underline{\underline{X}}_{p1} = (\acute{N}\underline{\underline{A}})_p \setminus \underline{\underline{T}}$ $\forall$ $p \in P$

$\underline{\underline{X}}_{02} = \acute{N}\underline{\underline{A}}$, $\underline{\underline{X}}_{12} = \underline{\underline{T}}$, $\underline{\underline{X}}_{p2} = (\acute{N}\underline{\underline{A}})_p$ for all $p \in P$

Then

A wr B $\in \acute{N}\underline{\underline{A}}$ <=> $\exists$ $i \in I$ such that $A \in \underline{\underline{X}}_{i1}$ and $B \in \underline{\underline{X}}_{i2}$

We require the following well known lemma.

<u>Lemma 3.5.3</u> [See e.g. [17] Volume 2] Let G be any group.

Then $G \in \underline{\underline{\acute{N}A}} \iff G \in \underline{\underline{LN}} \cap \underline{\underline{\acute{P}A}}$

<u>Proof of Theorem 3.5.1</u>

If $A = \{1\}$, $W \in \underline{\underline{\acute{N}A}} \iff B \in \underline{\underline{\acute{N}A}}$, trivially. Thus we obtain classes $\underline{\underline{X}}_0 = \underline{\underline{T}}$ and $\mathcal{X}_0 = \mathcal{T}$.

Now suppose A and (Λ, B) are not trivial. We show that $W \in \underline{\underline{\acute{N}A}} \iff \exists\ p \in P$ such that $A \in (\underline{\underline{\acute{N}A}})_p$ and $(\Lambda, B) \in L\bar{\mathcal{K}}_p \cap \mathcal{T}$

Suppose firstly that $W \in \underline{\underline{\acute{N}A}}$. Then W is locally nilpotent and so by Theorem 3.4.1 there exists prime p such that $A \in L\underline{\underline{\bar{N}}}_p$ and $(\Lambda, B) \in L\bar{\mathcal{K}}_p$. Since B is a subgroup of W, $B \in S(\underline{\underline{\acute{N}A}}) = \underline{\underline{\acute{N}A}}$, and so $(\Lambda, B) \in L\bar{\mathcal{K}}_p \cap \mathcal{T}$. Similarly, $A \in \underline{\underline{\acute{N}A}}$, and so $A \in \underline{\underline{\acute{N}A}} \cap L\underline{\underline{\bar{N}}}_p = (\underline{\underline{\acute{N}A}})_p$, since $L\underline{\underline{\bar{N}}}_p = (L\underline{\underline{N}})_p$, as periodic locally nilpotent groups are locally finite (see e.g. [10] p190]), and $\underline{\underline{\acute{N}A}} \leq L\underline{\underline{N}}$.

Now suppose that $B \in \underline{\underline{\acute{N}A}}$, and there exists prime p such that $A \in (\underline{\underline{\acute{N}A}})_p$ and $(\Lambda, B) \in L\bar{\mathcal{K}}_p$. Then $A \in L\underline{\underline{\bar{N}}}_p$ and $(\Lambda, B) \in L\bar{\mathcal{L}}_p$; therefore by Theorem 3.4.1, $W \in L\underline{\underline{N}}$.
$A \in \underline{\underline{\acute{N}A}} \Rightarrow A \in \underline{\underline{\acute{P}A}}$ by Lemma 3.5.3, and so $A^{\Lambda} \in D_p(\underline{\underline{\acute{P}A}}) = \underline{\underline{\acute{P}A}}$;
and $B \in \underline{\underline{\acute{N}A}} \Rightarrow B \in \underline{\underline{\acute{P}A}}$ by Lemma 3.5.3.
Hence $W \in P(\underline{\underline{\acute{P}A}}) = \underline{\underline{\acute{P}A}}$, and so $W \in L\underline{\underline{N}} \cap \underline{\underline{\acute{P}A}} = \underline{\underline{\acute{N}A}}$ by Lemma3.5.3.

So we have classes $\underline{\underline{X}}_p = (\underline{\underline{\acute{N}A}})_p$ and $\mathcal{X}_p = L\bar{\mathcal{K}}_p \cap \mathcal{T}$.

Suppose finally that (Λ, B) is trivial. Then by Proposition 3.1.5,

$W \in \underline{\underline{\acute{N}A}} \iff A \in \underline{\underline{\acute{N}A}}$ and $B \in \underline{\underline{\acute{N}A}}$; hence we obtain classes $X_1 = \underline{\underline{\acute{N}A}} \setminus \bigcup_{p \in P}(\underline{\underline{NA}})_p$, and $\mathcal{X}_1 = \{(\Lambda, B) : B \in \underline{\underline{\acute{N}A}}$ and $B = C_B(\Lambda)\}$.

Theorem 3.5.1 is now completed.

Section 3.6 $W \in \underline{\underline{\acute{Z}}}$

Recall that $\underline{\underline{\acute{Z}}}$ is the class of ZA - groups, i.e. groups with an ascending central series.

Let $\mathcal{Z} = \{ (\Lambda , B) : B \in \underline{\underline{\acute{Z}}} \}$.

We prove

Theorem 3.6.1 Let $I = \{0,1\} \cup P$, and define

$\underline{\underline{X}}_0 = \underline{\underline{T}}$, $\underline{\underline{X}}_1 = \underline{\underline{\acute{Z}}} \setminus \bigcup_{p \in P} \underline{\underline{\acute{Z}}}_p$, $\underline{\underline{X}}_p = \underline{\underline{\acute{Z}}}_p \setminus \underline{\underline{T}}$ $\forall$ $p \in P$

$\mathcal{X}_0 = \mathcal{Z}$, $\mathcal{X}_1 = \{ (\Lambda , B) : B \in \underline{\underline{\acute{Z}}}$ and $B = C_B(\Lambda) \}$

$\mathcal{X}_p = \mathcal{K}_p \cap \mathcal{Z}$ for all $p \in P$

Let A be a group and (Λ , B) be a pair. Let $W = A \text{ wr}^{\Lambda} B$.

Then

$W \in \underline{\underline{\acute{Z}}} \iff \exists$ $i \in I$ such that $A \in \underline{\underline{X}}_i$ and $(\Lambda , B) \in \mathcal{X}_i$

Putting $X_{i2} = \{ B : (B , B) \in \mathcal{X}_i \}$, we have

Theorem 3.6.2 Let A and B be groups. Let $I = \{0,1\} \cup P$, and let

$\underline{\underline{X}}_{01} = \underline{\underline{T}}$, $\underline{\underline{X}}_{11} = \underline{\underline{\acute{Z}}} \setminus \bigcup_{p \in P} \underline{\underline{\acute{Z}}}_p$, $\underline{\underline{X}}_{p1} = \underline{\underline{\acute{Z}}}_p \setminus \underline{\underline{T}}$ $\forall$ $p \in P$

$\underline{\underline{X}}_{02} = \underline{\underline{\acute{Z}}}$, $\underline{\underline{X}}_{12} = \underline{\underline{T}}$, $\underline{\underline{X}}_{p2} = \underline{\underline{F}}_p$ for all $p \in P$

Then

$A \text{ wr } B \in \underline{\underline{\acute{Z}}} \iff \exists$ $i \in I$ such that $A \in \underline{\underline{X}}_{i1}$ and $B \in \underline{\underline{X}}_{i2}$

This extends an unpublished result of J.D.P. Meldrum [12], viz

Theorem 3.6.3 Let A and B be non-trivial groups. Then

$A \text{ wr } B \in \underline{\underline{\acute{Z}}} \iff \exists$ prime p such that $A \in \underline{\underline{\acute{Z}}}_p$ and $B \in \underline{\underline{F}}_p$

To prove Theorem 3.6.1, we require the following preliminary results.

For any group G and subgroups H and K, denote the subgroup of K which centralises H by $C_K(H)$.

Let G be a group. Then the upper central series of G is given by

$$\zeta_0(G) = \{1\}$$

$$\zeta_1(G) = \{ g \in G : gh = hg \quad \text{for all } h \in G \}$$

$$\zeta_{n+1}(G)/\zeta_n(G) = \zeta_1(G/\zeta_n(G)) \quad \text{for all ordinals } n$$

$$\zeta_\lambda(G) = \bigcup_{\mu<\lambda} \zeta_\mu(G) \quad \text{for all limit ordinals } \lambda$$

Lemma 3.6.4 Let A be a non-trivial group, let (Λ, B) be a pair, and let $W = A \operatorname{wr}^{\Lambda} B$. Let H comprise all $f \in A^{\Lambda}$ with fixed value on each orbit of B on Λ, i.e. $f(\lambda b) = f(\lambda) \ \forall \ \lambda \in \Lambda$ and $b \in B$. Then

$$\zeta_1(W) = (\zeta_1(B) \cap C_B(\Lambda))(H \cap \zeta_1(A^{\Lambda}))$$

Proof: $\zeta_1(W) = C_W(B) \cap C_W(A^{\Lambda})$; we prove that $C_W(B) = \zeta_1(B)H$ and $C_W(A^{\Lambda}) = C_B(\Lambda)\zeta_1(A^{\Lambda})$, whence

$$\zeta_1(W) = (\zeta_1(B) \cap C_B(\Lambda))(H \cap \zeta_1(A^{\Lambda}))$$

Note firstly that $\zeta_1(B)H$ is a subgroup of W, since H is normalised by B. Now

$$
\begin{aligned}
& bf \in C_W(B) \\
\Longleftrightarrow \quad & \beta^{bf} = \beta \quad \forall \ \beta \in B \\
\Longleftrightarrow \quad & \beta^{b} f^{-\beta b} f = \beta \quad \forall \ \beta \in B \\
\Longleftrightarrow \quad & \beta^{b} = \beta \ \text{ and } \ f^{\beta b} = f \quad \forall \ \beta \in B \\
\Longleftrightarrow \quad & b \in \zeta_1(B) \ \text{ and } \ f(\lambda c) = f(\lambda) \quad \forall \ c \in B \\
\Longleftrightarrow \quad & b \in \zeta_1(B) \ \text{ and } \ f \in H
\end{aligned}
$$

Hence $C_W(B) = \zeta_1(B)H$.

$C_B(\Lambda)\zeta_1(A^{\Lambda})$ is clearly a subgroup of W since $C_B(\Lambda)$ and $\zeta_1(A^{\Lambda})$ are normal subgroups of W.

Then

$$bf \in C_W(A^\Lambda)$$

$$\Rightarrow \quad g^{bf} = g \quad \text{for all } g \in A^\Lambda$$

$$\Rightarrow \quad f^{-1}g^b f = g \quad \text{for all } g \in A^\Lambda$$

If $b \notin C_B(\Lambda)$, then there exists $\lambda \in \Lambda$ such that $\lambda b^{-1} \neq \lambda$. Let $1 \neq a \in A$ and let $g = a_\lambda$. Then $a = g(\lambda) = f^{-1}g^b f(\lambda) = 1$, which is a contradiction. Hence $b \in C_B(\Lambda)$, and so $g^b = g$ for all $g \in A^\Lambda$. Therefore $f^{-1}gf = g$ for all $g \in A^\Lambda$, and so $f \in \zeta_1(A^\Lambda)$. Hence $C_W(A^\Lambda) \leqslant C_B(\Lambda)\zeta_1(A^\Lambda)$.

Conversely, if $bf \in C_B(\Lambda)\zeta_1(A^\Lambda)$, then for all $g \in A^\Lambda$, $g^b = g$ and $g^f = g$, and so $bf \in C_W(A^\Lambda)$.

Hence $C_W(A^\Lambda) = C_B(\Lambda)\zeta_1(A^\Lambda)$.

<u>Lemma 3.6.5</u> [[13] (unpublished)] Let A be a group, let (Λ, B) be a pair, and let $W = A \text{ wr}^\Lambda B$. Let $W_f = A \text{ wr}^\Lambda \{B/C_B(\Lambda)\}$. Then $\zeta_m(W_f) \cap A^\Lambda = \zeta_m(W) \cap A^\Lambda$ for all ordinals m

<u>Proof:</u> The proof is by induction on m.

$\zeta_0(W) \cap A^\Lambda = \{1\} = \zeta_0(W_f) \cap A^\Lambda$ immediately.

Suppose that $m > 0$ and the result holds for all smaller ordinals.

If m-1 exists, then by the induction hypothesis

$$\zeta_{m-1}(W_f) \cap A^\Lambda = \zeta_{m-1}(W) \cap A^\Lambda$$

Let $f \in \zeta_m(W) \cap A^\Lambda$. Then $[f,b] \in \zeta_{m-1}(W) \cap A^\Lambda$ for all $b \in B$, and so $[f,b] \in \zeta_{m-1}(W_f) \cap A^\Lambda$ for all $b \in B$, by the induction hypothesis.

Hence $[f, bC_B(\Lambda)] = [f,b] \in \zeta_{m-1}(W_f) \cap A^\Lambda$ for all $b \in B$. Clearly $[f,g] \in \zeta_{m-1}(W) \cap A^\Lambda = \zeta_{m-1}(W_f) \cap A^\Lambda$ for all $g \in A^\Lambda$. Hence $f \in \zeta_m(W_f) \cap A^\Lambda$, and so $\zeta_m(W) \cap A^\Lambda \leqslant \zeta_m(W_f) \cap A^\Lambda$.

A similar argument shows that $\zeta_m(W_f) \cap A^\Lambda \leq \zeta_m(W) \cap A^\Lambda$, and so we have equality.

If m is a limit ordinal, we have

$$\begin{aligned}\zeta_m(W) \cap A^\Lambda &= (\bigcup_{\mu<m} \zeta_\mu(W)) \cap A^\Lambda \\ &= \bigcup_{\mu<m} (\zeta_\mu(W) \cap A^\Lambda) \\ &= \bigcup_{\mu<m} (\zeta_\mu(W_f) \cap A^\Lambda) \quad \text{by the induction hypothesis} \\ &= (\bigcup_{\mu<m} \zeta_\mu(W_f)) \cap A^\Lambda \\ &= \zeta_m(W_f) \cap A^\Lambda\end{aligned}$$

Hence result by induction.

<u>Lemma 3.6.6</u> [[12] (unpublished)] Let G be any group, let $H \triangleleft G$, and let $\overline{G} = G/H$. Suppose $H \leq \zeta_m(G)$ for some ordinal m. Then

$$\zeta_r(\overline{G}) \leq \{\zeta_{m+r}(G)\}/H \quad \text{for all ordinals } r$$

<u>Proof:</u> The proof is by induction on r.

The result is true by hypothesis for $r = 0$.

Suppose that $r > 0$ and the result holds for all smaller ordinals.

If $r-1$ exists, then for all $g \in G$,

$$\begin{aligned}gH \in \zeta_r(\overline{G}) &\Rightarrow [gH, \overline{G}] \leq \zeta_{r-1}(\overline{G}) \leq \{\zeta_{m+r-1}(G)\}/H \quad \text{by the induction hypothesis} \\ &\Rightarrow [g, G] \leq \zeta_{m+r-1}(G) \\ &\Rightarrow g \in \zeta_{m+r}(G)\end{aligned}$$

Hence $\zeta_r(\overline{G}) \leq \{\zeta_{m+r}(G)\}/H$ as required.

Suppose r is a limit ordinal. Then

$$\zeta_r(\overline{G}) = \bigcup_{s<r} \zeta_s(\overline{G})$$
$$\leq \bigcup_{s<r} \{\zeta_{m+s}(G)\}/H$$
$$= \{\bigcup_{s<r} \zeta_{m+s}(G)\}/H$$
$$= \{\zeta_{m+r}(G)\}/H$$

Hence result by induction.

Lemma 3.6.7 $S\overline{\mathcal{K}}_p = \overline{\mathcal{K}}_p$

Proof: Let $(\Lambda, B) \in \overline{\mathcal{K}}_p$ and let (Λ_1, B_1) be a subpair of (Λ, B). Let Θ be any orbit of B_1 acting on Λ_1. Then Θ is contained in some orbit Σ of B acting on Λ.

Then $C_B(\Sigma) \leq C_B(\Theta)$ and $C_{B_1}(\Sigma) \leq C_{B_1}(\Theta)$; and $C_B(\Sigma) \triangleleft B$ and $C_{B_1}(\Theta) \triangleleft B_1$. Also $C_{B_1}(\Sigma) = B_1 \cap C_B(\Sigma)$. Then

$B_1/C_{B_1}(\Theta) \cong \{B_1/C_{B_1}(\Sigma)\}/\{C_{B_1}(\Theta)/C_{B_1}(\Sigma)\}$ and

$B_1/C_{B_1}(\Sigma) = B_1/\{B_1 \cap C_B(\Sigma)\} \cong \{B_1C_B(\Sigma)\}/C_B(\Sigma) \leq B/C_B(\Sigma) \in \underline{\underline{F}}_p$

Hence since $\underline{\underline{F}}_p = QS\underline{\underline{F}}_p$, $B_1/C_{B_1}(\Theta) \in \underline{\underline{F}}_p$; further since there exists n such that $|B/C_B(\Sigma)| \leq p^n$ for all orbits Σ of B on Λ, we have that

$$|B_1/C_{B_1}(\Theta)| \leq p^n ;$$

Θ was any orbit, and so $(\Lambda_1, B_1) \in \overline{\mathcal{K}}_p$, as required.

Lemma 3.6.8 Let A be a group, let (Λ, B) be a pair, and let $W = A \text{ wr}^{\Lambda} B$. Let $G \triangleleft A$. Then $G^{\Lambda} \triangleleft W$ and

$$W/G^{\Lambda} \cong \{A/G\} \text{ wr}^{\Lambda} B$$

Proof: $G \triangleleft A \Rightarrow G^{\Lambda} \triangleleft A^{\Lambda}$; and if $f \in G^{\Lambda}$, $b \in B$, then $f^b(\lambda) \in G \;\; \forall \;\; \lambda \in \Lambda$, and so $f^b \in G$. Hence $G^{\Lambda} \triangleleft W$.

Define $\theta : W \to \{A/G\} \text{ wr}^{\Lambda} B$ by

$(bf)\theta = b(f\theta)$ for all $b \in B$ and all $f \in A^{\Lambda}$, where $f\theta : \Lambda \to A/G$ is given by $(f\theta)(\lambda) = f(\lambda)G \quad \forall \quad \lambda \in \Lambda$.

$f\theta$ is clearly well defined, as is θ.

θ is a homomorphism:

Let $b, c \in B$, $f, g \in A^{\Lambda}$. Then

$(f^{c}g)\theta(\lambda) = f^{c}g(\lambda)G = f(\lambda c^{-1})g(\lambda)G = (f\theta)^{c}(g\theta)(\lambda) \quad \forall \; \lambda \in \Lambda$

and hence $(bfcg)\theta = bc(f^{c}g)\theta = (bf)\theta(cg)\theta$

Ker $\theta = G^{\Lambda}$:

Let $b \in B$ and $f \in A^{\Lambda}$. Then

$bf\theta = 1 \iff b = 1$ and $f\theta = 1$

$\iff b = 1$ and $f(\lambda) \in G \quad \forall \quad \lambda \in \Lambda$

$\iff bf \in G^{\Lambda}$

θ is an epimorphism:

Let $bf \in \{A/G\} \text{ wr}^{\Lambda} B$. Let T be a transversal to G in A and define $g : \Lambda \to G$ by $g(\lambda) = t$ where $f(\lambda) = tG \quad \forall \quad \lambda \in \Lambda$ and $t \in T$. Then g is well defined, and $g\theta = f$; for

$g\theta(\lambda) = g(\lambda)G = f(\lambda) \quad \forall \quad \lambda \in \Lambda$

Hence $bg \in A \text{ wr}^{\Lambda} B$ and $(bg)\theta = bf$.

Hence $W/G^{\Lambda} \cong \{A/G\} \text{ wr}^{\Lambda} B$.

Proof of Theorem 3.6.1

If $A = \{1\}$, then $W \in \acute{\underline{\underline{Z}}} \iff B \in \acute{\underline{\underline{Z}}}$, clearly; thus we have classes $\underline{\underline{X}}_0 = \underline{\underline{T}}$ and $\mathfrak{X}_0 = \mathcal{Z}$.

Now suppose that neither A nor (Λ, B) is trivial. We prove that

$$W \in \acute{\underline{\underline{Z}}} \iff \exists \; p \in P \text{ such that } A \in \acute{\underline{\underline{Z}}}_p \text{ and } (\Lambda, B) \in \mathcal{K}_p \cap \mathcal{Z}$$

Suppose that $W \in \acute{\underline{\underline{Z}}}$. Then $W \in \underline{\underline{LN}}$. Let Θ be any non-trivial orbit and let $W_{\Theta f} = A \text{ wr}^{\Theta} \{B/C_B(\Theta)\} \in QS(W)$. Then $W_{\Theta f} \in L\underline{\underline{N}}$, and so by Theorem 3.4.1 there exists prime p such that $A \in L\bar{\underline{\underline{N}}}_p$ and $(\Theta , B/C_B(\Theta)) \in L\,\bar{\mathcal{K}}_p$

A a subgroup of W and $\bar{\underline{\underline{Z}}} = S\bar{\underline{\underline{Z}}}$ implies that $A \in \bar{\underline{\underline{Z}}}$, so A is a $\bar{\underline{\underline{Z}}}$ - p - group as required.

Suppose that Θ is infinite. Then

$$\{ f \in A^{\Theta} : f \text{ has fixed value } \} = \{1\}$$

and so $\zeta_1(W_{\Theta f}) = \{1\}$, by Lemma 3.6.4, which is a contradiction since $W_{\Theta f}$ is a $\bar{\underline{\underline{Z}}}$ - group. Hence Θ is finite, and so $B/C_B(\Theta)$ is finite. Hence ~~since $B\bar{\mathcal{K}}_p = \bar{\mathcal{K}}_p$ by Lemma 3.6.7~~, $(\Theta , B/C_B(\Theta)) \in \bar{\mathcal{K}}_p$, i.e. $B/C_B(\Theta)$ is a finite p-group. Therefore $(\Lambda , B) \in \mathcal{K}_p$. Since B is a subgroup of W, B is a $\bar{\underline{\underline{Z}}}$ - group, and so $(\Lambda , B) \in \mathcal{K}_p \cap \mathcal{Z}$.

Suppose conversely that there exists prime p such that $A \in \underline{\underline{Z}}_p$ and $(\Lambda , B) \in \mathcal{K}_p \cap \mathcal{Z}$. To prove that $W \in \bar{\underline{\underline{Z}}}$, we use essentially the argument used in [12] to prove Theorem 3.6.2.

Let A be a $\bar{\underline{\underline{Z}}}$ - p - group, say $\zeta_{\pi}(A) = A$, and let $(\Lambda , B) \in \mathcal{K}_p \cap \mathcal{Z}$. We show that there exists an ordinal π' such that for all orbits Θ,

$$A^{\Theta} \leqslant \zeta_{\pi'}(A \text{ wr}^{\Theta} B);$$

this implies that

$$A^{\Theta} \leqslant \zeta_{\pi'}(W) \quad \text{for all orbits } \Theta$$

since A^{Ψ} and A^{Σ} commute elementwise for distinct orbits Ψ and Σ.

Then $A^{\Lambda} \leqslant \zeta_{\pi'}(W)$, and hence $W \in \bar{\underline{\underline{Z}}}$, since $W/\zeta_{\pi'}(W)$ is then a factor of B, a $\bar{\underline{\underline{Z}}}$ - group.

Let Θ be any orbit and let $W_{\Theta f} = A \text{ wr}^{\Theta} \{B/C_B(\Theta)\}$. We prove by induction on β that for all ordinals β there exists an ordinal β' dependent on β such that

$$\zeta_\beta(A^\Theta) \leq \zeta_{\beta'}(W_{\Theta f})$$

Suppose $\beta = 1$. For integers $k \geq 0$, let A_k be the subgroup of $\zeta_1(A)$ consisting of all elements of order at most p^k. Let $W_k = A_k \text{ wr}^\Theta \{B/C_B(\Theta)\}$. Then W_k is nilpotent for all k; say W_k has class c_k.

Let $f \in \zeta_1(A^\Theta)$. Then there exists k such that $f \in A_k^\Theta$. Let $g_1,\ldots,g_m$ be a sequence of arbitrary elements in $W_{\Theta f}$, where $m \geq c_k$; say $g_i = b_i f_i$ for $1 \leq i \leq m$.
Then $b_i \in W_k$ for $1 \leq i \leq m$.
Now $[g,b_1f_1,\ldots,b_mf_m] = [g,b_1,\ldots,b_m]$ for all $g \in \zeta_1(A^\Lambda)$; so in particular,

$$[f,b_1f_1,\ldots,b_mf_m] = [f,b_1,\ldots,b_m] = 1$$

since $[f,b_1,\ldots,b_m] \in \gamma_{m+1}(W_k) = \{1\}$.
Hence $f \in \zeta_m(W_{\Theta f})$ for all $m \geq c_k$, and so $f \in \zeta_\omega(W_{\Theta f})$.
Hence $\zeta_1(A^\Theta) \leq \zeta_\omega(W_{\Theta f})$; so take $1' = \omega$.

Now let β be an ordinal ≥ 1, and suppose the result holds for all smaller ordinals.

Suppose β is a limit ordinal. Then

$$\begin{aligned} \zeta_\beta(A^\Theta) &= \bigcup_{\mu<\beta} \zeta_\mu(A^\Theta) \\ &\leq \bigcup_{\mu<\beta} \zeta_{\mu'}(W_{\Theta f}) && \text{by the induction hypothesis} \\ &= \zeta_{\beta'}(W_{\Theta f}) && \text{where } \beta' = \operatorname*{limit}_{\mu<\beta} \mu' \end{aligned}$$

Suppose $\beta = \mu + 1$. Then there exists μ' such that $\zeta_\mu(A^\Theta) \leq \zeta_{\mu'}(W_{\Theta f})$. By the first part of the proof, putting $\underline{W} = W_{\Theta f}/\zeta_\mu(A^\Theta) \cong \{A/\zeta_\mu(A) \text{ wr}^\Theta \{B/C_B(\Theta)\}$ by Lemma 3.6.8, we have

$$\zeta_1(\{A/\zeta_\mu(A)\}^\Theta) \leq \zeta_\omega(\underline{W})$$

i.e. $\{\zeta_{\mu+1}(A^{\Theta})\}/\{\zeta_{\mu}(A^{\Theta})\} \leq \zeta_{\omega}(\underline{W})$

Then by Lemma 3.6.6, putting $H = \zeta_{\mu}(A^{\Theta})$ and $m = \mu'$,

$$\zeta_{\omega}(\underline{W}) \leq \{\zeta_{\mu'+\omega}(W_{\Theta f})\}/\{\zeta_{\mu}(A^{\Theta})\} \qquad \text{and so}$$

$$\zeta_{\mu+1}(A^{\Theta}) \leq \zeta_{\mu'+\omega}(W_{\Theta f})$$

So take $(\mu + 1)' = \mu' + \omega$.

Hence result by induction. In particular, since $A^{\Lambda} = \zeta_{\pi}(A^{\Lambda})$, $A^{\Theta} = \zeta_{\pi}(A^{\Theta})$, and there exists π' such that

$$A^{\Theta} = \zeta_{\pi}(A^{\Theta}) \leq \zeta_{\pi'}(W_{\Theta f}).$$

By Lemma 3.6.5,

$$\zeta_{\pi'}(W_{\Theta f}) \cap A^{\Theta} = \zeta_{\pi'}(A \text{ wr}^{\Theta} B) \cap A^{\Theta} \quad ;$$

hence

$$A^{\Theta} \leq \zeta_{\pi'}(A \text{ wr}^{\Theta} B) \qquad \text{as required.}$$

Note that π' is independent of Θ.

This completes the proof, and so we have classes $\underline{\underline{X}}_p = \underline{\underline{\acute{Z}}}_p$, and $\mathfrak{X}_p = \mathcal{K}_p \cap \mathfrak{Z}$ for all primes p.

Finally, if (Λ , B) is trivial, by Proposition 3.1.5 $A \text{ wr}^{\Lambda} B \in \underline{\underline{\acute{Z}}} \iff A \in \underline{\underline{\acute{Z}}}$ and $B \in \underline{\underline{\acute{Z}}}$; so we obtain classes $\underline{\underline{X}}_1 = \underline{\underline{\acute{Z}}} \setminus \bigcup_{p \in P} \underline{\underline{\acute{Z}}}_p$ and $\mathfrak{X}_1 = \{ (\Lambda , B) : C_B(\Lambda) = B \in \underline{\underline{\acute{Z}}} \}$, and Theorem 3.6.1 is complete.

Section 3.7 Radicals of W

Recall

Theorem 2.3.1 Let $\underline{\underline{X}}$ be a $\langle Q, S_n, N_o, D_p \rangle$-closed class of groups, and suppose (*) holds for $\underline{\underline{X}}$. Let A be a group, let (Λ , B) be a pair, and let $W = A \text{ wr}^{\Lambda} B$.

If $A \notin \underline{\underline{X}}$, then

$$\rho(W : \underline{\underline{X}}) = \{\rho(B : \underline{\underline{X}}) \cap C_B(\Lambda)\}\rho(A^{\Lambda} : \underline{\underline{X}})$$

$$= \{\rho(B : \underline{\underline{X}}) \cap C_B(\underline{\underline{\Lambda}})\}\rho(A : \underline{\underline{X}})^{\Lambda}$$

On the other hand, if there exists $i \in I$ such that $A \in \underline{\underline{X}}_i$, then

$$\rho(W : X) = \rho((\Lambda , B) : \mathcal{X}_i)A^{\Lambda}$$

We now have hypothesis (*) holding for $\underline{\underline{X}} = \underline{\underline{N}}, L\underline{\underline{N}}, N\underline{\underline{A}}, \acute{N}\underline{\underline{A}}$, and $\acute{\underline{\underline{Z}}}$. Further, all the hypotheses of Theorem 2.3.1 on $\underline{\underline{X}}$ are satisfied for these classes.

Note that in each case we have a class $\underline{\underline{X}}_1 \leqslant \underline{\underline{X}}$ corresponding to $\mathcal{X}_1$ where $\mathcal{X}_1$ consists of trivial pairs with $B \in \underline{\underline{X}}$; hence in this case

$$\rho((\Lambda , B) : \mathcal{X}_1)A^{\Lambda} = \langle B_1 : B_1 \Delta B, (\Lambda , B_1) \in \mathcal{X}_1 \rangle \rho(A^{\Lambda} : \underline{\underline{X}})$$

$$= \{\rho(B : \underline{\underline{X}}) \cap C_B(\Lambda)\}\rho(A^{\Lambda} : \underline{\underline{X}})$$

since (Λ , B_1) is trivial if and only if $B_1 \leqslant C_B(\Lambda)$.
Also we have a class $\underline{\underline{X}}_0 = \underline{\underline{T}}$ corresponding to $\mathcal{X}_0$, the class of all pairs (Λ , B) with $B \in \underline{\underline{X}}$. Then

$$\rho((\Lambda , B) : \mathcal{X}_0)A^{\Lambda} = \rho(B : \underline{\underline{X}}).$$

Thus we have by Theorem 2.3.1 and the above remarks

<u>Theorem 3.7.1</u> Let A be a group, let (Λ , B) be a pair, and let $W = A \text{ wr}^{\Lambda} B$.

(a) Baer radical of W

(1) If $A = \{1\}$, $\rho(W : N\underline{\underline{A}}) = \rho(B : N\underline{\underline{A}})$

(2) If $A \notin \bigcup_{p \in P} \overline{\underline{\underline{S}}}_p$,

$$\rho(W : N\underline{\underline{A}}) = \{\rho(B : N\underline{\underline{A}}) \cap C_B(\Lambda)\}\rho(A^{\Lambda} : N\underline{\underline{A}})$$

(3) If $\exists \; p \in P$ such that $A \in \overline{\underline{\underline{S}}}_p \setminus \underline{\underline{T}}$

$$\rho(W : NA) = \rho((\Lambda , B) : \mathcal{B}_p)A^{\Lambda}$$

(b) Fitting radical of W

(1) If $A = \{1\}$, $\rho(W : \underline{\underline{N}}) = \rho(B : \underline{\underline{N}})$

(2) If $A \not\in \bigcup_{p \in P} \overline{\underline{\underline{N}}}_p$, then

$$\rho(W : \underline{\underline{N}}) = \{\rho(B : \underline{\underline{N}}) \cap C_B(\Lambda)\}\rho(A^{\Lambda} : \underline{\underline{N}})$$
$$= \{\rho(B : \underline{\underline{N}}) \cap C_B(\Lambda)\}\rho(A : \underline{\underline{N}})^{\Lambda}$$

(3) If $\exists\ p \in P$ such that $A \in \overline{\underline{\underline{N}}}_p \setminus \underline{\underline{T}}$,

$$\rho(W : \underline{\underline{N}}) = \rho(\ (\Lambda , B) : \overline{\mathcal{L}}_p)A^{\Lambda}$$

(c) Hirsch-Plotkin radical of W

(1) If $A = \{1\}$, $\rho(W : L\underline{\underline{N}}) = \rho(B : L\underline{\underline{N}})$

(2) If $A \not\in \bigcup_{p \in P} L\overline{\underline{\underline{N}}}_p$, then

$$\rho(W : L\underline{\underline{N}}) = \{\rho(B : L\underline{\underline{N}}) \cap C_B(\Lambda)\}\rho(A^{\Lambda} : L\underline{\underline{N}})$$
$$= \{\rho(B : L\underline{\underline{N}}) \cap C_B(\Lambda)\}\rho(A : L\underline{\underline{N}})^{\Lambda}$$

(3) If $\exists\ p \in P$ such that $A \in L\overline{\underline{\underline{N}}}_p \setminus \underline{\underline{T}}$,

$$\rho(W : L\underline{\underline{N}}) = \rho(\ (\Lambda , B) : L\overline{\mathcal{L}}_p)A^{\Lambda}$$

(d) Gruenberg radical of W

(1) If $A = \{1\}$, $\rho(W : \acute{N}\underline{\underline{A}}) = \rho(B : \acute{N}\underline{\underline{A}})$

(2) If $A \not\in \bigcup_{p \in P} (\acute{\underline{\underline{N}}}A)_p$, then

$$\rho(W : \acute{N}\underline{\underline{A}}) = \{\rho(B : \acute{N}\underline{\underline{A}}) \cap C_B(\Lambda)\}\rho(A^{\Lambda} : \acute{N}\underline{\underline{A}})$$
$$= \{\rho(B : \acute{N}\underline{\underline{A}}) \cap C_B(\Lambda)\}\rho(A : \acute{N}\underline{\underline{A}})^{\Lambda}$$

(3) If $\exists\ p \in P$ such that $A \in (\acute{N}\underline{\underline{A}})_p \setminus \underline{\underline{T}}$,

$$\rho(W : N\underline{\underline{A}}) = \rho(\ (\Lambda , B) : L\,\overline{\mathcal{K}}_p \cap \mathcal{J})A^{\Lambda}$$

(e) ZA - radical of W

(1) If $A = \{1\}$, $\rho(W : \acute{\underline{\underline{Z}}}) = \rho(B : \acute{\underline{\underline{Z}}})$

(2) If $A \not\in \bigcup_{p \in P} \acute{\underline{\underline{Z}}}_p$, then

$$\rho(W : \acute{\underline{\underline{Z}}}) = \{\rho(B : \acute{\underline{\underline{Z}}}) \cap C_B(\Lambda)\}\rho(A^{\Lambda} : \acute{\underline{\underline{Z}}})$$
$$= \{\rho(B : \acute{\underline{\underline{Z}}}) \cap C_B(\Lambda)\}\rho(A : \acute{\underline{\underline{Z}}})^{\Lambda}$$

(3) If $\exists\ p \in P$ such that $A \in \acute{\underline{\underline{Z}}}_p \setminus \underline{\underline{T}}$

$$\rho(W : \acute{\underline{\underline{Z}}}) = \rho(\ (\Lambda , B) : \mathcal{K}_p \cap \mathcal{Z}\ )$$

This result is valid for the standard case, but can also be written as follows.

Theorem 3.7.2 Let A and B be groups, and let $W = A \text{ wr } B$.

(a) Baer radical of W

(1) If $A = \{1\}$, $\rho(W : N\underline{\underline{A}}) = \rho(B : N\underline{\underline{A}})$

(2) If $A \notin \bigcup_{p \in P} \overline{\underline{\underline{S}}}_p$,

$$\rho(W : N\underline{\underline{A}}) = \rho(A^B : N\underline{\underline{A}}) = \rho(A : N\underline{\underline{A}})^B$$

(3) If $\exists \; p \in P$ such that $A \in \overline{\underline{\underline{S}}}_p \setminus \underline{\underline{T}}$,

$$\rho(W : N\underline{\underline{A}}) = \rho(B : (N\underline{\underline{A}})_p)A^B$$

(b) Fitting radical of W

(1) If $A = \{1\}$, $\rho(W : \underline{\underline{N}}) = \rho(B : \underline{\underline{N}})$

(2) If $A \notin \bigcup_{p \in P} \overline{\underline{\underline{N}}}_p$,

$$\rho(W : \underline{\underline{N}}) = \rho(A^B : \underline{\underline{N}}) = \rho(A : \underline{\underline{N}})^B$$

(3) If $\exists \; p \in P$ such that $A \in \overline{\underline{\underline{N}}}_p \setminus \underline{\underline{T}}$,

$$\rho(W : \underline{\underline{N}}) = \rho(B : \underline{\underline{F}}_p)A^B$$

(c) Hirsch-Plotkin radical of W

(1) If $A = \{1\}$, $\rho(W : L\underline{\underline{N}}) = \rho(B : L\underline{\underline{N}})$

(2) If $A \notin \bigcup_{p \in P} L\overline{\underline{\underline{N}}}_p$,

$$\rho(W : L\underline{\underline{N}}) = \rho(A^B : L\underline{\underline{N}}) = \rho(A : L\underline{\underline{N}})^B$$

(3) If $\exists \; p \in P$ such that $A \in L\overline{\underline{\underline{N}}}_p \setminus \underline{\underline{T}}$,

$$\rho(W : L\underline{\underline{N}}) = \rho(B : L\underline{\underline{F}}_p)A^B$$

(d) Gruenberg radical of W

(1) If $A = \{1\}$, $\rho(W : \acute{N}\underline{\underline{A}}) = \rho(B : \acute{N}\underline{\underline{A}})$

(2) If $A \notin \bigcup_{p \in P} (\acute{N}\underline{\underline{A}})_p$,

$$\rho(W : \acute{N}\underline{\underline{A}}) = \rho(A^B : \acute{N}\underline{\underline{A}}) = \rho(A : \acute{N}\underline{\underline{A}})^B$$

(3) If $\exists \; p \in P$ such that $A \in (\acute{N}\underline{\underline{A}})_p \setminus \underline{\underline{T}}$,

$$\rho(W : \acute{N}\underline{\underline{A}}) = \rho(B : (\acute{N}\underline{\underline{A}})_p)A^B$$

(e) ZA - radical of W

(1) If $A = \{1\}$, $\rho(W : \acute{\underline{\underline{Z}}}) = \rho(B : \acute{\underline{\underline{Z}}})$

(2) If $A \notin \bigcup_{p \in P} \acute{\underline{\underline{Z}}}_p$,

$$\rho(W : \acute{\underline{\underline{Z}}}) = \rho(A^B : \acute{\underline{\underline{Z}}}) = \rho(A : \acute{\underline{\underline{Z}}})^B$$

(3) If $\exists\ p \in P$ such that $A \in \acute{\underline{\underline{Z}}}_p \setminus \underline{\underline{T}}$

$$\rho(W : \acute{\underline{\underline{Z}}}) = \rho(B : \underline{\underline{F}}_p)A^B$$

<u>Note</u> Case (a) of Theorem 3.7.1 is very similar to work of B.I. Plotkin [16]; the results were, however, obtained independently. He proves in [16]

<u>Theorem</u> Let A be a group, let (Λ , B) be a faithful pair, and let $W = A \text{ wr}^{\Lambda} B$. Then

(1) If there exists prime p such that $A \in \underline{\underline{S}}_p$, then ~~$\rho(W : N\underline{\underline{A}}) \leq A^{\Lambda}$~~

~~(2) Otherwise,~~ $\rho(W : N\underline{\underline{A}}) = A^{\Lambda}\rho((\Lambda , B) : \mathcal{B}_p)$

(2) Otherwise, $\rho(W : N\underline{\underline{A}}) \leq A^{\Lambda}$

J.C. Lennox has obtained similar characterisations of the Fitting and Hirsch-Plotkin radicals in the standard case (unpublished).

Chapter 4 Residuals in wreath products

In this chapter we obtain some results on the $\underline{\underline{X}}$ - residual of $A \text{ wr}^{\Lambda} B$ for general classes $\underline{\underline{X}}$, and show that we may reduce the general case to some more specific cases.

We also characterise the residual for the special case $\underline{\underline{X}} = \langle Q, S, W_o, R_o \rangle \underline{\underline{X}}$, with some restrictions on A and $B/C_B(\Lambda)$.

Section 4.1 Reduction theorems

Let G be a group and let $\underline{\underline{X}}$ be a class of groups. Let $N \leqslant G$, and suppose $N \leqslant \rho_*(G : \underline{\underline{X}})$.

Then $N^G \leqslant \rho_*(G : \underline{\underline{X}})$ and hence

$$\begin{aligned}\rho_*(G/N^G) &= \cap\{M/N^G : M/N^G \triangle G/N^G, \{G/N^G\}/\{M/N^G\} \in \underline{\underline{X}}\} \\ &= \cap\{M/N^G : N^G \leqslant M \triangle G, G/M \in \underline{\underline{X}}\} \\ &= \rho_*(G)/N^G\end{aligned}$$

Hence to obtain $\rho_*(G)$ we need only look at G/N^G.

In particular we have

Lemma 4.1.1 Let G be a group and let $\underline{\underline{X}}$ be a class of groups. Let $H \leqslant G$. Then

(a) If $\underline{\underline{X}} = S\underline{\underline{X}}$, $\rho_*(H : X) \leqslant \rho_*(G : \underline{\underline{X}}) \cap H$

(b) If $\underline{\underline{X}} = S_n \underline{\underline{X}}$ and H sn G, $\rho_*(H : X) \leqslant \rho_*(G : \underline{\underline{X}}) \cap H$

Proof: Let $N \triangle G$, $G/N \in \underline{\underline{X}}$. Then $N \cap H \triangle H$ and $H/(N \cap H) \cong (HN)/N \leqslant G/N$.

If $\underline{\underline{X}} = S\underline{\underline{X}}$, $H/(N \cap H) \in S\underline{\underline{X}} = \underline{\underline{X}}$.

If $\underline{\underline{X}} = S_n \underline{\underline{X}}$, and H sn G, then $(HN)/N$ sn G/N and so $H/(N \cap H) \in S_n \underline{\underline{X}} = \underline{\underline{X}}$.

Hence in either case, $H/(N \cap H) \in \underline{\underline{X}}$; hence $\rho_*(H : X) \leqslant N \cap H$

N was any normal subgroup of G such that $G/N \in \underline{\underline{X}}$; hence

$$\rho_*(H : X) \leq \rho_*(G : X) \cap H$$

So we may factor out residuals of subgroups.

For wreath products in particular, we have the following reduction theorem

Theorem 4.1.2 Let A be a group, let (Λ , B) be a pair, and let $W = A \text{ wr}^{\Lambda} B$. Let $\underline{\underline{X}}$ be a class of groups.

(1) Suppose $\underline{\underline{X}} = S_n \underline{\underline{X}}$. Then $\rho_*(A : \underline{\underline{X}})^{\Lambda} \leq \rho_*(W : \underline{\underline{X}})$ and

$$W/\rho_*(A : X)^{\Lambda} \cong \{A/\rho_*(A : X)\} \text{ wr}^{\Lambda} B$$

(2) Suppose $\underline{\underline{X}} = QS\underline{\underline{X}}$, (Λ , B) is transitive, and $B/C_B(\Lambda) \notin \underline{\underline{X}}$. Then $(A')^{\Lambda} \leq \rho_*(W : \underline{\underline{X}})$ and

$$W/(A')^{\Lambda} \cong \{A/A'\} \text{ wr}^{\Lambda} B$$

(3) Suppose $\underline{\underline{X}} = S\underline{\underline{X}}$ and $A \in \underline{\underline{A}}$. Then $\rho_*(B : \underline{\underline{X}}) \leq \rho_*(W : \underline{\underline{X}})$ and

$$W/\{\rho_*(B : X)\}^W \cong A \text{ wr}^{\Lambda/\rho_*(B : X)} \{B/\rho_*(B : X)\}$$

i.e. in case (1) we need only consider $A \in R\underline{\underline{X}}$, in case (2) we need only consider $A \in \underline{\underline{A}}$, and in case (3) we need only consider $B \in R\underline{\underline{X}}$.

To prove the theorem we will need some preliminary results.

Lemma 4.1.3 Let M be some index set and let $\{ G_m : m \in M \}$ be a family of groups. Let $\underline{\underline{X}}$ be a group class such that $\underline{\underline{X}} = S_n \underline{\underline{X}}$. Then

$$\rho_*(\underset{m \in M}{\text{Dr}} G_m : \underline{\underline{X}}) = \underset{m \in M}{\text{Dr}} \rho_*(G_m : \underline{\underline{X}})$$

Proof: $\{ \underset{m \in M}{\text{Dr}} G_m \}/\{ \underset{m \in M}{\text{Dr}} \rho_*(G_m : \underline{\underline{X}}) \} \cong \underset{m \in M}{\text{Dr}} G_m/\rho_*(G_m : \underline{\underline{X}})$ under

$f \operatorname{Dr}_{m \in M} \rho_*(G_m : \underline{\underline{X}}) \mapsto f'$ where $f'(m) = f(m)\rho_*(G_m : \underline{\underline{X}})$ for all $m \in M$.

$\operatorname{Dr}_{m \in M} G_m/\rho_*(G_m : \underline{\underline{X}}) \in D(R\underline{\underline{X}}) = R\underline{\underline{X}}$; and so

$$\rho_*(\operatorname{Dr}_{m \in M} G_m : \underline{\underline{X}}) \leqslant \operatorname{Dr}_{m \in M} \rho_*(G_m : \underline{\underline{X}})$$

To prove the reverse inclusion we consider $\operatorname{Dr}_{m \in M} G_m$ as an internal direct product.

By Lemma 4.1.1, since $\underline{\underline{X}} = S_n \underline{\underline{X}}$, and $G_m \triangleleft \operatorname{Dr}_{m \in M} G_m$ for all $m \in M$,

$$\rho_*(G_m : \underline{\underline{X}}) \leqslant \rho_*(\operatorname{Dr}_{m \in M} G_m : \underline{\underline{X}}) \quad \text{for all } m \in M$$

$$\Rightarrow \quad \operatorname{Dr}_{m \in M} \rho_*(G_m : \underline{\underline{X}}) \leqslant \rho_*(\operatorname{Dr}_{m \in M} G_m : \underline{\underline{X}})$$

Hence $$\operatorname{Dr}_{m \in M} \rho_*(G_m : \underline{\underline{X}}) = \rho_*(\operatorname{Dr}_{m \in M} G_m : \underline{\underline{X}})$$

We may now prove Theorem 4.1.2 (1).

<u>Proof of Theorem 4.1.2 (1)</u>

Since $\underline{\underline{X}} = S_n \underline{\underline{X}}$, by Lemma 4.1.1 $\rho_*(A^\Lambda : \underline{\underline{X}}) \leqslant \rho_*(W : \underline{\underline{X}})$; therefore by Lemma 4.1.3,

$$\rho_*(A : \underline{\underline{X}})^\Lambda = \rho_*(A^\Lambda : \underline{\underline{X}}) \leqslant \rho_*(W : \underline{\underline{X}})$$

By Lemma 3.6.8,

$$W/\rho_*(A : X)^\Lambda \cong \{A/\rho_*(A : X)\} \operatorname{wr}^\Lambda B$$

Hence we have the result.

<u>Lemma 4.1.4</u> Let A be a group, let (Λ, B) be a pair, and let $W = A \operatorname{wr}^\Lambda B$. Let $\underline{\underline{X}}$ be a class of groups such that $\underline{\underline{X}} = QS\underline{\underline{X}}$, and suppose that $B/C_B(\Lambda) \notin \underline{\underline{X}}$. Let $N \triangleleft W$, with $W/N \in \underline{\underline{X}}$. Then $N \cap B \not\leqslant C_B(\Lambda)$.

Proof Suppose the contrary. Then $N \cap B \trianglelefteq C_B(\Lambda)$ and $B/(N \cap B) \cong (BN)/N \leqslant W/N \in \underline{\underline{X}}$; so $B/(N \cap B) \in S\underline{\underline{X}} = \underline{\underline{X}}$. Then $B/C_B(\Lambda) \cong \{B/N \cap B\}/\{C_B(\Lambda)/(N \cap B)\} \in Q\underline{\underline{X}} = \underline{\underline{X}}$, which is a contradiction. Hence $N \cap B \not\leqslant C_B(\Lambda)$.

Proof of Theorem 4.1.2 (2)

Suppose $\underline{\underline{X}} = QS\underline{\underline{X}}$, (Λ , B) is transitive, and $B/C_B(\Lambda) \not\leqslant \underline{\underline{X}}$. Then using Lemma 3.2.4 and Lemma 4.1.4, we see that

$$N \trianglelefteq W,\ W/N \in \underline{\underline{X}} \Rightarrow (A')^{\Lambda} \leqslant N$$

and so
$$(A')^{\Lambda} \leqslant \rho_*(W : \underline{\underline{X}}).$$

By Lemma 3.6.8,

$$W/(A')^{\Lambda} \cong \{A/A'\} \text{ wr}^{\Lambda} B \ ;$$

hence we have Theorem 4.1.2 (2).

In order to be able to factor out subgroups of B, we need some further properties of pairs.

Definition Let (Λ , B) be a pair and let $D \leqslant B$. Define Λ/D to be $\{ \lambda D : \lambda \in \Lambda \}$, i.e. the set of D - orbits of Λ .

Lemma 4.1.5 Let (Λ , B) be a pair and let $D \trianglelefteq B$. Then

(a) $(\Lambda , B)/D = (\Lambda/D , B/D)$ is a pair with action

$$(\lambda D)bD = \lambda bD \quad \text{for all } \lambda \in \Lambda \text{ and } b \in B$$

(b) $(\Lambda/D , B)$ is a pair with action

$$(\lambda D)b = \lambda bD \quad \text{for all } \lambda \in \Lambda \text{ and } b \in B$$

Proof: Note firstly that

$$\lambda D = \mu D \iff \lambda, \mu \text{ lie in the same D - orbit}$$
$$\iff \exists\ d \in D \text{ such that } \lambda d = \mu$$

(a) Let $b, \beta \in B$ and $\lambda, \mu \in \Lambda$ be such that $bD = \beta D$ and

$\lambda D = \mu D$. Then there exists $d_1, d_2 \in D$ such that $b = \beta d_1$ and $\lambda = \mu d_2$. Then

$(\lambda b)D = (\mu d_2 \, \beta d_1)D = (\mu\beta \, d_2^{\beta} d_1)D = (\mu\beta)D$ since $d_2^{\beta} d_1 \in D$.

Hence the action of B/D on Λ/D is well defined.

To show that bD is a permutation, note that $1D$ is the identity permutation on Λ/D, and that $(bD)(b^{-1}D) = 1D$ for all $b \in B$.

Finally, let $\lambda \in \Lambda$ and $b_1, b_2 \in B$. Then

$$(\lambda D)\{(b_1D)(b_2D)\} = (\lambda D)(b_1b_2D) = (\lambda b_1 b_2)D = (\lambda b_1 D)(b_2 D)$$
$$= \{(\lambda D)(b_1 D)\}(b_2 D)$$

(b) Let $\lambda, \mu \in \Lambda$ be such that $\lambda D = \mu D$. Then there exists $d \in D$ such that $\lambda = \mu d$; hence for all $b \in B$,

$(\lambda b)D = (\mu d b)D = (\mu b d^{b})D = (\mu b)D$ since $d^{b} \in D$.

Hence the action of B on Λ/D is well defined.

The proof that b acts as a permutation for all $b \in B$ is similar to that for part (a).

Finally, let $\lambda \in \Lambda$, $b_1, b_2 \in B$. Then

$(\lambda D)(b_1b_2) = \lambda b_1 b_2 D = (\lambda b_1 D)b_2 = \{(\lambda D)b_1\}b_2$

Hence we have the result.

<u>Lemma 4.1.6</u> Let (Λ, B) be a pair and let $D \triangleleft B$. Then

$$(\Lambda/D, B)/D = (\Lambda/D, B/D)$$

If $H \leqslant D$ and is a normal subgroup of B, then

$$\{(\Lambda, B)/H\}/\{D/H\} \cong (\Lambda, B)/D$$

<u>Proof:</u> $(\Lambda/D)/D = \{\alpha D : \alpha \in \Lambda/D\}$

$$= \{(\lambda D)D : \lambda \in \Lambda\}$$

$$= \{\lambda D : \lambda \in \Lambda\} = \Lambda/D$$

and $(\lambda D)DbD = (\lambda Db)D = (\lambda bD)D = \lambda bD = \lambda DbD$ for all $\lambda \in \Lambda$ and $b \in B$, i.e. the action is correct.

Hence $(\Lambda/D, B)/D = (\Lambda, B)/D$

$\{(\Lambda, B)/H\}/\{D/H\} = (\{\Lambda/H\}/\{D/H\}, \{B/H\}/\{D/H\})$

Now $\{\Lambda/H\}/\{D/H\} = \{ \alpha(D/H) : \alpha \in \Lambda/H \}$

$= \{ \lambda H(D/H) : \lambda \in \Lambda \}$

$= \{ \lambda D : \lambda \in \Lambda \}$

for $\mu \in \lambda D \Rightarrow \mu = \lambda d$ for some $d \in D$

$\Rightarrow \mu H = \lambda HdH$

$\Rightarrow \mu H \in \lambda H(D/H)$

and $\mu H \in \lambda H(D/H) \Rightarrow \mu H = \lambda HdH$ for some $d \in D$

$\Rightarrow \mu = \lambda dh$ for some $h \in H$

and $\lambda dh \in \lambda D$ since $H \leq D$.

So $\lambda D = \lambda H(D/H)$ for all $\lambda \in \Lambda$.

Also, for all $\lambda \in \Lambda$ and $b \in B$,

$(\lambda H(D/H))bH(D/H) = (\lambda HbH)(D/H)$

$= (\lambda bH)(D/H)$

$= \lambda bD$

$= \lambda DbD$

so the action is correct. Hence $\{(\Lambda, B)/H\}/\{D/H\} \cong (\Lambda, B)/D$ as permutation groups, via $\lambda D \mapsto \lambda D$ for all $\lambda \in \Lambda$ and $(bH)(D/H) \mapsto bD$ for all $b \in B$.

Let Ω be a set and let G be a permutation group on Ω. Let $\Omega' \subseteq \Omega$, and define

$$S_G(\Omega') = \{ g \in G : \Omega' g = \Omega' \}$$

Lemma 4.1.7 Let (Λ, B) be a pair, and let $D \Delta B$. Then

(a) (Λ, B) is transitive <=> $(\Lambda, B)/D$ is transitive

(b) If (Λ, B) is the right regular representation of B, then $(\Lambda, B)/D$ is the right regular representation of B/D.

(c) $C_{\{B/D\}}(\Lambda/D) = \{\bigcap_{\Sigma \text{ a } D\text{-orbit}} S_B(\Sigma)\}/D$

Proof: (a) Let (Λ, B) be transitive. Let $\lambda D, \mu D \in \Lambda/D$. Then there exists $b \in B$ such that $\lambda b = \mu$; so $\lambda DbD = \mu D$. Hence $(\Lambda, B)/D$ is transitive.

Let $(\Lambda, B)/D$ be transitive. Let $\lambda, \mu \in \Lambda$. Then there exists $b \in B$ such that $\lambda DbD = \mu D$, which implies that there exists $d \in D$ such that $\lambda bd = \mu$. Hence (Λ, B) is transitive.

(b) Let (Λ, B) be the right regular representation of B. Then $\Lambda = B$ and so $\Lambda/D = \{ \lambda D : \lambda \in \Lambda \} = \{ bD : b \in B \} = B/D$ and B/D acts by right multiplication as required.

(c) Let $b \in \bigcap_{\Sigma \text{ a } D\text{-orbit}} S_B(\Sigma)$. Then

$$\lambda DbD = (\lambda b)D = (\lambda D)b = \lambda D \quad \text{for all } \lambda \in \Lambda$$

$$\Rightarrow \quad bD \in C_{\{B/D\}}(\Lambda/D)$$

So $$\{\bigcap_{\Sigma \text{ a } D\text{-orbit}} S_B(\Sigma)\}/D \leq C_{\{B/D\}}(\Lambda/D)$$

Let $bD \in C_{\{B/D\}}(\Lambda/D)$. Then $\lambda bD = \lambda D$ for all $\lambda \in \Lambda$, and so λb and λ lie in the same D-orbit for all $\lambda \in \Lambda$; hence $b \in \bigcap_{\Sigma \text{ a } D\text{-orbit}} S_B(\Sigma)$. Hence $C_{\{B/D\}}(\Lambda/D) \leq \{\bigcap_{\Sigma \text{ a } D\text{-orbit}} S_B(\Sigma)\}/D$

Hence result.

Lemma 4.1.8 [cf [4], standard case] Let A be an abelian group, let (Λ, B) be a pair, and let $W = A \text{ wr}^{\Lambda} B$. Let $D \Delta B$. Then $W/D^W = \{A \text{ wr}^{\Lambda} B\}/\{[A^{\Lambda}, D]D\} \cong A \text{ wr}^{\Lambda/D} \{B/D\}$ and $W/[A^{\Lambda}, D] \cong A \text{ wr}^{\Lambda/D} B$.

Proof: $D^W = [W,D]D = \langle [A^\Lambda,D]^B, [B,D] \rangle D = [A^\Lambda,D]D$.

Hence $[A^\Lambda,D] = D^W \cap A^\Lambda \Delta W$ also.

Let b* be the image of $b \in B$ under the natural homomorphism from B onto B/D, and extend * to a map from W to $A \text{ wr}^{\Lambda/D} \{B/D\}$ by

$$bf \mapsto b^*f^* \quad \text{for all } b \in B \text{ and all } f \in A^\Lambda$$

where $f^* : \Lambda/D \to A$ is given by

$$f^*(\lambda D) = \prod_{\mu \in \lambda D} f(\mu) \qquad \forall \; \lambda \in \Lambda$$

f* is well defined since $A \in \underline{\underline{A}}$ and only a finite number of the $f(\mu)$ are not 1.

* is a homomorphism:

Let $b, c \in B$ and $f, g \in A^\Lambda$. Then

$$(f^c)^*(\lambda D) = \prod_{\mu \in \lambda D} f^c(\mu) \qquad \forall \; \lambda \in \Lambda$$

$$= \prod_{\mu \in \lambda D} f(\mu c^{-1}) \qquad \forall \; \lambda \in \Lambda$$

and

$$(f^*)^{cD}(\lambda D) = f^*(\lambda c^{-1} D) \qquad \forall \; \lambda \in \Lambda$$

$$= \prod_{\mu \in \lambda c^{-1} D} f(\mu) \qquad \forall \; \lambda \in \Lambda$$

Hence since $c^{-1}D = Dc^{-1}$,

$$(f^c)^* = (f^*)^{cD}$$

Also

$$(fg)^*(\lambda D) = \prod_{\mu \in \lambda D} fg(\mu) \qquad \forall \; \lambda \in \Lambda$$

$$= \prod_{\mu \in \lambda D} f(\mu) \prod_{\mu \in \lambda D} g(\mu) \qquad \forall \; \lambda \in \Lambda$$

since $A \in \underline{\underline{A}}$

$$= f^*g^*(\lambda D) \qquad \forall \; \lambda \in \Lambda$$

So

$$(fg)^* = f^*g^*$$

Hence

$$(bfcg)^* = (bc)^*(f^c g)^*$$

$$= b^*c^*(f^*)^{c^*} g^*$$

$$= (bf)^*(cg)^*$$

Hence * is a homomorphism.

$*$ is an epimorphism:

Let $b^*f \in A \text{ wr}^{\Lambda/D} \{B/D\}$. Let Δ be a set of D-orbit representatives.

Define $g : \Lambda \to A$ by

$$g(\delta) = f(\delta D) \qquad \forall\ \delta \in \Delta$$

$$g(\lambda) = 1 \qquad \forall\ \lambda \in \Lambda \setminus \Delta$$

Then

$$g^*(\delta D) = \prod_{\mu \in \delta D} g(\mu) \qquad \forall\ \delta \in \Delta$$

$$= g(\delta) \qquad \forall\ \delta \in \Delta$$

$$= f(\delta D) \qquad \forall\ \delta \in \Delta$$

Hence $\qquad g^* = f$

and so $\qquad (bg)^* = b^*g^* = b^*f$

Hence $*$ is an epimorphism.

Ker $* = [A^\Lambda, D]D$:

If $d \in D$ then $d^* = D$, and so $D \leqslant \text{Ker } *$; hence

$$[A^\Lambda, D]D = D^W \leqslant \text{Ker } *$$

Now let $bf \in \text{Ker } *$. Then $(bf)^* = b^*f^* = 1$, and so $b^* = 1 = f^*$; hence $b \in D$ and $f^* = 1$.

We prove that for any $f \in A^\Lambda$, $f^* = 1 \Rightarrow f \in [A^\Lambda, D]$. Let Δ be a set of D-orbit representatives. For $\lambda \in \sigma(f)$ let $d_\lambda \in D$ be such that $\lambda d_\lambda \in \Delta$.

If $\lambda d_\lambda = \lambda$, $[f_\lambda, d_\lambda] = 1$. ($f_\lambda$ is the λ-component of f)

Otherwise, $\sigma([f_\lambda, d_\lambda]) = \{\lambda, \lambda d_\lambda\}$.

Let $\quad g = f \prod_{\lambda \in \sigma(f)} [f_\lambda, d_\lambda]$.

Then $\sigma(g) \subseteq \Delta$; for

$$\sigma(g) \subseteq \sigma(f) \cup \bigcup_{\lambda \in \sigma(f)} \{\lambda, \lambda d_\lambda\}$$

$$= \sigma(f) \cup \bigcup_{\lambda \in \sigma(f)} \{\lambda d_\lambda\}$$

Let $\lambda \in \sigma(f) \setminus \Delta$. Then $\lambda \neq \lambda d_\lambda$ and $f(\lambda)[f_\lambda, d_\lambda](\lambda) = 1$

Let $\mu \in \sigma(f)$, $\mu \neq \lambda$. Then

$$[f_\mu, d_\mu](\lambda) = f_\mu(\lambda)^{-1} f_\mu(\lambda d_\mu^{-1})$$
$$= f_\mu(\lambda d_\mu^{-1})$$

If $\lambda d_\mu^{-1} = \mu$ then $\lambda = \mu d_\mu \in \Delta$, which is a contradiction.
So $\lambda d_\mu^{-1} \neq \mu$, and $[f_\mu, d_\mu](\lambda) = 1$.

Hence $$g(\lambda) = f(\lambda)[f_\lambda, d_\lambda](\lambda) \prod_{\mu \in \sigma(f) \setminus \{\lambda\}} [f_\mu, d_\mu](\lambda)$$
$$= 1$$

Hence $\sigma(g) \subseteq \bigcup_{\lambda \in \sigma(f)} \{\lambda d_\lambda\} \not\subseteq \Delta = \Delta$.

Also, $g^* = 1$; for $g^* = f^* \prod_{\lambda \in \sigma(f)} [f_\lambda, d_\lambda]^*$

$f^* = 1$ by hypothesis.

Let $\lambda \in \sigma(f)$.

(a) If $\lambda = \lambda d_\lambda$, $[f_\lambda, d_\lambda] = 1$, and so $[f_\lambda, d_\lambda]^* = 1$.

(b) Otherwise,

$$[f_\lambda, d_\lambda]^*(\mu D) = 1 \quad \text{if} \quad \mu D \neq \lambda D$$

$$[f_\lambda, d_\lambda]^*(\lambda D) = [f_\lambda, d_\lambda](\lambda)[f_\lambda, d_\lambda](\lambda d_\lambda)$$

$$= 1$$

So $[f_\lambda, d_\lambda]^* = 1$ for all $\lambda \in \sigma(f)$.

Hence $g^* = 1$.

Therefore $g = 1$ and so

$$f = \prod_{\lambda \in \sigma(f)} [f_\lambda, d_\lambda]^{-1} \in [A^\Lambda, D]$$

Hence $\operatorname{Ker} * = [A^\Lambda, D]D$, and

$$W/\{[A^\Lambda, D]D\} \cong A \text{ wr}^{\Lambda/D} \{B/D\}.$$

To show that

$$W/[A^\Lambda, D] \cong A \text{ wr}^{\Lambda/D} B,$$

define $\theta : W \to A \text{ wr}^{\Lambda/D} B$ by $(bf)\theta = bf^*$.

Then θ is well defined since * is.

θ is a homomorphism since if $f \in A^{\Lambda}$, $c \in B$,

$$(f^c)^*(\lambda D) = \prod_{\mu \in \lambda D} f(\mu c^{-1}) = \prod_{\mu \in \lambda c^{-1} D} f(\mu) = f^*(\lambda c^{-1} D) = (f^*)^c(\lambda D)$$

for all $\lambda \in \Lambda$.

θ is an epimorphism, for if $bg \in A \text{ wr}^{\Lambda/D} B$, by the first part of the proof there exists $f \in A \text{ wr}^{\Lambda} B$ such that $f^* = g$; then $bf \in A \text{ wr}^{\Lambda} B$ and $(bf)\theta = bg$.

Ker $\theta = [A^{\Lambda}, D]$: for let $bf \in W$. Then

$(bf)\theta = 1 \iff bf^* = 1 \iff b = 1 = f^* \iff b = 1$ and $f \in [A^{\Lambda}, D]$

Hence result.

<u>Note</u> For the standard case, the first part of the lemma gives $\{A \text{ wr } B\}/\{[A^B, D]D\} \cong A \text{ wr } \{B/D\}$ by Lemma 4.1.7 (b).

We may now complete the proof of Theorem 4.1.2.

<u>Proof of Theorem 4.1.2 (3)</u>

Suppose $\underline{\underline{X}} = S\underline{\underline{X}}$ and A is abelian. Then by Lemma 4.1.1

$$\rho_*(B : \underline{\underline{X}}) \leq \rho_*(W : \underline{\underline{X}})$$

$$\Rightarrow \quad [A^{\Lambda}, \rho_*(B : \underline{\underline{X}})]\rho_*(B : \underline{\underline{X}}) \leq \rho_*(W : \underline{\underline{X}})$$

and then by Lemma 4.1.8,

$$W/\{[A^{\Lambda}, \rho_*(B : X)]\rho_*(B : X)\} \cong A \text{ wr}^{\Lambda/\rho_*(B : X)} \{B/\rho_*(B : X)\}$$

So we have the result.

Theorem 4.1.2 is now proved.

Section 4.2 The case $\underline{\underline{X}} = W_0 \underline{\underline{X}}$

Let $\underline{\underline{X}}$ be a class of groups and define the closure operation W_0 by

$\underline{\underline{X}} = W_0 \underline{\underline{X}}$ <=> whenever $A \in \underline{\underline{X}}$, $B \in \underline{\underline{X}}$, and (Λ , B) is a pair,

then $W = A \text{ wr}^{\Lambda} B \in \underline{\underline{X}}$

Note that $W_0 \leq \langle D_p , P \rangle$; and if $\underline{\underline{X}} = S\underline{\underline{X}}$ also, then

$$A \text{ wr}^{\Lambda} B \in \underline{\underline{X}} \iff A \in \underline{\underline{X}} \text{ and } B \in \underline{\underline{X}}$$

An example of such a class is the class of soluble groups.

We require some properties of a special type of pair, namely a quasi-regular pair.

Definition: A pair (Λ , B) is quasi-regular if and only if given $\lambda, \mu \in \Lambda$ there exist only finitely many $b \in B$ such that $\lambda b = \mu$.

Let $\mathfrak{X}$ be a class of pairs. Then define a class $\underline{\underline{X}}(\mathfrak{X})$ of groups to be all groups B for which there exists a pair (Λ , B) in $\mathfrak{X}$.

Similarly, let $\underline{\underline{X}}$ be a class of groups. Then define a class $\mathfrak{X}(\underline{\underline{X}})$ of pairs to be all pairs (Λ , B) with $B \in \underline{\underline{X}}$.

Then if $\mathfrak{X}$ is any class of pairs, $\mathfrak{X} \subseteq \mathfrak{X}(\underline{\underline{X}}(\mathfrak{X}))$; and if $\underline{\underline{X}}$ is any class of groups, $\underline{\underline{X}}(\mathfrak{X}(\underline{\underline{X}})) \subseteq \underline{\underline{X}}$.

Now let $\underline{\underline{X}}$ be a class of groups and let $\mathfrak{X} = \mathfrak{X}(\underline{\underline{X}})$. Then we make the following definition.

Definition:

$(\Lambda , B) \in R\mathfrak{X} \iff$ (a) $B \in R\underline{\underline{X}}$ and

(b) given $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$, there exists $K \triangle B$, $B/K \in \underline{\underline{X}}$ such that $\mu \notin \lambda K$

Then we have

Lemma 4.2.1 Let $\underline{\underline{X}}$ be a class of groups such that $\underline{\underline{X}} = R_0 \underline{\underline{X}}$, and let (Λ, B) be a pair such that $(\Lambda, B)/\rho_*(B : \underline{\underline{X}})$ is quasi-regular. Let $\mathcal{X} = \mathcal{X}(\underline{\underline{X}})$. Then

$$(\Lambda, B)/\rho_*(B : \underline{\underline{X}}) \in R\mathcal{X}$$

Proof: Clearly $B/\rho_*(B : \underline{\underline{X}}) \in R\underline{\underline{X}}$.

We require to show that given $\lambda, \mu \in \Lambda$ such that $\lambda\rho_*(B : \underline{\underline{X}}) \neq \mu\rho_*(B : \underline{\underline{X}})$, there exists $K \Delta B$ with $\rho_*(B : \underline{\underline{X}}) \leq K$ and $B/K \in \underline{\underline{X}}$ such that

$$\mu\rho_*(B : \underline{\underline{X}}) \notin \lambda\rho_*(B : \underline{\underline{X}})\{K/\rho_*(B : \underline{\underline{X}})\}$$

Let $\{b_1\rho_*(B : \underline{\underline{X}}), \ldots, b_m\rho_*(B : \underline{\underline{X}})\}$ be those permutations such that $\lambda b_i\rho_*(B : \underline{\underline{X}}) = \mu\rho_*(B : \underline{\underline{X}})$. Then for $1 \leq i \leq m$, there exists $K_i \Delta B$ such that $B/K_i \in \underline{\underline{X}}$ and $\rho_*(B : \underline{\underline{X}}) \leq K_i$, with $b_i\rho_*(B : \underline{\underline{X}}) \notin K_i/\rho_*(B : \underline{\underline{X}})$.

Let $K = \bigcap_{1 \leq i \leq n} K_i$. Then $K \Delta B$, $\rho_*(B : \underline{\underline{X}}) \leq K$, and $B/K \in R_0 \underline{\underline{X}} = \underline{\underline{X}}$.

Suppose $\mu\rho_*(B : \underline{\underline{X}}) \in \lambda\rho_*(B : \underline{\underline{X}})\{K/\rho_*(B : \underline{\underline{X}})\}$. Then there exists $k \in K$ such that $\mu\rho_*(B : \underline{\underline{X}}) = \lambda k\rho_*(B : \underline{\underline{X}})$

$\Rightarrow$ there exists $i \in \{1, \ldots, m\}$ such that $k\rho_*(B : \underline{\underline{X}}) = b_i\rho_*(B : \underline{\underline{X}})$

$\Rightarrow$ there exists $i \in \{1, \ldots, m\}$ such that $b_i\rho_*(B : \underline{\underline{X}}) \in K_i/\rho_*(B : \underline{\underline{X}})$, which is a contradiction.

Hence $\mu\rho_*(B : \underline{\underline{X}}) \notin \lambda\rho_*(B : \underline{\underline{X}})\{K/\rho_*(B : \underline{\underline{X}})\}$, and we have the result.

The following is a generalisation of Lemma 9 from [8]; there it is proved for the standard case.

Let $\underline{\underline{X}}$ be a class of groups, and let (Λ, B) be a pair. Then the class $\underline{\underline{X}}\ \mathrm{wr}^{\Lambda} B$ is the class of all groups of the form

$A \text{ wr}^{\Lambda} B$ with $A \in \underline{\underline{X}}$.

Lemma 4.2.2 (i) Let $\underline{\underline{X}}$ be a class of groups. Let A be a group and let (Λ, B) be a pair; suppose $A \in R\underline{\underline{X}}$. Then

$$A \text{ wr}^{\Lambda} B \in R(\underline{\underline{X}} \text{ wr}^{\Lambda} B)$$

(ii) Let $\underline{\underline{X}}$ be a class of groups and let $\underline{\underline{Y}}$ be a class of groups such that $\underline{\underline{Y}} = R_0 \underline{\underline{Y}}$. Let $A \in R\underline{\underline{X}} \cap \underline{\underline{A}}$ and $(\Lambda, B) \in R\mathcal{Y}$ where $\mathcal{Y} = \mathcal{H}(\underline{\underline{Y}})$. Then given $1 \neq w \in W = A \text{ wr}^{\Lambda} B$, there exists $N \triangle W$ such that $w \notin N$ and $W/N \cong \underline{A} \text{ wr}^{\Sigma} \underline{B}$ for some $\underline{A} = \underline{A}(w) \in \underline{\underline{X}}$, $\underline{B} = \underline{B}(w) \in \underline{\underline{Y}}$, and pair $(\Sigma, \underline{B}) \in \mathcal{Y}$.

Proof: (i) Let $1 \neq bf \in W = A \text{ wr}^{\Lambda} B$. We require to find $K \triangle W$ such that $bf \notin K$ and $W/K \cong G \text{ wr}^{\Lambda} B$ for some $G \in \underline{\underline{X}}$. If $b \neq 1$, take $K = A^{\Lambda}$; for $bf \notin A^{\Lambda}$, and $W/A^{\Lambda} \cong B \cong \{1\} \text{ wr}^{\Lambda} B$.

Suppose $b = 1$. Then $f \neq 1$ and so there exists $\lambda \in \Lambda$ and $1 \neq a \in A$ such that $f(\lambda) = a$. Hence there exists $H \triangle A$ such that $a \notin H$ and $A/H \in \underline{\underline{X}}$. Then $H^{\Lambda} \triangle W$, $f \notin H^{\Lambda}$, and $W/H^{\Lambda} \cong \{A/H\} \text{ wr}^{\Lambda} B$, by Lemma 3.6.8 ; so take $K = H^{\Lambda}$.

(ii) By part (i), we may suppose $A \in \underline{\underline{X}} \cap \underline{\underline{A}}$. Let $1 \neq bf \in W$. If $b \neq 1$, there exists $K \triangle B$, $B/K \in \underline{\underline{Y}}$ such that $b \notin K$; so $b \notin K[A^{\Lambda},K]$ and $W/\{K[A^{\Lambda},K]\} \cong A \text{ wr}^{\Lambda/K} \{B/K\}$ by Lemma 4.1.8. So take $N = K[A^{\Lambda},K]$, $\Sigma = \Lambda/K$, $\underline{B} = B/K$.

Suppose $b = 1$. Then $f \neq 1$; let $\sigma(f) = \{\lambda_1, \dots, \lambda_n\}$. If $n = 1$, let $K = B$. If $n \geq 2$,

$(\Lambda, B) \in R\mathcal{Y} \Rightarrow$ there exists $K_2, \dots, K_n \triangle B$ such that $B/K_i \in \underline{\underline{Y}}$ and $\lambda_i \notin \lambda_1 K_i$ for $2 \leq i \leq n$

Let $K = \bigcap_{2 \leq i \leq n} K_i$. Then in either case, $K \triangle B$ and $B/K \in R_0 \underline{\underline{Y}} = \underline{\underline{Y}}$. Further, $\lambda_i \notin \lambda_1 K$ for $2 \leq i \leq n$, since $K \leq K_i$ for $1 \leq i \leq n$. Hence

$$\prod_{\mu \in \lambda_1 K} f(\mu) = f(\lambda_1) \neq 1$$

i.e. $f \notin K[A^{\Lambda}, K]$

Also $W/\{K[A^{\Lambda},K]\} \cong A \text{ wr}^{\Lambda/K} \{B/K\}$ by Lemma 4.1.8.

So take $N = K[A^{\Lambda},K]$, $\Sigma = \Lambda/K$, $\underline{B} = B/K$.

Hence result.

Now consider the case $\underline{\underline{X}} = W_0 \underline{\underline{X}}$.

Lemma 4.2.3 Let $\underline{\underline{X}}$ be a class of groups such that $\underline{\underline{X}} = \langle S, W_0 \rangle \underline{\underline{X}}$. Let A be a group and let (Λ, B) be a pair such that $B \in \underline{\underline{X}}$. Let $W = A \text{ wr}^{\Lambda} B$. Then $\rho_*(W : \underline{\underline{X}}) = \rho_*(A : \underline{\underline{X}})^{\Lambda}$.

Proof: Let $G \triangleleft A$, with $A/G \in \underline{\underline{X}}$. Then by Lemma 3.6.8, $W/G^{\Lambda} \cong \{A/G\} \text{ wr}^{\Lambda} B \in \underline{\underline{X}}$ since $\underline{\underline{X}} = W_0 \underline{\underline{X}}$.

Hence $\rho_*(W : \underline{\underline{X}}) \leq G^{\Lambda}$ for all such G ; and so

$$\rho_*(W : \underline{\underline{X}}) \leq \cap\{ G^{\Lambda} : G \triangleleft A, A/G \in \underline{\underline{X}} \}$$
$$= \rho_*(A : \underline{\underline{X}})^{\Lambda}$$

$\underline{\underline{X}} = S\underline{\underline{X}} \Rightarrow \rho_*(A : \underline{\underline{X}})^{\Lambda} \leq \rho_*(W : \underline{\underline{X}})$ by Theorem 4.1.2 (1)

Hence $\rho_*(W : \underline{\underline{X}}) = \rho_*(A : \underline{\underline{X}})^{\Lambda}$.

Lemma 4.2.4 Let $\underline{\underline{X}}$ be a class of groups such that $\underline{\underline{X}} = \langle S, W_0, R_0 \rangle \underline{\underline{X}}$, and let $\mathfrak{X} = \mathfrak{X}(\underline{\underline{X}})$. Let $A \in \underline{\underline{A}} \cap R\underline{\underline{X}}$ and $(\Lambda, B) \in R\mathfrak{X}$; let $W = A \text{ wr}^{\Lambda} B$. Then $W \in R\underline{\underline{X}}$.

Proof: By Lemma 4.2.2 (ii), given $1 \neq w \in W$, there exists $N \triangleleft W$ such that $w \notin N$ and $W/N \cong \underline{A} \text{ wr}^{\Sigma} \underline{B}$ where $\underline{A} \in \underline{\underline{X}}$ and $\underline{B} \in \underline{\underline{X}}$; then since $\underline{\underline{X}} = W_0 \underline{\underline{X}}$, $\underline{A} \text{ wr}^{\Sigma} \underline{B} \in \underline{\underline{X}}$. Hence $W \in R\underline{\underline{X}}$.

Hence we have

Proposition 4.2.5 Let $\underline{\underline{X}}$ be a class of groups such that

$\underline{\underline{X}} = \langle S, W_0, Q, R_0 \rangle \underline{\underline{X}}$, and let $\mathfrak{X} = \mathfrak{X}(\underline{\underline{X}})$. Let A be a group and let (Λ, B) be a transitive pair such that $(\Lambda, B)/\rho_*(B : \underline{\underline{X}})$ is quasi-regular. Define $\overline{\rho_*(A : \underline{\underline{X}})}^{\Lambda}$ by

$$\overline{\rho_*(A : \underline{\underline{X}})}^{\Lambda}/(A')^{\Lambda} = \rho_*(A/A' : \underline{\underline{X}})^{\Lambda}$$

Then if $A \in \underline{\underline{A}}$ or $B/C_B(\Lambda) \notin \underline{\underline{X}}$,

$$\rho_*(W : \underline{\underline{X}}) = \rho_*(B : \underline{\underline{X}})[A^{\Lambda}, \rho_*(B : \underline{\underline{X}})]\overline{\rho_*(A : \underline{\underline{X}})}^{\Lambda}(A')^{\Lambda}$$

Proof: If $A \in \underline{\underline{A}}$ then clearly $(A')^{\Lambda} \leq \rho_*(W : \underline{\underline{X}})$; and if $B/C_B(\Lambda) \notin \underline{\underline{X}}$, by Theorem 4.1.2 (2), $(A')^{\Lambda} \leq \rho_*(W : \underline{\underline{X}})$; so in either case

$$\rho_*(W : \underline{\underline{X}})/(A')^{\Lambda} \cong \rho_*(\{A/A'\} \text{ wr}^{\Lambda} B : \underline{\underline{X}})$$

So let $\underline{A} = A/A'$. Then since $\underline{\underline{X}} = S\underline{\underline{X}}$, by Theorem 4.1.2 (1)

$$\rho_*(\underline{A} : \underline{\underline{X}})^{\Lambda} \leq \rho_*(\underline{A} \text{ wr}^{\Lambda} B : \underline{\underline{X}})$$

and

$$\rho_*(\underline{A} \text{ wr}^{\Lambda} B : \underline{\underline{X}})/\rho_*(\underline{A} : \underline{\underline{X}})^{\Lambda} \cong \rho_*(\{\underline{A}/\rho_*(\underline{A} : \underline{\underline{X}})\} \text{ wr}^{\Lambda} B : \underline{\underline{X}})$$

Let $G = \underline{A}/\rho_*(\underline{A} : \underline{\underline{X}})$. $G \in \underline{\underline{A}}$; so by Theorem 4.1.2 (3)

$$\rho_*(B : \underline{\underline{X}})[G^{\Lambda}, \rho_*(B : \underline{\underline{X}})] \leq \rho_*(G \text{ wr}^{\Lambda} B : \underline{\underline{X}})$$

and

$$\rho_*(G \text{ wr}^{\Lambda} B : \underline{\underline{X}})/\{\rho_*(B : \underline{\underline{X}})[G^{\Lambda}, \rho_*(B : \underline{\underline{X}})]\}$$
$$\cong \rho_*(G \text{ wr}^{\Lambda/\rho_*(B : \underline{\underline{X}})} \{B/\rho_*(B : \underline{\underline{X}})\} : \underline{\underline{X}})$$

Since $(\Lambda, B)/\rho_*(B : \underline{\underline{X}})$ is quasi-regular, $(\Lambda, B)/\rho_*(B : \underline{\underline{X}}) \in R\mathfrak{X}$ by Lemma 4.2.1; hence by Lemma 4.2.4, since $G \in \underline{\underline{A}} \cap R\underline{\underline{X}}$,

$$\rho_*(G \text{ wr}^{\Lambda/\rho_*(B : \underline{\underline{X}})} \{B/\rho_*(B : X)\} : \underline{\underline{X}}) = \{1\}$$

Hence $\rho_*(G \text{ wr}^{\Lambda} B : \underline{\underline{X}}) = \rho_*(B : \underline{\underline{X}})[G^{\Lambda}, \rho_*(B : \underline{\underline{X}})]$

$$\Rightarrow \rho_*(\{A/A'\} \text{ wr}^{\Lambda} B : \underline{\underline{X}}) = \rho_*(B : \underline{\underline{X}})[\{A/A'\}^{\Lambda}, \rho_*(B : \underline{\underline{X}})]\rho_*(A/A' : \underline{\underline{X}})^{\Lambda}$$

$$\Rightarrow \rho_*(A \text{ wr}^{\Lambda} B : \underline{\underline{X}}) = \rho_*(B : \underline{\underline{X}})[A^{\Lambda}, \rho_*(B : \underline{\underline{X}})]\overline{\rho_*(A : \underline{\underline{X}})}^{\Lambda}(A')^{\Lambda}$$

as required.

Chapter 5 The nilpotent residual of $A \text{ wr}^{\Lambda} B$

In this chapter we obtain some results on $\rho_*(A \text{ wr}^{\Lambda} B : \underline{\underline{N}})$ for general A and (Λ , B), and characterise the residual completely for certain special cases.

In Section 5.1 we give preliminary results and show that we can reduce the general case to two special cases, viz $B/C_B(\Lambda) \in \underline{\underline{F}}_p$ for some prime p, and $B/C_B(\Lambda) \notin \bigcup_{p \in P} \underline{\underline{F}}_p$, with certain other conditions. In Section 5.2 we give some results on these special cases; we characterise the residual for the case $B/C_B(\Lambda) \in \underline{\underline{F}}_p$, give some lower bounds for the general case, and give some upper bounds for the standard case. In Section 5.3 we 'lift' some of these results back to the case W = A wr B where A/A' and $B/\gamma_\omega(B)$ are periodic. Section 5.4 deals with the case B a perfect group, when we characterise the residual completely.

Section 5.1 Preliminary results

We will denote the first limit ordinal by ω.

The following is well known.

Lemma 5.1.1 Let G be a group. Then $\rho_*(G : \underline{\underline{N}}) = \gamma_\omega(G)$.

Let A be an abelian group, written multiplicatively, and let (Λ , B) be a pair. Let $W = A \text{ wr}^{\Lambda} B$. Let ZB be the integral group ring of B, i.e. ZB is the set of formal sums $\sum_{b \in B} n_b b$ where $n_b \in Z$ for all $b \in B$ and all but a finite number of the n_b are zero. Addition and multiplication are defined by

$$\sum_{b \in B} n_b b + \sum_{b \in B} m_b b = \sum_{b \in B} (n_b + m_b) b \qquad \text{and}$$

$$\sum_{b\in B} n_b b \sum_{b\in B} m_b b = \sum_{\substack{b\in B\\ \beta\in B}} n_\beta m_b \beta b = \sum_{\substack{b\in B\\ \beta\beta'=b}} n_\beta m_{\beta'} b .$$

We make A^Λ into a right ZB - module in a standard way with the following action:

$$f^{\left(\sum_{b\in B} n_b b\right)} = \prod_{b\in B} (f^{n_b})^b$$

where f^β has its usual meaning in $A \text{ wr}^\Lambda B$.

Define $\underline{\underline{b}}$ to be the ideal of ZB generated as an ideal by $\{1 - b : b \in B\}$, i.e. $\underline{\underline{b}}$ is the augmentation ideal of ZB.

Then we have

<u>Lemma 5.1.2</u> [3] Let B be a group, and let ZB be the integral group ring of B. Then

$$\underline{\underline{b}} = \sum_{b\in B} Z(1 - b) = \{ \sum_{b\in B} r_b b : \sum_{b\in B} r_b = 0 \}$$

Define

$$A^{\Lambda\, \underline{\underline{b}}^0} = A^\Lambda$$

$$A^{\Lambda\, \underline{\underline{b}}} = \langle f^{(1-b)} : f \in A^\Lambda, b \in B \rangle$$

$$A^{\Lambda\, \underline{\underline{b}}^{\alpha+1}} = (A^{\Lambda\, \underline{\underline{b}}^\alpha})^{\underline{\underline{b}}} \qquad \text{for all ordinals } \alpha$$

$$A^{\Lambda\, \underline{\underline{b}}^\beta} = \bigcap_{\alpha<\beta} A^{\Lambda\, \underline{\underline{b}}^\alpha} \qquad \text{for all limit ordinals } \beta$$

Let G be any group, and let H and K be subgroups of G. Define

$$[H,K] = \langle [h,k] : h \in H, k \in K \rangle$$

$$[H,_{\alpha+1}K] = [[H,_\alpha K],K] \qquad \text{for all ordinals } \alpha$$

$$[H,_\lambda K] = \bigcap_{\mu<\lambda} [H,_\mu K] \qquad \text{for all limit ordinals } \lambda$$

Much of the following lemma is contained in the unpublished paper [13] and in [2].

<u>Lemma 5.1.3</u> Let A be a group, let (Λ , B) be a pair, and let

$W = A \text{ wr}^{\Lambda} B$. Then

(1) $\gamma_{n+1}(W) = [A^{\Lambda}, {}_n W]\gamma_{n+1}(B)$ for all $n \in N$

and $\gamma_{\omega}(W) = [A^{\Lambda}, {}_{\omega} W]\gamma_{\omega}(B)$

Further, if $A \in \underline{\underline{A}}$ then

(2) $\gamma_{n+1}(W) = [A^{\Lambda}, {}_n B]\gamma_{n+1}(B)$ for all $n \in N$

and $\gamma_{\omega}(W) = [A^{\Lambda}, {}_{\omega} B]\gamma_{\omega}(B)$

and

(3) $A^{\Lambda \underline{\underline{b}}^n} = [A^{\Lambda}, {}_n B] = \gamma_{n+1}(W) \cap A^{\Lambda}$ for all $n \in N$

and $A^{\Lambda \underline{\underline{b}}^{\omega}} = [A^{\Lambda}, {}_{\omega} B] = \gamma_{\omega}(W) \cap A^{\Lambda}$

<u>Proof:</u> (1) By Lemma 3.3.9 (a),

$$\gamma_{n+1}(W) = [A^{\Lambda}, {}_n W]\gamma_{n+1}(B) \quad \text{for all } n \in N$$

Hence
$$\begin{aligned}\gamma_{\omega}(W) &= \bigcap_{n \in Z^+} \gamma_n(W) \\ &= \bigcap_{n \in Z^+} [A^{\Lambda}, {}_n W]\gamma_{n+1}(B) \\ &= \bigcap_{n \in Z^+} [A^{\Lambda}, {}_n W] \bigcap_{n \in Z^+} \gamma_{n+1}(B) \quad \text{since } A^{\Lambda} \triangleleft W \text{ and } A^{\Lambda} \cap B = \{1\} \\ &= [A^{\Lambda}, {}_{\omega} W]\gamma_{\omega}(B)\end{aligned}$$

Hence result.

(2) Suppose $A \in \underline{\underline{A}}$. We show by induction on n that

$$[A^{\Lambda}, {}_n W] = [A^{\Lambda}, {}_n B] \quad \text{for all } n \in N$$

If $n = 0$ the result is trivial.

Suppose the result holds for some $n \geq 0$; then

$$\begin{aligned}[A^{\Lambda}, {}_{n+1} W] &= [A^{\Lambda}, {}_n W, A^{\Lambda} B] \\ &= \langle [A^{\Lambda}, {}_n B, B], [A^{\Lambda}, {}_n B, A^{\Lambda}]^B \rangle \\ &= [A^{\Lambda}, {}_{n+1} B] \quad \text{since } A \in \underline{\underline{A}}\end{aligned}$$

Hence result by induction.

Hence from part (1),

$$\gamma_{n+1}(W) = [A^{\Lambda}, {}_n B]\gamma_{n+1}(B) \quad \text{as required.}$$

Also, $\gamma_\omega(W) = \bigcap_{n\in Z^+} \gamma_n(W)$

$$= \bigcap_{n\in N} [A^\Lambda,{}_n B]\gamma_{n+1}(W)$$

$$= [A^\Lambda,{}_\omega B]\gamma_\omega(B)$$

(3) We prove the first equalities by induction on n ; the second equalities follow from part (2).

For n = 0, the result is trivial.

Suppose the result holds for some $n \geq 0$. Then

$$A^{\Lambda\,\underline{\underline{b}}^{n+1}} = (A^{\Lambda\,\underline{\underline{b}}^{n}})^{\underline{\underline{b}}}$$

$$= \langle f^{(1-b)} : f \in A^{\Lambda\,\underline{\underline{b}}^{n}},\ b \in B\rangle$$

$$= \langle [f,b] : f \in [A^\Lambda,{}_n B],\ b \in B\rangle \quad \text{by the induction hypothesis}$$

$$= [A^\Lambda,{}_{n+1}B]$$

Hence by induction,

$$A^{\Lambda\,\underline{\underline{b}}^{n}} = [A^\Lambda,{}_n B] \quad \text{for all } n \in N$$

Also, $A^{\Lambda\,\underline{\underline{b}}^{\omega}} = \bigcap_{n\in N} A^{\Lambda\,\underline{\underline{b}}^{n}} = \bigcap_{n\in N} [A^\Lambda,{}_n B] = [A^\Lambda,{}_\omega B]$

The next four lemmas will be required for several later results.

<u>Lemma 5.1.4</u> Let A be a group, let $(\Lambda\ ,\ B)$ be a non-trivial transitive pair, and let $W = A \text{ wr}^\Lambda B$. Let $G \leq A$. Then

$$[G^\Lambda, A^\Lambda] \leq [G^\Lambda, B]^{A^\Lambda}$$

<u>Proof:</u> Since $(\Lambda\ ,\ B)$ is non-trivial, there exists $\lambda \in \Lambda$ and $b \in B$ such that $\lambda b \neq \lambda$. We show that

$$[G_\lambda, A_\lambda] \leq [G^\Lambda, B]^{A^\Lambda}$$

Let $a \in G,\ \alpha \in A$. Then

$$[a_\lambda^{-1}, b, \alpha_\lambda] = [a_\lambda a_{\lambda b}^{-1}, \alpha_\lambda] = [a_\lambda, \alpha_\lambda] \qquad \text{since } \lambda b \neq \lambda$$

and so $[G_\lambda, A_\lambda] \leq [G^\Lambda, B, A^\Lambda] \leq [G^\Lambda, B]^{A^\Lambda}$

Let $\mu \in \Lambda$. Then there exists $\beta \in B$ such that $\mu = \lambda\beta$. Then

$[G_\mu, A_\mu] = [G_{\lambda\beta}, A_{\lambda\beta}] = [G_\lambda^\beta, A_\lambda^\beta] \leq [G_\lambda, A_\lambda]^B$ and

$[G_\lambda, A_\lambda]^B \leq [G^\Lambda, B]^{A^\Lambda B} = [G^\Lambda, B]^{A^\Lambda}$ since $[G^\Lambda, B]$ is normalised by B.

Hence $\underset{\lambda\in\Lambda}{\mathrm{Dr}}\, [G_\lambda, A_\lambda] \leq [G^\Lambda, B]^{A^\Lambda}$. We now prove that

$$[G^\Lambda, A^\Lambda] = \underset{\lambda\in\Lambda}{\mathrm{Dr}}\, [G_\lambda, A_\lambda]$$

Clearly $$\underset{\lambda\in\Lambda}{\mathrm{Dr}}\, [G_\lambda, A_\lambda] \leq [G^\Lambda, A^\Lambda]$$

Let $f \in G^\Lambda$, $g \in A^\Lambda$. We prove that $[f,g] \in \underset{\lambda\in\Lambda}{\mathrm{Dr}}\, [G_\lambda, A_\lambda]$.

If $\sigma(f) \cup \sigma(g) = \phi$, then $[f,g] = 1 \in \underset{\lambda\in\Lambda}{\mathrm{Dr}}\, [G_\lambda, A_\lambda]$, and we have nothing to prove.

Now suppose that $\sigma(f) \cup \sigma(g) = \{\lambda_1, \ldots, \lambda_n\}$, so that

$$[f,g] = [f_{\lambda_1} \cdot \ldots \cdot f_{\lambda_n}, g_{\lambda_1} \cdot \ldots \cdot g_{\lambda_n}]$$

We prove by induction on k that

$$[f_{\lambda_1} \cdot \ldots \cdot f_{\lambda_n}, g_{\lambda_1} \cdot \ldots \cdot g_{\lambda_n}] \in \langle [G_\lambda, A_\lambda] : \lambda \in \Lambda \rangle$$

Suppose $k = 1$. Then $[f_{\lambda_1}, g_{\lambda_1}] \in \langle [G_\lambda, A_\lambda] : \lambda \in \Lambda \rangle$ trivially.

Suppose the result holds for some k, $1 \leq k \leq n-1$. Then

$[f_{\lambda_1} \ldots f_{\lambda_{k+1}}, g_{\lambda_1} \ldots g_{\lambda_{k+1}}]$

$= [f_{\lambda_1} \ldots f_{\lambda_k}, g_{\lambda_{k+1}}]^{f_{\lambda_{k+1}}} [f_{\lambda_1} \ldots f_{\lambda_k}, g_{\lambda_1} \ldots g_{\lambda_k}]^{g_{\lambda_{k+1}} f_{\lambda_{k+1}}} \times$

$$[f_{\lambda_{k+1}}, g_{\lambda_{k+1}}][f_{\lambda_{k+1}}, g_{\lambda_1} \ldots g_{\lambda_k}]^{g_{\lambda_{k+1}}}$$

$= [f_{\lambda_1} \ldots f_{\lambda_k}, g_{\lambda_1} \ldots g_{\lambda_k}][f_{\lambda_{k+1}}, g_{\lambda_{k+1}}]$

$\in \langle [G_\lambda, A_\lambda] : \lambda \in \Lambda \rangle$ by the induction hypothesis.

and $\underset{\lambda\in\Lambda}{\mathrm{Dr}} [G_\lambda, A_\lambda] = \langle [G_\lambda, A_\lambda] : \lambda \in \Lambda \rangle$.

Hence $[G^\Lambda, A^\Lambda] = \underset{\lambda\in\Lambda}{\mathrm{Dr}} [G_\lambda, A_\lambda]$, and we have the result.

<u>Corollary 5.1.5</u> Let A be a group, let (Λ, B) be a non-trivial transitive pair, and let $W = A \mathrm{wr}^\Lambda B$. Then

$$\gamma_\omega(W) \leqslant [A^\Lambda, B]\gamma_\omega(B)$$

<u>Proof:</u> By Lemma 5.1.3, $\gamma_\omega(W) \leqslant [A^\Lambda, W]\gamma_\omega(B)$; we show that $[A^\Lambda, W] = [A^\Lambda, B]$.

We have $[A^\Lambda, W] = [A^\Lambda, A^\Lambda B] = [A^\Lambda, B](A^\Lambda)'$

Now $(A^\Lambda)' = [A^\Lambda, A^\Lambda] \leqslant [A^\Lambda, B]^{A^\Lambda}$ by Lemma 5.1.4

$= [A^\Lambda, B]$ since $[A^\Lambda, B] \triangleleft W$

Hence $[A^\Lambda, W] = [A^\Lambda, B]$.

<u>Lemma 5.1.6</u> Let A be a group, let (Λ, B) be a pair, and let $W = A \mathrm{wr}^\Lambda B$. Let $W_f = A \mathrm{wr}^\Lambda \{B/C_B(\Lambda)\}$. Then

$[A^\Lambda, {}_n W] = [A^\Lambda, {}_n W_f]$ for all $n \in N$

and $[A^\Lambda, {}_\omega W] = [A^\Lambda, {}_\omega W_f]$

<u>Proof:</u> By Lemma 3.3.9(b),

$[A^\Lambda, {}_n W] = [A^\Lambda, {}_n W_f]$ for all $n \in N$, and so

$$[A^\Lambda, {}_\omega W] = \bigcap_{n\in N} [A^\Lambda, {}_n W] = \bigcap_{n\in N} [A^\Lambda, {}_n W_f] = [A^\Lambda, {}_\omega W_f] .$$

<u>Lemma 5.1.7</u> Let A be an abelian group, let (Λ, B) be a pair, and let $W = A \mathrm{wr}^\Lambda B$. Let $N \leqslant A$ and $M \leqslant B$. Then

$$N^\Lambda \cap [A^\Lambda, M] = [N^\Lambda, M]$$

Also, if $\{N_i : i \in I\}$ is a set of subgroups of A, then

$$[\bigcap_{i\in I} N_i^\Lambda , M] = \bigcap_{i\in I} N_i^\Lambda \cap [A^\Lambda, M] = \bigcap_{i\in I} [N_i^\Lambda, M]$$

Proof: Working in $A \text{ wr}^{\Lambda} M$, recall that

$$f \in [A^{\Lambda},M] \iff f^* = 1$$

where $f^* : \Lambda/M \to A$ is given by

$$f^*(\lambda M) = \prod_{\mu \in \lambda M} f(\mu) \qquad \forall \ \lambda \in \Lambda \quad \text{(Lemma 4.1.8)}$$

Let $f \in N^{\Lambda} \cap [A^{\Lambda},M]$. Then $f \in N^{\Lambda}$ and $f^* = 1$; i.e. in $N \text{ wr}^{\Lambda} M$, f is an element of $[N^{\Lambda},M]$.

Hence $N^{\Lambda} \cap [A^{\Lambda},M] \leq [N^{\Lambda},M]$.

$(N^{\Lambda})^B \leq N^{\Lambda}$ for any subgroup N of A ; hence $[N^{\Lambda},M] \leq N^{\Lambda} \cap [A^{\Lambda},M]$, and so we have equality.

For any set $\{ N_i : i \in I \}$ of subgroups,

$$\bigcap_{i\in I} N_i^{\Lambda} = \{ \bigcap_{i\in I} N_i \}^{\Lambda} ; \text{ hence}$$

$$\begin{aligned}
[\bigcap_{i\in I} N_i^{\Lambda}, M] &= [\{ \bigcap_{i\in I} N_i \}^{\Lambda},M] \\
&= \{ \bigcap_{i\in I} N_i \}^{\Lambda} \cap [A^{\Lambda},M] && \text{by the above} \\
&= \bigcap_{i\in I} N_i^{\Lambda} \cap [A^{\Lambda},M] && \text{as required,} \\
\text{and further} \quad &= \bigcap_{i\in I} \{N_i^{\Lambda} \cap [A^{\Lambda},M]\} \\
&= \bigcap_{i\in I} [N_i^{\Lambda},M] && \text{by the first part.}
\end{aligned}$$

Reduction theorems

Let A be a group, let (Λ , B) be a pair, and let $W = A \text{ wr}^{\Lambda} B$. It was shown in Theorem 4.1.2 that, since $\underline{\underline{N}} = QS\underline{\underline{N}}$,

(1) If (Λ , B) is transitive and $B/C_B(\Lambda) \notin \underline{\underline{N}}$, then $(A')^{\Lambda} \leq \gamma_{\omega}(W)$ and so we need only consider $A \in \underline{\underline{A}}$.

(2) $\gamma_{\omega}(A)^{\Lambda} \leq \gamma_{\omega}(W)$ and so we need only consider $A \in R\underline{\underline{N}}$.

(3) If $A \in \underline{\underline{A}}$, $\gamma_{\omega}(B) \leq \gamma_{\omega}(W)$ and so we need only consider

$B \in \underline{\underline{RN}}$.

We show that we can improve (1) to

(4) If (Λ, B) is transitive and $B/C_B(\Lambda) \notin \bigcup_{p \in P} \underline{\underline{F}}_p$, then

$(A')^{\Lambda} \leq \gamma_\omega(W)$ and so we need only consider $A \in \underline{\underline{A}}$.

and we have the further reduction

(5) we need only consider (Λ, B) transitive.

(4) and (5) arise from Theorem 5.1.17 and Lemma 5.1.8 respectively.

Thus we have two basic cases to look at:

(1) $B/C_B(\Lambda) \in \underline{\underline{F}}_p$ for some prime p , (Λ, B) transitive, and $A \in \underline{\underline{RN}}$

(2) $B/C_B(\Lambda) \notin \bigcup_{p \in P} \underline{\underline{F}}_p$, (Λ, B) transitive, $B \in \underline{\underline{RN}}$, and $A \in \underline{\underline{A}}$.

We also show in the course of the proofs of (4) and (5) that if A is a periodic residually nilpotent group, we may in fact reduce to the case A a p-group for some prime p.

We look first at reduction (5). We require the following lemma.

<u>Lemma 5.1.8</u> Let A be a group, let (Λ, B) be a pair, and let $W = A \text{ wr}^{\Lambda} B$. Suppose $A = \mathop{\mathrm{Dr}}_{i \in I} A_i$ for some index set I and subgroups A_i . Then

$$\gamma_\omega(A \text{ wr}^{\Lambda} B) = \{ \mathop{\mathrm{Dr}}_{i \in I} \mathop{\mathrm{Dr}}_{\Theta \text{ an orbit}} [A_i^{\Theta},_{\omega} A_i^{\Theta} B] \} \gamma_\omega(B)$$

$$= \langle \gamma_\omega(A_i \text{ wr}^{\Theta} B) : i \in I, \Theta \text{ an orbit} \rangle$$

<u>Proof:</u> By Lemma 3.3.9,

$$[A^{\Lambda},_n W] = \mathop{\mathrm{Dr}}_{i \in I} \mathop{\mathrm{Dr}}_{\Theta \text{ an orbit}} [A_i^{\Theta},_n A_i^{\Theta} B] \quad \text{for all } n \in N$$

Now

$$\bigcap_{n\in N} \operatorname*{Dr}_{i\in I} \operatorname*{Dr}_{\Theta \text{ an orbit}} [A_i^\Theta ,_n A_i^\Theta B] = \operatorname*{Dr}_{i\in I} \operatorname*{Dr}_{\Theta \text{ an orbit}} \{ \bigcap_{n\in N} [A_i^\Theta ,_n A_i^\Theta B]\}$$

since

$$[A_i^\Theta ,_n A_i^\Theta B] \cap [A_j^\Sigma ,_n A_j^\Sigma B] = \{1\} \quad \text{if} \quad i \neq j \text{ or } \Theta \neq \Sigma$$

Hence

$$\begin{aligned} \gamma_\omega(W) &= \{ \bigcap_{n\in N} [A^\Lambda ,_n W]\}\gamma_\omega(B) && \text{by Lemma 5.1.3} \\ &= \{ \bigcap_{n\in N} \operatorname*{Dr}_{i\in I} \operatorname*{Dr}_{\Theta \text{ an orbit}} [A_i^\Theta ,_n A_i^\Theta B]\}\gamma_\omega(B) \\ &= \{ \operatorname*{Dr}_{i\in I} \operatorname*{Dr}_{\Theta \text{ an orbit}} [A_i^\Theta ,_\omega A_i^\Theta B]\}\gamma_\omega(B) && \text{by the above} \\ &= \langle \gamma_\omega(A_i \operatorname{wr}^\Theta B) : i \in I, \Theta \text{ an orbit} \rangle \end{aligned}$$

Hence result.

We have as an immediate corollary

Proposition 5.1.9 Let A be a group, let (Λ , B) be a pair, and let $W = A \operatorname{wr}^\Lambda B$. Then

$$\gamma_\omega(W) = \langle \gamma_\omega(A \operatorname{wr}^\Theta B) : \Theta \text{ an orbit} \rangle$$

Hence we need only consider (Λ , B) transitive.

We now consider reduction (4). We show firstly that if (Λ , B) is transitive and Λ is infinite, then $(A')^\Lambda \leq \gamma_\omega(W)$; and then we prove that if (Λ , B) is transitive and $B/C_B(\Lambda) \in \{\underline{\underline{F}} \cap \underline{\underline{N}}\} \setminus \bigcup_{p\in P} \underline{\underline{F}}_p$, $(A')^\Lambda \leq \gamma_\omega(W)$. These two results and reduction (1) then show that if (Λ , B) is transitive and $B/C_B(\Lambda) \notin \bigcup_{p\in P} \underline{\underline{F}}_p$, $(A')^\Lambda \leq \gamma_\omega(W)$.

The following proposition, which gives us the first part of the above, is proved for the standard case in [8].

<u>Proposition 5.1.10</u> Let A be a group and let (Λ, B) be a transitive pair with Λ infinite. Then $(A')^{\Lambda} \leqslant \gamma_{\omega}(W)$.

<u>Proof:</u> If A is trivial, then $(A')^{\Lambda} \leqslant \gamma_{\omega}(W)$ immediately. Suppose A is non-trivial.

Let $n \in N$, $\mu \in \Lambda$, and $1 \neq f, g \in A_{\mu}$. From the proof of Lemma 3.3.7, since $2^n < |\Lambda|$, there exist $b_1, \ldots, b_{n+1} \in B$ such that

$$[f, b_1, \ldots, b_{n+1}]^{b}(\mu) = f(\mu) \quad \text{where } b = b_{n+1}^{-1} \cdots b_1^{-1}$$

Then $$[f,g] = [[f, b_1, \ldots, b_{n+1}]^{b}, g] ;$$

for $$[[f, b_1, \ldots, b_{n+1}]^{b}, g](\mu) = [f(\mu), g(\mu)]$$

~~with appropriate choice of sign~~, and if $\lambda \neq \mu$,

$$[[f, b_1, \ldots, b_{n+1}], g](\lambda) = 1$$

since $g(\lambda) = 1$.

Hence $[f,g] \in \gamma_{n+3}(W)$. Hence $A'_{\mu} \leqslant \gamma_{n+3}(W)$, and so since (Λ, B) is transitive, $(A')^{\Lambda} \leqslant \gamma_{n+3}(W)$. n was any integer; so

$$(A')^{\Lambda} \leqslant \bigcap_{n \in N} \gamma_n(W) = \gamma_{\omega}(W)$$

To show that if $B/C_B(\Lambda) \in \{\underline{\underline{F}} \cap \underline{\underline{N}}\} \setminus \bigcup_{p \in P} \underline{\underline{F}}_p$, then $(A')^{\Lambda} \leqslant \gamma_{\omega}(W)$, we need several preliminary results; these results and their proofs are due mainly to ideas of Dr J. A. Hulse.

<u>Lemma 5.1.11</u> (cf [8]) Let B be a group and let ZB be the integral group ring of B. Let p be a prime and n a fixed positive integer. For $i \in N$ define $B_{i,n} = \langle x \in B : x^{p^i} \in \gamma_n(B) \rangle$.

Then $p\, \underline{\underline{b}}_{i,n} \leqslant \underline{\underline{b}}_{i-1,n} B_{i,n} + (\underline{\underline{b}}_{i,n})^p$ for all $i \in Z^+$

Proof: Throughout the proof we will write B_i for $B_{i,n}$ for brevity.

Let $x \in \{ x \in B : x^{p^i} \in \gamma_n(B) \}$; let $u = 1-x$. Then

$$x^p = (1-u)^p = 1 - pu + \sum_{i=2}^{p-1} \binom{p}{i} (-u)^i + (-u)^p \quad ; \text{ so}$$

$$p(1-x) = pu = 1 - x^p + \sum_{i=2}^{p-1} \binom{p}{i} (-u)^i + (-u)^p .$$

Now p divides $\binom{p}{i}$ for $1 \leq i \leq p-1$; so we have

$$p(1-x) = pu = 1 - x^p + p \sum_{i=2}^{p-1} \nu_i u^i \pm u^p \qquad (1)$$

for some $\nu_i \in Z^+$, $2 \leq i \leq p-1$.

We show by reverse induction on j that

$$p\, u^j \in \underline{\underline{b}}_{i-1} B_i + \underline{\underline{b}}_i^{\,p} \quad \text{for all } j \geq 1 .$$

If $j \geq p$, $p\, u^j \in \underline{\underline{b}}_i^{\,p} \leq \underline{\underline{b}}_{i-1} B_i + \underline{\underline{b}}_i^{\,p}$.

Now suppose that $1 \leq j < p$, and that

$$p\, u^k \in \underline{\underline{b}}_{i-1} B_i + \underline{\underline{b}}_i^{\,p} \quad \text{for all } k > j .$$

Then $$p\, u^j = (1-x^p)\, u^{j-1} + p \sum_{i=2}^{p-1} \nu_i u^{i+j-1} \pm u^{p+j-1}$$
from (1).

Since $x \in \{ x \in B : x^{p^i} \in \gamma_n(B) \}$, x^p is an element of B_{i-1} and so $(1-x^p)\, u^{j-1} \in \underline{\underline{b}}_{i-1} B_i$.

Since $j-1+i > j$ for $i \geq 2$, the induction hypothesis shows that

$$p \sum_{i=2}^{p-1} \nu_i u^{i+j-1} \in \underline{\underline{b}}_{i-1} B_i + \underline{\underline{b}}_i^{\,p} \quad ;$$

and finally, since $p+j-1 \geq p$, $u^{p+j-1} \in \underline{\underline{b}}_i^{\,p}$. Hence

$$p\, u^j \in \underline{\underline{b}}_{i-1} B_i + \underline{\underline{b}}_{i-1} B_i + \underline{\underline{b}}_i^p + \underline{\underline{b}}_i^p = \underline{\underline{b}}_{i-1} B_i + \underline{\underline{b}}_i^p$$

Hence result by induction.

In particular, for $j = 1$,

$$pu \in \underline{\underline{b}}_{i-1} B_i + \underline{\underline{b}}_i^p$$

By [3] Proposition 1, $\underline{\underline{b}}_i$ is the right ideal of ZB_i generated by $\{1-x : x^{p^i} \in \gamma_n(B)\}$; $p(1-x) \in \underline{\underline{b}}_{i-1} B_i + \underline{\underline{b}}_i^p$ for all such x ; hence

$$p\, \underline{\underline{b}}_i \leq \underline{\underline{b}}_{i-1} B_i + \underline{\underline{b}}_i^p$$

<u>Lemma 5.1.12</u> Let B be a group and let ZB be the integral group ring of B . Let p be a prime and let n be a positive integer. Let $B_{i,n} = \langle x : x^{p^i} \in \gamma_n(B) \rangle$ for all $i \in N$. Then

$$p^i\, \underline{\underline{b}}_{i,n} \leq \underline{\underline{b}}^n + (\underline{\underline{b}}_{i,n})^p$$

for all $i \in N$.

<u>Proof:</u> Let $i \in N$. If $i = 0$, the result holds since $B_{0,n} = \gamma_n(B)$, and hence $\underline{\underline{b}}_{0,n} \leq \underline{\underline{b}}^n$ from [3].

Suppose now that $i > 0$. We show by induction on r that $p^r\, \underline{\underline{b}}_{i,n} \leq \underline{\underline{b}}_{i-r,n} B_{i,n} + (\underline{\underline{b}}_{i,n})^p$ for $1 \leq r \leq i$.

The case $r = 1$ is given by Lemma 5.1.11 .

Suppose $2 \leq r \leq i$ and the result holds for $r-1$. Then

$$\begin{aligned} p^r\, \underline{\underline{b}}_{i,n} &= p\, p^{r-1}\, \underline{\underline{b}}_{i,n} \\ &\leq p\{ \underline{\underline{b}}_{i-(r-1),n} B_{i,n} + (\underline{\underline{b}}_{i,n})^p\} \quad \text{by the induction hypothesis} \\ &\leq \underline{\underline{b}}_{i-(r-1)-1,n} B_{i,n} + (\underline{\underline{b}}_{i,n})^p + p(\underline{\underline{b}}_{i,n})^p \quad \text{by Lemma 5.1.11} \\ &= \underline{\underline{b}}_{i-r,n} B_{i,n} + (\underline{\underline{b}}_{i,n})^p \end{aligned}$$

Hence result by induction.

Hence taking $r = i$,

$$p^{i}\, \underline{\underline{b}}_{i,n} \leq \underline{\underline{b}}_{0,n}\, B_{i,n} + (\underline{\underline{b}}_{i,n})^{p}$$

$$\leq \underline{\underline{b}}^{n}\, B + (\underline{\underline{b}}_{i,n})^{p} \qquad \text{as noted before}$$

$$\leq \underline{\underline{b}}^{n} + (\underline{\underline{b}}_{i,n})^{p}$$

<u>Lemma 5.1.13</u> Let B be a group and let ZB be the integral group ring of B. Let $n \in Z^{+}$ and let

$$B_{i,n} = \langle b \in B : b^{p^{i}} \in \gamma_{n}(B) \rangle \quad \text{for all } i \in N$$

Then

$$\underline{\underline{b}}_{i,n}(1 - y^{p^{im}}) \cup (1 - y^{p^{im}})\, \underline{\underline{b}}_{i,n} \leq \underline{\underline{b}}^{\min(n+1,m+2)}$$

for all $y \in B$, all $m \geq 0$, and all $n \geq 1$.

<u>Proof:</u> The proof is by induction on n .

The case $n = 1$ is immediate, as $\min(n+1,m+2) = 2$.

Suppose that $n > 1$ and

$$\underline{\underline{b}}_{i,n-1}(1 - y^{p^{im}}) \cup (1 - y^{p^{im}})\, \underline{\underline{b}}_{i,n-1} \leq \underline{\underline{b}}^{\min(n+1,m+2)} \quad \text{for all } m \in N.$$

We prove that

$$\underline{\underline{b}}_{i,n}(1 - y^{p^{im}}) \cup (1 - y^{p^{im}})\, \underline{\underline{b}}_{i,n} \leq \underline{\underline{b}}^{\min(n,m+2)} \quad \text{for all } m \geq 0$$

by induction on m .

The result is immediate for $m = 0$ since $\min(n+1,m+2) = 2$.

Suppose $m > 0$ and

$$\underline{\underline{b}}_{i,n}(1 - y^{p^{i(m-1)}}) \cup (1 - y^{p^{i(m-1)}})\, \underline{\underline{b}}_{i,n} \leq \underline{\underline{b}}^{\min(n+1,m+1)}$$

Let $x \in B_{i,n}$ and $y \in B$. Now

$$(1-x)(1 - y^{p^{im}}) - p^{i}(1-x)(1 - y^{p^{i(m-1)}})$$

$$= (1-x)(1 - y^{p^{i(m-1)}})\Big(\sum_{r=0}^{p^{i}-1} y^{rp^{i(m-1)}} - p^{i}\Big) \qquad (*)$$

The sum of the coefficients of $k = \sum_{r=0}^{p^i-1} y^{rp^{i(m-1)}} - p^i$ is zero, and hence $k \in \underline{\underline{b}}$.

So since $(1-x)(1-y^{p^{i(m-1)}}) \in \underline{\underline{b}}^{\min(n+1,m+1)}$ by the induction hypothesis,

$$(1-x)(1-y^{p^{i(m-1)}})\left(\sum_{r=0}^{p^i-1} y^{rp^{i(m-1)}} - p^i\right) \in \underline{\underline{b}}^{\min(n+2,m+2)}$$
$$\leqslant \underline{\underline{b}}^{\min(n+1,m+2)}$$

We now show that

$$p^i (1-x)(1-y^{p^{i(m-1)}}) \in \underline{\underline{b}}^{\min(n+1,m+2)}$$

By Lemma 5.1.12,

$$p^i (1-x) \in \underline{\underline{b}}^n + (\underline{\underline{b}}_{i,n})^p$$

Now $\underline{\underline{b}}^n (1-y^{p^{i(m-1)}}) \leqslant \underline{\underline{b}}^{n+1} \leqslant \underline{\underline{b}}^{\min(n+1,m+2)}$ and

$$(\underline{\underline{b}}_{i,n})^p (1-y^{p^{i(m-1)}}) \leqslant (\underline{\underline{b}}_{i,n})^{p-1} (\underline{\underline{b}}_{i,n} (1-y^{p^{i(m-1)}}))$$
$$\leqslant (\underline{\underline{b}}_{i,n})^{p-1} \underline{\underline{b}}^{\min(n+1,m+1)}$$

$p-1 \geqslant 1$, so

$$(\underline{\underline{b}}_{i,n})^p (1-y^{p^{i(m-1)}}) \leqslant \underline{\underline{b}}_{i,n} \underline{\underline{b}}^{\min(n+1,m+2)}$$
$$= \underline{\underline{b}}^{\min(n+1,m+2)}$$

Hence by induction on m, for all $m \geqslant 0$,

$$(1-x)(1-y^{p^{im}}) \in \underline{\underline{b}}^{\min(n+1,m+2)} \qquad \text{from } (*).$$

Hence we have the result by induction on n .

(The proof that $(1-y^{p^{im}}) \underline{\underline{b}}_{i,n} \leqslant \underline{\underline{b}}^{\min(n+1,m+2)}$ is similar.)

Let B be any group and p any prime. Define subgroup $B_{\omega p}$ by $B_{\omega p} = \{ b \in B : b \text{ has p-power order mod } \gamma_n(B) \text{ for all } n \in Z^+\}$.

Denote $\rho_*(B : \bar{\underline{\underline{N}}}_p)$ by $\bar{B}_p$.

We now have

<u>Proposition 5.1.14</u> Let A be an abelian group, let (Λ , B) be a pair, and let $W = A \text{ wr}^{\Lambda} B$. Let p be any prime. Then

$$[A^{\Lambda}, B_{\omega p}, \bar{B}_p] \leq \gamma_{\omega}(W)$$

and

$$[A^{\Lambda}, \bar{B}_p, B_{\omega p}] \leq \gamma_{\omega}(W)$$

<u>Proof:</u> Let ZB be the integral group ring of B ; A^{Λ} is a ZB - module as before.

Let $x \in B_{\omega p}$; say $x^{p^{i(n)}} \in \gamma_n(B)$ for all $n \in Z^+$. Then $x \in B_{i(n),n}$ for all $n \in Z^+$. By Lemma 5.1.13, taking m = n,

$$(1-x)(1 - y^{p^{i(n)n}}) \in \underline{\underline{b}}^{n+1} \quad \text{for all } y \in B \text{ and all } n \in Z^+$$

i.e. in multiplicative notation,

$$[f,x,y^{p^{i(n)n}}] \in [A^{\Lambda},_{n+1}B] \quad \text{for all } y \in B, \text{ all } f \in A^{\Lambda}, \text{ and all } n \in Z^+$$

By a well-known result [see e.g. [6]; the result is a corollary to Theorem 10.3.6],

$$[f,x,\gamma_{n+1}(B)] \leq [A^{\Lambda},_{n+1}B] \quad \text{for all } y \in B, \text{ all } f \in A^{\Lambda}, \text{ and all } n \in Z^+$$

and hence

$$[A^{\Lambda},x,\gamma_{n+1}(B)\ B^{p^{i(n)n}}] \leq [A^{\Lambda},_{n+1}B] \quad \text{for all } n \in Z^+$$

Now $\bar{B}_p \leq \gamma_{n+1}(B)\ B^{p^{i(n)n}}$ for all $n \in Z^+$ and so

$$[A\ ,x,\bar{B}_p] \leq \bigcap_{n \in Z^+} [A^{\Lambda},_{n+1}B] \leq \gamma_{\omega}(W)$$

x was any element of $B_{\omega p}$; so

$$[A^{\Lambda},B_{\omega p},\bar{B}_p] \leq \gamma_{\omega}(W)$$

Similarly,

$$[A^{\Lambda},\bar{B}_p,B\omega_p] \leq \gamma_{\omega}(W)$$

This gives us the following corollary.

Corollary 5.1.15 Let A be a group and let (Λ, B) be a faithful transitive pair. Suppose $B \in \{\underline{\underline{F}} \cap \underline{\underline{N}}\} \setminus \bigcup_{p \in P} \underline{\underline{F}}_p$. Then $(A')^{\Lambda} \leq \gamma_\omega(W)$.

Proof: Let $p \in P$ be such that $B_p \neq \{1\} \neq B_{p'}$, where for any set of primes Π , B_Π is the Sylow Π-subgroup of B , and $p' = P \setminus \{p\}$. Applying Proposition 5.1.14 to $\langle a \rangle \, wr^{\Lambda} B$ for $a \in A$, we have

$$[\langle a \rangle^{\Lambda}, B_p, B_{p'}] \leq \gamma_\omega(\langle a \rangle \, wr^{\Lambda} B) \leq \gamma_\omega(A \, wr^{\Lambda} B)$$

since $B_p = B_{\omega p}$ and $B_{p'} = \overline{B}_p$ because $B = B_p \times B_{p'}$.

We show now that we may pick $\lambda \in \Lambda$, $z \in B_p$, and $y \in B_{p'}$ such that $|\{\lambda, \lambda z, \lambda y, \lambda yz\}| \geq 3$.

Suppose firstly that (Λ, B_p) is transitive. $2 < |B| \leq |\Lambda|!$ and so $|\Lambda| \geq 3$. $B_{p'} \neq \{1\} \Rightarrow$ there exists $y \in B_{p'}$ and $\lambda \in \Lambda$ such that $\lambda y \neq \lambda$. Pick $\mu \in \Lambda \setminus \{\lambda, \lambda y\}$; then there exists $z \in B_p$ such that $\lambda z = \mu$. Then λ, λz, and λy are distinct.

Suppose (Λ, B_p) is intransitive. $B_p \neq \{1\} \Rightarrow$ there exists $\lambda \in \Lambda$ such that $\Lambda > \lambda B_p > \{\lambda\}$. Let $\nu \in \Lambda \setminus \lambda B_p$. Then there exists $x \in B_p$ and $y \in B_{p'}$ such that $\lambda xy = \nu$; replace λ by λx as $\lambda B_p = \lambda x B_p$. Then $\lambda y = \nu$. $\lambda B_p \neq \{\lambda\} \Rightarrow$ there exists $z \in B_p$ such that $\lambda z \neq \lambda$. If $\lambda z = \lambda y$, then $\nu = \lambda y \in \lambda B_p$, a contradiction. Hence λ, λz, and λy are distinct.

So $|\{\lambda, \lambda z, \lambda y, \lambda zy\}| \geq 3$.

Now let $a \in A$ and let $\mu \in \Lambda$. Then

$$[a_\mu , z^{-1}, y^{-1}](\lambda) = a_\mu^{-1}(\lambda z) a_\mu(\lambda) a_\mu^{-1}(\lambda y) a_\mu^{-1}(\lambda yz) .$$

Now choose $\mu \in \{\lambda, \lambda y, \lambda z, \lambda yz\}$ such that μ is distinct from the

Thus we see that we have

<u>Theorem 5.1.17</u> Let (Λ , B) be a transitive pair and suppose that $B/C_B(\Lambda) \not\in \bigcup_{p \in P} \underline{\underline{F}}_p$. Let A be a group and let $W = A \text{ wr}^{\Lambda} B$. Then

$$(A')^{\Lambda} \leq \gamma_{\omega}(W)$$

So we have reduction (4), i.e. in this case we need only consider $A \in \underline{\underline{A}}$.

other three elements; then $[a_\mu, z^{-1}, y^{-1}] = a^{\pm 1}$. Hence if α is any other element of A,

$$[a_\lambda, \alpha_\lambda] = [[a_\lambda, z^{-1}, y^{-1}]^{\pm 1}, \alpha_\lambda] \quad \text{(sign as appropriate)}$$
$$\in \gamma_\omega(A \text{ wr}^\Lambda B)$$

since $[a_\lambda, z^{-1}, y^{-1}] \in [\langle a \rangle^\Lambda, B_p, B_{p'}]$, and so $A'_\lambda \leq \gamma_\omega(A \text{ wr}^\Lambda B)$. Hence since (Λ, B) is transitive, $(A')^\Lambda \leq \gamma_\omega(A \text{ wr}^\Lambda B)$.

Thus we have

Proposition 5.1.16 Let A be a group, let (Λ, B) be a transitive pair, and let $W = A \text{ wr}^\Lambda B$. Suppose $B/C_B(\Lambda) \in \{\underline{\underline{F}} \cap \underline{\underline{N}}\} \setminus \bigcup_{p \in P} \underline{\underline{F}}_p$. Then $(A')^\Lambda \leq \gamma_\omega(A \text{ wr}^\Lambda B)$.

Proof: $(\Lambda, B/C_B(\Lambda))$ is transitive since (Λ, B) is, and so by Corollary 5.1.15

$$(A')^\Lambda \leq \gamma_\omega(A \text{ wr}^\Lambda \{B/C_B(\Lambda)\}) \cap A^\Lambda$$
$$= \gamma_\omega(A \text{ wr}^\Lambda B) \cap A^\Lambda \text{ by Lemma 5.1.6.}$$

Hence result.

←

We now show that we can make further reductions in the case A a periodic residually nilpotent group.

The next result is well-known; see for example [17] Volume 2.

Lemma 5.1.18 A periodic locally residually nilpotent group is the direct product of its Sylow subgroups.

In particular, if $A \in R\underline{\underline{N}}$ and A is periodic, then $A = \underset{p \in P}{\text{Dr}} A_p$ where A_p is the Sylow p-subgroup of A for $p \in P$.

Hence we have a further corollary to Lemma 5.1.8, viz

Proposition 5.1.19 Let A be a periodic residually nilpotent group and let (Λ , B) be a transitive pair. Let A_p be the Sylow p-subgroup of A. Then

$$\gamma_\omega(A \text{ wr}^\Lambda B) = \{ \underset{p\in P}{\text{Dr}} [A_p^\Lambda ,_\omega A_p^\Lambda B]\} \gamma_\omega(B)$$
$$= \langle \gamma_\omega(A_p \text{wr}^\Lambda B) : p \in P \rangle$$

Hence in this case we need only consider A a p-group.

Also from Lemma 5.1.8, if $A \in \underline{\underline{A}} \cap \underline{\underline{G}}$, it is clear that we need only consider A wr^Λ B where A is a cyclic p-group for some prime p, or A is the infinite cyclic group .

Section 5.2 The case $B/C_B(\Lambda) \in \underline{\underline{F}}p$, and some upper and lower bounds for the general case.

This section contains two main types of result; Theorems 5.2.1 and 5.2.2 give upper and lower bounds for $\gamma_\omega(W)$ for quite general cases, and Theorems 5.2.4 - 5.2.7 deal with more specific cases, chiefly for standard wreath products.

For the case $B/C_B(\Lambda) \in \underline{\underline{F}}_p$, we characterise the residual completely (Theorem 5.2.3).

For any prime p and group G, we will denote $\rho_*(G : \underline{\underline{\bar{N}}}_p)$ by $\bar{G}_p$.

Note that $(G_{\omega p})^\Lambda = (G^\Lambda)_{\omega p}$ for any set Λ and group G .

Theorem 5.2.1 Let A be a group, let (Λ, B) be a pair, and let $W = A \operatorname{wr}^{\Lambda} B$. Then

(a) $\gamma_\omega(B)[\bar{A}_p^{\Lambda}, B_{\omega p}] \leq \gamma_\omega(W)$ for all $p \in P$

(b) $\gamma_\omega(B)[A_{\omega p}^{\Lambda}, \bar{B}_p] \leq \gamma_\omega(W)$ for all $p \in P$

(c) If $A \in \underline{\underline{A}}$ or $B/C_B(\Lambda) \in \underline{\underline{\bar{N}}}_q$ for some prime q, then

$$\gamma_\omega(B).\bigcap_{n\in Z^+}[\{A^{p^n}\gamma_n(A)\}^{\Lambda}, B_{\omega p}]^{A^{\Lambda}} \leq \gamma_\omega(W) \quad \text{for all } p \in P \text{, and}$$

$$\gamma_\omega(B).\bigcap_{n\in Z^+}[A_{\omega p}^{\Lambda}, B^{p^n}\gamma_n(B)]^{A^{\Lambda}} \leq \gamma_\omega(W) \quad \text{for all } p \in P$$

Theorem 5.2.2 Let A be a group and let (Λ, B) be a non-trivial transitive pair such that $B/C_B(\Lambda) \in \underline{\underline{F}}_p$ for some prime p. Let $W = A \operatorname{wr}^{\Lambda} B$. Then

$$\gamma_\omega(W) \leq \bigcap_{n\in Z^+}[\{A^{p^n}\gamma_n(A)\}^{\Lambda}, B]^{A^{\Lambda}}\gamma_\omega(B)$$

These two results allow us to prove

Theorem 5.2.3 Let A be a group and let (Λ, B) be a pair such that $B/C_B(\Lambda) \in \underline{\underline{F}}_p$ for some prime p. Let $W = A \operatorname{wr}^{\Lambda} B$ and let $\Sigma = \{\sigma \in \Lambda : \{\sigma\} \text{ is an orbit of } B \text{ on } \Lambda\}$. Then

$$\gamma_\omega(W) = \langle \gamma_\omega(A)^{\Sigma}, \bigcap_{n\in Z^+}[\{A^{p^n}\gamma_n(A)\}^{\Theta}, B]^{A^{\Theta}}, \gamma_\omega(B) : \Theta \text{ a non-trivial orbit} \rangle$$

$$= [\bar{A}_p^{\Lambda}, B]\gamma_\omega(B) \qquad \text{if } A \in \underline{\underline{A}}$$

The following result provides the key to the proof of Theorem 5.2.1. Part (a) is stated in [5], but not proved; it is an extension of Lemma 1 of [11].

Lemma 5.2.4 Let G be a group and V a normal subgroup of G. Let

$v \in V$ and $g \in G$.

(a) If there exists $m \in N$ such that $[v^m, G] = \{1\}$, then

$[v, G^{m^{n-1}}] \leqslant [V,_n G]$ for all $n \geqslant 1$.

(b) If there exists $m \in N$ such that $[V, g^m] = \{1\}$, then

$[V^{m^{n-1}}, g] \leqslant [V,_n G]$ for all $n \geqslant 1$.

Proof: (a) The proof is by induction on n.

The result is clear for $n = 1$.

Now suppose the result holds for some $n \geqslant 1$. To show that $[v, G^{m^n}] \leqslant [V,_{n+1} G]$ it will be sufficient to show that $[v, g^{m^n}] \in [V,_{n+1} G]$ for all $g \in G$, as $[V,_{n+1} G] \Delta G$.

Let $g \in G$, and let $K = [v, G^{m^{n-1}}, G]$. Then modulo K, $[v, g^{m^{n-1}}]$ lies in the centre of G; hence

$$\begin{aligned} [v, g^{m^n}] &= [v, (g^{m^{n-1}})^m] \\ &\equiv [v, g^{m^{n-1}}]^m \bmod K \\ &\equiv [v^m, g^{m^{n-1}}] \bmod K \\ &\equiv 1 \bmod K \end{aligned}$$

Hence $[v, g^{m^n}] \in [v, G^{m^{n-1}}, G] \leqslant [V,_n G, G]$ by the induction hypothesis
$= [V,_{n+1} G]$

Hence result by induction.

(b) The proof is by induction on n.

The result is clear for $n = 1$.

Suppose the result holds for some $n \geqslant 1$; $V \Delta G \Rightarrow [V,_{n+1} G] \Delta G$ and so it will be sufficient to show $[v^{m^n}, g] \in [V,_{n+1} G]$ for all $v \in V$.

Let $v \in V$ and let $K = [V^{m^{n-1}}, g, G]$. Modulo K, $[v^{m^{n-1}}, g]$ lies in the centre of G; so

$$\begin{aligned}[v^{m^n},g] &= [(v^{m^{n-1}})^m,g] \\ &\equiv [v^{m^{n-1}},g]^m \bmod K \\ &\equiv [v^{m^{n-1}},g^m] \bmod K \\ &\equiv 1 \bmod K\end{aligned}$$

Hence $[v^{m^n},g] \in [V^{m^{n-1}},g,G] \leq [V,_n G,G]$ by the induction hypothesis

$$= [V,_{n+1}G]$$

Hence result by induction.

We may now prove Theorem 5.2.1.

<u>Proof of Theorem 5.2.1</u>

(a) Let p be any prime. Let $b \in B_{\omega p}$, say $b^{p^{r(n)}} \in \gamma_n(B)$ for all $n \in Z^+$. Then

$$[A^\Lambda, b^{p^{r(n)}}] \equiv 1 \bmod \gamma_{n+1}(W) \quad \text{for all } n \in Z^+$$

=> by Lemma 5.2.4(b),

$$\begin{aligned}[(A^\Lambda)^{p^{r(n)(n-1)}},b] &\leq [A^\Lambda,_n W]\gamma_{n+1}(W) && \text{for all } n \in Z^+ \\ &= \gamma_{n+1}(W) && \text{for all } n \in Z^+\end{aligned}$$

Now $\gamma_n(A)^\Lambda \leq \gamma_n(W)$ for all $n \in Z^+$ and so

$$[\{\gamma_n(A)\ A^{p^{r(n)(n-1)}}\}^\Lambda,b] \leq \gamma_{n+1}(W) \quad \text{for all } n \in Z^+$$

Then $\quad A^\Lambda/\{\gamma_n(A)\ A^{p^{r(n)(n-1)}}\}^\Lambda \in \underline{\underline{\overline{N}}}_p \quad$ for all $n \in Z^+$

=> $\quad \overline{A}_p^\Lambda \leq (\gamma_n(A)\ A^{p^{r(n)(n-1)}})^\Lambda \quad$ for all $n \in Z^+$

=> $\quad [\overline{A}_p^\Lambda, b] \leq \gamma_{n+1}(W) \quad$ for all $n \in Z^+$

=> $\quad [\overline{A}_p^\Lambda, b] \leq \gamma_\omega(W)$.

Hence since b was any element of $B_{\omega p}$, $[\overline{A}_p^\Lambda, B_{\omega p}] \leq \gamma_\omega(W)$.

(b) Let p be any prime. Let $f \in (A^\Lambda)_{\omega p}$, say $f^{p^{r(n)}} \in \gamma_n(A^\Lambda)$ for all $n \in Z^+$.

Then $[f^{p^{r(n)}}, W] \equiv 1 \bmod \gamma_{n+1}(W)$ for all $n \in Z^+$

=> by Lemma 5.2.4,

$$[f, W^{p^{r(n)(n-1)}}] \leq [A^\Lambda,{}_n W]\gamma_{n+1}(W) \quad \text{for all } n \in Z^+$$

$$= \gamma_{n+1}(W) \quad \text{for all } n \in Z^+$$

=> $$[f, B^{p^{r(n)(n-1)}}] \leq \gamma_{n+1}(W) \quad \text{for all } n \in Z^+$$

$\gamma_n(B) \leq \gamma_n(W)$ for all $n \in Z^+$ and so

$$[f, B^{p^{r(n)(n-1)}}\gamma_n(B)] \leq \gamma_{n+1}(W) \quad \text{for all } n \in Z^+$$

$B/\{B^{p^{r(n)(n-1)}}\gamma_n(B)\} \in \overline{\underline{\underline{N}}}_p$ for all $n \in Z^+$, and so

$$\overline{B}_p \leq B^{p^{r(n)(n-1)}}\gamma_n(B) \quad \text{for all } n \in Z^+$$

=> $$[f, \overline{B}_p] \leq \gamma_{n+1}(W) \quad \text{for all } n \in Z^+$$

=> $$[f, \overline{B}_p] \leq \gamma_\omega(W)$$

f was any element of $(A^\Lambda)_{\omega p} = A^\Lambda_{\omega p}$; hence

$$[A^\Lambda_{\omega p}, \overline{B}_p] \leq \gamma_\omega(W) .$$

(c) We prove part (c) in the two cases $A \in \underline{\underline{A}}$ and $B/C_B(\Lambda) \in \overline{\underline{\underline{N}}}_q$ separately.

(1) Suppose $V = B/C_B(\Lambda) \in \overline{\underline{\underline{N}}}_q$ for some prime q. We show first that $\bigcap_{n \in Z^+} [\{A^{p^n}\gamma_n(A)\}^\Lambda, B_{\omega p}]^{A^\Lambda} \leq \gamma_\omega(W)$. Let $W_f = A \text{ wr}^\Lambda V$.

We show that

$$\bigcap_{n \in Z^+} [\{A^{p^n}\gamma_n(A)\}^\Lambda, V_{\omega p}]^{A^\Lambda} \leq [A^\Lambda,{}_\omega W_f] \quad \text{for all primes } p$$

If $p \neq q$, $V_{\omega p} = \{1\}$ and so

$$\bigcap_{n \in Z^+} [\{A^{p^n}\gamma_n(A)\}^\Lambda, V_{\omega p}]^{A^\Lambda} \leq [A^\Lambda,{}_\omega W_f] \quad \text{trivially.}$$

$V \in \overline{\underline{\underline{N}}}_q$ ~~$V \neq V_{\omega q}$~~, and so there exists $k \in N$ such that $v^{q^k} \in \gamma_n(V)$ for all $n \in Z^+$, and all $v \in V$.

Hence as in the proof of part (a) and by using 3.3.9 (a), for all $v \in V$,

$$[(A^{\Lambda})^{qk(n-1)}, v] \leq [A^{\Lambda},_n W_f] \quad \text{for all } n \in Z^+$$

from Lemma 5.2.4; therefore

$$[(A^{\Lambda})^{q^{k(n-1)}}, V] \leq [A^{\Lambda},_n W_f] \quad \text{for all } n \in Z^+$$

$$\Rightarrow \qquad [\{\gamma_n(A)\ A^{q^{k(n-1)}}\}^{\Lambda}, V] \leq [A^{\Lambda},_n W_f] \quad \text{for all } n \in Z^+$$

Therefore as $kn \geq n$ and $kn \geq k(n-1)$ for all $n \in Z^+$, and $[A^{\Lambda},_n W_f] \,\Delta\, W_f$ for all $n \in Z^+$, we have

$$[\{\gamma_{kn}(A)\ A^{q^{kn}}\}^{\Lambda}, V]^{A^{\Lambda}} \leq [A^{\Lambda},_n W_f] \quad \text{for all } n \in Z^+$$

$$\Rightarrow \qquad \bigcap_{n \in Z^+} [\{\gamma_{kn}(A)\ A^{q^{kn}}\}^{\Lambda}, V]^{A^{\Lambda}} \leq [A\ ,_{\omega} W_f]$$

Since $kn \to \infty$ as $n \to \infty$, we may say

$$\bigcap_{n \in Z^+} [\{\gamma_n(A)\ A^{q^n}\}^{\Lambda}, V]^{A^{\Lambda}} \leq [A^{\Lambda},_{\omega} W_f]$$

i.e. $$\bigcap_{n \in Z^+} [\{\gamma_n(A)\ A^{q^n}\}^{\Lambda}, V_{\omega q}]^{A^{\Lambda}} \leq [A^{\Lambda},_{\omega} W_f]$$

Now for all primes p, $\{B_{\omega p} C_B(\Lambda)\}/C_B(\Lambda) \leq \{B/C_B(\Lambda)\}_{\omega p}$, and so for all primes p,

$$\begin{aligned}
\bigcap_{n \in Z^+} [\{\gamma_n(A)\ A^{p^n}\}^{\Lambda}, B_{\omega p}]^{A^{\Lambda}} &= \bigcap_{n \in Z^+} [\{\gamma_n(A)\ A^{p^n}\}^{\Lambda}, \{B_{\omega p} C_B(\Lambda)\}/C_B(\Lambda)]^{A^{\Lambda}} \\
&\leq \bigcap_{n \in Z^+} [\{\gamma_n(A)\ A^{p^n}\}^{\Lambda}, V_{\omega p}]^{A^{\Lambda}} \\
&\leq [A^{\Lambda},_{\omega} W_f] \\
&= [A^{\Lambda},_{\omega} W] \quad \text{by Lemma 5.1.6} \\
&\leq \gamma_{\omega}(W) \quad \text{as required.}
\end{aligned}$$

We now prove that

$$\bigcap_{n \in Z^+} [A^{\Lambda}_{\omega p}, B^{p^n} \gamma_n(B)]^{A^{\Lambda}} \leq \gamma_{\omega}(W) .$$

We show first that for allprimes p

$$\bigcap_{n \in Z^+} [A^{\Lambda}_{\omega p}, V^{p^n} \gamma_n(V)]^{A^{\Lambda}} \leq [A^{\Lambda},_{\omega} W_f]$$

The result is trivially true if $p = q$, for then $V^{p^n}\gamma_n(V) = \{1\}$ for large enough n .

Suppose $p \neq q$. Then $V^{p^n} = V$ for all $n \in Z^+$. Let $f \in A^\Lambda_{\omega p}$, say $f^{p^{r(n)}} \in \gamma_n(A^\Lambda)$ for all $n \in Z^+$. Then as in part (b),

$$[f, V^{p^{r(n)(n-1)}}\gamma_n(V)] \leq [A^\Lambda,_n W_f] \qquad \forall \ n \in Z^+$$

$$\Rightarrow \qquad [f, V\gamma_n(V)] \leq [A^\Lambda,_n W_f] \qquad \forall \ n \in Z^+$$

$$\Rightarrow \qquad [A^\Lambda_{\omega p}, V\gamma_n(V)] \leq [A^\Lambda,_n W_f] \qquad \forall \ n \in Z^+$$

$$\Rightarrow \qquad [A^\Lambda_{\omega p}, V\gamma_n(V)]^{A^\Lambda} \leq [A^\Lambda,_n W_f] \qquad \forall \ n \in Z^+$$

$$\Rightarrow \qquad \bigcap_{n \in Z^+} [A^\Lambda_{\omega p}, V^{p^n}\gamma_n(V)]^{A^\Lambda} = \bigcap_{n \in Z^+} [A^\Lambda_{\omega p}, V\gamma_n(V)]^{A^\Lambda}$$

$$\leq [A^\Lambda,_\omega W_f]$$

For all primes p, $\{B^{p^n}\gamma_n(B)C_B(\Lambda)\}/C_B(\Lambda) \leq V^{p^n}\gamma_n(V)$, and so for all primes p,

$$\bigcap_{n \in Z^+} [A^\Lambda_{\omega p}, B^{p^n}\gamma_n(B)]^{A^\Lambda} = \bigcap_{n \in Z^+} [A^\Lambda_{\omega p}, \{B^{p^n}\gamma_n(B)C_B(\Lambda)\}/C_B(\Lambda)]^{A^\Lambda}$$

$$\leq [A^\Lambda,_\omega W_f] = [A^\Lambda,_\omega W] \leq \gamma_\omega(W)$$

Hence result.

(2) Now suppose $A \in \underline{\underline{A}}$. Let p be a prime and define $A_p \leq A$ by $A_p = A_{\omega p} = \{ a \in A : a \text{ has p-power order} \}$. The required results now become

$$\bigcap_{n \in Z^+} [(A^{p^n})^\Lambda, B_{\omega p}] \leq \gamma_\omega(W) \qquad \text{for all primes p}$$

$$\bigcap_{n \in Z^+} [A^\Lambda_p, B^{p^n}\gamma_n(B)] \leq \gamma_\omega(W) \qquad \text{for all primes p}$$

The first result follows almost immediately from part (a). Let p be any prime. By Lemma 5.1.7, since A is abelian,

$$\bigcap_{n\in Z^+} [(A^{p^n})^\Lambda, B_{\omega p}] = [\bigcap_{n\in Z^+} (A^{p^n})^\Lambda, B_{\omega p}] = [\bar{A}_p^\Lambda, B_{\omega p}]$$

Hence from (a),

$$\bigcap_{n\in Z^+} [(A^{p^n})^\Lambda, B_{\omega p}] = [\bar{A}_p^\Lambda, B_{\omega p}] \leqslant \gamma_\omega(W) .$$

We now prove the second inequality. For $m \in N$ and any prime p, define $\Omega_{m,p}(A) = \{ a \in A : a^{p^m} = 1 \}$; then

$$A_p = A_{\omega p} = \bigcup_{m\in N} \Omega_{m,p}(A)$$

Let $m \in N$ and let $f \in \Omega_{m,p}(A)^\Lambda$. Then $[f^{p^m}, W] = 1$ and so as before,

$$[f, B^{p^{m(n-1)}}\gamma_n(B)] \leqslant [A^\Lambda, {}_nW] \qquad \forall\ n \in Z^+$$

Hence $\quad [\Omega_{m,p}(A)^\Lambda, B^{p^{m(n-1)}}\gamma_n(B)] \leqslant [A^\Lambda, {}_nW] \quad \forall\ n \in Z^+$

$$\Rightarrow \quad \bigcap_{n\in Z^+} [\Omega_{m,p}(A)^\Lambda, B^{p^{m(n-1)}}\gamma_n(B)] \leqslant [A^\Lambda, {}_\omega W]$$

Now for any $m \in Z^+$,

$$\bigcap_{n\in Z^+} [\Omega_{m,p}(A)^\Lambda, B^{p^{m(n-1)}}\gamma_n(B)] = \bigcap_{n\in Z^+} [\Omega_{m,p}(A)^\Lambda, B^{p^n}\gamma_n(B)]$$

For let f be an element of the left hand side, and let $n \in Z^+$. Then there exists $k \in Z^+$ such that $k > n$ and $m(k-1) > n$; so $f \in [\Omega_{m,p}(A)^\Lambda, B^{p^{k(n-1)}}\gamma_k(B)] \leqslant [\Omega_{m,p}(A)^\Lambda, B^{p^n}\gamma_n(B)]$. Hence $f \in \bigcap_{n\in Z^+} [\Omega_{m,p}(A)^\Lambda, B^{p^n}\gamma_n(B)]$.

Now let f be an element of the right hand side, and let $n \in Z^+$. Then there exists $k \in Z^+$ such that $k > n$ and $k > m(n-1)$; then

$f \in [\Omega_{m,p}(A)^\Lambda, B^{p^k}\gamma_k(B)] \leqslant [\Omega_{m,p}(A)^\Lambda, B^{p^{m(n-1)}}\gamma_n(B)]$. Hence $f \in \bigcap_{n\in Z^+} [\Omega_{m,p}(A)^\Lambda, B^{p^{m(n-1)}}\gamma_n(B)]$.

Therefore for all $m \in Z^+$, and clearly for $m = 0$ also,

$$\bigcap_{n\in Z^+} [\Omega_{m,p}(A)^{\Lambda}, B^{p^n}\gamma_n(B)] \leqslant \gamma_\omega(W) \qquad (*)$$

Now $\bigcap_{n\in Z^+} [A_p^{\Lambda}, B^{p^n}\gamma_n(B)]$

$= A_p^{\Lambda} \cap \bigcap_{n\in Z^+} [A^{\Lambda}, B^{p^n}\gamma_n(B)]$ by Lemma 5.1.7

$= \left\{ \bigcup_{m\in N} \Omega_{m,p}(A)^{\Lambda} \right\} \cap \bigcap_{n\in Z^+} [A^{\Lambda}, B^{p^n}\gamma_n(B)]$

$= \bigcup_{m\in N} \{\Omega_{m,p}(A)^{\Lambda} \cap \bigcap_{n\in Z^+} [A^{\Lambda}, B^{p^n}\gamma_n(B)]\}$

$= \bigcup_{m\in N} \{ \bigcap_{n\in Z^+} [\Omega_{m,p}(A)^{\Lambda}, B^{p^n}\gamma_n(B)]\}$

$\leqslant \gamma_\omega(W)$ from (*)

Hence result. This completes part (c) of Theorem 5.2.1.

Thus the proof of Theorem 5.2.1 is completed.

<u>Proof of Theorem 5.2.2</u> We have A any group, and (Λ, B) non-trivial and transitive such that $V = B/C_B(\Lambda) \in \underline{\underline{F}}_p$ for some prime p.

Let $W_f = A \text{ wr}^{\Lambda} V$. For $n \in Z^+$ write $A_n = A^{p^n}\gamma_n(A)$; then $A/A_n \in \overline{\underline{\underline{N}}}_p$ for all $n \in Z^+$.

Let $n \in Z^+$. Then $A_n^{\Lambda} \trianglelefteq W_f$ and

$W_f/A_n^{\Lambda} \cong \{A/A_n\} \text{ wr}^{\Lambda} V \in \underline{\underline{N}}$ by Theorem 3.3.1. Therefore there exists $k \in Z^+$ such that $\gamma_k(W_f) \leqslant A_n^{\Lambda}$, and so

$\gamma_{k+1}(W_f) \leqslant [A_n^{\Lambda}, W_f] = \langle [A_n^{\Lambda}, V], [A_n^{\Lambda}, A^{\Lambda}]^V \rangle$

$\leqslant [A_n^{\Lambda}, V]^{A^{\Lambda}}$ by Lemma 5.1.4

Therefore $\gamma_\omega(W_f) \leqslant [A_n^{\Lambda}, V]^{A^{\Lambda}} \quad \forall\ n \in Z^+$

$\Rightarrow \gamma_\omega(W) \leqslant [A_n^{\Lambda}, B]^{A^{\Lambda}}\gamma_\omega(B) \quad \forall\ n \in Z^+$ by Lemma 5.1.6

$\Rightarrow \gamma_\omega(W) \leqslant \bigcap_{n\in Z^+} [A_n^{\Lambda}, B]^{A^{\Lambda}}\gamma_\omega(B)$ as required.

Proof of Theorem 5.2.3   By Lemma 5.1.8,

$$\gamma_\omega(W) = \langle \gamma_\omega(A \text{ wr}^\Theta B) : \Theta \text{ an orbit} \rangle$$

Let $\{\sigma\}$ be a trivial orbit. Then $A \text{ wr}^{\{\sigma\}} B = A_\sigma \times B$ and so

$$\gamma_\omega(A \text{ wr}^{\{\sigma\}} B) = \gamma_\omega(A_\sigma) \times \gamma_\omega(B)$$

So $$\langle \gamma_\omega(A \text{ wr}^{\{\sigma\}} B) : \sigma \in \Sigma \rangle = \gamma_\omega(A)^\Sigma . \gamma_\omega(B)$$

Let Θ be a non-trivial orbit. Then

$B/C_B(\Theta) \cong \{B/C_B(\Lambda)\}/\{C_B(\Theta)/C_B(\Lambda)\} \in \underline{\underline{F}}_p$, and so by Theorem 5.2.2,

$$\gamma_\omega(A \text{ wr}^\Theta B) \leqslant \bigcap_{n \in Z^+} [(A^{p^n}\gamma_n(A))^\Theta, B]^{A^\Theta} \gamma_\omega(B)$$

But
$$\bigcap_{n \in Z^+} [(A^{p^n}\gamma_n(A))^\Theta, B]^{A^\Theta}$$
$$= \bigcap_{n \in Z^+} [(A^{p^n}\gamma_n(A))^\Theta, B/C_B(\Theta)]^{A^\Theta}$$
$$= \bigcap_{n \in Z^+} [(A^{p^n}\gamma_n(A))^\Theta, \{B/C_B(\Theta)\}_{\omega p}]^{A^\Theta}$$
$$\leqslant \gamma_\omega(A \text{ wr}^\Theta \{B/C_B(\Theta)\}) \cap A^\Theta \qquad \text{by Theorem 5.3.1 (c)}$$
$$= \gamma_\omega(A \text{ wr}^\Theta B) \cap A^\Theta \qquad \text{by Lemma 5.1.6}$$

Hence $\gamma_\omega(A \text{ wr}^\Theta B) = \bigcap_{n \in Z^+} [(A^{p^n}\gamma_n(A))^\Theta, B]^{A^\Theta} \gamma_\omega(B)$ and so

$$\gamma_\omega(W) = \langle \gamma_\omega(A)^\Sigma, \bigcap_{n \in Z^+} [(A^{p^n}\gamma_n(A))^\Theta, B], \gamma_\omega(B) : \Theta \text{ a non-trivial orbit} \rangle$$

as required.

If $A \in \underline{\underline{A}}$, then $\gamma_\omega(A) = \{1\}$ and

$$\bigcap_{n \in Z^+} [(A^{p^n}\gamma_n(A))^\Lambda, B] = [\bigcap_{n \in Z^+} (A^{p^n}\gamma_n(A))^\Lambda, B] \qquad \text{by Lemma 5.1.7}$$
$$= [\bar{A}_p^\Lambda, B]$$

Hence $\gamma_\omega(A \text{ wr}^\Lambda B) = [\bar{A}_p^\Lambda, B]\gamma_\omega(B)$ in this case.

We now consider standard wreath products. We prove

Theorem 5.2.4 Let A be an abelian group, and let B be any group. Let $W = A \text{ wr } B$. Then

$$\gamma_\omega(W) \leq [\bar{A}_p^B, B][A^B, \bar{B}_p]\gamma_\omega(B) \qquad \text{for all primes } p$$

Theorem 5.2.5 Let A be a p-group and let B be a non-trivial p'-group, where p is a prime. Let $W = A \text{ wr } B$. Then

$$\gamma_\omega(W) = [A^B, B]\gamma_\omega(B)$$

Theorem 5.2.6 Let A be an abelian p-group and let B be a p-group for some prime p. Then

$$\gamma_\omega(W) = [\bar{A}_p^B, B][A^B, \bar{B}_p]\gamma_\omega(B)$$

Theorem 5.2.7 Let p be a prime. Let A be an abelian p-group, and suppose ~~$B = B_p \times B_{p'}$, where B_p is a p-group and $B_{p'}$ is a p'-group.~~ B is a group such that $B/\bar{B}_p$ is a p-group. Then

$$\gamma_\omega(W) = [A^B, \bar{B}_p][\bar{A}_p^B, B]\gamma_\omega(B)$$

The following result provides the key to most of these theorems.

Theorem 5.2.8 [8] Let p be a prime. Let $A \in \underline{\underline{A}} \cap R\bar{\underline{\underline{N}}}_p$ and let $B \in R\bar{\underline{\underline{N}}}_p$. Then $A \text{ wr } B \in R\bar{\underline{\underline{N}}}_p$.

This gives us immediately

Proposition 5.2.9 Let p be a prime and let $A \in \underline{\underline{A}} \cap R\bar{\underline{\underline{N}}}_p$. Let B be a group and let $W = A \text{ wr } B$. Then

$$\gamma_\omega(W) \leq [A^B, \bar{B}_p]\gamma_\omega(B)$$

Proof: Since $A \in \underline{\underline{A}}$,

$$W/\{[A^B,\bar{B}_p]\bar{B}_p\} \cong A \text{ wr } \{B/\bar{B}_p\} \quad \text{by Lemma 4.1.8}$$

Then by Theorem 5.2.8, $A \text{ wr } \{B/\bar{B}_p\} \in \underline{\underline{R\bar{N}}}_p \leq \underline{\underline{RN}}$, and so

$$\gamma_\omega(W) \leq [A^B,\bar{B}_p]\bar{B}_p$$

But ~~$\gamma_\omega(W) \cap B = \gamma_\omega(B) \leq \bar{B}_p$; hence~~ $\gamma_\omega(W) \leq A^B\gamma_\omega(B)$; hence

$$\gamma_\omega(W) \leq [A^B,\bar{B}_p]\gamma_\omega(B)$$

We may now prove Theorem 5.2.4.

Proof of Theorem 5.2.4 Let p be a prime. Since $A/\bar{A}_p \in \underline{\underline{R\bar{N}}}_p$, by Proposition 5.2.9 we have

$$\gamma_\omega(\{A/\bar{A}_p\} \text{ wr } B) \leq [(A/\bar{A}_p)^B,\bar{B}_p]\gamma_\omega(B)$$

Under the isomorphism from $\{A/\bar{A}_p\}$ wr B to $W/\bar{A}_p^B$ constructed in Lemma 3.6.8, $[(A/\bar{A}_p)^B,\bar{B}_p]\gamma_\omega(B)$ maps onto $\{[A^B,\bar{B}_p]\bar{A}_p^B\gamma_\omega(B)\}/\bar{A}_p^B$, and so $\gamma_\omega(W) \leq [A^B,\bar{B}_p]\bar{A}_p^B\gamma_\omega(B)$.

Since $A \in \underline{\underline{A}}$, $\gamma_\omega(W)$ ~~$\cap A^B$~~ $\leq [A^B,B]\gamma_\omega(B)$; hence

$$\gamma_\omega(W) \leq \{[A^B,\bar{B}_p]\bar{A}_p^B \cap [A^B,B]\}\gamma_\omega(B)$$

$$= [A^B,\bar{B}_p]\{\bar{A}_p^B \cap [A^B,B]\}\gamma_\omega(B)$$

$$= [A^B,\bar{B}_p][\bar{A}_p^B,B]\gamma_\omega(B)$$

and so $\gamma_\omega(W) \leq [\bar{A}_p^B,B][A^B,\bar{B}_p]\gamma_\omega(B)$

Proof of Theorem 5.2.5 Let A be a p-group and let B be a p′-group for some prime p. Then $\bar{B}_p = B$ and $A_{\omega p} = A$, and so by Theorem 5.2.1 (b), $[A^B,B] = [A^B_{\omega p},\bar{B}_p] \leq \gamma_\omega(W)$.
Since (B , B) is non-trivial and transitive, by Lemma 5.1.5 $\gamma_\omega(W) \leq [A^B,B]\gamma_\omega(B)$; therefore $\gamma_\omega(W) = [A^B,B]\gamma_\omega(B)$.

Proof of Theorem 5.2.6 Since A and B are p-groups, $A_{\omega p} = A$ and $B_{\omega p} = B$. Hence by Theorem 5.2.1 (a) and (b),

$$[A^B, \bar{B}_p][\bar{A}_p^B, B] \leq \gamma_\omega(W) \text{ and so}$$

$$[A^B, \bar{B}_p][\bar{A}_p^B, B]\gamma_\omega(B) \leq \gamma_\omega(W)$$

But by Theorem 5.2.4,

$$\gamma_\omega(W) \leq [A^B, \bar{B}_p][\bar{A}_p^B, B]\gamma_\omega(B) \quad ; \text{ therefore}$$

$$\gamma_\omega(W) = [A^B, \bar{B}_p][\bar{A}_p^B, B]\gamma_\omega(B)$$

Proof of Theorem 5.2.7 Since A is a p-group, $A_{\omega p} = A$ ~~; and $B = B_p \times B_p$, $\Rightarrow B_{\omega p} = B_p$ and $\bar{B}_p = B_p$~~. Then by Theorem 5.2.1 (b),

$[A^B, \bar{B}_p] = [A^B_{\omega p}, \bar{B}_p] \leq \gamma_\omega(W)$, and so

$$\gamma_\omega(W)/[A^B, \bar{B}_p] = \gamma_\omega(W/[A^B, \bar{B}_p]) \cong \gamma_\omega(A \text{ wr}^{\{B/\bar{B}_p\}} B)$$

by Lemma 4.1.8.

Now $C_B(B/\bar{B}_p) = \bar{B}_p$, and so

$$\begin{aligned}
A^{\{B/\bar{B}_p\}} \cap \gamma_\omega(A \text{ wr}^{\{B/\bar{B}_p\}} B) &= \gamma_\omega(A \text{ wr}^{\{B/\bar{B}_p\}} \{B/\bar{B}_p\}) \cap A^{\{B/\bar{B}_p\}} && \text{by Lemma 5.1.6} \\
&= \gamma_\omega(A \text{ wr } \{B/\bar{B}_p\}) \cap A^{\{B/\bar{B}_p\}} && \text{by Lemma 4.1.7 (b)} \\
&= [A^{\{B/\bar{B}_p\}}, \overline{\{B/\bar{B}_p\}}_p][\bar{A}_p^{\{B/\bar{B}_p\}}, B/\bar{B}_p] && \text{by Theorem 5.2.6} \\
&= [\bar{A}_p^{\{B/\bar{B}_p\}}, B/\bar{B}_p] && \text{~~since } \bar{B}_p = B_p\text{~~} \\
&= [\bar{A}_p^{\{B/\bar{B}_p\}}, B] && \text{by Lemma 5.1.6}
\end{aligned}$$

Hence $\gamma_\omega(A \text{ wr}^{\{B/\bar{B}_p\}} B) = [\bar{A}_p^{\{B/\bar{B}_p\}}, B]\gamma_\omega(B)$ and so

$$\gamma_\omega(W) = [\bar{A}_p^B, B][A^B, \bar{B}_p]\gamma_\omega(B)$$

Section 5.3 The case A/A' and $B/\gamma_\omega(B)$ periodic, with $B \not\in \bigcup_{p\in P} \underline{\underline{F}}_p$

We characterise $\gamma_\omega(A \text{ wr } B)$ where A is a group such that A/A' is periodic, and B is a group such that $B \not\in \bigcup_{p\in P} \underline{\underline{F}}_p$ and $B/\gamma_\omega(B)$ is periodic.

Theorem 5.3.1 Let A and B be groups and let $W = A \text{ wr } B$. Suppose A/A' and $B/\gamma_\omega(B)$ are periodic, and that $B \not\in \bigcup_{p\in P} \underline{\underline{F}}_p$. For any prime p let A_p/A' be the Sylow p-subgroup of A/A', and for any set of primes Π let $B_\Pi/\gamma_\omega(B)$ be the Sylow Π-subgroup of $B/\gamma_\omega(B)$. Then

$$\gamma_\omega(W) = \{ \operatorname*{Dr}_{p\in P} [A_p^B, \overline{B}_p][\overline{(A_p)}_p^B, B]\} [A^B, \gamma_\omega(B)] (A')^B \gamma_\omega(B)$$

Proof: By Theorem 5.1.17, $(A')^B \leq \gamma_\omega(W)$; therefore

$\gamma_\omega(W)/(A')^B = \gamma_\omega(W/(A')^B) \cong \gamma_\omega(\{A/A'\} \text{ wr } B)$ by Lemma 3.6.8

Let $\underline{A} = A/A'$ and $\underline{W} = A/A' \text{ wr } B$. Then

$[\underline{A}^B, \gamma_\omega(B)]\gamma_\omega(B) \leq \gamma_\omega(\underline{W})$, and so

$$\gamma_\omega(\underline{W})/\{[\underline{A}^B,\gamma_\omega(B)]\gamma_\omega(B)\} = \gamma_\omega(\underline{W}/\{[\underline{A}^B,\gamma_\omega(B)]\gamma_\omega(B)\})$$

$$\cong \gamma_\omega(\underline{A} \text{ wr } \{B/\gamma_\omega(B)\}) \text{ by Lemma 4.1.8}$$

Let $B_* = B/\gamma_\omega(B)$ and let $\underline{W}_* = \underline{A} \text{ wr } B_*$; let $B_\Pi/\gamma_\omega(B) = B_{*\Pi}$ for all sets Π of primes.

Since A/A' and B_* are ~~is~~ periodic, $A/A' = \operatorname*{Dr}_{p\in P} A_p/A' = \operatorname*{Dr}_{p\in P} \underline{A}_p$ say, and $B_* = \operatorname*{Dr}_{p\in P} B_{*p}$, by Lemma 5.1.18.

Then $\overline{(B_*)}_p \geq \operatorname*{Dr}_{q\neq p} B_{*q}$, and so $B_*/\overline{(B_*)}_p$ is a p-group for all primes p.

And so by Lemma 5.1.8, and Theorem 5.3.7,

$$\gamma_\omega(\underline{W}_*) = \operatorname*{Dr}_{p\in P} (\underline{A}_p \text{ wr } B_*) = \operatorname*{Dr}_{p\in P} [\underline{A}_p^{B_*}, \overline{B}_{*p}][\overline{(\underline{A}_p)}_p^{B_*}, B_*]$$

~~by Theorem 5.2.7, since for each p, $B_* = B_{*p} \times \overline{B}_{*p}$ by Lemma 5.1.18.~~

Hence, since $\gamma_\omega(B) \leq \bar{B}_p \ \forall \text{primes } p$, $\bar{B}_p/\gamma_\omega(B) \cong (\overline{B_*})_p$,

$$\gamma_\omega(\underline{W}) = \{ \underset{p \in P}{\mathrm{Dr}} [\underline{A}_p^B, \bar{B}_p][\overline{(\underline{A}_p)}_p^B, B] \} [\underline{A}^B, \gamma_\omega(B)] \ \gamma_\omega(B)$$

and so

$$\gamma_\omega(W) = \{ \underset{p \in P}{\mathrm{Dr}} [A_p^B, \bar{B}_p][\overline{(A_p)}_p^B, B] \} [A^B, \gamma_\omega(B)] \ (A')^B \ \gamma_\omega(B)$$

Hence result.

Section 5.4 B a perfect group

We may characterise $\gamma_\omega(A \text{ wr}^\Lambda B)$ completely for the case B a perfect group. We prove

Theorem 5.4.1 Let A be a group, let (Λ, B) be a pair with B perfect, and let $W = A \text{ wr}^\Lambda B$. Define $\Sigma \subseteq \Lambda$ by $\Sigma = \{ \sigma \in \Lambda : \{\sigma\} \text{ is an orbit of } B \text{ on } \Lambda \}$. Then

$$\gamma_\omega(W) = \gamma_\omega(A)^\Sigma . [A^{\Lambda \setminus \Sigma}, B]B$$

Proof: Let Θ be an orbit and let $W_\Theta = A \text{ wr}^\Theta B$. If $\Theta = \{\sigma\}$ for some $\sigma \in \Sigma$, then (Θ, B) is trivial and $W_\Theta = A^\Theta \times B = A_\sigma \times B$; hence $\gamma_\omega(W_\Theta) = \gamma_\omega(A_\sigma) \times \gamma_\omega(B) = \gamma_\omega(A_\sigma) \times B$.

If $\Theta \neq \{\sigma\}$ for any $\sigma \in \Sigma$, then (Θ, B) is non-trivial and hence $B/C_B(\Theta) \notin \underline{\underline{N}}$, since B is perfect. We show that this implies that $[A^\Theta, B] \leq \gamma_\omega(W_\Theta)$, whence $[A^\Theta, B]B \leq \gamma_\omega(W_\Theta)$.

Let $N \trianglelefteq W_\Theta$, with $W_\Theta/N \in \underline{\underline{N}}$. Then by Lemma 4.1.4, $N \cap B \not\leq C_B(\Theta)$, and hence by Lemma 3.2.4, $(A')^\Theta \leq N$. We now show that $[A^\Theta, B] \leq N$.

Let $f \in A^\Theta$ and $b \in B$. $N.A^\Theta \geq \gamma_\omega(W_\Theta)A^\Theta = W_\Theta$ since $B \leq \gamma_\omega(W_\Theta)$, and so there exists $g \in A^\Theta$ such that $bg \in N$; hence $[f, bg] \in N$. Now $[f, bg] = [f, g][f, b]^g$ and so since $(A')^\Theta \leq N$, $[f, g] \in N$ and so $[f, b]^g \in N$. Hence $[f, b] \in N$, since $N \trianglelefteq W$. Hence

$[A^{\Theta},B] \leq N$.

N was any normal subgroup of W_{Θ} such that $W_{\Theta}/N \in \underline{\underline{N}}$; and so $[A^{\Theta},B] \leq \gamma_{\omega}(W_{\Theta})$ as required.

By Corollary 5.1.5,

$$\gamma_{\omega}(W_{\Theta}) \leq [A^{\Theta},B]\gamma_{\omega}(B) = [A^{\Theta},B]B \quad ; \text{ hence}$$

$$\gamma_{\omega}(W_{\Theta}) = [A^{\Theta},B]B \ .$$

Therefore by Lemma 5.1.8,

$$\begin{aligned}\gamma_{\omega}(W) &= \langle \gamma_{\omega}(W_{\Theta}) : \Theta \text{ an orbit} \rangle \\ &= \langle \gamma_{\omega}(A_{\sigma})B,\ [A^{\Theta},B]B : \sigma \in \Sigma,\ \Theta \text{ an orbit},\ \Theta \subseteq \Lambda \setminus \Sigma \rangle \\ &= \gamma_{\omega}(A)^{\Sigma}\ [A^{\Lambda \setminus \Sigma},B]B \ .\end{aligned}$$

Hence result.

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