Nonlinear Guided Waves In Fibre Optics

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Abstract

Optical fibres are widely used in optical communication systems because they can transmit signals in the form of extremely short pulses of quasi-monochromatic light over large distances with high intensities and negligible attenuation. A fibre that is monomode and axisymmetric can support both left- and right-handed circularly polarised modes having the same dispersion relation. The evolution equations are coupled nonlinear Schrödinger equations, the cubic terms being introduced by the nonlinear response of the dielectric material at the high optical intensities required.

In this thesis we analyse signal propagation in axisymmetric fibres both for a fibre with dielectric properties which vary gradually, but significantly, along the fibre and for a fibre which is curved and twisted but with material properties assumed not to vary along the fibre.

For fibres with axial inhomogeneities, we identify two regimes. When the axial variations occur on length scales comparable with nonlinear evolution effects, the governing equations are found to be coupled nonlinear Schrödinger equations with variable coefficients. Whereas for more rapid axial variations it is found that the evolution equations have constant coefficients, defined as appropriate averages of those associated with each cross-section. The results of numerical experiments show that a sech-envelope pulse and a more general initial pulse lose little amplitude even after propagating through many periods of an axial inhomogeneity of significant amplitude.

For a curved and twisted fibre, it is found that the pulse evolution is governed by a coupled pair of cubic Schrödinger equations with linear cross coupling terms having coefficients related to the local curvature and torsion of the fibre. These coefficients are not, in general, constant. However for the case of constant torsion and constant radius of curvature which is comparable to the nonlinear evolution length, numerical evidence is presented which shows that a nominally non-distorting pulse is unstable but the onset of instability is delayed for larger values of torsion.

The integrability of partial differential equations can be determined by identifying whether, or not, the equations have the Painlevé property. Using Painlevé analysis we show that the integrability of the coupled constant coefficient nonlinear Schrödinger equations depends on the value of the coupling coefficient and we identify values for this constant in order that the equations have the Painlevé property. The values we obtain agree with previously known results about the integrability of this system of equations. For the cases that the equations are not integrable, we use Painlevé analysis to find possible solutions to the equations, by considering a truncated Painlevé series. We show that Painlevé analysis can suggest general forms of solution for the coupled nonlinear Schrödinger equations, however the only solutions that are identified are ones that are already known.

Declaration

Except where specific reference is made to other sources, the work presented in this thesis is the original work of the author. The work has not been submitted, in whole or in part, for any other degree.. Certain parts of this thesis have been published.

Elaine Ryder

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Contents

Chapter 1

Introduction

1.1 Communication systems

Communication systems which allow information to be broadcast over long distances have been in use since ancient times. The early communication systems mainly involved optical or acoustical signals such as beacons, smoke signals or drums, for example, the Greeks were known to have used signal fires as warning alarms. These simple communication systems had several drawbacks. The amount of information that could be conveyed was limited because the meaning of the signal had to be prearranged between the sender and the receiver, and it was not possible to add to the message or change its meaning. To overcome this problem, systems were introduced which allowed the message to be encoded in some way, for example semaphore or signal flags. Factors which also affect the signal transmission are weather conditions, such as rain or fog, and obstacles in the signal path, these reduce the distance over which the signal can be transmitted. An increase in the length of the transmission path was obtained by the use of relay stations. However the transmission rates of these systems were low, and runners, riders or carrier pigeons were often the preferred method for sending information. All of these methods of communication were unreliable, and there has been a constant desire to eliminate chance interruptions to messages.

The invention of the telegraph by Morse in 1838 marked the beginning of a series of developments in electrical communication systems which increased the data transmission rate. The telegraph consisted of a wire cable along which electrical signals could be sent at rates which were much faster than any of the previous communication systems. With the discovery of electromagnetic radiation by Hertz in 1887 and the first demonstration of radio eight years later by Marconi, the communication rates increased as higher frequency electromagnetic waves were used as the carrier waves. Since the amount of information that can be transmitted using electromagnetic radiation is directly related to the frequency range of the carrier wave, research into new communication systems has mainly been concerned with using higher frequency carrier waves, such as radar and microwaves. Although optical frequencies can offer much higher communication rates than frequencies used in electrical communication systems, development of optical communication systems was prevented by technical problems described below which have only been overcome since the laser was invented. As well as increasing the communication rates, these communication systems have increased the reliability as special receiving equipment is required to decode the messages.

1.2 Development of optical communication systems

Towards the end of the nineteenth century Alexander Graham Bell gave the first demonstration of his 'photophone' (Bell, 1880), which operated on the same basic principle that is used in optical communication systems today. This instrument transmitted sound over a distance of 200 metres, using a beam of light of varying intensity as the carrier wave. However, it was not until Hondros and Debye (1910) suggested that light could be transmitted through dielectric rods or *waveguides* that light was considered as a possible signal carrier wave.

The first unclad glass fibres were manufactured during the 1920's, but were impractical for transmission of light over long distances due to high losses caused by impurities in the glass and discontinuities which occurred at the glass-air interface where the fibre was supported. In the 1950's it was proposed that the signal power could be better confined within the fibre if it was coated with a layer of material of a slightly lower refractive index (van Heel, 1954). The first cladding layers were plastic and although the transmission loss was lower than that which occurred in unclad fibres, these fibres still exhibited a high level of light loss because of imperfections in the fibre. Losses were further reduced by coating the dielectric fibre with a glass of a lower refractive index (Kapany, 1959).

The interest in optical communications was revived with the invention of the laser in 1960. The laser provided a coherent light source which could be modulated sufficiently rapidly at the high frequencies required for a carrier wave in an optical communication system. Despite this most of the work on dielectric waveguides remained theoretical until 1966 because the only glass fibres which were available exhibited transmission losses of approximately 1000dB/km, which were too large for communication systems. Then in a theoretical study, Kao and Hockham (1966) suggested that these high loss levels were mainly due to impurities in the glass and that if a low-loss dielectric material could be manufactured, with a loss of around 20dB/km, optical waveguides could be used as a communication medium.

During the next decade, glass refining processes were improved and the losses in silica fibres were reduced to the 20dB/km threshold (Kapron et al., 1970), which had been suggested by Kao and Hockham. Further improvements in the fabrication process and the use of high purity starting materials has allowed this loss to be further reduced to 0.2dB/km near the 1.55μm wavelength (Miya et al., 1979). There is a lower limit on the fibre loss caused by Rayleigh scattering, an intrinsic property of the glass, which for pure silica occurs near the 1.55μm wavelength. The lowest attenuation which has been reported at this wavelength is 0.154dB/km (Kanamori et al., 1986), although for practical purposes a loss of

0.2dB/km is acceptable.

Optical waveguides are made of dielectric materials whose refractive index varies with the intensity of the light source. For weak light sources this dependence is approximately linear but for stronger sources, such as lasers, the dependence becomes nonlinear. Hasegawa and Tappert (1973) proposed that the dispersion which causes the broadening of the signal could be balanced by the nonlinear effects of the material which cause a sharpening of the pulse and that this balancing could, in theory, allow a stable pulse to propagate over the transmission lengths required for long distance telecommunications systems. They were able to demonstrate this balancing effect both theoretically and numerically for a medium which has a cubic nonlinearity. It was not until low-loss fibres were manufactured that Mollenauer, Stolen and Gordon (1980) were able to make the first experimental observations of non-distorting pulses in optical fibres.

There has been much work undertaken to show mathematically how the two phenomena affect the propagation of waves in a fibre. Much of this work (Anderson and Lisak, 1983, Zakharov and Shabat, 1972, Potasek et al., 1986) has shown that the governing equation for the amplitude modulation of the signal is the cubic Schrödinger equation. However, the fundamental mode of an ideal cylindrically symmetric, isotropic, monomode fibre consists of two equivalent modes (Snyder and Love, 1983) which are orthogonally polarised, these modes are degenerate, that is, they have the same dispersion relation. In practice a fibre is never ideal, the core may be slightly elliptic, the material anisotropic, or the fibre bent, these perturbations from the ideal cause the polarisation to change along the fibre, the degeneracy between the two modes is destroyed and birefringence is introduced. Menyuk (1987) and Blow et al. (1987) have shown that in birefringent fibres there is an interaction between two orthogonal linearly polarised modes which have slightly differing phase speeds and have shown that the evolution of the pulse amplitudes is governed by a pair of coupled equations. However, even in an ideal single-mode fibre the existence of two equivalent modes means that signals with different polarisation interact nonlinearly and that two independent complex amplitudes are required to describe the signal. Parker and Newboult (1989) have shown that the equations which describe the signal amplitudes are a coupled pair of cubic Schrödinger equations. Unlike the single cubic Schrödinger equation, this system of equations is not completely integrable (Zakharov and Schulman, 1982), but it does possess a large family of non-distorting pulselike solutions and other families of generalised similarity solution (Parker, 1988).

Numerical study of both the nonlinear Schrödinger equation and the coupled nonlinear Schrödinger equations has been carried out. Desem and Chu (1987) have investigated the interaction of two closely separated solitons using the exact two soliton solution to the single nonlinear Schrödinger equation, while Parker and Newboult (1989) have investigated the interaction of two initially well separated solitons which individually are solutions to the coupled nonlinear Schrödinger equations. Blow et al. (1987) have investigated the stability of single solitons in birefringent fibres.

Although solitons may propagate in perfectly lossless fibres, in practice there will always be attenuation and broadening of a pulse, which will reduce the length over which the soliton will propagate. Communication systems require that the pulse be transmitted over long distances and some means of compensating for these effects is required. One method is to amplify and reshape the solitons periodically. This can be achieved either, by injecting a weak pump beam into the fibre in the direction opposite to that of the soliton propagation to induce the Raman effect or, by splicing short lengths of erbium doped fibre into existing optical fibres. By inducing the Raman effect in a fibre, pulses have been transmitted experimentally over distances greater than 4000km (Mollenauer and Smith, 1988), while propagation lengths of 12000km at a bit rate of 24Gbits/s have been achieved using erbium doped fibres (Mollenauer et al., 1990). Tajima (1987) suggested that instead of periodically amplifying the solitons, invariant solitons could be obtained by tapering the fibre core. A more rigorous treatment was presented by Kuehl (1988) who showed that Tajima's work was a special case of his theory. Both Tajima and Kuehl considered only the single nonlinear Schrödinger equation, and did not take into account the nonlinear interaction between the orthogonal polarisations of the fundamental mode.

1.3 Modern optical fibres

In its simplest form an optical fibre consists of a cylindrical, dielectric rod. Although this type of fibre will function as an optical waveguide, the electromagnetic fields are not wholly contained inside the dielectric region and will decay exponentially outside of the waveguide. This will cause high losses at any discontinuities in the silica-air interface, such as where the fibre is supported. To prevent this optical fibres usually consist of a central core region surrounded by a cladding layer, so that the electromagnetic field is confined substantially to the core. The refractive index in the cladding is chosen to be slightly lower than that in the core and it is this change in the refractive index that allows guided modes to propagate in the fibre. The core and the cladding are both made of silica with dopants added to the silica to obtain the refractive index variation across the fibre, GeO_2 and P_2O_5 are used to increase the refractive index, while fluorine decreases it. The simplest type of fibre is the step-index fibre which has a constant refractive index in the core, n_{co}, and in the cladding, n_{cl}, with $n_{co} > n_{cl}$ and a discontinuity at the boundary between the core and cladding. A second type is the graded-index fibre, in this type of waveguide the refractive index varies across the radius r, and may not have a well defined interface between the core and the cladding.

An optical fibre is capable of supporting a finite number of guided modes and an infinite number of unguided radiation modes at any given frequency. The number of guided modes which can propagate along a fibre depends mainly on the radius of the core. If the diameter of the core is very much greater than the wavelength of the guided radiation, a large number of guided modes can exist and the fibre is referred to as a *multimode* fibre, and has core radii of between 25μm and 30μm. The number of modes that can propagate can be increased by using a larger core diameter. However, each of the guided modes travels at a different speed, causing the pulse to spread out as it travels along the fibre, this effect is called intermodal dispersion. Hence, for long distance optical communications systems it is desirable to allow only a small number of guided modes to propagate.

In order that a fibre is *single-mode*, so only two orthogonally polarised guided modes propagate, the radius must be chosen to be very small ($\approx 0.5\mu$m), this causes problems when manufacturing the fibre or when connecting the fibre to other fibres or components of the communication system. However, by a careful choice of the values of the refractive indices of the core, n_{co}, and the cladding, n_{cl}, so that the ratio n_{co}/n_{cl} is very close to unity, the radius of the core can be increased ($\approx 2\mu$m-4μm) without increasing the number of guided modes, and the fibre is said to be *weakly guiding* (Gloge, 1971). For a fibre with a graded-index profile it is not always possible to distinguish between the core and the cladding, but this type of fibre will be weakly-guiding if the change in the refractive index is small. The radius of the cladding is chosen so that the field strengths at the outer edge of the cladding are so small that they can be assumed to be zero, for both single-mode and multimode fibres the cladding radius is normally between $50\mu\text{m} - 60\mu\text{m}$. In this thesis we shall be considering single-mode, weakly-guiding, graded-index fibres.

1.4 Maxwell's equations

Light propagating in an optical fibre is an electromagnetic wave whose electric and magnetic field intensities, **E** and **H**, are governed by Maxwell's equations. For a medium having no free currents and no free charges, these equations can be written as

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.1}$$

$$\nabla \wedge \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \tag{1.2}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{1.3}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{1.4}$$

For a non-magnetic material, the magnetic induction **B** is related to the magnetic field intensity **H** by

$$\mathbf{B} = \mu_0 \mathbf{H}, \tag{1.5}$$

where μ_0 is the magnetic permeability of free space. In dielectric materials the electric displacement **D** is given by

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \tag{1.6}$$

where ε_0 is the dielectric permittivity of free space and **P** is the electric polarisation which is caused by the interaction between the electric field and the molecules of the material.

For weak electric fields the induced polarisation is proportional to the magnitude of the applied field and, in particular, for an isotropic medium which responds instantaneously to the electric field, the polarisation is given by

$$\mathbf{P} = \varepsilon_0 \chi \mathbf{E}, \tag{1.7}$$

where χ is called the electric susceptibility and is a scalar (for anisotropic materials χ will be a tensor), this type of material is said to be linear. The light source used in optical communication systems is a laser, which emits light at very high intensities so that the polarisation will no longer be linearly proportional to the electric field. The polarisation can be expressed more generally as the series

$$\mathbf{P} = \mathbf{P}^{(1)} + \mathbf{P}^{(2)} + \mathbf{P}^{(3)} + \cdots$$

where $\mathbf{P}^{(j)}$ represents all terms of degree j in the electric field. $\mathbf{P}^{(1)}$ is the linear polarisation which is given in (1.7), and for isotropic materials which possess inversion symmetry, such as silica glass, $\mathbf{P}^{(2)} = 0$. If it is assumed that the response of the medium is instantaneous and that

$$\mathbf{P}^{(3)} = N|\mathbf{E}|^2\mathbf{E}$$

(Bendow et al., 1980), where N is the nonlinear coefficient which does not depend on the frequency, and any nonlinear effects due to higher order terms are negligible compared with the first and third order terms, then the electric displacement can be written as

$$\mathbf{D} = \varepsilon\mathbf{E} + N|\mathbf{E}|^2\mathbf{E}. \tag{1.8}$$

where $\varepsilon = \varepsilon_0(1 + \chi)$ is the dielectric permittivity, and both ε and N depend on the position within the fibre. A material which exhibits this type of nonlinear effect is called a Kerr medium and the fibres considered in this thesis are of this type.

1.5 Summary

In Chapter 2 we consider the effect of axial inhomogeneities on pulse evolution in a cylindrical single-mode lossless fibre. Corrections to the electromagnetic fields are obtained and it is shown that these corrections depend on the length scale of the inhomogeneity. The amplitude modulation equations which govern the pulse evolution are then derived for two different length scales of the longitudinal inhomogeneity. We then consider a pair of coupled variable-coefficient cubic

Schrödinger equations and find conditions on the coefficients such that the equations can be transformed to the constant coefficient cubic Schrödinger equations. In the final section, numerical calculations are presented for the variable-coefficient evolution equations for both an initial sech-envelope pulse and more general initial pulses.

A curved and twisted axially homogeneous fibre is considered in Chapter 3. Correction terms to the fields which are caused by the curvature of the fibre are found. The evolution equations are then derived and are shown to be a coupled pair of nonlinear Schrödinger equations with linear terms due to torsion effects and linear cross-coupling terms due to curvature effects. In the final section of this chapter we show that in some special cases these nonlinear evolution equations reduce to the single cubic Schrödinger equation. Numerical results for the evolution of more general non-distorting initial pulses are also presented for different values of curvature and torsion.

In Chapter 4 we consider the integrability of the coupled pair of constant coefficient cubic Schrödinger equations using Painlevé analysis. An introduction to the Painlevé property is given. The Painlevé partial differential equation test is then applied to the coupled nonlinear evolution equations to determine values of the coupling constant for which the coupled equations have the Painlevé property.

In Chapter 5 Painlevé analysis is used to obtain solutions of coupled pairs of constant coefficient cubic Schrödinger equations which do not have the Painlevé property, by considering solutions which are of the form of truncated Painlevé series.

In Chapter 6 we present a summary of the results obtained in the preceding chapters.

In Appendix A we derive an expression which relates the wavenumber of a guided mode to it's frequency and also obtain an expression for the group slowness of a guided mode.

Appendix B contains a description of the method of multiple scales with particular reference to its application in this thesis.

The equations relating the correction fields which arise in the study of a curved axially homogeneous fibre are given in Appendix C.

The coupled constant coefficient nonlinear Schrödinger equations can be reduced to a pair of coupled nonlinear ordinary differential equations by seeking non-distorting pulse solutions and in Appendix D we derive a series solution to these ordinary differential equations.

Chapter 2

Fibres with axial inhomogeneity

2.1 Preamble

On a perfect, lossless fibre, solitons governed by a single cubic Schrödinger equation can retain their shape and amplitude due to a balance between nonlinearity and dispersion. However on a real fibre there will be losses which produce attenuation and pulse broadening. To compensate for this, both Hasegawa (1984) and Mollenauer et al. (1986) have proposed that the solitons could be amplified periodically by installing amplifiers at certain points along the fibre. Another method is to use a fibre which is axially nonuniform, this was proposed by Tajima (1987), who suggested that invariant solitons could be obtained by tapering the fibre core by an amount which is directly proportional to the soliton attenuation and inversely proportional to the square of the effective core radius. Kuehl (1988) presented a more rigorous treatment of these ideas and showed that Tajima's work was a special case of his theory.

Both Tajima and Kuehl considered only the single cubic Schrödinger equation. However, a cylindrically symmetric, isotropic monomode fibre has two equivalent modes (Snyder and Love, 1983) so two independent complex amplitudes are required to describe the signal since signals with different polarisations interact nonlinearly. Parker and Newboult (1989) have shown that the evolution equations of the signal amplitudes in an axially homogeneous fibre are a coupled pair of cubic Schrödinger equations. In this chapter the effect of axial inhomogeneities in a cylindrical, single-mode lossless fibre are considered. It is shown that the pulse evolution is again governed by a coupled pair of cubic Schrödinger equations. If the axial inhomogeneities have a length scale much shorter than that associated with nonlinear effects, only average properties of these nonuniformities enter the nonlinear evolution equations and for the special case of periodic nonuniformities the equations reduce to those of an equivalent uniform fibre. If the scale of the axial inhomogeneities is comparable with the nonlinear evolution length, the evolution equations for the pulse amplitudes are a coupled pair of cubic Schrödinger equations with variable coefficients.

For the variable-coefficient equations, conditions are found for the existence of a transformation which reduces the equations to constant coefficient equations. The transformations which are found are natural generalisations of those obtained by Grimshaw (1979) for a single cubic Schrödinger equation with variable coefficients. For such cases, suitable sech-envelope pulses will propagate without radiation, although over the long lengths required for optical communication systems most of the cases correspond to non-physical behaviour of either the dispersive or nonlinear effects. Numerical calculations are performed to show that a sech-envelope pulse, or a more general initial pulse loses little amplitude even after propagating through 40 periods of an axial inhomogeneity of significant amplitude.

2.2 Field corrections due to the axial inhomogeneity

The electromagnetic fields in an axially-symmetric, non-magnetic, isotropic, dielectric waveguide are governed by Maxwell's equations (1.1)–(1.4) in cylindrical polar coordinates (r, θ, z). For a material whose properties depend on the radial coordinate r and *slowly* on the distance z along the fibre, the electric displacement is given, assuming Kerr-law nonlinearity (see Section 1.4), by

$$\mathbf{D} = \left(\varepsilon\left(r, \gamma z\right) + N\left(r, \gamma z\right)|\mathbf{E}|^2\right)\mathbf{E}. \tag{2.1}$$

Here, the permittivity ε and Kerr coefficient N are functions of radius and the distance along the fibre, and γ is a small parameter which characterises the fibre inhomogeneities and represents the reciprocal of a typical length over which the slow variations occur. The fields which occur in the guided modes decay rapidly in the cladding, so the equations are analysed in the region $(0 \leq r < \infty)$ and a good approximation to the boundary conditions is $\mathbf{E}, \mathbf{H}, \mathbf{D} \to 0$ as $r \to \infty$. The fields are also required to be finite at $r = 0$.

To solve the set of nonlinear differential equations (1.1)–(1.4), a second small parameter ν is introduced. This is an amplitude parameter which characterises the signal strength and is chosen so that effects due to cubic nonlinearity are comparable in magnitude with the modulation effects. The fields were expressed as leading order approximations and correction terms, as

$$\begin{aligned}
\mathbf{E} &= \nu\mathbf{E}^{(1)} + O(\nu^2), \\
\mathbf{H} &= \nu\mathbf{H}^{(1)} + O(\nu^2), \\
\mathbf{D} &= \nu\mathbf{D}^{(1)} + O(\nu^2),
\end{aligned} \tag{2.2}$$

where $\mathbf{D}^{(1)} = \varepsilon\mathbf{E}^{(1)}$. The nonlinear term $|\mathbf{E}|^2\mathbf{E}$ will be $O(\nu^3)$ and so will not enter the analysis at this stage. The relationship between the two small parameters γ and ν is assumed to be $\gamma/\nu = O(1)$ or $\gamma/\nu = o(1)$. Later in this section it will be shown that there are two possible order of magnitude comparisons, $\gamma = O(\nu^2)$ and $\gamma = O(\nu)$, which give rise to different equations for the pulse evolution.

If the expressions for the fields (2.2) are substituted into equations (1.1)–(1.4), the fields $\mathbf{E}^{(1)}$ and $\mathbf{H}^{(1)}$ will satisfy at least to $O(1)$ the linearized equations

$$\nabla \wedge \mathbf{E}^{(1)} = -\mu_0 \frac{\partial \mathbf{H}^{(1)}}{\partial t}, \tag{2.3}$$

$$\nabla \wedge \mathbf{H}^{(1)} = \varepsilon \frac{\partial \mathbf{E}^{(1)}}{\partial t}, \tag{2.4}$$

$$\varepsilon \nabla \cdot \mathbf{E}^{(1)} = 0, \tag{2.5}$$

$$\mu_0 \nabla \cdot \mathbf{H}^{(1)} = 0, \tag{2.6}$$

with the fields $\mathbf{E}^{(1)}, \mathbf{H}^{(1)}$ vanishing as $r \to \infty$ and finite at $r = 0$.

The linearized Maxwell's equations have travelling wave solutions, which for a cylindrical waveguide will propagate in the axial direction, and must be single-valued functions of position and hence must be 2π-periodic in θ. Using separation of variables we seek solutions in the form of circularly polarised modes

$$\begin{aligned} \mathbf{E}^{(1)}(r,\theta,z,t) &= \mathbf{E}^{\pm}(r,\theta)e^{\mathrm{i}(\pm l\theta + kz - \omega t)} + c.c., \\ \mathbf{H}^{(1)}(r,\theta,z,t) &= \mathbf{H}^{\pm}(r,\theta)e^{\mathrm{i}(\pm l\theta + kz - \omega t)} + c.c., \end{aligned} \tag{2.7}$$

where l is the azimuthal mode number ($l = 0, 1, 2, \ldots$), the $+$ and $-$ represent the left- and right- handed circularly polarised modes, *c.c.* denotes the complex conjugate and ψ is a phase variable having $-\partial\psi/\partial t = \omega$ the radian frequency and $\partial\psi/\partial z = k(\omega, \gamma z)$ the local wavenumber.

The modal fields $\mathbf{E}^{\pm}$ and $\mathbf{H}^{\pm}$ are resolved along the basis vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ of cylindrical polar coordinates and represented as

$$\begin{aligned} \mathbf{E}^{\pm} &= \mathrm{i}\widetilde{E}_1\mathbf{e}_r \pm \widetilde{E}_2\mathbf{e}_\theta + \widetilde{E}_3\mathbf{e}_z, \\ \mathbf{H}^{\pm} &= \pm\widetilde{H}_1\mathbf{e}_r + \mathrm{i}\widetilde{H}_2\mathbf{e}_\theta \pm \mathrm{i}\widetilde{H}_3\mathbf{e}_z, \end{aligned} \tag{2.8}$$

with components $\widetilde{E}_i, \widetilde{H}_i$ being real functions independent of θ and satisfying the system of equations

$$
\begin{aligned}
l\tilde{E}_3 - kr\tilde{E}_2 - \omega\mu_0 r\tilde{H}_1 &= 0, & l\tilde{H}_3 - kr\tilde{H}_2 + \omega\varepsilon r\tilde{E}_1 &= 0, \\
\frac{\partial \tilde{E}_3}{\partial r} + k\tilde{E}_1 - \omega\mu_0\tilde{H}_2 &= 0, & \frac{\partial \tilde{H}_3}{\partial r} - k\tilde{H}_1 - \omega\varepsilon\tilde{E}_2 &= 0, \qquad (2.9) \\
\frac{\partial}{\partial r}(r\tilde{E}_2) + l\tilde{E}_1 + \omega\mu_0 r\tilde{H}_3 &= 0, & \frac{\partial}{\partial r}(r\tilde{H}_2) - l\tilde{H}_1 + \omega\varepsilon r\tilde{E}_3 &= 0,
\end{aligned}
$$

arising from equations (2.3) and (2.4). Additionally , they satisfy the equations

$$
\frac{\partial}{\partial r}(\varepsilon r\tilde{E}_1) + \varepsilon l\tilde{E}_2 + \varepsilon rk\tilde{E}_3 = 0\,, \qquad \frac{\partial}{\partial r}(r\tilde{H}_1) - l\tilde{H}_2 - rk\tilde{H}_3 = 0\,, \qquad (2.10)
$$

which result from equations (2.5) and (2.6) correct to $O(1)$ and which are linear combinations of the six equations (2.9). Consequently equations (2.10) may be omitted. Since no derivatives with respect to γz occur, equations (2.9) may be solved by treating γz as a parameter and, to leading order, the fields $\mathbf{E}^{\pm}$ and $\mathbf{H}^{\pm}$ are governed by the same equations as for an equivalent axially-symmetric and axially-uniform waveguide with permittivity $\varepsilon(r)$. Thus, equations (2.9) are treated as an eigenvalue problem which is to be solved under the conditions that $\tilde{E}_i$, $\tilde{H}_i$ are finite at $r = 0$ and decay to zero at $r = \infty$. Hence, for specified ω, $\varepsilon(r, \gamma z)$ and integer mode number l, the allowable values of k emerge as the eigenvalues. For a fibre which has a step-index profile at each γz, equations (2.9) can be solved in terms of Bessel functions (Marcuse 1974). However, for general $\varepsilon(r, \gamma z)$ approximate or numerical methods have to be used to find solutions. The resulting relation between the wavenumber k and the frequency ω for each mode number l such that the system of equations (2.9) has non-trivial solutions is called the *dispersion relation*, which is discussed in Appendix A. For weakly-guiding fibres, only the ± 1 modes propagate ($l = 1$) and, since ε depends on γz, the field components $\tilde{E}_i$, $\tilde{H}_i$ and the local wavenumber k for each choice of ω and l will depend on γz and the corresponding dispersion relation can be written as

$$
k = k(\omega, \gamma z),
$$

and the phase ψ as

$$\psi = \gamma^{-1} \int k(\omega, \gamma z) d(\gamma z) - \omega t.$$

Since a single-mode fibre allows two orthogonally polarised modes to propagate, a solution to equations (2.3)–(2.6) can be written as

$$\begin{aligned} \mathbf{E}^{(1)} &= A^{+}\mathbf{E}^{+}e^{\mathrm{i}(\theta+\psi)} + A^{-}\mathbf{E}^{-}e^{\mathrm{i}(-\theta+\psi)} + c.c., \\ \mathbf{H}^{(1)} &= A^{+}\mathbf{H}^{+}e^{\mathrm{i}(\theta+\psi)} + A^{-}\mathbf{H}^{-}e^{\mathrm{i}(-\theta+\psi)} + c.c., \end{aligned} \tag{2.11}$$

where $A^{\pm}$ are independent complex amplitudes.

To obtain approximations at higher orders of ν, a multiple-scales method (see Appendix B) is applied to equations (1.1) and (1.2) by introducing two scaled variables

$$\chi = \nu\left[\gamma^{-1}\int s_g d(\gamma z) - t\right], \qquad Z = \gamma z\,,$$

where $s_g \equiv \partial k/\partial\omega$ is the group slowness, and by allowing any fluctuations in the amplitudes $A^{\pm}$ to depend on the two slow scales χ and Z so that $A^{\pm} = A^{\pm}(\chi, Z)$. The fields are treated as functions of the variables r, θ, ψ, χ, Z and are 2π–periodic in θ and ψ. The z and t derivatives are replaced by

$$\begin{aligned} \frac{\partial}{\partial z} &= k\frac{\partial}{\partial\psi} + \nu s_g\frac{\partial}{\partial\chi} + \gamma\frac{\partial}{\partial Z}\,, \\ \frac{\partial}{\partial t} &= -\omega\frac{\partial}{\partial\psi} - \nu\frac{\partial}{\partial\chi}\,, \end{aligned} \tag{2.12}$$

and the fields are written as leading order terms and a correction of the order ν^2,

$$\begin{aligned} \mathbf{E} &= \nu\mathbf{E}^{(1)} + \nu^2\widehat{\mathbf{E}}\,, \\ \mathbf{H} &= \nu\mathbf{H}^{(1)} + \nu^2\widehat{\mathbf{H}}\,, \\ \mathbf{D} &= \nu\mathbf{D}^{(1)} + \nu^2\widehat{\mathbf{D}}\,, \end{aligned} \tag{2.13}$$

where $\widehat{\mathbf{D}} = \varepsilon\widehat{\mathbf{E}} + O(\nu)$. By substituting (2.12) and (2.13) into equations (1.1) and (1.2) the correction fields, $\widehat{\mathbf{E}}$ and $\widehat{\mathbf{H}}$, are found to be governed by

$$\nabla' \wedge \widehat{\mathbf{E}} - \omega\mu_0 \frac{\partial \widehat{\mathbf{H}}}{\partial \psi} = \Big[A_X^+ (s_g \mathbf{E}^+ \wedge \mathbf{e}_z \ + \mu_0 \mathbf{H}^+) + \frac{\gamma}{\nu}(A^+ \mathbf{E}_Z^+ + A_Z^+ \mathbf{E}^+) \wedge \mathbf{e}_z \Big] e^{\mathrm{i}(\theta+\psi)}$$

$$+ \Big[A_X^- (s_g \mathbf{E}^- \wedge \mathbf{e}_z \ + \mu_0 \mathbf{H}^-) + \frac{\gamma}{\nu}(A^- \mathbf{E}_Z^- + A_Z^- \mathbf{E}^-) \wedge \mathbf{e}_z \Big] e^{\mathrm{i}(-\theta+\psi)}$$

$$+ c.c. + o(1) , \qquad (2.14)$$

$$\nabla' \wedge \widehat{\mathbf{H}} + \omega\varepsilon \frac{\partial \widehat{\mathbf{E}}}{\partial \psi} = \Big[A_X^+ (s_g \mathbf{H}^+ \wedge \mathbf{e}_z \ - \varepsilon \mathbf{E}^+) + \frac{\gamma}{\nu}(A^+ \mathbf{H}_Z^+ + A_Z^+ \mathbf{H}^+) \wedge \mathbf{e}_z \Big] e^{\mathrm{i}(\theta+\psi)}$$

$$+ \Big[A_X^- (s_g \mathbf{H}^- \wedge \mathbf{e}_z \ - \varepsilon \mathbf{E}^-) + \frac{\gamma}{\nu}(A^- \mathbf{H}_Z^- + A_Z^- \mathbf{H}^-) \wedge \mathbf{e}_z \Big] e^{\mathrm{i}(-\theta+\psi)}$$

$$+ c.c. + o(1) , \qquad (2.15)$$

where

$$\nabla' \equiv \mathbf{e}_r \frac{\partial}{\partial r} + \frac{1}{r}\mathbf{e}_\theta \frac{\partial}{\partial \theta} + k(\omega, Z)\mathbf{e}_z \frac{\partial}{\partial \psi} . \qquad (2.16)$$

The terms in (2.14) and (2.15) which involve $A_X^\pm$ are due to amplitude modulation of the signal envelope and occur in the case of an axially homogeneous fibre (Parker and Newboult, 1989). The remaining terms are due to the fibre nonuniformities.

Equations (2.14) and (2.15) can be written in the form

$$\nabla' \wedge \widehat{\mathbf{E}} - \omega\mu_0 \frac{\partial \widehat{\mathbf{H}}}{\partial \psi} \equiv \mathbf{G}, \qquad (2.17)$$

$$\nabla' \wedge \widehat{\mathbf{H}} + \omega\varepsilon \frac{\partial \widehat{\mathbf{E}}}{\partial \psi} \equiv \mathbf{F}, \qquad (2.18)$$

where $\mathbf{G}$ and $\mathbf{F}$ are $2\pi-$periodic in θ and ψ, bounded at $r = 0$ and decay exponentially as $r \to \infty$. If $\widehat{\mathbf{E}}$ and $\widehat{\mathbf{H}}$ are also to obey these conditions, then they must satisfy the compatibility condition

$$\int_R (\mathbf{P}.\mathbf{F} - \mathbf{Q}.\mathbf{G}) \, \mathrm{d}V = 0 , \qquad (2.19)$$

where $R \equiv [0, \infty) \times [0, 2\pi] \times [0, 2\pi]$, $\mathrm{d}V = r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\psi$ and $\mathbf{P}$, $\mathbf{Q}$ are the most general solution of the linear equations

$$\nabla' \wedge \mathbf{P} = \omega\mu_0 \frac{\partial \mathbf{Q}}{\partial \psi}, \qquad \nabla' \wedge \mathbf{Q} = -\omega\varepsilon \frac{\partial \mathbf{P}}{\partial \psi}, \qquad (2.20)$$

which are $2\pi-$periodic in θ and ψ with similar conditions on $r=0$ and as $r\to\infty$. These equations govern periodic, linearized fields travelling at the phase speed ω/k in an equivalent uniform fibre. The general solution of these equations is a linear combination of fields having wavenumber an integer multiple of k. Since the fibre under consideration is assumed to be weakly-guiding only the modes $l=1$ propagate. So assuming that no integer harmonics of the $l=1$ mode have the phase speed ω/k, the most general solution for $\mathbf{P}$ and $\mathbf{Q}$ is

$$\begin{aligned} \mathbf{P} &= \alpha_1\mathbf{E}^+e^{\mathrm{i}(\theta+\psi)}+\alpha_2\mathbf{E}^-e^{\mathrm{i}(-\theta+\psi)}+c.c.\,, \\ \mathbf{Q} &= \alpha_1\mathbf{H}^+e^{\mathrm{i}(\theta+\psi)}+\alpha_2\mathbf{H}^-e^{\mathrm{i}(-\theta+\psi)}+c.c.\,, \end{aligned} \tag{2.21}$$

where α_1 and α_2 are arbitrary complex constants. Substituting for $\mathbf{F}$, $\mathbf{G}$, $\mathbf{P}$ and $\mathbf{Q}$ into the compatibility condition (2.19) and recalling that $\widehat{\mathbf{E}}$ and $\widehat{\mathbf{H}}$ are $2\pi-$periodic in θ and ψ, we find that the only terms which give a non-zero contribution to the integral are those in which the exponential factor is $e^{\mathrm{i}0}$. Since the equation must hold for arbitrary α_1 and α_2, the coefficients of α_1 and α_2 (or α_1^* and α_2^*) must vanish separately. The equation obtained from the coefficient of α_1^* is

$$\frac{\gamma}{\nu}\int_R\left[(A^+\mathbf{E}^{+*}\wedge\mathbf{H}_Z^+ + A_Z^+\mathbf{E}^{+*}\wedge\mathbf{H}^+).\mathbf{e}_z+(A^+\mathbf{E}_Z^+\wedge\mathbf{H}^{+*}+A_Z^+\mathbf{E}^+\wedge\mathbf{H}^{+*}).\mathbf{e}_z\right]\,\mathrm{d}V$$
$$= -A_X^+\int_R\left[s_g(\mathbf{E}^{+*}\wedge\mathbf{H}^+ + \mathbf{E}^+\wedge\mathbf{H}^{+*}).\mathbf{e}_z-(\varepsilon\mathbf{E}^+.\mathbf{E}^{+*}+\mu_0\mathbf{H}^+.\mathbf{H}^{+*})\right]\,\mathrm{d}V+o(1). \tag{2.22}$$

From the expression for the group slowness s_g (A.15), equation (2.22) is automatically satisfied to leading order if $\gamma/\nu=o(1)$. If $\gamma/\nu=O(1)$, the γ/ν terms must be retained in (2.14) and (2.15) and equation (2.22) becomes

$$A^+\int_R(\mathbf{E}^{+*}\wedge\mathbf{H}_Z^+ + \mathbf{E}_Z^+\wedge\mathbf{H}^{+*}).\mathbf{e}_z\,\mathrm{d}V+A_Z^+\int_R(\mathbf{E}^{+*}\wedge\mathbf{H}^+ + \mathbf{E}^+\wedge\mathbf{H}^{+*}).\mathbf{e}_z\,\mathrm{d}V=o(1). \tag{2.23}$$

This has the form

$$\frac{1}{2}A^+\frac{\partial P}{\partial Z}+\frac{\partial A^+}{\partial Z}P=o(1), \tag{2.24}$$

where

$$P \equiv \int_R \left(\mathbf{E}^{\pm *}\wedge \mathbf{H}^{\pm} + \mathbf{E}^{\pm}\wedge \mathbf{H}^{\pm *}\right).\mathbf{e}_z \, \mathrm{d}V \tag{2.25}$$

is proportional to the power in either of the modes described by $\mathbf{E}^{+}e^{\mathrm{i}(\theta+\psi)} + c.c.$ or $\mathbf{E}^{-}e^{\mathrm{i}(-\theta+\psi)} + c.c.$. If the solutions to equations (2.9) are normalized so that $P = 4\pi^2$ for all Z and ω, then equation (2.24) reduces to

$$\frac{\partial A^{+}}{\partial Z} = o(1), \tag{2.26}$$

indicating that the leading order approximation for A^{+} may be taken as $A^{+}(\chi)$, independently of Z. Performing a similar analysis for the coefficients of α_2^* gives a leading order expression $A^{-}(\chi)$ for A^{-}. Equations (2.14) and (2.15) can then be replaced by the equations

$$\begin{aligned} \nabla' \wedge \widehat{\mathbf{E}} - \omega\mu_0 \frac{\partial \widehat{\mathbf{H}}}{\partial \psi} &= \left[A^{+}_{\chi}(s_g \mathbf{E}^{+}\wedge \mathbf{e}_z \ + \mu_0 \mathbf{H}^{+}) + \frac{\gamma}{\nu} A^{+}\mathbf{E}^{+}_Z \wedge \mathbf{e}_z\right] e^{\mathrm{i}(\theta+\psi)} \\ &\quad + \left[A^{-}_{\chi}(s_g \mathbf{E}^{-}\wedge \mathbf{e}_z \ + \mu_0 \mathbf{H}^{-}) + \frac{\gamma}{\nu} A^{-}\mathbf{E}^{-}_Z \wedge \mathbf{e}_z\right] e^{\mathrm{i}(-\theta+\psi)} \\ &\quad + c.c. + o(1)\,, \end{aligned} \tag{2.27}$$

$$\begin{aligned} \nabla' \wedge \widehat{\mathbf{H}} + \omega\varepsilon \frac{\partial \widehat{\mathbf{E}}}{\partial \psi} &= \left[A^{+}_{\chi}(s_g \mathbf{H}^{+}\wedge \mathbf{e}_z \ - \varepsilon \mathbf{E}^{+}) + \frac{\gamma}{\nu} A^{+}\mathbf{H}^{+}_Z \wedge \mathbf{e}_z\right] e^{\mathrm{i}(\theta+\psi)} \\ &\quad + \left[A^{-}_{\chi}(s_g \mathbf{H}^{-}\wedge \mathbf{e}_z \ - \varepsilon \mathbf{E}^{-}) + \frac{\gamma}{\nu} A^{-}\mathbf{H}^{-}_Z \wedge \mathbf{e}_z\right] e^{\mathrm{i}(-\theta+\psi)} \\ &\quad + c.c. + o(1)\,. \end{aligned} \tag{2.28}$$

The terms in $A^{\pm}_{\chi}$ also occur for an axially homogeneous fibre and give rise to fields proportional to $\mathbf{E}^{\pm}_{\omega}$ and $\mathbf{H}^{\pm}_{\omega}$ (Parker and Newboult 1989). This can be seen by substituting the fields $\mathbf{E}^{\pm}e^{\mathrm{i}(\pm\theta+\psi)}$ and $\mathbf{H}^{\pm}e^{\mathrm{i}(\pm\theta+\psi)}$ into the linearized form of equations (1.1) and (1.2), with ∇ replaced by ∇' and $\partial/\partial t$ replaced by $-\omega\partial/\partial\psi$, which gives

$$\begin{aligned} \nabla' \wedge \left(\mathbf{E}^{\pm}e^{\mathrm{i}(\pm\theta+\psi)}\right) &= \omega\mu_0 \frac{\partial}{\partial \psi}\left(\mathbf{H}^{\pm}e^{\mathrm{i}(\pm\theta+\psi)}\right), \\ \nabla' \wedge \left(\mathbf{H}^{\pm}e^{\mathrm{i}(\pm\theta+\psi)}\right) &= -\omega\varepsilon \frac{\partial}{\partial \psi}\left(\mathbf{E}^{\pm}e^{\mathrm{i}(\pm\theta+\psi)}\right). \end{aligned}$$

If these equations are differentiated with respect to ω, we obtain the equations

$$\begin{aligned}
\nabla' \wedge \left(\mathbf{E}^{\pm}_{\omega} e^{\mathrm{i}(\pm\theta+\psi)}\right) - \mu_0 \omega \frac{\partial}{\partial \psi}\left(\mathbf{H}^{\pm}_{\omega} e^{\mathrm{i}(\pm\theta+\psi)}\right) &= \mathrm{i}\left(s_g \mathbf{E}^{\pm} \wedge \mathbf{e}_z + \mu_0 \mathbf{H}^{\pm}\right) e^{\mathrm{i}(\pm\theta+\psi)}, \\
\nabla' \wedge \left(\mathbf{H}^{\pm}_{\omega} e^{\mathrm{i}(\pm\theta+\psi)}\right) + \varepsilon \omega \frac{\partial}{\partial \psi}\left(\mathbf{E}^{\pm}_{\omega} e^{\mathrm{i}(\pm\theta+\psi)}\right) &= \mathrm{i}\left(s_g \mathbf{H}^{\pm} \wedge \mathbf{e}_z - \varepsilon \mathbf{E}^{\pm}\right) e^{\mathrm{i}(\pm\theta+\psi)},
\end{aligned}$$

and by comparing these equations to (2.27) and (2.28) we observe that a solution for the fields $\widehat{\mathbf{E}}$ and $\widehat{\mathbf{H}}$ can be written as

$$\begin{aligned}
\widehat{\mathbf{E}} &= \left(-\mathrm{i}A^{+}_{X}\mathbf{E}^{+}_{\omega} + \frac{\gamma}{\nu}A^{+}\widehat{\mathbf{E}}^{+}\right) e^{\mathrm{i}(\theta+\psi)} + \left(-\mathrm{i}A^{-}_{X}\mathbf{E}^{-}_{\omega} + \frac{\gamma}{\nu}A^{-}\widehat{\mathbf{E}}^{-}\right) e^{\mathrm{i}(-\theta+\psi)} + c.c., \\
\widehat{\mathbf{H}} &= \left(-\mathrm{i}A^{+}_{X}\mathbf{H}^{+}_{\omega} + \frac{\gamma}{\nu}A^{+}\widehat{\mathbf{H}}^{+}\right) e^{\mathrm{i}(\theta+\psi)} + \left(-\mathrm{i}A^{-}_{X}\mathbf{H}^{-}_{\omega} + \frac{\gamma}{\nu}A^{-}\widehat{\mathbf{H}}^{-}\right) e^{\mathrm{i}(-\theta+\psi)} + c.c.,
\end{aligned} \tag{2.29}$$

where $\widehat{\mathbf{E}}^{\pm}$, $\widehat{\mathbf{H}}^{\pm}$ satisfy the equations

$$\begin{aligned}
\nabla' \wedge \left(\widehat{\mathbf{E}}^{\pm} e^{\mathrm{i}(\pm\theta+\psi)}\right) - \omega\mu_0 \frac{\partial}{\partial \psi}\left(\widehat{\mathbf{H}}^{\pm} e^{\mathrm{i}(\pm\theta+\psi)}\right) &= \mathbf{E}^{\pm}_{Z} \wedge \mathbf{e}_z\, e^{\mathrm{i}(\pm\theta+\psi)}, \\
\nabla' \wedge \left(\widehat{\mathbf{H}}^{\pm} e^{\mathrm{i}(\pm\theta+\psi)}\right) + \omega\varepsilon \frac{\partial}{\partial \psi}\left(\widehat{\mathbf{E}}^{\pm} e^{\mathrm{i}(\pm\theta+\psi)}\right) &= \mathbf{H}^{\pm}_{Z} \wedge \mathbf{e}_z\, e^{\mathrm{i}(\pm\theta+\psi)},
\end{aligned} \tag{2.30}$$

with $\widehat{\mathbf{E}}^{\pm}, \widehat{\mathbf{H}}^{\pm} \to 0$ as $r \to \infty$ and bounded at $r = 0$. As in the case for the modal fields $\mathbf{E}^{\pm}$, $\mathbf{H}^{\pm}$, the fields $\widehat{\mathbf{E}}^{\pm}$, $\widehat{\mathbf{H}}^{\pm}$ may be represented as

$$\begin{aligned}
\mathrm{i}\widehat{\mathbf{E}}^{\pm} &= \mathrm{i}\widehat{E}_1 \mathbf{e}_r \pm \widehat{E}_2 \mathbf{e}_\theta + \widehat{E}_3 \mathbf{e}_z\,, \\
\mathrm{i}\widehat{\mathbf{H}}^{\pm} &= \pm\widehat{H}_1 \mathbf{e}_r + \mathrm{i}\widehat{H}_2 \mathbf{e}_\theta \pm \mathrm{i}\widehat{H}_3 \mathbf{e}_z\,,
\end{aligned}$$

where $\widehat{E}_i = \widehat{E}_i(r;\omega, Z)$ and $\widehat{H}_i = \widehat{H}_i(r;\omega, Z)$ are real functions which satisfy the inhomogeneous ordinary differential equations

$$\begin{aligned}
&\widehat{E}_3 - kr\widehat{E}_2 - \omega\mu_0 r\widehat{H}_1 = r\frac{\partial \widetilde{E}_2}{\partial Z}, && \widehat{H}_3 - kr\widehat{H}_2 + \omega\varepsilon r\widehat{E}_1 = r\frac{\partial \widetilde{H}_2}{\partial Z}, \\
&\frac{\partial \widehat{E}_3}{\partial r} + k\widehat{E}_1 - \omega\mu_0 \widehat{H}_2 = -\frac{\partial \widetilde{E}_1}{\partial Z}, && \frac{\partial \widehat{H}_3}{\partial r} - k\widehat{H}_1 - \omega\varepsilon \widehat{E}_2 = -\frac{\partial \widetilde{H}_1}{\partial Z}, \\
&\frac{\partial}{\partial r}(r\widehat{E}_2) + \widehat{E}_1 + \omega\mu_0 r\widehat{H}_3 = 0, && \frac{\partial}{\partial r}(r\widehat{H}_2) - \widehat{H}_1 + \omega\varepsilon r\widehat{E}_3 = 0.
\end{aligned}$$

In principle, $\widehat{E}_i$ and $\widehat{H}_i$ may be constructed from $\widetilde{E}_i$ and $\widetilde{H}_i$ using variation of parameters, or direct numerical integration. However, we shall not require explicit formulae in the subsequent analysis.

2.3 Evolution equations

As observed in Section 2.2, there are two distinct cases for the order of magnitude comparison between the parameters γ and ν, either $\gamma/\nu = o(1)$ or $\gamma/\nu = O(1)$. The first case describes a fibre whose longitudinal inhomogeneities arise on the same length scale as nonlinear effects. In the second case we consider a fibre whose longitudinal inhomogeneities occur on a scale comparable with a pulse width.

2.3.1 Case 1: $\gamma = O(\nu^2)$

For the case $\gamma = O(\nu^2)$, it is possible to write, without loss of generality, $\nu = \gamma^{1/2}$ and to seek solutions for the fields **E**, **H**, **D** of the form

$$\begin{aligned} \mathbf{E} &= \gamma^{\frac{1}{2}}\mathbf{E}^{(1)} + \gamma\mathbf{E}^{(2)} + \gamma^{\frac{3}{2}}\bar{\mathbf{E}}\,, \\ \mathbf{H} &= \gamma^{\frac{1}{2}}\mathbf{H}^{(1)} + \gamma\mathbf{H}^{(2)} + \gamma^{\frac{3}{2}}\bar{\mathbf{H}}\,, \qquad (2.31) \\ \mathbf{D} &= \gamma^{\frac{1}{2}}\varepsilon\mathbf{E}^{(1)} + \gamma\varepsilon\mathbf{E}^{(2)} + \gamma^{\frac{3}{2}}\bar{\mathbf{D}}\,, \end{aligned}$$

where $\bar{\mathbf{E}}$, $\bar{\mathbf{H}}$ and $\bar{\mathbf{D}}$ are correction terms to the series solution. To $O(1)$, the correction to the electric displacement $\bar{\mathbf{D}}$ is

$$\bar{\mathbf{D}} = \varepsilon(r,Z)\bar{\mathbf{E}} + N(r,Z)\left|\mathbf{E}^{(1)}\right|^2\mathbf{E}^{(1)}. \qquad (2.32)$$

For this case the terms in $\hat{\mathbf{E}}^{\pm}$ and $\hat{\mathbf{H}}^{\pm}$ are omitted from (2.29), and the fields $\mathbf{E}^{(2)}$ and $\mathbf{H}^{(2)}$ are given by

$$\begin{aligned} \mathbf{E}^{(2)} &= -\mathrm{i}A_{\chi}^{+}\mathbf{E}_{\omega}^{+}e^{\mathrm{i}(\theta+\psi)} - \mathrm{i}A_{\chi}^{-}\mathbf{E}_{\omega}^{-}e^{\mathrm{i}(-\theta+\psi)} + c.c.\,, \\ \mathbf{H}^{(2)} &= -\mathrm{i}A_{\chi}^{+}\mathbf{H}_{\omega}^{+}e^{\mathrm{i}(\theta+\psi)} - \mathrm{i}A_{\chi}^{-}\mathbf{H}_{\omega}^{-}e^{\mathrm{i}(-\theta+\psi)} + c.c.\,. \end{aligned}$$

Since $\mathbf{E}^{(2)}$ and $\mathbf{H}^{(2)}$ do not involve the terms in (2.14) and (2.15) which are $O(\gamma/\nu) = O(\gamma^{1/2})$, the compatibility condition analogous to (2.19) is automatically satisfied and therefore the reasoning which led to equation (2.26) is inappropriate. The amplitudes A^+ and A^- should, in this case, be allowed to depend on both χ and Z. The derivative expansions (2.12) become

$$\begin{aligned}\frac{\partial}{\partial z} &= k\frac{\partial}{\partial \psi} + \gamma^{\frac{1}{2}} s_g \frac{\partial}{\partial \chi} + \gamma \frac{\partial}{\partial Z}, \\ \frac{\partial}{\partial t} &= -\omega \frac{\partial}{\partial \psi} - \gamma^{\frac{1}{2}} \frac{\partial}{\partial \chi},\end{aligned}$$

and the equations governing the correction fields $\bar{\mathbf{E}}$ and $\bar{\mathbf{H}}$ are obtained by substituting for $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, $\partial/\partial z$ and $\partial/\partial t$ into equations (1.1) and (1.2) and are found to be

$$\begin{aligned}\nabla' \wedge \bar{\mathbf{E}} - \omega\mu_0 \frac{\partial \bar{\mathbf{H}}}{\partial \psi} &= \left[A^+ \mathbf{E}_Z^+ \wedge \mathbf{e}_z + A_Z^+ \mathbf{E}^+ \wedge \mathbf{e}_z - \mathrm{i} A_{\chi\chi}^+ (s_g \mathbf{E}_\omega^+ \wedge \mathbf{e}_z + \mu_0 \mathbf{H}_\omega^+)\right] e^{\mathrm{i}(\theta+\psi)} \\ &\quad + \left[A^- \mathbf{E}_Z^- \wedge \mathbf{e}_z + A_Z^- \mathbf{E}^- \wedge \mathbf{e}_z - \mathrm{i} A_{\chi\chi}^- (s_g \mathbf{E}_\omega^- \wedge \mathbf{e}_z + \mu_0 \mathbf{H}_\omega^-)\right] e^{\mathrm{i}(-\theta+\psi)} \\ &\quad + c.c. + O(\gamma^{\frac{1}{2}}), \\ &\equiv \widehat{\mathbf{G}} \end{aligned} \tag{2.33}$$

$$\begin{aligned}\nabla' \wedge \bar{\mathbf{H}} + \omega\varepsilon \frac{\partial \bar{\mathbf{E}}}{\partial \psi} &= \left[A^+ \mathbf{H}_Z^+ \wedge \mathbf{e}_z + A_Z^+ \mathbf{H}^+ \wedge \mathbf{e}_z - \mathrm{i} A_{\chi\chi}^+ (s_g \mathbf{H}_\omega^+ \wedge \mathbf{e}_z - \varepsilon \mathbf{E}_\omega^+)\right] e^{\mathrm{i}(\theta+\psi)} \\ &\quad + \left[A^- \mathbf{H}_Z^- \wedge \mathbf{e}_z + A_Z^- \mathbf{H}^- \wedge \mathbf{e}_z - \mathrm{i} A_{\chi\chi}^- (s_g \mathbf{H}_\omega^- \wedge \mathbf{e}_z - \varepsilon \mathbf{E}_\omega^-)\right] e^{\mathrm{i}(-\theta+\psi)} \\ &\quad + c.c. - \omega \frac{\partial}{\partial \psi}(N|\mathbf{E}^{(1)}|^2 \mathbf{E}^{(1)}) + O(\gamma^{\frac{1}{2}}) \\ &\equiv \widehat{\mathbf{F}}. \end{aligned} \tag{2.34}$$

Explicit solutions to these equations cannot be easily found, but $\widehat{\mathbf{F}}$ and $\widehat{\mathbf{G}}$ are $2\pi-$periodic in θ and ψ, bounded at $r = 0$ and decay exponentially as $r \to \infty$. This situation is analogous to that of equations (2.17) and (2.18), so allowing the compatibility condition (2.19) to be used with (2.21) to obtain equations which govern the evolution of the amplitudes A^+ and A^-. Applying the same reasoning as before, we consider only those terms which give a non-zero contribution to the equation and compare coefficients of α_1^* to obtain the equation

$$\begin{aligned} A^+ \int_R (\mathbf{E}^{+*} \wedge \mathbf{H}_Z^+ + \mathbf{E}_Z^+ \wedge \mathbf{H}^{+*}).\mathbf{e}_z \ \mathrm{d}V + A_Z^+ \int_R (\mathbf{E}^{+*} \wedge \mathbf{H}^+ + \mathbf{E}^+ \wedge \mathbf{H}^{+*}).\mathbf{e}_z \ \mathrm{d}V \\ = \mathrm{i} A_{\chi\chi} \int_R \left[s_g (\mathbf{E}^{+*} \wedge \mathbf{H}_\omega^+ + \mathbf{E}_\omega^+ \wedge \mathbf{H}^{+*}).\mathbf{e}_z \ - (\varepsilon \mathbf{E}^{+*}.\mathbf{E}_\omega^+ + \mu_0 \mathbf{H}^{+*}.\mathbf{H}_\omega^+)\right] \mathrm{d}V \\ + \omega \int_R \mathbf{E}^{+*}. \frac{\partial}{\partial \psi}(N|\mathbf{E}^{(1)}|^2 \mathbf{E}^{(1)}) e^{-\mathrm{i}(\theta+\psi)} \, \mathrm{d}V + O(\gamma^{\frac{1}{2}}). \end{aligned} \tag{2.35}$$

The nonlinear term can be simplified (Parker and Newboult, 1989) as

$$\int_R \mathbf{E}^{+*}.\frac{\partial}{\partial\psi}\left(N|\mathbf{E}^{(1)}|^2\mathbf{E}^{(1)}\right)e^{-\mathrm{i}(\theta+\psi)}\,\mathrm{d}V =$$

$$\mathrm{i}4\pi^2|A^+|^2A^+\int_0^\infty\left[|\mathbf{E}^+.\mathbf{E}^+|^2+2|\mathbf{E}^+|^4\right]Nr\,\mathrm{d}r$$

$$+\mathrm{i}8\pi^2|A^-|^2A^+\int_0^\infty\left[|\mathbf{E}^+.\mathbf{E}^-|^2+|\mathbf{E}^+.\mathbf{E}^{-*}|^2+|\mathbf{E}^+|^2|\mathbf{E}^-|^2\right]Nr\,\mathrm{d}r. \quad (2.36)$$

The first term in (2.35) vanishes when the normalisation $P = 4\pi^2$ is used, where P is given in (2.25). Equation (2.35), to leading order approximation, can then be written in the form

$$\mathrm{i}A_Z^+ = gA_{XX}^+ + (f_2|A^+|^2 + f_3|A^-|^2)A^+, \quad (2.37)$$

where the coefficients

$$f_2 = -\omega\int_0^\infty\left[|\mathbf{E}^+.\mathbf{E}^+|^2+2|\mathbf{E}^+|^4\right]Nr\,\mathrm{d}r \equiv f_2(\omega, Z)\,,$$

$$f_3 = -2\omega\int_0^\infty\left[|\mathbf{E}^+.\mathbf{E}^-|^2+|\mathbf{E}^+.\mathbf{E}^{-*}|^2+|\mathbf{E}^+|^2|\mathbf{E}^-|^2\right]Nr\,\mathrm{d}r \equiv f_3(\omega, Z)\,, \quad (2.38)$$

$$g = -\int_0^\infty\left[s_g(\mathbf{E}^{+*}\wedge\mathbf{H}_\omega^+ + \mathbf{E}_\omega^+\wedge\mathbf{H}^{+*}).\mathbf{e}_z \;-(\varepsilon\mathbf{E}^{+*}.\mathbf{E}_\omega^+ + \mu_0\mathbf{H}^{+*}.\mathbf{H}_\omega^+)\right]r\,\mathrm{d}r$$

$$= \frac{1}{2}\frac{\partial s_g}{\partial\omega} \equiv g(\omega, Z)\,,$$

are related to the field distributions $\mathbf{E}^\pm$, $\mathbf{H}^\pm$ of circularly polarised modes as for an axially uniform fibre (Parker and Newboult, 1989). The dependence on the inhomogeneity is included through the $Z-$dependence of ε, $\mathbf{E}^\pm$, $\mathbf{H}^\pm$ and k.

By equating to zero the coefficients of α_2^*, an equation similar in form to (2.37) is obtained and the pair of equations can be written as

$$\mathrm{i}A_Z^\pm = gA_{XX}^\pm + (f_2|A^\pm|^2 + f_3|A^\mp|^2)A^\pm. \quad (2.39)$$

The coefficients of α_1 and α_2 give the complex conjugate of this pair of equations, so imposing no further constraints. Equations (2.39) are the same as the coupled cubic Schrödinger equations which are obtained for axially uniform fibres except that the coefficients g, f_2, f_3 given by (2.38) depend on the axial coordinate $\gamma z = \nu^2 z \equiv Z$, with the length scale of the axial inhomogeneities comparable with those over which the nonlinearity acts.

2.3.2 Case 2: $\gamma = O(\nu)$

For this case we take $\gamma = O(\nu)$ and without loss of generality, write $\nu = \gamma$. We then seek solutions for the fields in the form

$$\begin{aligned}
\mathbf{E} &= \gamma \mathbf{E}^{(1)} + \gamma^2 \mathbf{E}^{(2)} + \gamma^3 \bar{\mathbf{E}}, \\
\mathbf{H} &= \gamma \mathbf{H}^{(1)} + \gamma^2 \mathbf{H}^{(2)} + \gamma^3 \bar{\mathbf{H}}, \\
\mathbf{D} &= \gamma \varepsilon \mathbf{E}^{(1)} + \gamma^2 \varepsilon \mathbf{E}^{(2)} + \gamma^3 \bar{\mathbf{D}},
\end{aligned}$$

where $\bar{\mathbf{E}}$, $\bar{\mathbf{H}}$, $\bar{\mathbf{D}}$ are the correction terms, and $\mathbf{E}^{(2)}$ and $\mathbf{H}^{(2)}$ are as given in (2.29)

$$\begin{aligned}
\mathbf{E}^{(2)} &= (-\mathrm{i}A_\chi^+ \mathbf{E}_\omega^+ + A^+ \widehat{\mathbf{E}}^+) e^{\mathrm{i}(\theta+\psi)} + (-\mathrm{i}A_\chi^- \mathbf{E}_\omega^- + A^- \widehat{\mathbf{E}}^-) e^{\mathrm{i}(-\theta+\psi)}, \\
\mathbf{H}^{(2)} &= (-\mathrm{i}A_\chi^+ \mathbf{H}_\omega^+ + A^+ \widehat{\mathbf{H}}^+) e^{\mathrm{i}(\theta+\psi)} + (-\mathrm{i}A_\chi^- \mathbf{H}_\omega^- + A^- \widehat{\mathbf{H}}^-) e^{\mathrm{i}(-\theta+\psi)},
\end{aligned}$$

with $\mathbf{E}^\pm$ and $\mathbf{H}^\pm$ normalized so that P, given by (2.25), satisfies $P = 4\pi^2$. The inhomogeneities occur on the scale $Z = \gamma z$, so that the fields $\mathbf{E}^{(1)}$, $\mathbf{H}^{(1)}$ and $\mathbf{E}^{(2)}$, $\mathbf{H}^{(2)}$ depend also on Z. However, we have shown in equation (2.26) the relation $\partial A^\pm / \partial Z = O(\gamma)$. This suggests introducing a further scaled variable

$$\widehat{Z} = \gamma^2 z$$

and allowing for $O(\gamma)$ fluctuations in $A^\pm$ on the Z scale by writing

$$A^\pm = B^\pm(\chi, \widehat{Z}) + \gamma a^\pm(\chi, Z, \widehat{Z}). \tag{2.40}$$

The nonlinear effects occur on the length scale associated with $\widehat{Z}$, and the correction term for the electric displacement is given, correct to $O(1)$ by

$$\bar{\mathbf{D}} = \varepsilon \bar{\mathbf{E}} + N(r, \widehat{Z}) |\mathbf{E}^{(1)}|^2 \mathbf{E}^{(1)}.$$

With the introduction of the new scaled variable the derivative expansions become

$$\begin{aligned}
\frac{\partial}{\partial z} &= k \frac{\partial}{\partial \psi} + \gamma s_g \frac{\partial}{\partial \chi} + \gamma \frac{\partial}{\partial Z} + \gamma^2 \frac{\partial}{\partial \widehat{Z}}, \\
\frac{\partial}{\partial t} &= -\omega \frac{\partial}{\partial \psi} - \gamma \frac{\partial}{\partial \chi}.
\end{aligned}$$

The equations relating the correction fields $\bar{\mathbf{E}}$ and $\bar{\mathbf{H}}$ are again found by substituting the field and derivative expansions into (1.1) and (1.2), which to leading order are

$$\begin{aligned}\nabla'\wedge\,\bar{\mathbf{E}}-\omega\mu_0\frac{\partial\bar{\mathbf{H}}}{\partial\psi}=&\Big\{a^+\mathbf{E}^+_Z\wedge\mathbf{e}_z+a^+_Z\mathbf{E}^+\wedge\mathbf{e}_z+a^+_X(s_g\mathbf{E}^+\wedge\mathbf{e}_z+\mu_0\mathbf{H}^+)\\&+B^+_Z\mathbf{E}^+\wedge\mathbf{e}_z+B^+\widehat{\mathbf{E}}^+_Z\wedge\mathbf{e}_z+B^+_X(s_g\widehat{\mathbf{E}}^+\wedge\mathbf{e}_z+\mu_0\widehat{\mathbf{H}}^+-\mathrm{i}\mathbf{E}^+_{\omega Z}\wedge\mathbf{e}_z)\\&-\mathrm{i}B^+_{XX}(s_g\mathbf{E}^+_\omega\wedge\mathbf{e}_z\ +\mu_0\mathbf{H}^+_\omega)\Big\}\,e^{\mathrm{i}(\theta+\psi)}\\&+\Big\{a^-\mathbf{E}^-_Z\wedge\mathbf{e}_z+a^-_Z\mathbf{E}^-\wedge\mathbf{e}_z+a^-_X(s_g\mathbf{E}^-\wedge\mathbf{e}_z+\mu_0\mathbf{H}^-)\\&+B^-_Z\mathbf{E}^-\wedge\mathbf{e}_z+B^-\widehat{\mathbf{E}}^-_Z\wedge\mathbf{e}_z+B^-_X(s_g\widehat{\mathbf{E}}^-\wedge\mathbf{e}_z+\mu_0\widehat{\mathbf{H}}^--\mathrm{i}\mathbf{E}^-_{\omega Z}\wedge\mathbf{e}_z)\\&-\mathrm{i}B^-_{XX}(s_g\mathbf{E}^-_\omega\wedge\mathbf{e}_z\ +\mu_0\mathbf{H}^-_\omega)\Big\}\,e^{\mathrm{i}(-\theta+\psi)}+c.c.\\&\equiv\bar{\mathbf{G}}\,,\end{aligned}\tag{2.41}$$

$$\begin{aligned}\nabla'\wedge\,\bar{\mathbf{H}}+\omega\varepsilon\frac{\partial\bar{\mathbf{E}}}{\partial\psi}=&\Big\{a^+\mathbf{H}^+_Z\wedge\mathbf{e}_z+a^+_Z\mathbf{H}^+\wedge\mathbf{e}_z+a^+_X(s_g\mathbf{H}^+\wedge\mathbf{e}_z-\varepsilon\mathbf{E}^+)\\&+B^+_Z\mathbf{H}^+\wedge\mathbf{e}_z+B^+\widehat{\mathbf{H}}^+_Z\wedge\mathbf{e}_z+B^+_X(s_g\widehat{\mathbf{H}}^+\wedge\mathbf{e}_z-\varepsilon\widehat{\mathbf{E}}^+-\mathrm{i}\mathbf{H}^+_{\omega Z}\wedge\mathbf{e}_z)\\&-\mathrm{i}B^+_{XX}(s_g\mathbf{H}^+_\omega\wedge\mathbf{e}_z\ -\varepsilon\mathbf{E}^+_\omega)\Big\}\,e^{\mathrm{i}(\theta+\psi)}\\&+\Big\{a^-\mathbf{H}^-_Z\wedge\mathbf{e}_z+a^-_Z\mathbf{H}^-\wedge\mathbf{e}_z+a^-_X(s_g\mathbf{H}^-\wedge\mathbf{e}_z-\varepsilon\mathbf{E}^-)\\&+B^-_Z\mathbf{H}^-\wedge\mathbf{e}_z+B^-\widehat{\mathbf{H}}^-_Z\wedge\mathbf{e}_z+B^-_X(s_g\widehat{\mathbf{H}}^-\wedge\mathbf{e}_z-\varepsilon\widehat{\mathbf{E}}^--\mathrm{i}\mathbf{H}^-_{\omega Z}\wedge\mathbf{e}_z)\\&-\mathrm{i}B^-_{XX}(s_g\mathbf{H}^-_\omega\wedge\mathbf{e}_z\ -\varepsilon\mathbf{E}^-_\omega)\Big\}\,e^{\mathrm{i}(-\theta+\psi)}+c.c.\\&-\omega\frac{\partial}{\partial\psi}(N|\mathbf{E}^{(1)}|^2\mathbf{E}^{(1)})\\&\equiv\bar{\mathbf{F}}\,.\end{aligned}\tag{2.42}$$

As for case 1, explicit solutions cannot easily be found, but these are not necessary in order to deduce the evolution equations for $B^{\pm}$. Using the fact that $\bar{\mathbf{F}}$ and $\bar{\mathbf{G}}$ are 2π-periodic in θ and ψ, decay exponentially as $r\to\infty$ and are bounded at $r=0$, the compatibility condition (2.19) can again be used to find the condition for the existence of 2π-periodic fields $\bar{\mathbf{E}}$ and $\bar{\mathbf{H}}$.

Considering the terms multiplying α_1^* we obtain the equation

$$
\begin{aligned}
&a^+ \int_R \left(\mathbf{E}^{+*}\wedge \mathbf{H}_Z^+ + \mathbf{E}_Z^+\wedge \mathbf{H}^{+*}\right).\mathbf{e}_z \mathrm{d}V + a_Z^+ \int_R \left(\mathbf{E}^{+*}\wedge \mathbf{H}^+ + \mathbf{E}^+\wedge \mathbf{H}^{+*}\right).\mathbf{e}_z \mathrm{d}V \\
&+a_X^+ \int_R \left[s_g \left(\mathbf{E}^{+*}\wedge \mathbf{H}^+ + \mathbf{E}^+\wedge \mathbf{H}^{+*}\right).\mathbf{e}_z - \left(\varepsilon \mathbf{E}^{+*}.\mathbf{E}^+ + \mu_0 \mathbf{H}^{+*}.\mathbf{H}^+\right)\right] \mathrm{d}V \\
&+B_{\mathcal{Z}}^+ \int_R \left(\mathbf{E}^{+*}\wedge \mathbf{H}^+ + \mathbf{E}^+\wedge \mathbf{H}^{+*}\right).\mathbf{e}_z \mathrm{d}V + B^+ \int_R \left(\mathbf{E}^{+*}\wedge \widehat{\mathbf{H}}_Z^+ + \widehat{\mathbf{E}}_Z^+\wedge \mathbf{H}^{+*}\right).\mathbf{e}_z \mathrm{d}V \\
&+B_X^+ \int_R \Big[s_g \left(\mathbf{E}^{+*}\wedge \widehat{\mathbf{H}}^+ + \widehat{\mathbf{E}}^+\wedge \mathbf{H}^{+*}\right).\mathbf{e}_z - \left(\varepsilon \mathbf{E}^{+*}.\widehat{\mathbf{E}}^+ + \mu_0 \mathbf{H}^{+*}.\widehat{\mathbf{H}}^+\right) \\
&\qquad\qquad -\mathrm{i}\left(\mathbf{E}^{+*}\wedge \widehat{\mathbf{H}}_{\omega Z}^+ + \widehat{\mathbf{E}}_{\omega Z}^+\wedge \mathbf{H}^{+*}\right).\mathbf{e}_z\Big] \mathrm{d}V \\
&= \mathrm{i}B_{XX}^+ \int_R \left[s_g \left(\mathbf{E}^{+*}\wedge \mathbf{H}_\omega^+ + \mathbf{E}_\omega^+ + \wedge \mathbf{H}^{+*}\right).\mathbf{e}_z - \left(\varepsilon \mathbf{E}^{+*}.\mathbf{E}_\omega^+ + \mu_0 \mathbf{H}^{+*}.\mathbf{H}_\omega^+\right)\right] \mathrm{d}V \\
&\quad +\mathrm{i}\omega |B^+|^2 B^+ \int_R \left(|\mathbf{E}^+.\mathbf{E}^+|^2 + 2|\mathbf{E}^+|^2\right) N \,\mathrm{d}V \\
&\quad +2\mathrm{i}\omega |B^-|^2 B^+ \int_R \left(|\mathbf{E}^+.\mathbf{E}^-|^2 + |\mathbf{E}^+.\mathbf{E}^{-*}|^2 + |\mathbf{E}^+|^2|\mathbf{E}^-|^2\right) N \,\mathrm{d}V . \qquad (2.43)
\end{aligned}
$$

The expression for the group slowness s_g (A.15), shows that the third term in (2.43) vanishes and the normalisation $P = 4\pi^2$ means that the first term vanishes, so equation (2.43) can be written in the form

$$
a_Z^+ + B_{\mathcal{Z}}^+ + \mathrm{i}f_4 B^+ + \mathrm{i}f_5 B_X^+ + \mathrm{i}g B_{XX}^+ + \mathrm{i}(f_2|B^+|^2 + f_3|B^-|^2)B^+ = 0, \qquad (2.44)
$$

where f_2, f_3 and g are as given in Case 1 and

$$
f_4 = -\frac{i}{4\pi^2} \int_R \left(\mathbf{E}^{+*}\wedge \widehat{\mathbf{H}}_Z^+ + \widehat{\mathbf{E}}_Z^+\wedge \mathbf{H}^{+*}\right).\mathbf{e}_z \mathrm{d}V,
$$

$$
\begin{aligned}
f_5 = \frac{i}{4\pi^2} \int_R \Big[&s_g(\mathbf{E}^{+*}\wedge \widehat{\mathbf{H}}^+ + \widehat{\mathbf{E}}^+\wedge \mathbf{H}^{+*}).\mathbf{e}_z - (\varepsilon \mathbf{E}^{+*}.\widehat{\mathbf{E}}^+ + \mu_0 \mathbf{H}^{+*}.\widehat{\mathbf{H}}^+) \\
&-i(\mathbf{E}^{+*}\wedge \mathbf{H}_{\omega Z}^+ + \mathbf{E}_{\omega Z}^+\wedge \mathbf{H}^{+*}).\mathbf{e}_z\Big] \mathrm{d}V.
\end{aligned}
$$

By differentiating the governing equations for $\widehat{\mathbf{E}}^\pm$ and $\widehat{\mathbf{H}}^\pm$ (2.30) with respect to ω, we obtain the equations

$$
\nabla' \wedge \left(\widehat{\mathbf{E}}_\omega^\pm e^{\mathrm{i}(\pm\theta+\psi)}\right) - \omega\mu_0 \frac{\partial}{\partial\psi}\left(\widehat{\mathbf{H}}_\omega^\pm e^{\mathrm{i}(\pm\theta+\psi)}\right) = \left(\mathrm{i}k\omega \widehat{\mathbf{E}}_Z^\pm \wedge \mathbf{e}_z + \mathrm{i}\mu_0 \widehat{\mathbf{H}}^\pm + \mathbf{E}_{Z\omega}^\pm \wedge \mathbf{e}_z\right) e^{\mathrm{i}(\pm\theta+\psi)},
$$

$$
\nabla' \wedge \left(\widehat{\mathbf{H}}_\omega^\pm e^{\mathrm{i}(\pm\theta+\psi)}\right) + \omega\varepsilon \frac{\partial}{\partial\psi}\left(\widehat{\mathbf{E}}_\omega^\pm e^{\mathrm{i}(\pm\theta+\psi)}\right) = \left(\mathrm{i}k\omega \widehat{\mathbf{H}}_Z^\pm \wedge \mathbf{e}_z - \mathrm{i}\varepsilon \widehat{\mathbf{E}}^\pm + \mathbf{H}_{Z\omega}^\pm \wedge \mathbf{e}_z\right) e^{\mathrm{i}(\pm\theta+\psi)},
$$

with $\widehat{\mathbf{E}}^\pm, \widehat{\mathbf{H}}^\pm \to 0$ as $r \to \infty$. Since this situation is analogous to that of equations (2.17) and (2.18), the compatibility condition (2.19) with (2.21) can be used to show that the coefficient f_5 vanishes. In equation (2.44), the coefficients of f_2, f_3, f_4 and g depend on the intermediate axial scale Z but the amplitudes $B^\pm$ do not. Consequently a^+ must be chosen to absorb all the fluctuations on the Z scale without accumulating $O(\gamma^{-1})$ deviations as Z varies over ranges of $O(\gamma^{-1})$. This is achieved by making the mean values of both sides of (2.44) vanish over large ranges of Z. The simplest statement of this requirement occurs when the fibre inhomogeneities are periodic in Z, of some period Z_p. Then equation (2.44) can be written in the form

$$iB^+_{\hat{Z}} = F_4 B^+ + G B^+_{XX} + (F_2|B^+|^2 + F_3|B^-|^2)B^+ , \tag{2.45}$$

where the real coefficients F_2, F_3, F_4, G are averages of f_2, f_3, f_4, g over each period of length Z_p and are given by

$$\begin{aligned} F_j &= \frac{1}{Z_p}\int_{Z_0}^{Z_0+Z_p} f_i \, dZ \qquad j = 2, 3, 4; \\ G &= \frac{1}{Z_p}\int_{Z_0}^{Z_0+Z_p} g \, dZ. \end{aligned}$$

As in the previous case a similar equation is obtained from the terms with coefficient α_2^*, so that the pair of constant coefficient equations

$$\mathrm{i}B^\pm_{\hat{Z}} = F_4 B^\pm + G B^\pm_{XX} + (F_2|B^\pm|^2 + F_3|B^\mp|^2)B^\pm \tag{2.46}$$

is obtained. The term with coefficient F_4 may be absorbed by the substitution

$$B^\pm = C^\pm e^{-\mathrm{i}F_4\hat{Z}} ,$$

which shows that $\gamma^2 F_4$ corresponds to an averaged perturbation in the group slowness s_g. Using this substitution and rescaling the independent variables by defining

$$\tau = F_2\hat{Z}, \qquad x = \sqrt{\frac{F_2}{G}}\chi,$$

shows that equations (2.46) can be written as

$$\mathrm{i}C^\pm_\tau = C^\pm_{xx} + (|C^\pm|^2 + h|C^\mp|^2)C^\pm, \tag{2.47}$$

where $h = F_3/F_2$. These are identical in form to the equations for a fibre without longitudinal inhomogeneities, so demonstrating that when longitudinal variations are periodic and take place on a scale intermediate between the wavelength and the nonlinear evolution length, evolution is the same as in an "equivalent" longitudinally homogeneous fibre. The relevant coefficients are averaged over a period of the longitudinal variation. This implies that for relatively weak signals, with nonlinear evolution length much longer than the scale of the longitudinal inhomogeneities, non-distorting pulses should be able to propagate, provided that the launching conditions are those appropriate to the "equivalent" homogeneous fibre. Equations (2.46) govern the dominant part of the solution for the amplitude modulations $A^{\pm}$, the small periodic correction terms $a^{\pm}$ will introduce small ripples to $A^{\pm}$ over long distances of fibre. This is similar to the concept of a 'guiding centre soliton' introduced by Hasegawa and Kodama (1990) for long distance transmission systems involving many periodically spaced amplifiers designed to compensate for small losses in the intervening cable. The system (2.47) is not completely integrable (Zakharov and Schulman, 1982) but it possess a large family of non-distorting pulse-like solutions and other families of generalised similarity solutions (Parker, 1988).

2.4 Fibres allowing exact soliton solutions

Grimshaw (1979) discussed possibilities for determining closed form solutions of the variable-coefficient cubic Schrödinger equation

$$iu_t + gu_{xx} + f|u|^2 u = 0, \tag{2.48}$$

with real-valued coefficients $g = g(t)$, $f = f(t)$. He showed that equation (2.48) can be reduced to the constant coefficient nonlinear Schrödinger equation

$$ip_\sigma + p_{\xi\xi} + \mathrm{sgn}(gf)|p|^2 p = 0\,,$$

by the transformation

$$u = \left|\frac{f}{g}\right|^2 p e^{-\frac{i}{4}M\left|\frac{f}{g}\right|x^2},$$

$$\xi = \left|\frac{f}{g}\right| x, \qquad \sigma = \frac{1}{M}\left|\frac{f}{g}\right| - \frac{1}{M}\left|\frac{f}{g}\right|_{t=0},$$

provided that g and f are related by the constraint

$$\left(\frac{g}{f}\right)_t = -gM, \qquad M = \text{constant}.$$

Consequently, this reduction to the constant coefficient equation is possible for arbitrary smooth, one-signed $g(t)$ provided that $f(t)$ has the form

$$f(t) = -g(t)\left\{M\int^t g(s)\,\mathrm{d}s\right\}^{-1},$$

or, equivalently for arbitrary one-signed $f(t)$ with

$$g(t) = \pm f(t)\exp\left\{-M\int^t f(s)\,\mathrm{d}s\right\}. \tag{2.49}$$

To investigate whether transformations exist which allow the coupled cubic Schrödinger equations (2.39) with variable coefficients to be reduced similarly to constant coefficient equations, we investigate substitutions of the form

$$\begin{aligned} A^{\pm}(\chi, Z) &= m^{\pm}C^{\pm}(\xi,\sigma)e^{in^{\pm}}, \\ \xi = F(\chi, Z), &\qquad \sigma = G(\chi, Z), \end{aligned} \tag{2.50}$$

where F, G, $m^{\pm}$ and $n^{\pm}$ are real functions of χ and Z. These functions are chosen such that $C^{\pm}$ satisfy the equations

$$\mathrm{i}C^{\pm}_{\sigma} = C^{\pm}_{\xi\xi} + \left(|C^{\pm}|^2 + h|C^{\mp}|^2\right)C^{\pm}, \tag{2.51}$$

where h is a constant.

Substitution of equations (2.50) into (2.39) yields equations (2.51) only if the following conditions are satisfied:

$$G_\chi = 0\,, \tag{2.52}$$

$$m^\pm G_Z = m^\pm g F_\chi^2 = f_2(m^\pm)^3, \tag{2.53}$$

$$F_{\chi\chi} = 0\,, \tag{2.54}$$

$$f_3(m^\mp)^2 = h f_2(m^\pm)^2, \tag{2.55}$$

$$m_Z^\pm = g m^\pm n_{\chi\chi}^\pm\,, \tag{2.56}$$

$$n_Z^\pm = g(n_\chi^\pm)^2, \tag{2.57}$$

$$F_Z = 2gF_\chi n_\chi^\pm\,. \tag{2.58}$$

Equations (2.52) and (2.53) require G, m^+ and m^- to be independent of χ and equation (2.54) requires F to be linear in χ. The cases in which the system (2.52)-(2.58) is compatible may be reduced, without loss of generality, to

$$n^\pm = \chi^2 n(Z)\,, \qquad m^\pm = \alpha n^{\frac{1}{2}}, \qquad n(Z) = \frac{\alpha^2}{\beta^2}\frac{f_2(Z)}{g(Z)}\,,$$

with

$$\xi = \frac{\alpha^2 f_2 \chi}{\beta g}\,, \qquad \sigma = \frac{\alpha^2 f_2}{4g}\,,$$

provided that

$$n'(Z) = 4g(Z)n^2(Z)\,.$$

Here α and β are constants. Setting $\beta = \alpha^2 = 4M^{-1}$, gives the transformations

$$A^\pm = \left(\frac{f_2}{g}\right)^{\frac{1}{2}} C^\pm(\xi,\sigma)\exp\left\{\frac{\mathrm{i}}{4}\frac{f_2}{g}M\chi^2\right\}, \tag{2.59}$$

with

$$\xi = \frac{f_2}{g}\chi, \qquad \sigma = \frac{f_2}{4g}\,,$$

which reduce the special case

$$\mathrm{i}A_Z^\pm = g(Z)A_{\chi\chi}^\pm + f_2(Z)\{|A^\pm|^2 + h|A^\mp|^2\}A^\pm \tag{2.60}$$

of equations (2.39) to the form (2.51) whenever $g(Z)$ and $f_2(Z)$ are related by

$$g(Z) = f_2(Z)\exp\left\{-M\int^Z f_2(s)\,\mathrm{d}s\right\}, \tag{2.61}$$

which is analogous to (2.49).

Thus, Grimshaw's reduction extends to the coupled system (2.39) whenever $f_3(Z)/f_2(Z)$ is constant ($= h$) and when $g(Z)$ is related to $f_2(Z)$ in the manner required for the single equation. Consequently, when (2.61) is satisfied, exact solutions for the system (2.60) may be found corresponding to all the similarity solutions catalogued in Parker (1988) and especially to the uniform wavetrains and linearly and circularly polarised solitons. Moreover, the pulse collisions investigated in Parker and Newboult (1989) will correspond to collisions with negligible scattering when (2.61) is satisfied.

The condition (2.61) which relates g and f_2 includes the possibility

$$\frac{g(Z)}{f_2(Z)} = \text{constant} \qquad (M = 0).$$

Presuming that f_2 does not change sign along the fibre, then the argument of the exponential in (2.61) tends to $\pm\infty$ as $Z \to \infty$. If the argument tends to $-\infty$, the fibre becomes effectively dispersionless as $Z \to \infty$, and since $f_2/g \to \infty$ a solution of the constant constant coefficient nonlinear Schrödinger equations will become compressed in width and amplified in height. However if the argument tends to ∞, the fibre becomes infinitely dispersive as $Z \to \infty$, which does not correspond to physical behaviour over the long fibre lengths which are required for optical communications systems. In the following section the effect of sinusoidal fluctuations on a linearly polarised pulse and a more general 'non-distorting' pulse are studied numerically.

2.5 Numerical Results

For an axisymmetric fibre whose longitudinal inhomogeneities act over the same length scale as nonlinear effects, the equations which describe the pulse evolution are a pair of coupled cubic Schrödinger equations with varying coefficients (2.39). To investigate the effect of slow variations in the material properties the following change of variable

$$\tau = \int^{Z} g(Z')dZ' \qquad \text{with} \qquad g > 0$$

was made in equations (2.39), and the resulting equations written in the form

$$\mathrm{i}A^{\pm}_{\tau} = A^{\pm}_{\chi\chi} + (h_1(\tau)|A^{\pm}|^2 + h_2(\tau)|A^{\mp}|^2)A^{\pm}, \tag{2.62}$$

where $h_1(\tau) = f_2/g$ and $h_2(\tau) = f_3/g$.

For circularly polarised solitons ($A^{-} = 0$), equation (2.62) reduces to a single cubic Schrödinger equation which has only one variable coefficient h_1. It is known that when h_1 is constant this equation allows solutions of the form

$$A^{+}(\tau,\chi) = \sqrt{\frac{2}{h_1}}\Gamma e^{-\mathrm{i}\phi}\,\mathrm{sech}\,\Gamma(\chi - 2V\tau), \qquad \text{where} \qquad \phi = V\chi - (V^2 - \Gamma^2)\tau.$$

Here Γ is the pulse amplitude and V is a frequency shift which determines the speed of the pulse envelope. For $h_1'(\tau) \neq 0$, the evolution of a pulse which has this initial condition may be analysed numerically.

It is also seen that for initial conditions which are of the form $A^{-}(0,\chi) = e^{-2\mathrm{i}\alpha}A^{+}(0,\chi)$, there exist solutions $A^{-}(\tau,\chi) = e^{-2\mathrm{i}\alpha}A^{+}(\tau,\chi)$, for which equation (2.62) becomes

$$\mathrm{i}A^{+}_{\tau} = A^{+}_{\chi\chi} + [h_1(\tau) + h_2(\tau)]|A^{+}|^2A^{+}. \tag{2.63}$$

This will allow solutions of a form similar to those of a circularly polarised pulse, but which will have h_1 replaced by $(h_1 + h_2)$. Again solutions can be computed numerically for $h_1' + h_2' \neq 0$.

This shows that both circularly or linearly polarised sech pulses evolve according to a single variable-coefficient nonlinear Schrödinger equation, although the coefficient of the nonlinear term will differ in the two cases.

To illustrate the evolution for a linearly polarised signal, and also for more general non-distorting pulse-like solutions which are discussed later in this section, h_1 was taken to be a constant and h_2 was taken to have a sinusoidal variation about a fixed value h_0,

$$h_2 = h_0 + a \sin b\tau \,.$$

For the numerical results given in this paper, values for these parameters were taken to be $h_0 = 2$, $h_1 = 1$, $a = 0.2$ and $b = 2.75$.

Numerical integration of equations (2.62) was performed using a split-step spectral method, with a damping scheme applied at the edges of the integration region as described by Menyuk (1988). The edge damping is required because the periodic boundary conditions assumed by the Fast Fourier Transform could cause any radiation which has left the computational region to return to the region and cause effects which are due to the numerical scheme rather than the physical system. The values for the step lengths for the numerical discretization were $\Delta\chi = 0.1$ and $\Delta\tau = 5 \times 10^{-3}$ for the results shown in this section.

For a linearly polarised pulse, the initial conditions, at $\tau = 0$, were chosen to agree with

$$\begin{aligned} A^+ &= \sqrt{\frac{2}{1+h_2}}\,\Gamma\,\mathrm{sech}[\Gamma(\chi - \chi_0)]\,, \\ A^- &= \sqrt{\frac{2}{1+h_2}}\,\Gamma\,\mathrm{sech}[\Gamma(\chi - \chi_0)]e^{2i\alpha}, \end{aligned}$$

where α is the polarisation angle and the soliton is centred initially at $\chi = \chi_0$. The parameters were taken to be $\Gamma = 1$, $\alpha = 0$ and $\chi_0 = 25.6$ as reported in Ryder and Parker (1992).

From Figure 2.1, which is a graph of $|A^+|^2$ plotted against χ and τ, it is not possible to detect radiation away from the pulse. However if the maximum values of $|A^+|^2$ are plotted against τ, see Figure 2.2, it can be seen from the decrease in the value of $|A^+|^2$ that there is some radiation of energy away from the pulse. It can also be seen that the peak values of $|A^+|^2$ are no longer constant, as in the case of a constant coefficient nonlinear Schrödinger equation, but fluctuate almost periodically with a period similar to that of the material fluctuations. However, the loss of peak amplitude of the pulse is only $\simeq 3\%$ after 40 cycles of fluctuations

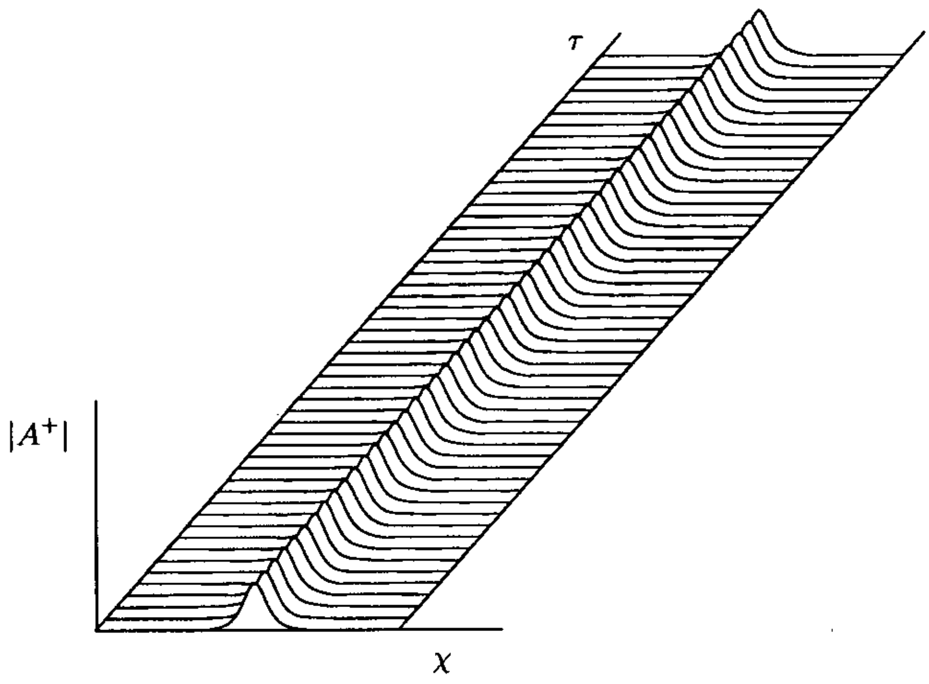

Figure 2.1: Evolution of a linearly polarised pulse governed by equation (2.63).

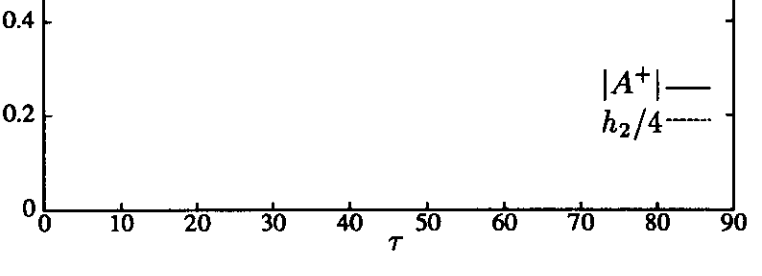

Figure 2.2: Peak values of $|A^+|$ and the material property $h_2(\tau)$ plotted against τ.

of h_2 having 10% variation, either side of its mean value. Figures (2.1) and (2.2) could alternatively show a circularly polarised pulse with h_1 varying.

More generally solutions will have to be computed numerically. To consider pulse propagation of a more general type, observe that the constant coefficient case has solutions of the form

$$\begin{aligned} A^+ &= e^{-i(\beta_1\tau+V\sigma)}F(\sigma)\,, \\ A^- &= e^{-i(\beta_2\tau+V\sigma)}G(\sigma)\,, \end{aligned} \tag{2.64}$$

leading to the ordinary differential equations

$$\begin{aligned} F'' &= (\beta_1 - V^2 - h_1F^2 - h_2G^2)F\,, \\ G'' &= (\beta_2 - V^2 - h_2F^2 - h_1G^2)G\,. \end{aligned} \tag{2.65}$$

Here β_1 and β_2 are real adjustable parameters, while F and G are real functions of $\sigma = \chi - 2V\tau$. To obtain initial conditions at $\tau = 0$ for equations (2.62), in the form of pulses with profiles $F(\sigma)$, $G(\sigma)$ satisfying equations (2.65), it is necessary to choose suitable values for the parameters $\beta_1 - V^2$ and $\beta_2 - V^2$, which are compatible with the conditions

$$F(0) = \cos\alpha, \qquad G(0) = \sin\alpha,$$

$$F'(0) = G'(0),$$

$$F, G, F', G' \to 0 \quad \text{as} \quad \sigma \to \pm\infty,$$

at $\tau = 0$. The frequency shift V was chosen to be zero without loss of generality. To find an approximation for β_1 and β_2 for a given value of α, a perturbation solution to the coupled ordinary differential equations (2.65) was sought (see Appendix D). The approximation obtained was found to give a pulse-like solution to equations (2.65) but was not sufficiently accurate as an initial pulse for equations (2.62). However McCabe (1990) has found values for β_1 and β_2

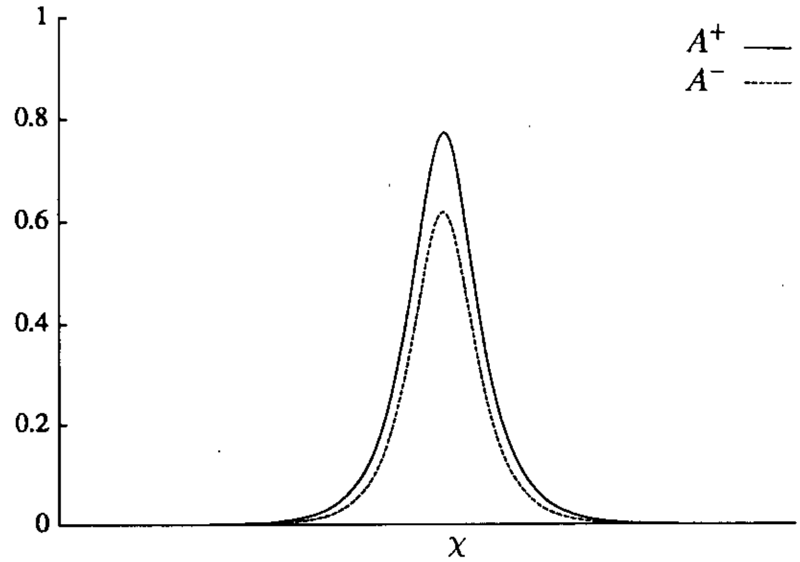

Figure 2.3: Initial pulse.

which give a pulse-like solution to equations (2.65) for some values of α. For the numerical results given in Figure 2.4–Figure 2.6, $\alpha = 0.67627220178$, which gives $\beta_1 = 0.67845594593$ and $\beta_2 = 0.83116876487$. The values of α, β_1, β_2 have to be calculated to this degree of accuracy in order that the expressions for $A^{\pm}$ given in (2.64) can be used as initial conditions for the partial differential equations (2.62). Equations (2.65) were solved using a Runge-Kutta fourth order method and Figure 2.3 is a graph of the initial pulse.

As for the case of a linearly polarised pulse, it is not possible to detect radiation away from the pulse from the graphs of $|A^{\pm}|$ plotted against χ and τ (Figures 2.4). However Figure 2.5, a graph of the maximum values of $|A^{\pm}|$ plotted against τ, shows that there is some radiation of energy away from the pulse and again the peak values of $|A^{\pm}|$ fluctuate almost periodically with period similar to that of the material fluctuations. Figure 2.6–Figure 2.8 show the numerical results for the parameter values $\alpha = 0.51683348354$, $\beta_1 = 0.59608140892$ and $\beta_2 = 0.95840572633$, These results exhibit the same trends as the previous non-distorting pulse.

These numerical experiments suggest that 'non-distorting pulses' are remarkably resilient to small fluctuations in the values of the 'coefficients' f_2/g

and f_3/g, which arise on length scales slow compared to those of the nonlinear evolution. It also suggests that these more general cases are as well behaved as the circularly or linearly polarised solitons.

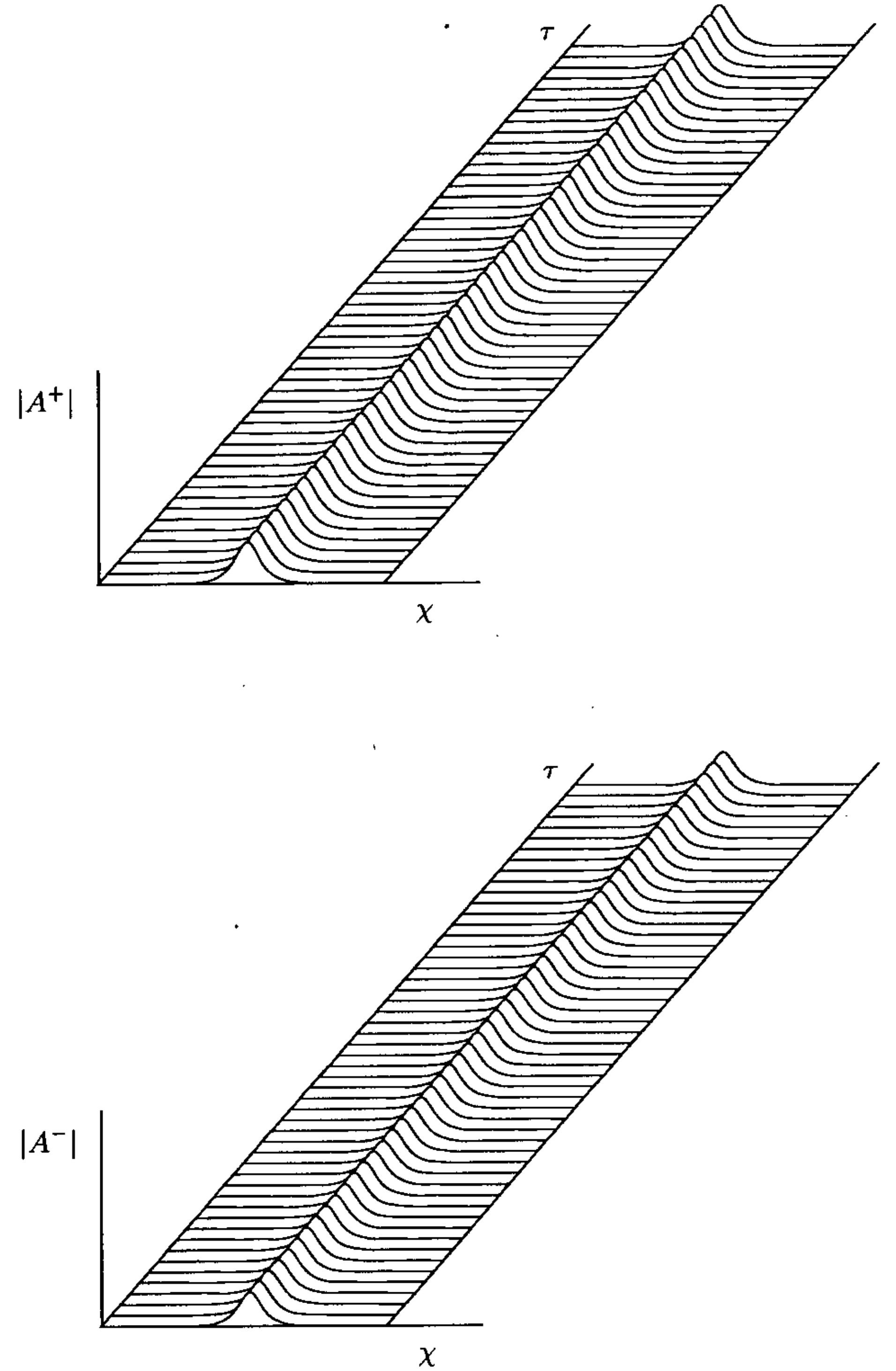

Figure 2.4: Evolution of a 'non-distorting' pulse.

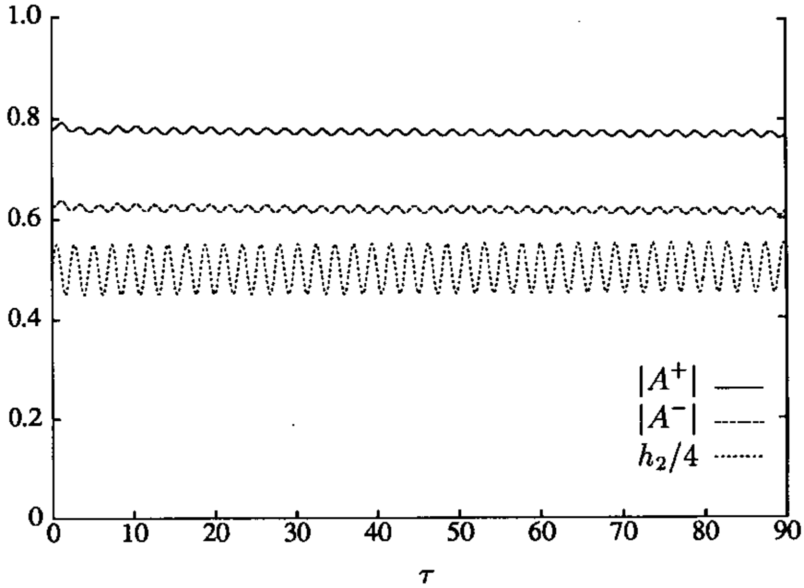

Figure 2.5: Peak values of $|A^+|$ and $|A^-|$ and the material property $h_2(\tau)$ plotted against τ.

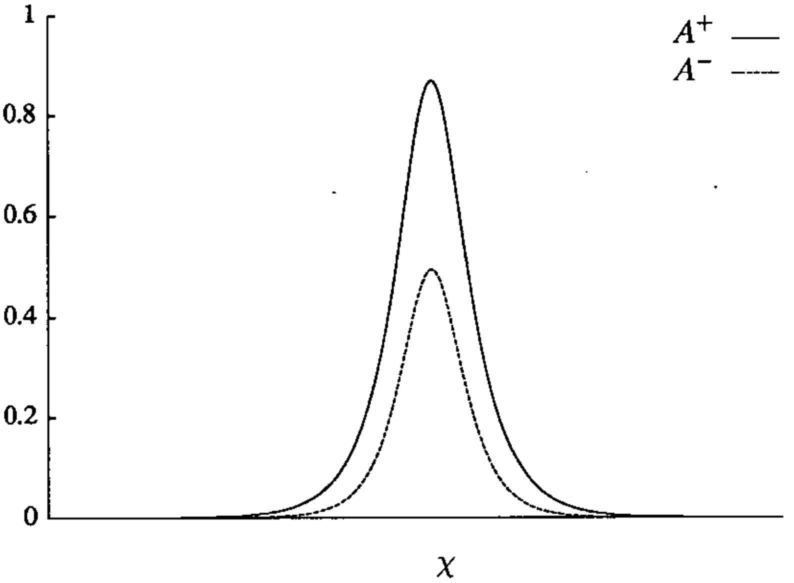

Figure 2.6: Initial pulse.

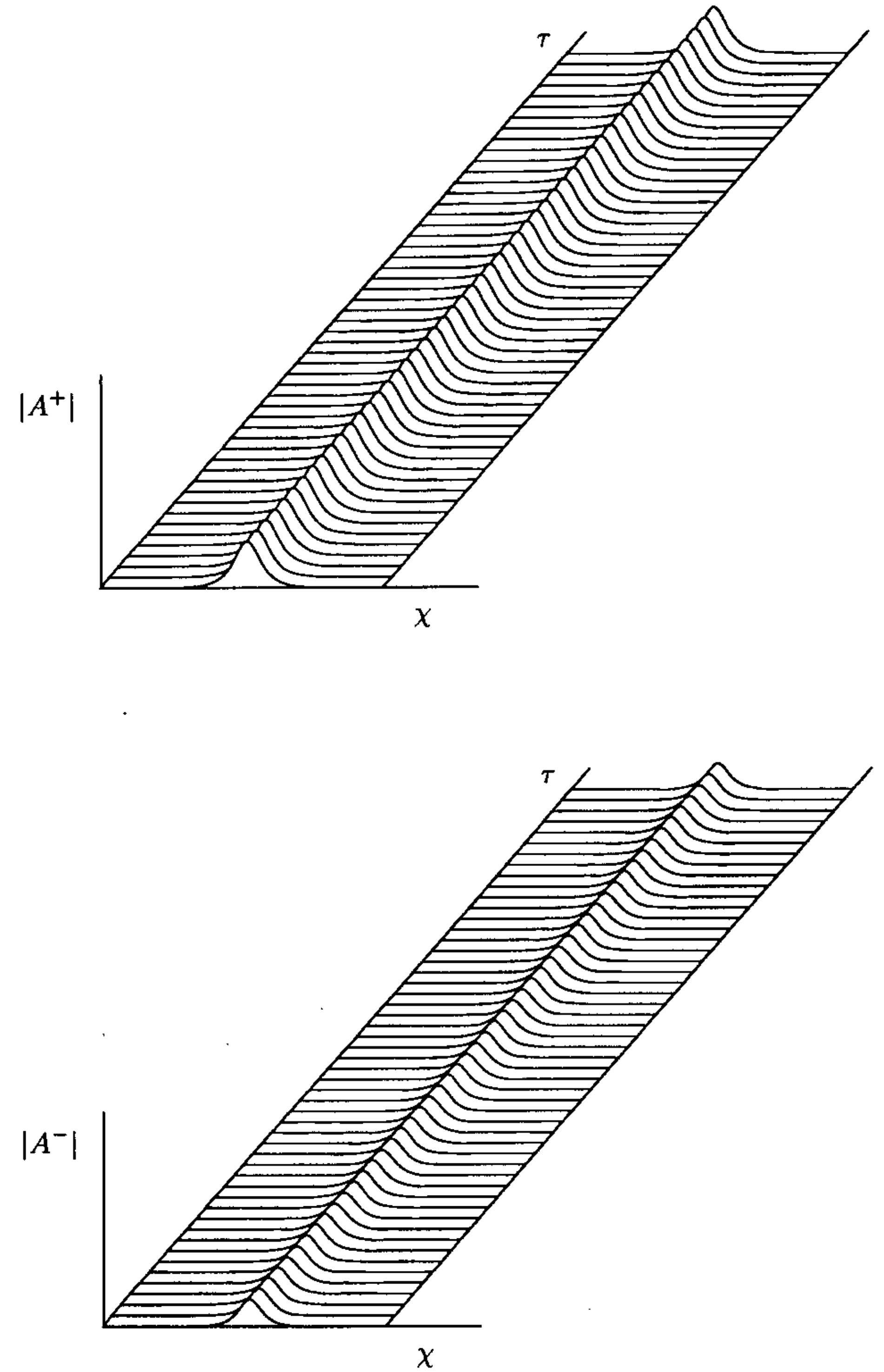

Figure 2.7: Evolution of a 'non-distorting' pulse.

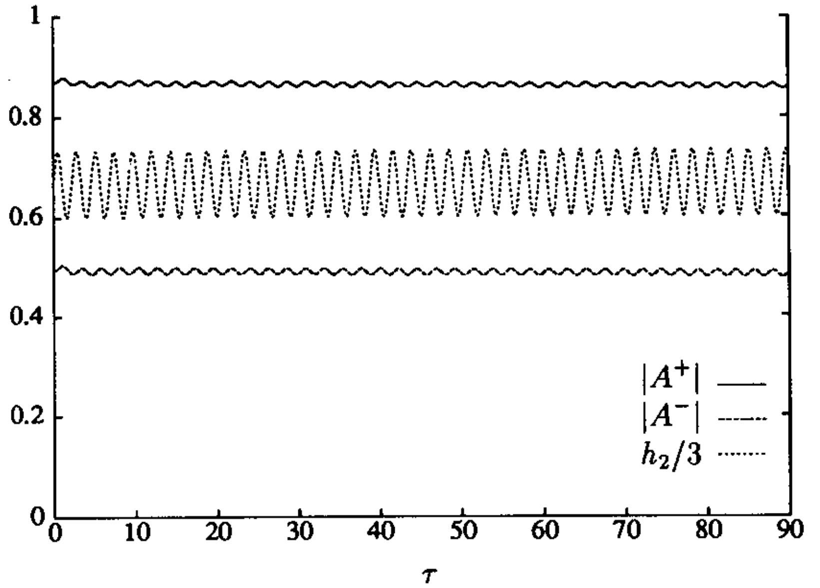

Figure 2.8: Peak values of $|A^+|$ and $|A^-|$ and the material property $h_2(\tau)$ plotted against τ.

Chapter 3

Curved and twisted fibres

3.1 Preamble

Although a light wave is guided by a curved single-mode fibre, some of the power is lost as radiation. A curved optical waveguide deforms the modal fields in such a way that they shift away from the plane of the bend and also become narrower (Gambling et al., 1976). There have been many studies concerned with the calculation of the radiation losses in optical waveguides. Marcuse (1976c) calculated the rate of energy loss for a fibre bent into a circular helix, and showed that the curvature loss formula that he obtained was equivalent to the curvature loss formula for a circularly deformed fibre (Chang and Kuester, 1976) if the radius of curvature of the circle is replaced by the radius of curvature of the helix. In several earlier papers, formulae were derived for the curvature loss of step-index fibres (for example Marcuse, 1976a, Snyder et al., 1975). However, in these derivations the field deformation which is caused by the fibre curvature has not been included. Marcuse (1976b) included this field deformation in his analysis, and found that for modes of low mode number the radiation losses may be much lower than that predicted by the more simple loss formulae, whereas the losses of modes with high mode numbers are increased.

The above derivations depended on knowing exact solutions for the fields, which can only be found for fibres which have a step-index profile and then only for a limited number of curvature geometries. Kath and Kriegsmann (1988) calculated the loss for arbitrary local curvature and torsion where the radiation loss was determined by the local curvature of the bend and not by the macroscopic shape of the bent fibre. They derived these formulae for a weakly-guiding fibre by approximating Maxwell's equations by the scalar wave equation. Loss formulae which are valid for arbitrary geometries and for different ranges of curvature size have been derived using the full set of Maxwell's equations (Hobbs and Kath, 1990).

Single-mode fibres allow two orthogonally polarised modes to propagate, in an ideal fibre these two modes propagate with identical phase velocity. However when a fibre is bent, linear birefringence is induced, while twisting of the fibre induces circular birefringence (Ulrich and Simon, 1979). Bend-induced birefringence is caused by core ellipticity, elastic strain and waveguide geometry, although the birefringence caused by the first two effects is too small to make any significant contribution to the overall bend-induced birefringence (Smith, 1980). However it has been shown that the field shift of the modal fields creates a geometrical or waveguide birefringence of the order $(\kappa a)^2$ where κ is the curvature and a is the core radius (Fang and Lin, 1985, Garth, 1988).

Effects of curvature and twisting have been included in a number of treatments of nonlinear fibre optics. Typically these studies consider a pair of coupled nonlinear Schrödinger equations, with linear terms describing birefringence and coupling between the basis modes, of the form

$$\mathrm{i}A_t^{\pm} + A_{xx}^{\pm} \pm \Delta A^{\pm} + \kappa A^{\mp} + \left(|A^{\pm}|^2 + h|A^{\mp}|^2\right) A^{\pm} = 0\,, \tag{3.1}$$

where A^{+}, A^{-} are the complex amplitudes in the circularly polarised basis and Δ, κ, h are real constants. For the case when twist induced birefringence is neglected ($\Delta = 0$), Trillo et al. (1989) performed numerical calculations and showed that for a circularly polarised input soliton ($A^{-} \equiv 0$), switching behaviour between A^{+} and A^{-} occurred. Also for the case $\Delta = 0$, analytic solutions for equations

(3.1) have been found in terms of Jacobian elliptic functions (Florjanczyk and Tremblay, 1989, Kostov and Uzunov, 1992). If $\Delta = 0$ and $h = 0$, equations (3.1) describe the amplitude modulation of pulses in directional couplers. For this special case Trillo et al. (1988) have considered the numerical solution of equations (3.1) and predicted soliton switching. Kivshar and Malomed (1989) have found analytical solutions for this case.

In this chapter coupled nonlinear Schrödinger equations describing the effects of curvature and torsion on an otherwise axisymmetric, single-mode, axially homogeneous fibre are derived from Maxwell's equations. An orthogonal coordinate system which follows the fibre as it bends is derived in the following section. Field corrections which are due to the curvature are then found and the pulse evolution equations are derived. The pulse evolution is governed by a coupled pair of cubic Schrödinger equations with linear cross coupling terms having coefficients related to the local curvature and torsion of the fibre. In general, these need not be constant. For constant radius of curvature which is comparable to the nonlinear evolution length and for constant torsion, numerical calculations are performed to show how the stability of a non-distorting pulse-like initial condition depends on the values of the curvature and torsion.

3.2 A coordinate system for curved fibres

Before studying the propagation of waves in a bent optical fibre, it is convenient to choose a coordinate system which follows the fibre as it bends (Hobbs and Kath, 1990). If the position of the centreline of the fibre is defined by $\mathbf{r}(z)$, a function of the arc length z, then the position of a point in the fibre can be written as

$$\mathbf{x} = \mathbf{r}(z) + x_1\hat{\mathbf{n}}(z) + x_2\hat{\mathbf{b}}(z)\,, \tag{3.2}$$

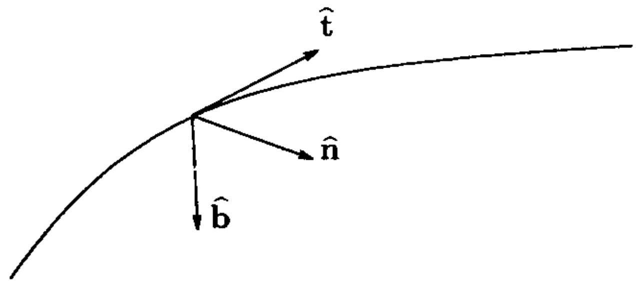

Figure 3.1: The Frenet-Serret coordinate system.

where x_1, x_2 are distances perpendicular to the centreline along the direction of the unit normal $\widehat{\mathbf{n}}$ and the unit binormal $\widehat{\mathbf{b}}$, respectively. The unit tangent vector $\widehat{\mathbf{t}}$, and the unit normal and binormal vectors are defined by the Frenet-Serret formulae (Hildebrand, 1976) (see Figure 3.1)

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}z} = \widehat{\mathbf{t}}\,, \quad \frac{\mathrm{d}\widehat{\mathbf{t}}}{\mathrm{d}z} = \kappa\widehat{\mathbf{n}}\,, \quad \frac{\mathrm{d}\widehat{\mathbf{n}}}{\mathrm{d}z} = \Delta\widehat{\mathbf{b}} - \kappa\widehat{\mathbf{t}}\,, \quad \frac{\mathrm{d}\widehat{\mathbf{b}}}{\mathrm{d}z} = -\Delta\widehat{\mathbf{n}}\,,$$

where κ is the curvature and Δ is the torsion of the centreline $\mathbf{r}$. Since

$$\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \left[(1-\kappa x_1)^2 + (x_2^2 + x_1^2)\Delta^2\right]\mathrm{d}z^2 + 2\Delta(x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1)\,\mathrm{d}z\,,$$

the coordinate system (x_1, x_2, z) is not orthogonal. This coordinate system can be transformed into an orthogonal curvilinear system by rotating the above system through an angle ϕ, where

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} = \Delta\,. \tag{3.3}$$

The position of a point on a fibre can be written as

$$\mathbf{x} = \mathbf{r}(z) + y_1\mathbf{e}_1(z) + y_2\mathbf{e}_2(z)\,,$$

where

$$\mathbf{e}_1 = \cos\phi\,\widehat{\mathbf{n}} - \sin\phi\,\widehat{\mathbf{b}}\,, \qquad \text{and} \qquad \mathbf{e}_2 = \sin\phi\,\widehat{\mathbf{n}} + \cos\phi\,\widehat{\mathbf{b}}\,.$$

The coordinate system (y_1, y_2, z) is orthogonal since

$$\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} = \mathrm{d}y_1^2 + \mathrm{d}y_2^2 + \left(1 - \kappa(y_1\cos\phi + y_2\sin\phi)\right)^2\,\mathrm{d}z^2\,,$$

and was proposed by Tang (1970) as an alternative to the Frenet-Serret system for analysis of curved waveguides and antennae.

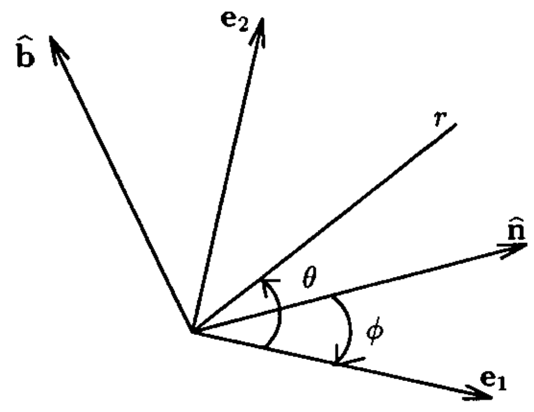

Figure 3.2: Relationship between the unit vectors (**b**,**n**) and ($\mathbf{e}_1$,$\mathbf{e}_2$).

The curved fibres that will be studied in this chapter are assumed to be axisymmetric and cylindrical which suggests using the orthogonal coordinate system (r, θ, z), where $y_1 = r\cos\theta$ and $y_2 = r\sin\theta$ and $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ are an orthogonal triad of vectors. For the orthogonal coordinate system (r, θ, z) described above

$$\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + (1 - \kappa r\cos(\theta - \phi))^2\,\mathrm{d}z^2\,,$$

thus giving the scale factors

$$h_r = 1\,, \qquad h_\theta = r\,, \qquad h_z = 1 - \kappa r\cos(\theta - \phi) = h_z(r, \theta, z)\,. \tag{3.4}$$

These scale factors are analogous to those of a cylindrical coordinate system except that the scale factor h_z is a function of the curvature of the fibre and of the polar angle relative to the direction of the principal normal. It may be noted that at points lying on the local binormal $\hat{\mathbf{b}}$, so that $\theta = \phi \pm \pi/2$, the scale factor is $h_z = 1$, as for the cylindrical coordinate system in a straight fibre.

3.3 Field corrections due to the curvature

As in Chapter 2, the optical properties of the fibre are assumed to be the same as for an axially symmetric, isotropic, non-magnetic and weakly-guiding fibre. This implies that effects due to straining of the fibre as it is bent or coiled are neglected. The material properties are also assumed not to vary along the fibre. The electromagnetic fields are governed by Maxwell's equations (1.1)–(1.4). When written in terms of the coordinate system (r, θ, z) derived in Section 3.2, with h_z given by (3.4), they become

$$\widetilde{\nabla}\wedge \mathbf{E} + \frac{h_z - 1}{h_z}\frac{\partial \mathbf{E}}{\partial z}\wedge \mathbf{e}_z + \frac{\widetilde{\nabla} h_z}{h_z}\wedge \mathbf{e}_z\ \mathbf{E}\cdot\mathbf{e}_z = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}, \tag{3.5}$$

$$\widetilde{\nabla}\wedge \mathbf{H} + \frac{h_z - 1}{h_z}\frac{\partial \mathbf{H}}{\partial z}\wedge \mathbf{e}_z + \frac{\widetilde{\nabla} h_z}{h_z}\wedge \mathbf{e}_z\ \mathbf{H}\cdot\mathbf{e}_z = \frac{\partial \mathbf{D}}{\partial t}, \tag{3.6}$$

$$\widetilde{\nabla}\cdot \mathbf{D} - \frac{h_z - 1}{h_z}\frac{\partial \mathbf{D}}{\partial z}\cdot \mathbf{e}_z = 0, \tag{3.7}$$

$$\widetilde{\nabla}\cdot \mathbf{H} - \frac{h_z - 1}{h_z}\frac{\partial \mathbf{H}}{\partial z}\cdot \mathbf{e}_z = 0, \tag{3.8}$$

where

$$\widetilde{\nabla} \equiv \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z},$$

has the same form as the standard operator '∇' for a cylindrical coordinate system. Assuming that the fibre has a Kerr type nonlinearity (see Section 1.4) and is axially homogeneous, the electric displacement can be written as

$$\mathbf{D} = \left(\varepsilon(r) + N(r)|\mathbf{E}|^2\right)\mathbf{E}.$$

As for the case of a straight axially inhomogeneous fibre, solutions to equations (3.5) and (3.6) can be found by writing the fields as leading order terms and corrections in a small parameter ν, as described in Section 2.2. If the curvature κ is also assumed to be small, then substitution of (2.2) into equations (3.5) and (3.6), gives to leading order, the linearized equations

$$\widetilde{\nabla}\wedge \mathbf{E}^{(1)} = -\mu_0 \frac{\partial \mathbf{H}^{(1)}}{\partial t}, \tag{3.9}$$

$$\widetilde{\nabla}\wedge \mathbf{H}^{(1)} = \varepsilon \frac{\partial \mathbf{E}^{(1)}}{\partial t}, \tag{3.10}$$

$$\widetilde{\nabla}\cdot \mathbf{E}^{(1)} = 0, \tag{3.11}$$

$$\widetilde{\nabla}\cdot \mathbf{H}^{(1)} = 0. \tag{3.12}$$

These equations are similar to the linearized equations (2.3)–(2.6) which were obtained in the leading order analysis of an axially inhomogeneous fibre except that ε is now a function of r only. The method of solution of equations (3.9)–(3.12) is analogous to that for equations (2.3)–(2.6). The travelling wave solutions to (3.9)–(3.12), for a weakly-guiding, single-mode fibre can therefore be written as

$$\begin{aligned} \mathbf{E}^{(1)} &= A^+\mathbf{E}^+e^{\mathrm{i}(\theta+\psi)} + A^-\mathbf{E}^-e^{\mathrm{i}(-\theta+\psi)} + c.c., \\ \mathbf{H}^{(1)} &= A^+\mathbf{H}^+e^{\mathrm{i}(\theta+\psi)} + A^-\mathbf{H}^-e^{\mathrm{i}(-\theta+\psi)} + c.c., \end{aligned} \tag{3.13}$$

where A^+ and A^- are complex amplitudes, $\psi = kz - \omega t$, where k is the local wavenumber and ω is the radian frequency. $\mathbf{E}^\pm$ and $\mathbf{H}^\pm$ are the modal fields which if resolved along the basis vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ can be represented as

$$\begin{aligned} \mathbf{E}^\pm &= \mathrm{i}\tilde{E}_1\mathbf{e}_r \pm \tilde{E}_2\mathbf{e}_\theta + \tilde{E}_3\mathbf{e}_z \ , \\ \mathbf{H}^\pm &= \pm\tilde{H}_1\mathbf{e}_r + \mathrm{i}\tilde{H}_2\mathbf{e}_\theta \pm \mathrm{i}\tilde{H}_3\mathbf{e}_z \ , \end{aligned} \tag{3.14}$$

where $\tilde{E}_i = \tilde{E}_i(r;\omega)$ and $\tilde{H}_i = \tilde{H}_i(r;\omega)$ are real functions of r which satisfy the system of equations (2.9).

Using the method of multiple scales (Appendix B), with ν identified as a parameter characterising times for pulse modulation, to obtain approximations to the fields at higher orders of ν, two scaled variables must be introduced

$$\chi = \nu(s_g z - t), \qquad Z = \nu^2 z,$$

where $s_g = \mathrm{d}k/\mathrm{d}\omega$ is the group-slowness. Any fluctuations in the amplitudes $A^\pm$ are allowed to depend on both of the slow scales χ and Z, so that $A^\pm = A^\pm(\chi, Z)$. The fields are treated as functions of the variables r, θ, ψ, χ, Z and are 2π-periodic in both θ and ψ. The z and t derivatives are replaced by

$$\begin{aligned} \frac{\partial}{\partial z} &= k\frac{\partial}{\partial\psi} + \nu s_g\frac{\partial}{\partial\chi} + \nu^2\frac{\partial}{\partial Z}, \\ \frac{\partial}{\partial t} &= -\omega\frac{\partial}{\partial\psi} - \nu\frac{\partial}{\partial\chi}. \end{aligned} \tag{3.15}$$

By writing the fields as leading order terms and corrections of the order ν^2

$$\begin{aligned} \mathbf{E} &= \nu\mathbf{E}^{(1)} + \nu^2\widehat{\mathbf{E}}\,, \\ \mathbf{H} &= \nu\mathbf{H}^{(1)} + \nu^2\widehat{\mathbf{H}}\,, \\ \mathbf{D} &= \nu\varepsilon\mathbf{E}^{(1)} + \nu^2\widehat{\mathbf{D}}\,, \end{aligned} \tag{3.16}$$

where $\widehat{\mathbf{D}} = \varepsilon\widehat{\mathbf{E}}$, and substituting (3.15) and (3.16) into equations (3.5) and (3.6) the correction fields, $\widehat{\mathbf{E}}$ and $\widehat{\mathbf{H}}$, are found to be governed by

$$\begin{aligned} \nabla'\wedge\widehat{\mathbf{E}} - \omega\mu_0\frac{\partial\widehat{\mathbf{H}}}{\partial\psi} = {} & e^{\mathrm{i}(\theta+\psi)}\Big\{A_X^+\big(s_g\mathbf{E}^+\wedge\mathbf{e}_z+\mu_0\mathbf{H}^+\big)+\frac{\kappa}{\nu}A^+\Big[\mathrm{i}kr\cos(\theta-\phi)\mathbf{E}^+\wedge\mathbf{e}_z \\ & -\left(\mathbf{e}_r\sin(\theta-\phi)+\mathbf{e}_\theta\cos(\theta-\phi)\right)\mathbf{E}^+.\mathbf{e}_z\Big]\Big\} \\ & +e^{\mathrm{i}(-\theta+\psi)}\Big\{A_X^-\big(s_g\mathbf{E}^-\wedge\mathbf{e}_z+\mu_0\mathbf{H}^-\big)+\frac{\kappa}{\nu}A^-\Big[\mathrm{i}kr\cos(\theta-\phi)\mathbf{E}^-\wedge\mathbf{e}_z \\ & -\left(\mathbf{e}_r\sin(\theta-\phi)+\mathbf{e}_\theta\cos(\theta-\phi)\right)\mathbf{E}^-.\mathbf{e}_z\Big]\Big\} \\ & +c.c. + o(1)\,, \end{aligned} \tag{3.17}$$

$$\begin{aligned} \nabla'\wedge\widehat{\mathbf{H}} + \omega\varepsilon\frac{\partial\widehat{\mathbf{E}}}{\partial\psi} = {} & e^{\mathrm{i}(\theta+\psi)}\Big\{A_X^+\big(s_g\mathbf{H}^+\wedge\mathbf{e}_z-\varepsilon\mathbf{E}^+\big)+\frac{\kappa}{\nu}A^+\Big[\mathrm{i}kr\cos(\theta-\phi)\mathbf{H}^+\wedge\mathbf{e}_z \\ & -\left(\mathbf{e}_r\sin(\theta-\phi)+\mathbf{e}_\theta\cos(\theta-\phi)\right)\mathbf{H}^+.\mathbf{e}_z\Big]\Big\} \\ & +e^{\mathrm{i}(-\theta+\psi)}\Big\{A_X^-\big(s_g\mathbf{H}^-\wedge\mathbf{e}_z-\varepsilon\mathbf{E}^-\big)+\frac{\kappa}{\nu}A^-\Big[\mathrm{i}kr\cos(\theta-\phi)\mathbf{H}^-\wedge\mathbf{e}_z \\ & -\left(\mathbf{e}_r\sin(\theta-\phi)+\mathbf{e}_\theta\cos(\theta-\phi)\right)\mathbf{H}^-.\mathbf{e}_z\Big]\Big\} \\ & +c.c. + o(1)\,, \end{aligned} \tag{3.18}$$

where

$$\nabla' \equiv \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\,k(\omega)\frac{\partial}{\partial z}\,.$$

If $\sin(\theta-\phi)$ and $\cos(\theta-\phi)$ are expressed in terms of exponentials and the representation for $\mathbf{E}^\pm$ and $\mathbf{H}^\pm$, given in (3.14), is used equations (3.17) and (3.18) can be written as

$$
\begin{aligned}
\nabla' \wedge \hat{\mathbf{E}} - \omega\mu_0 \frac{\partial \hat{\mathbf{H}}}{\partial \psi} = {} & e^{\mathrm{i}(\theta+\psi)} A_\chi^+ (s_g \mathbf{E}^+ \wedge \mathbf{e}_z + \mu_0 \mathbf{H}^+) \\
& + \frac{\kappa}{\nu} \frac{A^+}{2} \Big\{ e^{\mathrm{i}(2\theta+\psi-\phi)} \left[\mathrm{i}(kr\tilde{E}_2 + \tilde{E}_3)\mathbf{e}_r + (kr\tilde{E}_1 - \tilde{E}_3)\mathbf{e}_\theta \right] \\
& \qquad + e^{\mathrm{i}(\psi+\phi)} \left[\mathrm{i}(kr\tilde{E}_2 - \tilde{E}_3)\mathbf{e}_r + (kr\tilde{E}_1 - \tilde{E}_3)\mathbf{e}_\theta \right] \Big\} \\
& + e^{\mathrm{i}(-\theta+\psi)} A_\chi^- (s_g \mathbf{E}^- \wedge \mathbf{e}_z + \mu_0 \mathbf{H}^-) \\
& + \frac{\kappa}{\nu} \frac{A^-}{2} \Big\{ e^{\mathrm{i}(-2\theta+\psi+\phi)} \left[-\mathrm{i}(kr\tilde{E}_2 + \tilde{E}_3)\mathbf{e}_r + (kr\tilde{E}_1 - \tilde{E}_3)\mathbf{e}_\theta \right] \\
& \qquad + e^{\mathrm{i}(\psi-\phi)} \left[-\mathrm{i}(kr\tilde{E}_2 - \tilde{E}_3)\mathbf{e}_r + (kr\tilde{E}_1 - \tilde{E}_3)\mathbf{e}_\theta \right] \Big\} \\
& + c.c. + o(1) \,, \qquad (3.19)
\end{aligned}
$$

$$
\begin{aligned}
\nabla' \wedge \hat{\mathbf{H}} + \omega\varepsilon \frac{\partial \hat{\mathbf{E}}}{\partial \psi} = {} & e^{\mathrm{i}(\theta+\psi)} A_\chi^+ (s_g \mathbf{H}^+ \wedge \mathbf{e}_z - \varepsilon \mathbf{E}^+) \\
& + \frac{\kappa}{\nu} \frac{A^+}{2} \Big\{ e^{\mathrm{i}(2\theta+\psi-\phi)} \left[-(kr\tilde{H}_2 + \tilde{H}_3)\mathbf{e}_r - \mathrm{i}(kr\tilde{H}_1 + \tilde{H}_3)\mathbf{e}_\theta \right] \\
& \qquad + e^{\mathrm{i}(\psi+\phi)} \left[-(kr\tilde{H}_2 - \tilde{H}_3)\mathbf{e}_r - \mathrm{i}(kr\tilde{H}_1 + \tilde{H}_3)\mathbf{e}_\theta \right] \Big\} \\
& + e^{\mathrm{i}(-\theta+\psi)} A_\chi^- (s_g \mathbf{H}^- \wedge \mathbf{e}_z - \varepsilon \mathbf{E}^-) \\
& + \frac{\kappa}{\nu} \frac{A^-}{2} \Big\{ e^{\mathrm{i}(-2\theta+\psi+\phi)} \left[-(kr\tilde{H}_2 + \tilde{H}_3)\mathbf{e}_r + \mathrm{i}(kr\tilde{H}_1 + \tilde{H}_3)\mathbf{e}_\theta \right] \\
& \qquad + e^{\mathrm{i}(\psi-\phi)} \left[-(kr\tilde{H}_2 - \tilde{H}_3)\mathbf{e}_r + \mathrm{i}(kr\tilde{H}_1 + \tilde{H}_3)\mathbf{e}_\theta \right] \Big\} \\
& + c.c. + o(1) \,. \qquad (3.20)
\end{aligned}
$$

The terms in (3.19) and (3.20) involving $A_\chi^\pm$ are due to amplitude modulation of the signal envelope and occur in the absence of any curvature or torsion in the fibre and give rise to the fields $\mathbf{E}_\omega^\pm$, $\mathbf{H}_\omega^\pm$ (Parker and Newboult, 1989), while the remaining terms are due to the curvature of the fibre. A solution to equations (3.19) and (3.20) can be written as

$$
\begin{aligned}
\hat{\mathbf{E}} = {} & -\mathrm{i}A_\chi^+ \mathbf{E}_\omega^+ e^{\mathrm{i}(\theta+\psi)} + \frac{\kappa}{\nu} A^+ \left[\hat{\mathbf{E}}^+ e^{\mathrm{i}(2\theta+\psi-\phi)} + \check{\mathbf{E}}^+ e^{\mathrm{i}(\psi+\phi)} \right] \\
& -\mathrm{i}A_\chi^- \mathbf{E}_\omega^- e^{\mathrm{i}(-\theta+\psi)} + \frac{\kappa}{\nu} A^- \left[\hat{\mathbf{E}}^- e^{\mathrm{i}(-2\theta+\psi+\phi)} + \check{\mathbf{E}}^- e^{\mathrm{i}(\psi-\phi)} \right] \\
& + c.c. \,, \qquad (3.21)
\end{aligned}
$$

$$\begin{aligned}\widehat{\mathbf{H}} &= -\mathrm{i}A_\chi^+\mathbf{H}_\omega^+ e^{\mathrm{i}(\theta+\psi)} + \frac{\kappa}{\nu}A^+\left[\widehat{\mathbf{H}}^+ e^{\mathrm{i}(2\theta+\psi-\phi)} + \check{\mathbf{H}}^+ e^{\mathrm{i}(\psi+\phi)}\right] \\ &\quad -\mathrm{i}A_\chi^-\mathbf{H}_\omega^- e^{\mathrm{i}(-\theta+\psi)} + \frac{\kappa}{\nu}A^-\left[\widehat{\mathbf{H}}^- e^{\mathrm{i}(-2\theta+\psi+\phi)} + \check{\mathbf{H}}^- e^{\mathrm{i}(\psi-\phi)}\right] \\ &\quad + c.c.\,, \end{aligned} \tag{3.22}$$

where $\widehat{\mathbf{E}}^\pm$, $\widehat{\mathbf{H}}^\pm$ satisfy the equations

$$\begin{aligned} &\nabla'\wedge\left(\widehat{\mathbf{E}}^\pm e^{\mathrm{i}(\pm2\theta+\psi\mp\phi)}\right) - \omega\mu_0\frac{\partial}{\partial\psi}\left(\widehat{\mathbf{H}}^\pm e^{\mathrm{i}(\pm2\theta+\psi\mp\phi)}\right) \\ &\qquad = \frac{1}{2}\left[\pm\mathrm{i}(kr\widetilde{E}_2+\widetilde{E}_3)\mathbf{e}_r + (kr\widetilde{E}_1-\widetilde{E}_3)\mathbf{e}_\theta\right] e^{\mathrm{i}(\pm2\theta+\psi\mp\phi)}, \\ &\nabla'\wedge\left(\widehat{\mathbf{H}}^\pm e^{\mathrm{i}(\pm2\theta+\psi\mp\phi)}\right) + \omega\varepsilon\frac{\partial}{\partial\psi}\left(\widehat{\mathbf{E}}^\pm e^{\mathrm{i}(\pm2\theta+\psi\mp\phi)}\right) \\ &\qquad = \frac{1}{2}\left[-(kr\widetilde{H}_2+\widetilde{H}_3)\mathbf{e}_r \mp \mathrm{i}(kr\widetilde{H}_1+\widetilde{H}_3)\mathbf{e}_\theta\right] e^{\mathrm{i}(\pm2\theta+\psi\mp\phi)}, \end{aligned} \tag{3.23}$$

and $\check{\mathbf{E}}^\pm$, $\check{\mathbf{H}}^\pm$ satisfy the set of equations

$$\begin{aligned} &\nabla'\wedge\left(\check{\mathbf{E}}^\pm e^{\mathrm{i}(\psi\pm\phi)}\right) - \omega\mu_0\frac{\partial}{\partial\psi}\left(\check{\mathbf{H}}^\pm e^{\mathrm{i}(\psi\pm\phi)}\right) \\ &\qquad = \frac{1}{2}\left[\pm\mathrm{i}(kr\widetilde{E}_2-\widetilde{E}_3)\mathbf{e}_r + (kr\widetilde{E}_1-\widetilde{E}_3)\mathbf{e}_\theta\right] e^{\mathrm{i}(\psi\pm\phi)}, \\ &\nabla'\wedge\left(\check{\mathbf{H}}^\pm e^{\mathrm{i}(\psi\pm\phi)}\right) + \omega\varepsilon\frac{\partial}{\partial\psi}\left(\check{\mathbf{E}}^\pm e^{\mathrm{i}(\psi\pm\phi)}\right) \\ &\qquad = \frac{1}{2}\left[-(kr\widetilde{H}_2-\widetilde{H}_3)\mathbf{e}_r \mp \mathrm{i}(kr\widetilde{H}_1+\widetilde{H}_3)\mathbf{e}_\theta\right] e^{\mathrm{i}(\psi\pm\phi)}, \end{aligned} \tag{3.24}$$

with the fields satisfying the conditions $\widehat{\mathbf{E}}^\pm, \widehat{\mathbf{H}}^\pm, \check{\mathbf{E}}^\pm, \check{\mathbf{H}}^\pm \to 0$ as $r\to\infty$ and bounded at $r=0$. Comparison with the modal fields $\mathbf{E}^\pm$ and $\mathbf{H}^\pm$ given in (3.14), suggests that the fields $\widehat{\mathbf{E}}^\pm, \widehat{\mathbf{H}}^\pm$ and $\check{\mathbf{E}}^\pm, \check{\mathbf{H}}^\pm$, may be resolved along the basis vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ and represented as

$$\begin{aligned} \widehat{\mathbf{E}}^\pm &= \mathrm{i}\widehat{E}_1\mathbf{e}_r \pm \widehat{E}_2\mathbf{e}_\theta + \widehat{E}_3\mathbf{e}_z\,, \\ \widehat{\mathbf{H}}^\pm &= \pm\widehat{H}_1\mathbf{e}_r + \mathrm{i}\widehat{H}_2\mathbf{e}_\theta \pm \mathrm{i}\widehat{H}_3\mathbf{e}_z\,, \end{aligned} \tag{3.25}$$

and

$$\begin{aligned}\check{\mathbf{E}}^{\pm} &= \mathrm{i}\check{E}_1\mathbf{e}_r \pm \check{E}_2\mathbf{e}_\theta + \check{E}_3\mathbf{e}_z\,,\\ \check{\mathbf{H}}^{\pm} &= \pm\check{H}_1\mathbf{e}_r + \mathrm{i}\check{H}_2\mathbf{e}_\theta \pm \mathrm{i}\check{H}_3\mathbf{e}_z\,,\end{aligned} \tag{3.26}$$

where $\widehat{E}_i = \widehat{E}_i(r;\omega)$, $\widehat{H}_i = \widehat{H}_i(r;\omega)$, $\check{E}_i = \check{E}_i(r;\omega)$ and $\check{H}_i = \check{H}_i(r;\omega)$ are real functions which satisfy the inhomogeneous ordinary differential equations

$$2\widehat{E}_3 - kr\widehat{E}_2 - \omega\mu_0 r\widehat{H}_1 = \frac{r}{2}(kr\widetilde{E}_2 + \widetilde{E}_3)\,, \qquad 2\widehat{H}_3 - kr\widehat{H}_2 + \omega\varepsilon r\widehat{E}_1 = \frac{r}{2}(kr\widetilde{H}_2 + \widetilde{H}_3)\,,$$

$$\frac{\mathrm{d}\widehat{E}_3}{\mathrm{d}r} + k\widehat{E}_1 - \omega\mu_0\widehat{H}_2 = -\frac{1}{2}(kr\widetilde{E}_1 - \widetilde{E}_3)\,, \qquad \frac{\mathrm{d}\widehat{H}_3}{\mathrm{d}r} - k\widehat{H}_1 - \omega\varepsilon\widehat{E}_2 = \frac{1}{2}(kr\widetilde{H}_1 + \widetilde{H}_3)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(r\widehat{E}_2) + 2\widehat{E}_1 + \omega\mu_0 r\widehat{H}_3 = 0\,, \qquad \frac{\mathrm{d}}{\mathrm{d}r}(r\widehat{H}_2) - 2\widehat{H}_1 + \omega\varepsilon r\widehat{E}_3 = 0\,,$$

and

$$k\check{E}_2 + \omega\mu_0\check{H}_1 = -\frac{1}{2}(kr\widetilde{E}_2 - \widetilde{E}_3)\,, \qquad k\check{H}_2 - \omega\varepsilon\check{E}_1 = -\frac{1}{2}(kr\widetilde{H}_2 - \widetilde{H}_3)\,,$$

$$\frac{\mathrm{d}\check{E}_3}{\mathrm{d}r} + k\check{E}_1 - \omega\mu_0\check{H}_2 = -\frac{1}{2}(kr\widetilde{E}_1 - \widetilde{E}_3)\,, \qquad \frac{\mathrm{d}\check{H}_3}{\mathrm{d}r} - k\check{H}_1 - \omega\varepsilon\check{E}_2 = \frac{1}{2}(kr\widetilde{H}_1 + \widetilde{H}_3)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(r\check{E}_2) + \omega\mu_0 r\check{H}_3 = 0\,, \qquad \frac{\mathrm{d}}{\mathrm{d}r}(r\check{H}_2) + \omega\varepsilon r\check{E}_3 = 0\,.$$

For detailed evaluation of the coefficients which will arise in the evolution equations developed in the next section the above equations must be solved together with decay conditions as $r \to \infty$ and boundedness conditions at $r = 0$. However, for qualitative analysis of the evolution equations, we find it unnecessary to determine $\widehat{E}_i$, $\check{E}_i$, $\widehat{H}_i$, $\check{H}_i$ explicitly.

3.4 The evolution equations

We now seek solutions for the fields **E**, **H**, **D**, which are of the form

$$\begin{aligned}\mathbf{E} &= \nu\mathbf{E}^{(1)} + \nu^2\mathbf{E}^{(2)} + \nu^3\overline{\mathbf{E}}\,,\\ \mathbf{H} &= \nu\mathbf{H}^{(1)} + \nu^2\mathbf{H}^{(2)} + \nu^3\overline{\mathbf{H}}\,,\\ \mathbf{D} &= \nu\varepsilon\mathbf{E}^{(1)} + \nu^2\varepsilon\mathbf{E}^{(2)} + \nu^3\overline{\mathbf{D}}\,,\end{aligned}$$

where $\mathbf{E}^{(2)} = \hat{\mathbf{E}}$ and $\mathbf{H}^{(2)} = \hat{\mathbf{H}}$ are given in (3.21) and (3.22), and $\bar{\mathbf{E}}$, $\bar{\mathbf{H}}$ and $\bar{\mathbf{D}}$ are the total correction terms to the first two terms in a series solution, in which the electric displacement is related to $\mathbf{E}$ by

$$\bar{\mathbf{D}} = \varepsilon(r)\bar{\mathbf{E}} + N(r)|\mathbf{E}^{(1)}|^2\mathbf{E}^{(1)} + O(\nu).$$

It will be observed that the scaling choices involving ν have the familiar structure in which cubic nonlinearity has an effect on the long evolution scale associated with Z and will allow interaction with group dispersion effects. The equations, which govern the correction fields $\bar{\mathbf{E}}$ and $\bar{\mathbf{H}}$, are obtained by substituting for $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, $\partial/\partial z$ and $\partial/\partial t$ into equations (3.5) and (3.6) and are found to be

$$\nabla' \wedge \bar{\mathbf{E}} - \omega\mu_0 \frac{\partial \bar{\mathbf{H}}}{\partial \psi} = \bar{\mathbf{G}}, \tag{3.27}$$

$$\nabla' \wedge \bar{\mathbf{H}} + \omega\varepsilon \frac{\partial \bar{\mathbf{E}}}{\partial \psi} = \bar{\mathbf{F}}, \tag{3.28}$$

where $\bar{\mathbf{G}}$, $\bar{\mathbf{F}}$ are 2π−periodic in θ and ψ, decay exponentially as $r \to \infty$ and are bounded at $r = 0$, (expressions for $\bar{\mathbf{G}}$ and $\bar{\mathbf{F}}$ are given in Appendix C). Explicit solutions to these equations are unlikely to be tractable even for step index fibres. However, the form of (3.27) and (3.28) is analogous to that of equations (2.17) and (2.18), so that the compatibility condition given by (2.19) with (2.21) can be used to obtain equations which govern the evolution of the amplitudes A^+ and A^-. Applying the same reasoning as before requires consideration of only the terms which give a non-zero contribution to the equation. By comparing the coefficients of α_1^* in the expression analogous to (2.19), the equation

$$\mathrm{i} f_1 A_Z^+ = g A_{XX}^+ + \left(f_2|A^+|^2 + f_3|A^-|^2\right) A^+ + \frac{\kappa^2}{\nu^2}\left(f_4 A^+ + e^{-2\mathrm{i}\phi} f_5 A^-\right) \tag{3.29}$$

is obtained, where the coefficients are given by

$$f_1 = \int_0^\infty \left(\mathbf{E}^+ \wedge \mathbf{H}^{+*} + \mathbf{E}^{+*} \wedge \mathbf{H}^+\right) \cdot \mathbf{e}_z \, r \, \mathrm{d}r,$$

$$f_2 = -\omega \int_0^\infty \left[|\mathbf{E}^+ \cdot \mathbf{E}^+|^2 + 2|\mathbf{E}^+|^4\right] N r \, \mathrm{d}r,$$

$$f_3 = -2\omega\int_0^\infty \left[|\mathbf{E}^+\cdot\mathbf{E}^-|^2 + |\mathbf{E}^+\cdot\mathbf{E}^{-*}|^2 + |\mathbf{E}^+|^2|\mathbf{E}^-|^2\right] Nr\,\mathrm{d}r\,,$$

$$\begin{aligned} f_4 = \frac{1}{2}\int_0^\infty &\Big\{ r^2k(\mathbf{E}^{+*}\wedge\mathbf{H}^+ + \mathbf{E}^+\wedge\mathbf{H}^{+*})\cdot\mathbf{e}_z + rk(\mathbf{E}^{+*}\wedge\hat{\mathbf{H}}^+ + \hat{\mathbf{E}}^+\wedge\mathbf{H}^{+*})\cdot\mathbf{e}_z \\ &+ rk(\mathbf{E}^{+*}\wedge\check{\mathbf{H}}^+ + \check{\mathbf{E}}^+\wedge\mathbf{H}^{+*})\cdot\mathbf{e}_z + \mathrm{i}r\left[(\mathbf{H}^+\cdot\mathbf{e}_z)\mathbf{e}_\theta\cdot\mathbf{E}^{+*} - (\mathbf{E}^+\cdot\mathbf{e}_z)\mathbf{e}_\theta\cdot\mathbf{H}^{+*}\right] \\ &- \left[(\hat{\mathbf{H}}^+\cdot\mathbf{e}_z)\mathbf{E}^{+*} - (\hat{\mathbf{E}}^+\cdot\mathbf{e}_z)\mathbf{H}^{+*}\right]\cdot(\mathbf{e}_r - \mathrm{i}\mathbf{e}_\theta) \\ &+ \left[(\check{\mathbf{H}}^+\cdot\mathbf{e}_z)\mathbf{E}^{+*} - (\check{\mathbf{E}}^+\cdot\mathbf{e}_z)\mathbf{H}^{+*}\right]\cdot(\mathbf{e}_r + \mathrm{i}\mathbf{e}_\theta)\Big\}\, r\,\mathrm{d}r\,, \end{aligned}$$

$$\begin{aligned} f_5 = \frac{1}{2}\int_0^\infty &\Big\{ rk(\mathbf{E}^{+*}\wedge\check{\mathbf{H}}^- + \check{\mathbf{E}}^-\wedge\mathbf{H}^{+*})\cdot\mathbf{e}_z + \frac{r^2k}{2}(\mathbf{E}^{+*}\wedge\mathbf{H}^- + \mathbf{E}^-\wedge\mathbf{H}^{+*})\cdot\mathbf{e}_z \\ &+ \frac{r}{2}\left[(\hat{\mathbf{H}}^-\cdot\mathbf{e}_z)\mathbf{E}^{+*} - (\hat{\mathbf{E}}^-\cdot\mathbf{e}_z)\mathbf{H}^{+*}\right]\cdot(\mathbf{e}_r + \mathrm{i}\mathbf{e}_\theta) \\ &+ \left[(\check{\mathbf{H}}^-\cdot\mathbf{e}_z)\mathbf{E}^{+*} - (\check{\mathbf{E}}^-\cdot\mathbf{e}_z)\mathbf{H}^{+*}\right]\cdot(\mathbf{e}_r + \mathrm{i}\mathbf{e}_\theta)\Big\}\, r\,\mathrm{d}r\,, \end{aligned}$$

$$\begin{aligned} g &= -\int_0^\infty \left\{ s_g(\mathbf{E}^{+*}\wedge\mathbf{H}^+_\omega + \mathbf{E}^+_\omega\wedge\mathbf{H}^{+*})\cdot\mathbf{e}_z - (\varepsilon\mathbf{E}^{+*}\cdot\mathbf{E}^+_\omega + \mu_0\mathbf{H}^{+*}\cdot\mathbf{H}^+_\omega)\right\} r\,\mathrm{d}r \\ &= \frac{1}{2}\frac{\mathrm{d}s_g}{\mathrm{d}\omega}\,. \end{aligned}$$

By inspection of expressions (3.14), (3.25) and (3.26), it is readily found that all of f_1, f_2, f_3, f_4, f_5 and g are real quantities, independent of the curvature and torsion of the fibre. Equating the coefficients of α_2^* gives an equation similar to (3.29) so yielding the pair of equations

$$\mathrm{i}f_1A^\pm_Z = gA^\pm_{XX} + \left(f_2|A^\pm|^2 + f_3|A^\mp|^2\right)A^\pm + \frac{\kappa^2}{\nu^2}\left(f_4A^\pm + e^{\mp 2\mathrm{i}\phi}f_5A^\mp\right)\,. \quad (3.30)$$

The results that have been obtained are for small curvature but no relationship between the magnitude of the curvature and the magnitude of the amplitude parameter has been assumed. However, assuming that $\kappa = \nu\overline{\kappa}(Z)$, and using the change of variables

$$\begin{aligned} &A^\pm = B^\pm e^{\mp\mathrm{i}\phi}e^{-\mathrm{i}\int\overline{\kappa}^2\frac{f_4}{f_1}\mathrm{d}Z}\,, \\ &\tau = \frac{f_2}{f_1}Z\,, \qquad \xi = \sqrt{\frac{f_2}{g}}X\,, \end{aligned} \quad (3.31)$$

it is found that equations (3.30) can be written as

$$
\begin{aligned}
\mathrm{i}B^+_\tau &= B^+_{\xi\xi} - \tilde{\Delta}B^+ + \tilde{\kappa}B^- + \left(|B^+|^2 + h|B^-|^2\right)B^+\,, \\
\mathrm{i}B^-_\tau &= B^-_{\xi\xi} + \tilde{\Delta}B^- + \tilde{\kappa}B^+ + \left(h|B^+|^2 + |B^-|^2\right)B^-\,,
\end{aligned}
\tag{3.32}
$$

where

$$
\tilde{\Delta} = \Delta\frac{f_1}{f_2}\,, \qquad \tilde{\kappa} = \overline{\kappa}^2\frac{f_5}{f_2}\,,
$$

are real functions of Z and $h = f_3/f_2$ is a real constant. It should be noted that f_2/f_1 is the usual nonlinearity parameter for a straight optical fibre, while f_5 involves the perturbation fields due to fibre bending. The coefficients $\tilde{\Delta}$ and $\tilde{\kappa}$ are thus proportional to the torsion and square of curvature of the fibre respectively. Moreover, the relationship between the arguments of the left- and right-handed modal amplitudes $B^{\pm}$ and $A^{\pm}$ involve integrals of the torsion and squared curvature through (3.6) and (3.31).

Both numerical and analytical solutions to equations (3.32) have been presented previously when $\tilde{\Delta}$ and $\tilde{\kappa}$ are constants. For the case when the birefringence induced by twisting of the fibre is neglected, $\tilde{\Delta} = 0$, Trillo et al. (1989) have presented numerical solutions and predict that switching occurs between the two pulses, while Florjańczyk and Tremblay (1989) and Kostov and Uzunov (1992) have found analytic solutions which can be expressed in terms of Jacobian elliptic functions. For the special case $\tilde{\Delta} = 0$, $h = 0$ numerical calculations have been performed by Trillo et al. (1988) and analytical solutions have been found (Kivshar and Malomed, 1989).

3.5 Solutions of the evolution equations

In the previous section equations were derived which describe the evolution of the amplitude modulation of a pulse propagating along a curved and twisted fibre. In this section we consider both analytical solutions of (3.32) which arise for special cases of curvature and torsion, and more general numerical solutions of equations (3.32) when the coefficients $\widehat{\Delta}$ and $\widehat{\kappa}$ are constants.

For a straight, twisted fibre ($\kappa = 0$, $\Delta \neq 0$), the linear cross-coupling terms disappear and by making the substitution

$$B^{\pm}(\xi, \tau) = e^{\pm i\phi \frac{f_1}{f_2}} C^{\pm}(\xi, \tau), \tag{3.33}$$

equations (3.32) are reduced to the constant coefficient cubic Schrödinger equations

$$iC^{\pm}_{\tau} = C^{\pm}_{\xi\xi} + \left(|C^{\pm}|^2 + h|C^{\mp}|^2\right) C^{\pm}. \tag{3.34}$$

These are the pulse evolution equations for a straight, axially-homogeneous fibre, showing that for a straight fibre the effects of torsion can be removed by a rotation of the principal axes, since for zero curvature, the torsion is a consequence of the coordinate system that was chosen in Section 3.2. Equations (3.34) are known to have a number of analytical solutions (Parker and Newboult, 1989). For circularly polarised solitons, $C^- = 0$, equations (3.34) reduce to the single constant coefficient cubic Schrödinger equation which has solutions of the form

$$C^{+}(\xi, \tau) = \sqrt{2}\Gamma e^{-i\psi} \operatorname{sech} \Gamma(\xi - 2V\tau), \tag{3.35}$$

where $\psi = V\xi - (V^2 - \xi^2)\tau$, Γ is the pulse amplitude and V is a frequency shift. Equations (3.34) also allow linearly polarised solitons, $C^+ = C^- e^{-i2\alpha}$, which again reduce the coupled pair of equations to a single nonlinear Schrödinger equation, but this time with pulse (soliton) solutions

$$C^{+}(\xi, \tau) = \sqrt{\frac{2}{1+h}}\Gamma e^{-i\psi} \operatorname{sech} \Gamma(\xi - 2V\tau). \tag{3.36}$$

If the fibre is curved but not twisted, $\kappa \neq 0$ and $\Delta = 0$, equations (3.32) reduce to

$$iB^{\pm}_{\tau} = B^{\pm}_{\xi\xi} + \tilde{\kappa} B^{\mp} + \left(|B^{\pm}|^2 + h|B^{\mp}|^2\right) B^{\pm}. \tag{3.37}$$

Equations (3.37) allow solutions generalising the linearly polarised solutions (Parker and Newboult, 1989) of straight fibres. However they must have the form $B^- = \pm B^+(\xi,\tau)$;

$$B^+(\xi,\tau) = C^+(\xi,\tau)e^{\mp i\int\tilde{\kappa}\,d\tau},$$

where $C^+(\xi,\tau)$ satisfies

$$iC^+_\tau = C^+_{\xi\xi} + (1+h)|C^+|^2C^+ .$$

The conditions $B^- = \pm B^+$ show that these solutions are linearly polarised either in the principal plane of the fibre, or orthogonal to it. Allowable solutions include the soliton solutions of a form similar to

$$C^+(\xi,\tau) = \sqrt{\frac{2}{1+h}}\Gamma e^{-i\psi}\operatorname{sech}\Gamma(\xi - 2V\tau).$$

However, the effect of combined curvature and torsion can be described by considering solutions of the coupled pair of equations of the form

$$B^\pm = e^{-i(\beta\tau+V\sigma)}F_\pm(\sigma),$$

which describe non-distorting pulses. Here β is a real adjustable parameter, while F_+ and F_- are real functions of $\sigma = \xi - 2V\tau$ which satisfy the coupled ordinary differential equations

$$F''_\pm + (V^2 - \beta \mp \tilde{\Delta} - F^2_\pm - hF^2_\mp)F_\pm + \tilde{\kappa}F_\mp = 0, \tag{3.38}$$

for constant $\tilde{\kappa}$ and $\tilde{\Delta}$. These solutions thus relate to fibres in the form of a circular helix which is wound with constant curvature and torsion. By defining the new variables

$$F_\pm(\sigma) = \sqrt{\beta - V^2}\,\widehat{F}_\pm(\eta),$$

$$\eta = \sigma\sqrt{\beta - V^2},$$

equations (3.38) can be written as the pair of equations

$$\widehat{F}''_+ + (-1 - \widehat{\Delta} - \widehat{F}^2_+ - h\widehat{F}^2_-)\widehat{F}_+ + \widehat{\kappa}\widehat{F}_- = 0, \tag{3.39}$$

$$\widehat{F}''_- + (-1 + \widehat{\Delta} - h\widehat{F}^2_+ - \widehat{F}^2_-)\widehat{F}_- + \widehat{\kappa}\widehat{F}_+ = 0, \tag{3.40}$$

where $\widehat{\Delta} = \widetilde{\Delta}/(\beta - V^2)$ and $\widehat{\kappa} = \widetilde{\kappa}/(\beta - V^2)$. Since (3.39), (3.40) allow solutions in which both $\widehat{F}_+$, $\widehat{F}_-$ are even functions of η, values for $F_\pm(0)$, $\widehat{\Delta}$ and $\widehat{\kappa}$ are sought which allow solutions satisfying the conditions

$$\widehat{F}'_+(0) = \widehat{F}'_-(0) = 0 \,,$$

$$\widehat{F}_+, \widehat{F}_-, \widehat{F}'_+, \widehat{F}'_- \to 0 \quad \text{as} \quad \eta \to \pm\infty \,.$$

For solutions which decay as $\eta \to \pm\infty$, the linear terms in equations (3.39) and (3.40) will dominate, since $\widehat{F}_+, \widehat{F}_- \ll 1$. Thus, as $\eta \to \pm\infty$, equations (3.39) and (3.40) can be approximated by the linear equations

$$\begin{aligned} \widehat{F}''_+ &= (1 + \widehat{\Delta})\widehat{F}_+ - \widehat{\kappa}F_- \,, \\ \widehat{F}''_- &= (1 - \widehat{\Delta})\widehat{F}_- - \widehat{\kappa}F_+ \,. \end{aligned} \qquad (3.41)$$

These equations have solutions of the form $\widehat{F}_\pm \propto e^{\lambda\eta}$, where the squared eigenvalues λ^2 are given by the real quantities

$$\lambda^2 = 1 \pm \sqrt{\widehat{\Delta}^2 + \widehat{\kappa}^2} = \lambda^2_\pm \,.$$

Since we require solutions which decay exponentially, the eigenvalues must not be pure imaginary. The constraint $\widehat{\Delta}^2 + \widehat{\kappa}^2 < 1$ then allows purely real eigenvalues. With $\lambda_+, \lambda_- > 0$, a solution to equations (3.41) can be written as

$$\begin{aligned} \widehat{F}_+ &= m_+ e^{-\lambda_+\eta} + m_- e^{-\lambda_-\eta}, \\ \widehat{F}_- &= m_+ v_+ e^{-\lambda_+\eta} + m_- v_- e^{-\lambda_-\eta}. \end{aligned}$$

Here m_+, m_- are arbitrary constants, and $v_\pm$ are given by

$$v_+ = \frac{\widehat{\Delta} - \sqrt{\widehat{\Delta}^2 + \widehat{\kappa}^2}}{\widehat{\kappa}} = \frac{-\widehat{\kappa}}{\widehat{\Delta} + \sqrt{\widehat{\Delta}^2 + \widehat{\kappa}^2}} = -\frac{1}{v_-} \,.$$

For solutions which decay as $\eta \to +\infty$, the quantities

$$v_+ \left(\widehat{F}'_+ + \lambda_- \widehat{F}_+\right) - \left(\widehat{F}'_- + \lambda_- \widehat{F}_-\right) \,,$$

and

$$v_-\left(\widehat{F}'_+ + \lambda_+\widehat{F}_+\right) - \left(\widehat{F}'_- + \lambda_+\widehat{F}_-\right),$$

must vanish. To find suitable pairs of values of $\widehat{F}_+(0)$, $\widehat{F}_-(0)$ which give a decaying solution for fixed values of $\widehat{\Delta}$ and $\widehat{\kappa}$, we seek to minimise either

$$\begin{aligned}&\left[\left(\widehat{\Delta}-\sqrt{\widehat{\Delta}^2+\widehat{\kappa}^2}\right)\left(\widehat{F}'_+ + \lambda_-\widehat{F}_+\right) - \widehat{\kappa}\left(\widehat{F}'_- + \lambda_-\widehat{F}_-\right)\right]^2\\ &\quad+\left[\left(\widehat{\Delta}+\sqrt{\widehat{\Delta}^2+\widehat{\kappa}^2}\right)\left(\widehat{F}'_+ + \lambda_+\widehat{F}_+\right) - \widehat{\kappa}\left(\widehat{F}'_- + \lambda_+\widehat{F}_-\right)\right]^2,\end{aligned} \tag{3.42}$$

or, equivalently,

$$\begin{aligned}&\left[\widehat{\kappa}\left(\widehat{F}'_+ + \lambda_-\widehat{F}_+\right) + \left(\widehat{\Delta}+\sqrt{\widehat{\Delta}^2+\widehat{\kappa}^2}\right)\left(\widehat{F}'_- + \lambda_-\widehat{F}_-\right)\right]^2\\ &\quad+\left[\widehat{\kappa}\left(\widehat{F}'_+ + \lambda_+\widehat{F}_+\right) + \left(\widehat{\Delta}-\sqrt{\widehat{\Delta}^2+\widehat{\kappa}^2}\right)\left(\widehat{F}'_- + \lambda_+\widehat{F}_-\right)\right]^2,\end{aligned} \tag{3.43}$$

for $\eta \to \pm\infty$.

To obtain a first approximation for $\widehat{F}_+(0)$, $\widehat{F}_-(0)$, we consider equations (3.39) and (3.40), and use the transformation $\widehat{F}_\pm(\eta) = af_\pm(\widehat{\eta})$, where $\widehat{\eta} = a\eta$. These equations can then be written as

$$\begin{aligned}f''_+(\widehat{\eta}) + f_+\left(f_+^2 + hf_-^2 - \frac{1+\widehat{\Delta}}{a^2}\right) + \frac{\widehat{\kappa}}{a^2}f_- &= 0\,,\\ f''_-(\widehat{\eta}) + f_-\left(hf_+^2 + f_-^2 - \frac{1-\widehat{\Delta}}{a^2}\right) + \frac{\widehat{\kappa}}{a^2}f_+ &= 0\,.\end{aligned} \tag{3.44}$$

For the ordinary differential equations

$$\begin{aligned}f''_+ + f_+\left(f_+^2 + hf_-^2 - p_+^2\right) &= 0\,,\\ f''_- + f_-\left(hf_+^2 + f_-^2 - p_-^2\right) &= 0\,,\end{aligned}$$

for which $f_+(0) = \cos\alpha$ and $f_-(0) = \sin\alpha$, with f_+, f_- decaying exponentially as $\widehat{\eta} \to \pm\infty$, McCabe (1990) has obtained some sets of values for α, p_+ and p_-. By choosing $p_\pm = (1 \pm \widehat{\Delta})/a^2$, we find that McCabe's boundary value problem is equivalent to ours for (3.44) when $\widehat{\kappa} = 0$, with

$$a^2 = \frac{2}{p_+^2 + p_-^2}\,, \qquad \widehat{\Delta} = \frac{p_+^2 - p_-^2}{p_+^2 + p_-^2}\,,$$

$$\begin{aligned}
\widehat{F}_+(0) &= \sqrt{\frac{2}{p_+^2 + p_-^2}}\cos\alpha\,,\\
\widehat{F}_-(0) &= \sqrt{\frac{2}{p_+^2 + p_-^2}}\sin\alpha\,.
\end{aligned}$$

Using McCabe's values for α, p_+ and p_-, we can find initial conditions which satisfy (3.39) and (3.40) for a known value of $\widehat{\Delta}$, when $\widehat{\kappa} = 0$. To obtain initial conditions using this value of $\widehat{\Delta}$, for values of $\widehat{\kappa}$ other than zero, we can increase $\widehat{\kappa}$ in small steps from $\widehat{\kappa} = 0$ until the required value is reached and for each new value of $\widehat{\kappa}$ find values of $\widehat{F}_+(0)$ and $\widehat{F}_-(0)$ which minimise either (3.42) or (3.43) as $\eta \to \pm\infty$. Bounded solutions have

$$\frac{1}{4}\left(\widehat{F}_+^4 + \widehat{F}_-^4\right) + \frac{h}{2}\widehat{F}_+^2\widehat{F}_-^2 - \frac{1}{2}\left(\widehat{F}_+^2 + \widehat{F}_-^2\right) + \frac{\widehat{\Delta}}{2}\left(\widehat{F}_-^2 - \widehat{F}_+^2\right) + \widehat{\kappa}\widehat{F}_+\widehat{F}_- = 0 \quad (3.45)$$

at $\widehat{\eta} = 0$, where $\widehat{F}'_\pm(0) = 0$. Therefore, when $\widehat{F}_+(0)$ is chosen, $\widehat{F}_-(0)$ may be found from (3.45) using a Newton-Raphson iterative scheme. Hence we need only to perform a one-variable search procedure to find the values of $\widehat{F}_+(0)$ and $\widehat{F}_-(0)$ which minimise (3.42) or (3.43) at suitable large η, for given values of $\widehat{\Delta}$ and $\widehat{\kappa}$. Although we require that either (3.42) or (3.43) are minimised as $\eta \to \pm\infty$, it is sufficient to minimise these expressions at some value of $\eta = \eta_T$, where η_T is chosen to be suitably large. For each value of $\widehat{\kappa} > 0$, we first find values of $\widehat{F}_+(0)$ and $\widehat{F}_-(0)$ which minimise the appropriate expression for $\eta = 3$, using the values of $\widehat{F}_+(0)$ and $\widehat{F}_-(0)$ obtained for the previous value of $\widehat{\kappa}$ as a first approximation for the search. Having obtained values for $\widehat{F}_+(0)$ and $\widehat{F}_-(0)$ which minimise either (3.42) or (3.43), we increased η by a small amount and repeated the search procedure. This was repeated until $\eta = \eta_T$. For the pairs of values $(\widehat{\Delta}, \widehat{\kappa})$ considered in this section a value of $\eta_T = 10$ was sufficient for the resulting solutions to describe non-distorting pulses. It should be noted that the values obtained for $\widehat{F}_+(0)$ and $\widehat{F}_-(0)$ will not be so accurate as to define an isolated pulse if equations (3.39) and (3.40) are numerically integrated for values of η substantially greater than η_T.

For the numerical results presented in this section the curvature was taken to be $\widehat{\kappa} = 0.2$, while different values for the torsion were used. Figure 3.3–Figure 3.6 show the non-distorting pulses that were generated, using the method described above, for values of $\widehat{\Delta} = -0.10115946, -0.23308266, -0.30203382, -0.40127186$. These show that for large $|\widehat{\Delta}|$, the amplitudes of the left- and right-handed polarisation constituents become significantly unequal, though the pulse profile remains close to a sech-envelope.

Numerical integration of equations (3.32) was performed using a split-step spectral method, with the same step lengths for the numerical discretization as were used in Section 2.5. The coupling coefficient was taken to be $h = 2$. The non-distorting pulses shown in Figure 3.3–Figure 3.6 were used as the initial conditions. Figure 3.7–Figure 3.10 are graphs of the peak values of $|B^+|$ and $|B^-|$ plotted against τ for the input pulses Figure 3.3–Figure 3.6 respectively. From these graphs it can be seen that the pulse, although nominally a non-distorting pulse, is unstable but for larger values of the torsion the pulse evolution becomes more stable. The pulse is able to follow the curvature of the fibre for some distance before instabilities arise and the onset of instability is delayed by increasing $|\widehat{\Delta}|$ for fixed $\widehat{\kappa}$. It is noted that if the same values of $\widehat{\Delta}$ are used but $\widehat{\kappa}$ is decreased to 0.05 there does not appear to be any significant trend in the stability for decreasing $\widehat{\kappa}$. Figure 3.11–Figure 3.14 are plots of the real and imaginary parts and the modulus of the pulse amplitudes at four positions along the fibre for the values $\widehat{\kappa} = 0.2$ and $\widehat{\Delta} = -0.10115946$. The radiation tails can be seen in Figure 3.12–Figure 3.14, while for increasing values of τ, the pulses become asymmetric. This can be seen more clearly in Figure 3.15 and Figure 3.16 which show the phase of the B^+ and B^- pulses at the same values of τ as Figure 3.12 and Figure 3.14, where it should be noted that numerical accuracy in determining the phase diminishes rapidly in the tails on either side of the pulse. The pulses in Figure 3.13 and Figure 3.14 appear to show a pulse shape similar to that of the $N = 2$ soliton (Mollenauer, 1985), however in this case the pulse tail is caused by radiation due to the instability of the pulse.

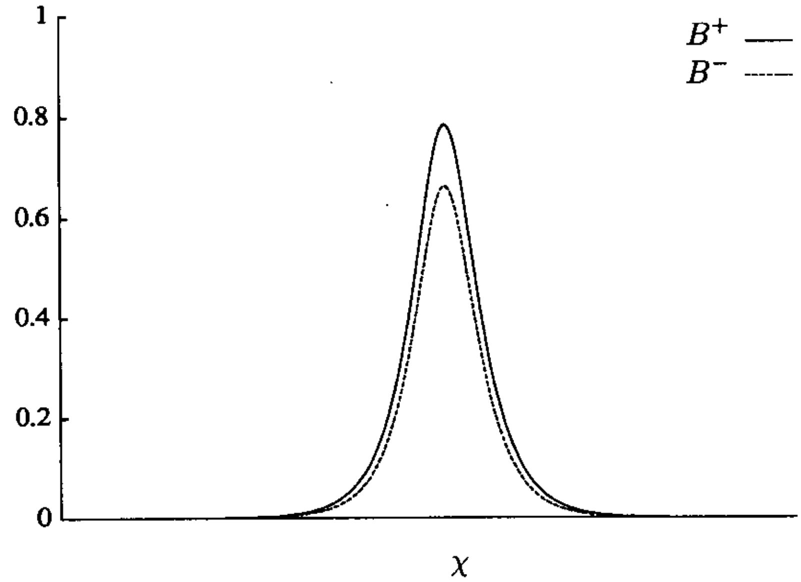

Figure 3.3: Initial pulses for $\hat{\kappa} = 0.2$, $\widehat{\Delta} = -0.10115946$

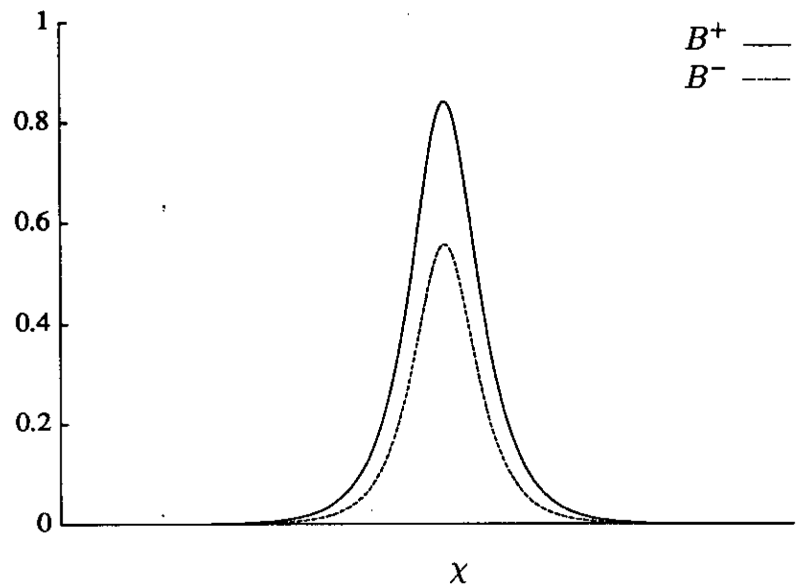

Figure 3.4: Initial pulses for $\hat{\kappa} = 0.2$, $\widehat{\Delta} = -0.23308266$

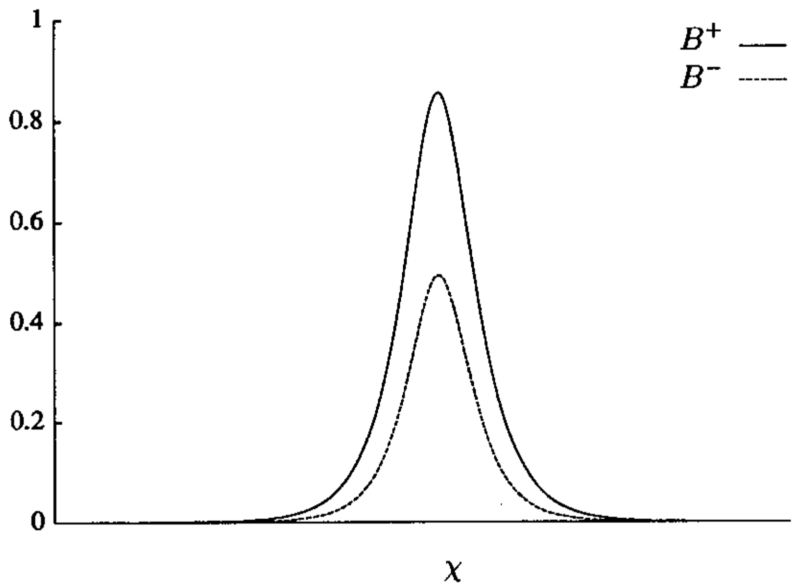

Figure 3.5: Initial pulses for $\widehat{\kappa} = 0.2$, $\widehat{\Delta} = -0.30203382$

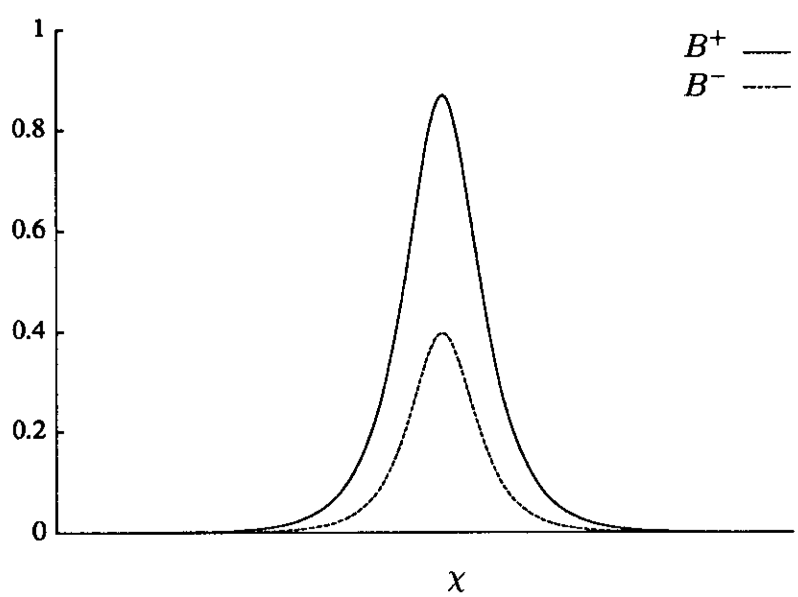

Figure 3.6: Initial pulses for $\widehat{\kappa} = 0.2$, $\widehat{\Delta} = -0.40127186$

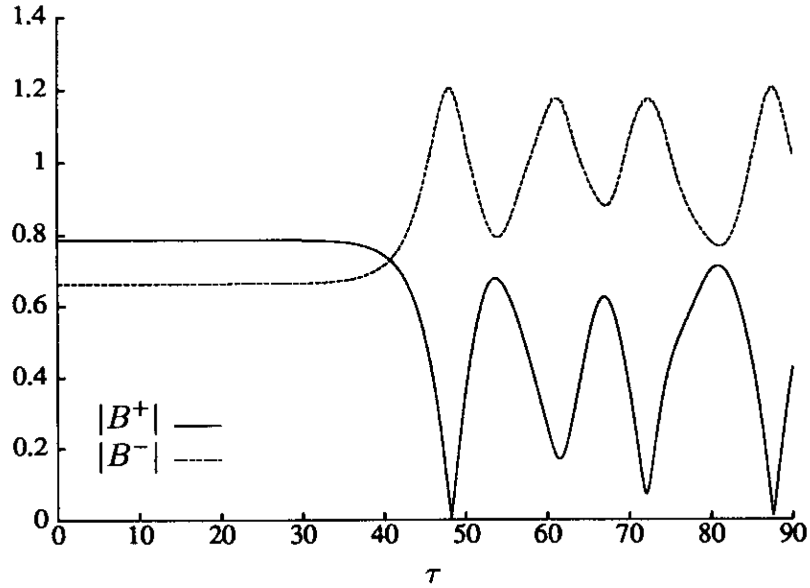

Figure 3.7: Peak values of $|B^+|$ and $|B^-|$ plotted against τ for $\hat{\kappa} = 0.2$, $\hat{\Delta} = -0.10115946$

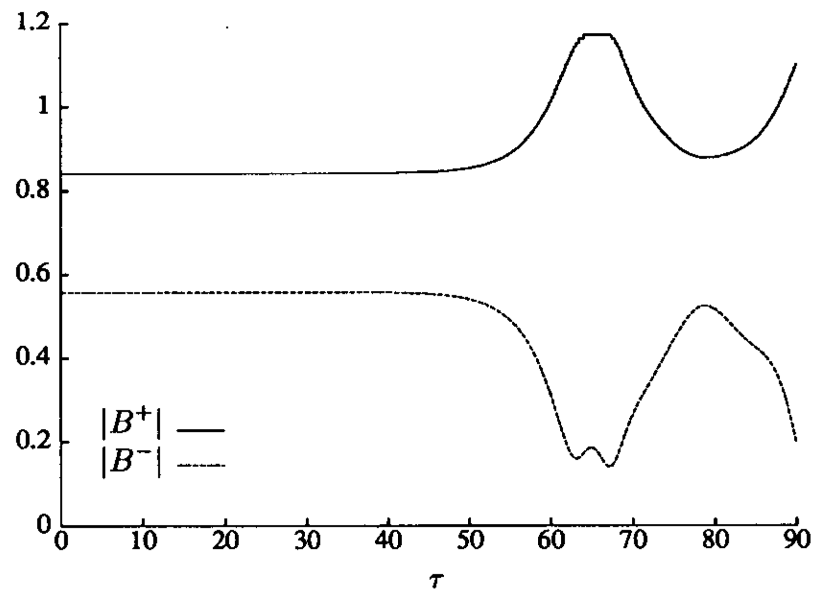

Figure 3.8: Peak values of $|B^+|$ and $|B^-|$ plotted against τ for $\hat{\kappa} = 0.2$, $\hat{\Delta} = -0.23308266$

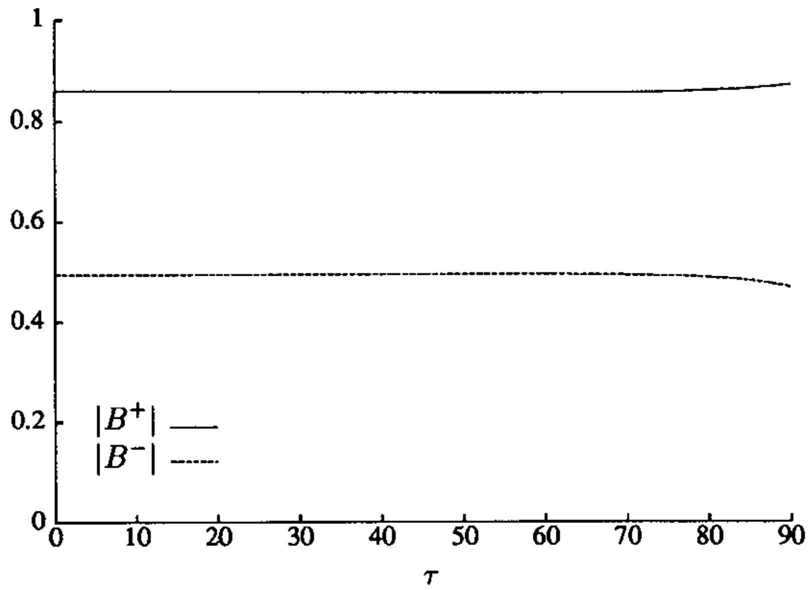

Figure 3.9: Peak values of $|B^+|$ and $|B^-|$ plotted against τ for $\widehat{\kappa} = 0.2$, $\widehat{\Delta} = -0.30203382$

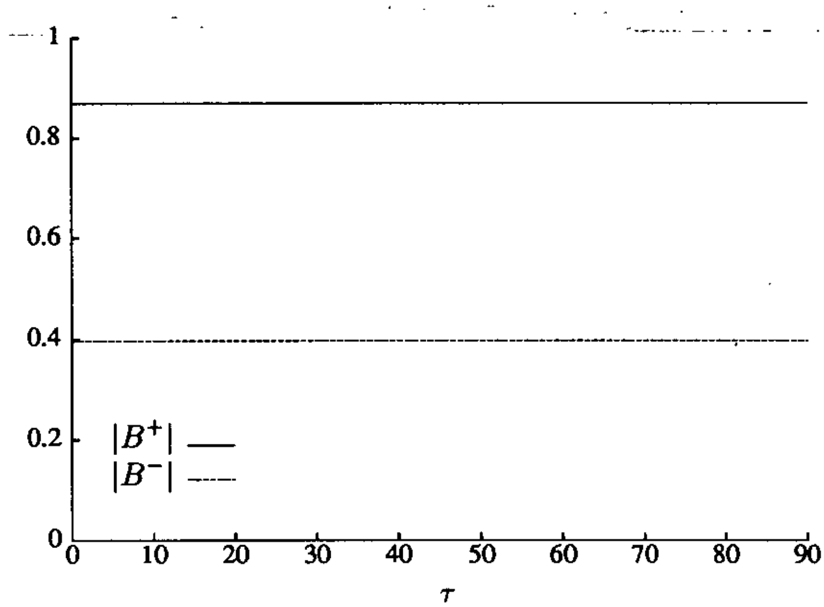

Figure 3.10: Peak values of $|B^+|$ and $|B^-|$ plotted against τ for $\widehat{\kappa} = 0.2$, $\widehat{\Delta} = -0.40127186$

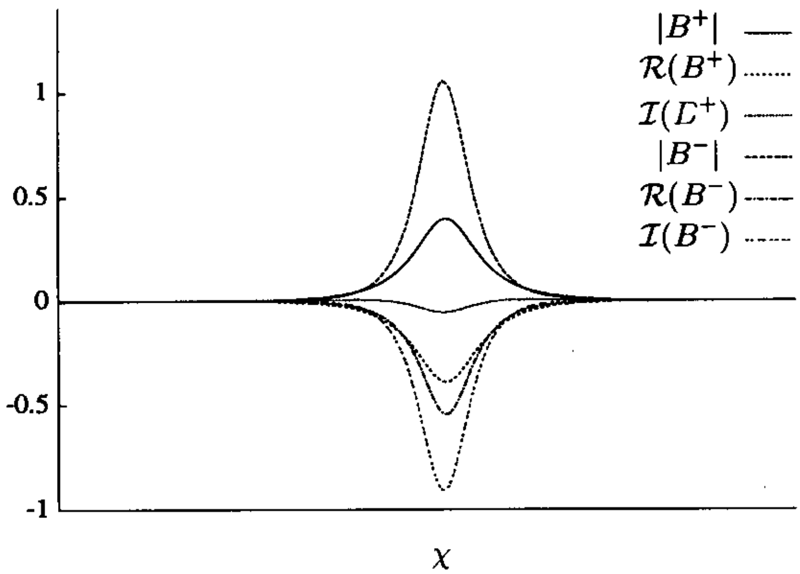

Figure 3.11: $B^{\pm}$ and $|B^{\pm}|$ at $\tau = 46$

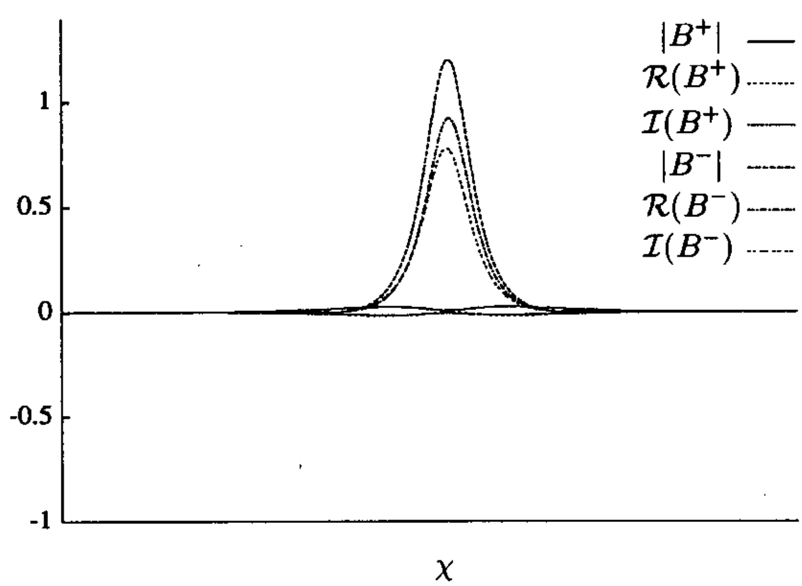

Figure 3.12: $B^{\pm}$ and $|B^{\pm}|$ at $\tau = 48.15$

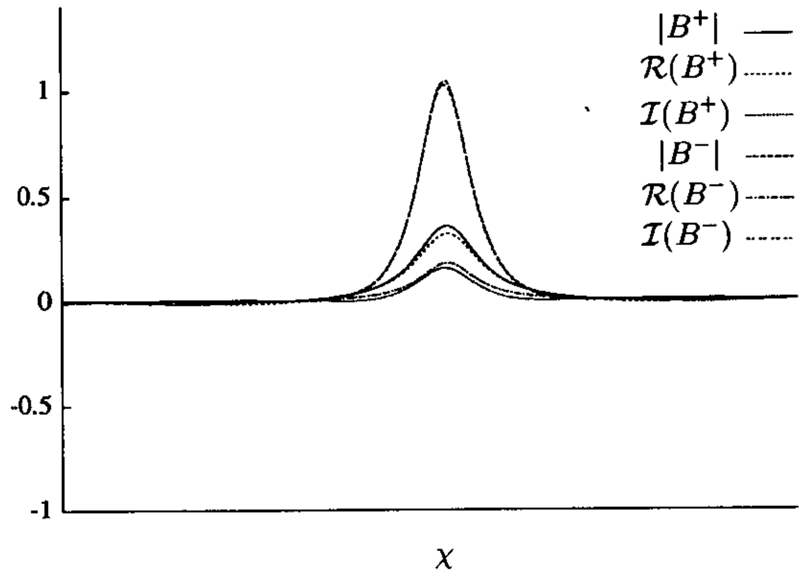

Figure 3.13: $B^{\pm}$ and $|B^{\pm}|$ at $\tau = 70$

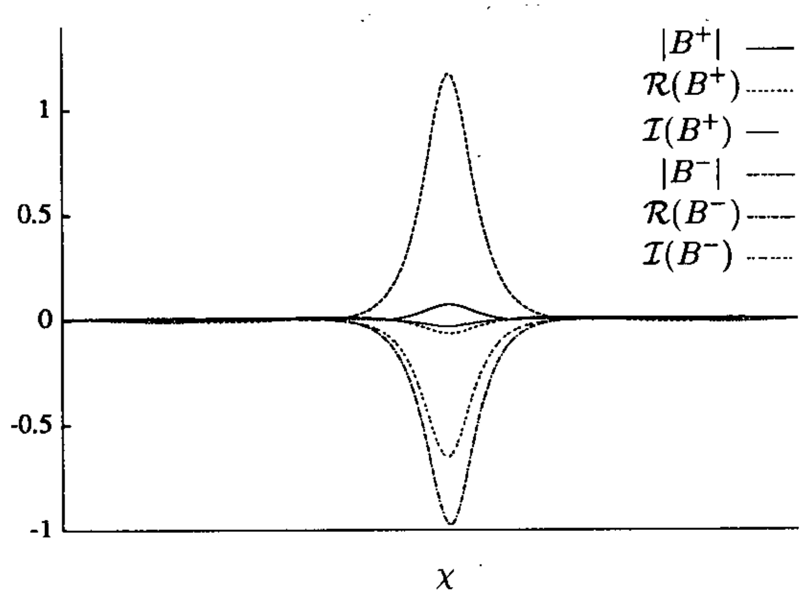

Figure 3.14: $B^{\pm}$ and $|B^{\pm}|$ at $\tau = 72$

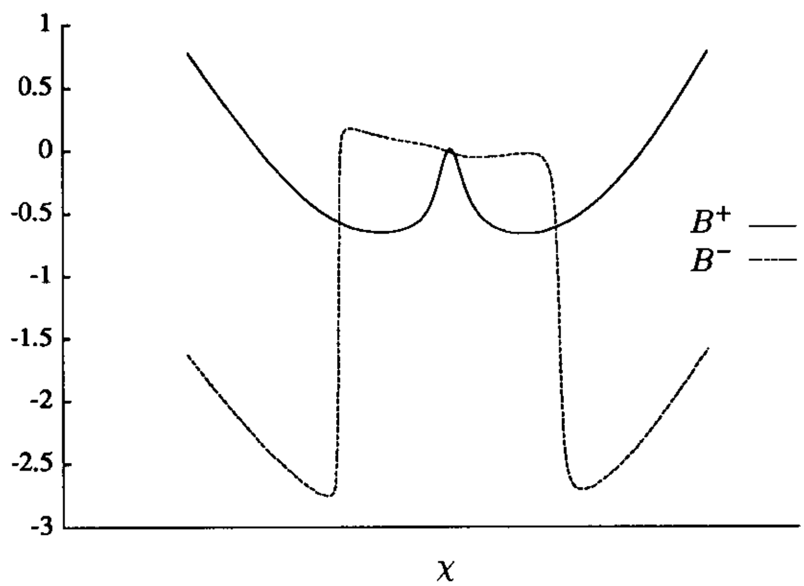

Figure 3.15: Phase of B^+ and B^- at $\tau = 48.15$

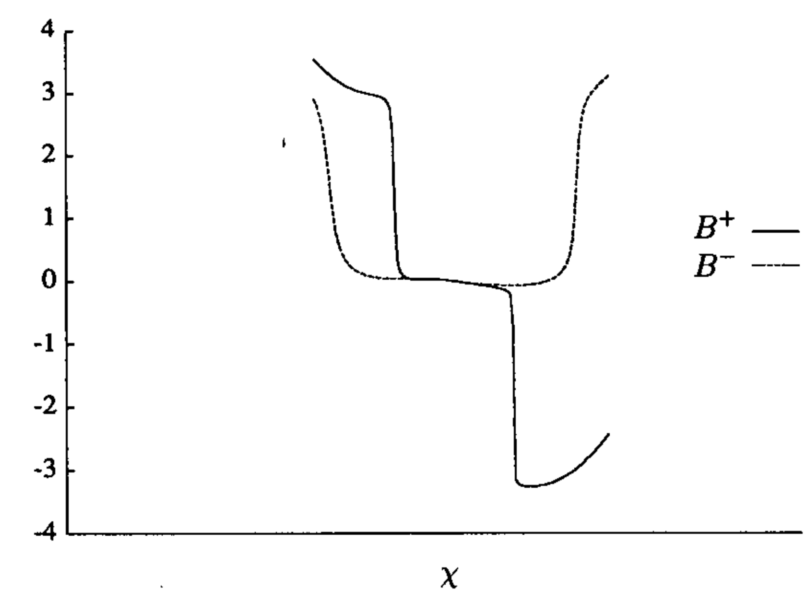

Figure 3.16: Phase of B^+ and B^- at $\tau = 72$

Chapter 4

Painlevé analysis of the coupled nonlinear Schrödinger equations

4.1 Preamble

The equations which govern the evolution of the two complex amplitudes of a pulse of light transmitted through an ideal, axisymmetric optical fibre with cubic nonlinearity are a coupled pair of nonlinear Schrödinger equations

$$\mathrm{i}A_t^{\pm} = A_{xx}^{\pm} + (|A^{\pm}|^2 + h|A^{\mp}|^2)A^{\pm}, \tag{4.1}$$

where A^+ and A^- are the complex amplitudes and h is the coupling parameter which is constant for an homogeneous waveguide (Parker and Newboult, 1989). For circular polarisation, $A^- = 0$, equations (4.1) reduce to

$$\mathrm{i}A_t^{+} = A_{xx}^{+} + |A^{+}|^2 A^{+}, \tag{4.2}$$

and for linearly polarised signals, $A^- = A^+ e^{\mathrm{i}2\alpha}$, equations (4.1) become

$$\mathrm{i}A_t^{+} = A_{xx}^{+} + (1+h)|A^{+}|^2 A^{+}. \tag{4.3}$$

Equations (4.2) and (4.3) are the single nonlinear Schrödinger equation which is known to be completely integrable by the inverse scattering method (Zakharov and Shabat, 1972). One consequence is that the solutions have no chaotic behaviour whatsoever, so the solutions are insensitive to the choice of initial conditions and do not display irregular behaviour over large scales of evolution.

Completely integrable partial differential equations typically possess many special properties, such as an inverse scattering transform, an infinite number of conservation laws, an auto-Bäcklund transform, a solution in terms of Hirota bilinear forms and soliton solutions. Also their similarity reductions give rise to ordinary differential equations of Painlevé type. For descriptions of these properties see, for example, Ablowitz and Segur (1981), Calogero and Degasperis (1982).

The inverse scattering method for solutions of partial differential equations of the type

$$\frac{\partial u}{\partial t} = F[u], \tag{4.4}$$

where F is a nonlinear differential operator in x, was discovered by Gardner, Green, Kruskal and Miura (1967), who showed that it was possible to reduce the Korteweg-de Vries equation to a linear integral equation.

Many nonlinear partial differential equations have been found to be solvable by the inverse scattering transform and hence are considered to be completely integrable. Ablowitz and Segur (1977) observed that all the similarity reductions obtained from partial differential equations that were known to be completely integrable led to ordinary differential equations of Painlevé type. The Painlevé conjecture, which provides a necessary condition for determining whether a partial differential equation is completely integrable, was first formulated by Ablowitz, Ramani and Segur (1978, 1980a,b). Weiss, Tabor and Carnevale (1983) then introduced the Painlevé property for partial differential equations, this replaced the Painlevé conjecture by a test that could be applied directly to partial differential equations or systems of partial differential equations, without the need to find all the similarity reductions leading to ordinary differential equations.

The Painlevé partial differential equation test was used by Steeb et al. (1984) to demonstrate that the single nonlinear Schrödinger equation (4.2), (4.3) was integrable by showing that the equation passed the Painlevé partial differential equation test. They also used the Painlevé approach to construct a Bäcklund transformation and solutions for the nonlinear Schrödinger equation. Painlevé analysis has also been applied to modified nonlinear Schrödinger equations to determine conditions for these equations to be completely integrable. For the generalised derivative nonlinear Schrödinger equation

$$\mathrm{i}u_t = u_{xx} + \mathrm{i}auu^*u_x + \mathrm{i}bu^2u_x^* + cu^3u^{*2}, \tag{4.5}$$

where a, b, c are real constants and u^* denotes the complex conjugate of u, Clarkson and Cosgrove (1987) found that this equation had the Painlevé property only if $c = b(2b-a)/4$. They also found that, if this condition on the coefficients was satisfied, then equation (4.5) could be transformed to the derivative nonlinear Schrödinger equation. Clarkson (1988) has also determined constraints on the coefficients of the damped, driven nonlinear Schrödinger equation

$$\mathrm{i}u_t + u_{xx} - 2|u|^2u = d(x,t)u + e(x,t)\,, \tag{4.6}$$

in order that it has the Painlevé property. These constraints are

$$\begin{aligned} d(x,t) &= x^2\left(\frac{1}{2}\frac{\mathrm{d}\beta}{\mathrm{d}t} - \beta^2\right) + \mathrm{i}\beta + x\alpha_1 + \alpha_0\,, \\ e(x,t) &= 0\,, \end{aligned}$$

where $\alpha_0(t)$, $\alpha_1(t)$ and $\beta(t)$ are arbitrary real functions of t. The variable coefficient nonlinear Schrödinger equation

$$\mathrm{i}u_t + g(t)u_{xx} + f(t)|u|^2u = 0 \tag{4.7}$$

has the Painlevé property if the condition

$$g(t) = f(t)\left[a_1\int^t g(s)ds + b_1\right] \tag{4.8}$$

is satisfied, where a_1, b_1 are constants (Joshi, 1988). This constraint for the integrability of the variable coefficient nonlinear Schrödinger equation had previously been shown (Grimshaw, 1979) to be equivalent to the condition (2.49) on the coefficients of equation (4.7) (i.e. (2.48)) in order that this equation could be reduced to the constant coefficient nonlinear Schrödinger equation. Likewise, the special cases of (4.6) and (4.7) which are completely integrable are those allowing transformations of the equations to the constant coefficient nonlinear Schrödinger equation.

The coupled constant coefficient nonlinear Schrödinger equations have been considered by Sahadevan, Tamizhmani and Lakshmanan (1986), who studied both a coupled pair of equations and a coupled system of N equations. For both of these cases they showed that the equations possessed the Painlevé property only for certain choices of the constant parameters. For the coupled pair this choice was identical to the restriction found by Zakharov and Schulman (1982).

In the following section we present a brief review of the developments which led to the definition of the Painlevé property. In the subsequent sections we apply the Painlevé test for partial differential equations to the coupled pair of constant coefficient cubic Schrödinger equations (4.1) to determine values for the coupling constant in order that the equations have the Painlevé property.

4.2 The Painlevé Property

At the beginning of this century, Painlevé and his colleagues sought to determine which nonlinear ordinary differential equations had only movable singularities which were poles. A review of their work is given in Ince (1944). A movable singularity is one in which the location of the singularity depends on the constant of integration, for example, consider the nonlinear ordinary differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} + w^2 = 0\,,$$

which has the general solution

$$w(z) = \frac{1}{z - z_0}\,,$$

with pole at $z = z_0$, where z_0 is an arbitrary constant. The other type of singularity, which is termed fixed, does not depend on the constants of integration, for example, the linear ordinary differential equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + a(z)\frac{\mathrm{d}w}{\mathrm{d}z} + b(z)w = 0$$

has the general solution

$$w(z) = Aw_1(z) + Bw_2(z)\,,$$

where A and B are arbitrary constants. In this case the location of any singularities will depend only on the particular form of the coefficients $a(z)$ and $b(z)$. Linear ordinary differential equations can have only fixed singularities whereas nonlinear ordinary differential equations can have both fixed and movable singularities.

For first order nonlinear ordinary differential equations the only equation which has no movable singularities except poles is the generalised Riccati equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} = f_0(z) + f_1(z)w + f_2(z)w^2\,,$$

where f_i, $(i = 0, 1, 2)$ are analytic in z. For second order nonlinear ordinary differential equations of the form

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = F\left(w, \frac{\mathrm{d}w}{\mathrm{d}z}, z\right)\,,$$

where F is rational in w and $\mathrm{d}w/\mathrm{d}z$ and analytic in z, Painlevé et al. found that there were fifty canonical equations whose only movable singularities were poles. Of these fifty, forty-four are integrable in terms of previously known functions. The remaining six, called the Painlevé transcendents, defined new transcendental functions.

In the 1970's Ablowitz and Segur (1977) demonstrated that all the similarity reductions obtained from several partial differential equations which were known to be completely integrable led to ordinary differential equations of Painlevé type, that is the only movable singularities of the solutions of the ordinary differential equations were poles. This observation led Ablowitz, Ramani and Segur (1980a,b) and McLeod and Olver (1983) to propose the following conjecture: 'every ordinary differential equation obtained by an exact reduction of a completely integrable partial differential equation is of Painlevé type', although a transformation of variables may be necessary. There has not been a full proof of this conjecture, although proofs have been given under certain conditions (Ablowitz et al., 1980a,b, McLeod and Olver, 1983).

For the Painlevé conjecture to be applied to a partial differential equation, *all* possible similarity reductions of the equation must be obtained and each of these reductions must be checked to see whether it is of Painlevé type. Weiss, Tabor and Carnevale (1983) defined the Painlevé property for partial differential equations. This allows the Painlevé conjecture to be applied directly to a partial differential equation and removes the need to find the similarity reductions. They proposed that a partial differential equation would possess the Painlevé property if the solutions of the partial differential equation are single-valued about a movable singularity manifold. Ward (1984) has shown that the singularity manifold must not be a characteristic. As in the case of the Painlevé conjecture, there have been no full proofs of the Painlevé property, although no failing cases have been identified so far.

For a partial differential equation with independent variables $z_1, z_2, \cdots, z_n$, it is assumed that in the vicinity of a singularity manifold

$$\phi(z_1, z_2, \cdots, z_n) = 0\,, \tag{4.9}$$

a solution $u(z_1, z_2, \cdots, z_n)$ may be expressed as a Laurent expansion

$$u(z_1, z_2, \cdots, z_n) = \phi^\alpha \sum_{j=0}^{\infty} u_j \phi^j, \tag{4.10}$$

where $\phi = \phi(z_1, z_2, \cdots, z_n)$ and $u_j = u_j(z_1, z_2, \cdots, z_n)$ are analytic functions of $z_1, z_2, \cdots, z_n$, in the neighbourhood of the singularity manifold (4.9). The partial differential equation is then said to possess the Painlevé property if α is a negative integer and the expansion (4.10) has as many arbitrary functions as are required by the Cauchy-Kowalevski theorem. The values of j in the series expansion (4.10) at which arbitrary functions arise are called resonances, and for each positive resonance one or more compatibility conditions are obtained. Satisfying these compatibility conditions may impose restrictions on the arbitrary functions and thereby reduce the number of these functions, if (4.10) is to be a solution of the partial differential equation. In such cases, the equation is said to fail the Painlevé test.

There are three main stages to be implemented when applying Painlevé analysis to a nonlinear partial differential equation: determining leading order behaviour, identifying the resonances, and verifying that the compatibility conditions which arise at the positive resonance values are identically satisfied so that the number of arbitrary functions is that required by the Cauchy-Kowalevski theorem.

4.3 Painlevé Analysis

In order that the Painlevé partial differential equation test can be applied to the coupled nonlinear Schrödinger equations

$$\begin{aligned} \mathrm{i}A_t &= A_{xx} + (|A|^2 + h|C|^2)A\,, \\ \mathrm{i}C_t &= C_{xx} + (h|A|^2 + |C|^2)C\,, \end{aligned} \tag{4.11}$$

where h is the coupling constant, all the variables must be complexified so that equations (4.11) can be written as the system of coupled equations

$$
\begin{aligned}
\mathrm{i}A_t &= A_{xx} + (AB + hCD)A\,,\\
-\mathrm{i}B_t &= B_{xx} + (AB + hCD)B\,,\\
\mathrm{i}C_t &= C_{xx} + (hAB + CD)C\,,\\
-\mathrm{i}D_t &= D_{xx} + (hAB + CD)D\,,
\end{aligned}
\tag{4.12}
$$

where $B = A^*$, $D = C^*$, and A, B, C, D are treated as independent complex functions of the complex variables x, t.

The Painlevé test of Weiss, Tabor and Carnevale (1983) defines the singularity manifold as

$$
\phi(x,t) = 0\,. \tag{4.13}
$$

Due to a simplification proposed by Kruskal the singularity manifold can be defined in the form

$$
\phi(x,t) = x - \psi(t) = 0\,, \tag{4.14}
$$

where $\psi(t)$ is an arbitrary analytic function of t (since $\phi = 0$ is not a characteristic (Ward, 1984), $\phi_x \neq 0$ and so, without loss of generality, it can be assumed that ϕ has the form (4.14)). A solution of the form

$$
\begin{aligned}
A(x,t) &= \phi^p \sum_{j=0}^{\infty} A_j(t)\phi^j, & B(x,t) &= \phi^q \sum_{j=0}^{\infty} B_j(t)\phi^j,\\
C(x,t) &= \phi^r \sum_{j=0}^{\infty} C_j(t)\phi^j, & D(x,t) &= \phi^s \sum_{j=0}^{\infty} D_j(t)\phi^j,
\end{aligned}
\tag{4.15}
$$

is then sought. For equations (4.12) to pass the Painlevé test p, q, r and s are required to be negative integers and the recursion relations which occur at the resonance values of j must be consistent so that the series (4.15) contain the correct number of arbitrary functions as required by the Cauchy-Kowalevski theorem. For the complexified nonlinear Schrödinger equations (4.12) the required number of arbitrary functions is eight.

Substituting (4.14) and (4.15) into the system of coupled equations (4.12) and comparing leading order terms gives the values $p = q = r = s = -1$ and

$$\begin{aligned} A_0(A_0B_0 + hC_0D_0 + 2) &= 0\,, \\ B_0(A_0B_0 + hC_0D_0 + 2) &= 0\,, \\ C_0(hA_0B_0 + C_0D_0 + 2) &= 0\,, \\ D_0(hA_0B_0 + C_0D_0 + 2) &= 0\,. \end{aligned} \tag{4.16}$$

From equations (4.16), it can be deduced that either $A_0 = B_0 = 0$ with $C_0D_0 = -2$, $C_0 = D_0 = 0$ with $A_0B_0 = -2$, or

$$\begin{aligned} A_0B_0 + hC_0D_0 &= -2\,, \\ hA_0B_0 + C_0D_0 &= -2\,. \end{aligned} \tag{4.17}$$

For the case $A_0 = B_0 = 0$, analysis of equations (4.12) at subsequent orders of ϕ gives $A_j = B_j = 0$ for $j \geq 1$ and hence $A = B = 0$, reducing the system of equations (4.11) to the single cubic Schrödinger equation which was shown to satisfy the Painlevé property by Steeb et al. (1984). A similar result is obtained for $C_0 = D_0 = 0$. For the third case equations (4.17) can be solved to give

$$A_0B_0 = C_0D_0 = \frac{-2}{1+h}\,, \tag{4.18}$$

if $h \neq \pm 1$. For $h = -1$, equations (4.17) are inconsistent, while for $h = 1$

$$A_0B_0 + C_0D_0 = -2\,. \tag{4.19}$$

By equating the coefficients of powers of ϕ^{j-3}, the general recursion relation

$$\begin{pmatrix} j^2 - 3j + A_0B_0 & A_0^2 & hA_0D_0 & hA_0C_0 \\ B_0^2 & j^2 - 3j + A_0B_0 & hB_0D_0 & hB_0C_0 \\ hB_0C_0 & hA_0C_0 & j^2 - 3j + C_0D_0 & C_0^2 \\ hB_0D_0 & hA_0D_0 & D_0^2 & j^2 - 3j + C_0D_0 \end{pmatrix} \begin{pmatrix} A_j \\ B_j \\ C_j \\ D_j \end{pmatrix} = \begin{pmatrix} a_j \\ b_j \\ c_j \\ d_j \end{pmatrix} \tag{4.20}$$

is obtained, where

$$a_j = -B_0\sum_{l=1}^{j-1} A_l A_{j-l} - \sum_{k=1}^{j-1}\sum_{l=0}^{k} A_l A_{k-l} B_{j-k} - hD_0\sum_{l=1}^{j-1} A_l C_{j-l} - h\sum_{k=1}^{j-1}\sum_{l=0}^{k} A_l C_{k-l} D_{j-k}$$
$$+\mathrm{i}\dot{A}_{j-2} - \mathrm{i}(j-2)A_{j-1}\dot{\psi}\,,$$

$$b_j = -A_0\sum_{l=1}^{j-1} B_l B_{j-l} - \sum_{k=1}^{j-1}\sum_{l=0}^{k} B_l B_{k-l} A_{j-k} - hC_0\sum_{l=1}^{j-1} B_l D_{j-l} - h\sum_{k=1}^{j-1}\sum_{l=0}^{k} B_l D_{k-l} C_{j-k}$$
$$-\mathrm{i}\dot{B}_{j-2} + \mathrm{i}(j-2)B_{j-1}\dot{\psi}\,,$$

$$c_j = -D_0\sum_{l=1}^{j-1} C_l C_{j-l} - \sum_{k=1}^{j-1}\sum_{l=0}^{k} C_l C_{k-l} D_{j-k} - hB_0\sum_{l=1}^{j-1} C_l A_{j-l} - h\sum_{k=1}^{j-1}\sum_{l=0}^{k} C_l A_{k-l} B_{j-k}$$
$$+\mathrm{i}\dot{C}_{j-2} - \mathrm{i}(j-2)C_{j-1}\dot{\psi}\,,$$

$$d_j = -C_0\sum_{l=1}^{j-1} D_l D_{j-l} - \sum_{k=1}^{j-1}\sum_{l=0}^{k} D_l D_{k-l} C_{j-k} - hA_0\sum_{l=1}^{j-1} D_l B_{j-l} - h\sum_{k=1}^{j-1}\sum_{l=0}^{k} D_l B_{k-l} A_{j-k}$$
$$-\mathrm{i}\dot{D}_{j-2} + \mathrm{i}(j-2)D_{j-1}\dot{\psi}\,,$$

for $j \geq 1$, and $A_j = B_j = C_j = D_j = 0$, for $j < 0$. Equations (4.20) uniquely define A_j, B_j, C_j, D_j unless

$$\begin{vmatrix} j^2 - 3j + A_0 B_0 & A_0^2 & hA_0 D_0 & hA_0 C_0 \\ B_0^2 & j^2 - 3j + A_0 B_0 & hB_0 D_0 & hB_0 C_0 \\ hB_0 C_0 & hA_0 C_0 & j^2 - 3j + C_0 D_0 & C_0^2 \\ hB_0 D_0 & hA_0 D_0 & D_0^2 & j^2 - 3j + C_0 D_0 \end{vmatrix} = 0\,.$$

Expanding out this determinant gives an equation for the resonance values of j,

$$j^2(j-3)^2\left[(j^2-3j)^2 + 2(A_0B_0 + C_0D_0)(j^2-3j) + 4(1-h^2)A_0B_0C_0D_0\right] = 0\,, \tag{4.21}$$

which applies for any value of h.

If $h = 1$, then equation (4.21) gives the resonance values

$$j = -1, 0, 0, 0, 3, 3, 3, 4\,.$$

For $h \neq 1$, we find that by using equations (4.18) in equation (4.21) the resonance values are given by

$$(j+1)j^2(j-3)^2(j-4)\left(j^2 - 3j - 4\frac{1-h}{1+h}\right) = 0\,,$$

and hence

$$j = -1, 0, 0, 3, 3, 4, \frac{3}{2} \pm \frac{1}{2}\sqrt{\frac{25 - 7h}{1 + h}},$$

for resonance. It can be seen that two of the resonance values of j will depend on the value of the coupling parameter h. For equations (4.12) to be integrable by the Painlevé property, j is required to be an integer, this means that $\sqrt{(25 - 7h)/(1 + h)}$ must be an odd integer. The values of h which give a positive integer for j are $h = 0$ (an uncoupled system of nonlinear Schrödinger equations), which have resonance values $j = -1, 0, 3, 4$ for each equation), and $h = 3$ with resonance values $j = -1, 0, 0, 1, 2, 3, 3, 4$. Values of $h < 0$ are not considered as they have no physical relevance. If h takes a value other than these outlined above then the final two resonance values are not integers so that the coupled nonlinear Schrödinger system will not, in general, be integrable as there are not enough arbitrary functions to satisfy the Painlevé partial differential equation test. Thus there are three cases to consider

(i) $h = 1$ with resonances at $j = -1, 0, 0, 0, 3, 3, 3, 4$,

(ii) $h = 3$ with resonances at $j = -1, 0, 0, 1, 2, 3, 3, 4$,

(iii) $h \neq 1, 3$ and $h > 0$ with resonances at $j = -1, 0, 0, 3, 3, 4$.

The resonance at $j = -1$ corresponds to the arbitrary function $\psi(t)$ and for this system of equations to have the Painlevé property all other resonance values must be non-negative integers. This means that case (iii), which has only six integer resonance values, cannot generate more than six arbitrary functions and therefore does not pass the Painlevé test and so equations (4.11) are not, in general, integrable. In the following section the first two cases are studied to identify whether or not they have the Painlevé property.

4.4 Arbitrary functions

To determine if the correct number of arbitrary functions exist for the Painlevé property to be satisfied, equations (4.20) are solved for values of j which are not resonance values. For values of j which are resonance values, conditions on a_j, b_j, c_j and d_j are found so that equations (4.20) are consistent, these compatibility conditions may impose constraints on some of the arbitrary functions.

4.4.1 Case (i): Coupling coefficient $h = 1$

The resonance values corresponding to $h = 1$ are $j = -1, 0, 0, 0, 3, 3, 3, 4$. Leading order analysis gives a single equation relating the functions A_0, B_0, C_0 and D_0,

$$A_0B_0 + C_0D_0 = -2\,, \tag{4.22}$$

so three of these functions B_0, C_0 and D_0, say, will be arbitrary.

For $j = 1$ or $j = 2$, equations (4.20) become

$$\begin{pmatrix} A_0B_0 - 2 & A_0^2 & A_0D_0 & A_0C_0 \\ B_0^2 & A_0B_0 - 2 & B_0D_0 & B_0C_0 \\ B_0C_0 & A_0C_0 & C_0D_0 - 2 & C_0^2 \\ B_0D_0 & A_0D_0 & D_0^2 & C_0D_0 - 2 \end{pmatrix} \begin{pmatrix} A_j \\ B_j \\ C_j \\ D_j \end{pmatrix} = \begin{pmatrix} a_j \\ b_j \\ c_j \\ d_j \end{pmatrix}. \tag{4.23}$$

For $j = 1$

$$\begin{aligned} a_1 &= \mathrm{i}A_0\dot{\psi}\,, & b_1 &= -\mathrm{i}B_0\dot{\psi}\,, \\ c_1 &= \mathrm{i}C_0\dot{\psi}\,, & d_1 &= -\mathrm{i}D_0\dot{\psi}\,, \end{aligned}$$

then solving equations (4.23) gives

$$\frac{A_1}{A_0} = -\frac{B_1}{B_0} = \frac{C_1}{C_0} = -\frac{D_1}{D_0} = -\frac{\mathrm{i}}{2}\dot{\psi}\,.$$

For $j = 2$

$$\begin{aligned} a_2 &= \frac{1}{2}A_0\dot{\psi}^2 + \mathrm{i}\dot{A}_0\,, & b_2 &= \frac{1}{2}B_0\dot{\psi}^2 - \mathrm{i}\dot{B}_0\,, \\ c_2 &= \frac{1}{2}C_0\dot{\psi}^2 + \mathrm{i}\dot{C}_0\,, & d_2 &= \frac{1}{2}D_0\dot{\psi}^2 - \mathrm{i}\dot{D}_0\,, \end{aligned}$$

then equations (4.23) give

$$
\begin{aligned}
\frac{A_2}{A_0} &= -\frac{1}{12}\dot{\psi}^2 - \frac{\mathrm{i}}{2}\frac{\dot{A}_0}{A_0} - \frac{\mathrm{i}}{12}(B_0\dot{A}_0 - A_0\dot{B}_0 + D_0\dot{C}_0 - C_0\dot{D}_0)\,, \\
\frac{B_2}{B_0} &= -\frac{1}{12}\dot{\psi}^2 + \frac{\mathrm{i}}{2}\frac{\dot{B}_0}{B_0} - \frac{\mathrm{i}}{12}(B_0\dot{A}_0 - A_0\dot{B}_0 + D_0\dot{C}_0 - C_0\dot{D}_0)\,, \\
\frac{C_2}{C_0} &= -\frac{1}{12}\dot{\psi}^2 - \frac{\mathrm{i}}{2}\frac{\dot{C}_0}{C_0} - \frac{\mathrm{i}}{12}(B_0\dot{A}_0 - A_0\dot{B}_0 + D_0\dot{C}_0 - C_0\dot{D}_0)\,, \\
\frac{D_2}{D_0} &= -\frac{1}{12}\dot{\psi}^2 + \frac{\mathrm{i}}{2}\frac{\dot{D}_0}{D_0} - \frac{\mathrm{i}}{12}(B_0\dot{A}_0 - A_0\dot{B}_0 + D_0\dot{C}_0 - C_0\dot{D}_0)\,.
\end{aligned}
$$

For $j = 3$, equations (4.20) become

$$
\begin{pmatrix} B_0 & A_0 & D_0 & C_0 \\ B_0 & A_0 & D_0 & C_0 \\ B_0 & A_0 & D_0 & C_0 \\ B_0 & A_0 & D_0 & C_0 \end{pmatrix}
\begin{pmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{pmatrix}
=
\begin{pmatrix} a_3/A_0 \\ b_3/B_0 \\ c_3/C_0 \\ d_3/D_0 \end{pmatrix}\,, \tag{4.24}
$$

where

$$
\frac{a_3}{A_0} = \frac{b_3}{B_0} = \frac{c_3}{C_0} = \frac{d_3}{D_0} = \frac{1}{2}\ddot{\psi}\,.
$$

Equations (4.24) are therefore consistent and the equation

$$
B_0A_3 + A_0B_3 + D_0C_3 + C_0D_3 = \frac{a_3}{A_0} = \frac{1}{2}\ddot{\psi} \tag{4.25}
$$

is the only restriction on the functions A_3, B_3, C_3, D_3. By introducing the three arbitrary functions α_3, β_3 and γ_3, a solution to equation (4.25) can be written as

$$
\begin{aligned}
A_3B_0 &= \frac{1}{8}\ddot{\psi} + \alpha_3 + \beta_3\,, \\
B_3A_0 &= \frac{1}{8}\ddot{\psi} + \alpha_3 - \beta_3\,, \\
C_3D_0 &= \frac{1}{8}\ddot{\psi} - \alpha_3 + \gamma_3\,, \\
D_3C_0 &= \frac{1}{8}\ddot{\psi} - \alpha_3 - \gamma_3\,.
\end{aligned}
$$

For $j = 4$, equations (4.20) become

$$
\begin{pmatrix} A_0B_0 + 4 & A_0^2 & A_0D_0 & A_0C_0 \\ B_0^2 & A_0B_0 + 4 & B_0D_0 & B_0C_0 \\ B_0C_0 & A_0C_0 & C_0D_0 + 4 & C_0^2 \\ B_0D_0 & A_0D_0 & D_0^2 & C_0D_0 + 4 \end{pmatrix}
\begin{pmatrix} A_4 \\ B_4 \\ C_4 \\ D_4 \end{pmatrix}
=
\begin{pmatrix} a_4 \\ b_4 \\ c_4 \\ d_4 \end{pmatrix}\,, \tag{4.26}
$$

where

$$
\begin{aligned}
\frac{a_4}{A_0} &= \frac{1}{12}\dot{G} + \frac{G^2}{24} + G\left(\frac{1}{6}\frac{\dot{A}_0}{A_0} - \frac{\mathrm{i}}{24}\ddot{\psi}\right) - \frac{1}{4}\left(\dot{A}_0\dot{B}_0 + \dot{C}_0\dot{D}_0\right) \\
&\quad - \frac{\mathrm{i}}{A_0B_0}\dot{\psi}\left[\left(\frac{A_0B_0}{4} - \frac{2}{3}\right)\ddot{\psi} - 2\alpha_3 - C_0D_0\beta_3 - A_0B_0\gamma_3\right] \\
&\quad - \frac{\mathrm{i}\dot{\psi}^2}{6}\frac{\dot{A}_0}{A_0} + \frac{1}{2}\frac{\ddot{A}_0}{A_0}, \\
\frac{b_4}{B_0} &= -\frac{1}{12}\dot{G} + \frac{G^2}{24} - G\left(\frac{1}{6}\frac{\dot{B}_0}{B_0} - \frac{\mathrm{i}}{24}\dot{\psi}^2\right) - \frac{1}{4}\left(\dot{A}_0\dot{B}_0 + \dot{C}_0\dot{D}_0\right) \\
&\quad - \frac{\mathrm{i}\dot{\psi}}{A_0B_0}\left[\left(\frac{A_0B_0}{4} - \frac{2}{3}\right)\ddot{\psi} + 2\alpha_3 + C_0D_0\beta_3 + A_0B_0\gamma_3\right] \\
&\quad + \frac{\mathrm{i}\dot{\psi}^2}{6}\frac{\dot{B}_0}{B_0} + \frac{1}{2}\frac{\ddot{B}_0}{B_0}, \\
\frac{c_4}{C_0} &= \frac{1}{12}\dot{G} + \frac{G^2}{24} + G\left(\frac{1}{6}\frac{\dot{C}_0}{C_0} - \frac{\mathrm{i}}{24}\ddot{\psi}\right) - \frac{1}{4}\left(\dot{A}_0\dot{B}_0 + \dot{C}_0\dot{D}_0\right) \\
&\quad - \frac{\mathrm{i}}{C_0D_0}\dot{\psi}\left[\left(\frac{C_0D_0}{4} - \frac{2}{3}\right)\ddot{\psi} - 2\alpha_3 - A_0B_0\beta_3 - C_0D_0\gamma_3\right] \\
&\quad - \frac{\mathrm{i}\dot{\psi}^2}{6}\frac{\dot{C}_0}{C_0} + \frac{1}{2}\frac{\ddot{C}_0}{C_0}, \\
\frac{d_4}{D_0} &= -\frac{1}{12}\dot{G} + \frac{G^2}{24} - G\left(\frac{1}{6}\frac{\dot{D}_0}{D_0} - \frac{\mathrm{i}}{24}\dot{\psi}^2\right) - \frac{1}{4}\left(\dot{A}_0\dot{B}_0 + \dot{C}_0\dot{D}_0\right) \\
&\quad - \frac{\mathrm{i}\dot{\psi}}{C_0D_0}\left[\left(\frac{C_0D_0}{4} - \frac{2}{3}\right)\ddot{\psi} + 2\alpha_3 + A_0B_0\beta_3 + C_0D_0\gamma_3\right] \\
&\quad + \frac{\mathrm{i}\dot{\psi}^2}{6}\frac{\dot{D}_0}{D_0} + \frac{1}{2}\frac{\ddot{D}_0}{D_0},
\end{aligned}
$$

with

$$G = \dot{A}_0B_0 - \dot{B}_0A_0 + \dot{C}_0D_0 - \dot{D}_0C_0\,.$$

Equations (4.26) are compatible if

$$B_0a_4 + A_0b_4 + D_0c_4 + C_0d_4 = 0\,.$$

Substituting a_4, b_4, c_4 and d_4 into the above condition, we find that it is identically satisfied. Hence, by introducing the arbitrary function α_4, a solution to the system (4.26) can be found as

$$A_4 = -\frac{1}{4}\frac{b_4}{B_0} + \alpha_4\,, \qquad B_4 = -\frac{1}{4}\frac{a_4}{A_0} + \alpha_4\,,$$

$$C_4 = -\frac{1}{4}\frac{d_4}{D_0} + \alpha_4\,, \qquad D_4 = -\frac{1}{4}\frac{c_4}{C_0} + \alpha_4\,.$$

Hence, in solving the system (4.23), (4.24) and (4.26) it is found that there are eight arbitrary functions ψ, B_0, C_0, D_0, α_3, β_3, γ_3 and α_4. Since the system (4.12) has order eight, so making (4.21) of degree eight, the coupled nonlinear Schrödinger equations (4.11) possess the Painlevé property for the special case $h = 1$. This agrees with the result of Sahadevan et al. (1986) that (4.11) is completely integrable for $h = 1$ and with the existence of an inverse scattering transform (Zakharov and Schulman, 1982) in this special case.

4.4.2 Case (ii): Coupling coefficient $h = 3$

For this case, the resonance values are $j = -1, 0, 0, 1, 2, 3, 3, 4$. The leading order analysis (4.17) gives

$$\begin{aligned} A_0B_0 + 3C_0D_0 &= -2\,, \\ 3A_0B_0 + C_0D_0 &= -2\,, \end{aligned}$$

which can be solved for the products A_0B_0 and C_0D_0 to give

$$A_0B_0 = C_0D_0 = -\frac{1}{2}\,.$$

This allows two arbitrary functions to be introduced, A_0 and C_0 say, which correspond to the two resonance values at $j = 0$. For $j = 1, 2$ equations (4.20) become

$$\begin{pmatrix} A_0B_0 - 2 & A_0^2 & 3A_0D_0 & 3A_0C_0 \\ B_0^2 & A_0B_0 - 2 & 3B_0D_0 & 3B_0C_0 \\ 3B_0C_0 & 3A_0C_0 & C_0D_0 - 2 & C_0^2 \\ 3B_0D_0 & 3A_0D_0 & D_0^2 & C_0D_0 - 2 \end{pmatrix} \begin{pmatrix} A_j \\ B_j \\ C_j \\ D_j \end{pmatrix} = \begin{pmatrix} a_j \\ b_j \\ c_j \\ d_j \end{pmatrix}, \qquad (4.27)$$

and for these equations to be consistent we require

$$\frac{a_j}{A_0} + \frac{b_j}{B_0} = \frac{c_j}{C_0} + \frac{d_j}{D_0} \qquad \text{for} \quad j = 1, 2 . \tag{4.28}$$

Since

$$\frac{a_1}{A_0} = -\frac{b_1}{B_0} = \frac{c_1}{C_0} = -\frac{d_1}{D_0} = \mathrm{i}\dot{\psi} ,$$

condition (4.28) is automatically satisfied for $j = 1$, giving the solution

$$\begin{aligned} \frac{A_1}{A_0} &= -\frac{\mathrm{i}}{2}\dot{\psi} + \alpha_1 , & \frac{B_1}{B_0} &= \frac{\mathrm{i}}{2}\dot{\psi} + \alpha_1 , \\ \frac{C_1}{C_0} &= -\frac{\mathrm{i}}{2}\dot{\psi} - \alpha_1 , & \frac{D_1}{D_0} &= \frac{\mathrm{i}}{2}\dot{\psi} - \alpha_1 , \end{aligned}$$

where $\alpha_1(t)$ is an arbitrary function. Using these expressions we find

$$\begin{aligned} \frac{a_2}{A_0} &= \frac{1}{2}\dot{\psi}^2 + \mathrm{i}\alpha_1\dot{\psi} + \mathrm{i}\frac{\dot{A}_0}{A_0} , \\ \frac{b_2}{B_0} &= \frac{1}{2}\dot{\psi}^2 - \mathrm{i}\alpha_1\dot{\psi} + \mathrm{i}\frac{\dot{A}_0}{A_0} , \\ \frac{c_2}{C_0} &= \frac{1}{2}\dot{\psi}^2 + \mathrm{i}\alpha_1\dot{\psi} + \mathrm{i}\frac{\dot{C}_0}{C_0} , \\ \frac{d_2}{D_0} &= \frac{1}{2}\dot{\psi}^2 - \mathrm{i}\alpha_1\dot{\psi} + \mathrm{i}\frac{\dot{C}_0}{C_0} , \end{aligned}$$

so showing that the compatibility condition (4.28) is satisfied for $j = 2$ only if the two free functions A_0 and C_0 satisfy

$$\frac{\dot{A}_0}{A_0} = \frac{\dot{C}_0}{C_0} .$$

This implies that

$$C_0(t) = kA_0(t) ,$$

where k is a constant. This restriction reduces the number of arbitrary functions which were introduced at $j = 0$ to one. If we introduce the arbitrary function $F(t)$, such that

$$A_0 = \frac{1}{\sqrt{2}}F , \qquad B_0 = -\frac{1}{\sqrt{2}}F^{-1}, \qquad C_0 = \frac{1}{\sqrt{2}}kF , \qquad D_0 = -\frac{1}{\sqrt{2}}(kF)^{-1}.$$

Then we find

$$\frac{\dot{A}_0}{A_0} = -\frac{\dot{B}_0}{B_0} = \frac{\dot{C}_0}{C_0} = -\frac{\dot{D}_0}{D_0} = \frac{\dot{F}}{F},$$

and hence solutions to (4.27) can be found in terms of an arbitrary function $\alpha_2(t)$ as

$$\begin{aligned}
\frac{A_2}{A_0} &= -\frac{\dot{\psi}^2}{12} - \frac{\mathrm{i}}{2}\dot{\psi}\alpha_1 - \frac{\mathrm{i}}{6}\frac{\dot{F}}{F} + \alpha_2\,, \\
\frac{B_2}{B_0} &= -\frac{\dot{\psi}^2}{12} + \frac{\mathrm{i}}{2}\dot{\psi}\alpha_1 - \frac{\mathrm{i}}{6}\frac{\dot{F}}{F} + \alpha_2\,, \\
\frac{C_2}{C_0} &= -\frac{\dot{\psi}^2}{12} + \frac{\mathrm{i}}{2}\dot{\psi}\alpha_1 - \frac{\mathrm{i}}{6}\frac{\dot{F}}{F} - \alpha_2\,, \\
\frac{D_2}{D_0} &= -\frac{\dot{\psi}^2}{12} - \frac{\mathrm{i}}{2}\dot{\psi}\alpha_1 - \frac{\mathrm{i}}{6}\frac{\dot{F}}{F} - \alpha_2\,.
\end{aligned}$$

Setting $j = 3$, equations (4.20) become

$$\begin{pmatrix} A_0B_0 & A_0^2 & 3A_0D_0 & 3A_0C_0 \\ B_0^2 & A_0B_0 & 3B_0D_0 & 3B_0C_0 \\ 3B_0C_0 & 3A_0C_0 & C_0D_0 & C_0^2 \\ 3B_0D_0 & 3A_0D_0 & D_0^2 & C_0D_0 \end{pmatrix} \begin{pmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{pmatrix} = \begin{pmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{pmatrix}, \tag{4.29}$$

and for these equations to be consistent, the conditions

$$\frac{a_3}{A_0} = \frac{b_3}{B_0}, \qquad \frac{c_3}{C_0} = \frac{d_3}{D_0},$$

must be satisfied, where

$$\begin{aligned}
\frac{a_3}{A_0} &= \frac{1}{2}\ddot{\psi} + \mathrm{i}\dot{\alpha}_1 - \frac{1}{2}\alpha_1\dot{\psi}^2 + 2\alpha_1^3 + \mathrm{i}\alpha_1\frac{\dot{F}}{F}\,, \\
\frac{b_3}{B_0} &= \frac{1}{2}\ddot{\psi} - \mathrm{i}\dot{\alpha}_1 - \frac{1}{2}\alpha_1\dot{\psi}^2 + 2\alpha_1^3 + \mathrm{i}\alpha_1\frac{\dot{F}}{F}\,, \\
\frac{c_3}{C_0} &= \frac{1}{2}\ddot{\psi} - \mathrm{i}\dot{\alpha}_1 + \frac{1}{2}\alpha_1\dot{\psi}^2 - 2\alpha_1^3 - \mathrm{i}\alpha_1\frac{\dot{F}}{F}\,, \\
\frac{d_3}{D_0} &= \frac{1}{2}\ddot{\psi} + \mathrm{i}\dot{\alpha}_1 + \frac{1}{2}\alpha_1\dot{\psi}^2 - 2\alpha_1^3 - \mathrm{i}\alpha_1\frac{\dot{F}}{F}\,.
\end{aligned}$$

In order that these compatibility conditions are satisfied a restriction on the arbitrary function α_1 is imposed

$$\dot{\alpha}_1 = 0\,.$$

A solution to equations (4.29) can be found by introducing two arbitrary functions $\alpha_3(t)$ and $\gamma_3(t)$, such that

$$\begin{aligned}
\frac{A_3}{A_0} &= -\frac{1}{8}\ddot{\psi} - \frac{1}{4}\alpha_1\dot{\psi}^2 + \alpha_1^3 + \mathrm{i}\frac{\alpha_1}{2}\frac{\dot{F}}{F} + \alpha_3\,,\\
\frac{B_3}{B_0} &= -\frac{1}{8}\ddot{\psi} - \frac{1}{4}\alpha_1\dot{\psi}^2 + \alpha_1^3 + \mathrm{i}\frac{\alpha_1}{2}\frac{\dot{F}}{F} - \alpha_3\,,\\
\frac{C_3}{C_0} &= -\frac{1}{8}\ddot{\psi} + \frac{1}{4}\alpha_1\dot{\psi}^2 - \alpha_1^3 - \mathrm{i}\frac{\alpha_1}{2}\frac{\dot{F}}{F} + \gamma_3\,,\\
\frac{D_3}{D_0} &= -\frac{1}{8}\ddot{\psi} + \frac{1}{4}\alpha_1\dot{\psi}^2 - \alpha_1^3 - \mathrm{i}\frac{\alpha_1}{2}\frac{\dot{F}}{F} - \gamma_3\,.
\end{aligned}$$

For $j = 4$, equations (4.20) become

$$\begin{pmatrix} A_0B_0+4 & A_0^2 & 3A_0D_0 & 3A_0C_0 \\ B_0^2 & A_0B_0+4 & 3B_0D_0 & 3B_0C_0 \\ 3B_0C_0 & 3A_0C_0 & C_0D_0+4 & C_0^2 \\ 3B_0D_0 & 3A_0D_0 & D_0^2 & C_0D_0+4 \end{pmatrix}\begin{pmatrix} A_4 \\ B_4 \\ C_4 \\ D_4 \end{pmatrix} = \begin{pmatrix} a_4 \\ b_4 \\ c_4 \\ d_4 \end{pmatrix}, \qquad (4.30)$$

where

$$\begin{aligned}
\frac{a_4}{A_0} &= \mathrm{i}\alpha_2\frac{\dot{F}}{F} - \frac{3}{2}\mathrm{i}\dot{\psi}(\alpha_3-\gamma_3) + \frac{1}{2}\dot{\psi}^2\alpha_2 + 6\alpha_1^2\alpha_2 + \frac{1}{2}\ddot{\psi}\alpha_1 + \frac{\mathrm{i}}{3}\dot{\psi}^3\alpha_1 - 2\mathrm{i}\dot{\psi}\alpha_1^3\\
&\quad -\mathrm{i}\dot{\psi}\alpha_1\alpha_2 + \frac{\mathrm{i}}{3}\dot{\psi}\ddot{\psi} + \mathrm{i}\dot{\alpha}_2 - \frac{1}{6}\left(\frac{\dot{F}}{F}\right)^2 + \frac{5}{6}\frac{\dot{F}}{F}\dot{\psi}\alpha_1 - 2\alpha_1\alpha_3 + \frac{1}{6}\frac{\ddot{F}}{F}\,,\\
\frac{b_4}{B_0} &= \mathrm{i}\alpha_2\frac{\dot{F}}{F} - \frac{3}{2}\mathrm{i}\dot{\psi}(\alpha_3-\gamma_3) + \frac{1}{2}\dot{\psi}^2\alpha_2 + 6\alpha_1^2\alpha_2 + \frac{1}{2}\ddot{\psi}\alpha_1 - \frac{\mathrm{i}}{3}\dot{\psi}^3\alpha_1 + 2\mathrm{i}\dot{\psi}\alpha_1^3\\
&\quad +\mathrm{i}\dot{\psi}\alpha_1\alpha_2 - \frac{\mathrm{i}}{3}\dot{\psi}\ddot{\psi} - \mathrm{i}\dot{\alpha}_2 + \frac{1}{6}\left(\frac{\dot{F}}{F}\right)^2 - \frac{5}{6}\frac{\dot{F}}{F}\dot{\psi}\alpha_1 + 2\alpha_1\alpha_3 - \frac{1}{6}\frac{\ddot{F}}{F}\,,\\
\frac{c_4}{C_0} &= -\mathrm{i}\alpha_2\frac{\dot{F}}{F} - \frac{3}{2}\mathrm{i}\dot{\psi}(\gamma_3-\alpha_3) - \frac{1}{2}\dot{\psi}^2\alpha_2 - 6\alpha_1^2\alpha_2 - \frac{1}{2}\ddot{\psi}\alpha_1 - \frac{\mathrm{i}}{3}\dot{\psi}^3\alpha_1 + 2\mathrm{i}\dot{\psi}\alpha_1^3\\
&\quad -\mathrm{i}\dot{\psi}\alpha_1\alpha_2 + \frac{\mathrm{i}}{3}\dot{\psi}\ddot{\psi} - \mathrm{i}\dot{\alpha}_2 - \frac{1}{6}\left(\frac{\dot{F}}{F}\right)^2 - \frac{5}{6}\frac{\dot{F}}{F}\dot{\psi}\alpha_1 + 2\alpha_1\alpha_3 + \frac{1}{6}\frac{\ddot{F}}{F}\,,\\
\frac{d_4}{D_0} &= -\mathrm{i}\alpha_2\frac{\dot{F}}{F} - \frac{3}{2}\mathrm{i}\dot{\psi}(\gamma_3-\alpha_3) - \frac{1}{2}\dot{\psi}^2\alpha_2 - 6\alpha_1^2\alpha_2 - \frac{1}{2}\ddot{\psi}\alpha_1 + \frac{\mathrm{i}}{3}\dot{\psi}^3\alpha_1 - 2\mathrm{i}\dot{\psi}\alpha_1^3\\
&\quad +\mathrm{i}\dot{\psi}\alpha_1\alpha_2 - \frac{\mathrm{i}}{3}\dot{\psi}\ddot{\psi} + \mathrm{i}\dot{\alpha}_2 + \frac{1}{6}\left(\frac{\dot{F}}{F}\right)^2 + \frac{5}{6}\frac{\dot{F}}{F}\dot{\psi}\alpha_1 - 2\alpha_1\alpha_3 - \frac{1}{6}\frac{\ddot{F}}{F}\,.
\end{aligned}$$

These equations are compatible if and only if

$$\frac{a_4}{A_0} + \frac{b_4}{B_0} + \frac{c_4}{C_0} + \frac{d_4}{D_0} = 0\,,$$

which is found to be identically satisfied on substituting for a_4, b_4, c_4 and d_4 in the above condition. By introducing an arbitrary function $\alpha_4(t)$, a solution to equations (4.30) can be written as

$$\begin{aligned}\frac{A_4}{A_0} = {} & \frac{\mathrm{i}}{2}\frac{\dot F}{F}\alpha_2 + \frac{\mathrm{i}}{12}\dot\psi^3\alpha_1 - \frac{\mathrm{i}}{2}\dot\psi\alpha_1^3 - \frac{\mathrm{i}}{4}\dot\psi\alpha_1\alpha_2 + \frac{\mathrm{i}}{12}\dot\psi\ddot\psi - \mathrm{i}\frac{3}{4}(\alpha_3 - \gamma_3)\dot\psi + \frac{\mathrm{i}}{4}\dot\alpha_2 \\ & - \frac{1}{24}\left(\frac{\dot F}{F}\right)^2 + \frac{5}{24}\frac{\dot F}{F}\dot\psi\alpha_1 + \frac{1}{4}\dot\psi^2\alpha_2 + 3\alpha_1^2\alpha_2 + \frac{1}{4}\ddot\psi\alpha_1 - \frac{1}{2}\alpha_1\alpha_3 + \frac{1}{24}\frac{\ddot F}{F} + \alpha_4\,,\end{aligned}$$

$$\begin{aligned}\frac{B_4}{B_0} = {} & \frac{\mathrm{i}}{2}\frac{\dot F}{F}\alpha_2 - \frac{\mathrm{i}}{12}\dot\psi^3\alpha_1 + \frac{\mathrm{i}}{2}\dot\psi\alpha_1^3 + \frac{\mathrm{i}}{4}\dot\psi\alpha_1\alpha_2 - \frac{\mathrm{i}}{12}\dot\psi\ddot\psi - \mathrm{i}\frac{3}{4}(\alpha_3 - \gamma_3)\dot\psi - \frac{\mathrm{i}}{4}\dot\alpha_2 \\ & + \frac{1}{24}\left(\frac{\dot F}{F}\right)^2 - \frac{5}{24}\frac{\dot F}{F}\dot\psi\alpha_1 + \frac{1}{4}\dot\psi^2\alpha_2 + 3\alpha_1^2\alpha_2 + \frac{1}{4}\ddot\psi\alpha_1 + \frac{1}{2}\alpha_1\alpha_3 - \frac{1}{24}\frac{\ddot F}{F} + \alpha_4\,,\end{aligned}$$

$$\begin{aligned}\frac{C_4}{C_0} = {} & -\frac{\mathrm{i}}{2}\frac{\dot F}{F}\alpha_2 - \frac{\mathrm{i}}{12}\dot\psi^3\alpha_1 + \frac{\mathrm{i}}{2}\dot\psi\alpha_1^3 - \frac{\mathrm{i}}{4}\dot\psi\alpha_1\alpha_2 + \frac{\mathrm{i}}{12}\dot\psi\ddot\psi - \mathrm{i}\frac{3}{4}(\gamma_3 - \alpha_3)\dot\psi - \frac{\mathrm{i}}{4}\dot\alpha_2 \\ & - \frac{1}{24}\left(\frac{\dot F}{F}\right)^2 - \frac{5}{24}\frac{\dot F}{F}\dot\psi\alpha_1 - \frac{1}{4}\dot\psi^2\alpha_2 - 3\alpha_1^2\alpha_2 - \frac{1}{4}\ddot\psi\alpha_1 + \frac{1}{2}\alpha_1\gamma_3 + \frac{1}{24}\frac{\ddot F}{F} + \alpha_4\,,\end{aligned}$$

$$\begin{aligned}\frac{D_4}{D_0} = {} & -\frac{\mathrm{i}}{2}\frac{\dot F}{F}\alpha_2 + \frac{\mathrm{i}}{12}\dot\psi^3\alpha_1 - \frac{\mathrm{i}}{2}\dot\psi\alpha_1^3 + \frac{\mathrm{i}}{4}\dot\psi\alpha_1\alpha_2 - \frac{\mathrm{i}}{12}\dot\psi\ddot\psi - \mathrm{i}\frac{2}{4}(\gamma_3 - \alpha_3)\dot\psi + \frac{\mathrm{i}}{4}\dot\alpha_2 \\ & + \frac{1}{24}\left(\frac{\dot F}{F}\right)^2 + \frac{5}{24}\frac{\dot F}{F}\dot\psi\alpha_1 - \frac{1}{4}\dot\psi^2\alpha_2 - 3\alpha_1^2\alpha_2 - \frac{1}{4}\ddot\psi\alpha_1 - \frac{1}{2}\alpha_1\gamma_3 - \frac{1}{24}\frac{\ddot F}{F} + \alpha_4\,.\end{aligned}$$

For this case there are only six arbitrary functions ψ, F, α_2, α_3, γ_3 and α_4 corresponding to the eight resonance values and according to the statement of the Painlevé property this system of coupled equations will not be completely integrable because the Cauchy-Kowalevski theorem requires eight arbitrary functions. Equations (4.11) are then said to have the conditional Painlevé property, and special solutions may exist.

If we seek non-distorting pulse solutions to equations (4.11) with $h = 3$, of the form

$$A = e^{-i(\lambda_1 t + V\sigma)} F_1(\sigma)\,,$$
$$C = e^{-i(\lambda_2 t + V\sigma)} F_2(\sigma)\,,$$

where λ_1, λ_2 are real adjustable parameters, V is a frequency shift and F_1, F_2 are real functions of $\sigma = x - 2Vt$ which satisfy the coupled ordinary differential equations

$$F_1'' + F_1^3 + 3F_2^2 F_1 - \beta_1 F_1 = 0\,, \tag{4.31}$$
$$F_2'' + F_2^3 + 3F_1^2 F_2 - \beta_2 F_2 = 0\,, \tag{4.32}$$

where $\beta_i = \lambda_i - V^2$ for $i = 1, 2$. This is analogous to the situation in (2.64) and (2.65) with $h_1 = 1$ and $h_2 = 3$. A conserved quantity for equations (4.31) and (4.32) can be found by adding (4.32) $\times\ F_2'$ to (4.31) $\times\ F_1'$ to obtain

$$F_1''F_1' + F_2''F_2' + F_1^3F_1' + F_2^3F_2' + 3(F_2^2F_1F_1' + F_1^2F_2F_2') - \beta_1F_1F_1' - \beta_2F_2F_2' = 0\,,$$

which can be integrated to give

$$\frac{1}{2}\left(F_1'^2 F_2'^2\right) + \frac{1}{4}\left(F_1^4 + F_2^4\right) + \frac{3}{2}\left(F_1^2F_2^2\right) - \frac{1}{2}\left(\beta_1F_1^2 + \beta_2F_2^2\right) = \text{constant}\,.$$

Furthermore by adding (4.31) $\times\ F_2'$ to (4.32) $\times\ F_1'$ we obtain

$$F_1''F_2' + F_2''F_1' + F_1^3F_2' + F_2^3F_1' + 3(F_2^2F_1F_2' + F_1^2F_2F_1') - \beta_1F_1F_2' - \beta_2F_2F_1' = 0\,,$$

which for $\beta_1 = \beta_2 = \beta$ can be integrated to give a second conserved quantity

$$F_1'F_2' + F_1^3F_2 + F_2^3F_1 - \beta F_1F_2 = \text{constant}\,.$$

Since there are two conserved quantities equations (4.31) and (4.32) are integrable for the case $\beta_1 = \beta_2$. For this case equations (4.31) and (4.32) are separable. By writing $F_1 + F_2 = G_1$ and $F_1 - F_2 = G_2$ we find that $\beta_1 = \beta_2$ equations (4.31) and (4.32) are separable and they uncouple to give

$$G_i'' + G_i^3 - \beta G_i = 0\,, \qquad i = 1, 2\,.$$

These equations have solutions

$$G_i = \sqrt{2\beta}\,\mathrm{sech}\,\sqrt{\beta}(\sigma - \sigma_i)\,, \qquad i = 1, 2$$

indicating that G_1, G_2 are isolated pulses with the equal amplitude, $\sqrt{2\beta}$, but independent pulse centres. Hence a solution to (4.31) and (4.32), for the case $\beta_1 = \beta_2$, is

$$\begin{aligned} F_1 &= \frac{1}{2}(G_1 + G_2) \\ &= \sqrt{\frac{\beta}{2}}\left\{\mathrm{sech}\,\sqrt{\beta}(\sigma - \sigma_1) + \mathrm{sech}\,\sqrt{\beta}(\sigma - \sigma_2)\right\}\,, \\ F_2 &= \frac{1}{2}(G_1 - G_2) \\ &= \sqrt{\frac{\beta}{2}}\left\{\mathrm{sech}\,\sqrt{\beta}(\sigma - \sigma_1) - \mathrm{sech}\,\sqrt{\beta}(\sigma - \sigma_2)\right\}\,, \end{aligned}$$

and non-distorting pulse solutions to equations (4.11), with $h = 3$ can be written as

$$\begin{aligned} A &= \sqrt{\frac{\beta}{2}}e^{-\mathrm{i}(\lambda t + V\sigma)}\left\{\mathrm{sech}\,\sqrt{\beta}(\sigma - \sigma_1) + \mathrm{sech}\,\sqrt{\beta}(\sigma - \sigma_2)\right\}\,, \\ C &= \sqrt{\frac{\beta}{2}}e^{-\mathrm{i}(\lambda t + V\sigma)}\left\{\mathrm{sech}\,\sqrt{\beta}(\sigma - \sigma_1) - \mathrm{sech}\,\sqrt{\beta}(\sigma - \sigma_2)\right\}\,, \end{aligned}$$

where $\lambda_1 = \lambda_2 = \lambda$. Other solutions allow one or both of G_1, G_2 to be bounded but periodic, and which can be written in terms of elliptic functions whose periods depend on amplitude.

Chapter 5

Truncated Painlevé expansions for the coupled nonlinear Schrödinger equations

5.1 Preamble

The Painlevé partial differential equation test, has been applied to many partial differential equations to determine whether or not they have the Painlevé property (Weiss et al., 1983), by seeking solutions to the equations which are in the form of Laurent expansions around a singularity manifold $\phi(x,t)$. Although in most cases the integrability of these equations had already been determined. For equations which are integrable, Bäcklund transformations and Lax pairs for these equations can be derived by using a truncated Painlevé expansion (see Weiss et al., 1983, Weiss, 1985). Bäcklund transformations for the single cubic Schrödinger equation have been presented by Steeb et al. (1984) and Weiss (1985). Sahadevan et al. (1986) have found Bäcklund transformations for a coupled pair of nonlinear Schrödinger equations which are integrable.

Equations which do not pass the Painlevé test are not, in general, integrable but may possess special solutions which can be identified by truncating the Painlevé expansion at the ϕ^0 term. For these cases, we find that ϕ is constrained to satisfy a set of consistency conditions which may suggest possible forms for ϕ. Cariello and Tabor (1989) and Halford and Vlieg-Hulstman (1992) have found solutions to several equations using truncated Painlevé expansions .

In the previous chapter, we have used the Painlevé partial differential equation test to determine the integrability of the coupled pair of constant coefficient cubic Schrödinger equations. It was found that when the the value of the coupling coefficient h was not equal to ± 1, the coupled equations were not integrable, although for $h = 3$ the equations were found to have the conditional Painlevé property. In the following sections we seek special solutions to the coupled constant coefficient nonlinear Schrödinger equations (4.11) which are of the form of a truncated Painlevé expansion, for values of h for which the equations do not have the Painlevé property. In Section 5.2 we consider the case $h \neq \pm 1, 3$, while in Section 5.3 we consider the special case $h = 3$.

5.2 The truncation procedure for $h \neq \pm 1, 3$

In this section Painlevé analysis is used to identify solutions of the constant coefficient nonlinear Schrödinger equations (4.11) for the case when the coupling constant h does not take the special values, $h = \pm 1, 3$ (see Section 4.3). Applying the Painlevé analysis as described in Chapter 4, we first complexify all the variables in equations (4.11) to obtain the system of coupled equations (4.12). Solutions of these equations are then sought which are of the form of Painlevé series (4.15) truncated at the $O(\phi^0)$ term, that is

$$A = \frac{A_0}{\phi} + A_1 , \qquad B = \frac{B_0}{\phi} + B_1 , \qquad C = \frac{C_0}{\phi} + C_1 , \qquad D = \frac{D_0}{\phi} + D_1 , \quad (5.1)$$

with $\phi = \phi(x,t)$. Substituting the truncated series (5.1) into equations (4.12) gives the system of equations

$$\mathrm{i}\left(-A_0\frac{\phi_t}{\phi^2}+\frac{A_{0t}}{\phi}+A_{1t}\right)=2A_0\frac{\phi_x^2}{\phi^3}-\frac{1}{\phi^2}\left(A_0\phi_{xx}+2A_{0x}\phi_x\right)+\frac{A_{0xx}}{\phi}+A_{1xx}$$
$$+\frac{A_0^2B_0}{\phi^3}+\frac{1}{\phi^2}\left(A_0^2B_1+2A_0B_0A_1\right)+\frac{1}{\phi}\left(A_1^2B_0+2A_0A_1B_1\right)+A_1^2B_1$$
$$+h\left[\frac{A_0C_0D_0}{\phi^3}+\frac{1}{\phi^2}\left(A_1C_0D_0+A_0C_1D_0+A_0C_0D_1\right)\right.$$
$$\left.+\frac{1}{\phi}\left(A_1C_1D_0+A_1C_0D_1+A_0C_1D_1\right)+A_1C_1D_1\right], \qquad (5.2)$$

$$-\mathrm{i}\left(-B_0\frac{\phi_t}{\phi^2}+\frac{B_{0t}}{\phi}+B_{1t}\right)=2B_0\frac{\phi_x^2}{\phi^3}-\frac{1}{\phi^2}\left(B_0\phi_{xx}+2B_{0x}\phi_x\right)+\frac{B_{0xx}}{\phi}+B_{1xx}$$
$$+\frac{B_0^2A_0}{\phi^3}+\frac{1}{\phi^2}\left(B_0^2A_1+2A_0B_0B_1\right)+\frac{1}{\phi}\left(B_1^2A_0+2B_0A_1B_1\right)+B_1^2A_1$$
$$+h\left[\frac{B_0C_0D_0}{\phi^3}+\frac{1}{\phi^2}\left(B_1C_0D_0+B_0C_1D_0+B_0C_0D_1\right)\right.$$
$$\left.+\frac{1}{\phi}\left(B_1C_1D_0+B_1C_0D_1+B_0C_1D_1\right)+B_1C_1D_1\right], \qquad (5.3)$$

$$\mathrm{i}\left(-C_0\frac{\phi_t}{\phi^2}+\frac{C_{0t}}{\phi}+C_{1t}\right)=2C_0\frac{\phi_x^2}{\phi^3}-\frac{1}{\phi^2}\left(C_0\phi_{xx}+2C_{0x}\phi_x\right)+\frac{C_{0xx}}{\phi}+C_{1xx}$$
$$+\frac{C_0^2D_0}{\phi^3}+\frac{1}{\phi^2}\left(C_0^2D_1+2C_0D_0C_1\right)+\frac{1}{\phi}\left(C_1^2D_0+2C_0C_1D_1\right)+C_1^2D_1$$
$$+h\left[\frac{C_0A_0B_0}{\phi^3}+\frac{1}{\phi^2}\left(C_1A_0B_0+C_0A_1B_0+C_0A_0B_1\right)\right.$$
$$\left.+\frac{1}{\phi}\left(C_1A_1B_0+C_1A_0B_1+C_0A_1B_1\right)+C_1A_1B_1\right], \qquad (5.4)$$

$$-\mathrm{i}\left(-D_0\frac{\phi_t}{\phi^2}+\frac{D_{0t}}{\phi}+D_{1t}\right)=2D_0\frac{\phi_x^2}{\phi^3}-\frac{1}{\phi^2}\left(D_0\phi_{xx}+2D_{0x}\phi_x\right)+\frac{D_{0xx}}{\phi}+D_{1xx}$$
$$+\frac{D_0^2C_0}{\phi^3}+\frac{1}{\phi^2}\left(D_0^2C_1+2C_0D_0D_1\right)+\frac{1}{\phi}\left(D_1^2C_0+2D_0C_1D_1\right)+D_1^2C_1$$
$$+h\left[\frac{D_0A_0B_0}{\phi^3}+\frac{1}{\phi^2}\left(D_1A_0B_0+D_0A_1B_0+D_0A_0B_1\right)\right.$$
$$\left.+\frac{1}{\phi}\left(D_1A_1B_0+D_1A_0B_1+D_0A_1B_1\right)+D_1A_1B_1\right]. \qquad (5.5)$$

The strategy for finding the functions ϕ, A_0, B_0, C_0, D_0, A_1, B_1, C_1, D_1 is to insist that terms of each degree ϕ^{-3}, ϕ^{-2}, ϕ^{-2}, ϕ^{0} are satisfied individually.

For terms of $O(\phi^{-3})$, the equations

$$
\begin{aligned}
A_0\left(A_0B_0 + hC_0D_0 + 2\phi_x^2\right) &= 0\,,\\
B_0\left(A_0B_0 + hC_0D_0 + 2\phi_x^2\right) &= 0\,,\\
C_0\left(hA_0B_0 + C_0D_0 + 2\phi_x^2\right) &= 0\,,\\
D_0\left(hA_0B_0 + C_0D_0 + 2\phi_x^2\right) &= 0\,,
\end{aligned}
$$

are obtained, from these equations it can be deduced that either, $A_0 = B_0 = 0$ with $C_0D_0 = -2\phi_x^2$, $C_0 = D_0 = 0$ with $A_0B_0 = -2\phi_x^2$, or

$$
\begin{aligned}
A_0B_0 + hC_0D_0 &= -2\phi_x^2\,,\\
hA_0B_0 + C_0D_0 &= -2\phi_x^2\,.
\end{aligned}
\tag{5.6}
$$

For the case $A_0 = B_0 = 0$, analysis of the equations obtained for subsequent orders of ϕ gives $A_1 = B_1 = 0$, and hence $A = B = 0$ which reduces the coupled equations (4.11) to the single cubic Schrödinger equation. Newell et al. (1987) consider truncated Painlevé expansions for this equation. A similar result is obtained when $C_0 = D_0 = 0$. These cases are integrable and will not be considered further. Equations (5.6) can be solved to give

$$A_0B_0 = C_0D_0 = -\frac{2\phi_x^2}{1+h} \qquad \text{for} \quad h \neq \pm 1\,.$$

If $h = -1$, equations (5.6) require $\phi_x = 0$ for consistency, however for Painlevé analysis the singularity manifold $\phi(x,t) = 0$ is not a characteristic and hence $\phi_x \neq 0$ (Ward, 1984) and there is not a solution of the form (5.1) when $h = -1$. The functions A_0, B_0, C_0 and D_0 can be written in the form

$$
\begin{aligned}
A_0 &= \sqrt{\frac{2}{1+h}}\phi_x e^{-P}\,, & B_0 &= -\sqrt{\frac{2}{1+h}}\phi_x e^{P}\,,\\
C_0 &= \sqrt{\frac{2}{1+h}}\phi_x e^{-Q}\,, & D_0 &= -\sqrt{\frac{2}{1+h}}\phi_x e^{Q}\,,
\end{aligned}
\tag{5.7}
$$

where $P(x,t)$ and $Q(x,t)$ are arbitrary functions.

Comparing terms of $O(\phi^{-2})$ in equations (5.2)–(5.5) and using the expressions given in (5.7), we obtain the equations

$$\begin{pmatrix} h+2 & 1 & h & h \\ 1 & h+2 & h & h \\ h & h & h+2 & 1 \\ h & h & 1 & h+2 \end{pmatrix} \begin{pmatrix} A_1B_0 \\ B_1A_0 \\ C_1D_0 \\ D_1C_0 \end{pmatrix} = \begin{pmatrix} 3\phi_{xx} - 2\phi_x P_x - \mathrm{i}\phi_t \\ 3\phi_{xx} + 2\phi_x P_x + \mathrm{i}\phi_t \\ 3\phi_{xx} - 2\phi_x Q_x - \mathrm{i}\phi_t \\ 3\phi_{xx} + 2\phi_x Q_x + \mathrm{i}\phi_t \end{pmatrix} . \quad (5.8)$$

For $h \neq 3$, the coefficient matrix on the left-hand side is non-singular so that (5.8) has the solution

$$\begin{aligned} \frac{A_1}{A_0} &= -\frac{1}{2\phi_x^2}\left(\phi_{xx} - 2\phi_x P_x - \mathrm{i}\phi_t\right) , \\ \frac{B_1}{B_0} &= -\frac{1}{2\phi_x^2}\left(\phi_{xx} + 2\phi_x P_x + \mathrm{i}\phi_t\right) , \\ \frac{C_1}{C_0} &= -\frac{1}{2\phi_x^2}\left(\phi_{xx} - 2\phi_x Q_x - \mathrm{i}\phi_t\right) , \\ \frac{D_1}{D_0} &= -\frac{1}{2\phi_x^2}\left(\phi_{xx} + 2\phi_x Q_x + \mathrm{i}\phi_t\right) . \end{aligned} \quad (5.9)$$

Terms which are of $O(\phi^{-1})$ in (5.2)–(5.5) give the system of equations

$$\begin{aligned} \mathrm{i}A_{0t} &= A_{0xx} + A_1^2 B_0 + 2A_0A_1B_1 + h\left[A_1(C_1D_0 + D_1C_0) + A_0C_1D_1\right] , \\ -\mathrm{i}B_{0t} &= B_{0xx} + B_1^2 A_0 + 2B_0B_1A_1 + h\left[B_1(C_1D_0 + D_1C_0) + B_0C_1D_1\right] , \\ \mathrm{i}C_{0t} &= C_{0xx} + C_1^2 D_0 + 2C_0C_1D_1 + h\left[C_1(A_1B_0 + B_1A_0) + C_0A_1B_1\right] , \\ -\mathrm{i}D_{0t} &= D_{0xx} + D_1^2 C_0 + 2D_0D_1C_1 + h\left[D_1(A_1B_0 + B_1A_0) + D_0A_1B_1\right] . \end{aligned} \quad (5.10)$$

Substitution of (5.7) and (5.9) into these equations gives

$$\begin{aligned} (1+h)&\left[\frac{\phi_{xxx}}{\phi_x} - \mathrm{i}\frac{\phi_{xt}}{\phi_x} + \mathrm{i}\frac{\phi_{xx}}{\phi_x}\frac{\phi_t}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 - \frac{1}{2}\left(\frac{\phi_t}{\phi_x}\right)^2 + \mathrm{i}P_t + P_x^2 - P_{xx}\right] \\ &= -2\mathrm{i}\frac{\phi_t}{\phi_x}(P_x + hQ_x) - 2(P_x^2 + hQ_x^2) , \end{aligned} \quad (5.11)$$

$$\begin{aligned} (1+h)&\left[\frac{\phi_{xxx}}{\phi_x} + \mathrm{i}\frac{\phi_{xt}}{\phi_x} - \mathrm{i}\frac{\phi_{xx}}{\phi_x}\frac{\phi_t}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 - \frac{1}{2}\left(\frac{\phi_t}{\phi_x}\right)^2 + \mathrm{i}P_t + P_x^2 + P_{xx}\right] \\ &= -2\mathrm{i}\frac{\phi_t}{\phi_x}(P_x + hQ_x) - 2(P_x^2 + hQ_x^2) , \end{aligned} \quad (5.12)$$

$$(1+h)\left[\frac{\phi_{xxx}}{\phi_x}-\mathrm{i}\frac{\phi_{xt}}{\phi_x}+\mathrm{i}\frac{\phi_{xx}}{\phi_x}\frac{\phi_t}{\phi_x}-\frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2-\frac{1}{2}\left(\frac{\phi_t}{\phi_x}\right)^2+\mathrm{i}Q_t+Q_x^2-Q_{xx}\right]$$
$$=-2\mathrm{i}\frac{\phi_t}{\phi_x}(hP_x+Q_x)-2(hP_x^2+Q_x^2), \qquad (5.13)$$

$$(1+h)\left[\frac{\phi_{xxx}}{\phi_x}+\mathrm{i}\frac{\phi_{xt}}{\phi_x}-\mathrm{i}\frac{\phi_{xx}}{\phi_x}\frac{\phi_t}{\phi_x}-\frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2-\frac{1}{2}\left(\frac{\phi_t}{\phi_x}\right)^2+\mathrm{i}Q_t+Q_x^2+Q_{xx}\right]$$
$$=-2\mathrm{i}\frac{\phi_t}{\phi_x}(hP_x+Q_x)-2(hP_x^2+Q_x^2). \qquad (5.14)$$

The solvability of this set of equations imposes restrictions on the functions ϕ, P and Q arising in (5.7) and (5.9).

By subtracting (5.12) from (5.11) and (5.14) from (5.13), we deduce that

$$\begin{aligned}
&\mathrm{i}\frac{\partial}{\partial x}\left(\frac{\phi_t}{\phi_x}\right)+P_{xx}=0, \qquad \text{giving} \qquad P_x=-\mathrm{i}\frac{\phi_t}{\phi_x}+\beta(t),\\
&\mathrm{i}\frac{\partial}{\partial x}\left(\frac{\phi_t}{\phi_x}\right)+Q_{xx}=0, \qquad \text{giving} \qquad Q_x=-\mathrm{i}\frac{\phi_t}{\phi_x}+\gamma(t).
\end{aligned} \qquad (5.15)$$

Together these imply that

$$\begin{aligned}
P-Q &= 2\alpha(t)x+2\delta(t),\\
P_x+Q_x &= -2\mathrm{i}\frac{\phi_t}{\phi_x}+2\Gamma(t),
\end{aligned} \qquad (5.16)$$

where $2\alpha=\beta-\gamma$ and $2\Gamma=\beta+\gamma$, with $\beta(t)$, $\gamma(t)$, $\delta(t)$ arbitrary functions. With P, Q satisfying (5.15) and (5.16), it is found that the system (5.11)–(5.14) reduces to

$$(1+h)\left[\frac{\phi_{xxx}}{\phi_x}-\frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2-\frac{1}{2}\left(\frac{\phi_t}{\phi_x}\right)^2+\mathrm{i}P_t+P_x^2\right]=-2\mathrm{i}\frac{\phi_t}{\phi_x}(P_x+hQ_x)-2(P_x^2+hQ_x^2). \qquad (5.17)$$

$$(1+h)\left[\frac{\phi_{xxx}}{\phi_x}-\frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2-\frac{1}{2}\left(\frac{\phi_t}{\phi_x}\right)^2+\mathrm{i}Q_t+Q_x^2\right]=-2\mathrm{i}\frac{\phi_t}{\phi_x}(hP_x+Q_x)-2(hP_x^2+Q_x^2). \qquad (5.18)$$

Subtracting these equations leads to

$$\frac{\phi_t}{\phi_x}=\frac{1+h}{4}\frac{\alpha'}{\alpha}x+\frac{1+h}{4}\frac{\delta'}{\alpha}+\frac{\mathrm{i}}{2}(h-3)\Gamma, \qquad (5.19)$$

from which it can be deduced that the loci $\phi =$ constant have the form

$$xp(t) + q(t) = \text{constant}\,,$$

where

$$p(t) = \alpha^{(1+h)/4} \quad \text{and} \quad q'(t) = \left(\frac{1+h}{4}\frac{\delta'}{\alpha} + \frac{\mathrm{i}}{2}(h-3)\Gamma\right) p(t)\,.$$

This suggests that we can write $\phi = \Phi(\eta)$ with $\eta = xp(t) + q(t)$. Then, the one remaining restriction from (5.11)–(5.14) is found, by adding (5.17) and (5.18), as

$$\mathrm{i}(P_t + Q_t) + 2\frac{\phi_{xxx}}{\phi_x} - 3\left(\frac{\phi_{xx}}{\phi_x}\right)^2 - 3\left(\frac{\phi_t}{\phi_x}\right)^2 - 8\mathrm{i}\Gamma\frac{\phi_t}{\phi_x} + 6(\alpha^2 + \Gamma^2) = 0\,. \quad (5.20)$$

Comparing terms of $O(\phi^0)$ in equations (5.2)–(5.5), we find that A_1, B_1, C_1, D_1 must also satisfy the complexified coupled nonlinear Schrödinger equations (4.12),

$$\begin{aligned} \mathrm{i}A_{1t} &= A_{1xx} + (A_1B_1 + hC_1D_1)A_1\,, & (5.21)\\ -\mathrm{i}B_{1t} &= B_{1xx} + (A_1B_1 + hC_1D_1)B_1\,, & (5.22)\\ \mathrm{i}C_{1t} &= C_{1xx} + (hA_1B_1 + C_1D_1)C_1\,, & (5.23)\\ -\mathrm{i}D_{1t} &= D_{1xx} + (hA_1B_1 + C_1D_1)D_1\,. & (5.24) \end{aligned}$$

If new functions u and v are introduced such that

$$u = \frac{\phi_{xx}}{\phi_x}\,, \qquad v = \frac{\phi_t}{\phi_x}\,,$$

then we find, from (5.9), that

$$A_1 = -\frac{1}{2}\sqrt{\frac{2}{1+h}}(u + \mathrm{i}v - 2\beta)e^{-P}\,, \qquad B_1 = \frac{1}{2}\sqrt{\frac{2}{1+h}}(u - \mathrm{i}v + 2\beta)e^{P}\,,$$

$$C_1 = -\frac{1}{2}\sqrt{\frac{2}{1+h}}(u + \mathrm{i}v - 2\gamma)e^{-Q}\,, \qquad D_1 = \frac{1}{2}\sqrt{\frac{2}{1+h}}(u - \mathrm{i}v + 2\gamma)e^{Q}\,.$$

If we substitute for A_1, B_1, C_1, D_1, in equation (5.21), rewritten in the form

$$\mathrm{i}\frac{A_{1t}}{A_1} = \frac{A_{1xx}}{A_1} + (A_1B_1 + hC_1D_1)\,,$$

and write

$$-\mathrm{i}P_t - (A_1B_1 + hC_1D_1) \equiv \Phi_1 ,$$

we obtain the equation

$$\begin{aligned}\Phi_1(u + \mathrm{i}v - 2\beta) &= v_t + 2\mathrm{i}\beta' + u_{xx} - 3vv_x - 4\mathrm{i}v_x\beta - v^2u - 2\mathrm{i}uv\beta + u\beta^2 \\ &\quad -\mathrm{i}u_t + \mathrm{i}v_{xx} + \mathrm{i}v_xu + 2\mathrm{i}vu_x - 2u_x\beta - \mathrm{i}v^3 + 4v^2\beta + 5\mathrm{i}v\beta^2 - 2\beta^3 .\end{aligned}$$

Similarly, from (5.22) we find

$$\begin{aligned}\Phi_1(u - \mathrm{i}v + 2\beta) &= v_t + 2\mathrm{i}\beta' + u_{xx} - 3vv_x - 4\mathrm{i}v_x\beta - v^2u - 2\mathrm{i}uv\beta + u\beta^2 \\ &\quad +\mathrm{i}u_t - \mathrm{i}v_{xx} - \mathrm{i}v_xu - 2\mathrm{i}vu_x + 2u_x\beta + \mathrm{i}v^3 - 4v^2\beta - 5\mathrm{i}v\beta^2 + 2\beta^3 .\end{aligned}$$

Adding and subtracting these equations gives the equivalent system

$$\Phi_1 u = v_t + 2\mathrm{i}\beta' + u_{xx} - 3vv_x - 4\mathrm{i}v_x\beta - v^2u - 2\mathrm{i}uv\beta + u\beta^2 , \tag{5.25}$$

$$\Phi_1(\mathrm{i}v - 2\beta) = -\mathrm{i}u_t + \mathrm{i}v_{xx} + \mathrm{i}v_xu + 2\mathrm{i}vu_x - 2u_x\beta - \mathrm{i}v^3 + 4v^2\beta + 5\mathrm{i}v\beta^2 - 2\beta^3 . \tag{5.26}$$

Similarly, from (5.23) and (5.24), with

$$-\mathrm{i}Q_t - (hA_1B_1 + C_1D_1) \equiv \Phi_2 ,$$

we obtain

$$\Phi_2 u = v_t + 2\mathrm{i}\gamma' + u_{xx} - 3vv_x - 4\mathrm{i}v_x\gamma - v^2u - 2\mathrm{i}uv\gamma + u\gamma^2 , \tag{5.27}$$

$$\Phi_2(\mathrm{i}v - 2\gamma) = -\mathrm{i}u_t + \mathrm{i}v_{xx} + \mathrm{i}v_xu + 2\mathrm{i}vu_x - 2u_x\gamma - \mathrm{i}v^3 + 4v^2\gamma + 5\mathrm{i}v\gamma^2 - 2\gamma^3 . \tag{5.28}$$

By subtracting (5.27) from (5.25) we obtain

$$(\Phi_1 - \Phi_2)u = 4\mathrm{i}\alpha' - 8\mathrm{i}v_x\alpha - 4\mathrm{i}uv\alpha + 4u\alpha\Gamma ,$$

and substituting for Φ_1, Φ_2, u and v, we find that this equation reduces to the simple statement

$$(1 - h)\alpha' = 0 .$$

As we are considering cases for which $h \neq 1$, this implies that $\alpha = \widehat{\alpha} =$ constant, while (5.19) then shows that ϕ may be written as $\phi = \Phi(\eta)$ with $\eta = x + q(t)$ (i.e. $p(t) = 1$) and

$$q'(t) = \frac{1+h}{4}\frac{\delta'}{\widehat{\alpha}} + \frac{\mathrm{i}}{2}(h-3)\Gamma\,. \tag{5.29}$$

Hence, we can write u and v as

$$u = \frac{\phi_{xx}}{\phi_x} = \frac{\Phi''}{\Phi'} = \Psi(\eta)\,, \qquad\qquad v = \frac{\phi_t}{\phi_x} = q'(t)\,,$$

which are functions of only η and t, respectively. The expressions for P_x and Q_x in (5.15) then become

$$\begin{aligned} P_x &= -\mathrm{i}q'(t) + \Gamma(t) + \widehat{\alpha}\,, \\ Q_x &= -\mathrm{i}q'(t) + \Gamma(t) - \widehat{\alpha}\,, \end{aligned}$$

which may be integrated with respect to x to yield

$$\begin{aligned} P &= (-\mathrm{i}q'(t) + \Gamma(t) + \widehat{\alpha})x + \delta(t) + r\,, \\ Q &= (-\mathrm{i}q'(t) + \Gamma(t) - \widehat{\alpha})x - \delta(t) + r\,, \end{aligned}$$

where r is a constant.

Subtracting equation (5.28) from (5.26) gives

$$(\Phi_1 - \Phi_2)(\mathrm{i}v - 2\Gamma) - 2\widehat{\alpha}(\Phi_1 + \Phi_2) = -4u_x\widehat{\alpha} + 8v^2\widehat{\alpha} + 20\mathrm{i}v\widehat{\alpha}\Gamma - 4(3\Gamma^2 + \widehat{\alpha}^2)\widehat{\alpha}\,,$$

which, after substituting for Φ_1, Φ_2, u and v, can be written as

$$2\Psi' - \Psi^2 - 2\mathrm{i}(\mathrm{i}q'' - \Gamma')\eta = J(t)\,, \tag{5.30}$$

with

$$J(t) = 3q'^2 + 2\mathrm{i}q(\mathrm{i}q'' - \Gamma') + 8\mathrm{i}q'\Gamma - 6(\Gamma^2 + \widehat{\alpha}^2)\,.$$

Since η and t are independent variables, differentiating equation (5.30) with respect to η gives

$$(2\Psi' - \Psi^2)' - 2\mathrm{i}(\mathrm{i}q'' - \Gamma') = 0\,.$$

Therefore we deduce that $iq'' - \Gamma'$ is a constant. Consequently, J cannot depend on t, so that (5.30) has the form

$$2\Psi' - \Psi^2 = 2L\eta + J\,, \tag{5.31}$$

where

$$L = \mathrm{i}(\mathrm{i}q'' - \Gamma') = \text{constant}\,, \tag{5.32}$$

$$3q'^2 + 2Lq + 8\mathrm{i}q'\Gamma - 6(\Gamma^2 + \widehat{\alpha}^2) = J\,. \tag{5.33}$$

By considering the sum of equations (5.25) and (5.27) and noting that $v_x = 0$ we obtain

$$(\Phi_1 + \Phi_2)u = 2v_t + 4\mathrm{i}\Gamma' + 2u_{xx} - 2v^2u - 4\mathrm{i}uv\Gamma + 2u(\Gamma^2 + \widehat{\alpha}^2)\,,$$

which reduces to

$$L = \mathrm{i}(\mathrm{i}q'' - 2\Gamma')\,.$$

Comparison with (5.32) shows that we must have $\Gamma' = 0$, from which we can write $\Gamma = \widehat{\Gamma}$ (a constant) and deduce that

$$q''(t) = -L\,.$$

Consequently, the most general allowable forms for $q(t)$ and $v = q'(t)$ are

$$q = q_0 + q_1 t - \frac{1}{2}Lt^2\,, \qquad v = q_1 - Lt\,,$$

where q_0 and q_1 are constants. Substitution into (5.33) then gives compatibility only for $L = 0$. Thus $v =$ constant and $q = vt + q_0$, while J is given by

$$J = 3v^2 + 8\mathrm{i}v\widehat{\Gamma} - 6(\widehat{\Gamma}^2 + \widehat{\alpha}^2)\,. \tag{5.34}$$

The sum of equations (5.26) and (5.28) gives

$$(\Phi_1 + \Phi_2)(\mathrm{i}v - 2\widehat{\Gamma}) - 2\widehat{\alpha}(\Phi_1 - \Phi_2) =$$
$$-2\mathrm{i}u_t + 4\mathrm{i}vu_x - 4u_x\widehat{\Gamma} - 2\mathrm{i}v^3 + 8v^2\widehat{\Gamma} + 10\mathrm{i}v(\widehat{\Gamma}^2 + \widehat{\alpha}^2) - 4\widehat{\Gamma}(\widehat{\Gamma}^2 + 3\widehat{\alpha}^2)\,,$$

which reduces to

$$\left(\Psi^2 - 2\Psi' + 3v^2 + 8\mathrm{i}v\widehat{\Gamma} - 6(\widehat{\Gamma}^2 + \widehat{\alpha}^2)\right)(\mathrm{i}v - 2\widehat{\Gamma}) = 0\,.$$

Use of (5.31) with (5.34) shows that this equation is identically satisfied. The remaining restriction (5.20) obtained from the analysis of $O(\phi^{-1})$ terms is also identically satisfied. This yields all the compatibility conditions for the complexified form of equations (4.11) to have solutions of the form (5.1).

Equation (5.31) reduces to the separable form

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\eta} = \frac{1}{2}(\Psi^2 + J)\,,$$

with solution

$$\Psi = -2b\tanh b(\eta - \eta_0)\,, \qquad J = -4b^2\,,$$

where b, η_0 are constants. Consequently,

$$\Phi''\cosh b(\eta - \eta_0) + 2b\Phi'\sinh b(\eta - \eta_0) = 0$$

gives

$$\Phi'\cosh^2 b(\eta - \eta_0) = d_0 = \text{constant}\,.$$

Further integration gives

$$\Phi = \frac{d_0}{b}\tanh b(\eta - \eta_0) + \Phi_0 = \phi(x,t)\,, \tag{5.35}$$

with Φ_0 a constant.

An expression for A is obtained by substituting (5.7), (5.9) and (5.35) in the truncated series (5.1) for A as

$$A = \sqrt{\frac{2}{1+h}}\left[b\left(\frac{(\Phi_0 b + d_0)e^{b(\eta-\eta_0)} - (\Phi_0 b - d_0)e^{-b(\eta-\eta_0)}}{(\Phi_0 b + d_0)e^{b(\eta-\eta_0)} + (\Phi_0 b - d_0)e^{-b(\eta-\eta_0)}}\right) - \mathrm{i}\frac{v}{2} + \widehat{\Gamma} + \widehat{\alpha}\right]e^{-P}\,.$$

By writing $\Phi_0 b - d_0 = (\Phi_0 b + d_0)e^{2bx_0}$, with x_0 a constant, we find that

$$A = \sqrt{\frac{2}{1+h}}\left[b\tanh b(x + vt + q_0 - \eta_0 - x_0) - \mathrm{i}\frac{v}{2} + \widehat{\Gamma} + \widehat{\alpha}\right]e^{-P}\,.$$

Similar expressions for B, C and D can be found. If we write $\zeta = b(x - \widehat{x} + vt)$ with $\widehat{x} = x_0 - q_0 + \eta_0$, we then find

$$\begin{aligned} A &= \sqrt{\frac{2}{1+h}}\,[b\tanh\zeta + U + \widehat{\alpha}]\,e^{-P}, \\ B &= \sqrt{\frac{2}{1+h}}\,[-b\tanh\zeta + U + \widehat{\alpha}]\,e^{P}, \\ C &= \sqrt{\frac{2}{1+h}}\,[b\tanh\zeta + U - \widehat{\alpha}]\,e^{-Q}, \\ D &= \sqrt{\frac{2}{1+h}}\,[-b\tanh\zeta + U - \widehat{\alpha}]\,e^{Q}, \end{aligned}$$

where $U = \widehat{\Gamma} - iv/2$. These are the most general solutions of the complexified coupled nonlinear Schrödinger equations (4.12) which have the form of truncated Painlevé expansions. They require additionally that $q' = v$ and $\Gamma = \widehat{\Gamma}$ appearing in P, Q, and U satisfy condition (5.34), with $J = -4b^2$.

To complete the construction of solutions to (4.11), we require that $B = A^*$ and $D = C^*$, when x and t are real. We thus write the expressions for A, B, C and D as

$$\begin{aligned} A &= \sqrt{\frac{2}{1+h}}\left[\frac{Re^{\zeta} + Se^{-\zeta}}{e^{\zeta} + e^{-\zeta}}\right] e^{-P}, \\ B &= \sqrt{\frac{2}{1+h}}\left[\frac{Se^{\zeta} + Re^{-\zeta}}{e^{\zeta} + e^{-\zeta}}\right] e^{P}, \\ C &= \sqrt{\frac{2}{1+h}}\left[\frac{\widehat{R}e^{\zeta} + \widehat{S}e^{-\zeta}}{e^{\zeta} + e^{-\zeta}}\right] e^{-Q}, \\ D &= \sqrt{\frac{2}{1+h}}\left[\frac{\widehat{S}e^{\zeta} + \widehat{R}e^{-\zeta}}{e^{\zeta} + e^{-\zeta}}\right] e^{Q}, \end{aligned} \qquad (5.36)$$

with $R = U + \widehat{\alpha} + b$, $S = U + \widehat{\alpha} - b$, $\widehat{R} = U - \widehat{\alpha} + b$ and $\widehat{S} = U - \widehat{\alpha} - b$. The conditions $B = A^*$ and $D = C^*$ then become

$$\begin{aligned} e^{P+P^*}\left[Se^{\zeta+\zeta^*} + Se^{\zeta-\zeta^*} + Re^{-(\zeta-\zeta^*)} + Re^{-(\zeta+\zeta^*)}\right] &= \\ R^*e^{\zeta+\zeta^*} + S^*e^{\zeta-\zeta^*} + R^*e^{-(\zeta-\zeta^*)} + S^*e^{-(\zeta+\zeta^*)}, & \qquad (5.37) \end{aligned}$$

$$e^{Q+Q^*}\left[\widehat{S}e^{\zeta+\zeta^*}+\widehat{S}e^{\zeta-\zeta^*}+\widehat{R}e^{-(\zeta-\zeta^*)}+\widehat{R}e^{-(\zeta+\zeta^*)}\right]=$$

$$\widehat{R}^*e^{\zeta+\zeta^*}+\widehat{S}^*e^{\zeta-\zeta^*}+\widehat{R}^*e^{-(\zeta-\zeta^*)}+\widehat{S}^*e^{-(\zeta+\zeta^*)}. \qquad (5.38)$$

We notice that the exponents all involve either $P+P^*=2P^+$, $Q+Q^*=2Q^+$, $\zeta+\zeta^*=2\zeta^+$ or $\zeta-\zeta^*=2\mathrm{i}\zeta^-$, which are linear in x and t, where $P=P^++\mathrm{i}P^-$, $Q=Q^++\mathrm{i}Q^-$, $\zeta=\zeta^++\mathrm{i}\zeta^-$ with P^+, P^-, Q^+, Q^-, ζ^+ and ζ^- all real.

If $\zeta^-\neq$ constant, we then find that

$$R^*=Re^{2P^+}, \qquad S=S^*e^{-2P^+},$$

from which $P^+=0$, with R and S both real. Since additionally $R^*=S$, we deduce that $b=0$ with $R=S=U+\widehat{\alpha}$ real arbitrary. Similarly we find $Q^+=0$, leading to $\widehat{R}=\widehat{S}=U-\widehat{\alpha}$ real and arbitrary so giving

$$A=\widehat{A}e^{-\mathrm{i}P^-(x,t)}, \qquad C=\widehat{C}e^{-\mathrm{i}Q^-(x,t)}, \qquad (5.39)$$

which describe continuous (unmodulated) solutions to equations (4.11), with independent and arbitrary real constant amplitudes $\widehat{A}$ and $\widehat{C}$. Direct substitution into (4.11) then requires only that

$$P^-(x,t)=k_1x-\omega_1t+\delta_1, \qquad Q^-(x,t)=k_2x-\omega_2t+\delta_2,$$

where

$$\omega_1=k_1^2-\widehat{A}^2-h\widehat{C}^2, \qquad \omega_2=k_2^2-h\widehat{A}^2-\widehat{C}^2, \qquad (5.40)$$

with k_1, k_2, δ_1, δ_2, $\widehat{A}$ and $\widehat{C}$ as arbitrary real constants. It should be noted that the expressions (5.40) are much less restrictive than those which result from $q'=v=$ constant and $J=0$ in (5.34).

For $\zeta^-=$ constant, there are two cases to consider, either $P^+=$ constant or $P^+\neq$ constant. The case $P^+=$ constant reduces to a subcase of the possibilities for $P^+=0$. For $P^+\neq$ constant, we find that (5.37) has only the trivial solution $R=0=S$ unless there is duplication between the real exponents on the left- and right- hand sides, implying that either $S=0$ or $R=0$. For the case $S=0$

equation (5.37) reduces to

$$Re^{2P^+}e^{-2i\zeta^-} + Re^{2(P^+-\zeta^+)} = R^*e^{2P^+} + R^*e^{2i\zeta^-},$$

from which we can deduce that $P^+ = \zeta^+$ with

$$R = R^*e^{2i\zeta^-} = R^*e^{-2i\zeta^-},$$

so giving

$$\zeta^- = \frac{n\pi}{2}, \qquad n \quad \text{integer}.$$

Without loss of generality, this yields just the two possibilities

$$A = R_1 e^{-iP^-}\operatorname{sech}\zeta^+, \qquad R_1 \quad \text{real;} \quad n \quad \text{even},$$

or (5.41)

$$A = iW_1 e^{-iP^-}\operatorname{cosech}\zeta^+, \qquad W_1 \quad \text{real;} \quad n \quad \text{odd},$$

with $R_1 = iW_1 = \sqrt{2/(1+h)}\,R/2$. Similarly, for $R = 0$ we obtain $P^+ = -\zeta^+$ with

$$S = S^*e^{2i\zeta^-} = S^*e^{-2i\zeta^-},$$

leading to just the two possibilities (5.41) with S replacing R. Correspondingly, equation (5.38) leads to $Q^+ = \zeta^+$, for $\widehat{S} = 0$, with

$$C = \widehat{R}_1 e^{-iQ^-}\operatorname{sech}\zeta^+, \qquad \widehat{R_1} \quad \text{real;} \quad n \quad \text{even},$$

or (5.42)

$$C = i\widehat{W}_1 e^{-iQ^-}\operatorname{cosech}\zeta^+, \qquad \widehat{W}_1 \quad \text{real;} \quad n \quad \text{odd},$$

with $\widehat{R}_1 = i\widehat{W}_1 = \sqrt{2/(1+h)}\,\widehat{R}/2$. Similarly, for $\widehat{R} = 0$ we obtain $Q^+ = -\zeta^+$ leading to the two possibilities (5.42) with $\widehat{S}$ replacing $\widehat{R}$. Since we require bounded solutions, and cosech is singular, we consider only the sech solutions for A and C. If we now try to restrict ζ^+, P^- and Q^- using the algebraic restrictions (5.29) and (5.34), for the case $S = 0$, it is found that $\zeta^+ = b(x + 2bt - \overline{x})$, $P^- = -bx + \delta_1$ and $Q^- = -bx + \widehat{\delta}_1$, where b, $\overline{x}$, δ_1 and $\widehat{\delta}_1$ are real arbitrary constants. Similar results are obtained for $R = 0$.

The solution obtained for A, and C is too restrictive, since more general solutions of a similar form have previously been identified. Although seeking solutions in the form of truncated Painlevé expansions yields possible solutions, the algebraic restrictions emanating from Painlevé matching of the various terms appear to be much too restrictive.

However the Painlevé analysis does suggest that there exist solutions of the form

$$A = \widehat{A}e^{-i\psi_1}\operatorname{sech}\zeta^+ , \qquad C = \widehat{C}e^{-i\psi_2}\operatorname{sech}\zeta^+ , \tag{5.43}$$

with $\zeta^+ = a(x - Vt + \mu)$, $\psi_1 = k_1x - \omega_1t + \delta_1$ and $\psi_2 = k_2x - \omega_2t + \delta_2$. Direct substitution into equations (4.11) then shows that these assumed forms do indeed give exact solutions subject to the conditions

$$k_1 = k_2 = k , \qquad \omega_1 = \omega_2 = k^2 - a^2 , \qquad V = 2k ,$$

with either $\widehat{A} = \widehat{C} = 2a^2/(1+h)$ or $\widehat{A} = 2a^2$ and $\widehat{C} = 0$. The first case describes a linearly polarised pulse while the second describes a circularly polarised pulse (Parker and Newboult, 1989). This analysis indicates that Painlevé expansions can be used to suggest possible general forms of solution, but that the only solutions identified are ones already known.

5.3 The truncation procedure for $h = 3$

In Chapter 4 the constant coefficient nonlinear Schrödinger equations (4.11) were shown to have the conditional Painlevé property when the coupling coefficient h equals 3. For this case equations (4.11) are not, in general, integrable although some special solutions exist (see Section 4.4). In this section we seek solutions to equations (4.11), for $h = 3$, which are of the form of the truncated Painlevé series given in (5.1). The analysis differs from that given in the previous section, however, since the matrix equivalent to that in (5.8) is singular. Substituting the truncated series (5.1) into the complexified nonlinear Schrödinger equations (4.12) gives the system of equations (5.2)–(5.5) with $h = 3$. Again the functions ϕ, A_0, B_0, C_0, D_0, A_1, B_1, C_1, D_1 are found by insisting that terms of each degree ϕ^{-3}, ϕ^{-2}, ϕ^{-1}, ϕ^0 are satisfied individually.

For terms of $O(\phi^{-3})$, the analysis is analogous to that given in Section 5.2 with $h = 3$ and hence we find

$$A_0B_0 = C_0D_0 = -\frac{\phi_x^2}{2}\,.$$

The functions A_0, B_0, C_0, D_0 can be written in the form

$$\begin{aligned} A_0 &= \frac{1}{\sqrt{2}}\phi_x e^{-P}\,, & B_0 &= -\frac{1}{\sqrt{2}}\phi_x e^{P}\,, \\ C_0 &= \frac{1}{\sqrt{2}}\phi_x e^{-Q}\,, & D_0 &= -\frac{1}{\sqrt{2}}\phi_x e^{Q}\,, \end{aligned} \tag{5.44}$$

where $P(x,t)$ and $Q(x,t)$ are arbitrary functions.

Comparing terms of $O(\phi^{-2})$ in equations (5.2)–(5.5) and using the expressions given in (5.44), we obtain equations analogous to (5.8), although in this case the coefficient matrix on the left-hand side is singular. By introducing the arbitrary function $R(x,t)$, solutions to the system (5.8) can be written as

$$\begin{aligned} \frac{A_1}{A_0} &= -\frac{1}{2\phi_x}\left(\frac{\phi_{xx}}{\phi_x} - 2P_x - \mathrm{i}\frac{\phi_t}{\phi_x} + R\right)\,, \\ \frac{B_1}{B_0} &= -\frac{1}{2\phi_x}\left(\frac{\phi_{xx}}{\phi_x} + 2P_x + \mathrm{i}\frac{\phi_t}{\phi_x} + R\right)\,, \\ \frac{C_1}{C_0} &= -\frac{1}{2\phi_x}\left(\frac{\phi_{xx}}{\phi_x} - 2Q_x - \mathrm{i}\frac{\phi_t}{\phi_x} - R\right)\,, \\ \frac{D_1}{D_0} &= -\frac{1}{2\phi_x}\left(\frac{\phi_{xx}}{\phi_x} + 2Q_x + \mathrm{i}\frac{\phi_t}{\phi_x} - R\right)\,. \end{aligned} \tag{5.45}$$

Terms which are of $O(\phi^{-1})$ in (5.2)–(5.5) give a system of equations analogous to (5.10). Substitution of (5.44) and (5.45) into these equations gives

$$3\left(\frac{\phi_{xx}}{\phi_x}\right)^2 + \left(\frac{\phi_t}{\phi_x}\right)^2 - 2\frac{\phi_{xxx}}{\phi_x} + 2\mathrm{i}\frac{\phi_{xt}}{\phi_x} - 2\mathrm{i}\frac{\phi_{xx}}{\phi_x}\frac{\phi_t}{\phi_x}$$
$$= 3\left(P_x^2 + Q_x^2\right) + \mathrm{i}\frac{\phi_t}{\phi_x}\left(P_x + 3Q_x\right) + 2\mathrm{i}P_t - 2P_{xx} - 2P_xR - \mathrm{i}\frac{\phi_t}{\phi_x}R\,, \tag{5.46}$$

$$3\left(\frac{\phi_{xx}}{\phi_x}\right)^2+\left(\frac{\phi_t}{\phi_x}\right)^2-2\frac{\phi_{xxx}}{\phi_x}-2\mathrm{i}\frac{\phi_{xt}}{\phi_x}+2\mathrm{i}\frac{\phi_{xx}}{\phi_x}\frac{\phi_t}{\phi_x}$$
$$=3\left(P_x^2+Q_x^2\right)+\mathrm{i}\frac{\phi_t}{\phi_x}\left(P_x+3Q_x\right)+2\mathrm{i}P_t+2P_{xx}+2P_xR+\mathrm{i}\frac{\phi_t}{\phi_x}R\,, \quad (5.47)$$

$$3\left(\frac{\phi_{xx}}{\phi_x}\right)^2+\left(\frac{\phi_t}{\phi_x}\right)^2-2\frac{\phi_{xxx}}{\phi_x}+2\mathrm{i}\frac{\phi_{xt}}{\phi_x}-2\mathrm{i}\frac{\phi_{xx}}{\phi_x}\frac{\phi_t}{\phi_x}$$
$$=3\left(P_x^2+Q_x^2\right)+\mathrm{i}\frac{\phi_t}{\phi_x}\left(3P_x+Q_x\right)+2\mathrm{i}Q_t-2Q_{xx}+2Q_xR+\mathrm{i}\frac{\phi_t}{\phi_x}R\,, \quad (5.48)$$

$$3\left(\frac{\phi_{xx}}{\phi_x}\right)^2+\left(\frac{\phi_t}{\phi_x}\right)^2-2\frac{\phi_{xxx}}{\phi_x}-2\mathrm{i}\frac{\phi_{xt}}{\phi_x}+2\mathrm{i}\frac{\phi_{xx}}{\phi_x}\frac{\phi_t}{\phi_x}$$
$$=3\left(P_x^2+Q_x^2\right)+\mathrm{i}\frac{\phi_t}{\phi_x}\left(3P_x+Q_x\right)+2\mathrm{i}Q_t+2Q_{xx}-2Q_xR-\mathrm{i}\frac{\phi_t}{\phi_x}R\,. \quad (5.49)$$

The solvability of this set of equations imposes restrictions on the functions ϕ, P, Q and R arising in (5.44) and (5.45). By adding (5.46) to (5.47) and (5.48) to (5.49), we obtain

$$3\left(\frac{\phi_{xx}}{\phi_x}\right)^2+\left(\frac{\phi_t}{\phi_x}\right)^2-2\frac{\phi_{xxx}}{\phi_x}=3\left(P_x^2+Q_x^2\right)+\mathrm{i}\frac{\phi_t}{\phi_x}\left(P_x+3Q_x\right)+2\mathrm{i}P_t\,, \quad (5.50)$$

$$3\left(\frac{\phi_{xx}}{\phi_x}\right)^2+\left(\frac{\phi_t}{\phi_x}\right)^2-2\frac{\phi_{xxx}}{\phi_x}=3\left(P_x^2+Q_x^2\right)+\mathrm{i}\frac{\phi_t}{\phi_x}\left(3P_x+Q_x\right)+2\mathrm{i}Q_t\,. \quad (5.51)$$

Subtracting these equations leads to

$$P_t-\frac{\phi_t}{\phi_x}P_x=Q_t-\frac{\phi_t}{\phi_x}Q_x\,,$$

and since

$$\frac{\phi_t}{\phi_x}=-\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{\phi=\mathrm{const}}\,,$$

it can be deduced that along loci $\phi(x,t)=$ constant we have

$$\frac{\mathrm{d}P}{\mathrm{d}t}=\frac{\mathrm{d}Q}{\mathrm{d}t}\,.$$

Consequently, it can be deduced that $P-Q=$ constant and hence

$$P-Q=2M(\phi)\,,$$

or

$$P = -T(x,t) + M\,, \qquad Q = -T(x,t) - M\,, \tag{5.52}$$

where $T(x,t)$ is an arbitrary function and $\phi = \Phi(M)$. Adding (5.50) to (5.51) leads to the restriction

$$3\left(\frac{\Phi''}{\Phi'}M_x + \frac{M_{xx}}{M_x}\right)^2 + \left(\frac{M_t}{M_x}\right)^2 - 2M_x^2\frac{\Phi'''}{\Phi'} - 6M_{xx}\frac{\Phi''}{\Phi'} - 2\frac{M_{xxx}}{M_x}$$
$$= 6\left(M_x^2 + T_x^2\right) - 4\mathrm{i}T_x\frac{M_t}{M_x} - 2\mathrm{i}T_t\,. \tag{5.53}$$

Further restrictions on the arbitrary functions can be found by subtracting (5.47) from (5.46) and (5.49) from (5.48) to give

$$\mathrm{i}\frac{M_{xt}}{M_x} - \mathrm{i}\frac{M_t}{M_x}\frac{M_{xx}}{M_x} = T_{xx} - M_{xx} + T_xR - M_xR - \frac{\mathrm{i}}{2}R\frac{M_t}{M_x}\,,$$

$$\mathrm{i}\frac{M_{xt}}{M_x} - \mathrm{i}\frac{M_t}{M_x}\frac{M_{xx}}{M_x} = T_{xx} + M_{xx} - T_xR - M_xR + \frac{\mathrm{i}}{2}R\frac{M_t}{M_x}\,.$$

Subtracting and adding this pair of equations gives the equivalent system

$$2M_{xx} = 2T_xR - \mathrm{i}R\frac{M_t}{M_x}\,, \tag{5.54}$$

$$\mathrm{i}\frac{\partial}{\partial x}\left(R\frac{M_t}{M_x}\right) = T_{xx} - M_xR\,. \tag{5.55}$$

Comparing terms of $O(\phi^0)$ in equations (5.2)–(5.5) we find that A_1, B_1, C_1 and D_1 must also satisfy the complexified coupled nonlinear Schrödinger equations (5.21)–(5.24) with $h = 3$. By applying analysis similar to that used in Section 5.2 we consider the combinations [(5.21) + (5.22) − (5.23) − (5.24)], [(5.21) + (5.22) + (5.23)+(5.24)], [(5.21)−(5.22)−(5.23)+(5.24)] and [(5.21)−(5.22)+(5.23)−(5.24)] to give the equivalent system

$$8\mathrm{i}M_{xt} - 2\mathrm{i}M_{xx}\frac{M_t}{M_x} - 12\left(M_{xx}T_x + T_{xx}M_x\right) - R^3 - 4\mathrm{i}RT_x\frac{M_t}{M_x}$$
$$+6R\left(M_x^2 + T_x^2\right) - 2\mathrm{i}RT_t - R\left(\frac{M_t}{M_x}\right)^2 + 2R_{xx} = 0\,, \tag{5.56}$$

$$2M_x^3\frac{\Phi''''}{\Phi'} - 6M_x^3\frac{\Phi'''}{\Phi'}\frac{\Phi''}{\Phi'} + 6M_xM_{xx}\frac{\Phi'''}{\Phi'} + 3M_x^3\left(\frac{\Phi''}{\Phi'}\right)^3 - 9M_xM_{xx}\left(\frac{\Phi''}{\Phi'}\right)^2$$

$$+\left(2M_{xxx} - \frac{M_t^2}{M_x} - 3\frac{M_{xx}^2}{M_x} - 4\mathrm{i}M_tT_x + 6M_x^3 - 2\mathrm{i}M_xT_t + 6M_xT_x^2\right)\frac{\Phi''}{\Phi'} + 2\frac{M_{xt}}{M_x}\frac{M_t}{M_x}$$

$$-4\mathrm{i}T_x\frac{M_{xt}}{M_x} - 2\frac{M_{tt}}{M_x} - \left(\frac{M_t}{M_x}\right)^2\frac{M_{xx}}{M_x} - 2\mathrm{i}T_{xx}\frac{M_t}{M_x} + 2\frac{M_{xxxx}}{M_x} - 6\frac{M_{xxx}}{M_x}\frac{M_{xx}}{M_x}$$

$$+3\left(\frac{M_{xx}}{M_x}\right)^3 + 18M_{xx}M_x - 2\mathrm{i}T_t\frac{M_{xx}}{M_x} + 6T_x^2\frac{M_{xx}}{M_x} - 4\mathrm{i}T_{xt} + 12T_{xx}T_x = 0\,, \tag{5.57}$$

$$4M_x^3\frac{\Phi'''}{\Phi'} - 6M_x^3\left(\frac{\Phi''}{\Phi'}\right)^2 + (2M_{xx}M_x - 2M_xT_xR + \mathrm{i}M_tR)\frac{\Phi''}{\Phi'}$$

$$-2\frac{M_t^2}{M_x} - 8\mathrm{i}M_tT_x + 8M_{xxx} - 4\frac{M_{xx}^2}{M_x} + 12M_x^3 - 4\mathrm{i}M_xT_t + 12M_xT_x^2$$

$$+\mathrm{i}R\frac{M_t}{M_x}\frac{M_{xx}}{M_x} - 2RT_x\frac{M_{xx}}{M_x} - 2M_xR^2 + 2\mathrm{i}R_t - 4R_xT_x - 2T_{xx}R = 0\,, \tag{5.58}$$

$$\left(2\mathrm{i}M_tM_x - 4M_x^2T_x\right)\frac{\Phi'''}{\Phi'} + \left(6M_x^2T_x - 3\mathrm{i}M_tM_x\right)\left(\frac{\Phi''}{\Phi'}\right)^2$$

$$+\left(2\mathrm{i}M_{xt} - 2\mathrm{i}M_{xx}\frac{M_t}{M_x} - 2M_xT_{xx} - 2M_x^2R\right)\frac{\Phi''}{\Phi'} + 4\mathrm{i}\frac{M_{xxt}}{M_x} - 6\mathrm{i}\frac{M_{xt}}{M_x}\frac{M_{xx}}{M_x}$$

$$-\mathrm{i}\left(\frac{M_t}{M_x}\right)^3 + 6T_x\left(\frac{M_t}{M_x}\right)^2 - 2\mathrm{i}\frac{M_t}{M_x}\frac{M_{xxx}}{M_x} + 3\mathrm{i}\frac{M_t}{M_x}\left(\frac{M_{xx}}{M_x}\right)^2 + 6\mathrm{i}M_tM_x$$

$$+2T_t\frac{M_t}{M_x} + 14\mathrm{i}T_x^2\frac{M_t}{M_x} - 4T_x\frac{M_{xxx}}{M_x} + 6T_x\left(\frac{M_{xx}}{M_x}\right)^2 - 2T_{xx}\frac{M_{xx}}{M_x} - 12M_x^2T_x$$

$$+4\mathrm{i}T_tT_x - 4T_{xxx} - 12T_x^3 - \mathrm{i}R^2\frac{M_t}{M_x} + 2T_xR^2 + 4M_{xx}R + 4M_xR_x = 0\,. \tag{5.59}$$

The functions Φ, M, T, R must satisfy equations (5.53)–(5.55) which were obtained from $O(\phi^{-1})$ terms and equations (5.56)–(5.59) obtained from $O(\phi^0)$ analysis. It is difficult to determine how to find the most general solution of this overdetermined system. However, there is considerable simplification in choosing M and T linear in x and t, so suggesting the choices

$$M = a(x + Vt) + \mu\,, \qquad T = cx + dt + \tau\,, \qquad \Phi = le^{gM} + \alpha\,, \qquad R = 0\,, \tag{5.60}$$

where a, V, c, d, μ, τ, l, g, α are all complex constants. Then we find that equations (5.54), (5.55) and (5.56) are identically satisfied. Equations (5.53), (5.57) and (5.58) reduce to the equation

$$a^2g^2 = 6(a^2 + c^2) - V^2 - 4\mathrm{i}Vc - 2\mathrm{i}d\,, \tag{5.61}$$

while equation (5.59) reduces to

$$(2c - iV)(a^2g^2 - 6(a^2 + c^2) + V^2 + 4\mathrm{i}Vc + 2\mathrm{i}d) = 0\,,$$

which is identically satisfied using (5.61).

An expression for A is obtained by substituting (5.44), (5.45) and (5.60) in the truncated series (5.1) for A as

$$A = \frac{1}{\sqrt{2}}e^{T-M}\left[\frac{ag}{2}\left(\frac{le^{gM} - \alpha}{le^{gM} + \alpha}\right) + \mathrm{i}\frac{V}{2} - c + a\right].$$

By writing $ag = 2G$ and $\alpha = le^{2Gx_0}$, with x_0 a constant, we find that

$$A = \frac{1}{\sqrt{2}}e^{T-M}\left[G\tanh\left[G(x - x_0 + Vt) + \frac{1}{2}g\mu\right] + \mathrm{i}\frac{V}{2} - c + a\right].$$

Similar expressions for B, C and D can be found. If we write $\zeta = G(x - \widehat{x} + Vt)$ with $G\widehat{x} = Gx_0 - g\mu/2$, we then find

$$\begin{aligned}
A &= \frac{1}{\sqrt{2}}e^{T-M}\left[G\tanh\zeta + \mathrm{i}\frac{V}{2} - c + a\right],\\
B &= \frac{1}{\sqrt{2}}e^{-(T-M)}\left[-G\tanh\zeta + \mathrm{i}\frac{V}{2} - c + a\right],\\
C &= \frac{1}{\sqrt{2}}e^{T+M}\left[G\tanh\zeta + \mathrm{i}\frac{V}{2} - c - a\right],\\
D &= \frac{1}{\sqrt{2}}e^{-(T+M)}\left[-G\tanh\zeta + \mathrm{i}\frac{V}{2} - c - a\right].
\end{aligned}$$

Since M, T and ζ are linear in x and t, these expressions are analogous to those in (5.36) with the identifications $c = -\widehat{\Gamma}$, $V = -v$, $a = \hat{\alpha}$, $G = b$, $M - T = P$ and $-M - T = Q$. Consequently we deduce that further analysis to ensure that $B = A^*$, $D = C^*$ and that the arbitrary constants satisfy (5.61) will be too restrictive despite having to satisfy only one algebraic condition instead of two conditions which are required in the more general case given in Section 5.2. Although this case produces nothing more general than the solutions (5.43) obtained in Section 5.2, the only way to obtain a solution was to assume a form for the arbitrary functions as (5.60) and it is conceivable that there may be other classes of solution than those from the choices (5.60).

Chapter 6

Summary of results

For a fibre with longitudinal inhomogeneity we have shown that the pulse evolution is governed by a coupled pair of cubic Schrödinger equations. The exact form of these nonlinear evolution equations is determined by the length scale of the inhomogeneity. If these axial inhomogeneities occur on a scale comparable with a pulse width, so their length scale is much shorter than that associated with nonlinear effects, then only average properties of the inhomogeneities occur in the evolution equations. We considered the simplest case of periodic nonuniformities and have deduced that the nonlinear evolution equations reduce to the coupled pair of constant coefficient cubic Schrödinger equations. These equations are identical in form to the evolution equations for an axially homogeneous fibre (Parker and Newboult, 1989). However, for a longer length scale of the axial inhomogeneity that is comparable with the nonlinear evolution length, we found that the evolution equations are a coupled pair of cubic Schrödinger equations with coefficients which vary with the axial coordinate. Results obtained from numerical experiments for the variable-coefficient evolution equations were presented which show that both sech-envelope pulses and more general non-distorting pulses lose little amplitude even after propagating through many periods of an axial inhomogeneity of significant amplitude.

For a curved and twisted fibre, we have shown that by seeking a solution to Maxwell's equations in the form of a perturbation expansion, the correction fields involve both terms which are due to the amplitude modulation of the signal envelope and terms which are proportional to the curvature of the fibre. The equations governing the pulse evolution are derived and are shown to be a coupled pair of nonlinear Schrödinger equations with linear cross-coupling terms. The coefficients of the linear cross-coupling terms are proportional to the square of the curvature, whereas the coefficients of the linear terms are proportional to the torsion. For special cases of the curvature and torsion, the evolution equations have both linearly and circularly polarised pulse solutions. Results of numerical experiments when the curvature and torsion are constants indicate that a non-distorting pulse is unstable, but for larger values of the torsion the pulse evolution becomes more stable. Hence by increasing the torsion we can delay the onset of instability.

The Painlevé partial differential equation test of Weiss, Tabor and Carnevale was applied to the coupled pair of constant coefficient nonlinear Schrödinger equations. Values for the coupling constant h were determined for which these equations are integrable by the Painlevé property and we found that there were three possible cases that required consideration. For $h = 1$ we found that the correct number of arbitrary functions existed to satisfy the Painlevé partial differential equation test. Hence the coupled cubic Schrödinger equations satisfy the Painlevé property for the special case $h = 1$, and are therefore completely integrable. This is compatible with the existence of an inverse scattering transform for this case. For the case $h = 3$, we found that the Painlevé test allowed only six arbitrary functions, corresponding to eight integer resonance values and hence the equations are not completely integrable. For this case the coupled nonlinear Schrödinger equations are said to have the conditional Painlevé property and we have shown that special solutions exist in the form of isolated sech-pulses. For values of $h \neq 1, 3$ we find that there are six integer resonance values indicating that at most there can be six arbitrary functions and hence the equations are not completely integrable by the Painlevé property. These results are consistent with those obtained by Sahadevan et al. (1986).

Solutions to the coupled pair of constant coefficient nonlinear Schrödinger equations, which did not have the Painlevé property, were sought in the form of truncated Painlevé expansions. It was found that, although truncated Painlevé expansions do yield possible solutions, the algebraic restrictions imposed by the Painlevé matching of the various terms is too restrictive. However, the Painlevé analysis does suggest possible forms for solutions of the coupled constant coefficient nonlinear Schrödinger equations which describe either unmodulated solutions or sech-profile solutions. These solutions are not however of a form for which the truncated Painlevé series yields term-by-term agreement. The solutions are ones whose existence was previously known, so a major conclusion must be that for systems as complicated as the coupled pair of nonlinear Schrödinger equations, the truncated Painlevé procedure is unlikely, despite the apparent considerable generality of the complexified solutions, to yield new explicit results.

Appendix A

Dispersion relation and group slowness for linearized modes

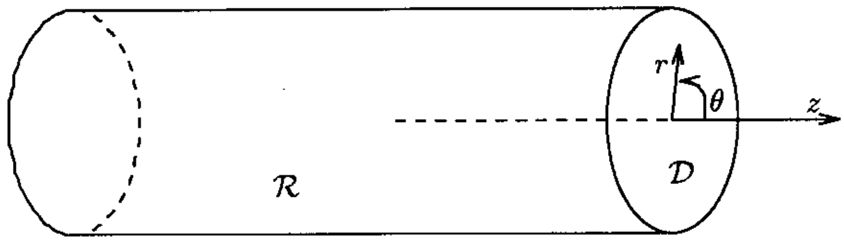

Figure A.1: Coordinate system for a straight fibre

The electromagnetic fields in a dielectric waveguide are governed by Maxwell's equations (1.1)–(1.4). By expressing the solution for the fields as a series in terms of a small amplitude parameter, a first approximation to the fields $\mathbf{E}$ and $\mathbf{H}$ is given by the solution of the linearized equations (see Section 2.2)

$$\nabla\wedge\mathbf{E} = -\mu_0\frac{\partial\mathbf{H}}{\partial t}\,, \tag{A.1}$$

$$\nabla\wedge\mathbf{H} = \varepsilon(r)\frac{\partial\mathbf{E}}{\partial t}\,, \tag{A.2}$$

$$\nabla\cdot(\varepsilon(r)\mathbf{E}) = 0\,, \tag{A.3}$$

$$\nabla\cdot\mathbf{H} = 0\,. \tag{A.4}$$

For an axially-symmetric fibre, we can seek solutions in the form of circularly polarised modes

$$\mathbf{E} = \mathbf{E}^{\pm}(r,\theta)e^{\mathrm{i}(\pm l\theta+\psi)}\,, \qquad \mathbf{H} = \mathbf{H}^{\pm}(r,\theta)e^{\mathrm{i}(\pm l\theta+\psi)}\,, \tag{A.5}$$

where l is the azimuthal mode number, the $+$ and $-$ represent the left- and right-handed circularly polarised modes, $\psi = kz - \omega t$ is a phase variable, k is the local wavenumber and ω is the radian frequency. The modal fields $\mathbf{E}^{\pm}$ and $\mathbf{H}^{\pm}$ satisfy the equations

$$\mathrm{L}\mathbf{E}^{\pm} \equiv \nabla\wedge \mathbf{E}^{\pm} - \frac{\mathrm{i}}{r}(\pm l)\mathbf{E}^{\pm}\wedge \mathbf{e}_{\theta} - \mathrm{i}k\mathbf{E}^{\pm}\wedge \mathbf{e}_z = \mathrm{i}\omega\mu_0\mathbf{H}^{\pm}\,, \qquad \text{(A.6)}$$

$$\mathrm{L}\mathbf{H}^{\pm} \equiv \nabla\wedge \mathbf{H}^{\pm} - \frac{\mathrm{i}}{r}(\pm l)\mathbf{H}^{\pm}\wedge \mathbf{e}_{\theta} - \mathrm{i}k\mathbf{H}^{\pm}\wedge \mathbf{e}_z = -\mathrm{i}\omega\varepsilon\mathbf{E}^{\pm}\,, \qquad \text{(A.7)}$$

with two other equations arising from (A.3) and (A.4) which are omitted from the following analysis because they can be shown to be consequences of (A.1) and (A.2). Here L is a first order differential operator. The complex conjugate equations of (A.6) and (A.7) are also true

$$\mathrm{L}^{*}\mathbf{E}^{\pm *} = \nabla\wedge \mathbf{E}^{\pm *} + \frac{\mathrm{i}}{r}(\pm l)\mathbf{E}^{\pm *}\wedge \mathbf{e}_{\theta} + \mathrm{i}k\mathbf{E}^{\pm *}\wedge \mathbf{e}_z = -\mathrm{i}\omega\mu_0\mathbf{H}^{\pm *}\,, \qquad \text{(A.8)}$$

$$\mathrm{L}^{*}\mathbf{H}^{\pm *} = \nabla\wedge \mathbf{H}^{\pm *} + \frac{\mathrm{i}}{r}(\pm l)\mathbf{H}^{\pm *}\wedge \mathbf{e}_{\theta} + \mathrm{i}k\mathbf{H}^{\pm *}\wedge \mathbf{e}_z = \mathrm{i}\omega\varepsilon\mathbf{E}^{\pm *}\,. \qquad \text{(A.9)}$$

The amplitudes of the fields of the guided modes decay rapidly away from the core (r fairly small) and it is usual not to impose boundary conditions over $\partial\mathcal{D}\times[0,\mathcal{L}]$ but to assume that $\mathbf{E}^{\pm}$ and $\mathbf{H}^{\pm}$ decay sufficiently rapidly that any 'flux' terms through $\partial\mathcal{D}\times[0,\mathcal{L}]$ may be neglected, where $\mathcal{L}$ is a representative length of fibre. The condition that equations (A.6) and (A.7) have non-trivial solutions which satisfy the decay condition as $r \to \infty$, is called the *dispersion relation.* To obtain an identity relating ω and k, consider any two pairs of functions $(\mathbf{u}(r,\theta), \mathbf{v}(r,\theta))$ and $(\mathbf{U}(r,\theta), \mathbf{V}(r,\theta))$ which satisfy the decay condition, then

$$\begin{aligned}
&\int_0^{\infty}\int_0^{2\pi}\int_0^{\infty}\{\mathbf{U}\cdot(\mathrm{L}\mathbf{u} - \mathrm{i}\omega\mu_0\mathbf{v}) + \mathbf{V}\cdot(\mathrm{L}\mathbf{v} + \mathrm{i}\omega\varepsilon\mathbf{u})\}\,r\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\
&= \int_0^{\infty}\int_0^{2\pi}\int_0^{\infty}\nabla\cdot\{\mathbf{u}\wedge\mathbf{U} + \mathbf{v}\wedge\mathbf{V}\}r\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \qquad \text{(A.10)} \\
&\quad + \int_0^{\infty}\int_0^{2\pi}\int_0^{\infty}\{\mathbf{u}\cdot(\mathrm{L}^{*}\mathbf{U} + \mathrm{i}\omega\varepsilon\mathbf{V}) + \mathbf{v}\cdot(\mathrm{L}^{*}\mathbf{V} - \mathrm{i}\omega\mu_0\mathbf{U})\}\,r\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z\,.
\end{aligned}$$

Noting that the fields $\mathbf{u}$, $\mathbf{v}$, $\mathbf{U}$ and $\mathbf{V}$ depend on only the transverse coordinates r, θ, application of the divergence theorem gives

$$\int_0^{\infty}\int_0^{2\pi}\int_0^{\infty}\nabla\cdot\{\mathbf{u}\wedge\mathbf{U} + \mathbf{v}\wedge\mathbf{V}\}r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = 0\,,$$

and (A.10) shows that the complex operator L^* is the operator adjoint to L.

If we choose

$$\mathbf{u} = \mathbf{E}^{\pm}, \qquad \mathbf{v} = \mathbf{H}^{\pm},$$

and

$$\mathbf{U} = \mathbf{H}^{\pm *}, \qquad \mathbf{V} = \mathbf{E}^{\pm *},$$

then we find that equations (A.6) and (A.7) give

$$\mathrm{L}\mathbf{u} - \mathrm{i}\omega\mu_0\mathbf{v} = \mathbf{0}, \qquad \mathrm{L}\mathbf{v} + \mathrm{i}\omega\varepsilon\mathbf{u} = \mathbf{0}\,,$$

and equations (A.8) and (A.9) give

$$\mathrm{L}^*\mathbf{U} + \mathrm{i}\omega\varepsilon\mathbf{V} = 2\mathrm{i}\omega\varepsilon\mathbf{E}^{\pm *}\,, \qquad \mathrm{L}^*\mathbf{V} - \mathrm{i}\omega\mu_0\mathbf{U} = -2\mathrm{i}\omega\mu_0\mathbf{H}^{\pm *}.$$

Then equation (A.10) reduces to

$$\int_0^{2\pi}\int_0^{\infty}\left(\varepsilon\mathbf{E}^{\pm}\cdot\mathbf{E}^{\pm *} - \mu_0\mathbf{H}^{\pm}\cdot\mathbf{H}^{\pm *}\right) r\,\mathrm{d}r\,\mathrm{d}\theta = 0\,, \tag{A.11}$$

which is one identity holding when (ω, k) satisfy the dispersion relation. From the representation of the modal fields (A.6), (A.7), it can be seen that the dispersion relation is independent of the sign of l. Since the modal fields $\mathbf{E}^{\pm}$ and $\mathbf{H}^{\pm}$ implicitly depend on k and ω, (A.11) can be written as

$$k = k(\omega) \tag{A.12}$$

for each value of l.

The field distribution given in (A.5) travels at the phase speed, whereas the power or the envelope of an amplitude modulated signal travels at the corresponding group speed v_g, where $v_g^{-1} = s_g$ the group slowness and

$$s_g = \frac{\mathrm{d}k}{\mathrm{d}\omega}$$

where k and ω are related by the dispersion relation (A.12). An expression for the group slowness can be found by differentiating equations (A.6) and (A.7) with respect to ω, to give

$$\mathrm{L}\frac{\partial \mathbf{E}^{\pm}}{\partial \omega} - \mathrm{i}\omega\mu_0\frac{\partial \mathbf{H}^{\pm}}{\partial \omega} = \mathrm{i}\frac{\mathrm{d}k}{\mathrm{d}\omega}\mathbf{E}^{\pm}\wedge \mathbf{e}_z + \mathrm{i}\mu_0\mathbf{H}^{\pm}\,, \tag{A.13}$$

$$\mathrm{L}\frac{\partial \mathbf{H}^{\pm}}{\partial \omega} + \mathrm{i}\omega\varepsilon\frac{\partial \mathbf{E}^{\pm}}{\partial \omega} = \mathrm{i}\frac{\mathrm{d}k}{\mathrm{d}\omega}\mathbf{H}^{\pm}\wedge \mathbf{e}_z - \mathrm{i}\varepsilon\mathbf{E}^{\pm}\,. \tag{A.14}$$

The choices

$$\mathbf{u} = \frac{\partial \mathbf{E}^{\pm}}{\partial \omega}\,, \qquad \mathbf{v} = \frac{\partial \mathbf{H}^{\pm}}{\partial \omega}\,, \qquad \mathbf{U} = \mathbf{H}^{\pm *}, \qquad \mathbf{V} = -\mathbf{E}^{\pm *},$$

in equation (A.10), combined with use of (A.8), (A.9), (A.13) and (A.14), then leads to

$$\frac{\mathrm{d}k}{\mathrm{d}\omega}\int_0^{2\pi}\int_0^{\infty}\left(\mathbf{E}^{\pm}\wedge\mathbf{H}^{\pm *} + \mathbf{E}^{\pm *}\wedge\mathbf{H}^{\pm}\right)\cdot\mathbf{e}_z\, r\,\mathrm{d}r\,\mathrm{d}\theta$$
$$= \int_0^{2\pi}\int_0^{\infty}\left(\varepsilon\mathbf{E}^{\pm}\cdot\mathbf{E}^{\pm *} + \mu_0\mathbf{H}^{\pm}\cdot\mathbf{H}^{\pm *}\right) r\,\mathrm{d}r\,\mathrm{d}\theta\,.$$

This gives an expression for the group slowness

$$s_g = \frac{\mathrm{d}k}{\mathrm{d}\omega} = \frac{\int_0^{2\pi}\int_0^{\infty}\left(\varepsilon\mathbf{E}^{\pm}\cdot\mathbf{E}^{\pm *} + \mu_0\mathbf{H}^{\pm}\cdot\mathbf{H}^{\pm *}\right) r\,\mathrm{d}r\,\mathrm{d}\theta}{\int_0^{2\pi}\int_0^{\infty}\left(\mathbf{E}^{\pm}\wedge\mathbf{H}^{\pm *} + \mathbf{E}^{\pm *}\wedge\mathbf{H}^{\pm}\right)\cdot\mathbf{e}_z\, r\,\mathrm{d}r\,\mathrm{d}\theta} \tag{A.15}$$

in the familiar form of a ratio of the average electromagnetic energy density to the average electromagnetic power flux over a fibre cross-section.

Appendix B

The multiple scales method

Optical pulses are the envelope of the amplitude modulations of the carrier wave with a width between $\sim$ 10ns to $\sim$ 10fs. Since the period of the carrier wave is much shorter than that of the pulse, each pulse contains several cycles of the carrier wave, and the envelope is said to be *slowly varying* with respect to the carrier wave.

Using the derivative-expansion version of the multiple scales method (Nayfeh, 1973), a set of *slow* space and time variables is introduced

$$\begin{aligned} Z_m &= \nu^m z, \\ T_m &= \nu^m t \end{aligned}$$

where $m = 1, 2, 3, \cdots$, and ν is a measure of the ratio of the wavelength of the carrier wave to the amplitude modulation length, such that $\nu \ll 1$. Slow variables in the radial direction do not need to be introduced, since the field is concentrated in the core (r fairly small) and decays rapidly as $r \to \infty$. The amplitudes of the fields are allowed to depend on the slow variables, so allowing equations which describe the envelope of the carrier wave to be derived.

The fields are expanded in terms of the small amplitude parameter ν

$$\begin{aligned} \mathbf{E} &= \nu \mathbf{E}^{(1)} + \nu^2 \mathbf{E}^{(2)} + \nu^3 \mathbf{E}^{(3)} + \cdots, \\ \mathbf{H} &= \nu \mathbf{H}^{(1)} + \nu^2 \mathbf{H}^{(2)} + \nu^3 \mathbf{H}^{(3)} + \cdots, \end{aligned}$$

and are assumed to be functions of z, t and the slow variables, such that

$$\mathbf{E}^{(m)} = \mathbf{E}^{(m)}(z, Z_1, Z_2, \cdots, t, T_1, T_2, \cdots),$$
$$\mathbf{H}^{(m)} = \mathbf{H}^{(m)}(z, Z_1, Z_2, \cdots, t, T_1, T_2, \cdots).$$

The z and t derivatives are replaced by

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z} + \nu\frac{\partial}{\partial Z_1} + \nu^2\frac{\partial}{\partial Z_2} + \cdots,$$
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \nu\frac{\partial}{\partial T_1} + \nu^2\frac{\partial}{\partial T_2} + \cdots.$$

By substituting the field expansions and the derivative expansions into Maxwell's equations and equating like powers of ν, equations which govern the fields $\mathbf{E}^{(m)}$ and $\mathbf{H}^{(m)}$ can be determined. These equations will have solutions which decay exponentially outside of the core region, $\mathbf{E}^{(m)}, \mathbf{H}^{(m)} \to 0$ as $r \to \infty$ and are finite at all points in the fibre if they satisfy the compatibility condition (2.19) (see Section 2.2). To $O(\nu)$, linearized versions of Maxwell's equations are obtained which automatically satisfy this condition. However, at $O(\nu^2)$ the compatibility condition is satisfied only if

$$\frac{\partial A}{\partial Z_1} + k_\omega\frac{\partial A}{\partial T_1} = 0,$$

where A is the slowly varying amplitude of the pulse and k_ω is the group slowness (s_g) (see Appendix A). This equation shows that to first order the envelope A moves with the group velocity of the carrier wave and suggests that the new scaled variables

$$\chi = \nu(k_\omega z - t), \qquad Z = \nu^2 z,$$

are introduced (see Hasegawa, 1989 and Newell and Moloney, 1992).

Appendix C

Correction fields for a curved fibre

The equations which govern the correction fields $\bar{\mathbf{E}}$ and $\bar{\mathbf{H}}$ of a curved axially homogeneous fibre are derived in Chapter 3, equations (3.27) and (3.28), and are found to be of the form

$$\begin{aligned} \nabla' \wedge \bar{\mathbf{E}} - \omega\mu_0 \frac{\partial \bar{\mathbf{H}}}{\partial \psi} &= \bar{\mathbf{G}}, \\ \nabla' \wedge \bar{\mathbf{H}} + \omega\varepsilon \frac{\partial \bar{\mathbf{E}}}{\partial \psi} &= \bar{\mathbf{F}}, \end{aligned}$$

where the expressions for $\bar{\mathbf{G}}$ and $\bar{\mathbf{F}}$ are given on the following two pages

$$
\begin{aligned}
\bar{\mathbf{G}} \;=\; & e^{\mathrm{i}(\theta+\psi)}\Big[\, A_Z^+\mathbf{E}^+\wedge\mathbf{e}_z - \mathrm{i}A_{XX}^+(s_g\mathbf{E}_\omega^+\wedge\mathbf{e}_z+\mu_0\mathbf{H}_\omega^+) \\
& \quad + \left(\frac{\kappa}{\nu}\right)^2\frac{A^+}{2}\Big\{\mathrm{i}kr^2\mathbf{E}^+\wedge\mathbf{e}_z - r(\mathbf{E}^+\cdot\mathbf{e}_z)\mathbf{e}_\theta + \mathrm{i}kr(\widehat{\mathbf{E}}^+ + \check{\mathbf{E}}^+)\wedge\mathbf{e}_z \\
& \quad\quad -(\widehat{\mathbf{E}}^+\cdot\mathbf{e}_z)(\mathrm{i}\mathbf{e}_r+\mathbf{e}_\theta) + (\check{\mathbf{E}}^+\cdot\mathbf{e}_z)(-\mathrm{i}\mathbf{e}_r+\mathbf{e}_\theta)\Big\} \\
& \quad + \left(\frac{\kappa}{\nu}\right)^2\frac{A^-}{2}e^{-2\mathrm{i}\phi}\Big\{\,\mathrm{i}\frac{kr^2}{2}\mathbf{E}^-\wedge\mathbf{e}_z - \frac{r}{2}(\mathbf{E}^-\cdot\mathbf{e}_z)(-\mathrm{i}\mathbf{e}_r+\mathbf{e}_\theta) \\
& \quad\quad +\mathrm{i}kr\,\check{\mathbf{E}}^-\wedge\mathbf{e}_z - (\check{\mathbf{E}}^-\cdot\mathbf{e}_z)(-\mathrm{i}\mathbf{e}_r+\mathbf{e}_\theta)\Big\}\Big] \\
& + e^{\mathrm{i}(-\theta+\psi)}\Big[\, A_Z^-\mathbf{E}^-\wedge\mathbf{e}_z - \mathrm{i}A_{XX}^-(s_g\mathbf{E}_\omega^-\wedge\mathbf{e}_z+\mu_0\mathbf{H}_\omega^-) \\
& \quad + \left(\frac{\kappa}{\nu}\right)^2\frac{A^-}{2}\Big\{\mathrm{i}kr^2\mathbf{E}^-\wedge\mathbf{e}_z - r(\mathbf{E}^-\cdot\mathbf{e}_z)\mathbf{e}_\theta + \mathrm{i}kr(\widehat{\mathbf{E}}^- + \check{\mathbf{E}}^-)\wedge\mathbf{e}_z \\
& \quad\quad -(\widehat{\mathbf{E}}^-\cdot\mathbf{e}_z)(-\mathrm{i}\mathbf{e}_r+\mathbf{e}_\theta) - (\check{\mathbf{E}}^-\cdot\mathbf{e}_z)(\mathrm{i}\mathbf{e}_r+\mathbf{e}_\theta)\Big\} \\
& \quad + \left(\frac{\kappa}{\nu}\right)^2\frac{A^+}{2}e^{2\mathrm{i}\phi}\Big\{\,\mathrm{i}\frac{kr^2}{2}\mathbf{E}^+\wedge\mathbf{e}_z - \frac{r}{2}(\mathbf{E}^+\cdot\mathbf{e}_z)(\mathrm{i}\mathbf{e}_r+\mathbf{e}_\theta) \\
& \quad\quad +\mathrm{i}kr\,\check{\mathbf{E}}^+\wedge\mathbf{e}_z - (\check{\mathbf{E}}^+\cdot\mathbf{e}_z)(\mathrm{i}\mathbf{e}_r+\mathbf{e}_\theta)\Big\}\Big] \\
& + e^{\mathrm{i}(2\theta+\psi-\phi)}\frac{\kappa}{\nu}\frac{A_X^+}{2}\Big[kr\mathbf{E}_\omega^+\wedge\mathbf{e}_z + rs_g\mathbf{E}^+\wedge\mathbf{e}_z \\
& \quad\quad +\mathrm{i}(\mathbf{E}_\omega^+\cdot\mathbf{e}_z)(-\mathrm{i}\mathbf{e}_r+\mathbf{e}_\theta) + 2s_g\widehat{\mathbf{E}}^+\wedge\mathbf{e}_z + 2\mu_0\widehat{\mathbf{H}}^+\Big] \\
& + e^{\mathrm{i}(-2\theta+\psi+\phi)}\frac{\kappa}{\nu}\frac{A_X^-}{2}\Big[kr\mathbf{E}_\omega^-\wedge\mathbf{e}_z + rs_g\mathbf{E}^-\wedge\mathbf{e}_z \\
& \quad\quad +\mathrm{i}(\mathbf{E}_\omega^-\cdot\mathbf{e}_z)(\mathrm{i}\mathbf{e}_r+\mathbf{e}_\theta) + 2s_g\widehat{\mathbf{E}}^-\wedge\mathbf{e}_z + 2\mu_0\widehat{\mathbf{H}}^-\Big] \\
& + e^{\mathrm{i}(\psi+\phi)}\frac{\kappa}{\nu}\frac{A_X^+}{2}\Big[kr\mathbf{E}_\omega^+\wedge\mathbf{e}_z + rs_g\mathbf{E}^+\wedge\mathbf{e}_z \\
& \quad\quad +\mathrm{i}(\mathbf{E}_\omega^+\cdot\mathbf{e}_z)(\mathrm{i}\mathbf{e}_r+\mathbf{e}_\theta) + 2s_g\,\check{\mathbf{E}}^+\wedge\mathbf{e}_z + 2\mu_0\,\check{\mathbf{H}}^+\Big] \\
& + e^{\mathrm{i}(\psi-\phi)}\frac{\kappa}{\nu}\frac{A_X^-}{2}\Big[kr\mathbf{E}_\omega^-\wedge\mathbf{e}_z + rs_g\mathbf{E}^-\wedge\mathbf{e}_z \\
& \quad\quad +\mathrm{i}(\mathbf{E}_\omega^-\cdot\mathbf{e}_z)(-\mathrm{i}\mathbf{e}_r+\mathbf{e}_\theta) + 2s_g\,\check{\mathbf{E}}^-\wedge\mathbf{e}_z + 2\mu_0\,\check{\mathbf{H}}^-\Big] \\
& + e^{\mathrm{i}(3\theta+\psi-2\phi)}\left(\frac{\kappa}{\nu}\right)^2\frac{A^+}{4}\Big[kr^2\mathbf{E}^+\wedge\mathbf{e}_z - r(\mathbf{E}^+\cdot\mathbf{e}_z)(-\mathrm{i}\mathbf{e}_r+\mathbf{e}_\theta) \\
& \quad\quad +2\mathrm{i}kr\widehat{\mathbf{E}}^+\wedge\mathbf{e}_z - 2(\widehat{\mathbf{E}}^+\cdot\mathbf{e}_z)(-\mathrm{i}\mathbf{e}_r+\mathbf{e}_\theta)\Big] \\
& + e^{\mathrm{i}(-3\theta+\psi+2\phi)}\left(\frac{\kappa}{\nu}\right)^2\frac{A^-}{4}\Big[kr^2\mathbf{E}^-\wedge\mathbf{e}_z - r(\mathbf{E}^-\cdot\mathbf{e}_z)(\mathrm{i}\mathbf{e}_r+\mathbf{e}_\theta) \\
& \quad\quad +2\mathrm{i}kr\widehat{\mathbf{E}}^-\wedge\mathbf{e}_z - 2(\widehat{\mathbf{E}}^-\cdot\mathbf{e}_z)(\mathrm{i}\mathbf{e}_r+\mathbf{e}_\theta)\Big] \\
& + c.c. + o(1)\,,
\end{aligned}
$$

$$
\begin{aligned}
\overline{\mathbf{F}} \;=\;& e^{\mathrm{i}(\theta+\psi)} \Big[A_Z^+ \mathbf{H}^+ \wedge \mathbf{e}_z - \mathrm{i} A_{XX}^+ (s_g \mathbf{H}_\omega^+ \wedge \mathbf{e}_z - \varepsilon \mathbf{E}_\omega^+) \\
& \quad + \left(\frac{\kappa}{\nu}\right)^2 \frac{A^+}{2} \Big\{ \mathrm{i}kr^2 \mathbf{H}^+ \wedge \mathbf{e}_z - r(\mathbf{H}^+ \cdot \mathbf{e}_z)\mathbf{e}_\theta + \mathrm{i}kr(\widehat{\mathbf{H}}^+ + \check{\mathbf{H}}^+) \wedge \mathbf{e}_z \\
& \qquad - (\widehat{\mathbf{H}}^+ \cdot \mathbf{e}_z)(\mathrm{i}\mathbf{e}_r + \mathbf{e}_\theta) - (\check{\mathbf{H}}^+ \cdot \mathbf{e}_z)(-\mathrm{i}\mathbf{e}_r + \mathbf{e}_\theta) \Big\} \\
& \quad + \left(\frac{\kappa}{\nu}\right)^2 \frac{A^-}{2} e^{-2\mathrm{i}\phi} \Big\{ \mathrm{i}\frac{kr^2}{2} \mathbf{H}^- \wedge \mathbf{e}_z - \frac{r}{2}(\mathbf{H}^- \cdot \mathbf{e}_z)(-\mathrm{i}\mathbf{e}_r + \mathbf{e}_\theta) \\
& \qquad + \mathrm{i}kr\, \check{\mathbf{H}}^- \wedge \mathbf{e}_z - (\check{\mathbf{H}}^- \cdot \mathbf{e}_z)(-\mathrm{i}\mathbf{e}_r + \mathbf{e}_\theta) \Big\} \Big] \\
& + e^{\mathrm{i}(-\theta+\psi)} \Big[A_Z^- \mathbf{H}^- \wedge \mathbf{e}_z - \mathrm{i} A_{XX}^- (s_g \mathbf{H}_\omega^- \wedge \mathbf{e}_z - \varepsilon \mathbf{E}_\omega^-) \\
& \quad + \left(\frac{\kappa}{\nu}\right)^2 \frac{A^-}{2} \Big\{ \mathrm{i}kr^2 \mathbf{H}^- \wedge \mathbf{e}_z - r(\mathbf{H}^- \cdot \mathbf{e}_z)\mathbf{e}_\theta + \mathrm{i}kr(\widehat{\mathbf{H}}^- + \check{\mathbf{H}}^-) \wedge \mathbf{e}_z \\
& \qquad - (\widehat{\mathbf{H}}^- \cdot \mathbf{e}_z)(-\mathrm{i}\mathbf{e}_r + \mathbf{e}_\theta) - (\check{\mathbf{H}}^- \cdot \mathbf{e}_z)(\mathrm{i}\mathbf{e}_r + \mathbf{e}_\theta) \Big\} \\
& \quad + \left(\frac{\kappa}{\nu}\right)^2 \frac{A^+}{2} e^{2\mathrm{i}\phi} \Big\{ \mathrm{i}\frac{kr^2}{2} \mathbf{H}^+ \wedge \mathbf{e}_z - \frac{r}{2}(\mathbf{H}^+ \cdot \mathbf{e}_z)(\mathrm{i}\mathbf{e}_r + \mathbf{e}_\theta) \\
& \qquad + \mathrm{i}kr\, \check{\mathbf{H}}^+ \wedge \mathbf{e}_z - (\check{\mathbf{H}}^+ \cdot \mathbf{e}_z)(\mathrm{i}\mathbf{e}_r + \mathbf{e}_\theta) \Big\} \Big] \\
& + e^{\mathrm{i}(2\theta+\psi-\phi)} \frac{\kappa}{\nu} \frac{A_X^+}{2} \Big[kr \mathbf{H}_\omega^+ \wedge \mathbf{e}_z + r s_g \mathbf{H}^+ \wedge \mathbf{e}_z \\
& \qquad - \mathrm{i}(\mathbf{H}_\omega^+ \cdot \mathbf{e}_z)(-\mathrm{i}\mathbf{e}_r + \mathbf{e}_\theta) + 2 s_g \widehat{\mathbf{H}}^+ \wedge \mathbf{e}_z - 2\varepsilon \widehat{\mathbf{E}}^+ \Big] \\
& + e^{\mathrm{i}(-2\theta+\psi+\phi)} \frac{\kappa}{\nu} \frac{A_X^-}{2} \Big[kr \mathbf{H}_\omega^- \wedge \mathbf{e}_z + r s_g \mathbf{H}^- \wedge \mathbf{e}_z \\
& \qquad - \mathrm{i}(\mathbf{H}_\omega^- \cdot \mathbf{e}_z)(\mathrm{i}\mathbf{e}_r + \mathbf{e}_\theta) + 2 s_g \widehat{\mathbf{H}}^- \wedge \mathbf{e}_z - 2\varepsilon \widehat{\mathbf{E}}^- \Big] \\
& + e^{\mathrm{i}(\psi+\phi)} \frac{\kappa}{\nu} \frac{A_X^+}{2} \Big[kr \mathbf{H}_\omega^+ \wedge \mathbf{e}_z + r s_g \mathbf{H}^+ \wedge \mathbf{e}_z \\
& \qquad - \mathrm{i}(\mathbf{H}_\omega^+ \cdot \mathbf{e}_z)(\mathrm{i}\mathbf{e}_r + \mathbf{e}_\theta) + 2 s_g \check{\mathbf{H}}^+ \wedge \mathbf{e}_z - 2\varepsilon \check{\mathbf{E}}^+ \Big] \\
& + e^{\mathrm{i}(\psi-\phi)} \frac{\kappa}{\nu} \frac{A_X^-}{2} \Big[kr \mathbf{H}_\omega^- \wedge \mathbf{e}_z + r s_g \mathbf{H}^- \wedge \mathbf{e}_z \\
& \qquad - \mathrm{i}(\mathbf{H}_\omega^- \cdot \mathbf{e}_z)(-\mathrm{i}\mathbf{e}_r + \mathbf{e}_\theta) + 2 s_g \check{\mathbf{H}}^- \wedge \mathbf{e}_z - 2\varepsilon \check{\mathbf{E}}^- \Big] \\
& + e^{\mathrm{i}(3\theta+\psi-2\phi)} \left(\frac{\kappa}{\nu}\right)^2 \frac{A^+}{4} \Big[kr^2 \mathbf{H}^+ \wedge \mathbf{e}_z - r(\mathbf{H}^+ \cdot \mathbf{e}_z)(-\mathrm{i}\mathbf{e}_r + \mathbf{e}_\theta) \\
& \qquad + 2\mathrm{i}kr\, \widehat{\mathbf{H}}^+ \wedge \mathbf{e}_z - 2(\widehat{\mathbf{H}}^+ \cdot \mathbf{e}_z)(-\mathrm{i}\mathbf{e}_r + \mathbf{e}_\theta) \Big] \\
& + e^{\mathrm{i}(-3\theta+\psi+2\phi)} \left(\frac{\kappa}{\nu}\right)^2 \frac{A^-}{4} \Big[kr^2 \mathbf{H}^- \wedge \mathbf{e}_z - r(\mathbf{H}^- \cdot \mathbf{e}_z)(\mathrm{i}\mathbf{e}_r + \mathbf{e}_\theta) \\
& \qquad + 2\mathrm{i}kr\, \widehat{\mathbf{H}}^- \wedge \mathbf{e}_z - 2(\widehat{\mathbf{H}}^- \cdot \mathbf{e}_z)(\mathrm{i}\mathbf{e}_r + \mathbf{e}_\theta) \Big] \\
& + \text{c.c.} - \omega \frac{\partial}{\partial \psi} \left(N |\mathbf{E}^{(1)}|^2 \mathbf{E}^{(1)} \right) + o(1) \,.
\end{aligned}
$$

Appendix D

Perturbation solution for solitary pulses of the coupled constant coefficient nonlinear Schrödinger equations

The coupled cubic Schrödinger equations

$$\begin{aligned} \mathrm{i}A_t^+ &= A_{xx}^+ + \left(|A^+|^2 + h|A^-|^2\right) A^+ , \\ \mathrm{i}A_t^- &= A_{xx}^- + \left(h|A^+|^2 + |A^-|^2\right) A^- , \end{aligned}$$

with constant coupling coefficient, h, arise from the study of pulse propagation in an axisymmetric, axially homogeneous optical fibre. Certain special solutions for these equations are discussed in (Parker and Newboult, 1989, Parker, 1988). More generally, these equations can have solutions of the form

$$\begin{aligned} A^+ &= e^{-\mathrm{i}(\beta_+ t + V\sigma)} F_+(\sigma) , \\ A^- &= e^{-\mathrm{i}(\beta_- t + V\sigma)} F_-(\sigma) , \end{aligned}$$

where β_+ and β_- are real adjustable parameters, V is a phase shift and F_+, F_- are real functions of $\sigma = x - 2Vt$, which satisfy the ordinary differential equations

$$F_+'' = (\gamma_+ - F_+^2 - hF_-^2)F_+ , \tag{D.1}$$

$$F_-'' = (\gamma_- - hF_+^2 - F_-^2)F_- , \tag{D.2}$$

where $\gamma_\pm = \beta_\pm - V^2$.

These equations possess solutions which are even in σ. Also they are invariant under the scalings $F_\pm(\sigma) \to kF_\pm(k\sigma)$, $\gamma_\pm \to k^2\gamma_\pm$. Consequently, to identify solutions describing isolated, symmetrical pulses it is sufficient to determine values γ_+, γ_- for which equations (D.1) and (D.2) possess solutions with

$$F_+(0) = \cos\alpha\,, \qquad F_-(0) = \sin\alpha\,,$$
$$F_+'(0) = F_-'(0) = 0\,, \tag{D.3}$$
$$F_+, F_-, F_+', F_-' \to 0 \qquad \text{as} \qquad \sigma \to \pm\infty\,.$$

By integrating the combination $(D.1) \times F_+' + (D.2) \times F_-'$, we find one relation between γ_+, γ_- and α is

$$\gamma_+ \cos^2\alpha + \gamma_- \sin^2\alpha = \frac{1}{2} + \frac{h-1}{4}\sin 2\alpha\,. \tag{D.4}$$

Since, for $h = 1$, equations (D.1)–(D.3) have solutions

$$F_+(\sigma) = \cos\alpha\,\mathrm{sech}(\sigma/\sqrt{2})\,, \qquad F_-(\sigma) = \sin\alpha\,\mathrm{sech}(\sigma/\sqrt{2})\,; \qquad \gamma_\pm = \frac{1}{2}\,, \tag{D.5}$$

we seek expansions for $\gamma_\pm$ and $F_\pm(\sigma)$ which reduce to (D.5) as $h \to 1$.

We first write (D.1) and (D.2) as

$$F_\pm''(\sigma) + \frac{1+h}{2}\left\{\frac{-2\gamma_\pm}{1+h} + F_+^2 + F_-^2\right\} F_\pm = \pm\frac{h-1}{2}\left(F_+^2 - F_-^2\right) F_\pm$$

which suggests introduction of $\varepsilon = (h-1)/(h+1)$ as the expansion parameter and use of the change of variable $\sigma = \eta\sqrt{1+h}/2$. Then, by writing

$$\gamma_\pm = \frac{1}{4}(1+h)(1+\varepsilon b_\pm)\,,$$

we obtain equations (D.1) and (D.2) as

$$F_+''(\eta) + \left[2\left(F_+^2 + F_-^2\right) - 1\right] F_+ = \varepsilon b_+ F_+ + 2\varepsilon\left(F_+^2 - F_-^2\right) F_+\,, \tag{D.6}$$
$$F_-''(\eta) + \left[2\left(F_+^2 + F_-^2\right) - 1\right] F_- = \varepsilon b_- F_- + 2\varepsilon\left(F_-^2 - F_+^2\right) F_-\,. \tag{D.7}$$

Then expanding F_+, F_-, b_+ and b_- as power series in the small parameter ε as

$$F_\pm(\eta) = \sum_{j=0}^{\infty} \varepsilon^j \overset{j}{F}_\pm\,, \qquad b_\pm = \sum_{j=0}^{\infty} \varepsilon^j \overset{j}{b}_\pm\,,$$

and substituting these expressions into equations (D.6) and (D.7), we find that leading order terms give the system of ordinary differential equations

$$\begin{aligned} \overset{0}{F}{}''_+ + \left[2\left(\overset{0}{F}{}^2_+ + \overset{0}{F}{}^2_-\right) - 1\right]\overset{0}{F}_+ &= 0\,, \\ \overset{0}{F}{}''_- + \left[2\left(\overset{0}{F}{}^2_+ + \overset{0}{F}{}^2_-\right) - 1\right]\overset{0}{F}_- &= 0\,, \end{aligned}$$

with

$$\overset{0}{F}_+(0) = \cos\alpha\,, \qquad \overset{0}{F}_-(0) = \sin\alpha\,,$$

$$\overset{0}{F}{}'_+(0) = \overset{0}{F}{}'_-(0) = 0\,,$$

$$\overset{0}{F}_+, \overset{0}{F}_-, \overset{0}{F}{}'_+, \overset{0}{F}{}'_- \to 0 \quad \text{as} \quad \eta \to \pm\infty\,.$$

Hence, to $O(\varepsilon^0)$, an approximation to $F_\pm$ is given by

$$\begin{aligned} \overset{0}{F}_+ &= \cos\alpha\,\mathrm{sech}\,\eta\,, \\ \overset{0}{F}_- &= \sin\alpha\,\mathrm{sech}\,\eta\,, \end{aligned} \tag{D.8}$$

which corresponds to expressions (D.5).

A closer approximation for F_+ and F_- is obtained by comparing coefficients of ε in equations (D.6) and (D.7), which gives the coupled ordinary differential equations for $\overset{1}{F}_+$ and $\overset{1}{F}_-$

$$\begin{aligned} &\overset{1}{F}{}''_+ + \left[(6\cos^2\alpha + 2\sin^2\alpha)\,\mathrm{sech}^2\,\eta - 1\right]\overset{1}{F}_+ + 2\sin 2\alpha\,\mathrm{sech}^2\,\eta\overset{1}{F}_- \\ &\qquad = \overset{0}{b}_+\cos\alpha\,\mathrm{sech}\,\eta + 2\cos\alpha\cos 2\alpha\,\mathrm{sech}^3\,\eta\,, \end{aligned} \tag{D.9}$$

$$\begin{aligned} &\overset{1}{F}{}''_- + \left[(2\cos^2\alpha + 6\sin^2\alpha)\,\mathrm{sech}^2\,\eta - 1\right]\overset{1}{F}_- + 2\sin 2\alpha\,\mathrm{sech}^2\,\eta\overset{1}{F}_+ \\ &\qquad = \overset{0}{b}_-\sin\alpha\,\mathrm{sech}\,\eta - 2\sin\alpha\cos 2\alpha\,\mathrm{sech}^3\,\eta\,. \end{aligned} \tag{D.10}$$

By writing

$$R_1 = \overset{1}{F}_+\cos\alpha + \overset{1}{F}_-\sin\alpha\,,$$

$$R_2 = \overset{1}{F}_+\sin\alpha - \overset{1}{F}_-\cos\alpha\,,$$

we uncouple equations (D.9) and (D.10) as the linear ordinary differential equations

$$R_1'' + \left(6\,\mathrm{sech}^2\,\eta - 1\right) R_1 = \left(\overset{\circ}{b}_+ \cos^2\alpha + \overset{\circ}{b}_- \sin^2\alpha\right) \mathrm{sech}\,\eta + 2\cos^2 2\alpha\,\mathrm{sech}^3\,\eta = J\,, \tag{D.11}$$

$$R_2'' + \left(2\,\mathrm{sech}^2\,\eta - 1\right) R_2 = \frac{1}{2}\left(\overset{\circ}{b}_+ - \overset{\circ}{b}_-\right) \sin 2\alpha\,\mathrm{sech}\,\eta + \sin 4\alpha\,\mathrm{sech}^3\,\eta = K\,. \tag{D.12}$$

The conditions (D.3) impose the requirements

$$R_j(0) = 0\,, \qquad R_j'(0) = 0\,; \qquad R_j \to 0 \quad \text{as} \quad \eta \to \pm\infty\,; \qquad j = 1, 2\,.$$

The general solution to the homogeneous form of (D.11) is

$$\begin{aligned} R_1 &= A_1\,\mathrm{sech}\,\eta\,\tanh\eta + A_2(3\eta\,\mathrm{sech}\,\eta\,\tanh\eta + \cosh\eta - 3\,\mathrm{sech}\,\eta) \\ &= A_1 S_1(\eta) + A_2 S_2(\eta)\,, \end{aligned}$$

where A_1 and A_2 are arbitrary constants. Using the method of variation of parameters, we can write solutions to (D.11) in the form

$$\begin{aligned} R_1 &= V_1\,\mathrm{sech}\,\eta\,\tanh\eta + V_2(3\eta\,\mathrm{sech}\,\eta\,\tanh\eta + \cosh\eta - 3\,\mathrm{sech}\,\eta) \\ &= V_1(\eta) S_1(\eta) + V_2(\eta) S_2(\eta)\,, \end{aligned} \tag{D.13}$$

in which $S_1(\eta)$ is odd and bounded while $S_2(\eta)$ is even but unbounded. We impose the usual condition

$$V_1' S_1 + V_2' S_2 = 0\,, \tag{D.14}$$

which leads to

$$R_1' = V_1 S_1' + V_2 S_2'\,.$$

Substitution into (D.11) implies that

$$V_1' S_1' + V_2' S_2' = J\,. \tag{D.15}$$

Equations (D.14) and (D.15) can be solved to give formulae for V_1' and V_2' as

$$V_1' = -\frac{1}{2}JS_2\,, \qquad\qquad V_2' = \frac{1}{2}JS_1\,. \tag{D.16}$$

which are most readily integrated for V_2, then $V_1+3\eta V_2$. Observing that $R_1(0)=0$ imposes the condition $V_2(0)=0$ we find that

$$V_2 = \frac{1}{4}(\mathring{b}_1\cos^2\alpha + \mathring{b}_2\sin^2\alpha + 2\cos^2 2\alpha)\tanh^2\eta - \frac{1}{4}\cos^2 2\alpha\tanh^4\eta\,,$$

so that $V_2(\eta)S_2(\eta)$ is even and vanishes at $\eta=0$. Since (D.16) gives

$$\begin{aligned}(V_1+3\eta V_2)' &= 3V_2 - \frac{1}{2}\left\{\left(\mathring{b}_+\cos^2\alpha + \mathring{b}_-\sin^2\alpha\right)\operatorname{sech}\eta\right.\\ &\qquad \left.+2\cos^2 2\alpha\operatorname{sech}^3\eta\right\}(\cosh\eta - 3\operatorname{sech}\eta)\,,\end{aligned}$$

which is an even polynomial in $\operatorname{sech}\eta$. Since we require that $V_1+3\eta V_2$ is odd, we find that on integrating the above expression

$$\begin{aligned}V_1+3\eta V_2 &= \frac{1}{4}(\mathring{b}_1\cos^2\alpha + \mathring{b}_2\sin^2\alpha)(\eta + 3\tanh\eta)\\ &\quad +\frac{1}{4}\cos^2 2\alpha(3\eta + 5\tanh\eta + 3\tanh^3\eta)\,.\end{aligned}$$

We notice from (D.13) that R_1 has the form

$$R_1 = (V_1+3\eta V_2)\operatorname{sech}\eta\tanh\eta + V_2(\cosh\eta - 3\operatorname{sech}\eta)\,.$$

and since we require that $R_1 \to 0$ as $\eta\to\pm\infty$, we can deduce that $V_2\to 0$ as $\eta\to\pm\infty$. This shows that $\mathring{b}_+$ and $\mathring{b}_-$ are related by

$$\mathring{b}_+\cos^2\alpha + \mathring{b}_-\sin^2\alpha = -\cos^2 2\alpha\,. \tag{D.17}$$

It is then found that R_1 can be written as

$$R_1 = \frac{1}{2}\cos^2 2\alpha\,\eta\operatorname{sech}\eta\tanh\eta\,.$$

Applying the same argument, a solution to (D.12) is sought of the form

$$\begin{aligned}R_2 &= V_3\operatorname{sech}\eta + V_4(\sinh\eta + \eta\operatorname{sech}\eta)\\ &= V_3S_3(\eta) + V_4S_4(\eta)\,,\end{aligned}$$

with condition

$$V_3' S_3 + V_4' S_4 = 0 \tag{D.18}$$

which converts (D.12) into

$$V_3' S_3' + V_4' S_4' = K\,. \tag{D.19}$$

Equations (D.18) and (D.19) can be solved to give the following formulae for V_3' and V_4' as

$$V_3' = -\frac{1}{2} K S_4\,, \qquad V_4' = \frac{1}{2} K S_3\,, \tag{D.20}$$

with $S_3(\eta)$ and $K_3(\eta)$ even and bounded, while $S_4(\eta)$ is odd and unbounded. This situation is analogous to (D.16) and these equations are most readily integrated for V_4 and $V_3 + \eta V_4$ as

$$V_4 = \frac{1}{4}(\mathring{b}_1 - \mathring{b}_2)\sin 2\alpha \tanh\eta + \frac{1}{2}\sin 4\alpha(\tanh\eta - \frac{1}{3}\tanh^3\eta) + c_4\,,$$
$$V_3 + \eta V_4 = -\frac{1}{3}\sin 4\alpha(\ln \operatorname{sech}\eta + \frac{1}{2}\tanh^2\eta) + c_4\eta + c_3\,.$$

We observe that R_2 can be written in the form

$$R_2 = (V_3 + \eta V_4)\operatorname{sech}\eta + V_4 \sinh\eta\,,$$

and since we require that R_2 is even and $R_2(0) = 0$, we find that $c_4 = 0$ and $c_3 = 0$. We also require that $R_2 \to 0$ as $\eta \to \pm\infty$, and hence it can be deduced that $V_4 \to 0$ as $\eta \to \pm\infty$. From this we obtain a second relationship between $\mathring{b}_+$ and $\mathring{b}_-$

$$\mathring{b}_+ - \mathring{b}_- = -\frac{8}{3}\cos 2\alpha\,. \tag{D.21}$$

It is then found that

$$R_2 = -\frac{1}{3}\sin 4\alpha \operatorname{sech}\eta \ln \operatorname{sech}\eta\,.$$

Equations (D.17) and (D.21) can be solved to give the expressions for $\mathring{b}_+$ and $\mathring{b}_-$ as

$$\begin{aligned} \mathring{b}_+ &= -\frac{1}{3}\cos 2\alpha\left(3 + 2\sin^2\alpha\right)\,, \\ \mathring{b}_- &= \frac{1}{3}\cos 2\alpha\left(3 + 2\cos^2\alpha\right)\,, \end{aligned} \tag{D.22}$$

while the corresponding solutions to equations (D.9) and (D.10) can be written as

$$
\begin{aligned}
\overset{1}{F}_+ &= \frac{1}{2}\cos\alpha\cos^2 2\alpha\,\eta\,\mathrm{sech}\,\eta\tanh\eta - \frac{1}{3}\sin\alpha\sin 4\alpha\,\mathrm{sech}\,\eta\ln\mathrm{sech}\,\eta\,, \\
\overset{1}{F}_- &= \frac{1}{2}\sin\alpha\cos^2 2\alpha\,\eta\,\mathrm{sech}\,\eta\tanh\eta + \frac{1}{3}\cos\alpha\sin 4\alpha\,\mathrm{sech}\,\eta\ln\mathrm{sech}\,\eta\,.
\end{aligned}
\tag{D.23}
$$

To improve the series approximation to F_+ and F_- we consider terms of $O(\varepsilon^2)$ in equations (D.6) and (D.7), which give the coupled ordinary differential equations for $\overset{2}{F}_+$ and $\overset{2}{F}_-$

$$
\begin{aligned}
&\overset{2}{F}{}''_+ + \left[(6\cos^2\alpha + 2\sin^2\alpha)\,\mathrm{sech}^2\eta - 1\right]\overset{2}{F}_+ + 2\sin 2\alpha\,\mathrm{sech}^2\eta\,\overset{2}{F}_- \\
&\quad= \overset{1}{b}_+\overset{0}{F}_+ + \overset{0}{b}_+\overset{1}{F}_+ + 6\overset{0}{F}{}^2_+\overset{1}{F}_+ - 4\overset{0}{F}_+\overset{0}{F}_-\overset{1}{F}_- - 2\overset{0}{F}{}^2_-\overset{1}{F}_+ \\
&\quad\quad -6\overset{0}{F}_+\overset{1}{F}{}^2_+ - 4\overset{0}{F}_-\overset{1}{F}_+\overset{1}{F}_- - 2\overset{0}{F}_+\overset{1}{F}{}^2_-\,,
\end{aligned}
\tag{D.24}
$$

$$
\begin{aligned}
&\overset{2}{F}{}''_- + \left[(2\cos^2\alpha + 6\sin^2\alpha)\,\mathrm{sech}^2\eta - 1\right]\overset{2}{F}_- + 2\sin 2\alpha\,\mathrm{sech}^2\eta\,\overset{2}{F}_+ \\
&\quad= \overset{1}{b}_-\overset{0}{F}_- + \overset{0}{b}_-\overset{1}{F}_- + 6\overset{0}{F}{}^2_-\overset{1}{F}_- - 4\overset{0}{F}_+\overset{0}{F}_-\overset{1}{F}_+ - 2\overset{0}{F}{}^2_+\overset{1}{F}_- \\
&\quad\quad -4\overset{0}{F}_+\overset{1}{F}_+\overset{1}{F}_- - 6\overset{0}{F}_-\overset{1}{F}{}^2_- - 2\overset{0}{F}_-\overset{1}{F}{}^2_+\,.
\end{aligned}
\tag{D.25}
$$

By writing

$$
\begin{aligned}
R_3 &= \overset{2}{F}_+\cos\alpha + \overset{2}{F}_-\sin\alpha\,, \\
R_4 &= \overset{2}{F}_+\sin\alpha - \overset{2}{F}_-\cos\alpha\,,
\end{aligned}
$$

equations (D.24) and (D.25) uncouple to give the ordinary differential equations

$$
R_3'' + \left(6\,\mathrm{sech}^2\eta - 1\right)R_3 = L\,, \tag{D.26}
$$

$$
R_4'' + \left(2\,\mathrm{sech}^2\eta - 1\right)R_4 = M\,, \tag{D.27}
$$

where

$$
\begin{aligned}
L &= \left(\overset{1}{b}_+\cos^2\alpha + \overset{1}{b}_-\sin^2\alpha\right)\mathrm{sech}\,\eta \\
&\quad -\frac{1}{2}\cos^4 2\alpha\left[\eta\,\mathrm{sech}\,\eta\tanh\eta(6\tanh^2\eta - 5) + 3\eta^2\,\mathrm{sech}^3\eta\tanh^2\eta\right] \\
&\quad +\frac{1}{9}\sin^2 4\alpha\left[\mathrm{sech}\,\eta\ln\mathrm{sech}\,\eta(9\tanh^2\eta - 7) - 2\,\mathrm{sech}^3\eta(\ln\mathrm{sech}\,\eta)^2\right]\,,
\end{aligned}
$$

$$
\begin{aligned}
M &= \frac{1}{2}\left(\overset{1}{b}_+ - \overset{1}{b}_-\right)\sin 2\alpha \operatorname{sech}\eta - \frac{8}{3}\sin^3 2\alpha \cos 2\alpha \operatorname{sech}^3\eta \ln \operatorname{sech}\eta \\
&\quad -\frac{1}{9}\sin 2\alpha \cos^3 2\alpha \left[3(9\tanh^2\eta - 7)\eta \operatorname{sech}\eta \tanh\eta \right. \\
&\quad \left. +2\operatorname{sech}\eta \ln \operatorname{sech}\eta (6\tanh^2\eta - 1) - 12\eta \operatorname{sech}^3\eta \tanh\eta \ln \operatorname{sech}\eta\right] .
\end{aligned}
$$

Equations (D.26) and (D.27) are of the same form as (D.11) and (D.12), and hence can be solved analogously by making the substitutions

$$
\begin{aligned}
R_3 &= W_1(\eta)S_1(\eta) + W_2(\eta)S_2(\eta) , \\
R_4 &= W_3(\eta)S_3(\eta) + W_4(\eta)S_4(\eta) .
\end{aligned}
$$

As before, we can obtain expressions for derivatives of these functions in terms of L and M as

$$
W_1' = -\frac{1}{2}LS_2 , \quad W_2' = \frac{1}{2}LS_1 , \tag{D.28}
$$

$$
W_3' = -\frac{1}{2}MS_4 , \quad W_4' = \frac{1}{2}MS_3 . \tag{D.29}
$$

and solve (D.28) and (D.29) in terms of W_2, respectively, $W_1 + 3\eta W_2$ and W_4, $W_3 + \eta W_4$, as we can write

$$
\begin{aligned}
R_3 &= (W_1 + 3\eta W_2)\operatorname{sech}\eta \tanh\eta + W_2(\cosh\eta - 3\operatorname{sech}\eta) , \\
R_4 &= (W_3 + \eta W_4)\operatorname{sech}\eta + W_4 \sinh\eta .
\end{aligned}
$$

Since we require that $R_3, R_4 \to 0$ as $\eta \to \pm\infty$, we must have $W_2 \to 0$ and $W_4 \to 0$ as $\eta \to \pm\infty$, which, as for the $O(\varepsilon)$ analysis, leads to two conditions relating $\overset{1}{b}_+$ and $\overset{1}{b}_-$. After substantial manipulation these are found to be

$$
\overset{1}{b}_+ \cos^2\alpha + \overset{1}{b}_- \sin^2\alpha = 0 , \tag{D.30}
$$

$$
\overset{1}{b}_+ - \overset{1}{b}_- = -\frac{16}{27}\cos 2\alpha \left[\cos^2 2\alpha + (5 - 6\ln 2)\right] , \tag{D.31}
$$

which can be solved to give

$$
\begin{aligned}
\overset{1}{b}_+ &= -\frac{16}{27}\sin^2\alpha \cos 2\alpha \left[\cos^2 2\alpha + (5 - 6\ln 2)\right] , \\
\overset{1}{b}_- &= \frac{16}{27}\cos^2\alpha \cos 2\alpha \left[\cos^2 2\alpha + (5 - 6\ln 2)\right] .
\end{aligned} \tag{D.32}
$$

Correspondingly we find

$$R_3 = -\frac{1}{4}\cos^4 2\alpha\, \eta \,\mathrm{sech}\,\eta \left(\eta(1-2\tanh^2\eta) - \tanh\eta\right)$$
$$-\frac{1}{18}\sin^2 4\alpha\,\mathrm{sech}\,\eta \left[(\ln \mathrm{sech}\,\eta)^2 + 2\ln\mathrm{sech}\,\eta + \tanh\eta\left(\eta + 4\int \ln\mathrm{sech}\,\eta\, d\eta\right)\right],$$

$$\text{(D.33)}$$

$$R_4 = -\frac{1}{18}\sin 4\alpha\cos^2 2\alpha\,\mathrm{sech}\,\eta\left(4(\ln\mathrm{sech}\,\eta)^2 + 3\eta\tanh\eta(1+4\ln\mathrm{sech}\,\eta) + \frac{10}{3}\ln\mathrm{sech}\,\eta\right)$$
$$-\frac{2}{27}\sin 4\alpha\sin^2 2\alpha\left(3\,\mathrm{sech}\,\eta(\eta^2 - (\ln\mathrm{sech}\,\eta)^2 + 3\sinh\eta(\eta - 2\ln 2\tanh\eta)\right.$$
$$\left. + 2\cosh\eta\ln\mathrm{sech}\,\eta(4-\tanh^2\eta)\right).$$

Therefore, the required solution to equations (D.26) and (D.27) is obtained from

$$\begin{aligned} \overset{2}{F}_+ &= -R_3\cos\alpha + R_4\sin\alpha, \\ \overset{2}{F}_- &= R_3\sin\alpha + R_4\cos\alpha. \end{aligned} \qquad \text{(D.34)}$$

Hence, correct to $O(\varepsilon^2)$, the solution of equations (D.1) and (D.2) is found by substituting (D.23), (D.33) and (D.34) into

$$F_+ = \cos\alpha\,\mathrm{sech}\,\eta + \varepsilon\overset{1}{F}_+ + \varepsilon^2\overset{2}{F}_+ + O(\varepsilon^3),$$

$$F_- = \sin\alpha\,\mathrm{sech}\,\eta + \varepsilon\overset{1}{F}_- + \varepsilon^2\overset{2}{F}_- + O(\varepsilon^3),$$

where

$$\sigma = \eta\frac{\sqrt{1+h}}{2}.$$

The corresponding expansions for the parameters $\gamma_\pm$ is

$$\begin{aligned} \gamma_\pm &= \frac{1+h}{4}\left(1 + \varepsilon\overset{0}{b}_\pm + \varepsilon^2\overset{1}{b}_\pm\right) + O(\varepsilon^3) \\ &= \frac{1+h}{4}\left\{1 \mp \frac{\varepsilon}{3}\cos 2\alpha(4 \mp \cos 2\alpha)\right. \\ &\qquad \left. + \frac{8\varepsilon^2}{23}(\cos 2\alpha \mp 1)(\cos^2\alpha + 5 - 6\ln 2)\right\} + O(\varepsilon^3). \end{aligned} \qquad \text{(D.35)}$$

Recalling that $\varepsilon = (h-1)/(h+1)$, it is possible to use (D.35) as an approximation for the parameters required in (D.1) and (D.2) when seeking solutions to the nonlinear eigenvalue problem with decay conditions (D.3). In principal this gives an approximation to $\gamma_\pm$ for a range of values of α. However,in practice we are interested in values of $\varepsilon \approx 1/3$ and except for values of $\cos\alpha \approx \pi/4$, the values obtained for $\gamma_\pm$ are insufficiently accurate, indicating that more terms in the expansion are required.

Appendix E

Coupled evolution equations for axially inhomogeneous optical fibres

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Coupled evolution equations for axially inhomogeneous optical fibres

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At 'monomode' frequencies, a uniform axisymmetric optical fibre can support left- and right-handed circularly polarized modes having the same dispersion relation. Nonlinearity introduces cubic terms into the evolution equations, which are coupled nonlinear Schrödinger equations (Newboult, Parker, and Faulkner, 1989). This paper analyses signal propagation in axisymmetric fibres for which the distribution of dielectric properties varies gradually, but significantly, along the fibre. At each cross-section, left- and right-handed modal fields are defined, but their axial variations introduce changes into the coupled evolution equations. Two regimes are identified. When axial variations occur on length scales comparable with nonlinear evolution effects, the governing equations are determined as coupled nonlinear Schrödinger equations with variable coefficients. On the other hand, for more rapid axial variations it is found that the evolution equations have constant coefficients, defined as appropriate averages of those associated with each cross-section. Situations in which the variable coefficient equations may be transformed into constant coefficient equations are investigated. It is found that the only possibilities are natural generalizations of those found by Grimshaw (1979) for a single nonlinear Schrödinger equation. In such cases, suitable sech-envelope pulses will propagate without radiation. Numerical evidence is presented that, in some other cases with periodically varying coefficients, a sech-envelope pulse loses little amplitude even after propagating through 40 periods of axial inhomogeneity of significant amplitude.

1. Introduction

Optical fibres are thin cylindrical glass waveguides with a core of slightly higher refractive index than the cladding. This allows signals at optical frequencies to propagate as guided modes, each mode having a 'cut-off frequency', below which it does not propagate. At frequencies for which only one mode shape can propagate, the fibre is known as a monomode fibre. Such fibres are capable of transmitting extremely short pulses of quasi-monochromatic light over large distances with high intensity and negligible attenuation. The rate at which data can be transmitted is limited by the pulse distortion due to the dispersive effects of the medium. Hasegawa & Tappert (1973) suggested that this dispersive effect could be balanced against the sharpening phenomenon due to the nonlinearity of the material, so allowing pulses of information to travel without distortion.

There has been much work undertaken to show mathematically how the two phenomena affect the propagation of waves in a fibre. Much of this work (Anderson & Lisak, 1983; Zakharov & Shabat, 1972; Potasek *et al.*, 1986) has shown that the governing equation for the amplitude modulations of the signal is the cubic Schrödinger equation. However, a cylindrically symmetric isotropic monomode fibre has two equivalent modes (Snyder & Love, 1983), so signals with different polarizations interact nonlinearly and therefore two independent complex amplitudes are required to describe the signal. Parker & Newboult (1989) have shown that the equations which describe the signal amplitudes are a coupled pair of cubic Schrödinger equations. Although, unlike the cubic Schrödinger equation, this system is not completely integrable (Zakharov & Schulman, 1982), it possesses a large family of nondistorting pulselike solutions, and other families of generalized similarity solutions (Parker, 1988). Numerical studies also suggest that these pulselike solutions are stable and have interesting collision properties (Parker & Newboult, 1989).

On a perfect lossless fibre, solitons governed by a single cubic Schrödinger equation retain their shape and amplitude, but on a real fibre there will be losses which produce attenuation. One method of compensating for this is to amplify the solitons periodically as has been advocated by Hasegawa (1984) and Mollenauer *et al.* (1986). Another method, proposed by Tajima (1987), is to make the fibre axially nonuniform. He suggested that, by tapering the fibre core by an amount which is directly proportional to the soliton attenuation and inversely proportional to the square of the effective core radius, invariant solitons could be obtained. Kuehl (1988) presented a more rigorous treatment of these ideas and showed that Tajima's work was a special case of his theory. Both Tajima and Kuehl considered only the single cubic Schrödinger equation.

In this paper, we consider more generally the effect of axially symmetric fibre inhomogeneities in a lossless fibre. We show that if the axial nonuniformities have a length scale much shorter than that associated with nonlinear effects then only average properties of these nonuniformities enter the nonlinear evolution equations. In particular, if the nonuniformities are periodic, the equations reduce to those of an equivalent uniform fibre. Explicit formulae for the appropriate fibre coefficients are given. If the scale of the axial inhomogeneities is comparable with the nonlinear evolution length the evolution equations for the two independent complex amplitudes are a coupled pair of cubic Schrödinger equations with variable coefficients. Again, formulae for these coefficients are given.

For the variable coefficient equations, conditions are found for the existence of a transformation reducing the equations to constant coefficient equations. This transformation is similar to that obtained by Grimshaw (1979) for a single cubic Schrödinger equation with variable coefficients. Clearly, under suitable conditions, a pulse corresponding to a nondistorting pulse of the coupled system travels with modulation of amplitude and width, but without radiation. However, if these conditions apply over an extensive fibre length, they require nonphysical behaviour of either the dispersive or nonlinear effects; numerical calculations are performed for other cases.

2. Field corrections due to inhomogeneity

In an axially symmetric optical fibre having cross-section $r \leqslant R$ of cylindrical polar coordinates (r, θ, z) the electric and magnetic field intensities are governed by

Maxwell's equations:

$$\nabla \wedge \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t}, \quad \nabla \wedge \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t}, \quad \mu_0 \nabla \cdot \boldsymbol{H} = 0, \quad \nabla \cdot \boldsymbol{D} = 0. \quad (2.1\text{–}2.4)$$

As is normal, the material is taken to be a nonmagnetic isotropic dielectric, with properties which depend on the radius and slowly on the distance along the fibre. Also the fibre is assumed to be a Kerr medium, thus giving

$$\boldsymbol{D} = [\epsilon(r, \gamma z) + N(r, \gamma z)\, |\boldsymbol{E}|^2]\boldsymbol{E}, \tag{2.5}$$

where ϵ is the linear permittivity, N is the coefficient of cubic nonlinearity, and the small parameter γ is the reciprocal of a typical length over which the slow variations occur.

Since the fields in the guided modes decay rapidly in the cladding, the equations are analysed over all radii $0 \leq r < \infty$, with the conditions $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D} \to 0$ as $r \to \infty$ being a good approximation to boundary conditions at $r = R$. It is required also that the fields are finite at $r = 0$.

For each position γz along the fibre, the field distribution across the fibre is, to leading order, obtained by linear theory for an axially uniform fibre. An amplitude parameter ν was introduced and the fields were expressed in terms of leading-order approximations and corrections, as

$$\boldsymbol{E} = \nu \boldsymbol{E}^{(1)} + O(\nu^2), \qquad \boldsymbol{H} = \nu \boldsymbol{H}^{(1)} + O(\nu^2), \qquad \boldsymbol{D} = \nu \boldsymbol{D}^{(1)} + O(\nu^2), \tag{2.6}$$

where $\boldsymbol{D}^{(1)} = \epsilon \boldsymbol{E}^{(1)}$. This assumes that the small parameter ν characterizing the signal strength and the small parameter γ characterizing the fibre inhomogeneities satisfy $\gamma/\nu = O(1)$ or $\gamma/\nu = o(1)$.

Then, by substitution into Maxwell's equations (2.1–2.4), we see that $\boldsymbol{E}^{(1)}$ and $\boldsymbol{H}^{(1)}$ may be taken as solutions to the linearized equations

$$\nabla \wedge \boldsymbol{E}^{(1)} = -\mu_0 \frac{\partial \boldsymbol{H}^{(1)}}{\partial t}, \quad \nabla \wedge \boldsymbol{H}^{(1)} = \epsilon \frac{\partial \boldsymbol{E}^{(1)}}{\partial t}, \quad \mu_0 \nabla \cdot \boldsymbol{H}^{(1)} = 0, \quad \epsilon \nabla \cdot \boldsymbol{E}^{(1)} = 0.$$
$$(2.7\text{–}2.10)$$

Even though ϵ depends on γz, solutions with azimuthal mode number l may be sought, with errors $o(1)$, using separation of variables in the form

$$\left.\begin{aligned} \boldsymbol{E}^{(1)} &= (A^+ \boldsymbol{E}^+ \mathrm{e}^{il\theta} + A^- \boldsymbol{E}^- \mathrm{e}^{-il\theta})\mathrm{e}^{i\psi} + \text{c.c.} \\ \boldsymbol{H}^{(1)} &= (A^+ \boldsymbol{H}^+ \mathrm{e}^{il\theta} + A^- \boldsymbol{H}^- \mathrm{e}^{-il\theta})\mathrm{e}^{i\psi} + \text{c.c.} \end{aligned}\right\} \tag{2.11}$$

where A^+ and A^- are complex amplitudes, c.c. denotes a complex conjugate, and ψ is a phase variable having $-\partial\psi/\partial t = \omega$, the radian frequency, and $\partial\psi/\partial z = k(\omega, \gamma z)$, the local wavenumber.

It is found that, as for an axially uniform fibre (Parker & Newboult, 1989), when the vectors $\boldsymbol{E}^{\pm}$ and $\boldsymbol{H}^{\pm}$ are resolved along the basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ of cylindrical polar coordinates, they may be represented as

$$\boldsymbol{E}^{\pm} = \mathrm{i}\tilde{E}_1 \mathbf{e}_r \pm \tilde{E}_2 \mathbf{e}_\theta + \tilde{E}_3 \mathbf{e}_z, \qquad \boldsymbol{H}^{\pm} = \pm \tilde{H}_1 \mathbf{e}_r + \mathrm{i}\tilde{H}_2 \mathbf{e}_\theta \pm \mathrm{i}\tilde{H}_3 \mathbf{e}_z, \tag{2.12}$$

where $\tilde{E}_i = \tilde{E}_i(r;\omega,\gamma z)$ and $\tilde{H}_i = \tilde{H}_i(r;\omega,\gamma z)$ are real functions which satisfy the system of equations

$$\left.\begin{aligned} l\tilde{E}_3 - kr\tilde{E}_2 - \omega\mu_0 r\tilde{H}_1 &= 0, & l\tilde{H}_3 - kr\tilde{H}_2 + \omega\epsilon r\tilde{E}_1 &= 0,\\ \frac{\partial \tilde{E}_3}{\partial r} + k\tilde{E}_1 - \omega\mu_0\tilde{H}_2 &= 0, & \frac{\partial \tilde{H}_3}{\partial r} - k\tilde{H}_1 - \omega\epsilon\tilde{E}_2 &= 0,\\ \frac{\partial}{\partial r}(r\tilde{E}_2) + l\tilde{E}_1 - \omega\mu_0 r\tilde{H}_3 &= 0, & \frac{\partial}{\partial r}(r\tilde{H}_2) - l\tilde{H}_1 + \omega\epsilon r\tilde{E}_3 &= 0. \end{aligned}\right\} \tag{2.13}$$

The equations resulting from (2.3) and (2.4) may be shown to be satisfied to $O(1)$ by solutions to the six equations (2.13), and so may be omitted. Since no derivatives with respect to γz occur, equations (2.13) may be solved treating γz as a parameter, so that the fields $\boldsymbol{E}^{\pm}$ and $\boldsymbol{H}^{\pm}$ of left- and right-handed modes respectively are, to leading order, governed by the same equations as for an equivalent axially symmetric and axially uniform waveguide having permittivity $\epsilon(r,\gamma z)$. The condition that this system of equations has nontrivial solutions, bounded at $r=0$ and satisfying the decay conditions as $r\to\infty$ is, for each modenumber l, the *dispersion relation*, relating the wavenumber k to the frequency ω. Since ϵ depends on γz, so also will the field components $\tilde{E}_i$ and $\tilde{H}_i$ and the local wavenumber k for each choice of ω and l. For 'weakly-guiding' fibres only the ± 1 modes propagate, so we need consider only $l=1$, and we may write the corresponding dispersion relation as $k=k(\omega,\gamma z)$. Consequently, the phase ψ may be written as

$$\psi = \gamma^{-1}\int k(\omega,\gamma z)\,\mathrm{d}(\gamma z) - \omega t.$$

We may now apply a multiple-scales method to equations (2.1) and (2.2) and introduce the scaled variables

$$\chi = \nu\left(\gamma^{-1}\int s_g\,\mathrm{d}(\gamma z) - t\right), \qquad Z = \gamma z,$$

where $s_g = \partial k/\partial\omega$ is the *group slowness*. The amplitude scaling ν is chosen, as usual, so that effects due to cubic nonlinearity are comparable in magnitude with the modulation effects. Any fluctuations in the amplitudes $A^{\pm}$ are allowed to depend on both of the scaled variables χ and Z, so that $A^{\pm} = A^{\pm}(\chi, Z)$. The fields will be treated as functions of the variables r, θ, ψ, χ, Z which are 2π-periodic in both θ and ψ. The z and t derivatives are replaced by

$$\frac{\partial}{\partial z} = k\frac{\partial}{\partial\psi} + \nu s_g\frac{\partial}{\partial\chi} + \gamma\frac{\partial}{\partial Z}, \qquad \frac{\partial}{\partial t} = -\omega\frac{\partial}{\partial\psi} - \nu\frac{\partial}{\partial\chi}. \tag{2.14}$$

Splitting the fields into their leading-order terms and corrections as

$$\boldsymbol{E} = \nu\boldsymbol{E}^{(1)} + \nu^2\hat{\boldsymbol{E}}, \qquad \boldsymbol{H} = \nu\boldsymbol{H}^{(1)} + \nu^2\hat{\boldsymbol{H}}, \qquad \boldsymbol{D} = \nu\boldsymbol{D}^{(1)} + \nu^2\hat{\boldsymbol{D}} \tag{2.15}$$

gives $\hat{\boldsymbol{D}} = \epsilon\hat{\boldsymbol{E}} + O(\nu)$, because the nonlinear term $|\boldsymbol{E}|^2\boldsymbol{E}$ is $O(\nu^3)$ and so will not enter the analysis at $O(\nu^2)$.

Bearing in mind that $E^{(1)}$ and $H^{(1)}$ satisfy the equations

$$\nabla' \wedge E^{(1)} = \omega\mu_0 \frac{\partial H^{(1)}}{\partial\psi}, \qquad \nabla' \wedge H^{(1)} + \omega\epsilon \frac{\partial E^{(1)}}{\partial\psi} = \mathbf{0}, \tag{2.16}$$

where

$$\nabla' \equiv \mathbf{e}_r \frac{\partial}{\partial r} + \frac{1}{r}\mathbf{e}_\theta \frac{\partial}{\partial\theta} + k(\omega, Z)\mathbf{e}_z \frac{\partial}{\partial\psi}, \tag{2.17}$$

substitution of (2.15) into equations (2.1) and (2.2) gives

$$\begin{aligned}\nabla' \wedge \hat{E} - \omega\mu_0 \frac{\partial \hat{H}}{\partial\psi} = &\left(A_\chi^+(s_g E^+ \wedge \mathbf{e}_z + \mu_0 H^+) + \frac{\gamma}{v}(A^+ E_Z^+ + A_Z^+ E^+) \wedge \mathbf{e}_z\right)e^{i(\theta+\psi)} \\ &+ \left(A_\chi^-(s_g E^- \wedge \mathbf{e}_z + \mu_0 H^-) + \frac{\gamma}{v}(A^- E_Z^- + A_Z^- E^-) \wedge \mathbf{e}_z\right)e^{i(-\theta+\psi)} \\ &+ \text{c.c.} + o(1),\end{aligned} \tag{2.18}$$

$$\begin{aligned}\nabla' \wedge \hat{H} + \omega\epsilon \frac{\partial \hat{E}}{\partial\psi} = &\left(A_\chi^+(s_g H^+ \wedge \mathbf{e}_z - \epsilon E^+) + \frac{\gamma}{v}(A^+ H_Z^+ + A_Z^+ H^+) \wedge \mathbf{e}_z\right)e^{i(\theta+\psi)} \\ &+ \left(A_\chi^-(s_g H^- \wedge \mathbf{e}_z - \epsilon E^-) + \frac{\gamma}{v}(A^- H_Z^- + A_Z^- H^-) \wedge \mathbf{e}_z\right)e^{i(-\theta+\psi)} \\ &+ \text{c.c.} + o(1).\end{aligned} \tag{2.19}$$

The terms in (2.18) and (2.19) involving $A_\chi^\pm$ are due to amplitude modulation of the signal envelope and occur in the absence of axial nonuniformities in the fibre. The remaining terms are due to the fibre nonuniformities.

We observe that equations (2.18) and (2.19) have the form

$$\nabla' \wedge \hat{E} - \omega\mu_0 \frac{\partial \hat{H}}{\partial\psi} \equiv G, \tag{2.20}$$

$$\nabla' \wedge \hat{H} + \omega\epsilon \frac{\partial \hat{E}}{\partial\psi} \equiv F, \tag{2.21}$$

where F and G are 2π-periodic in θ and ψ, bounded at $r = 0$, and decay exponentially as $r \to \infty$. We require that $\hat{E}$ and $\hat{H}$ also obey these conditions, which gives a compatibility condition

$$\int_{\mathscr{R}} (P \cdot F - Q \cdot G)\,\mathrm{d}V = 0, \tag{2.22}$$

where $\mathscr{R} \equiv [0, \infty) \times [0, 2\pi] \times [0, 2\pi]$, $\mathrm{d}V \equiv r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\psi$, and (P, Q) are the most general solution of the linear equations

$$\nabla' \wedge P - \omega\mu_0 \frac{\partial Q}{\partial\psi} = \mathbf{0}, \qquad \nabla' \wedge Q + \omega\epsilon \frac{\partial P}{\partial\psi} = \mathbf{0} \tag{2.23}$$

which are 2π-periodic in θ and ψ with similar conditions on $r = 0$ and as $r \to \infty$. Since these equations govern periodic linearized fields travelling at the phase speed ω/k in an equivalent uniform fibre, the general solution is a linear combination of fields having wavenumber an integer multiple of k. We have already assumed that modes $l>1$ do not propagate so that, assuming that no integer harmonics of the $l=1$ mode have the phase speed ω/k, the most general solution for $\boldsymbol{P}$ and $\boldsymbol{Q}$ is

$$\left.\begin{aligned}\boldsymbol{P}&=\alpha_1\boldsymbol{E}^+e^{i(\theta+\psi)}+\alpha_2\boldsymbol{E}^-e^{i(-\theta+\psi)}+\text{c.c.},\\ \boldsymbol{Q}&=\alpha_1\boldsymbol{H}^+e^{i(\theta+\psi)}+\alpha_2\boldsymbol{H}^-e^{i(-\theta+\psi)}+\text{c.c.},\end{aligned}\right\}\tag{2.24}$$

where α_1 and α_2 are arbitrary complex constants.

Substituting for $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{F}$, $\boldsymbol{G}$ into (2.22) and recalling that $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{H}}$ are 2π-periodic in θ and ψ, we find that the only terms which give a nonzero contribution to the integral are those in which the exponential factor is e^{i0}. Since the equations obtained must hold for arbitrary complex numbers α_1 and α_2, the coefficients of α_1 and α_2 (or equivalently α_1^* and α_2^*) must vanish separately. The equation obtained from the coefficients of α_1^* is

$$\frac{\gamma}{\nu}\int_{\Re}[(A^+\boldsymbol{E}^{+*}\wedge\boldsymbol{H}_Z^+ + A_Z^+\boldsymbol{E}^{+*}\wedge\boldsymbol{H}^+)\cdot\mathbf{e}_z+(A^+\boldsymbol{E}_Z^+\wedge\boldsymbol{H}^{+*}+A_Z^+\boldsymbol{E}^+\wedge\boldsymbol{H}^{+*})\cdot\mathbf{e}_z]\,dV$$

$$=-A_X^+\int_{\Re}[s_g(\boldsymbol{E}^{+*}\wedge\boldsymbol{H}^+ +\boldsymbol{E}^+\wedge\boldsymbol{H}^{+*})\cdot\mathbf{e}_z-(\epsilon\boldsymbol{E}^+\cdot\boldsymbol{E}^{+*}+\mu_0\boldsymbol{H}^+\cdot\boldsymbol{H}^{+*})]\,dV+o(1).$$

Now the group slowness may be shown to be

$$s_g=\frac{\partial k}{\partial\omega}=\frac{\displaystyle\int_{\Re}(\epsilon\boldsymbol{E}^+\cdot\boldsymbol{E}^{+*}+\mu_0\boldsymbol{H}^+\cdot\boldsymbol{H}^{+*})\,dV}{\displaystyle\int_{\Re}(\boldsymbol{E}^{+*}\wedge\boldsymbol{H}^+ +\boldsymbol{E}^+\wedge\boldsymbol{H}^{+*})\cdot\mathbf{e}_z\,dV},\tag{2.25}$$

so that, if $\nu^{-1}\gamma=o(1)$, the above equation is automatically satisfied to leading order. If $\nu^{-1}\gamma=O(1)$ the $\nu^{-1}\gamma$ terms must be retained in (2.18) and (2.19) and the above equation becomes

$$A^+\int_{\Re}(\boldsymbol{E}^{+*}\wedge\boldsymbol{H}_Z^+ +\boldsymbol{E}_Z^+\wedge\boldsymbol{H}^{+*})\cdot\mathbf{e}_z\,dV$$

$$+A_Z^+\int_{\Re}(\boldsymbol{E}^{+*}\wedge\boldsymbol{H}^+ +\boldsymbol{E}^+\wedge\boldsymbol{H}^{+*})\cdot\mathbf{e}_z\,dV=o(1).\tag{2.26}$$

This has the form

$$\tfrac{1}{2}A^+\frac{\partial P}{\partial Z}+\frac{\partial A^+}{\partial Z}P=o(1),\tag{2.27}$$

where $P\equiv\int_{\Re}(\boldsymbol{E}^{\pm*}\wedge\boldsymbol{H}^{\pm}+\boldsymbol{E}^{\pm}\wedge\boldsymbol{H}^{\pm*})\cdot\mathbf{e}_z\,dV$ is proportional to the power in either of the modes described by $\boldsymbol{E}^+e^{i(\theta+\psi)}+$ c.c. or $\boldsymbol{E}^-e^{i(-\theta+\psi)}+$ c.c..

By normalizing the solutions to equations (2.13) so that $P=4\pi^2$ for all Z and ω, equation (2.27) reduces to

$$\frac{\partial A^+}{\partial Z}=o(1),\tag{2.28}$$

is showing that the leading-order approximation for A^+ may be taken as (χ), independently of Z. Similar analysis of the coefficient of α_2^* gives a ding-order expression $A^-(\chi)$ for A^-. With this choice of P, equations (2.18) d (2.19) may be replaced by the simpler equations

$$\nabla' \wedge \hat{\boldsymbol{E}} - \omega\mu_0 \frac{\partial \hat{\boldsymbol{H}}}{\partial \psi} = \left(A_\chi^+(s_g \boldsymbol{E}^+ \wedge \mathbf{e}_z + \mu_0 \boldsymbol{H}^+) + \frac{\gamma}{\nu} A^+ \boldsymbol{E}_Z^+ \wedge \mathbf{e}_z\right) e^{i(\theta+\psi)} + \left(A_\chi^-(s_g \boldsymbol{E}^- \wedge \mathbf{e}_z + \mu_0 \boldsymbol{H}^-) + \frac{\gamma}{\nu} A^- \boldsymbol{E}_Z^- \wedge \mathbf{e}_z\right) e^{i(-\theta+\psi)} + \text{c.c.} + o(1), \tag{2.29}$$

$$\nabla' \wedge \hat{\boldsymbol{H}} + \omega\epsilon \frac{\partial \hat{\boldsymbol{E}}}{\partial \psi} = \left(A_\chi^+(s_g \boldsymbol{H}^+ \wedge \mathbf{e}_z - \epsilon \boldsymbol{E}^+) + \frac{\gamma}{\nu} A^+ \boldsymbol{H}_Z^+ \wedge \mathbf{e}_z\right) e^{i(\theta+\psi)} + \left(A_\chi^-(s_g \boldsymbol{H}^- \wedge \mathbf{e}_z + \epsilon \boldsymbol{E}^-) + \frac{\gamma}{\nu} A^- \boldsymbol{H}_Z^- \wedge \mathbf{e}_z\right) e^{i(-\theta+\psi)} + \text{c.c.} + o(1). \tag{2.30}$$

The terms in $A_\chi^\pm$ due to amplitude modulation are known to give rise to fields $(\boldsymbol{E}_\omega^\pm, \boldsymbol{H}_\omega^\pm)$ (see Parker & Newboult, 1989), so that, neglecting $o(1)$ terms, a lution to these equations can be written as

$$\hat{\boldsymbol{E}} = \left(-iA_\chi^+ \boldsymbol{E}_\omega^+ + \frac{\gamma}{\nu} A^+ \hat{\boldsymbol{E}}^+\right) e^{i(\theta+\psi)} + \left(-iA_\chi^- \boldsymbol{E}_\omega^- + \frac{\gamma}{\nu} A^- \hat{\boldsymbol{E}}^-\right) e^{i(-\theta+\psi)} + \text{c.c.}, \tag{2.31}$$

$$\hat{\boldsymbol{H}} = \left(-iA_\chi^+ \boldsymbol{H}_\omega^+ + \frac{\gamma}{\nu} A^+ \hat{\boldsymbol{H}}^+\right) e^{i(\theta+\psi)} + \left(-iA_\chi^- \boldsymbol{H}_\omega^- + \frac{\gamma}{\nu} A^- \hat{\boldsymbol{H}}^-\right) e^{i(-\theta+\psi)} + \text{c.c.}, \tag{2.32}$$

here $\hat{\boldsymbol{E}}^\pm$ and $\hat{\boldsymbol{H}}^\pm$ satisfy the equations

$$\left.\begin{aligned} \nabla' \wedge (\hat{\boldsymbol{E}}^\pm e^{i(\pm\theta+\psi)}) - \omega\mu_0 \frac{\partial}{\partial \psi}(\hat{\boldsymbol{H}}^\pm e^{i(\pm\theta+\psi)}) &= \boldsymbol{E}_Z^\pm \wedge \mathbf{e}_z e^{i(\pm\theta+\psi)}, \\ \nabla' \wedge (\hat{\boldsymbol{H}}^\pm e^{i(\pm\theta+\psi)}) + \omega\epsilon \frac{\partial}{\partial \psi}(\hat{\boldsymbol{E}}^\pm e^{i(\pm\theta+\psi)}) &= \boldsymbol{H}_Z^\pm \wedge \mathbf{e}_z e^{i(\pm\theta+\psi)}, \end{aligned}\right\} \tag{2.33}$$

d appropriate boundary conditions. It may be shown that $\hat{\boldsymbol{E}}^\pm$ and $\hat{\boldsymbol{H}}^\pm$ have the presentations

$$i\hat{\boldsymbol{E}}^\pm = i\hat{E}_1 \mathbf{e}_r \pm \hat{E}_2 \mathbf{e}_\theta - \hat{E}_3 \mathbf{e}_z, \qquad i\hat{\boldsymbol{H}}^\pm = \pm \hat{H}_1 \mathbf{e}_r + i\hat{H}_2 \mathbf{e}_\theta \pm i\hat{H}_3 \mathbf{e}_z,$$

here $\hat{E}_i = \hat{E}_i(r; \omega, Z)$ and $\hat{H}_i = \hat{H}_i(r; \omega, Z)$ are real functions satisfying in-mogeneous ordinary differential equations analogous to (2.13). Consequently e fields $\hat{\boldsymbol{E}}^\pm$ and $\hat{\boldsymbol{H}}^\pm$ may be represented explicitly in terms of integrals with spect to r involving $\boldsymbol{E}_Z^\pm$ and $\boldsymbol{H}_Z^\pm$, though the details for this are omitted.

Subsequent corrections, which incorporate into the evolution equations effects ie to the fibre inhomogeneity, are treated in Section 3.

3. The evolution equations

Two distinct cases arise, depending on whether or not $\gamma = O(\nu^2)$.

3.1 *Case* 1: $\gamma = O(\nu^2)$

If longitudinal inhomogeneities are sufficiently weak that they act on the sa length scale as nonlinear evolution, then $\hat{\boldsymbol{E}}^{\pm}$ and $\hat{\boldsymbol{H}}^{\pm}$ may be omitted from (2.3 and (2.32), we may take $\gamma = O(\nu^2)$, and, without loss of generality, write $\nu =$ and seek solutions for $\boldsymbol{E},\boldsymbol{H},\boldsymbol{D}$, of the form

$$\boldsymbol{E} = \gamma^{\frac{1}{2}}\boldsymbol{E}^{(1)} + \gamma\boldsymbol{E}^{(2)} + \gamma^{\frac{3}{2}}\bar{\boldsymbol{E}},$$
$$\boldsymbol{H} = \gamma^{\frac{1}{2}}\boldsymbol{H}^{(1)} + \gamma\boldsymbol{H}^{(2)} + \gamma^{\frac{3}{2}}\bar{\boldsymbol{H}},$$
$$\boldsymbol{D} = \gamma^{\frac{1}{2}}\epsilon\boldsymbol{E}^{(1)} + \gamma\epsilon\boldsymbol{E}^{(2)} + \gamma^{\frac{3}{2}}\bar{\boldsymbol{D}}.$$

Since in this case, the terms in $\hat{\boldsymbol{E}}^{\pm}$ and $\hat{\boldsymbol{H}}^{\pm}$ are omitted from (2.31) and (2.32), may take

$$\boldsymbol{E}^{(2)} = -\mathrm{i}A_{\chi}^{+}\boldsymbol{E}_{\omega}^{+}\mathrm{e}^{\mathrm{i}(\theta+\psi)} - \mathrm{i}A_{\chi}^{-}\boldsymbol{E}_{\omega}^{-}\mathrm{e}^{\mathrm{i}(-\theta+\psi)} + \text{c.c.}$$
$$\boldsymbol{H}^{(2)} = -\mathrm{i}A_{\chi}^{+}\boldsymbol{H}_{\omega}^{+}\mathrm{e}^{\mathrm{i}(\theta+\psi)} - \mathrm{i}A_{\chi}^{-}\boldsymbol{H}_{\omega}^{-}\mathrm{e}^{\mathrm{i}(-\theta+\psi)} + \text{c.c.}$$

We note that, correct to $O(1)$, equation (2.5) gives

$$\bar{\boldsymbol{D}} = \epsilon\bar{\boldsymbol{E}} + N(r, Z)\,|\boldsymbol{E}^{(1)}|^2\,\boldsymbol{E}^{(1)}.$$

Since $\boldsymbol{E}^{(2)}$ and $\boldsymbol{H}^{(2)}$ do not involve the terms in (2.18) and (2.19) which a $O(\gamma/\nu) = O(\gamma^{\frac{1}{2}})$, the compatibility condition analogous to (2.22) is automatical satisfied and the reasoning leading to equation (2.28) is inappropriate. T amplitudes A^+ and A^- should, in this case, be allowed to depend on both χ a Z. The derivative expansions can be obtained by replacing ν by $\gamma^{\frac{1}{2}}$ in expressio (2.14). Then substitution of $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{D}$, $\partial/\partial z$, $\partial/\partial t$ into equations (2.1) and (2. gives the equations for $\bar{\boldsymbol{E}}$ and $\bar{\boldsymbol{H}}$:

$$\begin{aligned}\nabla' \wedge \bar{\boldsymbol{E}} - \omega\mu_0\frac{\partial\bar{\boldsymbol{H}}}{\partial\psi} &= [A^{+}\boldsymbol{E}_Z^{+} \wedge \mathbf{e}_z + A_Z^{+}\boldsymbol{E}^{+} \wedge \mathbf{e}_z - \mathrm{i}A_{\chi\chi}^{+}(s_g\boldsymbol{E}_{\omega}^{+} \wedge \mathbf{e}_z + \mu_0\boldsymbol{H}_{\omega}^{+})]\mathrm{e}^{\mathrm{i}(\theta+\psi)}\\ &\quad + [A^{-}\boldsymbol{E}_Z^{-} \wedge \mathbf{e}_z + A_Z^{-}\boldsymbol{E}^{-} \wedge \mathbf{e}_z - \mathrm{i}A_{\chi\chi}^{-}(s_g\boldsymbol{E}_{\omega}^{-} \wedge \mathbf{e}_z + \mu_0\boldsymbol{H}_{\omega}^{-})]\mathrm{e}^{\mathrm{i}(-\theta+\psi)}\\ &\quad + \text{c.c.} + O(\gamma^{\frac{1}{2}})\\ &= \hat{\boldsymbol{G}},\end{aligned} \tag{3.1}$$

$$\begin{aligned}\nabla' \wedge \bar{\boldsymbol{H}} + \omega\epsilon\frac{\partial\bar{\boldsymbol{E}}}{\partial\psi} &= [A^{+}\boldsymbol{H}_Z^{+} \wedge \mathbf{e}_z + A_Z^{+}\boldsymbol{H}^{+} \wedge \mathbf{e}_z - \mathrm{i}A_{\chi\chi}^{+}(s_g\boldsymbol{H}_{\omega}^{+} \wedge \mathbf{e}_z - \epsilon\boldsymbol{E}_{\omega}^{+})]\mathrm{e}^{\mathrm{i}(\theta+\psi)}\\ &\quad + [A^{-}\boldsymbol{H}_Z^{-} \wedge \mathbf{e}_z + A_Z^{-}\boldsymbol{H}^{-} \wedge \mathbf{e}_z - \mathrm{i}A_{\chi\chi}^{-}(s_g\boldsymbol{H}_{\omega}^{-} \wedge \mathbf{e}_z - \epsilon\boldsymbol{E}_{\omega}^{-})]\mathrm{e}^{\mathrm{i}(-\theta+\psi)}\\ &\quad + \text{c.c.} - \omega\frac{\partial}{\partial\psi}(N\,|\boldsymbol{E}^{(1)}|^2\,\boldsymbol{E}^{(1)}) + O(\gamma^{\frac{1}{2}})\\ &= \hat{\boldsymbol{F}}.\end{aligned} \tag{3.2}$$

Explicit solutions to these equations cannot be easily found. However, $\hat{\boldsymbol{F}}$ and are 2π-periodic in θ and ψ, bounded at $r = 0$, and decay exponentially as $r \to$

Comparing the situation with that of equations (2.20) and (2.21), we see that we can use the compatibility condition (2.22) with (2.24) to obtain equations which govern the evolution of the amplitudes A^+ and A^-. Then considering the terms in α_1^*, as before, gives the equation

$$A^+\int_0^\infty (\boldsymbol{E}^{+*}\wedge \boldsymbol{H}_Z^+ + \boldsymbol{E}_Z^+\wedge \boldsymbol{H}^{+*})\cdot \mathbf{e}_z r\,\mathrm{d}r + A_Z^+\int_0^\infty (\boldsymbol{E}^{+*}\wedge \boldsymbol{H}^+ + \boldsymbol{E}^+\wedge \boldsymbol{H}^{+*})\cdot \mathbf{e}_z r\,\mathrm{d}r$$
$$-\mathrm{i}A_{\chi\chi}^+\int_0^\infty [s_g(\boldsymbol{E}^{+*}\wedge \boldsymbol{H}_\omega^+ + \boldsymbol{E}_\omega^+\wedge \boldsymbol{H}^{+*})\cdot \mathbf{e}_z - (\epsilon \boldsymbol{E}^{+*}\cdot \boldsymbol{E}_\omega^+ + \mu_0 \boldsymbol{H}^{+*}\cdot \boldsymbol{H}_\omega^+)]r\,\mathrm{d}r$$
$$-\frac{\omega}{4\pi^2}\int_{\mathcal{R}} \boldsymbol{E}^{+*}\cdot \frac{\partial}{\partial\psi}(N\,|\boldsymbol{E}^{(1)}|^2\boldsymbol{E}^{(1)})\mathrm{e}^{-\mathrm{i}(\theta+\psi)}\,\mathrm{d}V = O(\gamma^{\frac{1}{2}}). \quad (3.3)$$

This is formally similar to the case of a uniform fibre, all dependence on the inhomogeneity being incorporated in the Z dependence of ϵ, $\boldsymbol{E}^\pm$, $\boldsymbol{H}^\pm$, and k. The nonlinear term can be simplified (Parker & Newboult, 1989) as

$$\frac{1}{4\pi^2}\int_{\mathcal{R}} \boldsymbol{E}^{+*}\cdot \frac{\partial}{\partial\psi}(N\,|\boldsymbol{E}^{(1)}|^2\boldsymbol{E}^{(1)})\mathrm{e}^{-\mathrm{i}(\theta+\psi)}\,\mathrm{d}V$$
$$= \mathrm{i}\,|A^+|^2A^+\int_0^\infty (|\boldsymbol{E}^+\cdot \boldsymbol{E}^+|^2 + 2\,|\boldsymbol{E}^+|^2)Nr\,\mathrm{d}r$$
$$+2\mathrm{i}\,|A^-|^2A^+\int_0^\infty (|\boldsymbol{E}^+\cdot \boldsymbol{E}^-|^2 + |\boldsymbol{E}^+\cdot \boldsymbol{E}^{-*}|^2 + |\boldsymbol{E}^+|^2\,|\boldsymbol{E}^-|^2)Nr\,\mathrm{d}r.$$

The normalization $P = 4\pi^2$ of the fields implies that the first term in (3.3) vanishes, so that the leading-order approximation to equation (3.3) can be written in the form

$$\mathrm{i}A_Z^+ = gA_{\chi\chi}^+ + (f_2\,|A^+|^2 + f_3\,|A^-|^2)A^+, \quad (3.4)$$

where the coefficients

$$\left.\begin{aligned} f_2 &= -\omega\int_0^\infty (|\boldsymbol{E}^+\cdot \boldsymbol{E}^+|^2 + 2\,|\boldsymbol{E}^+|^2)Nr\,\mathrm{d}r \equiv f_2(\omega, Z)\\ f_3 &= -2\omega\int_0^\infty (|\boldsymbol{E}^+\cdot \boldsymbol{E}^-|^2 + |\boldsymbol{E}^+\cdot \boldsymbol{E}^{-*}|^2 + |\boldsymbol{E}^+|^2\,|\boldsymbol{E}^-|^2)Nr\,\mathrm{d}r \equiv f_3(\omega, Z)\\ g &= -\int_0^\infty [s_g(\boldsymbol{E}^{+*}\wedge \boldsymbol{H}_\omega^+ + \boldsymbol{E}_\omega^+\wedge \boldsymbol{H}^{+*})\cdot \mathbf{e}_z - (\epsilon \boldsymbol{E}^{+*}\cdot \boldsymbol{E}_\omega^+ + \mu_0 \boldsymbol{H}^{+*}\cdot \boldsymbol{H}_\omega^+)]r\,\mathrm{d}r\\ &= \tfrac{1}{2}\frac{\partial s_g}{\partial\omega} \equiv g(\omega, Z) \end{aligned}\right\} \quad (3.5)$$

are related to the field distributions $\boldsymbol{E}^\pm$ and $\boldsymbol{H}^\pm$ of circularly polarized modes just as in an axially uniform fibre (Parker & Newboult, 1989).

A similar equation arises from the terms in α_2^*, so yielding the pair of equations

$$\mathrm{i}A_Z^\pm = gA_{\chi\chi}^\pm + (f_2\,|A^\pm|^2 + f_3\,|A^\mp|^2)A^\pm. \quad (3.6)$$

Thus, when longitudinal inhomogeneities have a length scale comparable to the nonlinear evolution length, evolution equations for $A^{\pm}(\chi, Z)$ are the same as the coupled cubic Schrödinger equations for axially uniform fibres, except that the coefficients depend on Z. The coefficients are related to the material coefficients ϵ and N exactly as for an equivalent axially homogeneous fibre associated with the cross-section at position Z.

3.2 *Case* 2: $\gamma = O(\nu)$

If inhomogeneity causes changes on a scale comparable with a pulse width, we may, without loss of generality, take $\gamma = \nu$ and seek solutions $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}$ of the form

$$\boldsymbol{E} = \gamma \boldsymbol{E}^{(1)} + \gamma^2 \boldsymbol{E}^{(2)} + \gamma^3 \bar{\boldsymbol{E}},$$
$$\boldsymbol{H} = \gamma \boldsymbol{H}^{(1)} + \gamma^2 \boldsymbol{H}^{(2)} + \gamma^3 \bar{\boldsymbol{H}},$$
$$\boldsymbol{D} = \gamma \epsilon \boldsymbol{E}^{(1)} + \gamma^2 \epsilon \boldsymbol{E}^{(2)} + \gamma^3 \bar{\boldsymbol{D}}.$$

In this case, $\boldsymbol{E}^{(2)}$ and $\boldsymbol{H}^{(2)}$ have a form similar to (2.31) and (2.32):

$$\boldsymbol{E}^{(2)} = (-\mathrm{i}A_\chi^+ \boldsymbol{E}_\omega^+ + A^+ \hat{\boldsymbol{E}}^+)\mathrm{e}^{\mathrm{i}(\theta+\psi)} + (-\mathrm{i}A_\chi^- \boldsymbol{E}_\omega^- + A^- \hat{\boldsymbol{E}}^-)\mathrm{e}^{\mathrm{i}(-\theta+\psi)},$$
$$\boldsymbol{H}^{(2)} = (-\mathrm{i}A_\chi^+ \boldsymbol{H}_\omega^+ + A^+ \hat{\boldsymbol{H}}^+)\mathrm{e}^{\mathrm{i}(\theta+\psi)} + (-\mathrm{i}A_\chi^- \boldsymbol{H}_\omega^- + A^- \hat{\boldsymbol{H}}^-)\mathrm{e}^{\mathrm{i}(-\theta+\psi)},$$

with $(\boldsymbol{E}^{\pm}, \boldsymbol{H}^{\pm})$ again normalized so that $P \equiv 4\pi^2$. Although inhomogeneities occur on the scale of $Z = \gamma z$, so that the fields $(\boldsymbol{E}^{(1)}, \boldsymbol{H}^{(1)})$ and $(\boldsymbol{E}^{(2)}, \boldsymbol{H}^{(2)})$ depend on Z, equation (2.28) shows that derivatives $\partial A^+/\partial Z$ are $O(\gamma)$, with a similar result for $\partial A^-/\partial Z$. This suggests introduction of a further stretched variable $\hat{Z} = \gamma^2 z$ and allowance for $O(\gamma)$ fluctuations in $A^{\pm}$ on the Z scale by writing

$$A^{\pm} = B^{\pm}(\chi, \hat{Z}) + \gamma a^{\pm}(\chi, Z, \hat{Z}). \tag{3.7}$$

Nonlinearity has effect only over scales associated with $\hat{Z}$, and arises from the Kerr law (2.5), which gives $\bar{\boldsymbol{D}} = \epsilon \bar{\boldsymbol{E}} + N(r, Z)\,|\boldsymbol{E}^{(1)}|^2 \boldsymbol{E}^{(1)}$, correct to $O(1)$. The derivative expansions become

$$\frac{\partial}{\partial z} = k\frac{\partial}{\partial \psi} + \gamma s_g \frac{\partial}{\partial \chi} + \gamma \frac{\partial}{\partial Z} + \gamma^2 \frac{\partial}{\partial \hat{Z}},$$
$$\frac{\partial}{\partial t} = -\omega \frac{\partial}{\partial \psi} - \gamma \frac{\partial}{\partial \chi}.$$

Substitution of the fields and the derivative expansions into equations (2.1) and (2.2) will give equations for $\bar{\boldsymbol{E}}$ and $\bar{\boldsymbol{H}}$ which, correct to leading order, are

$$\begin{aligned}
\nabla' \wedge \bar{\boldsymbol{E}} - \omega\mu_0 \frac{\partial \bar{\boldsymbol{H}}}{\partial \psi} = {} & [a^+ \boldsymbol{E}_Z^+ \wedge \mathbf{e}_z + a_Z^+ \boldsymbol{E}^+ \wedge \mathbf{e}_z + a_\chi^+ (s_g \boldsymbol{E}^+ \wedge \mathbf{e}_z + \mu_0 \boldsymbol{H}^+) \\
& + B_{\hat{Z}}^+ \boldsymbol{E}^+ \wedge \mathbf{e}_z + B^+ \hat{\boldsymbol{E}}_Z^+ \wedge \mathbf{e}_z + B_\chi^+ (s_g \hat{\boldsymbol{E}}^+ \wedge \mathbf{e}_z + \mu_0 \hat{\boldsymbol{H}}^+ - \mathrm{i}\boldsymbol{E}_{\omega Z}^+ \wedge \mathbf{e}_z) \\
& - \mathrm{i}B_{\chi\chi}^+ (s_g \boldsymbol{E}_\omega^+ \wedge \mathbf{e}_z + \mu_0 \boldsymbol{H}_\omega^+)]\mathrm{e}^{\mathrm{i}(\theta+\psi)} \\
& + [a^- \boldsymbol{E}_Z^- \wedge \mathbf{e}_z + a_Z^- \boldsymbol{E}^- \wedge \mathbf{e}_z + a_\chi^- (s_g \boldsymbol{E}^- \wedge \mathbf{e}_z + \mu_0 \boldsymbol{H}^-) \\
& + B_{\hat{Z}}^- \boldsymbol{E}^- \wedge \mathbf{e}_z + B^- \hat{\boldsymbol{E}}_Z^- \wedge \mathbf{e}_z + B_\chi^- (s_g \hat{\boldsymbol{E}}^- \wedge \mathbf{e}_z + \mu_0 \hat{\boldsymbol{H}}^- - \mathrm{i}\boldsymbol{E}_{\omega Z}^- \wedge \mathbf{e}_z) \\
& - \mathrm{i}B_{\chi\chi}^- (s_g (\boldsymbol{E}_\omega^- \wedge \mathbf{e}_z + \mu_0 \boldsymbol{H}_\omega^-)]\mathrm{e}^{\mathrm{i}(-\theta+\psi)} + \text{c.c.} \\
\equiv {} & \bar{\boldsymbol{G}},
\end{aligned} \tag{3.8}$$

$$\nabla' \wedge \bar{H} + \omega\epsilon\frac{\partial \bar{E}}{\partial\psi} = [a^+ H_Z^+ \wedge \mathbf{e}_z + a_Z^+ H^+ \wedge \mathbf{e}_z + a_\chi^+(s_g H^+ \wedge \mathbf{e}_z - \epsilon E^+)$$
$$+ B_Z^+ H^+ \wedge \mathbf{e}_z + B^+ \hat{H}_Z^+ \wedge \mathbf{e}_z + B_\chi^+(s_g \hat{H}^+ \wedge \mathbf{e}_z - \epsilon\hat{E}^+ - \mathrm{i}H_{\omega Z}^+ \wedge \mathbf{e}_z)$$
$$- \mathrm{i}B_{\chi\chi}^+(s_g H_\omega^+ \wedge \mathbf{e}_z - \epsilon E_\omega^+)]\mathrm{e}^{\mathrm{i}(\theta+\psi)}$$
$$+ [a^- H_Z^- \wedge \mathbf{e}_z + a_Z^- H^- \wedge \mathbf{e}_z + a_\chi^-(s_g H^- \wedge \mathbf{e}_z - \epsilon E^-)$$
$$+ B_Z^- H^- \wedge \mathbf{e}_z + B^- \hat{H}_Z^- \wedge \mathbf{e}_z + B_\chi^-(s_g \hat{H}^- \wedge \mathbf{e}_z - \epsilon\hat{E}^- - \mathrm{i}H_{\omega Z}^- \wedge \mathbf{e}_z)$$
$$- \mathrm{i}B_{\chi\chi}^-(s_g H_\omega^- \wedge \mathbf{e}_z - \epsilon H_\omega^-)]\mathrm{e}^{\mathrm{i}(-\theta+\psi)} + \text{c.c.} - \omega\frac{\partial}{\partial\psi}(N\,|E^{(1)}|^2\,E^{(1)})$$
$$\equiv \bar{F}. \tag{3.9}$$

As for Case 1, it is unnecessary to construct solutions $\bar{E}$ and $\bar{H}$ to these equations in order to deduce evolution equations for $B^\pm$. Since $\bar{F}$ and $\bar{G}$ are 2π-periodic in θ and ψ, are finite at $r=0$, and decay as $r\to\infty$, we may again apply the compatibility condition (2.22) for the existence of 2π-periodic fields $\bar{E}$ and $\bar{H}$. By considering the terms multiplying α_1^* and recalling the identity (2.25) for s_g, the normalization $P=4\pi^2$ leading to $P_Z=0$, and the definitions of f_2, f_3, and g, we obtain

$$a_Z^+ + B_Z^+ + \mathrm{i}f_4 B^+ + \mathrm{i}f_5 B_\chi^+ + \mathrm{i}g B_{\chi\chi}^+ + \mathrm{i}f_2\,|B^+|^2\,B^+ + \mathrm{i}f_3\,|B^-|^2\,B^+ = 0,$$

where

$$f_4 = \frac{\mathrm{i}}{4\pi^2}\int_{\mathscr{R}} (H^{+*} \wedge \hat{E}_Z^+ + \hat{H}_Z^+ \wedge E^{+*})\cdot \mathbf{e}_z\,\mathrm{d}V,$$

$$f_5 = \frac{\mathrm{i}}{4\pi^2}\int_{\mathscr{R}} [s_g(H^{+*} \wedge \hat{E}^+ + \hat{H}^+ \wedge E^{+*})\cdot \mathbf{e}_z + (\epsilon E^{+*}\cdot\hat{E}^+ + \mu_0 H^{+*}\cdot\hat{H}^+)$$
$$+ \mathrm{i}(E^{+*} \wedge H_{\omega Z}^+ + E_{\omega Z}^+ \wedge H^{+*})\cdot\mathbf{e}_z]\,\mathrm{d}V.$$

However, equations (2.33) which define $(\hat{E}^\pm, \hat{H}^\pm)$ may be used to show that the coefficient f_5 vanishes identically, thus giving

$$\mathrm{i}a_Z^+ = -\mathrm{i}B_Z^+ + f_4 B^+ + g B_{\chi\chi}^+ + (f_2\,|B^+|^2 + f_3\,|B^-|^2)B^+. \tag{3.10}$$

In equation (3.10), the coefficients of f_2, f_3, f_4, and g depend on the intermediate axial scale Z but the amplitudes $B^\pm$ do not. Consequently a^+ must be chosen to absorb all the fluctuations on the Z scale without accumulating $O(\gamma^{-1})$ deviations as Z varies over ranges of $O(\gamma^{-1})$. This is achieved by making the mean values of both sides of (3.10) vanish over large ranges of Z. The simplest statement of this requirement occurs when the fibre inhomogeneities are periodic in Z, of some period Z_p. Then equation (3.10) can be written in the form

$$\mathrm{i}B_Z^+ = F_4 B^+ + G B_{\chi\chi}^+ + (F_2\,|B^+|^2 + F_3\,|B^-|^2)B^+ \tag{3.11}$$

where the real coefficients F_2, F_3, F_4, G are averages of f_2, f_3, f_4, g over each period of length Z_p and are given by

$$F_j = \frac{1}{Z_p}\int_{Z_0}^{Z_0+Z_p} f_j\,\mathrm{d}Z \quad (j=2,3,4), \qquad G = \frac{1}{Z_p}\int_{Z_0}^{Z_0+Z_p} g\,\mathrm{d}Z.$$

We obtain a similar equation from the terms in α_2^*, so yielding the pair of constant coefficient equations

$$iB^{\pm}_{\hat{Z}} = F_4 B^{\pm} + G B^{\pm}_{\chi\chi} + (F_2 |B^{\pm}|^2 + F_3 |B^{\mp}|^2) B^{\pm}. \tag{3.12}$$

The term with coefficient F_4 may be absorbed by the substitution

$$B^{\pm} = C^{\pm} e^{-iF_4 \hat{Z}},$$

which shows that $\gamma^2 F_4$ corresponds to an averaged perturbation in the group slowness s_g. Rescaling the independent variables by defining

$$\tau = F_2 \hat{Z}, \qquad x = (F_2/G)^{\frac{1}{2}} \chi$$

allows equations (3.12) to be written as

$$iC^{\pm}_{\tau} = C^{\pm}_{xx} + (|C^{\pm}|^2 + h\,|C^{\mp}|^2) C^{\pm}, \tag{3.13}$$

where $h = F_3/F_2$.

These are identical in form to the equations for a fibre without longitudinal inhomogeneities, so demonstrating that, when longitudinal variations are periodic and take place on a scale intermediate between the wavelength and the nonlinear evolution length, evolution is the same as in an 'equivalent' longitudinally homogeneous fibre. The relevant coefficients are averaged over a period of the longitudinal variation. This implies that for relatively weak signals, with nonlinear evolution length much longer than the scale of the longitudinal inhomogeneities, nondistorting pulses should propagate, provided that the launching conditions are those appropriate to the 'equivalent' fibre. This is similar to the concept of a 'guiding centre soliton' introduced by Hasegawa & Kodama (1990) for long-distance transmission systems involving many periodically spaced amplifiers designed to compensate for small losses in the intervening cable.

4. Fibres allowing exact soliton solutions

Grimshaw (1979) discussed possibilities for determining closed form solutions of the variable coefficient cubic Schrödinger equation

$$iu_t + \lambda u_{xx} + \nu\,|u|^2 u = 0, \tag{4.1}$$

with real-valued coefficients $\lambda = \lambda(t)$ and $\nu = \nu(t)$. He showed that equation (4.1) can be reduced to the constant coefficient nonlinear Schrödinger equation

$$ip_\sigma + p_{\xi\xi} + (\operatorname{sgn} \lambda\nu)\,|p|^2 p = 0,$$

by the transformation

$$u = |\nu/\lambda|^2 p e^{-\frac{1}{4} M |\nu/\lambda| x^2},$$

$$\xi = |\nu/\lambda|\,x, \qquad \sigma = \frac{1}{M}|\nu/\lambda| - \frac{1}{M}|\nu/\lambda|_{t=0},$$

provided that λ and ν are related by the constraint

$$(\lambda/\nu)_t = -\lambda M \quad (M = \text{const}).$$

Consequently, this reduction to the constant coefficient equation is possible for arbitrary smooth one-signed $\lambda(t)$ provided that v has the form

$$v(t) = -\lambda(t)\left(M\int^t \lambda(s)\,ds\right)^{-1},$$

or equivalently for arbitrary one-signed $v(t)$ with

$$\lambda(t) = \pm v(t)\exp\left(-M\int^t v(s)\,ds\right). \tag{4.2}$$

To investigate whether transformations exist allowing the coupled cubic Schrödinger equations (3.6) with variable coefficients to be reduced similarly to constant coefficient equations, we investigate substitutions of the form

$$\left.\begin{aligned} A^{\pm}(\chi, Z) &= m^{\pm}C^{\pm}(\xi, \sigma)e^{in^{\pm}}, \\ \xi = F(\chi, Z), &\qquad \sigma = G(\chi, Z), \end{aligned}\right\} \tag{4.3}$$

where F, G, $m^{\pm}$, and $n^{\pm}$ are real functions of χ and Z. These functions are chosen such that $C^{\pm}$ satisfy the equations

$$-iC^{\pm}_{\sigma} + C^{\pm}_{\xi\xi} + (|C^{\pm}|^2 + h\,|C^{\mp}|^2)C^{\pm} = 0, \tag{4.4}$$

where h is a constant.

Substitution of equations (4.3) into (3.6) yields equations (4.4) only if G, m^{+}, and m^{-} are independent of χ and, moreover, if

$$m^{\pm}_{Z} - gm^{\pm}n^{\pm}_{\chi\chi} = 0, \qquad n^{\pm}_{Z} - g(n^{\pm}_{\chi})^2 = 0,$$

$$F_Z - 2gF_{\chi}n^{\pm}_{\chi} = 0, \qquad G_Z = gF^2_{\chi} = f_2(m^{\pm})^2,$$

$$f_3/f_2 = h(m^{+}/m^{-})^2 = h(m^{-}/m^{+})^2.$$

The cases in which this system is compatible may be reduced, without loss of generality, to

$$n^{\pm} = \chi^2 n(Z), \qquad m^{\pm} = \alpha n^{\frac{1}{2}}, \qquad n(Z) = \frac{\alpha^2}{\upsilon^2}\frac{f_2(Z)}{g(Z)},$$

with

$$\xi = \alpha^2 f_2\chi/\upsilon g, \qquad \sigma = \alpha^2 f_2/4g,$$

provided that $n'(Z) = 4g(Z)n^2(Z)$. Here α and υ are constants. Setting $\upsilon = \alpha^2 = 4M^{-1}$ gives the transformations

$$A^{\pm} = (f_2/g)^{\frac{1}{2}}C^{\pm}(\xi, \sigma)\exp\left[\tfrac{1}{4}iM(f_2/g)\chi^2\right], \tag{4.5}$$

with

$$\xi = (f_2/g)\chi, \qquad \sigma = f_2/4g.$$

These reduce the special case

$$iA^{\pm}_{Z} = g(Z)A^{\pm}_{\chi\chi} + f_2(Z)(|A^{\pm}|^2 + h\,|A^{\mp}|^2)A^{\pm} \tag{4.6}$$

of equations (3.6) to the form (4.4) when $g(Z)$ and $f_2(Z)$ are related by

$$g(Z) = f_2(Z) \exp\left(-M\int^Z f_2(s)\, ds\right), \tag{4.7}$$

which is analogous to (4.2).

Thus, Grimshaw's reduction extends to the coupled system (3.6) wheneve $f_3(Z)/f_2(Z)$ is constant $(=h)$ and when $g(Z)$ is related to $f_2(Z)$ in the manne required for the single equation. Consequently, when (4.7) is satisfied, exac solutions for the system (4.6) may be found corresponding to all the similarit solutions catalogued in Parker (1988) and especially to the uniform wavetrain and linearly and circularly polarized solitons. Moreover, the pulse collision investigated by Parker & Newboult (1989) will correspond to collisions wit negligible scattering when (4.7) is satisfied.

The condition which relates g and f_2, equation (4.7), includes the possibilit $g(Z)/f(Z) = \text{constant}$ $(M=0)$. Otherwise it does not correspond to physica behaviour over the long fibre lengths which are required for optical communica tions systems if f_2 fluctuates without any change of sign. In this case, as $Z \to \infty$ the argument of the exponential in (4.7) will tend to $\pm\infty$, and will model dispersionless fibre, $g/f_2 \to 0$, when the argument of the exponential tends to $-\infty$ or a fibre with infinite dispersion, $g/f_2 \to \infty$, when the argument tends to ∞. In th following section the effects of sinusoidal fluctuations on a linearly polarized puls are studied numerically.

5. Numerical results

To investigate the effect of slow variations in the material properties, th following change of variable

$$\tau = \int^Z g(Z')\, dZ' \quad (g > 0)$$

was made in equations (3.6), and the resulting equations written in the form

$$iA^{\pm}_{\tau} = A^{\pm}_{\chi\chi} + [h_1(\tau)\,|A^{\pm}|^2 + h_2(\tau)\,|A^{\mp}|^2]A^{\pm}, \tag{5.1}$$

where $h_1(\tau) = f_2/g$ and $h_2(\tau) = f_3/g$.

For circularly polarized solitons $(A^- = 0)$, equation (5.1) reduces to a singl cubic Schrödinger equation which has only one variable coefficient h_1. It is know that when h_1 is constant this equation allows solutions of the form

$$A^+(\tau, \chi) = (2/h_1)^{\frac{1}{2}}\Gamma e^{-i\phi} \operatorname{sech} \Gamma(\chi - 2V\tau), \quad \text{where} \quad \phi = V\chi - (V^2 - \Gamma^2)\tau.$$

Here Γ is the pulse amplitude and V is a frequency shift which determines th speed of the pulse envelope. For $h_1'(\tau) \neq 0$, the evolution of a pulse which has thi initial condition may be analysed numerically.

It is also seen that for $A^-(0, \chi) = e^{-2i\alpha}A^+(0, \chi)$ there exist solution $A^-(\tau, \chi) = e^{-2i\alpha}A^+(\tau, \chi)$, for which equation (5.1) becomes

$$iA^+_{\tau} = A^+_{\chi\chi} + [h_1(\tau) + h_2(\tau)]\,|A^+|^2 A^+. \tag{5.2}$$

This will allow solutions of a form similar to those of a circularly polarized pulse, but which will have h_1 replaced by $h_1 + h_2$. Again solutions can be computed numerically for $h_1' + h_2' \neq 0$.

This shows that both circularly or linearly polarized sech pulses evolve according to a single variable coefficient nonlinear Schrödinger equation, although the coefficient of the nonlinear term will differ in the two cases. More general initial conditions could be used which will yield solutions with a stronger coupling between the equations.

To illustrate the evolution for a linearly polarized signal, h_1 was taken to be a constant and h_2 was taken to have a sinusoidal variation about a fixed value h_0,

$$h_2 = h_0 + a \sin b\tau.$$

For the numerical results given in this paper, values for these parameters were taken to be $h_0 = 2$, $h_1 = 1$, $a = 0.2$, and $b = 2.75$.

Numerical integration of equations (5.1) was performed using a split-step spectral method, with a damping scheme applied at the edges of the integration region as described by Menyuk (1988). The edge damping is required because the periodic boundary conditions assumed by the fast Fourier transform could cause any radiation which has left the computational region to return to the region and cause effects which are due to the numerical scheme rather than the physical

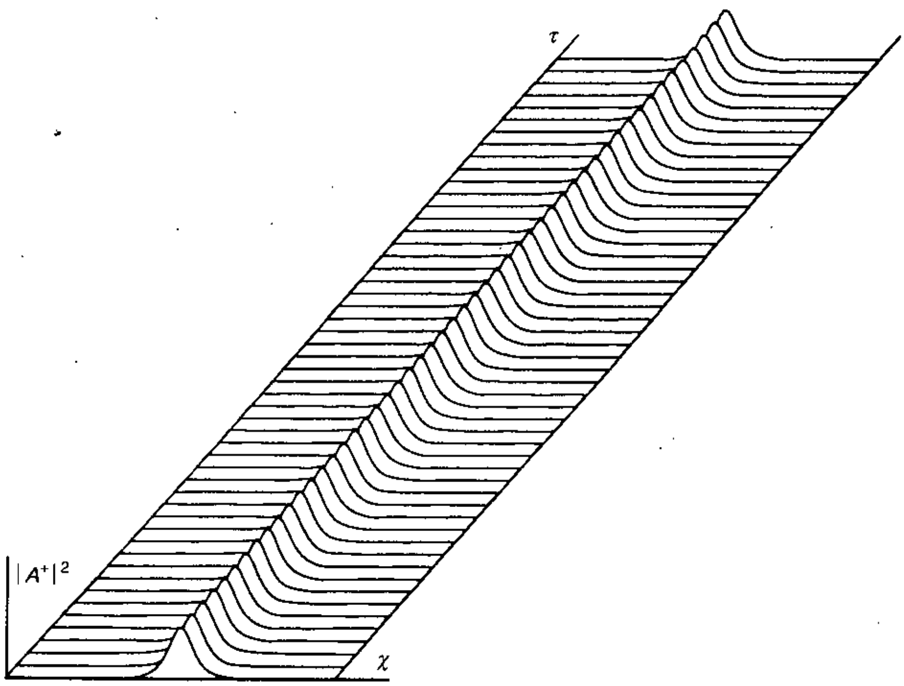

FIG. 1. Evolution of a linearly polarized pulse through a fibre whose material properties vary sinusoidally along the fibre. The propagation range is 40 cycles of the inhomogeneity.

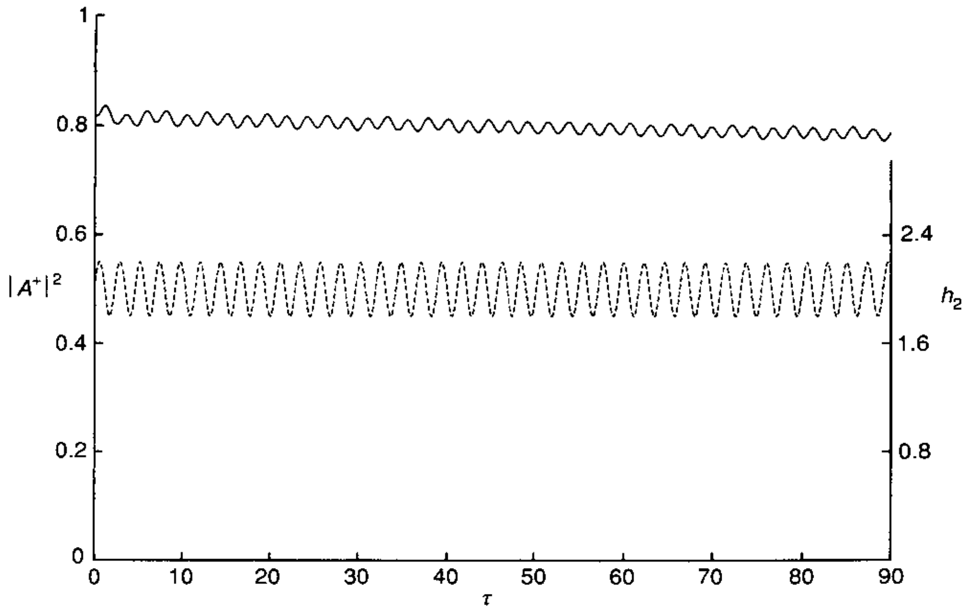

FIG. 2. Peak values of $|A^+|^2$ (——) and the material property $h_2(\tau)$ (- - - -) plotted against τ.

system. The initial conditions, at $\tau = 0$, were chosen to agree with

$$A^+ = \left(\frac{2}{1+h_2}\right)^{\frac{1}{2}} \Gamma \operatorname{sech} \Gamma(\chi - \chi_0), \qquad A^- = \left(\frac{2}{1+h_2}\right)^{\frac{1}{2}} \Gamma \operatorname{sech} \Gamma(\chi - \chi_0) e^{2i\alpha},$$

where α is the polarization angle and the soliton is centred initially at $\chi = \chi_0$. The parameters were taken to be $\Gamma = 1$, $\alpha = 0$, and $\chi_0 = 25.6$, and the values for the step lengths for the numerical discretization were $\Delta\chi = 0.1$ and $\Delta\tau = 5 \times 10^{-3}$.

From Fig. 1, which is a graph of $|A^+|^2$ plotted against χ and τ, it is not possible to detect radiation away from the pulse. However, if the maximum values of $|A^+|^2$ are plotted against τ (Fig. 2), it can be seen from the decrease in the value of $|A^+|^2$ that there is some radiation of energy away from the pulse. It can also be seen that the peak values of $|A^+|^2$ are no longer constant, as in the case of a constant coefficient nonlinear Schrödinger equation, but fluctuate almost periodically with a period similar to that of the material fluctuations. However, the loss of peak amplitude of the pulse is only about 3% after 40 cycles of fluctuations of h_2 having 10% variation either side of its mean value.

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