Realizability and Recursive Mathematics

David Charles McCarty
Department of Computer Science

and

Centre for Cognitive Science
University of Edinburgh

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Abstract

Section 1: Philosophy, logic and constructivity

Philosophy, formal logic and the theory of computation all bear on problems in the foundations of constructive mathematics. There are few places where these, often competing, disciplines converge more neatly than in the theory of realizability structures. Realizability applies recursion-theoretic concepts to give interpretations of constructivism along lines suggested originally by Heyting and Kleene. The research reported in the dissertation revives the original insights of Kleene—by which realizability structures are viewed as models rather than proof-theoretic interpretations—to solve a major problem of classification and to draw mathematical consequences from its solution.

Section 2: Intuitionism and recursion: the problem of classification

The internal structure of constructivism presents an interesting problem. Mathematically, it is a problem of classification; for philosophy, it is one of conceptual organization. Within the past seventy years, constructive mathematics has grown into a jungle of fully-developed "constructivities," approaches to the mathematics of the calculable which range from strict finitism through hyperarithmetic model theory. The problem we address is taxonomic: to sort through the jungle, set standards for classification and determine those features which run through everything that is properly "constructive."

There are two notable approaches to constructivity; these must appear prominently in any proposed classification. The most famous is Brouwer's *intuitionism.* Intuitionism relies on a complete constructivization of the basic mathematical objects and logical operations. The other is *classical recursive mathematics*, as represented by the work of Dekker, Myhill, and Nerode. Classical constructivists use standard logic in a mathematical universe restricted to coded objects and recursive operations.

The theorems of the dissertation give a precise answer to the classification problem for intuitionism and classical constructivism. Between these realms are connected semantically through a model of intuitionistic set theory. The intuitionistic set theory IZF encompasses all of the intuitionistic mathematics that does not involve choice sequences. (This includes all the work of the Bishop school.) IZF has as a model a recursion-theoretic structure, $V(Kl)$, based on Kleene realizability. Since realizability takes set variables to range over "effective" objects, large parts of classical constructivism appear over the model as interpreted subsystems of intuitionistic set theory. For example, the *entire* first-order classical theory of recursive cardinals and ordinals comes out as an *intuitionistic* theory of cardinals and ordinals under realizability. In brief, we prove that a satisfactory partial solution to the classification problem exists; theories in classical recursive constructivism are identical, under a natural interpretation, to intuitionistic theories. The interpretation is especially satisfactory because it is *not* a Gödel-style translation; the interpretation can be developed so that it leaves the classical logical forms unchanged.

Section 3: Mathematical applications of the translation

The solution to the classification problem is a bridge capable of carrying two-way mathematical traffic. In one direction, an identification of classical constructivism with intuitionism yields a certain *elimination of recursion theory* from the standard mathematical theory of effective structures, leaving pure set theory and a bit of model theory. Not only are the theorems of classical effective mathematics faithfully represented in intuitionistic set theory, but also the arguments that provide proofs of those theorems. Via realizability, one can find set-theoretic proofs of many effective results, and the set-theoretic proofs are often more straightforward than their recursion-theoretic counterparts. The new proofs are also more transparent, because they involve, rather than recursion theory plus set theory, at most the set-theoretic "axioms" of effective mathematics.

Working the other way, many of the negative ("cannot be obtained recursively") results of classical constructivism carry over immediately into strong independence results from intuitionism. The theorems of Kalantari and Retzlaff on effective topology, for instance, turn into independence proofs concerning the structure of the usual topology on the intuitionistic reals.

The realizability methods that shed so much light over recursive set theory can be applied to "recursive theories" generally. We devote a chapter to verifying that the realizability techniques can be used to good effect in the semantical foundations of computer science. The classical theory of effectively given computational domains à la Scott can be subsumed into the Kleene realizability universe as a species of countable noneffective domains. In this way, the theory of effective domains becomes a chapter (under interpretation) in an intuitionistic study of denotational semantics. We then show how the "extra information" captured in the logical signs under realizability can be used to give proofs of classical theorems about effective domains.

Section 4: Solutions to metamathematical problems

The realizability model for set theory is very tractible; in many ways, it resembles a Boolean-valued universe. The tractibility is apparent in the solutions it offers to a number of open problems in the metamathematics of constructivity. First, there is the perennial problem of finding and delimiting in the wide constructive universe those features that correspond to structures familiar from classical mathematics. In the realizability model, it is easy to locate the collection of classical ordinals and to show that they form, intuitionistically, a set rather than a proper class. Also, one interprets an argument of Dekker and Myhill to prove that the classical powerset of the natural numbers contains at least continuum-many distinct cardinals.

Second, a major tenet of Bishop's program for constructivity has been that constructive mathematics is "numerical:" all the properties of constructive objects, including the real numbers, can be represented as properties of the natural numbers. The realizability model shows that Bishop's numericalization of mathematics can, in principle, be accomplished. Every set over the model with decidable equality and every metric space is enumerated by a collection of natural numbers.

Introduction

Section 1: Recursive mathematics and realizability

What we would now call "recursive mathematics" was in attendance at the birth of recursion theory, although only in a very exclusionary way. If recursive mathematics is a field of mathematics all of whose objects and all of whose basic morphisms are computable, then the early undecidability results, for arithmetic and logical validity, can be viewed as setting limits to that field, as long as computability is understood as recursivity. The undecidability theorems tell us that certain mathematical operations are demonstrably excluded from the realm of recursive mathematics. The undecidability theorems show that functions which occur naturally in metamathematics, for instance, the characteristic function of the set of arithmetic truths, are not computable by Turing machines. Were we to look further on in the history, we would find even better examples of the exclusionary force of undecidability theorems; the undecidability of the word problem for groups keeps certain algebraic operations from falling within the limits of recursive mathematics.

Once undecidability results and the allied techniques for working with recursive functions had been discovered, it was natural to ask how much mathematics could be accomplished within the limits and to wonder what its character might be. More precisely, one asks, "If recursive mathematics directs primary attention to those domains which can be coded numerically and to operations on the domains which are recursive, what sort of mathematical theories can be developed within recursive mathematics?" Trivially, every area of traditional mathematics has its computable aspects and these are readily "recursivized." More significantly, the restriction to coded domains is not as severe as might at first appear. Fields such as the rationals, the domains of polynomials with rational coefficients and topologies with a countable base can be coded easily. Analysis on the recursive reals can be studied; a recursive real is a recursive sequence of rationals with a recursive modulus of convergence function. A recursive real is, therefore, represented as the pair whose first component is the machine index of the sequence and whose second is the index of the modulus.

Any subset of ω, the set of natural numbers, is, in this sense, automatically coded and set-theoretic operations on $P\omega$, the powerset of ω, afford prime candidates for "recursive" investigation. During the late 1950's, recursion theorists Myhill, Dekker, Nerode and others initiated the systematic study of elementary set-theoretic operations on $P\omega$ as a branch of recursive mathematics. The immediate goal of this research was to create, within recursive mathematics, a recognizable analogue of Cantor's arithmetic on the (finite and infinite) cardinals. Under the intended analogy, what takes the place of the class of sets is $\mathbf{P}(\omega)$ itself. Taking the place of Cantor's notion of one–to–one correspondence is the notion of *partial recursive isomorphism.* Sets A and B are partially recursive isomorphic (in symbols, $A \simeq B$) if and only if there is a partial recursive function f which is one–to–one, defined at least on A and taking A onto B. The recursive "cardinal numbers" are, then, the $\simeq$-equivalence classes on $\mathbf{P}(\omega)$. These are called 'recursive equivalence types,' or 'RETs' for short.

Realizability and recursive mathematics.

Our central thesis is twofold. First, we claim that the class of recursive equivalence types is *identical* to a natural, set-theoretically defined collection of nonrecursive *intuitionistic* cardinals in a model of full intuitionistic set theory. Second, we will prove that not only is there an enlightening ontological correspondence but that it parallels and underlies a perfect intertheoretical correspondence. The algebraic theory of the RETs turns out to be identical, under interpretation, with the theory of the intuitionistic cardinals. The interpretation is an extension to set theory of Kleene's number realizability for arithemtic.

The dual correspondence between RETs and cardinals allows for an economical two-way interchange between classical recursive and intuitionistic mathematics. In one direction, from intuitionistic mathematics to classical, one can determine precisely which of the theorems of intuitionistic cardinal arithmetic go over unchanged onto the RETs. We will show in detail how the basic results about RETs (as they were proved by Dekker and Myhill in Recursive Equivalence Types (1960)) arise as theorems of intuitionistic set theory *without any mention of recursion.* In the opposite direction, there is an effective routine for converting negative theorems about the recursive equivalence types into independence results for any known constructive set theory.

Later in this chapter and in Chapter Five, we will elaborate on the exact details of the correspondence and the techniques it underwrites. Meanwhile, we can indicate how all this works out in some simple instances. Specifically, we will consider the Cantor-Schroeder-Bernstein Theorem, the existence of Dedekind-finite cardinals and the cardinality of $\mathrm{P}(\omega)$.

Traditionally, the collection of RETs is called 'Ω'. On Ω, operations of cardinal addition and multiplication are defined in a natural way. One of the earliest negative results about Ω was that partial recursive injection (f partial recursive and one–to–one) is not the appropriate analogue to set-theoretic injection within the recursive cardinal numbers. Equivalently, one could say that the ordinary notion of subset is not the correct one for recursive mathematics. What shows all this is the failure of a particular recursivization of the Cantor-Schroeder-Bernstein theorem.

Myhill and Dekker proved that, where '$A \leq B$' means that A is mapped into B by a partial recursive injection,

$$(A \leq B \wedge A \leq B) \rightarrow A \simeq B$$

is not generally true. For a counterexample, take $A = \omega$ and $B = \overline{K}$, the complement of the "halting set." Dekker and Myhill discovered that a correct version of the Cantor-Bernstein theorem comes from identifying RET injection with "recursively detachable injection." A is mapped into B via a recursively detachable injection, in symbols $A \preceq B$, whenever $A \leq B$ via a partial recursive f and the range of f is decidable as a subset of B. With the $\preceq$ concept, Cantor–Bernstein goes through; it is provable that

$$(A \preceq B \wedge B \preceq A) \rightarrow A \simeq B$$

Under the correspondence mediated by realizability, Ω becomes the collection of cardinal numbers of the "ω-stable" part of $\mathrm{P}(\omega)$. A set is ω-stable just in case it is closed under $\neg\neg$ with respect to elements of ω. The "recursive" operations on Ω turn into the usual cardinal-theoretic operations on these ω-stable cardinals. We call the collection of ω-stables '$\mathrm{P}(\omega)^{st}$.'

One can give a proof that the notion $\preceq$ of strong recursion-theoretic subset coincides with a notion of strong subset familiar from the writings of the intuitionists. The intuitionistic notion is that of "separated subset," as studied by both Brouwer and Heyting. With this in hand, one easily devises a constructive proof of the Cantor-Schroeder-Bernstein Theorem for the elements of $\mathbf{P}(\bar{\omega})^{st}$ under separated subsets. Also, one sees quickly that the recursive failure of the strong version of the theorem (the one employing the ordinary notion of subset) proves that the corresponding nonrecursive theorem is independent of intuitionistic set theory.

1.1. Note. We see emerging, even at this elementary stage, a feature of recursive and of constructive mathematics which is pervasive. The success of a "recursive" or "constructive" analogue to a classical theory depends greatly on a correct choice of basic concepts. This choice is often difficult and delicate; suitable concepts are not always those that come first, or even second, to mind.

Before we go further, one should be cautioned about the equivocity of 'Ω'in constructive contexts. In recursive mathematics, 'Ω' is used, as above, to refer to the collection of RETs. In discussions of intuitionistic set theory, 'Ω' refers to $\mathrm{P}(\{0\})$, the collection of intuitionistic propositions. Both of these notational practices are so deeply entrenched that we were loathe to tamper with either. Instead, we will try to write around the problem and in such a way that confusion would be difficult.

Since the late 19th Century onwards, there have been at least two definitions of "finite set". The first, familiar from the writings of the logicists, takes finite sets to be sets of the same cardinality as some natural number. According to the other definition, that proposed by Dedekind, a set is finite if it has no subset of the same cardinality as ω. In classical mathematics, it takes the axiom of dependent choice (DC) to prove that the two definitions coincide in extension. In Cohen-forcing models where DC fails, finite and Dedekind finite can come apart. Because they were aware that the relevant forms of DC fail in recursive mathematics, Dekker and Myhill directed special attention to the subcollection of Ω made up of *isols*. Isols are the RETs of isolated sets, those sets which contain no infinite recursive subset, and, in the scheme of comparison with ordinary set theory, the isols play the role of *Dedekind finite* sets.

1.2. Note. It is worth noting that the collapse of Dedekind finite into finite caused by the onset of DC is irremediably *classical*. The presence of recursion theory in models of intuitionistic mathematics preserves the distinction even under the influence of DC. ∎

Emil Post had already proved the existence of infinite, isolated sets; therefore, the recursive version of DC fails and the finite RETs are distinct from the isols. Λ is the domain of isols, and, as an ideal in Ω, is closed under the operations of RET arithmetic. Cardinal arithmetic on Λ is in many ways similar to that on ω; for example, whenever X is an isol, $X + 1 > X$. Unfortunately, the parallel between isolic and ordinary arithmetic is not extensive; when X is an infinite isol, the sequence

$$X,\ X-1,\ X-2,\ \ldots$$

is not only infinite, but has no greatest lower bound in Ω !

It should now come as no surprise to discover, that, once embedded into the realizability model, the domain of isols is identical to the set of Dedekind-finite cardinals from $\mathbf{P}(\omega)^{st}$. Over the latter set, we will give constructive proofs for all the basic theorems of isolic arithmetic. The fact that the isols are not well-founded under their natural ordering can also be internalized. An upshot of the internalization is a proof that the well-foundedness of the natural order on the Dedekind-finites is independent of intuitionistic set theory.

Contrary to our experience with ordinary set theory, $\mathbf{P}(\omega)$ is by no means a meagre or trivial domain for a recursive cardinal arithmetic. One might say that Ω, even Λ, contains recursive cardinals up to ω_{ω_1}. But since the RETs are not linearly ordered by $\leq$ (if X is an infinite isol, then X and ω are $\leq$-incomparable), it would be more appropriate to express this fact by saying that Λ contains an $\leq$-chain of ω_1 distinct isols. If we add the continuum hypothesis to the external set theory, then the $\leq$-chain can be taken cofinal in Ω.

This fact from classical mathematics is yet another negative result that reappears over realizability in the guise of a constructive falsehood. As in classical set theory, an ordinal of intuitionistic set theory is a transitive set of transitive sets. Under realizability, there is an intuitionistic ordinal α that shares many of the properties of classical ω_1 and yet indexes an α-sequence of distinct cardinals from $\mathbf{P}(\omega)^{st}$.

Realizability and the extent of recursive mathematics.

The RETs are by no means the only, or even the premier, instances of systematic and self-conscious recursive mathematics. Even in the 1950's, the study of recursive equivalence types was but one segment of a discipline that was beginning to touch on every part of classical mathematics. By that time, Fröhlich and Shepherdson had already put recursion-theoretic tools to work on problems in field theory. At the same time, Specker was developing a recursive version of analysis and Lacombe was advancing the cause of recursive topology.

Recently, Metakides and Nerode have encouraged the use of priority arguments in recursive algebra; by employing finite-injury arguments, they refuted a recursive version of Steinitz's Theorem. (Steinitz's Theorem is the claim that every algebraically closed field has a transcendence base.) Kalantari and Retzlaff have also used priority arguments in their work on recursive point-set topology. Research in recursive mathematics continues apace up to the present day. A good idea of the scope and unity of the field can be got by perusing the volume Crossley (1981), **Aspects of Effective Algebra.**

Of these recent developments, we have the time and the space to say very little. In Chapter Eight, we show how realizability can be employed to give a nonrecursive, constructive analogue to a result of Kalantari and Retzlaff. For at least two reasons, this is not entirely satisfactory as it stands. First, the interest of the analogue will not be made clear until much more work is done on realizability analogues from recursive topology. Second, we have not shown how to internalize the priority method itself. We have carried out this internalization on certain brands of priority arguments and have proved that they correspond, under realizability, to (admittedly somewhat unusual) types of *forcing.* These results will, we hope, appear later.

Our general feeling is that only limitations of space prohibit us here from extending realizability methods into all fields of recursive mathematics. We trust that the future will afford an opportunity for us to demonstrate that the extension is humanly possible and can be carried out in a unitary way. In this introduction, we will draw a picture of each of the elements that play a part in the subsumption of recursive mathematics to realizability. We begin with intuitionism, and, in particular, with the intuitionism of Brouwer. After that, we can describe a route via Heyting's interpretation from traditional intuitionism to realizability. At the very end, there will be occasion to speak in more precise terms of realizability for set theory and of the set theory itself.

Section 2: Brouwer's intuitionism

If the demand for constructivity in mathematics is wholly characterized by an insistence that graspable mathematics be limited entirely to those portions of it that can be seen as concerned with operations, proofs and definitions which are explicitly "computable", there must be numerous species of constructivism. Among these are anthropologism, intuitionism and finitism, which are familiar to the philosophical audience as divergent visions of a more-or-less graspable mathematics. There is also recursive mathematics, a region of "computable" mathematics under the sway of classical logic. Of these, the premier form of constructivism is undeniably *intuitionism.*

Intuitionism as complete constructivization.

To give the essence of intuitionism its most compact expression, we might say that intuitionism represents the complete constructivization of the mathematical and logical activities. But without further explanation, this expression would be highly misleading. For example, use of the word 'constructivization' may suggest that there is some brand of mathematics which exists prior to intuitionism and upon which intuitionism is conceptually dependent. It would be from this prior mathematics that intuitionistic mathematics arises via acts of 'constructivization.' We want to do everything we can to block such a suggestion. As a matter of history, there was a mathematics prior to intuitionism and upon which it was causally dependent for its appearance; this is the classical mathematics of the late 19th Century. Within that mathematics were constructive tendencies, which we (in McCarty (1983)) styled 'classical constructivism.' For these portions of classical mathematics, the statement of dependence, when read historically or methodologically, is accurate. At the level of basic concepts, however, intuitionism is wholly autonomous. The intuitionist needs to look neither to classical mathematics nor at classical constructivism for his basic concepts or for the inspirations requisite to develop an intuitionistic mathematics. Rather, intuitionism is, from its very foundation, a free-standing mathematical edifice.

Intuitionism rests on a revolutionary semantico-mathematical idea: that the only sort of fact in virtue of which a mathematical statement can be true is a fact about *mathematical constructions.* Because this idea is so fundamental and because, once adopted, its effects on mathematics are so pervasive, the process of constructivization need be autonomous; intuitionistic mathematics and logic cannot be developed as an offshoot of any mathematics not informed by this idea. It would, therefore, have been more accurate to say that the essence of intuitionism is the mathematics that comes from complete constructivization.

This process of constructivization is, in intuitionism, complete because, ultimately, the interpretation of any bit of intuitionistically-intelligible mathematical language is to be given entirely in terms of constructions. The preferred intuitionistic interpretation of the statements of a mathematical language is molecular rather than, say, contextual. Hence, this talk of constructivization applies even to the logical signs. The sense of an intuitionistic logical sign is embodied in a grasp of operations on constructions, primarily, on those constructions which are constructive proofs.

For the moment, all we need say about constructions (above and beyond the superficial feature already noted, that they are allied to explicit calculation) is that constructions are mathematical operations which satisfy a recognition condition on their being given. A construction is completely and correctly given only if it can be recognized as such. The very paradigm of a nontrivial construction is, therefore, the Euclidean Algorithm. Here is an operation, which, on any two natural numbers (each of which is, in the intuitionist's metaphysic, itself a construction), calculates and outputs the number which is the greatest common divisor of the original inputs. The procedure of Euclid is an explicit calculation in terms of the inputs, and satisfies the recognition condition. Whenever the procedure is carried out on a number pair, the result is recognizably the requisite divisor.

Recognition, decidability and Benacerraf.

Notice that the recognition condition, as stated above, is not equivalent to the *decidability* of intuitionistic proof. There are those (e.g. Kreisel) who insist that truly constructive notions, like the intuitionistic proof predicate, be decidable. To the author, such an insistence is both ill-motivated and inessential to a correct interpretation of intuitionism. There are several metamathematical facts which should encourage one to look favorably on the idea that the intuitionistic truth predicate for arithmetic would have to be nonarithmetic. Among these is the fact that the realizability predicate is nonarithmetic. Also, whatever the complexity of the intuitionistic truth predicate, it seems that the classical truth predicate should be one-one reducible to it via Gödel-Gentzen stabilization.

As a digression, we would like to comment on a suggestion of Scott Weinstein. In his recent *The intended interpretation of intuitionistic logic* (1983), Weinstein suggests that, if the intuitionistic proof predicate is decidable, then intuitionism evades "Benacerraf's problem." In *Mathematical truth* (1973), Benacerraf argued that classical mathematics, as he would prefer to understand it, cannot be both semantically and epistemically tractible.

According to Benacerraf, mathematics (or any form of discourse, it would seem) is semantically tractible only if one can give a Tarski-style truth theory for it. It is epistemically tractible only if one can sketch some causal account of how we could have knowledge of the objects of the theory, as picked out by the base clauses of the truth definition. Apparently, Weinstein believes that the intuitionists rescue us from Benacerraf's difficulty by giving mathematics a semantics according to which mathematical knowledge is epistemologically tractible, that is, a semantics that makes it clear how we come to have the knowledge of abstract objects over which the semantics interprets mathematical utterance. The basic idea seems to be that, while the intuitionistic interpretation of arithmetic will support a Tarksian truth theory, mathematical knowledge is still open to those of us who speak "intuitionese" because knowledge flows from proof and the proof predicate is decidable.

Weinstein's response to Benacerraf is not open to us as we desire to remain agnostic about decidability. Even so, there seems to be very little reason to make heavy weather with Benacerraf's objections, even for classical mathematics. We can simply refuse outright to become entangled in the conceptual bind Benacerraf has constructed, a clash between Tarskian semantics and causal theories of reference. There are any number of reasons for thinking that the causal theory (or anything like it) *cannot apply* to the references made in mathematics. The very cogency of the causal theory requires that one be able, in principle, to grasp the references of terms in ways which are distinct from grasping the sentences we wish to interpret as containing those terms. In the case of mathematics, this seems impossible. To take just one example, there seems to be no grasping set-theoretic quantification by means independent of the sentences in which such quantification occurs. Abstract sets cannot, "by their very nature," be presented to someone in isolation from one or another mathematical statement in which reference to them occurs.

Constructive existence and logic.

Of course, we cannot leave the central semantical idea of constructivism, that even mathematical facts are constructions, just like that. Its relative unfamiliarity, together with its importance for logic and mathematics, demand for it a more extensive exploration and illumination. We prefer to do this by way of its application to logic and by means of a mild "rationalization" of the early history of intuitionism.

There were, in the late 19th Century, vague and, it seems to us, overly metaphysical "constructivisms." Among these was the classical constructivism of Kronecker and Hölder. We would not want to date the beginning of full constructivism with these mathematicians.

We would prefer to point to the early 20th Century, particularly, to the reaction of certain Paris mathematicians to Zermelo's "proof" of the Well-ordering Theorem. The members of the Paris school, later referred to by Heyting as 'semi-intuitionists,' adopted the idea of "constructive existence" as a conceptual rallying point. On the basis of that idea, the semi-intuitionists, headed by Poincaré, Borel and Lebesgue, rejected Zermelo's work and his proof.

The semi-intuitionists held that a pure existential conclusion $\exists x\ \phi(x)$ is mathematically correct only if it is, in its proof, constructive. A mathematician profferring an existential conclusion as constructive must be able to provide an explicit construction of an object a and a proof that $\phi(a)$. Patently, Zermelo was unable to give such a "correct" proof of the Well-ordering Theorem; he could offer no set-theoretical recipe which demonstrably defines a well-ordering on the reals. Hence, in the eyes of the semi-intuitionists, Zermelo's proof was part of an ill-fated metaphysics rather than a piece of mathematics.

Constructive existence, as it stood behind the semi-intuitionistic critique, was taken over entirely by the intuitionists. In fact, a penchant for constructive existence proofs seems to be a part of the basic methodology of most forms of non-intuitionistic constructivism. It would, therefore, be worth examining the idea as applied to a simple example.

$\pi(n)$ is a property of natural numbers defined as follows. First, in the expansion of π, we will say that a sequence of consecutive decimal digits forms a "7-7 string" iff it is a sequence consisting entirely of '7's and is seven decimal places long. Also, we will want to say that $1\text{st}(m)$ holds of a natural number m just in case, at the mth place in π's expansion, there begins the first 7-7 string in the expansion. Then, the predicate $\pi(n)$ is defined by

$$\pi(n) \Leftrightarrow \begin{cases} n = m + 1 \text{ and } 1\text{st}(m) & \text{if there is any 7-7 string} \\ 0 & \text{if otherwise} \end{cases}$$

For a classical mathematician unattracted by constructive existence, the route to the conclusion $\exists n\ \pi(n)$ is direct and free of obstacles. He might reason as follows: there is either a 7-7 string in the expansion of π or none ever appears. In the first case, there is some place m at which the earliest such string appears. Take $n = m + 1$, and $\pi(n)$. On the other hand, if there is no such string, $\pi(0)$ holds. Therefore, in any case, $\exists n\ \pi(n)$.

As a matter of mathematical fact, this proof is demonstrably nonconstructive. At the time of writing, it is unknown whether or not π contains a 7-7 string in its decimal expansion. There is no construction, therefore, of a natural number p that can come equipped with a proof that $\pi(p)$. If there were such a construction, we could remedy our ignorance on the subject of the distribution of '7's in the expansion of π by performing a trivial check. To see whether there is a 7-7 string, we need only look to see whether p is 0.

The recognition condition on constructions blocks a seemingly attractive objection to our use of this particular "nonconstructive" proof. Someone might say that the very definition of $\pi(n)$, together with the proof of $\exists x\ \pi(x)$, yields up a trivial construction of a p such that $\pi(p)$. Therefore, the proof is not really nonconstructive. One can merely take p to be that natural number which is $m+1$ when $1\text{st}(m)$, if there is a 7-7 string, and is 0 otherwise. This kind of response is to be rejected. If taken seriously, it would trivialize the requirement that a construction is given only when it is recognizably correct. In this case, the only route to recognizing that the purported construction of p successfully picks out a natural number is a circular one. To recognize the construction of p as correct would require us to have already accepted some nonconstructive proof of $\exists n\ \pi(n)$ like the one sketched above.

A demand for constructive existence proofs is indeed characteristic of a wide range of possible positions in the constructivistic spectrum, among them, intuitionism. (Notably, the recursive mathematician, as a classical mathematician, makes no such demand.) By contrast with that of the intuitionist, however, the attitude of the semi-intuitionist seems to incur considerable internal tension, if not incoherence.To see this, first recall that the semi-intuitionists were, on the whole, unabashedly classical mathematicians. The suitability of classical logic to mathematical use was not called by them into serious question. We would argue that a call for constructive existence imposes other demands on the shape of acceptable mathematics, and that perhaps the likeliest way to meet these demands is to go along with the intuitionists in tailoring classical logic to fit constructive needs. Hence, the Parisians must have felt that their attitude toward existence could be held in isolation from the rest of mathematics and its traditional logic. Arguably, such an isolation cannot be maintained.

An argument to the conclusion that constructive existence infects classical logic is best set out in terms of the sample "nonconstructive existence" proof. In the passage to constructivism, one wants to retain at least the external form of the classical account of validity; this is especially true of the semi-intuitionists, who desired to retain even the content of the classical account. In particular, one wants to continue to hold that an inference $\Gamma \vdash \phi$ is valid whenever the truth of the premises Γ insures the truth of the conclusion ϕ. If, from the nonconstructive proof of $\exists n\ \pi(n)$, the conclusion is rejected, then it is incumbent on the purveyor of constructive existence either to reject one of the premises of the "proof" or to refuse one of its primitive inferences. In the case of our proof, the inferential steps are so simple as to be beyond question. As is familiar, the intuitionist prefers to take issue with the first premise, that either there is some 7-7 string in π's expansion or there is none. In general, the intuitionist will object to many instances of the classical *tertium non datur* (TND). It is also incumbent on the semi-intuitionist to make some such rejection and it seems that a rejection of the instance of TND is indicated. It is the failure of the semi-intuitionist to make, or even to see the need for, such a rejection that makes their position unstable. The proponent of "constructive existence" will not reasonably be able to isolate the interpretation of existential quantification which he prefers from principles of logic which do not involve $\exists n$, not, at least, if he wishes to retain a notion of validity recognizable as such.

The likeliest course of action is to refrain from adopting TND as a universally valid principle of logic. It would not be wholly inaccurate to conceive of the advent of intuitionism, in the early writings of L.E.J. Brouwer, as contemporaneous with a perception that the call for constructive existence proofs cannot be isolated from logic, and, hence, that severe conceptual strains lie within semi-intuitionism. The rejection of TND for mathematical contexts was the immediate instigation of Brouwer's "programme:" the critique of classical mathematics and the development *ab initio* of a form of mathematics independent of TND.

The move in Brouwer's thought that goes directly from the rejection of TND to a fullscale transformation of classical mathematics would surely give offense to contemporary philosophical sensibilities. The rejection of TND seems, at least from the foregoing presentation, a relatively unprincipled expedient. One rejects TND to get out of trouble. And it is not at all clear how far the trouble extends. One should ask "Once we are convinced that 'constructive existence' imposes constraints on logic, how can we proceed into constructive mathematics with any confidence before we have circumscribed in some way the effects of those constraints?" In asking this question, one is asking after, first, a collection of logical principles which are intuitively constructive, and, second, after a means for assessing the constructive acceptability of putative principles.

Both these requests could be satisfied by the provision of a formal semantics that accords with the constructivist insight. Such semantics would be a "survey" of the effects of the basic ideas of constructivism on all the logical signs. With respect to this semantics, one can assess the constructive correctness of various principles. One might even hope to use the semantics as a means to axiomatizing a specifically intuitionistic logic.

Brouwer was, however, both in personal preference and in doctrine, wholly opposed to the idea that formal semantics and axiomatization afford a necessary prolegomena to constructivization. Brouwer conceived of intuitionism in a way vastly different from that in which one would naturally think of it today. Intuitionism was seen by its founder as an (or even the only) appropriate response to the pressing problem of certainty in mathematics.

Mathematics had been thought to be, among all the branches of human knowledge, that which is most certain and immediate. The set-theoretical paradoxes had, according to Brouwer, brought full classical mathematics down from its epistemic primacy. It was a philosophical commonplace of the day, and one which Brouwer accepted, that the appearance of the paradoxes had shown that this assessment of classical mathematics had to be withdrawn. For Brouwer, the logicistic diagnosis and response to the paradoxes represented a double error. First, Brouwer saw the paradoxes as far removed from the "core" of mathematics, the mental activity of mathematical construction; this core of mathematics retains its epistemic primacy. Second, the paradoxes are not totally irrelevant to mathematics, however. They are symptoms, on this view, of the deep malaise of *classical* mathematics, a malaise that had infected a great bulk of the subject. For Brouwer, therefore, it was one mistake to follow Russell in thinking of the paradoxes as themselves a great problem for mathematicians to solve and yet another, and compounding mistake, to try to begin again the task of basing mathematics on the foundation of *classical* logic.

For Brouwer, the paradoxes would be harmless to a mathematics which is correctly understood. The constructive core is immune to contradiction; hence, any mathematics that retains full reliability must have a logic which is drawn out by reflection on the core. Therefore, logicism, on Brouwer's view, could not possibly achieve the epistemological goals which it set itself. Brouwer's unimpreachable core was the mental activity of mathematical construction, and any segment of classical mathematics which is adequately interpretable in terms of mentally accessible constructions is thereby *reliable*; the interpretable mathematics can draw its certainty from that stored in the core. Under constructivistic interpretation, each graspable mathematical proposition p is to be understood as the posing of a mathematical problem, or, better, as the supposition of a hypothetical construction, the accessibility of which gives a solution to the problem posed by *p*. Famously, Brouwer rejected the validity of the classical TND because it is not universally valid when so interpreted.

Brouwer was also convinced that the ideas on the epistemology of arithmetic expressed by Kant in the early parts of the *Kritique* were fundamentally correct. Brouwer's picture of intuitionism, as the only correct mathematics, is attached to the transcendental framework of the Kantian faculty of *Anschauung*. Somehow, Brouwer made the move from this vision of constructive mathematics to the idea that the facts about constructions are always on inner display. There is, however, no making the obtaining of any sophisticated constructive mathematical fact coextensive with our inner awareness of it. Perhaps the idea that such a coextensiveness is possible came from an implicit likening of the conceptual "field" of constructive mathematics with the private visual field. Admittedly, the latter is a realm in which factuality is coextensive with our awareness of it. Brouwer was unaware of the dangers of encouraging an assimilation of conceptual items to visual ones. The sort of dangers we have in mind are those apparent in the egregious errors of Humean empiricism.

It may have been on the basis of some such visual analogy that Brouwer rejected formalization along with the (dubious) goals of the logicism of his day. He felt that constructive mathematics needs no formal devices to underwrite its correctness, that any construction, when correct, is obviously so. Therefore, no formalization was required to block the gaps out of which certainty might leak. Brouwer was to continue in this anti-formalistic bent throughout his long career. He would never admit the possibility that formalization might bring with it other benefits.

Section 3: Heyting's interpretation

It was Brouwer's student and, later, colleague, Arend Heyting, who first adopted a reasonably comprehensive formal and semantical approach to intuitionism. Heyting introduced and championed what is now recognized as the standard "proof-theoretic" interpretation of the intuitionistic logical signs. In Heyting's interpretation, the constructive semantical idea manifests itself in the definition of constructive truth. The sort of construction in virtue of which a mathematical statement is true is a *constructive proof.*

Constructive proofs are, first and foremost, constructions. As such, constructive proofs must satisfy the constraint on recognition. When the application of the constraint is taken together with Heyting's definition of constructive truth, the result is a semantics for intuitionism which is verificationist or "nonrealist." A verificationist semantics is to be contrasted with the standard realist semantics, with which it breaks sharply. On the realist conception, which comes to us from Frege via Tarski, the understanding of a mathematical proposition is analyzed in terms of a grasp of its abstract "truth conditions". For the realists, the truth conditions of p are, if you will, the "possible facts" so associated with each proposition that, if any of these were to obtain, then p would be true. The heart of the realist vision is embodied in the idea that the relevant possible facts can obtain or not in total independence of anyone's mathematical ability to discern the truth value of p.

This description of the conditions under which p holds is patently inapplicable to Heyting's nonrealist semantics for intuitionism. If the condition under which a proposition is true is the availability of a proof of that proposition, then the intuitionistic truth conditions of the proposition cannot obtain in a way wholly independent of our abilities to see that it does. If p is proved, then it must be possible in principle for us to access the proof of p and to assess its cogency.

A suspicion of circularity may lurk about this exposition of the intuitionistically preferred semantics. It may seem that the very setting of the task of giving such a semantics already presupposes the achievement of the task, and, hence, that proof-theoretic semantics can give rise to no coherent account of "coming to understand" a mathematical proposition. A grasp of a mathematical proposition p is here a grasp of its truth conditions, and these are, in turn, given via a description of what it is to prove p. But how could one even contemplate the proving of p as a possible task, if p is not, as yet, even understood? How can one, in giving the basic explanation of the sense of p, speak of p as the conclusion of intelligible proofs?

Anyone who asks such questions, we think, belies a misunderstanding of the working-out of the constructivistic semantical idea. Knowing the constructive sense of p is not something *derivable* from a preëxistent knowledge of what it is to be given a direct proof of p. Rather, that knowledge is *identical* with it. Beyond specifying the proof conditions, there need be nothing more that a constructivist must say about the sense of a mathematical proposition. In Heyting's unfolding of the constructivist idea, the possible circle is cut by giving, simultaneously and recursively, specifications of the sense of p and of those constructions which count as direct proofs of p. The clauses of Heyting's interpretation provide the required specifications. What follows is a paraphrase, in contemporary terms, of the original Heyting explanations.

Heyting's explanations of the connectives are usually illustrated for a first-order language of arithmetic which includes as primitives signs for addition, multiplication, successor and zero, and we will follow suit. For n a natural number, $\overline{n}$ is the numeral representing n in the language. In the metalanguage, variables ρ and σ range over constructions; the application $\rho(\sigma)$ is the construction output when the operation ρ is given σ as argument. The domain of constructions is assumed to be closed under a binary pairing operation and the corresponding "unpairings", denoted ' ρ_0' and 'ρ_1.'

The recursion works not by defining, as in Tarski's semantics, satisfaction as a property of sentences with respect to sequences, but by defining constructive proof as a property of sentences. In the clauses of the interpretation, we read '$\Pi(\rho, \phi)$', as 'ρ is a constructive proof of proposition ϕ.'

3.1. Definition.

(1) $\Pi(\rho, \phi \wedge \psi)$ iff $\Pi(\rho_0, \phi)$ and $\Pi(\rho_1, \psi)$

(2) $\Pi(\rho, \phi \vee \psi)$ iff $\Pi(\rho_0, \phi)$ or $\Pi(\rho_1, \psi)$

(3) $\Pi(\rho, \neg \phi)$ iff, for all σ, if $\Pi(\sigma, \phi)$, then $\Pi(\rho(\sigma), \overline{0} = \overline{1})$

(4) $\Pi(\rho, \phi \rightarrow \psi)$ iff, for all σ, if $\Pi(\sigma, \phi)$, then $\Pi(\rho(\sigma), \psi)$

(5) $\Pi(\rho, \exists x\ \phi)$ iff $\Pi(\rho_0, \phi(x/\overline{\rho_1}))$

(6) $\Pi(\rho, \forall x\ \phi)$ iff, for all $n, \Pi(\rho(n), \phi(x/\overline{n}))$

Finally, we say that a sentence ϕ is *intuitionistically true* iff $\exists\rho\ \Pi(\rho, \phi)$. ∎

Don't be taken aback by the fact that (1) through (6) make no provision for atomic statements. As is the case with Tarski's truth definition, the meanings of the signs that enter into atomic statements have no bearing on the interpretation of the logical apparatus but are specific to the universe of discourse. Since we have chosen arithmetic to illustrate Heyting's work, a specification of the proof conditions of the atomic statements, which are polynomial equations, is straightforward. ρ is a proof of atomic ϕ iff ρ is a combination of the simple sums and products that we learned in grammar school to take as a verification of the equation ϕ.

The individual content of each clause of the interpretation can be motivated by examining certain salient features of inferential practice with the basic semantical idea in mind. (For more details, cf. McCarty (1983)). Clause (3), for instance, describes the central proof-theoretic grounds for a denial of ϕ. It says that ρ shows ϕ to be constructively false iff, whenever σ proves ϕ, ρ, as a constructive function, takes σ as an argument and produces $\rho(\sigma)$ as a proof of '$0 = 1$.'

In much the same way, one can come to see each clause of the interpretation and the entire truth definition as a plausible regimentation of what has here been touted as the central idea of constructive semantics. *In fine*, Heyting's interpretation is the very semantical picture of complete constructivization. The only kinds of objects and the only sorts of facts in virtue of which a statement of intuitionistic mathematics can be true are constructions. Moreover, the "truthifying" facts are one and all accessible; whenever such a fact obtains, we can recognize that it does so.

Logic and logicism.

From the Heyting interpretation, one sees immediately that Brouwer was right in thinking that the very idea of "intuitionistic logicism" is *totally incoherent*. (This is not to say that Brouwer held this belief for the right reasons.) The success of any strand of classical logicism depends upon the claim that the senses of the logical signs can be grasped independently of and prior to any nontrivial bit of mathematics. This claim is just flatly false for the intuitionistic logical signs. These signs can only be grasped if the Heyting interpretation is grasped, and that is understood only on the basis of a prior understanding of the constructions over which the metaparameters of the interpretation range. However, according to the intuitionist, the constructions are *echt* mathematical items that are grasped, in part, by understanding the simple applicative algebra on them. In a word, for the intuitionists, some mathematics must preceed all logic.

Once first-order logic is formalized, one can see that there is a specifically intuitionistic logic which is sound with respect to (a version of) Heyting's interpretation of the logical constants. It would be seriously misleading to say that intuitionistic logic is just classical logic minus TND or even classical logic independent of TND. The line taken in some of Brouwer's papers may have encouraged an idea like this. However, the presence of various intermediate calculi show that intuitionistic logic is not adequately characterized by "independence from TND." Not only does the intuitionistic logic lack $\phi \vee \neg \phi$, but it is also independent of these classical laws:

$$\neg\neg\phi \to \phi$$

$$\neg(\neg\phi \wedge \neg\psi) \to (\phi \vee \psi)$$

$$\neg\forall x\, \phi \to \exists x\, \neg\phi$$

Intuitionistic logic, at least on the coarsest measure of logical strength, is weaker than the classical. Nonetheless, this form of weakness can be a virtue. It is a part of the plan of our work to show exactly how this is and to *exploit* the virtues of intuitionistic logic to complete a task which is fully specifiable within classical mathematics but of which classical mathematics seems either wholly incapable or able to do only at considerable mathematical expense.

Two aspects of intuitionism.

There are two important features or "faces" of intuitionism. One has already received considerable attention: the semantical tie to the basic constructivist idea given by Heyting's interpretation. This feature alone gives to intuitionism a dimension wholly absent from classical mathematics. The dimension is the "evidential" or proof-theoretic one. Its mathematical manifestation is in the presence of "extra parameters;" by its very interpretation, every statement of intuitionistic mathematics carries with it one or more parameters ranging over the "evidence," the collection of possible proofs of the statement.

The other feature of intuitionism is one which has drawn more attention (and fire) in the literature. This is intuitionism's liberality toward higher-order abstractions. Intuitionism sets itself off from other flavors of philosophical constructivism in its ontologically liberal admission standards. The range of the intuitionistic quantifiers extends far beyond those of the finitists and even of Bishop's "new constructivists." The traditional intuitionist is willing to quantify over choice sequences and over species, objects which, *per* *se*, seem wholly nonconstructive. (One can think of *species* as the presentations, or intentional correlates, of predicative sets.) The incorporation into the constructivist universe of sensible collections of "nonconstructions" was perhaps Brouwer's greatest contribution to the subject. Unfortunately, it is one face of intuitionism which is too extensive to be treated adequately in the pages of an introduction. For a detailed discussion of the abstractions of intuitionistic mathematics, we refer the reader to Troelstra (1983). This face of intuitionism, the tolerance for abstraction, must be mentioned, because it is one which we will need to exploit to the fullest in our dealings with set theory.

Our outlook, as schematized by realizability, relies on both features of intuitionism. On the one hand, realizability is a nonstandard, although nonbizarre, model of Heyting's explanation, and, as such, gives numerous insights into the discrete or "highly evidential" aspects of intuitionism. At the same time, realizability's nonstandard account of the proof relation extends beyond arithmetic to an understanding of full intuitionistic set theory. Against this abstract higher-order backdrop, we can display the consequences for both classical and constructive mathematics of adopting nonstandard set-theoretic principles.

Section 4: Kleene realizability

There have been several moments in the history of recursive mathematics when there surfaced indications of the existence of some fairly large conceptual area that serves as an interface between constructive and recursive mathematics. It had even been suggested, at least in an offhand way, that the interface is semantical and that an open channnel lies in the region of Kleene's *recursive realizability* (cf. Kleene (1945)). Actually, the very status of realizability for arithmetic is itself some kind of indication. It would not be wholly inaccurate to say that arithmetic realizability is a scheme for translating intuitionistic arithmetic into a tiny fragment of recursive mathematics. But this is a very limited indication of an interface that is remarkable in extent.

A vague and perhaps accidental suggestion of the whole connection occurred as early as 1968, in Kreisel's Zentralblatt review (1968) of J.N. Crossley's article *Constructive Order Types* I. As a field within recursive mathematics, the study of the constructive order types (or COTs) stands to the study of the classical ordinals much as RETs stands to Cantor's cardinals. In his review, Kreisel opined (correctly, it turns out) that Crossley's theory of COTs would lose its somewhat unnatural cast were it to arise, not as a subfield of classical recursive mathematics, but as a relativization of some naturally-occurring theory of order. Kreisel also argued that, whatever that natural theory of order might turn out to be, it is unlikely to appear within classical mathematics. Kreisel believed (once again, correctly) that the natural theory would be formalized in the intuitionistic predicate calculus. Finally, the soundness of intuitionistic logic with respect to Kleene realizability was advanced by Kreisel as rationale for this proposal.

In Chapter Five, Kreisel's suggestions will be filled out and shown to be accurate. There we prove that the classical theory of Ω is precisely a relativization plus a realizability interpretation of a thoroughly natural theory of constructive cardinals. The latter theory is one which could well have existed prior to and independently of that of Ω. It is merely a misfortune of history that it did not. Even though the realizability interpretation in question is not that of arithmetic, but an extension of it to set theory, our progress will be easier if we begin with arithmetic anyway.

Realizability for arithmetic.

If we think of Heyting's interpretation of intuitionistic mathematics as a truth definition akin to that provided for classical mathematics by Tarski, there come immediately to mind two means to its full mathematical development. One approach is via an axiomatization of a "standard interpretation" of the notion of construction. Here, one tries to embed Heyting's work into an axiomatization of the theory of constructions, a formal theory of the "mathematical springs" of the interpretation. The parallel in the case of Tarski is the expression of the Tarski clauses within a standard set theory. Unfortunately, this idea, applied to intuitionistic (or "Heyting") arithmetic, has proved more than a little difficult in execution. The logical pitfalls which were exposed by the work of Kreisel and Goodman are well known. In the face of the constructions' apparent resistance to straightforward axiomatization, one might say that our intuitions are, as yet, insufficient to the task of "filling out" Heyting's interpretation *directly*.

If the direct axiomatic path to our goal is temporarily blocked, the obvious alternative is to try to score on an end run. By this, we mean that it is possible to exploit nonstandard interpretations of the clauses of the Heyting "truth definition." On this approach, one looks about for a domain which is more tractible than that of the constructions and yet has enough of the right structure to serve as a collection of "pseudoconstructions." In 1941, S. C. Kleene discovered that the natural numbers, under the operation $\{e\}(n)$ of Turing application and with primitive recursive pairing $\langle n, m\rangle$, embodies enough of the right structure. Fortunately, Kleene had already devoted much of the preceding ten years to making this structure mathematically tractible. Kleene interpreted Heyting arithmetic using these recursion theoretic constructs and called the resulting reinterpretation of Heyting's clauses 'recursive realizability.' The earliest metamathematical applications of realizability appeared in Kleene (1945).

Under Kleene's operations, ω has enough of the "right structure" to act, in the eyes of arithmetic, like the collection of constructions. Can we characterize the "rightness" of this structure independently of Heyting's constructions? There are two answers to this question, each one of which is correct. The first is a longer and more abstract answer; the second is short and intuitively appealing. All of Chapter Two and the beginning of Chapter Three supply the longer answer. There, we will point out that each structure composed of "pseudoconstructions" participates in an abstract model-theoretic property; they are one and all models of the axioms APP of "applicative systems."

There is no need to linger over APP and its models here; these things will receive full treatment later. Instead, we prefer to concentrate on the second answer, which is a shortcut route to realizability. If one accepts Church's Thesis, the idea of recursive realizability comes effortlessly. First, one remarks that, since constructive proofs contain only a finite amount of information, nothing will be lost by replacing proofs of atomic statements of arithmetic with "codes." Think of the codes as Gödel numbers of the elementary computations that verify (or falsify) atomic statements. Then, *á la* Heyting, number-theoretic pairs of codes represent constructive proofs of conjunctions, disjunctions and existentials. A coded proof of the existential $\exists x\phi$ is a pair $\langle n, m\rangle$ where m codes a proof that $\phi(\overline{n})$. What about proofs of $\rightarrow$ and $\forall$ statements? Constructively, these sorts of proofs are operations carrying proofs into proofs. Once possible proofs of the antecedent and consequent of $\phi \rightarrow \psi$ are coded, proofs of $\phi \rightarrow \psi$ emerge as effective operations on the natural numbers. By Church's Thesis, the effective operations coincide with the general recursive. Therefore, each proof of a conditional has as presentation an index e of a recursive function $\{e\}(x)$ such that, whenever n codes a proof of the antecedent, then $\{e\}(n)$ codes a proof of the consequent. The very same coding scheme applies, *mutatis mutandis,* to give as presentations of proofs of $\forall$ statements indices $\{e\}$ such that, for any n, $\{e\}(n)$ "proves" $\phi(\overline{n})$.

The precise representation of Kleene's idea is as a recursive definition.

4.1. Definition. Read $R(n, \phi)$ as 'n (Kleene) realizes ϕ:'

(i) For atomic ϕ, $R(n, \phi)$ iff ϕ is true

(ii) $R(n, \phi \vee \psi)$ iff either $n_0 = 0$ and $R(n_1, \phi)$
or $\neg n_0 = 0$ and $R(n_1, \psi)$

(iii) $R(\phi \wedge \psi)$ iff $R(n_0, \phi)$ and $R(n_1, \psi)$

(iv) $R(n, \neg\phi)$ iff, for all m, $\neg R(m, \phi)$

(v) $R(n, \phi \rightarrow \psi)$ iff, for all m, if $R(m, \phi)$,then
$\{n\}(m)$ is defined and $R(\{n\}(m), \psi)$

(vi) $R(n, \exists x\phi)$ iff $R(n_1, \phi(x/\overline{n_0}))$

(vii) $R(n, \forall x\phi)$ iff, for all m, $\{n\}(m)$ is defined
and $R(\{n\}(m), \phi(x/\overline{m}))$

ϕ is (Kleene) realized iff $\exists n\, R(n, \phi)$.

In general, $R(n, \phi)$ is anything but recursive; the predicate (of n) $R(n, \forall x\,(x = x))$ is already complete Π^0_2. However, to realize "disjunction-elimination":

$$(p \vee q) \rightarrow ((p \rightarrow r) \rightarrow ((q \rightarrow r) \rightarrow r))$$

we need to determine recursively, for n such that $R(n, p \vee q)$, which of p or q is realized. This explains the extra complications appearing on the righthand side of our (ii), the clause governing $\vee$. These decidability conditions enforce the requisite disjunctwise decidability of realizability for $\vee$-statements.

Kleene's realizability interpretation, as applied to arithmetic, does demonstrable justice to the senses of the logical signs, at least as they are expressed there. Kleene proved that realizability is sound with respect to HA, first-order Heyting arithmetic. Kleene proved that, for sentences ϕ,

4.2. Theorem. *If* $HA \vdash \phi$, *then* $\exists n\ R(n, \phi)$.

4.3. Note. Although the high vantage of hindsight makes the transition, via coding, from the Heyting interpretation to Kleene realizability seem extremely natural, our sketch of the transition is neither historically nor psychologically correct. Kleene, in his *Realizability: a retrospective survey* (1971), denied that Heyting's "proof interpretation," which was known to him in the early forties, had exerted anything but a retarding effect on the genesis of realizability. Actually, Kleene derived his principal inspirations from the writings of Hilbert and Bernays. In particular, Kleene's thoughts were shaped by the finitist conception of existential mathematical propositions as "incomplete communications." This, according to Kleene, served as the major preformal intimation of realizability. ∎

Section 5: Realizability for set theory

By invoking two old ideas, one due to Poincaré and the other suggested by some early work of Brouwer, we can clear a conceptual path from arithmetic to realizability for set theory. Naturally, the path will take us through a partial explanation of what it is to talk constructively about one particular conception of set. Poincaré's contribution is just that: a picture, painted in broad strokes, of a universe of abstract sets which is both mathematically attractive and recognizably constructive. We will interpret early work of Brouwer as a precedent for introducing an account of quantification over sets so conceived; we call this notion of quantification 'generic.'

Abstract constructive sets.

In **Mathematics and Science: Last Essays (1963)**, Poincaré put forward a description of a characteristically constructive notion of set. The notion includes the requirement that a constructive set be given by a *double* specification. Presenting a set A constructively is setting out, for constructions ρ and items a from some domain, the conditions under which ρ proves that $a \in A$. This stands in sharp contrast with the way in which sets are presented in classical mathematics. There, a set is fully described in a *single* specification. To describe a set A classically, one need provide only those abstract conditions on which arbitrary a satisfies '$x \in A$'. We put the requirements on a constructive set most succinctly by saying that a constructive set can be pictured as a collection of pairs $\langle \rho, a \rangle$ such that $\Pi(\rho, a \in A)$; taken together, all the evidence-item pairs given in specifying A comprise the constructive membership conditions that define A.

This way of speaking about sets can easily be translated into realizability-theoretic terms by an extension of the coding idea. For subsets of ω, the translation is especially easy to work out; each constructive subset A of ω can be represented as a "realizability set" $\overline{A}$, where $\overline{A}$ is a collection of number pairs $\langle n, m \rangle$. The idea behind this is familiar: for $\langle n, m \rangle \in A$, we think of n as coding a proof that $m \in A$. If we allow arbitrary iterations of the powerset operation, this idea can even be extended by analogy to a full hierarchy of realizability sets. A cumulative hierarchy of "pure" realizability sets is then generable. Consider the following ordinal-recursive definition:

$$\mathbf{V}(Kl)_0 = \emptyset$$

$$\mathbf{V}(Kl)_{\alpha+1} = \mathbf{P}(\omega \times \mathbf{V}(Kl)_\alpha)$$

$$V(Kl)_\lambda = \bigcup_{\beta<\lambda} V(Kl)_\beta$$

The universe of realizability sets, $V(Kl)$ ('Kl' stands for 'Kleene'), is the union of the $V(Kl)_\alpha$'s over all the ordinals α. More succinctly, $V(Kl)$ is the least class closed under the formation of pure realizability sets. It follows from the second of the above equations that this succinct description is accurate; each $a \in V(Kl)$ is a collection of pairs $\langle n, b\rangle$ that is most naturally thought of as a realizability version of a Poincaré set. Think of the first item of the pair as a coded proof, a bit of constructive information or a realizability "witness" for the fact that the second element, b, belongs to the set a. One might say that the first place of each of the pairs is a repository for the values of the "extra parameters' of which we spoke earlier. In this way, the "proof information" of the original Heyting interpretation (or Kleene's coded version of it) can be given a mathematical place within the very notion of set itself.

Genericity.

Once the basic notion of constructive set has been isolated, or, more formally, once we circumscribe those constructions that can act as proofs of particular atomic statements about sets, the only remaining task is that of giving a sensible account of quantification over the universe of constructive sets itself.

Prior to the discovery of *choice sequences*, Brouwer favored a picture of the continuum which was "wholistic" (cf. Troelstra (1982)). The primary focus in the picture of wholistic reals was on the idea that there may be no constructive analysis of what it is to be an individual real number *qua* real number. At some level, this means that there is no finer detail to the intuitionistic notion of the collection of real numbers than that given by some set of axioms for analysis, supplemented perhaps by examples of particular reals. We might say that the wholistic picture is intended to display the grasp of "real number" as no more than purely generic. Apparently, Brouwer adopted the generic picture in order to skirt the apparent measure-theoretic consequences of seeing the reals as given, one and all, by lawlike sequences of rationals. (The subject of lawlike sequences will appear again in Section 7.)

From our present standpoint, we need elaborate no further Brouwer's idea of wholistic real as a chapter in the history of intuitionistic mathematics. All we want to do here is to use the idea to suggest two themes that point toward an intelligible account of quantification over the collection of abstract Poincaré sets. The first theme is negative; it imposes tight strictures on the places at which information about a collection can come into play in a proof. The second theme is compensatory and positive; it tells us where certain information can be lodged.

The first theme is concerned with what might be called 'intuitionistic metaphysics.' According to intuitionistic semantic ideology, our concept of a mathematical collection must be fully captured in the explanations of the proofs of statements with quantifiers and parameters restricted to members of that collection. This applies to the generic conception as to any other. When our grasp of a collection is generic, then the objects of the collection *as objects of that very collection* carry no mathematical information. When we remember that, on the intuitionistic view, the only package in which mathematical information can arrive is a proof, then we must say that, when a collection is generic, no member of the collection has a presentation, an intensional aspect, that can make any difference to a proof. This stricture applies, principally, to the interpretation of quantification over the generic collection; it is in terms of the understanding of quantification that our concept of the collection as collection comes strongly to the fore.

The second theme is that of axiomatization. If a domain is generically conceived, then our positive mathematical knowledge of the structure on the domain can be taken up in a grasp of a set of axioms describing that structure. This grasp is direct; it is not mediated by a prior understanding of a certain class of proofs. This second theme cannot, however, be independent of the first: under the notion of "presentationless" proof set out above, the grasped axioms must be provable.

As applied to set theory, genericity is also explainable in terms of this duality between quantification (proofs of quantifier statements) and axioms. Fundamentally, sets are grasped directly in terms of a collection of axioms for set theory. (In our case, these will be the axioms for IZF, intuitionistic Zermelo-Fraenkel.) Second, we understand quantification over the class of sets in such a way that it does not affect the informational parameters of realizability sets. The generic notion of sets is most easily illustrated for second-order arithmetic. Here, we say that we can understand axioms for a theory of unbounded quantification over species on the natural numbers and that quantification over species is nugatory in its proof-theoretic effects. Specifically, we will want to say that

$$R(n, \forall X\ \phi) \text{ iff, for all } \overline{A}, R(n, \phi[X/\overline{A}]) \text{ and}$$

$$R(n, \exists X\ \phi) \text{ iff, for some } \overline{A}, R(n, \phi[X/\overline{A}]).$$

In these conditions, $\overline{A}$ designates the realizability set associated with a species A. Clauses like these, which shunt set-theoretic quantification outside of the realizability operator R, do give constructive sense to the "genericity" picture. First, in order to manipulate the clauses mathematically, we need already to have grasped (in the language in which the clauses are written) some bit of set-theoretic machinery; certainly, a set of axioms would do. Second, one sees that set parameters are completely isolated from proof parameters. Therefore, the only way in which one could prove an assertion about *all sets* is to prove it in a wholly generic fashion, by using axioms for set theory, and in total ignorance of the (metatheoretic) fact that various realizability sets might carry mathematically interesting sorts of information as the witnesses of their realizability elements.

5.1. Note. The *cognoscenti* will recognize the clauses just displayed as coincident with those of "Kreisel-Troelstra realizability" for HAS, standard second-order intuitionistic arithmetic. ∎

When the "underlying matter" of the mathematics is the domain of abstract sets, the form of genericity manifests itself in recursion clauses like these:

$$R(n, \forall x\ \phi) \text{ iff, for all realizability sets } a, R(n, \phi(a))$$
$$R(n, \exists x\ \phi) \text{ iff, for some realizability set } a, R(n, \phi(a)).$$

One can tell a story analogous to that told above to explain why these clauses give constructive form to the idea of generic set as applied to abstract sets. As it affects presentations and proofs, the genericity idea is clear from the first condition (that interpreting universal quantification): a proof of a universal quantification is coded as a number n that serves, uniformly, as a proof of any instance of the quantification. The proof is unaffected by any possible information about particular sets. What allows us to verify that universal quantifications are provable is a grasp of axioms of set theory in the metatheory. With those axioms, one can show (cf. Chapter Two) that all the axioms of the set theory are provable.

With respect to sets, the genericity idea can be expressed more concisely in terms of a contrast between presentations of membership and presentations of sethood. The notion of constructive set that comes from Poincaré tells us that individual statements of membership do have presentations: these are the "codes," the first elements of the pairs that comprize realizability sets. In contrast, the generic account of quantification tells us that no set has a presentation of sethood; by being told that some item is an abstract set, we are being given no mathematical information. What supplies the requisite mathematical information about the class of all sets is a collection of axioms of set theory.

5.2. Note. A tentative (but not altogether satisfactory) representation of the sorts of proofs that give proof conditions for universally quantified set assertions is in terms of "schematic" or "free variable" proofs about sets. In this case, a free variable proof makes reference to sets only as unrestricted values of free set parameters. The free variable proof is recognized as valid when it employs only constructively valid reasoning from the axioms for sets and, otherwise, makes no reference to any structural analysis of sets. Simply put, a free variable proof of a universal conclusion about sets is seen to be correct because one can see that, regardless of the replacement of the free variables by terms for sets, the conclusion of the proof remains correct. ∎

Although quantification over sets conceived generically has the effect of driving a firm wedge between the notion of set quantification and *echt* constructive mathematical information, there is no implication from this fact to the conclusion that this understanding of 'set' has no appreciable mathematical consequences. In the realizability structure $\mathbf{V}(Kl)$, one quickly finds that a strong uniformity holds between sets and natural numbers. $\mathbf{V}(Kl)$ satisfies

$$\forall x\ \exists n\ \phi(x,n) \rightarrow \exists n\ \forall x\ \phi(x,n).$$

In the literature on models for constructive mathematics, the principle in question is conventionally called 'UP,' for 'Uniformity Principle.'

From a classical standpoint, the truth of UP can only be a source of bewilderment. From a viewpoint on which constructive set quantification is generic, the truth of UP is readily explicable. Remember that, on the generic approach, no two sets are (at least at the level of the interpretation of quantification) distinguished by means of proof information. Also, recall that the *premier ideè* of realizability is that the members of (the domain of an applicative structure on) ω encapsulate all there is to say about constructive proof, and that, conversely, mathematical information about members of ω is wholly taken up in talk of constructive proof. Then, given that $\forall x\ \exists n\ \phi(x,n)$, we have an assignment of a natural number to each member of $V(Kl)$. If from the proof of this assertion, there were derivable the means to assign two distinct natural numbers to two distinct sets, then we would possess proof- (or realizability-)theoretic means, at the level of set quantification, for distinguishing between sets. By genericity, this is something that we cannot do. Consequently, a simple analysis of the proof conditions of $\forall x \, \exists n \, \phi$ shows that a proof of this can only occur by way of assigning a single natural number uniformly to all sets. It follows that we have a constructive proof of $\exists n \, \forall x \, \phi(x, n)$.

5.3. Note. We make no pretense that the discussion of the above paragraph gives anything other than a motivation, addressed to a classical audience, for the inevitability of UP under realizability. Constructive proofs of UP are forthcoming; see Chapter Three. ∎

When the time comes, it will be clear that the truth of UP has some important mathematical consequences. In Chapter Three, we give a direct proof of the fact that, constructively, whenever a set A is nonempty, UP implies that the powerset of A is unsubcountable. (A set B is subcountable whenever it can be enumerated by a subset of ω; for more details, see Section 6.)

Defining realizability for set theory.

Extensional realizability for set theory comes from fitting together the salient parts of each of the preceeding sections: Poincaré's notion of set, the Heyting-Kleene interpretation of propositional connectives and the generic account of quantification over sets.

Let a, b, c and d stand for elements of $V(Kl)$ and let e, f and g range over ω. We presuppose a primitive recursive pairing function on ω for which x_0 and x_1 are the unpairing functions. As usual, '$\{e\}(n)$' represents the result of applying the Turing machine with index e to numeral n. In giving the interpretation, we assume that, whenever we write an application term, it is defined. The language over which realizability is defined is that of classical ZF augmented by names for the elements of $V(Kl)$.

Realizability is defined recursively on the subformula tree of ϕ; for assertions of atomic form, it is defined by transfinite recursion on $\in$. For sentences ϕ of the augmented language, read $e \Vdash \phi$ as "e realizes ϕ."

5.4. Definition.

$e \Vdash a \in b$ iff $\exists c\,(\langle e_0, c\rangle \in b$ and $e_1 \Vdash a = c)$ [1]

$e \Vdash a = b$ iff $\forall c, f\,(\langle f, c\rangle \in a$ implies that $\{e_0\}(f) \Vdash c \in b$ and [2]
$\langle f, c\rangle \in b$ implies that $\{e_1\}(f) \Vdash c \in a)$

$e \Vdash \phi \wedge \psi$ iff $e_0 \Vdash \phi$ and $e_1 \Vdash \psi$ [3]

$e \Vdash \phi \vee \psi$ iff either $e_0 = 0$ and $e_1 \Vdash \phi$ or [4]
$e_0 \neq 0$ and $e_1 \Vdash \psi$

$e \Vdash \phi \rightarrow \psi$ iff $\forall f\,(f \Vdash \phi$ implies that $\{e\}(f) \Vdash \psi)$ [5]

$e \Vdash \neg\phi$ iff $\forall f\ \neg f \Vdash \phi$ [6]

$e \Vdash \forall x\,\phi$ iff $\forall a\ e \Vdash \phi(a)$ [7]

$e \Vdash \exists x\,\phi$ iff $\exists a\ e \Vdash \phi(a)$ [8]

We say that $V(Kl)$ satisfies ϕ (in symbols, $V(Kl) \models \phi$) whenever $\exists n\ n \Vdash \phi$. ∎

The relative complexity of clauses [1] and [2] is required to guarantee that EXT, the axiom of extensionality, is satisfied. Conditions [3] through [6] reprise Kleene's original interpretation of the propositional connectives. [7] and [8], for the quantifiers, show that the interpretation of constructive quantification over the class of sets is *generic*.

One should not conclude from all this that the adoption of genericity for sets amounts to or entails the abdication of Heyting's and Kleene's insights into constructive quantification. On the latter two accounts, proof information has direct impact on the understanding of quantification, and although this stands in contrast with generic quantification, it does anything but exclude it. Rather, we might say that, given the realizability interpretation of implication, genericity for the class of sets entails specificity for individual sets. It follows from clauses [5] and [7] that, whenever quantification is restricted to a particular set, the proof-theoretic characterization of the contents of the set plays a full Heyting-Kleene role in quantification over it. One can give a formal proof that, once schematic quantification comes to interact with the realizability conditions on proofs of conditionals, the (realizability form of the) Heyting interpretation reappears.

As an example, we can sketch such a proof in the case when quantification is restricted to ω. From the definition of $\Vdash$, we know that

$$n \Vdash \forall x\,(x \in \omega \rightarrow \phi(x)) \text{ iff, for all } a, n \Vdash (a \in \omega \rightarrow \phi(a)).$$

The righthand side of this condition will hold only if, for all realizability sets a and all natural numbers m, we have that

$$\text{if } m \Vdash a \in \omega, \text{ then } \{n\}(m) \Vdash \phi(a).$$

This clause has already a reasonable (albeit abstract) resemblance to the Heyting-Kleene clauses for quantification, on which that the truth of a quantified formula is conditioned by facts about constructive proof. However, the exigencies of $V(Kl)$ allow one to go further and bring these conditons into perfect accord with those of Kleene's original realizability model. Later on, we will prove that there is, up to the realizability of $=$, a single internal representative of ω which we call '$\overline{\omega}$.' We will also prove that quantification restricted to $\overline{\omega}$ is merely number-theoretic quantification. Roughly speaking, for any realizability set a,

$$m \Vdash a \in \overline{\omega} \text{ iff } a = m.$$

Knowing only this much, one can see that the last but one of the displayed equations reduces to give the result that

$$n \Vdash \forall x\, (x \in \omega \rightarrow \phi) \text{ iff, for all } m, \{n\}(m) \Vdash \phi(m).$$

It will follow from this, plus sundry details, that $\forall x \in \omega\ \phi$ is realizably true in set theory just in case it is realizably true in the sense of Kleene. And this holds just in case $\forall x \in \omega\ \phi$ is "recursively true" in the sense of Heyting. Chapter Four contains a proof that this correspondence is global: for ϕ a sentence of arithmetic,

$$V(Kl) \models \phi \text{ iff } \exists n\, R(n, \phi).$$

Arithmetic sentences are realized over $V(Kl)$ just in case they are realized in the original Kleene sense. Hence, realizability for set theory is truly a generalization plus a "recursivization" of the insights of Heyting.

Section 6: Intuitionistic set theory

The reduced or lawlike continuum is the structure on the reals that derives from an analysis of individual reals as completely presented by rule-governed sequences. This conception stands at the very antipodes from that of the generic reals; here every real number has a presentation which is given as a constructive rule which, in part, specifies a sequence of rationals approximating the number. We believe that, before 1967, a majority of constructivists would have accepted Brouwer's attitude toward the reduced continuum. Brouwer held that the reduced continuum was, for measure-theoretic reasons, geometrically inadequate as a picture of the reals, and that, for a constructive analysis to be at all respectable as an analysis of the continuum, it would have to enlarge its vision to include reals not given by laws.

In 1967, Errett Bishop's Foundations of Constructive Analysis appeared; there, Bishop provided a development of a purely constructive analysis so successful that it forced a thorough reëvaluation of Brouwer's attitude. Bishop showed how to elaborate, given only the means of "lawlike analysis," a full constructive analysis, up through measure theory. As prolegomena to his "New Constructivism," Bishop made an appeal to an informal constructive set theory. As interpretation of that theory, Bishop adumbrated a concept of set which is, in some respects, similar to that which we discovered in Poincaré. More significantly, Bishop's reliance on a "constructive set theory" established a new foundational task: the axiomatization of a set theory sufficient unto the needs of Bishop's analysis.

Beginning about 1971, a number of researchers, including Myhill, Feferman, Friedman, Powell and Beeson, made detailed proposals on the form of the requisite set theory. Records of the early investigations appear in Myhill (1973), Myhill (1975) and Friedman (1973a). The strongest of the theories surveyed was the intuitionistic set theory IZF, which derives from classical Zermelo-Fraenkel by modifying axioms in accord with some extremely minimal constraints. Briefly, IZF results from ZF by dropping full AC, formulating Fraenkel's replacement axiom as a scheme of collection and putting foundation in the form of transfinite induction. Needless to say, the logic of IZF is the intuitionistic predicate logic of Heyting. (For an apologia for IZF and a discussion of the "minimal constraints," see Chapter One.)

In particular, IZF is formulated in a single-sorted first-order language with $\in$ and $=$ as its only primitive nonlogical predicates. The axioms of IZF are all instances of (1) through (8):

6.1. Axioms of IZF.

$$(1)\quad \forall x\, \forall y\, (\forall z\, (z \in x \leftrightarrow z \in y) \rightarrow x = y) \qquad \text{[EXT]}$$

$$(2)\quad \forall x\, \forall y\, \exists z\, (x \in z \wedge y \in z) \qquad \text{[PAIR]}$$

$$(3)\quad \forall x\, \exists y\, \forall z\, \forall u \in x\, (z \in u \rightarrow z \in y) \qquad \text{[UN]}$$

$$(4)\quad \forall x\, \exists y\, \forall z\, (z \in y \leftrightarrow (z \in x \wedge \phi)) \qquad \text{[SEP]}$$

$$(5)\quad \forall x\, \exists y\, \forall z\, (\forall u \in z\, (u \in x) \rightarrow z \in y) \qquad \text{[POW]}$$

$$(6)\quad \exists x\, ((\exists u \in x\, \forall y\;\; y \notin u) \wedge \forall y \in x\, \exists z \in x\;\; y \in z) \qquad \text{[INF]}$$

$$(7)\quad \forall x\, (\forall y \in x\, \exists z\, \phi) \rightarrow \exists u\, \forall y \in x\, \exists z \in u\, \phi) \qquad \text{[COLL]}$$

$$(8)\quad \forall x\, (\forall y \in x\, \phi(y) \rightarrow \phi) \rightarrow \forall x\, \phi \qquad \text{[IND]}$$

∎

In classical logic, the axioms of IZF are equivalent to those of traditional Zermelo-Fraenkel as they are usually formulated. In intuitionistic logic, this equivalence fails; IZF derives neither the the general law of excluded third nor the classical axiom of foundation.

Section 7: On the internal mathematics of realizability, a preview

We have said that there must be a consonance between the two themes—axiomatic and quantificational—that meld to form the generic conception of set. Under the account of quantification, the axioms of the relevant theory must be correct. The soundness proof for IZF with respect to extensional realizability shows that the consonance requirement is satisfied. Just as Kleene proved that HA is sound with respect to his realizability, so we can prove

7.1. Theorem. *If* IZF $\vdash \phi$, *then* $\mathbf{V}(Kl) \models \overline{\forall x}\ \phi$.

Proof. Here, $\overline{\forall x}\ \phi$ is a universal closure of ϕ. For complete details, consult Chapter Two. This proof, just as Kleene's, proceeds by induction of the length of formal derivations in IZF. ∎

From Section 1, we know that recursive mathematics as elaborated by Dekker, Myhill, Crossley, Nerode, Ellentuck *et al.* bears a surprizingly close semantical relation to fields of mathematics over $\mathbf{V}(Kl)$. All the notable subfields of recursive set theory, the theories of RETs, isols, isolic integers, COTs and losols are subtheories of pure nonrecursive set theory over $\mathbf{V}(Kl)$. Moreover, the "realizability correlates" of each of these domains is easily definable in the language of IZF over $\mathbf{V}(Kl)$. Even the theory of effectively given (eg) information systems fits snugly into the same pattern. Under realizability, the external category of eg systems is isomorphic to a category of noneffective information systems internal to $\mathbf{V}(Kl)$. We might summarize this situation by saying that $\mathbf{V}(Kl)$ is truly a universe for recursive mathematics.

The appropriateness of this summary is borne out in the "recursive" axioms that also hold in $\mathbf{V}(Kl)$. (Some of these are proved using classical logic in the metatheory.)

7.2. Theorem. $\mathbf{V}(Kl) \models \mathrm{MP} \wedge \mathrm{CT} \wedge \mathrm{AC}^{\omega} \wedge \mathrm{DC}$.

MP is *Markov's Principle*:

$$\forall x \in \omega\, (\phi \vee \neg \phi) \rightarrow (\neg\neg \exists x \in \omega\ \phi \rightarrow \exists x \in \omega\ \phi)$$

CT is *Church's Thesis*:

$$f \in (\omega \Rightarrow \omega) \rightarrow \exists e \in \omega\ \forall x \in \omega\ \exists y, u \in \omega\ (T(e, x, u) \wedge U(u, y) \wedge f(x) = y)$$

T represents the Kleene "T predicate" and U is the upshot function. CT says that every total function from ω into ω is general recursive. AC^{ω} is the *Axiom of Countable Choice*:

$$\forall x \in \omega\ \exists y \in A\ \phi(x,y) \rightarrow \exists f \in (\omega \Rightarrow A)\ \forall x \in \omega\ \phi(x, f(x)).$$

Finally, DC is the *Principle of Dependent Choice*:

$$\forall x \in A\ \exists y \in A\ \phi(x,y) \rightarrow \forall x \in A\ \exists f \in (\omega \Rightarrow A)\ (f(0) = x \wedge \forall x \in \omega\ \phi(f(x), f(x+1))).$$

Even old-fashioned analysis in $\mathrm{V}(Kl)$ takes on a pleasant form. For one thing, the category of realizability metric spaces is *small*:

7.3. Theorem. *In* $\mathrm{V}(Kl)$, *every metric space is subcountable.*

where a set is subcountable just in case it is enumerated by a subset of ω.

Theorem 7.3 is, in fact, corollary to a far more general result: subcountability extends to the cartesian closed category of sets that admit strict apartness.

7.4. Theorem. *Every set admitting strict apartness is subcountable.*

For nonempty sets in classical set theory, countability and subcountability coincide; any nonempty set is enumerated by ω just in case it is enumerated by a proper subset of ω. In intuitionistic set theory and even in some of its strong extensions, this equivalence may fail. It is then possible for the notion of subcountability to take on an independent interest. In $\mathrm{V}(Kl)$, the distinction between countability and subcountability is crucial and amounts to the distinction between r.e. and arbitrary subsets of ω. All subsets of (external) ω reappear in $\mathrm{V}(Kl)$ as (stable) subsets of ω in $\mathrm{V}(Kl)$; each of these internal sets is trivially subcountable. However, only the representatives of r.e. sets are countable.

We will think of Chapter Eight as, in part, a plea for the notion of subcountability. In certain topological spaces it is trivially true that the "subcountable opens" of the space form a Heyting algebra, but it is, in $\mathrm{V}(Kl)$, false that the countable opens do so.

The axiom of infinity is obviously true in $\mathrm{V}(Kl)$, thanks to the fact that ω has a particularly salient internal representation as the realizability set $\overline{\omega}$. For each $n \in \omega$, let $\overline{n} = \{\langle m, \overline{m}\rangle : m \in n\}$. Then,

$$\overline{\omega} = \{\langle n, \overline{n}\rangle : n \in \omega\}$$

With respect to this internal version of ω, $\mathbf{V}(Kl)$ mediates a strict correspondence between relations which are classically recursive and those which the intuitionist would call 'decidable'.

7.5. Definition. A subset B of A is *decidable* on A iff $\forall x \in A\, (x \in B \vee x \notin B)$.

∎

Since the intuitionistic law of excluded middle fails of universal validity, the assumption of decidability is a significant *mathematical constraint* on the structure of a set. It is easily seen that a subset of the internal natural numbers ω is decidable in $\mathbf{V}(Kl)$ just in case, when viewed from without, it is recursive. To see this, assume that

$$\mathbf{V}(Kl) \models Y \subseteq \overline{\omega} \text{ is decidable.}$$

By definition, this means that there is an index e such that $e \Vdash \forall x \in \overline{\omega}\, (x \in Y \vee x \notin Y)$. By clauses [4] and [6] of 5.4, e decides, for each internal number $\overline{n}$, which of $\overline{n} \in Y$ or $\overline{n} \notin Y$ holds. Specifically, we can assume without loss of generality that $\{e\}$ is total, $\{e\} : \omega \to 2$ and $\mathbf{V}(Kl) \models \overline{n} \in Y$ if $\{e\}(n) = 0$ while $\mathbf{V}(Kl) \models \overline{n} \notin Y$ if $\{e\}(n) = 1$. Conversely, every recursive set can be injected into $\mathbf{V}(Kl)$ as a decidable set. If (external) Y is recursive, just "internalize" its characteristic function, c_Y, as a realizability set. Then, $\mathbf{V}(Kl) \models \overline{c_Y} : \overline{\omega} \rightarrowtail 2$. Therefore, $\mathbf{V}(Kl) \models \forall x \in \omega\, (\overline{c_Y}(x) = 0 \vee \overline{c_Y}(x) = 1)$ and $\mathbf{V}(Kl) \models (\overline{c_Y})^{-1}(0)$ is decidable.

$\mathbf{V}(Kl)$, therefore, faithfully manifests the mathematical duality we found inherent in Brouwer's intuitionism. Not only does the internal mathematics of realizability provide a setting for a full higher-order constructive theory of abstract sets, but, as we have seen, it reflects internally many of the salient relations on the domain of proofs. When the domain of proofs has been coded as a structure on ω, the face that $\mathbf{V}(Kl)$ directs toward the proofs is an expression of fragments of recursion theory.

7.6. Note. The internal "recursiveness" of $\mathbf{V}(Kl)$, as evidenced by the internal truth of Church's Thesis, is bound up not so much with these fortuitous correspondences but, rather, with a basic model-theoretic fact about realizability. This fact underlies both the correspondences and Church's Thesis. Our version of this fact is called 'Kleene absoluteness' and it is the claim that recursive relations on ω are absolute over $\mathbf{V}(Kl)$. This means that, if appropriately defined, the recursive relations are altered neither intensionally nor extensionally by passage under the realizability operator. Chapter Four contains all the pertinent details. ∎

The face tht $V(Kl)$ directs toward the domain of (coded) constructive proofs, the face whose lineaments are recursive, is not that of a total stranger, even for the traditional constructivist. Given the correspondences set out in the last paragraph, one sees that simple unsolvability theorems of classical recursion theory can carry directly over into independence results for IZF (plus whatever other principles hold over $V(Kl)$). In Chapter Two, we apply this idea to calculate internal "realizability" versions of the halting problem and to show that there is a discernible array of these versions that relate naturally to Brouwer's method of *weak counterexamples.* The unsolvability of the halting problem, once expressed over $V(Kl)$, is sufficient to capture, over $V(Kl)$, all of Brouwer's traditional weak counterexamples as *strict falsehoods.* Our method for capturing counterexamples is completely uniform and applies not only to those weak counterexamples that reduce a mathematical problem to the question of intuitionistic decidability, but also to those that reduce a problem to intuitionistic testability. For a discussion of the method of weak counterexamples, see McCarty (1983).

Section 8: What is constructivism? A philosophical postscript

In constructivism, as elsewhere in philosophy, the *phainomena* seem, at first, mixed and confusing. If we ignore finitism for a moment, intuitionistic and recursive mathematics stand poles apart in the the world of "constructive" mathematics. Recursive mathematics is obviously a form of constructivism, but, in comparison with the *complete* constructivism represented by intuitionism, it seems skewed and fragmentary. It seems skewed because the recursive mathematician does not constructivize his logic; recursive mathematics lives under the sway of classical logic. It seems fragmentary because only certain features of its objects have been constructivized, or better, "recursivized." If we remain solely on the level of superficial comparison, we can only conclude that the term 'constructivist' as applied to intuitionist and to recursive mathematician is maddeningly equivocal.

One should, however, never be overwhelmed by appearances. Our investigations show that there is an underlying semantic structure, that embodied in realizability, common to both forms of mathematics, intuitionistic and recursive. We have proved that recursive mathematics must appear, on the semantic level, as constructive a mathematics as the intuitionistic. We see various fields of recursive set theory as nothing more (nor less) than nonstandard interpretations of natural theories in intuitionistic set theory. Recursive mathematics is, therefore, a semantically disguised form of intuitionism.

Perhaps a few details are in order. For example, in Chapter Five is a proof that $\underline{\Omega}$, the first-order algebraic structure on the collection Ω of RETs, is embedded into $\mathbf{V}(Kl)$ in such a way that both (1) and (2) hold:

(1) Under realizability, $\underline{\Omega}$ becomes exactly the structure given by the natural constructive cardinal arithmetic on the ω-stable elements of $\mathbf{P}(\omega)$. The embedding works so that the "recursive" content of each of the relations and operations on $\underline{\Omega}$ *disappears* into realizability. In particular, partial recursive equivalence $\simeq$ in $\underline{\Omega}$ is identified with pure cardinal equivalence over $\mathbf{V}(Kl)$, and the partial recursive inclusion $\preceq$ becomes the notion of "separable subset" familiar from the writings of Brouwer.

(2) More importantly, *none* of the first-order classical theory of $\underline{\Omega}$ gets lost along the way. (1) shows that this already holds for the atomic formulae of the relevant languages; (2) shows that the result extends to arbitrary formulae. There is a simple, recursive translation *tr* from the language L_{Ω} for $\underline{\Omega}$ into the language of IZF, such that, for each sentence ϕ of L_{Ω}, ϕ^{tr} is just what one would want to be the expression of the "nonrecursive" content of

ϕ. Moreover, if classical logic is properly set up, tr can be taken to be *invariant* on the logical constants of L_Ω. With ϕ^{tr} properly defined, we can prove that

$$\underline{\Omega} \models \phi \text{ iff } V(Kl) \models \phi^{tr}.$$

This assertion, which amounts to a kind of "isomorphism theorem" for realizability structures, is understood and proved as a statement of classical mathematics.

It follows from the isomorphism theorems that recursive set theory is really a classical mathematician's view of Kleene's recursivization of the Heyting interpretation. Once we see Heyting through the eyes of Kleene, we have no difficulty in locating recursive mathematics within a notion of constructivity that includes intuitionism, initial "appearances" notwithstanding.

On methodological reduction.

There looms a potential misunderstanding that we hasten to dispel. Underlying these theorems and all the attendant verbiage, there is no hidden suggestion that one should opt either for a "reduction" of recursive mathematics or for a methodological elimination of large parts of recursive constructivity in favor of interpreted intuitionistic mathematics.

This kind of elimination would serve only to eliminate a fair amount of the metamathematical gains that accrue to our theorems. One is encouraged to remember that isomorphism theorems like those above do their work in two directions and one of these is from recursive mathematics *into* intuitionistic set theory. Via the isomorphisms, negative results about the RETs, isols and isolic integers go directly and uniformly, without fuss about the logic, into independence results for IZF. The reader will find a plethora of such results proved in the final section of Chapter Five. We believe that the subject would be greatly depleted should the possibility of such interaction be eliminated.

The status of the relation between analytic and synthetic geometry is paradigmatic for the multiple relations between recursive mathematics and mathematics internal to $V(Kl)$. There are *two* dualities associated with the name of Descartes and, luckily, one of them still holds good. This is the duality between analytic and synthetic Euclidean geometries. Here is a mathematical situation similar in form to that between recursive and intuitionistic set theory, and a situation that eschews reduction and grants working rights to each of the fields involved. Synthetic geometry remains to enliven algebraic relations with visual imagery; the analytic geometry underwrites *inter alia* the transfer into Greek geometry of powerful algebraic methods. One need only reflect on the details of Lindeman's proof of the impossibility of trisection to see the power of nonreductionism as it applies here. We would prefer to view recursive and intuitionistic mathematics as living in a similar symbiosis, one on which each is allowed to shed some nonreflected light onto the darker side of the other.

8.1. Note. Mention should be made of a methodological fact which is not unconnected with this discussion. Thanks to the isomorphism theorems, even the pure constructivist can, at times, find classical proofs of both positive and negative results from recursive mathematics extremely useful. The foundational work on RETs done by Dekker and Myhill (1960) continually suggested to us proof methods which, although inspired by classical recursion theory, suggest purely constructive results which otherwise might have been missed. An attitude of nonreductionism leaves the classical theory of recursive cardinals available to the constructivist as sort of exemplar. ∎

Intuitionism as a foundation.

Philosophers of mathematics are almost unanimous in their rejection of intuitionism, not to mention constructivism generally, as a satisfactory answer to the "foundational questions" of mathematics. (Admittedly, the sense of the word 'foundation' on which some branch of logic or mathematics can be viewed as a foundation for the rest has always been obscure to us. We would be loathe to offer any suggestion as to what this sense might be.) Among the reasons given for the rejection, there is a clear statistical favorite. We are referring to the oft-cited "weakness" of intuitionistic logic. In support of the charge of weakness, most purveyors of the objection point to one or another of some highly superficial "facts". One such "fact" is that cherished theorems of classical analysis, like the intermediate value theorem, are not intuitionistically provable.

The theorems of the chapters to follow serve, in part, as a reply to this sort of objection. Our results show just how superficial a collection of "facts" these nonderivability results are. They also prove that, for classically specifiable enterprises like recursive mathematics, mathematics in constructive logic is in some ways superior to its classical analogue.

One straightforward, but limited, response to the objection is already available. The response we have in mind rests on the Gödel-Gentzen translations, not only of classical into intuitionistic logic, but also of the classical into the intuitionistic set theory. (In this regard, cf. Friedman (1973b) and Powell (1975).) The existence and character of the translations do something to expose the superficiality of the facts cited above. The existence of the translation shows that there is, for each theorem of classical analysis, a uniformly-specifiable version of it that can be obtained among the hereditarily-stable sets in a pure intuitionistic set theory.

Moreover, the uniform character of the translations and the fact that all the respective entailments are preserved affords the intuitionist a plausible explanation of the relative success of classical mathematics. To the "speaker" of intuitionistic set theory, classical set theory works as well as it does because it has drawn its mathematical horizons within severe limits. The classical mathematician restricts the available mathematical universe to those sets which are in the image of the translation, the hereditarily stable sets. This limitation advances the classical enterprise, but, at the same time, bars the classical mathematician from investigating sets and notions that lie outside his limits. None of the results we obtain about recursive mathematics could be obtained directly over the hereditarily stable fragment of the class of realizability sets. The class of hereditarily stable sets is precisely that class on which the "proof parameter" of realizability is mathematically otiose. Hence, in so restricting himself, the classical mathematician ignores the extra mathematical information which might be available to him if he were to reflect on the possible evidence for mathematical claims. (See our discussion of the hereditarily stable sets in Chapter Three.)

Unfortunately, the Gödel-Gentzen maneuver may not afford a final answer; it certainly does not afford the most compelling answer. This is because the translation not only "stabilizes" the mathematical subject matter but also some of the logical operations. Hence, an anti-constructivist may have a reply open to him. He might argue that, once logical operations like existential quantification have been stabilized, the basic concepts of mathematics have themselves been altered. After stabilization, it seems that the notions of fundamental concern to the mathematician: set, function and number, have been distorted along with the logical forms of expression for them. The Gödel translation of the notion of set is not the full notion of set, the translation of the notion of (total) functions does not necessarily coincide with the intuitionists' idea of function. Shortly put, it might be said that the intuitionistic mathematician cannot, using this translation scheme, account for classical mathematics as recognizable *mathematics*. Hence, it is only in a Pickwickian sense that the intuitionist can "derive" all the theorems of classical set theory. If mathematics as a subject is identified by any of its perennial concerns, intuitionistic reconstrual along Gödel's lines will entail that, in intuitionistic eyes, classical mathematics ceases to be mathematics.

The isomorphism theorems and their character offer to the pro-constructivist an impressive response to the "weakness" complaint. He can reply to the charge of weakness by providing translation that alters neither the logic nor the basic mathematical "character" of the fundamental concepts of classical recursive set theory. As we have seen, the elements of recursive mathematics are captured over $\mathbf{V}(Kl)$ without change in logical form; *tr* can be defined so as to leave the force of the classical connectives unchanged. In the case of RETs (and this applies equally elsewhere), the basic nature of the recursive cardinality concepts is not obscured, but, we might claim, illuminated in the transition to $\mathbf{V}(Kl)$. The classical recursive relation $\simeq$ of equivalence is seen as a restriction of pure cardinal equivalence over $\mathbf{V}(Kl)$. Therefore, we can reply to our objectors by saying that significant portions of classical mathematics can be incorporated into interpreted constructive mathematics in a way that seems mathematically unimpeachable. Furthermore, all this is due to the soundness of realizability, so, if it's weakness on the part of intuitionistic logic that permits these kinds of interpretations, so be it.

From our standpoint, the mistake that lay in the charge of weakness is easily diagnosed. The mistake derives from an insistence, on the part of the objector, in pressing certain invidious comparisons between classical and intuitionistic theories. In light of the isomorphism theorems, we would want to press alternative comparisons. Instead of pitting classical against intuitionistic analysis, we would encourage a comparsion of intuitionistic analysis with classical recursive analysis and of intuitionistic set theory with recursive set theory. It seems to us that the usual style of comparison already takes one well along the path to prejuding the issue against intuitionism and should be resisted at all costs. That style of comparison can only seem fair when the extra resources of the intuitionistic connectives, as understood by Heyting and captured (to some extent) in realizability, have been ignored.

A final note. Superficial weakness in intuitionistic logic allows for a situation which we call "axiomatic freedom." The freedom comes with the recognition of the fact that a fully expressive set theory like IZF is consistent with mathematically useful but classically false axioms of tremendous consequence. One need only mention Extended Church's Thesis, Brouwer's Theorem for number-theoretic functions and the "Brouwer Theorem" for constructive information systems. Each of these is classically false yet true in $\mathbf{V}(Kl)$. At the same time, each is of value and interest to the classical mathematician. Of these, we will only have the opportunity to detail the classical consequences of axiomatic freedom as they flow from the Brouwer Theorem for information systems. For that, we refer the reader to Chapter Seven.

Section 9: Prospectus

A synopsis of the ensuing chapters will round off our introduction. Chapter One is a short course in and apologia for the intuitionistic set theory IZF. The short course is largely of the "hands on" variety; the reader is confronted with a sampler of results from that theory, together with their proofs. The choice of results is, in part, inspired by the uses to which the set theory will later be put. Our primary concern is with the basic facts provable in IZF about cardinals and Dedekind-finite cardinals from $\mathbf{P}(\omega)$. The realizability interpretation of these facts gives the basic facts about the RETs and isols.

Realizability makes a formal entrance in Chapter Two, where it appears in the very general setting afforded by models of the APP axioms. The resultant "abstract" realizability is called by us 'general extensional realizability;' the more apposite term, 'abstract realizability,' has already been attached, in the literature, to other interpretations. Sadly, we have little time to linger over realizability at so general a level and move quickly on to the APP-model discovered by Kleene. The use of APP and the attendant realizability is by no means an idle generalization. Were the investigative net spread no wider than Kleene realizability, APP would still be the appropriate place to start. It turns out that the proofs of the soundness theorems for intuitionistic logic and set theory with respect to general realizability are not only much prettier than those in recursive realizability, but also, since the possible realizability witnesses are constrained only by APP, the proofs are much less encumbered with extraneous details and are easier to "see."

Chapter Three is an extended discussion of the internal mathematics of set-theoretic realizability. Perforce, this chapter is a patchwork of old and new. The realizability of dependent choices, e.g., is very easy and has been known for many years. By contrast, we have included some new results on the subcountability of metric spaces that solve problems posed by Beeson. Chapter Four is a survey of metamathematical relations: those between set theory and first-order arithmetic and between set-theory and second-order arithmetic. We prove that Kleene's original realizability is the submodel of $\mathbf{V}(Kl)$ obtained by restricting quantifiers to ω and that Kreisel-Troelstra realizability for the second-order arithmetic HAS is precisely the submodel of $\mathbf{V}(Kl)$ culled out by restricting attention to ω within $\mathbf{P}(\omega)$.

In Chapter Five, all material is completely new. It is here that we begin to draw the picture of recursive set theory as a subtheory of "realizability set theory." The RETs, isols and isolic integers are each treated as we have indicated above. Needless to say, Chapter Five is the mathematical centerpiece. Chapter Six is properly an appendix to Chapter Five; only its length requires that it have independent standing. The root issue of this chapter is a technical question internal to the workings of Chapter Five: "Is the relativization of quantifiers from $\mathbf{P}(\omega)$ to $\mathbf{P}(\omega)^{st}$ actually required in the working out of the constructive versions of theories from recursive set theory? If so, why isn't it apparent from the proofs in Chapter One?" Our answer to the first question is 'No' and takes the form of yet another realizability interpretation, one in which the necessary relativization has been discretely "hidden." The relativization of quantifiers is hidden in much the same way as the recursive machinery of the external recursive mathematics had, in Chapter Five, been hidden behind the realizability interpretations of the constructive logical signs.

Chapter Seven is a reasonably complete display of the advantages of realizability for the foundations of denotational semantics. Scott's notion of *information system* is shown to have a natural constructive formulation, and, when that formulation is given a realizability interpretation, a "Brouwer's Theorem" holds for information systems. The Brouwer's Theorem for information systems is the statement that every set morphism between information systems is continuous and monotone. (This answers a question posed some time ago by Dana Scott.) Realizability is also exploited to show how classical information about effective indexing can be drawn painlessly out of constructive proofs. (This answers a question about domains posed originally by Gordon Plotkin.)

If we think of Chapter Seven as a study in the applied topology of "realizability mathematics," then Chapter Eight is a study in its pure topology. There, we offer a brief but promising look into the relations between recursive point-set topology and its constructive analogue. The *modus operandi* is much as it was with the RETs; we prove that results obtained in a classical setting by Kalantari and Retzlaff coincide with intelligible independence theorems concerning IZF.

Some general remarks.

(1) In Chapters One through Four, the logic of the object language (be it IZF or HA or HAS) is uniformly intuitionistic. And, unless there is indication to the contrary, the metalogic will be intuitionstic as well.

In the remaining Chapters (Five through Eight), the logic of the metalanguage will be uniformly classical. After all, we are here hunting after relations between realizability models and *classical structures*. The various object languages fall where they may: the logic of the theory of RETs is classical, but that of HAS^{st} is intuitionistic. The reader must rely on contextual clues as a guide.

(2) Once in the realm of recursive realizability, we make full and repeated use of the notational distinction (due, we believe, to Kleene) between "λ" and "Λ" specifications of functions. Whenever ϕ is an applicative context, $\lambda x\phi$ denotes the function whose value at each permissible argument y is $\phi[x/y]$. When ϕ specifies a partial recursive number-theoretic function, then $\Lambda x\phi$ denotes the Gödel number (or Turing machine index) of the function $\lambda x\phi$.

(3) Finally, we take the capital Greek letters 'Φ', 'Ψ' and 'Θ' as metavariables ranging over the partial recursive number-theoretic functions.

A Primer of Intuitionistic Set Theory

Section 1: Prefatory and historical remarks

The purpose of this chapter is to provide the reader with a brief but serviceable introduction to constructive mathematics at the level of full set theory. On the approach we prefer, we present (and try to motivate) the axioms, detail some samples of work in the theory and illuminate the strengths and weaknesses of this form of intuitionistic mathematics by reference to the metamathematical literature. All this is expressed in continuous prose. Consequently, the details need form something of a *potpourri*; formalized mathematics and informal metamathematics are freely mixed; we trust that the reader will make the necessary metatheoretic gesalt shifts along with us.

One of our not-so-ulterior motives will be to encourage the idea and to advance the argument set out in Chapter Zero to the conclusion that there is no univocal response to the question "Is constructive set theory weaker than its classical counterpart?" Should our efforts fail and should the reader persist in thinking of constructive logic and mathematics as too weak to serve any honest foundational purpose, we may yet hope for him to see that, in certain quarters, weakness is a virtue.

Relatively widespread interest in constructive set theory dates back only to 1971. A number of seminal papers in the field appeared in that year in the Cambridge Summer School in Mathematical Logic, and were published by Springer in number 337 of their Lecture Notes in Mathematics series. Foremost among these are Friedman (1973) and Myhill (1973).

Chapter Zero contains a brief historical introduction to constructive set theories. Beeson (1979) makes a survey of the constructive set theories of current interest. In working out the details of this chapter, we have relied on two excellent sources, Grayson (1978) and Minio (1974). These works together serve as a recommendable guide to the rudiments of mathematics within constructive set theories.

Section 2: Axioms and aetiologies

An aetiological account for IZF can begin with the classical set theory ZFC, which is Zermelo-Fraenkel set theory plus Zermelo's axiom of choice. Needless to say, such an account will be more heuristic than psychogenetic or historical and may even encourage a certain misunderstanding. We do not mean to suggest that ZF is somehow conceptually prior to IZF; one can find an independently intelligible concept of constructive set for which IZF would, recognizably, provide a partial codification. It was one of the points of Chapter Zero to argue that the hierarchies of pure Poincaré or of realizability sets provide such a concept. Rather, in beginning with ZF, we wish merely to move to the unfamiliar from the familiar; the slightly nonstandard formulation of some of the IZF axioms, plus the absence of AC, can be motivated by starting with classical set theory and applying a simpleminded constraint.

We do not intend for anyone to think of this constraint as an indefeasible criterion which any expressive constructive set theory *must* satisfy. Notoriously, attempts to set limits on what is "constructively intelligible," just as with attempts to limit "scientific intelligibility," have a poor track record. Rather, we want only to point out that, first impressions notwithstanding, IZF represents a notion of set that comes naturally from ZF by imposing a minimal amount of constructivization. IZF has the advantage that its axioms can be introduced on the basis of a very general knowledge of constructivism rather than only at the end of some lengthy analysis of the "constructive set" concept.

At the risk of tedium, we will commence with a review of the axioms of ZFC.

Axioms for classical ZF.

2.1. Classical ZFC. ZFC is a theory in the one-sorted predicate language L_{zf} with $\in$

and = as primitive binary predicates. The logic is classical and the axioms are as follows:

$$\text{(1)} \quad \forall x\, \forall y\, (\forall z\, (z \in x \leftrightarrow z \in y) \rightarrow x = y) \qquad \text{[EXT]}$$

$$\text{(2)} \quad \forall x\, \forall y\, \exists z\, (x \in z \wedge y \in z) \qquad \text{[PAIR]}$$

$$\text{(3)} \quad \forall x\, \exists y\, \forall z\, \forall u \in x\, (z \in u \rightarrow z \in y) \qquad \text{[UN]}$$

$$\text{(4)} \quad \forall x\, \exists y\, \forall z\, (z \in y \leftrightarrow (z \in x \wedge \phi)) \qquad \text{[SEP]}$$

$$\text{(5)} \quad \forall x\, \exists y\, \forall z\, (\forall u \in z\, (u \in x) \rightarrow z \in y) \qquad \text{[POW]}$$

$$\text{(6)} \quad \exists x\, ((\exists u \in x\, \forall y\;\; y \notin u) \wedge (\forall y \in x\, \exists z \in x\;\; y \in z)) \qquad \text{[INF]}$$

$$\text{(7)} \quad \forall x\, (\forall y \in x\, \exists! z\, \phi) \rightarrow \exists u\, \forall y \in x\, \exists z \in u\, \phi \qquad \text{[REP]}$$

$$\text{(8)} \quad \forall x\, (\exists y\;\; y \in x \rightarrow \exists z \in x\, \forall y \in z\;\; y \notin x) \qquad \text{[REG]}$$

$$\text{(9)} \quad \forall x\, ((\forall y \in x\, \exists z\;\; z \in y) \rightarrow \exists u\, \forall y \in x\, \exists! z\, (z \in y \wedge z \in u)) \qquad \text{[AC]}$$

∎

The axioms SEP and REP are really axiom-schemes. In SEP, ϕ ranges over all the formulae of the language not containing y free. Similarly, ϕ in REP should not have u free. As always, $\exists! z \phi$ means that there is a unique z such that ϕ.

A minimal constraint.

Proceeding naively, we should say that the first constraint any formal system must satisfy in order to be counted as constructive is a purely formal one. The system has to stay within the confines of intuitionistic logic. An axiom system for the intuitionistic first-order predicate calculus will appear in the next chapter; for the present, it suffices to say that intuitionistic logic is a segment of classical logic which is independent of *tertium non datur* or TND:

$$\phi \vee \neg \phi$$

for arbitrary formulae ϕ. ϕ may, of course, include free parameters. (Cf. Chapter Zero or McCarty (1983) for more information on intuitionistic logic.)

To be quite honest, the TND constraint is just too simpleminded to serve as sufficient condition for the constructivity of a system, but, as a necessary condition, it has an obvious justification. In most formalizations, classical logic is just what you get when you add TND to intuitionistic logic, and classical logic is not sound with respect to Heyting's concept of intuitionistic entailment. Admittedly, the absence of TND is the grossest measure of the constructivity of a system, but, for what are largely historical reasons, we light on it as an expectation of intuitionistic set theory. We will take it as established that, for any system S in standard formalization to qualify as a system for intuitionistic set theory, the law of excluded middle must be independent of S.

2.2. Note. Doubtless, the "criterion" for constructivity set out in the preceding paragraph is too rough-and-ready. In actual fact, intuitionistic logic is neither necessary nor sufficient for the constructivity of a system, or better, for the constructivity of the notions which the system is intended to express. It is clearly not necessary: primitive recursive arithmetic in classical logic is a perfectly reasonable constructive theory of natural numbers. On the other hand, it is unquestionably the case that any number of the intermediate calculi fail of direct and adequate constructive interpretation; to see this one need only reflect on the system axiomatized by intuitionistic propositional calculus plus

$$(\neg p \rightarrow (q \vee r)) \rightarrow (\neg p \rightarrow q) \vee (\neg p \rightarrow r).$$

Hence, the TND constraint is not sufficient. ∎

In the case of full set theory, the way to satisfy the TND constraint is not the metamathematically obvious one. One cannot, in attempting to satisfy the TND constraint, merely do unto ZFC what is usually done unto PA in the move to HA. One cannot simply exchange the classical logic of set theory for the intuitionistic. Myhill (1973) has shown that this simple strategem is unavailable: the classical axiom of foundation, even in intuitionistic logic, entails TND.

For the moment, let IZF be the theory axiomatized by (1) through (4) in intuitionistic logic.

2.3. Note. A certain notion (and notation) is indigenous to discussions of intuitionistic set theory. This is the notion of *restricted singletons.* For any set a and proposition ϕ, the set which contains a just in case ϕ holds is a restricted singleton and is denoted '$\{a : \phi\}$.' With classical logic, a restricted singleton will reduce either to a singleton or to $\emptyset$. In strictly constructive contexts, such reductions are not obtainable; in fact, the reduction procedure amounts to the adoption of TND. ∎

2.4. Proposition. IZF $\vdash$ REG$\rightarrow$TND

Proof. Let $x = \{0 : \phi\} \cup \{1\}$. Use REG to get z as an $\in$-minimal element of x. Then, either $z = 0$ or $z = 1$. If the former, then ϕ. If the latter, $\neg 0 \in x$, and $\neg\phi$ follows. ∎

In order to satisfy the constraint and yet deal with this unfortunate result, we replace REG by IND, its classically-equivalent "positivization:"

$$[\text{IND}] \qquad \forall x\,(\forall y \in x\,\phi(y) \rightarrow \phi) \rightarrow \forall x\,\phi$$

Notice that we've begun to use the helpful bounded-quantifier abbreviations: '$\forall y\ (y \in x \rightarrow$' is abbreviated '$\forall y \in x$.' The dual abbreviation for $\exists$ contexts will also appear.

IND is, of course, the principle of transfinite induction on $\in$. In classical set theory, it is equivalent to REG, but, as the soundness theorem of Chapter Three will show, the equivalence fails intuitionistically. The equivalence fails in intuitionistic set theory for much the same reasons that, in the intuitionistic arithmetic HA, the arithmetical least number principle is independent of the scheme of induction. To be convinced that IND is a satisfactory replacement for REG in a constructive setting, i.e., that IND can do many of the jobs assigned to REG, see Grayson (1975).

The TND constraint also mandates another change, as was shown in Diaconescu (1975). Using 'IZF' as stipulated before the last proposition, one can prove that

2.5. Proposition. IZF $\vdash$ AC$\rightarrow$TND

Proof. Take $x = \{\{0 : \phi\} \cup \{1\}, \{0\} \cup \{1 : \phi\}\}$, set $a = \{0\} \cup \{1 : \phi\}$ and let $b = \{0 : \phi\} \cup \{1\}$. By AC, there is a "choice function" f on x which selects, from a and from b respective elements of each. If $f(a) = 1$, then ϕ; if $f(b) = 0$ then ϕ. The only other possibility is $f(a) = 0$ and $f(b) = 1$. In this case, $f(a) \neq f(b)$, so $a \neq b$ and $\neg\phi$ holds. Therefore, the existence of the appropriate choice function implies TND. ∎

From a constructive standpoint, failure of full AC represents no substantial loss. The only forms of choice which the Bishop-style constructivist ever needs are AC^{ω}, choice over the natural numbers, and RDC, relativized dependent choice. All these are obviously consistent with IZF and, as we prove in Chapter Three, true in $V(Kl)$ and independent of TND. Moreover, Myhill has, in Myhill (1975), proffered general reasons for thinking that, for situations which are extensional, AC is not constructively plausible.

The modifications enjoined by adherence to the TND constraint yield a theory which is "constructive" in a minimal sense and perfectly useful. (The system has even made several appearances in the literature under the title 'ZFI.') However, we prefer to adopt a further slight modification, one encouraged not by a constraint but by a state of noninformation. This state attaches to forms of REP, the Fraenkel-Skolem Axiom of Replacement. As is familiar, in the presence of classical logic and REG, REP is equivalent in ZF to the axiom scheme of collection, COLL:

$$\forall x \in a \exists y \phi(x,y) \rightarrow \exists b \forall x \in a \exists y \in b \phi(x,y)$$

Even intuitionistically, COLL implies REP. Unfortunately, although the converse implication surely seems irredeemably classical, it is unknown whether REG can be eliminated and whether COLL and REG are equivalent within an intuitionistic set theory. Hence, since there are a number of places where COLL would be useful, and we don't know whether it can be obtained constructively from REP, it is a matter of prudence to take COLL over REP. In Chapter Three, we will prove that COLL does not entail TND.

Axioms for IZF.

What follows is the terminus of all this heuristic: the system IZF of intuitionistic set theory. IZF is the deductive closure in Heyting's predicate calculus of the axioms:

2.6. Intuitionistic ZF.

(1) $\forall x\, \forall y\, (\forall z\, (z \in x \leftrightarrow z \in y) \rightarrow x = y)$ [EXT]

(2) $\forall x\, \forall y\, \exists z\, (x \in z \wedge y \in z)$ [PAIR]

(3) $\forall x\, \exists y\, \forall z\, \forall u \in x\, (z \in u \rightarrow z \in y)$ [UN]

(4) $\forall x\, \exists y\, \forall z\, (z \in y \leftrightarrow (z \in x \wedge \phi))$ [SEP]

(5) $\forall x\, \exists y\, \forall z\, (\forall u \in z\, (u \in x) \rightarrow z \in y)$ [POW]

(6) $\exists x\, ((\exists u \in x\, \forall y\ \ y \notin u) \wedge \forall y \in x\, \exists z \in x\ \ y \in z)$ [INF]

(7) $\forall x\, ((\forall y \in x\, \exists z\, \phi) \rightarrow \exists u\, \forall y \in x\, \exists z \in u\, \phi)$ [COLL]

(8) $\forall x\, (\forall y \in x\, \phi(y) \rightarrow \phi) \rightarrow \forall x\, \phi$ [IND]

∎

The satisfaction of the TND constraint goes very little distance (if any) to making IZF palatable to a traditionally-minded constructivist. For one thing, IZF includes POW and SEP, full powerset and full separation, rather than the more constructively acceptable versions of those axioms, weak power set and Δ_0-separation. A majority of constructivists would likely object to POW and SEP on the grounds that they are impredicative or even "wildly impredicative." It would be worth pausing momentarily to entertain these objections and to point out the ways in which we might skirt them.

Remarks on impredicativity and powerset.

In our concessive moods, we might allow that a full dosage of powerset and separation sends one into the realm of higher infinities and of impredicative specifications, a realm which any right-thinking disciple of Brouwer should avoid. Our liberality is not, however, a disguised attempt to rescue the contested principles from constructivist opprobrium, but derives from the character of the task at hand. We are under no delusion that POW, SEP and COLL are each somehow more constructive than weak powerset, predicative separation and replacement, rather it is part of our project to derive independence results for constructive systems and to do it economically. On the side of syntax, this means that we should adopt theories which are the strongest conformable to the realizability structures. Such theories will always include POW and SEP, which we will prove in Chapter Two. It is also worth noting that any known constructive set theory can be embedded into IZF, so IZF is thoroughly economical from a metamathematical standpoint.

When in a more bellicose frame of mind, we try to dismiss forms of the argument from impredicativity by severing the historical tie between it and constructivity. First, there is, so far as we can see, no serious argument from premises justified by the basic constructivist scheme to the conclusion that impredicativity is constructively unworthy. Of course, impredicativity has been barred by fiat, but we do not take the pronouncements on impredicativity by, e.g., Heyting, to count among the basic ideas of constructivity. In fact, since Heyting's specification of the intuitionistic "proof conditions" of $\forall x\phi$ and of $\phi \rightarrow \psi$ are impredicative, there is a considerable price set on giving any general argument to the conclusion that impredicativity is constructively unacceptable.

The very idea that there is something reprehensible about impredicativity *per se* goes back to Poincaré (cf. Poincaré (1963)) and seems to be attached to a particular picture of the constructive universe as an incomplete totality. According to Poincaré's picture, an incomplete totality is one which "grows over time as more members are added." On the basis of this picture, Poincaré objected to impredicative specifications because, when the totality over which the specification takes place is incomplete, the reference of the specification may shift over time. Strangely, Poincaré insisted that logic is applicable only to terms and predicates whose extensions will not change over time.

Independently of any rationale for holding such a peculiar view about logic, there is no compulsion to adopt Poincaré's vision of the mathematical universe as a domain expanding over time. This is true even if we are strict constructivists; one can give a complete presentation of the Heyting interpretation without having to rely on an assumption that the membership of certain classes is time-dependent.

Next, on intuitionist thinking, *all* infinite domains, including the domain of natural numbers, are incomplete. There is, however, no consequent demand that we look askance at impredicative specifications of natural numbers; no intuitionist believes that "least number such that" and "greatest number such that" contexts should be stricken from the rosters of the sensible just because the natural numbers form an incomplete totality. Therefore, there is no direct inference from incompleteness to the meaninglessness of impredicative specifications.

One of the leading ideas of intuitionism is that the nature of a domain is not expressed primarily in terms of a metaphysical portrait from which the properties of the domain are "read off." Instead, if there is a portrait of a domain, it appears in the analysis of quantification over the domain and in the quantificational principles which, consequent on the analysis, either hold or fail there. The "incompleteness" of the natural numbers is registered, in part, in the failure of the classical inference from $\neg\forall x\phi$ to $\exists x\neg\phi$. There seems to be no direct line from a reasoned approach to this failure to the condemnation of impredicativity.

Finally, our account of quantification over $\mathbf{V}(Kl)$, the preferred generic understanding of set quantifiers, leaves no jumping-off place for a Poincaré-style attack on impredicativity from a constructivist metaphysics. Generic quantification countenances impredicativity without relinquishing any of the usual marks of "incompleteness." Moreover, the generic understanding rescues us from an objection that could arise from one well-known traditional account of the constructivist's grasp of an abstract set. (I believe that this account can be found to underlie some of the remarks in Gödel (1964).) On this way of thinking, the constructivist's grasp of a particular set A must be mediated by or even consist entirely in an understanding of a class abstract $\{x : \phi\}$ such that $A = \{x : \phi\}$. The interpretation of universal set quantification which would naturally be allied to this view is one that takes proofs of $\forall x\ \psi$ to be intentional functions, which, for each possible class abstract $\{x : \phi\}$, produce a proof of $\psi(x/\{x : \phi\})$. This sort of semantical standpoint might well be interpreted so as to prohibit impredicative instances of SEP. If A were defined via a class abstract $\{x : \phi\}$ containing unrestricted quantification, then a provision of the proof conditions of $x \in A$ would make reference to ϕ, and hence, by the interpretation of $\forall$, to the abstract $\{x : \phi\}$ again. Therefore, a vicious circularity bars any attempt to comprehend an impredicatively specified class. (We should point out that we do not believe that an antipathy to impredicativity is mandated by such a stand on quantification. We are sketching what we take to be a view which, historically, has been held.)

Generic quantification doesn't give this line a chance to get started. Sets enter into proofs as "bare" sets; our constructive understanding of them is unmediated by abstracts. This metaphysics of sets is allied to an account of set quantification involving functions which are invariant under changes in the way we might elect to specify sets. Every proof of a universal set quantification is completely schematic; the proof relies only on the broad notion of set embodied in the axioms and ignores the possibility that sets might be distinguished by presentations.

Personally, we have always been more than a little puzzled by "in principle" constructivist objections to the powerset operation. We can well understand the force of certain pragmatic considerations: the presence of full powerset often impedes metamathematical progress. In fact, most of our Chapter Five can be seen as a demonstration that, over $\mathbf{V}(Kl)$, the full powerset of ω is far too strange a place to be of much real mathematical use. The success of the theories of RETs and of isols becomes, under the isomorphism theorems, reason to restrict set-theoretic consideration to well-controlled segments of the powerset like $\mathbf{P}(\omega)^{st}$, the collection of ω-stable subsets.

When it comes to the powerset axiom itself, a constructivist who allows variables to range over arbitrary constructions could have very little to say by way of objection. Given a set A, a constructive subset S of A is just an assignment, to each element a of A, of the constructions which, if available, would prove that $a \in S$. In keeping with the basic constructivist metaphysics, the assignment is just a constructive function from A into the domain of constructive propositions. The powerset of A, then, is simply the function-type containing all such functions. A constructivist who allows quantification over the domain of all constructions and allows one to form function-types $A \Rightarrow B$ for any A and B could have objection neither to this idea of powerset nor to quantification over the powerset. (Once we give our realizability models for IZF, you can see that this is, modulo a coding with natural numbers, just the idea of powerset which we adopt.)

Finally, there is a technical point which can be made in favor of POW. In Myhill (1975), Myhill explains how the admission of the full resources of POW allows a straightforward approach to measure theory, an approach which permits quantification over all measurable sets. Consequently, with POW, we need no circumlocutions to accomodate the foundations of the classical theory of measure and integration.

A note on another formulation of IZF.

In passing, we remark that the IZF axioms receive an unnecessarily redundant formulation in Beeson's **Constructive Formal Systems** (1984), p.108. The axiom INF appears with an extra clause, as follows:

$$\exists x(0 \in x \wedge \forall y \in x(y \cup \{y\} \in x) \wedge \forall z((0 \in z \wedge \forall y \in z(y \cup \{y\} \in z)) \rightarrow x \subseteq z))$$

The extra clause makes explicit guarantee that the postulated infinite set x is the $\subseteq$-least set hereditary with respect to 0 and successor. However, this guarantee is, for us, unnecessary. Given that definition by recursion on well-founded relations is available in our version of IZF, we can establish the existence of such a set by the procedure familiar to classical set theorists and without the aid of Beeson's clause. For all the details, see Grayson (1975).

On page 107 of the same manuscript, Beeson misassesses the strength of IZF. Beeson writes "we cannot [in IZF] prove the existence of any functions, even of plus and times on the integers." With POW and SEP, however, one easily demonstrates the existence and functionality of any number of relations on ω, among them the identity and the predecessor functions. One can also use Frege's notion of hereditary set, plus induction, to define the addition relation and to prove its totality and functionality.

Section 3: Varieties of finitude and infinity

The remainder of this chapter is devoted to elucidating the inner workings of IZF. Although we will not restrict expression to the purely formal, all our arguments are easily formalizable in IZF. Also, we make selective mention of independence results which show that various classical theorems cannot be reproduced directly in IZF. We make no promise that all the independence results can be obtained constructively; the independence proofs may well be classical. Acquaintance with the conventions and notation of informal set theory is presupposed.

3.1. Definition. When used as notation for functional relations, '$\rightarrow$,' '$\twoheadrightarrow$,' '$\rightarrowtail$' and '$\rightarrowtail\!\!\!\!\twoheadrightarrow$' take their familiar meanings:

(1) $f : A \rightarrow B$ iff f is a function from A into B

(2) $f : A \twoheadrightarrow B$ iff f is a function from A onto B

(3) $f : A \rightarrowtail B$ iff f is a one-to-one function from A into B

(4) $f : A \rightarrowtail\!\!\!\!\twoheadrightarrow B$ iff f is a one-to-one correspondence between A and B.

∎

Essential distinctions.

Here is a list of the notions basic to an intuitionistic theory of cardinality:

3.2. Definition. Let ω be the set of natural numbers. Let X be a set.

(1) X is *strictly finite* iff $\exists n \in \omega \exists f (f : n \rightarrowtail\!\!\!\!\twoheadrightarrow X)$

(2) X is *finite* iff $\exists n \in \omega \exists f (f : n \twoheadrightarrow X)$

(3) X is *subfinite* iff $\exists n \in \omega \exists A \subseteq n \exists f (f : A \twoheadrightarrow X)$

(4) X is *countable* iff $\exists f (f : \omega \twoheadrightarrow X)$

(5) X is *subcountable* iff $\exists A \subseteq \omega \exists f (f : A \twoheadrightarrow X)$

(6) X is *infinite* iff $\exists f (f : \omega \rightarrowtail X)$

(7) X is *Dedekind finite* (or just *D-finite*) iff X is not infinite

∎

A set is finite whenever it can be enumerated by the elements of a natural number. It would be far too stringent to insist that the only finite sets are those in one-to-one correspondence with a natural number. Only discrete sets (*vide infra*) are finite in that sense. Adopting such a notion of finiteness would, therefore, prohibit us from asserting, when ϕ is undecided, that the set $\{\{0\}, \{0 : \phi \vee \neg\phi\}\}$ is finite. This would be highly counterintuitive. Naturally, a set is subfinite when it is enumerated by a subset of a finite set, or, equivalently, by a subset of some number. The finite-subfinite distinction is necessary; $\{0 : \phi \vee \neg\phi\}$ is trivially subfinite but, unless ϕ is decided, it is not finite. The same sort of examples encourage the parallel distinction between countable and subcountable.

Once we delve into recursive set theory, these distinctions will be of paramount importance. Within that framework, every subset of ω is subcountable, but only the r.e. sets are countable. In fact, the familiar nonempty r.e. sets of classical recursion theory are precisely the countable sets of realizability set theory. Although the ideas of finite and of D-finite sets are taken directly from traditional set theory, the distinction between them parallels a crucial distinction in recursive set theory. This is the distinction between isolated and nonisolated sets. (For details, see Chapters Three and Five.)

All these distinctions can be formulated as weak counterexamples. The weak counterexample procedure is a pervasive feature of intuitionstic practice; for an intuitive explanation, see McCarty (1983). For a discussion of the weak counterexample procedure in the light of realizability, see Chapter Three.

3.3. Proposition.

(1) $\forall x\,(x \textit{ is finite} \rightarrow x \textit{ is strictly finite}) \rightarrow \mathrm{TND}$

(2) $\forall x(x \textit{ is subfinite} \rightarrow x \textit{ is finite}) \rightarrow \mathrm{TND}$

Proof. One can manipulate strange sets like $\{0 : \phi \vee \neg\phi\}$ to get the appropriate counterexamples. For example, the set $\{0 : \phi\}$ is surely subfinite; were it to be finite, $\phi \vee \neg\phi$ would have to hold. ∎

3.4. Definition. A set X is *discrete* iff equality is decidable on X:

$$\forall x, y \in X\ (x = y \vee x \neq y).$$

3.5. Proposition. *If A is finite and discrete, then A is strictly finite.*

Proof. Just use induction on ω. ∎

It follows from this little proposition that every finite subset X of the natural numbers has a unique cardinality, $num(X)$, which is a natural number.

With a little help, we can even show that *prima facie* plausible and weaker connections between these notions are intuitionistically unacceptable. Let MP stand for "Markov's Principle:"

$$\forall x \in \omega(\phi \vee \neg\phi) \rightarrow (\neg\neg\exists x \in \omega\phi \rightarrow \exists x \in \omega\phi)$$

Strictly speaking, MP is intuitionistically incorrect and is independent of IZF. However, if we assume that MP holds in the metatheory, MP will hold in all the models we consider. (MP can be given suitable justification if logical signs like $\exists$ are interpreted over ω by reference to search procedures carried out by Turing machines.)

The classical notion of nonempty set (like so many classical notions) splits up in intuitionistic contexts. The strong intuitionistic notion of nonempty set is that of *inhabited* set:

3.6. Definition. A set X is *inhabited* iff $\exists a \ a \in X$ ∎

3.7. Theorem.

(1) $\forall x((x \textit{ is inhabited} \wedge x \textit{ is subfinite}) \rightarrow x \textit{ is finite}) \rightarrow$ TND

(2) MP$\rightarrow(\forall x((x \textit{ is inhabited} \wedge x \textit{ is subcountable}) \rightarrow x \textit{ is countable}) \rightarrow$ TND)

Proof. For the first implication, consider $x = \{0\} \cup \{1 : \phi\}$. x is an inhabited set and is subfinite. If x is finite, then it is enumerated by 1 or by some n with $n \geq 2$. If the former, then $\neg\phi$. If the latter, let f be an enumerating function. Either $\forall m < nf(m) = 0$ or $\exists m < nf(m) = 1$. If the former, then $\neg\phi$. If the latter, ϕ.

For the second statement, take x to be any inhabited subset of ω. x is trivially subcountable. If x is countable,

$$\exists f \forall n \in \omega(n \in x \text{ iff } \exists m f(m) = n).$$

If we apply MP to this last condition, we see that

$$\forall n \in \omega(n \in x \text{ iff } \neg\neg n \in x).$$

It follows that $\neg\neg\phi \rightarrow \phi$ and that TND holds. ∎

In traditional set theory, the presence of DC collapses any extensional distinction between finite and D-finite sets. DC is the principle of dependent choices:

$$\forall x \in S \exists y \in S \phi(x,y) \rightarrow \forall x \in S \exists f : \omega \rightarrow S(f(0) = x \wedge \forall n \in \omega \phi(f(n), f(n+1))).$$

However, the collapse does not take place in IZF, even in the presence of DC. In fact, we can add all our favorite choice principles: $\mathrm{AC}^{\omega,X}$, PAx, and RDC, and still preserve the distinction between finite and D-finite. In recursive set theory, the distinction is maintained in the face of the choice principles by the presence of *isolated* sets of natural numbers. When the time comes (Chapters Four and Five), we will explain all this in detail. Until then, the following is a promissory note:

3.8. Proposition.

(1) $\forall x(x \textit{ is infinite} \rightarrow \neg x \textit{ is finite})$

(2) $\mathrm{IZF+DC} \nvdash \forall x \subseteq \omega(\neg x \textit{ is subfinite} \rightarrow x \textit{ is infinite})$

(3) $\mathrm{IZF+DC} \nvdash \forall x \subseteq \omega(\neg x \textit{ is finite} \rightarrow x \textit{ is infinite})$

Proof. The first statement is trivially true. The proof of the second will be apparent from the theorems in Chapter Three on $\mathrm{V}(Kl)$. The third follows directly from the second. ∎

Finite, subfinite, countable, subcountable.

In intuitionistic settings, there are a multitude of alternatives to the notion of finite as we understand it. Minio (1974) contains a complete rundown. The definition of 'finite' we prefer is unquestionably the most useful. For one thing, it's just what one needs to formulate the idea of a "finite element" in an algebraic lattice. This sort of finiteness has been studied in Kock, Mikkelson and Lecouturier (1975) and corresponds to what is there called 'Kuratowski finiteness.'

Grayson has verified that finiteness has a number of pleasant properties. Here, we say that the *finite power* of a set X is the collection of finite subsets of its powerset, $\mathbf{P}(X)$.

3.9. Proposition. *Quotients, finite unions, products and finite powers of finite sets are finite. All singletons are finite.*

Proof. See Grayson (1978). ∎

In general, the powerset of a finite set is maddeningly uncountable. Within the realizability model, it is easily seen that the powerset of $\{0\}$ is neither counted nor subcounted by the natural numbers, by the reals, by $N \Rightarrow N$, or by any reasonably "classical" ordinal.

Our notion of subfinite is the weakest of the finiteness concepts surveyed by Grayson and Minio. Subfiniteness has some lovely closure properties of its own:

3.10. Proposition. *Subsets, quotients, products and subfinite unions of subfinite sets are subfinite. Every subset of a singleton set is subfinite.*

Proof. See Grayson (1978). ∎

The next proposition does something to illuminate our nomenclature.

3.11. Theorem. *A set is subfinite iff it is a subset of a finite set. In other words, "subfinite" is precisely the closure of "finite" under subsets.*

Proof. The implication from right to left is trivial. For the converse, assume that x is subfinite. By definition,

$$\exists n \exists A \subseteq n \exists f (f : A \twoheadrightarrow x)$$

Take $\overline{f}$ to be the extension of f such that, for $m \in n$, $\overline{f}(m) = \bigcup\{f(m) : m \in A\}$. $\overline{f}$ obviously extends f: if $m \in A$, then $\overline{f} = \bigcup\{f(m)\} = f(m)$. Therefore, $x \subseteq range(\overline{f})$ and $range(\overline{f})$ is finite. ∎

There are results for countable and subcountable sets that stand in perfect analogy to the above.

3.12. Proposition. *Quotients, finite unions and products of countable sets are countable. Given* $AC^{\omega,X}$*, countable unions of countable sets are countable. Every finite set is countable.*

Proof. Again, we refer the reader to Grayson (1978). ∎

$AC^{\omega,X}$ is the axiom of choice over the natural numbers:

$$\forall n \in \omega \; \exists x \in X \; \phi(x, y) \rightarrow \exists f \in (\omega \Rightarrow X) \; \forall n \in \omega \; \phi(n, f(n)).$$

We will see that $\mathbf{V}(Kl) \models AC^{\omega,X}$ provided that $AC^{\omega,X}$ is assumed to hold in the ground model.

3.13. Proposition. *Quotients, subsets, subfinite unions and products of subcountable sets are subcountable. With* $\mathrm{AC}^{\omega,X}$, *subcountable unions of subcountable sets are subcountable. Every subfinite set is subcountable.*

Proof. Again, we refer to Grayson (1978). ∎

Subcountability bears just the same sort of closure relation to countability that subfiniteness bears to finiteness:

3.14. Proposition. *"Subcountable" is the closure of "countable" under subsets.*

Proof. Just as for the finite-subfinite case. ∎

Section 4: Cardinals from $\mathbf{P}(\omega)$

$\mathbf{P}(\omega)$ is the powerset of ω, and the theory of cardinals of subsets of ω gives an interesting and well-controlled sample of what can be done with cardinal arithmetic in IZF. Classically, the cardinals from $\mathbf{P}(\omega)$ would be anything but interesting, since there are only $\aleph_0$ such cardinals: $0, 1, 2, \ldots, \omega$. Intuitionistically, there is plenty of room to work. In Chapter Five, we prove that there are at least "ω_1" distinct cardinals in $\mathbf{P}(\omega)$. There we will also explain the need for the scare quotes around 'ω_1.'

4.1. Definition. Let $A, B \in \mathbf{P}(\omega)$ and let $\langle\ ,\ \rangle$ be a primitive recursive number-theoretic pairing operation.

(1) A is (*cardinally*) *equivalent* to B iff $\exists f(f : A \rightarrowtail\!\!\twoheadrightarrow B)$. When A and B are equivalent, we write $A \approx B$.

(2) $A + B = \{\langle 0, n\rangle : n \in A\} \cup \{\langle 1, m\rangle : m \in B\}$

(3) $A \times B = \{\langle n, m\rangle : n \in A \wedge n \in B\}$

(4) A is a *finite number* (in symbols, $\mathrm{N}(A)$) iff A is strictly finite.

(5) A is (*cardinally*) *less than* B (in symbols, $A \leq B$) iff $\exists C \in \mathbf{P}(\omega)\ A + C \approx B$.

∎

4.2. Note. In Chapter Zero, the symbol '$\preceq$' was used to express the relation "cardinally less than" in the form appropriate to recursion theory. Since this relation gets far more play in our work than the usual order relation on cardinals, we will use the standard symbol '$\leq$' to denote it. ∎

With these relations and operations we can develop a reasonable intuitionistic arithmetic of cardinals on $\mathbf{P}(\omega)$:

4.3. Proposition.

(1) $\approx$ *is an equivalence relation on* $\mathbf{P}(\omega)$.

(2) *As relations on* $\mathbf{P}(\omega)$, N *and* $\leq$ *are congruences with respect to* $\approx$.

(3) *As operations on* $\mathbf{P}(\omega)$, $+$ *and* $\times$ *respect* $\approx$.

Proof. Trivial. ∎

4.4. Definition. For sets A and B, A is a *decidable* subset of B iff

$$\forall x \in B\, (x \in A \vee \neg\, x \in A).$$

▮

In his Cambridge lectures (Brouwer (1981)), Brouwer used the term 'removable' for our 'decidable.' The expression 'detachable' is also in common use. We will discover that, as far as cardinal arithmetic is concerned, the removable or decidable subset relation is far more important than the bare subset notion. In recursive set theory, it is with the relation of decidable subset that a version of the Cantor-Bernstein Theorem is provable.

This result shows how the idea of decidable subset interacts with the basic notions:

4.5. Proposition. *For* $A, B \in \mathbf{P}(\omega)$, $A \leq B$ *iff*

$$\exists C \in \mathbf{P}(\omega)(\ C \textit{ is a decidable subset of } B \textit{ and } A \approx C\ ).$$

Proof. Immediate. ▮

The arithmetic operations on cardinals from $\mathbf{P}(\omega)$ constitute a quite reasonable algebra of cardinals. The structure $\langle \mathbf{P}(\omega), +, \leq, \approx \rangle$ will prove to be a partially-ordered commutative semigroup with refp, the refinement property.

4.6. Definition. A structure $\langle \Phi, +, \leq, \approx \rangle$ is a *partially-ordered commutative semigroup* iff

(1) $\approx$ is an equivalence relation on Φ, $+$ respects $\approx$ and $\leq$ is a $\approx$-congruence.

(2) $+$ is commutative and $\langle \Phi, + \rangle$ is a semigroup with respect to $\approx$.

(3) For $a, b \in \Phi, a \leq b$ iff $\exists c \in \Phi(a + c \approx b)$.

(4) $\leq$ is a partial order on Φ with respect to $\approx$.

▮

4.7. Theorem. $\langle P(\omega), +, \leq, \approx \rangle$ *is a partially-ordered commutative semigroup.*

Proof. (1) through (3) are either too easy or have already been dealt with. To get (4), one shows that $\leq$ is (a) reflexive, (b) transitive and (c) anti-symmetric. (a) and (b) are very easy but (c) presents something of a difficulty.

(a) Since, for all $A \in \mathbf{P}(\omega), A + 0 \approx A$, reflexivity follows from (3).

(b) Assume that $A \leq B$ and $B \leq C$. According to (3), there are D and E such that $A + D \approx B$ and $B + E \approx C$. It follows from (1) that $(A + D) + E \approx C$. Since $\mathbf{P}(\omega)$ is a commutative semigroup under the operations, $A + (D + E) \approx C$. Therefore, $A \leq C$.

(c) This proof is of a different magnitude entirely. Not only is it far and away more involved than the others, but the result seems to depend essentially on the properties of the natural numbers. Among other things, we need to make judicious use of the minimum operator. Therefore, the procedure will not generalize to any set which is too different from ω. The proof to follow is a greatly abbreviated version of a proof of an analogous result, due to Dekker and Myhill, for the domain of recursive equivalence types. It would be enlightening for the reader to compare our work with that on pages 74 through 77 of Dekker and Myhill (1960). In what sense this proof is a "version" of an "analogous" result for the recursive equivalence types and in what sense the Dekker-Myhill proof allows of systematic "abbreviation" will be revealed in the sequel, specifically, in Chapter Five.

The territory of our proof is subdivided into three lemmata:

4.8. Lemma. $(\omega \times B) + B \approx \omega \times B$.

Proof. Trivial. ∎

4.9. Lemma. $\omega \times B \leq A \Rightarrow A + B \approx A$.

Proof. This is just a matter of concatenation of definitions plus a bit of the previous lemma:

$$\omega \times B \leq A \Rightarrow \exists C \; (\omega \times B) + C \approx A$$

$$\Rightarrow A + B \approx ((\omega \times B) + C) + B$$

$$\Rightarrow \text{(using (2))} \; A + B \approx ((\omega \times B) + B) + C$$

$$\Rightarrow \text{(using preceding lemma)} \; A + B \approx A$$

∎

4.10. Lemma. $A + B \approx A \Rightarrow \omega \times B \leq A$.

Proof. Let $f : A \rightarrowtail\!\!\!\twoheadrightarrow A + B$. For $x \in A + B$, define f^* as

$$f^*(x) = \begin{cases} f(x_1) & \text{if } x_0 = 0 \\ x & \text{if } x_0 = 1 \end{cases}$$

Here, $(\ )_0$ is the first projection function relative to $\langle\ ,\ \rangle$ and $(\ )_1$ is the second. From its definition, $f^* : A + B \twoheadrightarrow A + B$. Now, for each $x \in A$, define f^x by ω–recursion:

$$f^x(0) = \langle 0, x \rangle$$

$$f^x(n+1) = f^*(f^x(n))$$

Let $\Phi(i)$ hold of i just in case

$$f^x(i)_1 \in B \wedge \forall j, k < i\,(j \neq k \rightarrow f^x(j)_1 \neq f^x(k)_1).$$

Then, let

$$B^* = \{x \in A : \exists i < x + 2\ \Phi(i) \wedge \forall j \leq i\ f^x(j)_1 \leq f^x(0)_1)\}.$$

B^* is a decidable subset of A. Therefore, if we can show that $B^* \approx \omega \times B$, we are done.

To that end, construct a map $g : B^* \rightarrow \omega \times B$ by setting, for $x \in B^*$,

$$g(x) = \langle t(x), f^x(\mu(x)) \rangle$$

where

$$\mu(x) = \mu n < x + 2\ (f_1^x(n) \in B)$$

$$t(x) = num(\{m < \mu(x) : \forall j (m < j \leq \mu(x) \rightarrow f^x(m)_1 > f^x(j)_1\})$$

μn stands for 'the least n such that'; '$num(A)$' stands for the finite cardinality of A. As the set in question is finite and decidable, and equality on it is decidable, the num operator in the above clause succeeds in picking out a unique natural number. This follows from the Proposition 3.5 of the last section.

Since f is bijective, so is g. To check that g is surjective, we consider a collection of maps g^b for $b \in B$ and a map h. For $b \in B$, set

$$g^b(0) = \langle 1, b \rangle$$

$$g^b(n+1) = \langle 0, f^{-1}(g^b(n)) \rangle$$

And let

$$\mu(n,b) = \mu m\,[\,num(\{0 < i \le m : \forall j(0 < j < i \to g^b(j)_1 < g^b(i)_1)\}) = n\,].$$

$\mu(n,b)$ is well-defined because, for each $\langle n,b\rangle \in \omega \times B$,

$$\mu(0,b) = 0$$

$$\mu(n+1,b) \le g^b(\mu(n,b))_1 + 2$$

Finally, set $h(\langle n,b\rangle) = g^b(\mu(n,b))$ and check that $p \circ g =$ id $\restriction \omega \times B$. id is the identity function on A.

∎

The antisymmetry of $\le$ follows directly from the last two lemmas:

$$A \le B \wedge B \le A \Rightarrow \exists C, D\ A + C \approx B \wedge B + D \approx A$$

$$\Rightarrow A + (C + D) \approx A$$

$$\Rightarrow \omega \times (C + D) \le A$$

$$\Rightarrow \omega \times C \le A$$

$$\Rightarrow A \approx A + C \approx B$$

∎

Antisymmetry for $\le$ on $\mathbf{P}(\omega)$ gives a weak version of the classical Cantor-Schroeder-Bernstein theorem:

$$X \le Y \wedge Y \le X \Rightarrow X \approx Y$$

The appellation 'weak' is correct because the $\le$ notion is, intuitively, stronger than the classical notion of cardinal inclusion. For X to stand in the $\le$ relation with Y, X must be equivalent to a decidable subset of Y, not just an arbitrary subset. The independence from IZF of some "strong" versions of Cantor-Schroeder-Bernstein will be proved in Chapter Five.

We present one last definition and theorem about cardinal addition before we move on.

4.11. Definition. A structure $\langle \Phi, + \approx \rangle$ has the *refinement property* (*refp*) iff for all $a, b, c, d \in \Phi$, whenever $a + b \approx c \pm d \Rightarrow$, then

$$\exists e_1, e_2, e_3, e_4 \in \Phi \; a \approx e_1 + e_2 \wedge b \approx e_3 + e_4 \wedge c \approx e_1 + e_3 \wedge d \approx e_2 + e_4$$

∎

4.12. Theorem. $\langle \mathbf{P}(\omega), +, \approx \rangle$ *has refp.*

Proof. Let $f : A + B \rightarrowtail\!\!\!\!\twoheadrightarrow C + D$. Take

$$E_1 = \{n \in A : f(\langle 0, n \rangle)_1 \in C\}$$

$$E_2 = \{n \in A : f(\langle 0, n \rangle)_1 \in D\}$$

$$E_3 = \{n \in B : f(\langle 1, n \rangle)_1 \in C\}$$

$$E_4 = \{n \in B : f(\langle 1, n \rangle)_1 \in D\}$$

The theorem follows immediately. ∎

Once we know that $\langle \mathbf{P}(\omega), +, \leq, \approx \rangle$ is a partially-ordered semigroup with refp, we know that it has all manner of pleasant algebraic properties. For a listing of some of them, one may consult pages 67 through 86 of Dekker and Myhill (1960).

The operation of multiplication on the RETs is also very pleasant.

4.13. Definition. For $A, B \in \mathbf{P}(\omega)$, $A \mid B$ iff $\exists C \; A \times C \approx B$. ∎

4.14. Theorem.

(1) *In* $\langle \mathbf{P}(\omega), \times, |, \approx \rangle$, $\times$ *respects* $\approx$ *and* $|$ *is a congruence for* $\approx$.

(2) $\langle \mathbf{P}(\omega), \times, |, \approx \rangle$ *is (up to stability) a partially-ordered semigroup.*

(3) $\times$ *distributes over* $+$.

Proof. (1) and (2) are easy. By 'up to stability' in (2), we mean that the structure $\langle \mathbf{P}(\omega), \times, |, \approx \rangle$ would be an intuitionistic partially-ordered semigroup except for the intrusion of a double negative, $\neg\neg$. In particular, the divisibility relation $|$ is reflexive and transitive, but only antisymmetric up to stability. It is intuitionistically the case that

$$\forall A, B \in \mathbf{P}(\omega)(A \mid B \wedge B \mid A) \rightarrow \neg\neg A \approx B.$$

To prove this, suppose that $A \mid B$ and $B \mid A$ and make the assumption that

$$\exists x\ x \in B \vee \neg \exists x\ x \in B.$$

If the second disjunct holds, then A is also empty and $A \approx \emptyset \approx B$. If the first disjunct is true, $\exists C\ B \times C \approx A$.

Now, we make a second temporary assumption, that

$$\exists x\ x \in C \vee \neg \exists x\ x \in C.$$

Again, on the second disjunct, A is empty and $A \approx \emptyset \approx B$. On the assumption of the other disjunct, we can let $y \in C$ and set $D = C \sim \{y\}$. Then,

$$A \approx B \times (D + \{y\}) \approx B \times D + B \times \{y\} \approx B \times D + B \geq B.$$

A symmetrical argument, starting from $A \mid B$, will show that $B \geq A$. Hence, under the stated assumptions and since $\leq$ is a partial order on $\mathbf{P}(\omega)$, we have that $A \approx B$.

It only remains to do away with the two assumptions. Granted, we cannot prove them in intuitionistic logic, but we can prove their double negations. Therefore, we have, by intuitionistic contraposition, that $\neg\neg A \approx B$ holds without qualification. It follows that $|$ is antisymmetric up to stability. ▮

For other properties of multiplication, the reader can consult the chapters on the multiplication of recursive equivalence types in the Dekker and Myhill monograph.

Section 5: D-finite cardinals

5.1. Definition. For $A, B \in \mathbf{P}(\omega)$,

(1) A is D*-finite iff $\neg \exists f \exists x (x \in A \wedge f : A \rightarrowtail A \wedge \forall y \neg x = f(y))$

(2) $A < B$ iff $A \leq B \wedge \neg A \approx B$

∎

Our D*-finite is classically equivalent to Dedekind's original definition of finite set. We want to show that, even intuitionistically, it is equivalent to what we call 'D-finite.'

5.2. Lemma. *For $A \in \mathbf{P}(\omega)$, if f takes ω injectively into A, then there is a g such that g also takes ω injectively into A and $Rng(g)$ is a decidable subset of ω.*

Proof. Given $f : \omega \rightarrowtail A$, simply define g so that

$$g(0) = f(0)$$
$$g(n+1) = f(\mu y . g(n) < f(y))$$

To be sure that $Dom(g) \supseteq \omega$, we need only show that $\forall n \exists m\, n < f(m)$. One can easily prove that by checking that $\forall n\ \exists m < n+1\ \ n < f(m)$. Since g is strictly increasing, $Rng(g)$ is decidable. ∎

5.3. Theorem. *For $A \in \mathbf{P}(\omega)$, the following are equivalent:*

(1) A *is* D*-*finite*

(2) A *is* D-*finite*

(3) $\forall C \in \mathbf{P}(\omega)(C$ *is inhabited* $\rightarrow A < A + C)$

(4) $A < A + 1$

Proof. (1)⇒(2): By the lemma, if $f : \omega \rightarrowtail A$, then there is a g such that $g : \omega \rightarrowtail A$ and $Rng(g)$ is decidable. It follows that there is an h such that $h : A \rightarrowtail A$ and an x such that $x \notin Rng(h)$. If we take j to be

$$j(x) = \begin{cases} x & \text{if } x \notin Rng(g) \\ g(n+1) & \text{if } x = g(n) \end{cases}$$

then we can get the requisite h as $h = j \restriction A$.

(2)⇒(3): Let C be inhabited and assume that $A \approx A + C$. Then, $\exists f\ f : A + C \rightarrowtail\!\!\!\twoheadrightarrow A$. Let $a \in C$. Consider $g : \omega \rightarrowtail A$, where

$$g(0) = f(\langle 1, a\rangle)$$

$$g(n+1) = f(\langle 0, g(n)\rangle)$$

(3)⇒(4): Trivial.

(4)⇒(1): Assume that $f : A \rightarrowtail A$ and that $x \in A$ such that $\forall y \neg x = f(y)$. Define $g : \omega \rightarrowtail A$ so that

$$g(0) = x$$

$$g(n+1) = f(g(n))$$

Use the lemma to get an h with $h : \omega \rightarrowtail A$ and $Rng(h)$ decidable and then define $j : A+1 \twoheadrightarrow A$:

$$j(x) = \begin{cases} x_1 & \text{if } x_0 = 0 \wedge x_1 \notin Rng(h) \\ h(n+1) & \text{if } x_0 = 0 \wedge x_1 = h(n) \\ g(0) & \text{if } x_0 = 1 \end{cases}$$

∎

5.4. Corollary. *For $A \in \mathbf{P}(\omega)$, A is D-finite iff A is D*-finite.*

Proof. ∎

Arithmetic operations on the D-finites.

Our ultimate goal is to demonstrate that, up to stability, realizability captures the classical theory of isols as the theory of D-finite sets and that, if constructive reasoning is at all natural, working constructively over the realizability model gives the theory its natural deductive structure. In so doing, we can make use of any of the principles that hold with respect to realizability. MP is one of these principles; MP seems necessary to the proof that the D-finite sets are closed under the operations of cardinal arithmetic.

5.5. Definition. For $A \in \mathbf{P}(\omega)$, A has the *cancellation property* iff

$$\forall B, C \in \mathbf{P}(\omega)(A + B \approx A + C \rightarrow B \approx C)$$

∎

5.6. Theorem (with MP). *For $A \in \mathbf{P}(\omega)$, A is D-finite iff A has the cancellation property.*

Proof. If A has the cancellation property, then $A < A+1$, so, by the previous theorem, A is D-finite.

On the other hand, assume that A is not infinite and that $f : A+B \rightarrowtail\!\!\twoheadrightarrow A+C$. For each $x \in B$, one can use ω-recursion to define a function $F^x : \omega \to A+C$ where

$$F^x(0) = f(\langle 1, x\rangle)$$

$$F^x(n+1) = \begin{cases} f(F^x(n)) & \text{if } F^x(n) \in \{0\} \times A \\ F^x(n) & \text{if } F^x(n) \in \{1\} \times C \end{cases}$$

We want to show that, with MP, $F^x(n) \in \{1\} \times C$ for some n. To that end, assume that $\forall n\, F^x(n) \in \{0\} \times A$. It follows directly that $F^x : \omega \rightarrowtail A$. Therefore, since A is not infinite, the assumption is false and $\neg \forall n\, F^x(n) \in \{0\} \times A$. Now, if MP is applied, we obtain

$$\forall x \in B \exists n\, F^x(n) \in \{1\} \times C.$$

Take $\mu(x) = \mu n. F^x(n) \in \{1\} \times C$ and set, for $x \in B$,

$$h(x) = F^x(\mu(x))_1$$

To construct a function inverse to h, just work through this same routine again. It follows from the availability of these constructions that $B \approx C$. ∎

5.7. Corollary. *The D-finite sets are closed under +.*

Proof. ∎

The cancellation property for D-finite sets shows that a subtraction operation is well-defined for those D-finite A and B such that $A \leq B$.

5.8. Definition. If $A, B \in \mathbf{P}(\omega)$ and A and B are D-finite, $C \approx B - A$ iff $A + C \approx B$. ∎

5.9. Corollary. *Subtraction is, with respect to $\approx$, well-defined.*

Proof. ∎

5.10. Note. The *cognoscenti* may well remark that a proof of the preceding results could be obtained without the intrusion of MP if a stronger definition of D-finite were adopted. Some such stronger definition might be

(1) X is D-finite iff $\forall f((f : \omega \to X) \to \exists n\, n \approx Rng(f))$

Unfortunately, this definition does not suit our purpose. We would like it to be the case, classically, that, for any classical set X,

(2) X is isolated in $\mathbf{V}$ iff $\mathbf{V}(Kl) \models \phi(\overline{X})$

where $\overline{X}$ is the "internalization" (cf. Chapter Three) of X in $\mathbf{V}(Kl)$, and ϕ is the expression of D-finiteness in IZF. However, if we adopt the right-hand side of (1) as our definition of D-finite, no infinite set satisfies (2). To see this (although we get well ahead of the story), assume that X is infinite and isolated and that $\mathbf{V}(Kl) \models \phi(\overline{X})$. By the definition of realizability, there is a partial recursive function g such that if $\{e\} : \omega \to X$, then $g(e) = num(Rng\{e\})$. g is obviously extensional, but g is not the restriction of any continuous operator. Therefore, the fulfillment of our desires with respect to a definition of D-finite is blocked by the Myhill-Shepherdson theorem. ▮

To round out our incursion into the realm of the D-finite, we offer a proof that the D-finite sets are closed under $\times$. Our proof calls for the introduction of yet one more notion of "finite set," that of *bounded set.* The boundedness concept does derive from traditional intuitionism and is constructively useful; however, this proof is the only place where it will feature in our work.

5.11. Definition. A set X is *bounded* iff $\exists m\, X \subseteq m$. ▮

For collections of natural numbers, boundedness falls strictly between finiteness and subfiniteness. Intuitionistically, every finite set of natural numbers is bounded, but not conversely; some bounded sets are no more than subfinite. Every bounded set is subfinite, but a simple realizability argument will show that the statement

$$\forall X \in \mathbf{P}(\omega)(X \text{ is subfinite} \to X \text{ is bounded})$$

is independent of IZF.

With that out of the way, we can show that

5.12. Theorem (with MP). *The* D-*finite sets are closed under* $\times$.

Proof. Let A and B be D-finite. Assume that $f : \omega \rightarrowtail A \times B$. Take f_0 to be $(\,)_0 \circ f$ and take A_0 to be $Rng(f_0)$.

Claim: $\neg\neg\, A_0$ is bounded.

Assume, for contradiction's sake, that A_0 is not bounded. Then, $\neg \exists m \forall x (x \in A_0 \rightarrow x \leq m)$. Since A_0 is counted by f_0, we can write this as $\neg \exists m \forall x (f_0(x) \leq m)$. By MP, we have $\forall m \exists x (f_0(x) > m)$. It follows that A_0 is infinite, and so, *a fortiori*, is A. But A was assumed to be D-finite. Therefore, the claim is correct.

Assume, for the sake of argument, that A_0 actually is bounded. Define a map g on ω as follows:

$$g(0) = f(0)_1$$

$$g(n+1) = f(\mu y. \forall m < n+1\ f(y)_1 \neq f(m)_1)_1$$

Under this latest assumption on A_0, we can prove that g is total. Assume that, for some n,

$$\neg \exists y \forall m < n+1 f(y)_1 \neq f(m)_1$$

Then, $\forall y \exists m < n+1\ f(y)_1 = f(m)_1$. It follows that $B_0 = Rng((\,)_1 \circ f)$ is bounded. Hence, $A_0 \times B_0 \supseteq Rng(f)$ is bounded. But this contradicts the assumption that f is injective. Therefore,

$$\neg\neg \exists y \forall m < n+1\ f(y)_1 \neq f(m)_1$$

Now, we use MP again to get

$$\exists y \forall m < n+1\ f(y)_1 \neq f(m)_1$$

It follows from this that g is total. Therefore, we have that $g : \omega \rightarrowtail B$ and B is infinite. Therefore, A_0 is not bounded, in contradiction to the claim. ∎

5.13. Corollary. $\langle$ D-*finite*, $+, \leq, \approx\rangle$ *is a partially-ordered commutative semigroup with refp and the cancellation properties.*

Proof. ∎

5.14. Corollary. $\langle$ D-*finite*, $\times, |, \approx\rangle$ *is, up to stability, a partially-ordered commutative semigroup.*

Proof. ∎

Section 6: ω-stable $\mathrm{P}(\omega)$

6.1. Definition. A set $A \in \mathrm{P}(\omega)$ is *ω-stable* iff

$$\forall n \in \omega\, (\neg\neg n \in A \rightarrow n \in A).$$

$\mathrm{P}(\omega)^{st}$ is the collection of all ω-stable members of $\mathrm{P}(\omega)$. ▮

The results of Sections Three and Four show that there is a substantial cardinal arithmetic on full $\mathrm{P}(\omega)$, provided that some care is taken in the formulation of definitions. It is not at all difficult to check that those same results, with much the same proofs, hold over the slightly more restrictive domain $\mathrm{P}(\omega)^{st}$. Actually, one need verify little more than this: that, when A and B are ω-stable, so are the sets constructed from them by pairing and by projection on their elements.

As Chapter Zero more than intimates, our concern with $\mathrm{P}(\omega)^{st}$, come Chapter Five, will be considerable. $\mathrm{P}(\omega)^{st}$ is precisely the domain of a structure which, under realizability, becomes the structure determined by the algebra on the recursive equivalence types. With this identity of structure, the theorems of these sections turn into constructive proofs of the fundamental properties of the RETs. These remarks likewise apply, *mutatis mutandis*, to the D-finites and to the isols.

CHAPTER 2

General Extensional Realizability

Section 1: Prefatory and historical remarks

Kleene-style realizabilities for set theory are nowise new. Our interpretation is the immediate descendant of the interpretations of Friedman (1973a) and of Beeson (1979). The idea of using models of APP came to us from a remark of Solomon Feferman in his paper *A language and axioms for explicit mathematics* (1975). In our case, the ultimate investigative goals–that of limning the exact logical relations between classical recursive mathematics and constructive mathematics over V(Kl)–influenced both the form of the interpretation and our attitude toward it. Unlike the realizabilities of Beeson, realizability over V(Kl) applies directly to *extensional* IZF; for what we wanted to do, we found it a nuisance to interpret the extensional theory into the intensional before realizing. Also, our interpretations are not designed with proof-theoretic ends in view, but are considered model-theoretically. We think of realizability structures in much the same way as one thinks intuitively of Boolean-valued models. In fact, since our models work directly on the extensional theories, our clauses for $\in$ and $=$ bear a marked formal analogy to the correlative clauses over Boolean-valued universes. Incidentally, the "model-theoretic attitude" is, we think, more in keeping with the attitudes expressed by Kleene in 1945 than the attitudes expressed in more recent work. The model-theoretic attitude adopted in Hyland (1982) stands as an exception.

The form in which our realizability appears was the product of a joint effort expended in Oxford during Michaelmas term 1980. The effort had at least contingent connection with Dana Scott's seminar *Sheaves and logic*. Among the many individuals who made notable contributions, foremost were Guiseppe Rosolini, Simon Thompson and Dana Scott.

As Chapter Zero suggests, general realizability is defined over a von Neumann-like hierarchy, V(A), which is constructed with the aid of a model A of APP. We will refer to the models of APP as *applicative structures.* As we will show, these structures are precise abstract representations of the minimal limits one would have to set on the structure of the domain of constructions so that intuitionistic logic is sound with respect to the Heyting interpretation.

Elements from the domain $|A|$ of A serve as the realizability witnesses. When a sentence ϕ is realized over $V(A)$ by an element e of $|A|$, we will write $e \Vdash \phi$. There will be no subscript attached to the '$\Vdash$' sign to indicate its dependence on A. No confusions will arise, since all the work in these chapters really goes on over no more than one realizability structure at a time. Whenever $\exists e \in |A|\ e \Vdash \phi$, we write '$V(A) \models \phi$.'

Section 2: Language, axioms and models of APP

Terms and pseudoterms.

2.1. Definition. The language L_{app} is a first-order language with the usual logical signs and with a ternary predicate, $App(x,y,z)$, and equality, $=$, as primitives. The language has an infinite collection of variables, denoted x, y, z, ..., and seven distinguished constants: **k, s, d, p, l, r** and **0**. ∎

Economical presentation of the APP system requires the definition of the set PT of *pseudoterms* and the correlative collection of *pseudoformulae.*

2.2. Definition. PT is defined by recursion on the strings of L_{app}: the variables and constants of L_{app} are members of PT and, whenever τ and $\sigma \in$ PT, so is $\tau(\sigma)$. $\tau(\sigma)$ is an application term and we always parse application terms by associating to the left. All formulae of L_{app} are pseudoformulae, and, whenever $\tau, \sigma \in$ PT, so are $\tau \simeq \sigma$ and $\tau \downarrow$. When ϕ is a pseudoformula, so is $\phi[x/\tau]$, for each $\tau \in$ PT. ∎

Each pseudoformula is a metalinguistic abbreviation for one of the official formulae of L_{app}. Here are the abbreviations:

2.3. Definition.

$$\tau \simeq x = \begin{cases} \tau = x & \text{if } \tau \text{ is a variable or constant} \\ \exists y, z\,(\sigma \simeq y \wedge \theta \simeq z \wedge \mathrm{App}(y, z, x)) & \text{if } \tau = \sigma(\theta) \end{cases}$$

$$\tau \simeq \sigma = \forall x\,(\tau \simeq x \leftrightarrow \sigma \simeq x)$$

$$\tau\downarrow = \exists x\,(\tau \simeq x)$$

$$\phi[x/\tau] = \exists x\,(\tau \simeq x \wedge \phi)$$

▮

Each pseudoformula has a *definiens* suggested by the notation for it. '$\tau(\sigma)$' is (partial) application; '$\tau\downarrow$' means that τ is defined or, equivalently, that τ takes a value in models for APP. We note that, if τ is defined, so is every subterm of τ. '$\tau \simeq \sigma$' means that τ and σ are equal whenever they are defined.

Axioms of APP.

It is in terms of pseudoformulae that the five axioms of APP are presented:

2.4. Axioms of APP.

(1) $\mathbf{k}\,xy\downarrow \wedge\, \mathbf{k}\,xy \simeq x$

(2) $\mathbf{s}\,xy\downarrow \wedge\, \mathbf{s}\,xyz \simeq xz(yz)$

(3) $\mathbf{p}\,xy\downarrow \wedge\, \mathbf{l}\,x\downarrow \wedge\, \mathbf{r}\,x\downarrow \wedge\, \mathbf{l}\,(\mathbf{p}\,xy) \simeq x \wedge \mathbf{r}\,(\mathbf{p}\,xy) \simeq y$

(4) $((x = 0) \rightarrow \mathbf{d}\,uvx \simeq u) \wedge ((x \neq 0) \rightarrow \mathbf{d}\,uvx \simeq v)$

(5) $\mathbf{p}\,xy \neq 0$

When A is a model of APP, we write $A \models$ APP. ▮

k and s are familiar items; they are the combinators that make for the combinatorial completeness of every APP model (*vide infra*). Of course, partiality is allowed; the APP axioms guarantee that k be total on $|A|$, but s may well be undefined on its arguments. p makes for total pairing, while l and r are its left and right projections, respectively. d gives definition-by-cases on 0. We insist that $\mathbf{p}\,xy \neq 0$; ultimately, this will insure the realizability of one of the '$\vee$-introduction' axioms of intuitionistic logic.

Completeness and recursion.

Both the combinatorial completeness and the recursion theorem hold for models of APP. These will make our work much easier when it comes to proving the realizability soundness theorem for IZF. There is nothing original about either of the following theorems or their proofs; they are standard results from combinatory logic.

2.5. Theorem (Combinatorial completeness). *If $\tau \in \mathrm{PT}$ and x is any variable, there is a pseudoterm $\lambda x.\tau \in \mathrm{PT}$ whose free variables do not include x and which is such that* $\mathrm{APP} \vdash (\lambda x.\tau)\downarrow \wedge (\lambda x.\tau)x \simeq \tau$.

Proof. $\lambda x.\tau$ is defined by recursion (in the metatheory) on the applicative structure of τ:

(1) if $\tau = x$, $\lambda x.\tau = \mathbf{s\,k\,k}$

(2) if $\tau = y$ and $y \neq x$, $\lambda x.\tau = \mathbf{k}\,y$

(3) if $\tau = \theta(\sigma)$, $\lambda x.\tau = \mathbf{s}\,(\lambda x.\theta)(\lambda x.\sigma)$

By axiom (1) of 2.4, $\mathbf{s\,k\,k}\downarrow$, $\mathbf{s\,k\,k}$ does not have x free and $\mathbf{s\,k\,k}\,x \simeq \mathbf{k}\,x(\mathbf{k}\,x) \simeq x$. Axiom (2) guarantees that, since $\mathbf{k}\,xy\downarrow$, $\mathbf{k}\,y\downarrow$. $\mathbf{k}\,y$ does not have x free and $\mathbf{k}\,yx \simeq y$. By axiom (2) and structural induction, $\mathbf{s}\,(\lambda x.\sigma)(\lambda x.\theta)\downarrow$ and does not contain x free. Also, we know that

$$\mathbf{s}\,(\lambda x.\sigma)(\lambda x.\theta) \simeq (\lambda x.\sigma)x((\lambda x.\theta)x) \simeq \sigma\theta$$

∎

Each pseudoterm σ has a fixed-point under application. Moreover, for each model A of APP, there is a single element of $|A|$, called 'τ^{fix}', which computes a fixed-point for each σ.

2.6. Theorem (Recursion). *There is a pseudoterm $\tau^{\mathrm{fix}} \in \mathrm{PT}$ such that*

$$\mathrm{APP} \vdash \tau^{\mathrm{fix}}\downarrow \wedge \tau^{\mathrm{fix}}\sigma \simeq \sigma(\tau^{\mathrm{fix}}\sigma)$$

Proof. Take τ^{fix} to be the pseudoterm

$$\lambda z.((\lambda y.z(yy))(\lambda y.z(yy))).$$

Theorem 2.5 guarantees that this term exists and that it is defined. A simple calculation shows that τ^{fix} gives the requisite fixed point:

$$\begin{aligned}\tau^{\mathrm{fix}}\sigma &\simeq (\lambda y.\sigma(yy))(\lambda y.\sigma(yy))\\ &\simeq \sigma((\lambda y.\sigma(yy))(\lambda y.\sigma(yy)))\\ &\simeq \sigma(\tau^{\mathrm{fix}}\sigma)\end{aligned}$$

∎

Finally, we prove a simple "double recursion" theorem, which will come into play when we verify the substitutivity of identity over $\mathbf{V}(A)$.

2.7. Corollary (Double recursion). *For $\sigma_1, \sigma_2 \in \mathrm{PT}$, there are $\tau_1, \tau_2 \in \mathrm{PT}$ such that*

$$\mathrm{APP} \vdash \tau_1 \simeq \sigma_1 \tau_1 \tau_2 \wedge \tau_2 \simeq \sigma_2 \tau_1 \tau_2.$$

Proof. Consider the pseudoterm

$$\lambda x.\, \mathbf{p}\,(\sigma_1(\mathbf{l}\,x)(\mathbf{r}\,x))(\sigma_2(\mathbf{l}\,x)(\mathbf{r}\,x)).$$

One application of the recursion theorem gives a term τ_3 such that

$$\tau_3 \simeq \mathbf{p}\,(\sigma_1(\mathbf{l}\,\tau_3)(\mathbf{r}\,\tau_3))(\sigma_2(\mathbf{l}\,\tau_3)(\mathbf{r}\,\tau_3)).$$

Now, we simply set τ_1 equal to $\mathbf{l}\,\tau_3$ and τ_2 equal to $\mathbf{r}\,\tau_3$ and the proof is complete. ∎

Section 3: The general realizability structure

The notion of $V(A)$, the cumulative realizability structure over A, is inspired by the conceptual combination of two significant insights. The first is the semantical idea behind realizability: the evidence for a statement of constructive mathematics is finite and algorithmic and could well be encoded as the index of a Turing machine. This was Kleene's idea and it dates back to the introduction of number realizability in Kleene (1945). (For a nontechnical exposition of realizability and its relation to Heyting's fundamental work on the interpretation of the logical signs of intuitionistic mathematics, see McCarty (1983) or Chapter Zero.) The other idea came along much earlier and in the context of interpreting classical set theory. This is the von Neumann-Mirimanoff-Zermelo picture of a hierarchical universe of sets organized by the membership relation itself. The fundamental insight is that the universe of mathematically significant sets is subsumed by the collection of sets on which membership is well-founded and that the latter sets appear in a natural hierarchy articulated by the ordinals.

For structures A such that $A \models \mathrm{APP}$, $V(A)$ is a cumulative universe on the ordinals. However, at each stage α, $V(A)_{\alpha+1}$ is formed not by taking the collection of all subsets of $V(A)_\alpha$ as in the von Neumann universe, but by throwing in all subsets of $|A| \times V(A)_\alpha$. In this way, each "realizability set" y of the universe is a collection of pairs $\langle e, x\rangle$ where x is hereditarily a realizability set and (up to extensionality) e codes the evidence that $x \in y$.

3.1. Definition. For $A \models \mathrm{APP}$,

$$V(A)_\alpha = \bigcup_{\beta<\alpha} P(|A| \times V(A)_\beta).$$

$$V(A) = \bigcup_\alpha V(A)_\alpha.$$

∎

3.2. Proposition. *$V(A)$ is cumulative: for $\beta < \alpha$, $V(A)_\beta \subseteq V(A)_\alpha$.*

Proof. Immediate. ∎

Section 4: Defining realizability

As is standard in setting up the machinery of Boolean- or of Heyting-valued models, we expand $\mathrm{L_{zf}}$ to a proper-class-sized language L_A. L_A has a distinct primitive constant a for each $a \in \mathrm{V}(A)$; one can even think of the constants as the elements of $\mathrm{V}(A)$, used autonymously. Lowercase Roman letters from the beginning of the alphabet will range over $\mathrm{V}(A)$. Letters from the middle of the same alphabet, e, f, g, ..., denote members of $|A|$. This represents a minor change with respect to our previous usage. Up to this point, x, y, z and their ilk have ranged over $|A|$. But since we don't want to confuse quantification over $|A|$ with formal quantification within $\mathrm{L_{app}}$, we have switched conventions. Needless to say, there are never enough of the right letters to go round and subscripts will be used to compensate. Realizability, $\Vdash$, is first defined only on sentences of L_A.

The Definition of Realizability.

4.1. Definition.

$$\text{(1)}\quad e \Vdash a \in b \text{ iff } \exists c\, \langle \mathbf{l}\, e, c\rangle \in b \wedge \mathbf{r}\, e \Vdash a = c$$

$$\text{(2)}\quad e \Vdash a = b \text{ iff } \forall f, d\, (\langle f, d\rangle \in a \rightarrow \mathbf{l}\, e f \Vdash d \in b \wedge \langle f, d\rangle \in b \rightarrow \mathbf{r}\, e f \Vdash d \in a)$$

$$\text{(3)}\quad e \Vdash \phi \wedge \psi \text{ iff } \mathbf{l}\, e \Vdash \phi \wedge \mathbf{r}\, e \Vdash \psi$$

$$\text{(4)}\quad e \Vdash \phi \vee \psi \text{ iff } (\mathbf{l}\, e = \mathbf{0} \wedge \mathbf{r}\, e \Vdash \phi) \vee (\mathbf{l}\, e \neq \mathbf{0} \wedge \mathbf{r}\, e \Vdash \phi)$$

$$\text{(5)}\quad e \Vdash \neg \phi \text{ iff } \forall f\, \neg f \Vdash \phi$$

$$\text{(6)}\quad e \Vdash \phi \rightarrow \psi \text{ iff } \forall f\, (f \Vdash \phi \rightarrow (ef\downarrow \wedge\, ef \Vdash \psi))$$

$$\text{(7)}\quad e \Vdash \forall x\, \phi \text{ iff } \forall a\, e \Vdash \phi[x/a]$$

$$\text{(8)}\quad e \Vdash \exists x\, \phi \text{ iff } \exists a\, e \Vdash \phi[x/a]$$

∎

Clauses (1) and (2) certainly deviate from the obvious. The obvious thing would have been to write clauses which directly express the notions which we touted as basic insights, namely, that

$$e \Vdash a \in b \text{ iff } \langle e, a\rangle \in b \text{ and}$$

$$e \Vdash a = b \text{ iff } a = b$$

Unfortunately, this will not do if we wish to realize *extensional* IZF directly. The simple clauses would work if we removed axiom EXT. Under the alternative definition of $\Vdash$, one which replaces our (1) and (2) with the two clauses above, there would exist realizability sets $a, b \in \mathrm{V}(A)$ which are distinct, when viewed from outside $\mathrm{V}(A)$, but which, on the basic conception, contain the same elements "from the inside." More precisely, on the alternative, there would exist a and b such that, as sets, a and b may consist of distinct pairs, and so are distinct sets from the metatheoretic viewpoint. However, there may still be $|A|$ elements e and f which interchange the evidence relative to a and b: e carries coded proofs g of $c \in a$ into proofs eg of $c \in b$ and f does the same, but in reverse. If e and f work for all $c \in \mathrm{V}(A)$, then, as far as the coded evidence is concerned, a and b would be the same realizability set. Our clause (2) guarantees that this situation will never arise; sets which are identical in the way of realizability elements are always taken to be identical realizability sets. Clause (1) then insures that membership in realizability sets is closed under these identifications.

Clauses (3) through (6) express the realizability interpretation of propositional intuitionistic logic just as Kleene saw it in 1945. The last two clauses show that unbounded set quantification is interpreted generically, as described in Chapter Zero. In the universal case, we say that e is evidence that proves $\forall x\ \phi$ if and only if e represents a proof schema that proves $\phi(a)$ uniformly for all $a \in \mathrm{V}(A)$. Think of the proof of $\phi(a)$ as given by "inserting" a into an argument place in the proof e. There is nothing about a schematic proof that is keyed into either the structure or the presentations of individual a's. The generic picture of higher-order quantification first took concrete form in the realizability interpretation for second-order Heyting arithmetic provided by Kreisel and Troelstra and expounded by Troelstra in his paper (1973b).

Section 5: The soundness theorem for HPL

'HPL' stands for Heyting's predicate logic, the formal system which codifies the intuitionistic logic we used in Chapter One. The axioms and rules of HPL are sound with respect to realizability in the sense that the universal closures of each axiom is realized and that, whenever a closure of the antecedent of a rule holds, then so do all the closures of the rule's conclusion. Our presentation of HPL is axiomatic.

Heyting's predicate logic.

5.1. The axioms of HPL.

(1) $\phi \rightarrow (\psi \rightarrow \phi)$

(2) $(\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$

(3) $\phi \rightarrow (\psi \rightarrow (\phi \wedge \psi))$

(4) $(\phi \wedge \psi) \rightarrow \phi$

(5) $(\phi \wedge \psi) \rightarrow \psi$

(6) $\phi \rightarrow (\phi \vee \psi)$

(7) $\psi \rightarrow (\phi \vee \psi)$

(8) $(\phi \vee \psi) \rightarrow ((\phi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow \chi))$

(9) $(\phi \rightarrow \psi) \rightarrow ((\phi \rightarrow \neg \psi) \rightarrow \neg \phi)$

(10) $\phi \rightarrow (\neg \phi \rightarrow \psi)$

(11) $\forall x\, \phi \rightarrow \phi[x/y]$

(12) $\phi[x/y] \rightarrow \exists x\, \phi$

∎

As usual, y must be free for x in ϕ in axioms (11) and (12). $\phi[x/y]$ indicates the result of substituting y for x uniformly throughout ϕ.

There are three inference rules:

5.2. Inference rules.

[DET] $\phi, \phi \rightarrow \psi \vdash \psi$

[UG] $\phi \rightarrow \psi[x/y] \vdash \phi \rightarrow \forall x\, \psi$

$$[\text{EI}]\ \phi[x/y] \to \psi \vdash \exists x\, \phi \to \psi$$

∎

In UG and EI, y is free for x in ϕ and occurs free in neither ϕ nor ψ.

Now we can state and prove the realizability soundness of HPL. '$\overline{\forall}\phi$' indicates universal closure of ϕ.

The soundness theorem for HPL.

5.3. . *If* $\text{HPL} \vdash \phi$, *then* $\text{V}(A) \models \overline{\forall}\phi$.

Proof. The verification of realizability for the axioms and rules of HPL is straightforward; we restrict consideration to three examples. For the reader desiring further information, a complete proof appears in the last chapter of Kleene's (1952a). The results of Theorem 2.5 on combinatorial completeness will be used without explicit mention.

For Axiom (3): We show that $g = \lambda x \lambda y.\, \mathbf{p}\, xy \Vdash (3)$. Assume that $e \Vdash \phi$. Then, $ge \downarrow$ and $ge \simeq \lambda y.\, \mathbf{p}\, ey$. Assume also that $f \Vdash \psi$. Then, $gef \simeq \mathbf{p}\, ef$, which is defined, and, by clause (3) of 4.1, $\mathbf{p}\, ef \Vdash (\phi \wedge \psi)$.

For Axiom (8): This one is only slightly more taxing than the others. With h set equal to

$$\lambda x \lambda y \lambda z.\, \mathbf{d}\, (y(\mathbf{r}\, x))(z(\mathbf{r}\, x))(\mathbf{l}\, x),$$

one can easily check that $h \Vdash (8)$. Assume that $e \Vdash \phi \vee \psi$. By (5) of 4.1, either $\mathbf{l}\, e = 0 \wedge \mathbf{r}\, e \Vdash \phi$ or $\mathbf{l}\, e \neq 0 \wedge \mathbf{r}\, e \Vdash \psi$. $he \downarrow$ and he equals

$$\lambda y \lambda z.\, \mathbf{d}\, (y(\mathbf{r}\, e))(z(\mathbf{r}\, e))(\mathbf{l}\, e)$$

Next, assume that $f \Vdash \phi \to \chi$ and $g \Vdash \psi \to \chi$. Then, $hefg \downarrow$ and $hefg$ equals

$$\mathbf{d}\, (f(\mathbf{r}\, e))(g(\mathbf{r}\, e))(\mathbf{l}\, e)$$

Now we use the properties of $\mathbf{d}$ in APP. If $\mathbf{l}\, e = 0$ and $\mathbf{r}\, e \Vdash \phi$, then $hefg \simeq f(\mathbf{r}\, e)$, $f(\mathbf{r}\, e) \downarrow$ and $f(\mathbf{r}\, e) \Vdash \chi$. Similarly, if $\mathbf{l}\, e \neq 0$ and $\mathbf{r}\, e \Vdash \psi$, then $g(\mathbf{r}\, e) \downarrow$ and it realizes χ.

For [EI]: Assume that $e \Vdash \forall y\ (\phi[x/y] \to \psi)$. One shows that there is an $f \in |A|$ such that $fe \Vdash \exists x\, \phi \to \psi$. Without loss of generality, we can assume that y is the only variable free in $\phi[x/y] \to \psi$. From clause (7) of 4.1, we conclude that, for all $a \in \text{V}(A)$, $e \Vdash \phi[x/a] \to \psi$. Now, we assume that $g \Vdash \exists x\, \phi$. Using clause (8) of 4.1, we can say that $\exists b\ \ g \Vdash \phi[x/b]$. Therefore, $eg \downarrow$ and $eg \Vdash \psi$. Altogether then, we see that $e \Vdash \exists x\, \phi \to \psi$ and we can take the desired f to be $\mathbf{s\,k\,k}$. ∎

Section 6: Realizing the axioms of identity

In preparation for the full soundness theorem for IZF, we prove that realizability is sound for the identity axioms of L_A.

The identity axioms.

6.1. Axioms for identity.

(1) $x = x$

(2) $x = y \to y = x$

(3) $(x = y \wedge y = z) \to x = z$

(4) $(x = y \wedge y \in z) \to x \in z$

(5) $(x = y \wedge z \in x) \to z \in y$

We will use 'ID' to refer to the set of closures of these axioms. ∎

The soundness proof for these axioms is only slightly more delicate than the proof of the last theorem. Since realizability for atomic statements has been defined recursively on the membership relation, induction on the ordinals will be required. The work is easier if we first take time out to prove two lemmas on the closure properties, under membership and equality, of the levels in the $V(A)$ hierarchy.

The closure lemma.

6.2. Lemma.

(1) $b \in V(A)_\alpha \to \exists \beta < \alpha \, \forall c \, (V(A) \models c \in b \to c \in V(A)_\beta)$

(2) $(a \in V(A)_\alpha \wedge V(A) \models a = b) \to b \in V(A)_\alpha$

Proof. (1) and (2) are proved simultaneously by induction on the ordinals. Assume that, for $\beta < \alpha$,

$$(d \in V(A)_\beta \wedge V(A) \models c = d) \to c \in V(A)_\beta$$

Let $b \in V(A)_\alpha$. From the definition of $V(A)$,

$$\exists \beta < \alpha \text{ such that } b \subseteq |A| \times V(A)_\beta$$

Let $e \Vdash c \in b$. By clause (1) of 4.1, the definition of realizability,

$$\exists d\, \langle \mathbf{l}\, e, d\rangle \in b \wedge \mathbf{r}\, e \Vdash c = d$$

Since $b \subseteq |A| \times \mathbf{V}(A)_\beta$, $d \in \mathbf{V}(A)_\beta$, so, by assumption, $c \in \mathbf{V}(A)_\beta$ and we have (1).

To prove (2), take $a \in \mathbf{V}(A)_\alpha$ and suppose that $e \Vdash a = b$. Choose a $\beta < \alpha$ such that, if $\mathbf{V}(A) \models c \in a$, then $c \in \mathbf{V}(A)_\beta$. Now, for any $d \in \mathbf{V}(A)$, if $\langle k, d\rangle \in b$, then $\mathbf{r}\, ek \models d \in a$. Hence, $d \in \mathbf{V}(A)_\beta$. Therefore, $b \in \mathrm{P}(|A| \times \mathbf{V}(A)_\beta)$ and $\beta < \alpha$, so $b \in \mathbf{V}(A)_\alpha$. This gives (2) for $a \in \mathbf{V}(A)_\alpha$. ∎

The soundness theorem for identity.

6.3. Theorem (Soundness for identity). $\mathbf{V}(A) \models \mathrm{ID}$.

Proof. For Axiom (1): even this simple axiom calls for induction on the ordinals. The proof will show that there is an $i \in |A|$ such that, given the assumption that for all $\beta < \alpha$ and all $b \in \mathbf{V}(A)_\beta$, $i \Vdash b = b$, then $i \Vdash a = a$ for $a \in \mathbf{V}(A)_\alpha$. i is constructed using the recursion theorem of 2.6.

First, use the recursion theorem to find an $i \in |A|$ such that

$$i \simeq \mathbf{p}\,(\lambda y.\, \mathbf{p}\, yi)(\lambda y.\, \mathbf{p}\, yi)$$

Then, assume that for all $\beta < \alpha$ and $b \in \mathbf{V}(A)_\beta, i \Vdash b = b$. If $\langle f, b\rangle \in a$, then $\mathbf{l}\, if \simeq \mathbf{p}\, fi$, which is defined and, since $\langle f, b\rangle \in a$ and $i \Vdash b = b$, $\mathbf{p}\, fi \Vdash b \in a$.

For Axiom (2): No induction is necessary here. Let $g = \lambda x.\, \mathbf{p}\,(\mathbf{l}\, x)(\mathbf{r}\, x)$. Intuitively, g is given by the λ-pseudoterm that interchanges the left and right members of any pair. The check that $g \Vdash$ (2) is easy.

For Axioms (3) and (4): Let $\tau_1, \ldots, \tau_6 \in \mathrm{PT}$ be as follows:

$$\tau_1 = \mathbf{r}\,(\mathbf{l}\,(\mathbf{l}\, y)x)$$

$$\tau_2 = \mathbf{l}\,(\mathbf{r}\, y)$$

$$\tau_3 = \mathbf{r}\,(\mathbf{r}\,(\mathbf{r}\, y)x)$$

$$\tau_4 = \mathbf{r}\,(\mathbf{l}\, y)(\mathbf{l}\,(\mathbf{r}\,(\mathbf{r}\, h)x))$$

$$\tau_5 = \tau_2(\mathbf{l}\,(\mathbf{l}\,(\mathbf{l}\, y)x))$$

$$\tau_8 = \mathbf{p}\,(\mathbf{l}\,y)(\mathbf{r}\,(\mathbf{r}\,y))$$

By the Lemma 2.7 on double recursion, there are $|A|$-elements e and f such that

$$e = \lambda y.\, \mathbf{p}\,(\tau_2(f\tau_8))$$

$$f = \lambda y.\, \mathbf{p}\,(\lambda x.e(\mathbf{p}\,\tau_1\tau_5))(\lambda x.e(\mathbf{p}\,\tau_3\tau_4)).$$

There is a proof, by simultaneous ordinal-induction, that $f \Vdash (3)$ and $e \Vdash (4)$. For ease o exposition, the proof is presented in two parts.

Part I: Assume that, for all $\beta < \alpha$ and $d_1 \in \mathrm{V}(A)_\beta$,

$$f \Vdash (d = d_1 \wedge d_1 = d_2) \rightarrow d = d_2.$$

On this assumption, we prove that, for $c \in \mathrm{V}(A)_\alpha$,

$$e \Vdash (a = b \wedge b \in c) \rightarrow a \in c$$

Let $h \Vdash a = b \wedge b \in c$. Then $\mathbf{l}\,h \Vdash a = b$ and $\mathbf{r}\,h \Vdash b \in c$. The latter implies that

$$\exists d\, \langle \mathbf{l}\,(\mathbf{r}\,h), d\rangle \in c \wedge \mathbf{r}\,(\mathbf{r}\,h) \Vdash b = d$$

By the lemma 6.2 on the closure of $\mathrm{V}(A)$, we can assume that $b \in \mathrm{V}(A)_\beta$ for some $\beta < \alpha$ Therefore, by the inductive assumption, $f(\tau_8[y/h]) \Vdash a = d$ and by 4.1,

$$\mathbf{p}\,(\tau_2[y/h])(f(\tau_8[y/h])) \Vdash a \in c.$$

Hence, we have that $eh\downarrow$ and $eh \Vdash a \in c$.

This concludes *Part I*; we have shown that, for $c \in \mathrm{V}(A)_\alpha$,

$$e \Vdash (a = b \wedge b \in c) \rightarrow a \in c.$$

Part II: With the result of *Part I* as assumption, we want to prove that, for b $\mathrm{V}(A)_\alpha$,

$$f \Vdash (a = b \wedge b = c) \rightarrow a = c$$

Assume that $h \Vdash a = b \wedge b = c$ and let $\langle g, d\rangle \in a$. Then,

$$\mathbf{l}(\mathbf{l}\, h)g \Vdash d \in b.$$

By clause (1) of the definition of realizability,

$$\exists i\, \langle \mathbf{l}(\mathbf{l}(\mathbf{l}\, h)g), i\rangle \in b \wedge \tau_1[y/h][x/g] \Vdash d = i$$

Then, $\tau_5[y/h][x/g] \Vdash i \in c$. By closure, we can assume that $c \in \mathrm{V}(A)_\alpha$.

With e as in *Part I*,

$$e(\mathbf{p}\,(\tau_1[y/h][x/g])(\tau_5[y/h][x/g])) \Vdash d \in c$$

A similar argument shows that, on the assumption that $\langle g, d\rangle \in c$,

$$e(\mathbf{p}\,(\tau_3[y/h][x/g])(\tau_5[y/h][x/g])) \Vdash d \in a.$$

Finally, with λ-abstraction and pairing, we see that

$$f \Vdash (a = b \wedge b = c) \to a = c$$

This concludes *Part II.*

For Axiom (5): With e as in *Part I* of the preceding, set

$$g = \lambda x.e(\mathbf{p}\,(\mathbf{r}\,(\mathbf{r}\, x)(\mathbf{l}\,(\mathbf{l}\, h)(\mathbf{l}\,(\mathbf{r}\, x)))))$$

With the result of *Part I*, it is easy to check that $g \Vdash (5)$. ∎

6.4. Note. There are numerous times when we will have need for a fixed witness for $\forall x\; x = x$. We set aside the letter '$\mathbf{i}$' as notation for such a witness. Hence, whenever an unannounced '$\mathbf{i}$' suddenly appears in a proof, the reader can presume that the intrusive '$\mathbf{i}$' stands for this fixed witness. ∎

Section 7: The soundness theorem for IZF

The last section is devoted to the only remaining task of this chapter—that of proving the soundness of full IZF with respect to general realizability.

The soundness theorem for IZF.

7.1. . $\mathrm{V}(A) \models \mathrm{IZF}$

Proof. We treat the axioms in the order in which they were introduced in Chapter One section 2, beginning with extensionality. Again, we fix $i \in |A|$ such that $i \Vdash \forall x\,(x = x)$.

(1) [EXT] It is a simple matter to check that, with

$$e = \lambda y.\, \mathbf{p}\,(\lambda x.\, \mathbf{l}\, y(\mathbf{p}\, xi))(\lambda x.\, \mathbf{r}\, y(\mathbf{p}\, xi)),$$

$e \Vdash$ EXT. The matter is so simple because we took pains to see that extensionality wa "built into" the very definition of $\Vdash$.

(2) [PAIR] We need to guarantee the existence of an $e \in |A|$ such that

$$\forall a, b\; \exists c \;\; e \Vdash a \in c \wedge b \in c.$$

Set $e = \mathbf{p}\,(\mathbf{p}\,0\,i)(\mathbf{p}\,0\,i)$ and, for $a, b \in \mathrm{V}(A)$, take $c = \{\langle 0\,, a\rangle, \langle 0\,, b\rangle\}$.

Since $\mathrm{V}(A)$ is cumulative (cf. Proposition 3.2), if $a \in \mathrm{V}(A)_\alpha$ and $b \in \mathrm{V}(A)_\beta$, then and $b \in \mathrm{V}(A)_{\bigcup\{\alpha+1,\beta+1\}}$. Hence, for $\gamma = \bigcup\{\alpha+1, \beta+1\} + 1$, $c \in \mathrm{V}(A)_\gamma$. Trivially, then $e \Vdash$ PAIR.

7.2. Note. If we wish to remain within a strictly constructive metatheory, we canno make the simplifying assumption that $\forall \alpha, \beta\; (\alpha \in \beta \vee \beta \in \alpha \vee \alpha = \beta)$. As Grayso has shown in his *Heyting-valued models* (1979), the assumption of trichotomy for ordinal implies TND in IZF. In the same paper, Grayson describes a Kripke model for IZF o which ordinal trichotomy is demonstrably false. In the next chapter, we will prove no only that ordinal trichotomy is both weakly and strongly false over $\mathrm{V}(A)$, but also that i is consistent with IZF to assume that the collection of ordinals on which trichotomy hold comprises a *relatively small set.* ∎

(3) [UN] Let $e = \lambda x.\, \mathbf{p}\, xi$ and let $b = \mathrm{Un}^A(a)$, where, for each $a \in \mathrm{V}(A)$,

$$\mathrm{Un}^A(a) = \{\langle e, c\rangle : e \Vdash \exists x\,(c \in x \wedge x \in a)\}$$

Because of part (1) of 6.2, the closure lemma, Un^A is well-defined on $\mathrm{V}(A)$, and one can quickly see that

$$e \Vdash \forall b\, \forall c\, ((c \in b \wedge b \in a) \rightarrow c \in \mathrm{Un}^A(a)).$$

(4) [SEP] This time we need to find an $e \in |A|$ such that

$$\forall a\, \exists b\, \forall c\quad e \Vdash (c \in b \leftrightarrow (c \in a \wedge \phi[x/c]))$$

We define the operator $\mathrm{Sep}^A(a, \phi)$ to be

$$\{\langle e, c\rangle : e \Vdash (c \in a \wedge \phi[x/c])\}.$$

The operator is well-defined on $\mathrm{V}(A)$ for much the same reason that $\mathrm{Un}^A(a)$ is.

Starting with the soundness of identity, (cf. 6.3), and using induction in the metalanguage, one can prove that there is a $j_\phi \in |A|$ such that

$$j_\phi \Vdash (z \in a \wedge \phi[x/z] \wedge z = y) \rightarrow (y \in a \wedge \phi[x/y]).$$

Take $e = \mathbf{p}\, j_\phi(\lambda x.\, \mathbf{p}\, xi)$. Then

$$e \Vdash c \in \mathrm{Sep}^A(a, \phi) \leftrightarrow (c \in a \wedge \phi[x/c]).$$

(5) [POW] For $a \in \mathrm{V}(A)_\alpha$, set

$$\mathrm{P}^A(a) = \{\langle e, c\rangle : e \Vdash c \subseteq a\}.$$

To see that $\mathrm{P}^A(a)$ is well-defined, assume that $e \Vdash c \subseteq a$. If $\langle f, d\rangle \in c$, then $\mathbf{p}\, fi \Vdash d \in c$ and $e(\mathbf{p}\, fi) \Vdash d \in a$. By 3.2, part (1), there is a $\beta < \alpha$ such that for all f and d, if $\langle f, d\rangle \in c$, then $d \in \mathrm{V}(A)_\beta$. It follows that $c \in \mathrm{V}(A)_\alpha$. Therefore, $\mathrm{P}^A(a)$ is well-defined and takes $\mathrm{V}(A)_\alpha$ into $\mathrm{V}(A)_{\alpha+1}$.

Finally, if $e = \lambda x.\, \mathbf{p}\, xi$, then $e \Vdash (c \subseteq a \rightarrow c \in \mathrm{P}^A(a))$.

(6) [INF] Our favorite of the many representatives of ω in $\mathrm{V}(A)$ is $\overline{\omega}$, which is given via a metatheoretic recursive injection of ω into $\mathrm{V}(A)$. For $n \in \omega$, set $\underline{n+1} = \mathbf{p}\, \underline{n}\, \mathbf{0}$ and set $\overline{n} = \{\langle \underline{m}, \overline{m}\rangle : m \in n\}$. (Recall that $\mathbf{0}$ is already given—with the APP axioms.) Then, we take $\overline{\omega} = \{\langle \underline{n}, \overline{n}\rangle : n \in \omega\}$, and $\overline{\omega} \in \mathrm{V}(A)_{\omega+1}$.

To verify INF, note both that

$$\overline{0} = \emptyset \in \mathrm{V}(A) \text{ and } \mathbf{p}\,0\,i \Vdash \overline{0} \in \overline{\omega}$$

and that

$$\text{if } g \Vdash c \in \overline{\omega}, \text{ then } \exists n \in \omega \;\; \mathbf{r}\,g \Vdash c = \overline{n} \text{ and } \mathbf{l}\,g = \underline{n}.$$

It follows that $\mathbf{p}\,(\mathbf{l}\,g)i \Vdash \overline{n} \in \overline{n+1}$, while

$$\mathbf{p}\,\underline{n+1}\,i \Vdash \overline{n+1} \in \overline{\omega} \text{ and}$$

$$\underline{n+1} = \mathbf{p}\,(\mathbf{p}\,(\mathbf{l}\,g))\,0\,.$$

Therefore, if we fix $j \in |A|$ so that

$$j \Vdash x = y \to (y \in z \to x \in z), \text{ then}$$

$$\mathbf{p}\,(j(\mathbf{r}\,g)(\mathbf{p}\,(\mathbf{l}\,g)i))(\mathbf{p}\,(\mathbf{p}\,(\mathbf{l}\,g)\,0)i) \Vdash \exists y\,(c \in y \wedge y \in \overline{\omega}).$$

(7) [COLL] Let $g \Vdash \forall x \in a\; \exists y\; \phi$. Then, for $h \Vdash b \in a$, there is a c such that $gh \Vdash \phi(b, c)$. Again, the closure lemma for $\mathrm{V}(A)$ proves that $\{\langle h, b\rangle : h \Vdash b \in a\} \in \mathrm{V}(A)$. With collection in the metatheory, there is a β such that if $h \Vdash b \in a$, then $\exists c \in \mathrm{V}(A)_\beta \;\; gh \Vdash \phi(b, c)$.

Take $d = |A| \times \mathrm{V}(A)_\beta$; $d \in \mathrm{V}(A)_{\beta+1}$. If $e = \lambda x \lambda y.\, \mathbf{p}\,(\mathbf{p}\,0\,i)(xy)$, then $e \Vdash$ COLL. To see this, let

$$g \Vdash \forall x \in a\; \exists y\; \phi \text{ and let } h \Vdash b \in a.$$

Then, $egh = \mathbf{p}\,(\mathbf{p}\,0\,i)(gh)$, and there is a c such that $gh \Vdash \phi(b, c)$. We can assume that $\langle 0\,, c\rangle \in d$. Hence, $\mathbf{p}\,0\,i \Vdash c \in d$.

(8) [IND] Since IND captures that aspect of the universe of sets that underlies definition by recursion on $\in$, one can safely predict that we will call on the second recursion theorem for APP, 2.6, to construct an $e \in |A|$ to realize IND. To begin with, we assume that for all $a \in \mathrm{V}(A)$, $g \Vdash \forall y\; (y \in a \to \phi[x/y]) \to \phi(a)$ and that, for all $\beta < \alpha$ and $b \in \mathrm{V}(A)_\beta$, $e \Vdash \phi(b)$.

Take $a \in \mathrm{V}(A)_\alpha$. Then $g \Vdash \forall y\,(y \in a \rightarrow \phi[x/y]) \rightarrow \phi(a)$ and, by closure, if $h \Vdash b \in a$, then $\exists \beta < \alpha\; b \in \mathrm{V}(A)_\beta$. Hence, $e \Vdash \phi(b)$. Therefore,

$$\lambda x.\, \mathrm{k}\, ex \Vdash \forall y\,(y \in a \rightarrow \phi[x/y]) \text{ and}$$
$$g(\lambda x.\, \mathrm{k}\, ex) \Vdash \phi(a).$$

By the second recursion theorem, there is an $f \in |A|$ that fixes $j = g(\lambda x.\, \mathrm{k}\, yx)$, i.e.,

$$fj = g(\lambda x.\, \mathrm{k}\,(fj)x).$$

Then, with $e = \lambda g.fj$, $e \Vdash \mathrm{IND}$. ∎

Section 8: Final remarks

(1) All the notions that went into constructing $\mathrm{V}(A)$ can be expressed in IZF, even the recursive definition of $\Vdash$ for atomic sentences. Therefore, if A is ZF-definable, we can associate with each ϕ from $\mathrm{L_{zf}}$ a formula $\phi^{\Vdash}$ of $\mathrm{L_{zf}}$, the latter saying 'x realizes ϕ with respect to A.' Also, all of the work that went into proving the soundness theorems is constructive. It follows that, if $\mathrm{IZF} \vdash (A \models \mathrm{APP})$, then $\mathrm{IZF} \vdash (\mathrm{V}(A) \models \mathrm{IZF})$.

(2) Anyone familiar with the Scott-Solovay approach to Boolean-valued models (cf. Bell (1977)) will have noticed the analogy between our construction of realizability over $\mathrm{V}(A)$ and the construction of models of set theory over complete Boolean algebras. The analogy can be made good mathematically within the category-theoretic framework provided by the theory of triposes (cf. Pitts (1981)).

(3) In the work reported here, Kleene (or "number") realizability for set theory will afford the primary focus of interest. General extensional realizability, as we have defined it in this chapter, is not, however, a useless generalization. For one thing, the soundness theorem for IZF is "visually easier" when presented generally than when proved directly for Kleene realizability. The use of combinators and lambda terms makes the abstract relations between formulae and their realizers much more vivid. For another thing, the general approach will allow one to apply APP-based realizability directly to other structures upon the mere recognition that the structure in question provides a model of APP. We will have very little opportunity to make these applications here; we hope that further writings will fill this lacuna.

Introducing Kleene Realizability

Section 1: Prefatory and historical remarks

Very few of the results of this chapter are wholly new; even fewer, we fear, are very startling. At best, some of the proofs are new and some of the results are new (so far as we know) for *extensional* intuitionistic set theory. The material on the Presentation Axiom and the applications of uniformity and subcountability stand as exceptions; these things are completely new.

The fact that this chapter is largely a reworking of the literature is in keeping with its intention. want here to augment the work of Chapter One with an *overview* of set theory under realizability. The reader will recall that, in Chapter Zero, we went to some length to impress upon the reader the fact that intuitionism and realizability have *two faces*. One face is turned toward the proof relation and is combinatorial in character; the other addresses the realm of higher-order entities (sets, species, sequences and so forth). This chapter is an introduction to the realizability investigation of the latter face.

Those results from this chapter that have appeared before in print can be found (perhaps in slightly different forms) in Beeson (1979), Friedman (1973a) or Hyland (1982). Everything here (both new and old) was known to the author prior to November 1981.

Section 2: The structure *Kl*

2.1. Definition.

(1) Let $\lambda x \lambda y \langle x, y\rangle$ be a primitive recursive pairing function from $\omega \times \omega$ into ω such that

$$\forall x, y\ \langle x, y\rangle \neq 0.$$

(2) Let $\lambda x.x_0$ and $\lambda x.x_1$ be total left and right unpairing operations with respect to $\langle x, y\rangle$; for all x and y from ω,

$$\langle x, y\rangle_0 = x \text{ and } \langle x, y\rangle_1 = y.$$

(3) Let G be the function given by

$$G(x, y, z) = \begin{cases} x & \text{if } z = 0 \\ y & \text{if } z \neq 0 \end{cases}.$$

and let

$$g = \Lambda x \Lambda y \Lambda z G(x, y, z).$$

▮

2.2. Definition. Kl is the realization of L_{app} in which the universe $|Kl|$ is ω and over which $=$ from L_{app} is interpreted as equality. $\mathrm{App}^{Kl}(x, y, z)$ is Turing machine application:

$$\mathrm{App}^{Kl}(x, y, z) \text{ iff } z \simeq \{x\}(y).$$

The primitive constants of L_{app} are interpreted over Kl as follows:

(1) $\mathbf{k}^{Kl} = \Lambda x \Lambda y.x$

(2) $\mathbf{s}^{Kl} = \Lambda x \Lambda y \Lambda z.\{x\}(z)(\{y\}(z))$

(3) $\mathbf{p}^{Kl} = \Lambda x \Lambda y \langle x, y\rangle$

(4) $\mathbf{l}^{Kl} = \Lambda x.x_0$

(5) $\mathbf{r}^{Kl} = \Lambda x.x_1$

(6) $\mathbf{d}^{Kl} = g$

(7) $\mathbf{0}^{Kl} = 0.$

▮

All the recursion theory required to construct a Kleene-style number realizability model for full IZF comes neatly packed into this little theorem:

2.3. Theorem. $Kl \models$App.

Proof. The classical s-m-n theorem proves that τ^{Kl} is well-defined for each primitive term τ of L_{App}. The reader can easily use the definitions from 2.1 to prove that the interpreted constants have the properties required by the App axioms. ▮

Now, all of Chapter Two can be brought to bear on Kl. $V(Kl)$ is the universe of realizability sets, as described in Chapter Zero, and IZF is interpreted soundly over it.

2.4. Theorem. $V(Kl) \models$IZF.

Proof. See Theorem 7.1 of Chapter Two. ▮

Section 3: Internalizing basic set theory

The natural first step in the investigation of set theory over $V(Kl)$ is the isolation of the basic constructs: singletons, pairing, successors and natural numbers. The $V(Kl)$-internal versions of each of these notions will run through and organize our treatment of the subject.

3.1. Notation. We utilize the perspicuous "overbar" convention to distinguish between the external and internal versions of each of the basic notions. For example, the favored internal version of the natural numbers, ω, is a realizability set named '$\overline{\omega}$'. Once we can presume some facility in working between internal and external versions, much of the "overbarring" will be omitted. Like any convenient notation, ours becomes cumbersome when taken to extremes; in extremity, we simply drop the notation and put faith in the reader's talents. ∎

Set-theoretic pairing.

3.2. Lemma. *For $a, b \in V(Kl)$, $\overline{\{a,b\}} = \{\langle 0,a\rangle, \langle 1,b\rangle\}$ is the internal unordered pair of a and b, i.e.,*

$$V(Kl) \models x \in \overline{\{a,b\}} \leftrightarrow x = a \vee y = b.$$

There is a realizability witness for the above which is uniform in a and b.

Proof. From the very definition of $\overline{\{a,b\}}$, we see that

$$e \Vdash c \in \overline{\{a,b\}} \text{ iff either } e_0 = 0 \wedge e_1 \Vdash c = a \text{ or } e_0 = 1 \wedge e_1 \Vdash c = b.$$

The desired result then follows immediately from the clauses in the definition of $\Vdash$ which govern the interpretations of the symbols '$\in$' and '$\vee$'. ∎

3.3. Definition. For $a \in V(Kl)$, $\overline{\{a\}} = \overline{\{a,a\}}$. ∎

3.4. Lemma. *For $a, b \in V(Kl)$,*

$$\overline{\langle a,b\rangle} = \{\langle 0, \overline{\{a\}}\rangle, \langle 1, \overline{\{a,b\}}\rangle\}$$

is the internal ordered pair of a and b, i.e.,

$$V(Kl) \models x \in \overline{\langle a,b\rangle} \leftrightarrow x = \overline{\{a\}} \vee x = \overline{\{a,b\}}.$$

There is a witness for the above which is independent of a and b.

Proof. This follows directly from the preceding lemma and from the realizability clauses for '$\in$' and '$\vee$.' ∎

3.5. Lemma. *For $a, b \in \mathbf{V}(Kl)$,*

$$\overline{a \cup b} = \{\langle\langle 0, n\rangle, c\rangle : \langle n, c\rangle \in a\} \cup \{\langle\langle 1, m\rangle, d\rangle : \langle m, d\rangle \in b\}$$

is the internal union of realizability sets a and b. Again, a witness can be chosen which is uniform over all a and b.

Proof. $e \Vdash g \in \overline{a \cup b}$ implies that either there is an $\langle n, c\rangle \in a$ for which $e_0 = \langle 0, n\rangle$ while $e_1 \Vdash g = c$, or there is an $\langle m, d\rangle \in b$ for which $e_0 = \langle 1, m\rangle$ and $e_1 \Vdash g = d$. It follows that

$$\langle e_{00}, \langle e_{01}, e_1\rangle\rangle \Vdash g \in a \vee g \in b.$$

Therefore, if $h = \Lambda e.\langle e_{00}, \langle e_{01}, e_1\rangle\rangle$, then

$$h \Vdash \forall x\, (x \in \overline{a \cup b} \rightarrow x \in a \vee x \in b).$$

For the converse implication, let $e \Vdash g \in a \vee g \in b$. Then, either $e_0 = 0$ or $e_0 = 1$. If $e_0 = 0$,

$$\exists c\, \langle e_{10}, c\rangle \in a \wedge e_{11} \Vdash g = c.$$

Hence, $\langle\langle e_0, e_{10}\rangle, e_{11}\rangle \Vdash g \in \overline{a \cup b}$. In case $e_0 = 1$, there is a parallel argument to the same conclusion. ∎

ω and $\overline{\omega}$.

Now we can offer a natural internalization of ω and check that, in $\mathbf{V}(Kl)$, the internalization has all the familiar properties of the set of natural numbers. Salient among the possible internalizations of the natural number concept is the realizability set $\overline{\omega}$:

3.6. Definition. For each $n \in \omega$, let

$$\overline{n} = \{\langle m, \overline{m}\rangle : m \in n\}.$$

Then set

$$\overline{\omega} = \{\langle n, \overline{n}\rangle : n \in \omega\}.$$

∎

$\overline{\omega}$ is our candidate for internal ω, and, for each external n, $\overline{n}$ is its internal version.

3.7. . $\mathbf{V}(Kl) \models \overline{\omega} = \omega$.

Proof. It will suffice to show that, over $\mathbf{V}(Kl)$, $\overline{\omega}$ is the $\subseteq$-minimum successor-hereditar class that contains $\emptyset$ as a member. In terms of the basic formalisms and of the interna izations just constructed, we need to prove that

$$\mathbf{V}(Kl) \models \overline{0} \in \overline{\omega} \wedge \forall x \in \overline{\omega}\ (\overline{x \cup \overline{\{x\}}} \in \overline{\omega}) \wedge ((\phi(\overline{0}) \wedge \forall z\ (\phi(z) \rightarrow \phi(\overline{z \cup \overline{\{z\}}}))) \rightarrow \forall x \in \overline{\omega}\ \phi($$

Here, ϕ is assumed to be a formula of the augmented language $\mathbf{L}_{Kl}$.

By definition, $\langle 0, i\rangle \Vdash \overline{0} \in \overline{\omega}$. The easiest way to verify that $\overline{\omega}$ is closed under intern successors in $\mathbf{V}(Kl)$ is to prove two lemmas: the first is a lemma which characteriz quantification over $\overline{\omega}$ in terms of external recursive functions and the second reveals t accord between the internal and the external versions of successor.

3.8. Note. The following is a precise version of a characterization of realizability f arithmetic which we accepted on informal grounds in Chapter Zero. ∎

3.9. Lemma. $\mathbf{V}(Kl) \models \forall x \in \overline{\omega}\ \phi$ *iff there is a total recursive* $\{e\}$ *such that, for all* $\{e\}(n) \Vdash \phi[x/\overline{n}]$.

Proof. On the one hand, let $e \Vdash \forall x \in \overline{\omega}\ \phi$ and take $n \in \omega$. By the definition of $\langle n, i\rangle \Vdash \overline{n} \in \overline{\omega}$. Therefore, $\{e\}(\langle n, i\rangle)\downarrow$ and $\{e\}(\langle n, i\rangle) \Vdash \phi[x/\overline{n}]$.

On the other hand, assume that there is a total recursive $\{e\}$ such that, for all extern n, $\{e\}(n) \Vdash \phi[x/\overline{n}]$. Let $g \Vdash a \in \overline{\omega}$; then, $g_1 \Vdash a = \overline{g_0}$. By the assumption on $\{e\}(g_0) \Vdash \phi[x/\overline{g_0}]$. The realizability of the substitutivity of identity now gives the resu

∎

3.10. Lemma. *For each* n*, there is a realizer for*

$$\overline{\overline{n} \cup \overline{\{\overline{n}\}}} = \overline{n+1}$$

which depends effectively on n.

Proof. Once one unpacks all the definitions of internal notions, it is a trivial matter to see that the lemma holds. First, $\overline{\overline{n} \cup \overline{\{\overline{n}\}}}$ is defined to be

$$\{\langle\langle 0, m\rangle, \overline{m}\rangle : m \in n\} \cup \{\langle\langle 1, 0\rangle, \overline{n}\rangle\}.$$

(This is not, strictly speaking, true. We have assumed here that $\overline{\{\overline{n}\}}$ is defined to be $\{\langle 0, \overline{n}\rangle\}$. This is not externally the same as our definition of singleton, but is easily seen to be internally the same.)

$\overline{n+1}$ is defined to be

$$\{\langle m, \overline{m}\rangle : m \in n\} \cup \{\langle n, \overline{n}\rangle\}.$$

One can now see that, given n, there is, in accord with the interpretation of '$=$,' an effective procedure for interchanging realizability elements of the one set with those of the other. Therefore, the lemma follows by the realizability of $=$. ∎

We return to the proof of theorem 3.7. Thanks to the lemmas, we see now that, to prove that $\overline{\omega}$ is realizably closed under successor, it will suffice to provide a total index e such that for all n,

$$\{e\}(n) \Vdash \overline{n+1} \in \overline{\omega}.$$

But that's easy (after all, $\overline{\omega}$ was defined just to make these things easy); we take $e = \Lambda n.\langle n+1, i\rangle$.

Finally, we need the realizability of the fact that $\overline{\omega}$ is the least set hereditary with respect to successor. Assume tht $e \Vdash \phi(\overline{0})$ and $g \Vdash \phi(a) \to \phi(\overline{a \cup \{a\}})$ for all $a \in V(Kl)$. Let j provide the witnessing guaranteed by the last lemma, i.e., there is a number j such that, for every n,

$$\{j\}(n) \Vdash \overline{\overline{n} \cup \overline{\{\overline{n}\}}} = \overline{n+1}.$$

Let k_ϕ effect the substitutivity of identity:

$$k_\phi \Vdash (\phi(y) \wedge y = z) \to \phi(z)$$

Clearly, the desired realizer will be constructed by primitive recursion on these indices. Define $G(n)$ so that

$$G(0) = e$$

$$G(n+1) = k_\phi(\langle \{g\}(G(n)), j \rangle).$$

Set $h = \Lambda n.G(n)$. A simple induction on external ω shows that, for all n, $\{h\}(n) \Vdash \phi(\overline{n})$. Since there is an effective routine which, given indices e and g, produces an index for h, we are done. ∎

$\overline{\omega}$, therefore, is an acceptable version of the set of natural numbers in $\mathbf{V}(Kl)$. There are, of course, innumerable realizability sets in $\mathbf{V}(Kl)$ which are "ω-like". Some of these "crypto-ω's" are actually equal to ω in the model; the others are strange and wonderful variants of ω that are highly useful for giving counterexamples and independence results. An example of the former is the set $\overline{\overline{\omega}}$:

$$\overline{\overline{\omega}} = \{\langle n+1, n \rangle : n \in \omega\}.$$

There is no question that $\mathbf{V}(Kl) \models \overline{\overline{\omega}} = \omega$; it is easy to transform witnesses for membership in $\overline{\omega}$ into witnesses for membership in $\overline{\overline{\omega}}$ and vice versa.

Among the latter, sets which are "close to but not quite" ω are sets like ω^0:

$$\omega^0 = \{\langle 0, \overline{n} \rangle : n \in \omega\}.$$

As we shall see, ω^0 is a *uniformity set*, a set over which the relevant axiom of choice holds with such a vengence that constant functions are the *only* number-theoretic functions defined over it.

But ω^0 is even more remarkable than that. In $\mathbf{V}(Kl)$, $\overline{\omega} \subseteq \omega^0$ and ω^0 is an ordinal, but $\overline{\omega} \neq \omega^0$. Also (and this is obvious) $\mathbf{V}(Kl) \models \overline{\omega} \notin \omega^0 \wedge \omega^0 \notin \overline{\omega}$. Therefore, the existence of ω^0 provides a *simple* proof that not only does trichotomy fail in the class of realizability ordinals but that it can fail "way down" in the cumulative hierarchy. We note that, in terms of ranks, trichotomy cannot fail for any sets lower down. Trichotomy is provable in IZF for elements of ω. We will prove later in this chapter that the class of ordinals in $\mathbf{V}(Kl)$ on which trichotomy holds forms a (relatively small) set. In fact, every ordinal that is sufficiently classical that trichomtomy holds of its elements is *subcountable*.

There are also unusual variants of $\overline{\omega}$ on which the axiom of choice fails. A sample of this sort of phenomenon appears at the very end of Chapter Six.

Absoluteness for $\overline{\omega}$.

In the next chapter, there will be extended discussion of the possibility and application of various *absoluteness properties* for predicates defined over the natural numbers. For the present, we need only prove the most basic of the *absoluteness theorems*. Eventually, these turn out to be the simplest cases of some very general results. We give these cases now because they will prove helpful in assessing the material of the present chapter.

3.11. Theorem. *Equality and membership are realizability absolute for $\overline{\omega}$. This means that for all $n, m \in \omega$,*

$$\mathbf{V} \models m = n \textit{ iff } \mathbf{V}(Kl) \models \overline{m} = \overline{n} \textit{ and}$$
$$\mathbf{V} \models m \in n \textit{ iff } \mathbf{V}(Kl) \models \overline{m} \in \overline{n}$$

Proof. On the appropriate definitions, the implications from left to right are obvious. The converse implications require simultaneous induction on both types of statement. Nevertheless, the requisite induction is very easy and we leave it to the trusty reader to reconstruct a complete proof from the following judicious example.

If $e \Vdash \overline{m} \in \overline{n}$, then, by definition,

$$e_1 \Vdash \overline{m} = e_0 \text{ and } \mathbf{V} \models e_0 \in n.$$

By the induction hypothesis applied to equality statements, it follows that, in $\mathbf{V}$, m is really equal to e_0. Therefore, $\mathbf{V} \models m \in n$. ∎

Section 4: Intuitionistic counterexamples

The Heyting universe.

One measure of the success of realizability as an interpretation of intuitionism is an assessment of the extent to which V(Kl) validates principles and methods that one would find natural, as a strict intuitionist, to accept. One can put the same point in model-theoretic terms by asking for a characterization of the structure-preserving relations that obtain between the universe V(Kl) of realizability sets and the universe V(Ht) of constructive sets.

If the constructivist were allowed the powerset operation and recursion on the class of ordinals, he could specify a "standard model" for IZF within "Heyting's universe" of proofs and constructions. This specification would proceed along lines identical to those which serve to introduce V(Kl); the only real difference is that, in defining a "constructive set," the first coordinates of the "elements" need not be just natural numbers but may be arbitrary constructions. (We would like to argue that there is *nothing* among the basic principles of constructive mathematics that bars the constructivist from the consideration of powersets and ordinal recursion. For the present, we will insist on such consideration only as a hypothetical.) We might call the resultant class model 'V(Ht);' it is conceived as a von Neumann-style hierarchy built over the universe on which Heyting first interpreted the intuitionistic logical signs. In V(Ht), we could interpret the logical signs as usual and give set quantification its *generic* interpretation. On a first impression, we can say that V(Kl) is, in some sense, an "approximation" to V(Ht). Part of the goal of this chapter will be to show to what extent this impression is not deceptive. We hope to exhibit those features of V(Kl) that make it a reflection of V(Ht).

First, we shall look to see how V(Kl) reflects one of the pervasive features of work over V(Ht), the method of weak counterexamples. We shall prove that V(Kl) "supports" weak counterexamples. By "supports," we mean that, whenever IZF provides a weak counterexample to a claim, realizability converts the counterexample argument into a proof that the claim is (strongly) false over V(Kl). Hence, weak counterexamples in IZF give independence results and this fact follows directly from the soundness theorem.

4.1. Theorem. *If ψ is closed and if* $\mathrm{IZF} \vdash \psi \rightarrow \forall x\,(\phi \vee \neg\phi)$ *for arbitrary ϕ (or even for arbitrary number-theoretic Σ^0_1 ϕ), then* $\mathrm{V}(Kl) \models \neg\psi$ *and ψ is independent of* IZF.

Proof. The task is, obviously, to falsify the quantified *tertium non datur.* To that end, we will define a realizability set which acts as the $\mathrm{V}(Kl)$-internal analogue of the halting problem. If K represents the set of "solutions" to the halting problem:

$$K = \{n \in \omega : \{n\}(n)\downarrow\},$$

let

$$\overline{K} = \{\langle n, \overline{n}\rangle : n \in K\}.$$

(Incidentally, this is a sample of a procedure for "injecting" sets from V into $\mathrm{V}(Kl)$ that will be central in obtaining our results about RETs and isols in Chapter Five.)

$\mathrm{V}(Kl)$, like V, is very generous in its admission standards, so there's no question but that $\overline{K}$ is a realizability set:

$$\overline{K} \in \mathrm{V}(Kl).$$

We pause to prove some helpful lemmas:

4.2. Lemma. *For all $n \in \omega$,* $\mathrm{V}(Kl) \models \overline{n} \in \overline{K}$ *iff* $\mathrm{V} \models n \in K$.

Proof. If $n \in K$, then the definition of $\overline{K}$ shows that

$$\langle n, i\rangle \Vdash \overline{n} \in \overline{K}\,.$$

(Here, and elsewhere, i is taken to be a fixed realizer for the first identity axiom: $\forall x\; x = x$.)

On the other hand, if $e \Vdash \overline{n} \in \overline{K}$, then $e_1 \Vdash \overline{n} = \overline{e_0}$ and $e_0 \in K$. It follows from the absoluteness of $=$ over ω that $n \in K$. ∎

4.3. Lemma. $\mathrm{V}(Kl) \models \forall n \in \omega\,(\exists m \in \omega\; T(n,n,m) \leftrightarrow n \in \overline{K})$.

Proof. T is given its usual expression in set theory. We will not prove this lemma now, but will defer it until we can treat it in a more general context. At that time, we shall see it as an manifestation of the absoluteness phenomena. ∎

Now back to the main theorem. Assume that $V(Kl) \models \psi$. Then, by the assumption of the theorem and the definition of realizability for $\rightarrow$,

$$V(Kl) \models \forall x \in \overline{\omega}\, (x \in \overline{K} \vee \neg x \in \overline{K}).$$

From the latter and the definition of $\Vdash$ for $\vee$, it follows that there is a partial recursive index g such that $\{g\}$ is total, $\{g\} : \omega \rightarrow 2$ and for each n,

$$\text{either } \{g\}(n) = 0 \text{ and } V(Kl) \models \overline{n} \in \overline{K}$$
$$\text{or } \{g\}(n) = 1 \text{ and } V(Kl) \models \neg \overline{n} \in \overline{K}.$$

By the last lemma, $\{g\}$ solves the halting problem. But this is impossible. Hence,

$$V(Kl) \models \neg \psi.$$

By the result of the second lemma, we see that a restriction to formulae ϕ which are number-theoretic Σ_1^0 will do: the lemma shows that the predicate $x \in \overline{K}$ is coextensive, over $V(Kl)$, with the "halting predicate," $\exists m\, T(x, x, m)$.

∎

There is an easy corollary that should go almost without mention:

4.4. Corollary. TND *is independent of* IZF

Proof. ∎

4.5. Remark. Since every detail of the preceding proof is fully constructive—including the proof of the unsolvability of the halting problem—the method of weak counterexamples is supported in $V(Kl)$ for the intuitionist and even for possible inhabitants of $V(Kl)$.

We note that we can, at best, falsify quantified instances of TND over $V(Kl)$. If the logic of the ground model is classical, then all closed instances of TND hold over $V(Kl)$.
∎

The conditions on this result clearly include all the weak counterexamples derived in Chapter One and a majority of those which appear in the recent literature on IZF. It follows that the assertions of full AC, the classical axiom of regularity, that successor is not strictly increasing on the ordinals and that every ordinal is an aleph are all strongly false in V(Kl). (Cf. Grayson (1979)) Also, it is false in V(Kl) that all inhabited subfinite sets are finite and that all inhabited subcountable sets are countable. We constructed weak counterexamples to these last claims in Chapter One.

Fleeing properties.

We can find support in V(Kl) for much more than these "higher order" counterexamples. V(Kl) can handle traditional intuitionistic counterexamples as devised by Brouwer and Heyting, despite the fact that these counterexamples seemingly lack the generality required by Theorem 4.1. The counterexamples to which we refer are not the strong counterexamples derived from the Continuity Principles nor those dependent on the "creative subject," but the conceptually unencumbered counterexamples based on "fleeing properties."

4.6. Definition. An number-theoretic predicate ψ is (expresses) a property fleeing in n just in case ψ is intuitionistically decidable on n

$$\forall n\, \psi(n) \vee \neg \psi(n)$$

but it is unknown whether or not $\exists n\, \psi(n)$. ∎

In the traditional literature, Goldbach's Conjecture or Fermat's Last Theorem often provide material for the construction of fleeing properties. For instance, since the truth of Goldbach's conjecture has yet to be decided, the predicate 'is even but is not the sum of two primes' expresses a fleeing property. In "recursive mathematics," Kleene's T predicate supplies the relevant fleeing properties. In particular, since the T predicate is p.r., we can use the above lemmas, plus the intuitionists' "fleeing property" constructions, to obtain over V(Kl) strong counterexamples to intuitionistically unacceptable statements of classical analysis.

To be assured of the truth of this claim, it behooves one to sketch the traditional counterexamples more carefully in general terms and then to pencil in the fine details of one of the counterexamples—by way of a verification of the accuracy of the general picture. Many of the "fleeing property" constructions proceed along the following lines: let $\phi(m,n)$ be a numerical predicate which is fleeing in n for every m. With A a species, $\forall x \in A\ \Psi$ is seen not to be constructively true because there is a construction that shows that $\forall x \in A\ \Psi(x)$ entails

$$\forall m\, (\exists n\, \phi(m,n) \vee \neg \exists n\, \phi(m,n)).$$

The construction is given by a function $\rho(m)$ such that, for each m, ρ constructs a member of the species A for which

$$\Psi(\rho(m)) \rightarrow (\exists n\, \phi(m,n) \vee \neg \exists n\, \phi(m,n)).$$

Therefore,

$$\forall x \in A\, \Psi(x) \rightarrow \forall m\, (\exists n\, \phi(m,n) \vee \neg \exists n\, \phi(m,n)).$$

Finally, on the assumption that $\phi(m,n)$ truly expresses a property fleeing in n, there can be no constructive proof of the consequent. Therefore, $\forall x \in A\, \Psi(x)$ is not constructively true.

If all the reasoning that went into deriving the counterexample is purely constructive, that is, if the reasoning can be expressed and carried out in IZF, then realizability over $V(Kl)$ will convert the counterexample into a falsehood over $V(Kl)$. To see this, we assume that, once A and Ψ have been expressed in IZF,

$$\text{IZF} \vdash \forall x \in A\, \Psi(x) \rightarrow \forall m\, (\exists n\, \phi(m,n) \vee \neg \exists n\, \phi(m,n)).$$

Since no constraint was imposed on ϕ other than that it be fleeing in n for every m, then the expression of the halting problem in terms of the T predicate will do as well as ϕ. We might even say that the halting problem, expressed in terms of Kleene's T, specifies a "recursively fleeing property."

Therefore, we have that

$$\text{IZF} \vdash \forall x \in A\, \Psi(x) \rightarrow \forall m\, (\exists n\, T(m,m,n) \vee \neg \exists n\, T(m,m,n))$$

Now, by the soundness theorem and the second lemma,

$$V(Kl) \models \forall x \in A\, \Psi(x) \rightarrow \forall n\, (n \in \overline{K} \vee \neg n \in \overline{K}).$$

Previous considerations have shown that

$$V(Kl) \models \neg \forall n\, (n \in \overline{K} \vee \neg n \in \overline{K}).$$

Therefore,

$$\neg \forall x \in A\, \Psi(x)$$

and this weak counterexample is supported by $\mathrm{V}(Kl)$. Furthermore, if all "fleeing property counterexamples can be made to fit this mould, all of these will be so supported.

4.7. Remark. It can also be shown that $\mathrm{V}(Kl)$ supports those "fleeing property" coun terexamples that reduce the constructive truth statements $\forall x \in A\ \Psi(x)$ to the *testabilit* rather than to the decidability of ϕ. We say that ϕ is testable in n for every m iff it intuitionistically the case that

$$\forall m\,(\neg\exists n\ \phi(n,m) \vee \neg\neg\exists n\ \phi(n,m)).$$

To show that testability counterexamples are supported as well, one need only point ou that the very same considerations adduced above to prove that

$$\mathrm{V}(Kl) \models \neg\forall n\,(n \in \overline{K} \vee \neg n \in \overline{K})$$

also show that

$$\mathrm{V}(Kl) \models \neg\forall n\,(\neg n \in \overline{K} \vee \neg\neg n \in \overline{K})$$

∎

It remains only to prove that historical Brouwerian counterexamples can be so analyze as to fit our paradigm. We limit ourselves to one example, Brouwer's counterexample to th monotonic convergence theorem of analysis. Even though every lawlike Cauchy sequence reals converges to a lawlike real, it is (weakly) not the case that every bounded monoton lawlike sequence of reals converges.

Let $\phi(m,n)$ be a property fleeing in n for every m. For each m, we define a lawlik sequence $\{x_n^m\}_{n\in\omega}$ of Cauchy reals where

$$x_n^m = \begin{cases} 0 & \text{if } \neg\exists p \le n\ \phi(m,p) \\ 1 & \text{if } \exists p \le n\ \phi(m,p) \end{cases}$$

Since the case distinction of the definition is p.r., $\{x_n^m\}_{n\in\omega}$ is well-defined and lawlike fo each m.

Now, assume that the convergence "theorem" holds: that, for each m, there is an x^m such that $\{x_n^m\}$ converges to x^m. By the basic properties of apartness on the reals, eithe x^m is apart from 0 or x^m is apart from 1. If the first alternative holds, the definition o convergence shows that $\exists n\ x_n^m = 1$ or $\exists n\ \phi(n,m)$. On the other hand, if x^m is actuall apart from 1, then $\neg\exists n\ x_n^m = 1$ and $\neg\exists n\ \phi(n,m)$. Therefore, there is a constructive entailment from the classical convergence theorem to the conclusion that

$$\forall m\,(\exists n\ \phi(n,m) \vee \neg\exists n\ \phi(n,m)).$$

This stands in contradiction with the assumption that ϕ is fleeing. Since one could use this same procedure with ϕ instantiated by any suitably "undecidable" p.r. predicate, the intuitionists' actual procedure fits our description, and it is false over $\mathrm{V}(Kl)$ that every bounded monotone sequence of reals converges.

4.8. Theorem. *Each of the following statements is false over* $\mathrm{V}(Kl)$:

the decidability of equality on the reals

the totality of the multiplicative inverse on the reals

the intermediate value theorem

the maximum theorem for continous functions on compact intervals

Proof. Brouwer and Heyting constructed "fleeing property" counterexamples to each of these assertions and each can be treated just as we have treated the classical bounded convergence theorem. ∎

A Note on the Brouwer-Kripke Schema.

Whilst on the subject of counterexamples, we would prefer to linger a moment over the fate in $\mathrm{V}(Kl)$ of the strong "creative subject" counterexamples. In making this examination, it will be convenient to anticipate a few of the developments of later sections from this chapter and of sections from Chapter Four. Conventionally, the mathematical essence of the creative subject is isolated in the form of BKS, the Brouwer-Kripke Schema. In its most familiar form, the schema is

$$\forall p\,\exists f \in (\omega \Rightarrow 2)\,(p \leftrightarrow \exists y \in \omega\ f(y) = 1).$$

p is a variable ranging over Ω, the species of constructive propositions. On the usual identifications of Ω with $\mathrm{P}(\{0\})$, and of the truth of a proposition with the habitation of a subset of $\{0\}$, BKS becomes

$$\forall x \subseteq 1\,\exists f \in (\omega \Rightarrow 2)\,(\exists y\ y \in x \leftrightarrow \exists y \in \omega\ f(y) = 1).$$

In this form, BKS fails over $V(Kl)$. Perhaps this is to be expected, because the logical signs are interpreted in $V(Kl)$ as recursive, rather than as constructive, operations and under such interpretation, the right side of the BKS biconditional is r.e. while the left side is arbitrary. To check this in detail, one can either work directly over $V(Kl)$, or (and here we anticipate) one can show that $\neg$BKS follows from Church's Thesis, CT_0, together with the Uniformity Principle, $UP^{\Omega,\omega}$. Both CT_0 and $UP^{\Omega,\omega}$ hold in $V(Kl)$.

4.9. Proposition. $V(Kl) \models CT_0 \wedge UP^{\Omega,\omega}$.

Proof. See the final sections of this chapter and the second section of the next. ∎

CT_0 is the claim that every total number-theoretic function is general recursive. $UP^{\Omega,\omega}$ is the nonclassical choice principle

$$\forall x \in \Omega\, \exists y \in \omega\, \phi \rightarrow \exists y \in \omega\, \forall x \in \Omega\, \phi.$$

4.10. Theorem. $IZF \vdash (CT_0 \wedge UP^{\Omega,\omega}) \rightarrow \neg BKS$

Proof. Assume that BKS holds. From CT_0, we know that

$$\forall x \subseteq 1\, \exists e \in \omega\, (\exists y\, y \in x \leftrightarrow \exists y \in \omega\, \{e\}(y) = 1).$$

The Uniformity Principle allows one to interchange the leading quantifiers to get

$$\exists e \in \omega\, \forall x \subseteq 1\, (\exists y\, y \in x \leftrightarrow \exists y \in \omega\, \{e\}(y) = 1).$$

This is clearly contradictory. Therefore, given the above proposition, that

$$V(Kl) \models CT_0 \wedge UP^{\Omega,\omega},$$

we conclude that the creative subject method is not, at least in the straightforward way, available over $V(Kl)$: $V(Kl) \models \neg BKS$. ∎

As we said, this result is predictable. However, a mildly restricted form of BKS does offer some service over $V(Kl)$. If BKS is restricted to certain numerical statements, then it characterizes (internally) the *countable* subsets of ω and (externally) the nonempty r.e. sets. Let BKS^A be the statement

$$\forall x \in \omega\, \exists f \in (\omega \Rightarrow 2)\, (x \in A \leftrightarrow \exists y \in \omega\, f(y) = 1).$$

For a complete proof of the characterization, we appeal to CT_0 and to $AC^{\omega,\omega}$. The latter is the simple choice principle for the quantifier combination $\forall x \in \omega\ \exists y \in \omega$:

$$\forall x \in \omega\ \exists y \in \omega\ \phi(x,y) \to \exists f \in (\omega \Rightarrow \omega)\ \forall x \in \omega\ \phi(x, f(x)).$$

We will prove later in this chapter that

4.11. Proposition. $V(Kl) \models AC^{\omega,\omega}$.

Proof. *Vide infra.* ∎

4.12. Theorem. *In* $V(Kl)$, $A \subseteq \omega$ *is countable iff* BKS^A *holds and* A *is inhabited.*

Proof. In one direction, the proof is almost trivial. Let $A \subseteq \omega$ be countable and let f count A, i.e., $f : \omega \twoheadrightarrow A$. For each $x \in \omega$, define f^x as follows:

$$f^x(y) = \begin{cases} 0 & \text{if } f(y) \neq x \\ 1 & \text{if } f(y) = x \end{cases}$$

$f^x \in (\omega \Rightarrow 2)$ and $\exists y\ f^x(y) = 1$ iff $x \in A$. Therefore, BKS^A holds.

For the converse, we begin with BKS^A and apply CT_0 to get

$$\forall x \in \omega\ \exists e \in \omega\ (x \in A \leftrightarrow \exists y\ \{e\}(y) = 1).$$

If we use $AC^{\omega,\omega}$ and CT_0 again, then

$$\exists g \in \omega\ \forall x \in \omega\ (x \in A \leftrightarrow \exists y\ \{\{g\}(x)\}(y) = 1)$$

results. Finally, since A is assumed to be inhabited, we can fix $a \in A$ and define $f \in (\omega \Rightarrow \omega)$ so that

$$f(x) = \begin{cases} x_0 & \text{if } \{g(x_0)\}(x_1) = 1 \\ a & \text{if o.w.} \end{cases}$$

It is clear that f will count A. ∎

4.13. Remark. One closing comment about the role of BKS in $V(Kl)$. In Minio (1974), it is shown that, constructively, the statement

$$\forall x\ (x \text{ is discrete} \to \exists y \subseteq \omega\ (y \text{ is decidable} \wedge x \approx y))$$

implies BKS. Therefore, the displayed statement fails over $V(Kl)$. There is also a straightforward argument to the negation of the displayed statement from CT_0 and a Uniformity Principle for $P(\omega)$, $UP^{P(\omega),\omega}$. $UP^{P(\omega),\omega}$ also holds in $V(Kl)$.

This result should be contrasted with the fact (to be proved later in this chapter) that every discrete set is subcountable. ∎

Section 5: Axioms of choice; constructive methods

The ways in which realizability handles choice principles is another intimation of the close kinship that obtains between V(Kl) and V(Ht). The interpretation of the $\forall\exists$ quantifier combination over V(Ht) forces one to make a distinction between bases, species over which a principle of choice holds, and nonbases, species over which it fails. Just the same kind of distinction must be made in V(Kl), for the same reasons and along the same lines. Moreover, in V(Kl), one can prove an interesting characterization theorem for the class of bases: every internalized realizability set is a base and an arbitrary set is a base iff it forms a retract of its own internalized realizability set.

Before launching into the technicalities of "internalized realizability sets," however we want to go after easier quarry: an assessment (in V(Kl)) of the more familiar choice principles AC and DC.

Constructive choices.

Working even in a constructive ground model, one can show that, in V(Kl), $\overline{\omega}$ satisfies the $\omega \Rightarrow \omega$ axiom of choice, $AC^{\omega,\omega}$:

$$\forall x \in \omega\ \exists y \in \omega\ \phi(x,y) \rightarrow \exists f \in (\omega \Rightarrow \omega)\ \forall x \in \omega\ \phi(x, f(x)).$$

5.1. Theorem. V(Kl) $\models AC^{\omega,\omega}$

Proof. Assume that e realizes the antecedent of $AC^{\omega,\omega}$. From the above discussion (particularly Lemma 3.9), it is clear that this means that there is a total recursive $\{g\}$ such that for all n,

$$\{g_0\}(n) \Vdash \phi(\overline{n}, \overline{\{g_1\}(n)}).$$

In fact, g is calculable effectively from e and some (coded) information on the syntactical features of ϕ. Let $\overline{g_1}$ be the following realizability collection of internal pairs:

$$\{\langle n, \overline{\langle \overline{n}, \overline{\{g_1\}(n)} \rangle} \rangle : n \in \omega\}.$$

It is now a simple matter to check that V(Kl) believes that $\overline{g_1}$ is the required choice function. We will give a proof that $\overline{g_1}$ does the "choosing" and leave the proof of $\overline{g_1}$'s functionality to the reader.

By the Lemma 3.9, it will be enough to construct from g an h such that condition (A) holds

(A) $\{h\}$ is total and, for all n,

$$\{h\}(n) \Vdash \exists y\,(y \in \overline{\omega} \wedge \langle \overline{n}, y\rangle \in \overline{g_1} \wedge \phi(\overline{n}, y)).$$

To that end, we note that for each n,

(1) $\langle \{g_1\}(n), i\rangle \Vdash \overline{\{g_1\}(n)} \in \overline{\omega}$,
(2) $\langle n, i\rangle \Vdash \overline{\langle \overline{n}, \overline{\{g_1\}(n)}\rangle} \in \overline{g_1}$ and
(3) $\{g_0\}(n) \Vdash \phi(\overline{n}, \overline{\{g_1\}(n)})$.

Therefore, with

$$h = \Lambda n.\langle\langle \{g_1\}(n), i\rangle, \langle n, i\rangle, \{g_0\}(n)\rangle,$$

we know that h satisfied condition (A). ▮

From the above it follows immediately and constructively that $V(Kl)$ satisfies the principle DC^ω of dependent choices over the natural numbers.

5.2. Corollary. *$V(Kl) \models DC^\omega$, where DC^ω is*

$$\forall x \in \omega\; \exists y \in \omega\; \phi(x,y) \rightarrow \forall x \in \omega\; \exists f \in (\omega \Rightarrow \omega)\; (f(0) = z \wedge \forall x \in \omega\; \phi(f(x), f(x+1)))).$$

Proof. In IZF, $AC^{\omega,\omega}$ implies DC^ω. Grayson (1975) has already shown that definition by recursion is as simple in IZF as it is in ZF. Assume that $\forall x \in \omega\; \exists y \in \omega\; \phi$, the antecedent of DC^ω, holds. By $AC^{\omega,\omega}$, there is a choice function G. Now, we can define F by ω-recursion:

$$F(0) = z$$

$$F(n+1) = G(F(n)).$$

z is any fixed element of ω. An argument by induction shows that F is the sort of function required for the consequent of DC^ω to hold. ▮

At this stage, the natural question to ask is "For what other realizability sets X and Y can $AC^{X,Y}$ and DC^X be proved to hold in $V(Kl)$ by using constructive means alone?" We are not in a position to provide a complete answer, but we can set boundary conditions on the correct answer by showing that, for X countable and discrete and fo subcountable, it is provable in IZF that $\mathrm{V}(Kl) \models \mathrm{AC}^{X,Y}$. We also show that, for counta sets, discreteness and choice are equivalent.

5.3. Theorem.

(1) For X countable and discrete and for Y subcountable, $\mathrm{V}(Kl) \models \mathrm{AC}^{X,Y}$.

(2) IZF $\vdash$ For X countable, $\mathrm{AC}^{X,\omega}$ holds just in case X is discrete.

Proof. Only a proof of (2) is required; a proof of (1) follows, in IZF, from a proof of together with the fact that $\mathrm{V}(Kl) \models \mathrm{AC}^{\omega,\omega}$.

If X is countable and $\mathrm{AC}^{X,\omega}$ holds, then there is an f that counts X, $f : \omega \twoheadrightarrow$ and f splits over X: there is a $g : X \rightarrowtail \omega$ such that g is a right inverse to f. Theref for x and y from X,

$$x = y \leftrightarrow g(x) = g(y)$$

Then, since ω is discrete, so is X.

On the other hand, if X is countable and discrete, we can split any counting f us the minimum operator. Simply set

$$g(x) = \mu n \, (f(n) = x).$$

Use of the μ-operator is allowed because discreteness makes the matrix $f(n) = x$ decida Clearly, g is a right inverse to f. ∎

Section 6: Dependent choices and nonconstructive methods

With measured amounts of choice in the metatheory, we can prove that a general choice axiom, $AC^{\omega,X}$, holds for the set of natural numbers in $V(Kl)$. $AC^{\omega,X}$ is the principle

$$\forall x \in \omega\ \exists y \in X\ \phi(x,y) \rightarrow \exists f \in (\omega \Rightarrow X)\ \forall x \in \omega\ \phi(x, f(x)).$$

This, in turn, is proved by showing that a related principle, DC^X, is realizable. DC^X is the axiom of unrestricted dependent choice:

$$\forall x \in X\ \exists y \in X\ \phi(x,y) \rightarrow \forall x \in X\ \exists f \in (\omega \Rightarrow X)\ (f(0) = x \wedge \forall x \in \omega\ \phi(f(x), f(x+1))).$$

6.1. Theorem. $V(Kl) \models DC^X$.

Proof. This proof exemplifies the simplest possible "externalization—internalization" argument. The argument begins with the "externalization" of the realizability conditions for membership in an internal set a and the expression of those conditions as a set $a^{\Vdash}$ in V, the "actual" world. The next step is to use a version of DC in V to prove an external choice theorem about $a^{\Vdash}$. Finally, we "internalize" the choice function from the external choice theorem and return to the confines of $V(Kl)$ to check that the presence of the internalized function guarantees that DC holds in $V(Kl)$.

For $a \in V(Kl)$, set $a^{\Vdash} = \{\langle n, b\rangle : n \Vdash b \in a\}$. Now, $a^{\Vdash}$ is a set in V and, in V, it is assumed that $DC^{a^{\Vdash}}$ already holds. To apply this form of external dependent choice, we assume that the antecedent of DC^a is realized by e:

$$e \Vdash \forall x \in a\ \exists y \in a\ \phi(x,y).$$

By the definition of $\Vdash$, we know that, for all $n \in \omega$ and all $b \in V(Kl)$, if $\langle n, b\rangle \in a^{\Vdash}$, then $\{e\}(n)\downarrow$ and there is a $c \in V(Kl)$ such that

$$\langle \{e\}(n)_0, c\rangle \in a^{\Vdash} \text{ and } \{e\}(n)_1 \Vdash \phi(b,c).$$

It will also be necessary to externalize the realizability-relation determined in $V(Kl)$ by ϕ. Let $\phi^{\Vdash}$ be such that, externally,

$$\phi^{\Vdash}(\langle n, b\rangle, \langle m, c\rangle) \text{ holds iff } m = \{e\}(n)_0 \text{ and } \{e\}(n)_1 \Vdash \phi(b,c).$$

Next, $\mathrm{DC}^{a^{\Vdash}}$ is applied to $a^{\Vdash}$ and $\phi^{\Vdash}$ in the ground model. Fix g and d such that g
$d \in a$. Then, $\langle g, d\rangle \in a^{\Vdash}$. By external DC, there is an F such that

$$F \in \mathbf{V} \wedge F \in (\omega \Rightarrow a^{\Vdash}),$$
$$F(0) = \langle g, d\rangle \qquad \text{and}$$
$$\forall n\, \phi^{\Vdash}(F(n), F(n+1)).$$

Finally, we internalize F and prove that it supplies the function required for the tru
of internal DC. The appropriate internalization of F is $\overline{F}$:

$$\overline{F} = \{\langle\langle n, F(n)_0\rangle, \overline{\langle \overline{n}, F(n)_1\rangle}\rangle : n \in \omega\}.$$

Obviously, $\overline{F}$ belongs to $\mathbf{V}(Kl)$. It remains to check that $\overline{F}$ is internally a function fr
ω into a and that, given a realizability witness for the antecedent of DC, $\overline{F}$ makes
consequent of DC realizable as well.

First, because of the properties of internal pairing in $\mathbf{V}(Kl)$ (cf. Lemma 3.4), V(
believes that $\overline{F}$ is a binary relation with domain ω and range a subset of a and this ho
with a witness obtainable independently of e and g. To see that $\overline{F}$ is realizably function
assume that

$$h \Vdash \overline{\langle a, b\rangle} \in \overline{F} \text{ and } j \Vdash \overline{\langle a, b\rangle} \in \overline{F}.$$

Then,

$$h_1 \Vdash \overline{\langle a, b\rangle} = \overline{\langle h_{00}, F(h_{00})_1\rangle} \text{ and}$$
$$j_1 \Vdash \overline{\langle a, c\rangle} = \overline{\langle j_{00}, F(j_{00})_1\rangle}.$$

This holds strictly in virtue of the definition of $\overline{F}$ and of the $\Vdash$ conditions on statemen
of membership.

From the absoluteness of $\in$ and $=$ on ω (Theorem 3.11), we know that

$$h_{00} = j_{00} \text{ and that } F(h_{00})_1 = F(j_{00})_1.$$

Therefore, there is a partial recursive Θ such that $\Theta(h, j) \Vdash b = c$. Θ confirms that $\overline{F}$
realizably functional.

Next, to see that $V(Kl) \models \overline{F} \subseteq \overline{\omega} \times a$, let

$$h \Vdash \overline{\langle a, b \rangle} \in \overline{F}.$$

As above,

$$h_1 \Vdash \overline{\langle a, b \rangle} = \overline{\langle h_{00}, F(h_{00})_1 \rangle}.$$

Therefore, $\langle h_{00}, i \rangle \Vdash \overline{h_{00}} \in \overline{\omega}$, and, by the definition of $\overline{F}$, $h_{01} \Vdash F(h_{00})_1 \in a$.

$\langle\langle 0, g\rangle, i\rangle \Vdash \overline{\langle 0, d \rangle} \in \overline{F}$. Hence, $V(Kl) \models \overline{F}(0) = d$.

Finally, we have to check on the realizability of $\forall x \in \omega \; \phi(\overline{F}(x), \overline{F}(x+1))$. Since, all n, $\phi^{\Vdash}(F(n), F(n+1))$, we have for all n,

$$\{e\}(F(n)_0)_0 = F(n+1)_0 \text{ and}$$
$$\{e\}(F(n)_0)_1 \Vdash \phi(F(n)_1, F(n+1)_1).$$

Define the number-theoretic function $\Psi(n)$ so that

$$\Psi(0) = g \quad \text{and}$$
$$\Psi(n+1) = \{e\}(\Psi(n))_0$$

The classical s-m-n theorem shows that an index for Ψ is calculable from e and g. Th one can use induction over ω to check that, for all n,

$$\langle\langle n, \Psi(n)\rangle, i\rangle \Vdash \overline{\langle \overline{n}, F(n)_1 \rangle} \in \overline{F} \text{ while}$$
$$\{e\}(\Psi(n))_1 \Vdash \phi(F(n)_1, F(n+1)_1).$$

This completes the proof. ∎

Realizability for a general axiom of choice now follows directly from the realizabi of DC^X and the realizability soundness theorem for IZF.

6.2. Corollary. $V(Kl) \models AC^{\omega,a}$.

Proof. Let a be any set and let $b = \bigcup_{m\in\omega} a^m$. Then

$$IZF \vdash DC^b \rightarrow AC^{\omega,a}.$$

∎

ω-retracts and bases.

We round out our discussion of the axiom of choice with a characterization of those in V(Kl) which are "ω-retracts." The notion of ω-retract is important for an understan of $\underline{ISys}$, an internal category of Scott information systems over which a Brouwer Theo holds (cf. Chapter Seven). Happily, a countable set S is an ω-retract just in case it an axiom of choice that holds with respect to it and the latter holds just in case S has "logical property" of discreteness. (The reader will recall that a set S is discrete prov that equality on the set satisfies TND, i.e., it is intuitionistically true that $\forall x, y \in S$ ($y \vee x \neq y$).)

Integral to our characterization is the idea of a *base*. A set is a base just in cas axiom of choice holds over it:

6.3. Definition. A set X is a *base* iff, for all Y, every subset of $X \times Y$ which is tota X is uniformized by a function which is total on X. ∎

We will see, in the next subsection, that every internal realizability set is a base that, up to retractions, the realizability sets are the only bases. If a set is a retract we say that it is an *ω-retract*:

6.4. Definition. A set S is an ω-retract whenever there are functions $i : S \rightarrowtail \omega$ $j : \omega \twoheadrightarrow S$ such that $j \circ i = id_S$. ∎

The following theorem characterizes the ω-retracts. It is interesting to note that seemingly "external" property of being a retract of ω is equivalent (for countable sets the "internal" or "logical" property of discreteness.

6.5. Proposition. V(Kl) $\models$ *For S countable, S is an ω-retract iff S is discrete iff a base.*

Proof. For S countable, it is easy to show that S is an ω-retract iff S is a base; one only reference the fact that ω is itself a base, which is what the last corollary showed.

If S is an ω-retract, then there is an f such that $f : S \rightarrowtail \omega$. Therefore, since discrete, so is S. Conversely, if S is discrete and countable, then, for some f, $f : \omega \twoheadrightarrow$ Again, since S is discrete, minimization with respect to S is possible. Hence, the func $g : S \rightarrowtail \omega$, where $g(y) = \mu x. f(x) = y$, is a right inverse to f. ∎

For sets which are already subsets of ω, the characterization becomes quite neat. Th is due to the fact that every subobject of ω inherits ω's discreteness.

6.6. Corollary. $\mathrm{V}(Kl) \models$ *For* $S \subseteq \omega$, *S is countable iff S is an* ω*-retract.*

Proof. As we just said, every subobject of ω is discrete. ∎

6.7. Corollary. $\mathrm{V}(Kl) \models$ *For* $S \subseteq \omega$, BKS^S *holds iff S is an* ω*-retract.*

Proof. See Theorem 4.12. ∎

We shall return to these topics when we come to treat information systems intuitio istically in Chapter Seven.

Section 7: The Presentation Axiom

The most general choice principle whose validity we assess over $\mathrm{V}(Kl)$ is Aczel's sentation Axiom, PAx (cf. Azcel (1982)).

PAx is the assertion that every set is the functional image of a base:

$$\forall x\, \exists y\, \exists f\, (y \text{ is a base } \wedge f : y \twoheadrightarrow x).$$

7.1. Remarks. (1) PAx is a weak form of choice: every set is a functional image set over which a full choice principle holds. PAx is weak because $\mathrm{V}(Kl) \models$PAx will that IZF+PAx$\nvdash$ AC fails. Also, $\mathrm{V}(Kl) \models$PAx shows that TND is independent of IZF PAx.

(2) The proof of the following theorem is further confirmation of our assertion that over $\mathrm{V}(Kl)$ mimics certain features of work over $\mathrm{V}(Ht)$. In $\mathrm{V}(Ht)$, choice will hold sets which have a canonical proof function, where a constructive function G is a cano proof function for a species X whenever G is total on X, and, for each $y \in X$, $G(y$ constructive proof that $y \in X$. Our proof of PAx shows that certain realizability sets natural canonical proof functions "built-in". Hence, each of these is a base. Moreov is easily seen that every set is a quotient of one of these sets. Intuitively, we can thi the surjection that gives the quotient as "destroying" the kind of information contain the base that allows one to choose, over it, canonical proofs. ∎

7.2. Definition. For each $a \in \mathrm{V}(Kl)$, let $a^{\Vdash}$ be

$$\{\langle n, \overline{\langle \overline{n}, b\rangle}\rangle : n \Vdash b \in a\}.$$

∎

Here, '$a^{\Vdash}$' is playing a slightly different role from that given it above; for e realizability set a, $a^{\Vdash}$ is the internalization of the realizability conditions for member in a. In other words, $a^{\Vdash}$ is the reinternalization of the externalization of a (the $a^{\Vdash}$ o last section).

7.3. Proposition. $a^{\Vdash} \in \mathrm{V}(Kl)$ *whenever* $a \in \mathrm{V}(Kl)$.

Proof. This follows immediately from the closure lemma, 6.2 of Chapter Two. ∎

The collection of internal $a^{\Vdash}$ sets is the prospective internal class of bases. H one must check that every set a is realizably the image of its $a^{\Vdash}$.

7.4. Lemma. *There is a g such that, for all $a \in \mathrm{V}(Kl)$,*

$$g \Vdash \underline{a} \text{ is an image of } a^{\Vdash}.$$

Proof. Let

$$\overline{F} = \{\langle n, \overline{\langle \overline{\langle \overline{n}, b \rangle}, b \rangle} \rangle : n \Vdash b \in a\}.$$

(1) Obviously, $\overline{F}$ is internally a relation and this is realized uniformly in a.

(2) If $e \Vdash \overline{\langle b, c \rangle} \in \overline{F}$, then

$$e_1 \Vdash \overline{\langle b, c \rangle} = \overline{\langle \overline{\langle \overline{e_0}, d \rangle}, d \rangle}$$

where $e_0 \Vdash d \in a$. Therefore, $\mathrm{V}(Kl) \models \overline{F} \subseteq a^{\Vdash} \times a$ and this holds uniformly in a.

(3) To see that $\overline{F}$ is total on $a^{\Vdash}$, let

$$e \Vdash \overline{\langle c, d \rangle} \in a^{\Vdash}.$$

Then, where $e_0 \Vdash b \in a$,

$$e_1 \Vdash \overline{\langle c, d \rangle} = \overline{\langle \overline{e_0}, b \rangle}.$$

It follows immediately that $\langle e_0, i \rangle \Vdash \overline{\langle \overline{\langle \overline{e_0}, b \rangle}, b \rangle} \in \overline{F}$.

(4) For the functionality of $\overline{F}$, let $e \Vdash \overline{\langle b, c \rangle} \in \overline{F}$ and let $h \Vdash \overline{\langle b, d \rangle} \in \overline{F}$. Then, for $r, s \in \mathrm{V}(Kl)$,

$$e_1 \Vdash \overline{\langle b, c \rangle} = \overline{\langle \overline{\langle \overline{e_0}, r \rangle}, r \rangle} \text{ and}$$
$$h_1 \Vdash \overline{\langle b, d \rangle} = \overline{\langle \overline{\langle \overline{h_0}, s \rangle}, s \rangle}.$$

Given the properties of internal pairing, one easily constructs partial recursive Θ a so that

$$\Theta(e, h) \Vdash r = s \text{ and } \Psi(e, h) \Vdash c = d.$$

ence,

(5) Finally, we check that $\overline{F}$ maps $a^{\Vdash}$ onto a in $V(Kl)$. But this is trivial: if $e \Vdash$ then

$$\langle e, i\rangle \Vdash \overline{\langle \overline{e}, b\rangle} \in a^{\Vdash} \text{ and } \langle e, i\rangle \Vdash \overline{\langle \overline{\langle \overline{e}, b\rangle}, b\rangle} \in \overline{F}.$$

∎

There is also a straightforward proof that, in $V(Kl)$, the $a^{\Vdash}$'s have the ca proof functions mentioned earlier.

7.5. Lemma. *There is an e such that, for all $a \in V(Kl)$,*

$$e \Vdash a^{\Vdash} \text{ is a base.}$$

Proof. Assume that

$$g \Vdash \forall x \in a^{\Vdash}\ \exists y \in b\ \phi(x, y).$$

Let n and c be such that $n \Vdash c \in a$. Then, $\langle n, i\rangle \Vdash \overline{\langle \overline{n}, c\rangle} \in a^{\Vdash}$, and, for some $V(Kl)$,

$$\{g\}(\langle n, i\rangle)_0 \Vdash d \in b \text{ and}$$
$$\{g\}(\langle n, i\rangle)_1 \Vdash \phi(\overline{\langle \overline{n}, c\rangle}, d).$$

Collection and choice over V are applied to this data to insure the existence of a fu F such that, for all $\langle n, c\rangle$ such that $n \Vdash c \in a$,

$$\{g\}(\langle n, i\rangle)_0 \Vdash F(\langle n, c\rangle) \in b \text{ and}$$
$$\{g\}(\langle n, i\rangle)_1 \Vdash \phi(\overline{\langle \overline{n}, c\rangle}, F(\langle n, c\rangle)).$$

Now, F is, as usual, internalized as $\overline{F}$:

$$\overline{F} = \{\langle n, \overline{\langle \overline{\langle \overline{n}, c\rangle}, F(\langle n, c\rangle)\rangle}\rangle : n \Vdash c \in a\}.$$

One can use the same arguments as in the preceding lemma to show that, interna is a function and is a subset of $a^{\Vdash} \times b$. It remains only to show that there is un witnessing for

$$\forall x \in a^{\Vdash}\ \exists y\ (\langle x, y\rangle \in \overline{F} \wedge \phi(x, y)).$$

To that end, let $e \Vdash \overline{\langle r, s\rangle} \in a^{\Vdash}$. Then,

$$e_1 \Vdash \overline{\langle r, s\rangle} = \overline{\langle \overline{e_0}, c\rangle},$$

where $e_0 \Vdash c \in a$. On the basis of these considerations, one sees that

$$\{g\}(\langle e_0, i\rangle)_0 \Vdash F(\langle e_0, c\rangle) \in b \text{ while}$$
$$\{g\}(\langle e_0, i\rangle)_1 \Vdash \phi(\overline{\langle \overline{e_0}, b\rangle}, F(\langle e_0, b\rangle)).$$

From the definition of $\overline{F}$,

$$e_0 \Vdash \overline{\langle \overline{\langle \overline{e_0}, b\rangle}, F(\langle e_0, b\rangle)\rangle} \in \overline{F}$$

and we are done. ▮

Together, the two lemmas 7.4 and 7.5 show that

7.6. Theorem. $\mathbf{V}(Kl) \models$PAx

Proof. ▮

Given PAx in $\mathbf{V}(Kl)$, it is a simple matter to characterize those realizability sets are bases under realizability. Recall that AC is equivalent to the statement that epimorphism splits. It turns out that any $a \in \mathbf{V}(Kl)$ is realizably a base iff the cano map $f : a^{\Vdash} \twoheadrightarrow a$ splits over a. Along the way, it will also be shown that, over V arbitrary a is a base just in case a is a base "with respect to ω."

7.7. Definition. A set X is a *base with respect to* ω iff, for all $Y \subseteq X \times \omega$ whicl total on X, Y is uniformizable. ▮

7.8. Theorem. *Let $f : a^{\Vdash} \twoheadrightarrow a$ be the canonical epimorphism.* $\mathbf{V}(Kl) \models$ "*a base" iff for some g, $g : a \rightarrowtail a^{\Vdash}$ and $f \circ g = id{\upharpoonright}a$.*

Proof. We work in IZF over $\mathbf{V}(Kl)$. If a is assumed to be a base, then there is a g uniformizes the total relation

$$\{\langle x, y\rangle : x \in a \wedge y \in a^{\Vdash} \wedge f(y) = x\}.$$

Clearly, g splits f.

Conversely, assume that g splits the canonical f over a and that $y \subseteq a \times x$ on a. Then,

$$\forall z \in a\ \exists v \in x\ (\langle z, v\rangle \in y).$$

A fortiori, $\forall z \in a^{\Vdash}\ \exists v \in x\ (\langle f(z), v\rangle \in y)$. Since $a^{\Vdash}$ is a base in $\mathrm{V}(Kl)$,

$$\exists F\ \forall z \in a^{\Vdash}\ (\langle f(z), F(z)\rangle \in y).$$

Now, g splits f, so

$$\exists F\ \forall z \in a\ (\langle f(g(z)), F(g(z))\rangle \in y) \text{ or}$$

$$\exists F\ \forall z \in a\ (\langle z, F(g(z))\rangle \in y).$$

Now, we set $G = F \circ g$, and recognize G as the desired choice function. ∎

7.9. Theorem. $\mathrm{V}(Kl) \models$ *a is a base iff a is a base with respect to ω.*

Proof. The implication from left to right is trivial. For the converse, assume t
a base with respect to ω in $\mathrm{V}(Kl)$. If $e \Vdash b \in a$, then $e \Vdash \overline{\langle \overline{e}, b\rangle} \in a^{\Vdash}$. Therefore,

$$\mathrm{V}(Kl) \models \forall x \in a\ \exists y \in \omega\ \langle y, x\rangle \in a^{\Vdash}.$$

a is a base with respect to ω, so, in $\mathrm{V}(Kl)$,

$$\exists F\ \forall x \in a\ \langle F(x), a\rangle \in a^{\Vdash}.$$

Therefore, F splits f and a is a base. ∎

7.10. Remark. The relationship between these two theorems is a reflection, on a
scopic scale, of a feature of $\mathrm{V}(Kl)$ which is macroscopic (and to which we have ac
repeatedly). As we have said before, $\mathrm{V}(Kl)$ has a sort of Janus-like "double nature
of these natures provides a quite accurate meter of the state of set-theory over V(H
is $\mathrm{V}(Kl)$'s "higher order" side. We might say that the realizability universe is a "nc
dard model" of Heyting's interpretation. To our eyes, this aspect is largely indepen
the fact that, in $\mathrm{V}(Kl)$, it is the collection of *general recursive functions* that goes
for the species of constructive functions. This fact underlies realizability's other r
the nature that is recursive mathematics.

In the first theorem, we have an accurate reflection, in $\mathrm{V}(Kl)$, of a situation tha exist in $\mathrm{V}(Ht)$: for given a, the species y whose elements are of the form $\langle\rho, x\rangle$, w proves that $x \in a$, is a base. Moreover, bases like this are so well-scattered in $\mathrm{V}(H$ they "cover" all the other sets as functional images. This claim—that the first theo an accurate reflection of the state of $\mathrm{V}(Ht)$—is fully justified because the proof of th theorem is wholly constructive and employs none of the specifically recursion-the underpinnings of $\mathrm{V}(Kl)$.

In contrast, the second of the above theorems (on bases with respect to ω) parta large measure of the recursive nature of $\mathrm{V}(Kl)$. This theorem holds only in virtue of t that, in the transition from $\mathrm{V}(Ht)$ to $\mathrm{V}(Kl)$, constructive proofs are replaced by n number codes. Hence, the latter theorem tells us more about $\mathrm{V}(Kl)$'s abilities to recursive mathematics than about its abilities to shed light on traditional intuitioni

Incidentally, reflections like these go a long way to explaining why ω is so promi base. It is easily seen that, as far as sets and functions are concerned, ω is indistingui from $\omega^{\Vdash}$:

$$\mathrm{V}(Kl) \models \omega \approx \omega^{\Vdash}.$$

In short, much of the prominence of ω in $\mathrm{V}(Kl)$ lies in the fact that ω can be ide with the canonical base over it. ∎

7.11. Note. One last remark concerning versions of the axiom of choice. It known that $\mathrm{AC}((\omega \Rightarrow \omega) \Rightarrow \omega)$ is inconsistent with Church's Thesis in extensional intuitionistic arithmetic in all finite types. The argument is quite simple and appe Troelstra (1973a). Since the principles embodied in HA^{ω} are easily interpretable in and Church's Thesis holds in $\mathrm{V}(Kl)$, we know that the axiom of choice does not hol the function space $\omega \Rightarrow \omega$. ∎

Section 8: Uniformity and subcountability

Our next subject involves a logical property of the universe of realizability sets is extraordinary from a classical standpoint, but perfectly ordinary in a constructive work. We are referring to UP, the Uniformity Principle, which is expressed as an scheme:

$$\forall x\, \exists n \in \omega\; \phi \rightarrow \exists n \in \omega\; \forall x\; \phi.$$

For an informal discussion of uniformity, the reader can consult Chapter Zero. T cussions of uniformity under realizability which appear in the literature are in Fri (1973a) and in Troelstra (1973b).

8.1. Remark. In the latter reference, there is an intuitive "explanation" of the ability of (a version of) UP which suggests that UP is correctly understood as the accompaniment of a certain concept of species. We will argue that UP should derstood, primarily, as the natural accompaniment of the generic interpretation t give to unbounded set quantification over $V(Kl)$, and only secondarily as arising fr "qualities" of realizability sets. This is in perfect accord with the basic insights of co tivism, on which the "metaphysical properties" of various species are captured prin in the explanation of quantification over them. ∎

Uniformity and realizability.

First, we will give the (elementary) proof that UP holds over $V(Kl)$ and dra some of the immediate consequences of this fact. Later, there will be an opportu test realizability for more delicate versions of UP.

8.2. Theorem. $V(Kl) \models \mathrm{UP}$.

Proof. Let $e \Vdash \forall x\; \exists y\; (y \in \overline{\omega} \wedge \phi)$. Since quantification over $V(Kl)$ is generic $\exists y \in \overline{\omega}\; \phi$ uniformly for all $a \in V(Kl)$. It follows that, uniformly in a, there is a b suc

$$e_1 \Vdash \phi(a, b) \text{ and } e_{01} \Vdash b = \overline{e_{00}}\,.$$

Therefore, there is a fixed g, mechanically calculable from e, such that, uniformly,

$$g_1 \Vdash \phi(a, \overline{g_0}).$$

Hence, $\langle\langle g_0, i\rangle, g_1\rangle \Vdash \exists n \in \overline{\omega}\; \forall x\; \phi$. ∎

The following corollary, albeit easy to prove, is a fair indication of stronger resul come. Recall that 'Ω' is a symbol for the species of constructive propositions. It is that, for purposes of set theory, Ω is readily identifiable with $\mathrm{P}(\{0\})$; the constru function that takes proposition ϕ into $\{0 : \phi\}$ is obviously a natural bijection betweer space of propositions and the powerset of the singleton.

8.3. Corollary. $\mathrm{V}(Kl) \models \Omega$ *is not subcountable.*

Proof. We work within IZF, taking UP as an assumption. If Ω is subcountable, th a function f for which $\forall x \in \Omega\ \exists y \in \omega\ \langle y, x\rangle \in f$. It follows that

$$\forall x\ \exists y \in \omega\ \langle y, x \cap \{0\}\rangle \in f.$$

Let UP fix such a y. Then, we consider $x = \emptyset$ and $x = \{0\}$ as possible values of f on fixed y and derive a contradiction to the functionality of f. ∎

8.4. Corollary. $\mathrm{V}(Kl) \models$ *If x is inhabited, then* $\mathrm{P}(x)$ *is not subcountable.*

Proof. If x is inhabited, then $\exists y\ y \in x$ and, for such a y,

$$\Omega \approx \mathrm{P}(\{y\}) \subseteq \mathrm{P}(x).$$

∎

Since the conclusion of the last corollary ("not subcountable") is negative, we make a considerable improvement in the generality of this result:

8.5. Theorem. $\mathrm{V}(Kl) \models$ *If x is nonempty, then* $\mathrm{P}(x)$ *is not subcountable.*

Proof. In HPL, x is nonempty iff $\neg\neg\, x$ is inhabited. Since "$\mathrm{P}(x)$ is not subcountab a negative property, the result follows immediately by a judicious double negating o premiss and conclusion of the preceding corollary. ∎

8.6. Remark. The first corollary presents the easiest proof we know of the fact, origi proved by a kind of "effective diagonalization" in Myhill (1975), that strong constru set theories are consistent with the claim "Ω is not subcountable." In our termin (*vide infra*), Myhill proved, in effect, that

$$\mathrm{IZF} + \mathrm{CT}_0 + \mathrm{AC}^{\omega,X} \vdash \Omega \text{ is not subcountable.}$$

CT_0 is Church's Thesis for total number-theoretic functions. Myhill was aware
and the axiom of choice over ω are consistent with his constructive set theory CS
ally, Myhill had proved the stronger result that

$$\text{IZF} + (\omega \Rightarrow \omega) \text{ is subcountable } + \text{AC}^{\omega,X} \vdash \Omega \text{ is not subcountable}$$

We have shown already that the relevant choice principle, $\text{AC}^{\omega,X}$, holds in $V(Kl)$
later show that CT_0 and "$(\omega \Rightarrow \omega)$ is subcountable" hold there also. Hence, all of
work is available over the realizability universe. For its succintness, however, one
prefer the proof via Uniformity. ▮

Before we move on to further refinements of UP, there is an obvious gener
and a further remark to be made.

8.7. Corollary. $V(Kl) \models$ *If y is subcountable, then* $\forall x\, \exists z \in y\, \phi \rightarrow \exists z \in y\, \forall x$

Proof. The proof of this fact is obvious and constructive from UP. ▮

8.8. Remark. This is the appropriate place to begin a reply to a philosophical
and to mention a problem about our treatment of $V(Kl)$. We are always tempt
in fact, we often succumb to the temptation) to refer to $V(Kl)$ as a *model* or *cla*
of IZF. It has been objected (by Solomon Feferman, among others) that this rep
serious transgression of the rules of usage among logicians. The idea behind this
seems to be that interpretation over $V(Kl)$ is a mere interpretation and that
interpretation of a language over a "real" model goes by the familiar Tarski recu
truth conditions.

It would go some distance in answering the objection to prove that "modell
the sense of the objection depends on one's point of view and, if one takes up a
within $V(Kl)$, the realizability interpretation is as good a model as one would care
Specifically, one would like to show that, internally, if truth is defined as realizab

$$T(\phi) \text{ iff } \exists e\; e \Vdash \phi,$$

then the familiar clauses of the Tarski scheme are provable. If the reader cares
writing out the Tarski clauses, he will find that UP and CT_0 go a long way toward o
this result. Unfortunately, the author has yet to find the time to discover a comple

▮

Unzerlegbarkeit.

Lapsing into metaphor, one could say that the realizability of UP shows that, i are at most (sub)countably many colors in your box, the only way to color all the the $V(Kl)$ universe is to make everything the same color! It is an immediate conse of this that, at least internally, the realizability universe cannot be colored with twc unless the color scheme is monochromatic. The absence of nontrivial "two-coloring been called '*unzerlegbarkeit*.' Without the metaphor, a universe of sets is *unzerle* UZ:

$$\forall x\, (\phi \vee \neg \phi) \rightarrow \forall x\, \phi \vee \forall x\, \neg \phi$$

holds over the universe.

In Brouwer's development of intuitionistic analysis as based on *choice sequenc* reals are provably unzerlegbar. This happens thanks to the presence of strong con principles for the reals. The most well known of the consequences of the continuity ples is Brouwer's Theorem: every total real-valued function of a real variable is cont Unrestricted UP is likewise a kind of strong continuity principle; it follows imme from UP that every class function from the universe into ω is constant, and, henc tinuous relative to ω's discrete topology.

8.9. Theorem. $V(Kl) \models \mathrm{UZ}$

Proof. Since $\{0,1\}$ is subcountable, this is immediate from Corollary 8.7. ∎

Let x be any set; $\mathrm{UZ}^{P(x)}$ asserts that the unzerlegbarkeit property holds when fiers are restricted to $P(x)$, the powerset of x. It is easily seen that $\mathrm{UZ}^{P(x)}$ is a const consequence of UZ, so we have

8.10. Corollary. $V(Kl) \models \forall x\, \mathrm{UZ}^{P(x)}$.

Proof. The map $y \mapsto y \cap x$ is a surjective class function that takes the univers $P(x)$. ∎

Characterizing uniformity.

There is a fine internal structure to the realizability sets over which uniformity erties hold. It is this structure that now comes under examination. The result examination will be a set-theoretic characterization of the "uniformity sets" as qu of sets which are stable. In the process, we will give a quite general form of the u principle, which we call 'the extended uniformity princple' or EUP. For simi from the tripos-theoretic setting, see Hyland (1982).

8.11. Definition. A set $a \in \mathrm{V}(Kl)$ is (*realizably*) *uniform* iff

$$\mathrm{V}(Kl) \models \forall x \in a\ \exists n\ \phi(x,n) \to \exists n\ \forall x \in a\ \phi(x,n)$$

for all ϕ in the extended language. ∎

8.12. Definition. A set $a \in \mathrm{V}(Kl)$ has *fixed realizability* iff there is a number which "fixes" realizbility for a, i.e., there is a $\langle j, n\rangle$ such that

$$\forall m\ \forall b\ (m \Vdash b \in a \to \exists c\ (n \Vdash c \in a \wedge \{j\}(m) \Vdash b = c)).$$

∎

8.13. Theorem. *For $a \in \mathrm{V}(Kl)$, a is uniform iff a has fixed realizability.*

Proof. (1) Assume, on the one hand, that $e \Vdash \forall x \in a\ \exists n\ \phi$ and that a realizability. Let n be the second component of the fixing pair. One can s $\{e\}(n)_{00}$ gives a number m that makes for uniformity. In other words, one can p a witness for

$$\mathrm{V}(Kl) \models \forall x \in a\ \phi(x, \overline{m})$$

is obtainable effectively from e and n.

(2) Assume that a is realizably uniform. Then, for some $e \in \omega$,

$$e \Vdash \forall x \in a\ \exists n\ \phi \to \exists n\ \forall x \in a\ \phi.$$

As before, the internalized realizability set for a, $a^{\Vdash}$, is a member of $\mathrm{V}(Kl)$. Si

$$\mathrm{V}(Kl) \models \forall x \in a\ \exists n\ \langle n, x\rangle \in a^{\Vdash},$$

there is a $j \in \omega$ such that

$$j \Vdash \exists n\ \forall x \in a\ \langle n, x\rangle \in a^{\Vdash}.$$

It follows that there is, calculable from j, an n such that for all m and all b from

$$m \Vdash b \in a \rightarrow \{j\}(m) \Vdash \overline{\langle \overline{n}, a \rangle} \in a^{\Vdash}.$$

A fixing pair and a proof of the requisite fixing property is now derivable direc
this condition. ∎

8.14. Definition. A set $a \in \mathbf{V}(Kl)$ is (*realizably*) *degenerate* iff there is an $n \in$
that, for all m and all $b \in \mathbf{V}(Kl)$, if $m \Vdash b \in a$, then $n \Vdash b \in a$. ∎

8.15. Lemma. *A set* $a \in \mathbf{V}(Kl)$ *is degenerate iff it is realizably stable, i.e., iff*

$$\mathbf{V}(Kl) \models \forall x\,(\neg\neg x \in a \rightarrow x \in a).$$

Proof. This is obvious, given classical logic and the realizability interpretatio
∎

The following theorem presents our attempt to characterize the realizability se
are uniform. We prove that uniformity holds over a set a just in case a is, in V
image of a set which is stable.

8.16. Theorem. $a \in \mathbf{V}(Kl)$ *is uniform iff*

$$\mathbf{V}(Kl) \models \exists y\, \exists f\, (y \textit{ is stable } \wedge f : y \twoheadrightarrow a).$$

Proof. (1) Given the right half of the biconditional, we will prove that a h
realizability. Assume that there are b, f, j such that $b, f \in \mathbf{V}(Kl)$, $j \in \omega$ and

$$j \Vdash \exists f\, (b \text{ is stable} \wedge f : b \twoheadrightarrow a).$$

Let n be the number associated with the degeneracy of b, as guaranteed by Lem
Now, if $m \Vdash c \in a$, then

$$\mathbf{V}(Kl) \models \exists z \in b\; f(z) = c.$$

So, for some d,

$$\mathbf{V}(Kl) \models d \in b \wedge f(d) = c.$$

By degeneracy, $n \Vdash d \in b$. Hence, there is a fixed e such that

$$\{e_0\}(n) \Vdash f(d) = c \text{ and } \{e_1\}(n) \Vdash f(d) = a.$$

Therefore, a has fixed realizability.

(2) For the converse, we give a straightforward construction of a stable set of given a is a quotient. Let

$$a^{st} = \{\langle 0, b\rangle : 0 \Vdash \neg\neg b \in a\}.$$

Clearly, $a^{st} \in \mathrm{V}(Kl)$. Also, a^{st} is stable: for, if $m \Vdash b \in a^{st}$, then (by definitio

$$\exists c\, \langle m_0, c\rangle \in a^{st} \wedge m_1 \Vdash b = c.$$

It follows that there are $p, q \in \omega$ such that $p \Vdash c \in a$ while $q \Vdash b \in a$. $\langle 0, i\rangle \Vdash b \in a^{st}$.

Next, we construct the map f^{st}, where

$$f^{st} = \{\langle 0, \overline{\langle b, b\rangle}\rangle : \exists n\, n \Vdash b \in a\}.$$

It is a simple matter to prove that

$$\mathrm{V}(Kl) \models f^{st} : a^{st} \twoheadrightarrow a, \text{ and}$$

this completes the proof. ∎

Extended Uniformity.

We offer the remark that, as far as uniformity is concerned, any subcou would do as well as ω:

8.17. Definition. A set $a \in \mathrm{V}(Kl)$ is *generally uniform* iff, for all subcountab

$$\forall x \in a\, \exists z \in y\, \phi(x, z) \to \exists z \in y\, \forall x \in a\, \phi(x, z)$$

holds of a. ∎

8.18. Theorem. *In* $\mathrm{V}(Kl)$, *we have*

$$\forall x\, (x \textit{ is uniform iff } x \textit{ is generally uniform}).$$

Proof. Just work in IZF; the proof is elementary. ▮

All the ground has now been prepared for the most general uniformity EUP. If x is a quotient of a stable y, we write 'qs(x,y).' With this understood, th *uniformity principle* (EUP) is the assertion

$$\forall x\, \forall y\, (\mathrm{qs}(x, y) \rightarrow x \text{ is generally uniform }).$$

8.19. Theorem. $\mathrm{V}(Kl) \models \mathrm{EUP}$.

Proof. The realizability of EUP follows directly from the preceding results.

8.20. Note. As we predicted earlier, the set

$$\omega^0 = \{\langle 0, \overline{n}\rangle : n \in \omega\}$$

is a uniformity set. One can verify this in a few seconds by showing that ω' realizability. ▮

8.21. Remark. Were space and time to permit, one would like to investigate t tions between EUP and a particularly memorable theorem from Troelstra (198(to the statement and the proof of that theorem was the notion of "interpolati species X is an *interpolation set* iff for each pair a and b of elements of X and e sition p from Ω, there is an "interpolant," an element c of X which is equal to far as" p is true and equal to b "in so far as" p is false. Specifically, c is an for a and b relative to p iff p implies that $a = c$ and $\neg p$ implies that $b = c$. S interpolation set just in case, for every pair a, b of elements of X and for each p, there is a $c \in X$ such that p implies that $a = c$ while $\neg p$ implies that $b = c$. that every powerset is an interpolation set: for sets a and b from $\mathrm{P}(x)$ and pro an interpolant is easily defined; just set

$$c = \{x \in a : p\} \cup \{x \in b : \neg p\}.$$

With the notion of interpolation set, Troelstra proved that if a is an interpolati if b is a set with apartness (*vide infra*), then every map from a into b is constan a proof and a discussion of Troelstra's theorem in Greenleaf (1981). ▮

8.22. Remark. To confirm our earlier remark that uniformity is to be closel[y] with the conception of set marked by generic quantification, one notes that the version of realizability designed by the author which is based on *set-* or . Allied to this realizability is a conception of constructive set which is highly Mathematically, nongenericity under E-realizability will be apparent from t[h] quantification is highly sensitive to the set-recursive internal structure of the sets. Sets are conceived as specific entities with graspable individual charac[t] the model given by E-recursion, UP is demonstrably false. ∎

Section 9: Subcountability, apartness and metric spaces

We begin here much as we have in previous sections—by presenting an acterization of those sets which satisfy some crucial set-theoretic property the property of concern is *subcountability.*

9.1. Definition. $a \in V(Kl)$ is (*realizably*) *separated* iff there is a $j \in \omega$ su $b, c \in V(Kl)$, and all m,

$$\text{if } m \Vdash b \in a \text{ and } m \Vdash c \in a \text{ then } \{j\}(m) \Vdash b = c.$$

▮

9.2. Lemma. *$a \in V(Kl)$ is separated iff a is subcountable in $V(Kl)$.*

Proof. Assume that a is realizably separated and consider

$$a^{\Vdash} = \{\langle n, \overline{\langle \overline{n}, b\rangle}\rangle : n \Vdash b \in a\}.$$

To show that a is subcountable, it will suffice to show that, internally, $a^{\Vdash}$ To that end, assume that $m \Vdash \overline{\langle \overline{p}, b\rangle} \in a^{\Vdash}$ while $n \Vdash \overline{\langle \overline{p}, c\rangle} \in a^{\Vdash}$. Then, c j given by the assumption of separation to show that $b = c$.

For the converse, assume that a is internally subcountable and that m $m \Vdash c \in a$. Since a is subcountable, there is a $g \in \omega$ and an internal functi

$$g \Vdash \forall x \in a\ \exists y \in \overline{\omega}\ f(y) = x.$$

The conclusion is now immediate from the realizability properties of g, anc ing" j can be obtained effectively from it. ▮

In his *Sheaves and Logic* Seminar, Michaelmas Term 1980, Dana Scott p tion: "Is it consistent with IZF to assume that the class of ordinals on wh holds comprises a set, rather than a proper class?" Before the end of the sub Term, Guiseppe Rosolini and the author had devised a version of the real pretation over $V(Kl)$, and had proved that, under that intepretation, IZF with a claim that amounts to a positive reply to Scott's question.

9.3. Definition. A set x is an *ordinal* iff x is a transitive set of transitive of ordinals is denoted 'Ord.' ▮

9.4. Definition. An ordinal x is *trichotomous* ($\mathrm{Tri}(x)$) iff

$$\forall y, z \in x\, (y \in z \vee z \in y \vee z = y)$$

∎

9.5. Lemma. IZF$\vdash$ *$x \in$***Ord** *is trichotomous only if x is discrete.*

Proof. This is immediate from the axiom of foundation. For complete
Grayson (1979). ∎

9.6. Theorem. $\mathrm{V}(Kl) \models \exists x \subseteq \mathrm{Ord}\, \forall y\, (y \in x \leftrightarrow \mathrm{Tri}(x))$.

Proof. By the lemma, it is sufficient to show that, over $\mathrm{V}(Kl)$, every
is subcountable. Given what has gone before, we only need show that if
$\mathrm{V}(Kl)$, then x is separated.

Let $e \Vdash$ "a is discrete" and take m such that $m \Vdash b \in a$ while $m \Vdash$

$$\{e\}(\langle m, m\rangle) \Vdash b = b \vee b \neq b.$$

Since $\mathrm{V}(Kl)$ cannot force $b \neq b$,

$$\{e\}(\langle m, m\rangle)_0 = 0 \text{ and } \{e\}(\langle m, m\rangle)_1 \Vdash b = b.$$

By the same token,

$$\{e\}(\langle m, m\rangle) \Vdash a = b \vee a \neq b,$$

so, $\{e\}(\langle m, m\rangle) \Vdash a = b$. Therefore, a is realizably separated and is subcou

This result admits an obvious generalization. The fact that a is an ordi
nothing special to the realizability calculation; any discrete set would have
Therefore, we can claim that

9.7. Corollary. $\mathrm{V}(Kl) \models$ *Every discrete set is subcountable.*

Proof. ∎

9.8. Remark. (1) The theorem answers a question raised by Beeson
Beeson referred to the statement "Every discrete set is subcountable" as

asked whether SCDS is consistent with IZF. We can now say that SCDS is c
IZF (plus all manner of choice, continuity and effectivity theorems).

(2) Notice that SCDS is not specific to $\mathbf{V}(Kl)$. Any model in which the A
modeled over the natural numbers realizes SCDS. Hence, every discrete set i
in the models given by the hyperarithmetic indices, indices of functions recu
degree, and so on. ∎

This kind of reasoning admits an even further generalization. There i
here with discreteness that could not have been done with the weaker and,
more interesting, notion of apartness.

9.9. Definition. $R \subseteq x \times x$ is a (*strict*) *apartness* on x iff for all y and z

$$\forall y, z \in x\, (y = z \leftrightarrow \neg R(y, z)) \quad \text{and } \forall v \in x\, (R(y, z) \rightarrow (R(y, v$$

9.10. Theorem. *In* $\mathbf{V}(Kl)$, *every set with a strict apartness is subcount*

Proof. Suppose $e \Vdash$ "R is an apartness on a". We will show that, effect
separated. Assume that $m \Vdash b \in a$ and $m \Vdash c \in a$; assume also that $n \Vdash$
the definition of strict apartness, there is a partial recursive Θ such that

$$\Theta(m, n, e) \Vdash R(b, c) \vee R(c, c).$$

It follows that $\Theta(m, n, e)_0 = 0$ and $\Theta(m, n, e)_1 \Vdash R(b, c)$. By the same tok

$$\Theta(m, n, e) \Vdash R(b, b) \vee R(c, b).$$

Therefore, $\mathbf{V}(Kl) \models R(b, b)$. But this is a contradiction; hence, our origin
that $R(b, c)$ is realized, is false.

We now have that $0 \Vdash \neg R(b, c)$. It follows from the definition of s
that $\mathbf{V}(Kl) \models b = c$, with witness computable from e. Therefore, every
apartness is realizably subcountable.

9.11. Remark. The following corollary settles another (not unrelated) qu
Beeson (1981). Beeson asked whether IZF might be consistent with SCMS,
that every metric space has a subcountable basis. The realizability of SC
corollary of the following (much stronger) theorem. ∎

9.12. Corollary. *In* V(*Kl*), *every metric space is subcountable.*

Proof. It is easy to prove, in IZF, that the strict apartness on the reals i
metric space, a strict apartness relation. ∎

9.13. Note. (1) As with SCDS, there is nothing special here about *recurs*
Any of the general realizabilities that use natural numbers as indices wil
result.

(2) It is fairly obvious that the converse to SCDS fails over V(*Kl*): IZF
Thesis shows that $(\omega \Rightarrow \omega)$ is subcountable. Therefore, the considerations
will show that $(\omega \Rightarrow \omega)$ is subcountable in V(*Kl*). However, the undecidab

$$\{\langle n, m\rangle : \text{if } \{n\} \text{ and } \{m\} \text{ are total, then } \{n\} = \{m\}\}$$

reduces to the realizability of the discreteness of $(\omega \Rightarrow \omega)$:

$$\forall x \in (\omega \Rightarrow \omega)\,(x = y \vee x \neq y).$$

Therefore, over V(*Kl*), we have both that every discrete set is subcountable
is a nondiscrete subcountable set. Neither of these is, of course, provable i

Section 10: Injecting the classical sets

When one deals with Boolean-valued models, the classical ground mo[...]
be injected into a Boolean-valued class structure built over it. The injec[...]
familiar, takes the sets of the ground model and embeds them into the Bo[...]
as sets whose "membership values" are constant. In almost the very same [...]
embedded into the realizability universe V(Kl); the relevant injection, x^{st} [...]
recursion on membership. As the notation suggests, x^{st} takes the entire cla[...]
into the stable part of V(Kl).

10.1. Definition. For $x \in \mathrm{V}$,

$$x^{st} = \{\langle 0, y^{st}\rangle : y \in x\}.$$

∎

In the Boolean-valued case, the appropriate injection turns out to [...]
injection of class models. In the case of V(Kl), x^{st} will not give an isomorph[...]
tinkers slightly with the logic; after all, the virtues of realizability accrue tha[...]
that realizability conditions differ markedly from the truth-conditional int[...]
the logical signs in the external world. Even so, the requisite tinkering is re[...]
and quite predictable. The sort of tinkering that has to be done results [...]
that we call "isomorphism up to stability;" this sort of logical relation will [...]
important) play when we begin to do internal recursive mathematics in Ch[...]

The first stage in plumbing the precise relations between logic over V [...]
the image of x^{st} in V(Kl) is an assessment of the effect of the injection [...]
statements. In this case, the effect is negligible.

10.2. Lemma. *For* $a, b \in \mathrm{V}$,

$$\mathrm{V} \models a \in b \textit{ iff } \mathrm{V}(Kl) \models a^{st} \in b^{st}$$

$$\mathrm{V} \models a = b \textit{ iff } \mathrm{V}(Kl) \models a^{st} = b^{st}.$$

Proof. Each of the implications from left to right is trivial. As befits the [...]
x^{st} is defined, we prove the converse implication by a simultaneous transfin[...]

First, assume that $e \Vdash a^{st} \in b^{st}$. By definition, there is a d such that (
$e_1 \Vdash a^{st} = d$. By definition of x^{st}, $d = g^{st}$ for some g from b. Hence, b
hypothesis, $a \in b$ holds in $\mathbf{V}$.

For the other statement, assume that $\mathbf{V}(Kl) \models a^{st} = b^{st}$ and let c
of $a \in \mathbf{V}$. Then, $\mathbf{V}(Kl) \models c^{st} \in a^{st}$ and $\mathbf{V}(Kl) \models c^{st} \in b^{st}$. If we us
hypothesis, $c \in b$ will follow. ∎

Naturally, the next step is the provision of an internal set-theoretic c
of the range of x^{st} in $\mathbf{V}(Kl)$. We note that Robin Grayson (1975) has ve
construction of transfinite closures can be carried out just as handily in IZ
with the same inductive effect: the transfinite closure of x is the $\subseteq$-least
containing x.

10.3. Definition. A set x is said to be *hereditarily stable* (*hs*) iff x is st
element of the transfinite closure of x is stable. ∎

We prove that the hereditarily stable sets of $\mathbf{V}(Kl)$ are precisely those
injections of classical sets.

10.4. Theorem. *There is a $j \in \omega$ such that for all $a \in \mathbf{V}(Kl)$, if $e \Vdash$ hs*
is a $b \in \mathbf{V}$ such that

$$\{j\}(e) \Vdash a = b^{st}.$$

Proof. This is a straightforward application of Kleene's second recursion
parallel with transfinite induction. To start the induction, assume that, for
$e \Vdash \mathrm{hs}(a)$. The inductive hypothesis gives us the following: there is a part
such that, if $g \Vdash c \in a$, then, for some $d \in \mathbf{V}$, $\{j\}(g) \Vdash c = d^{st}$.

Now, we can use collection over $\mathbf{V}$ to gather together all and only the
to realizability elements of a. Call this set of d's '$\underline{a}$.' Then, we form the set

$$a^{st} = \{\langle 0, d^{st}\rangle : d \in \underline{a}\}.$$

From the definition of a^{st} and some simple facts, it follows immediatel

$$\mathbf{V}(Kl) \models a \subseteq a^{st}$$

with witness calculable from j. For, if $g \Vdash c \in a$, then

$$\langle \{j\}(g), 0\rangle \Vdash c \in a^{st}.$$

To prove the converse inclusion, assume that $g \Vdash c \in a^{st}$. Then, by
clause governing membership, there is a $d \in \underline{a}$ such that

$$\langle 0, d^{st}\rangle \in a^{st} \text{ while } g_1 \Vdash c = d^{st}.$$

It is a consequence of the conditions set on $\underline{a}$ that

$$0 \Vdash \neg\neg c \in a.$$

But, since $\mathrm{V}(Kl) \models \mathrm{hs}(a)$, we have that $\mathrm{V}(Kl) \models c \in a$ with witness calcu
The desired result now follows from one application of the second recursi

We can now be quite precise about the logical relations between V
lization", the image of st, in $\mathrm{V}(Kl)$. Let L_{V} be the language of IZF,
autonymous names for elements of V. We will define a translation (also c
takes sentences of L_{V} into sentences of L_{Kl}.

10.5. Definition. The translation st is defined inductively as follows:

$$(a \in b)^{st} = (a^{st} \in b^{st}) \text{ and } (a = b)^{st} = (a^{st} = b^{st}).$$

st is taken to commute with $\wedge, \vee, \neg, \rightarrow$, and

$$(\exists x\, \phi)^{st} = \exists x\, (\mathrm{hs}(x) \wedge \phi^{st}) \text{ while}$$
$$(\forall x\, \phi)^{st} = \forall x\, (\mathrm{hs}(x) \rightarrow \neg\neg\, \phi^{st}).$$

∎

10.6. Theorem. *For sentences ϕ of L_{V},*

$$\mathrm{V} \models \phi \textit{ iff } \mathrm{V}(Kl) \models \phi^{st}.$$

Proof. The proof goes via a straightforward structural induction on ϕ
logic freely in the metatheory. Theorem 10.4 is required to handle the uni
case. ∎

10.7. Remark. (1) To a great extent, this theorem anticipates the notion up to stability" which we introduce and apply in Chapter Five. There, we basic idea to display the logical relations between certain areas of recurs and natural subtheories of the theory of $\mathbf{V}(Kl)$.

(2) Since the image of *st* in $\mathbf{V}(Kl)$ coincides (even internally) with the cla stable sets, the classical universe reappears in $\mathbf{V}(Kl)$ as a portion of the verse which is thoroughly uniform. Therefore, the class of hereditarily st uniform, as is the image of each classical set. Since, to the intuitionist, th are largely interesting pathologies, we might be moved to think of classi something of a pathology itself. ∎

Section 11: Recursion, MP and IP

This is the point at which we begin to direct attention exclusively V(Kl) directs toward recursive mathematics. Our approach will be gra with some very general (even trite) propositions. The first of these recor in V(Kl), sets which are decidable over ω (or sets which are sufficiently like collections which are externally recursive. The second records a similar between r.e. sets and sets which are internally countable. These constitut primitive intimations of some of the absoluteness results from Chapter Fo

11.1. Proposition. $V \models$ *"$S \subseteq \omega$ is recursive" iff* $V(Kl) \models$ *"$\overline{S}$ is decid there is an effective correspondence between recursive indices for externa of '$\overline{S}$ is decidable.'*

Proof. This is a trivial application of the realizability interpretation of

11.2. Proposition. $V \models$ *"$S \subseteq \omega$ is r.e. and nonempty" iff* $V(Kl) \models$ *Again, there is an effective map which interchanges canonical re indices for for '$\overline{S}$ is countable.'*

Proof. Once again, this is immediate. ∎

Markov's Principle.

Markov's Principle (MP) is closely associated with the work of the s constructivists. For these mathematicians, the acceptance of MP is justifie a classical conception of a universe of mathematical objects thought of as c of possible inputs for Turing machines. On the constructivist view of intuitionists, however, MP is clearly false.

The version of MP most appropriate to our concerns is

$$\forall n\, (\phi(n) \vee \neg \phi(n)) \rightarrow (\neg\neg \exists n\, \phi(n) \rightarrow \exists n\, \phi(n))).$$

The consistency of IZF with MP is confirmed by showing that

11.3. Theorem. $V(Kl) \models$MP.

Proof. For purposes of this proof, we suppose that V models classical l that

$$(1) \qquad e \Vdash \forall n\, (\phi(n) \vee \neg \phi(n))$$

and that

$$g \Vdash \neg\neg\exists n\, \phi(n).$$

By classical logic in **V**, it follows from the latter that

$$(2) \quad \exists m\, m \Vdash \exists n\, \phi.$$

It is a consequence of (1) that, for some partial recursive Θ and for each

$$\Theta(e,n)_0 = 0 \wedge \Theta(e,n)_1 \Vdash \phi(\overline{n}) \text{ or}$$

$$\Theta(e,n)_0 = 1 \wedge \Theta(e,n)_1 \Vdash \neg\phi(\overline{n}).$$

From (2), we know that there are p and q from ω such that

$$p \Vdash \phi(\overline{q}).$$

Then, with $r = \mu n.\Theta(e,n)_0 = 0$,

$$\Theta(e,r)_1 \Vdash \phi(\overline{r}).$$

Therefore, $\mathbf{V}(Kl) \models \mathrm{MP}$. ∎

11.4. Note. (1) A brief glance at the foregoing will reveal that we can
MP iff $\mathbf{V}(Kl) \models \mathrm{MP}$.

(2) For those familiar with realizability for arithmetic, the situation with
as no surprise. Troelstra (1971) showed that MP, together with the ex
Thesis $\mathrm{ECT_0}$, axiomatizes classical realizability in the follow sense:

$$\mathrm{HA{+}TND} \vdash \phi \text{ iff } \mathrm{HA{+}ECT_0{+}MP} \vdash \neg\neg\phi$$

∎

Independence of Premisses.

Over $\mathbf{V}(Kl)$, we have a limited version of IP, the principle of independe

$$(\neg\phi \rightarrow \exists x\, \psi) \rightarrow \exists x\, (\neg\phi \rightarrow \psi).$$

Here, we presuppose that ϕ is closed.

11.5. Theorem. $V(Kl) \models \mathrm{IP}$

Proof. Again, V is taken to be classical. Assume that $e \Vdash \neg\phi \to \exists x\, \psi$. $\neg\phi$, $0 \Vdash \neg\phi$ and $\{e\}(0) \Vdash \exists x\, \phi$. Therefore, there is an $a \in V(Kl)$ such $\psi[x/a]$.

Hence, if ϕ is not realized, $\{e\}(0)$ is defined and, for some a from $V(K$

$$\Lambda n.\{e\}(0) \Vdash \neg\phi \to \psi[x/a].$$

So, in this case, $V(Kl) \models \mathrm{IP}$.

On the other hand, should ϕ be realized, then $\neg\phi$ is never realized, so

$$\Lambda n.n \Vdash \mathrm{IP}.$$

∎

Certain sharper versions of IP hold in $V(Kl)$ in virtue of the same sort of as those adduced in support of general IP. Consider the principle we call 'I

$$\forall x\,(\neg\phi \to \exists y \in \omega\ \psi) \to \exists y \in \omega\ \forall x\,(\neg\phi \to \psi).$$

The following theorem is an application both of external classical logic UP.

11.6. Theorem. $V(Kl) \models \mathrm{IP}^{\omega}$.

Proof. First, assume that for some $a \in V(Kl)$ and for some n,

$$n \Vdash \neg\phi[x/a].$$

In this case, the class $\{x : \neg\phi(x)\}$ satisfies the conditions on a uniformity s this may be a proper class over $V(Kl)$, but the uniformity considerations can accordingly.) Hence, IP^{ω} follows directly from uniformity.

On the other hand, assume that nothing serves as a witness for $\neg\phi($ regardless of the value of x. In that case, the realizability of IP^{ω} is immedi

11.7. Note. More restrictive versions of IP, in particular, those in wh quantifier is restricted to ω and in which free variables of ϕ are bounde V(Kl). Consider the principle $\mathrm{IP}^{\omega,\omega}$, where the variables are so restrict

$$(\neg\phi \to \exists n\, \psi) \to \exists n\, (\neg\psi \to \psi)$$

$\mathrm{IP}^{\omega,\omega}$ is not available over V(Kl). In effect, this has been proved by (1971). There, Troelstra notes that the triad consisting of Church's The is inconsistent with the intuitionistic first-order arithmetic HA.

Troelstra's argument can be paraphrased as follows: Let '$T(x,y,$ expression in HA of Kleene's T-predicate. In HA plus MP, we have that

$$\forall x\, (\neg\neg\exists y\, T(x,x,y) \to \exists y\, T(x,x,y)).$$

Given $\mathrm{IP}^{\omega,\omega}$, it is a consequence of the above that

$$\forall x\, \exists y\, (\neg\neg\exists y\, T(x,x,y) \to T(x,x,y)).$$

Church's Thesis now applies, and it provides a *recursive* number-theoret for the quantifier combination $\forall x\, \exists y$. Therefore,

$$\exists u\, \forall x\, [\{u\}(x)\downarrow \wedge (\exists y\, T(x,x,y) \leftrightarrow T(x,x,\{u\}(x)))]$$

holds. This outcome clearly flouts the (provable in HA) unsolvability of lem.

(2) A close examination of the proof of the last theorem will convince th in the ground model will suffice to give IP in the realizability model. M have that

$$\mathrm{V} \models \mathrm{IP} \text{ implies that } \mathrm{V}(Kl) \models \mathrm{IP}.$$

A similar remark applies to IP^{ω}. ∎

Section 12: A final theorem

We close Chapter Three with a simple exercise on undecidability and
Considered as a sample of what can be done, the following theorem gives
claim that undecidability theorems from recursive mathematics can be t
less mechanically, into independence theorems for IZF. The corollary ans
posed by Dana Scott.

12.1. Theorem. $\mathbf{V}(Kl)$ *satisfies*

$$\neg \forall x \in \mathbf{P}(\omega)\ (x \textit{ is decidable} \rightarrow x \textit{ is finite} \vee x \textit{ is not finite}).$$

Proof.

Let 'Φ' stand for the above statement, less its negation:

$$\forall x \in \mathbf{P}(\omega)\ (x \text{ is decidable} \rightarrow x \text{ is finite} \vee x \text{ is not finite}).$$

Assume that $e \Vdash \Phi$. We assume externally that $A \subseteq \omega$ is recursive and
characteristic function for A. Then, there is a primitive recursive Θ such

$$\Theta(g) \Vdash \overline{A} \in \mathbf{P}(\omega) \text{ and } \overline{A} \text{ is decidable}.$$

As always, $\overline{A} = \{\langle n, \overline{n}\rangle : n \in A\}$. Given the realizability conditions fo
$\{e\}(\Theta(g))\downarrow$ and either

$$\Psi(e,g)_0 = 0 \text{ and } \mathbf{V}(Kl) \models \overline{A} \text{ is finite or}$$
$$\Psi(e,g)_0 \neq 0 \text{ and } \mathbf{V}(Kl) \models \overline{A} \text{ is not finite}.$$

It is easy to see that, if the former condition obtains, then A is externally
the latter, A is externally infinite.

Therefore, $\Lambda g \Psi(e,g)_0$ represents an effective procedure which, given
recursive characteristic function, determines whether or not the set asso
function is finite. However, no such procedure can exist; the halting pr
reduced to the problem solved by $\Lambda g \Psi(e,g)$. ∎

12.2. Corollary. *" $\forall x \in \mathbf{P}(\omega)$ (x is decidable $\rightarrow x$ is finite $\vee\ x$ is not*
pendent of IZF.

Proof. ∎

Section 1: Prefatory and historical remarks

The budget of results in this chapter lay buried in various dusty corne realizability "folklore." We believe that our compilation of this folklo elegant and that our proofs are the most efficient possible. It seems th the realizability of Church's Thesis is new. Originality aside, all of these proofs, were known to the author prior to November 1981.

Absoluteness, according to Sacks (1962), is to be considered the pr traditional model theory. The primacy of absoluteness is nowise diminis to the nontraditional situation of $\mathrm{V}(Kl)$. Unsurprisingly, the sort of abs $\mathrm{V}(Kl)$ supports is implicit in the work of Kleene and can be derived fr Troelstra. We call instances of these "realizability absoluteness" phenomen of "Kleene absoluteness."

Section 2: Kleene absoluteness

2.1. Note. (1) For this chapter, the symbols 'n', 'm' and that ilk will be u but, one hopes, intelligibly. In the metatheoretic dialect as applied to arit 'm' either range over the elements of ω in V or stand for the apposite for some system of arithmetic. The precise meaning should be clear from cont to formal languages for set theory, 'n' and 'm' become set-theoretic para over ω.

(2) For purposes of exposition in this chapter, we restrict discussion and ex number-theoretic relations which are binary. The truth of our theorems a of their proofs suffer no such restriction. ∎

2.2. Definition. For $n \in \omega$, we take $\overline{\overline{n}}$ to be the set-theoretic term which is c
used to denote n. Officially, $\overline{\overline{n}}$ is defined by metatheoretic ω-recursion:

$$\overline{\overline{0}} = \emptyset$$

$$\overline{\overline{n+1}} = \overline{\overline{n}} \cup \{\overline{\overline{n}}\}$$

∎

Our entrée to Kleene absoluteness is by way of one of the oldest and m
ideas of mathematical logic, that of *numeralwise representation* in a forma
is well known, this idea carries a significant portion of the conceptual burd
Incompleteness Theorems.

2.3. Definition. Let $R(m,n)$ be a number-theoretic predicate. A formula
language of ZF *numeralwise represents* R iff, for all m and n,

$$\mathbf{V} \models R(m,n) \Rightarrow \mathrm{IZF} \vdash \phi(\overline{\overline{m}}, \overline{\overline{n}}) \text{ and}$$

$$\mathbf{V} \models \neg R(m,n) \Rightarrow \mathrm{IZF} \vdash \neg \phi(\overline{\overline{m}}, \overline{\overline{n}}).$$

∎

Trivially, if ϕ numeralwise represents R, then ϕ defines R over $\mathbf{V}$.
the following proposition is a direct application of the coding techniques (e
function) that lie at the heart of Gödel's approach to incompleteness.

2.4. Proposition. *R is numeralwise represented iff R is recursive in* $\mathbf{V}$.

Proof. For complete details, the reader may consult any standard text for
logic. Shoenfield (1967) is an excellent source. ∎

Numeralwise representation establishes a scheme which associates with
relation a natural set-theoretic expression defining it.

2.5. Definition. For each recursive predicate $R(m,n)$, let $\phi_R(x,y)$ be a f
language of ZF which numeralwise represents it. For the sake of definitene
take ϕ_R to be the formula which would be constructed in the course of the s
of the last proposition. ∎

We are now in a position to give an exceedingly simple proof of Klee
At bottom, the proof relies on little more than the soundness theorem for
ability. We note that this theorem extends and incorporates the results o
on the absoluteness of membership and equality statements on ω in $\mathbf{V}(Kl$

2.6. Theorem (Kleene absoluteness). *Let $R(m,n)$ be recursive. T* *e_R such that, if $\mathbf{V} \models R(m,n)$, then $\{e_R\}(m,n) \Vdash \phi_R(\overline{m},\overline{n})$. Also, if $\mathbf{V}(K$* *then $\mathbf{V} \models R(m,n)$.*

Proof. Two moments of reflection will be enough to see that this is a di of the preceding proposition. One spends the first moment reflecting on proof of the soundess theorem for IZF is thoroughly effective. By this, we is a uniform effective procedure which, given a (coded) proof of a sentence axioms, produces an index e_ϕ such that $e_\phi \Vdash \phi$. During the second mon that there is a total index g such that, for each n, $\{g\}(n) \Vdash \overline{n} = \overline{\overline{n}}$. ∎

2.7. Corollary. *When R is recursive, and $m,n \in \omega$,*

$$\mathbf{V} \models \phi_R(m,n) \text{ iff } \mathbf{V}(Kl) \models \phi_R(\overline{m},\overline{n}).$$

This last corollary should be compared with Theorem 3.11 of Chapte

Section 3: Varieties of Church's Thesis

The proof that every number-theoretic function in $V(Kl)$ is general re us, the primary application of Kleene absoluteness. The claim that, in som context, every number-theoretic function is recursive is called (historical notwithstanding) 'Church's Thesis.' In truth, one can consider any one o variants on Church's Thesis. Some of these deal with partial functions, some on specific definable collections of sets, others, like "Weak Church's Thes constructive connections between the contributory notions by interjecting $\neg\neg$. For a survey of the variants, see Beeson (1979). We will assay the reali of several variants in this section.

Church's Thesis and total functions.

At present, our interest lies in that form of the thesis that pertains to and which holds in $V(Kl)$ without restriction. We refer to this form as 'CT_0' it as

$$\forall n\, \exists m\, \phi(n,m) \rightarrow \exists e\, \forall n\, \exists m\, \exists p\, (T(e,n,p) \wedge U(p,m) \wedge \phi(n,m)).$$

We have expressed CT_0 as a strengthening of $AC^{\omega,\omega}$; this is common in models of IZF. T and U are set-theoretic predicates which numeralwise rep tively, Kleene's T and result-extraction predicates.

3.1. Theorem. $V(Kl) \models CT_0$.

Proof. We already know that $AC^{\omega,\omega}$ holds constructively on $V(Kl)$, so by assuming that there is a total index g such that, for all n,

$$\{g_0\}(n) \Vdash \phi(\overline{n}, \overline{\{g_1\}(n)}).$$

$\{g_1\}$ is total in V. Hence, for each n, there is a least number-theoretic pa that

$$V \models T(g_1, n, p) \wedge U(p, m).$$

Let h index a general recursive function which calculates such a least pa Certainly, h can be found effectively from g.

By absoluteness, there are indices e_T and e_U such that, for all n,

$$\{e_T\}(g_1, n, \{h\}(n)_0) \Vdash T(\overline{g_1}, \overline{n}, \overline{\{h\}(n)_0}) \text{ and}$$
$$\{e_U\}(\{h\}(n)_0, \{h\}(n)_1) \Vdash U(\overline{\{h\}(n)_1}, \overline{\{h\}(n)_0}).$$

Set j equal to

$$\Lambda n.\langle \{h\}(n)_0, \{h\}(n)_1, \{e_T\}(g_1, n, \{h\}(n)_0), \{e_U\}(\{h\}(n)_0, \{$$

Without question, j can be calculated from g, given e_T and e_U. Then, a
adjustment to j yields a witness for

$$\forall n\, \exists m\, \exists p\, [T(e, n, p) \wedge U(p, m) \wedge \phi(n, m)].$$

∎

3.2. Remark. (1) Close attention to the essential details of the proof
shows that the proof can be constructivized without loss. Therefore, IZF⊢

(2) The remarkable strength of CT_0 is, we think, not often recognized.
of CT_0 shows that, regardless of the higher-order or impredicative appar
constructive proofs in IZF of the totality of a number-theoretic functio
with full extensional IZF to assume that the function is recursive. Mo
tiveness of the soundness proof shows that there is a uniform method
constructive proof of totality directly into a machine table that compute

(3) The last point of (2) is worth underscoring. The effective character
theorem plus the realizability of CT_0 show that there is a single machin
vides "automatic programming" for *all* classically provable recursive fur
a single index which, given a constructive proof of the totality of a de
converts the proof into a program (in ALGOL, say) that computes the va
tion. This index will work for all classically provable recursive functio
Kreisel-Friedman theorem: the classically provable and intuitionistically
sive functions coincide. The Kreisel-Friedman theorem is discussed in T
Hence, whenever a described function is known intuitionistically to be tot
chanically computes an ALGOL program for it. Moreover, every classicall
function is known to have a description under which it is intuitionisticall

Church's Thesis and partial functions.

Church's Thesis for partial functions does not fare nearly so well unde In Chapter Six, we will describe a version of realizability under which a Thesis holds. Full Church's Thesis is the claim that *all* partial functions ar the meantime, we shall prove that full Church's Thesis, CT_1, fails under rea respect to $\mathbf{V}(Kl)$. In what follows, let $\mathbf{F}(\omega)$ be the collection comprised of subsets of $\mathbf{P}(\omega \times \omega)$. CT_1 is the statement

$$\forall f \in \mathbf{F}(\omega) \exists e \, \forall n, m \, (\langle n, m \rangle \in f \leftrightarrow \exists p \, (T(e, n, p) \wedge U(p, m))).$$

3.3. Theorem. $\mathbf{V}(Kl) \models \neg \mathrm{CT}_1$.

Proof. The halting problem is manifested extensionally in the set

$$K = \{n : \{n\}(n)\downarrow\}.$$

Let 'K' refer also to the (classically conceived) characteristic function of K the characteristic function is 'classically conceived' is not to say that this pr structive. We desire merely to mark the fact that the characteristic functio confabulation which holds little intrinsic interest for the intuitionist. Th mathematical fact about K which we require is that the two-place relati ing to K is functional.) As usual, '$\overline{K}$' denotes the straightforward interna function K:

$$\overline{K} = \{\overline{\langle n, \overline{\langle \overline{n}, \overline{K(n)} \rangle} \rangle} : n \in \omega\}.$$

It is easy to confirm that $\mathbf{V}(Kl) \models \overline{K} \in \mathbf{F}(\omega)$ and that, for arbitrary n,

$$\langle n, i \rangle \Vdash \overline{\langle \overline{n}, \overline{K(n)} \rangle} \in \overline{K} \,.$$

Now, we assume that $\mathbf{V}(Kl) \models \mathrm{CT}_1$. From this assumption and facts K, we conclude that there is an index g and a number e such that

$$\{g\}(n) \Vdash \exists p \, T(\overline{e}, \overline{n}, p) \wedge U(p, \overline{K(n)}).$$

From the definition of realizability for $\exists x$, we know that there is a j such

$$\{j\}(n)_0 \Vdash T(\overline{e}, \overline{n}, \overline{\{j\}(n)_1}) \wedge U(\overline{\{j\}(n)_1}, \overline{K(n)}).$$

Let u be the external result-extraction function. With Kleene absoluten
relations, one can prove that

$$\lambda n.u(\{j\}(n)_1)$$

is equal to K. This conflicts with the unsolvability of the halting problen

$$\mathbf{V}(Kl) \models \neg \mathrm{CT}_1.$$

∎

3.4. Remark. The above proof can be fully constructivized. ∎

Fundamentally, CT_1 fails because $\mathbf{P}(\omega)$ is, under realizability, a ni
Most likely, the fact that $\mathbf{P}(\omega)$ in $\mathbf{V}(Kl)$ is a uniformity set would alone su
the reader of the difficulties lurking there. Since $\mathbf{P}(\omega)$ is so thoroughly
will contain functions whose domains are from a classical standpoint ano
a simple instance, the internal domain D of the function K is a subset o
$\mathbf{V}(Kl)$, $D \neq \omega$ but $\forall n \ \neg\neg n \in D$.

Partial functions and ω-stability.

It is not terribly difficult to concoct a realizably correct version of full
further constraints on the possible domains of functional relations. Th
impose here is that of ω-stability. To recall notions introduced in Chapte
A of ω is ω-stable whenever

$$\forall x \in \omega \, (\neg\neg x \in A \rightarrow x \in A).$$

The collection of ω-stable sets is denoted '$\mathbf{P}(\omega)^{st}$.' As will be apparent fro
the concept of ω-stability holds sway over the "reduction" of effective to
theory. The ω-stable version of full Church's Thesis is CT_2: for X an ele

$$\forall n \in X \, \exists m \, \phi(n,m) \rightarrow \exists e \, \forall n \in X \, \exists m \, \exists p \, (T(e,m,p) \wedge U(p,n) \wedge \phi(n,m$$

3.5. Theorem. $\mathbf{V}(Kl) \models \mathrm{CT}_2$.

Proof. Assume that $g \Vdash A \in \mathbf{P}(\omega)^{st}$ and that

$$e \Vdash \forall n \in A \, \exists m \, \phi(n,m).$$

Let $\underline{A} = \{m : \mathrm{V}(Kl) \models \overline{m} \in A\}$. If $m \in \underline{A}$, then $0 \Vdash \neg\neg\overline{m} \in A$. g witnesses the fact that A is internally ω-stable. Hence, with

$$\Phi(m) = \{\{g\}(\langle m, i\rangle)\}(0),$$

$\Phi(m) \Vdash \overline{m} \in A$.

We also know that $\{e\}(\Phi(m)) \Vdash \exists n \in \omega\ \phi(m,n)$. Set $h = \lambda m.\{e\}(\Phi(m))_{00}$. It is a straightforward calculation to confirm that a witness for

$$\exists e\ \forall m \in X\ \exists n, p\ (T(e,m,p) \wedge U(p,n) \wedge \phi(m,n)$$

is calculable from h. For this, we need to rely on Kleene absoluteness once again. ∎

Church's Thesis: a final generalization.

As a corollary to the above, one can prove to be realizable a simple generalization of Church's Thesis which first appeared in Hyland (1982). For this, we need a pair of definitions and a lemma:

3.6. Definition. Let X be subcountable, and let $X^* \subseteq \omega$ be such that X^* "counts" X, i.e., X^* is such that, for some f, $f : X^* \twoheadrightarrow X$. X^* is called a '*presentation*' of X and *X^* presents X via f.* ∎

3.7. Definition. If X is subcountable and X^* is a presentation of X which is ω-stable, then we will say that X^* is a *canonical presentation* of X. ∎

3.8. Lemma. *In $\mathrm{V}(Kl)$, if X is subcountable, then X has a presentation which is canonical.*

Proof. Assume that, over $\mathrm{V}(Kl)$, X is subcountable, and Y presents X via function f. Let $X^{\Vdash}$ be the internal realizability set corresponding to X,

$$X^{\Vdash} = \{\langle n, \overline{\langle \overline{n}, a\rangle}\rangle : n \Vdash a \in X\}.$$

The techniques of the last chapter suffice to prove that, since X is realizably subcountable, then it is realizably separated and $X^{\Vdash}$ is a functional relation on $\mathrm{V}(Kl)$.

Now, set $\overline{X} = \{\langle n, \overline{n}\rangle : \exists a\ n \Vdash a \in X\}$. Clearly, $\overline{X}$ is ω-stable under realizability. To complete the proof, it suffices to show that $\overline{X}$ is, in $\mathrm{V}(Kl)$, none other than $\mathrm{Dom}(X^{\Vdash})$. But this is straightforward. ∎

Here is our final generalization of CT_2:

3.9. Theorem. *In* $V(Kl)$, *if* A *and* B *are subcountable, and if* $h : A \to B$, *then for any canonical presentations* A^* *of* A *and* B^* *of* B *and functions* $f : A^* \twoheadrightarrow A$ *and* $g : B^* \twoheadrightarrow B$, *there is a partial recursive function* j *which agrees with* h *and preserves presentations. In other words,* h *is such that, for any* $n \in A^*$,

$$h(f(n)) = g(j(n)).$$

Proof. This is a direct consequence of the definition of canonicity and of Theorem 3.5. ∎

3.10. Note. (1) Admittedly, it is rather unhelpful, from the constructive standpoint, to think of partial functions as single-valued relations on $\omega \times \omega$. So conceived, the implicit connection between computability and functionality disappears entirely. One could well recapture the connection by internalizing the notion of partial function more carefully, that is, by trying to express in set-theoretic terms a bit more of the intuition behind *constructive* partial functions. One can reinstate the "Church's Thesis" connection by insisting that a function which is partial on ω is a *countable* two-place relation. There is no difficulty in showing that, with this more constructive notion of partial function, every partial function in $V(Kl)$ is precisely a partial recursive function.

(2) This would, of course, be the natural place to enter into a discussion of abstract analysis over $V(Kl)$. We could show that Brouwer's Theorem,

Every function from the reals into the reals is continuous

holds over $V(Kl)$. We would also want to realize generalizations of Brouwer's Theorem, e.g., that every function from a complete metric space into a separable metric space is continuous. Moreover, there is the work of Moschovakis (1964) on recursive metric spaces and any complete discussion would show how Moschovakis' theorems can be viewed under realizability on a natural class of constructive metric spaces.

Much of this material has already received a thorough treatment in the literature. Besides, the goal of our work is the charting of the less-traveled portions of recursive mathematics, RETs and isols. Consequently, we would prefer to push on to that and not to wander into real analysis. For information on analysis, we refer the reader to Beeson (1979) and Hyland (1982). ∎

Section 4: Interpreting arithmetic and Kleene realizability

4.1. Definition. *Heyting arithmetic*, HA, is the formal system which is conventionally accepted as the formalization of the first-order fragment of the intutionistic arithmetic. The language of HA is a single-sorted predicate language with $=$ as a logical sign and with, for each n, infinitely-many function parameters f_i^n of arity n. We abbreviate f_0^0 as $\overline{0}$ and f_0^1 as S, and take these as symbols for the zero element and for the successor function, respectively. In this language, Heyting arithmetic is axiomatized as is Peano arithmetic, using $\overline{0}$ and S, except that the underlying logic is generated not by CPL but by HPL. For each n, f_i^n is to represent F_i^n, the i-th primitive recursive function of arity n. In keeping with this idea, HA includes, for each f_i^n, an array of functions that give a primitive recursive axiomatization of F_i^n. ∎

There is close accord between arithmetic as interpreted over V(Kl) and the original realizability interpretation of Kleene (1945). For those who are *au fait* with the latter, there have been plenty of indications that this is the case. See, for instance, Lemma 3.9 of Chapter Three. For those who are not acquainted with Kleene's ideas, we provide a brief resumé.

Kleene realizability for arithmetic.

4.2. Definition. For sentences ϕ of the language of HA and numbers n, we specify recursively the arithmetic conditions under which n *Kleene realizes* ϕ. For (this notion of) Kleene realizability, we will write $n \Vdash^1 \phi$ to distinguish it from its set-theoretic generalization.

ϕ atomic, $n \Vdash^1 \phi$ iff $n = 0 \wedge \phi$ is true [1]

$n \Vdash^1 (\phi \wedge \psi)$ iff $n_0 \Vdash^1 \phi$ and $n_1 \Vdash^1 \psi$ [2]

$n \Vdash^1 (\phi \vee \psi)$ iff $n_0 = 0$ and $n_1 \Vdash^1 \phi$ or $n_0 \neq 0$ and $n_1 \Vdash^1 \psi$ [3]

$n \Vdash^1 \neg\phi$ iff, for all $m, \neg\, m \Vdash^1 \phi$ [4]

$n \Vdash^1 (\phi \to \psi)$ iff, for all m, if $m \Vdash^1 \phi$ then $\{n\}(m) \Vdash^1 \psi$ [5]

$n \Vdash^1 \exists n\, \phi$ iff $n_1 \Vdash^1 \phi[x/\overline{n_0}]$ [6]

$n \Vdash^1 \forall n\, \phi$ iff, for all m, $\{n\}(m) \Vdash \phi[x/\overline{m}]$ [7]

We refer to the structure of this interpretation as '$1(Kl)$'; and we shall say that

$$1(Kl) \models \phi \text{ iff } \exists n \,.\, n \Vdash^1 \phi.$$

∎

For a presentation of $1(Kl)$ and of Kleene realizability which is more heuristic, the reader can consult McCarty (1983).

The soundness theorem for HA with respect to $1(Kl)$ was proved by Kleene; an elegant presentation of the proof appears in the final chapter of Kleene (1952a).

4.3. Theorem. HA$\vdash \phi$ *only if* $1(Kl) \models \overline{\forall}\, \phi$.

Proof. This can be proved directly as an arithmetic analogue of our soundness proof for IZF with respect to $\mathbf{V}(Kl)$. Alternatively, we can first interpret HA into IZF and use the latter soundness theorem. It is to the alternative that we direct our efforts. ∎

In his (1971), Troelstra proved that, when classical logic is allowed into the metatheory, we can show that

4.4. Theorem. $1(Kl) \models \mathrm{ECT}_0 \wedge \mathrm{MP}^{\omega} \wedge \neg \mathrm{IP}^{\omega,\omega}$.

ECT_0 is the "Extended Church's Thesis" for arithmetic "almost negative" ϕ.

4.5. Definition. An formula of the language of HA is *almost negative* (*a.n.*) iff $\vee$ does not appear in ϕ and instances of $\exists n$ appear only as prefixed to atomic subformulae of ϕ. ∎

ECT_0 is expressed, for ϕ a.n., as

$$\forall n\, (\phi(n) \rightarrow \exists m\, \psi(n,m)) \rightarrow \exists e\, \forall n\, (\phi(n) \rightarrow \exists m,p\, [T(e,n,p) \wedge U(p,m) \wedge \psi(n,m)]).$$

MP^{ω} is Markov's Principle for specifically arithmetic properties. Let $\phi(n)$ be any formula from the language for HA; MP^{ω} is the scheme

$$\forall n\, (\phi(n) \vee \neg \phi(n)) \rightarrow (\neg\neg \exists n\, \phi(n) \rightarrow \exists n\, \phi(n)).$$

The reader will recall (from Chapter Three) that $\mathrm{IP}^{\omega,\omega}$ is the independence of premises principle with all quantifiers restricted to ω:

$$(\neg \phi \rightarrow \exists n\, \psi) \rightarrow \exists n\, (\neg \phi \rightarrow \psi(n)).$$

The most obvious route to a proof of the last theorem is the direct one: we can check that the realizability conditions set out above for $\Vdash^{1}$ hold (or fail to hold) on the principle in question. It would be more informative, however, to approach the theorem indirectly by way of a sketch of the logical relations between realizability over $1(Kl)$ and that over $V(Kl)$. The first step in the indirect route is, of course, a specification of an interpretation of the language of HA into that of IZF.

Interpreting arithmetic into set theory.

4.6. Note. The theorems of this section and the next, if proved in all their glorious detail, would become unbearably long. Rather than give detailed proofs, we will indicate how the relevant proofs can be constructed from contributory lemmas. ∎

4.7. Definition. (1) Let $x \to \overline{x}$ be an injection of the variables from HA into the variables of IZF.

(2) For each f_i^n, there is a p.r. function F_i^n to which f_i^n "refers". We also take 'F_i^n' to refer to the $n+1$-place relation which is the graph of F_i^n. With this understood, we let for each f_i^n, Ψ_i^n be the set-theoretic formula $(F_i^n)_{\acute{R}}$, given by the proof of the representability theorem. (Cf. Definition 3.5)

(3) We will translate the terms of HA into IZF by associating with each term τ an expression $\overline{\tau}$. If $\tau = x$ and x is a variable, then $\overline{\tau} = \overline{x}$. For terms $\tau = f_i^n(\tau_1, \tau_2)$, $\overline{\tau}$ is

$$\exists y_1, y_2 \in \omega\ (\overline{\tau_1} \wedge \overline{\tau_2} \wedge \Psi_i^n(y_1, y_2)).$$

We assume that the variables y_1 and y_2 have been chosen to avoid clashes. ∎

4.8. Definition. With each formula ϕ of the language of HA we associate a formula Φ of the language of IZF in the following way

(1) for $\phi = (\tau_1 = \tau_2), \Phi = \forall y \in \omega\ (\overline{\tau_1} \leftrightarrow \overline{\tau_2})$

(2) the translation commutes with $\wedge, \vee, \neg$, and $\rightarrow$

(3) for $\phi = \exists x\ \psi, \Phi = \exists \overline{x} \in \omega\ \Psi$

(4) for $\phi = \forall x\ \psi, \Phi = \forall \overline{x} \in \omega\ \Psi$

∎

Now we can state the main result of the section:

4.9. Theorem. *For sentences ϕ from the language of HA,*

$$1(Kl) \models \phi \text{ iff } \mathbf{V}(Kl) \models \Phi.$$

Given the above, the proof of the main theorem is straightforward and relies entirely on the proof of this stronger result:

4.10. Lemma. *For each formula $\phi(x_1, x_2)$ from the language of HA, there are partial recursive indices e_ϕ and g_ϕ such that, for all m_1, m_2,*

$$\text{if } p \Vdash^1 \phi(m_1, m_2), \text{ then } \{e_\phi\}(m_1, m_2, p) \Vdash \Phi(\overline{m_1}, \overline{m_2}) \text{ and}$$

$$\text{if } p \Vdash \Phi(\overline{m_1}, \overline{m_2}), \text{ then } \{g_\phi\}(m_1, m_2, p) \Vdash^1 \phi(m_1, m_2).$$

Proof. One uses Kleene absoluteness for the atomic case. The propositional connectives are trivial. For the quantifiers, use Lemma 3.9 from Chapter Three. ∎

On the basis of the main theorem, it can now be concluded that Kleene realizability has the properties discovered by Troelstra. To treat ECT_0, we apply the

4.11. Lemma. *If ϕ is a.n., then $\mathbf{V}(Kl) \models$ "Φ is ω-stable."*

Proof. In general, Φ defines over $\mathbf{V}(Kl)$ an n-place relation with $n \neq 1$; 'Φ is ω-stable' means that

$$\mathbf{V}(Kl) \models \forall x_1, \ldots, x_n \in \omega\, (\neg\neg\,\Phi(x_1, \ldots, x_n) \rightarrow \Phi(x_1, \ldots, x_n)).$$

The lemma is proved by structural induction on the a.n. formulae of HA. The only possible obstacle is the realizability of $\exists n$; for that, the realizability over $\mathbf{V}(Kl)$ of MP is required. ∎

4.12. Corollary. $1(Kl) \models \mathrm{ECT}_0 \wedge \mathrm{MP}^\omega \wedge \neg \mathrm{IP}^{\omega,\omega}$.

Proof. Given the lemma, the fact that $\mathbf{V}(Kl) \models \mathrm{CT}_2$ suffices to show that $1(Kl) \models \mathrm{ECT}_0$. One also notes that

$$\mathbf{V}(Kl) \models \mathrm{MP} \wedge \neg\, \mathrm{IP}^{\omega,\omega}.$$

∎

It follows immediately from the main theorem that all of set-theory arithmetic in IZF is Kleene realized over $1(Kl)$.

4.13. Corollary. *For ϕ a sentence of the language of* HA, *if* $\mathrm{IZF} \vdash \Phi$, *then* $1(Kl) \models \phi$.

Proof. ∎

We can now give our alternative proof of Kleene's soundness theorem for arithmetic, Theorem 4.3 above.

4.14. Corollary. $\mathrm{HA} \vdash \phi$ *only if* $1(Kl) \models \overline{\forall n}\, \phi$.

Proof. The proof now comes from the main theorem, the realizability soundness theorem for IZF and the remark that

$$\text{whenever } \mathrm{HA} \vdash \phi,\ \mathrm{IZF} \vdash \Phi.$$

∎

By the way, the former of the above corollaries, shows that, *inter alia*, $1(Kl) \models$ Consis(HA).

Section 5: Realizability for second-order arithmetic

5.1. Definition. *Second-order Heyting arithmetic*, HAS, is the accepted formalization of intuitionistic arithmetic with variables over species. The language appropriate to HAS is assumed to contain species variables for unary species only. This is no real restriction on the expressiveness of the language, since the usual codings of number-theoretic tuples are available. We also assume that, for each binary $A \subseteq \omega \times \omega$, there is a unary predicate constant A in the language; these will be the only predicate constants. Finally, we assume that the only well-formed formulae of the form $X(\tau)$ for X a species variable or constant are such that τ is an individual variable or canonical numeral. In this language, HAS has the same axiomatization as the corresponding version of full classical second-order arithmetic. ∎

Kreisel-Troelstra realizability.

Kreisel-Troelstra realizability is the first of the realizability interpretations presented in Troelstra (1973b). To our knowledge, this was the first time that intuitionistic quantifers were given what we have called a "generic" interpretation.

For sentences ϕ of the language of HAS and for natural numbers n, the second-order conditions under which n realizes ϕ in the sense of Kreisel and Troelstra are given recursively. When n realizes ϕ, we write '$n \Vdash^2 \phi$.'

5.2. Definition.

ϕ atomic first-order, $n \Vdash^2 \phi$	iff $n = 0 \wedge \phi$ is true	[1]
if $\phi = A(m), n \Vdash^2 \phi$	iff $\langle n, m\rangle \in A$	[2]
$n \Vdash^2 \exists X\, \phi$	iff for some $A \subseteq \omega \times \omega, n \Vdash^2 \phi[X/A]$	[3]
$n \Vdash^2 \forall X\, \phi$	iff for all $A \subseteq \omega \times \omega, n \Vdash^2 \phi[X/A]$	[4]
on $\vee, \wedge, \neg, \rightarrow, \exists n, \forall n$, $\Vdash^2$ agrees with $\Vdash^1$		[5].

The structure of this interpretation we call '2(*Kl*)' and, for sentences ϕ, we say that

$$2(Kl) \models \phi \text{ iff, for some } n,\ n \Vdash^2 \phi.$$

∎

Troelstra (1973b) contains a constructive proof that HAS is sound with respect to $\Vdash^2$.

5.3. Theorem. HAS $\vdash \phi$ *only if* $2(Kl) \models \overline{\forall}\, \phi$.

Proof. The standard proof proceeds by induction on the length of proofs from the axioms of HAS. Our proof goes via realizability for IZF, as in the last section, and is the subject of the following subsection.

Interpreting second-order arithmetic.

There is a translation $\phi \mapsto \Phi$ from the language of HAS into that of $\mathbf{V}(Kl)$, which we call '$\mathbf{L}_{Kl}$.' The translation extends that of the last section and mediates the logical transition from $\Vdash^2$ to $\Vdash$ and vice versa.

5.4. Definition. For each $A \subseteq \omega \times \omega$, let $\overline{A}$ be

$$\{\langle\langle m, n\rangle, n\rangle : \langle m, n\rangle \in A\}.$$

∎

In the clauses below, we have prevented variable clashes explicitly by formulating IZF with the usual variables $x, y, z, \ldots$ plus a disjoint set of new variables $x^X, x^Y, \ldots$, one x^X for each second-order variables X of HAS.

5.5. Definition. With each formula ϕ from the language of HAS, we associate a formula Φ of $\mathbf{L}_{Kl}$ so that, for ϕ first-order, Φ is just that given by the translation of Definition 4.8 from the preceding section and

[1] for $\phi = Y(n), \Phi = \overline{n} \in x^Y$

[2] for $\phi = Y(x), \Phi = x \in x^Y$

[3] for $\phi = A(n), \Phi = \overline{n} \in A$

[4] for $\phi = A(x), \Phi = x \in A$

[5] for $\phi = \forall X\, \psi, \Phi = \forall x^X \in \mathbf{P}(\omega)\, \Psi$

[6] for $\phi = \exists X\, \psi, \Phi = \exists x^X \in \mathbf{P}(\omega)\, \Psi$.

∎

With $\phi \mapsto \Phi$ as above, the following is easily provable.

5.6. Theorem. *For sentences ϕ of the language of HAS,*

$$2(Kl) \models \phi \text{ iff } \mathbf{V}(Kl) \models \Phi.$$

Proof. As with realizability $\Vdash^1$, the theorem is a consequence of a more inclusive result for arbitrary formulae:

5.7. Lemma. *For each formula $\phi(x, X)$ of the language of HAS, there are partial recursive indices e_ϕ and g_ϕ such that, for $m \in \omega$ and $A \subseteq \omega \times \omega$,*

$$\text{if } p \Vdash^2 \phi(m, A), \text{ then } \{e_\phi\}(m,p) \Vdash \Phi(\overline{m}, \overline{A}) \text{ and}$$

$$\text{if } p \Vdash \Phi(\overline{m}, \overline{A}), \text{ then } \{g_\phi\}(m,p) \Vdash^2 \phi(m, A).$$

Proof. The proof of the lemma is by induction on the complexity of the formulae ϕ. For the quantifier cases of the induction, we require a lemma.

5.8. Lemma. *There is a $j \in \omega$ such that, for all $e \in \omega$ and $b \in \mathbf{V}(Kl)$, if $e \Vdash b \in \mathbf{P}(\omega)$, then there is an $A \subseteq \omega \times \omega$ such that $\{j\}(e) \Vdash b = \overline{A}$.*

The theorem of Troelstra, Theorem 5.3, is an immediate consequence of a corollary to Theorem 5.6.

5.9. Corollary. *For sentences ϕ of* HAS, *if* $\text{IZF} \vdash \Phi$, *then* $2(Kl) \models \phi$.

EXT is the principle of "extensionality for species":

$$\forall n, m\, ((Xn \land n = m) \rightarrow Xm).$$

UP^2 is the Uniformity Principle as it would naturally by formulated in a second-order language:

$$\forall X\, \exists n\, \phi \rightarrow \exists n\, \forall X\, \phi.$$

In the same paper, (1973b), Troelstra also showed that

5.10. Theorem. $2(Kl) \models \text{EXT} \land \text{UP}^2$.

Proof. The result is an immediate consequence of the Theorem 5.6 and the theorems of Chapter Three. ∎

Finally, if we use the soundness theorem for IZF with respect to $\mathbf{V}(Kl)$, we can show that all of set-theoretic second-order arithmetic is realized in the Kreisel-Troelstra sense:

We know, therefore, that $2(Kl) \models$ Consis(HAS).

Realizability and Recursive Set Theory

Section 1: Prefatory and historical remarks

This chapter is the centerpiece of our work; as suggested by later sections of Chapter Zero, its intention is twofold. First, we hope to exhibit what should now be an unsurprizing correspondence: that the traditional classical objects and structures of recursive set theory correspond exactly to certain nontraditional sets definable in pure set theory over $\mathrm{V}(Kl)$. Despite the initial strangeness of $\mathrm{V}(Kl)$, the correspondence is perfectly natural; the realizability set in $\mathrm{V}(Kl)$ associated with a given "recursive" set from the classical universe satisfies the condition one would obtain by removing from the classical definition of the latter any explicit reference to recursion. Second, and more significantly, this ontological congruence underlies a parallel congruence between theories in recursive mathematics and subtheories of $\mathrm{V}(Kl)$. Just as the congruence in ontology marks a philosophical economy, the congruence in theory promises a nonnegligible mathematical economy, not only in mathematical formulations and proofs, but also in our conception of the resources of constructivism. In particular, the latter correspondence tells us that proofs in pure constructive set theory become, under interpretation, proofs in the classical development of recursive mathematics and without *slippage*. By 'without slippage,' we mean that there is no classical truth about recursive set theory that cannot, in principle, be recaptured over $\mathrm{V}(Kl)$.

First, we will treat the correspondence between classical structures and realizability sets. We prove that there is a natural injection $x \mapsto \overline{x}$ from the classical ground model V into $\mathrm{V}(Kl)$ such that, if A is a member of any of the structures in the column on the left below, then $\overline{A}$ is an object which, over $\mathbf{V}(Kl)$, satisfies the respective description in the column on the right:

recursive equivalence types	cardinal numbers in $\mathbf{P}(\omega)^{st}$
isols	D-finite cardinals in $\mathbf{P}(\omega)^{st}$
isolic integers	integers generated from D-finites

In moving from left to right between the columns, the recursion-theoretic content of each classical notion is absorbed into the logical or set-theoretic aspects of the notion to its right. In the right column, the recursion theory is confined entirely to the interpretation of logic and set theory over $\mathbf{V}(Kl)$. A recursive equivalence type, for example, becomes a Cantorean cardinal number over the ω-stable subsets of $\mathbf{P}(\omega)$.

From their very conception, the isols have been thought of as the rough analogues, relative to recursive correspondences, of the Dedekind finite cardinals of choice-free classical set theory. Interpretation over $\mathbf{V}(Kl)$ not only makes that this conception perfectly correct, but also uncovers the precise mathematical facts upon which the idea rests. Once the recursion theory is "built into" the interpretations of the logical signs, isols reappear as Dedekind finite cardinals *simpliciter*.

For each of the cited correspondences, we prove that there is an isomorphism theorem with the consequence that the classical first-order theory of each of the structures on the left of the diagram is virtually identical to the $\mathbf{V}(Kl)$-elementary theory of its correspondent on the right. By 'virtually,' we mean that no real adjustment need be made to go from the classical theory to its realizability correlate. Universal quantification is the only logical operation not srictly invariant under the translation, and, if one is willing to adopt a specific understanding of $\forall$, even this difficulty is obviated. This shows that, e.g., although the usual first-order structure of the isols fails to be a cardinal algebra in the classical sense, it is precisely the cardinal algebra of a salient subset of $\mathbf{P}(\omega)$ within constructive logic.

This translation yields a simple and effective routine for passing between the relevant theories; under translation, a theorem in pure set theory over $\mathbf{V}(Kl)$ comes to express a classical truth about, e.g., the isols. At the same time, any of the numerous "impossibility" results of recursive set theory go effectively into strong independence results for cardinal arithmetic in IZF (not to mention IZF+MP+ECT$_0$+AC$^{\omega,X}$+EUP).

A short sampler of possible independence results will be canvassed at the end of this chapter. There, structure theorems on the "size" of the collection of RETs and Dedekind-finites will be employed, under translation, to give results on the cardinality and internal workings of $P(\omega)$ in $V(Kl)$. The reader may think of this last section as an answer to a natural question, "Now that we know about the straightforward correspondences between the first-order theories of recursion-theoretic and of realizability structures, what can we do for the theorems about these structures which are not first-order expressible over them?" After perusing the last section, the reader will see that the answer must be "Provided that certain uniformity conditions hold, useful versions of these theorems carry also carry over into $V(Kl)$."

As we mentioned in section 4 of Chapter Zero, intimations of the existence of the correspondences and translations have surfaced occasionally in the history of the subject, most notably in the review (1968) of Kreisel. The remarks of Kreisel, together with suggestions from Dana Scott, encouraged us to attempt the theorems of this chapter; all the results documented here were obtained in January 1982.

Section 2: Fundamental correspondences

Rather than rushing into the proofs of the most general results that govern the possible correspondences between recursive and realizability mathematics, we prefer to work straight through the correspondence theorems for a specific case. The case we have in mind concerns the relation between Ω, the collection of recursive equivalence types (RETs), and the cardinals of ω-stable $\mathbf{P}(\omega)$ in $\mathbf{V}(Kl)$. The mathematical materials needed to treat the other cases, the isols and the isolic integers, can be extracted directly from this one.

It is interesting to note that the main ideas of the isomorphism theorem are, in essence, the hereditary descendants of two old ideas which have been somewhat neglected in recent work on realizability. The first old idea appears as a brief aside in the original realizability paper, Kleene (1945). Kleene noted that, as far as propositional logic is concerned, the logic of his realizability structure (when treated classically on the outside) is just classical logic itself. In fact, although this was not noted by Kleene, this property of classical realizability extends to the $\forall$–free fragment of classical predicate logic. This idea is heavily exploited in the proof of our isomorphism theorem. The second idea is that of *Kleene absoluteness*: there is an effective invariance theorem for the recursive predicates under realizability.

Stability.

It is worthwhile taking a moment to recall the definition of recursive equivalence type à la Dekker and Myhill:

2.1. Definition.

In $\mathbf{V}$, two subsets A and B of ω represent the same RET iff there is a partial recursive function p such that $A \subseteq \mathrm{Dom}(p)$, p is injective and p takes A onto B. When A and B represent the same RET, we write '$A \simeq B$' and, when p is responsible for the relation, '$p : A \simeq B$.' ▮

There is an injection $x \mapsto \overline{x}$ from $\mathbf{P}(\omega)$ in $\mathbf{V}$ into $\mathbf{P}(\omega)$ in $\mathbf{V}(Kl)$ such that $A \simeq B$ in $\mathbf{V}$ iff $\mathbf{V}(Kl) \models \overline{A} \approx \overline{B}$, where the latter relation, $\approx$, is cardinal-theoretic equivalence. First, we specify the injection, which is our old friend from Chapter Three:

2.2. Definition. For $A \subseteq \omega$, $\overline{A} = \{\langle n, \overline{n}\rangle : n \in A\}$. ▮

Since $\mathbf{V}(Kl)$ is so liberal in its admission standards, it is obvious that, for every A, $\overline{A} \in \mathbf{V}(Kl)$. For purposes at hand, we also need the more specific information that $\overline{A}$ satisfies, in $\mathbf{V}(Kl)$, the conditions "$\overline{A} \subseteq \overline{\omega}$" and "$\overline{A}$ is ω-stable."

A short sampler of possible independence results will be canvassed at the end of this chapter. There, structure theorems on the "size" of the collection of RETs and Dedekind-finites will be employed, under translation, to give results on the cardinality and internal workings of $P(\omega)$ in $V(Kl)$. The reader may think of this last section as an answer to a natural question, "Now that we know about the straightforward correspondences between the first-order theories of recursion-theoretic and of realizability structures, what can we do for the theorems about these structures which are not first-order expressible over them?" After perusing the last section, the reader will see that the answer must be "Provided that certain uniformity conditions hold, useful versions of these theorems carry also carry over into $V(Kl)$."

As we mentioned in section 4 of Chapter Zero, intimations of the existence of the correspondences and translations have surfaced occasionally in the history of the subject, most notably in the review (1968) of Kreisel. The remarks of Kreisel, together with suggestions from Dana Scott, encouraged us to attempt the theorems of this chapter; all the results documented here were obtained in January 1982.

Section 2: Fundamental correspondences

Rather than rushing into the proofs of the most general results that govern the possible correspondences between recursive and realizability mathematics, we prefer to work straight through the correspondence theorems for a specific case. The case we have in mind concerns the relation between Ω, the collection of recursive equivalence types (RETs), and the cardinals of ω-stable $\mathbf{P}(\omega)$ in $\mathrm{V}(Kl)$. The mathematical materials needed to treat the other cases, the isols and the isolic integers, can be extracted directly from this one.

It is interesting to note that the main ideas of the isomorphism theorem are, in essence, the hereditary descendants of two old ideas which have been somewhat neglected in recent work on realizability. The first old idea appears as a brief aside in the original realizability paper, Kleene (1945). Kleene noted that, as far as propositional logic is concerned, the logic of his realizability structure (when treated classically on the outside) is just classical logic itself. In fact, although this was not noted by Kleene, this property of classical realizability extends to the $\forall$-free fragment of classical predicate logic. This idea is heavily exploited in the proof of our isomorphism theorem. The second idea is that of *Kleene absoluteness*: there is an effective invariance theorem for the recursive predicates under realizability.

Stability.

It is worthwhile taking a moment to recall the definition of recursive equivalence type à la Dekker and Myhill:

2.1. Definition.

In $\mathbf{V}$, two subsets A and B of ω represent the same RET iff there is a partial recursive function p such that $A \subseteq \mathrm{Dom}(p)$, p is injective and p takes A onto B. When A and B represent the same RET, we write '$A \simeq B$' and, when p is responsible for the relation, '$p : A \simeq B$.' ∎

There is an injection $x \mapsto \overline{x}$ from $\mathbf{P}(\omega)$ in $\mathbf{V}$ into $\mathbf{P}(\omega)$ in $\mathrm{V}(Kl)$ such that $A \simeq B$ in $\mathbf{V}$ iff $\mathrm{V}(Kl) \models \overline{A} \approx \overline{B}$, where the latter relation, $\approx$, is cardinal-theoretic equivalence. First, we specify the injection, which is our old friend from Chapter Three:

2.2. Definition. For $A \subseteq \omega$, $\overline{A} = \{\langle n, \overline{n}\rangle : n \in A\}$. ∎

Since $\mathrm{V}(Kl)$ is so liberal in its admission standards, it is obvious that, for every A, $\overline{A} \in \mathrm{V}(Kl)$. For purposes at hand, we also need the more specific information that $\overline{A}$ satisfies, in $\mathrm{V}(Kl)$, the conditions "$\overline{A} \subseteq \overline{\omega}$" and "$\overline{A}$ is ω-stable."

2.3. Lemma.

(1) $\mathrm{V}(Kl) \models \overline{A} \subseteq \overline{\omega}$ *and*

(2) $\mathrm{V}(Kl) \models \overline{A} \in \mathrm{P}(\omega)^{st}$.

Proof. Actually, the proof we give will be of a stronger result. We show that a realizability witness for each of the above statements is strictly uniform over all $A \in \mathrm{P}(\omega)$.

(1) $e \Vdash a \in \overline{A}$ only if $e_1 \Vdash a = \overline{e_0}$ and $e_0 \in A$. Hence, $e \Vdash a \in \overline{A}$ implies that $e \Vdash a \in \overline{\omega}$. Therefore, $\Lambda x.x \Vdash \overline{A} \subseteq \overline{\omega}$.

(2) Assume that $e_0 \Vdash a \in \overline{\omega}$ and $e_1 \Vdash \neg\neg a \in \overline{A}$. Then,

$$e_{01} \Vdash a = \overline{e_{00}} \text{ and } \exists m\, .m \Vdash a \in \overline{A}.$$

From the latter it follows that there is a witness m such that $m_1 \Vdash a = \overline{m_0}$ and $m_0 \in A$. Now, by the absoluteness of equality on ω, $m_0 = e_{00}$ and $e_{00} \in A$. Consequently, $e_0 \Vdash a \in \overline{A}$ and $\Lambda x.x_0$ will realize that $\overline{A}$ is stable. ∎

The cardinality correspondence.

2.4. Note. As is our practice, we fix *ab initio* an $i \in \omega$ to serve as a witness for self-identity over $\mathrm{V}(Kl)$. ∎

It is a tedious business to check that $\mathrm{V}(Kl)$ mediates the predicted relations between RETs and cardinals, so we have divided the work of the following lemma up into a series of parts.

2.5. Lemma. $A \simeq B$ *in* V *iff* $\mathrm{V}(Kl) \models \overline{A} \approx \overline{B}$.

Proof. In Part (1), we will prove the implication from left to right, and, in Part (2), we deal with the converse. Each part has, in turn, three subparts, (a), (b) and (c).

(1) Given $p : A \simeq B$, we embed p into $\mathrm{V}(Kl)$ as

$$\overline{p} = \{\langle n, \overline{\langle \overline{n}, \overline{m} \rangle} \rangle : p(n) \simeq m \wedge n \in A\}.$$

It is no more than routine to check that, over $\mathrm{V}(Kl)$, $\overline{p}$ is a single-valued relation. It only remains to check that $\overline{p}$ is injective and takes $\overline{A}$ onto $\overline{B}$.

(a) First, we check that $\mathrm{V}(Kl) \models \forall x \in \overline{A}\, \exists y \in \overline{B}\, (\langle x, y\rangle \in \overline{p})$. We take an e such that $e \Vdash a \in \overline{A}$. Then, $e_1 \Vdash a = \overline{e_0}$ and $e_0 \in A$. By the definition of $\overline{p}$, we know that

$$\langle e_0, i\rangle \Vdash \overline{\langle \overline{e_0}, \overline{p(e_0)}\rangle} \in \overline{p} \qquad \text{and } \langle p(e_0), i\rangle \Vdash \overline{p(e_0)} \in \overline{B}.$$

Therefore,

$$\langle\langle p(e_0), i\rangle, \langle e_0, i\rangle\rangle \Vdash \exists y \in \overline{B}\, (\overline{\langle \overline{e_0}, y\rangle} \in \overline{p}).$$

(b) Next, we see that $\mathrm{V}(Kl) \models \overline{p} : \overline{A} \rightarrowtail \overline{B}$. Assume that

$$h \Vdash \overline{\langle a, c\rangle} \in \overline{p} \wedge \overline{\langle b, c\rangle} \in \overline{p}.$$

From the definition of $\overline{p}$, we know that

$$h_{01} \Vdash \overline{\langle a, c\rangle} = \overline{\langle \overline{h_{00}}, \overline{p(h_{00})}\rangle} \text{ while } h_{11} \Vdash \overline{\langle b, c\rangle} = \overline{\langle \overline{h_{10}}, \overline{p(h_{10})}\rangle}.$$

By the properties of pairing and by the absoluteness of equality on $\overline{\omega}$, $h_{00} = h_{10}$. Hence, a witness for $a = b$ is obtainable effectively from h and from the fixed witness i.

(c) Finally, we can see that $\overline{p}$ is realizably onto $\overline{B}$, i.e., that

$$\mathrm{V}(Kl) \models \forall x \in \overline{B}\, \exists y \in \overline{A}\, (\langle y, x\rangle \in \overline{p}).$$

But this proof is really the same as that of (a). One need only use the partial recursive inverse to p in place of p to carry information across the implication.

(a), (b) and (c) together show that if $p : A \simeq B$, then $\mathrm{V}(Kl) \models \overline{A} \approx \overline{B}$. Just for the record, we note that the witnesses calculated here depend only on an index for p and neither on A nor on B.

(2) This time, we begin with the assumption that $\mathrm{V}(Kl) \models \overline{A} \approx \overline{B}$. This means that there is a witness for

$$\exists f \in (\overline{A} \Rightarrow \overline{B})\, f : \overline{A} \rightarrowtail\!\!\!\twoheadrightarrow \overline{B}.$$

It follows that there are numbers e and g such that

$$e \Vdash \forall x \in \overline{A}\, \exists y \in \overline{B}\, \overline{\langle x, y\rangle} \in f \text{ and}$$
$$g \Vdash \forall x \in \overline{B}\, \exists y \in \overline{A}\, \overline{\langle y, x\rangle} \in f.$$

Now, we shift attention to V and set

$$\underline{e} = \{\langle n, \{e\}(\langle n, i\rangle)_{00}\rangle : \{e\} \downarrow (\langle n, i\rangle)\} \text{ and}$$
$$\underline{g} = \{\langle n, \{g\}(\langle n, i\rangle)_{00}\rangle : \{g\} \downarrow (\langle n, i\rangle)\}.$$

$\underline{e}$ and $\underline{g}$ are partial recursive and one would like to prove that they can be "glued together" in the appropriate way to define a p (in V) such that $p : A \simeq B$.

(a) The clauses of the definition of realizability that govern the components of the statements realized by e and by g show that $\underline{e}$ is defined on A and $\underline{g}$ on B.

(b) One sees easily that, if $n \in A$, then $\underline{g}\,\underline{e}(n) = n$: For $n \in A$, $\langle n, i\rangle \Vdash \overline{n} \in \overline{A}$, so

$$\exists b\ \{e\}(\langle n, i\rangle) \Vdash b \in \overline{B} \wedge \overline{\langle \overline{n}, b\rangle} \in f.$$

Consequently,

$$\{e\}(\langle n, i\rangle)_{01} \Vdash b = \overline{\underline{e}(n)} \text{ and } \mathrm{V}(Kl) \models \overline{\langle \overline{n}, \overline{\underline{e}(n)}\rangle} \in f.$$

Now, working the same line (but in reverse) on the second statement, we obtain

$$\langle \underline{e}(n), i\rangle \Vdash \overline{\underline{e}(n)} \in \overline{B}\,, \text{ and } \mathrm{V}(Kl) \models \overline{\langle \overline{\underline{g}\,\underline{e}(n)}, \overline{\underline{e}(n)}\rangle} \in f.$$

Since $\mathrm{V}(Kl) \models f$ is injective, $n = \underline{g}\,\underline{e}(n)$.

(c) By parity of reasoning, for $n \in B$, we have that $\underline{e}\,\underline{g}(n) = n$. Now, we know that $\underline{e}$ is injective on (at least) all of A. Therefore,

$$\underline{e} \restriction \{n : \underline{g}\,\underline{e}(n) \simeq n\}$$

is our candidate for p; it is a partial recursive function, it is injective and it takes A onto B.

This completes the proof of part (2), and, together with part (1), it provides a proof of the lemma. ∎

Operations on Ω.

Now that we have in hand an exact correspondence between the atomic statements over Ω and those over $\mathrm{P}(\omega)^{st}$ in $\mathrm{V}(Kl)$, we proceed just as if we were to use the injection $x \mapsto \overline{x}$ to prove that a classical isomorphism holds between the algebraic structure of the RETs and the cardinal algebra on $\mathbf{P}(\omega)^{st}$. The lemma above gives the basis of the isomorphism; it shows that the injection preserves the relevant equality relations "on the nose." The lemma to follow shows that the recursion-theoretic operations of $+$ and $\times$ on Ω are mirrored precisely by the set-theoretically defined operations of the cardinal arithmetic. Prerequisite to this is a reprise of the definitions of $+$ and $\times$ on Ω.

2.6. Definition. For A, $B \in \Omega$,

(1) $A+B=\{\langle 0,n\rangle : n\in A\}\cup\{\langle 1,m\rangle : m\in B\}$.

(2) $A\times B=\{\langle n,m\rangle : n\in A\wedge m\in B\}$.

Here, $\langle\ ,\rangle$ represents primitive recursive number-theoretic pairing. ∎

Just as in standard model-theoretic isomorphism theorems, one proves that both $+$ and $\times$ commute with the injection $\overline{x}$. As you would expect, the '$+$' and '$\times$' on the left of the equalities in the lemma statement below are the natural operations of cardinal addition and of multiplication as defined set-theoretically over $\mathbf{P}(\omega)^{st}$.

2.7. Lemma.

(1) $\mathrm{V}(Kl)\models \overline{A}+\overline{B}=\overline{A+B}$.

(2) $\mathrm{V}(Kl)\models \overline{A}\times\overline{B}=\overline{A\times B}$.

Proof. *Ad* (1). First, we prove that $\mathrm{V}(Kl)$ satisfies

$$\overline{A}+\overline{B}=\{\langle\overline{0},x\rangle : x\in\overline{A}\}\cup\{\langle\overline{0},y\rangle : y\in\overline{B}\}\subseteq\overline{A+B}.$$

We will rely heavily on the absoluteness of the recursive relations. On the one hand, if $n \Vdash a\in\overline{A}+\overline{B}$, then

$$\begin{array}{l}\text{either } n_0=0 \text{ and } n_1 \Vdash \exists y\in\overline{A}\,(a=\langle\overline{0},y\rangle)\\ \text{or } n_0\neq 0 \text{ and } n_1 \Vdash \exists y\in\overline{B}\,(a=\langle\overline{0},y\rangle).\end{array}$$

In the first case, one can calculate, effectively from n, numbers m and p such that

$$m \Vdash a=\langle\overline{0},\overline{p}\rangle,$$

where $p\in A$. By the absoluteness of primitive recursive pairing,

$$\mathrm{V}(Kl)\models\overline{\langle 0,p\rangle}=\langle\overline{0},\overline{p}\rangle,$$

with witness calculable from p. Therefore, there is no loss of generality in claiming that

$$m \Vdash a = \overline{\langle 0, p\rangle}, \text{ and } \langle\langle 0, p\rangle, i\rangle \Vdash \overline{\langle 0, p\rangle} \in \overline{A+B}.$$

Of course, we would come to the same conclusion in the case $n_0 \neq 0$. It follows that $V(Kl) \models \overline{A} + \overline{B} \subseteq \overline{A+B}$.

On the other hand, if $n \Vdash a \in \overline{A+B}$, then

$$n_1 \Vdash a = \overline{\langle n_{00}, n_{01}\rangle},$$

where either $n_{00} = 0$ and $n_{01} \in A$, or $n_{00} = 1$ and $n_{01} \in B$. By the absoluteness of the recursive relations, there is a p, calculable from n, such that

$$p \Vdash \overline{\langle n_{00}, n_{01}\rangle} = \langle \overline{n_{00}}, \overline{n_{01}}\rangle.$$

Hence, either

$$n_{00} = 0 \text{ and } \langle n_{01}, i\rangle \Vdash \overline{n_{01}} \in \overline{A} \text{ or}$$
$$n_{00} = 1 \text{ and } \langle n_{01}, i\rangle \Vdash \overline{n_{01}} \in \overline{B}.$$

Consequently, there is a partial recursive Θ such that $\Theta(n)$ realizes

$$(\exists b\; a = \langle \overline{0}, b\rangle \wedge b \in \overline{A}) \vee (\exists b\; a = \langle \overline{1}, b\rangle \wedge b \in \overline{B}).$$

Hence,

$$\overline{A+B} \subseteq \overline{A} + \overline{B}$$

holds in $V(Kl)$ and we are done.

The proof of (2) is virtually identical to that of (1) and is omitted. ∎

The order on Ω.

There is no reason to exclude from consideration the canonical partial order on the RETs, since it comes out so naturally over $V(Kl)$. Strong recursion-theoretic inclusion on the RETs coincides, over $V(Kl)$, with the traditional intuitionistic notion of "decidable subset."

2.8. Definition. For B, $C \in \mathbf{P}(\omega)$, $C \subseteq B$ is *partial recursive* on B iff there is a partial recursive f such that $\mathrm{Dom}(f) \supseteq B$, $\mathrm{Ran}(f) \subseteq \{0,1\}$ and $\forall x \in B\ (x \in C \text{ iff } f(x) = 0)$. In this situation, we say that f *decides* C on B. ∎

2.9. Definition. For A, $B \in \Omega$, $A \leq B$ iff there is a subset C of B, C partial recursive on B, such that $A \simeq C$. ∎

The set-theoretic construct that fits into the analogy with partial recursivity for subsets is that of decidability (Brouwer's "removability") for subsets. See section 4 of Chapter One. The availability in intuitionistic mathematics of the latter notion motivates our (perhaps remarkable) interest in the notion $\leq$ for RETs.

2.10. Definition. When $A \approx C$ and C is a decidable subset of B, we will write $A \preceq B$. ∎

2.11. Lemma. *$A \leq B$ in $\mathbf{V}$ iff $\mathbf{V}(Kl) \models \overline{A} \preceq \overline{B}$.*

Proof. Once again, we have divided the proof up into two parts, one for each half of the 'if and only if.' We will consider the first part now and return to the second following an interlude on quantification.

(1) If $A \leq B$ in $\mathbf{V}$, there is a $C \in \mathbf{P}(\omega)$ such that $A \simeq C$ and C is partial recursive on B. From the previous lemmas, we know that $\mathbf{V}(Kl) \models \overline{A} \approx \overline{C}$. It suffices to show that $\mathbf{V}(Kl) \models \overline{C}$ is a decidable subset of $\overline{B}$. Let e index a partial recursive function deciding C on B. For $n \in B$, either $\{e\}(n) = 0$ and $n \in C$ or $\{e\}(n) = 1$ and $n \notin C$. We look to construct from e a witness for

$$\forall x \in B\ (x \in C \vee x \notin C)$$

Let $n \Vdash a \in B$. Then, $n_1 \Vdash a = n_0$ with $n_0 \in B$. It is a consequence of this that,

$$\text{either } \{e\}(n_0) = 0 \text{ and } \langle n_0, i\rangle \Vdash \overline{n_0} \in \overline{C}$$
$$\text{or } \{e\}(n_0) \neq 0 \text{ and } 0 \Vdash \neg\, \overline{n_0} \in \overline{C}$$

Therefore, there is a partial recursive Θ such that $n \Vdash a \in \overline{B}$ imples that $\Theta(n)\downarrow$ and either

$$\{e\}(n_0) = 0 \text{ and } \Theta(n) \Vdash a \in \overline{C}$$
$$\text{or } \{e\}(n_0) \neq 0 \text{ and } \Theta(n) \Vdash a \notin \overline{C}.$$

That is,

$$\mathbf{V}(Kl) \models \overline{C} \text{ is a decidable subset of } \overline{B}.$$

This concludes the first half of the theorem. ∎

The second half of the theorem is marginally more difficult, because of the need to attend to quantification over $\mathbf{P}(\omega)$ in $\mathbf{V}(Kl)$. We will break off the main line of proof to present some lemmas on quantification before returning to the proof proper. This is hardly wasted effort; the lemmas we prove here are essential to discerning the relation between quantification in Ω and quantification over $\mathbf{P}(\omega)^{st}$.

2.12. Lemma. *If* $\mathbf{V}(Kl) \models A \in \mathbf{P}(\omega)^{st}$, *then there is a* $B \in \mathbf{V}$ *such that* $\mathbf{V}(Kl) \models A = \overline{B}$.

Proof. Assuming that $\mathbf{V}(Kl) \models A \in \mathbf{P}(\omega)^{st}$, we have, by definition, that

$$\mathbf{V}(Kl) \models A \subseteq \overline{\omega} \wedge \forall x \in \overline{\omega}\, (\neg\neg x \in A \rightarrow x \in A).$$

In particular, assume that $g \Vdash \forall x \in \overline{\omega}\, (\neg\neg x \in A \rightarrow x \in A)$. Consider

$$\underline{A} = \{n : \mathbf{V}(Kl) \models \overline{n} \in A\}.$$

Our claim is that $\mathbf{V}(Kl) \models A = \overline{\underline{A}}$. (This means that, if $\mathbf{V}(Kl)$ sees that the membership condition on A is ω-stable, then A is equal in $\mathbf{V}(Kl)$ to the re-embedding of the set of its "realizability members.") To see that this is so, let $e_1 \Vdash \overline{e_0} \in A$. Immediately, $\langle e_0, i\rangle \Vdash \overline{e_0} \in \overline{\underline{A}}$. Hence, regardless of the stability of A, $\mathbf{V}(Kl) \models A \subseteq \overline{\underline{A}}$. Conversely, if $e \Vdash a \in \overline{\underline{A}}$ then

$$e_1 \Vdash a = \overline{e_0} \text{ and } \mathbf{V}(Kl) \models \overline{e_0} \in A.$$

Then, $\langle e_0, i\rangle \Vdash \overline{e_0} \in \overline{\omega}$, while

$$0 \Vdash \neg\neg\overline{e_0} \in A, \text{ and } \{g\}(\langle\langle e_0, i\rangle, 0\rangle) \Vdash \overline{e_0} \in A.$$

Therefore,

$$\mathbf{V}(Kl) \models \overline{\underline{A}} \subseteq A.$$

∎

To some extent, this lemma explains our fascination with the stable subsets in $\mathbf{P}(\omega)$. A less self-interested explanation for the fascination appears in Section 3 of this Chapter.

2.13. Lemma. IZF $\vdash$ (*If* $C \in \mathbf{P}(\omega)$ *and* $B \in \mathbf{P}(\omega)^{st}$ *and* C *is a decidable subset of* B, *then* $C \in \mathbf{P}(\omega)^{st}$).

Proof. We work in IZF. Let $n \in \omega$ and assume that $\neg\neg n \in C$. Since $C \subseteq B$, $\neg\neg n \in B$. $B \in \mathbf{P}(\omega)^{st}$, so $n \in B$. C is decidable on B and $\neg\neg n \in C$, so $n \in C$. ∎

2.14. Corollary. *If* $\mathbf{V}(Kl) \models (C \in \mathbf{P}(\omega) \wedge C$ *is a decidable subset of* $\overline{B})$, *then* $\mathbf{V}(Kl) \models$ ($\overline{C}$ *is a decidable subset of* $\overline{B}$).

Proof. This follows directly from the above and from the soundness theorem for IZF with respect to realizability. ∎

We return now to the proof of Lemma 2.11. Assume that $\mathbf{V}(Kl) \models \overline{A} \preceq \overline{B}$. Then,

$$\exists C \in \mathbf{V}(Kl)\,.\mathbf{V}(Kl) \models C \in \mathbf{P}(\omega)^{st} \wedge \overline{A} \approx C \wedge C \text{ is a decidable subset of } \overline{B}.$$

From the corollary, we have that, for some C in $\mathbf{P}(\omega)$,

$$\mathbf{V}(Kl) \models \overline{A} \approx \overline{C} \wedge \overline{C} \text{ is a decidable subset of } \overline{B}.$$

From the previous lemmas, $A \simeq C$ in $\mathbf{V}$, and it's obvious that $C \subseteq B$. It remains only to show that C is partial recursive on B. If $\mathbf{V}(Kl) \models \overline{C}$ is a decidable subset of $\overline{B}$, then, for some e,

$$e \Vdash \forall x \in \overline{B}\,(x \in \overline{C} \vee x \notin \overline{C}).$$

Then, if $n \in B$, $\langle n, i\rangle \Vdash \overline{n} \in B$, so

$$\text{either } \{e\}(\langle n,i\rangle)_0 = 0 \text{ and } \{e\}(\langle n,i\rangle)_1 \Vdash \overline{n} \in \overline{C}$$
$$\text{or } \{e\}(\langle n,i\rangle)_0 \neq 0 \text{ and } \{e\}(\langle n,i\rangle)_1 \Vdash \overline{n} \notin \overline{C}.$$

Clearly, if $n \in C$, then $\mathbf{V}(Kl) \models \overline{n} \in \overline{C}$. Therefore, $\{e\}(\langle n,i\rangle)_0$ decides C on B. ∎

Section 3: The Isomorphism Theorem

The next (and final) stage in our plan is to exploit Kleene's remarks on the logic of classical realizability to prove that Ω in V and the cardinal structure on $\mathbf{P}(\omega)^{st}$ in $\mathbf{V}(Kl)$ are isomorphic "up to stability."

Let L_Ω be a first-order language one of whose realizations is the algebraic structure on the RETs:

$$\underline{\Omega} = \langle \Omega, +, \times, \leq, \simeq, 0, \omega, A\rangle_{A\in\Omega}.$$

We assume that L_Ω contains a distinct constant A for each $A \in \Omega$. For each $\phi \in \mathrm{Form}_{\mathrm{L}_\Omega}$, we define its translation tr into the language of ZF as follows, beginning with the terms of L_Ω.

3.1. Definition.

(1) $0^{tr} = \overline{0}$, $\omega^{tr} = \overline{\omega}$, $A^{tr} = \overline{A}$.

(2) For $\tau, \sigma \in \mathrm{Term}_{\mathrm{L}_\Omega}$, $(\tau+\sigma)^{tr} = \tau^{tr} + \sigma^{tr}$, $(\tau \times \sigma)^{tr} = \tau^{tr} \times \sigma^{tr}$.

∎

Again, the + and × on the right sides of the equations in part (2) stand for the set-theoretically defined terms, e.g.,

$$\tau^{tr} + \sigma^{tr} = \{\langle \overline{0}, x\rangle : x \in \tau^{tr}\} \cup \{\langle \overline{1}, x\rangle : x \in \sigma^{tr}\}.$$

Next, tr extends almost homophonically to all ϕ from L_Ω:

3.2. Definition.

(1) $(\tau \leq \sigma)^{tr} = \tau^{tr} \preceq \sigma^{tr}$

(2) $(\tau \simeq \sigma)^{tr} = \tau^{tr} \approx \sigma^{tr}$

(3) tr commutes with $\wedge, \vee, \neg$, and $\rightarrow$.

(4) $(\exists x\, \phi)^{tr} = \exists x \in \mathbf{P}(\omega)^{st}\, \phi^{tr}$

(5) $(\forall x\, \phi)^{tr} = \forall x \in \mathbf{P}(\omega)^{st}\, \neg\neg\phi^{tr}$

3.3. Note. In the clauses defining tr on the quantifiers, we take it as given that there is a correspondence of variables that prohibits clashes or redundancies in translation. ∎

3.4. Theorem. *For $\phi \in \text{Sent}_{L_\Omega}$, we have $\underline{\Omega} \models \phi$ iff $\mathbf{V}(Kl) \models \phi^{tr}$.*

Proof. The proof proceeds in seven stages, one for each clause in the structural induction on ϕ.

(1) The preceding lemmas of this chapter give the proof for the base or "atomic" clause of the inductive argument.

(2) $\underline{\Omega} \models (\phi \wedge \psi)$ iff $\underline{\Omega} \models \phi$ and $\underline{\Omega} \models \psi$

$$\mathbf{V}(Kl) \models \phi^{tr} \text{ and } \mathbf{V}(Kl) \models \psi^{tr} \text{ iff}$$

$$\mathbf{V}(Kl) \models (\phi \wedge \psi)^{tr}.$$

(3) $\underline{\Omega} \models (\phi \vee \psi)$ iff either $\underline{\Omega} \models \phi$ or $\underline{\Omega} \models \psi$. If the former, then $\exists e \,.\, e \Vdash \phi^{tr}$ and $\langle 0, e\rangle \Vdash (\phi \vee \psi)^{tr}$. If the latter, $\langle 1, e\rangle \models (\phi \vee \psi)^{tr}$. The converse is trivial.

(4) That $\underline{\Omega} \models \neg\phi$ iff $\mathbf{V}(Kl) \models (\neg\phi)^{tr}$ is trivial.

(5) $\underline{\Omega} \models \phi \to \psi$ iff, either it is not the case that $\underline{\Omega} \models \phi$, or $\underline{\Omega} \models \psi$. If the former, then $\lambda x.0 \Vdash (\phi \to \psi)^{tr}$. If the latter, then, for some m, $m \Vdash \psi^{tr}$, so $\lambda x.m \Vdash (\phi \to \psi)^{tr}$.

On the other hand, assume that $\mathbf{V}(Kl) \models (\phi^{tr} \to \psi^{tr})$ and that $\underline{\Omega} \models \phi$. Then, $\mathbf{V}(Kl) \models \phi^{tr}$, so $\mathbf{V}(Kl) \models \psi^{tr}$. Hence, $\underline{\Omega} \models \psi$.

(6) If $\underline{\Omega} \models \exists x\, \phi$, then $\exists A \in \mathbf{P}(\omega)\ \underline{\Omega} \models \phi[x/\mathring{A}]$. The latter implies that $\mathbf{V}(Kl) \models \phi[x/A]^{tr}$. From this it follows that

$$\mathbf{V}(Kl) \models \overline{A} \in \mathbf{P}(\omega)^{st} \wedge \phi^{tr}[x/\overline{A}] \text{ and } \mathbf{V}(Kl) \models (\exists x\, \phi)^{tr}.$$

Conversely, that $\mathbf{V}(Kl) \models \exists x \in \mathbf{P}(\omega)^{st}\, \phi^{tr}$ implies that $\exists A \in \mathbf{P}(\omega)$ such that $\mathbf{V}(Kl) \models \phi^{tr}[x/\overline{A}]$. Therefore,

$$\mathbf{V}(Kl) \models \phi[x/A]^{tr} \text{ and } \underline{\Omega} \models \exists x\, \phi.$$

(7) If $\underline{\Omega} \models \forall x\, \phi$, then, for all $A \in \mathbf{P}(\omega)$, $\underline{\Omega} \models \phi[x/A]$. Now, assume that $e \Vdash B \in \mathbf{P}(\omega)^{st}$. Then, $\mathbf{V}(Kl) \models B = \overline{\underline{B}}$ and $\underline{\Omega} \models \phi[x/\underline{B}]$. Hence, $\mathbf{V}(Kl) \models \phi^{tr}[x/\overline{\underline{B}}]$, and, by the soundness of equality, $\mathbf{V}(Kl) \models \phi^{tr}[x/B]$. Therefore, $0 \Vdash \neg\neg\phi^{tr}[x/B]$ and $\Lambda x.0 \Vdash (\forall x\, \phi)^{tr}$.

In the opposite direction, assume that $\mathbf{V}(Kl) \models \forall x \in \mathbf{P}(\omega)^{st}\ \neg\neg\phi^{tr}$ and that $A \in \mathbf{P}(\omega)$ in $\mathbf{V}$. Then, $\mathbf{V}(Kl) \models \phi^{tr}[x/\overline{A}]$. Therefore, $\underline{\Omega} \models \phi[x/A]$. Since A was arbitrary, $\underline{\Omega} \models \forall x\, \phi$. ∎

Clause (5) of the definition of *tr* explains why we say that the relation between $\underline{\Omega}$ and $\mathbf{P}(\omega)^{st}$ in $\mathbf{V}(Kl)$ is one of isomorphism *up to stability.* Universal quantification is not preserved precisely; instead, a double negation is inserted to break down the strong realizability condition on relativized quantification. An example will be given later to prove that the $\neg\neg$ cannot be eliminated.

However, if one is careful in defining the classical quantifiers, then the intrusive $\neg\neg$ can be removed and we can say, without qualification, that the classical theory of RETs is *precisely* the realizability theory of ω-stable cardinals. One can simply insist that the classical universal quantifer $\forall x$ over $\underline{\Omega}$ be defined as $\neg\exists x\ \neg$. Then, we may take *tr* to be homophonic on $\forall x$, for $\neg\exists x\ \neg$ is equivalent intuitionistically to $\forall x\ \neg\neg$ and the "stabilization" of $\forall x$ is given automatically via its classical definition.

Section 4: The Preservation Theorems

There is no great difficulty in characterizing the class of L_Ω-formulae which are preserved in the passage from $V(Kl)$ to $\underline{\Omega}$ without alteration of the "logic."

4.1. Definition. The translation *pr* is defined just as *tr*, Definition 3.2, except that

$$(\forall x\, \phi)^{pr} = \forall x \in P(\omega)^{st}\ \phi^{pr}$$

. ∎

4.2. Definition. Let Γ be the class of formulae of Form_{L_Ω} such that $\phi \in \Gamma$ iff there is in ϕ no occurrence of $\forall x$ in the scope of $\neg$ or in the antecedent of $\rightarrow$. ∎

4.3. Note. Γ should be compared with the class Γ_0 as defined in Troelstra (1971). ∎

4.4. Theorem. *If* $\phi \in \Gamma \cap \mathrm{Sent}_{L_\Omega}$, *and* $V(Kl) \models \phi^{pr}$, *then* $\underline{\Omega} \models \phi$.

Proof. It is clear from the isomorphism theorem that, on the $\forall x$-free fragment of L_Ω, *pr* agrees with *tr* and that $\underline{\Omega} \models \phi$ if and only if $V(Kl) \models \phi^{pr}$. The theorem follows immediately by induction on Γ. ∎

The Theory of RETs.

One knows precisely which algebraic properties of the stable cardinals in $P(\omega)$ in $V(Kl)$ transfer directly, without the intermediary $\neg\neg$, into properties of the RETs. The point of all the earlier work on pure cardinal arithmetic should now be clear. First, since $P(\omega)^{st}$ is obviously closed under cardinal addition, the propositions 3.6 (and following) of Chapter One show that all the fundamental properties of the RETs can be obtained from constructively acceptable axioms via the preservation theorem, and, hence, without the intervention of explicit recursion theory.

4.5. Proposition. $\underline{\Omega}$ *is a partially-ordered commutative semigroup with respect to* $+$, $\leq$ *and* $\approx$.

Proof. The assertion that $P(\omega)^{st}$ is a partially-ordered commutative semigroup is in Γ and the proof that $P(\omega)^{st}$ has this property can be carried out in IZF. See the relevant sections of Chapter One. ∎

4.6. Proposition. *$\underline{\Omega}$ has refp.*

Proof. One readily checks that the proof of Proposition 3.10 can be carried out over $P(\omega)^{st}$. ∎

4.7. Proposition. *$\underline{\Omega}$ is a partially-ordered commutative semigroup with respect to $\times$, $|$, and $\approx$.*

Proof. We proved, in Proposition 3.12 of Chapter One, that $P(\omega)$ (and, hence, $P(\omega)^{st}$) is, under $\times$ and $|$, almost a partially-ordered commutative semigroup. Then, since all these assertions naturally lie in Γ, Ω has these properties. Since Ω is classical, it is not *almost*, but truly a partially-ordered semigroup. ∎

The separation of recursion theory from set theory is seen to be highly successful. In proving all the basic properties of the RETs, no recursion-theoretic techniques more sophisticated than those required to define $V(Kl)$, to verify the soundness theorem, and to set up the isomorphism are required. Now, the recursive mathematician can work in pure constructive set theory, undistracted by recursion, and leave the rest to realizability.

Adding the natural numbers.

We remark that there is no obstacle to extending the isomorphism to include any of the conventional predicates defined over Ω. For instance, the theorem can incorporate the apposite 'is a natural number' notion:

4.8. Definition. For $A \in \Omega$, A is a *natural number* ($N(A)$) iff $\exists n \in \omega\ A \simeq n$. ∎

For its correlate in $V(Kl)$, we adopt the definition of 'finite number' discussed in Chapter One; cf. Definition 3.1. To extend the isomorphism theorem, it suffices to prove that the atomic formulae of the extended language are preserved.

4.9. Proposition. *For $A \in \Omega$, $\underline{\Omega} \models N(A)$ iff $V(Kl) \models \overline{A}$ is a finite number.*

Proof. Immediate from the lemma 2.5 and from the absoluteness of ω. ∎

It follows that we can extend the isomorphism and preservation theorems to the structure $\underline{\Omega}^N$, which is Ω augmented by the natural numbers:

4.10. Corollary. *If $\underline{\Omega}^N = \langle \Omega, +, \times, \leq, N, \simeq, 0, \omega \rangle$ and ϕ is a sentence of a suitable language, then $\underline{\Omega}^N \models \phi$ iff $V(Kl) \models \phi^{tr}$. tr is defined in accord Definitions 3.1 and 3.2.*

Proof. ∎

Section 5: On a Mathematical Property of Stable Sets

This section represents a useful digression from the main argument of the chapter. Here we want to show that there are good mathematical reasons (independent of interest in the isomorphism theorem) to prefer a cardinal arithmetic on the *stable* subsets of ω. For one thing, unlike some of their recalcitrant brethern in $\mathrm{V}(Kl)$, the stable subsets always have "enough functions" defined on them. The sufficiency of functions is guaranteed by the fact that the stable subsets of ω satisfy the relevant axioms of choice.

5.1. Proposition.

$$\mathrm{V}(Kl) \models A \in \mathbf{P}(\omega)^{st} \rightarrow (\forall x \in A\ \exists y \in X\ \phi \rightarrow \exists F \in (A \Rightarrow X)\ \forall x \in A\ \phi[y/F(x)])$$

Proof. Assume that $e \Vdash \forall x \in \overline{\omega}\,(\neg\neg x \in A \rightarrow x \in A)$, that

$$g \Vdash \forall x\,(x \in A \rightarrow \exists y \in B\ \phi(x,y))$$

and that $h \Vdash A \subseteq \overline{\omega}$.

There is a partial recursive Θ such that, if there is an m such that $m \Vdash \overline{n} \in A$, then $\Theta(n)\downarrow$ and

$$\exists b\ \{g\}(\Theta(n)) \Vdash b \in B \wedge \phi(\overline{n}, b).$$

We consider the set $\{n : \mathrm{V}(Kl) \models \overline{n} \in A\}$ and use the axiom of choice in the metatheory to guarantee the existence of a choice function F defined on this set and such that

$$\mathrm{V}(Kl) \models \overline{n} \in A \text{ only if } \{g\}(\Theta(n)) \Vdash F(n) \in B \wedge \phi(\overline{n}, F(n)).$$

Now, we inject F into $\mathrm{V}(Kl)$ as $\overline{F}$:

$$\overline{F} = \{\langle \{g\}(\Theta(n)), \overline{\langle \overline{n}, F(n)\rangle}\rangle : \mathrm{V}(Kl) \models \overline{n} \in A\}.$$

It is easily seen that $\overline{F}$ is single-valued: let $n \Vdash \overline{\langle a, b\rangle} \in \overline{F}$ and $m \Vdash \overline{\langle a, c\rangle} \in \overline{F}$. By definition,

$$n_1 \Vdash \overline{\langle a, b\rangle} = \overline{\langle \overline{p}, F(p)\rangle} \text{ while } m_1 \Vdash \overline{\langle a, c\rangle} = \overline{\langle \overline{q}, F(q)\rangle}.$$

Absoluteness for the basic relations on ω gives $p = q$, and, hence, $F(p) = F(q)$. It only remains to check that, with a witness calculable in e, g and h,

$$\mathrm{V}(Kl) \models \forall x \in A\, \exists y\, (\langle x, y\rangle \in \overline{F} \wedge \phi(x, y)).$$

Assume that $n \Vdash a \in A$. Then, $\{h\}(n) \Vdash a \in \overline{\omega}$, in other words, $\{h\}(n)_1 \Vdash a = \overline{\{h\}(n)_0}$. Hence, $\mathrm{V}(Kl) \models \overline{\{h\}(n)_0} \in A$. Consequently,

$$\langle \{g\}(\Theta(\{h\}(n)_0)), i\rangle \Vdash \overline{\langle \overline{\{h\}(n)_0}, F(\{h\}(n)_0\rangle} \in \overline{F} \text{ and}$$

$$\{g\}(\Theta(\{h\}(n)_0))_1 \Vdash \phi(\overline{\{h\}(n)_0}, F(\{h\}(n_0))).$$

The result now follows by the substitutivity of equality. ∎

In our case, having "enough functions" means that each ω-stable set A is a retract of its internal realizability set:

$$\{\langle n, \langle n, a\rangle\rangle : n \Vdash a \in A\}.$$

The fact used in the proof, that $\mathrm{V}(Kl) \models \underline{A} = \overline{A}$, is a direct expression of the "enough functions" property. Our proofs in earlier sections would have been impossible without this. We might say, then, that "reducing" the theory of RETs to that of pure cardinals in $\mathrm{V}(Kl)$ not only depends on the realizability interpretation of the logical signs, but on selecting the right concept of set—that of internal realizability set—from out of the amazing variety of intuitionistic sets.

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$$\mathbf{V}(Kl) \models A \in \mathbf{P}(\omega)^{st} \to (\forall x \in A\, \exists y \in X\, \phi \to \exists F \in (A \Rightarrow X)\, \forall x \in A\, \phi[y/F(x)])$$

Proof. Assume that $e \Vdash \forall x \in \overline{\omega}\, (\neg\neg x \in A \to x \in A)$, that

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$$\langle \{g\}(\Theta(\{h\}(n)_0)), i \rangle \Vdash \overline{\langle \overline{\{h\}(n)_0}, F(\{h\}(n)_0 \rangle} \in \overline{F} \text{ and}$$

$$\{g\}(\Theta(\{h\}(n)_0))_1 \Vdash \phi(\overline{\{h\}(n)_0}, F(\{h\}(n_0))).$$

The result now follows by the substitutivity of equality. ∎

In our case, having "enough functions" means that each ω-stable set A is a retract of its internal realizability set:

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Section 6: Isols and Dedekind-finite cardinals

The isols were originally conceived as the analogues (in choice-free recursive mathematics) of the Dedekind-finite cardinals in choice-free set theory. Accordingly, a subset of $P(\omega)$ in V is an isol if and only if it contains no infinite re subset.

6.1. Definition. $A \in \Omega$ is an *isol* iff $\neg \exists f \ (f : \omega \rightarrowtail A \wedge f$ is recursive). Speaking classically, if A is not an isol, we say that A is *recursively infinite*. The collection of isols is denoted 'Λ'. ∎

6.2. Definition. Let $\underline{\Lambda} = \langle \Lambda, +, \times, \mathrm{N}, \leq, \simeq, A \rangle_{A \in \Lambda}$ be the algebraic structure on the isols, with relations and operations defined as for $\underline{\Omega}$. ∎

It should come as no surprise that the notion of isol, when put over $\mathrm{V}(Kl)$, comes to correspond with Dedekind's original notion of 'finite set'. The latter is the concept we called 'D*-finite' in Section 4 of Chapter One. Since all the results of Section 4 of Chapter One are constructively correct, they hold over $\mathrm{V}(Kl)$ and a subset of $\mathrm{P}(\omega)$ is finite in Dedekind's sense if and only if it is not infinite and if and only if it has the cancellation property. Moreover, the Dedekind-finites are closed under the cardinal operations of + and $\times$, so that, if $\mathrm{P}(\omega)^d$ is the collection of stable Dedekind-finites of $\mathrm{P}(\omega)$, then $\langle \mathrm{P}(\omega)^d, +, \times, \approx \rangle$ is a cardinal algebra. This is, up to stability, the algebra $\underline{\Lambda}$ of isols.

6.3. Lemma. *$A \in \Omega$ is an isol iff $\mathrm{V}(Kl) \models \overline{A}$ is D-finite.*

Proof. What we prove is this statement: $A \in \Omega$ is recursively infinite iff $\mathrm{V}(Kl) \models \overline{A}$ is infinite.

First, assume that $f : \omega \rightarrowtail A$, f recursive. Then, as in Lemma 2.5, $\mathrm{V}(Kl) \models \overline{f} : \overline{\omega} \rightarrowtail \overline{A}$.

On the other hand, assume that $\mathrm{V}(Kl) \models f : \overline{\omega} \rightarrowtail A$. *Inter alia*, there is an $e \in \omega$ such that

$$e \Vdash \forall x \, (x \in \overline{\omega} \rightarrow \exists y \, (y \in \overline{A} \wedge \langle x, y \rangle \in f))$$

We consider the function

$$\underline{f} = \{\langle n, \{e\}(\langle n, i \rangle)_{00} \rangle : n \in \omega\}.$$

in IZF, the conventional construction of the ring of pairs over the D-finites and check that the construction gives precisely $\underline{\mathbf{P}(\omega)^{d^*}}$. ∎

7.4. Theorem. IZF+MP $\vdash$ $\underline{\mathbf{P}(\omega)^d}$ *is isomorphic to the semiring* $\langle \Phi, +, \times, \approx \rangle$ *in* $\underline{\mathbf{P}(\omega)^{d^*}}$, *where* $\Phi = \{\langle C, D\rangle \in \mathbf{P}(\omega)^{d^*} : \exists A \in \mathbf{P}(\omega)^d \, \langle C, D\rangle = \langle A, \emptyset\rangle\}$.

Proof. Once again, we work the analogy between the natural numbers and the integers. For $A \in \mathbf{P}(\omega)^d$, one sets $F(A) = \langle A, \emptyset\rangle$ and shows that F is injective and a homomorphism of $+$ and $\times$. ∎

7.5. Theorem. *For the appropriate languages and translation tr,* $\underline{\Lambda^*}$ *is isomorphic up to stability with* $\underline{\mathbf{P}(\omega)^{d^*}}$ *in* $\mathbf{V}(Kl)$.

Proof. We need only check that

$$\langle A, B\rangle \in \Lambda^* \text{ iff } \mathbf{V}(Kl) \models \overline{\langle \overline{A}, \overline{B}\rangle} \in \mathbf{P}(\omega)^{d^*}$$

and for A, $B \in \Lambda$,

$$\mathbf{V}(Kl) \models \overline{\langle A, B\rangle + \langle C, D\rangle} = \overline{\langle \overline{A}, \overline{B}\rangle} + \overline{\langle \overline{C}, \overline{D}\rangle}$$

while

$$\mathbf{V}(Kl) \models \overline{\langle A, B\rangle \times \langle C, D\rangle} = \overline{\langle \overline{A}, \overline{B}\rangle} \times \overline{\langle \overline{C}, \overline{D}\rangle}$$

∎

With *pr* defined as above, we have the appropriate preservation properties:

7.6. Corollary. *For sentences of the class* Γ *(of the appropriate languages),* $\Lambda^* \models \phi$ *if* $\mathbf{V}(Kl) \models \phi^{pr}$.

Proof. ∎

Thanks to a clever construction on the infinite retraceable sets (cf. Dekker and Myhill (1960), pp. 148–152), it is demonstrable that $\underline{\Lambda^*}$ fails to be an integral domain. Therefore,

$$\underline{\Lambda^*} \models \exists X \, \exists Y \, (X \times Y = 0 \wedge X \neq 0 \wedge Y \neq 0).$$

By the preceeding theorem,

$$\mathbf{V}(Kl) \models \exists X \, \exists Y \in \mathbf{P}(\omega)^{d^*} (X \times Y = 0 \wedge X \neq 0 \wedge Y \neq 0).$$

Consequently, the following independence result is trivial:

7.7. Corollary. IZF+MP *does not derive "*$\underline{\mathbf{P}(\omega)^{d^*}}$ *is an integral domain".*

Section 8: Consequences of Isomorphism: Some Independence Results

Let T_{Kl} be the theory of $V(Kl)$. CSB is the strong classical Cantor-Schroeder-Bernstein theorem:

$$(X \ll Y \wedge Y \ll X) \rightarrow X \approx Y.$$

Here, $\ll$ represents the weak inclusion relation of Cantor's cardinal arithmetic:

$$X \ll Y \text{ if and only if } \exists f \; f : X \rightarrowtail Y.$$

As indicated in Chapter Zero, we can easily prove the following

8.1. Theorem. *T_{Kl} does not derive CSB.*

Proof. In fact, we prove a stronger result:

$$V(Kl) \models \exists X \, \exists Y \in P(\omega)^{st}(Y \text{ is a decidable subset of } \omega \wedge X \ll Y \wedge Y \ll X \wedge \neg X \approx Y).$$

The idea is to exploit the isomorphism theorem (and its proof) to embed the Dekker-Myhill counterexample to strong CSB into $V(Kl)$. To get the counterexample, let $A = \omega \sim K$, where K encodes the halting problem. Set $B = \omega$. Since A is productive, $B \ll A$. Since B is ω, $A \ll B$. However, A is not r.e., so it is not the case that $A \simeq B$.

Now, we apply the isomorphism theorem, noting that the proof of that theorem shows that for $C, D \in \Omega$, $C \ll D$ iff $V(Kl) \models \overline{C} \ll \overline{D}$. ∎

8.2. Note. Feferman has, using a realizability technique, obtained a similar result for his theory T_0. Cf. Feferman (1975). ∎

The correspondence between the structures of recursive set theory and of cardinal algebras in $V(Kl)$ gives easy access to methods for transforming the failures of the classical results in recursive cardinal arithmetic into independence results from T_{Kl}.

8.3. Theorem. *In $V(Kl)$, $\ll$ is not well-founded, even over $P(\omega)^d$. There is, in fact, an infinitely-descending ω-sequence in $P(\omega)^d$.*

Proof. According to Dekker and Myhill (1960), there is an ω-sequence $F = \langle X_n \rangle_{n \in \omega}$ in Λ such that

$$\forall n \in \omega \; X_{n+1} < X_n.$$

($<$ is the strict inclusion correlative to $\leq$.) The classical proof of this result shows that the partial recursive functions in virtue of which $X_{n+1} < X_n$ can be chosen uniformly in n. In fact, an index j of the identity on ω can be used for all n.

We can put F into $\mathbf{V}(Kl)$ as the $\overline{\omega}$-sequence $\overline{F}$ of stable, D-finite sets:

$$\overline{F} = \{\overline{\langle n, \langle \overline{n}, \overline{X_n} \rangle \rangle} : n \in \omega\}$$

and then show that the range of F has, in $\mathbf{V}(Kl)$, no minimal element.

Because of the cited uniformity, $\mathbf{V}(Kl)$ satisfies

$$\overline{F} : \omega \longrightarrow \mathbf{P}(\omega)^d \wedge \forall n \in \omega \; \overline{F}(n+1) \ll \overline{F}(n).$$

The proof that the first conjunct is realized is trivial. To see that the second obtains, let $e \Vdash a \in \overline{\omega}$. Then, $e_1 \Vdash a = \overline{e_0}$ and

$$\langle e_0, i \rangle \Vdash \overline{\langle \overline{e_0}, \overline{X_{e_0}} \rangle} \in \overline{F} \text{ and}$$

$$\langle e_0 + 1, i \rangle \Vdash \overline{\langle \overline{e_0 + 1}, \overline{X_{e_0+1}} \rangle} \in \overline{F}.$$

As the proof of the isomorphism theorem shows, there is a partial recursive Θ such that

$$\Theta(j) \Vdash \overline{X_{e_0+1}} \ll \overline{X_{e_0}}.$$

The latter holds independently of e_0.

Finally, a simple application of the preservation theorem proves that $0 \Vdash \neg \overline{X_{e_0+1}} \approx \overline{X_{e_0}}$. ∎

8.4. Theorem. *In $\mathbf{V}(Kl)$, there is an infinite $\prec$-ascending chain from $\mathbf{P}(\omega)^d$ which has, in $\mathbf{P}(\omega)^d$, no least upper bound with respect to $\prec$.*

Proof. Dekker and Myhill proved that there is, externally, a sequence $F = \langle X_n \rangle_{n \in \omega}$ from Λ such that, for all n, $X_n < X_{n+1}$ but such that $\mathrm{Rng}(F)$ has no lub in Λ. Again, we can assume that it is the identity that guarantees $X_n < X_{n+1}$, uniformly in n.

As before, we embed F into $\mathbf{V}(Kl)$, and then check that

$$\mathbf{V}(Kl) \models \overline{F} : \omega \longrightarrow \mathbf{P}(\omega)^d \wedge \forall n \in \omega \; (\overline{F}(n) \prec \overline{F}(n+1)).$$

To show that there is no lub for $\overline{F}$, we assume that, in $\mathbf{V}(Kl)$,

$$U \in \mathbf{P}(\omega)^d \wedge \forall n \in \omega\ \overline{F}(n) \preceq U \wedge \forall A \in \mathbf{P}(\omega)^d\,(\forall n \in \omega\ \overline{F}(n) \preceq A \rightarrow U \preceq A).$$

From the Dekker-Myhill proof one can see that, in $\mathbf{V}$, $X_{n+1} = X_n + 1$. Now, if we work in IZF and take $\overline{F}(n)$ arbitrary, then $\overline{F}(n+1) \preceq U$. Let f be the constructive function in virtue of which this is true and take $U^- = U \sim \{f(\langle 1, 0\rangle)\}$. By cancellation, $\overline{F}(n) \preceq U^-$ and this holds for arbitrary n. Therefore, $\forall n\ \overline{F}(n) \preceq U^-$. But $U^- \prec U$, since U is D-finite. ∎

8.5. Theorem. *In $\mathbf{V}(Kl)$, there are infinitely-descending $\ll$-chains from $\mathbf{P}(\omega)^d$ that lack greatest lower bounds both in $\mathbf{P}(\omega)^d$ and in $\mathbf{P}(\omega)$.*

Proof. Externally, if X_0 is an infinite isol, then the sequence F of Lemma 6.3 has a range in Λ which has a glb neither in Λ nor in Ω. We proceed as before and consider $\overline{F}$. $\mathbf{V}(Kl) \models (\overline{F}$ enumerates an infinitely descending ω-sequence from $\mathbf{P}(\omega)^d$.) Since $\mathbf{P}(\omega)^d$ is downward-closed under $\preceq$, it would be sufficient to prove that the range of $\overline{F}$ has no glb in $\mathbf{P}(\omega)^d$.

Without loss of generality, we can assume that $\overline{F}(n+1) \approx \overline{F}(n) \sim 1$. Let L be a lower bound for the $\overline{F}(n)$'s in $\mathbf{P}(\omega)^d$. Then, for any n, $L \preceq \overline{F}(n) \sim 1$, so $L + 1 \preceq \overline{F}(n)$. Therefore, $L+1$ is also a lower bound for $\mathrm{Rng}(\overline{F})$, but $L \prec L+1$. ∎

8.6. Theorem. *$\preceq$ is not a total ordering, even in $\mathbf{P}(\omega)^d$. In fact, there is a collection $\overline{\Delta}$ from $\mathbf{P}(\omega)^d$ such that the members of $\overline{\Delta}$ are mutually incomparable and $\overline{\Delta}$ is in one-to-one correspondence with $\mathbf{P}(\omega)^{st}$.*

Proof. Dekker and Myhill have proved that there is a collection $\Delta \subseteq \Lambda$ such that Δ has $|\mathbf{P}(\omega)|$-many members and the members of Δ are mutually incomparable. (Cf. p. 103 of *Recursive equivalence types.*)

We embed Δ into $\mathbf{V}(Kl)$ as

$$\overline{\Delta} = \{\langle 0, \overline{A}\rangle : A \in \Delta\}.$$

It is easily seen that $\mathbf{V}(Kl) \models \overline{\Delta} \subseteq \mathbf{P}(\omega)^d$. To see that the members of $\overline{\Delta}$ are pairwise incomparable in $\mathbf{V}(Kl)$, one first assumes that

$$e \Vdash A \in \overline{\Delta} \text{ and } g \Vdash B \in \overline{\Delta}.$$

Then, for some $C, D \in \Delta$,

$$e_1 \Vdash A = \overline{C} \text{ and } g_1 \Vdash B = \overline{D}.$$

Since, C and D are incomparable in $\mathbf{V}$, the preservation theorem shows that

$$\langle 0,0\rangle \Vdash \neg \overline{C} \preceq \overline{D} \wedge \neg \overline{D} \preceq \overline{C}.$$

Therefore,

$$\mathbf{V}(Kl) \models \forall X, Y \in \overline{\Delta}\ \neg X \preceq Y \wedge \neg Y \preceq X.$$

As for the cardinality of $\overline{\Delta}$, let $F : \mathbf{P}(\omega) \rightarrowtail\!\!\!\twoheadrightarrow \Delta$ in $\mathbf{V}$. Embed F as $\overline{F}$, where $\overline{F}$ is

$$\{\langle 0, \overline{\langle \overline{A}, \overline{B}\rangle}\rangle : B = F(A)\}.$$

The goal is to show that, in $\mathbf{V}(Kl)$, $\overline{F} : \mathbf{P}(\omega)^{st} \rightarrowtail\!\!\!\twoheadrightarrow \overline{\Delta}$.

To see that $\overline{F}$ is single-valued in its second component, we assume that

$$e \Vdash \overline{\langle X, Y\rangle} \in \overline{F} \text{ and } g \Vdash \overline{\langle X, Z\rangle} \in \overline{F}.$$

Then, for appropriately chosen A, B, C, D in $\mathbf{P}(\omega)$,

$$e_1 \Vdash \overline{\langle X, Y\rangle} = \overline{\langle \overline{A}, \overline{B}\rangle} \text{ and } g_1 \Vdash \overline{\langle X, Z\rangle} = \overline{\langle \overline{C}, \overline{D}\rangle}.$$

It follows that $\mathbf{V}(Kl) \Vdash \overline{A} = \overline{C}$ and that $A = C$ in $\mathbf{V}$. Therefore, $B = D$ and $i \Vdash \overline{B} = \overline{D}$.

Next, we check that $\mathbf{V}(Kl) \models \mathrm{Dom}(\overline{F}) \supseteq \mathbf{P}(\omega)^{st}$. This is a direct consequence of a fact we unearthed much earlier—that there is a partial recursive Θ such that, if $e \Vdash A \in \mathbf{P}(\omega)^{st}$, then $\Theta(e)\downarrow$ and $\Theta(e) \Vdash A = \overline{\underline{A}}$. We leave a complete verification to the reader.

Finally, we want to be assured that $\mathbf{V}(Kl) \models \mathrm{Ran}(\overline{F}) \supseteq \Delta$. To that end, we note that $e \Vdash C \in \overline{\Delta}$ only if $e_1 \Vdash C = \overline{B}$, for some $B \in \Delta$. Then, when $F(A) = B$, $\langle 0, i\rangle \Vdash \overline{\langle \overline{A}, \overline{B}\rangle} \in \overline{F}$. Since there is a $j \in \omega$ such that $j \Vdash \overline{A} \in \mathbf{P}(\omega)^{st}$ uniformly in A, we are finished. ∎

8.7. Definition. $B \in \Omega$ is an *immediate successor* of A iff $A < B$ and for no $C \in \Omega$ is it the case that $A < C < B$. Immediate successor is defined analogously for $\mathbf{P}(\omega)^{st}$. ∎

8.8. Theorem. *There are $X \in \mathbf{P}(\omega)^{st}$ and $\overline{\Delta} \subseteq \mathbf{P}(\omega)^{st}$ in $\mathbf{V}(Kl)$ such that $\overline{\Delta}$ is a collection of immediate successors of X and is in one-to-one correspondence with $\mathbf{P}(\omega)^{st}$.*

Proof. Every nonisolic RET in $\mathbf{V}$ has precisely $|\mathbf{P}(\omega)|$ immediate successors. Let X be any of these RETs. By König's Lemma, there is a set Δ of immediate successors of X such that Δ is in one-to-one correspondence with $\mathbf{P}(\omega)$ and there is an $e \in \omega$ such that for all $Y \in \Delta$, $X < Y$ holds in virtue of the partial recursive function $\{e\}$. We insert Δ into $\mathbf{V}(Kl)$ as

$$\overline{\Delta} = \{\langle e, \overline{A}\rangle : A \in \Delta\}.$$

Now, $\mathbf{V}(Kl) \models \overline{X} \in \mathbf{P}(\omega)^{st}$. If $g \Vdash B \in \overline{\Delta}$, then $g_1 \Vdash B = \overline{A}$, where $A \in \Delta$ and $g_0 = e$. From the proof of the isomorphism theorem, we know that there is a fixed partial recursive function Σ such that

$$\Sigma(g_0) \Vdash \overline{X} \prec \overline{A}.$$

Therefore, $\mathbf{V}(Kl) \models$

$$\forall Y \in \overline{\Delta}\ \overline{X} \prec \overline{Y} \wedge \neg \exists Z \in \mathbf{P}(\omega)\ \overline{X} \prec Z \prec \overline{Y}.$$

The last clause is realized uniformly, since

$$0 \Vdash \neg \exists Z \in \mathbf{P}(\omega)\ \overline{X} \prec Z \prec \overline{Y}.$$

The remainder of the theorem, that Δ is, internally, of the same cardinality as $\mathbf{P}(\omega)^{st}$ is proved precisely as in Theorem 8.6. ∎

The previous theorem shows that, in terms of cardinality, $\mathbf{P}(\omega)^{st}$ in $\mathbf{V}(Kl)$ is exceedingly "wide:" a sizeable collection of members of $\mathbf{P}(\omega)^{st}$ each have at least $\mathbf{P}(\omega)^{st}$-many cardinals serving as immediate successors. Another result of the classical theory of RETs (also due to Dekker and Myhill) can be employed to show that $\mathbf{P}(\omega)^{st}$ is extremely "tall." One can show that there is a "version" of $\aleph_{\omega_1}$ in $\mathbf{P}(\omega)^{st}$, indeed, even in $\mathbf{P}(\omega)^{d}$. The proof works by internalizing the classical fact, proved by Dekker and Myhill, that

$$\exists F : \omega_1 \rightarrowtail \Omega \wedge \forall \alpha \in \beta \in \omega_1\ F(\alpha) < F(\beta).$$

First, we prove that ω_1 can be embedded into $\mathrm{V}(Kl)$ as an ω_1-analogue. For each $\alpha \leq \omega_1$, set $\overline{\alpha}$ equal to

$$\{\langle 0, \overline{\beta}\rangle : \beta \in \alpha\}.$$

8.9. Lemma. $\mathrm{V}(Kl) \models \overline{\alpha} = \overline{\beta}$ *iff* $\alpha = \beta$.

Proof. Assume that $\mathrm{V}(Kl) \models \overline{\alpha} = \overline{\beta}$ and that $\gamma \in \alpha$. Then, $\mathrm{V}(Kl) \models \overline{\gamma} \in \overline{\alpha} \wedge \overline{\gamma} \in \overline{\beta}$. It follows that there is a $\delta \in \beta$ such that $\mathrm{V}(Kl) \models \overline{\delta} = \overline{\gamma}$. By induction, $\gamma \in \beta$. ▮

Here is a cardinality concept that bears an obvious relation to traditional recursion theory. Grayson used this notion, in the context of pure constructive set theory, in his dissertation (1978).

8.10. Definition. A set X is ω-productive iff $\forall f \in (\omega \Rightarrow X)\ \exists x \in X\ x \notin \mathrm{Rng}(f)$. ▮

8.11. Lemma. $\mathrm{V}(Kl) \models \overline{\omega_1}$ *is a nonsubcountable, ω-productive, regular cardinal.*

Proof. (1) By its very definition, $\overline{\omega_1}$ is a *uniformity set.* If $e \Vdash a \in \overline{\omega_1}$, then $e_1 \Vdash a = \overline{\beta}$, while $\langle 0, i\rangle \Vdash \overline{\beta} \in \overline{\omega_1}$. Therefore, $\overline{\omega_1}$ is not subcountable.

(2) To see that $\overline{\omega_1}$ is ω-productive, assume that $e \Vdash f \in (\omega \Rightarrow \overline{\omega_1})$. Then, with g computable from e, g realizes

$$\forall x \in \overline{\omega}\ \exists y \in \overline{\omega_1}\ \langle x, y\rangle \in f.$$

It follows that there is a partial recursive Θ such that, for each $n \in \omega$, there is an $\alpha \in \omega_1$ for which

$$\Theta(e,n) \Vdash \langle \overline{n}, \overline{\alpha}\rangle \in f.$$

By countable choice in the metatheory, there is a function $F : \omega \to \omega_1$ such that, for all n,

$$\Theta(e,n) \Vdash \langle \overline{n}, \overline{F(n)}\rangle \in f.$$

Let β be the least element of $\omega_1 \sim \mathrm{Rng}(F)$ and let $g \Vdash \overline{\langle \overline{n}, \overline{F(n)}\rangle} \in f$. Then, $\langle 0, i\rangle \Vdash \overline{F(n)} \in \overline{\beta}$. Assume that $\mathrm{V}(Kl) \models \overline{\langle \overline{m}, \overline{\beta}\rangle} \in f$, for some $m \in \omega$. Then, $\mathrm{V}(Kl) \models \overline{\langle \overline{m}, \overline{F(m)}\rangle} \in f$. Since f is realizably a function, and since, by the previous theorem, the equality on the ordinals is absolute, $F(m) = \beta$. This contradicts the choice of β. Therefore, $\overline{\omega}$ is internally ω-productive and $\mathbf{V}(Kl) \models$ no function on ω is cofinal in $\overline{\omega_1}$.

(3) Assume that $\mathbf{V}(Kl) \models f \in (\overline{\alpha} \Rightarrow \overline{\omega_1})$, where $\alpha \in \omega_1$. Then, for some $e \in \omega$,

$$e \Vdash \forall x \in \overline{\alpha}\ \exists y \in \overline{\omega_1}\ \langle x, y \rangle \in f.$$

It follows that there is a partial recursive Θ such that for all $\beta \in \alpha$, there is a $\gamma \in \omega_1$ such that

$$\Theta(e) \Vdash \langle \overline{\beta}, \overline{\gamma} \rangle \in f.$$

Now, it's only a matter of repeating the argument of (2) to prove that f is not cofinal in $\overline{\omega_1}$. ∎

Next, we look to embed the F of the original Dekker-Myhill theorem into $\mathbf{V}(Kl)$ as $\overline{F}$. A cardinality argument over ω_1 shows that we may assume there to be a single $e \in \omega$ such that for all $\alpha \in \beta \in \omega_1$, $\{e\} : F(\alpha) < F(\beta)$. Set $\overline{F} = \{\langle 0, \overline{\langle \overline{\alpha}, \overline{F(\alpha)} \rangle} \rangle : \alpha \in \omega_1\}$.

8.12. Lemma. $\mathbf{V}(Kl) \models \overline{F} : \overline{\omega_1} \rightarrowtail \mathbf{P}(\omega)^{st}$.

Proof. This is absolutely straightforward; we need only exploit the absoluteness properties of equality on the $\overline{A}$'s in $\mathbf{P}(\omega)^{st}$ and on the $\overline{\alpha}$'s in $\overline{\omega_1}$. ∎

8.13. Lemma. $\mathbf{V}(Kl) \models \forall x, y \in \omega_1\ (x \in y \rightarrow \overline{F}(x) \prec \overline{F}(y))$.

Proof. For α, $\beta \in \omega_1$, let $h \Vdash \overline{\alpha} \in \overline{\beta}$. Then, $\alpha \in \beta$ in reality, and there is an m (provided by the abovementioned cardinality considerations) which, uniformly in α and β, realizes $\overline{F(\alpha)} \prec \overline{F(\beta)}$. ∎

These lemmas, taken together, prove that

8.14. Theorem. *In $\mathbf{V}(Kl)$, there is a nonsubcountable ω-productive, regular cardinal κ, and a function f such that f takes κ injectively into $\mathbf{P}(\omega)^{st}$ and such that, for all $\alpha, \beta \in \kappa$, $f(\alpha) \prec f(\beta)$.*

Myhill and Dekker proved that the f of the theorem can even be assumed to map κ into $\mathbf{P}(\omega)^d$ with the same results. Therefore, we also have

8.15. Theorem. *In $\mathbf{V}(Kl)$, there is a nonsubcountable ω-productive, regular cardinal κ, and a function f such that f takes κ injectively in $\mathbf{P}(\omega)^d$ and such that, for all $\alpha, \beta \in \kappa$, $f(\alpha) \prec f(\beta)$.*

Section 9: Notes on exponentiation

We have yet to treat the Dekker-Myhill notion of exponentiation for the RETs and the reader may well have wondered at this. It is not that there is some conceptual barrier to the ready incorporation of exponentiation into the "realizability scheme." The reader should, by now, have had sufficient contact with the mechanisms underlying our approach to be morally certain that this could easily be done.

It is not a question of feasibility, but one of naturalness. If we do not alter the Dekker-Myhill notion of RET exponentiation, one can show that its internal version corresponds to no natural constructive notion of "power." If, however, we permit ourselves some liberty to tamper with the classical definition, we can prove that a slightly revised concept of exponentiation agrees with a notion of "finitely indexed" exponentiation for constructive cardinals.

In large part, the unnaturalness arises in the very attempt to perform an exponentiation operation on sets of natural numbers in a constructive setting. However, some measure of the "unnaturalness" of the Dekker-Myhill notion resides in the details of the specific coding scheme upon which their definition of exponentiation depends.

9.1. Definition. Let p_x be the x-th prime in ω. For each pair $n, x \in \omega$,

$$r_n(x) = \mu y \leq n(p_x^{y+1} \text{ does not divide } n+1).$$

∎

Every finite function on ω is then indexed by the n such that $\mathrm{Dom}(f) = \{x : r_n(x) > 0\}$ and $r_n(x)$ gives the value of $f(x)$.

9.2. Definition. For $A, B \in \mathbf{P}(\omega)$, r_n *maps* A *into* B ($r_n : A \to B$) iff, whenever $r_n(x) > 0$, $x \in A$ and $r_n(x) \in B$. ∎

With this mapping concept, Dekker and Myhill define the RET-exponentiation operation $\mathrm{Exp}(A, B)$:

9.3. Definition. For $A, B \in \mathbf{P}(\omega)$, $\mathrm{Exp}(A, B) = \{n : r_n : A \to B\}$. ∎

9.4. Remark. '$\mathrm{Exp}(A, B)$' is our own terminology for the canonical RET exponentiation operation. In light of the facts which we are about to uncover, we find our terminology less misleading than the usual: 'A^B'. ∎

Negative results on exponentiation.

The authors of *Recursive equivalence types* offered no explanation for their choice of definitions, although their exposition suggests that, somehow, Exp is "constructively correct." This section is designed to put that suggestion to a realizability-theoretic test. Our test will show that any interesting or straightforward conception of exponentiation in $\mathbf{P}(\omega)^{st}$ does not give rise to anything amenable to RET-style treatment. This result turns on the fact that nontrivial exponentiation objects cannot be (equivalent to) the range or domain of a constructive number-theoretic function in $\mathbf{V}(Kl)$. In fact, no natural function-space object on an infinite subset of ω can be injected into a discrete space in $\mathbf{V}(Kl)$.

As prelude, we prove that some of the absoluteness properties of elements of ω extend to elements of $(\omega \Rightarrow \omega)$.

9.5. Lemma. *Let g and h index partial recursive function on ω. If $\overline{g}$ is defined as*

$$\{\langle n, \overline{\langle \overline{n}, \overline{\{g\}(n)} \rangle} \rangle : \{g\}(n)\downarrow\}$$

and $\overline{h}$ is defined analogously, then

$$\{g\} = \{h\} \textit{ iff } \mathbf{V}(Kl) \models \overline{g} = \overline{h}.$$

Proof. The implication from left-to-right is obvious. For the other direction, assume that $\mathbf{V}(Kl) \models \overline{g} = \overline{h}$. Then, for any n from ω such that $\{g\}(n)\downarrow$,

$$\langle n, i \rangle \Vdash \overline{\langle \overline{n}, \overline{\{g\}(n)} \rangle} \in \overline{g} \text{ and}$$
$$\mathbf{V}(Kl) \models \overline{\langle \overline{n}, \overline{\{g\}(n)} \rangle} = \overline{\langle \overline{m}, \overline{\{h\}(m)} \rangle}$$

for some $m \in \omega$.

By the absoluteness conditions on ω, $n = m$ and $\{g\}(n) = \{h\}(m)$. Hence, $\{g\} \subseteq \{h\}$. By parity of reasoning, $\{h\} \subseteq \{g\}$ holds also. ∎

9.6. Lemma. *There is a $j \in \omega$ such that, if g indexes a total recursive function, then $\{j\}(g)\downarrow$ and $\{j\}(g) \Vdash \overline{g} \in (\omega \Rightarrow \omega)$.*

Proof. A proof of this is easily abstracted from a proof of Lemma 2.5. ∎

9.7. Lemma. $V(Kl) \models \neg \forall x, y \in (\omega \Rightarrow \omega)(x = y \vee \neg x = y)$.

Proof. We can use Lemma 9.5 and the unsolvability of the halting problem. The proof is easy. ∎

9.8. Theorem. $V(Kl) \models \neg \exists a \subseteq \omega \, (\omega \Rightarrow \omega) \approx a$.

Proof. Reasoning in set theory over $V(Kl)$, we assume that $f : (\omega \Rightarrow \omega) \approx a$. It follows that

$$\forall x, y \in (\omega \Rightarrow \omega) \, (x = y \leftrightarrow f(x) = f(y)).$$

Since equality is provably decidable on a, we have, in $V(Kl)$,

$$\forall x, y \in (\omega \Rightarrow \omega) \, (x = y \vee \neg x = y).$$

This contradicts the conclusion of Lemma 9.7.

9.9. Remark.

(1) Since $V(Kl) \models \mathrm{CT}_0$, Theorem 9.8 yields an explicit proof that

$$V(Kl) \models \neg \mathrm{AC}^{(\omega \Rightarrow \omega), \omega}.$$

(2) Much the same technique could be harnessed to show that

$$V(Kl) \models \neg \exists a \subseteq \omega \, (\omega \Rightarrow 2) \approx a.$$

This sharpens the very simple result that

$$V(Kl) \models \neg \exists f \, . f : \omega \twoheadrightarrow (\omega \Rightarrow 2),$$

where the latter is proved via diagonalization. ∎

To extend the discouraging results of Theorem 9.8 to all infinite RETs, we need relativizations of Lemmas 9.5 and 9.6:

9.10. Lemma. *Let $A \subseteq \omega$. Let g and h index partial recursive function on ω. If $\overline{g}_A$ is defined as*

$$\overline{\{\langle n, \langle \overline{n}, \overline{\{g\}(n)} \rangle \rangle : n \in A \wedge \{g\}(n)\downarrow\}}$$

and $\overline{h}_A$ is defined analogously, then

$$\{g\}\upharpoonright A = \{h\}\upharpoonright A \text{ iff } \mathbf{V}(Kl) \models \overline{g}_A = \overline{h}_A.$$

Proof. One simply repeats the proof of Lemma 9.5. ∎

9.11. Lemma. *There is a $j \in \omega$ such that, if g indexes a total recursive function on ω and $A \subseteq \omega$, then*

$$\{j\}(g)\downarrow \text{ and } \mathbf{V}(Kl) \models \overline{g}_A \in (\overline{A} \Rightarrow \omega).$$

Proof. Just as in Lemma 9.6. ∎

9.12. Theorem. *For any $A \in \mathbf{P}(\omega)$, A infinite,*

$$\mathbf{V}(Kl) \models \neg \exists x \subseteq \omega\, (\overline{A} \Rightarrow 2) \approx x.$$

Proof. Just as before, if $\mathbf{V}(Kl) \models (\overline{A} \Rightarrow 2) \approx c \wedge c \subseteq \omega$, then there is an $e \in \omega$ such that for all $a, b \in \mathbf{V}(Kl)$,

$$e \Vdash a \in (\overline{A} \Rightarrow 2) \wedge b \in (\overline{A} \Rightarrow 2) \rightarrow a = b \vee \neg a = b.$$

By Lemmas 9.10 and 9.11, there is a $k \in \omega$ such that, if g and h index total recursive functions, then $\{k\}(g,h)\downarrow$ and

$$\{k\}(g,h) = 0 \text{ iff } \{g\}\upharpoonright A = \{h\}\upharpoonright A.$$

Hence, $\{k\}$ indexes an effective procedure which, given a pair of total indices as arguments, decides whether or not the indexed functions agree on the infinite set A. With k in hand, one can solve the halting problem, as follows. there is a $p \in \omega$ which, given x, produces an index of $f \in (\omega \Rightarrow 2)$ where

$$f(y) = 0 \text{ iff } \neg \exists u < y\, T(x,x,u).$$

Then, $\{k\}(\{p\}(x), \Lambda x.0) = 0$ iff $f = \lambda x.0$ iff $\neg \exists u\, T(x,x,u)$. ∎

It follows from our theorems 9.8 and 9.12 that there can be no definition of exponentiation that does full justice to the usual constructive notion and yet fits securely into the "world" of RETs. The latter is a world in which sets are fully represented as collections of natural numbers. This conclusion stands in conflict with the claim of Dekker and Myhill that

> [*t*]*he operation* 2^A [Dekker-Myhill exponentiation] *can be considered as the constructive analogue to the operation* 2^Γ *in cardinal arithmetic, provided that we restrict it to* Λ.

Theorem 9.12 applies *a fortiori* to all infinite isols and shows that there can be no such analogue in the fullest sense.

Dekker-Myhill exponentiation.

In the face of all this and with the desire to extend the isomorphism theorems to exponentiation, one is left with searching out best possible alternatives. There is a notion which reflection on the isomorphism theorem might suggest as a plausible candidate for internal "effective exponentiation." The notion comes from relativizing exponentials to decidable subsets:

9.13. Definition. $\mathbf{D}(y, x) = \{f : \exists z\, z \preceq y \wedge f \in (z \Rightarrow x)\}$. ∎

Unfortunately, the plausibility of the notion as an internalization of Exp is merely apparent. We have immediately that

$$\mathrm{IZF} \vdash \forall y \subseteq \omega\ [(y \Rightarrow 2) \subseteq \mathbf{D}(x, 2)].$$

We already know that $\mathbf{V}(Kl) \models \neg\, \exists x \subseteq \omega\, (\overline{A} \Rightarrow 2) \approx x$ and this implies that

$$\mathbf{V}(Kl) \models \neg\, \exists x \subseteq \omega\, \mathbf{D}(A, 2) \approx x.$$

D will not, therefore, provide both a constructively viable notion of exponentiation and a set over which a number-theoretic "RET" function can be defined.

There is an obvious, albeit unnatural, internalization of the original Dekker-Myhill notion and it is to this that we now turn. This internalization puts everything in terms of finite functions, so, it agrees with exponentiation on (classical) finite sets and does not fall afoul of the halting problem. 'IExp' is our notation for this internal concept.

9.14. Definition. For $x, y \in \mathbf{P}(\omega)$, set

$$\mathbf{IExp}(x, y) = \{f : \exists z \subseteq x\, \exists n \in \omega\, n \approx z \wedge f \in (z \Rightarrow y) \wedge \neg\, \exists m\, f(m) = 0\}.$$

∎

We assert that **Exp** is amenable to the treatment undergone by $+$ and $\times$ on $\underline{\Omega}$ by proving that

$$\mathbf{V}(Kl) \models \mathbf{IExp}(\overline{A}, \overline{B}) \approx \overline{\mathbf{Exp}(A, B)}$$

for any $A, B \in \mathbf{P}(\omega)$. We will not, in fact, carry out the full treatment here. By this stage, the reader should well be able to do this by himself. Suffice it to say that this shows how, by coding, the isomorphism theorem could be extended to allow for Dekker-Myhill exponentiation.

9.15. Theorem. *For all $A, B \in \mathbf{P}(\omega)$,*

$$\mathbf{V}(Kl) \models \overline{\mathbf{Exp}(A, B)} = \mathbf{IExp}(\overline{A}, \overline{B})$$

and a witness for the claim can be found independently of A and B.

The proof is divided into a series of lemmas:

9.16. Lemma. *There is an $e \in \omega$ such that if $\{f\} : n \approx C$, $C \subseteq B$, $\{g\} : C \to A$ and $\forall n\,(\{g\}(n)\downarrow \to \{g\}(n) > 0)$, then*

$$\{e\}(\langle f, n, g\rangle) \Vdash \overline{g} \in \mathbf{IExp}(\overline{B}, \overline{A}).$$

Proof. This is a straightforward and tedious application of the earlier techniques of this chapter. ∎

9.17. Lemma. *There is a j from ω such that for all $e \in \omega$ and $f \in \mathbf{V}(Kl)$, if $e \Vdash f \in \mathbf{IExp}(\overline{B}, \overline{A})$, then there is a $C \subseteq A$ such that*

(1) $\{j\}(e)_0 \in \omega$,

(2) $\{\{j\}(e)_1\} : \{j\}(e)_0 \approx C$,

(3) $\{\{j\}(e)_2\} : C \to A$,

(4) $\{j\}(e)_3 \Vdash f = \overline{\{\{j\}(e)_2\}}$ *and*

(5) $\forall n\,(\{\{j\}(e)_2\}(n)\downarrow \to \{\{j\}(e)_2\}(n) > 0)$.

Proof. (1) is immediate from the realizability conditions on the initial numerical quantifier of the internal definition of **IExp**.

(2) From (1), there is a $k \in \omega$ and $b \in \mathrm{V}(Kl)$ for which

$$\{k\}(e) \Vdash \overline{\{j\}(e)_0} \approx b \wedge b \subseteq \overline{B}.$$

$\mathrm{IZF} \vdash (\exists n \in \omega\ (n \approx b \wedge b \in \mathbf{P}(\omega)) \rightarrow b$ is ω-stable). Therefore, by Theorem 2.12, there is an $l \in \omega$ and a $C \in \mathbf{P}(\omega)$ of V such that

$$\{l\}(e) \Vdash \overline{\{j\}(e)_0} \approx \overline{C}.$$

By our previous proofs, there is an f such that $f : \{j\}(e)_0 \approx C$ in V and an index for f is calculable from $\{l\}(e)$.

(3) By (2), there is a $p \in \omega$ such that

$$\{p\}(e) \Vdash f \in (\overline{C} \Rightarrow \overline{A}).$$

By examining the realizer of

$$\forall x \in \overline{C}\ \exists y \in \overline{A}\ \langle x, y\rangle \in f,$$

one can obtain from $\{p\}(e)$ the desired $\{\{j\}(e)_2\}$.

(4) This follows almost immediately from (3).

(5) Here, we commit only a few details to paper to serve as an illustration. If given in full detail, this proof would be excruciatingly long. Let $h = \{\{j\}(e)_2\}$. From (4), we know that

$$\mathrm{V}(Kl) \models \forall n\ \overline{h}(n) > 0.$$

Now, if $h(n)\downarrow$, then $\langle n, i\rangle \Vdash \langle \overline{n}, \overline{h(n)}\rangle \in \overline{h}$. Hence,

$$\mathrm{V}(Kl) \models \overline{h(n)} \neq 0.$$

By absoluteness, $h(n) > 0$ and we are done. ∎

At this point in the proof, it helps to define an intermediate internal concept, $\overline{\overline{\mathrm{E}}}$:

9.18. Definition. For $A, B \in \mathbf{P}(\omega)$ in V, $\overline{\overline{\mathrm{E}}}(B, A)$ is the element of $\dot{\mathrm{V}}(Kl)$ which is

$$\{\langle\langle f, n, g\rangle, \overline{g}\rangle : \exists C \subseteq B\ \{f\} : n \approx C \text{ and } \{g\} \in (C \Rightarrow A) \text{ and } \forall n\ (\{g\}(n)\downarrow \rightarrow \{g\}(n) > 0$$

$\overline{\overline{\mathrm{E}}}$ is a simple-minded internalization of the RET-theoretic notion of exponentiation. In terms of $\overline{\overline{\mathrm{E}}}$, we want to prove that

$$\mathbf{V}(Kl) \models \mathbf{IExp}(\overline{A}, \overline{B}) \approx \overline{\overline{\mathrm{E}}}(A, B) \approx \overline{\mathbf{Exp}(A, B)},$$

and this will complete the proof. To fulfill this intention, we prove another lemma.

9.19. Lemma. $\exists k\, \forall A, B \in \mathbf{P}(\omega)\ k \Vdash \overline{\overline{E}}(A, B) = \mathbf{IExp}(\overline{A}, \overline{B})$.

Proof. First, if $n \Vdash g \in \overline{\overline{\mathrm{E}}}(A, B)$, then $n_1 \Vdash g = \overline{n_{01}}$ and for some $C \subseteq B$,

$$\{n_{000}\} : n_{001} \approx C \wedge \{n_{01}\} \in (C \Rightarrow A).$$

And, with e as in Lemma 9.16, we know that

$$\{e\}(\langle n_{000}, n_{001}, n_{01}\rangle) \Vdash n_{01} \in \mathbf{IExp}(\overline{A}, \overline{B}).$$

An application of the realizability of the substitutivity of identity concludes this part of the proof.

On the other hand, if $n \Vdash g \in \mathbf{IExp}(\overline{A}, \overline{B})$, then 9.17 applies and there is a $C \subseteq A$ such that $\{j\}(n)_0 \in \omega$, $\{\{j\}(n)_1\} : \{j\}(n)_0 \approx C$, $\{\{j\}(n)_2\} : C \to A$ and $\{j\}(n)_3 \Vdash g = \{j\}(n)_2$. It follows that there is a partial recursive Θ such that $\langle \Theta(n), i\rangle \Vdash \{j\}(n)_2 \in \overline{\overline{\mathrm{E}}}(A, B)$. Once more, a simple substitution is required to complete the proof. ∎

Finally, we can return to the main proof. Thanks to the last lemma, all we need to show is that

$$\exists e\, \forall A, B \in \mathbf{P}(\omega)\ e \Vdash \overline{\overline{\mathrm{E}}}(A, B) \approx \overline{\mathbf{Exp}(A, B)}.$$

9.20. Definition. For f, g partial functions on ω into ω, we say that $f =_0 g$ iff

$$\forall n\, ((f(n)\downarrow \wedge f(n) > 0) \leftrightarrow (g(n)\downarrow \wedge g(n) > 0)) \text{ and}$$
$$\forall n\, (f(n) > 0 \to f(n) = g(n)).$$

∎

Integral to our proof are two obvious facts of special note:

Fact (1): There is a $k \in \omega$ such that, if

$$\langle\langle f, n, g\rangle \overline{g}\rangle \in \overline{\overline{\mathrm{E}}}(A, B) \text{ then}$$

$$r_{\{k\}(\langle f,n,g\rangle)} =_0 \{g\} \text{ and } \{k\}(\langle f, n, g\rangle) \in \mathbf{Exp}(A, B).$$

Fact (2): There is an $l \in \omega$ such that, if $r_n : A \to B$, then

$$\{\{l\}(n)_2\} : \{l\}(n)_1 \approx \{x : r_n(x) \neq 0\} \subseteq A,$$

$$\{\{l\}(n)_3\} \in (\{x : r_n \neq 0\} \Rightarrow B), \text{ and}$$

$$\{\{l\}(n)_3\} =_0 r_n.$$

In short, if $r_n : A \to B$, then, with $\Theta(n) = \langle\{l\}(n)_2, \{l\}(n)_1, \{l\}(n)_3\rangle$,

$$\langle\Theta(n), \overline{\{l\}(n)_3}\rangle \in \overline{\overline{\mathrm{E}}}(A, B).$$

To realize the purported internal isomorphism between $\overline{\overline{\mathrm{E}}}$ and $\overline{\mathbf{Exp}(A, B)}$, we define a function $\overline{H}$:

9.21. Definition. For $A, B \in \mathbf{P}(\omega)$, set

$$\overline{H} = \{\langle\langle f, n, g\rangle, \overline{\langle\overline{g}, \overline{\{k\}(\langle f, n, g\rangle)}\rangle}\rangle : \langle\langle f, n, g\rangle, \overline{g}\rangle \in \overline{\overline{\mathrm{E}}}(A, B)\}.$$

The index k is that of Fact (1). ∎

There is no question but that $\overline{H} \in \mathbf{V}(Kl)$.

9.22. Lemma. $\mathbf{V}(Kl) \models \overline{H} : \overline{\overline{E}}(A, B) \approx \overline{\mathbf{Exp}(A, B)}$ *uniformly in A and B.*

Proof.

(1) $\mathbf{V}(Kl) \models \overline{H} \subseteq \overline{\overline{\mathrm{E}}}(A, B) \times \overline{\mathbf{IExp}(A, B)}$ and $\mathbf{V}(Kl) \models \overline{H}$ is functional: this follows directly from our earlier results.

(2) $\mathbf{V}(Kl) \models \overline{H}$ is injective: this is an immediate consequence of Fact (1).

(3) $\mathbf{V}(Kl) \models \overline{\overline{\mathrm{E}}}(A, B) \subseteq \mathrm{Dom}(\overline{H})$: once again, this is obtained from Fact (1).

(4) $\mathbf{V}(Kl) \models \overline{\mathbf{Exp}(A, B)} \subseteq \mathrm{Rng}(\overline{H})$: Assume that $e \Vdash a \in \overline{\mathbf{Exp}(A, B)}$. Then, $e_1 \Vdash a = \overline{e_0}$ and $r_{e_0} : A \to B$. By Fact (2),

$$\langle\Theta(e_0), i\rangle \Vdash \overline{\{l\}(e_0)_3} \in \overline{\overline{\mathrm{E}}}(A, B) \text{ and}$$

$$\{\{l\}(e_0)\} = r_{e_0}.$$

Since $r_n =_0 r_m$ implies that $n = m$, Fact (1) shows that

$$\{k\}(\Theta(e_0)) = e_0 \text{ and that}$$

$$\langle \Theta(e_0), \langle \overline{\{l\}(e_0)_3}, e_0\rangle\rangle \in \overline{\overline{\mathrm{E}}}(A,B).$$

Barring the obligatory applications of substitutivity, we are done. ▮

It follows from the various lemmas that IExp is the correct pure analogue of recursion-theoretic exponentiation. The last group of theorems puts us well on the way to extending the isomorphism and preservation theorems to cover Dekker-Myhill exponentiation. Even though (or, even, thanks to the fact that) its definition is unnatural, we know that

$$\mathbf{IExp}(\overline{A}, \overline{B}) \approx \overline{\mathbf{Exp}(A,B)} \subseteq \omega$$

in $\mathbf{V}(Kl)$. That is, unlike the natural function-space concepts, IExp is representable as the internalization of an RET, in fact, as the internalization of the Dekker-Myhill notion Exp.

Cardinals away from zero.

A large measure of the unnaturalness of the Dekker-Myhill notion of exponentiation comes from our general negative results on exponentiation over $\mathbf{V}(Kl)$. It is impossible to find any reasonably natural exponentiation operation on cardinals from $\mathbf{P}(\omega)^{st}$ to fit into the existing scheme. What makes for the impossibility is the discreteness of ω under realizability plus the unsolvability of the halting problem.

The unnaturalness that remains comes from the coding scheme chosen by Dekker and Myhill. Under that scheme, the number 0 becomes an intrusive singularity. Should a different coding be adopted, or should we choose to work on cardinals away from zero, that singularity would disappear.

9.23. Definition. For x, y from $\mathbf{P}(\omega)^{st}$, set

$$\mathbf{JExp} = \{f : \exists z \subseteq x\ \exists n\,.f : z \to y \wedge n \approx f\}.$$

▮

JExp is clearly the set of "finitely indexed" functions from subsets of x into y. If we restrict consideration to x and y "away from zero," then JExp captures the RET notion of exponentiation.

9.24. Theorem. *For $A, B \in \Omega$, if $0 \neq A \cup B$, then*

$$\mathbf{V}(Kl) \models \mathbf{JExp}(\overline{A}, \overline{B}) = \overline{\mathbf{Exp}(A, B)}.$$

Proof. This follows immediately from the lemmas above. ∎

Again, much the same effect could be achieved by altering the coding that goes into the Dekker-Myhill definition of exponentiation. In any case, we know that, with some circumlocution, the isomorphism and preservation theorems can be extended to incorporate exponentiation.

CHAPTER 6

Eliminating Stability: An Extended Note

Section 1: Prefatory and historical remarks

This chapter serves as an appendix to or extended note on the labors of the previous chapter. Were it not for its length, it would have been attached directly to some inconspicuous niche in Chapter Five. The reader should, therefore, think of these two chapters as thematically one. Or better, the reader might well press on to Chapter Seven and leave this note for later.

The concern of this chapter is really with two details. These details seem to be independent, but, happily, they can be dealt with simultaneously. The first concerns a point of mathematical economy as pertains to *stabilization*. As we saw in Chapter Five, the classical theory of recursive equivalence types will disappear into interpreted constructive mathematics, so long as we work over the ω-stable fragment of $\mathbf{P}(\omega)$. From a semantical point of view, stabilization is quite natural. After all, Gödel has already shown that classical mathematics results when constructive mathematics is factored wholesale through a double negation. From a mathematical point of view, however, it all seems quite mysterious.

The mystery comes to light when we reflect on the results of Chapter One. There, we showed that the entire basic theory of RETs can be obtained within IZF+MP by working over $\mathbf{P}(\omega)$. Never once did we have to assume that the sets on which cardinal operations are defined are actually ω-stable! Unlike the stabilization required by Gödel's theorem, which interposes itself continually in any attempt to work out the classical mathematics in an intuitionistic setting, the move from $\mathbf{P}(\omega)$ to $\mathbf{P}(\omega)^{st}$ seems irrelevant to working out the classical mathematics of RETs in a constructive setting. The mathematics of RETs appears over $V(Kl)$ without any reference to stabilization of sets; that comes in only when we apply the Preservation Theorem. So much for mystery.

The second detail pertains to the sheer "size" of IZF. IZF, like its classical cousin ZF, reigns over a luxuriant domain of mathematical objects. Among these are ordinals, Hilbert spaces, function spaces of arbitrarily high types; all of which seem largely irrelevant to the relatively pedestrian concerns of the mathematics of equivalence types of sets of natural numbers. The fundamental theory of the latter seems to require, at most, talk of sets of natural numbers and constructive functions on them. A desire for parsimony of expression might well motivate one to ask after a formal theory for the constructive cardinal arithmetic which is less capacious than full IZF and which treats primarily such sets and functions. It would be desirable if such a theory could link directly with the classical theory of RETs in the fashion of Chapter Five. Such a formal theory would, most likely, be a version of the second-order arithmetic HAS.

This note addresses both details simultaneously. We describe a theory HAS^{st}, which is a modification of HAS. In that theory, all of the work of Chapter One on cardinals in $P(\omega)$ can be carried out. This takes care of the second detail: HAS^{st} is constrained syntactically to direct its attention to numbers, sets and functions. Then, we present a realizability model for that theory. The model is a variation on the Kreisel-Troelstra idea of Chapter Four. Using the realizability as an interim technique, we can prove that HAS^{st} so-interpreted offers a solution for the first detail as well. We prove that, over the realizability structure, the entire classical theory of RETs appears as an interpreted subtheory—without relativization of set quantifiers to stable sets. As a result, one sees that the framework provided by HAS^{st} is an excellent place within which to develop the constructive analogue of RET-theory. The realizability result shows that, by adding extra axioms to those of HAS^{st}, the classical theory can be rebuilt constructively without loss and without set-stabilization. Where did the stabilization go? It has been built into the new realizability interpretation itself.

There is another, more unified, way in which to think of the themes of this note. One can think of it as a "narrow circumscription" of syntax and semantics. The usual approach to the metamathematics of constructive arithemtic provides a good example of this circumscription. Even though both arithmetic and Kleene realizability are interpretable into set theory and $V(Kl)$, respectively, each of them retains their usefulness as a "narrow circumscription." We mean that, if one cares to work in constructive arithmetic with a realizability interpretation attached, $\mathbf{V}(Kl)$ is just too big. One needs a narrow view that focuses on that circumscribed part of $\mathbf{V}(Kl)$ that pertains only to arithmetic. Original Kleene realizability does just that. It would be far too confusing to look to set-theoretic realizability whenever one is concerned only with constructive arithmetic. Far better to be limited by the realizability interpretation that pertains properly to arithmetic.

One might have similar thoughts about recursive equivalence types. If our desire was to concentrate efforts on constructive cardinal arithmetic, then it would be better to take a narrow view. Here, a narrow view would be on a syntax adequate for RET-theory and a realizability that applies directly to that syntax, without the intermediary of set theory. The syntax we propose is that of HAS^{st}. The realizability will be that of $\Vdash^{s}$. This realizability is an unmediated picture of the $\mathbf{V}(Kl)$-interpretation relativized to $\mathbf{P}(\omega)^{st}$, just as Kleene realizability is an unmediated picture of $\mathbf{V}(Kl)$ as relativized in ω. In effect, $\Vdash^{s}$ is a capsule summary of what $\mathbf{V}(Kl)$ makes true over $\mathbf{P}(\omega)^{st}$, and, together with HAS^{st}, it gives a narrow circumscription of the mathematics of RETs.

Therefore, in circumscribing realizability over $\mathbf{P}(\omega)^{st}$, we are paring things down considerably. The benefit of the circumscription is that the "building in" process of Chapter Five is complete. In $\Vdash^{s}$, not only is all the explicit recursion of classical RET-theory built into the logical signs of constructive mathematics, but so is all the classical set theory. That is, we need not relativize explicitly to the stable part of $\mathbf{P}(\omega)$. Hence, a fair bit of tedium is avoided.

However, this paring-down of realizability relative to classical quantification does carry a price tag. The paring-down eliminates from the realizability universe all of those pathological sets that live in $\mathbf{P}(\omega)$ outside of $\mathbf{P}(\omega)^{st}$. These sets are attached, under $\mathbf{V}(Kl)$-realizability, to comprehension terms defined over ω. Therefore, comprehension terms which would naturally specify subsets of ω will, under the realizability described in this section, come to delineate *proper classes*.

Section 2: Stable Heyting arithmetic and realizability

Full second-order Heyting arithmetic, HAS, should now be a familiar creature. *In fine*, HAS incorporates the mechanisms for first and second-order number-theoretic quantification, together with the axioms for full arithmetic which are the heritage of Dedekind and Peano. For a description of the language, the reader can refer to section 5 of Chapter Five or to the relevant sections of Troelstra (1973b). We recall that our version of HAS contains, as a primitive, a distinct function sign for each primitive-recursive function scheme. We also assume that the language contains second-order variables for unary species only. Finally, we presume that the only well-formed formulae of the form $X(\tau)$, for X a species variable or constant, are those in which τ is an individual variable or canonical numeral.

Axioms for HAS include all the axioms characteristic of HA and those specifying the graphs of the primitive recursive function signs. The induction scheme of elementary arithmetic has been replaced by full second-order induction. There is also the usual comprehension scheme:

$$\exists X\, \forall x\, (X(x) \leftrightarrow \phi(x)).$$

To go from HAS to the more restrictive HAS^{st}, we will replace full induction by an induction scheme and allow comprehension only with respect to almost negative formulae of the language.

2.1. Definition. For ϕ from the language of HAS, ϕ is *second-order almost negative* (*a.n.2*) iff no $\vee$ occurs in ϕ and $\exists x$ occurs only immediately prior to atomic first-order subformulae. ∎

a.n.2 is the notion of "almost negative" appropriate to second-order arithmetic. The reader should recall the correlative first-order notion of a.n. from Section 4 of Chapter Four.

2.2. Definition. HAS^{st} is defined to be the formal system which varies from HAS in that full second-order induction is replaced by the induction scheme and comprehension is restricted to those second-order formulae which are a.n.2. Otherwise HAS^{st} and HAS are identical. ∎

By restricting comprehension, we can, in the realizability model, implicitly restrict the second-order variables to ranging only over ω-stable subsets. The imposition of the induction scheme represents our admission that, in some models of HAS^{st}, not every comprehension term will specify a set. HAS^{st} is as expressive as HAS, so it can specify collections which, under realizability, fail to be ω-stable. However, even in realizability models, we wish ω to retain its full inductive properties.

Stable realizability.

We go directly to defining the desired "nonstandard" realizability for HAS^{st}. Realizability for HAS^{st} is defined precisely as was realizability for HAS, except that quantification is restricted to the realizability sets in the image of the injection $x \mapsto \overline{x}$. With this restriction, quantification in HAS^{st} is restricted automatically to the ω-stable sets.

2.3. Definition. For each unary number-theoretic relation A in $\mathbf{V}$, we set

$$\overline{A} = \{\langle n, n\rangle : \mathbf{V} \models A(n)\}.$$

∎

The apposite realizability notion, $\Vdash^s$, is defined as follows:

2.4. Definition. Assume that the language of HAS^{st} includes autonymous names for each $n \in \omega$ and each $\overline{A}$. $\Vdash^s$ is defined directly for the sentences ϕ of this expanded language. The definition of $\Vdash^s$ agrees with that of Kreisel-Troelstra realizability (cf. Section 5 of Chapter Four) except that

(1) for ϕ atomic and second-order,

$$e \Vdash^s \overline{A}(n) \text{ iff } e = n \text{ and } \mathbf{V} \models A(n).$$

(2) $e \Vdash^s \forall X\, \phi$ iff, for all $\overline{A}, e \Vdash^s \phi(X/\overline{A})$ and

(3) $e \Vdash^s \exists X\, \phi$ iff, for some $\overline{A}, e \Vdash^s \phi(X/\overline{A})$.

The structure for this interpretation we call '$2^{st}(Kl)$' and, for sentences ϕ, we say that $2^{st}(Kl) \models \phi$ whenever $\exists n\, n \Vdash^s \phi$. ∎

It is now a simple exercise to check that HAS^{st} provides a base for the right set theory under this notion of realizability:

2.5. Theorem. $\mathrm{HAS}^{st} \vdash \overline{\forall} \phi$ *only if* $2^{st}(Kl) \models \overline{\forall} \phi$.

Proof. In the theorem statement '$\overline{\forall}$' represents universal closure on all free variables—both first- and second-order. Very little manipulation is required to see that $2^{st}(Kl)$ satisfies all of the second-order consequences of the HA axioms, including all instances of the induction scheme. It remains only to check on the validity of the restricted comprehension axiom, and, for that, we isolate a crucial lemma:

2.6. Lemma. *If ϕ is a.n.2, then ϕ is ω-stable over $2^{st}(Kl)$:*

$$2^{st}(Kl) \models \forall x\,(\neg\neg\phi \to \phi).$$

Proof. In the statement of the lemma, we presumed that, for the sake of illustration, ϕ contains at most one free first-order variable.

Since Markov's principle MP holds in $2^{st}(Kl)$, it will suffice to show (1) that the relevant second-order atomic formulae are ω-stable and (2) that ω-stability is closed under second-order existential quantification.

(1) Given n, assume that, for some m, $m \Vdash^{s} \overline{A}(n)$. By definition, $m = n$ and $A(n)$ holds. Hence, $\Lambda n \Lambda e.n$ realizes that $\overline{A}$ is ω-stable.

(2) Given n, assume that, for some m, $m \Vdash^{s} \exists X\ \phi$. By definition, $\exists \overline{A}\ \exists m\ \ m \Vdash^{s} \phi(X/\overline{A})$. The inductive hypothesis tells us $\phi(X/\overline{A})$ is ω-stable, so there is an $e \in \omega$ such that

$$\{e\}(n) \Vdash^{s} \phi(X/\overline{A}).$$

Therefore,

$$\{e\}(n) \Vdash^{s} \exists X\ \phi,$$

and the proof is complete. ∎

Now we can return to the proof of the theorem. Given ϕ a.n.2, we consider the ϕ-instance of the comprehension axiom. Let

$$\overline{A} = \{\langle n, n\rangle : 2^{st}(Kl) \models \phi(n)\}.$$

All one needs to show is that

$$2^{st}(Kl) \models \forall x\,(\neg\neg\overline{A}(x) \leftrightarrow \neg\neg\phi(x)),$$

which, by the lemma and the earlier parts of the proof, will suffice. But the truth of the displayed line follows directly from the definition of $\overline{A}$, since

$$2^{st}(Kl) \models \overline{A}(n) \text{ iff } 2^{st}(Kl) \models \phi(n).$$

∎

We should also record the fact that, because of the presence of MP in $2^{st}(K\ell)$, a.n.2 comprehension is coextensive with ω-stable comprehension:

2.7. Proposition. *If ϕ is ω-stable over $2^{st}(K\ell)$, then ϕ is semantically equivalent to an a.n.2 formula of* HAS^{st}.

Proof. This is immediate by intuitionistic logic. ∎

Section 3: Mathematics in $2^{st}(K\ell)$

At first sight, $2^{st}(K\ell)$ supports what appears to be a number of mathematical anomalies. For instance, there are comprehension terms which are classically identical to ω but which, when evaluated over $2^{st}(K\ell)$, specify proper classes. $2^{st}(K\ell)$ also satisfies a *full* Church's Thesis. On a wider view, however, one sees that the anomalies are merely representative of regularities characteristic of $\mathbf{P}(\omega)^{st}$, when $\mathbf{P}(\omega)^{st}$ is considered as a universe of sets. The wider view is one that includes the translation theorems of the next section. With those translations, the anomalies can all be identified with results already known to us.

3.1. Proposition. *With T as the unary Kleene "T" predicate, the comprehension term*

$$\{n : \exists m\, T(n,n,m) \vee \neg \exists m\, T(n,n,m)\}$$

specifies, in $2^{st}(K\ell)$, a proper class.

Proof. Set $\phi(x) = \exists m\, T(x,x,m) \vee \neg \exists m\, T(x,x,m)$. By logic, we have

$$\mathrm{HAS}^{st} \vdash \forall x\ \neg\neg\phi.$$

By the preceding lemma, $2^{st}(K\ell)$ satisfies

$$\forall X\, \forall n\, (\neg\neg X(n) \rightarrow X(n)).$$

Therefore, were ϕ to specify a value for a second-order variable over $2^{st}(K\ell)$, then we would have $2^{st}(K\ell) \models \exists X\, \forall x\, (X(x) \leftrightarrow \phi)$. It then follows from the two displayed lines that

$$2^{st}(K\ell) \models \forall x\, \phi.$$

But that would clearly flout the unsolvability of the halting problem. Therefore, $\{x : \phi(x)\}$ represents a proper class in $2^{st}(Kl)$. ∎

$2^{st}(Kl)$ also satisfies what seems to be a very strong Church's Thesis, CT_3, a principle asserting that functions between *arbitrary* sets are computable. From the satisfaction of CT_3, it is a trivial matter to prove that ECT_0 holds in $2^{st}(Kl)$. Moreover, from the translation results of the final section, it will be easy to transfer this result to $V(Kl)$ and, thence, to see that CT_3 has the same realizability conditions as CT_2 (cf. Chapter Four, Section 3).

3.2. Definition. CT_3 is the statement

$$\forall X\ \forall Y\ [\exists F : (X \to Y)\ \to \exists e\ \forall x\ (X(x) \to \exists y\ T(e,x,y) \land U(y, F(x)))].$$

∎

3.3. Note. Since all the primitive second-order variables and constants are unary, the binary 'F' of CT_3 can be taken as shorthand for a "coded" unary sign. ∎

3.4. Proposition. $2^{st}(Kl) \models CT_3$.

Proof. Assume that $e \Vdash^{s} \forall x\ (\overline{A}(x) \to \exists y\ (\overline{B}(y) \land F(x,y)))$. Then, for $n \in A$,

$$\{\{e\}(n)\}(n)_1 \Vdash^{s} \overline{B}(\{\{e\}(n)\}(n)_0) \land F(n, \{\{e\}(n)\}(n)_0).$$

Let $g \in \omega$ be such that, for all n, $\{g\}(n) \simeq \{\{e\}(n)\}(n)_0$. By the classical *s-m-n* Theorem, g is $\Theta(e)$ for some primitive recursive Θ. Our intention is to show that a realizing witness for

$$\forall x\ (\overline{A}(x) \to \exists y\ (T(g,x,y) \land U(y, F(x))))$$

is calculable effectively from e. To that end, assume that $e \Vdash^{s} \overline{A}(n)$. Then, $e = n$ and $e \in A$. Therefore, $\{g\}(n)\downarrow$ and for some m,

$$\models T(g,n,m) \land U(m, \{g\}(n)).$$

m is, of course, effectively calculable from g and n. Moreover, T and U are atomic, so the proof is complete. ∎

3.5. Corollary. $2^{st}(Kl) \models \mathrm{ECT}_0$.

Proof. One uses a.n.2 comprehension to show that $\mathrm{HAS}^{st} \vdash \mathrm{CT}_3 \to \mathrm{ECT}_0$. ∎

In contrast with the above, the second-order version of CT_1 does not hold; once again, CT_1 fails because of the unsolvability of the halting problem. (Cf. Chapter Four.) Expressed in the second-order idiom,

3.6. Definition. CT_1 is

$$\forall x\, \forall y\, \forall z\, [(F(x,y) \wedge F(x,z) \to y = z) \to \exists e\, \forall x\, \forall y\, (F(x,y) \leftrightarrow \{e\}(x)\downarrow \wedge F(x, \{e\}(x))].$$

∎

To see the failure of CT_1 under $\Vdash^{s}$, we take F to be the coded graph of the characteristic function of the halting problem and set $\overline{F}$ accordingly. Now, we know that

$$2^{st}(Kl) \models \forall x\, \forall y\, \forall z\, (\overline{F}(x,y) \wedge \overline{F}(x,z) \to y = z).$$

This is due simply to the fact that F is externally functional and that the natural numbers bear the same absoluteness relations to $2^{st}(Kl)$ that they do to $\mathrm{V}(Kl)$. However, were $2^{st}(Kl)$ to satisfy CT_3, there would be an $e \in \omega$ such that

$$2^{st}(Kl) \models [\overline{F}(x,y) \leftrightarrow \{e\}(x)\downarrow \wedge \overline{F}(x, \{e\}(x))].$$

Since $2^{st}(Kl) \models \mathrm{MP}$, $2^{st}(Kl)$ also satisfies

$$\forall x\, (\neg\neg \exists y\ \overline{F}(x,y) \to \exists y\ \overline{F}(x,y)).$$

Classically, F is total, so, over realizability in $2^{st}(Kl)$, $\overline{F}$ is almost total:

$$2^{st}(Kl) \models \neg\neg \exists y\ \overline{F}(x,y).$$

Therefore,

$$2^{st}(Kl) \models \forall x\, \exists y\ \overline{F}(x,y).$$

This would mean that F is total recursive, which it most decidedly is not. We have, then

3.7. Theorem. $2^{st}(Kl) \models \neg\mathrm{CT}_1$.

$\Vdash^s$ supports a quite general version of AC, which we call 'AC^X'. This choice principle should be compared with that of Chapter Five, Section 5.

3.8. Definition. AC^X is

$$\forall x\,(X(x) \to \exists y\,\phi) \to \exists F\,\forall x\,(X(x) \to \phi(x/F(x))).$$

▮

3.9. Theorem. $2^{st}(Kl) \models \mathrm{AC}^X$.

Proof. When a form of AC holds in the "Heyting universe," the domain of all proofs and constructions, it does so in virtue of the proof conditions of the statement's antecedent. This situation is perfectly reflected in $2^{st}(Kl)$: it is from the realizability witness of the antecedent that we extract the requisite choice functions. Here is a case in which realizability (and intuitionism) make an old saw *literally* true: whenever an inference obtains, the truth conditions of the consequent are contained in those of the antecedent.

We begin with the assumption that

$$e \Vdash^s \forall x\,(\overline{A}(x) \to \exists y\,\phi).$$

Then, if $n \in A$,

$$\{\{e\}(n)\}(n)_1 \Vdash^s \phi(n, \{g\}(n))$$

where $g = \Lambda n.\{\{e\}(n)\}(n)_0$. Again, g is effectively calculable from e.

The required choice function is the obvious internalization of $\{g\}$. Let

$$F = \{\langle n, \{g\}(n)\rangle : n \in A\}.$$

It is clear that a realizer for $\forall x\,(\overline{A}(x) \to \phi(y/\overline{F}(x)))$ is obtainable from e. ▮

3.10. Note. We have already (in Chapter Five) uncovered the fact that a version of full AC holds over the ω-stable subsets of ω in $\mathbf{V}(Kl)$. Under the translation of the next section, the fact that $2^{st}(Kl) \models \mathrm{AC}^X$ will come to coincide precisely with the former fact. This "coincidence" is an indication that the claims of earlier sections of this chapter are correct: that HASst under $2^{st}(Kl)$ embodies the realizability mathematics true over $\mathbf{P}(\omega)^{st}$.

Section 4: Translation theorems

Given our ultimate designs, any more of this piecemeal checking of axioms would not only be dreary but otiose. We set out to see how results about $2^{st}(Kl)$ might be obtained by relativization from $\mathbf{V}(Kl)$. In particular, we want to make the realizability version of RET theory carry over to the new structure. In keeping with that, we ought to eschew further evaluation of specific principles and turn toward a translation theorem that will allow us to calculate the realizability conditions of HASst-statements automatically by relativization to $\mathbf{P}(\omega)^{st}$ over $\mathbf{V}(Kl)$.

For purposes of the present chapter (and for no other), we will use '$\overline{\overline{A}}$' to stand for the result of injecting $A \in \mathbf{P}(\omega)$ into $\mathbf{V}(Kl)$ in the standard fashion. We coin the new notation only because $\overline{A}$ is already performing a duty in the service of $2^{st}(Kl)$.

4.1. Definition. For each $A \in \mathbf{P}(\omega)$ in $\mathbf{V}$, we set

$$\overline{\overline{A}} = \{\langle n, \overline{n}\rangle : A(n)\}.$$

4.2. Assumption. In order to avoid clashes in translating the two-sorted second-order language into the single-sorted language of IZF, we assume that there is an extra, hithertofore untapped source of set variables, one variable for each second-order variable X. For X, the corresponding "extra" set parameter is denoted 'x^X.' ∎

At this stage, the reader might consult Section 5 of Chapter Four to refresh his memory on the translation under consideration there.

4.3. Definition. The translation $\phi \stackrel{\cdot}{\mapsto} \Phi$ is defined on the formulae of the language of HASst by associating with each formula ϕ of HASst a formula Φ of the language of set theory. The translation agrees with that of Chapter Four except that

[1] for $\phi = \overline{A}(n), \Phi = \overline{n} \in \overline{\overline{A}}$

[2] for $\phi = \overline{A}(x), \Phi = x \in \overline{\overline{A}}$

[3] for $\phi = \forall X\ \psi, \Phi = \forall x^X \in \mathbf{P}(\omega)^{st}\ \Psi$

[4] for $\phi = \exists X\ \psi, \Phi = \exists x^X \in \mathbf{P}(\omega)^{st}\ \Psi$.

∎

Now we can prove a theorem relating $2^{st}(Kl)$, $\mathrm{V}(Kl)$ and the above translation.

4.4. Theorem. *Let ϕ be a sentence of the language of* HAS^{st}. $2^{st}(Kl) \models \phi$ *iff* $\mathrm{V}(Kl) \models \Phi$.

This proof mimics those of correlative translation theorems in Chapter Four. The proof arises by way of what is, in effect, an isomorphism between realizability structures. This isomorphism makes for perfect parity of atomic formulae. As before, the correspondence engendered by isomorphism must be carried up the hierarchy of formulae by effective maps.

What follows are the lemmas that do the "carrying." We give only the merest indications of their proofs.

4.5. Lemma. *There is a $j \in \omega$ such that, for all A, $j \Vdash \overline{\overline{A}} \in \mathbf{P}(\omega)^{st}$.*

Proof. This is precisely the lemma 2.3 from Chapter Five. ∎

4.6. Lemma. *There is a $j \in \omega$ such that for all $b \in \mathrm{V}(Kl)$, if $e \Vdash b \in \mathbf{P}(\omega)^{st}$, then $\{j\}(e)\downarrow$ and, for some A from* V, *$\{j\}(e) \Vdash b = \overline{\overline{A}}$.*

Proof. One need only apply the proof of Lemma 2.12 in Chapter Five. ∎

For ease of notation, we will now assume that ϕ from the language of HAS^{st} has at most two numerical and two unary set parameters. The truth of our result suffers no such restriction.

4.7. Lemma. *Let $\phi(x, y, X, Y)$ be a formula of the* HAS^{st} *language. There are partial recursive indices e_ϕ and g_ϕ such that*

whenever $p \Vdash^{s} \phi(\overline{n}, \overline{m}, \overline{A}, \overline{B})$, $\{e_\phi\}(p, n, m)\downarrow$ and $\{e_\phi\}(p, n, m) \Vdash \Phi(\overline{n}, \overline{m}, \overline{\overline{A}}, \overline{\overline{B}})$, and

whenever $p \Vdash \Phi(\overline{n}, \overline{m}, \overline{\overline{A}}, \overline{\overline{B}})$, $\{g_\phi\}(p, n, m)\downarrow$ and $\{g_\phi\}(p, n, m) \Vdash^{s} \phi(\overline{n}, \overline{m}, \overline{A}, \overline{B})$.

Proof. After all the work we did in Chapter Four, there's very little left to do. Actually, the clauses of the translation governing second-order quantification need be our sole concern.

We consider existential second-order quantification: if $p \Vdash^{s} \exists X\, \phi(\overline{n}, \overline{m}, \overline{A}, \overline{B})$, then, for some C from $\mathbf{P}(\omega)$ in V, $p \Vdash^{s} \phi(X/\overline{C})(\overline{n}, \overline{m}, \overline{A}, \overline{B})$. Hence, for some such C,

$$\{e_\phi\}(p, n, m)\downarrow \text{ and } \{e_\phi\}(p, n, m) \Vdash \Phi(X/\overline{\overline{C}})(\overline{n}, \overline{m}, \overline{\overline{A}}, \overline{\overline{B}}).$$

It follows that

$$\langle j, \{e_\phi\}(p,n,m)\rangle \Vdash (\exists x^X \in \mathbf{P}(\omega)^{st}\ \Phi)(\overline{n}, \overline{m}, \overline{\overline{A}}, \overline{\overline{B}}),$$

where j is given by Lemma 4.5.

Conversely, $p \Vdash \exists x^X\ (x^X \in \mathbf{P}(\omega^n)^{st} \wedge \Phi)(\overline{n}, \overline{m}, \overline{\overline{A}}, \overline{\overline{B}})$ only if

$$\text{for some } C, \rho(p) \Vdash \Phi(x^X / \overline{\overline{C}})(\overline{n}, \overline{m}, \overline{\overline{A}}, \overline{\overline{B}}).$$

ρ is given by applying Lemma 4.6 and the substitutivity of identity for Φ. From this, it follows immediately that

$$\{g_\phi\}(\rho(p), n, m) \Vdash^s \exists X\ \phi(\overline{n}, \overline{m}, \overline{A}, \overline{B}).$$

A similar argument gives the proof for the universal quantifier. This completes the proofs of the lemma and of the main theorem. ∎

The desired result is now a corollary:

4.8. Corollary. *For sentences of ϕ of the augmented* HAS^{st} *language,*

$$2^{st}(Kl) \models \phi \text{ iff } \mathbf{V}(Kl) \models \Phi.$$

Proof. ∎

In summary, $2^{st}(Kl)$ is precisely as advertised. $\Vdash^s$-realizability for HAS^{st} is just the relativization of $\Vdash$ to $\mathbf{P}(\omega)^{st}$. We now see that the realizability of CT_3 is a register of the fact that, in $\mathbf{V}(Kl)$, a version of Church's Thesis holds on the ω-stable sets. Similarly, AC^X mirrors in $2^{st}(Kl)$ the choice principle for ω-stables over $\mathbf{V}(Kl)$. Hence, we need not have checked these results directly in $2^{st}(Kl)$ but could have obtained them via translation from $\mathbf{V}(Kl)$.

Section 5: Hiding the relativization

We now have at our disposal all the equipment required to check the claims of the first section of this chapter. We first indicate how the constructive mathematics of cardinal arithmetic on ω-stable subsets can be developed in HAS^{st}. This development will take place without relativizing quantifiers to stable subsets. Second, we will show that, under realizability in $2^{st}(Kl)$, all of the classical theory of Ω can be captured in the language of HAS^{st}. Again, we can hide the recursivity of the RETs in the logical signs as interpreted over realizability. Moreover, we can now hide the relativization of quantifiers to $\mathbf{P}(\omega)^{st}$ within $2^{st}(Kl)$ and leave pure constructive set quantification in its stead.

We recall that $\underline{\Omega}$, the elementary structure of RETs, is

$$\langle \Omega, +, \times, \leq, \simeq, 0, \omega \rangle$$

and that L_Ω is a diagram language for $\underline{\Omega}$. (Cf. Section 3 of Chapter Five.) Each $\phi \in \mathrm{Form}_{\mathrm{L}_\Omega}$ will, by structural induction, be assigned a translation *str* into the augmented language of HAS^{st}. First, *str* is defined for primitive constants. Given that HAS^{st} allows a.n.2 comprehension, we can make use of comprehension terms in specifying the interpretation *str*.

5.1. Definition. $0^{str} = \overline{0}$, $\omega^{str} = \{x : x = x\}$, $A^{str} = \overline{A}$. ∎

For interpreting the compound terms, one needs to be apprised of the existence in $2^{st}(Kl)$ of certain abstracts. The following proposition shows that the required abstract will exist. Again, $(\ )_0$ and $(\ )_1$ denote projections relative to a convenient primitive recursive pairing.

5.2. Proposition.

$$\mathrm{HAS}^{st} \vdash \exists Z\, \forall n\, [Z(n) \leftrightarrow ((n_0 = 0 \wedge X(n_1)) \vee (n_0 = 1 \wedge Y(n_1)))]$$
$$\mathrm{HAS}^{st} \vdash \exists Z\, \forall n\, [Z(n) \leftrightarrow (X(n_0) \wedge Y(n_1))].$$

Proof. The second entailment is immediate because the right hand side of the defining biconditional is a.n.2. The first is almost as easy; the right hand side of its biconditional is HAS^{st}-provably equivalent to the a.n.2 predicate

$$(n_0 < 2) \wedge (n_0 = 0 \rightarrow X(n_1)) \wedge (n_0 = 1 \rightarrow Y(n_1)).$$

∎

5.3. Definition. For $\sigma, \tau \in \mathrm{Term}_{L_\Omega}$, *str* is defined inductively on the term structure using comprehension terms liberally:

(1) $(\sigma+\tau)^{str} = \{x : (x_0 = 0 \wedge \sigma^{str}(x_1)) \vee (x_0 = 1 \wedge \tau^{str}(x_1))\}$

(2) $(\sigma \times \tau)^{str} = \{x : \sigma^{str}(x_0) \wedge \tau^{str}(x_1)\}$

∎

It is easy to check that all the comprehension terms which correspond to terms from the language of Ω exist as sets in $2^{st}(Kl)$.

5.4. Proposition. *For all* $\tau \in \mathrm{Term}_{L_\Omega}$,

$$2^{st}(Kl) \models \exists X\, \forall n\, (X(n) \leftrightarrow \tau^{str}(n)).$$

Proof. We use structural induction, the preceding proposition and the definition. ∎

Next, the sentences of L_Ω are interpreted by *str* so that, for each atomic sentence, its "purely set theoretic component" is retained in its obvious expression over $2^{st}(Kl)$. For instance, $A \leq B$ has, as its *str*-translate, the sentence

$$\exists C\, \exists F\, (F : \overline{A} \rightarrowtail\!\!\twoheadrightarrow \overline{B} \wedge C \text{ is a decidable subset of } \overline{B}).$$

5.5. Definition.

(1) $(\sigma \simeq \tau)^{str} = \exists F \;\; F : \sigma^{str} \rightarrowtail\!\!\twoheadrightarrow \tau^{str}$

(2) $(\sigma \leq \tau)^{str} = \exists C\, \exists F \;\; F : \sigma^{str} \rightarrowtail\!\!\twoheadrightarrow C \wedge C$ is a decidable subset of τ^{str}

(3) *str* commutes with $\wedge$, $\vee$, $\neg$ and $\rightarrow$

(4) $(\exists x\, \phi)^{str} = \exists X\, \phi^{str}$

(5) $(\forall x\, \phi)^{str} = \forall X\, \neg\neg\phi^{str}$

∎

It should go without saying that in the clauses pertaining to the quantifiers, variables are so chosen that clashes and redundancies do not occur. The tendency of all this should be clear; the idea is to prove that, for $\phi \in \mathrm{Sent}_{L_\Omega}$,

$$\Omega \models \phi \text{ iff } 2^{st}(Kl) \models \phi^{str}.$$

Given the main theorem of the last section, we can prove this result and yet circumvent the tedium that made Chapter Five so monstrous a read. To prove the above, it will suffice to show that translation *str* results when the language of Ω is translated into set theory as in Chapter Five and then set theory is "translated back" into the language of HASst by reversing the translation of the last section. In short, using '*st*' for the translation of the last section, all we need show is

5.6. Theorem. *For each* $\phi \in \mathrm{Sent}_{L_\Omega}$,

$$\mathbf{V}(Kl) \models (\phi^{str})^{st} \leftrightarrow \phi^{tr}.$$

Proof. The proof of this theorem is nothing more than a simple check. The only "new fact" that one needs is a fact about functions in $\mathbf{V}(Kl)$: if f is, in $\mathbf{V}(Kl)$, a map between ω-stable sets, then the graph of f is itself ω-stable. ∎

Finally, combining Corollary 4.8 with the above, we obtain the desired theorem. This is truly the desired theorem because of the clauses that govern quantification under *str*. We note that, under *str*, we need not relativize to the ω-stable sets.

5.7. Corollary. $\underline{\Omega} \models \phi$ *iff* $2^{st}(Kl) \models \phi^{str}$.

To complete the foray into nonstandard set theory and "stable realizability," one merely checks that the theory of RETs can actually be developed in HASst over $2^{st}(Kl)$. This amounts to ascertaining that the restrictions on comprehension do not stand in the way of any of the proofs in pure cardinal arithmetic surveyed in Chapter One. Indeed, should one care to consult the proofs in Section Four of that Chapter, he will find that no comprehension term even gives the appearance of calling for expression in non-a.n.2 style. Therefore, we can say with confidence that all our labors on constructive cardinals could have been carried out in HASst without restriction to stable sets. Similar remarks apply to Dedekind-finite cardinals, because a subset of ω is infinite in IZF if and only if it is infinite thanks to an ω-stable function:

$$\mathrm{IZF} \vdash \forall x \in \mathbf{P}(\omega)\, [\exists f\, (f : \omega \rightarrowtail x) \leftrightarrow \exists f \in \mathbf{P}(\omega^2)^{st}\;\; f : \omega \rightarrowtail x).$$

Section 6: Assessing the alternatives

In a way, our elimination of relativization is still somewhat unsatisfactory, because, in so doing, we have moved outside the set-theoretic framework. It would have been far more elegant and consonant with our general perspective (the set-theoretic one) to have carried out the elimination of stability by restriction not of comprehension in HAS but of separation in full IZF.

Sadly, the proper restriction to set theory is not immediately apparent and it is the task of this section to indicate why this is so. Presumably, an appropriate generalization of ω-stability would suggest the commensurate restriction of separation. It is unfortunate that the concept of ω-stability is very closely attached to ω, so much so that it is nowise obvious what the right generalization should be. There are things one might try (we won't pursue them fully here). For one, there is the aforementioned affinity between ω-stable sets and the axiom of choice; the ω-stable sets are those which enjoy full number-theoretic choice.

There is an intuitive explanation of this choice phenomenon: that the properties of ω-stable sets are predicated more on the arrangement of the elements of the set than on the arrangement of the possible "proofs." We elaborate on this explanation later in the section. General ω-stable sets may well be those for which, as realizability sets, there is an E-recursive choice function from the elements into the evidence. The function we have in mind would, given a realizing witness, choose an element for which the witness is evidence of membership. Unfortunately, this approach will have to await the development of E-realizability.

Many other natural alternatives to eliminating stability by restriction of comprehension do not come rushing up. There is one alternative which seems favorable. Given the number and variety of "copies" and "versions" of normal sets in $\mathbf{V}(Kl)$, we might well ask after an "ω-like" set whose subsets are just the ω-stable sets. However, the very feature of $\mathbf{V}(Kl)$ on which this suggestion relies undercuts it immediately. With unrestricted separation, there simply is no such ω-variant. ω is, of course, ω-stable in itself. Hence, if A is the desired ω-variant, $\omega \subseteq A$. Therefore, by full separation, the entire multitude of nonstable subsets of ω live in $\mathbf{P}(A)$.

On the other hand, perhaps the mathematical facts about RETs tell us something important for semantics. The details of cardinal arithmetic from Chapter One encourage the formulation of an even more radical alternative to elimination. None of the proofs from Sections 3 and 4 of Chapter One called explicitly for the assumption of ω-stability. Isn't it possible that no conceivable proof from the theory of RETs as pure cardinals over $\mathbf{V}(Kl)$ need make such a call? Isn't it possible that ω-stability, although making smooth the way of the isomorphism and preservation theorems, has no *real* semantic relevance? Sadly, answers to these questions are not forthcoming; at this stage, we know of no RET- or isol-theoretic property not shared by both $\mathbf{P}(\omega)$ and $\mathbf{P}(\omega)^{st}$.

However, there is some evidence for the conclusion that, as far as RET-properties are concerned, $\mathbf{P}(\omega)^{st}$ and $\mathbf{P}(\omega)$ diverge. It is a fact that the obvious model-theoretic strategy for linking the cardinal-theoretic universes of $\mathbf{P}(\omega)$ and $\mathbf{P}(\omega)^{st}$ fails. $\mathbf{P}(\omega)^{st}$ is clearly a retract of $\mathbf{P}(\omega)$, but the retraction does not preserve cardinal-theoretic properties. It seems, therefore, that the relevant properties of the domains are not the same.

In terms of sets and of identity, $\mathbf{P}(\omega)^{st}$ is a retract of $\mathbf{P}(\omega)$ under the operation $\omega^{\neg\neg}$, where for $A \subseteq \omega$,

$$\omega^{\neg\neg}(A) = \{x \in \omega : \neg\neg x \in A\}.$$

This retraction demonstrably fails to carry the right mathematical properties along with it. Specifically, there are realizability sets A and B which are subsets of ω such that

$$\mathbf{V}(Kl) \models A \approx B \text{ but } \mathbf{V}(Kl) \models \neg(\omega^{\neg\neg}(A) \approx \omega^{\neg\neg}(B)).$$

6.1. Theorem. *There are* $A, B \in \mathbf{V}(Kl)$ *such that* $\mathbf{V}(Kl) \models A, B \in \mathbf{P}(\omega) \wedge A \approx B$, *but* $\mathbf{V}(Kl) \models \neg(\omega^{\neg\neg}(A) \approx \omega^{\neg\neg}(B))$.

Proof. Reasoning externally, let K be the Kleene set:

$$\{n \in \omega : \{n\}(n)\downarrow\}$$

and let $\neg K$ be its classical complement. Now, we set

$$A = \{\langle\langle k_n, n\rangle, \overline{n}\rangle : n \in \omega\},$$

where $\langle k_n\rangle_{n\in\omega}$ is a (nonrecursive) listing of $\neg K$ in increasing order. Take

$$B = \{\langle\langle k_n, n\rangle, \overline{k_n}\rangle : n \in \omega\}.$$

First, $\Lambda e\langle e_1, e_{01}\rangle \Vdash A \subseteq \overline{\omega}$. For, if $e \Vdash a \in A$, then $e_1 \Vdash a = \overline{e_{01}}$. Next, it is just as easy to see that $\Lambda e\langle e_1, e_{00}\rangle \Vdash B \subseteq \overline{\omega}$. Third, $\mathbf{V}(Kl) \Vdash A \approx B$. To see this, we take

$$f = \{\langle\langle k_n, n\rangle, \overline{\langle \overline{n}, \overline{k_n}\rangle}\rangle : n \in \omega\}.$$

Clearly, $\mathbf{V}(Kl)$ satisfies

$$f \subseteq A \times B \wedge f \text{ is functional} \wedge f \text{ is injective.}$$

To see that f is total on A, let $e \Vdash a \in A$. Then, $e_0 = \langle k_n, n\rangle$ and $e_1 \Vdash a = \overline{n}$. To finish, we simply remark that

$$\langle e_0, i\rangle \Vdash \overline{\langle \overline{n}, \overline{k_n}\rangle} \in f.$$

In just the same way, f is seen to be onto B.

Finally, $0 \Vdash \neg(\omega^{\neg\neg}(A) \approx \omega^{\neg\neg}(B))$. This follows immediately from each of these:

$$\mathbf{V}(Kl) \models \omega^{\neg\neg}(A) = \overline{\omega} \text{ and}$$

$$\mathbf{V}(Kl) \models \omega^{\neg\neg}(B) = \overline{\neg K}$$

and a consequence of the Preservation Theorem of Chapter Five:

$$\mathbf{V}(Kl) \models \neg(\overline{\omega} \approx \overline{\neg K}).$$

∎

These internal phenomena strongly suggest that $\mathbf{P}(\omega)$ in $\mathbf{V}(Kl)$ is not the right place to look for an analogue of the theory of RETs. The theorem exemplifies the fact that, among *arbitrary* realizability subsets of ω, cardinal equivalence may have more to do with the "evidential" component of the realizability set than with the *echt* elements of the set. In more detail: if one cares to think somewhat inaccurately and picture a realizability set as a collection of evidence-element pairs $\langle e, a\rangle$, then the proof shows that a one-to-one correspondence in $\mathbf{V}(Kl)$ can have little to do with the a's and can depend for its subsistence almost entirely on the e's. In the particular case at hand, there is a trivial correspondence between the evidential components and no constructive functional relation of the right sort holding between the collections of internal elements of A and of B.

Without question, a cardinal arithmetic mirroring the theory of RETs will have to treat mainly of the recursion-theoretic "sizes" of the sets of elements, and use the bits of evidence to encapsulate the recursion theory. Hence, it seems that ω-stable sets are indicated. It is precisely because AC holds over the ω-stables that we are assured that these relations between element and evidence hold; one can use AC to prove that a correspondence between bits of evidence will always carry over into one on the sets of elements. Therefore, reference to the ω-stables can be "hidden," as we have in this chapter, but the passage from recursive mathematics to recursive realizability cannot skirt that reference altogether.

CHAPTER 7

Continuity, Realizability and Information Systems

Section 1: Prefatory and historical remarks

There is a widespread and injurious impression—that working in constructive rather than in classical mathematics is *mathematically expensive*. Perhaps the impression derives from the feeling that one pays too high a price for relinquishing those logical laws, like TND, that lapse on the passage into constructivism. Usually, the price is quoted in terms of the comforting theorems from analysis, like the mean value theorem, that lapse along with classical logic. On the other hand, a benefit of working in realizability is the perception that this impression is truly a prejudice, one fostered by lingering over invidious comparisons between constructive and classical theories.

With realizability, we find that the initial price of constructivism brings considerable profit. Net gain accrues from the fact that lost mathematics is more than countered by an *axiomatic freedom* that attends intuitionistic logic. Granted, intuitionistic logic is, when measured along certain dimensions, weaker than its classical counterpart. This very weakness, however, holds tremendous potential value. Axiomatic freedom is the recognition that intuitionistic logic allows axioms which are classically false but mathematically efficient to be consistent with powerful theories.

The results of this chapter afford a case study in and demonstration of axiomatic freedom. First, the full intuitionistic set theory IZF is proved to be consistent with the assumption that there is an extensive category of information systems, <u>ISys</u>, in which *every* function is approximable. As you might imagine, the assumption that every function on information systems is approximable becomes an extremely useful and extremely powerful nonclassical axiom for the theory of domains. Its presence in constructive set theory turns some moderately difficult results in the theories of domains and effective domains into trivialities. Next, we show that it is consistent with IZF to assume that ISys is at the same time the category of effectively given information systems (eg systems) with computable maps.

In each case, consistency is seen by working over the realizability model V(*Kl*). The profit from our venture into constructivity is revealed when ISys, as interpreted over V(*Kl*), is examined from the conventional mathematical standpoint of V. So seen, ISys is precisely the external category ESys of effectively given information systems, computable elements and computable morphisms. This ontic correspondence extends to one between the respective theories: the V(*Kl*)-theory of ISys incorporates all the usual theory of classical effectively given systems. Consequently, the burden of working constructively with axioms true in V(*Kl*) is pure profit; the theory of effectively given domains is captured in a streamlined theory of purely intuitionistic sets *without effective superstructure.* In other words, we will prove that the smooth isomorphism between recursive and realizability set theories can be extended to cover recursive and realizability *domain theories.*

These are not the only benefits to axiomatic freedom; there are some mathematical benefits that fall out along the way. Incidental to the representation of ESys as ISys is a complete answer to a question of G. Plotkin. Plotkin asked (in Plotkin (1978)) whether the tedious calculations characteristic of work over eg systems—the calculations of recursive indices of products, of exponentials and of other domain constructs in terms of their components' indices—are eliminable in favor of constructive mathematics over some form of realizability. Our answer to Plotkin is affirmative and uniform for a variety of domain constructs that includes all those conventionally considered.

All the results of this chapter were known to the author in July 1982 and were proved in complete detail in March 1983. The reader is advised to compare our consistency results with the theorems on sequential continuity of Hyland (1982).

Section 2: Wouldn't it be lovely?

Wouldn't it be lovely if the cartesian closed category of information systems and approximable maps (à la Scott (1982)) were a *full* subcategory of the category of sets? Proofs in denotational semantics would be much easier (and shorter) if the semanticist could freely assume that every map is continuous and monotone. To take a simple example, it would then be trivial to show that the least fixed-point operator, fix, is approximable as a function from $\underline{A} \Rightarrow \underline{A}$ into $\underline{A}$. If morphisms of information systems were just set maps, it would suffice to prove that fix provides a *map*. More generally, it would be child's play to prove that every term of the typed λ-calculus defines an approximable map; one need only check that each term defines a function. Indeed, with a "Brouwer's Theorem" for information systems in a set-theoretic context, we can say that any set-theoretically definable relation between information systems which is intuitionistically functional is automatically continuous and monotone! Hence, any functional construction available to the most liberal constructivist, even ones involving uncountable ordinals, various forms of AC or full powerset, give continuous functions on information systems.

Once one assumes that there is such a lovely category and that it is cartesian closed, there is a simple proof that the loveliness extends from the morphisms in the category to certain endomorphisms of the category. Specifically, *every endofunctor* on the category of systems is provably approximable as a functor on the correlative category of systems with embeddings. To see this, we make the assumption that a Brouwer's Theorem for information systems is already in place and deduce from the assumption the desired property of endofunctors.

2.1. Assumption. $\underline{ISys}$ is a cartesian closed category of information systems in which every set map is approximable. ∎

2.2. Definition. If $\underline{ISys}$ is a category of information systems, $\underline{ISys}^E$ is the *category of systems with embeddings.* $\prec^i_j$ is an *embedding* of system $\underline{A}$ into system $\underline{B}$ just in case i and j are approximable maps, $i : \underline{A} \to \underline{B}$ and $j : \underline{B} \to \underline{A}$, such that $j \circ i = id_{\underline{A}}$ and $i \circ j \subseteq id_{\underline{B}}$. ∎

2.3. Definition. A unary endofunctor $F : \underline{ISys} \to \underline{ISys}$ is *approximable* whenever it is continuous and monotone on $\underline{ISys}^E$. F is *monotone* when it carries embeddings into embeddings, i.e., when $\underline{A} \prec^i_j \underline{B}$ implies that $F(\underline{A}) \prec^{F(i)}_{F(j)} F(\underline{B})$. F is *continuous* when it commutes with direct limits, i.e., when $F(lim\langle \underline{A}_i , e_{ij}\rangle) = lim\langle F(\underline{A}_i), F(e_{ij})\rangle$. ∎

2.4. Theorem. *On the above assumption, every endofunctor of* $\underline{ISys}$ *is approximable when considered as a functor over* $\underline{ISys}^E$.

Proof. To prove that every such functor F is monotone and continuous, we assume that $i : \underline{A} \to \underline{B}$ and $j : \underline{B} \to \underline{A}$ are such that $j \circ i = id_{\underline{A}}$ and $i \circ j \subseteq id_{\underline{B}}$. First, F is a functor, so $F(i) : F(\underline{A}) \to F(\underline{B})$, $F(j) : F(\underline{B}) \to F(\underline{A})$ and $F(j) \circ F(i) = id_{F(\underline{A})}$. But F is also a map on function spaces; in particular, $F : (\underline{A} \Rightarrow \underline{B}) \to (F(\underline{A}) \Rightarrow F(\underline{B}))$. By assumption,

$\underline{ISys}$ is cartesian closed and every map is monotone. Therefore, $F(i) \circ F(j) \subseteq id_{F(\underline{B})}$ and, consequently, F is itself monotone.

Next, if $\mathrm{Lim} = lim\langle \underline{A}_i, e_{ij} \rangle$ is a direct limit, then $id_{\mathrm{Lim}} = \bigcup (f_i \circ f_i^r)$, where, for each i,

$$\underline{A}_i \prec^{f_i}_{f_i^r} \mathrm{Lim}.$$

Since F is approximable as a mapping from ($\mathrm{Lim} \Rightarrow \mathrm{Lim}$) into ($F(\mathrm{Lim}) \Rightarrow F(\mathrm{Lim})$),

$$id_{F(\mathrm{Lim})} = F(\bigcup (f_i \circ f_i^r)) = \bigcup F(f_i) \circ F(f_i^r).$$

It follows (cf. Plotkin (1978)) that $F(\mathrm{Lim})$ is the direct limit of $\langle F(\underline{A}_i), F(e_{ij}) \rangle$. Therefore, the functor F is continuous. Consequently, even though $\underline{ISys}$ is nothing more than a category of sets and set maps, there are solutions to recursive domain equations defined over it. ∎

Once we've come this far, not even propriety prohibits us from becoming even more imaginative. Can we also assume that $\underline{ISys}$ coincides with the category of effectively given information systems with computable maps? If the answer were 'Yes,' all manner of tiresome calculations become superfluous. Dispensible would be the check that λ-terms always define computable maps on effectively given systems. Also, the knowledge that recursive domain equations are solvable over the effectively given systems would come automatically from the elementary considerations of the preceding theorem.

We will prove that, over $\mathbf{V}(Kl)$, the answer to our (classically) fanciful question can be 'Yes.' If one insists on ordinary set theory in classical logic, the category $\underline{ISys}$ is a pure fancy; our assumption 2.1 is outrageously false. Even in classical arithmetic, one can define demonstrably nonapproximable maps on information systems. However, in intuitionistic set theory, we attain axiomatic freedom: there are no *counterexamples* to our assumption. In fact, the requisite intuitionistic set theory is nowise nonstandard; it is our old friend IZF. It is in IZF that we think of $\underline{ISys}$ as formalized. In describing the consistency proof, we will have reference to the following fact about $\overline{\omega}$: there is an index i such that, whenever f is a total recursive function with index e, $\{i\}(e) \Vdash \overline{f} \in (\omega \Rightarrow \omega)$. $\overline{f}$ is the usual internal representation of f defined in terms of pairing:

$$\overline{f} = \{\langle n, \overline{\langle \overline{n}, \overline{m} \rangle} \rangle : f(n) = m\}$$

This is a fact easily abstracted from the lengthy considerations of Chapter Five. One should also bear in mind the results of Chapters Three and Four, primarily the fact that a subset of the internal natural numbers is decidable in $\mathrm{V}(Kl)$ just in case, when viewed from without, it is recursive. (Of course, such a subset is also internally recursive, thanks to the presence of CT_0.)

Section 3: A category of constructive information systems

Just as IZF is in many ways a natural set theory, so *ISys* is a quite natural category of information systems. In fact, *ISys* is properly conceived as the result of a straightforward constructivization of Scott's notion of "information system." And that is just how we conceive it and how we will motivate its definition, starting from the conventional definition of an information system:

3.1. Definition. Let $P^{<\omega}(A)$ be the finite powerset of A. A quadruple

$$\underline{A} = \langle A, \mathrm{Cons}_A, \vdash_A, \Delta_A \rangle$$

is an *information system* iff Cons_A is a unary relation on $P^{<\omega}(A)$ and $\vdash_A$ is a binary relation on $P^{<\omega}(A) \times P^{<\omega}(A)$ such that

$$\text{whenever } u \subseteq v \text{ and } \mathrm{Cons}_A(v),\ \mathrm{Cons}_A(u) \qquad [1]$$

$$\text{for all } x \in \mathrm{A},\ \mathrm{Cons}_A(\{x\}), \quad \text{and} \qquad [2]$$

$$\text{whenever } u \vdash_A v \text{ and } \mathrm{Cons}_A(u),\ \mathrm{Cons}_A(u \cup v). \qquad [3]$$

Also, whenever $\mathrm{Cons}_A(u)$ and $\mathrm{Cons}_A(v)$, we assume that

$$u \vdash_A \{\Delta_A\} \qquad [4]$$

$$u \vdash_A v \text{ if } v \subseteq u \qquad [5]$$

$$u \vdash_A v \text{ and } v \vdash_A w \text{ implies that } u \vdash_A w, \text{ and} \qquad [6]$$

$$u \vdash_A v \text{ and } u \vdash_A w \text{ implies that } u \vdash_A v \cup w. \qquad [7]$$

∎

When $\underline{A}$ is an information system, A is thought of as the set of tokens or "atomic bits of information" about some interpreted computations. Con_A and $\vdash_A$ are, respectively, a notion of consistency on the finite subsets of A and a relation of entailment holding between the finite subsets. Δ_A is the null bit, the token incorporating no information. Under the implied analogy, one thinks of A as a "space" of all possible propositions or coherent bits of information about a collection of computations.

Of all the available bits of information, the reasonably complete collections of bits are the *elements* of $\underline{A}$ and these are collected into the set A^*. Officially, $x \in A^*$ iff $x \subseteq A$, all finite subsets of x satisfy Cons_A, and, whenever $u \subseteq x$ and $u \vdash_A v$, then $v \subseteq x$. One can extend the obvious analogy between the information-systems concept of consistency and consistency in logic by saying that any subset of A, all of whose finite subsets are consistent, is also called consistent. Moreover, as far as the elements of $\underline{A}$ are concerned, such consistent sets are all that matter. The members of A^* are, then, precisely the closures of consistent sets under $\vdash_A$. Hence, we can refer to each consistent subset of A, finite or otherwise, as a *basis* (for some element). Furthermore, (speaking classically) if A is countable, so is every basis, as is the set of all finite subsets of a basis.

The move from the classical notion of information system to the notion appropriate for constructive contexts calls for the addition of some extra structure to the classical notion. This is no cause for alarm; the weakness of intuitionistic logic itself issues a general permit for structural improvements. (Cf. Bishop (1967).) Aside from this, one can offer good reasons, both prudential and philosophical, for the structure we propose to add.

The first addition attaches to the notion of element. Working as constructivists, we will identify the elements with the countable bases. In constructive contexts, it is generally a matter of prudence to refrain from talk about arbitrary subsets of (even finite) structures. So, one should avoid arbitrary elements of systems as classically construed. The IZF axioms exert even less control over powersets than do their classical counterparts. This is apparent from realizability: as we have seen, $\mathbf{P}(\omega)$ in $\mathbf{V}(Kl)$ contains cardinal numbers up to (a version of) ω_{ω_1}. On the philosophical side, insisting that bases be enumerable preserves the computational metaphors that enliven Scott's ideas. When the basis is enumerable, the finite consistent subsets of it literally form a *series* of approximations to the corresponding element.

Second, in *ISys*, Cons_A and $\vdash_A$ will be assumed to be decidable relations. This restriction shares its prudential motivation with the first: arbitrary relations on a structure are just too uncontrolled. But again, decidability is mandated if we're to follow the advice of Scott in conceiving of the building blocks of systems:

The best advice is to think of the members of $[A]$ as consisting of finite *data objects, some of which are more informative than others. The word "finite" should be taken in the sense of "fully circumscribed"—as regards what is given in $[A]$ these data objects can be comprehended in "one step."*

If the tokens are *thoroughly finite* informational bits, one should be able to determine, in a finite number of steps, the entailment and consistency relations holding on finite collections of them. Hence, the decidability requirement is extremely natural.

Lastly, we assume that the set of tokens of an <u>*ISys*</u> system is an ω-retract. In that regard, we pause to review some results from Section 6 of Chapter Three.

3.2. Definition. A set S is an ω-retract whenever there are functions $i : S \rightarrowtail \omega$ and $j : \omega \twoheadrightarrow S$ such that $j \circ i = id_S$. ▮

Admittedly, this is a nontrivial constraint. In the presence of the axiom of choice AC, even in intuitionistic logic, all and only countable sets are ω-retracts. As we know (cf. Chapter One, Proposition 2.5), full choice is not constructively consistent with IZF. However, the connection with AC points the way to an acceptable motivation for the requirement, for one can prove that

3.3. Proposition. $V(Kl) \models$ *For S countable, S is an ω-retract iff S is discrete iff AC holds on S.*

Proof. See Chapter Three, Proposition 6.5. ▮

Insisting on ω-retracts, then, is in keeping with Scott's advice; it guarantees that identity on the sets of bits is decidable. Also, AC will make sure that functions exist over the system whenever they are needed. It is a simple consequence of this characterization that, for subsets of ω, ω-retraction is no restriction beyond countability.

3.4. Corollary. $V(Kl) \models$ *For $S \subseteq \omega$, S is countable iff S is an ω-retract.*

Proof. See Chapter Three, Corollary 6.6. ▮

A point worth noting: to the classical mathematician schooled in AC, all this extra structure is nugatory. In ZF+AC, <u>*ISys*</u> is merely the category of countable information systems.

Now that all the conceptual software is mounted, we can give the official description of <u>*ISys*</u>.

3.5. Definition. $\underline{A}$ is an object of <u>*ISys*</u> iff $\underline{A} = \langle A, \mathrm{Cons}_A, \vdash_A, \Delta_A \rangle$ is an information system, Cons_A and $\vdash_A$ are decidable on $\mathrm{P}^{<\omega}(A)$ and $\mathrm{P}^{<\omega}(A) \times \mathrm{P}^{<\omega}(A)$, respectively, and A is an ω-retract. ▮

3.6. Definition. f is an *element* of $\underline{A}$ iff $f \in (\omega \Rightarrow \mathrm{P}^{<\omega}(A))$, and

$$\forall n\ \mathrm{Cons}_A(f(0) \cup f(1) \cup \ldots \cup f(n)).$$

The set of elements of $\underline{A}$ is denoted 'A^*.'

∎

Intuitively, equality on bases should be equality on elements, and it is for this that the next definition provides. When f and g are equal as elements, we will say that $f \approx_A g$.

3.7. Definition. For $f, g \in A^*$, $f \subseteq_A g$ iff

$$\forall n\ \exists m\ g(0) \cup g(1) \cup \ldots \cup g(m) \vdash_A f(n)$$

and $f \approx_A g$ iff

$$f \subseteq_A g \text{ and } g \subseteq_A f.$$

∎

Strictly speaking, the set of elements of $\underline{A}$ is the collection of bases, plus the appropriately defined equality:

3.8. Definition. If $\underline{A}$ is an object of $\underline{ISys}$, the *set of elements* of $\underline{A}$ is the pair $\langle A^*, \approx_A \rangle$.

∎

Finally, there is no difficulty in proving constructively that $\underline{ISys}$ is truly a category of domains. The traditional notion of domain, as in Plotkin (1978), is that of consistently complete, ω-algebraic cpo.

3.9. Proposition. *For $\underline{A}$ from $\underline{ISys}$, $\langle A^*, \subseteq_A, \approx_A \rangle$ is a consistently complete, ω-algebraic cpo.*

Proof. Immediate from the definitions of the constituent notions. ∎

As an ω-algebraic cpo, $\underline{A}$ has finite elements. These are sequences of finite consistent subsets from A which are, up to $\approx_A$, constant.

3.10. Definition. $f \in A^*$ is *finite* iff $\exists g \in A^*$ g is a constant function and $f \approx_A g$. ∎

3.11. Theorem. *$f \approx g$ just in case for all finite h,*

$$h \subseteq f \text{ iff } h \subseteq g.$$

Proof. Immediate. ▮

As a category, *ISys* has a perspicuous and useful skeleton—a collection of *presented information systems.* Roughly, the presented systems are those having ω itself as a set of tokens. For present purposes, we let '$\{x_0, x_1, \ldots x_n\}$' represent a surjective primitive recursive coding into ω of finite subsets consisting of elements x_0, x_1, ... x_n of ω. Under this coding, $n \cup m$ represents the code of the finite set which is the union of the set coded by n with that coded by m. $n \subseteq m$ is the primitive recursive relation of set-theoretic inclusion holding between the sets coded by n and by m.

3.12. Definition. A pair $\langle \text{Cons}, \vdash \rangle$ is a *presentation* iff Cons is a decidable unary relation on ω and $\vdash$ is a decidable binary relation such that

whenever $n \subseteq m$ and Cons(m), Cons(n) [1]

for all n, Cons($\{n\}$), and [2]

whenever $n \vdash m$ and Cons(n), Cons($n \cup m$). [3]

Also, whenever Cons(n) and Cons(m), we must have

$n \vdash 0$ [4]

$n \vdash p$ if $p \subseteq n$ [5]

$m \vdash n$ and $n \vdash p$ implies that $m \vdash p$, and [6]

$n \vdash m$ and $n \vdash p$ implies that $n \vdash m \cup p$. [7]

▮

This definition makes explicit what it takes to be an information system in terms of the coding on the natural numbers. Note that, in presented systems, 0 always plays the role of $\{\Delta\}$. The constant m-valued ω-sequence is denoted '$[m]$' and finite elements are specified in accord with this notational convention. Hence, if $\underline{S}$ is presented, then $f \in S^*$ is finite iff $\exists m \in \omega \; f \approx_S [m]$.

One readily sees that the presented systems really do provide a domain-theoretic skeleton for *ISys*.

3.13. Definition. Information systems $\underline{A}$ and $\underline{B}$ are equivalent iff $\langle A^*, \subseteq_A, \approx_A\rangle$ is isomorphic (in the order-theoretic sense) to $\langle B^*, \subseteq_B, \approx_B\rangle$. ∎

3.14. Definition. An information system $\underline{A}$ is *presented* iff there is a presentation $\langle \mathrm{Cons}, \vdash\rangle$ such that, when $\underline{S} = \langle \omega, \mathrm{Cons}, \vdash, 0\rangle$, $\langle A^*, \subseteq_A, \approx_A\rangle$ is equivalent to $\langle S^*, \subseteq_S, \approx_S\rangle$. ∎

3.15. Theorem. *Every object of* ISys *is equivalent to some presented system.*

Proof. Given $\underline{A}$ in *ISys*, A is an ω-retract. Hence, there are functions $i : A \rightarrowtail \omega$ and $j : \omega \twoheadrightarrow A$ such that $j \circ i = id_A$. Define a presentation on ω so that it accords with consistency and entailment in A, as mediated by the pair $\langle i, j\rangle$. Specifically, we take

$$\mathrm{Cons}(n) \text{ iff } \mathrm{Cons}_A\{j(m) : m \in n\} \text{ and}$$

$$m \vdash n \text{ iff } \{j(p) : p \in m\} \vdash_A \{j(q) : q \in m\}.$$

It is a simple matter to show that this presented system is equivalent to $\underline{A}$. ∎

Section 4: Consistency theorems

This section is devoted to proving that, for the intuitionistic information systems of $\underline{ISys}$, axiomatic freedom is attainable. We will prove that, in V(Kl), the assumption that every set map in $\underline{ISys}$ is approximable holds. In keeping with the set-theoretic paradigm, a map between objects of $\underline{ISys}$ is just a binary relation that is single-valued on its second place and respects the relevant equalities. More formally,

4.1. Definition. If $\underline{A}$ and $\underline{B}$ are in $\underline{ISys}$, and $\approx_A$ and $\approx_B$ are the respective equalities, F is a function from $\underline{A}$ to $\underline{B}$ ($F : \underline{A} \to \underline{B}$) iff $F \subseteq A^* \times B^*$ which is invariant under $\approx_A$ in its first coordinate, invariant with respect to $\approx_B$ in its second, and which is total and functional:

$$\langle a, b \rangle \in F \wedge \langle a, c \rangle \in F \to b \approx_B c$$

∎

It is also worth recalling the definitions of monotonicity and of continuity for systems maps. Again, these are written in "set-theoreticalese:"

4.2. Definition. For $\underline{A}$ and $\underline{B} \in \underline{ISys}$, $F : \underline{A} \to \underline{B}$ is *monotone* iff

$$\forall a, c \in A^* \; (a \subseteq_A c \to \forall b, d \in B^* \; ((\langle a, b \rangle \in F \wedge \langle c, d \rangle \in F) \to b \subseteq_B d))$$

∎

4.3. Definition. $F : \underline{A} \to \underline{B}$ is *continuous* iff

$$\forall n \, \forall a \in A^* \; (\forall b \in B^* \; (\langle a, b \rangle \in F \to [n] \subseteq_B b) \to$$

$$\exists m \, ([m] \subseteq_A a \wedge \forall b \in B^* \; (\langle [m], b \rangle \in F \to [n] \subseteq_B b)))$$

∎

In giving these definitions, we have taken an excusable liberty. Strictly speaking, an arbitrary $\underline{A}$ from $\underline{ISys}$ does not *contain* the finite elements $[n]$ of a presented system. But, since every object of $\underline{ISys}$ is equivalent to a presented system, we can use the same notation for the finites of any system. Conjoining these two definitions gives the definition of systems morphism:

4.4. Definition. $F : \underline{A} \to \underline{B}$ is *approximable* iff F is continuous and monotone. ▮

All the machinery is now in place for the proofs of the consistency theorems.

4.5. Theorem. $\mathrm{V}(Kl) \models$ *If* $F : \underline{A} \to \underline{B}$, *then* F *is continuous.*

Proof. Since every member of $\underline{ISys}$ is equivalent to a presented system, we are free to restrict consideration to $\underline{A}$ and $\underline{B}$ presented. There will be no notational distinction made between the basic relations Cons and $\vdash$ for $\underline{A}$ and the corresponding relations for $\underline{B}$. This will not be a source of confusion.

F is, by definition, total on A^*, so we can assume that there is an e_1 such that

$$e_1 \Vdash \forall f\,(f \in A^* \to \exists g\,(g \in B^* \wedge \langle f, g\rangle \in F)).$$

Now, we evaluate the definition of continuity over $\mathrm{V}(Kl)$: let $e_3 \Vdash f \in A^*$ while

$$\{e_2\}(\langle n, e_3\rangle) \Vdash \forall g\,(\langle f, g\rangle \in F \to \exists m \in \omega\ g^*(m) \vdash \overline{n}). \qquad [1]$$

Here, we are abbreviating $g(0) \cup \ldots \cup g(m)$ as $g^*(m)$.

We now think of i as an index for a Turing machine. Take ϕ to be a total recursive function for which $\{\phi(i)\}$ outputs 0 on 0 and such that, for $n > 0$,

$$\{\phi(i)\}(n) \simeq \begin{cases} \{i\}(n) & \text{if } \forall x \leq n\ \{i\}(x)\downarrow \wedge\, \mathrm{V}(Kl) \models \mathrm{Cons}\,\overline{\{i\}}^*(n) \\ \uparrow & \text{if } \forall x \leq n\ \{i\}(x)\downarrow \wedge \neg\, \mathrm{V}(Kl) \models \mathrm{Cons}\,\overline{\{i\}}^*(n) \\ \uparrow & \text{if otherwise} \end{cases}$$

Given that $\mathrm{V}(Kl)$ mediates a close relation between decidability and recursivity, it is clear that ϕ exists. Let $\psi(i)$ index the total recursive function enumerating the range of $\{\phi(i)\}$ by dovetailing. Because of the way $\phi(i)$ is defined, there is a total recursive θ such that

$$\theta(i) \Vdash \overline{\{\psi(i)\}} \in A^*.$$

By clause (6) of the definition of $\Vdash$ (cf. Definition 4.1 of Chapter Zero), we infer that

$$\{e_1\}(\theta(i)) \Vdash \exists g\,(g \in B^* \wedge \langle \overline{\{\psi(i)\}}, g\rangle \in F).$$

Because total functions on ω in $\mathrm{V}(Kl)$ are recursive, there are partial recursive ρ, σ_1, σ_2 such that

$\{\rho(i)\}$ is total,

$$\sigma_1(i) \Vdash \overline{\{\rho(i)\}} \in B^*, \text{ while}$$

$$\sigma_2(i) \Vdash \langle\overline{\{\psi(i)\}}, \overline{\{\rho(i)\}}\rangle \in F.$$

From [1] we know that

$$\{\{e_2\}(\langle n, e_3\rangle)\}(\sigma_2(i)) \Vdash \exists m \ \overline{\{\rho(i)\}}^*(m) \vdash \overline{n}.$$

Then, for each $n \in \omega$, let

$$U_n = \{i : \mathrm{V}(Kl) \models \exists m \ \overline{\{\rho(i)\}}^*(m) \vdash \overline{n}\}.$$

Our intention is to apply the classical Rice-Shapiro theorem to U_n. For that purpose, we have to check that U_n is r.e. and extensional on indices.

Because $\underline{A}$ is presented and $\vdash$ is decidable in $\mathrm{V}(Kl)$, U_n is clearly r.e. For extensionality, assume that $\{i\} \simeq \{j\}$. Then $\{\phi(i)\} \simeq \{\phi(j)\}$ and

$$\mathrm{V}(Kl) \models \overline{\{\psi(i)\}} \approx \overline{\{\psi(j)\}}.$$

As $F : \underline{A} \to \underline{B}$ is a function in $\mathrm{V}(Kl)$,

$$\mathrm{V}(Kl) \models \overline{\{\rho(i)\}} \approx \overline{\{\rho(j)\}}.$$

This asserts that $\{\rho(i)\}$ and $\{\rho(j)\}$ determine bases generating the same element of $\underline{B}$. Hence, if $i \in U_n$, then so is j. Therefore, U_n is extensional. (Note that an r.e. index for U_n can be calculated uniformly in n.)

Now, suppose that $i \Vdash f \in A^*$. Without loss of generality, we can assume that $\mathrm{V}(Kl) \models f = \overline{\{i\}}$. From our work above, it follows that

$$\theta(i) \Vdash \overline{\{\psi(i)\}} \in A^*,$$

and that there is a σ_3 such that

$$\sigma_3(i) \Vdash \overline{\{\psi(i)\}} \approx f.$$

Working just as above and using [1], we get

$$\{\{e_2\}(\langle n, \theta(i)\rangle)\}(\sigma_2(i)) \Vdash \exists m \ \overline{\{\rho(i)\}}^*(m) \vdash \overline{n}.$$

Therefore, every index for $\{i\}$ lies in U_n. By the Rice-Shapiro theorem, there is a finite subfunction g of $\{i\}$ which is defined on an initial segment of ω and which has an index in U_n. Given the conditions on i and n, one can find, effectively in i and n, a canonical index for g. Consequently, there is available a partial recursive π such that $\pi(i,n)$ indexes the total constant function whose value is m, where

$$m = \bigcup_{i \in \mathrm{Dom}(g)} g(i).$$

Since $\mathrm{V}(Kl) \models \overline{\{i\}} \in A^*$, $\mathrm{V}(Kl) \models \mathrm{Cons}(\overline{m})$. Also, there is an effective routine which, from i and n, calculates a j such that

$$j \Vdash [\overline{m}] \subseteq f.$$

If k is a Turing index for g, calculable from i and n, then $k \in U_n$ or

$$\mathrm{V}(Kl) \models \exists m \ \overline{\{\rho(k)\}}^{\bullet}(m) \vdash \overline{n}.$$

Again, it is easy to calculate a realizing number for the statement above.

To complete the proof, it suffices to calculate a witness for the assertion

$$\overline{\langle[\overline{m}], \overline{\{\rho(k)\}}\rangle} \in F. \qquad [2]$$

This will suffice, since we already know that, in $\mathrm{V}(Kl)$,

$$\overline{n} \subseteq \overline{\{\rho(k)\}}.$$

To find a realizing number for [2], it is sufficient to find one for

$$[\overline{m}] \approx \overline{\{\psi(k)\}}.$$

But that is easy—one runs through the dovetailing procedure that specifies $\overline{\{\psi(k)\}}$. The proof is now complete. ∎

4.6. Theorem. $\mathrm{V}(Kl) \models$ *If* $F : \underline{A} \to \underline{B}$, *then F is monotone.*

Proof. Assume that $\mathrm{V}(Kl) \models F : \underline{A} \to \underline{B}$. Take e_1, e_2, e_3, $e_4 \in \omega$ such that

$$e_1 \Vdash \forall f \in A^* \, \exists h \in B^* \, (\langle f, h\rangle \in F)$$

$$e_3 \Vdash f \in A^*$$

$$\{e_2\}(\langle n, e_3\rangle) \Vdash \forall h\, (\langle f, h\rangle \in F \to \exists m\, h^*(m) \vdash \overline{n}) \quad \text{and}$$

$$e_4 \Vdash \forall n\, \exists m\, g^*(m) \Vdash f(n).$$

e_1 realizes that F is total, $e_2(\langle n, e_3\rangle)$ realizes that $[\overline{n}] \subseteq F(f)$ and e_4 realizes that $f \subseteq g$. The plan is to prove monotonicity for F in $\mathrm{V}(Kl)$ by checking that

$$\mathrm{V}(Kl) \models ((f \subseteq g \wedge [\overline{n}] \subseteq F(f)) \to [\overline{n}] \subseteq F(g)).$$

e_1 is used to instigate the same construction as that of the preceding theorem. As before, we form the collection

$$U_n = \{i : \mathrm{V}(Kl) \models \exists m\ \overline{\{\rho(i)\}}^*\,(m) \vdash \overline{n}\}.$$

Again, it's easily provable that, if $i \Vdash f \in A^*$, then $i \in U_n$. By the Rice-Shapiro theorem, there is a finite subfunction g_1 of $\{i\}$ all of whose indices belong to U_n. We continue as in Theorem 4.5 to find an $m \in \omega$ and a realizing number for $[\overline{m}] \subseteq g$. To complete the proof, it will suffice to show that there is a $j \in \omega$ such that $j \Vdash g \in A^*$ and $j \in U_n$.

To that end, assume that $j \Vdash g \in A^*$ and $\mathrm{V}(Kl) \models g = \overline{\{j\}}$. We can locate a function h and a subfunction h_1 such that $\mathrm{V}(Kl) \models h \approx g$ and h_1 has its indices in U_n. Again, this is very easy:

$$\mathrm{V}(Kl) \models [\overline{m}] \subseteq g.$$

This means that

$$\mathrm{V}(Kl) \models \exists k\, g^*(k) \vdash \overline{m}.$$

Let h be a function on ω such that for all $p \in \omega$,

$$h(p) = \begin{cases} m & \text{if } p \leq k \\ \{i\}(p-(k+1)) & \text{if } p > k \end{cases}$$

Obviously, $\mathrm{V}(Kl) \models \overline{h} \approx g$. Take $h_1 = h \restriction (k+1)$ and let r be an index for this finite function. It follows that

$$\mathrm{V}(Kl) \models \exists\, k\, \overline{\{\rho(r)\}}^*\,(k) \vdash \overline{m}$$

and, hence, that r has an index for h_1 that falls into U_n. ▮

4.7. Note. Certainly, other proofs of the consistency theorems are available. We opted to present one that employs the Rice-Shapiro Theorem explicitly to emphasize the accord (which will be plainly apparent from later sections) between the objects of *ISys* and the classical eg systems. Using the Rice-Shapiro Theorem in this way shows that the consistency theorems derive from a realizability-theoretic application of known results about effective domains, specifically the Myhill-Shepherdson Theorem for eg systems. (Cf. Plotkin (1978).) ▮

Section 5: Properties of the category

In the eyes of V(Kl), every set map between members of $\underline{ISys}$ is continuous and monotone. Therefore, the category of $\underline{ISys}$ objects and set maps is precisely a category of information systems and approximable maps. Now, IZF can take over almost entirely; from the definitions and these nonstandard axioms one can show that $\underline{ISys}$ has all the properties of a rich collection of computational domains. In fact, we devote this section to cataloging some of these properties.

Every map between $\underline{A}$ and $\underline{B}$ in $\underline{ISys}$ is determined by its graph:

5.1. Definition. For $F : \underline{A} \to \underline{B}$, $gh(F)$, the *graph* of F, is the binary relation on ω for which

$$\langle m, n\rangle \in gh(F) \text{ iff } [n] \subseteq_B F([m]).$$

∎

5.2. Theorem. V(Kl) $\models$ *If* $F : \underline{A} \to \underline{B}$, *then* F *is uniquely determined by* $gh(F)$.

Proof. Assume that both $\underline{A}$ and $\underline{B}$ are presented and take F approximable. We work in IZF. For each $f \in A^*$, let $H(f)$ enumerate

$$\{n \in \omega : \exists m\,([m] \subseteq_A f \wedge \langle m, n\rangle \in gh(F))\}.$$

Since all the basic relations are decidable, $H(f)$ exists. It will suffice to show that $\langle f, H(f)\rangle \in F$, and, for that, it will suffice in turn to prove that, if $\langle f, g\rangle \in F$, then $g \approx_B H(f)$.

Assume that $\langle f, g\rangle \in F$. Then, by continuity of F, if $[n] \subseteq_B g$ and $\langle f, g\rangle \in F$, then $H(f)$ eventually outputs n. Hence, $g \subseteq_B H(f)$. On the other hand, let $[n] \subseteq_B H(f)$. From the definition of $H(f)$ and the monotonicity of F, $[n] \subseteq_B g$. Therefore, $g \approx_B H(f)$. ∎

5.3. Theorem. V(Kl) $\models$ *If* $F : \underline{A} \to \underline{B}$, *then* $gh(F)$ *is countable.*

Proof. This is immediate, in IZF, from the decidability of Cons and $\vdash$. ∎

Thanks to the recursion theory which is "built into" realizability, $\underline{ISys}$ is, in V(Kl), precisely the category of effectively given domains. First, the fact that decidable and recursive relations coincide in $\mathbf{V}(Kl)$ has already received considerable attention. Moreover, since CT_0 holds in $\mathbf{V}(Kl)$, every countable set of natural numbers is r.e. It follows directly from this that $\underline{ISys}$ coincides with $\underline{ESys}$ and that every $\underline{ISys}$ morphism is computable.

5.4. Corollary. $\mathbf{V}(Kl) \models$ *$\underline{ISys}$ is equivalent to the category of eg information systems and computable morphisms.*

Again, because of CT_0, only the computable elements of a system exist in $\mathbf{V}(Kl)$:

5.5. Corollary. $\mathbf{V}(Kl) \models$ *If $\underline{A} \in \underline{ISys}$ and $f \in A^*$, then f is computable.*

The standard proof that $\underline{Sys}$, the classical category of information systems, is cartesian closed is fully constructive. Hence, it is reproducible in IZF and, to show that $\underline{ISys}$ is cartesian closed, it suffices to show that products and exponentials of presented systems are presented. In truth, one can show that the category is closed under any of the usual constructs.

5.6. Theorem. $\mathbf{V}(Kl) \models$ *$\underline{ISys}$ is closed under products, sums, exponentiation, and the formation of Hoare and of Smyth powerdomains.*

Proof. All the requisite verifications are obtainable in IZF. We restrict ourselves here to a sample, the proof of closure under products.

Recall that $\langle\ ,\ \rangle$ is a surjective p.r. pairing function with x_0 and x_1 as the corresponding projections. Let m^0 and m^1 be the p.r. functions defined by

$$m^0 = \{n_0 : n \in m\} \text{ and } m^1 = \{n_1 : n \in m\}.$$

For this pairing, we assume that $0^0 = 0 = 0^1$. Let $\underline{A}$ and $\underline{B}$ be presented elements of $\underline{ISys}$. We define relations $\mathrm{Cons}_{A\times B}$ and $\vdash_{A\times B}$ as follows:

$$\mathrm{Cons}_{A\times B}(n) \text{ iff } \mathrm{Cons}_A(n^0) \text{ and } \mathrm{Cons}_B(n^1)$$
$$m \vdash_{A\times B} n \text{ iff } m^0 \vdash_A n^0 \text{ and } m^1 \vdash_B n^1$$

These new relations are clearly decidable and satisfy the conditions on presentations. It is then straightforward to check that $\langle \mathrm{Cons}_{A\times B}, \vdash_{A\times B}\rangle$ is a presentation for $\underline{A} \times \underline{B}$. ∎

Section 6: On eg systems: a question of Plotkin

In V, ESys is the category whose objects are the eg information systems and whose morphisms are the computable approximable maps. Specifically,

6.1. Definition. $\underline{S} = \langle \omega, \mathrm{Cons}_S, \vdash_S, 0\rangle$ is an *eg information system* (an object of ESys) iff $\underline{S}$ is an information system and Cons_S and $\vdash_S$ are recursive relations on (coded) $\mathrm{P}^{<\omega}(\omega)$ and on $\mathrm{P}^{<\omega}(\omega)\times\mathrm{P}^{<\omega}(\omega)$, respectively. As is the case with ISys, when $\underline{S}$ is in ESys, we say that $\langle \mathrm{Cons}_S, \vdash_S\rangle$ is a *presentation* of $\underline{S}$. ∎

6.2. Definition. $i \in \omega$ indexes $\underline{S}$ in ESys iff i_0 in an index for Cons_S and i_1 is an index for $\vdash_S$. ∎

The proof of the last theorem of the preceding section, not to mention the effectiveness of all set maps of ISys in $\mathrm{V}(Kl)$, point to the existence of a close connection between ISys and ESys. As a first step toward illuminating the connection, we note that the objects of ISys in $\mathrm{V}(Kl)$ stand in a correspondence with the objects of ESys in V given by the respective presentations. A construction on the familiar injection $\underline{S} \mapsto \overline{\underline{S}}$ takes each presentation from ESys into the presentation of a presented system of ISys.

6.3. Definition. Let $\langle \mathrm{Cons}_S, \vdash_S\rangle$ be a presentation from ESys. Then, $\overline{\langle \mathrm{Cons}_S, \vdash_S\rangle} = \langle \overline{\mathrm{Cons}_S}, \overline{\vdash_S}\rangle$ where, as usual,

$$\overline{\mathrm{Cons}_S} = \{\langle n, \overline{n}\rangle : n \in \mathrm{Cons}_S\} \text{ and}$$

$$\overline{\vdash_S} = \{\langle\langle n, m\rangle, \langle \overline{n}, \overline{m}\rangle\rangle : n \vdash_S m\}.$$

∎

Of special note is the fact that the correspondence is effective:

6.4. Lemma. *There is an $e \in \omega$ with the property that, if i indexes $\underline{S}$, then $\{e\}(i)\downarrow$ and*

$$\{e\}(i) \Vdash \overline{\langle Cons_S, \vdash_S\rangle} \textit{ is a presentation in ISys}.$$

Conversely, there is an $h \in \omega$ with the property that, if

$$j \Vdash \overline{\langle Cons_S, \vdash_S\rangle} \textit{ is a presentation in ISys},$$

then $\{h\}(j)\downarrow$ *and* $\{h\}(j)$ *indexes* $\overline{S}$.

Proof. This is a straightforward application of the absoluteness theorems of Chapter Four. One need only remark that the properties Cons and $\vdash$ are a.n. ∎

The lemma justifies the following definition.

6.5. Definition. If $\underline{S}$ is an eg system whose presentation is $\langle \text{Cons}, \vdash \rangle$, then $\overline{S}$ is the system of $\underline{ISys}$ whose presentation is $\overline{\langle \text{Cons}, \vdash \rangle}$. ∎

Plotkin has asked whether a realizability construction could be employed to eliminate from the classical development of $\underline{ESys}$ theory the explicit calculation of indices. With the above information, we propose to answer the question of Plotkin affirmatively in a simple case, that of products. Then, from the simple case, we can extrapolate to a general answer. Since the usual proof of the closure of $\underline{ESys}$ under products is constructive, there is no need to calculate on indices to insure that the product construct is effective. The realizability interpretation of intuitionistic logic then gives the indexing calculations automatically.

The mathematical core of the proof that $\underline{ISys}$ is, intuitionistically, closed under products is the provision of a presentation for the product, $\langle \text{Cons}_{A\times B}, \vdash_{A\times B} \rangle$, in terms of the presentations $\langle \text{Cons}_A, \vdash_A \rangle$ and $\langle \text{Cons}_B, \vdash_B \rangle$ of its component systems. This same construction is also the mathematical core of the proof that $\underline{ESys}$ is closed under products. The very identification of these two "cores" has itself a mathematical content, which is expressed in the proof that, over $\mathbf{V}(Kl)$, the results of the two constructions are identical.

Let $\underline{A}$ and $\underline{B}$ belong to $\underline{ESys}$. Let $\text{Cons}_A \times \text{Cons}_B$ represent $\text{Cons}_{A\times B}$ as it is defined arithmetically in terms of Cons_A and Cons_B. Similarly for $\vdash_A \times \vdash_B$.

6.6. Lemma. *Let* $\langle Cons_A, \vdash_A \rangle$ *and* $\langle Cons_B, \vdash_B \rangle$ *present* $\underline{A}$ *and* $\underline{B}$ *as objects from* $\underline{ESys}$. $\mathbf{V}(Kl)$ *satisfies*

$$\langle \overline{Cons_{A\times B}}, \overline{\vdash_{A\times B}} \rangle = \langle \overline{Cons_A} \times \overline{Cons_B}, \overline{\vdash_A} \times \overline{\vdash_B} \rangle.$$

A realizability witness can be found for the latter statement independently of $\underline{A}$, $\underline{B}$ *and of their respective presentations.*

Proof. We will check that

$$\overline{\vdash_{A\times B}} = \overline{\vdash_A} \times \overline{\vdash_B}$$

and leave the correlative check on Cons for the reader. First,

$$e \Vdash \overline{\langle \overline{m}, \overline{n} \rangle \in \overline{\vdash_{A\times B}}}$$

implies that $m = e_0$, $n = e_1$ and $e_0 \vdash_{A\times B} e_1$. By the internal definition of $\vdash_{A\times B}$, this means that

$$e_0^0 \vdash_A e_1^0 \text{ and } e_0^1 \vdash_B e_1^1.$$

Hence, $\langle\langle e_0^0, e_1^0\rangle, i\rangle \Vdash \overline{\langle \overline{m^0}, \overline{n^0} \rangle \in \overline{\vdash_A}}$ and $\overline{\langle \overline{m^1}, \overline{n^1} \rangle \in \overline{\vdash_B}}$.

The absoluteness of the recursive functions shows that there is a partial recursive Θ such that

$$\Theta(e) \Vdash \langle \overline{m}^0, \overline{n}^0 \rangle \in \overline{\vdash_A} \text{ and } \langle \overline{m}^1, \overline{n}^1 \rangle \in \overline{\vdash_B}.$$

Second, if $e \Vdash \overline{\langle \overline{n}, \overline{m} \rangle \in \overline{\vdash_A} \times \overline{\vdash_B}}$, then

$$e_0 \Vdash \overline{\langle \overline{n}^0, \overline{n}^0 \rangle} \in \overline{\vdash_A} \text{ and}$$

$$e_1 \Vdash \overline{\langle \overline{n}^1, \overline{n}^1 \rangle} \in \overline{\vdash_B}.$$

Then, the above reasoning can easily be worked in reverse to prove that there is a partial recursive Ψ such that

$$\Psi(e) \Vdash \overline{\langle \overline{n}, \overline{m} \rangle \in \overline{\vdash_{A\times B}}}.$$

This completes the proof. ▮

With the above lemmas in place, the elimination of indices for products is an easy exercise:

6.7. Theorem. *The explicit calculation of indices is eliminable from a complete proof that* <u>*ESys*</u> *is closed under products. The calculation is eliminated in favor of working constructively over* $\mathbf{V}(Kl)$.

Proof. Assume that $\langle \mathrm{Cons}_A, \vdash_A \rangle$ and $\langle \mathrm{Cons}_B, \vdash_B \rangle$ present $\underline{A}$ and $\underline{B}$ as objects of <u>*ESys*</u> and have indices i_A and i_B, respectively. Thanks to the preceding lemmas, for $X = A$ or $X = B$, $\{e\}(i_X)\downarrow$ and

$$\{e\}(i_X) \Vdash \langle \overline{\mathrm{Cons}_X}, \overline{\vdash_X} \rangle \text{ is a presentation of an object of } \underline{ISys}.$$

As we have seen, IZF $\vdash$ "$\underline{ISys}$ is closed under $\times$," so the soundness of realizability produces a j such that $j \Vdash$ "$\langle \overline{\mathrm{Cons}_A}, \overline{\vdash_A} \rangle$ and $\langle \overline{\mathrm{Cons}_B}, \overline{\vdash_B} \rangle$ are presentations from $\underline{ISys}$ only if $\langle \overline{\mathrm{Cons}_A} \times \overline{\mathrm{Cons}_B}, \overline{\vdash_A} \times \overline{\vdash_B} \rangle$ presents their product."

By the definition of realizability for implication,

$$\{j\}(\langle \{e\}(i_A), \{e\}(i_B) \rangle) \Vdash \langle \overline{\mathrm{Cons}_A} \times \overline{\mathrm{Cons}_B}, \overline{\vdash_A} \times \overline{\vdash_B} \rangle \text{ presents their product.}$$

Next, there is a $k \dot{\in} \omega$ such that $\{k\}(\{j\}(\langle \{e\}(i_A), \{e\}(i_B) \rangle))$ realizes that

$$\langle \overline{\mathrm{Cons}_{A \times B}}, \overline{\vdash_{A \times B}} \rangle$$

presents the product of $\overline{\underline{A}}$ and $\overline{\underline{B}}$ in $\underline{ISys}$.

At last, we apply the second half of the first lemma to obtain the result that

$$\{h\}(\{k\}(\{j\}(\langle \{e\}(i_A), \{e\}(i_B) \rangle)))$$

indexes $\langle \mathrm{Cons}_{A \times B}, \vdash_{A \times B} \rangle$ in $\underline{ESys}$.

∎

The theorem shows that all indexing calculations can be removed from the consideration of products in $\underline{ESys}$. This is the most instructive elementary demonstration we know of the profits of constructivity. The only price for automatic index calculation is the observation that the conventional proof of closure is constructive and that constructive mathematics is sound with respect to realizability. But that's not all: the pleasures of realizability are not limited to products. Any of the conventional operations on $\underline{ESys}$ admit of the same treatment because the definitions of the operations can take a particularly simple form. The form in question is that which allows, for each operation, a version of Lemma 6.6 to go through.

6.8. Definition. Let **P** be a set of decidable number-theoretic predicates. Let L^P be the language of Peano arithmetic with predicates from **P**. $\phi \in \mathrm{Form}(\mathrm{L}^P)$ is *almost negative* (a.n.) in **P** iff $\vee$ does not occur in ϕ and $\exists x$ occurs only before decidable subformulae of ϕ. ∎

This definition should be compared with that of "almost negative" formulae as it appears in Chapter Four. For purposes of exposition, we pretend that the predicates of **P** are all binary.

6.9. Definition. For $\phi \in \mathrm{Form}(\mathrm{L}^P)$, $\overline{\phi}$ is obtained by replacing each appearance of any P from $\mathbf{P}$ in ϕ by $\overline{\mathrm{P}}$ where $\overline{\mathrm{P}} =$

$$\{\langle\langle m,n\rangle, \overline{\langle \overline{m}, \overline{n}\rangle}\rangle : \mathbf{V} \models \mathrm{P}(m)\},$$

and then by expressing ϕ "in the natural way" in set theory (Cf. Chapter Four.) ∎

For purposes of exposition in the next lemma, we assume that ϕ has at most two free variables.

6.10. Lemma. *If ϕ is a.n. in $\mathbf{P}$, then there is an $e_\phi \in \omega$ such that if $\mathbf{V} \models \phi(m,n)$, then $\{e_\phi\}(m,n)\downarrow$ and $\{e_\phi\}(m,n) \Vdash \overline{\phi}(\overline{m}, \overline{n})$. Also, if $\mathbf{V}(Kl) \models \overline{\phi}(\overline{m}, \overline{n})$, then $\mathbf{V} \models \phi(m,n)$.*

Proof. One need only reproduce, modulo $\mathbf{P}$, the proof that the a.n. definable relations are effectively absolute. Again, the reader should consult Chapter Four. ∎

6.11. Lemma. *Let $F(\underline{A}, \underline{B})$ be an operation on $\underline{ESys}$ such that the presentation of $F(\underline{A}, \underline{B})$ is a.n. in the presentations of $\underline{A}$ and of $\underline{B}$. Specifically, let ϕ_0 be an a.n. formula defining $Cons_{F(A,B)}$ in terms of $Cons_A$, $Cons_B$, $\vdash_A$ and $\vdash_B$ and let ϕ_1 do the same for $\vdash_{F(A,B)}$. Then*

$$\mathbf{V}(Kl) \models \overline{Cons_{F(A,B)}} = \overline{\phi_0} \text{ and}$$
$$\mathbf{V}(Kl) \models \overline{\vdash_{F(A,B)}} = \overline{\phi_1}\,.$$

Also, realizability witnesses for each of the above is obtainable independently of $\underline{A}$ and $\underline{B}$.

Proof. We use the preceding lemma and the usual injection manipulations. The proof is straightforward but tedious. ∎

This, finally, is our general theorem on the elimination of indices:

6.12. Theorem. *Assume that $F(\underline{A}, \underline{B})$ is an operation on $\underline{ESys}$ such that a presentation of $F(\underline{A}, \underline{B})$ is a.n. in those of $\underline{A}$ and $\underline{B}$. Let T be any extension of IZF such that both $\mathbf{V} \models T$ and $\mathbf{V}(Kl) \models T$ and assume that $T \vdash$ "$\underline{ESys}$ is closed under F". Then, $\mathbf{V}(Kl) \models$ "$\underline{ISys}$ is closed under F" and all the indexing calculations relevant to F are effective and eliminable in favor of realizability.*

Proof. Use the lemmas and work just as in the case of products. ∎

The following corollary provides a general answer to Plotkin's question.

6.13. Corollary. *In* $\mathbf{V}(Kl)$, *$\underline{ISys}$ is closed under products, exponentials, sums, Hoare powerdomains, and Smyth powerdomains. The operations corresponding to the constructions are effective and all the indexing calculations are eliminable.*

Proof. **One checks that all the operations have presentations which are a.n. in those of their components and that all the relevant proofs can be carried out in IZF.** ▮

Section 7: Eliminating the first-order theory of eg systems

One might say that, in eliminating index calculations, we have shown that the effective *form* of the theory of eg systems is fully captured by the logic of the theory of <u>*ISys*</u>. The phrase 'eliminating the theory of <u>*ESys*</u>' means more than this; it means that the mathematical *content* of the theory of eg systems can be wholly replaced by that of <u>*ISys*</u> over $V(Kl)$. As rationale for the suggestion, we give a "small scale" isomorphism result linking <u>*ESys*</u> with <u>*ISys*</u>. The result shows that, mathematically speaking, nothing will be lost in taking up this suggestion. The isomorphism is "small scale" because it does not treat certain logical features, such as quantification over objects from the various categories. These features could only be considered on a much grander scale than that available here. (In a future writing, we will show how the isomorphism can be made to work on the "grand scale.")

7.1. Definition. For $\underline{A}$ from <u>*ESys*</u>, the language $\mathbf{L}^e$ ('e' for 'effective') is a two-sorted first-order language with sorts E and M and with the following predicates as primitive:

$$\subseteq \text{ of sort } E \times E$$

$$\text{App of sort } M \times E \times E$$

As a matter of convenience, we assume that each computable $f \in A^*$ and computable $F : \underline{A} \to \underline{A}$ appears in $\mathbf{L}^e$ as an autonymous name. ∎

Under the natural interpretation of $\mathbf{L}^e$ over <u>*ESys*</u>, E represents the collection of computable elements of $\underline{A}$ and M the collection of computable morphisms from $\underline{A}$ into $\underline{A}$. Specifically,

$$\underline{ESys} \models f \subseteq g$$

iff computable f is contained in computable g as $\underline{A}$-elements and

$$\underline{ESys} \models \text{App}(F, f, g)$$

iff $F : \underline{A} \to \underline{A}$, f and g are computable, and $F(f) = g$.

To carry out the proposed elimination, each of the objects of the interpreted sorts is injected into $V(Kl)$. There each reappears, under the appropriate description, as a feature of $\overline{\underline{A}}$ in <u>*ISys*</u>. In the following, we let e_f be any index for recursive f.

7.2. Definition. For $f \in (\omega \Rightarrow \omega)$, let

$$\overline{f} = \{\langle n, \overline{\langle \overline{n}, \overline{m} \rangle} \rangle : \langle n, m \rangle \in f\}.$$

For $F : \underline{A} \to \underline{A}$, and F computable let

$$\overline{F} = \{\langle e_f, \overline{\langle \overline{f}, \overline{g} \rangle} \rangle : \langle f, g \rangle \in F\}.$$

∎

The next lemma is intended to show that the "overlining" injection underlies a perfect semantic accord—at least as far as the sorts and atomic sentences are concerned—between $\underline{ESys}$ and $\underline{ISys}$ over $\mathbf{V}(Kl)$.

7.3. Lemma.

$$\underline{ESys} \models E(f) \quad \textit{iff} \quad \mathbf{V}(Kl) \models \overline{f} \in \overline{A}^{\bullet} \quad [1].$$

$$\underline{ESys} \models M(F) \quad \textit{iff} \quad \mathbf{V}(Kl) \models \overline{F} : \overline{\underline{A}} \to \overline{\underline{A}} \quad [2]$$

$$\underline{ESys} \models f \subseteq g \quad \textit{iff} \quad \mathbf{V}(Kl) \models \overline{f} \subseteq_A \overline{g} \quad [3]$$

$$\underline{ESys} \models \mathrm{App}(F, f, g) \ \textit{iff} \ \mathbf{V}(Kl) \models \langle \overline{f}, \overline{g} \rangle \in \overline{F} \quad [4]$$

Proof. [1] and [3] follow immediately from the definitions of the pertinent notions and from our oft cited reflections on decidability in $\mathbf{V}(Kl)$. [2] follows from [1] and [3]. We will prove [4] explicitly:

If $\mathrm{App}(F, f, g)$ holds in $\underline{ESys}$, then F is computable and takes f into g. By the definition of $\overline{F}$, $\langle e_f, i \rangle \Vdash \overline{\langle \overline{f}, \overline{g} \rangle} \in \overline{F}$. Conversely, if $e \Vdash \overline{\langle \overline{f}, \overline{g} \rangle} \in \overline{F}$, then

$$e_0 = e_h \text{ and } e_1 \Vdash \overline{\langle \overline{f}, \overline{g} \rangle} = \overline{\langle \overline{h}, \overline{k} \rangle},$$

where $\langle h, k \rangle$ is in F. By the absoluteness properties for the natural numbers, $f = h$ and $g = k$, so $\langle f, g \rangle \in F$. ∎

As should now be familiar, the neat correspondence of this lemma extends to a full translation ϕ^{tr}. Let $\mathbf{S}^e$ be the set of sentences of $\mathbf{L}^e$.

7.4. Definition. ϕ^{tr} is defined for ϕ from S^e. If ϕ is atomic, let ϕ^{tr} be given by the correspondence of the preceding lemma. ϕ^{tr} commutes with $\wedge$, $\vee$, $\neg$, $\rightarrow$. For X and x ranging, respectively, over the computable endomorphisms and elements of $\underline{A}$,

$$(\forall X\ \phi)^{tr} = \forall X\ (X : (\overline{\underline{A}} \rightarrow \overline{\underline{A}}) \rightarrow \neg\neg\phi^{tr})$$

$$(\forall x\ \phi)^{tr} = \forall x\ (x \in \overline{A}^{*} \rightarrow \neg\neg\phi^{tr})$$

$$(\exists X\ \phi)^{tr} = \exists X\ (X : (\overline{\underline{A}} \rightarrow \overline{\underline{A}}) \wedge \phi^{tr})$$

$$(\exists x\ \phi)^{tr} = \exists x\ (x \in \overline{A}^{*} \wedge \phi^{tr}).$$

∎

7.5. Theorem. *For $\phi \in \mathrm{S}^e$, $\underline{ESys} \models \phi$ iff $\mathrm{V}(Kl) \models \phi^{tr}$.*

Given the previous lemmas and the properties of realizability, there is no difficulty in proving that if ϕ is quantifier-free, then the theorem holds. For quantified expressions, the following lemmas are necessary. The proof of each lemma is strictly analogous to that of the correlative theorem of Chapter Five.

7.6. Lemma. *For $f \in \mathrm{V}(Kl)$, if $\mathrm{V}(Kl) \models f \in \overline{A}^{*}$, then, for some $g \in A^{*}$, we have $\mathrm{V}(Kl) \models f = \overline{g}$.*

7.7. Lemma. *For $F \in \mathrm{V}(Kl)$, if $\mathrm{V}(Kl) \models F : \overline{\underline{A}} \rightarrow \overline{\underline{A}}$, then $\mathrm{V}(Kl) \models F = \overline{G}$, for some computable $G : \underline{A} \rightarrow \underline{A}$.*

Now we return to the proof of the theorem.

Proof. The essential idea can be conveyed by presenting the proof for one of the universal quantifiers.

Assume that $\underline{ESys} \models \forall X\ \phi$ and suppose that $\mathrm{V}(Kl) \models F : \overline{\underline{A}} \rightarrow \overline{\underline{A}}$. By the lemma, there is a $G : \underline{A} \rightarrow \underline{A}$ such that, in $\mathrm{V}(Kl)$, $F = \overline{G}$. $\phi(X/G)$ holds in $\underline{ESys}$, so

$$\mathrm{V}(Kl) \models \phi^{tr}(X/F) \text{ and } 0 \Vdash \neg\neg\phi^{tr}(X/F).$$

Therefore, $\mathrm{V}(Kl) \models (\forall X\ \phi)^{tr}$.

On the other hand, if

$$\mathrm{V}(Kl) \models \forall X\ (X : (\overline{\underline{A}} \rightarrow \overline{\underline{A}}) \rightarrow \neg\neg\phi^{tr}),$$

and $F : \underline{A} \to \underline{A}$ in V is computable, then

$$\mathrm{V}(Kl) \models \overline{F} : \overline{\underline{A}} \to \overline{\underline{A}}.$$

It follows that $\mathrm{V}(Kl) \models \phi^{tr}(X/\overline{F})$ and, hence, by the inductive hypothesis, $\underline{ESys} \models \phi(X/F)$. ▮

The theorem on ϕ^{tr} shows that, in studying the effective aspects of *ESys*, no mathematical fact is ever lost by restricting our researches to *ISys* and using the mathematical principles holding over V(*Kl*). The next theorem is a straightforward preservation result; as with the similar result for RETs, it shows the ease with which constructive truth about *ISys* can be transformed into classical truth for *ESys*.

First, we pick out those formulae of L^e which are naturally preserved in this transition.

7.8. Definition. **A formula ϕ of L^e is in Γ iff $\phi \in \mathrm{S}^e$, and, in ϕ, occurrences of $\forall$ appear neither in the scope of $\neg$ nor in the antecedent of $\to$.** ▮

7.9. Definition. **The translation ϕ^{pr} is defined just as ϕ^{tr}, except that the double negatives are removed from the cases governing the quantifiers.** ▮

7.10. Theorem. ***For $\phi \in \Gamma$, $\mathrm{V}(Kl) \models \phi^{pr}$ only if $\underline{ESys} \models \phi$.***

Proof. **This is immediate by induction on the structure of ϕ.** ▮

Section 8: Conclusion and prospectus

Nothing now stands in the way of giving elegant and informative but decidedly "nonstandard" derivations of classical results about $\underline{ESys}$. For example, one can prove the full effective fixed-point theorem without the use of recursion theory by proving the pure fixed-point theorem constructively over $\mathbf{V}(Kl)$. The usual proof of the noneffective version of the fixed-point theorem is fully constructive, so there is no difficulty in carrying it out over $\underline{ISys}$ in $\mathbf{V}(Kl)$. It is an immediate consequence of this trivial remark that the following holds in $\mathbf{V}(Kl)$ for every $\underline{A}$ from $\underline{ESys}$:

$$\forall X\ (X : (\overline{\underline{A}} \to \overline{\underline{A}}) \to \exists y \in \overline{A}^{\bullet}\ (X(y) = y \wedge \forall z \in \overline{A}^{\bullet}\ (X(z) = z \to y \subseteq_A z))).$$

Since this expression of the fixed-point result is formulable in $\mathbf{L}^{e}$ as a sentence in Γ, the *effective* fixed-point theorem holds automatically and without further ado in $\underline{ESys}$.

Moreover, because the constructive proof of the fixed-point theorem is interpreted over realizability, even more information is forthcoming. The realizability conditions for '$\to$' show that there is a uniform effective procedure which, given an index for computable $F : \underline{A} \to \underline{A}$ in $\underline{ESys}$, outputs an index of the computable fixed-point of F. But that is not all. The proof of the soundness theorem for realizability gives one even more: an index for this uniform procedure is itself effectively calculable from the code of a constructive proof of the fixed-point theorem together with the index of a presentation of $\underline{A}$. We want to emphasize that this is not a further result that one labors to derive in addition to the proof of the pure fixed-point result—as is the practice in the theory of classical eg systems. Rather, "automatic effectiveness" is merely part of the profit earned by constructive mathematics.

This is just an instance of a general phenomenon on which we have already had occasion to remark. We are refering to the ability of IZF under realizability to offer automatic "program specification." The conclusions of the preceding paragraph should be compared with our remarks in Chapter Four on Church's Thesis. One may take it as given that, whenever a universally-quantified conditional statement in the language of set theory is realized, there is something that one could say about automatic programming.

In $\mathbf{V}(Kl)$, $\underline{ISys}$ represents exactly what one would want from a constructive category of information systems. First, we've seen that $\underline{ISys}$ has a kind of domain-theoretic Brouwer's Theorem: every set map of systems is continuous. It also satisfies a kind of Church's Thesis: that every set map is computable. These highly nonclassical axioms make for the axiomatic freedom encouraged earlier; in $V(Kl)$, domain-theoretic life is blissful. But that's not all. Since set theory is interpreted over $V(Kl)$ as realizability, *ISys* stands in an illuminating semantical relationship with the classical category *ESys*. Thanks to this relationship, proofs of "effective facts" about *ESys*, like the effective fixed-point theorem, can be obtained from simple, wholly noneffective set-theoretic arguments over *ISys*. In this way, the theory of *ESys* is eliminated, and all that remains are realizability and the constructive mathematics of *ISys*.

CHAPTER 8

Recursive Point-Set Topology: A Final Application

Section 1: Prefatory and historical remarks

The primary result of this chapter is that certain sorts of forcing constructions cannot be carried out in IZF and in some of its extensions. We make no pretense that this in itself answers some pressing or longstanding problem in the foundations of intuitionistic mathematics. Rather, our designs are, in a way, much less ambitious. First, our hope is to reply to those who claim that, under realizability, recursive topology does not represent any constructively interesting mathematics. Our contention is that work in recursive topology often represents a contribution to the foundations of nonrecursive constructive mathematics, and that attention to realizability might increase the frequency of positive contributions from this direction.

Second, we hope to suggest that this feature of classical recursive mathematics is ubiquitous. Independently of topology and of polemics, this chapter offers a hint of the ways in which our work in the past chapters can be extended. There is every indication that *any* of the fields of recursive mathematics—recursive topology, recursive algebra, recursive analysis and even recursively saturated models—might be examined under realizability for their contributions to pure constructive mathematics. Consequently, we close the present exposition with a very brief indication of the work which is open to the future.

All the results of this chapter were obtained in June 1982.

Section 2: cHa's and forcing in IZF

Just as a complete Boolean algebra is a natural algebraic representative of classical elementary logic, the category of *complete Heyting algebras* (cHa's) is the category of first-order intuitionistic "theories." A cHa is a distributive, complemented lattice with arbitrary infs and sups and with the infinitary distributive law:

$$p \wedge \bigvee_i q_i = \bigvee_i (p \wedge q_i).$$

The quantificational correlate of the distributive law is

$$p \wedge \exists x\, \phi \leftrightarrow \exists x\, (p \wedge \phi),$$

which is intuitionistically valid, as is the logical correlate of any law of Heyting algebra. Ω, the set of constructive propositions, and $P(A)$, for any set A, afford natural examples of Heyting algebras.

The dual distributive law

$$p \vee \bigwedge_i q_i = \bigwedge_i (p \vee q_i)$$

does not hold in all cHa's and has as its logical correlate

$$p \vee \forall x\, \phi \leftrightarrow \forall x\, (p \vee \phi).$$

This principle is not intuitionistically correct. Instances of the latter are false on finite Kripke structures. Addition of the dual law gives algebraic formulation to the logic of "constant domains." (For the logic of constant domains, see Görneman (1971).)

Forcing in IZF is relatively easy. It is intrinsically no more difficult than in ZF. All one needs to do is locate an IZF-provably cHa A—the opens of an IZF-topological space will do—and then run through the construction of the usual Heyting-valued universe $V(A)$ over it. Just as in the classical world, if IZF $\vdash$ "A is a cHa," then IZF $\vdash$ "$V(A) \models$ IZF." The details of this construction are fully explained in Grayson (1975) and applied in Grayson (1979). Forcing in IZF can even be sufficient unto the needs of TND: if A happens to be (IZF-)Boolean, then IZF $\vdash$ "$V(A) \models$ ZF." In IZF, there are plenty of Boolean algebras; the $\neg\neg$-closed subsets of $P(\{0\})$ comprise, constructively, a Boolean algebra with the Boolean operations defined using a double negation translation.

We will prove that some work in recursive topology presents, under realizability, a theorem that sets limits to forcing in IZF. A pair of enumeration arguments from the work of Kalantari and Retzlaff (in particular, from Kalantari and Retzlaff (1979)) can be transformed into surprizingly strong theorems on the structure of cHa's and on the limits of forcing over $\mathbf{V}(Kl)$. The theorems are all the more surprizing because of their universality; we prove that all objects of an extensive category of topological spaces over $\mathbf{V}(Kl)$ exhibit certain "bad" cHa behaviors. The theorems also partake of a fair measure of naturalness; there is no call to go searching wildly through the conceptual warehouse to locate the constructive concepts appropriate to expressing the classical theorems.

Classically, it is trivially true that the countable opens of a space with a countable basis form a cHa. If an open is countable whenever it is a union of a countable collection of basis elements, then, classically, every open in a space with countable basis is countable (or empty). However, this sort of reasoning is, to all appearances, highly nonconstructive. The present chapter shows to what a great extent these appearances are not deceiving. In $\mathbf{V}(Kl)$, for every ω-presented topological space, the lattice of countable opens is not complemented, and, hence, cannot form a cHa. Incidentally, this fact about $\mathbf{V}(Kl)$ provides a powerful argument for the importance of the concept of subcountability; it shows that, for some important applications of constructive topology, countability had best be replaced by *subcountability*.

Roughly, a space $\langle X, \Lambda \rangle$ is ω-presented if and only if there are $\Delta \subseteq \Lambda$ and a function f such that Δ is a basis for Λ and $f : \omega \approx \Delta$. The rationals and the reals, under their usual topologies, provide paradigmatic ω-presented spaces. In $\mathbf{V}(Kl)$, in none of these familiar spaces does the lattice of countable opens (relative to the presentation) form a cHa. As independence results for IZF go, this is extremely strong; it is not the case that there is some *recherché* space for which the lattice of countable opens is nonHeyting. Rather, there is a constructively effective procedure, which, given the realizability "data" on $\langle X, \Lambda \rangle$ as an ω-presented space in $\mathbf{V}(Kl)$, enumerates an open U of X which is noncomplemented in $\mathbf{V}(Kl)$. The set U can even be placed under a maximum of logical control. In terms of the basis, U can be chosen to be a *decidable* open. For us, an open of $\langle X, \Lambda \rangle$ is decidable just in case there is a constructive function which determines the inclusion of basis elements vis-a-vis U. In short, an open U is decidable just in case

$$\forall \delta \in \Delta \, (\delta \subseteq U \vee \neg \delta \subseteq U) \text{ holds.}$$

Section 3: Noncomplemented opens in recursive topology

There are certain known facts from recursive topology which can be exploited in proving the desired independence theorems; this section provides a rapid summary of them. The reader is refered to Kalantari and Retzlaff (1979) for further details.

3.1. Definition. A topological space $\langle X, \Lambda \rangle$ is *recursively presented* if and only if there is a $\Delta \subseteq \Lambda$ and an f such that $f : \omega \approx \Delta$, Δ is a base for Λ and

(1) Δ is closed under finite unions and intersections,

(2) $\cup$ and $\cap$ are, after being pulled back onto ω along f, recursive,

(3) $f(0) = \emptyset$ and

(4) $\forall \delta \in \Delta\, (\delta \neq \emptyset \rightarrow \exists \epsilon_1, \epsilon_2 \in \Delta\, (\delta \supseteq \epsilon_1 \cup \epsilon_2 \wedge \epsilon_1 \cap \epsilon_2 = \emptyset \wedge \epsilon_1 \neq \emptyset \wedge \epsilon_2 \neq \emptyset))$.

∎

We remark that (4) implies that the recursively presented spaces contain no isolated points.

3.2. Definition. If $\langle X, \Lambda \rangle$ is a recursively presented space in virtue of the correspondence $f : \omega \approx \Delta$, then f is a *presentation* of $\langle X, \Lambda \rangle$. ∎

3.3. Definition. For $\langle X, \Lambda \rangle$ recursively presented with presentation f, an open $U \in \Lambda$ is *r.e.* (in $\langle \langle X, \Lambda \rangle, f \rangle$) iff there is a g such that g is total recursive and $U = \bigcup_{i \in \omega} f(g(i))$. ∎

3.4. Definition. For $\langle X, \Lambda \rangle$ recursively presented with presentation f, $\mathbf{RE}(\langle \langle X, \Lambda \rangle, f \rangle) = \{U \in \Lambda : U \text{ is r.e. }\}$. ∎

Henceforth, we will adopt a studied carelessness about terminology; '$\mathbf{RE}(\langle \langle X, \Lambda \rangle, f \rangle)$' will often be reduced to '$\mathbf{RE}(X)$.'

3.5. Definition. $U \in \mathbf{RE}(X)$ is *complemented* in $\mathbf{RE}(X)$ iff $\exists V \in \mathbf{RE}(X)$ such that

(1) $U \cap V = \emptyset$ and

(2) $U \cup V$ is dense in $\langle X, \Lambda \rangle$.

At first glance, the members of $\mathbf{RE}(X)$ play a role in topology analogous to that played in ordinary recursion theory by the r.e. sets. The analogy is borne out to some extent by the behavior of the (classical) finite sets; just as all finite sets are r.e. in the recursion-theoretic sense, all finite unions of basic opens are r.e. in the topological sense.

However, the next lemma destroys any hope of such across-the-board parity between r.e. opens and r.e. sets.

3.6. Note. Kalantari and Retzlaff have themselves adopted no simple expression for the spaces that we call 'recursively presented.' Our terminology is not, however, without precedent in the literature. ▮

3.7. Lemma. *For recursively presented $\langle X, \Lambda\rangle$, there is a $U \in \mathrm{RE}(X)$ which is not complemented.*

Proof. A full proof appears as Theorem 2.1 of Kalantari and Retzlaff (1979); as noted above, our terminology differs slightly from theirs. The essence of the proof is a simple enumeration technique; the technique and the attendant reasoning could well be treated in IZF. For present purposes, it is enough to extract from the proof the knowledge that, given indices for $\cup$ and $\cap$ as recursive functions on ω, an index for an r.e. but noncomplemented U is effectively calculable. ▮

3.8. Definition. For $\langle X, \Lambda\rangle$ recursively presented with presentation f, $U \in \Lambda$ is *recursive* (in $\langle\langle X, \Lambda\rangle, f\rangle$) if and only if there is an $E \subseteq \omega$ such that E is recursive and $U = \bigcup_{i \in E} f(i)$. ▮

3.9. Lemma. *For $\langle X, \Lambda\rangle$ recursively presented, there is a recursive $U \in \Lambda$ which is noncomplemented in $\mathrm{RE}(X)$.*

Proof. This is Corollary 2.4 of Kalantari and Retzlaff. Again, from the construction of Kalantari and Retzlaff, there is obtainable an effective procedure, which, given indices for $\cup$ and $\cap$, produces an index for the required U.

Section 4: Noncomplemented opens in constructive topology

Naturally, the concept of "recursively presented topological space" is related to some nonrecursive notion via the interpretation of the logical signs given by $\mathbf{V}(Kl)$. The appropriate nonrecursive notion is that of "ω-presented" space:

4.1. Definition. A topological space $\langle X, \Lambda\rangle$ is *ω-presented* if and only if there is a $\Delta \subseteq \Lambda$ and an f such that $f : \omega \approx \Delta$ and

(1) Δ is a basis for Λ,

(2) Δ is closed under finite unions and intersections,

(3) $f(0) = \emptyset$ and

(4) $\forall \delta \in \Delta\, (\delta \neq \emptyset \rightarrow \exists \epsilon_1, \epsilon_2 \in \Delta\, (\delta \supseteq \epsilon_1 \cup \epsilon_2 \wedge \epsilon_1 \cap \epsilon_2 = \emptyset \wedge \epsilon_1 \neq \emptyset \wedge \epsilon_2 \neq \emptyset))$.

∎

4.2. Definition. When $\langle X, \Lambda\rangle$ is ω-presented via f, where $f : \omega \approx \Delta$, we say that f is a *presentation* for $\langle X, \Lambda\rangle$. ∎

4.3. Definition. If $\langle X, \Lambda\rangle$ is ω-presented, an open $U \in \Lambda$ is *countable* (relative to $\langle\langle X, \Lambda\rangle, f\rangle$) iff $\exists g \in (\omega \Rightarrow \omega)\; U = \bigcup_{i\in\omega} f(g(i))$. ∎

4.4. Definition. When $\langle X, \Lambda\rangle$ is ω-presented via f, $\mathbf{C}(\langle\langle X, \Lambda\rangle, f\rangle)$ is the set of all countable U from Λ. ∎

Here, we will lapse into an abbreviated idiom: '$\mathbf{C}(X)$' will be used for '$\mathbf{C}(\langle\langle X, \Lambda\rangle, f\rangle)$' when Λ and f are already understood.

4.5. Definition. Let $U, V \in \Lambda$. $U \cup V$ is *dense* in ω-presented $\langle X, \Lambda\rangle$ if and only if

$$\forall \delta \in \Delta\, (\delta \neq \emptyset \rightarrow (\exists \epsilon_1 \subseteq U\, (\epsilon_1 \cap \delta \neq \emptyset) \vee \exists \epsilon_2 \subseteq V\, (\epsilon_2 \cap \delta \neq \emptyset))).$$

∎

4.6. Definition. $U \in \mathbf{C}(X)$ is *complemented* in $\mathbf{C}(X)$ if and only if there is a $V \in \mathbf{C}(X)$ such that $U \cap V = \emptyset$ and $U \cup V$ is dense in $\langle X, \Lambda\rangle$. ∎

4.7. Remark. From a constructive standpoint, our definition of density is a rather stringent one. We are insisting that, when $A \cup B$ is dense in space X, one can apply a constructive function, which, given a nonempty basis element, determines which of A or B it intersects. Later, we will show that a weaker definition would have done just as well. At that point, we will prove that the theorems are not altered by a reasonable weakening of the density notion. At this point, we prefer to retain the strict notion because it lies closer to classical sensibilities. ∎

4.8. Lemma. *There is a partial recursive Θ such that, for $f \in \mathrm{V}(Kl)$, if*

$$e \Vdash f \text{ is a binary function on } \omega,$$

then $\Theta(e)\downarrow$ and $\Theta(e)$ indexes $\underline{f}$, where

$$\underline{f} = \{\langle n, m\rangle : \mathrm{V}(Kl) \models \overline{\langle \overline{n}, \overline{m}\rangle} \in f\}.$$

Proof. This should be compared with the proof of Part (2) of Lemma 2.5, Chapter Five.

Assume that $e \Vdash$ "f is a binary function on ω." First, it is clear that $\underline{f}$ is total and functional on ω. Second, there is a partial recursive Ψ such that

$$\Psi(e) \Vdash \forall x \in \overline{\omega}\ \exists y \in \overline{\omega}\ \overline{\langle x, y\rangle} \in f.$$

If we can prove that

$$\underline{f} = \{\langle n, \{\Psi(e)\}(\langle n, i\rangle)_{00}\rangle : n \in \omega\},$$

then the proof will be complete.

Let $n \in \omega$. Then, as always, $\langle n, i\rangle \Vdash \overline{n} \in \overline{\omega}$, and $\exists b \in \mathrm{V}(Kl)$ such that

$$\{\Psi(e)\}(\langle n, i\rangle) \Vdash b \in \overline{\omega} \wedge \overline{\langle \overline{n}, b\rangle} \in f.$$

By definition of $\overline{\omega}$, we know that

$$\{\Psi(e)\}(\langle n, i\rangle)_{01} \Vdash b = \overline{\{\Psi(e)\}(\langle n, i\rangle)_{00}}.$$

Hence, $\mathrm{V}(Kl) \models$

$$\overline{\langle \overline{n}, \overline{\{\Psi(e)\}(\langle n, i\rangle)_{00}}\rangle} \in f.$$

∎

This lemma shows that ω-presentations of topological spaces can be "excised" from $\mathbf{V}(Kl)$ and will reappear in $\mathbf{V}$ as ω-structures under appropriate recursive operations. The central idea of the proof of the next (and main) theorem exploits this fact. An ω-presentation can first be excised as a recursive structure which is provably a presentation for a recursively presented topological space. Then, the construction of Lemma 3.7 is performed in $\mathbf{V}$. Finally, the results of the construction are reëmbedded into $\mathbf{V}(Kl)$ and we prove that the resulting object indexes a noncomplemented set there.

4.9. Theorem. $\mathbf{V}(Kl) \models$ *For every ω-presented space $\langle X, \Lambda\rangle$, there is a noncomplemented $U \in \mathbf{C}(X)$.*

Proof. Let $e \Vdash$ "$\langle X, \Lambda\rangle$ is an ω-presented topological space." Then, by the preceding lemma, there are partial recursive Θ and Ψ such that, in $\mathbf{V}$,

$$\Theta(e) \text{ indexes } \underline{\cap} \text{ and } \Psi(e) \text{ indexes } \underline{\cup}.$$

The constructive lattice-theoretic structure on (internal) $\langle X, \Delta\rangle$ now appears in $\mathbf{V}$ as a recursion-theoretic structure on ω. Moreover, this structure specifies a unique recursively presented topological space. This is almost obvious; the only real question is whether

$$\forall \delta \in \Delta\, (\delta \neq \emptyset \rightarrow \exists \epsilon_1, \epsilon_2 \in \Delta\, (\delta \supseteq \epsilon_1 \cup \epsilon_2 \wedge \epsilon_1 \cap \epsilon_2 = \emptyset \wedge \epsilon_1 \neq \emptyset \wedge \epsilon_2 \neq \emptyset)).$$

holds for this external topology.

In terms of the excised presentation, the latter condition becomes

$$\forall n \in \omega\, (n \neq 0 \rightarrow \exists p, q \in \omega\, (n \underline{\cap} (p \underline{\cup} q) = p \underline{\cup} q \wedge p \underline{\cap} q = 0 \wedge p \neq 0 \wedge q \neq 0).$$

But it is trivial to check that, given that $\langle X, \Lambda\rangle$ is presented via f in $\mathbf{V}(Kl)$, this statement holds effectively in $\mathbf{V}$. Therefore, there are $\langle X, \Lambda\rangle$ and f such that $\langle X, \Lambda\rangle$ is presented via f and $\underline{\cap}$ and $\underline{\cup}$ are the pullbacks of the intersection and union operations on the presentation basis Δ.

The strategy is now clear. We construct noncomplemented U in the external $\langle X, \Lambda\rangle$ with $K \subseteq \omega$ such that $U = \bigcup_{k \in K} f(k)$. K is then embedded back into $\mathbf{V}(Kl)$ in the usual way as an ω-stable set $\overline{K}$. It is then easy—because the requisite conditions are negative—to check that, internally, $V = \bigcup_{k \in \overline{K}} f(k)$ where V is noncomplemented in $\mathbf{V}(Kl)$. As adverted to in the proof of Lemma 3.7, we can, given indices for $\Theta(e)$ and $\Psi(e)$, effectively calculate an index for a $K \subseteq \omega$ such that K is noncomplemented in $\mathrm{RE}(X)$. Hence,

$$\mathrm{V}(Kl) \models \overline{K} \text{ is countable and}$$

$$\mathrm{V}(Kl) \models \bigcup f(\overline{K}) \text{ is a countable open .}$$

Witnesses for these are obtainable effectively from e. It only remains to show that $\bigcup f(\overline{K})$ is not complemented in internal $\mathrm{C}(X)$.

To that end, assume that $\mathrm{V}(Kl)$ satisfies the statement

$$\exists Z \in \mathrm{C}(X)\, (\bigcup f(\overline{K})) \cap Z = \emptyset \wedge (\bigcup f(\overline{K})) \cup Z \text{ is dense in } \langle X, \Lambda \rangle.$$

From this it follows that $\mathrm{V}(Kl) \models$

$$\exists L \subseteq \overline{\omega}\; L \text{ is countable } \wedge$$
$$(1)\;\; \neg \exists m\, \exists p\, (m \in \overline{K} \wedge p \in L \wedge m \cap p \neq 0) \wedge$$
$$(2)\;\; \forall n \in \omega\, (n \neq 0 \rightarrow (\exists m \in \overline{K}\; n \cap m \neq 0 \vee \exists m \in L\; n \cap m \neq 0)).$$

IZF+MP $\vdash \forall L \in \mathrm{P}(\omega)$ (L is countable $\rightarrow L$ is $\omega-$stable), so L in the above can be taken to be ω-stable. By Lemma 2.3 of Chapter Five, L is realizably identical to the injection of a classical subset of ω. Hence, there is a set $\underline{L}$ in $\mathrm{P}(\omega)$ of V such that $\mathrm{V}(Kl) \models \overline{\underline{L}} = L$ and (1) and (2) *supra* hold in $\mathrm{V}(Kl)$ with $\overline{\underline{L}}$ replacing L.

Since $\mathrm{V}(Kl) \models$ "L is countable," $\underline{L}$ is r.e. in V. We then prove that $\underline{L}$ is disjoint from K. Assume that $\exists m \in K\; \exists p \in \underline{L}\; m \underline{\cap} p = 0$. Then, given the definitions of $\overline{K}$, $\overline{\underline{L}}$ and $\underline{\cap}$, $\mathrm{V}(Kl)$ satisfies

$$\exists m\, \exists p\, (m \in \overline{K} \wedge p \in \overline{\underline{L}} \wedge m \cap p \neq 0).$$

But this contradicts assumption (1).

Finally, $K \underline{\cup} \underline{L}$ is provably dense in external $\langle X, \Lambda \rangle$. Take $n \in \omega$, $n \neq 0$. Then,

$$(1)\;\; \mathrm{V}(Kl) \models \overline{n} \in \overline{\omega} \wedge \overline{n} \neq 0.$$

From (2), using the realizability interpretations of $\rightarrow$ and $\vee$, we know that either

$$\exists m \in K\; \mathrm{V}(Kl) \models \overline{n} \cap \overline{m} \neq 0 \text{ or}$$

$$\exists m \in \underline{L}\, \mathrm{V}(Kl) \models \overline{n} \cap \overline{m} \neq 0.$$

Therefore, there is some m, either from K or from $\underline{L}$, such that $\mathrm{V} \models n \underline{\cap} m \neq 0$. It follows that $\bigcup f(K \cup \underline{L})$ is dense in external $\langle X, \Lambda \rangle$. But this contradicts the conditions set by the Kalantari-Retzlaff construction of K. Consequently, our original assumption was false and the desired conclusion, that

$$\mathrm{V}(Kl) \models \bigcup f(\overline{K}) \text{ is not complemented in } \mathrm{C}(X),$$

is true. ∎

4.10. Note. As it happens, the "excision and reëmbedding" strategy is not entirely necessary for this theorem. Because the proof of Lemma 3.7 is constructive, the whole proof could have been conducted internally However, doing everything internally seems to require unnecessary circumlocution. Also, our strategy permits the use of full classical logic (externally) in obtaining results about IZF and does not restrict our dealings to those portions of recursive mathematics that just happen to be constructively acceptable. After all, the point of our entire enterprise is to plumb the relations between parts of classical mathematics and mathematics over $\mathrm{V}(Kl)$, so, our strategy is in keeping with our metamathematical *Weldbild*. ∎

As mentioned before, the logical strength of our density condition:

$$\forall \delta \in \Delta\, (\delta \neq 0 \rightarrow (\exists \epsilon_1 \in U\; \delta \cap \epsilon_1 \neq \emptyset \vee \exists \epsilon_2 \in V\; \delta \cap \epsilon_2 \neq \emptyset)).$$

is more than sufficient to the task. In the preceding proof, we could well have adopted a weaker notion:

$$\forall \delta \in \Delta\, (\delta \neq \emptyset \rightarrow \neg(\forall \epsilon \in U\; \delta \cap \epsilon = \emptyset \wedge \forall \epsilon \in V\; \delta \cap \epsilon = \emptyset)).$$

To see this, return to stage (3) of the proof and assume that

$$\mathrm{V} \models \forall m \in K\; n \underline{\cap} m = 0 \wedge \forall m \in \underline{L}\; n \underline{\cap} m = 0.$$

Then, if $e \Vdash \overline{m} \in \overline{K}$, $e_0 = m \in K$. So, $n \underline{\cap} e_0 = 0$ and

$$\langle\langle n, e_0\rangle, i\rangle \Vdash \langle \overline{n}, \overline{m}, \overline{0}\rangle \in \underline{\cap}.$$

Therefore, $V(Kl) \models \forall m \in \overline{K}\ n \cap m = 0$. (Remember that $V(Kl) \models \overline{\cap} = \cap$). Similarly,

$$V(Kl) \models \forall m \in L\, n \cap m = 0.$$

But this would contradict the above assumption, so

$$V \models \neg \forall m \in K\ n \underline{\cap} m = 0 \wedge \forall m \in \underline{L}\, n \cap m = 0.$$

and $K \cup \underline{L}$ is dense externally. Therefore, our proof would have gone through untroubled for a much weaker notion of density.

4.11. Corollary. $V(Kl) \models$ *If* $\langle X, \Lambda \rangle$ *is* ω*-presented, then* $C(X)$ *is not a Heyting algebra.*

Proof. We work in IZF. We already know that, in $V(Kl)$, if $\langle X, \Lambda \rangle$ is ω-presented, there is a $U \in C(X)$ such that U is not complemented. The intention is to prove that this U from $C(X)$ is not complemented in the "Heyting algebra" sense, so that $C(X)$ does not, under the usual operations, constitute the domain of a Heyting algebra.

Assume that U is Heyting-complemented. Then there is a $\subseteq$-maximum V in $C(X)$ disjoint from U. Since the space is ω-presented, $V = f(L)$ for some $L \in P(\omega)$. We take $n \in \omega$, $n \neq 0$ and assume that

$$\forall m \in K\ n \cap m = 0 \wedge \forall m \in L\, n \cap m = 0.$$

Then, $V \cup f(n)$ is disjoint from U and $V \cup f(n) \subseteq V$.

Now, we assume that $f(n)$ is inhabited. Then, for some $m \in L$, there is an $x \in f(n) \cap f(m)$. Therefore, $\exists m \in L\ \ n \cap m \neq 0$. This contradicts our previous assumption, so

$$\neg \exists x\, (x \in f(n)),$$

or $n = 0$. But this contradicts the original assumption on n. Hence,

$$\neg(\forall m \in K\ n \cap m = 0 \wedge \forall m \in L\, n \cap m = 0).$$

Therefore, U is complemented in $C(X)$ and this contradicts the proved property of U. ∎

Finally, the second of the lemmas drawn from recursive topology (3.9), when internalized using excision and reëmbedding, shows that the sets for which complementation fails in $V(Kl)$ can be maximally well-controlled—they can be taken to satisfy TND.

4.12. Theorem. $\mathbf{V}(Kl) \models$ *If* $\langle X, \Lambda\rangle$ *is* ω*-presented, then there is a decidable* $V \in \mathbf{C}(X)$ *with* V *noncomplemented in* $\mathbf{C}(X)$.

Proof. Apply the techniques of Theorem 4.9 to Lemma 3.9. ▮

In summary, we have shown that, in the project of forcing over topologies in IZF, we cannot force with the countable opens of the reals. Should one care to force with opens from the reals which are indexed by natural numbers, one must make do with sets which are subcountable. For this weaker notion, there is no such failure of complementation. We have come to this information by way of the same realizability techniques we have applied repeatedly in gleaning purely constructive information from the RETs, isols, isolic integers and eg systems. We claim that there is no in principle barrier to carrying such techniques into every realm of effective mathematics.

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