Conformal Killing spinors in supergravity and related aspects of spin geometry

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Abstract

The aims of this thesis are two-fold: finding a geometric realisation for Nahm's conformal superalgebras and generalising the concept of a conformal Killing spinor to supergravity, in particular M-theory.

We introduce the necessary tools of conformal geometry and construct a conformal Killing superalgebra (that turns out not to be a Lie superalgebra in general) out of the conformal Killing vectors and the conformal Killing spinors of a semi-Riemannian spin manifold and investigate a natural definition of the spinorial Lie derivative that differs from the more commonly used Kosmann-Schwarzbach Lie derivative. We then attempt to generalise the definition of conformal Killing spinors to M-theory and characterise M-theory backgrounds admitting such spinors. We also construct a M-theory analogue of the conformal Killing superalgebra. We show that further examples can be constructed in type IIA and in the massive IIA theory of Howe, Lambert and West via Kaluza-Klein reduction. We also comment on a curious identity involving the Penrose operator in type IIB supergravity.

Finally — building on known results about the relationship between the dimension of the space of conformal Killing spinors on a non-simply connected manifold and the choice of spin structure — we explore the importance of the choice of spin structure in determining the amount of supersymmetry preserved by a symmetric M-theory background constructed by quotienting a supersymmetric Hpp-wave with a discrete subgroup in the centraliser of its isometry group.

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Isälle ja äidille: kiitos, että toivoitte, luotitte ja uskoitte.

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Introduction

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Riemannin metrinen tensori on
Meidän Braillemme todellisuudelle

It took a lot of thought
to conclude
The human brain
is blind to the the four-dimensional
The Riemann metric tensor is
Our Braille for reality

—*A. W. Yrjänä,* Tesserakti

It is widely acknowledged that one of the key components in any theory that attempts to formulate physics beyond the Standard Model must be *supersymmetry.* During the last three decades the study of supersymmetric field theories and superstring theories has grown into a vast subject. Supersymmetry has appeared in many guises both in pure mathematics (inspiring intense study of Calabi-Yau manifolds and manifolds of exceptional holonomy) and theoretical physics, and the hope of finding supersymmetric partners of known particles continues to drive experimental particle physics as well.

In 1975, Haag, Lopuszański, and Sohnius [1] showed that under relatively weak assumptions, the only possible symmetries of the S-matrix of a quantum field theory in addition to the standard Poincaré symmetries and "internal" symmetries related to conserved quantum numbers are those which mix *bosonic* symmetries with

fermionic ones. Soon afterwards, Nahm [2] gave a classification of possible supersymmetry algebras, based on Kac's classification of simple Lie superalgebras [3].
In the mid-70s, study of supersymmetric field theories led to the realisation that in addition to global supersymmetry invariance, it is possible to construct theories with *local* supersymmetries, that is, theories with supersymmetry invariance where the fermionic generators are allowed to depend on spacetime coordinates. This led to the discovery of the supergravity zoo — a bewildering number of theories in spacetime dimensions ranging up to eleven, all incorporating Einstein's gravity and a variety of other bosonic and fermionic fields. Many of these theories were constructed by "gauging" [4] one of Nahm's supersymmetry algebras — basically by requiring that the ground state of the theory should naturally admit a given symmetry algebra.
A particularly interesting discovery was the eleven-dimensional supergravity in 1976 [5], which was found to be essentially unique: for a short while in the early 80s (together with its Kaluza-Klein compactifications) it was even a contender for the coveted title of Theory of Everything [6]. However, it was soon overtaken by its younger and hungrier siblings, so-called *superstring* theories [7, 8]. Superstring theories approach the problem of quantum gravity by quantising a one-dimensional extended object — the string — instead of a pointlike particle. They have attracted an enormous amount of attention from the mid-80s onwards, experiencing an explosive renaissance during the past decade or so.
It was during this recent burst of activity that eleven-dimensional supergravity again rose into prominence. The five consistent superstring theories — Type I, Type IIA, Type IIB, Heterotic $E_8 \times E_8$ and Heterotic Spin$/\mathbb{Z}_2$ — all feature spacetime local supersymmetry in ten dimensions. In fact, their low energy limits correspond to known ten-dimensional supergravity theories. It is a long-standing conjecture that there is an eleven-dimensional quantum theory, tentatively called *M-theory* that underlies all the known string theories and relates them to each other via a complex web of *dualities*. The low-energy limit of M-theory is believed to be no other than eleven-

dimensional supergravity, and these days the terms are often used interchangeably in the literature.

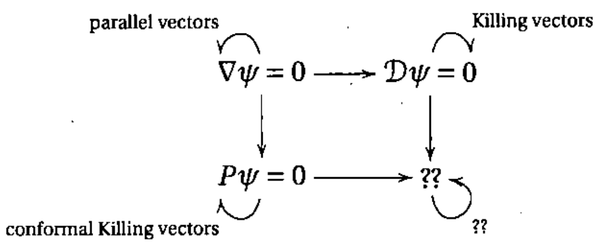
Since the study of non-perturbative sectors of these theories is at present extremely difficult, much of recent research has focused on studying supersymmetric solutions of eleven-dimensional supergravity and lower-dimensional supergravities. Since the bosonic sectors of supergravity theories resemble Einstein's gravity coupled to Maxwell-like p-form fields, they can be studied using classical tools of differential geometry and spin geometry. In particular, it turns out that supersymmetry of a supergravity background can be characterised geometrically: it corresponds to the existence of so-called *supergravity Killing spinors*, spinors which are parallel with respect to a special connection induced from the supersymmetry variation of the gravitino. In this sense, supergravity Killing spinors are a supergravity generalisation of *geometric* parallel spinors[1]. Parallel spinors have, of course, also played a crucial role in string theory in the context of realistic compactifications of string theories and M-theory to four dimensions due to their relationship with manifolds of special holonomy.

There are many parallels between the study of supergravity Killing spinors and the study of special spinors (that is, spinors annihilated by some natural differential operator) in spin geometry. Parallel spinor fields are in fact special cases of a more general class of objects called *conformal Killing spinors* or *twistor spinors*. Conformal Killing spinors were originally introduced by Penrose in the context of general relativity [9] and appeared in pure mathematics as integrability conditions for the complex structure of a four-dimensional Riemannian manifold [10]. In the late 80s Lichnerowicz started a systematic investigation of conformal Killing spinors on Riemannian spin manifolds in the context of conformal differential geometry [11], and since then, a body of strong structure results, examples and a partial classification has developed, both in the Riemannian and in the Lorentzian setting (see e.g. [12, 13, 14, 15] and references therein).

[1]A better name might be "superparallel spinors", to avoid confusion with *geometric* Killing spinors.

The space of parallel vector fields and parallel spinors on a semi-Riemannian spin manifold can be given the structure of a Lie superalgebra, with the Lie bracket between a vector field and a spinor given by the spinorial Lie derivative (which is actually trivial in this case). The odd bracket between two spinors can be defined as the so-called *Dirac current* associated to the spinors, which is also parallel if the spinors are. Similarly, one natural object one can associate to a supergravity background is its *supersymmetry superalgebra*. This is a Lie superalgebra constructed out of the Killing spinors of the background and the Killing vectors of the background's metric which also preserve the p-form fields: guaranteeing the closure of these algebras also usually requires imposing the equations of motion of the theory. The supersymmetry superalgebras of many supergravity backgrounds have been computed [16, 17, 18, 19] and it has been found that many of them correspond to Lie superalgebras on Nahm's list. Thus, many of the supersymmetry algebras have a manifest geometric origin.

The space of conformal Killing spinors and conformal Killing vectors on a semi-Riemannian manifold also admits a natural algebraic structure, first investigated by Habermann[20], which we call a *conformal Killing superalgebra*. It would thus be natural to be able to fill in the question marks in the lower right corner of the following diagram



where ∇ is the Levi-Cività connection, P is the so-called *Penrose operator* whose kernel defines conformal Killing spinors, and $\mathcal{D}$ is the supercovariant connection.

The primary goal of this thesis is to construct the supergravity analogue of conformal Killing spinors in eleven-dimensional supergravity and type IIA and IIB supergravities and to see if supergravity backgrounds admitting such spinors can be characterised geometrically. A secondary — although not unrelated — goal is more algebraic in flavor. Many of the superalgebras on Nahm's list incorporate conformal symmetry algebras in their even part. Although some of these *superconformal algebras* have been realised as symmetry superalgebras of supergravity backgrounds with an AdS factor, others still lack a manifest geometric origin. We will construct conformal Killing superalgebras both in geometric and supergravity context and try to see if these objects could provide a geometric origin for known Lie superalgebras involving conformal symmetries.

Along the way we are naturally led to reconsider the definition of the spinorial Lie derivative and introduce other machinery such as Weyl connections and Kaluza-Klein reductions: we use the latter as a tool to construct a number of explicit examples of conformal Killing superalgebras.

Chapter 2

Preliminaries

In this chapter we introduce the notation and some basic algebraic and geometric tools that we will need in the following chapters. The material herein is mostly standard, although presented with a view towards our applications and utilising some non-standard but useful notation.

2.1 Algebraic preliminaries

2.1.1 Natural properties of vector spaces with an inner product

Let V be a n-dimensional vector space equipped with an inner product $\langle -, - \rangle$. There are natural "musical" isomorphisms $\flat : V \to V^*$ and $\sharp : V^* \to V$ relating V and its dual, given by

$$\begin{aligned} X^\flat(Y) &= \langle X, Y \rangle \\ \langle \alpha^\sharp, Y \rangle &= \alpha(Y) \,. \end{aligned}$$

where $\alpha \in V^*$ and $X, Y \in V$. The isomorphisms also induce an inner product on V^*, which we will similarly denote $\langle -, - \rangle$. Any endomorphism of V also acts naturally on V^*. Given $A \in \mathrm{End}(V)$ and $\beta \in V^*$, $A\beta = -\beta \circ A$, or

$$(A\beta)(Y) = -\beta(AY) \,. \tag{2.1}$$

Now let $\mathfrak{so}(V)$ be the skew-symmetric endomorphisms of V with respect to the inner product. If $X \in V$ is a vector field and $\alpha \in V^*$ is a 1-form, then we define $X \curlywedge \alpha \in \mathfrak{so}(V)$

by

$$\langle (X \curlywedge \alpha)Y, Z\rangle = \alpha(Y)\langle X, Z\rangle - \langle X, Y\rangle \alpha(Z) \tag{2.2}$$

for all $Y, Z \in V$.

There is also a natural isomorphism $\mathfrak{so}(V) \cong \Lambda^2 V^*$. Given a skew-symmetric endomorphism $A \in \mathfrak{so}(V)$, we can find a corresponding two-form ω_A via

$$\omega_A(X, Y) = \langle X, A(Y)\rangle . \tag{2.3}$$

Correspondingly, any $\omega \in \Lambda^2 V^*$ defines a skew-symmetric endomorphism A^ω by

$$A^\omega(X) = -(\iota_X \omega)^\sharp , \tag{2.4}$$

where the sign on the right-hand side turns out to be convenient later on.

Let $\mathfrak{co}(V)$ denote those endomorphisms whose symmetric part is proportional to the identity — in other words, $\mathfrak{co}(V) = \mathfrak{so}(V) \oplus \mathbb{R}\,\mathrm{Id}_V$. We define a natural map

$$\bullet : V \otimes V^* \to \mathfrak{co}(V) \tag{2.5}$$

by $X \bullet \alpha = X \curlywedge \alpha + \alpha(X)\,\mathrm{Id}$, or more explicitly

$$(X \bullet \alpha)(Y) = \alpha(Y)X + \alpha(X)Y - \langle X, Y\rangle \alpha^\sharp \tag{2.6}$$

Note that this is manifestly symmetric in X, Y, so in fact $(X \bullet \alpha)(Y) = (Y \bullet \alpha)(X)$. Similarly, we can exhibit the action of $X \bullet \alpha$ on V^*:

$$\big((X \bullet \alpha)(\beta)\big)(Y) = -\beta((X \bullet \alpha)(Y)) = -\alpha(X)\beta(Y) - \beta(X)\alpha(Y) + \langle X, Y\rangle \langle \alpha, \beta\rangle ,$$

which is also symmetric in α, β and hence $(X \bullet \alpha)(\beta) = (X \bullet \beta)(\alpha)$.

We collect these observations and other properties of the $\bullet$ operator into the following useful

Lemma 1. *The following identities hold for all $X, Y \in V$, $\alpha, \beta \in V^*$ and $A \in \mathfrak{co}(V)$:*

(a) $(X \bullet \alpha)Y = (Y \bullet \alpha)X$

(b) $(X \bullet \alpha)\beta = (X \bullet \beta)\alpha$

(c) $[A, X \bullet \alpha] = A(X) \bullet \alpha + X \bullet A(\alpha)$

(d) $[X \bullet \alpha, Y \bullet \alpha]\alpha = 0$,

where $[-,-]$ *denotes the natural commutator of two endomorphisms.*

Proof. We've already established (a) and (b).

(c): Both sides of the equation are linear, so it is sufficient to check it for $A \in \mathfrak{so}(V)$ — obviously, the result holds if $A = \mathrm{Id}_V$, since then both sides vanish identically. Therefore, we assume that A is skew-symmetric and compute the left-hand side:

$$[A, X \bullet \alpha](Y) = \alpha(Y)A(X) - \langle X, Y\rangle A(\alpha^\sharp) - \alpha(A(Y))\,X + \langle X, A(Y)\rangle \alpha^\sharp\,,$$

On the other hand, the first term on the right-hand side gives

$$\begin{aligned}(A(X) \bullet \alpha)(Y) &= \alpha(Y)A(X) + \alpha(A(X))\,Y - \langle A(X), Y\rangle \alpha^\sharp \\ &= \alpha(Y)A(X) + \alpha(A(X))\,Y + \langle X, A(Y)\rangle \alpha^\sharp\,,\end{aligned}$$

whereas the second term yields

$$\begin{aligned}(X \bullet A(\alpha))(Y) &= A(\alpha)(Y)X + A(\alpha)(X)Y - \langle X, Y\rangle A(\alpha)^\sharp \\ &= -\alpha(A(Y))X - \alpha(A(X))Y - \langle X, Y\rangle A(\alpha^\sharp)\,.\end{aligned}$$

Adding the last two equations gives the result.

(d): We define the one-form $\omega(X, Y) \in V^*$ by

$$\omega(X, Y) = [X \bullet \alpha, Y \bullet \alpha]\alpha\,,$$

which is manifestly antisymmetric in X, Y. We now use (c), (a) and (b), in that order, to find

$$\begin{aligned}\omega(X,Y) &= \tfrac{1}{2}(\omega(X,Y) - \omega(X,Y)) \\ &= \tfrac{1}{2}(X \bullet \alpha)Y \bullet \alpha)\alpha + (Y \bullet (X \bullet \alpha)\alpha - ((Y \bullet \alpha)X \bullet \alpha)\alpha - (X \bullet (Y \bullet \alpha)\alpha)\alpha) \\ &= \tfrac{1}{2}((Y \bullet (X \bullet \alpha)\alpha)\alpha - (X \bullet (Y \bullet \alpha)\alpha)\alpha) \\ &= \tfrac{1}{2}((Y \bullet \alpha)(X \bullet \alpha)\alpha - (X \bullet \alpha)(Y \bullet \alpha)\alpha) \\ &= \tfrac{1}{2}[Y \bullet \alpha, X \bullet \alpha]\alpha \\ &= -\tfrac{1}{2}\omega(X,Y)\end{aligned}$$

so $\omega(X,Y)=0$, which proves the result. $\square$

2.1.2 The Möbius Lie algebra of V

Consider the vector space $V\oplus \mathfrak{co}(V)\oplus V^*$. We can make it into a Lie algebra $\mathfrak{mo}(V)$, called the *Möbius Lie algebra of V*, by introducing the Lie bracket given by

$$\left[\begin{pmatrix}X\\A\\\beta\end{pmatrix},\begin{pmatrix}Y\\B\\\gamma\end{pmatrix}\right]=\begin{pmatrix}AY-BX\\ [A,B]+X\bullet\gamma-Y\bullet\beta\\ A\gamma-B\beta\end{pmatrix}, \tag{2.7}$$

which is manifestly antisymmetric. It is a straightforward calculation to show that the Jacobi identity of this bracket vanishes. Using (2.7), the vanishing of

$$\left[\begin{pmatrix}X\\A\\\alpha\end{pmatrix},\left[\begin{pmatrix}Y\\B\\\beta\end{pmatrix},\begin{pmatrix}Z\\C\\\gamma\end{pmatrix}\right]\right]+\left[\begin{pmatrix}Z\\C\\\gamma\end{pmatrix},\left[\begin{pmatrix}X\\A\\\alpha\end{pmatrix},\begin{pmatrix}Y\\B\\\beta\end{pmatrix}\right]\right]+\left[\begin{pmatrix}Y\\B\\\beta\end{pmatrix},\left[\begin{pmatrix}Z\\C\\\gamma\end{pmatrix},\begin{pmatrix}X\\A\\\alpha\end{pmatrix}\right]\right]$$

is equivalent to

$$\begin{aligned}
0&=-(X\bullet\beta)Z+(Y\bullet\alpha)Z-(Z\bullet\alpha)Y+(X\bullet\gamma)Y\,,\\
0&=[A,Y\bullet\beta]-[A,Z\bullet\beta]+X\bullet B\gamma-X\bullet C\beta-BZ\bullet\alpha+CY\bullet\alpha\\
&\quad+[B,Z\bullet\alpha]-[B,X\bullet\gamma]+Y\bullet C\alpha-Y\bullet A\gamma-CX\bullet\beta+AZ\bullet+[C,X\bullet\beta]\\
&\quad-[C,Y\bullet\alpha]+Z\bullet A\beta-Z\bullet B\alpha-AY\bullet\gamma+BX\bullet\gamma\,,\\
0&=(X\bullet\beta)\gamma-(Y\bullet\alpha)\gamma-(Z\bullet\alpha)\beta+(X\bullet\gamma)\beta-(Y\bullet\gamma)\alpha+(Z\bullet\beta)\alpha\,,
\end{aligned}$$

which can be seen to hold by using the first three identities in Lemma 1.

The Möbius Lie algebra $\mathfrak{mo}(V)$ is in fact isomorphic to $\mathfrak{so}(\hat V)$, where $\hat V=V\oplus\mathbb{R}^{1,1}$, where the inner product on $\hat V$ (which we also denote by $\langle -,-\rangle$) extends the one on V.

Let e_0,e_1 span $\mathbb{R}^{1,1}$ and define $e_\pm=\frac{1}{\sqrt2}(e_0\pm e_1)$. Then $e_\pm\perp V$ and $\langle e_+,e_-\rangle=1$. We decompose an arbitrary vector $\hat Y\in\hat V$ as

$$\hat Y=Y+y^+e_++y^-e_-\,,$$

where $Y\in V$, and a two-form $\hat\omega\in\Lambda^2\hat V^*$ as

$$\hat\omega=\omega+\alpha\wedge e_++e_-\wedge X^\flat+he_+\wedge e_-\,,$$

where $\alpha \in V^*$, $\omega \in \Lambda^2 V^*$ and $X \in V$. Then the action of a skew-symmetric endomorphism $\hat{S}^{\hat{\omega}}$ on $\hat{Y}$ is given by (2.4):

$$\begin{aligned}\hat{S}^{\hat{\omega}}(\hat{Y}) &= S^{\omega}(Y) + y^+ X - y^- \alpha^{\sharp} + (\alpha(Y) - hy^+)e_+ + (hy^- - \langle X, Y\rangle)e_- \\ &= \begin{pmatrix} S^{\omega} & X & -\alpha^{\sharp} \\ \alpha & -h & 0 \\ -X^{\flat} & 0 & h \end{pmatrix}\begin{pmatrix} Y \\ y^+ \\ y^- \end{pmatrix}.\end{aligned}$$

This gives us the identification

$$(X, A, \alpha) \mapsto \begin{pmatrix} S & X & -\alpha^{\sharp} \\ \alpha & -h & 0 \\ -X^{\flat} & 0 & h \end{pmatrix} \in \mathfrak{so}(\hat{V}) , \tag{2.8}$$

where $A = S + h\,\mathrm{Id}_V$ and $S \in \mathfrak{so}(V)$. It is easy to see that using this identification, the matrix commutator on $\mathfrak{so}(\hat{V})$ induces the Lie bracket (2.7) on $\mathfrak{mo}(V)$, so equation (2.8) is an explicit isomorphism $\mathfrak{mo}(V) \to \mathfrak{so}(\hat{V})$.

2.1.3 The Clifford algebras of V and $\hat{V}$

Let $C\ell(V)$ denote the Clifford algebra of $(V, \langle -, - \rangle)$, defined by the relation

$$X \cdot Y + Y \cdot X = -2\langle X, Y\rangle \,\mathrm{Id} . \tag{2.9}$$

We will frequently need the following formulas for the Clifford product between a vector $X \in V$ and a p-form $\eta \in \Lambda^p V^*$:

$$X \cdot \eta = X^{\flat} \wedge \eta - \iota_X \eta \tag{2.10}$$

$$\eta \cdot X = (-1)^p (X^{\flat} \wedge \eta + \iota_X \eta) \tag{2.11}$$

We also introduce the gamma matrices Γ_i as the generators of the Clifford algebra, corresponding to the image of the pseudo-orthonormal frame e_i on $\hat{V}$ under the embedding $V \hookrightarrow C\ell(V)$. The following Clifford product identities are frequently useful:

$$\sum_i \Gamma^i \cdot \iota_{e_i} \eta = p\eta, \tag{2.12}$$

$$\sum_i \Gamma^i \cdot \theta^i \wedge \eta = -(n-p)\eta, \tag{2.13}$$

The associative algebra $C\ell(V)$ can be made into a Lie algebra using the Clifford commutator, and the embedding $\mathfrak{so}(V) \hookrightarrow C\ell(V)$ defined by $A \mapsto -\frac{1}{2}\omega_A$ is a Lie algebra homomorphism. Furthermore, if $X \in V \subset C\ell(V)$, using (2.10) we see that

$$[-\tfrac{1}{2}\omega_A, X] = A(X)\,. \tag{2.14}$$

This follows because

$$\begin{aligned} [-\tfrac{1}{2}\omega_A, X] &= -\tfrac{1}{2}(\omega_A \cdot X - X \cdot \omega_A) \\ &= -\iota_X \omega_A \\ &= A(X)\,. \end{aligned}$$

Thus, any $C\ell(V)$-module $\mathfrak{M}$ restricts to a $\mathfrak{so}(V)$-module, giving rise to a spinor representation of $\mathfrak{so}(V)$. We recall that there is a natural isomorphism between $\Lambda^* V^*$ and $C\ell(V)$, which allows us to define the Clifford action of a p-form on $\mathfrak{M}$. When there is no chance of confusion, we will also denote this action by $\cdot$.

Let $\mathfrak{S}$ be an irreducible $C\ell(V)$-module. It is possible to extend it into a $\mathfrak{co}(V)$-module by introducing a *weight*: if $\sigma : \mathfrak{so}(V) \to \mathrm{End}(\mathfrak{S})$ denotes the representation map, we define $\sigma^w : \mathfrak{co}(V) \to \mathrm{End}(\mathfrak{S})$ for all $w \in \mathbb{R}$ by $\sigma^w(A) = \sigma(A)$ for $A \in \mathfrak{so}(V)$ and $\sigma^w(\mathrm{Id}_V) = w\,\mathrm{Id}_{\mathfrak{S}}$ and extending linearly. We denote the corresponding $\mathfrak{co}(V)$-module by $\mathfrak{S}^{[w]}$. Using this representation, we can express the Clifford algebra element identified with $X \bullet \alpha$ in terms of Clifford products:

$$\sigma^w(X \bullet \alpha) = -\tfrac{1}{2} X \cdot \alpha^\sharp + (w - \tfrac{1}{2})\alpha(X)\,\mathrm{Id}_{\mathfrak{S}}\,.$$

2.1.4 Spinor representations of the Möbius algebra

We can also relate the Clifford algebras of V and $\hat{V}$. As associative algebras, $C\ell(\hat{V}) \cong C\ell(V) \otimes \mathrm{End}(\mathbb{R}^2)$. An irreducible $C\ell(\hat{V})$-module $\hat{\mathfrak{S}}$ decomposes into a direct sum of $C\ell(V)$-modules: $\hat{\mathfrak{S}} = \mathfrak{S}_+ \oplus \mathfrak{S}_-$, where $\mathfrak{S}_\pm = \mathrm{Ker}\,\Gamma_\pm$. (See e.g. [21] for a proof of the isomorphism.)

In fact, $\mathfrak{S}_\pm$ are isomorphic as $C\ell(V)$-modules: the isomorphism $\imath : \mathfrak{S}_- \to \mathfrak{S}_+$ is given by the restriction of Γ_+ to $\mathfrak{S}_-$. Since $\Gamma_+\Gamma_- = -\Gamma_-\Gamma_+ - 2\,\mathrm{Id}$, the inverse isomorphism $\imath^{-1}$ is the restriction of $-\frac{1}{2}\Gamma_-$ to $\mathfrak{S}_+$.

The module $\hat{\mathfrak{S}}$ also restricts to a $\mathfrak{co}(V)$-module since $\mathfrak{co}(V) \subset \mathfrak{mo}(V) \cong \mathfrak{so}(\hat{V})$. The element of $C\ell(\hat{V})$ corresponding to $A = S + h\mathrm{Id}_V \in \mathfrak{co}(V)$ is $-\frac{1}{4}S^{ij}\Gamma_{ij} - \frac{1}{4}h(\Gamma_-\Gamma_+ - \Gamma_+\Gamma_-)$, and thus $\hat{\mathfrak{S}} = \mathrm{Ker}\,\Gamma_+ \oplus \mathrm{Ker}\,\Gamma_- = \mathfrak{S}^{[\frac{1}{2}]} \oplus \mathfrak{S}^{[-\frac{1}{2}]}$ as a $\mathfrak{co}(V)$-module.

Composing the isomorphism (2.8) with the embedding $\mathfrak{so}(\hat{V}) \hookrightarrow C\ell(\hat{V})$, we obtain the following embedding $\mathfrak{mo}(V) \hookrightarrow C\ell(\hat{V})$:

$$\begin{pmatrix} X \\ A \\ \alpha \end{pmatrix} \mapsto -\frac{1}{4}S^{ij}\Gamma_{ij} - \frac{1}{4}h(\Gamma_-\Gamma_+ - \Gamma_+\Gamma_-) + \tfrac{1}{2}\alpha^i\Gamma_i\Gamma_+ - \tfrac{1}{2}X^i\Gamma_i\Gamma_- \tag{2.15}$$

The spinorial representation $\varrho : \mathfrak{mo}(V) \to \mathrm{End}(\mathfrak{S}_- \oplus \mathfrak{S}_+)$ induced by the embedding is thus given explicitly by

$$\varrho(X, A, \alpha) = \begin{pmatrix} -\frac{1}{4}S^{ij}\Gamma_{ij} + \frac{1}{2}h\mathrm{Id} & -X^i\Gamma_i\iota^{-1} \\ \frac{1}{2}\alpha^i\Gamma_i\iota & -\frac{1}{4}S^{ij}\Gamma_{ij} - \frac{1}{2}h\mathrm{Id} \end{pmatrix} \tag{2.16}$$

Note that this representation automatically gives the right weights to the representations of $A \in \mathfrak{co}(V)$, so we can rewrite ϱ as

$$\varrho(X, A, \alpha) = \begin{pmatrix} \sigma^{\frac{1}{2}}(A) & -X\cdot \\ \frac{1}{2}\alpha\cdot & \sigma^{-\frac{1}{2}}(A) \end{pmatrix}. \tag{2.17}$$

We remark that the isomorphism ι could be rescaled by any scalar, although here we have taken it to be the identity.

2.2 Geometric preliminaries

We now fix our geometric conventions and introduce a number of useful tensors. Our conventions are consistent with [22].

Let (M^n, g) be an n-dimensional pseudo-riemannian manifold and let ∇ denote the Levi-Cività connection. Apart from TM and T^*M (whose sections will be denoted $\mathscr{X}(M)$ and $\Omega^1(M)$, respectively), we will be somewhat cavalier about the distinction between bundles and their sections, often taking $A \in \mathrm{End}(TM)$ to mean that A is a smooth section of $\mathrm{End}(TM)$.

The algebraic machinery introduced in Section 2.1 naturally carries over to this global setting if we now take the vector space V to be T_pM, where $p \in M$, and operations such as • naturally extend to $\mathscr{X}(M)$ and $\Omega^1(M)$ as well if we replace the inner product $\langle -,-\rangle$ with the metric g. Let $\mathfrak{so}(TM) \subset \mathrm{End}(TM)$ denote those endomorphisms of TM which are skew-symmetric relative to g, and $\mathfrak{co}(TM) = \mathfrak{so}(TM) \oplus \langle \mathrm{Id} \rangle$ denote those endomorphisms whose symmetric part is proportional to the identity. We can redefine $X \wedge \alpha \in \mathfrak{so}(TM)$ as

$$g((X \wedge \alpha)Y, Z) = \alpha(Y) g(X, Z) - g(X, Y)\alpha(Z) \tag{2.18}$$

Similarly we define $X \bullet \alpha := X \wedge \alpha + \alpha(X)\,\mathrm{Id} \in \mathfrak{co}(TM)$.

We note that since equation (2.18) is homogenenous in g, it is apparent that $X \wedge \alpha$ and $X \bullet \alpha$ depend *only on the conformal class of the metric.* Indeed, introducing the musical isomorphisms $\sharp : T^*M \to TM$ and $\flat : TM \to T^*M$ defined by the metric g, we can rewrite $X \wedge \alpha$ as

$$X \wedge \alpha = X \otimes \alpha - \alpha^\sharp \otimes X^\flat \in TM \otimes T^*M \cong \mathrm{End}(TM) ,$$

which is manifestly invariant under conformal rescalings of the metric.

It is also clear that any section (X, A, α) of $TM \oplus \mathfrak{co}(TM) \oplus T^*M$ defines an endomorphism of the bundle $\mathbb{S} \oplus \mathbb{S}$ (where $\mathbb{S}$ is the spinor bundle on M, which we take to be a bundle of irreducible $C\ell(TM)$-modules over M) — sometimes known as the "local twistor bundle" — via the global version of (2.17).

2.2.1 The Riemann curvature tensor and its relatives

The Riemann curvature operator of the Levi-Cività connection ∇ is defined by

$$R(X, Y)Z = \nabla_{[X,Y]} Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z . \tag{2.19}$$

and the corresponding curvature tensor is

$$R(X, Y, Z, U) = g(R(X, Y)Z, U) . \tag{2.20}$$

The curvature operator satisfies the algebraic Bianchi identity

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0 . \tag{2.21}$$

The Ricci tensor is defined as the trace with respect to the the second argument of the curvature operator:

$$r(X,Y) = \mathrm{tr}(Z \mapsto R(X,Z)Y)\,. \tag{2.22}$$

It turns out to be convenient to introduce the Ricci operator $\mathrm{Ric}: TM \to TM$ via

$$g(\mathrm{Ric}(X),Y) = r(X,Y)\,, \tag{2.23}$$

whose trace gives the scalar curvature s. The Riemann curvature tensor has a natural decomposition

$$R(X,Y,Z,U) = W(X,Y,Z,U) + (\ell \odot g)(X,Y,Z,U) \tag{2.24}$$

where W is the (conformally invariant) Weyl curvature tensor and $\odot$ stands for *the Kulkarni–Nomizu product* of two symmetric tensors which guarantees that $a \odot b$ has the symmetries of a curvature tensor:

$$(a\odot b)(X,Y,Z,U) = a(X,Z)b(Y,U)+a(Y,U)b(X,Z)-a(Y,Z)b(X,U)-a(X,U)b(Y,Z)\,, \tag{2.25}$$

and ℓ is the *Schouten tensor*

$$\ell(X,Y) = \frac{1}{n-2}\left(r(X,Y) - \frac{s}{2(n-1)}g(X,Y)\right), \tag{2.26}$$

which can be thought as the first quotient of the division of the Riemann tensor R by the metric.

Alternatively, one may define the Schouten tensor via its associated map $L: TM \to T^*M$, by

$$R(X,Y) = W(X,Y) - X \bullet L(Y) + Y \bullet L(X) \in \mathrm{End}(TM)\,, \tag{2.27}$$

Another useful tensor will be the Cotton-York tensor $C: \Lambda^2 TM \to T^*M$, defined as

$$C(X,Y) := (\nabla_X L)(Y) - (\nabla_Y L)(X)\,. \tag{2.28}$$

Taking the appropriate traces in the differential Bianchi identity we can show that the Cotton-York tensor is nothing more than the divergence of the Weyl tensor:

$$(\mathrm{div}\, W)(X,Y) = (n-3)C(X,Y)\,, \tag{2.29}$$

Chapter 3

The Lie algebra of conformal Killing vectors

In this chapter we introduce the notion of *conformal Killing transport* and study the Lie algebra of conformal Killing vector fields. We also define the Weyl structure on a semi-Riemannian manifold and present manifestly Weyl-invariant versions of both the conformal Killing transport equations and the conformal Lie algebra.

3.1 Conformal Killing transport

Definition 2. A vector field X on (M, g) is a *conformal Killing vector* if $\mathscr{L}_X g = -2h_X g$ for some smooth function $h_X \in C^\infty(M)$.

The equation $\mathscr{L}_X g = -2h_X$ is equivalent to

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = -2h_X g(Y, Z) \tag{3.1}$$

for all vector fields Y, Z. In local coordinates this equation is often called the *conformal Killing equation*

$$\nabla_a X_b + \nabla_b X_a = -2h_X g_{ab}\,, \tag{3.2}$$

from which we see that $h_X = \frac{1}{n}\operatorname{div} X$, where $\operatorname{div} X = -\nabla_a X^a$. Equivalently, conformal Killing vectors are characterized in terms of the endomorphism $A_X \in \operatorname{End}(TM)$, defined by

$$A_X Y = -\nabla_Y X \qquad \text{for all } Y \in C^\infty(M, TM). \tag{3.3}$$

Indeed, we have

Lemma 3. *A vector field X is a conformal Killing vector if and only if $A_X \in \mathfrak{co}(TM)$.*

Proof. Let $S_X \in \mathfrak{so}(TM)$ denote the skew-symmetric part of A_X. Then if $A_X \in \mathfrak{co}(TM)$, we have

$$A_X = S_X + h_X \,\mathrm{Id}\,, \tag{3.4}$$

or in local coordinates

$$\nabla_a X_b = S_{ab} - h_X g_{ab}\,. \tag{3.5}$$

Then

$$g(A_X Y, Z) + g(Y, A_X Z) = -g(\nabla_Y X, Z) - g(Y, \nabla_Z X) = 2h_X g(Y,Z)\,, \tag{3.6}$$

so obviously X satisfies the conformal Killing equation (3.1). The converse is obvious, since if X satisfies (3.1), then clearly A_X can be written in the form (3.4) and is thus an element of $\mathfrak{co}(TM)$. □

Differentiating A_X further we obtain $\nabla A_X \in \Omega^1(M, \mathfrak{co}(TM))$.

Lemma 4.

$$\nabla_Y A_X = R(Y,X) + Y \bullet \alpha_X \in \mathfrak{co}(TM)\,,$$

where $\alpha_X = dh_X$.

Proof. We can write the covariant derivative of A_X as

$$\begin{aligned}(\nabla_Z A_X)Y &= \nabla_Z(A_X Y) - A_X \nabla_Z Y \\ &= -\nabla_Z \nabla_Y X + \nabla_{\nabla_Z Y} X\end{aligned}$$

so

$$\begin{aligned}(\nabla_Z A_X)Y - (\nabla_Y A_X)Z &= R(Z,Y)X \\ &= R(Z,X)Y = R(Y,X)Z\,,\end{aligned}$$

where the last line follows from the algebraic Bianchi identity (2.21). This means that if we define

$$\mathscr{F}(Z,Y,U) := g((\nabla_Z A_X - R(Z,X))Y, U)\,,$$

then $\mathscr{F}$ is symmetric in the first two arguments:

$$\mathscr{F}(Z,Y,U) = \mathscr{F}(Y,Z,U) \tag{3.7}$$

We can also take ∇ of the conformal Killing equation (3.6), obtaining

$$g((\nabla_Z A_X)Y,U) + g((\nabla_Z A_X)U,Y) = 2\alpha_X(Z)g(Y,U)\ ,$$

where $\alpha_X = dh_X$. It follows that

$$\mathscr{F}(Z,Y,U) = -\mathscr{F}(Z,U,Y) + \alpha(Z)g(Y,U)\ , \tag{3.8}$$

and combining this with (3.7), we have

$$\mathscr{F}(Z,Y,U) = \alpha(Z)g(Y,U) - \alpha(U)g(Z,Y) + \alpha(Y)g(U,Z)\ . \tag{3.9}$$

Comparing this with (2.18), we conclude that

$$\nabla_Y A_X = R(Y,X) + Y \bullet \alpha_X\ . \tag{3.10}$$

□

In local coordinates,

$$\nabla_a S_{bc} = X^d R_{dabc} + g_{ab}\alpha_c - g_{ac}\alpha_b \qquad \text{and} \qquad \nabla_a h_X = \alpha_a\ . \tag{3.11}$$

In other words,

$$\nabla_a \nabla_b X_c = X^d R_{dabc} + g_{ab}\alpha_c - g_{ac}\alpha_b - \alpha_a g_{bc}\ , \tag{3.12}$$

whence tracing with g^{ab}, we obtain

$$\nabla^2 X_c = (n-2)\alpha_c - R_{cd}X^d\ , \tag{3.13}$$

or

$$\alpha_a = \frac{1}{n-2}\left(\nabla^2 X_a + R_{ab}X^b\right)\ . \tag{3.14}$$

Differentiating further we find

$$\nabla_a \alpha_b = (\nabla_c L_{ab})X^c + L_a{}^c S_{bc} + L_b{}^c S_{ac} - 2L_{ab}h_X\ , \tag{3.15}$$

where L_{ab} are the components of the Schouten tensor.

Now note that the Lie derivative of a vector field can be written as $\mathscr{L}_X Y = [X, Y] = \nabla_X Y - \nabla_Y X = \nabla_X Y + A_X Y$, where A_X is the endomorphism associated to the covariant derivative of X as defined in Equation 3.3. (In fact, the Lie derivative of any tensor can be written as $\mathscr{L}_X = \nabla_X + \varrho(A_X)$, where ϱ is the representation of A_X acting on the appropriate bundle.)

Rewriting Equation 3.15 using this observation gives

Lemma 5.

$$\nabla_Y \alpha_X = (\nabla_X L)(Y) - L(Y) \circ A_X - L(A_X Y) = (\nabla_X L)(Y) + (A_X L)(Y) = (\mathscr{L}_X L)(Y)\,.$$

Putting Lemmas 4 and 5 together, we arrive at the characterization of conformal Killing vectors in terms of *conformal Killing transport.*

Proposition 6. *Conformal Killing vectors are in bijective correspondence with sections of the bundle* $TM \oplus \mathfrak{co}(TM) \oplus T^*M$ *which are parallel relative to the following connection which we call the* Geroch connection*:*

$$\mathscr{D}_Y \begin{pmatrix} X \\ A \\ \alpha \end{pmatrix} := \begin{pmatrix} \nabla_Y X + AY \\ \nabla_Y A + R(X, Y) - Y \bullet \alpha \\ \nabla_Y \alpha - (\nabla_X L)(Y) + L(Y) \circ A + L(AY) \end{pmatrix}. \tag{3.16}$$

Indeed a Killing vector X determines and is determined uniquely by a parallel section (X, A_X, α_X).

We remark that in terms of $\beta := \alpha - L(X)$, the Killing transport equations can be rewritten in terms of the Weyl, Schouten and the (normalized) Cotton–York tensor:

$$\mathscr{D}_Y \begin{pmatrix} X \\ A \\ \beta \end{pmatrix} := \begin{pmatrix} \nabla_Y X + AY \\ \nabla_Y A + W(X, Y) - Y \bullet \beta - X \bullet L(Y) \\ \nabla_Y \beta + C(X, Y) + L(Y) \circ A \end{pmatrix}. \tag{3.17}$$

3.2 The conformal Lie algebra

If X, Y are conformal Killing vectors, so is their Lie bracket; indeed,

$$\mathscr{L}_{[X,Y]} g = -2 h_{[X,Y]} g\,,$$

where

$$h_{[X,Y]} = \alpha_Y(X) - \alpha_X(Y) = \beta_Y(X) - \beta_X(Y) . \tag{3.18}$$

This means that conformal Killing vectors span a Lie subalgebra of the Lie algebra of vector fields, which we call the *conformal Lie algebra of* (M, g).

Proposition 6 implies that a conformal Killing vector is uniquely determined by its conformal Killing transport data (X, A_X, α_X) or (X, A_X, β_X) at a point $p \in M$. This exhibits the conformal Lie algebra of (M, g) at a point as a vector subspace of $\mathfrak{mo}(T_pM)$. We will now determine the Lie bracket for this algebra.

The conformal Killing transport data for $[X, Y]$ is $([X, Y], A_{[X,Y]}, \alpha_{[X,Y]})$, or with $\beta_{[X,Y]}$ replcing $\alpha_{[X,Y]}$. Since the Levi-Cività connection is torsion-free,

$$[X, Y] = \nabla_X Y - \nabla_Y X = A_X Y - A_Y X .$$

Differentiating $[X, Y]$ we obtain

$$A_{[X,Y]} = [A_X, A_Y] + X \bullet \alpha_Y - Y \bullet \alpha_X + R(X, Y) , \tag{3.19}$$

or, in terms of β,

$$A_{[X,Y]} = [A_X, A_Y] + X \bullet \beta_Y - Y \bullet \beta_X + W(X, Y) . \tag{3.20}$$

Differentiating $h_{[X,Y]}$ we obtain

$$\alpha_{[X,Y]} = \alpha_X \circ A_Y - \alpha_Y \circ A_X - C(X, Y) + L(A_X Y - A_Y X) + L(Y) \circ A_X - L(X) \circ A_Y \tag{3.21}$$

or, in terms of β,

$$\beta_{[X,Y]} = \beta_X \circ A_Y - \beta_Y \circ A_X - C(X, Y) . \tag{3.22}$$

In summary, we have the following Lie brackets for the conformal Killing transport data:

$$\left[\begin{pmatrix} X \\ A_X \\ \beta_X \end{pmatrix}, \begin{pmatrix} Y \\ A_Y \\ \beta_Y \end{pmatrix} \right] = \begin{pmatrix} A_X Y - A_Y X \\ [A_X, A_Y] + X \bullet \beta_Y - Y \bullet \beta_X + W(X, Y) \\ -\beta_Y \circ A_X + \beta_X \circ A_Y - C(X, Y) \end{pmatrix} , \tag{3.23}$$

which shows that the Weyl curvature measures the failure of this Lie bracket to agree with the algebraic Lie bracket on sections of $TM \oplus \mathfrak{co}(TM) \oplus T^*M$, given by equation (2.7).

3.3 Conformal changes of the metric

Let $\overline{g} = e^{2f} g$, where $f \in C^\infty(M)$ is a smooth function, be a conformal rescaling of the metric. If X is a conformal Killing vector for g, it is also a conformal Killing vector for $\overline{g}$. Indeed,

$$\mathscr{L}_X \overline{g} = -2\overline{h}_X \overline{g} ,$$

where $\overline{h}_X = h_X - df(X)$. As this calculation already shows, the conformal Killing transport data for X does depend on the metric and not just on its conformal class. Let $(X, \overline{A}_X, \overline{\alpha}_X)$ denote the conformal Killing transport data associated to X relative to the conformally rescaled metric $\overline{g}$. To relate $(X, \overline{A}_X, \overline{\alpha}_X)$ to the conformal Killing transport data (X, A_X, α_X) relative to the original metric, we need to see how certain geometric objects behave under conformal rescalings of the metric. The Levi-Cività connection changes by [22]

$$\overline{\nabla}_X = \nabla_X + X \bullet df , \tag{3.24}$$

whence

$$\overline{A}_X = A_X - X \bullet df . \tag{3.25}$$

Finally,

$$\overline{\alpha}_X = d\overline{h}_X = \alpha_X - d\iota_X df = \alpha_X - \mathscr{L}_X df . \tag{3.26}$$

In summary, under a conformal rescaling of the metric, the conformal Killing transport data associated to a conformal Killing vector X changes by

$$\begin{pmatrix} X \\ \overline{A}_X \\ \overline{\alpha}_X \end{pmatrix} = \begin{pmatrix} X \\ A_X \\ \alpha_X \end{pmatrix} - \begin{pmatrix} 0 \\ X \bullet df \\ \mathscr{L}_X df \end{pmatrix} . \tag{3.27}$$

Things are a little more complicated in terms of β, since the Schouten tensor has more complicated transformation laws under conformal rescalings of the metric. The $(4,0)$ Riemann curvature tensor transforms as [22]

$$\overline{R} = e^{2f} R - \overline{g} \odot \left(\nabla df - (df)^2 + \tfrac{1}{2}|df|^2 g\right) , \tag{3.28}$$

with ∇df the Hessian of f. On the other hand, from (2.24), we have that

$$\overline{R} = \overline{W} + \overline{g} \odot \overline{L} = e^{2f} W + \overline{g} \odot \overline{L} ,$$

whereas on the other hand, inserting (2.24) into (3.28), we have that

$$\overline{R} = e^{2f} W + \overline{g} \odot \left(L - \nabla df + (df)^2 - \tfrac{1}{2}|df|^2 g \right) .$$

Comparing the two expressions, we can read off how the Schouten tensor transforms:

$$\overline{L} = L - \nabla df + (df)^2 - \tfrac{1}{2}|df|^2 g . \tag{3.29}$$

In local coordinates,

$$\overline{L}_{ab} = L_{ab} - \nabla_a \nabla_b f + \nabla_a f \nabla_b f - \tfrac{1}{2} g^{cd} \nabla_c f \nabla_d f g_{ab} . \tag{3.30}$$

The Cotton-York tensor transforms in a particularly simple way under Weyl transformations:

$$\overline{C}(X,Y) = C(X,Y) + W(X,Y) df . \tag{3.31}$$

We note that $\nabla_X df - d\iota_X df = \nabla_X df - \mathscr{L}_X df = -A_x df$, whence

$$\overline{\beta}_X = \beta_X - A_X df - df(X) df - \tfrac{1}{2}|df|^2 X^\flat . \tag{3.32}$$

Since the conformal Lie algebra is an invariant of the conformal structure, we would like to find a version of the Geroch connection (3.16) which is manifestly invariant under conformal rescalings of the metric. This requires introducing a *Weyl connection.*

3.4 Weyl connections

By a Weyl connection we mean a torsion-free connection D on TM preserving the conformal class of the metric; that is, a connection which obeys, for any vector field X,

$$D_X g = 2\theta(X) g , \tag{3.33}$$

where θ is a 1-form. This connection is invariant under conformal rescaling of the metrics $\overline{g} = e^{2f} g$, provided that the one-form θ transforms as $\overline{\theta} = \theta + df$. We call such a transformation a *Weyl transformation*.

A manifold M equipped with a conformal class of metrics and a Weyl connection is often said to have a *Weyl structure*: in fact, it can be realised as a reduction of the frame bundle of M to $CO(TM)$. We will work with a fixed representative metric g, but one frequently encounters the more general viewpoint in conformal geometry literature [23].

One can derive an explicit formula for D in terms of the Levi-Cività connection of g:

$$D_X = \nabla_X - X \bullet \theta \,. \tag{3.34}$$

The curvature R^D of the Weyl connection is defined by

$$R^D(X,Y) = D_{[X,Y]} - D_X D_Y + D_Y D_X \,,$$

and using the above expression for D_X, can be related to the Riemann curvature R of g by

$$R^D(X,Y) = R(X,Y) - X \bullet \nabla_Y \theta + Y \bullet \nabla_X \theta - [X \bullet \theta, Y \bullet \theta] \,. \tag{3.35}$$

Inserting (2.27) into this equation and decomposing the result in a way similar to (2.27) itself, we find

$$R^D(X,Y) = W(X,Y) - X \bullet L^D(Y) + Y \bullet L^D(X) \,, \tag{3.36}$$

where

$$L^D(X) = L(X) + \nabla_X \theta + \theta(X)\theta - \tfrac{1}{2}|\theta|^2 X^\flat \,, \tag{3.37}$$

or, equivalently, using that $D_X\theta = \nabla_X \theta - 2\theta(X)\theta + |\theta|^2 X^\flat$,

$$L^D(X) = L(X) + D_X \theta - \theta(X)\theta + \tfrac{1}{2}|\theta|^2 X^\flat \,. \tag{3.38}$$

Unlike the Schouten tensor L, the map L^D is not symmetric, and we can make this manifest by rewriting (3.36) as follows

$$R^D(X,Y) = W(X,Y) + F^D(X,Y)\,\mathrm{Id} - X \curlywedge L^D(Y) + Y \curlywedge L^D(X) \,, \tag{3.39}$$

where we have introduced the *Faraday 2-form* $F^D = d\theta$, which is invariant under Weyl transformations. It follows from Equation 3.39 that L^D is also Weyl-invariant, a fact which can also be checked directly from equations (3.29) and (3.34) and using equation (3.24).

Similarly, we can construct the Weyl-invariant analogue of the Cotton-York tensor. From the Weyl transformation law (3.31), we immediately see that

$$C^D(X,Y) := C(X,Y) - W(X,Y)\theta \tag{3.40}$$

is Weyl-invariant. Note that this is in fact equal to the "naive" Weyl-covariantisation of the Cotton-York tensor, i.e. obtained simply by replacing the Levi-Cività connection with the Weyl connection in equation (2.28) and the Schouten tensor with its Weyl-invariant analogue:

$$C^D(X,Y) = (D_X L^D)(Y) - (D_Y L^D)(X)\,.$$

Naturally, one can also show that C^D can be obtained as the D-divergence of the Weyl tensor.

3.5 Weyl-invariant conformal Killing transport

Defining manifestly Weyl-invariant conformal Killing transport requires redefining the conformal Killing transport data itself — although the vector field X itself is conformally invariant, because of (3.24) A_X and α_X are not. We will remedy the situation by adding θ-dependent terms to them in such a way that the resulting data (X, A_X^D, α_X^D) is Weyl-invariant.

Taking into account the transformation property of θ and equation (3.25), it is easy to see that

$$A_X^D := A_X + X \bullet \theta \tag{3.41}$$

is Weyl-invariant. However, the one-form α_X actually has a one-parameter family of Weyl-invariant extensions, since we can always add the manifestly Weyl-invariant term $\iota_X F^D$ to it. From equation (3.26) it is apparent that

$$\alpha_X^D := \alpha_X + \mathscr{L}_X\theta + t\iota_X F^D \tag{3.42}$$

is Weyl-invariant for all t. Similarly, for any choice of t,

$$\beta_X^D := \alpha_X^D - L^D(X) \tag{3.43}$$

is manifestly Weyl-invariant.

It remains to be shown that (X, A_X^D, α_X^D) is also parallel with respect to a suitable connection on the Möbius bundle $TM \oplus \mathfrak{co}(TM) \oplus T^*M$. Using equation (3.34) and (3.41), we observe that in fact

$$A_X^D Y = A_X Y + (X \bullet \theta) Y = -\nabla_Y X + (Y \bullet \theta) X = -D_Y X\,, \tag{3.44}$$

which would suggest attempting a naive covariantisation of the conformal Killing transport equations (3.16) simply by replacing ∇ by D. Indeed, applying D to (3.41) and using (3.34) and (3.35) yields

$$D_Y A_X^D = R^D(Y, X) + Y \bullet (\alpha_X + \mathscr{L}_X \theta)\,, \tag{3.45}$$

which in comparison with (3.42) and (3.16) suggests setting the parameter $t = 0$ in the definition of α_X^D. This allows us to rewrite equation (3.45) in a more familiar form:

$$D_Y A_X^D = R^D(Y, X) + Y \bullet \alpha_X^D\,, \tag{3.46}$$

which can also be rewritten in terms of β_X^D with the help of equation (3.35) as

$$D_Y A_X^D = W(Y, X) + Y \bullet \beta_X^D + X \bullet L^D(Y)\,. \tag{3.47}$$

Similarly, we can calculate $D_Y \alpha_X^D$ and find

$$D_Y \alpha_X^D = (D_X L^D)(Y) + (A_X^D L^D)(Y) = (\mathscr{L}_X L^D)(Y)\,, \tag{3.48}$$

or using β_X^D,

$$D_Y \beta_X^D = -C^D(X, Y) + A_X^D L^D(Y)\,. \tag{3.49}$$

Combining these results allows us to define the manifestly Weyl-invariant version of the Geroch connection, whose parallel sections are in one-to-one correspondence with conformal Killing vectors on M:

$$\mathscr{D}_Y^D \begin{pmatrix} X \\ A \\ \beta \end{pmatrix} = \begin{pmatrix} D_Y X + AY \\ D_Y A + W(X, Y) - Y \bullet \beta - X \bullet L^D(Y) \\ D_Y \beta + C^D(X, Y) - A L^D(Y) \end{pmatrix}. \tag{3.50}$$

3.6 The conformal Lie algebra of a Weyl structure

The conformal Lie algebra of a Weyl structure (M, g, θ) can now be determined with the help of the Weyl-invariant Geroch connection (3.50). Since D is torsion-free, the TM-component of the bracket does not change:

$$[X, Y] = D_X Y - D_Y X = A^D_X Y - A^D_Y X \, .$$

Proceeding in a similar fashion as before, we find

$$A^D_{[X,Y]} = [A^D_X, A^D_Y] + X \bullet \beta^D_Y - Y \bullet \beta^D_X + W(X, Y) \tag{3.51}$$

and

$$\beta^D_{[X,Y]} = A^D_X \beta^D_Y - A^D_Y \beta^D_X - C^D(X, Y) \, . \tag{3.52}$$

Combining these, we obtain the Weyl-invariant version of equation (3.23):

$$\left[\begin{pmatrix} X \\ A^D_X \\ \beta^D_X \end{pmatrix}, \begin{pmatrix} Y \\ A^D_Y \\ \beta^D_Y \end{pmatrix} \right] = \begin{pmatrix} A^D_X Y - A^D_Y X \\ [A^D_X, A^D_Y] + X \bullet \beta^D_Y - Y \bullet \beta^D_X + W(X, Y) \\ \beta^D_{[X,Y]} = A^D_X \beta^D_Y - A^D_Y \beta^D_X - C^D(X, Y) \end{pmatrix} , \tag{3.53}$$

where the Weyl curvature again measures the failure of this bracket to agree with the natural Lie bracket of the Möbius algebra.

3.7 Normal conformal Killing vectors

Naturally, when (M, g) is conformally flat and thus $W = 0$, the bracket (3.53) agrees with the algebraic one. However, even in general, (M, g) may possess a Lie subalgebra of conformal Killing vectors whose Lie bracket *does* agree with the Möbius algebra bracket.

Definition 7. If X is a conformal Killing vector field of (M, g, θ) and in addition

$$W(X, Y) = C(X, Y) = 0 \tag{3.54}$$

for any vector field $Y \in \mathscr{X}(M)$, we call X a *normal conformal Killing vector field.*

While we will content ourselves with the above definition for the purposes of this thesis, the term "normal" here is motivated by the fact that normal conformal Killing vector fields arise as parallel sections of $\mathfrak{mo}(TM)$ with respect to not just the Geroch connection but a connection induced from the so-called *normal conformal Cartan connection* of M [24, 25, 26], an important tool in conformal geometry. In our notation, this connection can be written as [24]

$$\mathscr{D}_Y^{NC} = \nabla_Y + \operatorname{ad} Y + \operatorname{ad} L(Y) \,, \tag{3.55}$$

where the connection acts on the Möbius bundle and the adjoint action is with respect to the bracket (2.7). More explicitly, we can write

$$\mathscr{D}_Y^{NC} \begin{pmatrix} X \\ A \\ \beta \end{pmatrix} = \begin{pmatrix} \nabla_Y X + AY \\ \nabla_Y A - Y \bullet \beta - X \bullet L(Y) \\ \nabla_Y \beta - AL(X) \end{pmatrix} . \tag{3.56}$$

Comparing equation (3.56) with (3.17), we see that a conformal Killing vector which is also parallel with respect to the normal conformal Cartan connection satisfies equation (3.54).

Chapter 4

Conformal Killing spinors

Let (M, g) be a n-dimensional Lorentzian spin manifold[1], $\mathbb{S}$ its spinor bundle and ∇ the spin connection acting on $\mathbb{S}$ induced from the Levi-Civitá connection. We denote the Clifford multiplication of a spinor by a p-form — the Clifford action of the Clifford bundle $C\ell(TM)$ on $\mathbb{S}$ — by $\cdot$. The spin connection respects the Clifford product in the following way: if Y is any vector field and ψ is a spinor, then

$$\nabla_Z(Y\cdot\psi) = \nabla_Z Y\cdot\psi + Y\cdot\nabla_Z\psi\,. \tag{4.1}$$

Note that the following relationship holds between the Riemann curvature tensor and the curvature of the spin connection:

$$R(X,Y)\psi = \nabla_{[X,Y]}\psi - [\nabla_X,\nabla_Y]\psi = -\tfrac{1}{2}R(X,Y)\cdot\psi\,, \tag{4.2}$$

where the right-hand side means the Clifford action of the Riemann curvature tensor $R(X, Y)$ considered as a two-form.

There are two natural first-order operators acting on $\mathbb{S}$ derived from the connection and the Clifford product. The *Dirac operator* $\not\nabla$ is the connection ∇ composed with the Clifford product. Given a spinor $\psi\in\mathbb{S}$, if e_a is a local pseudo-orthonormal frame and e^a is the coframe defined by $g(e_a, e^b) = \delta_a^{\ b}$, then the Dirac operator acting on ψ can be expressed as

$$\not\nabla\psi = \sum_a e^a\cdot\nabla_{e_a}\psi\,. \tag{4.3}$$

[1]Henceforth we will always assume that g has Lorentzian signature unless explicitly stated otherwise.

The *Penrose operator* P is complementary to the Dirac operator in the following sense. Let $\mu : TM \otimes S \to S$ denote the Clifford multiplication by a vector. Then $\mathrm{Ker}\,\mu$ is a subbundle of $TM \otimes S$. Let $p : TM \otimes S \to \mathrm{Ker}\,\mu$ be the projection on the kernel. Then P is defined to be the composition of the spin connection and the projection p.

$$P : \Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \xrightarrow{p} \mathrm{Ker}\,\mu. \tag{4.4}$$

Much of our treatment is concerned with spinors that lie in the kernel of the Penrose operator.

Definition 8. A spinor $\psi \in S$ is called a *conformal Killing spinor* iff $P_X\psi = 0$ for all $X \in \mathscr{X}(M)$. Equivalently, ψ satisfies the differential equation

$$\nabla_X\psi + \tfrac{1}{n} X \cdot \not{\nabla}\psi = 0\,. \tag{4.5}$$

S admits a Spin-invariant inner product $(-,-)$ whose properties depend on dimension and signature of g. Details can be found in standard texts, for example [21], but the only properties we will need are the following:

$$(\psi, \chi) = \epsilon(\chi, \psi) \tag{4.6}$$

$$(X \cdot \psi, \chi) = \epsilon(\psi, X \cdot \chi) \tag{4.7}$$

$$X(\psi, \chi) = (\nabla_X\psi, \chi) + (\psi, \nabla_X\chi)\,, \tag{4.8}$$

where ϵ is a sign. Now given any two spinors $\begin{pmatrix}\psi\\ \chi\end{pmatrix}$, one can define a vector field $V_{\psi,\chi}$ using the spinor inner product, sometimes known as the *Dirac current*.
It is defined by the following equation:

$$g(Y, V_{\psi,\chi}) = (\psi, Y \cdot \chi) \qquad \text{for all vector fields } Y. \tag{4.9}$$

The reason why spinors in the kernel of P are called conformal Killing spinors[2] is explained by the following proposition.

Proposition 9. *If ψ, χ are conformal Killing spinors, $V_{\psi,\chi}$ is a conformal Killing vector.*

[2] Conformal Killing spinors are also often known as *twistor spinors* in the literature due to their relationship with the twistor bundle in four dimensions[15]

Proof. Taking ∇_Z of equation (4.9), we find:

$$g(\nabla_Z V_{\psi,\chi}, Y) + g(V_{\psi,\chi}, \nabla_Z Y) = (\nabla_Z \psi, Y\cdot\chi) + (\psi, \nabla_Z Y\cdot\chi) + (\psi, Y\cdot\nabla_Z\chi)\,, \quad (4.10)$$

so

$$\begin{aligned} g(\nabla_Z V_{\psi,\chi}, Y) &= (\nabla_Z\psi, \chi) + (\psi, Y\cdot\nabla_Z\chi) \\ &= -\tfrac{1}{n}(Z\cdot \not{\nabla}\psi, Y\cdot\chi) - \tfrac{1}{n}(\psi, Y\cdot Z\cdot\not{\nabla}\chi) \\ &= \frac{(-1)^{\epsilon+1}}{n}(Y\cdot Z\cdot\not{\nabla}\psi, \chi) - \tfrac{1}{n}(\psi, Y\cdot Z\cdot\not{\nabla}\chi)\,, \end{aligned}$$

whence

$$g(\nabla_Z V_{\psi,\chi}, Y) + g(\nabla_Y V_{\psi,\chi}, Z) = \frac{2}{n}\left((\psi, \not{\nabla}\chi) + \epsilon(\not{\nabla}\psi, \chi)\right) g(Y,Z)\,, \quad (4.11)$$

where we have used the defining relation of the Clifford algebra (2.9). Equation (4.11) shows that in fact $V_{\psi,\chi}$ satisfies the conformal Killing equation (3.1), with $h_{V_{\psi,\chi}} = \frac{1}{n}\left((\psi, \not{\nabla}\chi) + \epsilon(\not{\nabla}\psi, \chi)\right)$. □

4.1 Spinorial conformal Killing transport

We saw in the previous chapter that conformal Killing vectors define parallel sections of the Möbius bundle $TM \oplus \mathfrak{co}(TM) \oplus T^*M$. Similarly, there exists a characterisation of conformal Killing spinors as parallel sections of $\mathbb{S} \oplus \mathbb{S}$ with respect to a certain connection. We will determine this connection by rewriting the conformal Killing spinor equation (4.5) as a first-order system.

Lemma 10. *If ψ is a conformal Killing spinor, then the following identities are satisfied for every vector field X:*

(a) $\not{\nabla}^2\psi = \frac{1}{4}\frac{n}{n-1}s\psi\cdot$

(b) $\nabla_X \not{\nabla}\psi = -\frac{n}{2}L(X)\cdot\psi$,

where $L(X)$ is the Schouten map defined in equation (2.27).

Proof. For this calculation, it is convenient to assume that the pseudo-orthonormal frame e_a arises from a basis of T_pM for some $p \in M$ via parallel transport along geodesics, so that

$$\begin{aligned} \nabla e_a(p) &= 0 \\ [e_a, e_b](p) &= 0 . \end{aligned}$$

(a): Differentiating the conformal Killing spinor equation (4.5), using the property (4.1) and taking the trace, we obtain (at the point $p \in M$):

$$\begin{aligned} 0 &= \sum_a \nabla_a \nabla_a \psi + \tfrac{1}{n} \nabla_a \left(e_a \cdot \not\nabla \psi \right) \\ &= -\Delta \psi + \tfrac{1}{n} \not\nabla^2 \psi , \end{aligned}$$

where Δ is the spin connection Laplacian. We now apply the Weitzenböck formula [22] $\not\nabla^2 = \Delta + \frac{1}{4} s$ to obtain

$$\not\nabla^2 \psi = \frac{1}{4} \frac{n}{n-1} s \psi .$$

(b): The conformal Killing spinor equation implies

$$R(X, e_i) \psi = -\tfrac{1}{n} e_i \cdot \nabla_X \not\nabla \psi + \tfrac{1}{n} X \cdot \nabla_i \not\nabla \psi ,$$

and taking the Clifford trace, we obtain

$$\begin{aligned} \mathrm{Ric}(X) \cdot \psi &= -2 \nabla_X \not\nabla \psi - \tfrac{2}{n} \sum_i e_i \cdot X \cdot \nabla_i \not\nabla \psi \\ &= -2 \nabla_X \not\nabla \psi + \tfrac{2}{n} X \cdot \not\nabla^2 \psi + \frac{4}{} \nabla_X \not\nabla \psi , \end{aligned}$$

and substituting the result of (a) and using the definition of L in equation (2.26), we arrive at the result. □

Using Lemma 10, we can immediately see what the spinorial analogue of Proposition 6 is.

Proposition 11. *Conformal Killing spinors are in one-to-one correspondence with sections of the bundle* $\mathrm{S} \oplus \mathrm{S}$ *which are parallel with respect to the following connection:*

$$\mathscr{P}_X \begin{pmatrix} \psi \\ \chi \end{pmatrix} = \begin{pmatrix} \nabla_X & X \cdot \\ \frac{1}{2} L(X) & \nabla_X \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix}$$

A conformal Killing spinor determines and is determined uniquely by a parallel section $(\psi, \frac{1}{n} \nabla \psi)$. *Furthermore, the connection* $\mathscr{P}$ *has curvature*

$$R^{\mathscr{P}}(X,Y) = \mathscr{P}_{[X,Y]} - [\mathscr{P}_X, \mathscr{P}_Y] = -\frac{1}{2}\begin{pmatrix} W(X,Y) & 0 \\ C(X,Y) & W(X,Y) \end{pmatrix} . \tag{4.12}$$

Proof. The first part follows immediately from Lemma 10. As for the curvature, we simply compute

$$\begin{aligned}
\mathscr{P}_{[X,Y]} - [\mathscr{P}_X, \mathscr{P}_Y] &= \begin{pmatrix} \nabla_{[X,Y]} & [X,Y]\cdot \\ \frac{1}{2}L([X,Y])\cdot & \nabla_{[X,Y]} \end{pmatrix} \\
&\quad - \begin{pmatrix} [\nabla_X, \nabla_Y] + \frac{1}{2}X\cdot L(Y) - \frac{1}{2}Y\cdot L(X) & \nabla_X Y - \nabla_Y X \\ \frac{1}{2}(\nabla_X L)(Y) - \frac{1}{2}(\nabla_Y L)(X) + \frac{1}{2}L([X,Y]) & [\nabla_X, \nabla_Y] + \frac{1}{2}X\cdot L(Y) - \frac{1}{2}Y\cdot L(X) \end{pmatrix} \\
&= -\frac{1}{2}\begin{pmatrix} W(X,Y) & 0 \\ C(X,Y) & W(X,Y) \end{pmatrix} ,
\end{aligned}$$

where we have used equation (2.27). □

4.2 Conformal covariance of the Penrose operator

An important property of the Penrose operator P is *conformal covariance* under Weyl transformations. This implies that $\operatorname{Ker} P$ — the space of conformal Killing spinors — is an invariant of the conformal structure on M. In this section we show how spinors transform under Weyl transformations.

Let (M, g) and $(M, \overline{g} = e^{2f} g)$ be conformally related pseudo-riemannian spin manifolds. Given a a pseudo-orthonormal frame (e_a) for g, we can readily construct a frame $(\overline{e}_a)$ for $\overline{g}$ by setting $\overline{e}_a = e^{-f} e_a$. This defines a bundle isomorphism between the two frame bundles $\xi : P_{\mathrm{SO}}(M,g) \to P_{\mathrm{SO}}(M,\overline{g})$, which in turn lifts to a bundle isomorphism of the spin bundles $\tilde{\xi} : P_{\mathrm{Spin}}(M,g) \to P_{\mathrm{Spin}}(M,\overline{g})$, provided that we choose the same topological spin structure for both manifolds. Now let S and $\overline{\mathrm{S}}$ be the corresponding spinor bundles constructed as associated bundles of the spinor representation $\sigma : \mathrm{Spin} \to \mathrm{GL}(\mathfrak{S})$ — in other words, $\mathrm{S} = P_{\mathrm{Spin}}(M,g) \times_\sigma \mathfrak{S}$ and $\overline{\mathrm{S}} = P_{\mathrm{Spin}}(M,\overline{g}) \times_\sigma \mathfrak{S}$. The bundle isomorphism $\tilde{\xi}$ is Spin-equivariant, and thus induces a further bundle isomorphism $\Xi : \mathrm{S} \to \overline{\mathrm{S}}$, which obeys

$$\Xi(e_a \cdot \psi) = \xi(e_a) \cdot \Xi(\psi) . \tag{4.13}$$

To keep the notation from getting out of hand, we often simply denote $\Xi(\psi) = \overline{\psi}$, rewriting the previous equation as

$$\overline{e_a \cdot \psi} = \overline{e_a} \cdot \overline{\psi} \,.$$

This means that for all vector fields $X \in \mathscr{X}(M)$, $\overline{X \cdot \psi} = e^{-f} X \cdot \overline{\psi}$.

It is also convenient to introduce bundle isomorphisms $\Xi_w : S \to \overline{S}$ for every $w \in \mathbb{R}$, defined by

$$\Xi_w(\psi) = e^{wf}\,\Xi(\psi) = e^{wf}\overline{\psi} \,. \tag{4.14}$$

Using these isomorphisms, we can now compute what happens to the spin connection, the Dirac operator and the Penrose operator under a Weyl transformation.

Proposition 12. *The spin connection, Dirac and Penrose operators of* (M, g) *and* $(M, \overline{g})$ *are related as follows:*

$$\begin{aligned} \overline{\nabla}_X \circ \Xi &= \Xi \circ (\nabla_X - \tfrac{1}{2} X \cdot \operatorname{grad} f - \tfrac{1}{2} df(X)\,\mathrm{Id}) \\ \overline{\nabla\!\!\!/} \circ \Xi &= \Xi_{-1} \circ (\nabla\!\!\!/ + \tfrac{1}{2}(n-1)\operatorname{grad} f) \\ \overline{P}_X \circ \Xi &= \Xi \circ (P_X - \tfrac{1}{2} df(X)\,\mathrm{Id} - \frac{1}{2n} X \cdot \operatorname{grad} f) \,. \end{aligned}$$

Proof. Beginning with the spin connection, it is easy to see that the transformation law for ∇ follows using the structure equations:

$$\begin{aligned} d\overline{e}^{\,a} + \overline{\omega}^{\,a}_{\ b} \wedge \overline{e}^{\,b} &= 0 \\ &= d(e^f e^a) + \overline{\omega}^a_{\ b} \wedge \overline{e}^b \\ &= e^f df \wedge e^a + e^f de^a + \overline{\omega}^a_{\ b} \wedge \overline{e}^b \\ &= df \wedge \overline{e}^{\,a} - e^f \omega^a_{\ b} \wedge e^b + \overline{\omega}^{\,a}_{\ b} \wedge \overline{e}^{\,b} \\ &= -\partial_b f \overline{e}^{\,a} \wedge \overline{e}^{\,b} - \omega^a_{\ b} \wedge \overline{e}^{\,b} + \overline{\omega}^{\,a}_{\ b} \wedge \overline{e}^{\,b} \,, \end{aligned}$$

which implies that

$$\overline{\omega}^{\,ab} = \omega^{ab} + \partial^a f\, \overline{e}^{\,b} \,. \tag{4.15}$$

Substituting this identity into the expression for the spin connection in local coordinates, we have

$$\begin{aligned}
\overline{\nabla}_X &= X^a\partial_a + \frac{1}{4}\overline{\omega}^{ab}\overline{\Sigma}_{ab} \\
&= \nabla_X - \tfrac{1}{2}\partial^b f X^a \overline{\Gamma}_{ab} \\
&= \nabla_X - \tfrac{1}{2}X\cdot \operatorname{grad} f \cdot - \tfrac{1}{2}df(X)\,\mathrm{Id}\ ,
\end{aligned}$$

where we have used $\overline{\Sigma}_{ab} = -\frac{1}{2}\overline{\Gamma}_{ab}$ and $\overline{\Gamma}_{ab} = \overline{\Gamma}_a\overline{\Gamma}_b + \eta_{ab}$. The expression for the Dirac operator now follows simply by taking the Clifford trace, and combining the two gives the expression for the transformed Penrose operator. □

It follows that the Dirac and Penrose operators are covariant, provided that they act on spinors with the correct weight.

Corollary 13.

$$\begin{aligned}
\overline{\nabla\!\!\!/} \circ \Xi_{-\frac{n-1}{2}} &= \Xi_{\frac{n+1}{2}} \circ \nabla\!\!\!/ \\
\overline{P}_{\overline{X}} \circ \Xi_{\frac{1}{2}} &= \Xi_{-\frac{1}{2}} \circ P_X\ ,
\end{aligned}$$

where $\overline{X} = e^{-f}X$.

In particular, it follows that if ψ is a conformal Killing spinor on (M,g), then $e^{\frac{1}{2}f}\overline{\psi}$ is a conformal Killing spinor on $(M,\overline{g})$.

4.3 Weyl-invariant spinorial conformal Killing transport

As in the case of conformal Killing vectors, we would like to modify the spinorial conformal Killing transport equation and make it transform covariantly under Weyl transformations. This requires determining how the Weyl connection D acts on spinors: that is, we must specify how spinors transform under the action of $\mathfrak{co}(TM)$. A spinor is said to have a weight $w \in \mathbb{R}$ if for any $A = S + h\,\mathrm{Id} \in \mathfrak{co}(TM)$, $A\psi = -\frac{1}{2}S\cdot\psi + wh\psi$, where $S \in \mathfrak{so}(TM)$ has been identified with the corresponding two-form in $\Lambda^2 T^*M$ and we use the identification of the spin representation of S with

$-\frac{1}{2}$ times the Clifford multiplication by S. As before, we will denote the bundle of spinors with weight w by $\mathrm{S}^{[w]}$ and the Weyl connection acting on it by D^w. Using these definitions and equation (3.34), it is easy to see how D^w acts on $\mathrm{S}^{[w]}$:

$$\begin{aligned} D^w_X\psi &= \nabla_X\psi + \tfrac{1}{2}X\wedge\theta\cdot\psi - w\theta(X)\psi \\ &= \nabla_X\psi + \tfrac{1}{2}X\cdot\theta\cdot\psi + (\tfrac{1}{2}-w)\theta(X)\psi\,. \end{aligned}$$

The Clifford trace of the last equation gives an expression for the corresponding Dirac operator:

$$D\!\!\!\!/^{\,w}\psi = \nabla\!\!\!\!/\,\psi + (\tfrac{1}{2}-\tfrac{n}{2}-w)\theta\cdot\psi\,. \tag{4.16}$$

Finally, combining these two results, we obtain the Penrose operator $P^{D,w}$ associated to the Weyl connection D.

$$P^{D,w}_X\psi = P_X\psi + (\tfrac{1}{2}-w)(\tfrac{1}{n}X\cdot\theta\cdot\psi + \theta(X)\psi)\,. \tag{4.17}$$

It is apparent from this expression that in fact, $P^{D^{\frac{1}{2}}}_X = P_X$. This and Corollary 13 then give rise to the following result:

Proposition 14. *Conformal Killing spinors are in one-to-one correspondence with sections of the bundle* $\mathrm{S}^{[\frac{1}{2}]}\oplus\mathrm{S}^{[-\frac{1}{2}]}$ *which are parallel with respect to the following connection:*

$$\mathscr{P}^D_X\begin{pmatrix}\psi\\ \chi\end{pmatrix} = \begin{pmatrix} D^{\frac{1}{2}}_X & X\cdot \\ \frac{1}{2}L^D(X)\cdot & D^{-\frac{1}{2}}_X\end{pmatrix}\begin{pmatrix}\psi\\ \chi\end{pmatrix}, \tag{4.18}$$

with a conformal Killing spinor ψ *determining a unique parallel section* $(\psi, \frac{1}{n}D\!\!\!\!/^{\,\frac{1}{2}}\psi)$. *In addition,* $\mathscr{P}^D$ *has curvature*

$$R^{\mathscr{P}^D}(X,Y) = \mathscr{P}^D_{[X,Y]} - [\mathscr{P}^D_X, \mathscr{P}^D_Y] = -\frac{1}{2}\begin{pmatrix} W(X,Y) & 0 \\ C^D(X,Y) & W(X,Y)\end{pmatrix}. \tag{4.19}$$

Chapter 5

Conformal Killing superalgebras

In this section, our aim is to construct a superalgebra $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ associated to (M, g, θ) which is also a conformal invariant. The natural object turns out to be the algebra for which $\mathfrak{h}_0$ consists of the *normal* conformal Killing vectors of (M, g) and $\mathfrak{h}_1$ is the space of conformal Killing spinors. In this section we define the natural product structure of this algebra. Unlike in the analogous Killing superalgebra case, we will see that $\mathfrak{h}$ is not in general a *Lie* superalgebra: one of the Jacobi identities of the algebra can fail. Nevertheless, we find that there are well-defined maps $\mathfrak{h}_0 \times \mathfrak{h}_0 \to \mathfrak{h}_0$, $\mathfrak{h}_0 \times \mathfrak{h}_1 \to \mathfrak{h}_1$ and $\mathfrak{h}_1 \times \mathfrak{h}_1 \to \mathfrak{h}_0$. In Chapter 3, we have already defined the natural Lie bracket of conformal Killing vectors. In this chapter we define the two remaining maps and study the structure of the superalgebra we thus obtain.

5.1 From conformal Killing spinors to conformal Killing vectors

We begin by determining the odd-odd bracket $[-,-] : \mathfrak{h}_1 \times \mathfrak{h}_1 \to \mathfrak{h}_0$. In other words, we want to define a map $[-,-] : S^2\left(\mathrm{S}^{[\frac{1}{2}]} \oplus \mathrm{S}^{[-\frac{1}{2}]}\right) \to \mathfrak{mo}(TM)$ which preserves parallel sections with respect to $\mathscr{P}$ and the Geroch connection $\mathscr{D}$.

Proposition 15. *The map* $[-,-] : \begin{pmatrix} \psi \\ \chi \end{pmatrix} \mapsto (X, A, \beta)$ *is defined by*

$$g(X, Y) = (\psi, Y \cdot \psi) \tag{5.1}$$

$$g(Y, AZ) = 2(\psi, Y \cdot Z \cdot \chi) \tag{5.2}$$

$$\beta(Y) = 2(Y \cdot \chi, \chi)\,, \tag{5.3}$$

when $(\psi,\chi) \in \operatorname{Ker}\mathscr{P}$.

Proof. A natural way to define X is to take it to be the Dirac current $X = V_{\psi,\psi}$. To obtain A, we differentiate it, obtaining

$$\begin{aligned} \nabla_Z g(X,Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \\ &= (\nabla_Z \psi, \psi) + (\psi, \nabla_Z Y \cdot \psi) + (\psi, Y \cdot \nabla_Z \psi)\,, \end{aligned}$$

and since $\nabla_Z X = -A_X Z$ and $\nabla_Z \psi = -Z \cdot \chi$,

$$\begin{aligned} g(A_X Z, Y) &= (Z \cdot \chi, Y \cdot \psi) + (\psi, Y \cdot Z \cdot \chi) \\ &= 2(\psi, Y \cdot\cdot Z\chi) \\ &= 2(\psi, Y \wedge Z \cdot \chi) - 2g(Y,Z)(\psi,\chi)\,, \end{aligned}$$

which also implies that if $A_X = S_X + h_X \operatorname{Id}$, $h_X = -2(\psi,\chi)$ and $\omega_{S_X}(Y,Z) = 2(\psi, Y \wedge Z \cdot \chi)$.

Now

$$\begin{aligned} \alpha_X(Y) &= \nabla_Y h_X \\ &= -2(\nabla_Y \psi, \chi) - 2(\Psi, \nabla_Y \chi) \\ &= 2(Y \cdot \chi, \chi) + (\psi, L(Y) \cdot \psi)\,, \end{aligned}$$

and since $\beta_X = \alpha_X - L(X)$, we have

$$\beta_X(Y) = 2(Y \cdot \chi, \chi)\,. \tag{5.4}$$

Proposition 9 guarantees that (X, A, β) is parallel with respect to the Geroch connection $\mathscr{D}$. □

We extend the map defined in the previous Proposition to $S^2\left(\mathrm{S}^{\left[\frac{1}{2}\right]} \oplus \mathrm{S}^{\left[-\frac{1}{2}\right]}\right)$ using a standard polarisation argument.

5.2 Spinorial Lie derivatives

The general question of constructing a Lie derivative for spinors was first studied by Kosmann-Schwarzbach [27] and by Bourguignon and Gauduchon [28]. Bourguignon and Gauduchon construct the so-called *metric Lie derivative* for spinor fields. Computing the Lie derivative of a spinor with respect to a conformal Killing vector, however, requires comparing spinors on manifolds with different metrics. Using the isomorphism for identifying the spinor bundles on manifolds with conformally related metrics which we introduced in Chapter 4, Section 4.2 makes it possible to define a Lie derivative in a classical manner — that is, along a parametrized curve generated by the vector field, and this is the approach taken in [27, 20].

For us, however, this approach is not completely natural as we want to emphasise the underlying algebraic structure of conformal Killing spinors and conformal Killing vectors. Furthermore, there seems to be some confusion about the correct definition of the spinorial Lie derivative and its properties. Therefore, we choose to work from first principles and define our Lie derivative in terms of the connection $\mathscr{D}^D$ and the natural spinorial representation of the conformal Killing data ϱ. We will show that the action of this Lie derivative agrees with the Kosmann-Schwarzbach Lie derivative.

Definition 16. By a *spinorial Lie derivative* we mean an endomorphism $\mathscr{L}_X$ of sections of the local twistor bundle $S^{[\frac{1}{2}]} \oplus S^{[-\frac{1}{2}]}$ associated to any conformal Killing vector X, satisfying the following properties when X and Y are conformal Killing vectors and Z is any vector field:

(a) $\mathscr{L}_X(f\psi) = X(f)\psi + f\mathscr{L}_X\psi$, i.e. $\mathscr{L}_X$ is a derivation

(b) $[\mathscr{L}_X, \mathscr{L}_Y] = \mathscr{L}_{[X,Y]}$, so the map $X \mapsto \mathscr{L}_X$ is a homomorphism from the Lie algebra of conformal isometries to the Lie algebra of endomorphisms of $S^{[\frac{1}{2}]} \oplus S^{[-\frac{1}{2}]}$;

(c) $[\mathscr{L}_X, \mathscr{D}^D_Y] = \mathscr{D}^D_{[X,Y]}$, so that $\mathscr{L}_X$ preserves the space of conformal Killing spinors.

A Lie derivative of a section of any vector bundle with respect to a vector field can be written as a sum of a connection and a suitable representation acting on the section. Acting on the bundle $S^{[\frac{1}{2}]} \oplus S^{[-\frac{1}{2}]}$, a natural candidate is thus

$$\mathscr{L}_X = \mathscr{D}^D_X + \varrho(X, A^D_X, \beta^D_X)\,, \tag{5.5}$$

where $\varrho(X)$ is the spinor representation of $\mathfrak{mo}(TM)$ defined in (2.16). Clearly, this Lie derivative satisfies property (a) in Definition 16. The other two properties can be rewritten in the following way.

Proposition 17. *The operator $\mathscr{L}_X$ defined above is a spinorial Lie derivative if for every X, Y conformal Killing vectors, the following equalities hold:*

$$[\mathscr{P}^D_X, \varrho(Y)] = R^{\mathscr{P}^D}(X,Y) = R^{\varrho}(X,Y)\,, \tag{5.6}$$

where

$$R^{\varrho}(X,Y) := \varrho([X,Y]) - [\varrho(X), \varrho(Y)]\,. \tag{5.7}$$

Proof. The properties that need to be checked are (b) and (c). We begin with (c) and compute

$$\begin{aligned}[\mathscr{L}_X, \mathscr{P}^D_Y] - \mathscr{P}^D_{[X,Y]} &= [\mathscr{P}^D_X, \mathscr{P}^D_Y] - \mathscr{P}^D_{[X,Y]} + [\varrho(X), \mathscr{P}^D_Y] \\ &= R^{\mathscr{P}^D}(Y,X) - [\mathscr{P}^D_Y, \varrho(X)]\,,\end{aligned}$$

which clearly vanishes if the first equality in (5.6) is satisfied.
Similarly, for the property (b) we can compute

$$\mathscr{L}_{[X,Y]} - [\mathscr{L}_X, \mathscr{L}_Y] = \mathscr{P}^D_{[X,Y]} + \varrho([X,Y]) - [\mathscr{P}^D_X, \mathscr{P}^D_Y] - [\mathscr{P}^D_X, \varrho(Y)] - [\varrho(X), \mathscr{P}^D_Y] - [\varrho(X), \varrho(Y)]$$

which also vanishes provided equation (5.6) holds. □

Using the definition of ϱ and the Weyl-invariant bracket of conformal vector fields (3.53), one finds that $R^{\varrho}(X,Y) = R^{\mathscr{P}^D}(X,Y)$ as required. However, the first equality *fails* to hold. Instead, we have:

$$[\mathscr{P}^D_X, \varrho(Y)] = -\tfrac{1}{2}\begin{pmatrix} W(X,Y) & 0 \\ -C^D(X,Y) & W(X,Y)\end{pmatrix},$$

The offending term is the lower left-hand corner, which has the wrong sign. This means that neither property (b) or (c) are satisfied unless $C^D(X,Y) = 0$. Since we are primarily interested in conformal Killing spinors in supergravity, we might be content with this, since in the absence of fluxes and cosmological constant terms, the supergravity field equations require (M,g) to be Ricci-flat, in which case this condition is automatically satisfied. It seems counterintuitive that such a natural candidate for the spinorial Lie derivative should fail in the general case.

However, equation (5.6) is satisfied for *normal* conformal Killing vectors: this is obvious since (recalling equation (3.40)) $C^D(X,Y) = C(X,Y) - W(X,Y)\theta$. Thus, it is natural to define $\mathfrak{h}_0$ as the Lie algebra of normal conformal Killing vectors on (M,g). This is admittedly a strong restriction, but without it we have little hope of finding a well-defined conformal Killing superalgebra.

We now show that conformal Killing vectors arising as Dirac currents of conformal Killing spinors are actually normal, so that the bracket $[-,-] : S^2\mathfrak{h}_1 \to \mathfrak{h}_0$ is well-defined.

Proposition 18. *Let $X = V_{\psi,\psi}$ be a conformal Killing vector obtained as the Dirac current of a conformal Killing spinor ψ. Then X is a normal conformal Killing vector: that is, $W(X,Y) = 0$, where Y is any vector field (not necessarily conformal).*

Proof. We begin by showing that $W(X,Y) = 0$. From the conformal Killing transport equations (3.17) we know that for any conformal Killing vector X,

$$W(X,Y) = -\nabla_Y A_X + Y \bullet \beta_X + X \bullet L(Y) . \tag{5.8}$$

We will show that if X is the Dirac current of a conformal Killing spinor ψ, the right-hand side of this equation vanishes.

Recall that if $X = V_{\psi,\psi}$, $g(A_X Z, U) = 2(\psi, U \cdot Z \cdot \chi)$. Differentiating this, we find

$$g((\nabla_Y A_X)Z, U) = 2(Y \cdot \chi, U \cdot Z \cdot \chi) + (\psi, U \cdot Z \cdot L(Y) \cdot \psi) . \tag{5.9}$$

Similarly — again using Proposition 15 — for the second term in (5.8) we have

$$\begin{aligned} g(U,(Y \bullet \beta_X)) &= g(U, \beta_X(Z)Y + \beta_X(Y)Z - g(Y,Z)\beta_X^\sharp) \\ &= g(U,Y)\beta_X(Z) + g(U,Z)b_X(Y) - g(Y,Z)\beta_X(U) \\ &= 2g(U,Y)(Z \cdot \chi, \chi) + 2g(U,Z)(Y \cdot \chi, \chi) - 2g(Y,Z)(Y \cdot \chi, \chi) . \end{aligned}$$

Finally, the last term in equation (5.8) can be rewritten as

$$\begin{aligned} g(U,(X \bullet L(Y))Z) &= L(Y,Z)g(U,X) + L(X,Y)g(U,Z) - L(U,Y)g(X,Z) \\ &= L(Y,Z)(\psi, U\cdot\psi) - L(U,Y)(\psi, Z\cdot\psi) + g(U,Z)(\psi, L(Y)\cdot\psi) . \end{aligned}$$

The right-hand side of equation (5.9) can be rewritten using the fact that

$$U\cdot Z\cdot L(Y) = U\wedge Z\wedge L(Y) - g(U,Z)L(Y) + L(Y,U)Z - L(Y,Z)U ,$$

and

$$Z\cdot U\cdot Y = Z\wedge U\wedge Y - g(U,Z)Y + g(Y,Z)U - g(U,Y)Z ,$$

where we have repeatedly used the formula (2.10). Substituting these results into equation (5.9) and into (5.8), we find that

$$g(W(X,Y)Z,U) = (\psi, U\wedge Z\wedge L(Y)\cdot\psi) + 2(Z\wedge U\wedge Y\cdot\chi,\chi) . \tag{5.10}$$

It is possible to show that the right-hand side of this equation vanishes by symmetry. For any spinor ψ, it holds that

$$\begin{aligned} (\psi, \Gamma_{abc}\psi) &= (-1)^{3\epsilon}(\Gamma_{cba}\psi,\psi) \\ &= -(-1)^{3\epsilon}(\Gamma_{abc}\psi,\psi) \\ &= -(-1)^{4\epsilon}(\psi,\Gamma_{abc}\psi) \\ &= -(\psi,\Gamma_{abc}\psi) , \end{aligned}$$

so in fact — given the assumptions we made about the spinor inner product — any three-form constructed out of a spinor must vanish. Applying this to equation (5.10), we see that

$$W(X,Y) = 0 ,$$

as required.

Similarly, from the conformal Killing transport equations (3.17) we know that when X is a conformal Killing vector,

$$C(X,Y) = -\nabla_Y \beta_X + L(Y)\circ A_X . \tag{5.11}$$

When X is the Dirac current of a conformal Killing spinor ψ, we compute (using Proposition 15 as before):

$$
\begin{aligned}
(\nabla_Y \beta_X)(Z) &= 2(Z\cdot\nabla_Y\chi,\chi)+2(Z\cdot\chi,\nabla_Y\chi) \\
&= -2(\psi, L(Y)\cdot Z\cdot\chi) \\
&= -2\ell(Y,e_a)(\psi,e_a\cdot Z\cdot\chi) \\
&= \ell(Y,A_X Z)\,.
\end{aligned}
$$

But the second term in (5.11) gives

$$
(L(Y)\circ A_X)(Z)=\ell(Y,A_X Z)\,,
$$

so the RHS of equation (5.11) vanishes and $C(X,Y)=0$ as required. □

We remark that it can be shown [13] that the normal conformal Cartan connection induces the connection $\mathscr{P}$ on the bundle $S^{[\frac{1}{2}]}\oplus S^{[-\frac{1}{2}]}$ as well as the connection defining normal conformal Killing vectors we mentioned in section 3.6. In this sense, both $\mathfrak{h}_0$ and $\mathfrak{h}_1$ originate as parallel sections of the same connection acting on different vector bundles on M, and thus it is not surprising that there exists a natural algebraic structure involving them both. Since there is no analogue of the normal conformal Cartan connection in the supergravity case which we will discuss later, we forego presenting this unified viewpoint in more detail and refer the interested reader to the literature [29, 30].

Note that because of the Weyl-invariant way we have defined both conformal Killing transport, its spinorial counterpart and the Lie derivative $\mathscr{L}$, $\mathfrak{h}$ is manifestly a conformal invariant of (M,g).

5.3 The Kosmann-Schwarzbach Lie derivative

We now introduce the definition of the spinorial Lie derivative that has become standard in the literature [20, 27, 28].

Definition 19. The *Kosmann-Schwarzbach* Lie derivative $\hat{\mathscr{L}}_X$ of a spinor ψ with respect to a conformal Killing vector field X is defined as follows [27]:

$$\hat{\mathscr{L}}_X\psi = \nabla_X\psi - \tfrac{1}{2}S_X\cdot\psi + \tfrac{1}{2}h_X\psi = \nabla_X + \sigma^{\frac{1}{2}}(A_X)\,. \tag{5.12}$$

The Kosmann-Schwarzbach Lie derivative fails to respect the Clifford product, as the following Lemma shows.

Lemma 20. *The Kosmann-Schwarzbach Lie derivative has the following properties with respect to Clifford multiplication when X is a conformal Killing vector, ψ is a conformal Killing spinor, Y is an arbitrary vector field and η is a p-form.*

(a) $\hat{\mathscr{L}}_X(Y\cdot\psi) = \mathscr{L}_X Y\cdot\psi + Y\cdot\hat{\mathscr{L}}_X\psi - h_X Y\cdot\psi$

(b) $\hat{\mathscr{L}}_X(\eta\cdot\psi) = \mathscr{L}_X\eta\cdot\psi + \eta\cdot\hat{\mathscr{L}}_X\psi + ph_X\eta\cdot\psi$

Proof. (a): Recall that we can write $\mathscr{L}_X Y = [X,Y] = \nabla_X Y - \nabla_Y X = \nabla_X Y + A_X Y$, where $A_X = S_X + h_X\,\mathrm{Id}$. Using Definition 19, the properties of the spin connection and equation (2.14), we compute

$$\begin{aligned}
\hat{\mathscr{L}}_X(Y\cdot\psi) &= \nabla_X Y\cdot\psi + Y\cdot\nabla_X\psi - \tfrac{1}{2}S_X\cdot Y\cdot\psi + \tfrac{1}{2}h_X Y\cdot\psi \\
&= \nabla_X Y\cdot\psi + [-\tfrac{1}{2}S_X, Y]\cdot\psi + Y\cdot\hat{\mathscr{L}}_X\psi \\
&= \nabla_X Y\cdot\psi + S_X Y\cdot\psi + Y\cdot\hat{\mathscr{L}}_X\psi \\
&= \nabla_X Y\cdot\psi + A_X Y\cdot\psi - h_X Y\cdot\psi + Y\cdot\hat{\mathscr{L}}_X\psi \\
&= \mathscr{L}_X Y\cdot\psi + +Y\cdot\hat{\mathscr{L}}_X\psi - h_X Y\cdot\psi\,.
\end{aligned}$$

(b): Now equation (2.1) implies that if β is a one-form, $\mathscr{L}_X\beta = \nabla_X\beta - \beta\circ A_X$. This leads to an almost identical calculation as above, except that now the remaining h_X term has a different sign. It is straightforward to extend the result to p-forms by linearity. □

Given the suggestive name of this operator, it is not surprising that we can prove the following proposition.

Proposition 21. *The Kosmann-Schwarzbach Lie derivative $\widehat{\mathscr{L}}$ is a spinorial Lie derivative in the sense of Definition 16.*

Proof. Property (a) in Definition 16 is again obvious.

(b): We simply compute

$$\begin{aligned}
[\widehat{\mathscr{L}}_X, \widehat{\mathscr{L}}_Y] &= [\nabla_X, \nabla_Y] + [\nabla_X, \sigma^{\frac{1}{2}}(A_Y)] - [\nabla_Y, \sigma^{\frac{1}{2}}(A_X)] + \sigma^{\frac{1}{2}}([A_X, A_Y]) \\
&= -R(X,Y) + \nabla_{[X,Y]} + \sigma^{\frac{1}{2}}(\nabla_X A_Y) - \sigma^{\frac{1}{2}}(\nabla_Y A_X) + \sigma^{\frac{1}{2}}([A_X, A_Y]) \\
&= -R(X,Y) + \nabla_{[X,Y]} \\
&\quad + \sigma^{\frac{1}{2}}\left(R(X,Y) + X \bullet \alpha_Y - R(Y,X) - Y \bullet \alpha_X + A_{[X,Y]} - R(X,Y) - X \bullet \alpha_Y + Y \bullet \alpha_X\right) \\
&= \nabla_{[X,Y]} + \sigma^{\frac{1}{2}}(A_{[X,Y]}) \\
&= \widehat{\mathscr{L}}_{[X,Y]} ,
\end{aligned}$$

where we have used the identity $-\frac{1}{2}R^{\nabla}(X,Y)\cdot = R(X,Y)$, the conformal Killing transport equations and the Lie bracket for conformal Killing data defined in equation (3.23).

(c): We begin by computing the commutator of $\widehat{\mathscr{L}}$ and the spin connection ∇. Now

$$\begin{aligned}
[\widehat{\mathscr{L}}_X, \nabla_Y] &= [\nabla_X, \nabla_Y] + [\sigma^{\frac{1}{2}}(A_X), \nabla_Y] \\
&= -R(X,Y) + \nabla_{[X,Y]} - \sigma^{\frac{1}{2}}(\nabla_Y A_X) \\
&= -R(X,Y) + \nabla_{[X,Y]} - \sigma^{\frac{1}{2}}(R(X,Y) + Y \bullet \alpha_X) \\
&= \nabla_{[X,Y]} + \tfrac{1}{2} Y \cdot \alpha_X \cdot .
\end{aligned}$$

Using Lemma 20, we can now show that

$$[\widehat{\mathscr{L}}_X, Z^\flat \cdot \nabla_Y] = \mathscr{L}_X Z^\flat \cdot \nabla_Y + Z^\flat \cdot [\widehat{\mathscr{L}}_X, \nabla_Y] + h_X Z \cdot \nabla_Y , \tag{5.13}$$

and taking the trace of this equation over Y and Z, we obtain the following result for the Dirac operator $\not{\nabla}$:

$$[\widehat{\mathscr{L}}_X, \not{\nabla}] = \mathscr{L}_X e^a \cdot \nabla_a + e^a \cdot \nabla_{[X,e_a]} - \tfrac{n}{2}\alpha_X \cdot + h_X \not{\nabla} . \tag{5.14}$$

Combining these results and using Lemma 20 again, it follows that

$$[\widehat{\mathscr{L}}_X, P_Y] = P_{[X,Y]} . \tag{5.15}$$

Thus, $\widehat{\mathscr{L}}$ preserves the space of conformal Killing spinors on M. □

Because of Proposition 21 , the Kosmann-Schwarzbach Lie derivative would also appear to be a natural candidate for a spinorial Lie derivative. However, it only contains the TM- and the $\mathfrak{co}(TM)$-components of the conformal Killing data defining X.

Given that we have shown that there is a well-defined action of (X, A, β) on the local twistor bundle $S^{[\frac{1}{2}]} \oplus S^{[-\frac{1}{2}]}$ — including the one-form part — our definition appears more natural, even if there is an obstruction (proportional to the Cotton-York tensor C^D) for it to be a spinorial Lie derivative.

We now show that in fact the actions of $\widehat{\mathscr{L}}_X$ and $\mathscr{L}_X$ on $S^{[\frac{1}{2}]}$ agree when X is normal, so that there is no ambiguity in the way we define the even-odd bracket of the conformal Killing superalgebra $\mathfrak{h}$.

Suppose that X is normal and (ψ, χ) is any section of $S^{[\frac{1}{2}]} \oplus S^{[-\frac{1}{2}]}$. Then

$$\mathscr{L}_X \begin{pmatrix} \psi \\ \chi \end{pmatrix} = \begin{pmatrix} D_X^{\frac{1}{2}} \chi + X \cdot \chi + \sigma^{\frac{1}{2}}(A_X^D)\psi - X \cdot \chi \\ \frac{1}{2} L^D(X) + D_X^{-\frac{1}{2}} \chi + \frac{1}{2}\beta_X^D \cdot \psi + \sigma^{-\frac{1}{2}}(A_X^D)\chi \end{pmatrix}$$
$$= \begin{pmatrix} \nabla_X \psi + \sigma^{\frac{1}{2}}(A_X)\psi \\ \nabla_X \chi + \sigma^{-\frac{1}{2}}(A_X)\chi + \frac{1}{2}\alpha_X \cdot \psi \end{pmatrix},$$

where we have used the definitions of the Weyl-invariant conformal Killing data and the Weyl connection from Chapter 3, Section 3.5. The action on the $S^{\frac{1}{2}}$-component clearly agrees with the Kosmann-Schwarzbach Lie derivative. Since $\widehat{\mathscr{L}}$ and $\mathscr{L}$ both preserve the kernel of P, it is also clear that the induced action on the $S^{[-\frac{1}{2}]}$-component agrees with that of $\mathscr{L}$ when $\begin{pmatrix} \psi \\ \chi \end{pmatrix}$ defines a conformal Killing spinor.

When we have explicit expressions for conformal Killing spinors, the Kosmann-Schwarzbach Lie derivative $\widehat{\mathscr{L}}$ is often more convenient, whereas the "natural" spinorial Lie derivative $\mathscr{L}$ makes the unified origin of conformal Killing vectors and conformal Killing spinors as parallel sections of bundles with a natural algebraic Lie bracket more manifest. Since their actions agree on conformal Killing spinors, for the remainder of this thesis we will denote both Lie derivatives simply by $\mathscr{L}$, trusting that it will be apparent from context which one we are using.

5.4 The Jacobi identities

The Jacobi identity for a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ can be written as

$$[\boldsymbol{X},[\boldsymbol{Y},\boldsymbol{Z}]] = [[\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z}] + (-1)^{|\boldsymbol{X}||\boldsymbol{Y}|}[\boldsymbol{Y},[\boldsymbol{X},\boldsymbol{Z}]] , \tag{5.16}$$

where $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z} \in \mathfrak{g}$ are homogeneous elements and $|\boldsymbol{X}|$ denotes the degree of $\boldsymbol{X}$. In this section we show that the conformal Killing superalgebra $\mathfrak{h}$ is not a Lie superalgebra in general. We will examine each of the four possible Jacobi identities in turn. We already know that the natural Lie bracket on $\mathfrak{h}_0$ satisfies the Jacobi identity, since for normal conformal Killing vectors the Lie bracket (3.53) reduces to the natural algebraic Lie bracket on $\mathfrak{mo}(TM)$. We have also shown that the even-even-odd Jacobi identity holds: it is easy to see that this is equivalent to property (b) in Definition 16. Assume that $\Psi := (\psi,\chi)$ defines a conformal Killing spinor. Checking the even-odd-odd Jacobi identity amounts to showing that

$$[X,[\Psi,\Psi]] = [\mathscr{L}_X\Psi,\Psi] + [\Psi,\mathscr{L}_X\Psi] . \tag{5.17}$$

It is enough to check this identity for the TM-component, since we are dealing with $\mathscr{D}$-parallel sections of $\mathfrak{mo}(TM)$. The other components are then fixed by the conformal Killing transport equations. We denote the Dirac current of ψ by $V := V_{\psi,\psi}$ and the TM-component of $[\mathscr{L}_X\Psi,\Psi]$ by $V_{\mathscr{L}_X\psi,\psi}$.

Let us begin by computing the TM-component of the left-hand side of equation (5.17). We recall that $[X,V] = A_XV - B_VX$ (where $B_V = -\nabla V$) and obtain

$$\begin{aligned} g(A_XV,Z) - g(B_VX,Z) &= g(A_XV,Z) - 2(\psi, Z\cdot X\cdot\chi) \\ &= -g(V,S_XZ) + h_Xg(V,Z) - 2(\psi,Z\cdot X\cdot\chi) \\ &= -(\psi,S_XZ\cdot\psi) + h_X(\psi,Z\cdot\psi) - 2(\psi,Z\cdot X\cdot\chi) . \end{aligned}$$

On the other hand, computing the right-hand side of (5.17) yields

$$\begin{aligned} g(V_{\mathscr{L}_X\psi},Z) + g(V_{\psi,\mathscr{L}_X\psi}) &= (\sigma^{\frac{1}{2}}(A_X)\psi, Z\cdot\psi) - (X\cdot\chi, Z\cdot\psi) + (\psi, Z\cdot\sigma^{\frac{1}{2}}(A_X)\psi) - (\psi, Z\cdot X\cdot\chi) \\ &= -(\psi,[-\tfrac{1}{2}S_X,Z]\cdot\psi) + h_X(\psi,Z\cdot\psi) - 2(\psi,Z\cdot X\cdot\chi) , \end{aligned}$$

where we have used the fact that the adjoint of a two-form η with respect to the spinor inner product is $-\eta$ and equation (2.14). The even-odd-odd Jacobi identity is thus satisfied.

By a standard polarisation argument, the vanishing of the odd-odd-odd Jacobi identity would be equivalent to

$$\mathscr{L}_{V_{\psi,\psi}} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = 0. \tag{5.18}$$

Unfortunately, we find that

$$\mathscr{L}_{V_{\psi,\psi}} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = \begin{pmatrix} \sigma^{\frac{1}{2}}(A_V)\psi - V\cdot\chi \\ -\frac{1}{2}\beta_V\cdot\psi + \sigma^{-\frac{1}{2}}(A_V)\chi \end{pmatrix}.$$

The vanishing of this expression is equivalent to

$$\begin{aligned} (\psi,\Gamma_a\psi)\Gamma_a\chi + (\psi,\Gamma_{ab}\chi)\psi + (\psi,\chi)\psi &= 0 \\ (\Gamma_a\chi,\chi)\Gamma_a\psi + (\psi,\Gamma_{ab}\chi)\Gamma_{ab}\chi - (\psi,\chi)\chi &= 0\,. \end{aligned}$$

However, the expressions on the left-hand side do not vanish in general for arbitrary spinors ψ, χ, and thus we are forced to conclude that $\mathfrak{h}$ is not necessarily a Lie superalgebra. This result is not new: Habermann [20] studies the algebra of conformal Killing vectors and conformal Killing spinors and presents an explicit (non-complete) example where the fourth Jacobi identity is not satisfied. However, from the above considerations it is evident that the fourth Jacobi identity may fail purely for algebraic reasons.

5.5 The Minkowski conformal Killing superalgebra

In this section, we exhibit the simplest possible example of a conformal Killing superalgebra: that of the flat Minkowski space $(\mathbb{R}^{1,n-1}, \eta_{ab})$. For convenience, we compute the brackets of the algebra using the Kosmann-Schwarzbach Lie derivative and the Lie bracket of vector fields, although of course we could have used the natural bracket of $\mathfrak{mo}(\mathbb{R}^{1,n-1})$ and the spinorial Lie derivative $\mathscr{L}$ as well.

The conformal algebra of the Minkowski space $\mathbb{R}^{1,n-1}$ with coordinates x^a and the standard flat metric η_{ab} is generated by P_a, M_{ab}, D, K_a — corresponding to translations, rotations, the dilatation and the special conformal transformations.

$$\begin{aligned} P_a &= \partial_a, & (5.19)\\ M_{ab} &= x_a\partial_b - x_b\partial_a, & (5.20)\\ D &= x^a\partial_a, & (5.21)\\ K_a &= 2x_a x^b\partial_b - (x,x)\partial_a. & (5.22) \end{aligned}$$

The even part of the algebra is given by

$$\begin{aligned} [P_a, P_b] &= 0, & (5.23)\\ [M_{ab}, P_c] &= \eta_{bc}P_a - \eta_{ac}P_b, & (5.24)\\ [M_{ab}, M_{cd}] &= \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc}, & (5.25)\\ [P_a, D] &= P_a, & (5.26)\\ [K_a, D] &= -K_a, & (5.27)\\ [P_a, K_b] &= 2\eta_{ab}D - 2M_{ab}, & (5.28)\\ [M_{ab}, K_c] &= \eta_{bc}K_a - \eta_{ac}K_b. & (5.29) \end{aligned}$$

The conformal Killing spinor equation is readily solved using Lemma 10: $\nabla(\nabla\!\!\!/\,\psi) = 0$, so $\nabla\!\!\!/\,\psi = \chi$, a constant spinor. Substituting this into the equation $P_X\psi = 0$, we have

$$\psi = \psi_0 + x\cdot\chi, \qquad (5.30)$$

where $x\cdot\chi := x^a\Gamma_a\chi$ and ψ_0, χ are arbitrary constant spinors. It is straightforward to compute the action of the conformal algebra generators on ψ via the Kosmann-Schwarzbach spinorial Lie derivative. This yields

$$\begin{aligned} \mathscr{L}_{P_a}\psi &= \Gamma_a\psi_2, & (5.31)\\ \mathscr{L}_{M_{ab}}\psi &= -\tfrac{1}{2}M_{ab}\cdot\psi_1 - \tfrac{1}{2}x\cdot M_{ab}\cdot\psi_2, & (5.32)\\ \mathscr{L}_D\psi &= -\tfrac{1}{2}\psi_1 + \tfrac{1}{2}x\cdot\psi_2, & (5.33)\\ \mathscr{L}_{K_a}\psi &= x\cdot\Gamma_a\psi_1\,. & (5.34) \end{aligned}$$

Finally, we compute the Dirac current $V := V_{\psi,\psi}$ associated to a conformal Killing spinor ψ. In local coordinates, we have

$$\begin{aligned} V^a &= (\psi, \Gamma^a \psi) \\ &= (\psi_0, \Gamma^a \psi_0) + x^b (\psi_0, \Gamma^a \Gamma_b \chi) + x^b (\Gamma_b \chi, \Gamma^a \psi_0) + x^b x^c (\Gamma_b \chi, \Gamma^a \Gamma_c \chi) \\ &= (\psi_0, \Gamma^a \psi_0) + 2x^b (\psi_0, \Gamma^a{}_b \chi) - 2x^a (\psi_0, \chi) + x^b x^c (\Gamma_b b \chi, \Gamma^a \Gamma_c \chi) \\ &= (\psi_0, \Gamma^a \psi_0) + 2x^b (\psi_0, \Gamma^a{}_b \chi) - 2x^a (\psi_0, \chi) - 2x^a x^c (\Gamma_c \chi, \chi) + |x|^2 (\Gamma^a \chi, \chi) , \end{aligned}$$

where we have used the Clifford algebra identity $\Gamma_{abc} = \Gamma_a \Gamma_b \Gamma_c - \eta_{ac} \Gamma_b + \eta_{ab} \Gamma_c + \eta_{cb} \Gamma_a$ and the fact that the three-form constructed out of the spinor χ vanishes due to the symmetry properties of the spinor inner product. We can rewrite this in terms of the conformal Lie algebra generators as

$$V_{\psi,\psi} = (\psi_0, \Gamma^a \psi_0) \partial_a + 2(\psi_0, \Gamma^{ab} \chi) M_{ab} - 2(\psi_0, \chi) D - (\Gamma^a \chi, \chi) K_a . \tag{5.35}$$

In particular, this shows that all the conformal Killing vectors of Minkowski space are normal, since they arise as Dirac currents of conformal Killing spinors.

Chapter 6

Conformal Killing spinors in M-theory

In this chapter we generalise the concept of conformal Killing spinors to eleven-dimensional supergravity. We show that M-theory backgrounds that admit a *supergravity conformal Killing spinor* distinct from supergravity Killing spinors and geometric conformal Killing spinors must be of a very particular type: the metric must be one of the so-called *Bryant metrics* and the four-form must satisfy a strong integrability condition.

6.1 M-theory backgrounds

A (bosonic) background of eleven-dimensional supergravity is a triple (M, g, F), where M is an 11-dimensional Lorentzian spin manifold with metric g, and F is a closed four-form subject to the following equation:

$$d \star F = \tfrac{1}{2} F \wedge F \tag{6.1}$$

There is also an Einstein-type equation relating the Ricci curvature of g to the stress-energy tensor of F.

$$r(X,Y) = \tfrac{1}{2}\langle \iota_X F, \iota_Y F\rangle - \tfrac{1}{6} g(X,Y)|F|^2 , \tag{6.2}$$

where $\langle -, - \rangle$ is the inner product on p-forms induced by the metric g.

We remark that equations (6.1) and (6.2) actually arise as the Euler-Lagrange equations of the eleven-dimensional supergravity action: we will not need the explicit form of the action here, and refer the interested readed to e.g. [5] for details.

For future reference we note that equation (6.2) also implies that the Schouten tensor L can also be expressed in terms of F. A straightforward computation using the definition of L and equation (6.2) gives

$$L(X,Y) = \tfrac{1}{18}\langle \iota_X F, \iota_Y F\rangle - \tfrac{7}{360} g(X,Y)|F|^2 . \tag{6.3}$$

We remark that since the geometric conformal Killing spinors are conformally covariant objects, one might worry that the lack of conformal invariance in M-thery might prevent one from defining their supergravity analogue. It is known, however, that the M-theory equations of motion (6.2), (6.1) admit a scaling symmetry

$$\begin{aligned} g &\mapsto \lambda^{-2} g \\ F &\mapsto \lambda^{-3} F , \end{aligned} \tag{6.4}$$

where λ is a constant. It is easy to see that this transformation maps M-theory backgrounds to other M-theory backgrounds since the equations of motion transform homogeneously under (6.4). One might therefore expect that a M-theory background admits some sort of scale-invariant structure, characterised by scale-invariant spinorial objects.

The spinors in M-theory are real and 32-dimensional. There are two possible Clifford modules, both isomorphic to $\mathbb{R}^{32}$; we choose the one for which the action of the eleven-dimensional volume element is nontrivial. The spinor inner product $(-,-)$ is now symplectic and obeys $(\psi, Y\cdot\chi) = -(Y\cdot\psi, \chi)$ for any vector field Y. Moreover, if η is a p-form, its Clifford adjoint (considered as a spinor endomorphism) with respect to the spinor inner product is $\eta^\star = (-1)^{\frac{p(p+1)}{2}}\eta$.

With these conventions, the supercovariant connection acting on $\mathbb{S}$ is given by

$$\mathcal{D}_X = \nabla_X + \tfrac{1}{6}\iota_X F + \tfrac{1}{12} X^\flat \wedge F := \nabla_X + \Omega_X . \tag{6.5}$$

The curvature of $\mathcal{D}$ is defined in the usual way:

$$R^{\mathcal{D}}(X,Y) = \mathcal{D}_{[X,Y]} - [\mathcal{D}_X, \mathcal{D}_Y] . \tag{6.6}$$

It is an important fact that the vanishing of the Clifford trace of $R^{\mathcal{D}}$ considered as a Clifford endomorphism is equivalent to the equations of motion. [31, 32]. In other words,the identity

$$\sum_a e^a \cdot R^{\mathcal{D}}(X, e_a) = 0 . \tag{6.7}$$

is equivalent to (6.2),(6.1).

Unlike the Levi-Cività spin connection ∇, $\mathcal{D}$ does *not* respect the Clifford product. Instead, we have the identity

$$\mathcal{D}_Y\left(Z \cdot \psi\right) = \nabla_Y Z \cdot \psi + Z \cdot \mathcal{D}_Y \psi - \tfrac{1}{3} Z^\flat \wedge \iota_Y F \cdot \psi - \tfrac{1}{6} Z^\flat \wedge Y^\flat \wedge F \cdot \psi . \tag{6.8}$$

6.2 The M-theory Penrose operator

We wish to consider spinors in the kernel of a Penrose-type operator defined using the supercovariant connection. Note that this can be thought of as the composition of the projection to the kernel of Clifford multiplication with the supercovariant connection. We therefore define

$$\mathcal{P}_Y = \mathcal{D}_Y + \tfrac{1}{11} Y \cdot \not{\mathcal{D}} , \tag{6.9}$$

where $\not{\mathcal{D}} = \sum_a e^a \cdot \mathcal{D}_a = \not{\nabla} + \frac{1}{12} F \cdot$ is the Dirac operator associated to the supercovariant connection. We note that $\mathcal{P}$ can be written as a sum of the geometric Penrose operator and the F-dependent terms as

$$\mathcal{P}_X \psi = P_X \psi + \Omega_X \psi + \tfrac{1}{121} X \cdot F \cdot \psi .$$

We call the spinors in the kernel of $\mathcal{P}$ *supergravity conformal Killing spinors* (SCKS)[1]. Since this is a somewhat unwieldy term, for the rest of the chapter we refer to them simply as conformal Killing spinors, making the distinction between them and geometric conformal Killing spinors when necessary.

The reason for this nomenclature is that spinors in the kernel of $\mathcal{P}$ have the properties that we would expect supergravity generalisations of conformal Killing spinors to have. The D-parallel spinors – the supergravity Killing spinors – are obviously in $\operatorname{Ker}\mathcal{P}$, for instance. In addition, when $F = 0$, supergravity conformal Killing spinors

[1] With some reluctance, we refrain from introducing the acronym SUCKS, suggested by José Figueroa-O'Farrill. Nevertheless, look out for a forthcoming paper [33].

reduce to geometric conformal Killing spinors. We now show that the Dirac currents of supergravity twistor spinors are conformal Killing vectors, obtaining a M-theory analogue of Proposition 9.

Proposition 22. *Suppose that* $\psi, \chi \in \operatorname{Ker} \mathcal{P}$ *are supergravity conformal Killing spinors. Then the Dirac current* $V_{\psi,\chi}$ *is a conformal Killing vector.*

Proof. We proceed in a similar fashion as in the geometric case, taking the covariant derivative of the Dirac current.

$$\begin{aligned} g(\nabla_Y V_{\psi,\chi}, Z) &= (\nabla_Y \psi, Y \cdot \chi) + (\psi, Z \cdot \nabla_Y \chi) \\ &= -\tfrac{1}{6}(\iota_Y F \cdot \psi, Z \cdot \chi) - \tfrac{1}{6}(\psi, Z \cdot \iota_Y F \cdot \chi) - \tfrac{1}{11}(Y^\flat \wedge F \cdot \psi, Z \cdot \chi) \\ &= -\tfrac{1}{12}(\psi, Z \cdot Y^\flat \wedge F \cdot \chi) - \tfrac{1}{11}(Y \cdot \mathcal{D}\psi, Z \cdot \chi) - \tfrac{1}{11}(\psi, Z \cdot Y \cdot \mathcal{D}\chi) \\ &= -\tfrac{1}{3}(\iota_Z \iota_Y F \cdot \varphi_1, \chi) + \tfrac{1}{6}(Z^\flat \wedge Y^\flat \wedge F \cdot \psi, \chi) \\ &= +\tfrac{1}{11}\big((Z \cdot Y \cdot \mathcal{D}\psi, \chi) - (Y \cdot Z \cdot \psi, \mathcal{D}\chi)\big) \end{aligned}$$

where we have made use of the fact that the Clifford adjoints of the terms appearing in Ω_Y are $(\iota_Y F)^\star = \iota_Y F$ and $(Y^\flat \wedge F)^\star = -Y^\flat \wedge F$. The first two terms in the final expression are manifestly antisymmetric, so antisymmetrising it gives

$$g(\nabla_Y V_{\psi,\chi}, Z) + g(Y, \nabla_Z V_{\psi,\chi}) = -2h_{V_{\psi,\chi}} g(Y, Z) \,,$$

where $h_{V_{\psi,\chi}} = -\frac{1}{11}\left[(\psi, \mathcal{D}\chi) - (\mathcal{D}\psi, \chi)\right]$. In other words, the conformal Killing equation is satisfied. □

6.3 Supercovariant conformal Killing transport

We will now find a connection that allows us to identify $\operatorname{Ker} \mathcal{P}$ with parallel sections of $S \oplus S$.

Lemma 23. *If* ψ *is a conformal Killing spinor,* $\mathcal{D}\psi$ *satisfies the following identity:*

$$\mathcal{D}_X \mathcal{D}\psi = \tfrac{1}{9}\mathcal{D}^2\psi + \tfrac{7}{27}\iota_X F \cdot \mathcal{D}\psi + \tfrac{2}{27} X^\flat \wedge F \cdot \mathcal{D}\psi. \tag{6.10}$$

Proof. Suppose e_a is again a geodesic frame as in the proof of Lemma 10. We differentiate the conformal Killing spinor equation at a point $p \in M$, obtaining

$$\begin{aligned} \mathcal{D}_X(\mathcal{D}_a\psi) + \tfrac{1}{11}\mathcal{D}_X(e_a \cdot \mathcal{D}\psi) &= 0, \\ \mathcal{D}_a(\mathcal{D}_X\psi) + \tfrac{1}{11}\mathcal{D}_i(X \cdot \mathcal{D}\psi) &= 0. \end{aligned}$$

Subtracting the second equation from the first and applying (6.8),we obtain

$$\begin{aligned} 0 = R^{\mathcal{D}}(X, e_a)\psi + \tfrac{1}{11}(e_a \cdot \mathcal{D}_X \mathcal{D}\psi - \tfrac{1}{3}e^a \wedge \iota_X F \cdot \mathcal{D}\psi \\ - \tfrac{1}{6}e^a \wedge X^b \wedge F \cdot \mathcal{D}\psi - X \cdot \mathcal{D}_a \mathcal{D}\psi - \tfrac{1}{3}X^b \wedge \iota_a F \cdot \mathcal{D}\psi - \tfrac{1}{6}X^b \wedge e^a \wedge F \cdot \mathcal{D}\psi). \end{aligned}$$

Taking the Clifford trace of this equation and using equations (2.12) and (6.7), this yields

$$\mathcal{D}_X \mathcal{D}\psi = \tfrac{1}{9}\mathcal{D}^2\psi + \tfrac{7}{27}\iota_X F \cdot \mathcal{D}\psi + \tfrac{2}{27}X^b \wedge F \cdot \mathcal{D}\psi. \tag{6.11}$$

We can compute $\mathcal{D}^2\psi$ by taking the Clifford trace of this equation:

$$\mathcal{D}^2\psi = \tfrac{14}{60}F \cdot \mathcal{D}\psi \tag{6.12}$$

and substituting this expression back to (6.11) gives

$$\nabla_X \mathcal{D}\psi = \tfrac{1}{15}\iota_X F \cdot \mathcal{D}\psi + \tfrac{1}{60}X^b \wedge F \cdot \mathcal{D}\psi.$$

We can also write this as

$$\tilde{\mathcal{D}}_X \mathcal{D}\psi = 0\,,$$

where $\tilde{\mathcal{D}}_X = \nabla_X - \frac{1}{15}\iota_X F - \frac{1}{60}X^b \wedge F$ – a connection similar to the supercovariant connection, but with different numerical coefficients. □

This immediately gives us the supercovariant version of Proposition 11.

Proposition 24. *M-theory conformal Killing spinors are in one-to-one with with parallel sections of* $S \oplus S$ *with respect to the connection*

$$\wp_X \begin{pmatrix} \psi \\ \chi \end{pmatrix} = \begin{pmatrix} \mathcal{D}_X & X\cdot \\ 0 & \tilde{\mathcal{D}}_X \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \tag{6.13}$$

with a conformal Killing spinor ψ uniquely determining a parallel section $(\psi, \frac{1}{11}\mathcal{D}\psi)$. Furthermore, $\wp$ has curvature

$$R^{\wp}(X,Y) = \begin{pmatrix} R^{\mathcal{D}}(X,Y) & -\frac{1}{5}X^\flat \wedge \iota_Y F + \frac{1}{5}Y^\flat \wedge \iota_X F - \frac{14}{30}\iota_X\iota_Y F - \frac{2}{15}X^\flat \wedge Y^\flat \wedge F \\ 0 & R^{\tilde{\mathcal{D}}}(X,Y) \end{pmatrix}. \tag{6.14}$$

Proof. The first part follows from Lemma 23. Obtaining the curvature of $\wp$ is a straightforward calculation using equation (6.8) and the usual Clifford product identities (2.10). □

We observe that knowing the expression for the curvature (6.14) immediately allows us to determine when a (simply connected) M-theory background admits a *maximal* number of conformal Killing spinors. This happens when $R^{\wp} = 0$, which in turn implies that $R^{\mathcal{D}}$ and $R^{\tilde{\mathcal{D}}}$ must vanish as well. The vanishing of $R^{\mathcal{D}}$ means that the background must be maximally supersymmetric. Such eleven-dimensional backgrounds have been classified [34], up to local isometry, and in fact the only possibilities are the $AdS_4 \times S^7$ and $AdS_7 \times S^4$ Freund-Rubin solutions, the maximally supersymmetric Hpp-wave and flat Minkowski space.
The vanishing of the remaining component of the curvature gives

$$\tfrac{1}{5}X^\flat \wedge \iota_Y F - \tfrac{1}{5}Y^\flat \wedge \iota_X F + \tfrac{14}{30}\iota_X\iota_Y F + \tfrac{2}{15}X^\flat \wedge Y^\flat \wedge F = 0.$$

But this implies that $F = 0$ and hence the only remaining possibility is Minkowski space.
What about the non-maximal case? We now show that an important corollary of Proposition 24 is a very strong integrability condition.

Corollary 25. *Let $(\psi, \chi) \in \operatorname{Ker}\wp$ define a conformal Killing spinor on (M, g, F). Then the spinor χ must in fact be parallel with respect to the Levi-Cività spin connection ∇. In addition,*

$$X^\flat \wedge F \cdot \chi = \iota_X F \cdot \chi = F \cdot \chi = 0\,. \tag{6.15}$$

Proof. Since (ψ, χ) is parallel with respect to $\wp$, we have that

$$R^{\wp}(X,Y)\begin{pmatrix}\psi\\ \chi\end{pmatrix} = 0$$

for any $X, Y \in \mathscr{X}(M)$. This is equivalent to

$$\begin{aligned} R^{\mathcal{D}}(X,Y)\psi &= \left(\tfrac{1}{5}X^\flat \wedge \iota_Y F - \tfrac{1}{5}Y^\flat \wedge \iota_X F + \tfrac{14}{30}\iota_X\iota_Y F + \tfrac{2}{15}X^\flat \wedge Y^\flat \wedge F\right)\cdot \chi\,, \\ R^{\tilde{\mathcal{D}}}(X,Y)\chi &= 0\,. \end{aligned}$$

Taking the Clifford trace of the first equation with respect to Y and using (6.7) again gives (after some manipulation)

$$X^\flat \wedge F \cdot \chi = 0\,.$$

Taking a Clifford trace of this equation and using (2.12) then gives $F\cdot\chi = 0$. This tells us that for any vector field X, $X\cdot F\cdot\chi = (X^\flat \wedge F - \iota_X F)\cdot\chi = \iota_X F\cdot\chi = 0$. Using equation (6.10), we obtain

$$\tilde{\mathcal{D}}_Z\chi = \mathcal{D}_Z\chi = \nabla_Z\chi = 0\,.$$

□

M-theory backgrounds that admit conformal Killing spinors must thus be rather special. Not only must they admit solutions of the conformal Killing spinor equation, they must also possess (supergravity) Killing spinors χ which are *parallel.*

We thus have a set of necessary (but not sufficient) conditions for a M-theory background M to admit supergravity conformal Killing spinors. If $\chi = \frac{1}{11}\mathcal{D}\psi$ vanishes, the supergravity conformal Killing equation reduces to the usual supergravity Killing equation. We already know that if $F = 0$, the supergravity Penrose operator agrees with the geometric Penrose operator. The remaining possibility (that allows for the existence of supergravity conformal Killing spinors distinct from geometric conformal Killing spinors or supergravity Killing spinors) is that χ is nonzero and parallel (so that M has constrained holonomy), F is nonzero and moreover satisfies the integrability condition (6.15).

We summarise these findings in the following proposition.

Corollary 26. *If* $(\psi,\chi) \in \operatorname{Ker}\wp$ *is a conformal Killing spinor on a M-theory background* (M,g,F)*, then one of the following holds.*

(a) *M is Ricci-flat and ψ is a* geometric *conformal Killing spinor. This occurs when* $X^\flat \wedge F = 0$ *for all* X*, which implies* $F = 0$.

(b) $\chi = \frac{1}{11}\mathcal{D}\psi = 0$ *and thus* $\mathcal{P}_X\psi = \mathcal{D}_X\psi = 0$*; in other words, ψ is a supergravity* Killing *spinor.*

(c) *The spinor χ is parallel,* $F \neq 0$ *and in addition* $F\cdot\chi = X^\flat \wedge F \cdot \chi = \iota_X F \cdot \chi = 0$ *for any vector field* X.

In the sequel we will mostly be interested in case (c) since the two other cases have been studied extensively in the literature.

6.4 M-theory backgrounds admitting conformal Killing spinors

Using Corollary 25, we will now attempt to characterise M-theory backgrounds which admit non-trivial (i.e. distinct from geometric conformal Killing and supergravity Killing) solutions to the SCKS equation.

Suppose that (M,g,F) is a M-theory background and that in addition, (M,g) admits a parallel spinor χ. Metrics of this type have been studied extensively for example in [35, 36] and in supergravity context in [37].

It is well known that the existence of a parallel spinor constraints the holonomy of a manifold. In eleven dimensions, the subgroups $H \subset \mathrm{Spin}(1,10)$ that leave a spinor invariant have been classified by Bryant [35, 36]. There are two possibilities, distinguished by the type of the Dirac current V_χ of χ. Note that since χ is parallel, V_χ is parallel as well.

As Bryant shows, if V_χ is time-like, $\mathrm{Hol}(M)$ must be contained in $\mathrm{SU}(5) \subset \mathrm{Spin}(10)$. This means that (M,g) is locally isometric to a product $\mathbb{R} \times N$ with metric

$$g = -dt^2 + h\,, \tag{6.16}$$

where (N,h) is any Calabi-Yau 5-fold. Such spacetimes are automatically Ricci-flat (as the product of a flat direction with a Ricci-flat manifold), which means that from equation (6.2) we must have

$$\begin{aligned}\langle \iota_X F, \iota_Y F\rangle &= 0\,,\\ |F|^2 &= 0\end{aligned}$$

for all vector fields X, Y.

Considering the possible non-vanishing components that F can have in this case, it is not hard to show that in fact $F = 0$. In a pseudo-orthonormal frame $e_\pm, e_i$, the possible components of F are $F_{+-ij}, F_{-ijk}, F_{+ijk}, F_{ijkl}$. The condition $\langle \iota_X F, \iota_Y F\rangle = 0$ then implies that (with summation over repeated indices implied):

$$\begin{aligned}F_{+ijk}F_{+}{}^{ijk} &= 0\\ F_{-ijk}F_{+}{}^{ijk} &= 0\\ F_{+-ij}F_{+-}{}^{ij} &= 0\\ F_{ijkl}F^{ijkl} &= 0\,.\end{aligned}$$

Since these are sums of squares, each term must in fact vanish separately and thus all components of F vanish.

In other words, $F = 0$ and any solutions of the SCKS equation are again *geometric* conformal Killing spinors.

The remaining possibility is that V_χ is null and $\mathrm{Hol}(M) \subset \left(\mathrm{Spin}(7) \ltimes \mathbb{R}^8\right) \times \mathbb{R}$. In this case it can be shown [12] that

$$V_\chi \cdot \chi = 0\,. \tag{6.17}$$

Since χ is parallel, we have $\nabla_X \chi = 0$ for any vector field X, and iterating this equation we find the following integrability condition:

$$R(X,Y)\cdot\chi = 0 \tag{6.18}$$

for any vector fields X, Y. Taking the Clifford trace of this equation and using the algebraic Bianchi identity (2.21), we find that or, using the Ricci map defined in (2.23),

$$\mathrm{Ric}(X)\cdot\chi = 0\,, \tag{6.19}$$

which in turn implies that $|\mathrm{Ric}(X)|^2\chi = 0$. Since $g(\mathrm{Ric}(X), \mathrm{Ric}(Y)) = 0$ for all vector fields X, Y, metrics of this type are often called *Ricci-null.*

The most general local metric in eleven dimensions admitting a parallel null spinor is given by [36] :

$$g = 2dx^+dx^- + a(dx^-)^2 + (dx^9)^2 + h_{ij}dx^i dx^j \ , \tag{6.20}$$

where $i, j = 1\ldots9$ and and a is a function satisfying $\partial_+ a = 0$ but otherwise arbitrary. h_{ij} is now an x^--dependent family of metrics with holonomy contained in Spin(7) and with the property

$$\partial_- \Upsilon = \lambda \Upsilon + \Psi \ , \tag{6.21}$$

where Υ is the self-dual Spin(7)-invariant Cayley 4-form, λ a smooth function of (x^-, x^i) and Ψ is an anti-selfdual 4-form. Bryant calls such metrics *conformal anti-selfdual* and shows that any one-parameter family of Spin(7)-metrics can be made to satisfy this property using diffeomorphisms[36].

Following [37], we want to couple the metric (6.20) to a four-form F satisfying the M-theory equations of motion (6.1), (6.2) in addition to the SCKS integrability conditions 25. Note that for the metric (6.20), the vector ∂_+ is parallel and null and in fact $V_\chi \propto \partial_+$.

The only nonzero component of the Ricci tensor is R_{--}, which follows from the Ricci-null propety[37] and the fact that ∂_+ is parallel. Satisfying the Einstein equation then requires that the four-form F must be null. It is easy to see that the only nonzero components of F are F_{-ijk}; in other words, F must be of the form

$$F = dx^- \wedge \Theta \ , \tag{6.22}$$

where Θ is a 3-form on the transverse space N_9 with coordinates x^9, x^i: we refer to [37] for details. The requirement $dF = 0$ is satisfied provided that $\partial_+\Theta = 0$ and that Θ is closed as a three-form on N_9; note that it may still have x^--dependence. The Maxwell equation (6.1) is satisfied [37] if $d\star_9\Theta = 0$ on N_9, where $\star_9$ is the Hodge dual operator on N_9.

Finally, we want to make sure that the integrability conditions (6.15), necessary for the SCKS equation to admit solutions, are satisfied. Using the condition $\iota_X F \cdot \chi = 0$ and contracting F with ∂_- implies that

$$\Theta \cdot \chi = 0 \, .$$

In addition, since $V_\chi \propto \partial_+$ and $V_\chi \cdot \chi = 0$, we know that $\chi \in \operatorname{Ker}\Gamma_+$.

In summary:

Theorem 27. *Let $(\psi, \chi) \in \operatorname{Ker}\wp$ define a (non-Killing, non-geometric) SCKS on a M-theory background (M, g, F). Then the vector V_χ is null, (M, g) is Ricci-null and $\operatorname{Hol}(M) \subset \left(\operatorname{Spin}(7) \ltimes \mathbb{R}^8\right) \times \mathbb{R}$. Furthermore, locally the metric g can be written in the form (6.20). The four-form F can be written as $F = dx^- \wedge \Theta$, with Θ an x^--dependent family of 3-forms which are closed and coclosed on N_9. In addition, $\chi \in \Gamma_+$ and $\Theta \cdot \chi = 0$.*

There are plentiful examples of this type[37], so provided that we equip them with suitable four-forms satisfying the integrability conditions (6.15), we expect to find examples of non-Killing, non-geometric supergravity conformal Killing spinors out of which we also hope to construct a supergravity version of the conformal Killing superalgebra.

6.5 Conformal Killing spinors of Hpp-waves

We are now ready to solve the SCKS equation $\mathcal{P}_X \psi = 0$ on Hpp-waves, that is, the equations

$$\partial_+ \psi = 0 \, , \tag{6.23}$$

$$\partial_- \psi + \tfrac{1}{2} x^i A_{ij} \Gamma_i \Gamma_+ \psi + \tfrac{1}{6} \Theta \cdot \psi + \tfrac{1}{12} \Gamma^+ \Gamma^- \Theta \psi + \Gamma_- \chi = 0 \, , \tag{6.24}$$

$$\partial_k \psi + \tfrac{1}{8} \Gamma_+ \Theta \Gamma_k + \tfrac{1}{24} \Gamma_+ \Gamma_k \Theta \cdot \psi + \Gamma_k \chi = 0 \, , \tag{6.25}$$

where $\chi \in \operatorname{Ker}\Gamma_+$ is a constant spinor that also satisfies $\Theta \cdot \chi = 0$. The first equation simply implies that $\psi = \psi(x^-, x^i)$. It is convenient to decompose ψ as $\psi = \psi_+ + \psi_-$, where $\psi_\pm \in \operatorname{Ker}\Gamma_\pm$. Thus we can rewrite the last equation as

$$\partial_k\psi_+ + \partial_k\psi_- + \tfrac{1}{8}\Theta\Gamma_k\Gamma_+\psi_- + \tfrac{1}{24}\Gamma_k\Theta\Gamma_+\psi_- + \Gamma_k\chi = 0\,.$$

It immediately follows that $\partial_k\psi_- = 0$. Taking ∂_j of the previous equation also shows that $\partial_j\partial_k\psi_+ = 0$, which means that ψ_+ depends at most linearly on the transverse coordinates x^i. We can therefore decompose it as

$$\psi_+(x^-, x^i) = \varphi(x^-) - \tfrac{1}{8}\Theta\cdot x^\perp\cdot\Gamma_+\psi_- - \tfrac{1}{24}x^\perp\cdot\Theta\Gamma_+\psi_- - x^\perp\cdot\chi\,, \tag{6.26}$$

where $x^\perp\cdot := x^i\Gamma_i$ and $\varphi \in \operatorname{Ker}\Gamma_+$.

Next, we look at the $\operatorname{Ker}\Gamma_-$-component of the $\partial_-\psi$ equation. We find

$$\partial_-\psi_- = -\tfrac{1}{6}\Theta\cdot\psi_- - \tfrac{1}{12}\Gamma_-\Gamma_+\Theta\cdot\psi_- - \Gamma_-\chi = -\Gamma_-\chi\,,$$

because $\Gamma_-\Gamma_+\psi_- = -2\psi_-$. We can immediately integrate this equation and find

$$\psi_- = \zeta_0 - x^-\Gamma_-\chi\,,$$

where $\zeta_0 \in \operatorname{Ker}\Gamma_-$ is a constant spinor. We substitute the expressions for ψ_+, ψ_- into the remaining equation and obtain

$$\begin{aligned}0 &= \varphi' + \tfrac{1}{6}\Theta\cdot\varphi - \tfrac{1}{4}\Theta\cdot x^\perp\cdot\chi + \tfrac{1}{2}x^i A_{ij}\Gamma_i\Gamma_+\zeta_0\\ &\quad + x^i A_{ij}\Gamma_j x^-\chi - \tfrac{1}{48}\Theta^2\cdot x^\perp\cdot\Gamma_+\zeta_0 - \tfrac{1}{24}\Theta^2 x^\perp\cdot x^-\chi\\ &\quad - \tfrac{1}{144}\Theta\cdot x^\perp\cdot\Theta\Gamma_+\zeta_0 - \tfrac{1}{6}\Theta\cdot x^\perp\cdot\chi\,.\end{aligned}$$

This equation contains three kinds terms: those depending only on x^-, those depending on x^i and those depending on both. Taking derivatives, it is easy to see that all three must vanish separately. The terms depending solely on x^- give an equation for φ,

$$\varphi' = -\tfrac{1}{6}\Theta\cdot\varphi\,,$$

which is readily solved to give

$$\varphi = e^{-\frac{x^-}{6}\Theta}\varphi_0\,,$$

where $\varphi_0 \in \operatorname{Ker}\Gamma_+$ is a constant spinor.

The vanishing of the terms depending on x^- and x^- both implies that

$$A_{ij}\Gamma_j\chi = \tfrac{1}{24}\Theta^2\Gamma_i\chi\,, \tag{6.27}$$

and the vanishing of the remaining terms requires that

$$-\tfrac{5}{6}\Theta\Gamma_i\chi + A_{ij}\Gamma_j\Gamma_+\zeta_0 - \tfrac{1}{24}\Theta^2\Gamma_i\Gamma_+\zeta_0 - \tfrac{1}{72}\Theta\Gamma_i\Theta\Gamma_+\zeta_0 = 0\,. \tag{6.28}$$

Recall that the M-theory equations of motion are satisfied when $\operatorname{Tr} A = -\frac{1}{2}|\Theta|^2$. Taking the trace of equation (6.27), we find that

$$\begin{aligned} |\Theta|^2\chi &= \tfrac{1}{96}\Theta_{jkn}\Theta_{nlm}\Gamma_i\Gamma_{jklm}\Gamma_i\chi \\ &= -\tfrac{1}{96}\Theta_{jkn}\Theta_{nlm}\Gamma_{jklm}\chi\,, \end{aligned} \tag{6.29}$$

where we have used a $C\ell(9)$ Fierz identity $\Gamma_i\Gamma_{jklm}\Gamma_i = -\Gamma_{jklm}$. But recall that $\Theta\chi = 0$. This means that the Clifford square of Θ acting on χ must vanish as well, which implies

$$|\Theta|^2\chi = \tfrac{1}{4}\Theta_{jkn}\Theta_{nlm}\Gamma_{jklm}\,,$$

since Θ is a 3-form. Clearly, this contradicts (6.29), unless either $\Theta = 0$ or $\chi = 0$. We have thus estabilished the following somewhat disappointing result:

Proposition 28. *Let $(M, g, F = dx^- \wedge \Theta)$ be a supersymmetric Hpp-wave solution of M-theory and (ψ, χ) define a SCKS. Then one of the following holds.*

(a) $\Theta = 0$ *and ψ is a* geometric *conformal Killing spinor.*

(b) $\chi = 0$ *and ψ is a (supergravity) Killing spinor ($\mathcal{D}\psi = 0$).*

Chapter 7

Conformal Killing superalgebras in M-theory

We now turn to the question of defining a supergravity analogue of the conformal Killing superalgebra introduced in Chapter 5. In particular, we will define a supercovariant conformal Killing superalgebra associated to a M-theory background (M, g, F) consisting of supergravity conformal Killing spinors and so-called *supernormal* conformal Killing vectors of M.

7.1 M-theory conformal Killing spinors

As in the geometric case, we would also like to construct a map from $S^2(\mathrm{S} \oplus \mathrm{S})$ to sections of $\mathfrak{mo}(TM)$ which maps parallel sections with respect to $\wp$ to parallel sections of the Geroch connection $\mathscr{D}$.

We can easily formulate the analogue of Proposition 15 with slight modifications involving F-dependent terms.

Proposition 29. *The map* $[-,-] : (\psi, \chi) \mapsto (X, A, \beta)$ *is defined by the following equations*

$$
\begin{aligned}
g(X, Y) &= (\psi, Y \cdot \psi) \\
g(A_X Z, Y) &= 2(\psi, Y \cdot Z \cdot \chi) + 2(\psi, Y \cdot \Omega_Z \cdot \psi) \\
\beta_X(Y) &= 2(Y \cdot \chi, \chi) - L(X, Y)\,.
\end{aligned}
$$

Proof. Let $(\psi, \chi) \in \operatorname{Ker} \wp$. As before, we take X to be the Dirac current of ψ and differentiate it, obtaining

$$\begin{aligned} g(\nabla_Z X, Y) &= (\nabla_Z \psi, Y \cdot \psi) + (\psi, Y \cdot \nabla_Z \psi) \\ &= -(Z \cdot \chi, Y \cdot \psi) - (\psi, Y \cdot Z \cdot \chi) - (\Omega_Z \cdot \psi, Y \cdot \psi) - (\psi, Y \cdot \Omega_Z \cdot \psi) . \end{aligned}$$

and using the properties of the symplectic spinor inner product gives the result. Now $h_X = -2(\psi, \chi)$, and thus we find the one-form α_X by differentiating

$$\alpha_X(Y) = \nabla_Y h_x = 2(Y \cdot \chi, \chi) + 2(\Omega_Y \cdot \psi, \chi) ,$$

since χ is parallel, and in fact taking the Clifford adjoint of Ω_Y and using the integrability condition in Corollary 25, we see the last term in the previous equation vanishes. Thus, $\alpha_X(Y) = 2(Y \cdot \chi, \chi)$ and using the definition of β, we arrive at the desired result. □

7.2 Supernormal conformal Killing vectors

In order to be able to construct a well-defined superalgebra from M-theory conformal Killing spinors, we must again show that there is a special ideal of conformal Killing vectors X for which $C(X, Y) = 0$ for any vector field Y — recall that this is required for $\mathscr{L}$ to be a homomorphism from the algebra of vector fields to a subalgebra of $\mathfrak{mo}(TM)$. To get an idea of what we should require from these vectors, we now determine if Dirac currents of M-theory conformal Killing spinors satisfy this condition.

Recall that the conformal Killing transport equations (3.16) imply that if X is a conformal Killing vector and Y is any vector field,

$$C(X, Y) = -\nabla_Y \beta_X - L(Y) \circ A_X . \tag{7.1}$$

Now suppose $(\psi, \chi) \in \operatorname{Ker} \wp$ define a conformal Killing spinor and X is the Dirac current of ψ. Then we have

$$\begin{aligned} C(X,Y)Z &= -\nabla_Y\left[2(Z\cdot\chi,\chi)-L(X,Z)\right]-L(X,\nabla_Y Z)-L(Y,A_X Z) \\ &= (\nabla_Y L)(X,Z)+L(\nabla_Y X,Z)+L(Y,\nabla_Z X) \\ &= (\nabla_Y L)(X,Z)+(A_X L)(Y,Z) \\ &= (\mathscr{L}_X L)(Y,Z)+(\nabla_Y L)(X,Z)-(\nabla_X L)(Y,Z)\,, \end{aligned}$$

where we recognise the last two terms as $-C(X,Y)Z$. Thus, we see that

$$C(X,Y)Z=\tfrac{1}{2}(\mathscr{L}_X L)(Y,Z)\,. \tag{7.2}$$

This suggests that a natural choice for the analogue of normal conformal Killing vectors in supergravity context would be the CKVs for which $\mathscr{L}_X L=0$. We can state this requirement in a slightly more M-theoretic way by rewriting expression (6.3) in local coordinates as

$$L_{ab}=\tfrac{1}{108}g^{mn}g^{pq}g^{rs}F_{mpra}F_{nqsb}-\frac{7}{4!\times 360}g^{kl}g^{mn}g^{pq}g^{rs}F_{kmnpr}F_{lnqs}\,, \tag{7.3}$$

where g^{ab} are the components of the inverse metric. Since $\mathscr{L}_X g^{ab}=2h_X g^{ab}$ (which is easy to see by taking the Lie derivative of $\mathrm{Id}=gg^{-1}$), the Lie derivative of L_{ab} vanishes if $\mathscr{L}_X F=-3h_X F$. This motivates the following definition.

Definition 30. Let (M,g,F) be a M-theory background. If X is a conformal Killing vector of (M,g) and in addition,

$$\mathscr{L}_X F=-3h_X F\,, \tag{7.4}$$

we call X a *supernormal* conformal Killing vector.

Note that a supernormal conformal Killing vector is not necessarily normal in the same sense as in the geometric case, since it might not correspond to a parallel section with respect to the normal conformal Cartan connection (even if it does define a parallel section of $\mathfrak{co}(TM)$ with respect to the Geroch connection).

It is clear that supernormal conformal Killing vector fields form a subalgebra of the conformal Lie algebra of M, since if X and Y are supernormal, $\mathscr{L}_{[X,Y]}L=(\mathscr{L}_X\mathscr{L}_Y-$

$\mathscr{L}_Y\mathscr{L}_X)L = 0$. Note, however, that unlike for normal conformal Killing vectors, their Lie bracket does *not* reduce to the algebraic one since $W(X,Y)$ does not necessarily vanish even when X and Y are supernormal.

We can motivate this definition further with the following proposition.

Proposition 31. *Let (M,g,F) be a M-theory background and X a conformal Killing vector. Then X preserves the kernel of $\mathcal{P}$, that is,*

$$[\mathscr{L}_X, \mathcal{P}_Y] = \mathcal{P}_{[X,Y]}$$

if and only if X is supernormal.

Proof. Using equation (9.6), we can rewrite $\mathcal{P}_Y$ as

$$\mathcal{P}_Y = P_Y + \tfrac{1}{8}F\cdot Y - \tfrac{3}{88}X\cdot F\,. \tag{7.5}$$

We know that $[\mathscr{L}_X, P_Y] = P_{[X,Y]}$ by equation (5.15). Now note that

$$\begin{aligned}[\mathscr{L}_X, F\cdot Y] &= \mathscr{L}_X F\cdot Y\cdot\psi + 4h_X F\cdot Y\cdot\psi + F\cdot\mathscr{L}_X(Y\cdot\psi) - X\cdot F\cdot\mathscr{L}_X\psi \\ &= \mathscr{L}_X F\cdot Y\cdot\psi + 3h_X F\cdot Y\cdot\psi + F\cdot[X,Y]\cdot\psi\,,\end{aligned}$$

where we have used the properties of the Kosmann-Schwarzbach Lie derivative from Lemma 20. In other words, $[\mathscr{L}_X, F\cdot Y] = F\cdot[X,Y]$ if and only if $\mathscr{L}_X F = -3h_X F$; similarly for $[\mathscr{L}_X, Y\cdot F]$. □

7.3 Jacobi identities in the M-theory conformal Killing superalgebra

As before, we need to check the Jacobi identities of the conformal Killing superalgebra we have construced. Again, the even-even-odd Jacobi identity follows from the fact that for supernormal conformal Killing vectors, $X \mapsto \mathscr{L}_X$ is a Lie algebra homomorphism. Let $\Psi = (\psi,\chi) \in \operatorname{Ker}\wp$ define a SCKS and let X be a supernormal conformal Killing vector. The even-odd-odd Jacobi identity can be written as

$$[X,[\Psi,\Psi]] = [\mathscr{L}_X\Psi,\Psi] + [\Psi,\mathscr{L}_X\Psi]\,. \tag{7.6}$$

Again, we have

$$
\begin{aligned}
g(A_X V_{\psi,\psi}, Z) - g(B_{V_{\psi,\psi}} X, Z) &= -g(V_{\psi,\psi}, S_X Z) + h_X g(V_{\psi,\psi}, Z) - 2(\psi, Z\cdot X\cdot\chi) - 2(\psi, X\cdot\Omega_Z\cdot\psi) \\
&= -(\psi, S_X Z\cdot\psi) + h_X(\psi, Z\cdot\psi) - 2(\psi, Z\cdot X\cdot\chi) - 2(\psi, X\cdot\Omega_Z\cdot\psi) .
\end{aligned}
$$

Note that when $(\psi,\chi)\,\mathrm{Ker}\,\wp$, the spinorial Lie derivative $\mathscr{L}$ acts as follows:

$$
\begin{aligned}
\mathscr{L}_X \begin{pmatrix}\psi\\ \chi\end{pmatrix} &= \left(\mathscr{P}_X + \varrho(X)\right)\begin{pmatrix}\psi\\ \chi\end{pmatrix} \\
&= \begin{pmatrix}\nabla_X\psi + X\cdot\chi \\ \frac{1}{2}L(X)\cdot\chi\end{pmatrix} + \begin{pmatrix}\sigma^{\frac{1}{2}}(A_X)\psi - X\cdot\chi \\ \frac{1}{2}\beta_X\cdot\psi + \sigma^{-\frac{1}{2}}(A_X)\chi\end{pmatrix} \\
&= \begin{pmatrix}\sigma^{\frac{1}{2}}(A_X)\psi - X\cdot\chi - \Omega_X\cdot\psi \\ \frac{1}{2}\alpha_X\cdot\psi + \sigma^{-\frac{1}{2}}(A_X)\chi\end{pmatrix} .
\end{aligned}
$$

We can thus write the right-hand side of equation (7.6) as

$$
\begin{aligned}
g(V_{\mathscr{L}_X\psi,\psi}, Z) + g(V_{\psi,\mathscr{L}_X\psi}, Z) &= (\sigma^{\frac{1}{2}}(A_X)\psi, Z\cdot\psi) - (\Omega_X\cdot\psi, Z\cdot\psi) + (\psi, Z\cdot\sigma^{\frac{1}{2}}\psi) - (\psi, Z\cdot\Omega_X\cdot\psi) \\
&= -(\psi, S_X Z\cdot\psi) + h_X(\psi, Z\cdot\psi) - 2(\psi, Z\cdot X\cdot\chi) - 2(\psi, X\cdot\Omega_Z\cdot\psi) ,
\end{aligned}
$$

which thus agrees with the left-hand side and the Jacobi identity is satisfied. As for the odd-odd-odd Jacobi identity, the same comments as in the geometric case apply. Since geometric conformal Killing spinors are special cases of M-theory conformal Killing spinors, the fourth Jacobi identity does not vanish in general.

Chapter 8

Conformal Killing spinors in type IIA and IIB supergravities

One way to generate supergravity solutions starting from purely geometrical backgrounds is *Kaluza-Klein reduction.* In this procedure one exploits a symmetry of the background corresponding to a Killing vector ξ, which generates a 1-parameter subgroup Γ of the isometry group of M. If we take M to be a principal Γ-bundle, we can construct a metric on the base $N = M/\Gamma$. This is a special case of a *semi-Riemannian submersion*[22] . In addition to the metric, there will also be other fields on N, arising from the curvature of the bundle and the norm of the vector field ξ. For supergravity backgrounds without flux, the field equations amount to the Ricci-flatness of the metric on M, and the corresponding equations on N can be derived e.g. using standard formulas relating the Ricci curvatures of the total space and the base of a semi-Riemannian submersion [38]. Any other objects on M (such as differential forms and Killing or conformal Killing spinors) left invariant by the action of ξ will also induce corresponding objects on N.

In particular, it is well known that the 10-dimensional type IIA supergravity can be obtained as a Kaluza-Klein reduction of M-theory. Many of the supersymmetric reductions of M-theory backgrounds to type IIA solutions have been classified recently, including reductions of flat space (leading to so-called fluxbranes), the M-wave, the Kaluza-Klein monopole and the M-branes[39, 40, 41, 42].

For our purposes, the utility of the Kaluza-Klein procedure lies in the fact that the connection in the lower-dimensional theory now includes fluxes even if we start with a purely geometric background of the higher-dimensional theory.

In this chapter we show that starting from geometric M-theory backgrounds, we can use the Kaluza-Klein procedure to construct 10-dimensional supergravity backgrounds with *supergravity* conformal Killing spinors. We also compute the associated conformal Killing superalgebras. Finally, we make a brief comment on the role of conformal Killing spinors in type IIB supergravity.

8.1 The Kaluza-Klein ansatz

Let (M,g) be an eleven-dimensional Lorentzian manifold that admits a Killing vector ξ which is everywhere spacelike. Note that if we regard (M,g) as a M-theory background with $F=0$, it must actually be Ricci-flat by virtue of the field equations (6.2).

Now suppose that ξ generates a 1-parameter group Γ. We think of M as a principal Γ-bundle

$$M \xrightarrow{\pi} N = M/\Gamma\,,$$

where π is the projection hat maps points in M to their Γ-orbits. We also have the derivative of this map: $\pi_* : T_pM \to T_qN$, where $q=\pi(p)$.

For any point $p \in M$ we have a split $T_pM = \mathscr{H}_p \oplus \mathscr{V}_p$ of the tangent space into horizontal and vertical subspaces, where $\mathscr{V}_p = \operatorname{Ker}\pi_*$. This split is orthogonal with respect to g, and the vertical subspace $\mathscr{V}_p$ is spanned by ξ. Now let $\alpha = \frac{1}{\|\xi\|^2}\xi^\flat$. Clearly $\alpha(\xi)=1$ and $\alpha(X)=0 \leftrightarrow X \perp \xi$ — that is, $\mathscr{H}_p = \operatorname{Ker}\alpha$. We call a vector field X *horizontal* if $\alpha(X)=0$.

For every $X \in T_qN$, there exists a horizontal lift $\tilde{X} \in \mathscr{H}_p$, defined via $\pi_*\tilde{X} = X$. There is a unique metric h on N for which the map π_* is an isometry, defined via $h(X,Y) = g(\tilde{X},\tilde{Y})$. We can write g in the following form:[1]

$$g = \pi^* h + \|\xi\|^2 \alpha \otimes \alpha\,. \tag{8.1}$$

[1] In string theory literature, this is often called the Einstein-frame Kaluza-Klein ansatz: this can be related to the standard IIA string-frame ansatz via a conformal rescaling.

We can characterise geometric objects induced on N with the help of the following definition.

Definition 32. Let η be a p-form on M. We say that η is *horizontal* if $\iota_\xi \eta = 0$, and *invariant* if $\mathscr{L}_\xi \eta = 0$. If η satisfies both of these properties, we call it *basic.* A basic form is a pull-back of a form on N.

Applying this definition to the objects defined above, we find:

Proposition 33. *Let (M, g) be a M-theory background with a Killing vector ξ as above. Then $\|\xi\|^2$ and $d\alpha$ are basic.*

Proof. The squared norm of ξ is clearly basic since ξ is Killing: $\mathscr{L}_\xi g(\xi, \xi) = 0$. Another natural basic form is $d\alpha$. It is easy to show that it is horizontal. For any vector field X,

$$\begin{aligned}\iota_\xi d\alpha &= d\alpha(\xi, X) \\ &= \xi\alpha(X) - X\alpha(\xi) - \alpha([\xi, X]) \\ &= \xi \frac{g(\xi, X)}{\|\xi\|^2} - \frac{g([\xi, X], \xi)}{\|\xi\|^2},\end{aligned}$$

which vanishes since $\mathscr{L}_\xi g(X, \xi) = 0$. Since $d\alpha$ is closed, $\mathscr{L}_\xi d\alpha = d\iota_\xi \alpha = 0$, so $d\alpha$ is invariant as well, and hence basic. □

We express these fields as $\|\xi\|^2 = e^{2\pi^*\phi}$, where ϕ is a function on N — usually called the *dilaton* — and $d\alpha = \pi^* F_2$, where F_2 is a 2-form on N. In the sequel we usually omit explicit pull-backs.

Definition 34. Let (M, g) and (N, h, F_2, ϕ) be as above. We say that (N, h, F_2, ϕ) is a *Kaluza-Klein reduction* of (M, g).

In summary, via the Kaluza-Klein reduction we can identify (M, g) with a background (N, h, F_2, ϕ) of type IIA supergravity. The Ricci-flatness of M naturally gives rise to the IIA field equations which can be readily derived by the standard submersion formulas for the Riemann curvature [38, 22]. In particular, the Ricci tensor of (N, h, F_2, ϕ) can be written in our conventions as

$$r(X, Y) = \tfrac{1}{2} e^{2\phi} \langle \iota_X F_2, \iota_Y F_2 \rangle + \tfrac{1}{4} e^{2\phi} |F|^2 h(X, Y) \,. \tag{8.2}$$

We remark that it is possible to include the four-form F in this construction by further demanding that $\mathscr{L}_\xi F = 0$. Then it is easy to show that we can write $F = \alpha\wedge\pi^* H_3 + \pi^* H_4$, where H_3, H_4 a 3-form and a 4-form on N, respectively. However, in this thesis we will only consider Kaluza-Klein reductions of M-theory backgrounds with $F = 0$.

8.2 Kaluza-Klein reduction of the conformal Killing spinor equation

We begin by describing the connection that the Levi-Cività connection on (M, g) induces on (N, h, F_2, ϕ).

A natural coframe for the metric (8.1) is $\{\tilde{e}^\alpha, \tilde{e}^i\}$, where $\tilde{e}^\alpha = \tilde{e}^\phi \alpha$. We can define a coframe on (N, h) via $e^i = \tilde{e}^i$. Note that the 11-dimensional volume element dvol_M can be written as

$$\mathrm{dvol}_M = \mathrm{dvol}_N \wedge e^\alpha, \tag{8.3}$$

Since we're focusing on the Clifford module on M for which the action of the center of the Clifford algebra is non-trivial,

$$\mathrm{dvol}_M \cdot \psi = \mathrm{dvol}_N \cdot e^\alpha \cdot \psi = -\psi \,,$$

Now $\mathrm{dvol}_N^2 = -\mathrm{Id}$, so the last equation implies that $e^\alpha \cdot \psi := \Gamma^\alpha \psi = \mathrm{dvol}_N \cdot \psi$. In other words, e^α acts on spinors like the 10-dimensional volume element. As a representation of Spin(1,9), the 11-dimensional spinor bundle S_{11} breaks up into $\mathrm{S}^+_{10} \oplus \mathrm{S}_{10} m$, where $\mathrm{S}^\pm_{10}$ are distinguished by chirality. For the purposes of this section, we prefer not to break 11-dimensional spinors explicitly into 10-dimensional spinors, leaving the 10-dimensional volume element manifest in the expressions below.

Proposition 35. *The spin connection $\widetilde{\nabla}$ on (M, g) induces the following connection on (N, h, F_2, ϕ).*

$$\mathcal{D}_X = \nabla_X - \tfrac{1}{2} e^\phi \alpha(X) \,\mathrm{grad}\phi \cdot \mathrm{dvol}_N \cdot - \tfrac{1}{8} e^\phi \iota_X F_2 \cdot \mathrm{dvol}_N \cdot \,, \tag{8.4}$$

where ∇ *is the Levi-Cività connection of* h*. We call* $\mathcal{D}$ *the* IIA supercovariant connection *on* (N, h, F_2, ϕ).

Proof. Recall that the spin connection acting on a spinor ψ is $\tilde{\nabla}_X\psi = X(\psi) + \frac{1}{4}\tilde{\omega}(X)^{ab}\tilde{\Gamma}_{ab}\psi$, where $\tilde{\omega}^{ab}$ are the connection one-forms on (M, g). To determine the connection on (N, h) induced from the spin connection of M we look at the structure equations.

$$\begin{aligned} de^\alpha + \tilde{\omega}^\alpha_{\ i} \wedge e^i &= 0, \\ de^i + \tilde{\omega}^i_{\ j} \wedge e^j &= 0 . \end{aligned}$$

The latter equation implies that $\omega^i_{\ j} = \tilde{\omega}^i_{\ j}$. To determine the remaining connection one-forms, we compute

$$\begin{aligned} de^\alpha &= d\left(e^\phi \alpha\right) \\ &= e^\phi d\phi \wedge \alpha + e^\phi d\alpha, \end{aligned}$$

and, writing $F_2 = d\alpha$,

$$\tilde{\omega}^\alpha_{\ i} \wedge e^i = 2e^\phi \alpha \wedge d\phi + \tfrac{1}{2} e^\phi (F_2)_{ij} e^j \wedge e^i$$

which becomes

$$\tilde{\omega}^\alpha_{\ i} = 2e^\phi \partial_i \phi \alpha + \tfrac{1}{2} e^\phi F_{ij} e^j .$$

Now $\iota_X F_2 = \iota_X\left(\frac{1}{2}(F_2)_{ij} e^i \wedge e^j\right) = -(F_2)_{ij} e^j(X) e^i$, so

$$\tilde{\omega}^{\alpha i}(X) = 2e^\phi \partial^i \phi \alpha(X) - \tfrac{1}{2} e^\phi \left(\iota_X F_2\right)^i , \tag{8.5}$$

and the result follows. □

Using Proposition 35, we can now work out what happens to the conformal Killing spinor equation under Kaluza-Klein reduction. To begin with, we can decompose the Dirac operator.

$$\begin{aligned} \tilde{\nabla\!\!\!/}\psi &= \Gamma^\alpha \tilde{\nabla}_\alpha \psi + \Gamma^i \tilde{\nabla}_i \psi & (8.6) \\ &= \Gamma^\alpha \tilde{\nabla}_\alpha \psi + \nabla\!\!\!/\, \psi - \tfrac{1}{4} e^\phi F_2 \cdot \mathrm{dvol}_N \cdot \psi, & (8.7) \end{aligned}$$

where we have used (2.10).

Now suppose ψ is a conformal Killing spinor on M which is left invariant by ξ; that is, $\mathscr{L}_\xi\psi = 0$, so that it can be identified with a pullback of a spinor on N. We wish to know what equation it satisfies on N.

The vertical component of the conformal Killing spinor equation implies that

$$\Gamma^\alpha \widetilde{\nabla}_\alpha \psi = \tfrac{1}{n}\widetilde{\nabla}\!\!\!/\,\psi. \tag{8.8}$$

Inserting this into (8.6), we obtain

$$\tfrac{1}{n}\widetilde{\nabla}\!\!\!/\,\psi = \tfrac{1}{n-1}\nabla\!\!\!/\,\psi - \tfrac{1}{4(n-1)}e^{\phi}F_2\cdot \mathrm{dvol}_N\cdot\psi\,. \tag{8.9}$$

The horizontal component of the twistor equation can then be written as

$$\nabla_X\psi - \tfrac{1}{2}e^{\phi}\iota_X F_2\cdot\mathrm{dvol}_N\cdot\psi + \tfrac{1}{n-1}X\cdot\nabla\!\!\!/\,\psi - \tfrac{1}{4(n-1)}e^{\phi}X\cdot F_2\cdot\mathrm{dvol}_N\,\psi = 0\,, \tag{8.10}$$

where X is any horizontal vector field.

We have thus proven the following:

Proposition 36. *Let (M,g) be a vacuum M-theory background with a Killing vector ξ and (N,h,F_2,ϕ) its Kaluza-Klein reduction with respect to ξ. Furthermore, let ψ be a conformal Killing spinor satisfying $\mathscr{L}_\xi\psi = 0$. Then $\mathcal{P}\psi = 0$, where for any horizontal vector field X,*

$$\mathcal{P}_X = \nabla_X - \tfrac{1}{2}e^{\phi}\iota_X F_2\cdot\mathrm{dvol}_N\cdot + \tfrac{1}{n-1}X\cdot\nabla\!\!\!/ - \tfrac{1}{4(n-1)}e^{\phi}X\cdot F_2\cdot\mathrm{dvol}_N\cdot$$

is the Penrose operator associated to the IIA supercovariant connection on (N,h,F_2,ϕ).

8.3 Kaluza-Klein reduction and conformal Killing superalgebras

As we have seen, we can associate a conformal Killing superalgebra $\tilde{\mathfrak{h}} = \tilde{\mathfrak{h}}_0 \oplus \tilde{\mathfrak{h}}_1$ to (M,g), constructed out of its normal conformal Killing vectors and conformal Killing spinors. In this section we show that via Kaluza-Klein reduction we can construct a *supergravity* conformal Killing superalgebra $\mathfrak{h}$ from $\tilde{\mathfrak{h}}$.

We define $\mathfrak{h}_1$ to be the space of ξ-invariant conformal Killing spinors of M, that is

$$\mathfrak{h}_1 = \{\psi \in \operatorname{Ker} \tilde{P} | \mathscr{L}_\xi \psi = 0\} .$$

As for $\mathfrak{h}_0$, suppose that $\mathfrak{h}_0 \subset \tilde{\mathfrak{h}}_0$ is a subalgebra. Let $\psi \in \mathfrak{h}_1$ and $X \in \mathfrak{h}_0$ Then it follows from the even-even-odd Jacobi identity of $\tilde{\mathfrak{h}}$ that

$$\mathscr{L}_\xi \mathscr{L}_X \psi = \mathscr{L}_{[\xi,X]} \psi + \mathscr{L}_X \mathscr{L}_\xi \psi .$$

But the last term vanishes, since $\mathscr{L}_\xi \psi = 0$. Therefore, for $\mathscr{L}_X \psi$ to lie in $\mathfrak{h}_1$, we must require that $\mathscr{L}_{[\xi,X]} \psi = 0$, which implies that $[\xi, X]$ leaves ψ invariant. Since we are only considering Kaluza-Klein reductions with respect to one Killing vector, we assume $[\xi, X]$ must be proportional to ξ. In other words, we must have $X \in \operatorname{Norm}(\xi)$, where $\operatorname{Norm}(\xi)$ is the normaliser of ξ in $\tilde{\mathfrak{h}}_0$.

Furthermore, we can prove the following:

Proposition 37. *Let (M, g) be a vacuum M-theory background, ξ a Killing vector and $\tilde{X} \in \operatorname{Norm}(\xi)$ a normal conformal Killing vector with $[\tilde{X}, \xi] = c\xi$ with $c \in \mathbb{R}$ a constant. Then $\tilde{X}$ induces a conformal Killing vector X of (N, h, F_2, ϕ), and furthermore it holds that*

$$\mathscr{L}_X \phi = -f_X + c$$
$$\mathscr{L}_X F_2 = -cF_2 .$$

Proof. Suppose that $\tilde{X}$ is a conformal Killing vector on M, with $f_{\tilde{X}} = \frac{1}{11} \operatorname{div} \tilde{X}$. Furthermore, assume that $X \in \operatorname{Norm}(\xi)$, with $[\tilde{X}, \xi] = c\tilde{X}$ for some constant c. We can decompose $\tilde{X}$ as $\tilde{X} = a\xi + X$, where a is a function on M and X is the horizontal component of $\tilde{X}$.

As a preliminary, let us compute the Lie derivative of $\xi^\flat$ along $\tilde{X}$. Let Y be any vector field on N and $\tilde{Y}$ its horizontal lift. Then

$$\begin{aligned}
\mathscr{L}_{\tilde{X}}\xi^\flat(\tilde{Y}) &= \tilde{X}\left(\xi^\flat(\tilde{Y})\right) - \xi^\flat([\tilde{X},\tilde{Y}]) \\
&= g(\tilde{\nabla}_{\tilde{X}}\xi,\tilde{Y}) + g(\xi,\tilde{\nabla}_{\tilde{X}}\tilde{Y}) - g(\xi,[\tilde{X},\tilde{Y}]) \\
&= g(\tilde{\nabla}_{\xi}\tilde{X},\tilde{Y}) + g(\xi,\tilde{\nabla}_{\tilde{Y}}\tilde{X}) + cg(\xi,\tilde{Y}) \\
&= (-2f_{\tilde{X}} + c)\, g(\xi,\tilde{Y}) \\
&= (-2f_{\tilde{X}} + c)\,\xi^\flat(\tilde{Y})\,,
\end{aligned}$$

where we have used the Koszul formula and the conformal Killing equation (3.1). Another useful quantity is the Lie derivative of $\|\xi\|^2$: this is simply

$$\begin{aligned}
\mathscr{L}_{\tilde{X}}\left(\|\xi\|^2\right) &= \tilde{X}\left(g(\xi,\xi)\right) \\
&= 2g(\tilde{\nabla}_{\tilde{X}}\xi,\xi) \\
&= 2g(\tilde{\nabla}_{\xi}\tilde{X},\xi) + 2cg(\xi,\xi) \\
&= 2\left(-f_{\tilde{X}} + c\right) g(\xi,\xi) \\
&= 2\left(-f_{\tilde{X}} + c\right)\|\xi\|^2\,.
\end{aligned}$$

From this we can immediately see the action of X on the dilaton ϕ, for

$$\tilde{X}\left(g(\xi,\xi)\right) = 2e^{2\phi}X(\phi)\,, \tag{8.11}$$

and combined with the previous result this implies that $\mathscr{L}_X\phi = X(\phi) = -f_X + c$. It remains to compute the action of $\tilde{X}$ on α and $F_2 = d\alpha$. Using the above,

$$\begin{aligned}
\mathscr{L}_{\tilde{X}}\alpha &= \mathscr{L}_{\tilde{X}}\left(\frac{1}{\|\xi\|^2}\xi^\flat\right) \\
&= -\frac{1}{\|\xi\|^4}\mathscr{L}_{\tilde{X}}\left(\|\xi\|^2\right) + \frac{1}{\|\xi\|^2}\mathscr{L}_{\tilde{X}}\xi^\flat \\
&= -\frac{2\left(-f_{\tilde{X}} + c\right)}{\|\xi\|^2}\xi^\flat + \frac{\left(-2f_{\tilde{X}} + c\right)}{\|\xi\|^2}\xi^\flat \\
&= -\frac{c}{\|\xi\|^2}\xi^\flat \\
&= -c\alpha\,.
\end{aligned}$$

What about $d\alpha$? The last equation immediately gives

$$\begin{aligned}
\mathscr{L}_{\tilde{X}} d\alpha &= \left(d\iota_{\tilde{X}} + \iota_{\tilde{X}} d\right) d\alpha \\
&= d\left(\iota_{\tilde{X}} d\alpha\right) \\
&= d\left(\mathscr{L}_{\tilde{X}}\alpha - d\iota_{\tilde{X}}\alpha\right) \\
&= -c\, d\alpha \\
&= -cF_2,
\end{aligned}$$

since $\mathscr{L}_{\tilde{X}}\alpha = -c\alpha$ and $d^2 = 0$.

With these observations in mind, we use the Kaluza-Klein ansatz (8.1) to determine the action of X on the lower-dimensional metric h:

$$\begin{aligned}
\mathscr{L}_{\tilde{X}} g &= \mathscr{L}_X h + \mathscr{L}_{\tilde{X}}\left(\|\xi\|^2 \alpha \otimes \alpha\right) \\
&= \mathscr{L}_X h + \left(\mathscr{L}_{\tilde{X}}\|\xi\|^2\right)\alpha \otimes \alpha + 2\|\xi\|^2\left(\mathscr{L}_{\tilde{X}}\alpha\right) \otimes \alpha \\
&= \mathscr{L}_X h + \left(-2f_{\tilde{X}} + 2c\right)\|\xi\|^2 \alpha \otimes \alpha - 2c\|\xi\|^2 \alpha \otimes \alpha \\
&= \mathscr{L}_X h - 2f_{\tilde{X}}\|\xi\|^2 \alpha \otimes \alpha \\
&= -2f_{\tilde{X}}\, g\,.
\end{aligned}$$

For this equality to hold, we must have $\mathscr{L}_X h = -2f_{\tilde{X}} h = -2f_X h$, so X is indeed a conformal Killing vector of (N, h). □

We can in fact do slightly better and show that conformal Killing vectors on N induced by normal conformal Killing vectors of the M-theory background satisfy a property analogous to Definition 30.

Definition 38. Let X be a conformal Killing vector field of (N, h, F_2, ϕ). Then we say that X is *IIA supernormal* if

$$\mathscr{L}_X L = 0\,,$$

where L is the Schouten tensor of (N, h).

Proposition 39. *Let (N, h, F_2, ϕ) be a Kaluza-Klein reduction of a vacuum M-theory background (M, g) and X a conformal Killing vector field induced by a normal conformal Killing vector of (M, g). Then X is IIA supernormal.*

Proof. As an immediate consequence of Proposition 37 we have $\mathcal{L}_X(e^{\phi}F_2) = -f_X e^{\phi}F_2$. Using a similar argument as in Section 7.2, Chapter 7, it is clear from equation (8.2) that $\mathcal{L}_X r = 0$. It follows that $\mathcal{L}_X L$ must vanish as well. □

Finally, we show that a conformal Killing vector X on N inherited from $\tilde{X} \in \mathrm{Norm}(\xi)$ on M preserves the space of supergravity conformal Killing spinors on N.

Proposition 40. *Let (N, h, F_2, ϕ) be a Kaluza-Klein reduction of a vacuum M-theory background (M, g) and X a conformal Killing vector field induced by a normal conformal Killing vector of (M, g). Then $\mathcal{L}_X$ preserves the space of superconformal Killing spinors on N, that is*

$$[\mathcal{L}_X, \mathcal{P}_Y] = \mathcal{P}_{[X,Y]} \,.$$

Proof. Obviously $\mathcal{P}$ can be writen as $P + \Phi(F_2, \phi)$, where

$$\Phi_Y = -\tfrac{1}{2} e^{\phi}\iota_Y F_2 - \tfrac{1}{4(n-1)} e^{\phi} Y \cdot F_2 \,, \tag{8.12}$$

Then we have

$$[\mathcal{L}_X, \mathcal{P}_Y] = P_{[X,Y]} + [\mathcal{L}_X, \Phi_Y] \,. \tag{8.13}$$

Using Proposition 37 and the properties of Kosmann-Schwarzbach Lie derivative, we compute

$$\begin{aligned}
[\mathcal{L}_X, e^{\phi}\iota_Y F_2]\cdot\psi &= \mathcal{L}_X\left(e^{\phi}\iota_Y F_2\right)\cdot\psi + e^{\phi}\iota_Y F_2\cdot\mathcal{L}_X\psi - f_X e^{\phi}\iota_Y F_2\cdot\psi - e^{\phi}\iota_Y F_2\cdot\mathcal{L}_X\psi \\
&= 2e^{\phi}X(\phi)\iota_Y F_2\cdot\psi + e^{\phi}\iota_{[X,Y]}F_2\cdot\psi + e^{\phi}\iota_Y\mathcal{L}_X F_2\cdot\psi - f_X e^{\phi}\iota_Y F_2\cdot\psi \\
&= e^{\phi}\iota_{[X,Y]}F_2\cdot\psi \,,
\end{aligned}$$

since $\mathcal{L}_X F_2 = -cF_2$ and $X(\phi) = -f_X + c$. A similar calculation for the remaining term yields

$$\begin{aligned}
[\mathcal{L}_X, e^{\phi}Y\cdot F_2]\cdot\psi &= \mathcal{L}_X\left(e^{\phi}Y\right)\cdot F_2\cdot\psi + e^{\phi}Y\cdot\mathcal{L}_X\left(F_2\cdot\psi\right) - f_X e^{\phi}Y\cdot F_2\cdot\psi - e^{\phi}Y\cdot F_2\cdot\psi \\
&= 2e^{\phi}X(\phi)Y\cdot F_2\cdot\psi + e^{\phi}Y\cdot\mathcal{L}_X F_2\cdot\psi + e^{\phi}[X,Y]\cdot F_2\cdot\psi - f_X e^{\phi}Y\cdot F_2\cdot\psi \\
&= e^{\phi}[X,Y]\cdot F_2\cdot\psi \,,
\end{aligned}$$

because $\mathcal{L}_X\left(F_2\cdot\psi\right) = \mathcal{L}_X F_2\cdot\psi + 2h_X F_2\cdot\psi$. It follows that $[\mathcal{L}_X, \Theta_Y] = \Theta_{[X,Y]}$. □

The results of this section guarantee that we can associate a well-defined conformal Killing superalgebra $\mathfrak{h}$ to a IIA background (N, h, F_2, ϕ) obtained as a quotient of the *geometric* conformal Killing superalgebra of a vacuum M-theory background (M, g). We devote the rest of this chapter to presenting a number of explicit examples.

8.4 Conformal Killing spinors and Kaluza-Klein reductions of flat space

A generic Killing vector of Minkowski space $\mathbb{R}^{1,n}$ can be written in the form

$$\xi = \tau + \lambda\,, \tag{8.14}$$

where τ is a translation and λ is a Lorentz transformation. Requiring that a conformal Killing spinor $\psi = \psi_0 + x \cdot \chi$ of the flat space given in equation (5.30) is invariant under ξ implies that

$$\lambda \cdot \chi = 0\,, \tag{8.15}$$

$$\tau \cdot \chi - \tfrac{1}{2}\lambda \cdot \psi_0 = 0\,. \tag{8.16}$$

By imposing the requirement that ξ be everywhere spacelike (so that its action on Minkowski space is free), it can be shown [39] that there are two families of spacelike Killing vectors which give rise to smooth quotients and preserve some spinors. There exists a coordinate system $(z, x^i, x^\pm)$ in which the flat metric takes the form

$$g_{\mathbb{E}^{1,10}} = 2dx^+ x^- + \sum_{i=1}^{8} dx^i dx^i + dz^2, \tag{8.17}$$

and in this metric (up to a scale), the normal forms of the relevant Killing vectors are given by

$$\xi = \partial_z + R_{12}(\beta_1) + R_{34}(\beta_2) + R_{56}(\beta_3) + R_{78}(\beta_4)\,, \tag{8.18}$$

$$\sum_i \beta_i = 0 \tag{8.19}$$

and by

$$\xi = \partial_z + N_{+1}(u) + R_{34}(\beta_1') + R_{56}(\beta_2') + R_{78}(\beta_3')\,, \tag{8.20}$$

$$\sum_i \beta_i' = 0. \tag{8.21}$$

Here $R_{ij}(\beta)$ are rotations in the ij plane with parameters β and $N_{+1}(u)$ is a null rotation in the ith direction with parameter u. In the second case (8.18) the condition (8.15) can be written as

$$(N+S)\chi = 0\,, \tag{8.22}$$

$$\Gamma_z\chi - \tfrac{1}{2}(N+S)\psi_0 = 0\,, \tag{8.23}$$

where N is nilpotent and S is semisimple (since the R_{ij}'s all commute and are therefore simultaneously diagonalisable); furthermore, N and S commute. The first condition then implies that $N\varphi_2$ and $S\chi = 0$ separately. In addition, N and S commute with Γ_z and $\Gamma_z^2 = -\mathrm{Id}$.

Let us look at these possibilities separately. First, note that it follows from equation (8.15)that a conformal Killing spinor with $\psi_0 = 0$ can never be invariant, since equation (8.15) then implies that χ must vanish as well: therefore, both ψ_0, χ must be nonzero, constrained by the invariance condition.

Consider the normal form (8.18). There are several possibilities depending on which of the β's vanish. At most two of the β_i's can be zero, since if three vanish, the remaining one must vanish as well. Therefore, let us consider the case when two are nonzero; without loss of generality we can choose them to be β_1, β_2. Equations (8.15) then become

$$\beta_1\Gamma_{12}\chi + \beta_2\Gamma_{34}\chi = 0\,, \tag{8.24}$$

$$\Gamma_z\chi - \tfrac{1}{2}\left(\beta_1\Gamma_{12} + \beta_2\Gamma_{34}\right)\psi_0 = 0\,. \tag{8.25}$$

Since $R = \beta_1\Gamma_{12} + \beta_2\Gamma_{34}$ commutes with Γ_z, Clifford multiplying the last equation again by R we obtain

$$R\cdot R\cdot\psi_0 = 0\,, \tag{8.26}$$

which implies that

$$2\beta_1\beta_2\Gamma_{1234}\psi_0 = \left(\beta_1^2 + \beta_2^2\right)\psi_0,$$

and using the fact that $\beta_1 + \beta_2 = 0$ and writing $\beta_1^2 + \beta_2^2 = \left(\beta_1 + \beta_2\right)^2 - 2\beta_1\beta_2$, we get

$$\Gamma_{12}\psi_0 = \Gamma_{56}\psi_0\,,$$

in other words, $R \cdot \psi_0 = 0$, which implies that $\chi = 0$ and the conformal Killing spinor φ is simply a parallel spinor invariant under the action of ξ.

In fact, one can show in general that for a semisimple element S, $S \cdot S \cdot \varphi$ implies that $S \cdot \varphi = 0$. Complexifying the spinors, if necessary, we can diagonalise the matrix $S\cdot$ by which we mean the matrix that gives the *Clifford action* of S on spinors, not the usual rotation matrix. The eigenvalues of $S^2\cdot$ are the squares of the eigenvalues of $S\cdot$, so the zero eigenvalues of both matrices coincide. This means that the only conformal Killing spinors of flat space left invariant by the action of a Killing vector of the form (8.18) are (a subset of) the parallel spinors.

Next, let us consider the case involving the null rotation. Now (8.15) becomes

$$\Gamma_z \chi - \tfrac{1}{2}(N+S)\psi_0 = 0 \,, \tag{8.27}$$

from which we can deduce that

$$\begin{aligned} N \cdot S \cdot \psi_0 &= 0 \,, \\ S^2 \psi_0 &= 0 \,, \end{aligned}$$

since N is nilpotent. Using a similar argument as before, we can again conclude that $S \cdot \varphi_1 = 0$. Now we can solve (8.27) to obtain

$$\chi = \tfrac{1}{2}\Gamma_z N \cdot \psi_0 \,, \tag{8.28}$$

whence the conformal Killing spinors invariant under the action of a Killing vector of the form (8.20) are given by

$$\psi(\zeta) = \zeta + \tfrac{1}{2} x \cdot \Gamma_z N \cdot \zeta \,, \tag{8.29}$$

and ζ is an arbitrary constant spinor.

8.5 Conformal superalgebras of nullbranes

In [39] the reductions of flat space with respect to Killing vectors involving a null rotation are called *nullbranes*, which have recently attracted much interest in the context of time-dependent string solutions and cosmological toy models [43, 44, 45].

In the previous section we showed that in addition to parallel spinors, certain non-parallel conformal Killing spinors are also invariant under the action of the Killing vector used in the reduction. Therefore, it is possible to associate a conformal Killing superalgebra to each nullbrane solution. There are three distinct cases: the null-brane solution and two solutions which interpolate between the nullbrane and fluxbrane solutios: we consider each in turn, although it turns out that the form of the conformal Killing superalgebras of the latter follow largely from the computation of the former.

8.5.1 The nullbrane

When the Killing vector (8.20) used in the reduction is a pure null rotation (i.e. all the β's vanish, with $u \neq 0$), one obtains a IIA solution which the authors of [39] call a *nullbrane.* This is a $\frac{1}{2}$-BPS solution with the following metric, dilaton and Ramond-Ramond 2-form:

$$
\begin{aligned}
g &= \Lambda^{\frac{1}{2}}\left[2dx^{+}dx^{-} - \left(x^{1}\right)^{2}(dx^{-})^{2} + ds^{2}\left(\mathrm{E}^{7}\right)\right] + \Lambda^{-\frac{1}{2}}\left(dx^{1} + x^{1}x^{-}dx^{-}\right)^{2}, \\
F_2 &= \frac{2}{\Lambda^{2}}dx^{-} \wedge dx^{1}, \\
\phi &= \tfrac{3}{4}\log\Lambda,
\end{aligned}
$$

where $\Lambda = 1 + (x^{-})^{2}$.

Note that the metric does not depend on the parameter u: it has been absorbed by a rescaling $x^{\pm} \to u^{\pm 1}x^{\pm}$.

It is straightforward to calculate the normaliser of $\xi = \partial_z + uM_{+1}$. Norm($\xi$) is generated by

$$
\begin{aligned}
X &:= M_{1z} - \tfrac{1}{2}uK_{+} - \tfrac{1}{u}P_{-}, \\
Y &:= P_1 + uM_{+z}, \\
Z &:= M_{+-} - D, \\
& \qquad P_i,\ P_{+},\ M_{ij},\ M_{i+},\ P_z,\ M_{+1},
\end{aligned}
$$

where $i = 2\ldots 8$. As with the metric, it is possible to absorb the parameter u into the translation generators $P_\pm$. Also, because we're quotienting by ξ and set $\xi = 0$ in the quotient algebra, we can eliminate one generator, leaving $P_z = -uM_{+1} := W$, say. We can also compute the Dirac current of two arbitrary ξ-invariant twistor spinors, $\psi_1(\zeta) = \zeta + \frac{u}{2} x \cdot \Gamma_{z+1}\zeta$ and $\psi_2(v) = v + \frac{u}{2} x \cdot \Gamma_{z+1} v$.

$$V(\psi_1, \psi_2) = (\Gamma_z\zeta, v)\xi + (\zeta, \Gamma_1 v)Y + (\Gamma_+\zeta, v)X + \sum_{i,j} (\Gamma_{z+1}\zeta, \Gamma^{ij} v) M_{ij} + \sum_i (\Gamma_{+-}\Gamma_{z1}\zeta, \Gamma^i v) M_{+i} + (\zeta, \Gamma_- v) P_+ + \sum_{i=2}^{9} (\zeta, \Gamma^i v) P_i \,. \quad (8.30)$$

For the even-odd bracket of the superalgebra we also need to compute the Lie derivative of $\psi(\zeta)$ with respect to elements of N_ξ. This yields

$$\mathscr{L}_{P_i}\psi(\zeta) = \psi\left(\tfrac{u}{2}\Gamma_{iz+1}\zeta\right) \qquad \mathscr{L}_{M_{ij}}\psi(\zeta) = \psi\left(-\tfrac{1}{2}\Gamma_{ij}\zeta\right)$$

$$\mathscr{L}_{P_+}\psi(\zeta) = 0 \qquad \mathscr{L}_X\psi(\zeta) = \psi\left(\tfrac{1}{2}(\mathrm{Id} + \Gamma_+\Gamma_-)\Gamma_{1z}\zeta\right)$$

$$\mathscr{L}_Y\psi(\zeta) = \psi(u\Gamma_{z+}\zeta) \qquad \mathscr{L}_Z\psi(\zeta) = \psi\left(-\tfrac{1}{2}\Gamma_+\Gamma_-\zeta\right)$$

$$\mathscr{L}_W\psi(\zeta) = \psi\left(\tfrac{u}{2}\Gamma_{+1}\zeta\right) ,$$

where we have used (5.31): To exhibit the form of the nullbrane conformal Killing superalgebra, it is convenient to write $\zeta = \zeta_+ + \zeta_-$, where $\zeta_\pm \in \operatorname{Ker}\Gamma_\pm$. To write down the algebra in a form that is more in line with notation used in the physics literature, we introduce "fermionic" generators Q_+, S_-, which generate infinitesimal shifts in the direction of ζ_+ and ζ_-, respectively. Note that Q_+ corresponds to the supersymmetry generator of the usual nullbrane supersymmetry algebra.

Thus, the (non-trivial) brackets of the nullbrane conformal Killing superalgebra are:

$$[X, Y] = \xi \; (= 0) \qquad [Z, Y] = Y$$

$$[W, Z] = -W \qquad [W, X] = Y$$

$$[W, Y] = -uP_+ \qquad [M_{ij}, P_k] = \delta_{jk}P_i - \delta_{ik}P_j$$

$$[M_{i+}, P_j] = -\delta_{ij}P_+ \qquad [M_{i+}, X] = -\tfrac{1}{u}P_i$$

$$[M_{i+}, Z] = -M_{i+} \qquad [P_+, Z] = -2P_+$$

$$
\begin{aligned}
[P_i, Q_+] &= \tfrac{u}{2}\Gamma_{iz+1}S_- & [M_{ij}, Q_+] &= -\tfrac{1}{2}\Gamma_{ij}Q_+ \\
[M_{ij}, S_-] &= -\tfrac{1}{2}\Gamma_{ij}S_- & [M_{i+}, Q_+] &= -\tfrac{1}{2}\Gamma_{i+}S_- \\
[X, Q_+] &= \left(\tfrac{1}{2}\,\mathrm{Id} - \Gamma_{1z}\right)Q_+ & [X, S_-] &= \tfrac{1}{2}S_- \\
[Y, Q_+] &= u\Gamma_{z+}S_- & [Z, Q_+] &= Q_+ \\
[W, Q_+] &= \tfrac{u}{2}\Gamma_{+1}S_- & [Q_+, Q_+] &= \Gamma_- C^{-1}P_+
\end{aligned}
$$

and

$$
\begin{aligned}
[S_-, S_-] &= \Gamma_+ C^{-1}X + \Gamma^{ij}\Gamma_{z+1}C^{-1}M_{ij} \\
[Q_+, S_-] &= \Gamma_z C^{-1}\xi + \Gamma_1 C^{-1}Y + \sum_i \Gamma^i\Gamma_{z1}C^{-1}M_{+i} + \sum_i \Gamma^i C^{-1}P_i\,,
\end{aligned}
$$

where C is the "charge conjugation matrix" used to define the spinor inner product as $(\psi, \chi) = \psi^t C\chi$, notation often favored by physicists. Note that the even part of the algebra contains a natural $\mathfrak{iso}(8)$ subalgebra generated by M_{ij}, M_{i+}, P_i and P_+.

8.5.2 Interpolating solutions

When the Killing vector also contains rotations, one obtains solutions that interpolate between the nullbrane and the supersymmetric fluxbrane solutions described in [39]. There are two possible cases:

1. $\beta_1 = 0, \beta_2 = -\beta_3 = \beta$
2. $\beta_1, \beta_2, \beta_3 \neq 0, \beta_1 + \beta_2 + \beta_3 = 0$

The explicit metrics can be found in [39]. The conformal Killing superalgebras of these solutions are actually subsuperalgebras of the nullbrane superalgebra.

In case (1) an $\mathfrak{iso}(4)$ subalgebra remains. The brackets are[2]

$$[X,Y] = \xi\ (=0) \qquad [W,X] = Y$$
$$[W,Y] = -uP_+ \qquad [M_{ij},P_k] = \delta_{jk}P_i - \delta_{ik}P_j$$
$$[M_{i+},P_j] = -\delta_{ij}P_+ \qquad [M_{i+},X] = -\tfrac{1}{u}P_i$$
$$[P_i,Q_+] = \tfrac{u}{2}\Gamma_{iz+1}S_- \qquad [M_{ij},Q_+] = -\tfrac{1}{2}\Gamma_{ij}Q_+$$
$$[M_{ij},S_-] = -\tfrac{1}{2}\Gamma_{ij}S_- \qquad [M_{i+},Q_+] = -\tfrac{1}{2}\Gamma_{i+}S_-$$
$$[X,Q_+] = \left(\tfrac{1}{2}\mathrm{Id}-\Gamma_{1z}\right)Q_+ \qquad [X,S_-] = \tfrac{1}{2}S_-$$
$$[Y,Q_+] = u\Gamma_{z+}S_- \qquad [W,Q_+] = \tfrac{u}{2}\Gamma_{+1}S_-$$
$$[Q_+,Q_+] = \Gamma_- C^{-1}P_+$$

and

$$[S_-,S_-] = \Gamma_+C^{-1}X + \Gamma^{ij}\Gamma_{z+1}C^{-1}M_{ij}$$
$$[Q_+,S_-] = \Gamma_z C^{-1}\xi + \Gamma_1 C^{-1}Y + \sum_i \Gamma^i\Gamma_{z1}C^{-1}M_{+i} + \sum_i \Gamma^i C^{-1}P_i\,,$$

where index i can now only take values 2,3,4.

In the case (2) we only have an $\mathfrak{iso}(2)$ subalgebra: the only possible value of i is 2, otherwise the form of the algebra remains the same.

8.6 Conformal Killing spinors in HLW massive IIA supergravity

It has been shown by Howe, Lambert and West [46] that any M-theory background (M,g,F) can be viewed as a Weyl structure (M,g,F,θ), where the Weyl one-form $\theta = dl$ is exact. The M-theory equations of motion written using the Weyl connection are then equivalent to the standard ones, and spinors are taken to be sections of $S^{[\frac{1}{2}]}$. This fact has been used in [47, 46, 48] to produce a variant of the Kaluza-Klein construction we presented in Section 8.1.

[2]Note that this is the algebra in the *quotient*, so we set $\xi = 0$.

Lemma 41. *Suppose (M,g,F) is an M-theory background that admits a homothety ξ (in other words, $\mathscr{L}_\xi g = -2mg$ for some constant m) which is also supernormal, i.e. $\mathscr{L}_\xi F = -3m$. Then there exists a M-theory background with a Weyl connection $(M,\overline{g},\overline{F},\theta = dl)$ for some function l for which $\mathscr{L}_\xi \overline{g} = 0$ and $\mathscr{L}_\xi \overline{F} = 0$.*

Proof. We simply perform a Weyl transformation $g \mapsto \overline{g} = e^{2l} g$ and in addition rescale the four-form: $F \mapsto \overline{F} = e^{3l} F$. Provided that $\xi(l) = m$, the above conditions are then satisfied. We can always find such an l, for instance by solving the equation $\xi(l) = m$ in adapted coordinates where $\xi = \partial_z$. □

We can now use the usual Kaluza-Klein procedure and find that $(M,\overline{g},\overline{F},dl)$ gives rise to the data (N,h,F_2,H_2,H_3,ϕ,m). The submersion formulas for curvature again give rise to 10-dimensional equations of motion which are distinct from those of the IIA theory since they now involve m-dependent terms. These equations define a theory called the *HLW massive IIA supergravity.* Note that m is actually a free parameter since we can send it to any value by rescaling ξ, and if $m = 0$ we recover the usual IIA equations of motion.

We call (N,h,F_2,H_2,H_3,ϕ,m) a *homothetic Kaluza-Klein reduction* of (M,g,F). As before, we will only consider reductions of M-theory backgrounds for which F vanishes.

Since acting on $S^{[\frac{1}{2}]}$, $P^D = P$, we immediately have

Proposition 42. *Let (M,g) be a vacuum M-theory background with homothety ξ so that $\mathscr{L}_\xi g = -2mg$ and let (N,h,F_2,ϕ,m) be the corresponding homothetic Kaluza-Klein reduction. The Penrose operator $\tilde{P}$ on M induces the following Penrose-type operator on N:*

$$\mathcal{P}_X\psi = \nabla_X\psi - \tfrac{1}{2}e^{\phi}\iota_X F_2 \cdot \mathrm{dvol}_N \cdot \psi + \tfrac{1}{n-1} X \cdot \nabla\psi - \tfrac{1}{4(n-1)} e^{\phi} X \cdot F_2 \cdot \mathrm{dvol}_N \psi\,, \qquad (8.31)$$

We remark that although $\mathcal{P}$ agrees formally with (8.10), the fields appearing in the expression now satisfy different equations of motion.

8.7 A HLW massive IIA conformal Killing superalgebra

In this section we construct an example of a conformal Killing superalgebra associated to a background of HLW massive IIA supergravity as a quotient of eleven-dimensional Minkowski conformal Killing superalgebra we introduced in Chapter 5, Section 5.5.

We proceed in a fashion similar to [47]. Consider a homothety

$$\xi = D + R,$$

where $D = x^a \partial_a$ is the dilation vector field and $R = R_a^b x^a \partial_b$ is a rotation. Let $\psi = \psi_0 + x \cdot \chi$ be an arbitrary conformal Killing spinor. The action of ξ on ψ is given by

$$\begin{aligned} \mathscr{L}_\xi \psi &= D(\psi) + R(\psi) - \tfrac{1}{4} dD^\flat \cdot \psi - \tfrac{1}{4} dR^\flat \cdot \psi - \tfrac{1}{2}\psi \\ &= x \cdot \chi + (Rx) \cdot \chi - \tfrac{1}{4} R \cdot \psi - \tfrac{1}{4} R \cdot x \cdot \chi - \tfrac{1}{2}\psi - \tfrac{1}{2} x \cdot \chi \end{aligned}$$

Note that

$$R \cdot x \cdot \chi = R_{ab} \Gamma^{ab} x^c \Gamma_c \chi = x \cdot R \cdot \chi + 4\,(Rx) \cdot \chi\,, \tag{8.32}$$

where we have used the commutation relation

$$[\Gamma^{ab}, \Gamma^c] = 2\eta^{ac}\Gamma^b - 2\eta^{bc}\Gamma^a. \tag{8.33}$$

The x-dependent part and the constant part of the expression must vanish separately, so R must satisfy

$$R \cdot \psi = -2\psi$$

$$R \cdot \chi = 2\chi$$

Without loss of generality we can take $R = 2M_{0\natural}$. Half of the conformal Killing spinors of the 11-dimensional Minkowski space satisfy the above condition. Hence, the space of ξ-invariant conformal Killing spinors is 32-dimensional.

For the following computations we prefer to use lightcone coordinates $x^\pm = \frac{1}{\sqrt{2}}\left(x^0 \pm x^\natural\right)$, where $x^\natural$ is the 10th spacelike direction. This implies that $\psi \in \operatorname{Ker}\Gamma_-$ and $\chi \in \operatorname{Ker}\Gamma_+$, so we write a conformal Killing spinor as $\psi(\zeta) = \zeta_- + x\cdot\zeta_+$. In these coordinates $\xi = D + M_{-+}$.

We also want to exhibit the HLW solution itself. The reduction ansatz for the eleven-dimensional metric g is usually written in the so-called *string frame* as [47]:

$$g = e^{(2mz-\frac{2\phi}{3})}\left(h + e^{2\phi}(dz+A)^2\right) , \tag{8.34}$$

and $F_2 = dA$.

We start by writing the flat eleven-dimensional flat metric in the form

$$g = -(dx^0)^2 + (dx^\natural)^2 + \left(dr^2 + r^2 d\Omega_8^2\right) \tag{8.35}$$

where $d\Omega_8^2$ is the metric on the 8-sphere. We choose new coordinates adapted to the vector field $\xi = D + 2M_{0\natural}$ such that $\xi = \partial_z$.

$$\begin{aligned} x^0 &= \tfrac{1}{2}y_2\left(e^{2z} + e^{-2y_1}\right), \\ x^\natural &= \tfrac{1}{2}y^2\left(e^{2z} - e^{-2y_1}\right), \\ r &= e^{z-y^1} . \end{aligned}$$

In these coordinates, the eleven-dimensional metric becomes

$$\begin{aligned} g = e^{2(z-y^1)}[&(dz - (1-2(y^2)^2)dy_1 - y^2 dy^2)^2 + 4y_2^2(1-y_2^2)(dy^1)^2 \\ &- (1+(y^2)^2)(dy^2)^2 + 4(y^2)^3 dy^1 dy^2 + d\Omega_8^2\,] . \end{aligned}$$

From the reduction ansatz (8.34) we can then read off the HLW solution:

$$\begin{aligned} &g = e^{-\frac{9}{4}y^1}\left(4(y^2)^2(1-(y^2)^2)(dy^1)^2 - (1+(y^2)^2)(dy^2)^2 + 4(y^2)^3 dy^1 dy^2 + d\Omega_8^2\right), \\ &F_2 = -4y^2 dy^1 \wedge dy^2, \\ &\phi = \tfrac{3}{2}y^1 . \end{aligned}$$

Note that the metric h has a singularity at $y^2 = 0$. Given the form of the adapted coordinates, at a first glance one could imagine that this is a coordinate singularity, but computation of the Ricci scalar R and the fully contracted Riemann tensor $R_{abcd}R^{abcd}$ shows that they both diverge as $y^2 \to 0$, so it is in fact a genuine curvature singularity.

We are now ready to calculate the structure of the superconformal algebra associated to this HLW IIA background. We proceed in a similar fashion as in Section 8.5. It is a straightforward calculation to show that the normaliser Norm(ξ) is generated by M_{-+}, D, M_{ij}, P_- and K_+. In fact, this agrees with the centraliser of ξ. As we set ξ to zero in the quotient, we can eliminate one of the generators, leaving $D = -M_{-+}$ in the algebra.

As before, we calculate the Dirac current V_{ψ_1,ψ_2} associated to two arbitrary ξ-invariant conformal Killing spinors $\psi_1(\zeta) = \zeta_- + x\cdot\zeta_+$ and $\psi_2(v) = v_- + x\cdot v_+$. After some Γ-matrix algebra and repeated use of the antisymmetry of the spinor inner product with respect to the Clifford multiplication by a vector field (that is, $(\psi_1, X\cdot\psi_2) = -(X\cdot\psi_1,\psi_2)$) we obtain

$$V_{\psi_1,\psi_2} = (\zeta_+,\Gamma_- v_+)K_+ + [(\zeta_+,v_-) - (\zeta_-,v_+)]\,(D+M_{-+}) + \left((\zeta_-,\Gamma^{ij}v_+) - (\zeta_+,\Gamma^{ij}v_-)\right)M_{ij} + (\zeta_-,\Gamma_+ v_-)P_- \,.$$

We can also compute the action of the generators of Norm(ξ) on a ξ-invariant conformal Killing spinor $\psi(\zeta_-,\zeta_+) = \zeta_- + x\cdot\zeta_+$ via the spinorial Lie derivative:

$$\mathscr{L}_{K_+}\psi(\zeta_-,\zeta_+) = \psi(0,\Gamma_+\zeta_-) \qquad \mathscr{L}_{P_-}\psi(\zeta_-,\zeta_+) = \psi(\Gamma_-\zeta_+,0)$$
$$\mathscr{L}_{M_{ij}}\psi(\zeta_-,\zeta_+) = \psi\left(-\tfrac{1}{2}\Gamma_{ij}\zeta_-,-\tfrac{1}{2}\Gamma_{ij}\zeta_+\right) \qquad \mathscr{L}_{D}\psi(\zeta_-,\zeta_+) = \psi\left(-\tfrac{1}{2}\zeta_-,\tfrac{1}{2}\zeta_+\right)$$

To exhibit the structure of the superalgebra we again introduce fermionic generators Q_-, S_+ which generate shifts along the spinors ζ_-,ζ_+, respectively. In terms of these, the non-trivial brackets of the algebra are

$$[P_-,K_+] = \xi \ (=0) \qquad [P_-,D] = P_-$$
$$[K_+,D] = -K_+ \qquad [P_-,S_+] = Q_- \,,$$
$$[K_+,Q_-] = S_+ \,, \qquad \left[M_{ij},Q_-\right] = -\tfrac{1}{2}\Gamma_{ij}Q_- \,,$$
$$\left[M_{ij},S_+\right] = -\tfrac{1}{2}\Gamma_{ij}S_+ \,, \qquad [D,Q_-] = -\tfrac{1}{2}Q_- \,,$$
$$[D,S_+,] = \tfrac{1}{2}S_+ \,, \qquad [S_+,S_+] = \Gamma_- C^{-1}K_+ \,,$$
$$[S_+,Q_-] = \xi + \Gamma^{ij}C^{-1}M_{ij} \,, \qquad [Q_-,Q_-] = \Gamma_+ C^{-1}P_- \,,$$

where C is the charge conjugation matrix as before.

8.8 Conformal Killing spinors in type IIB supergravity

The spinors in type IIB supergravity [34] are real representations of $\mathrm{Spin}(1,9)\times SL(2,\mathbb{R})$. It is convenient to consider them as sections of the bundle $S_+ \otimes \Delta$, where S_+ is the 16-dimensional positive chirality representation of Spin(1,9) and Δ is the standard representation of $SL(2,\mathbb{R})$. The IIB backgrounds are given by the data (M,g,H), where H is a self-dual closed 5-form that also satisfies an Einstein-type equation along with the Lorentzian metric g. The full theory admits other fields, but we are only interested in this truncated version.

Acting on the sections of $\mathcal{S} = S_+ \otimes \Delta$, the supercovariant connection is

$$\mathcal{D}_X = \nabla_X + \iota_X H \otimes J,$$

where J is a complex structure on Δ. We can consider $\mathcal{S}$ as a complexification of S_+ and write this as

$$\mathcal{D}_X = \nabla_X + i\iota_X H$$

Now as in the M-theory case, consider the twistor operator defined using the supercovariant connection:

$$\mathcal{P}_X = \mathcal{D}_X + \tfrac{1}{10} X \cdot \mathcal{D},$$

where $\mathcal{D} = \sum_i e_i \cdot \mathcal{D}_i$.

As observed by Leitner [49], $\mathcal{P}_X$ actually agrees with the *geometric* Penrose operator P_X due to a happy accident of Clifford product identities involving the self-dual five-form in ten dimensions, provided that only the self-dual 5-form flux is turned on. It holds that

$$\begin{aligned} X \cdot H &= -2\iota_X H \\ \star(X^\flat \wedge H) &= -\iota_X H, \end{aligned}$$

where the forms in these identities are understood to be acting on spinors via the

Clifford product. Now

$$
\begin{aligned}
\mathcal{P}_X &= \mathcal{D}_X + \tfrac{1}{10} X \cdot \mathcal{D} \\
&= \nabla_X + \tfrac{1}{10} X \cdot \nabla + i\iota_X H + \tfrac{1}{2} X \cdot iH \\
&= P_X + i\iota_X H + \tfrac{1}{2} X \cdot iH \\
&= P_X ,
\end{aligned}
$$

so $\operatorname{Ker}\mathcal{P} = \operatorname{Ker} P$. In particular, supergravity Killing spinors are *geometric* conformal Killing spinors.

The simplest example of a IIB solution that admits conformal Killing spinors is again the flat space $\mathbb{R}^{1,9}$. Studying the existing classification of Lorentzian manifolds admitting conformal Killing spinors [12], we can find other examples. Perhaps the most interesting is the IIB conformally flat pp-wave that has received much attention recently in the context of BMN correspondence and the Freund-Rubin solution $AdS_5 \times S^5$ — in fact, the former is the Penrose limit of the latter [50]. The conformal Killing superalgebras of them both are isomorphic to the Minkowski conformal Killing superalgebra we described in Chapter 5.

It has been suggested by [49] that this curious identity between geometric and supergravity Penrose operators might be useful in finding new supersymmetric backgrounds of type IIB supergravity, perhaps among the 10-dimensional pseudo-Hermitian Einstein spaces described in [51].

Chapter 9

Killing spinors, discrete quotients and spin structures

The relationship between the choice of the spin structure of a Lorentzian symmetric space and the dimension of the space of its conformal Killing spinors was analysed in detail in [14]. In the case of non-conformally flat symmetric spaces, the conformal Killing spinors actually agree with parallel spinors, corresponding to supergravity Killing spinors when $F = 0$. It is a natural question to ask whether the dimension of $\operatorname{Ker} \mathcal{D}$ also depends on the choice of spin structure in the case of nonzero four-form flux.

The authors of [52] observed that it is possible to construct examples of non-simply connected isometric M-theory backgrounds that have the same geometry and four-form F but admit different fractions of supersymmetry depending on the choice of the spin structure. Therefore, it would seem necessary to include the choice of spin structure in the data defining a M-theory background as well.

In this chapter we illustrate this point further by considering backgrounds that are Lorentzian symmetric spaces (Cahen-Wallach spaces), as opposed to the Freund-Rubin solutions involving spherical space forms that were treated in [52]. We will show that at least for known *symmetric* M-theory backgrounds with with more than 16 Killing spinors the choice of spin structure that preserves any supersymmetry appears to be unique. In particular, this includes symmetric discrete quotients of M-theory pp-wave solutions.

Orbifolds of 11-dimensional pp-wave solutions have also been considered previously in e. g. [53], but only for a very particular solution with 26 supersymmetries.

9.1 Discrete quotients and spin structures

Let (M,g) be a simply connected n-dimensional Lorentzian spin manifold. We denote its isometry group by $I(M,g)$. Suppose $D \subset I(M,g)$ is a discrete, orientation-preserving subgroup, and let e_B, $B = 0 \ldots n-1$ be a pseudo-orthonormal frame on M. Then for any $\gamma \in D$ at a point $x \in M$, $d\gamma_x \in \mathrm{SO}(1,n-1)$ corresponds to the linear map that transforms $e_A(x)$ to $e_A(\gamma(x))$. There are now two possible lifts of $d\gamma_x$ to $\mathrm{Spin}(1,n-1)$ since the covering map $\mathrm{Spin}(1,n-1) \rightarrow \mathrm{SO}(1,n-1)$ is two-to-one: we denote these by $\pm\Gamma(x)$.

Now let $\mathcal{E}(D)$ be the set of all left actions of D on $M \times \mathrm{Spin}(1,n-1)$ satisfying

$$\epsilon(\gamma)(x,a) = \left(\gamma(x), \epsilon(\gamma,x)\cdot a\right) , \tag{9.1}$$

where $\epsilon(\gamma,x) = \pm\Gamma(x)$.

Elements of $\mathcal{E}(D)$ correspond to spin structures on $N = M/D$. The spinor bundle corresponding to $\epsilon \in \mathcal{E}(D)$ is given by

$$S_\epsilon = \left(M \times \Delta_{1,n-1}\right)/\epsilon , \tag{9.2}$$

Here $\Delta_{1,n-1}$ is the spinor module and $\epsilon(\gamma)\left(x,\psi(x)\right) = \left(\gamma(x), \epsilon(\gamma,x)\cdot\psi(x)\right)$. It follows that the spinor fields ψ on N are the spinor fields on M that satisfy

$$\psi\left(\gamma(x)\right) = \epsilon(\gamma,x)\cdot\psi(x) . \tag{9.3}$$

In particular, when M is a M-theory background and D also preserves the four-form F, the Killing spinors on $N = M/D$ are the ϵ-invariant Killing spinors of M.

9.2 Hpp-waves in M-theory

The M-theory Hpp-waves are supersymmetric solutions of eleven-dimensional supergravity equipped with the metric of a Lorentzian symmetric space of the Cahen-Wallach type [18] and a null homogeneous four-form. In the light-cone coordinates

$x^{\pm}, x^i, i = 1 \ldots 9$ the Cahen-Wallach metric can be written as

$$g = 2dx^+ dx^- + \sum_{i,j} A_{ij} x^i x^j (dx^-)^2 + \sum_i \left(dx^i\right)^2 \tag{9.4}$$

where A_{ij} is a real 9×9 symmetric matrix. If A_{ij} is nondegenerate, (M, g) is indecomposable; otherwise it decomposes to a product of a lower-dimensional indecomposable CW space and an Euclidean space. If A_{ij} is zero, (9.4) is simply the metric on flat space. .

The moduli space of indecomposable CW metrics agrees with the space of unordered eigenvalues of A_{ij} up to a positive scale: this space is diffeomorphic to S^8/Σ_9, where Σ_9 is the permutation group of nine objects[18]. In particular, a positive rescaling of A_{ij} can always be absorbed by a coordinate transformation. It is, of course, also possible to exhibit (M, g) as a symmetric space by constructing its transvection group G_A for which (9.4) is the invariant metric. We refer the reader to [18] for details.

A natural choice of F is a four-form preserved by the symmetries of the CW metric that also satisfies the field equations. As explained e.g. in [54, 18], the natural choice is a parallel form

$$F = dx^- \wedge \Theta ,$$

where Θ is a 3-form with constant coefficients on $\mathbb{R}^9$. The equations of motion (6.2) are satisfied iff $\operatorname{Tr} A = -\frac{1}{2}|\Theta|^2$.

We will make use of a global pseudo-orthonormal frame:

$$\begin{aligned} e_+ &= \partial_+, \\ e_i &= \partial_i, \\ e_- &= \partial_- - \sum_{i,j} \tfrac{1}{2} A_{ij} x^i x^j \partial_+ \end{aligned}$$

and the corresponding coframe

$$\begin{aligned} e^+ &= dx^+ + \tfrac{1}{2} A_{ij} x^i x^j dx^- \\ e^- &= dx^- \\ e^i &= dx^i . \end{aligned}$$

For our purposes, there is no need to distinguish between coordinate and frame indices.

The only nonzero connection forms for the metric (9.4) are

$$\omega^{+i} = A^i{}_j x^j dx^- . \tag{9.5}$$

To solve the SCKS equation for the Hpp-waves, we will also need the explicit form of their parallel spinors that satisfy the integrability conditions (6.15). Observing that $\nabla_a \chi = 0$ implies that

$$\begin{aligned} 0 = \partial_+ \chi &= \partial_i \chi , \\ \partial_- \chi &= -\tfrac{1}{2} x^i A_{ij} \Gamma_{j+} \chi , \end{aligned}$$

and keeping in mind that since $F \cdot \chi = 0$, we must have $\chi \in \operatorname{Ker}\Gamma_+$, this implies that χ is a constant spinor in the kernel of Γ_+. We also require that $\Theta \cdot \chi = 0$. It is clear that there are many possible choices of Θ for which this condition is satisfied. For example, we could choose $\Theta = dx^{129} - dx^{349}$, for which the integrability condition would be satisfied if

$$\Gamma_{1234} \chi = -\chi ,$$

a condition which is generally satisfied by half of the spinors, since Γ_{1234} squares to the identity. In summary, the possible χ lie in $\operatorname{Ker}\Gamma_+ \cap \operatorname{Ker}\Theta$.

Before tackling the SCKS equation, we also determine the amount of supersymmetry preserved by the Hpp-wave solutions. As previously mentioned, supergravity Killing spinors are parallel sections of the supercovariant connection

$$\mathcal{D}_X = \nabla_X + \tfrac{1}{6}\iota_X F + \tfrac{1}{12} X^\flat \wedge F := \nabla_X + \Omega_X ,$$

It is convenient to rewrite Ω involving Clifford products as

$$\Omega_X = \tfrac{1}{8} F \cdot X - \tfrac{1}{24} X \cdot F . \tag{9.6}$$

Thus, we find that for the Hpp-waves,

$$\begin{aligned}\Omega_+ &= 0\,,\\ \Omega_- &= \tfrac{1}{6}\Theta+\tfrac{1}{12}dx^+\wedge dx^-\wedge\Theta\,,\\ \Omega_i &= -\tfrac{1}{24}\Gamma_i\cdot dx^-\wedge\Theta+\tfrac{1}{8}dx^-\wedge\Theta\cdot\Gamma_i\,.\end{aligned}$$

For convenience, we denote the fraction of supersymmetry preserved by $\nu=\frac{1}{32}\dim\mathrm{Ker}\,\mathcal{D}$. For the generic Hpp solution with arbitrary A_{ij} and Θ one finds ([18]) that the solutions to the Killing spinor equation

$$\mathcal{D}\varepsilon=0 \tag{9.7}$$

take the form

$$\varepsilon(\psi_+)=\exp\left(\tfrac{1}{24}\Theta_{ijk}\Gamma^{ijk}\right)\psi_+\,, \tag{9.8}$$

where $\psi_+\in\mathrm{Ker}\,\Gamma_+$ is a constant spinor. In other words, for the generic Hpp solution, $\nu=\frac{1}{2}$.

There is, however, a special point in the moduli space with

$$\Theta = \mu dx^1\wedge dx^2\wedge dx^3\,, \tag{9.9}$$

$$A_{ij} = \begin{cases}-\frac{1}{9}\mu^2\delta_{ij} & i,j=1,2,3\\ -\frac{1}{36}\mu^2\delta_{ij} & i,j=4\ldots 9\end{cases} \tag{9.10}$$

which preserves *all* supersymmetry. The explicit expression for the Killing spinors of this background was given in [18]:

$$\begin{aligned}\varepsilon_{\psi_+,\psi_-}(x)=&\left(\cos\left(\frac{\mu}{4}x^-\right)\mathrm{Id}-\sin\left(\frac{\mu}{4}x^-\right)I\right)\psi_+\\ &+\left(\cos\left(\frac{\mu}{12}x^-\right)\mathrm{Id}-\sin\left(\frac{\mu}{12}x^-\right)I\right)\psi_-\\ &-\tfrac{1}{6}\mu\left(\sum_{i\le 3}x^i\Gamma_i-\tfrac{1}{2}\sum_{i\ge 4}x^i\Gamma_i\right)\left(\sin\left(\frac{\mu}{12}x^-\right)\mathrm{Id}-\cos\left(\frac{\mu}{12}x^-\right)I\right)\Gamma_+\psi_-\,,\end{aligned} \tag{9.11}$$

where $I=\Gamma_{123}$, $I^2=\mathrm{Id}$ and $\psi_\pm\in\mathrm{Ker}\,\Gamma_\pm$ are arbitrary constant spinors.

In addition to the maximally supersymmetric Hpp-solution and the generic $\frac{1}{2}$-BPS solution with arbitrary A, there are a number of other interesting loci in the Hpp-wave moduli space. In [55] Gauntlett and Hull constructed Hpp-solutions admitting "exotic" values of $\nu=\frac{9}{16},\frac{5}{8},\frac{11}{16},\frac{3}{4}$. These solutions possess Killing spinors that lie in $\mathrm{Ker}\,\Gamma_-$(often referred to as supernumerary Killing spinors) in addition to the "generic" ones given in (9.8). As we will see, in these cases the 3-form Θ takes a very particular form.

We briefly outline how one obtains the form of the Killing spinors in these cases. Since $\Omega_i\Omega_j = 0$ for all i, j, a Killing spinor ε can always be written in the form

$$\varepsilon_{\psi_+,\psi_-}(x) = \left(\mathrm{Id} + x^i\Omega_i\right)\varphi\,, \tag{9.12}$$

where $\Omega_i = -\frac{1}{24}(\Gamma_i\Theta + 3\Theta\Gamma_i)\Gamma_+$ and

$$\begin{aligned} \varphi_+ &= \exp\left(-\tfrac{1}{6}x^-\Theta\right)\psi_+\,, \\ \varphi_- &= \exp\left(-\tfrac{1}{12}x^-\Theta\right)\psi_-\,. \end{aligned} \tag{9.13}$$

As in the generic case, ψ_+ is an arbitrary constant spinor annihilated by Γ_+. However, now the $\psi_- \in \mathrm{Ker}_-$ are not arbitrary. Since Ω_i always involves Γ_+, substituting (9.12) into the Killing spinor equation (9.7) imposes no extra conditions on φ_+. But further analysis reveals that (9.7) can be split into independent x^- - and x^i - dependent parts, the former of which gives the form of φ_- and the latter can be written as

$$\left(-144\sum_j A_{jk}\Gamma_k + X_j^2\Gamma_j\right)\varphi_- = 0\,, \tag{9.14}$$

for each j, where $X_j = \Gamma_j\Theta\Gamma_j + 3\Theta$. Finding solutions with supernumerary Killing spinors amounts to finding solutions to this equation.

Let us assume that A has been brought to a diagonal form via an orthogonal transformation so that $A = \mathrm{diag}(\mu_1, \mu_2 \ldots \mu_9)$, $\mu_i \in \mathbb{R}$. Then in order to find solutions to equation (9.14), we must ensure that the action of X_j^2 on spinors is diagonalisable. Since X_i involves the 3-form Θ, the natural next step is to find a diagonalisable ansatz for Θ.

To proceed, we will utilise the following Lemma.

Lemma 43. *The Lie algebra of* $\mathrm{SO}(16)$ *is isomorphic to* $\Lambda^2\mathbb{R}^9 \oplus \Lambda^3\mathbb{R}^9 \simeq \mathfrak{so}(9) \oplus \Lambda^3\mathbb{R}^9$.

Sketch of proof. Representations of $\mathrm{C}\ell(9)$ are real and 16-dimensional, and thus there exists a map

$$c : \mathrm{C}\ell(9) \longrightarrow \mathrm{End}(\mathbb{R}^{16})$$

There is a $C\ell(9)$-invariant inner product with respect to which we can break up $\mathrm{End}(\mathbb{R}^{16})$ into skew-symmetric and symmetric endomorphisms[21]. The space of skew-symmetric endomorphisms of $\mathbb{R}^{16}$ is naturally isomorphic to $\mathfrak{so}(16)$.
As a vector space, $C\ell(9) \simeq \Lambda^* \mathbb{R}^9$, but since the volume form acts as $\pm\mathrm{Id}$, $c(\Lambda^p \mathbb{R}^9) = c(\Lambda^{(9-p)}\mathbb{R}^9)$. This means that as endomorphisms of $\mathbb{R}^{16}$,

$$C\ell(9) \simeq \mathbb{R} \oplus \mathbb{R}^9 \oplus \Lambda^2\mathbb{R}^9 \oplus \Lambda^3\mathbb{R}^9 \oplus \Lambda^4\mathbb{R}^9 .$$

It remains to be determined which of these components give rise to skew-symmetric endomorphisms with respect to the $C\ell(9)$-invariant inner product. This can be done e.g. by utilising an explicit matrix representation of $C\ell(9)$. We find that $\Lambda^2\mathbb{R}^9 \oplus \Lambda^3\mathbb{R}^9$ are skew under the inner product, giving $\mathfrak{so}(16) \simeq \Lambda^2\mathbb{R}^9 \oplus \Lambda^3\mathbb{R}^9$, i.e. the desired result. □

In particular, given a Cartan subalgebra $\mathfrak{c} \subset \mathfrak{so}(16)$ (generated by skew-diagonal matrices with real skew eigenvalues), there is a decomposition $\mathfrak{c} = \mathfrak{c}_2 \oplus \mathfrak{c}_3$, where $\mathfrak{c}_2 \subset \mathfrak{so}(9)$ and $\mathfrak{c}_3 \subset \bigwedge^3 \mathbb{R}^9$. Since the $\mathfrak{so}(16)$ has rank 8 and $\mathfrak{so}(9)$ has rank 4, we can associate $n \leq 4$ 2-form generators and $8-n$ 3-form generators to every Cartan subalgebra $\mathfrak{c}$ via the isomorphism.
Now obviously

$$\begin{aligned} [\mathfrak{c}_2, \mathfrak{c}_2] &\subset \mathfrak{c}_2 , \\ [\mathfrak{c}_2, \mathfrak{c}_3] &\subset \mathfrak{c}_3 , \end{aligned}$$

since elements of $\mathfrak{c}_2$ act as infinitesimal $SO(9)$-rotations. Note that this doesn't necessarily imply that $\mathfrak{c}_3$ is commutative.
If we further assume that $\mathfrak{c}_3$ is also a Cartan subalgebra (so that $[\mathfrak{c}_3, \mathfrak{c}_3] = 0$ and hence $[\mathfrak{c}, \mathfrak{c}_3] = 0$ as well), a direct calculation [56] shows that only cases that occur are $n = 1$ and $n = 3$, and a convenient choices for 2-form and 3-form generators in terms of gamma matrix monomials are

$$\Gamma_{12}, \Gamma_{34}, \Gamma_{56}, \Gamma_{78}$$

$$\Gamma_{129}, \Gamma_{349}, \Gamma_{569}, \Gamma_{789}$$

in the $n = 1$ case and

$$\Gamma_{78}$$

$$\Gamma_{123}, \Gamma_{145}, \Gamma_{167}, \Gamma_{246}, \Gamma_{257}, \Gamma_{347}, \Gamma_{356}$$

in the $n = 3$ case.

These two orbit types give rise to 3-form ansätze whose action on spinors can be diagonalised. The ansatz for Θ is then

$$\Theta = \alpha_1 dx^{129} + \alpha_2 dx^{349} + \alpha_3 dx^{569} + \alpha_4 dx^{789} \tag{9.15}$$

or

$$\Theta = \beta_1 dx^{123} + \beta_2 dx^{145} + \beta_3 dx^{167} + \beta_4 dx^{246} + \beta_5 dx^{257} + \beta_6 dx^{347} + \beta_7 dx^{356}, \tag{9.16}$$

where the α's and β's are real parameters. As pointed out in [57], if the α's are set to be equal, Θ is proportional to $dx^9 \wedge \omega$, where ω is the Kähler form on $\mathbb{R}^8$. Similarly, if the β's agree in the second ansatz, Θ is proportional to the G_2-invariant associative 3-form on $\mathbb{R}^7$. In both cases each of the three-form terms $\Gamma_{i_1 i_2 i_3}$ for $i_1, i_2, i_3 \in \{1\ldots 9\}$ is a real structure on the spinor bundle, so when diagonalised they act as $\pm\mathrm{Id}$. The skew eigenvalues of Θ in the four-parameter case are $\lambda_a, a = 1\ldots 8$, where

$$\begin{aligned}
\lambda_1 &= \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 \\
\lambda_2 &= \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 \\
\lambda_3 &= \alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 \\
\lambda_4 &= -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 \\
\lambda_5 &= -\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\
\lambda_6 &= \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 \\
\lambda_7 &= \alpha_1 - \alpha_2 + \alpha_3 + \alpha_4 \\
\lambda_8 &= -\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4
\end{aligned}$$

Similarly the skew eigenvalues in the seven-parameter case are given by $\lambda'_a, a =$

$1\ldots8$, where

$$
\begin{aligned}
\lambda_1' &= -\beta_1-\beta_2-\beta_3-\beta_4+\beta_5+\beta_6+\beta_7\\
\lambda_2' &= -\beta_1+\beta_2+\beta_3+\beta_4-\beta_5+\beta_6+\beta_7\\
\lambda_3' &= \beta_1+\beta_2-\beta_3-\beta_4-\beta_5-\beta_6+\beta_7\\
\lambda_4' &= \beta_1-\beta_2+\beta_3+\beta_4+\beta_5-\beta_6+\beta_7\\
\lambda_5' &= \beta_1-\beta_2+\beta_3-\beta_4-\beta_5+\beta_6-\beta_7\\
\lambda_6' &= \beta_1+\beta_2-\beta_3+\beta_4+\beta_5+\beta_6-\beta_7\\
\lambda_7' &= \beta_1+\beta_2+\beta_3-\beta_4+\beta_5-\beta_6-\beta_7\\
\lambda_8' &= -\beta_1-\beta_2-\beta_3+\beta_4-\beta_5-\beta_6-\beta_7
\end{aligned}
$$

Note that by choosing suitable α's or β's, some of the eigenvalues can be made to vanish, i.e. Θ can have a nontrivial kernel. Equation (9.8) then implies that the subspace of Killing spinors that lies in $\operatorname{Ker}\Theta$ will be independent of x^-.

We can now work out the possible A that can occur in these ansätze, using the procedure explained in [57]. In the 4-parameter case, $X_9 = 4\Theta$ acting on χ_-, and thus

$$\mu_9^2 = \tfrac{1}{9}\lambda_a^2 \tag{9.17}$$

for some choice of λ_a. That is, *a priori* we can choose φ_- to be any eigenspinor of Θ, and this choice in turn determines the rest of the μ_i. For example, consider the direction $i=1$. To determine μ_1, we need to solve the equation $(X_1-\kappa_1)\Gamma_1\varphi_- = 0$. Substituting $X_1 = \Gamma_1\Theta\Gamma_1 + 3\Theta$, we find that

$$\lambda_a\varphi_- + 3\Gamma_1\Theta\Gamma_1\varphi_- - \kappa_1\varphi_- = 0\,. \tag{9.18}$$

Looking at the form of Θ, it is easy to see that the eigenvalues of $\Gamma_1\Theta\Gamma_1$ obtained from those of Θ by reversing the signs of α_2, α_3 and α_4 (since Γ_1 anticommutes with these terms). Applying this to λ_a is sufficient to solve the previous equation. Following this procedure we can solve the rest of the μ_i. A similar argument works in the 7-parameter case: now $X_8 = X_9 = 2\Theta$. The possible metrics that can occur can be found in Appendix A. Note that in the four-parameter case $\mu_1^2 = \mu_2^2$, $\mu_3^2 = \mu_4^2$, $\mu_5^2 = \mu_6^2$, $\mu_7^2 = \mu_8^2$ and in the seven-parameter case $\mu_8^2 = \mu_9^2$.

The degeneracy of eigenspinors of Θ satisfying (9.14) gives the dimension of supernumerary Killing spinors. In the generic case where the coefficients of Θ are arbitrary there are 2 supernumerary Killing spinors, but there can be more if the coefficients are chosen so that some of the λ_a's or λ'_a's agree. The conditions for degeneracy are worked out in detail in [55].

In both cases the supernumerary Killing spinors are independent of x^- if and only if $\mu_9 = 0$. Furthermore, if one or more of the μ_i vanish, the Killing spinors will be independent of the corresponding transverse coordinates x^i. For these solutions the metric will be decomposable: the product of a lower-dimensional pp-wave with an Euclidean space.

9.3 Symmetric discrete quotients of Hpp-waves

Considering all possible quotients of Hpp-solutions by discrete subgroups of $I(M,g,F) \subset I(M,g)$ (the subgroup of the isometry group of M,g that also preserves the four-form F) is somewhat intractable since we have no classification of the crystallographic subgroups of Hpp-wave isometry groups at hand. Therefore, we will restrict ourselves to quotients that are also symmetric. It is known [58] that a quotient of a symmetric space $M = G/H$ (where G is a Lie group and H is the stabiliser subgroup of a point) by a discrete subgroup $D \subset I(M,g)$ is also symmetric if and only if D lies in the centraliser of $I(M,g)$ inside the transvection group G. In other words, we want to study quotients by discrete subgroups $D \subset Z$, where

$$Z = [x \in I(M,g) \mid xh = hx \; \forall h \in G] \,. \tag{9.19}$$

The isometries and conformal symmetries of Cahen-Wallach spaces were investigated by Cahen and Kerbrat in [59]. They also give expressions for the centralisers that can occur.

There are two possibilities:

Case 1. One of the eigenvalues $\mu_i > 0$ for some i, *or* $\frac{\mu_i}{\mu_j} \notin \mathbb{Q}$ for some (i, j). Then

$$Z \simeq \mathbb{R} = [\gamma_\alpha \mid \gamma_\alpha\left(x^+, x^-, x^i\right) = \left(x^+ + \alpha, x^-, x^i\right)] , \tag{9.20}$$

where $\alpha \in \mathbb{R}$.

Case 2. All the eigenvalues μ_i are negative and $\frac{\mu_i}{\mu_j} \in \mathbb{Q}$ for all (i, j). We write $\mu_i = -k_i^2$ for all i. Then

$$Z = [\gamma_{\alpha,\underline{m}} \mid \gamma_{\alpha,\underline{m}}\left(x^+, x^-, x^i\right) = \left(x^+ + \alpha, x^- + \beta, (-1)^{m_i} x^i\right)] , \tag{9.21}$$

where $m_i \in \mathbb{Z}$ and $\beta = \frac{\pi m_i}{k_i}$ for all i.

The ratio of m_i and k_i is the same for all i, and for any (i, j) we can write

$$m_i = \frac{k_i m_j}{k_j} . \tag{9.22}$$

The values of all m_i are determined by any one of them, so in fact $Z \simeq \mathbb{Z} \oplus \mathbb{R}$ in this case. Also note that for N to be orientable, $\sum_{i=1}^{9} m_i$ must be even: otherwise the volume form dvol $= dx^+ \wedge dx^- \wedge dx^1 \wedge \cdots \wedge dx^9$ would not be left invariant.

Qualitatively speaking, in all cases quotienting by the action of Z consists of periodic identifications of the light-cone coordinates and $\mathbb{Z}_2$-orbifoldings of the transverse coordinates. We observe that all the pp-wave solutions with supernumerary supersymmetries are examples of Case 2, provided that the ratios of the coefficients appearing in Θ are also rational.

We are also interested in spinors (in particular Killing spinors) that are left invariant by the quotient. In Case 2, a discrete subgroup $D_{\alpha,\underline{m}} \subset Z$ is generated by the elements $\gamma_{\alpha,0}$ and $\gamma_{0,\underline{m}}$. Their derivatives acting on the frame bundle are

$$\begin{aligned} d\gamma_{\alpha,0} &= \mathrm{Id}_{11\times 11} , \\ d\gamma_{0,k} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & (-1)^{m_1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & (-1)^{m_2} & 0 & \cdots & 0 \\ \vdots & 0 & 0 & 0 & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & (-1)^{m_9} \end{pmatrix} \end{aligned}$$

Now suppose that $m_{s_1}, m_{s_2} \ldots, m_{s_r}$ are the odd m_i. The fact that we want $D_{\alpha,\underline{m}}$ to preserve orientation means that r is even, as we mentioned previously. Then the quotient $N = M/D_{\alpha,\underline{m}}$ has four possible spin structures, corresponding to the two possible lifts of $d\gamma_{0,\underline{m}}$: $\Gamma_{0,\underline{m}} = \pm\Gamma_{s_1 s_2 \ldots s_r} \in \mathrm{Spin}(9)$ and $\Gamma_{\alpha,0} = \pm\mathrm{Id}$ [14]. Using these expressions and equation (9.3), we can then study the existence of invariant spinors explicitly.

It is easy to see that quotients of solutions for which Case 1 applies are rather trivial: there are only two possible spin structures and since generic Killing spinors do not depend on x^+, only the trivial lift of $\gamma_{\alpha,0}$ will preserve any (and in fact all) Killing spinors. Therefore, in the sequel we will focus on the solutions that admit supernumerary Killing spinors.

9.3.1 The maximally supersymmetric case

To begin with a simple example before studying the generaly supernumerary case in detail, let us analyse the symmetric discrete quotients of the maximally supersymmetric solution (9.9) and see which choices of spin structure preserve Killing spinors. Now A_{ij} is diagonal and all eigenvalues of are negative,so Case 2 above applies. To obtain a quotient isometric to (M, g, F) as a supergravity solution, we want to focus on a subgroup $Z_F \subset Z$ that also preserves the four-form F. Looking at the form of F in (9.9), we observe that an element $\gamma_{\alpha,\underline{m}} \in Z$ will preserve F if and only if *none* of (m_1, m_2, m_3) are odd or if *two* of them are: $\gamma_{\alpha,\underline{m}}$ acts trivially on dx^-. Now

$$k_i = \begin{cases} \frac{1}{3}\mu\,, i = 1\ldots 3 \\ \frac{1}{6}\mu\,, i = 4\ldots 9 \end{cases}$$

But since the k_i are equal for $i = 1\ldots 3$, equation (9.22) implies that $m_1 = m_2 = m_3$. Therefore, for $\gamma_{0,\underline{m}}$ to preserve F, $m_1 = m_2 = m_3 := 2k$ for some $k \in \mathbb{Z}$. Equation (9.22) also implies that $m_4 = \ldots = m_9 := k$. We conclude that

$$Z_F = [\gamma_{\alpha,k} \in Z \mid \gamma_{\alpha,k}(x^+, x^-, x^i) = \left(x^+ + \alpha, x^- + \beta, x^{1,2,3}, (-1)^k x^{4,\ldots,9}\right)]\,, \qquad (9.23)$$

where $\beta = \frac{6\pi k}{\mu}$ and $\alpha \in \mathbb{R}$.

A discrete subgroup $D_{\alpha,k}$ of Z_F is generated by elements $\gamma_{\alpha,0}$ and $\gamma_{0,1}$. Looking at the condition (9.3), it is again obvious that if $d\gamma_{\alpha,0}$ lifts to $-\mathrm{Id}$, no spinors will be left invariant. But provided that $\Gamma_{\alpha,0} = \mathrm{Id}$, there are two possible spin structures depending on the choice of sign for $\pm\Gamma_{0,1}$.

The derivative of $\gamma_{0,1}$ lifts to the spinor bundle as $\pm\Gamma_{4\ldots9}$. Using the familiar trigonometric identities $\cos\left(\frac{\pi}{2}-\theta\right) = -\sin\theta$ and $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$, we obtain

$$\begin{aligned}\varepsilon_{\psi_+,\psi_-}\left(\gamma_{(0,1)}(x)\right) = &\left(-\cos\left(\tfrac{\mu}{4}x^-\right)\mathrm{Id} + \sin\left(\tfrac{\mu}{4}x^-\right)I\right)\psi_+ \\ &+ \left(\cos\left(\tfrac{\mu}{4}x^-\right)I + \sin\left(\tfrac{\mu}{4}x^-\right)\mathrm{Id}\right)\psi_- \\ &+ \tfrac{1}{6}\mu\left(\sum_{i\le3} x^i\Gamma_i + \tfrac{1}{2}\sum_{\ge4} x^i\Gamma_i\right)\left(\cos\left(\tfrac{\mu}{4}x^-\right)\mathrm{Id} + \sin\left(\tfrac{\mu}{4}x^-\right)I\right)\Gamma_+\psi_- ,\end{aligned} \tag{9.24}$$

Comparing this expression with (9.8) and noting that $\Gamma_{0,1}$ anticommutes with Γ_i for $i = 4\ldots9$, we find that we can write this as

$$\varepsilon_{\psi_+,\psi_-}\left(\gamma_{0,1}(x)\right) = \varepsilon_{-I\psi_+,I\psi_-}(x) .$$

Thus, the action of $\gamma_{(0,1)}$ leaves ε invariant if

$$\begin{aligned}\Gamma_{0,1}\psi_+ &= -I\psi_+ , \\ \Gamma_{0,1}\psi_- &= I\psi_- ,\end{aligned}$$

Recall that we have chosen the spinor module for which the action of the centre of the Clifford algebra (and thus the volume element) agrees with $-\mathrm{Id}$. Then

$$\Gamma_{-+}\Gamma_{1\ldots9}\psi_+ = -\psi_+ , \tag{9.25}$$

implies that (since $\Gamma_{-+} = \Gamma_-\Gamma_+ + \mathrm{Id}$)

$$\Gamma_{4\ldots9}\psi_+ = I\psi_+ .$$

Correspondingly, $\Gamma_{4\ldots9}\psi_- = -I\psi_-$. Thus, equation (9.24) is satisfied if and only if $\Gamma_{0,1} = -\Gamma_{4\ldots9}$. We can therefore conclude that for all symmetric discrete quotients of the maximally supersymmetric Hpp-wave, out of the four possible choices of spin structure there is precisely one that preserves any (and indeed, all) supersymmetry.

9.3.2 The four-parameter case

Going through all possible quotients of supernumerary pp-wave solutions listed in Appendix A using the explicit form of the Killing spinors would be somewhat tedious, so instead we use a method that can be implemented more easily using a computer program (in our case, a *Mathematica* notebook).

To study the moduli space of possible quotients, we could take the coefficients of Θ as the data, allow them to vary and examine the consequences, as in [55]. However, for computational purposes it is actually more useful to take the 9-tuple $(m_1, \ldots, m_9)$ and the eigenvalue λ_{a_0} (where λ_{a_0} is the eigenvalue chosen to appear in equation (9.17), that is, $\mu_9 = \frac{1}{9}\lambda_{a_0}^2$) as the data defining a quotient. Looking at the different metrics appearing in Table A, we find that it is always possible to express the α's — and thus the eigenvalues λ_a — in terms of k_i (recall that the k_i are related to the eigenvalues of the matrix A by $\mu_i = -k_i^2$). In other words, we can write $\lambda_a = \sum_i c_i k_i$ for each λ_a and for some coefficients c_i. Given $(m_1, \ldots, m_9)$, we can use equation (9.22) to determine the k_i and hence the coefficients $\alpha_1, \ldots, \alpha_4$. Knowing $m_1 \ldots, m_9, \lambda_{a_0}$ for the quotient is thus sufficient to determine the original solution.

Restricting to Z_F, the subgroup of Z that also preserves the four-form F, we observe that m_9 must always be even. Using the equation (9.22) and the remarks in section 9.2, we also know that $m_1 = m_2$, $m_3 = m_4$, $m_5 = m_6$ and $m_7 = m_8$. To preserve orientation, $\sum_i m_i$ has to be even as well, but in this case this imposes no further restrictions, since there is always an even number of odd m_i.

It is convenient to express the Killing spinors using the eigenspinors of Θ as a basis. Note that acting on the λ_a-eigenspace, $J_{\lambda_a} = \frac{1}{\lambda_a}\Theta$ is a real structure. Thus, we can write the exponentials appearing in $\chi_\pm$ explicitly as

$$\chi_+ = \sum_{a=1}^{8}\left(\cosh\left(\tfrac{\lambda_a x^-}{4}\right) + \sinh\left(\tfrac{\lambda_a x^-}{4}\right)\right)\psi_+(\lambda_a)\,, \tag{9.26}$$

$$\chi_- = \left(\cosh\left(\tfrac{\lambda_{a_0} x^-}{12}\right) + \sinh\left(\tfrac{\lambda_{a_0} x^-}{12}\right)\right)\psi_+(\lambda_{a_0}) \tag{9.27}$$

where $\Theta \cdot \psi_\pm(\lambda_a) = i\lambda_a\psi_\pm(\lambda_a)$.

To determine the fraction of supersymmetry preserved by a symmetric discrete quotient, we need to work out how $\gamma_{0,\underline{m}}$ and $\Gamma_{0,\underline{m}}$ act on $\chi_\pm$. Recall that $\gamma_{0,\underline{m}}(x^-) = x^- + \frac{m_i}{k_i}\pi$ for some i. Computing the action of this shift on $\chi_\pm$ is straightforward. As mentioned above, we can express the α's — and thus the eigenvalues λ_a — in terms of k_i. Thus,

$$\lambda_a \frac{m_j}{k_j} = \sum_i c_i \frac{m_j k_i}{kj} \tag{9.28}$$

$$= \sum_i c_i m_i \,, \tag{9.29}$$

so under the action of the isometry, $\lambda_a x^- \to \lambda_a x^- + \pi \sum_i c_i m_i$. Using this observation and usual trigonometric identities, we can work out how the trigonometric functions in (9.26) transform under $\gamma_{0,\underline{m}}$.

We also need to know how $\Gamma_{0,\underline{m}}$ acts on the eigenspinors. Since we're taking the m_i to be our data, it is not hard to enumerate the possibilities. In the four-parameter case, each of m_1, m_3, m_4, m_5 and m_7 can be even or odd.

If we write the 3-form in this ansatz as $\Theta = \alpha_1 I_1 + \alpha_2 I_2 + \alpha_3 I_3 + \alpha_4 I_4$, we can express any λ as

$$\lambda = \epsilon_1(\lambda)\alpha_1 + \epsilon_2(\lambda)\alpha_2 + \epsilon_3(\lambda)\alpha_3 + \epsilon_4(\lambda)\alpha_4 \,, \tag{9.30}$$

where $I_p \psi_\pm(\lambda) = i\epsilon_p(\lambda)\psi_\pm(\lambda)$, $p = 1\ldots4$ and $\epsilon_p(\lambda) = \pm1$. It is not hard to see that any $\Gamma_{0,\underline{m}}$ can be written as a product of the I_p's or identified with such a product via the identity $\Gamma_{1\ldots9}\psi_\pm = \pm\psi$ that relates the Clifford action of a form on $\mathbb{R}^9$ to that of its dual. We can thus always write

$$\Gamma_{0,\underline{m}} = \epsilon(q) I_{p_1} I_{p_2} \cdot \ldots \cdot I_{p_q}$$

acting on ψ_+ and

$$\Gamma_{0,\underline{m}} = -\epsilon(q) I_{p_1} I_{p_2} \cdot \ldots \cdot I_{p_q}$$

acting on ψ_-, where $1 \le q \le 4$ and $\epsilon(q) = -1$ if $q = 1$ and 1 otherwise.

Since the action of each of the I_p on $\psi_\pm(\lambda)$ is fixed by equation (9.30), the action of $\Gamma_{0,\underline{m}}$ on $\psi_\pm(\lambda)$ is given by

$$\Gamma_{0,\underline{m}} \cdot \psi_\pm(\lambda) = \mp\epsilon(q)\epsilon_{p_1}\epsilon_{p_2}\ldots\epsilon_{p_q} i^q \psi_+(\lambda) \,,$$

With these observations, the problem becomes essentially algorithmic and can be easily implemented in a symbolic computation environment. We have written a *Mathematica* notebook[1] that goes through the elements $\gamma_{\alpha,\underline{m}}$ that generate discrete subgroups D and computes their action on the Killing spinors. It turns out that for every quotient, the result is similar to what occurs in the maximally supersymmetric case: out of the four possible spin structures, there is only one that preserves any of the original Killing spinors.

9.3.3 The seven-parameter case

The method we described in the previous section also works in the seven-parameter case, but now we must take care to ensure that the discrete subgroups D also preserve the four-form F. The most convenient way to express this condition is to require that for each term Γ_{ijk} appearing in Θ, $m_i + m_j + m_k$ must be even. In other words, we have the equations

$$\begin{aligned} m_1 + m_2 + m_3 &= 0 \\ m_1 + m_4 + m_5 &= 0 \\ m_1 + m_6 + m_7 &= 0 \\ m_2 + m_4 + m_6 &= 0 \\ m_2 + m_5 + m_7 &= 0 \\ m_3 + m_4 + m_7 &= 0 \\ m_3 + m_5 + m_6 &= 0 \,. \end{aligned}$$

modulo 2. This system of equations is not hard to solve over $\mathbb{Z}_2$, and thus we find that the possible forms that $\Gamma_{0,\underline{m}}$ can take are:

$$\pm\Gamma_{1247},\ \pm\Gamma_{1256},\ \pm\Gamma_{1346}\,,$$
$$\pm\Gamma_{1357},\ \pm\Gamma_{2345},\ \pm\Gamma_{2367},\ \pm\Gamma_{4567}\,.$$

[1] Available upon request from the author.

Again, we note that all these terms can be written as products of terms in Θ, and thus the method described in the previous section works, provided that we take the above constraints into account.

Examining all the possible quotients yields the same result as in the four-parameter case: for all possible quotients, there is only *one* choice of spin structure that preserves any (and indeed all) of the supersymmetry of the original background. Furthermore, we do not obtain any new fractions of supersymmetry in either case.

9.4 A conjecture

The results of the previous section lead to the curious observation that all symmetric quotients of known symmetric M-theory backgrounds with more than 16 Killing spinors possess a unique spin structure that preserves supersymmetry – in contrast to the supersymmetric space forms described in [52]. The only such backgrounds we haven't yet considered are Freund-Rubin -type solutions of the form AdS $\times S/\mathbb{Z}_2$, since the only symmetric spherical space form is the projective space[60], and it is easy to see that there is no ambiguity about the choice of spin structure in this case – this situation only arises if $|D| \geq 4$, where D is the discrete group used in the quotient[52]. Thus we arrive at a

Conjecture. All symmetric quotients of symmetric M-theory backgrounds for which $\nu > \frac{1}{2}$ possess a unique spin structure which preserves all of the original supersymmetry.

We now show that the requirement $\nu > \frac{1}{2}$ is in fact necessary.

Provided that we drop the requirement of supernumerary Killing spinors, it is not hard to exhibit examples of symmetric quotients of Hpp-waves that admit more than one spin structure preserving some Killing spinors. For example, consider a solution of the form

$$\Theta = \mu\left(dx^{129} + dx^{349}\right) \tag{9.31}$$

$$A_{ij} = \begin{cases} -\frac{\mu^2}{9}\delta_{ij},\ i,j = 1,\dots 4 \\ -\frac{\mu^2}{36}\delta_{ij},\ i,j = 5\dots 8 \\ -\frac{4\mu^2}{9},\ i = j = 9\,. \end{cases} \tag{9.32}$$

The solution is obtained by taking a solution preserving 24 supersymmetries given by taking $\alpha_1 = \alpha_2$, $\alpha_3 = \alpha_4 = 0$ in the four-parameter case and permuting the values of the k_i: equation (9.14) is no longer satisfied, and thus this solution only admits 16 supersymmetries.

Let us analyse the centraliser of the isometry group. Now $k_1 = k_2 = k_3 = k_4 = \frac{\mu}{3}$, $k_5 = \dots = k_8 = \frac{\mu}{6}$ and $k_9 = \frac{2\mu}{3}$. Equation (9.22) then implies that $m_1 = \dots = m_4$ and $m_5 = \dots = m_8 := m$. Moreover, $m_1 = 2m_5$ and $m_9 = 4m_5$: in other words, $m_1, \dots m_4$ and m_9 will always be even, and since they correspond to the transverse directions that appear in the form of Θ, all elements of Z will preserve F as well. Thus, Z is of the form

$$Z = [\gamma_{\alpha,k} \in Z \mid \gamma_{\alpha,k}(x^+, x^-, x^i) = (x^+ + \alpha, x^- + \beta, x^{1,2,3,4}, (-1)^k x^{5,\dots,8}, x^9)]\,. \tag{9.33}$$

Again, Z is generated by $\gamma_{\alpha,0}$ and $\gamma_{0,1}$. These elements lift to the spinor bundle as $\Gamma_{\alpha,0} = \pm\mathrm{Id}$ and $\Gamma_{0,1} = \pm\Gamma_{5678}$. The Killing spinors are of the form given in equation (9.13). We could, of course, use the method described in the previous section and decompose the spinorial parameter ψ_+ into eigenspinors of Θ and work out precisely how $\Gamma_{0,1}$ acts on them, but it is sufficient to observe that (9.3) becomes

$$\varepsilon_{\psi_+}\left(\gamma_{0,1}(x)\right) = \varepsilon_{\Gamma_{1234}\psi_+}(x)\,.$$

For this equation to be satisfied we must have $\Gamma_{5678}\psi_+ = \pm\Gamma_{1234}\psi_+$. In other words, ψ_+ must lie in the $\pm$-eigenspaces of $\Gamma_{12\dots8}$, depending on which lift of $\gamma_{0,1}$ we choose. Half of the ψ_+ satisfy this additional condition; $\Gamma_{12\dots8}$ commutes with X_i^2, so demanding that its eigenspinors belong to the $\pm$-eigenspace of $\Gamma_{12\dots8}$ is an independent constraint.

We conclude that out of the four possible spin structures on the quotient, two admit no Killing spinors and two preserve 8 of the original sixteen supersymmetries, thus showing that the inequality in our conjecture must be sharp.

Chapter 10

Conclusions

Heitän kirjan luotani
Ennen kuin halu antautua
Valoa nopeamman matkustamisen
Pohdiskelemiseen kokopäiväisesti
Käy ylitsepääsemättömäksi

I cast the book away
Before the desire to be consumed
By thoughts
Of faster than light travel
Becomes unbearable

—*A. W. Yrjänä,* Tesserakti

In this thesis we have explored a natural conformal invariant associated to a semi-Riemannian spin manifold: the conformal Killing superalgebra. We constructed this object from first principles in a manifestly Weyl-invariant way and were naturally led to introduce a spinorial Lie derivative. However, we were forced to conclude that the resulting object is not in general a Lie superalgebra.

We have also seen that it is possible to generalise the concept of conformal Killing spinors to eleven-dimensional supergravity and other supergravity theories. We have also singled out a subspace of conformal Killing vectors of supergravity backgrounds — the *supernormal* conformal Killing vectors — that can be used alongside the conformal Killing spinors to construct a supergravity conformal Killing superalgebra. We showed that M-theory backgrounds that admit a supergravity conformal Killing spinor distinct from Killing spinors and geometric conformal Killing spinors must be of a very particular type: the metric must be one of the Bryant metrics and the four-form must satisfy a strong integrability condition. We have also exhausted one possible class of examples, namely the supersymmetric Hpp-wave solutions of M-theory. Nevertheless, we were able to find examples of supergravity conformal Killing superalgebras in type IIA and the HLW massive IIA supergravities, via Kaluza-Klein reduction and homothetic Kaluza-Klein reduction, respectively.

Finally — deviating slightly from the main line of development — we saw that as in the geometric conformal Killing spinor case, there is a relationship between the dimension of the space of Killing spinors of a non-simply connected M-theory background and its spin structure. In particular, we examined the symmetric discrete quotients of all the known symmetric M-theory backgrounds with more than 16 Killing spinors. In all cases, we found that there is a unique spin structure that preserves all of the original supersymmetry.

All the three threads in this thesis provide ample material for future work. For example, it would be interesting to study the interplay between spin structure, symmetry anda supersymmetry and find a formal proof of the conjecture presented in section 9.4. Failing that, as we mentioned in section 9.2, the known Hpp-wave solutions with supernumerary supersymmetries are very special and a more careful study of the moduli space of these solutions might reveal loci for which the matrix A is not diagonal but which still admit more than 16 Killing spinors. It would be interesting to see if these solutions could provide counterexamples to Conjecture 9.4.

While we have met our primary goal, we appear to have failed to provide Nahm's superconformal algebras with a geometric realisation. The algebras on Nahm's list [2] certainly are Lie superalgebras, but in general conformal Killing superalgebras are not. This is somewhat puzzling since in analogue with the Killing supersymmetry algebra case (as mentioned in the introduction), one would expect at least *some* of Nahm's algebras to have a geometric orgin.

We now make a few speculative remarks to explain why this failure occurs. Namely, some of the superconfomal algebras appear to have a component which has no direct geometric analogue, namely the so-called *R-symmetry.* The R-symmetry generators commute with the even part of the algebra but act nontrivially on the odd part. R-symmetry does not seem to arise naturally in the context of conformal Killing superalgebras, and it is likely that it is this fact that is responsible for the failure of the odd-odd-odd Jacobi identity (5.18) to vanish.

An example that occurs in the context of so-called AdS/CFT duality [61] provides some hints as to how one might hope to remedy the situation. It is widely believed that type IIB supergravity on the Freund-Rubin background $AdS_5 \times S^5$ is dual to a conformal field theory that admits a superconformal symmetry algebra living on the conformal boundary of AdS_5 — that is, $\mathbb{R}^3 \times S^1$. In the IIB setting, the supersymmetries of the theory correspond to supergravity Killing spinors on $AdS_5 \times S^5$. In the Freund-Rubin ansatz, these are actually tensor products of *geometric* Killing spinors on AdS_5 and S^5 [61, 62]. The generators of the odd part of the field theory superconformal algebra correspond to the conformal Killing spinors of the boundary, which are geometric Killing spinors from the AdS_5 point of view. The natural action of $\mathfrak{so}(6)$ (the isometry algebra of S^5) on the S^5 part of the IIB Killing spinors then induces the R-symmetry in the four-dimensional CFT. Obviously, the $\mathfrak{so}(6)$-action commutes with the action of isometries of AdS_5, the latter generating the conformal symmetries of the field theory superalgebra.

Motivated by this example, one might imagine making a conformal Killing superalgebra into a Lie superalgebra by adding a central extension to the even part and tensoring the odd part with the (bundle of) appropriate representations. Consider a superalgebra $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ which is not a Lie superalgebra but satisfies all Jacobi identities apart from the odd-odd-odd one: that is, we have a nonzero map

$$\mathcal{J} : S^3\mathfrak{h}_1 \to \mathfrak{h}_1 \tag{10.1}$$

defined by the fourth Jacobi identity. Now let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{h}_0 \times \mathfrak{k}_0$ and $\mathfrak{g}_1 = \mathfrak{h}_1 \otimes V$, where V is a $\mathfrak{k}_0$-module and $[\mathfrak{h}_0, \mathfrak{k}_0] = 0$ and furthermore now the Jacobi map $J : S^3\mathfrak{g}_1 \to \mathfrak{g}_1$ defined by the fourth Jacobi identity vanishes so that $\mathfrak{g}$ is a Lie superalgebra. In this construction, $\mathfrak{k}_0$ plays the role of R-symmetry. Unfortunately, given the above assumptions for $\mathfrak{h}$, there does not appear to be a general recipe for carrying out this construction as $\mathfrak{k}_0$ and V have to be put in by hand.

We note that there is a geometric formalism in which one extends M to a *supermanifold* by introducing fermionic coordinates [63]. It is then possible [64, 65] to realise some of Nahm's superconformal algebras directly as algebras of superisometries on the superspace: the R-symmetries then correspond to rotations of the fermionic coordinates. However, introducing supermanifold formalism and associated machinery is beyond the scope of the present treatment.

In spite of the R-symmetry problem, we have presented a variety of conformal Killing superalgebras in this thesis with the hope that they could perhaps be extended to Lie superalgebras using the procedure outlined above.

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Appendix A

Metrics of pp-waves with supernumerary supersymmetries

$\mu_8 = \mu_9$	$\mu_i,\ i = 1\ldots 7$
$-\frac{1}{36}(-\beta_1-\beta_2-\beta_3-\beta_4+\beta_5+\beta_6+\beta_7)^2$	$\mu_1 = -\frac{1}{36}(-2\beta_1-2\beta_2-2\beta_3+\beta_4-\beta_5-\beta_6-\beta_7)^2$ $\mu_2 = -\frac{1}{36}(-2\beta_1+\beta_2+\beta_3-2\beta_4+2\beta_5-\beta_6-\beta_7)^2$ $\mu_3 = -\frac{1}{36}(-2\beta_1+\beta_2+\beta_3+\beta_4-\beta_5+2\beta_6+2\beta_7)^2$ $\mu_4 = -\frac{1}{36}(\beta_1-2\beta_2+\beta_3-2\beta_4-\beta_5+2\beta_6-\beta_7)^2$ $\mu_5 = -\frac{1}{36}(\beta_1-2\beta_2+\beta_3+\beta_4+2\beta_5-\beta_6+2\beta_7)^2$ $\mu_6 = -\frac{1}{36}(\beta_1+\beta_2-2\beta_3-2\beta_4-\beta_5-\beta_6+2\beta_7)^2$ $\mu_7 = -\frac{1}{36}(\beta_1+\beta_2-2\beta_3+\beta_4+2\beta_5+2\beta_6-\beta_7)^2$
$-\frac{1}{36}(-\beta_1+\beta_2+\beta_3+\beta_4-\beta_5+\beta_6+\beta_7)^2$	$\mu_1 = -\frac{1}{36}(-2\beta_1+2\beta_2+2\beta_3-\beta_4+\beta_5-\beta_6-\beta_7)^2$ $\mu_2 = -\frac{1}{36}(-2\beta_1-\beta_2-\beta_3+2\beta_4-2\beta_5-\beta_6-\beta_7)^2$ $\mu_3 = -\frac{1}{36}(-2\beta_1-\beta_2-\beta_3-\beta_4+\beta_5+2\beta_6+2\beta_7)^2$ $\mu_4 = -\frac{1}{36}(\beta_1+2\beta_2-\beta_3+2\beta_4+\beta_5+2\beta_6-\beta_7)^2$ $\mu_5 = -\frac{1}{36}(\beta_1+2\beta_2-\beta_3-\beta_4-2\beta_5-\beta_6+2\beta_7)^2$ $\mu_6 = -\frac{1}{36}(\beta_1-\beta_2+2\beta_3+2\beta_4+\beta_5-\beta_6+2\beta_7)^2$ $\mu_7 = -\frac{1}{36}(\beta_1-\beta_2+2\beta_3-\beta_4-2\beta_5+2\beta_6-\beta_7)^2$
$-\frac{1}{36}(\beta_1+\beta_2-\beta_3-\beta_4-\beta_5-\beta_6+\beta_7)^2$	$\mu_1 = -\frac{1}{36}(2\beta_1+2\beta_2-2\beta_3+\beta_4+\beta_5+\beta_6-\beta_7)^2$ $\mu_2 = -\frac{1}{36}(2\beta_1-\beta_2+\beta_3-2\beta_4-2\beta_5+\beta_6-\beta_7)^2$ $\mu_3 = -\frac{1}{36}(2\beta_1-\beta_2+\beta_3+\beta_4+\beta_5-2\beta_6+2\beta_7)^2$ $\mu_4 = -\frac{1}{36}(-\beta_1+2\beta_2+\beta_3-2\beta_4+\beta_5-2\beta_6-\beta_7)^2$ $\mu_5 = -\frac{1}{36}(-\beta_1+2\beta_2+\beta_3+\beta_4-2\beta_5+\beta_6+2\beta_7)^2$ $\mu_6 = -\frac{1}{36}(-\beta_1-\beta_2-2\beta_3-2\beta_4+\beta_5+\beta_6+2\beta_7)^2$ $\mu_7 = -\frac{1}{36}(-\beta_1-\beta_2-2\beta_3+\beta_4-2\beta_5-2\beta_6-\beta_7)^2$
$-\frac{1}{36}(\beta_1-\beta_2+\beta_3+\beta_4+\beta_5-\beta_6+\beta_7)^2$	$\mu_1 = -\frac{1}{36}(2\beta_1-2\beta_2+2\beta_3-\beta_4-\beta_5+\beta_6-\beta_7)^2$ $\mu_2 = -\frac{1}{36}(2\beta_1+\beta_2-\beta_3+2\beta_4+2\beta_5+\beta_6-\beta_7)^2$ $\mu_3 = -\frac{1}{36}(2\beta_1+\beta_2-\beta_3-\beta_4-\beta_5-2\beta_6+2\beta_7)^2$

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$\mu_8 = \mu_9$	$\mu_i,\ i = 1\ldots 7$
	$\mu_4 = -\frac{1}{36}(-\beta_1 - 2\beta_2 - \beta_3 + 2\beta_4 - \beta_5 - 2\beta_6 - \beta_7)^2$ $\mu_5 = -\frac{1}{36}(-\beta_1 - 2\beta_2 - \beta_3 - \beta_4 + 2\beta_5 + \beta_6 + 2\beta_7)^2$ $\mu_6 = -\frac{1}{36}(-\beta_1 + \beta_2 + 2\beta_3 + 2\beta_4 - \beta_5 + \beta_6 + 2\beta_7)^2$ $\mu_7 = -\frac{1}{36}(-\beta_1 + \beta_2 + 2\beta_3 - \beta_4 + 2\beta_5 - 2\beta_6 - \beta_7)^2$
$-\frac{1}{36}(\beta_1 - \beta_2 + \beta_3 - \beta_4 - \beta_5 + \beta_6 - \beta_7)^2$	$\mu_1 = -\frac{1}{36}(2\beta_1 - 2\beta_2 + 2\beta_3 + \beta_4 + \beta_5 - \beta_6 + \beta_7)^2$ $\mu_2 = -\frac{1}{36}(2\beta_1 + \beta_2 - \beta_3 - 2\beta_4 - 2\beta_5 - \beta_6 + \beta_7)^2$ $\mu_3 = -\frac{1}{36}(2\beta_1 + \beta_2 - \beta_3 + \beta_4 + \beta_5 + 2\beta_6 - 2\beta_7)^2$ $\mu_4 = -\frac{1}{36}(-\beta_1 - 2\beta_2 - \beta_3 - 2\beta_4 + \beta_5 + 2\beta_6 + \beta_7)^2$ $\mu_5 = -\frac{1}{36}(-\beta_1 - 2\beta_2 - \beta_3 + \beta_4 - 2\beta_5 - \beta_6 - 2\beta_7)^2$ $\mu_6 = -\frac{1}{36}(-\beta_1 + \beta_2 + 2\beta_3 - 2\beta_4 + \beta_5 - \beta_6 - 2\beta_7)^2$ $\mu_7 = -\frac{1}{36}(-\beta_1 + \beta_2 + 2\beta_3 + \beta_4 - 2\beta_5 + 2\beta_6 + \beta_7)^2$
$-\frac{1}{36}(\beta_1 + \beta_2 - \beta_3 + \beta_4 + \beta_5 + \beta_6 - \beta_7)^2$	$\mu_1 = -\frac{1}{36}(2\beta_1 + 2\beta_2 - 2\beta_3 - \beta_4 - \beta_5 - \beta_6 + \beta_7)^2$ $\mu_2 = -\frac{1}{36}(2\beta_1 - \beta_2 + \beta_3 + 2\beta_4 + 2\beta_5 - \beta_6 + \beta_7)^2$ $\mu_3 = -\frac{1}{36}(2\beta_1 - \beta_2 + \beta_3 - \beta_4 - \beta_5 + 2\beta_6 - 2\beta_7)^2$ $\mu_4 = -\frac{1}{36}(-\beta_1 + 2\beta_2 + \beta_3 + 2\beta_4 - \beta_5 + 2\beta_6 + \beta_7)^2$ $\mu_5 = -\frac{1}{36}(-\beta_1 + 2\beta_2 + \beta_3 - \beta_4 + 2\beta_5 - \beta_6 - 2\beta_7)^2$ $\mu_6 = -\frac{1}{36}(-\beta_1 - \beta_2 - 2\beta_3 + 2\beta_4 - \beta_5 - \beta_6 - 2\beta_7)^2$ $\mu_7 = -\frac{1}{36}(-\beta_1 - \beta_2 - 2\beta_3 - \beta_4 + 2\beta_5 + 2\beta_6 + \beta_7)^2$
$-\frac{1}{36}(-\beta_1 + \beta_2 + \beta_3 - \beta_4 + \beta_5 - \beta_6 - \beta_7)^2$	$\mu_1 = -\frac{1}{36}(-2\beta_1 + 2\beta_2 + 2\beta_3 + \beta_4 - \beta_5 + \beta_6 + \beta_7)^2$ $\mu_2 = -\frac{1}{36}(-2\beta_1 - \beta_2 - \beta_3 - 2\beta_4 + 2\beta_5 + \beta_6 + \beta_7)^2$ $\mu_3 = -\frac{1}{36}(-2\beta_1 - \beta_2 - \beta_3 + \beta_4 - \beta_5 - 2\beta_6 - 2\beta_7)^2$ $\mu_4 = -\frac{1}{36}(\beta_1 + 2\beta_2 - \beta_3 - 2\beta_4 - \beta_5 - 2\beta_6 + \beta_7)^2$ $\mu_5 = -\frac{1}{36}(\beta_1 + 2\beta_2 - \beta_3 + \beta_4 + 2\beta_5 + \beta_6 - 2\beta_7)^2$ $\mu_6 = -\frac{1}{36}(\beta_1 - \beta_2 + 2\beta_3 - 2\beta_4 - \beta_5 + \beta_6 - 2\beta_7)^2$ $\mu_7 = -\frac{1}{36}(\beta_1 - \beta_2 + 2\beta_3 + \beta_4 + 2\beta_5 - 2\beta_6 + \beta_7)^2$

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$\mu_8 = \mu_9$	$\mu_i,\ i = 1\ldots 7$
$-\frac{1}{36}(\beta_1+\beta_2+\beta_3-\beta_4+\beta_5+\beta_6+\beta_7)^2$	$\mu_1 = -\frac{1}{36}(2\beta_1+2\beta_2+2\beta_3+\beta_4-\beta_5-\beta_6-\beta_7)^2$ $\mu_2 = -\frac{1}{36}(2\beta_1-\beta_2-\beta_3-2\beta_4+2\beta_5-\beta_6-\beta_7)^2$ $\mu_3 = -\frac{1}{36}(2\beta_1-\beta_2-\beta_3+\beta_4-\beta_5+2\beta_6+2\beta_7)^2$ $\mu_4 = -\frac{1}{36}(-\beta_1+2\beta_2-\beta_3-2\beta_4-\beta_5+2\beta_6-\beta_7)^2$ $\mu_5 = -\frac{1}{36}(-\beta_1+2\beta_2-\beta_3+\beta_4+2\beta_5-\beta_6+2\beta_7)^2$ $\mu_6 = -\frac{1}{36}(-\beta_1-\beta_2+2\beta_3-2\beta_4-\beta_5-\beta_6+2\beta_7)^2$ $\mu_7 = -\frac{1}{36}(-\beta_1-\beta_2+2\beta_3+\beta_4+2\beta_5+2\beta_6-\beta_7)^2$

Table A.1: Metrics associated to different eigenvalues of Θ for the 7-parameter ansatz.

μ_9	$\mu_i, i = 1\ldots8$
$-\frac{1}{9}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^2$	$\mu_1=\mu_2=-\frac{1}{36}(\alpha_1-\alpha_2-\alpha_3-\alpha_4)^2$
	$\mu_3=\mu_4=-\frac{1}{36}(\alpha_1-2\alpha_2+\alpha_3+\alpha_4)^2$
	$\mu_5=\mu_6=-\frac{1}{36}(\alpha_1+\alpha_2-2\alpha_3+\alpha_4)^2$
	$\mu_7=\mu_8=-\frac{1}{36}(\alpha_1+\alpha_2+\alpha_3-2\alpha_4)^2$
$-\frac{1}{9}(-\alpha_1-\alpha_2+\alpha_3-\alpha_4)^2$	$\mu_1=\mu_2=-\frac{1}{36}(2\alpha_1-\alpha_2+\alpha_3-\alpha_4)^2$
	$\mu_3=\mu_4=-\frac{1}{36}(\alpha_1-2\alpha_2-\alpha_3+\alpha_4)^2$
	$\mu_5=\mu_6=-\frac{1}{36}(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)^2$
	$\mu_7=\mu_8=-\frac{1}{36}(\alpha_1+\alpha_2-\alpha_3-2\alpha_4)^2$
$-\frac{1}{9}(\alpha_1+\alpha_2-\alpha_3-\alpha_4)^2$	$\mu_1=\mu_2=-\frac{1}{36}(2\alpha_1-\alpha_2+\alpha_3+\alpha_4)^2$
	$\mu_3=\mu_4=-\frac{1}{36}(\alpha_1-2\alpha_2-\alpha_3-\alpha_4)^2$
	$\mu_5=\mu_6=-\frac{1}{36}(\alpha_1+\alpha_2+2\alpha_3-\alpha_4)^2$
	$\mu_7=\mu_8=-\frac{1}{36}(\alpha_1+\alpha_2-\alpha_3+2\alpha_4)^2$
$-\frac{1}{9}(\alpha_1+\alpha_2+\alpha_3-\alpha_4)^2$	$\mu_1=\mu_2=-\frac{1}{36}(2\alpha_1-\alpha_2-\alpha_3+\alpha_4)^2$
	$\mu_3=\mu_4=-\frac{1}{36}(\alpha_1-2\alpha_2+\alpha_3-\alpha_4)^2$
	$\mu_5=\mu_6=-\frac{1}{36}(\alpha_1+\alpha_2-2\alpha_3-\alpha_4)^2$
	$\mu_7=\mu_8=-\frac{1}{36}(\alpha_1+\alpha_2+\alpha_3+2\alpha_4)^2$
$-\frac{1}{9}(-\alpha_1+\alpha_2+\alpha_3+\alpha_4)^2$	$\mu_1=\mu_2=-\frac{1}{36}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^2$
	$\mu_3=\mu_4=-\frac{1}{36}(\alpha_1+2\alpha_2-\alpha_3-\alpha_4)^2$
	$\mu_5=\mu_6=-\frac{1}{36}(\alpha_1-\alpha_2+2\alpha_3-\alpha_4)^2$
	$\mu_7=\mu_8=-\frac{1}{36}(\alpha_1-\alpha_2-\alpha_3+2\alpha_4)^2$
$-\frac{1}{9}(\alpha_1-\alpha_2-\alpha_3+\alpha_4)^2$	$\mu_1=\mu_2=-\frac{1}{36}(2\alpha_1+\alpha_2+\alpha_3-\alpha_4)^2$
	$\mu_3=\mu_4=-\frac{1}{36}(\alpha_1+2\alpha_2-\alpha_3+\alpha_4)^2$
	$\mu_5=\mu_6=-\frac{1}{36}(\alpha_1-\alpha_2+2\alpha_3+\alpha_4)^2$
	$\mu_7=\mu_8=-\frac{1}{36}(\alpha_1-\alpha_2-\alpha_3-2\alpha_4)^2$
$-\frac{1}{9}(\alpha_1-\alpha_2+\alpha_3+\alpha_4)^2$	$\mu_1=\mu_2=-\frac{1}{36}(2\alpha_1+\alpha_2-\alpha_3-\alpha_4)^2$
	$\mu_3=\mu_4=-\frac{1}{36}(-\alpha_1-2\alpha_2-\alpha_3-\alpha_4)^2$
	$\mu_5=\mu_6=-\frac{1}{36}(-\alpha_1+\alpha_2+2\alpha_3-\alpha_4)^2$
	$\mu_7=\mu_8=-\frac{1}{36}(-\alpha_1+\alpha_2-\alpha_3+2\alpha_4)^2$
$-\frac{1}{9}(-\alpha_1+\alpha_2-\alpha_3+\alpha_4)^2$	$\mu_1=\mu_2=-\frac{1}{36}(-2\alpha_1-\alpha_2+\alpha_3-\alpha_4)^2$
	$\mu_3=\mu_4=-\frac{1}{36}(\alpha_1+2\alpha_2+\alpha_3-\alpha_4)^2$
	$\mu_5=\mu_6=-\frac{1}{36}(\alpha_1-\alpha_2-2\alpha_3-\alpha_4)^2$
	$\mu_7=\mu_8=-\frac{1}{36}(\alpha_1-\alpha_2+\alpha_3+2\alpha_4)^2$

Table A.2: Metrics associated to different eigenvalues of Θ for the 4-parameter ansatz