LEAST SQUARE SMOOTHING BY LINEAR COMBINATION.

by

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INTRODUCTION.

The problem of fitting a polynomial to a set of observational data so that the sum of the squared residuals is a minimum has been frequently investigated. A.C. Aitken, in an appendix to his paper, "On the Graduation of Data by the Orthogonal Polynomials of Least Squares", (Proc. Roy. Soc. Edin. Vol. LIII (1933) pp. 77-78,) provides a list of the more important papers on this subject. Tchebychef* and Gram* were the first to expound, more than fifty years ago, the method of fitting my means of orthogonal polynomials. Their work has been followed up in more recent times by several writers, including Jordan, R.A. Fisher and A.C. Aitken. W.F. Sheppard* and C.W.M. Sherriff* develop the equivalent method of linear combination of data. However, in the latter method, the fitted value of the central observation only is considered in detail, and an odd number of data is therefore required.

It/

* Tchebychef, P.L. 1858. Oeuvres. Vol.I. pp.203-230, 381-384, 473-498, 541-560, 701-702.
Vol.II. pp.219-242.

Gram, J.P. 1882. J. für Math. Vol. XCIV. pp.41-73.

Sheppard, W.F. 1912 (a). Proc. Fifth Intern. Congr. Math. (Cambridge) Vol. II. p.348.
1914.(b). Proc. Lond. Math. Soc. (2). Vol. XIII. p.97.
1914, 1915. (c). Journ. Inst. Act. Vol. XLVIII. pp. 171-185, 390-412. Vol. XLIX. pp. 148-157.

Sherriff, C.W.M. 1920. Proc. Roy. Soc. Edin. Vol. XL. pp. 112-128.

It has been shown by W.F. Sheppard, and recently, with much more conciseness by G.J. Lidstone* that the methods of Least Square Fitting and Linear Combination of Minimal Reduction Co-efficient lead to identical results.

It is proposed, in the following investigation, to express all the fitted values as linear combinations of the observed values. The data considered are either odd or even in number, equidistant, unweighted and uncorrelated. Each fitted value v_x is expressed as a linear combination of the observed values $u_0, u_1, u_2, u_3, \ldots \ldots u_{n-1}$; and so there arise n equations of the form

$$v_x = c_0 u_0 + c_1 u_1 + \ldots \; c_{n-1} u_{n-1} ,$$

the co-efficients being certain bilinear functions of Tchebychef's orthogonal polynomials. Thus for every value of n , (the number of data) and of r , (the degree of the fitted polynomial) a matrix of co-efficients, C , exists, connecting the v's and the u's .

In Chapter I we shall investigate the form of the matrix C , and its properties, and shall give examples of its construction and practical use. The differences of the v's are similarly found to be expressible as linear combinations of the differences of the u's . The corresponding matrices of co-efficients, obtained in this way, of lower and lower order,/

* Lidstone, G.J. 1933. Journ. Inst. Act. Vol. LXIV. pp. 153-158.

order, are discussed in Chapter II, and appropriate examples given. The properties of these matrices are very similar to, and often identical with those of the matrix C . Chapter III contains alternative methods of fitting, with simple checks on accuracy of working. Examples of each method are given.

The appendix consists of tables of the numerical values of the matrices connecting observed and fitted values, for numbers of data equal to 4, 5, 6, 7 - - - 15, and for fitted polynomials of degree 0, 1, 2, 3, 4, 5, together with the corresponding matrices connecting the differences. There is also a short bibliography.

Methods of fitting involving the matrices discussed here are particularly suitable for rapid calculation with a machine. Indeed the use of a machine is taken for granted.

The same example is used throughout to simplify the comparison of the various methods. It is taken from Dr A.C. ~~Arthur's~~ Aitken's paper referred to above.

Finally I wish to express my very deep indebtedness to Dr A.C. Aitken of the University of Edinburgh for his constant help and encouragement; and, in particular, for suggesting alternative proofs for the rank of the matrices, Chapter I § 2 (VI) and Chapter II § 5 (V), and the method of fitting given in Chapter III § 7.

CHAPTER I.

§ 1. MATRICES CONNECTING THEORETICAL AND OBSERVED VALUES.

Let there be given n equidistant, unweighted, uncorrelated data u_0, u_1, u_2, . . . u_{n-1} . Suppose a polynomial of degree r is fitted to the data by the principle of Least Squares. Let v_0, v_1, v_2, . . . v_{n-1} denote the corresponding fitted values. If the fitting is done by means of Tchebychef's orthogonal polynomials, we have

$$v_x = a_0 + a_1 T_1(x) + a_2 T_2(x) \ldots + a_r T_r(x) \,, \quad *$$

where x can take the values 0, 1, 2, 3 --- $n-1$; $T_h(x)$ or T_h denotes the Tchebychef polynomial of degree h , and

$$a_h = \frac{\sum_s u_s T_h(s)}{\sum T_h^2}$$

$$\therefore\ v_x = \frac{\sum_s u_s}{\sum T_0^2} + \frac{\sum_s u_s T_1(s)}{\sum T_1^2} \cdot T_1(x) + \frac{\sum_s u_s T_2(s)}{\sum T_2^2} \cdot T_2(x) + \ldots + \frac{\sum_s u_s T_r(s)}{\sum T_r^2} \cdot T_r(x)$$

$$= \left\{ \frac{1}{\sum T_0^2} + \frac{T_1(0) T_1(x)}{\sum T_1^2} + \frac{T_2(0) T_2(x)}{\sum T_2^2} + \cdots + \frac{T_r(0) T_r(x)}{\sum T_r^2} \right\} u_0$$

$$+ \left\{ \frac{1}{\sum T_0^2} + \frac{T_1(1) T_1(x)}{\sum T_1^2} + \cdots + \frac{T_r(1) T_r(x)}{\sum T_r^2} \right\} u_1$$

$$+ \ . \ .$$

$$+ \left\{ \frac{1}{\sum T_0^2} + \frac{T_1(n-1) T_1(x)}{\sum T_1^2} + \ \ldots \ + \frac{T_r(n-1) T_r(x)}{\sum T_r^2} \right\} u_{n-1} \,.$$

* A.C. Aitken. Proc. Roy. Soc. Edin. Vol. LIII (1933) p. 57. Tables of the Tchebychef polynomials pp. 61-63.

There are n equations of this type, one for each value of x, expressing each v as a linear combination of all the u's. In matrix notation they may be written:

$$\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\Sigma_1 T_0^2} + \frac{T_1(0)T_1(0)}{\Sigma_1 T_1^2} + \cdots + \frac{T_r(0)T_r(0)}{\Sigma_1 T_r^2} & \cdot & \cdot & \frac{1}{\Sigma_1 T_0^2} + \frac{T_1(n-1)T_1(0)}{\Sigma_1 T_1^2} + \cdots + \frac{T_r(n-1)T_r(0)}{\Sigma_1 T_r^2} \\ \frac{1}{\Sigma_1 T_0^2} + \frac{T_1(1)T_1(0)}{\Sigma_1 T_1^2} + \cdots + \frac{T_r(1)T_r(0)}{\Sigma_1 T_r^2} & & & \\ \frac{1}{\Sigma_1 T_0^2} + \frac{T_1(2)T_1(0)}{\Sigma_1 T_1^2} + \cdots + \frac{T_r(2)T_r(0)}{\Sigma_1 T_r^2} & & & \\ \cdot & \cdot & & \\ & & & \\ \frac{1}{\Sigma_1 T_0^2} + \frac{T_1(n-1)T_1(0)}{\Sigma_1 T_1^2} + \cdots + \frac{T_r(n-1)T_r(0)}{\Sigma_1 T_r^2} & \cdot & \cdot & \frac{1}{\Sigma_1 T_0^2} + \frac{T_1(n-1)T_1(n-1)}{\Sigma_1 T_1^2} + \cdots + \frac{T_r(n-1)T_r(n-1)}{\Sigma_1 T_r^2} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \\ \\ u_{n-1} \end{bmatrix}$$

Let $C \equiv [c_{ij}]$ denote the matrix of the coefficients, so that C is the square matrix

$$\begin{bmatrix} \frac{1}{\Sigma_1 T_0^2} + \frac{T_1(0)T_1(0)}{\Sigma_1 T_1^2} + \cdots + \frac{T_r(0)T_r(0)}{\Sigma_1 T_r^2} & \cdot & \cdot & \frac{1}{\Sigma_1 T_0^2} + \frac{T_1(n-1)T_1(0)}{\Sigma_1 T_1^2} + \cdots + \frac{T_r(n-1)T_r(0)}{\Sigma_1 T_r^2} \\ \frac{1}{\Sigma_1 T_0^2} + \frac{T_1(1)T_1(0)}{\Sigma_1 T_1^2} + \cdots \frac{T_r(1)T_r(0)}{\Sigma_1 T_r^2} & & & \\ \cdot & \cdot & & \\ & & & \\ \frac{1}{\Sigma_1 T_0^2} + \frac{T_1(n-1)T_1(0)}{\Sigma_1 T_1^2} + \cdots + \frac{T_r(n-1)T_r(0)}{\Sigma_1 T_r^2} & & & \frac{1}{\Sigma_1 T_0^2} + \frac{T_1(n-1)T_1(n-1)}{\Sigma_1 T_1^2} + \cdots + \frac{T_r(n-1)T_r(n-1)}{\Sigma_1 T_r^2} \end{bmatrix}$$

Tables of this matrix for n = 4, 5, 6, 7 ----- 15 and r = 0, 1, 2, 3, 4, 5 will be found in the appendix.

§ 2. PROPERTIES OF THE MATRIX C.

(I) It is axi-symmetric.

The $(p+1, q+1)^{\text{th}}$ element is

$$\frac{1}{\Sigma T_0^2} + \frac{T_1(q)\,T_1(p)}{\Sigma T_1^2} + \frac{T_2(q)\,T_2(p)}{\Sigma T_2^2} + \cdots + \frac{T_r(q)\,T_r(p)}{\Sigma T_r^2}$$

This is symmetrical in p and q, and is therefore the $(q+1, p+1)^{\text{th}}$ element of the matrix also. Thus the matrix is axisymmetric.

(II) The sum of the elements in any row (or column) of C equals unity.

The sum of the elements in the $(p+1)^{\text{th}}$ row is

$$\frac{n}{\Sigma T_0^2} + \frac{T_1(p)}{\Sigma T_1^2}\left\{T_1(0)+T_1(1)+\cdots+T_1(n-1)\right\} + \frac{T_2(p)}{\Sigma T_2^2}\left\{T_2(0)+T_2(1)+\cdots+T_2(n-1)\right\} + \cdots$$
$$+ \frac{T_r(p)}{\Sigma T_r^2}\left\{T_r(0)+T_r(1)+\cdots+T_r(n-1)\right\}.$$

But $T_s(0) + T_s(1) + \ldots + T_s(n-1) = 0 \quad (s > 0)$, by the orthogonal properties of these polynomials.

and $\Sigma T_0^2 = n$

$\therefore$ the sum equals unity.

This is also obvious from the fact that the least square graduation of $u = \{1, 1, \ldots 1\}$ will simply reproduce $v = \{1, 1, \ldots 1\}$.

(III) C is centro-symmetric.

Since an arbitrary set of data could equally well be graduated in the reverse order, the matrix C must be centro-symmetric.

This/

This result also follows immediately from the fact[*1] that

$$T_r(x) = (-)^r T_r(n-x-1).$$

(IV). $\underline{C^2 = C}$

The graduating operation may be written

$$v = Cu$$

where u are the observed and v the fitted values.
If the values v are themselves regraduated, they will merely be reproduced, since they already lie on the polynomial.

Hence

$$Cv = v$$
$$\therefore C^2 u = v$$
$$= Cu.$$

Since u is an arbitrary set of data,

$$\therefore C^2 = C.$$

$\therefore$ Any power of C is identical with C itself. Thus the matrix C is idempotent.

(V). The latent roots of C are 1 and 0.

Since $C^2 = C$,

$\therefore C$ satisfies the equation.

$$\lambda^2 - \lambda = 0.$$

This is, in general, its reduced characteristic equation.[*2]

(In/

*1 A.C. Aitken. Proc. Roy. Soc. Edin. Vol. LIII (1933) p.64.
*2 "Canonical Matrices", Turnbull and Aitken. p.48.

(In the particular case when $C = I$, the reduced characteristic equation is $\lambda - 1 = 0$).

$\therefore$ The latent roots of C are 1 and 0 .

(VI.) <u>The number of latent roots equal to unity is $(r+1)$, where r is the degree of the fitted polynomial.</u>

If $(i+1)$ rows are linearly dependent, values of A, B, C ---- K can be found such that

$$\frac{1}{\Sigma T_0^2} + \frac{T_1(h_i)T_1(q)}{\Sigma T_1^2} + \dots + \frac{T_r(h_i)T_r(q)}{\Sigma T_r^2} = A\left\{\frac{1}{\Sigma T_0^2} + \frac{T_1(h_{i-1})T_1(q)}{\Sigma T_1^2} + \dots + \frac{T_r(h_{i-1})T_r(q)}{\Sigma T_r^2}\right\}$$
$$+ B\left\{\frac{1}{\Sigma T_0^2} + \frac{T_1(h_{i-2})T_1(q)}{\Sigma T_1^2} + \dots + \frac{T_r(h_{i-2})T_r(q)}{\Sigma T_r^2}\right\}$$
$$+ \dots$$
$$\cdot$$
$$+ K\left\{\frac{1}{\Sigma T_0^2} + \frac{T_1(h_0)T_1(q)}{\Sigma T_1^2} + \dots + \frac{T_r(h_0)T_r(q)}{\Sigma T_r^2}\right\} \quad \dots (1)$$

where the h_0+1, h_1+1, h_2+1, ... h_i+1 th rows and the $q+1$ th column are considered.

Equation (1) holds for n values of q , $q = 0, 1, 2 \dots n-1$ (n columns).

Also $r \leq (n-1)$, since the polynomial of degree $(n-1)$ gives a perfect fit.

$\therefore$ (1) is an identity.

Equating/

This theorem might have been expected from other considerations.

The first or indeed any set of $(r+1)$ rows of C operating on an arbitrary vector u, produce $(r+1)$ values of a polynomial v of degree r. Now these, and no lesser number, are sufficient to determine the polynomial, and any other values of the polynomial, such as those given by graduation of the u's by means of later rows of C, are expressible linearly in terms of these $(r+1)$ values. But u is arbitrary. Hence the later rows of C themselves are expressible linearly in terms of the first $(r+1)$ rows, from which it follows that C is of rank $(r+1)$.

Hence, as above, the number of latent roots equal to unity is $(r+1)$, and the remaining latent roots are zeros.

(VII). Sum of the diagonal elements.

(a) A deduction from (VI) is that the "trace" of the matrix C, i.e. the sum of the diagonal elements, is $r+1$, for this is invariant, being the sum of the latent roots.

For example, the sum of the diagonal elements in the matrix C for $n=11$, $r=4$ (p. ~~22~~ 21) is $\frac{2145}{429} = 5$.

This serves as a useful check on the calculations.

An alternative proof is as follows:

Sum of the elements in the principal diagonal

$$= \sum_{k=0}^{n-1} \left\{ \frac{1}{\Sigma T_0^2} + \frac{T_1^2(k)}{\Sigma T_1^2} + \cdots + \frac{T_r^2(k)}{\Sigma T_r^2} \right\}$$

$$= r+1.$$

Equating co-efficients of $\frac{T_s(q)}{\sum T_s^2}$ we have

$$A + B + C \qquad + K = 1$$

$$AT_1(h_{i-1}) + BT_1(h_{i-2}) + \dots \qquad + KT_1(h_0) = T_1(h_i)$$

$$AT_r(h_{i-1}) + BT_r(h_{i-2}) + \dots \qquad + KT_r(h_0) = T_r(h_i)$$

There are i unknowns A, B, ---- K , and $r+1$ equations. If $i \geq r+1$ one or more sets of values for A, B, C ---- K can be found, so that (1) holds.

$\therefore$ the rows in a set containing $(r+2)$ or more rows are linearly dependent.

$\therefore$ the rank of $C < r+2$ .... (2)

If $i = r$ no values can be found for A, B --- K to satisfy all the equations, since no linear relation exists between $T_0(x)$, $T_1(x)$, ... $T_r(x)$.

$\therefore$ $(r+1)$ rows are linearly independent, and from (2) no more than $(r+1)$ rows are linearly independent.

$\therefore$ the rank of C is $(r+1)$.

Since the matrix C can be reduced to a purely diagonal form with the latent roots as diagonal elements, therefore the number of latent roots equal to unity is $(r+1)$, the remaining ones by (v) being zeros.

This/

(b). The sum of the elements in the secondary diagonal equals 0 or 1, according as the rank is even or odd.

The sum of the elements in the secondary diagonal

$$= \frac{1}{\Sigma T_0^2} + \frac{T_1(n-1)T_1(0)}{\Sigma T_1^2} + \cdots + \frac{T_r(n-1)\,T_r(0)}{\Sigma T_r^2}$$

$$+ \frac{1}{\Sigma T_0^2} + \frac{T_1(n-2)T_1(1)}{\Sigma T_1^2} + \cdots + \frac{T_r(n-2)T_r(1)}{\Sigma T_r^2}$$

$$+ \cdots$$

$$+ \frac{1}{\Sigma T_0^2} + \frac{T_1(0)T_1(n-1)}{\Sigma T_1^2} + \cdots + \frac{T_r(0)T_r(n-1)}{\Sigma T_r^2}$$

But $T_k(x) = (-)^k\, T_k(n-x-1)$

$$\therefore \text{sum} = 1 - \frac{1}{\Sigma T_1^2}\left\{\Sigma T_1^2(x)\right\} + \frac{1}{\Sigma T_2^2}\left\{\Sigma T_2^2(x)\right\} + \cdots$$

$$+ (-)^r \frac{1}{\Sigma T_r^2}\left\{\Sigma T_r^2(x)\right\}$$

= 1 or 0 according as r is even or odd.

i.e. = 0 or 1 according as the rank is even or odd.

For example, the sum of the elements in the secondary diagonal of the matrix C for $n = 11$, $r = 4$ (p. ~~22~~ 21) is $\frac{429}{429} = 1$, the rank being odd, equal to 5.

(VIII)./

(VIII). Sum of the diagonal minors of C .

(a). Sum of the principal diagonal minors.

Let p_t denote the sum of the diagonal minors of order t of the matrix C .

Then

$$\lambda^n - p_1\lambda^{n-1} + p_2\lambda^{n-2} - \cdots\cdots + (-)^n p_n = 0 \qquad \ldots (1)$$

is the characteristic equation of C .

But the roots of (1) are $(r+1)$ units, and $(n-r-1)$ zeros.

$\therefore$ the equation

$$\lambda^{r+1} - p_1\lambda^r + p_2\lambda^{r-1} - \cdots + (-)^{r+1} p_{r+1} = 0 \qquad \ldots (2)$$

has all its roots equal to unity.

$$\therefore (2) \equiv (\lambda - 1)^{r+1} = 0$$

$\therefore p_t = {}^{r+1}c_t$ for values of t up to $r+1$,

where ${}^{r+1}c_t$ denotes the combination of $r+1$ things taken t at a time.

Since the rank of C is $r+1$,

$\therefore p_t = 0$ for $t > r+1$.

(b). Sum of the secondary diagonal minors.

The secondary diagonal minors of C are the principal diagonal minors of J.C.*

We shall investigate the principal diagonal minors of J.C.

Now/

* J is the matrix with units in the secondary diagonal, zeros everywhere else.
Turnbull and Aitken, "Theory of Canonical Matrices", p.11.

Now $(J\,C)^3 = J\,C$

since the matrix C is centro-symmetric, and idempotent.

Hence $J\,C$ satisfies the equation $\lambda^3 - \lambda = 0$

which is its reduced characteristic equation.

$\therefore$ the latent roots of $J\,C$ are $0, \pm 1$.

The rank of $J\,C$ = rank of C = $r+1$.

$\therefore$ there are $r+1$ latent roots of $J\,C$ equal to $+1$ or -1

and $n-r-1$ equal to zero.

The characteristic equation of $J\,C$ is

$$\lambda^n - q_1 \lambda^{n-1} + q_2 \lambda^{n-2} \ldots \quad + (-)^n q_n = 0 \qquad \text{- - - } (1)$$

where q_t denotes the sum of diagonal minors of order t.

Removing the $n-r-1$ roots equal to zero,

$$\therefore \lambda^{r+1} - q_1 \lambda^r + q_2 \lambda^{r-1} \ldots \quad (-)^{r+1} q_{r+1} = 0 \qquad \text{- - - } (2).$$

has roots equal to + 1 and - 1.

But the sum of the latent roots of $J\,C$, which equals the sum of the diagonal elements, equals 0 or 1 according as the rank is even or odd, by (VII), (b).

$\therefore$ Firstly, if the rank is even, the number of latent roots of $J\,C$ equal to + 1 equals the number equal to - 1.

$$\therefore (2) \equiv (\lambda^2 - 1)^{\frac{r+1}{2}} = 0. \qquad \text{- - } (3)$$

$$\therefore q_{2t-1} = 0.$$

$$q_{2t} = (-)^t \, {}^{\frac{r+1}{2}}C_t, \quad \text{when } t = 1, 2, 3 \ldots \tfrac{r+1}{2}.$$

Since the rank of $J\,C$ is $r+1$, the sum of diagonal minors of any order $> r+1$ equals 0.

Secondly,/

Secondly, if the rank is odd, the number of latent roots of $J\,C$ equal to $+1$ is one greater than the number equal to -1.

$$\therefore (2) \equiv (\lambda-1)(\lambda^2-1)^{\frac{r}{2}} = 0 \qquad (4).$$

$$\therefore q_{2t+1} = q_{2t} = (-)^{t}\, {}^{\frac{r}{2}}c_t \,, \text{ for } t = 0, 1, 2 \ldots \tfrac{r}{2}$$

As above, the sum of diagonal minors of any order $> r+1$ equals 0.

(IX). <u>Any row of C multiplied by another, or the same row, in which the element belonging to the principal diagonal is reduced by unity, has a zero product.</u>

Since $C^2 = C$

$\therefore C(C-I) = 0$

$\therefore$ a row of C multiplied by a column (which, by symmetry, is the same as a row) of $C - I$ $= 0$

Hence the result.

(X). <u>Latent vectors of C.</u>

Since any set of values v already lying on a polynomial of degree r will be reproduced on graduation, we have

$$Cv = v.$$

$\therefore$ a set of values v lying on the polynomial constitutes a latent vector of the matrix C.

In particular, the set $\{0^r, 1^r, 2^r, \ldots (n-1)^r\}$ is latent, i.e. transformed into itself.

(XI). Each row of C , after the first $(r+1)$ rows, is formed by multiplying the previous $(r+1)$ rows by the first $(r+1)$ co-efficients in the expansion of $(-)^r\ (1-1)^{r+1}$ respectively and adding.

Consider the elements in the $(q+1)^{th}$ column, where q may take any value from the 0 to $n-1$.

Take $p > r$.

Multiply elements in the $(p-r+1)^{th}, (p-r+2)^{th} \ldots (p+1)^{th}$ rows by the first $r+1$ co-efficients of $(-)^r\ (1-1)^{r+1}$ respectively and add.

We have, writing terms in the reverse order,

$$(r+1)\left\{\frac{1}{\Sigma T_0^2} + \frac{T_1(p)\,T_1(q)}{\Sigma T_1^2} + \ldots + \frac{T_r(p)\,T_r(q)}{\Sigma T_r^2}\right\}$$

$$-\frac{(r+1)r}{2!}\left\{\frac{1}{\Sigma T_0^2} + \frac{T_1(p-1)\,T_1(q)}{\Sigma T_1^2} + \ldots \quad \frac{T_r(p-1)\,T_r(q)}{\Sigma T_r^2}\right\}$$

$$+ \ldots$$

$$+ (-)^r\left\{\frac{1}{\Sigma T_0^2} + \frac{T_1(p-r)\,T_1(q)}{\Sigma T_1^2} + \ldots + \frac{T_r(p-r)\,T_r(q)}{\Sigma T_r^2}\right\}$$

$$= \sum_{s=0}^{r}\left[\frac{T_s(q)}{\Sigma T_s^2}\left\{(r+1)T_s(p) - \frac{(r+1)r}{2!}T_s(p-1) \ldots (-)^r T_s(p-r)\right\}\right] \quad \text{--- } (1)$$

Consider $T_s(p+1) - (r+1)T_s(p) + \frac{(r+1)r}{2!}T_s(p-1) + \ldots + (-)^{r+1}T_s(p-r)$

where $s = 0, 1, 2 \ldots r$.

This/

This equals $\Delta^{r+1} T_s(h-r)$

$= 0$ since $r+1 > s$.

$\therefore$ (1) equals $\frac{1}{\Sigma T_0^2} + \frac{T_1(q)}{\Sigma T_1^2} . T_1(h+1) + \ldots + \frac{T_r(q)}{\Sigma T_r^2} . T_r(h+1)$

$= (h+2,\ q+1)^{\underline{th}}$ element of the matrix.

Hence the theorem is proved.

Similarly it may be shown that each row of C equals the sum of the subsequent $(r+1)$ rows multiplied respectively by the first $r+1$ co-efficients of $(-)^r(1-1)^{r+1}$ in the reverse order.

(XII). <u>The sum of the squared elements in the $(h+1)^{\underline{th}}$ row of C equals the $(h+1,\ h+1)^{\underline{th}}$ element.</u>

Sum of the squared elements in the $(h+1)^{\underline{th}}$ row

$$= \sum_{q=0}^{n-1} \left\{ \frac{1}{\Sigma T_0^2} + \frac{T_1(h)T_1(q)}{\Sigma T_1^2} + \ldots + \frac{T_r(h)T_r(q)}{\Sigma T_r^2} \right\}^2$$

$$= \sum_{q=0}^{n-1} \left\{ \frac{1}{(\Sigma T_0^2)^2} + \frac{\{T_1(h)T_1(q)\}^2}{\{\Sigma T_1^2\}^2} + \ldots + \frac{\{T_r(h)T_r(q)\}^2}{\{\Sigma T_r^2\}^2} \right\}$$

the other terms vanishing owing to the orthogonal properties of the polynomials

$$= \frac{1}{\Sigma T_0^2} + \frac{\{T_1(h)\}^2}{\Sigma T_1^2} + \ldots \frac{\{T_r(h)\}^2}{\Sigma T_r^2}$$

$= (h+1,\ h+1)^{\underline{th}}$ element.

This also follows from the facts that C is axisymmetric, and $C^2 = C$. This result has been given, for the case of an odd number of data, by W.F. Sheppard and G.J. Lidstone in the form $c_x(x) = \sum_t c_t(x)^2$

where $v_x = c_{-m}(x) u_{-m} + \dots \quad + c_m(x) u_m$.

For example, when $n = 11$, $r = 4$ (p.21), the sum of the squared elements in the fifth row of C

$$= \frac{54483}{184041} = \frac{127}{429}$$

= element in the fifth row, fifth column.

It follows at once from this and the fact that the sum of the diagonal elements equals $r+1$ that <u>the sum of the squares of all the elements of C equals $r+1$</u>.

The labour of calculating the matrices C is much reduced owing to their axi- and centro- symmetric properties, while the fact that the sum of the elements in each row equals unity provides a useful and very simple check on the calculations.

<u>Example I.</u> Derive the matrix C for the case $n = 11$, $r = 4$.

From A.C. Aitken's tables of Tchebychef polynomials, we have

$$T_0(x) = 1 \qquad \sum T_0^2 = 11$$

$x =$	0	1	2	3	4	5	6	7	8	9	10
$T_1(x)$	-10	-8	-6	-4	-2	0	2	4	6	8	10
$T_1(x)/2$	-5	-4	-3	-2	-1	0	1	2	3	4	5
$T_1(0)T_1(x)/4$	25	20	15	10	5	0	-5	-10	-15	-20	-25
$T_1(1)T_1(x)/4$		16	12	8	4	0	-4	-8	-12	-16	
$T_1(2)T_1(x)/4$			9	6	3	0	-3	-6	-9		
$T_1(3)T_1(x)/4$				4	2	0	-2	-4			
$T_1(4)T_1(x)/4$					1	0	-1				
$T_1(5)T_1(x)/4$						0					

$\Sigma T_1^2 = 440 \quad \therefore \quad \Sigma T_1^2/4 = 110$

$x =$	0	1	2	3	4	5	6	7	8	9	10
$T_2(x)$	45	18	-3	-18	-27	-30	-27	-18	-3	18	45
$T_2(x)/3$	15	6	-1	-6	-9	-10	-9	-6	-1	6	15
$T_2(0)T_2(x)/9$	225	90	-15	-90	-135	-150	-135	-90	-15	90	225
$T_2(1)T_2(x)/9$		36	-6	-36	-54	-60	-54	-36	-6	36	
$T_2(2)T_2(x)/9$			1	6	9	10	9	6	1		
$T_2(3)T_2(x)/9$				36	54	60	54	36			
$T_2(4)T_2(x)/9$					81	90	81				
$T_2(5)T_2(x)/9$						100					

$\Sigma T_2^2 = 7722 \quad \therefore \quad \Sigma T_2^2/9 = 858$

$x =$	0	1	2	3	4	5	6	7	8	9	10
$T_3(x)$	-120	24	88	92	56	0	-56	-92	-88	-24	120
$T_3(x)/4$	-30	6	22	23	14	0	-14	-23	-22	-6	30
$T_3(0)T_3(x)/16$	900	-180	-660	-690	-420	0	420	690	660	180	-900
$T_3(1)T_3(x)/16$		36	132	138	84	0	-84	-138	-132	-36	
$T_3(2)T_3(x)/16$			484	506	308	0	-308	-506	-484		
$T_3(3)T_3(x)/16$				529	322	0	-322	-529			
$T_3(4)T_3(x)/16$					196	0	-196				
$T_3(5)T_3(x)/16$						0					

$\Sigma T_3^2 = 68640 \quad \therefore \quad \Sigma T_3^2/16 = 4290$

$x =$	0	1	2	3	4	5	6	7	8	9	10
$T_4(x)$	210	-210	-210	-35	140	210	140	-35	-210	-210	210
$T_4(x)/35$	6	-6	-6	-1	4	6	4	-1	-6	-6	6
$T_4(0)T_4(x)/1225$	36	-36	-36	-6	24	36	24	-6	-36	-36	36
$T_4(1)T_4(x)/1225$		36	36	6	-24	-36	-24	6	36	36	
$T_4(2)T_4(x)/1225$			36	6	-24	-36	-24	6	36		
$T_4(3)T_4(x)/1225$				1	-4	-6	-4	1			
$T_4(4)T_4(x)/1225$					16	24	16				
$T_4(5)T_4(x)/1225$						36					

$$\Sigma T_4^2 = 350350 \qquad \therefore \ \Sigma T_4^2/1125 = 286$$

Owing to the axi- and centro- symmetric properties of C we need to calculate only a V- shaped portion of the matrix, bounded by the two principal diagonals.

Adding $\frac{1}{\Sigma T_0^2}$ and rows from the above tables, previously dividing the elements by the appropriate $\Sigma T_s^2/c$, we obtain a V- shaped portion of the required matrix.

The working is as follows, the common denominator being 4290

+	390	390	390	390	390	390	390	390	390	390	390	
	975	780	585	390	195	0						
	1125	450								450	1125	
	900											
	540						420	690	660			
					360	540	360				540	
	3930	1620	975	780	945	930	1170	1080	1050	1020	2055	
−			75	450	675	750	195	390	585	780	975	
		180	660	690	420		675	450	75		900	
		540	540	90								
								90	540	540		
		720	1275	1230	1095	750	870	930	1200	1320	1875	
	3930	900	−300	−450	−150	180	300	150	−150	−300	180	--- (1)

+	390	390	390	390	390	390	390	390	390	
	624	468	312	156	0					
	180								180	
	36	132	138	84	0					
	540	540	90				90	540	540	
	1770	1530	930	630	390	390	480	930	1110	
−		30	180	270	300	156	312	468	624	
						270	180	30		
						84	138	132	36	
				360	540	360				
	0	30	180	630	840	870	630	630	660	
	1770	1500	750	0	−450	−480	−150	300	450	--- (2)

+	390	390	390	390	390	390	390	
	351	234	117					
	5	30	45	50	45	30	5	
	484	506	308					
	540	90				90	540	
	1770	1250	860	440	435	510	935	
−					117	234	351	
			360	540	308	506	484	
					360			
			360	540	785	740	835	
	1770	1250	500	−100	−350	−230	100	--- (3)

+	390	390	390	390	390
	156	78			
	180	270	300	270	180
	529	322			15
	15				
	1270	1060	690	660	585
−				78	156
		60	90	322	529
				60	
		60	90	460	685
	1270	1000	600	200	−100

$$- - - (4)$$

+	390	390	390
	39		
	405	450	405
	196		
	240	360	240
	1270	1200	1035
−			39
			196
			235
	1270	1200	800

$$- - - (5)$$

+	390
	500
	540
	1430

$$- - - (6)$$

The common factor 10 is removed, and the remaining elements filled in from the symmetric properties.

Thus finally for $n = 11$, $r = 4$ we have for C

$$\frac{1}{429}\begin{bmatrix} 393 & 90 & -30 & -45 & -15 & 18 & 30 & 15 & -15 & -30 & 18 \\ 90 & 177 & 150 & 75 & 0 & -45 & -48 & -15 & 30 & 45 & -30 \\ -30 & 150 & 177 & 125 & 50 & -10 & -35 & -23 & 10 & 30 & -15 \\ -45 & 75 & 125 & 127 & 100 & 60 & 20 & -10 & -23 & -15 & 15 \\ -15 & 0 & 50 & 100 & 127 & 120 & 80 & 20 & -35 & -48 & 30 \\ 18 & -45 & -10 & 60 & 120 & 143 & 120 & 60 & -10 & -45 & 18 \\ 30 & -48 & -35 & 20 & 80 & 120 & 127 & 100 & 50 & 0 & -15 \\ 15 & -15 & -23 & -10 & 20 & 60 & 100 & 127 & 125 & 75 & -45 \\ -15 & 30 & 10 & -23 & -35 & -10 & 50 & 125 & 177 & 150 & -30 \\ -30 & 45 & 30 & -15 & -48 & -45 & 0 & 75 & 150 & 177 & 90 \\ 18 & -30 & -15 & 15 & 30 & 18 & -15 & -45 & -30 & 90 & 393 \end{bmatrix}$$

The sum of each row is seen to be unity.

To calculate a matrix for $r + 1$, when that for r has already been calculated we merely add the appropriate fractions to each element of the matrix for r.

The matrices discussed above might have been expressed in terms of central Tchebychef polynomials. The $(p+1,\ q+1)^{\text{th}}$ element would be

$$\frac{1}{\Sigma \hat{T}_0^2} + \frac{\hat{T}_1(-m+\frac{1}{2}+q)\,\hat{T}_1(-m+\frac{1}{2}+p)}{\Sigma \hat{T}_1^2} + \cdots + \frac{\hat{T}_r(-m+\frac{1}{2}+q)\,\hat{T}_r(-m+\frac{1}{2}+p)}{\Sigma \hat{T}_r^2}$$

$$\text{or } \frac{1}{\Sigma \hat{T}_0^2} + \frac{\hat{T}_1(-m+q)\,\hat{T}_1(-m+p)}{\Sigma \hat{T}_1^2} + \cdots + \frac{\hat{T}_r(-m+q)\,\hat{T}_r(-m+p)}{\Sigma \hat{T}_r^2}$$

according as the matrix is of even or odd order, i.e. of order $2m$ or $2m+1$; while $\hat{T}_s$ denotes a central Tchebychef polynomial.

The numerical values of the elements will be the same as before.

Tables of the central Tchebychef polynomials are contained in the paper by A.C. Aitken, Proc. Roy. Soc. Edin. Vol. LIII. (1933) pp. 69 - 71.

Example II. Fit a polynomial of the fourth degree to the data

17 40 47 49 52 69 111 123 127 115 35.

This example is taken from A.C. Aitken's paper.

Using the matrix for $n = 11$, $r = 4$, we have

$$\frac{1}{429}\begin{bmatrix} 393 & 90 & -30 & -45 & -15 & 18 & 30 & 15 & -15 & -30 & 18 \\ 90 & 177 & 150 & 75 & 0 & -45 & -48 & -15 & 30 & 45 & -30 \\ -30 & 150 & 177 & 125 & 50 & -10 & -35 & -23 & 10 & 30 & -15 \\ -45 & 75 & 125 & 127 & 100 & 60 & 20 & -10 & -23 & -15 & 15 \\ -15 & 0 & 50 & 100 & 127 & 120 & 80 & 20 & -35 & -48 & 30 \\ 18 & -45 & -10 & 60 & 120 & 143 & 120 & 60 & -10 & -45 & 18 \\ 30 & -48 & -35 & 20 & 80 & 120 & 127 & 100 & 50 & 0 & -15 \\ 15 & -15 & -23 & -10 & 20 & 60 & 100 & 127 & 125 & 75 & -45 \\ -15 & 30 & 10 & -23 & -35 & -10 & 50 & 125 & 177 & 150 & -30 \\ -30 & 45 & 30 & -15 & -48 & -45 & 0 & 75 & 150 & 177 & 90 \\ 18 & -30 & -15 & 15 & 30 & 18 & -15 & -45 & -30 & 90 & 393 \end{bmatrix} \begin{bmatrix} 17 \\ 40 \\ 47 \\ 49 \\ 52 \\ 69 \\ 111 \\ 123 \\ 127 \\ 115 \\ 35 \end{bmatrix}$$

$$= \frac{1}{429}\begin{bmatrix} 7578 \\ 16992 \\ 19325 \\ 20542 \\ 24304 \\ 31968 \\ 42587 \\ 52910 \\ 57382 \\ 48144 \\ 15033 \end{bmatrix} = \begin{bmatrix} 17.7 \\ 39.6 \\ 45.0 \\ 47.9 \\ 56.7 \\ 74.5 \\ 99.3 \\ 123.3 \\ 133.8 \\ 112.2 \\ 35.0 \end{bmatrix}$$

taking values correct to one decimal place.

Comparing the observed and theoretical values we have

												Σ
u	17	40	47	49	52	69	111	123	127	115	35	785
v	17.7	39.6	45.0	47.9	56.7	74.5	99.3	123.3	133.8	112.2	35.0	785.0

With modern calculating machines the process of linear combination is not only rapid but very easy on the computer.

---

§ 3. ERRORS.

If u is the vector of observed values, v that of fitted values, $(u-v)$ is the vector of errors.

Denote it by ϵ, so that $\epsilon = (u-v)$

We have $Cu = v$

and $Cv = v$

$\therefore C(u-v) = 0$

or $C\epsilon = 0$

This implies that the quadratic form $\epsilon' C \epsilon = 0$ which is an interesting consequence of the condition that $\epsilon'\epsilon$ is to be a minimum.

Now $\epsilon = u - v$

$= u - Cu$

$= (I - C)u$

$\therefore$ the errors might be obtained by premultiplying u by the matrix $(I-C)$.

The matrix $(I-C)$ is

(I) Axi-symmetric.

(II) Centro-symmetric.

(III) The sum of the elements in each row (or column) equals zero.

(IV) $(I-C)^2 = (I-C)$

(V) Its latent roots are $0, 1$ the number of zeros being one greater than r, the degree of the fitted polynomial.

These properties follow at once from those of the matrix C given above.

To prove (V) we have the latent roots of

$(I-C)$ given by

$$|I - C - \lambda' I| = 0$$

where λ' is a latent root.

$$\therefore |(1-\lambda')I - C| = 0$$

$$\text{or } |C - (1-\lambda')I| = 0$$

which is the same equation as that for finding the latent roots of C, if $(1-\lambda')$ is replaced by λ.

$$\therefore \lambda' = 1 - \lambda$$

Hence the latent roots of $I - C$ are 0 and 1, the number of zeros being one greater than r.

The trace, being equal to the sum of the latent roots $= (n-r-1)$.

<u>The sum of the squared errors equals the sum of the products of each observed value with the corresponding error.</u>

Sum of the squared errors $= \epsilon'\epsilon$

$$= \{(I-C)u\}'\{(I-C)u\}$$

$$= u'(I-C)'(I-C)u$$

$$= u'(I-C)u \quad \text{since } C^2 = C \quad \text{--- (1)}$$

$$= u'\epsilon$$

This curious result, that the summed squares of residuals can be found alternatively by multiplying each residual by the corresponding datum u and summing, has little practical significance. It might be used, however, as an additional check. If, for example, the last fitted value v_{n-1} were found to agree by two different methods (alternative methods given in Chapters II and III), and $\sum t_x^2$ equalled $\sum u_x t_x$ one would feel certain the calculations were accurate.

Example. Taking the values on p. 24 $\sum u_x t_x = 248.3$.

Squaring the errors and adding we have $\sum t_x^2 = 249.3$, and in good agreement with $\sum u_x t_x$.

Again, since from (1)

$$(u-v)'(u-v) = u'(u-v)$$

$$\therefore u'u - 2u'v + v'v = u'u - u'v$$

$$\therefore v'v = u'v \qquad --- \quad (2)$$

But from (1)

$$t't = u'u - u'v$$

$$\therefore t't = u'u - v'v \text{ from (2)} \qquad --- \quad (3)$$

Examples. Taking the values on p. 24 we have

$u'v = 71744.7$

$v'v = 71745.7$

and these are approximately equal.

Also $u'u - v'v = 247.34$ which agrees with

$\sum t_x^2$ and $\sum u_x t_x$ above.

CHAPTER II.

MATRICES CONNECTING THE DIFFERENCES OF THE THEORETICAL VALUES WITH THE DIFFERENCES OF THE OBSERVED VALUES.

§ 4. Not only can we express the v's as linear combinations of the u's , as shown in the previous chapter, but we can also express the differences of the v's as linear combinations of the differences of the u's . We shall obtain in this way matrices of lower and lower order, many of whose properties are identical with those of the matrices from which they are derived.

Let the $(h,q)^{th}$ element of C be denoted by c_{hq} and let the x^{th} row of C be denoted by C_x .

$$\Delta v_x = v_{x+1} - v_x$$

$$= C_{x+2}\, u - C_{x+1}\, u$$

$$= \left(C_{x+2} - C_{x+1}\right) u$$

$$= \left(c_{x+2,1} - c_{x+1,1}\right) u_0 + \left(c_{x+2,2} - c_{x+1,2}\right) u_1 + \cdots$$

$$+ \left(c_{x+2,n-1} - c_{x+1,n-1}\right) u_{n-2} + \left(c_{x+2,n} - c_{x+1,n}\right) u_{n-1} \quad \text{--- } (1)$$

$$= \left(c_{x+1,1} - c_{x+2,1}\right)\Delta u_0 + \left(\overline{c_{x+1,2} - c_{x+2,2}} + \overline{c_{x+1,1} - c_{x+2,1}}\right)\Delta u_1 + \cdots$$

$$\cdots + \left(\overline{c_{x+1,n-1} - c_{x+2,n-1}} + \overline{c_{x+1,n-2} - c_{x+2,n-2}} + \cdots + \overline{c_{x+1,1} - c_{x+2,1}}\right)\Delta u_{n-2}$$

$$\text{--- } (2)$$

The coefficient of $\Delta u_{n-2} = (1 - c_{x+1,n}) - (1 - c_{x+2,n})$
$= c_{x+2,n} - c_{x+1,n}$

since the sum of the elements in any row of C equals unity. Thus every term of (1) is exactly accounted for in (2), and we have the first difference of v_x expressed as a linear combination of the first differences of the u's. There are $(n-1)$ equations of the form (2), for x takes the values 0, 1, 2 ---- $n-2$. We can thus write down the matrix of the co-efficients connecting the Δv's with the Δu's. It is a square matrix of order $n-1$, the $(h,q)^{th}$ element being:

$$c_{hq} - c_{h+1,q} + c_{h,q-1} - c_{h+1,q-1} + c_{h,q-2} - c_{h+1,q-2} + \cdots + c_{h,2} - c_{h+1,2} + c_{h,1} - c_{h+1,1}.$$

In an exactly similar manner we can express the second differences of the v's in terms of the second differences of the u's, and obtain a square matrix of order $n-2$, derived from that connecting the first differences precisely as it was derived from C. Similarly for the third, fourth, etc. differences.

If the fitted polynomial is of the r^{th} degree the r^{th} differences of the v's are constant. Thus each row of the matrix connecting the r^{th} differences of the v's with the r^{th} differences of the u's is the same, for the u's are arbitrary. Therefore the matrix connecting $(r+1)^{th}$ differences is null.

Denote the matrix connecting the h^{th} differences of v with the h^{th} differences of u by D_h.

From the method of formation we find that the $(h+1,q+1)^{th}$ element of the matrix D_h is

$$(-)^{k} \sum_{s=1}^{r} \left[\frac{\sum_{q=0}^{q} {}^{k}T_s(q)}{\sum T_s^2} \left\{ \Delta^{k} T_s(p) \right\} \right]$$

Tables of the matrices connecting the differences are found in the appendix for n = 4, 5, 6 ---- 15 and r = 0, 1, 2, 3, 4, 5.

§ 5. PROPERTIES OF D_k.

(I) The sum of the elements in any row of a matrix D_{k+1} equals 1 or 0, according as the degree of the fitted polynomial is greater than or equal to k.

Let the matrix used for fitting k^{th} differences be

$$D_k = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & & a_{2m} \\ \vdots & & & & \\ \vdots & & & & \\ a_{m1} & a_{m2} & & & a_{mm} \end{bmatrix}$$

(This matrix is C if $k = 0$).

Consider first the case where $k < r$.
Assume the result for the matrix D_k . We shall prove it then holds for the matrix D_{k+1} .

The sum of the elements in the p^{th} row of D_{k+1} is

$$\sum_{q=1}^{m-1}\left(a_{hq}-a_{h+1,q}+a_{h,q-1}-a_{h+1,q-1}+\cdots\ a_{h,1}-a_{h+1,1}\right)$$

$$=(m-1)\left(a_{h,1}-a_{h+1,1}\right)+(m-2)\left(a_{h,2}-a_{h+1,2}\right)+\cdots+\left(a_{h,m-1}-a_{h+1,m-1}\right)$$

$$=m\left(a_{h,1}-a_{h+1,1}+a_{h,2}-a_{h+1,2}+\cdots+a_{h,m-1}-a_{h+1,m-1}\right)$$

$$-\left\{\left(a_{h,1}-a_{h+1,1}\right)+2\left(a_{h,2}-a_{h+1,2}\right)+\cdots+(m-1)\left(a_{h,m-1}-a_{h+1,m-1}\right)\right\}$$

$$=\left(a_{h+1,1}-a_{h,1}\right)+2\left(a_{h+1,2}-a_{h,2}\right)+\cdots+(m-1)\left(a_{h+1,m-1}-a_{h,m-1}\right)$$

$$+\ m\left(a_{h+1,m}-a_{h,m}\right)$$

since we assume the result for the matrix D_k.

Take the particular set of data 1, 2, ---- m as k^{th} differences, and let the corresponding fitted differences be $\Delta^k v_0'$, $\Delta^k v_1'$, ... $\Delta^k v_{m-1}'$.

Then the above sum $= \Delta^k v_h' - \Delta^k v_{h-1}'$.

But with the values 1, 2, ---- m as k^{th} differences, for $r \geq 1+k$ we get a perfect fit.

$$\therefore\ \Delta^k v_h' - \Delta^k v_{h-1}' = 1 \qquad \text{for } r \geq 1+k\,.$$

$\therefore$ the sum of the elements in any row of D_{k+1} equals unity if $r > k$.

Thus/

Thus if the result holds for D_k it also holds for D_{k+1}.
But we proved in Chapter I that the result holds for $k = 0$.
$\therefore$ it holds for k = 1, 2, ----- etc.

If $r = k$ the sum of the elements in a row of the matrix D_{k+1} equals zero, since by the method of formation, we have the null matrix.

(II) D_k is centro-symmetric.

This follows for D_1, from the properties (II) and (III) of C, and for D_2 from the properties of D_1, etc., and so for D_k from the properties of D_{k-1}, from which it is derived.

(III) $D_k^2 = D_k$

$$D_k \Delta^k u = \Delta^k v$$

Suppose the $\Delta^k v$ are regraduated. They must be reproduced.

$$\therefore D_k \Delta^k v = \Delta^k v$$
$$\therefore D_k^2 \Delta^k u = D_k \Delta^k u$$

But the u's and therefore the $\Delta^k u$'s are arbitrary.

$$\therefore D_k^2 = D_k.$$

(IV) Latent roots of D_k are 1, 0.

Since $D_k^2 = D_k$

$\therefore$ D_k satisfies the equation

$$\lambda^2 - \lambda = 0$$

This/

This is, in general, its reduced characteristic equation. (In the particular cases when $D_k = I$, $D_k = 0$ the reduced characteristic equations are $\lambda - 1 = 0$, $\lambda = 0$ respectively.)

$\therefore$ The latent roots of D_k are 1 and 0.

(V) <u>The rank of a matrix D_{k+1} is one less than the rank of D_k , from which D_{k+1} has been derived, and hence the number of latent roots of D_{k+1} equal to unity is $(r-k)$</u> .

Let the matrix connecting the k^{th} differences of the v's with the k^{th} differences of the u's be

$$D_k = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & & \cdot & a_{2m} \\ \vdots & & & & \\ a_{m1} & a_{m2} & & \cdots & a_{mm} \end{bmatrix}$$

(This matrix is C if $k = 0$)

Assume that the rank of D_k is i , with the first i rows linearly independent.

We shall prove that the rank of D_{k+1} is $i-1$, with the first $(i-1)$ rows linearly independent.

Since each row of D_{k+1} is formed from two consecutive rows of D_k , whose first i rows are linearly independent, therefore the first $(i-1)$ rows of D_{k+1} are linearly independent. — (1)

We shall now show that any other row of D_{k+1} depends on/

on these first $i-1$ rows.

Let $a_{pq} = P_1 a_{1q} + P_2 a_{2q} + \dots + P_i a_{iq}$, for $q = 1, 2 \dots m$ --- (2)

and $p > i$. If $p = i$, $P_1 = P_2 = \dots P_{i-1} = 0$ and $P_i = 1$

Also let

$$a_{p+1,q} = Q a_{1q} + Q_2 a_{2q} + \dots + Q_i a_{iq} \quad \text{--- (3)}$$

Writing (2) and (3) for q = 1, 2, --- m and adding we have

$$1 = \sum_{s=1}^{i} P_s = \sum_{s=1}^{i} Q_s \qquad (4)$$

The $(p,q)^{\text{th}}$ element of D_{k+1} is

$$a'_{pq} = \left(a_{pq} - a_{p+1,q} + \dots + a_{p,1} - a_{p+1,1}\right)$$

Consider an element in the p^{th} row of D_{k+1}, where $p > i-1$

$$a'_{pq} = \sum_{q=1}^{q} \left(P_1 a_{1q} + P_2 a_{2q} + \dots + P_i a_{iq} - Q_1 a_{1q} - Q_2 a_{2q} \dots - Q_i a_{iq}\right)$$

$$= \sum_{q=1}^{q} \left\{ (P_1 - Q_1) a_{1q} + (P_2 - Q_2) a'_{2q} + \dots + (P_i - Q_i) a_{iq} \right\}$$

$$= (P_1 - Q_1) a'_{1q} + (P_2 - Q_2 + P_1 - Q_1) a'_{2q} + \dots$$

$$+ (P_{i-1} - Q_{i-1} + P_{i-2} - Q_{i-2} \dots + P_1 - Q_1) a'_{i-1,q} + (P_i - Q_i + P_{i-1} - Q_{i-1} + \dots + P_1 - Q_1) a_{iq}$$

But the coefficient of a_{iq} in this expression equals zero from (4).

$\therefore a'_{pq}$ can be expressed as a linear combination of

$a'_{1q}, a'_{2q}, \dots a'_{i-1,q}$.

Hence/

Hence the i^{th}, $(i+1)^{th}$, ... $(m-1)^{th}$ rows of D_{k+1} are linearly dependent on the first $i-1$ rows. — — (5)

$\therefore$ from (1) and (5) the rank of D_{k+1} is $(i-1)$ with the first $(i-1)$ rows linearly independent.

But the rank of C is $r+1$ with the first $r+1$ rows independent

$\therefore$ the rank of D_1 is r with the first r rows independent

$\therefore$ the rank of D_2 is $r-1$ with the first $r-1$ rows independent

etc.

$\therefore$ the rank of D_{k+1} is $r-k$ with the first $r-k$ rows independent.

Since the matrix D_{k+1} can be put in diagonal form, with the latent roots as diagonal elements, therefore the number of latent roots equal to unity is $r-k$ and the remaining latent roots are zeros.

An <u>alternative proof</u> is as follows:

If the values v lie on a polynomial of degree r, the $(k+1)^{th}$ differences of v lie on a polynomial of degree $(r-k-1)$.

The first (or any set of) $r-k$ rows of D_{k+1} operating on the $(k+1)^{th}$ differences of an arbitrary vector u produce $r-k$ values of the fitted differences lying on a polynomial of degree $r-k-1$. These $r-k$ values, and no lesser number, are sufficient to determine the polynomial, and any other values of the polynomial, such as those given by operating on the $(k+1)^{th}$ differences of the u's with the later rows of D_{k+1} are expressible linearly in terms of these $r-k$ values.

But/

But u, and hence the $(h+1)^{\underline{a}}$ differences of u are arbitrary. Therefore the later rows of D_{h+1} themselves are vectors expressible linearly in terms of the first $r-h$ rows, from which it follows that D_{h+1} is of rank $(r-h)$.

Hence, as above, the number of latent roots equal to unity is $(r-h)$, and the remaining latent roots are zeros.

(VI) Sum of the diagonal elements.

(a) The trace of D_{h+1} being the sum of the latent roots is $(r-h)$.

For example, when $n = 11, r = 4$ the trace (sum of diagonal elements) of D_1 (p. 44) $= \frac{1716}{429} = 4.$
This is a useful check on the calculations.

(b) The sum of the elements in the secondary diagonal is equal to 1 or 0 according as the rank of D_{h+1}, equal to $r-h$, is odd or even.

Assume the result holds for the matrix D_h.
We shall prove it then holds for D_{h+1}.

$$\text{Let } D_h = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & & \cdots & a_{2m} \\ \vdots & & & & \\ a_{m1} & a_{m2} & & \cdots & a_{mm} \end{bmatrix}.$$

Now $a_{1m} + a_{2,m-1} + a_{3,m-2} + \cdots a_{m,1} = 0 \text{ or } 1$
according as the rank of D_h is even or odd. $\quad ---(1)$

Sum/

Sum of the elements in the secondary diagonal of D_{k+1}

$$
\begin{aligned}
&= a_{11}+a_{12}+\cdots+a_{1,m-1}-(a_{21}+a_{22}+\cdots+a_{2,m-1})\\
&\quad +a_{21}+a_{22}+\cdots+a_{2,m-2}-(a_{31}+a_{32}+\cdots+a_{3,m-2})\\
&\quad +a_{31}+a_{32}+\cdots+a_{3,m-3}-(a_{41}+a_{42}+\cdots+a_{4,m-3})\\
&\quad +\cdots\\
&\quad +a_{m-1,1}-a_{m,1}\\
&= a_{11}+a_{12}+\cdots+a_{1,m-1}-(a_{2,m-1}+a_{3,m-2}+\cdots+a_{m-1,2}+a_{m,1})\\
&= (a_{11}+a_{12}+\cdots+a_{1,m-1}+a_{1,m})-(a_{1m}+a_{2,m-1}+\cdots+a_{m,1})
\end{aligned}
$$

= 1 - (0 or 1, according as the rank of D_k is even or odd) from (1), and the fact that the sum of the elements in a row of D_k equals unity (provided D_k is not null).

= 1 or 0 according as the rank of D_{k+1} is odd or even, since the rank of D_{k+1} is one less than the rank of D_k.

Hence if the result holds for D_k it holds for D_{k+1}.

But the result holds for the matrix C, i.e. D_0.

$\therefore$ it holds for $D_1, D_2 \ldots D_k$.

For example, the sum of the elements in the secondary diagonal of D_1 for $n=11$, $r=4$ (p.44) = 0, while the rank is 4 i.e. even.

(VII)/

(VII) Sum of the diagonal minors of D_k.

(a) Sum of the principal diagonal minors.

Let p_t denote the sum of the diagonal minors of order t of the matrix D_k.

Then

$$\lambda^{n-k} - p_1 \lambda^{n-k-1} + p_2 \lambda^{n-k-2} - \dots + (-)^{n-k} p_{n-k} = 0 \quad \text{---} (1)$$

is the characteristic equation of D_k.

But the roots of (1) are $r-k+1$ units, and $n-r-1$ zeros.

$\therefore$ the equation

$$\lambda^{r-k+1} - p_1 \lambda^{r-k} + p_2 \lambda^{r-k-1} - \dots + (-)^{r-k+1} p_{r-k+1} = 0 \quad \text{--} \quad (2)$$

has all its roots equal to unity.

$$\therefore (2) \equiv (\lambda - 1)^{r-k+1}$$

$\therefore p_t = {}^{r-k+1}c_t$ for values of t up to $r-k+1$,

where ${}^{r-k+1}c_t$ is the combination of $r-k+1$ things, taken t at a time.

Since the rank of D_k is $r-k+1$,

$$\therefore p_t = 0, \quad \text{for} \quad t > r-k+1.$$

(b) Sum of the secondary diagonal minors.

The secondary diagonal minors of D_k are the principal diagonal minors of JD_k.

Therefore we shall investigate the principal diagonal minors of JD_k.

Now $(JD_k)^3 = JD_k$, since the matrix D_k is centro-symmetric and idempotent.

Hence/

Hence JD_k satisfies the equation

$$\lambda^3 - \lambda = 0$$

which is its reduced characteristic equation.

$\therefore$ the latent roots of JD_k are 0 ± 1.

The rank of JD_k = the rank of D_k = $r-k+1$.

$\therefore$ there are $r-k+1$ latent roots of JD_k equal to $+1$ or -1, and $n-r-1$ equal to zero.

The characteristic equation of JD_k is

$$\lambda^{n-k} - q_1 \lambda^{n-k-1} + q_2 \lambda^{n-k-2} - \dots + (-)^{n-k} q_{n-k} = 0 \quad --- \quad (1)$$

where q_t denotes the sum of diagonal minors of order t.

Removing the $n-r-1$ roots equal to zero,

$$\therefore \lambda^{r-k+1} - q_1 \lambda^{r-k} + q_2 \lambda^{r-k-1} - \dots (-)^{r-k+1} q_{r-k+1} = 0 \quad --- \quad (2)$$

has roots equal to $+1$ and -1.

But the sum of the latent roots of JD_k which equals the sum of the diagonal elements, equals 0 or 1 according as the rank is even or odd, by (VI) (b).

$\therefore$ Firstly, if the rank is even, the number of latent roots of JD_k equal to $+1$ equals the number equal to -1.

$$\therefore (2) \equiv (\lambda^2 - 1)^{\frac{r-k+1}{2}} = 0 \quad --- \quad (3)$$

$$\therefore q_{2t-1} = 0$$

$$q_{2t} = (-)^t \, {}^{\frac{r-k+1}{2}}c_t \quad \text{for } t = 1, 2, 3 \dots \tfrac{r-k+1}{2}.$$

Since/

Since the rank of $\mathbb{D}_h$ is $r-h+1$, the sum of diagonal minors of any order $> r-h+1$ equals 0.

Secondly, if the rank is odd, the number of latent roots of $\mathbb{D}_h$ equal to + 1 is one greater than the number equal to - 1.

$$\therefore (2) \equiv (\lambda-1)(\lambda^2-1)^{\frac{r-h}{2}} = 0 \qquad \text{- - -} \qquad (4)$$

$$\therefore q_{2t+1} = q_{2t} = (-)^t \, {}^{\frac{r-h}{2}}c_t \quad \text{for } t = 0, 1, 2 \cdots \tfrac{r-h}{2}.$$

As above, the sum of diagonal minors of any order $> r-h+1$ equals 0.

(VIII) <u>Any row of D_h multiplied by a column in which the element belonging to the principal diagonal is reduced by unity has a product equal to zero.</u>

Since $D_h{}^2 = D_h$

$\therefore D_h(D_h - I) = 0$

$\therefore$ a row of D_h multiplied by a column of $D_h - I$ equals zero.

Hence the result.

(IX.) <u>Latent vectors of D_h.</u>

Since any set of differences $\Delta^h \sigma$ which equal the h^{th} differences of a polynomial of degree r, and thus lie on a polynomial of degree $r-h$, will be reproduced if graduated/

graduated again, we have

$$D_k \Delta^k v = \Delta^k v$$

$\therefore$ The values $\Delta^k v$ constitute a latent vector of the matrix D_k .

Since any set of $(n-k)$ values lying on a polynomial of degree $r-k$ constitute a latent vector of D_k , we have, in particular, $\{0, 1, 2^{r-k}, \dots (n-k-1)^{r-k}\}$ as a latent vector.

(X) <u>Each row of the matrix D_k after the $(r-k+1)^{th}$ row is formed by multiplying the previous $r-k+1$ rows by the first $r-k+1$ co-efficients of $(-)^{r-k}(1-1)^{r-k+1}$ respectively and adding.</u>

Consider the elements in the $(q+1)^{th}$ column, where q can have any value from 0 to $n-k$.

Take $p > r-k+1$

Multiply elements in the $(p-r+k+1)^{th}$, $(p-r+k+2)^{th}$ ... $(p+1)^{th}$ rows by the first $(r-k+1)$ co-efficients of $(-)^{r-k}(1-1)^{r-k+1}$ respectively and add.

Since the $(p+1, q+1)^{th}$ element of D_k is

$$(-)^k \sum_{s=1}^{r} \left[\frac{\sum_{q=0}^{q} \Delta^k T_s(q)}{\sum T_s^2} \left\{ \Delta^k T_s(p) \right\} \right]$$

we have, writing terms in the reverse order

$$(r-k+1)(-)^k \sum_{s=1}^{r} \left[\frac{\sum_{q=0}^{q} {}^k T_s(q)}{\Sigma T_s^2} \left\{ \Delta^k T_s(p) \right\} \right]$$

$$- \frac{(r-k+1)(r-k)}{2!} (-)^k \sum_{s=1}^{r} \left[\frac{\sum_{q=0}^{q} {}^k T_s(q)}{\Sigma T_s^2} \left\{ \Delta^k T_s(p-1) \right\} \right]$$

$$+ \cdots$$

$$+ (-)^{r-k} \cdot (-)^k \cdot \sum_{s=1}^{r} \left[\frac{\sum_{q=0}^{q} {}^k T_s(q)}{\Sigma T_s^2} \left\{ \Delta^k T_s(p-r+k) \right\} \right]$$

$$= (-)^k \sum_{s=1}^{r} \left[\frac{\sum_{q=0}^{q} {}^k T_s(q)}{\Sigma T_s^2} \left\{ (r-k+1)\Delta^k T_s(p) - \frac{(r-k+1)(r-k)}{2!} \Delta^k T_s(p-1) + \cdots + (-)^{r-k} \Delta^k T_s(p-r+k) \right\} \right] \quad \text{--- (1)}$$

But $\Delta^k T_s(p+1) - (r-k+1)\Delta^k T_s(p) + \frac{(r-k+1)(r-k)}{2!} \Delta^k T_s(p-1) + \cdots + (-)^{r-k+1} \Delta^k T_s(p-r+k)$

$$= \Delta^{r+1} T_s(p-r+k)$$

$$= 0 \quad \textit{since} \quad r+1 > s$$

$$\therefore (1) = (-)^k \sum_{s=1}^{r} \left[\frac{\sum_{q=0}^{q} {}^k T_s(q)}{\Sigma T_s^2} \left\{ \Delta^k T_s(p+1) \right\} \right]$$

$$= (p+2,\, q+1)^{\underline{th}} \text{ element of } D_k .$$

Hence the theorem is proved.

Similarly it may be shown that each row of D_k equals the sum of the subsequent $r-k+1$ rows multiplied respectively by the first $r-k+1$ co-efficients of $(-)^{r-k}(1-1)^{r-k+1}$ in the reverse order.

Example I. Construct the matrix used for fitting first differences in the case $n=11$, $r=4$.

This matrix is of the tenth order, but owing to its centro-symmetric property we shall calculate only the first five rows. Subtract the $(k+1)^{th}$ row from the k^{th}, then add the first entry to the second, the answer to the third and so on.

Only the first ten columns are required, and a check on accuracy is that the final entry should equal $c_{k+1}-c_k$.

The working is as follows:

303	−87	−180	−120	−15	63	78	30	−45	−75
	303	216	38	−84	−99	−36	42	72	27
303	216	36	−84	−99	−36	42	72	27	−48
120	27	−27	−50	−50	−35	−13	8	20	15
	120	147	120	70	20	−15	−28	−20	0
120	147	120	70	20	−15	−28	−20	0	15
15	75	52	−2	−50	−70	−55	−13	33	45
	15	90	142	140	90	20	−35	−48	−15
15	90	142	140	90	20	−35	−48	−15	30
−30	75	75	27	−27	−60	−60	−30	12	33
	−30	45	120	147	120	60	0	−30	−18
−30	45	120	147	120	60	0	−30	−18	15
−33	45	60	40	7	−23	−40	−40	−25	−3
	−33	12	72	112	119	96	56	16	−9
−33	12	72	112	119	96	56	16	−9	−12

Hence the matrix for fitting 1st differences is

$$\frac{1}{429}\begin{bmatrix} 303 & 216 & 36 & -84 & -99 & -36 & 42 & 72 & 27 & -48 \\ 120 & 147 & 120 & 70 & 20 & -15 & -28 & -20 & 0 & 15 \\ 15 & 90 & 142 & 140 & 90 & 20 & -35 & -48 & -15 & 30 \\ -30 & 45 & 120 & 147 & 120 & 60 & 0 & -30 & -18 & 15 \\ -33 & 12 & 72 & 112 & 119 & 96 & 56 & 16 & -9 & -12 \\ -12 & -9 & 16 & 56 & 96 & 119 & 112 & 72 & 12 & -33 \\ 15 & -18 & -30 & 0 & 60 & 120 & 147 & 120 & 45 & -30 \\ 30 & -15 & -48 & -35 & 20 & 90 & 140 & 142 & 90 & 15 \\ 15 & 0 & -20 & -28 & -15 & 20 & 70 & 120 & 147 & 120 \\ -48 & 27 & 72 & 42 & -36 & -99 & -84 & 36 & 216 & 303 \end{bmatrix}$$

The difference matrices may be used for fitting. For we may difference the u's, find a value v_x and its differences, using the appropriate matrices, and then build back the difference table of the v's. We may begin with any v_x, but, in general, it is more convenient to use v_0. The method is illustrated in the following example.

Example II. Fit the data of Example II in Chapter I by this method.

The/

The difference table of the u's is

u	Δ	Δ^2	Δ^3	Δ^4
17				
	23			
40		−16		
	7		11	
47		−5		−5
	2		6	
49		1		7
	3		13	
52		14		−2
	17		11	
69		25		−66
	42		−55	
111		−30		77
	12		22	
123		−8		−30
	4		−8	
127		−16		−44
	−12		−52	
115		−68		
	−80			
35				

$$v_0 = \frac{1}{429}\Big(393\times 17 + \overset{90}{9}\times 40 - 30\times 47 - 45\times 49 - 15\times 52 + 18\times 69 + 30\times 111$$
$$+ 15\times 123 - 15\times 127 - 30\times 115 + 18\times 35\Big)$$

$$= \frac{7578}{429}$$

$$\Delta v_0 = \frac{1}{429}\Big\{303\times 23 + 216\times 7 + 36\times 2 - 84\times 3 - 99\times 17 - 36\times 42$$
$$+ 42\times 12 + 72\times 4 + 27\times(-12) - 48\times(-80)\Big\}$$

$$= \frac{9414}{429}$$

$$\Delta^2 v_0 = \frac{1}{429}\Big\{183\times(-16) + 252\times(-5) + 168\times 1 + 14\times 14 - 105\times 25 - 126\times(-30) - 56\times(-8) + 36\times(-16) + 63\times(-68)\Big\}$$

$$= \frac{-7081}{429}.$$

$$\Delta^3 v_0 = \frac{1}{429}\Big\{78\times 11 + 168\times 6 + 196\times 13 + 140\times 11 + 35\times(-55) - 56\times 22 - 84\times(-8) - 48\times(-52)\Big\}$$

$$= \frac{5965}{429}$$

$$\Delta^4 v_0 = \frac{1}{143}\Big\{6\times(-5) + 18\times 7 + 30\times(-2) + 35\times(-66) + 30\times 77 + 18\times(-30) + 6\times(-44)\Big\}$$

$$= \frac{-768}{143} = -\frac{2304}{429}.$$

Building back the difference table we have

$429v$	Δ	Δ^2	Δ^3	Δ^4
7578				
	9414			
16992		-7081		
	2333		5965	
19325		-1116		-2304
	1217		3661	
20542		2545		-2304
	3762		1357	
24304		3902		-2304
	7664		-947	
31968		2955		-2304
	10619		-3251	
42587		-296		-2304
	10323		-5555	
52910		-5851		-2304
	4472		-7859	
57382		-13710		-2304
	-9238		-10163	
48144		-23873		
	-33111			
15033				

Hence, dividing out, and taking values correct to one decimal place, we have

$v =$ 17.7 39.6 45.0 47.9 56.7 74.5 99.3 123.3 133.8 112.2 35.0

We might check our answer by working out another v_x , say v_{n-1} .

This method, provided the linear combinations for v_0, Δv_0, $\Delta^2 v_0$ . . . . in terms of the u's, Δu's, $\Delta^2 u$'s . . . are tabulated once and for all, is rapid and efficacious. The tables in question might be given in such a form as e.g.

$n = 11 \quad r = 4$

Each element to be divided by 429.

u	393	9	−30	−45	−15	18	30	15	−15	−30	18
Δu	303	216	36	−84	−99	−36	42	72	27	−48	
$\Delta^2 u$	183	252	168	14	−105	−126	−56	36	63		
$\Delta^3 u$	78	168	196	140	35	−56	−84	−48			
$\Delta^4 u$	18	54	90	105	90	54	18				

Again, we might find v_x , Δv_{x-1} , $\Delta^2 v_{x-2}$, . . . $\Delta^r v_{x-r}$ and build back the difference table from these values.
In particular, we might find v_{n-1} , Δv_{n-2} , $\Delta^2 v_{n-3}$ . . . $\Delta^r v_{n-r-1}$, using the last rows of the matrices C and D_k .
These values would enable us to find v_n , v_{n+1} . . . . . if required, without finding v_0 . . . v_{n-2} .

CHAPTER III.

VARIATIONS OF THE METHOD OF SMOOTHING BY LINEAR COMBINATION.

§ 6. Fitting by calculating $r+1$ values by linear combination, and the remaining values from a difference table.

To fit a polynomial of the r^{th} degree, calculate $r+1$ consecutive values of v_x as in Chapter I. The r^{th} difference of v_x is constant, and is calculated from the $r+1$ values of v_x already obtained. It is then repeated in the difference table, and the remaining values of v_x obtained by summation.

Any $r+1$ consecutive values of v_x might be found as a beginning, but in general it is more convenient to find $v_0, v_1, v_2 \ldots v_r$ or $v_{n-r-1}, v_{n-r}, \ldots v_{n-1}$ if $v_n, v_{n+1} \ldots$ only are required

As a check we can calculate another value of v_x, as most convenient, say v_{n-1} and compare it with the value obtained from the difference table.

If the value of n is large this method will involve less labour than working out n linear combinations, as in Chapter I, and is one of the simplest methods of obtaining the graduated values, besides providing a check.

Example I. Fit the data of Example II, Chapter I by this method.

Using the first five rows of the matrix C for $n=11$ $r=4$ we have:

$$\frac{1}{429}\begin{bmatrix} 393 & 90 & -30 & -45 & -15 & 18 & 30 & 15 & -15 & -30 & 18 \\ 90 & 177 & 150 & 75 & 0 & -45 & -48 & -15 & 30 & 45 & -30 \\ -30 & 150 & 177 & 125 & 50 & -10 & -35 & -23 & 10 & 30 & -15 \\ -45 & 75 & 125 & 127 & 100 & 60 & 20 & -10 & -23 & -15 & 15 \\ -15 & 0 & 50 & 100 & 127 & 120 & 80 & 20 & -35 & -48 & 30 \end{bmatrix} \begin{bmatrix} 17 \\ 40 \\ 47 \\ 49 \\ 52 \\ 69 \\ 111 \\ 123 \\ 127 \\ 115 \\ 35 \end{bmatrix} = \frac{1}{429}\begin{bmatrix} 7578 \\ 16992 \\ 19325 \\ 20542 \\ 24304 \end{bmatrix}$$

The difference table is

$429v$	Δ	Δ^2	Δ^3	Δ^4
7578				
	9414			
16992		-7081		
	2333		5965	
19325		-1116		-2304
	1217		3661	
20542		2545		-2304
	3762		1357	
24304		3902		-2304
	7664		-947	
31968		2955		-2304
	10619		-3251	
42587		-296		-2304
	10323		-5555	
52910		-5851		-2304
	4472		-7859	
57382		-13710		-2304
	-9238		-10163	
48144		-23873		
	-33111			
15033				

Hence, dividing out, we have the graduated values

17.7 39.6 45.0 47.9 56.7 74.5 99.3 123.3 133.8 112.2 35.0

as before.

§ 7. Fitting involving co-efficients of terminal linear combinations only.

Let a polynomial of degree r be fitted.

As above, we have

$$v_{n-1} = c_{n-1,0}\,u_0 + c_{n-1,1}\,u_1 + c_{n-1,2}\,u_2 + \cdots + c_{n-1,n-1}\,u_{n-1}$$ where $c_{h,q}$ is $(h+1, q+1)^{th}$ element of C.

Now, extrapolate the v's, which all lie on the curve of the r^{th} degree, to v_n, and take a corresponding u_n, such that $u_n = v_n$.

Since $(u_n - v_n) = 0$, the summed squares of deviations is unaltered by the addition of this value.

Therefore the Least Square polynomial of $u_0, u_1, u_2 \ldots u_{n-1}, u_n$ is the same as that of $u_0, u_1, u_2 \ldots u_{n-1}$.

To find the value of v_n : let the co-efficients for terminal combination of $(n+1)$ data be

$$c'_{n,0},\ c'_{n,1},\ c'_{n,2} \ \ldots\ c'_{n,n}.$$

Then

$$v_n = c'_{n,0}\,u_0 + c'_{n,1}\,u_1 + \ldots + c'_{n,n}\,u_n$$

$$= c'_{n,0}\,u_0 + c'_{n,1}\,u_1 + \ldots + c'_{n,n}\,v_n$$

$$\therefore v_n = \frac{c'_{n,0}\,u_0 + c'_{n,1}\,u_1 + \cdots\ c'_{n,n-1}\,u_{n-1}}{1 - c'_{n,n}}$$

Similarly we can extrapolate to $v_{n+1} = u_{n+1}$, using the terminal co-efficients for $n+2$ data, and we can continue in this way to $v_{n+r-1} = u_{n+r-1}$ using the terminal co-efficients for $n+r$ data.

From/

From the $r+1$ values $v_{n-1}, v_n, v_{n+1}, \ldots v_{n+r-1}$ we can construct a difference table, repeat the r^{th} difference, which is constant, and build back the table to find the required values of v_x .

As a check one could calculate v_{n+r} as well; the r^{th} difference for $v_n, v_{n+1}, v_{n+2}, \ldots \quad v_{n+r}$ should be the same as that for $v_{n-1}, v_n, \ldots \; v_{n+r-1}$.

This method is not so well adapted for calculation as the previous one, for the fractional co-efficients of the matrices for n, $(n+1)$, $\ldots$ $(n+r+1)$ are different and this fact renders the work rather more laborious.

Alternatively we might extrapolate to v_{-1}, obtaining

$$v_{-1} = \frac{c_{01}u_0 + c_{02}u_1 + \cdots + c_{0n}u_{n-1}}{1 - c_{00}}$$

where $c_{00}, c_{01}, c_{02} \ldots \; c_{0n}$ are the co-efficients in the first row of the matrix for $n+1$ data.
Similarly we can extrapolate $v_{-2}, \ldots v_{-r}$, and build the difference table from the values $v_0, v_{-1}, \ldots v_{-r}$

Example II. Fit the data of Example II, Chapter I using only terminal values of the co-efficients.

Using the last lines of the matrices for $n = 11, 12, 13, 14, 15$ and $r = 4$ we have

$$v_{11} = \frac{1}{429}\left\{18\times17 - 30\times40 - 15\times47 + 15\times49 + 30\times52 + 18\times69 - 15\times111 - 45\times123 - 30\times127 + 90\times115 + 393\times35\right\}$$

$$= \frac{15033}{429} = \frac{5011}{143}$$

$$v_{12} = \frac{\frac{1}{10296}\left\{495\times17 - 693\times40 - 495\times47 + 165\times49 + 660\times52 + 660\times69 + 132\times111 - 660\times123 - 1155\times127 - 495\times115 + 2475\times35\right\}}{\left(1 - \frac{9207}{10296}\right)}$$

$$= \frac{-138138}{1089} = \frac{-4186}{33}$$

$$v_{13} = \frac{\frac{1}{34034}\left\{1815\times17 - 2145\times40 - 1947\times47 - 55\times49 + 1760\times52 + 2420\times69 + 1540\times111 - 572\times123 - 2915\times127 - 3795\times115 - 825\times35 + 9075\left(-\frac{4186}{33}\right)\right\}}{\left(1 - \frac{29678}{34034}\right)}$$

$$= -\frac{1776720}{4356} = -\frac{13460}{33}$$

$$v_{14} = \frac{\frac{1}{136136}\left\{7865\times17 - 7865\times40 - 8580\times47 - 2288\times49 + 5005\times52 + 9295\times69 + 8580\times111 + 2860\times123 - 5863\times127 - 13585\times115 - 14300\times35 + 0\cdot\left(-\frac{4186}{33}\right) + 39325\left(-\frac{13460}{33}\right)\right\}}{\left(1 - \frac{115687}{136136}\right)}$$

$$= \frac{-52013104}{3\times20449} = -\frac{363728}{429}$$

$$v_{15} = \frac{\dfrac{1}{11639628}\left\{\begin{aligned}&715715\times17-605605\times40-770770\times47-340340\times49+251251\times52\\&+695695\times69+810810\times111+540540\times123-450450\times127-749749\times115\\&-1251250\times35-1101100\left(-\frac{4186}{33}\right)+275275\left(-\frac{13460}{33}\right)+3578575\left(-\frac{363728}{429}\right)\end{aligned}\right\}}{\left(1-\dfrac{9635626}{11639628}\right)}$$

$$= \frac{-2989844858}{2004002} = -\frac{213347}{143}$$

From these five values we have the difference table

$429v$	Δ	Δ^2	Δ^3	Δ^4
7578				
	9414			
16992		-7081		
	2333		5965	
19325		-1116		-2304
	1217		3661	
20542		2545		-2304
	3762		1357	
24304		3902		-2304
	7664		-947	
31968		2955		-2304
	10619		-3251	
42587		-296		-2304
	10323		-5555	
52910		-5851		-2304
	4472		-7859	
57382		-13710		-2304
	-9238		-10163	
48144		-23873		-2304
	-33111		-12467	
15033		-36340		-2304
	-69451		-14771	
-54418		-51111		-2304
	-120562		-17075	
-174980		-68186		-2304
	-188748		-19379	
-363728		-87565		
	-276313			
-640041				

Thus we obtain, as before, the values
17.7, 39.6, 45.0, 47.9, 56.7, 74.5, 99.3, 123.3, 133.8, 112.2, 35.0, together with certain extrapolated values - 126.9, - 407.9, - 847.9, - 1491.9.

In passing, it may be observed that since statistical prediction (e.g. for an ensuing year), involves precisely this extrapolation to v_n, the method described in this section can be used to find v_n from $u_0, u_1 \ldots u_{n-1}$ without finding the other v's at all.

APPENDIX.

I. Numerical values of the matrices C and D_h for $n = 4, 5, 6$ ---- 15 and $r = 0, 1, 2, 3, 4, 5$.

II. Bibliography.

I. Tables.

Q denotes the matrix composed entirely of units.
$\underline{I}$ denotes the unit matrix.
O denotes the null matrix.

$\underline{\underline{n = 4 \qquad r = 0}}$.

$$C = \tfrac{1}{4} Q$$

$$D_1 = O$$

$\underline{\underline{n = 4 \qquad r = 1}}$

$$C = \frac{1}{10}\begin{bmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{bmatrix}$$

$$D_1 = \frac{1}{10}\begin{bmatrix} 3 & 4 & 3 \\ \text{''} & & \text{''} \\ \text{''} & & \text{''} \end{bmatrix}$$

$$D_2 = O$$

$\underline{\underline{n = 4 \qquad r = 2}}$

$$C = \frac{1}{20}\begin{bmatrix} 19 & 3 & -3 & 1 \\ 3 & 11 & 9 & -3 \\ -3 & 9 & 11 & 3 \\ 1 & -3 & 3 & 19 \end{bmatrix}$$

$n = 4 \quad r = 2$ (continued).

$$D_1 = \frac{1}{10}\begin{bmatrix} 8 & 4 & -2 \\ 3 & 4 & 3 \\ -2 & 4 & 8 \end{bmatrix}$$

$$D_2 = \tfrac{1}{2} Q$$

$$D_3 = 0$$

$n = 4 \quad r = 3$

$$C = I$$

$$D_1 = I$$

$$D_2 = I$$

$$D_3 = I$$

$n = 5 \quad r = 0$

$$C = \tfrac{1}{5} Q$$

$$D_1 = 0$$

$n = 5 \quad r = 1$

$$C = \frac{1}{10}\begin{bmatrix} 6 & 4 & 2 & 0 & -2 \\ 4 & 3 & 2 & 1 & 0 \\ 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 2 & 3 & 4 \\ -2 & 0 & 2 & 4 & 6 \end{bmatrix}$$

$$D_1 = \frac{1}{10}\begin{bmatrix} 2 & 3 & 3 & 2 \\ & '' & '' & '' \\ & '' & '' & '' \\ & '' & '' & '' \end{bmatrix}$$

$$D_2 = 0$$

$$\underline{\underline{n = 5 \quad r = 2}}$$

$$C = \frac{1}{35}\begin{bmatrix} 31 & 9 & -3 & -5 & 3 \\ 9 & 13 & 12 & 6 & -5 \\ -3 & 12 & 17 & 12 & -3 \\ -5 & 6 & 12 & 13 & 9 \\ 3 & -5 & -3 & 9 & 31 \end{bmatrix}$$

$$D_1 = \frac{1}{35}\begin{bmatrix} 22 & 18 & 3 & -8 \\ 12 & 13 & 8 & 2 \\ 2 & 8 & 13 & 12 \\ -8 & 3 & 18 & 22 \end{bmatrix}$$

$$D_2 = \frac{1}{7}\begin{bmatrix} 2 & 3 & 2 \\ & '' & '' \\ & '' & '' \end{bmatrix}$$

$$D_3 = 0$$

$$\underline{\underline{n = 5 \quad r = 3}}$$

$$C = \frac{1}{70}\begin{bmatrix} 69 & 4 & -6 & 4 & -1 \\ 4 & 54 & 24 & -16 & 4 \\ -6 & 24 & 34 & 24 & -6 \\ 4 & -16 & 24 & 54 & 4 \\ -1 & 4 & -6 & 4 & 69 \end{bmatrix}$$

$$D_1 = \frac{1}{14}\begin{bmatrix} 13 & 3 & -3 & 1 \\ 2 & 8 & 6 & -2 \\ -2 & 6 & 8 & 2 \\ 1 & -3 & 3 & 13 \end{bmatrix}$$

$$D_2 = \frac{1}{14}\begin{bmatrix} 11 & 6 & -3 \\ 4 & 6 & 4 \\ -3 & 6 & 11 \end{bmatrix}$$

$$D_3 = \frac{1}{2} Q$$

$$D_4 = 0$$

$$n = 5 \quad r = 4$$

$$C = I$$
$$D_1 = I$$
$$D_2 = I$$
$$D_3 = I$$
$$D_4 = I$$

$$n = 6 \quad r = 0$$

$$C = \tfrac{1}{6} Q$$
$$D_1 = 0$$

$$n = 6 \quad r = 1$$

$$C = \frac{1}{105}\begin{bmatrix} 55 & 40 & 25 & 10 & -5 & -20 \\ 40 & 31 & 22 & 13 & 4 & -5 \\ 25 & 22 & 19 & 16 & 13 & 10 \\ 10 & 13 & 16 & 19 & 22 & 25 \\ -5 & 4 & 13 & 22 & 31 & 40 \\ -20 & -5 & 10 & 25 & 40 & 55 \end{bmatrix}$$

$$D_1 = \frac{1}{35}\begin{bmatrix} 5 & 8 & 9 & 8 & 5 \\ & '' & & '' & '' \\ & '' & & '' & '' \\ & '' & & '' & '' \\ & '' & & '' & '' \end{bmatrix}$$

$$D_2 = 0$$

$$n = 6 \quad r = 2$$

$$C = \frac{1}{140}\begin{bmatrix} 115 & 45 & 0 & -20 & -15 & 15 \\ 45 & 43 & 36 & 24 & 7 & -15 \\ 0 & 36 & 52 & 48 & 24 & -20 \\ -20 & 24 & 48 & 52 & 36 & 0 \\ -15 & 7 & 24 & 36 & 43 & 45 \\ 15 & -15 & -20 & 0 & 45 & 115 \end{bmatrix}$$

$\underline{\underline{n = 6 \quad r = 2}}$ (continued.)

$$D_1 = \frac{1}{140}\begin{bmatrix} 70 & 72 & 36 & -8 & -30 \\ 45 & 52 & 36 & 12 & -5 \\ 20 & 32 & 36 & 32 & 20 \\ -5 & 12 & 36 & 52 & 45 \\ -30 & -8 & 36 & 72 & 70 \end{bmatrix}$$

$$D_2 = \frac{1}{28}\begin{bmatrix} 5 & 9 & 9 & 5 \\ & " & " & \\ & " & " & \\ & " & " & \end{bmatrix}$$

$$D_3 = 0$$

$\underline{\underline{n = 6 \quad r = 3}}$

$$C = \frac{1}{126}\begin{bmatrix} 121 & 16 & -14 & -4 & 11 & -4 \\ 16 & 73 & 52 & 2 & -28 & 11 \\ -14 & 52 & 58 & 32 & 2 & -4 \\ -4 & 2 & 32 & 58 & 52 & -14 \\ 11 & -28 & 2 & 52 & 73 & 16 \\ -4 & 11 & -4 & -14 & 16 & 121 \end{bmatrix}$$

$$D_1 = \frac{1}{126}\begin{bmatrix} 105 & 48 & -18 & -24 & 15 \\ 30 & 51 & 45 & 15 & -15 \\ -10 & 40 & 66 & 40 & -10 \\ -15 & 15 & 45 & 51 & 30 \\ 15 & -24 & -18 & 48 & 105 \end{bmatrix}$$

$$D_2 = \frac{1}{126}\begin{bmatrix} 75 & 72 & 9 & -30 \\ 40 & 51 & 30 & 5 \\ 5 & 30 & 51 & 40 \\ -30 & 9 & 72 & 75 \end{bmatrix}$$

$$D_3 = \frac{1}{18}\begin{bmatrix} 5 & 8 & 5 \\ " & " & \\ " & " & \end{bmatrix}$$

$$D_4 = 0$$

$n = 6 \quad r = 4$

$$C = \frac{1}{252}\begin{bmatrix} 251 & 5 & -10 & 10 & -5 & 1 \\ 5 & 227 & 50 & -50 & 25 & -5 \\ -10 & 50 & 152 & 100 & -50 & 10 \\ 10 & -50 & 100 & 152 & 50 & -10 \\ -5 & 25 & -50 & 50 & 227 & 5 \\ 1 & -5 & 10 & -10 & 5 & 251 \end{bmatrix}$$

$$D_1 = \frac{1}{252}\begin{bmatrix} 246 & 24 & -36 & 24 & -6 \\ 15 & 192 & 90 & -60 & 15 \\ -20 & 80 & 132 & 80 & -20 \\ 15 & -60 & 90 & 192 & 15 \\ -6 & 24 & -36 & 24 & 246 \end{bmatrix}$$

$$D_2 = \frac{1}{36}\begin{bmatrix} 33 & 9 & -9 & 3 \\ 5 & 21 & 15 & -5 \\ -5 & 15 & 21 & 5 \\ 3 & -9 & 9 & 33 \end{bmatrix}$$

$$D_3 = \frac{1}{18}\begin{bmatrix} 14 & 8 & -4 \\ 5 & 8 & 5 \\ -4 & 8 & 14 \end{bmatrix}$$

$$D_4 = \tfrac{1}{2}Q$$

$$D_5 = O$$

$n = 6 \quad r = 5$

$$C = I$$

$$D_1 = I$$

$$D_2 = I$$

$$D_3 = I$$

$$D_4 = I$$

$$D_5 = I$$

$$\underline{\underline{n = 7 \quad r = 0}}$$

$$C = \tfrac{1}{7} Q$$

$$D_1 = 0$$

$$\underline{\underline{n = 7 \quad r = 1}}$$

$$C = \frac{1}{28}\begin{bmatrix} 13 & 10 & 7 & 4 & 1 & -2 & -5 \\ 10 & 8 & 6 & 4 & 2 & 0 & -2 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ -2 & 0 & 2 & 4 & 6 & 8 & 10 \\ -5 & -2 & 1 & 4 & 7 & 10 & 13 \end{bmatrix}$$

$$D_1 = \frac{1}{28}\begin{bmatrix} 3 & 5 & 6 & 6 & 5 & 3 \\ '' & & '' & '' & '' & \\ '' & & '' & '' & '' & \\ '' & & '' & '' & '' & \\ '' & & '' & '' & '' & \\ '' & & '' & '' & '' & \end{bmatrix}$$

$$D_2 = 0$$

$$\underline{\underline{n = 7 \quad r = 2}}$$

$$C = \frac{1}{42}\begin{bmatrix} 32 & 15 & 3 & -4 & -6 & -3 & 5 \\ 15 & 12 & 9 & 6 & 3 & 0 & -3 \\ 3 & 9 & 12 & 12 & 9 & 3 & -6 \\ -4 & 6 & 12 & 14 & 12 & 6 & -4 \\ -6 & 3 & 9 & 12 & 12 & 9 & 3 \\ -3 & 0 & 3 & 6 & 9 & 12 & 15 \\ 5 & -3 & -6 & -4 & 3 & 15 & 32 \end{bmatrix}$$

$$D_1 = \frac{1}{42}\begin{bmatrix} 17 & 20 & 14 & 4 & -5 & -8 \\ 12 & 15 & 12 & 6 & 0 & -3 \\ 7 & 10 & 10 & 8 & 5 & 2 \\ 2 & 5 & 8 & 10 & 10 & 7 \\ -3 & 0 & 6 & 12 & 15 & 12 \\ -8 & -5 & 4 & 14 & 20 & 17 \end{bmatrix}$$

$n = 7 \quad r = 2$ (continued.)

$$D_2 = \frac{1}{42}\begin{bmatrix} 5 & 10 & 12 & 10 & 5 \\ \text{''} & & \text{''} & & \\ \text{''} & & \text{''} & & \\ \text{''} & & \text{''} & & \\ \text{''} & & \text{''} & & \end{bmatrix}$$

$$D_3 = 0$$

$n = 7 \quad r = 3$

$$C = \frac{1}{42}\begin{bmatrix} 39 & 8 & -4 & -4 & 1 & 4 & -2 \\ 8 & 19 & 16 & 6 & -4 & -7 & 4 \\ -4 & 16 & 19 & 12 & 2 & -4 & 1 \\ -4 & 6 & 12 & 14 & 12 & 6 & -4 \\ 1 & -4 & 2 & 12 & 19 & 16 & -4 \\ 4 & -7 & -4 & 6 & 16 & 19 & 8 \\ -2 & 4 & 1 & -4 & -4 & 8 & 39 \end{bmatrix}$$

$$D_1 = \frac{1}{42}\begin{bmatrix} 31 & 20 & 0 & -10 & -5 & 6 \\ 12 & 15 & 12 & 6 & 0 & -3 \\ 0 & 10 & 17 & 15 & 5 & -5 \\ -5 & 5 & 15 & 17 & 10 & 0 \\ -3 & 0 & 6 & 12 & 15 & 12 \\ 6 & -5 & -10 & 0 & 20 & 31 \end{bmatrix}$$

$$D_2 = \frac{1}{42}\begin{bmatrix} 19 & 24 & 12 & -4 & -9 \\ 12 & 17 & 12 & 3 & -2 \\ 5 & 10 & 12 & 10 & 5 \\ -2 & 3 & 12 & 17 & 12 \\ -9 & -4 & 12 & 24 & 19 \end{bmatrix}$$

$$D_3 = \frac{1}{6}\begin{bmatrix} 1 & 2 & 2 & 1 \\ \text{''} & & \text{''} & \\ \text{''} & & \text{''} & \\ \text{''} & & \text{''} & \end{bmatrix}$$

$$D_4 = 0$$

$\underline{\underline{n = 7 \quad r = 4}}$

$$C = \frac{1}{462}\begin{bmatrix} 456 & 25 & -35 & 10 & 20 & -19 & 5 \\ 25 & 356 & 155 & -60 & -65 & 70 & -19 \\ -35 & 155 & 212 & 150 & 25 & -65 & 20 \\ 10 & -60 & 150 & 262 & 150 & -60 & 10 \\ 20 & -65 & 25 & 150 & 212 & 155 & -35 \\ -19 & 70 & -65 & -60 & 155 & 356 & 25 \\ 5 & -19 & 20 & 10 & -35 & 25 & 456 \end{bmatrix}$$

$$D_1 = \frac{1}{462}\begin{bmatrix} 431 & 100 & -90 & -20 & 65 & -24 \\ 60 & 261 & 204 & -6 & -96 & 39 \\ -45 & 170 & 232 & 120 & -5 & -10 \\ -10 & -5 & 120 & 232 & 170 & -45 \\ 39 & -96 & -6 & 204 & 261 & 60 \\ -24 & 65 & -20 & -90 & 100 & 431 \end{bmatrix}$$

$$D_2 = \frac{1}{66}\begin{bmatrix} 53 & 30 & -12 & -14 & 9 \\ 15 & 28 & 24 & 6 & -7 \\ -5 & 20 & 36 & 20 & -5 \\ -7 & 6 & 24 & 28 & 15 \\ 9 & -14 & -12 & 30 & 53 \end{bmatrix}$$

$$D_3 = \frac{1}{33}\begin{bmatrix} 19 & 20 & 2 & -8 \\ 10 & 14 & 8 & 1 \\ 1 & 8 & 14 & 10 \\ -8 & 2 & 20 & 19 \end{bmatrix}$$

$$D_4 = \frac{1}{11}\begin{bmatrix} 3 & 5 & 3 \\ & \text{''} & \text{''} \\ & \text{''} & \text{''} \end{bmatrix}$$

$$D_5 = 0$$

$\underline{\underline{n = 7 \quad r = 5}}$

$$C = \frac{1}{924}\begin{bmatrix} 923 & 6 & -15 & 20 & -15 & 6 & -1 \\ 6 & 888 & 90 & -120 & 90 & -36 & 6 \\ -15 & 90 & 699 & 300 & -225 & 90 & -15 \\ 20 & -120 & 300 & 524 & 300 & -120 & 20 \\ -15 & 90 & -225 & 300 & 699 & 90 & -15 \\ 6 & -36 & 90 & -120 & 90 & 888 & 6 \\ -1 & 6 & -15 & 20 & -15 & 6 & 923 \end{bmatrix}$$

$n = 7 \quad r = 5$ (continued.)

$$D_1 = \frac{1}{132}\begin{bmatrix} 131 & 5 & -10 & 10 & -5 & 1 \\ 3 & 117 & 30 & -30 & 15 & -3 \\ -5 & 25 & 82 & 50 & -25 & 5 \\ 5 & -25 & 50 & 82 & 25 & -5 \\ -3 & 15 & -30 & 30 & 117 & 3 \\ 1 & -5 & 10 & -10 & 5 & 131 \end{bmatrix}$$

$$D_2 = \frac{1}{66}\begin{bmatrix} 64 & 8 & -12 & 8 & -2 \\ 4 & 50 & 24 & -16 & 4 \\ -5 & 20 & 36 & 20 & -5 \\ 4 & -16 & 24 & 50 & 4 \\ -2 & 8 & -12 & 8 & 64 \end{bmatrix}$$

$$D_3 = \frac{1}{22}\begin{bmatrix} 20 & 6 & -6 & 2 \\ 3 & 13 & 9 & -3 \\ -3 & 9 & 13 & 3 \\ 2 & -6 & 6 & 20 \end{bmatrix}$$

$$D_4 = \frac{1}{22}\begin{bmatrix} 17 & 10 & -5 \\ 6 & 10 & 6 \\ -5 & 10 & 17 \end{bmatrix}$$

$$D_5 = \tfrac{1}{2} Q$$

$$D_6 = O$$

$n = 8 \quad r = 0$

$$C = \tfrac{1}{8} Q$$

$$D_1 = O$$

$n = 8 \quad r = 1$

$$C = \frac{1}{84}\begin{bmatrix} 35 & 28 & 21 & 14 & 7 & 0 & -7 & -14 \\ 28 & 23 & 18 & 13 & 8 & 3 & -2 & -7 \\ 21 & 18 & 15 & 12 & 9 & 6 & 3 & 0 \\ 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 \\ 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 0 & 3 & 6 & 9 & 12 & 15 & 18 & 21 \\ -7 & -2 & 3 & 8 & 13 & 18 & 23 & 28 \\ -14 & -7 & 0 & 7 & 14 & 21 & 28 & 35 \end{bmatrix}$$

$\underline{n = 8 \quad r = 1}$ (continued)

$$D_1 = \frac{1}{84}\begin{bmatrix} 7 & 12 & 15 & 16 & 15 & 12 & 7 \\ & \cdot\cdot & & \cdot\cdot & & \cdot\cdot & \\ & \cdot\cdot & & \cdot\cdot & & \cdot\cdot & \\ & & & & & \text{7 rows} & \end{bmatrix}$$

$$D_2 = 0$$

$\underline{n = 8 \qquad r = 2}$

$$C = \frac{1}{168}\begin{bmatrix} 119 & 63 & 21 & -7 & -21 & -21 & -7 & 21 \\ 63 & 47 & 33 & 21 & 11 & 3 & -3 & -7 \\ 21 & 33 & 39 & 39 & 33 & 21 & 3 & -21 \\ -7 & 21 & 39 & 47 & 45 & 33 & 11 & -21 \\ -21 & 11 & 33 & 45 & 47 & 39 & 21 & -7 \\ -21 & 3 & 21 & 33 & 39 & 39 & 33 & 21 \\ -7 & -3 & 3 & 11 & 21 & 33 & 47 & 63 \\ 21 & -7 & -21 & -21 & -7 & 21 & 63 & 119 \end{bmatrix}$$

$$D_1 = \frac{1}{84}\begin{bmatrix} 28 & 36 & 30 & 16 & 0 & -12 & -14 \\ 21 & 28 & 25 & 16 & 5 & -4 & -7 \\ 14 & 20 & 20 & 16 & 10 & 4 & 0 \\ 7 & 12 & 15 & 16 & 15 & 12 & 7 \\ 0 & 4 & 10 & 16 & 20 & 20 & 14 \\ -7 & -4 & 5 & 16 & 25 & 28 & 21 \\ -14 & -12 & 0 & 16 & 30 & 36 & 28 \end{bmatrix}$$

$$D_2 = \frac{1}{84}\begin{bmatrix} 7 & 15 & 20 & 20 & 15 & 7 \\ & \cdot\cdot & & \cdot\cdot & \cdot\cdot & \\ & \cdot\cdot & & \cdot\cdot & \cdot\cdot & \\ & & & \text{6 rows} & & \end{bmatrix}$$

$$D_3 = 0$$

$\underline{n = 8 \qquad r = 3}$

$$C = \frac{1}{462}\begin{bmatrix} 413 & 112 & -28 & -56 & -21 & 28 & 42 & -28 \\ 112 & 173 & 152 & 84 & 4 & -53 & -52 & 42 \\ -28 & 152 & 193 & 144 & 54 & -28 & -53 & 28 \\ -56 & 84 & 144 & 145 & 108 & 54 & 4 & -21 \\ -21 & 4 & 54 & 108 & 145 & 144 & 84 & -56 \\ 28 & -53 & -28 & 54 & 144 & 193 & 152 & -28 \\ 42 & -52 & -53 & 4 & 84 & 152 & 173 & 112 \\ -28 & 42 & 28 & -21 & -56 & -28 & 112 & 413 \end{bmatrix}$$

$n = 8 \quad r = 3$ (continued.)

$$D_1 = \frac{1}{462}\begin{bmatrix} 301 & 240 & 60 & -80 & -105 & -24 & 70 \\ 140 & 161 & 120 & 60 & 10 & -15 & -14 \\ 28 & 96 & 145 & 144 & 90 & 8 & -49 \\ -35 & 45 & 135 & 172 & 135 & 45 & -35 \\ -49 & 8 & 90 & 144 & 145 & 96 & 28 \\ -14 & -15 & 10 & 60 & 120 & 161 & 140 \\ 70 & -24 & -105 & -80 & 60 & 240 & 301 \end{bmatrix}$$

$$D_2 = \frac{1}{462}\begin{bmatrix} 161 & 240 & 180 & 40 & -75 & -84 \\ 112 & 177 & 152 & 68 & -12 & -35 \\ 63 & 114 & 124 & 96 & 51 & 14 \\ 14 & 51 & 96 & 124 & 114 & 63 \\ -35 & -12 & 68 & 152 & 177 & 112 \\ -84 & -75 & 40 & 180 & 240 & 161 \end{bmatrix}$$

$$D_3 = \frac{1}{66}\begin{bmatrix} 7 & 16 & 20 & 16 & 7 \\ '' & & '' & & '' \\ '' & & '' & & '' \\ & & & \text{5 rows} & \\ & & & & \end{bmatrix}$$

$$D_4 = 0$$

$n = 8 \quad r = 4$

$$C = \frac{1}{1848}\begin{bmatrix} 1799 & 175 & -175 & -35 & 105 & 49 & -105 & 35 \\ 175 & 1199 & 725 & -15 & -335 & -95 & 299 & -105 \\ -175 & 725 & 799 & 495 & 135 & -85 & -95 & 49 \\ -35 & -15 & 495 & 823 & 675 & 135 & -335 & 105 \\ 105 & -335 & 135 & 675 & 823 & 495 & -15 & -35 \\ 49 & -95 & -85 & 135 & 495 & 799 & 725 & -175 \\ -105 & 299 & -95 & -335 & -15 & 725 & 1199 & 175 \\ 35 & -105 & 49 & 105 & -35 & -175 & 175 & 1799 \end{bmatrix}$$

$$D_1 = \frac{1}{924}\begin{bmatrix} 812 & 300 & -150 & -160 & 60 & 132 & -70 \\ 175 & 412 & 375 & 120 & -115 & -120 & 77 \\ -70 & 300 & 452 & 288 & 18 & -92 & 28 \\ -70 & 90 & 270 & 344 & 270 & 90 & -70 \\ 28 & -92 & 18 & 288 & 452 & 300 & -70 \\ 77 & -120 & -115 & 120 & 375 & 412 & 175 \\ -70 & 132 & 60 & -160 & -150 & 300 & 812 \end{bmatrix}$$

$n = 8 \quad r = 4$ (continued.)

$$D_2 = \frac{1}{132}\begin{bmatrix} 91 & 75 & 0 & -40 & -15 & 21 \\ 35 & 51 & 40 & 16 & -3 & -7 \\ 0 & 30 & 56 & 48 & 12 & -14 \\ -14 & 12 & 48 & 56 & 30 & 0 \\ -7 & -3 & 16 & 40 & 51 & 35 \\ 21 & -15 & -40 & 0 & 75 & 91 \end{bmatrix}$$

$$D_3 = \frac{1}{132}\begin{bmatrix} 56 & 80 & 40 & -16 & -28 \\ 35 & 56 & 40 & 8 & -7 \\ 14 & 32 & 40 & 32 & 14 \\ -7 & 8 & 40 & 56 & 35 \\ -28 & -16 & 40 & 80 & 56 \end{bmatrix}$$

$$D_4 = \frac{1}{44}\begin{bmatrix} 7 & 15 & 15 & 7 \\ '' & '' & '' & \\ '' & '' & '' & \\ '' & '' & '' & \end{bmatrix}$$

$$D_5 = 0$$

$n = 8 \quad r = 5$

$$C = \frac{1}{1716}\begin{bmatrix} 1709 & 36 & -69 & 50 & 15 & -48 & 29 & -6 \\ 36 & 1529 & 366 & -285 & -40 & 219 & -138 & 29 \\ -69 & 366 & 969 & 660 & -75 & -306 & 219 & -48 \\ 50 & -285 & 660 & 941 & 450 & -75 & -40 & 15 \\ 15 & -40 & -75 & 450 & 941 & 660 & -285 & 50 \\ -48 & 219 & -306 & -75 & 660 & 969 & 366 & -69 \\ 29 & -138 & 219 & -40 & -285 & 366 & 1529 & 36 \\ -6 & 29 & -48 & 15 & 50 & -69 & 36 & 1709 \end{bmatrix}$$

$$D_1 = \frac{1}{1716}\begin{bmatrix} 1673 & 180 & -255 & 80 & 135 & -132 & 35 \\ 105 & 1268 & 665 & -280 & -245 & 280 & -77 \\ -119 & 532 & 841 & 560 & 35 & -196 & 63 \\ 35 & -210 & 525 & 1016 & 525 & -210 & 35 \\ 63 & -196 & 35 & 560 & 841 & 532 & -119 \\ -77 & 280 & -245 & -280 & 665 & 1268 & 105 \\ 35 & -132 & 135 & 80 & -255 & 180 & 1673 \end{bmatrix}$$

$n = 8 \quad r = 5$ (continued)

$$D_2 = \frac{1}{858}\begin{bmatrix} 784 & 240 & -220 & -40 & 150 & -56 \\ 112 & 480 & 392 & -28 & -168 & 70 \\ -77 & 294 & 452 & 224 & -21 & -14 \\ -14 & -21 & 224 & 452 & 294 & -77 \\ 70 & -168 & -28 & 392 & 480 & 112 \\ -56 & 150 & -40 & -220 & 240 & 784 \end{bmatrix}$$

$$D_3 = \frac{1}{286}\begin{bmatrix} 224 & 144 & -60 & -64 & 42 \\ 63 & 125 & 105 & 21 & -28 \\ -21 & 84 & 160 & 84 & -21 \\ -28 & 21 & 105 & 125 & 63 \\ 42 & -64 & -60 & 144 & 224 \end{bmatrix}$$

$$D_4 = \frac{1}{286}\begin{bmatrix} 161 & 180 & 15 & -70 \\ 84 & 125 & 70 & 7 \\ 7 & 70 & 125 & 84 \\ -70 & 15 & 180 & 161 \end{bmatrix}$$

$$D_5 = \frac{1}{26}\begin{bmatrix} 7 & 12 & 7 \\ '' & & '' \\ '' & & '' \end{bmatrix}$$

$$D_6 = 0$$

$n = 9 \quad r = 0$

$$C = \frac{1}{9} Q$$

$$D_1 = 0$$

$n = 9 \quad r = 1$

$$C = \frac{1}{180}\begin{bmatrix} 68 & 56 & 44 & 32 & 20 & 8 & -4 & -16 & -28 \\ 56 & 47 & 38 & 29 & 20 & 11 & 2 & -7 & -16 \\ 44 & 38 & 32 & 26 & 20 & 14 & 8 & 2 & -4 \\ 32 & 29 & 26 & 23 & 20 & 17 & 14 & 11 & 8 \\ 20 & 20 & 20 & 20 & 20 & 20 & 20 & 20 & 20 \\ 8 & 11 & 14 & 17 & 20 & 23 & 26 & 29 & 32 \\ -4 & 2 & 8 & 14 & 20 & 26 & 32 & 38 & 44 \\ -16 & -7 & 2 & 11 & 20 & 29 & 38 & 47 & 56 \\ -28 & -16 & -4 & 8 & 20 & 32 & 44 & 56 & 68 \end{bmatrix}$$

$n = 9 \quad r = 1$ (continued.)

$$D_1 = \frac{1}{60}\begin{bmatrix} 4 & 7 & 9 & 10 & 10 & 9 & 7 & 4 \\ & \text{''} & & \text{''} & & \text{''} & \text{''} & \\ & \text{''} & & \text{''} & & \text{''} & \text{''} & \\ & & & & \text{8 rows} & & & \end{bmatrix}$$

$$D_2 = 0$$

$n = 9 \quad r = 2$

$$C = \frac{1}{2310}\begin{bmatrix} 1526 & 882 & 378 & 14 & -210 & -294 & -238 & -42 & 294 \\ 882 & 644 & 441 & 273 & 140 & 42 & -21 & -49 & -42 \\ 378 & 441 & 464 & 447 & 390 & 293 & 156 & -21 & -238 \\ 14 & 273 & 447 & 536 & 540 & 459 & 293 & 42 & -294 \\ -210 & 140 & 390 & 540 & 590 & 540 & 390 & 140 & -210 \\ -294 & 42 & 293 & 459 & 540 & 536 & 447 & 273 & 14 \\ -238 & -21 & 156 & 293 & 390 & 447 & 464 & 441 & 378 \\ -42 & -49 & -21 & 42 & 140 & 273 & 441 & 644 & 882 \\ 294 & -42 & -238 & -294 & -210 & 14 & 378 & 882 & 1526 \end{bmatrix}$$

$$D_1 = \frac{1}{2310}\begin{bmatrix} 644 & 882 & 819 & 560 & 210 & -126 & -343 & -336 \\ 504 & 707 & 684 & 510 & 260 & 9 & -168 & -196 \\ 364 & 532 & 549 & 460 & 310 & 144 & 7 & -56 \\ 224 & 357 & 414 & 410 & 360 & 279 & 182 & 84 \\ 84 & 182 & 279 & 360 & 410 & 414 & 357 & 224 \\ -56 & 7 & 144 & 310 & 460 & 549 & 532 & 364 \\ -196 & -168 & 9 & 260 & 510 & 684 & 707 & 504 \\ -336 & -343 & -126 & 210 & 560 & 819 & 882 & 644 \end{bmatrix}$$

$$D_2 = \frac{1}{462}\begin{bmatrix} 28 & 63 & 90 & 100 & 90 & 63 & 28 \\ & \text{''} & & \text{''} & & \text{''} & \\ & \text{''} & & \text{''} & & \text{''} & \\ & & & & \text{7 rows} & & \end{bmatrix}$$

$$D_3 = 0$$

$$n = 9 \quad r = 3$$

$$C = \frac{1}{1386}\begin{bmatrix} 1190 & 392 & -28 & -168 & -126 & 0 & 112 & 112 & -98 \\ 392 & 455 & 392 & 252 & 84 & -63 & -140 & -98 & 112 \\ -28 & 392 & 515 & 432 & 234 & 12 & -143 & -140 & 112 \\ -168 & 252 & 432 & 435 & 324 & 162 & 12 & -63 & 0 \\ -126 & 84 & 234 & 324 & 354 & 324 & 234 & 84 & -126 \\ 0 & -63 & 12 & 162 & 324 & 435 & 432 & 252 & -168 \\ 112 & -140 & -143 & 12 & 234 & 432 & 515 & 392 & -28 \\ 112 & -98 & -140 & -63 & 84 & 252 & 392 & 455 & 392 \\ -98 & 112 & 112 & 0 & -126 & -168 & -28 & 392 & 1190 \end{bmatrix}$$

$$D_1 = \frac{1}{1386}\begin{bmatrix} 798 & 735 & 315 & -105 & -315 & -252 & 0 & 210 \\ 420 & 483 & 360 & 180 & 30 & -45 & -42 & 0 \\ 140 & 280 & 363 & 360 & 270 & 120 & -35 & -112 \\ -42 & 126 & 324 & 435 & 405 & 243 & 21 & -126 \\ -126 & 21 & 243 & 405 & 435 & 324 & 126 & -42 \\ -112 & -35 & 120 & 270 & 360 & 363 & 280 & 140 \\ 0 & -42 & -45 & 30 & 180 & 360 & 483 & 420 \\ 210 & 0 & -252 & -315 & -105 & 315 & 735 & 798 \end{bmatrix}$$

$$D_2 = \frac{1}{1386}\begin{bmatrix} 378 & 630 & 585 & 300 & -45 & -252 & -210 \\ 280 & 483 & 480 & 300 & 60 & -105 & -112 \\ 182 & 336 & 375 & 300 & 165 & 42 & -14 \\ 84 & 189 & 270 & 300 & 270 & 189 & 84 \\ -14 & 42 & 165 & 300 & 375 & 336 & 182 \\ -112 & -105 & 60 & 300 & 480 & 483 & 280 \\ -210 & -252 & -45 & 300 & 585 & 630 & 378 \end{bmatrix}$$

$$D_3 = \frac{1}{198}\begin{bmatrix} 14 & 35 & 50 & 50 & 35 & 14 \\ '' & & '' & & '' & \\ '' & & '' & & '' & \\ & & & \text{6 rows} & & \\ & & & & & \end{bmatrix}$$

$$D_4 = 0$$

$n = 9 \quad r = 4$

$$C = \frac{1}{2574}\begin{bmatrix} 2462 & 350 & -250 & -150 & 90 & 162 & 10 & -170 & 70 \\ 350 & 1412 & 1025 & 225 & -330 & -360 & 37 & 385 & -170 \\ -250 & 1025 & 1112 & 675 & 180 & -105 & -110 & 37 & 10 \\ -150 & 225 & 675 & 912 & 810 & 405 & -105 & -360 & 162 \\ 90 & -330 & 180 & 810 & 1074 & 810 & 180 & -330 & 90 \\ 162 & -360 & -105 & 405 & 810 & 912 & 675 & 225 & -150 \\ 10 & 37 & -110 & -105 & 180 & 675 & 1112 & 1025 & -250 \\ -170 & 385 & 37 & -360 & -330 & 225 & 1025 & 1412 & 350 \\ 70 & -170 & 10 & 162 & 90 & -150 & -250 & 350 & 2462 \end{bmatrix}$$

$$D_1 = \frac{1}{2574}\begin{bmatrix} 2112 & 1050 & -225 & -600 & -180 & 342 & 315 & -240 \\ 600 & 987 & 900 & 450 & -60 & -315 & -168 & 180 \\ -100 & 700 & 1137 & 900 & 270 & -240 & -245 & 152 \\ -240 & 315 & 810 & 912 & 648 & 243 & -42 & -72 \\ -72 & -42 & 243 & 648 & 912 & 810 & 315 & -240 \\ 152 & -245 & -240 & 270 & 900 & 1137 & 700 & -100 \\ 180 & -168 & -315 & -60 & 450 & 900 & 987 & 600 \\ -240 & 315 & 342 & -180 & -600 & -225 & 1050 & 2112 \end{bmatrix}$$

$$D_2 = \frac{1}{2574}\begin{bmatrix} 1512 & 1575 & 450 & -600 & -720 & -63 & 420 \\ 700 & 987 & 750 & 300 & -30 & -105 & -28 \\ 140 & 525 & 852 & 840 & 462 & -21 & -224 \\ -168 & 189 & 756 & 1020 & 756 & 189 & -168 \\ -224 & -21 & 462 & 840 & 852 & 525 & 140 \\ -28 & -105 & -30 & 300 & 750 & 987 & 700 \\ 420 & -63 & -720 & -600 & 450 & 1575 & 1512 \end{bmatrix}$$

$$D_3 = \frac{1}{1287}\begin{bmatrix} 406 & 700 & 550 & 100 & -245 & -224 \\ 280 & 511 & 460 & 190 & -56 & -98 \\ 154 & 322 & 370 & 280 & 133 & 28 \\ 28 & 133 & 280 & 370 & 322 & 154 \\ -98 & -56 & 190 & 460 & 511 & 280 \\ -224 & -245 & 100 & 550 & 700 & 406 \end{bmatrix}$$

$$D_4 = \frac{1}{143}\begin{bmatrix} 14 & 35 & 45 & 35 & 14 \\ & " & " & " & \\ & " & " & " & \\ & & & \text{5 rows} & \end{bmatrix}$$

$$D_5 = 0$$

$n = 9 \quad r = 5$

$$C = \frac{1}{1716}\begin{bmatrix} 1700 & 72 & -108 & 32 & 60 & -24 & -52 & 48 & -12 \\ 72 & 1385 & 522 & -213 & -220 & 123 & 186 & -187 & 48 \\ -108 & 522 & 800 & 582 & 120 & -202 & -132 & 186 & -52 \\ 32 & -213 & 582 & 905 & 540 & -27 & -202 & 123 & -24 \\ 60 & -220 & 120 & 540 & 716 & 540 & 120 & -220 & 60 \\ -24 & 123 & -202 & -27 & 540 & 905 & 582 & -213 & 32 \\ -52 & 186 & -132 & -202 & 120 & 582 & 800 & 522 & -108 \\ 48 & -187 & 186 & 123 & -220 & -213 & 522 & 1385 & 72 \\ -12 & 48 & -52 & -24 & 60 & 32 & -108 & 72 & 1700 \end{bmatrix}$$

$$D_1 = \frac{1}{1716}\begin{bmatrix} 1628 & 315 & -315 & -70 & 210 & 63 & -175 & 60 \\ 180 & 1043 & 765 & -30 & -370 & -45 & 273 & -100 \\ -140 & 595 & 813 & 490 & 70 & -105 & -35 & 28 \\ -28 & -21 & 441 & 806 & 630 & 63 & -259 & 84 \\ 84 & -259 & 63 & 630 & 806 & 441 & -21 & -28 \\ 28 & -35 & -105 & 70 & 490 & 813 & 595 & -140 \\ -100 & 273 & -45 & -370 & -30 & 765 & 1043 & 180 \\ 60 & -175 & 63 & 210 & -70 & -315 & 315 & 1628 \end{bmatrix}$$

$$D_2 = \frac{1}{858}\begin{bmatrix} 724 & 360 & -180 & -200 & 90 & 144 & -80 \\ 160 & 384 & 360 & 100 & -120 & -90 & 64 \\ -56 & 252 & 438 & 280 & 0 & -84 & 28 \\ -56 & 63 & 252 & 340 & 252 & 63 & -56 \\ 28 & -84 & 0 & 280 & 438 & 252 & -56 \\ 64 & -90 & -120 & 100 & 360 & 384 & 160 \\ -80 & 144 & 90 & -200 & -180 & 360 & 724 \end{bmatrix}$$

$$D_3 = \frac{1}{286}\begin{bmatrix} 188 & 180 & 0 & -100 & -30 & 48 \\ 72 & 116 & 90 & 30 & -10 & -12 \\ 0 & 63 & 125 & 105 & 21 & -28 \\ -28 & 21 & 105 & 125 & 63 & 0 \\ -12 & -10 & 30 & 90 & 116 & 72 \\ 48 & -30 & -100 & 0 & 180 & 188 \end{bmatrix}$$

$$D_4 = \frac{1}{286}\begin{bmatrix} 116 & 180 & 90 & -40 & -60 \\ 72 & 125 & 90 & 15 & -16 \\ 28 & 70 & 90 & 70 & 28 \\ -16 & 15 & 90 & 125 & 72 \\ -60 & -40 & 90 & 180 & 116 \end{bmatrix}$$

$$D_5 = \frac{1}{26}\begin{bmatrix} 4 & 9 & 9 & 4 \\ & '' & '' & \\ & '' & '' & \\ & '' & '' & \end{bmatrix}$$

$$D_6 = 0$$

$n = 10 \quad r = 0$

$$C = \frac{1}{10} Q$$

$$D_1 = 0$$

$n = 10 \quad r = 1$

$$C = \frac{1}{165}\begin{bmatrix} 57 & 48 & 39 & 30 & 21 & 12 & 3 & -6 & -15 & -24 \\ 48 & 41 & 34 & 27 & 20 & 13 & 6 & -1 & -8 & -15 \\ 39 & 34 & 29 & 24 & 19 & 14 & 9 & 4 & -1 & -6 \\ 30 & 27 & 24 & 21 & 18 & 15 & 12 & 9 & 6 & 3 \\ 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 \\ 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 & 30 \\ -6 & -1 & 4 & 9 & 14 & 19 & 24 & 29 & 34 & 39 \\ -15 & -8 & -1 & 6 & 13 & 20 & 27 & 34 & 41 & 48 \\ -24 & -15 & -6 & 3 & 12 & 21 & 30 & 39 & 48 & 57 \end{bmatrix}$$

$$D_1 = \frac{1}{165}\begin{bmatrix} 9 & 16 & 21 & 24 & 25 & 24 & 21 & 16 & 9 \\ & '' & & '' & & & '' & & \\ & '' & & '' & & & '' & & \\ & & & & & & \text{9 rows.} & & \end{bmatrix}$$

$$D_2 = 0$$

$n = 10 \quad r = 2$

$$C = \frac{1}{660}\begin{bmatrix} 408 & 252 & 126 & 30 & -36 & -72 & -78 & -54 & 0 & 84 \\ 252 & 184 & 126 & 78 & 40 & 12 & -6 & -14 & -12 & 0 \\ 126 & 126 & 121 & 111 & 96 & 76 & 51 & 21 & -14 & -54 \\ 30 & 78 & 111 & 129 & 132 & 120 & 93 & 51 & -6 & -78 \\ -36 & 40 & 96 & 132 & 148 & 144 & 120 & 76 & 12 & -72 \\ -72 & 12 & 76 & 120 & 144 & 148 & 132 & 96 & 40 & -36 \\ -78 & -6 & 51 & 93 & 120 & 132 & 129 & 111 & 78 & 30 \\ -54 & -14 & 21 & 51 & 76 & 96 & 111 & 121 & 126 & 126 \\ 0 & -12 & -14 & -6 & 12 & 40 & 78 & 126 & 184 & 252 \\ 84 & 0 & -54 & -78 & -72 & -36 & 30 & 126 & 252 & 408 \end{bmatrix}$$

$n = 10 \quad r = 2$ (continued.)

$$D_1 = \frac{1}{660}\begin{bmatrix} 156 & 224 & 224 & 176 & 100 & 16 & -56 & -96 & -84 \\ 126 & 184 & 189 & 156 & 100 & 36 & -21 & -56 & -54 \\ 96 & 144 & 154 & 136 & 100 & 56 & 14 & -16 & -24 \\ 66 & 104 & 119 & 116 & 100 & 76 & 49 & 24 & 6 \\ 36 & 64 & 84 & 96 & 100 & 96 & 84 & 64 & 36 \\ 6 & 24 & 49 & 76 & 100 & 116 & 119 & 104 & 66 \\ -24 & -16 & 14 & 56 & 100 & 136 & 154 & 144 & 96 \\ -54 & -56 & -21 & 36 & 100 & 156 & 189 & 184 & 126 \\ -84 & -96 & -56 & 16 & 100 & 176 & 224 & 224 & 156 \end{bmatrix}$$

$$D_2 = \frac{1}{132}\begin{bmatrix} 6 & 14 & 21 & 25 & 25 & 21 & 14 & 6 \\ & & " & & " & & " & \\ & & " & & " & & " & \\ & & & & & \text{8 rows} & & \end{bmatrix}$$

$$D_3 = 0$$

$n = 10 \qquad r = 3$

$$C = \frac{1}{4290}\begin{bmatrix} 3534 & 1344 & 84 & -456 & -486 & -216 & 144 & 384 & 294 & -336 \\ 1344 & 1294 & 1064 & 724 & 344 & -6 & -256 & -336 & -176 & 294 \\ 84 & 1064 & 1399 & 1264 & 834 & 284 & -211 & -476 & -336 & 384 \\ -456 & 724 & 1264 & 1319 & 1044 & 594 & 124 & -211 & -256 & 144 \\ -486 & 344 & 834 & 1044 & 1034 & 864 & 594 & 284 & -6 & -216 \\ -216 & -6 & 284 & 594 & 864 & 1034 & 1044 & 834 & 344 & -486 \\ 144 & -256 & -211 & 124 & 594 & 1044 & 1319 & 1264 & 724 & -456 \\ 384 & -336 & -476 & -211 & 284 & 834 & 1264 & 1399 & 1064 & 84 \\ 294 & -176 & -336 & -256 & -6 & 344 & 724 & 1064 & 1294 & 1344 \\ -336 & 294 & 384 & 144 & -216 & -486 & -456 & 84 & 1344 & 3534 \end{bmatrix}$$

$$D_1 = \frac{1}{858}\begin{bmatrix} 438 & 448 & 252 & 16 & -150 & -192 & -112 & 32 & 126 \\ 252 & 298 & 231 & 123 & 25 & -33 & -42 & -14 & 18 \\ 108 & 176 & 203 & 192 & 150 & 88 & 21 & -32 & -48 \\ 6 & 82 & 168 & 223 & 225 & 171 & 77 & -22 & -72 \\ -54 & 16 & 126 & 216 & 250 & 216 & 126 & 16 & -54 \\ -72 & -22 & 77 & 171 & 225 & 223 & 168 & 82 & 6 \\ -48 & -32 & 21 & 88 & 150 & 192 & 203 & 176 & 108 \\ 18 & -14 & -42 & -33 & 25 & 123 & 231 & 298 & 252 \\ 126 & 32 & -112 & -192 & -150 & 16 & 252 & 448 & 438 \end{bmatrix}$$

$n = 10 \quad r = 3$ (continued)

$$D_2 = \frac{1}{858}\begin{bmatrix} 186 & 336 & 357 & 250 & 75 & -84 & -154 & -108 \\ 144 & 266 & 294 & 225 & 100 & -21 & -84 & -66 \\ 102 & 196 & 231 & 200 & 125 & 42 & -14 & -24 \\ 60 & 126 & 168 & 175 & 150 & 105 & 56 & 18 \\ 18 & 56 & 105 & 150 & 175 & 168 & 126 & 60 \\ -24 & -14 & 42 & 125 & 200 & 231 & 196 & 102 \\ -66 & -84 & -21 & 100 & 225 & 294 & 266 & 144 \\ -108 & -154 & -84 & 75 & 250 & 357 & 336 & 186 \end{bmatrix}$$

$$D_3 = \frac{1}{858}\begin{bmatrix} 42 & 112 & 175 & 200 & 175 & 112 & 42 \\ & \text{"} & & \text{"} & \text{"} & & \\ & \text{"} & & \text{"} & \text{"} & & \\ & & & & \text{7 rows} & & \end{bmatrix}$$

$$D_4 = 0$$

$n = 10 \quad r = 4$

$$C = \frac{1}{1716}\begin{bmatrix} 1608 & 300 & -150 & -150 & 0 & 108 & 90 & -30 & -120 & 60 \\ 300 & 808 & 650 & 250 & -100 & -240 & -142 & 90 & 220 & -120 \\ -150 & 650 & 733 & 475 & 150 & -70 & -115 & -17 & 90 & -30 \\ -150 & 250 & 475 & 533 & 450 & 270 & 55 & -115 & -142 & 90 \\ 0 & -100 & 150 & 450 & 608 & 540 & 270 & -70 & -240 & 108 \\ 108 & -240 & -70 & 270 & 540 & 608 & 450 & 150 & -100 & 0 \\ 90 & -142 & -115 & 55 & 270 & 450 & 533 & 475 & 250 & -150 \\ -30 & 90 & -17 & -115 & -70 & 150 & 475 & 733 & 650 & -150 \\ -120 & 220 & 90 & -142 & -240 & -100 & 250 & 650 & 808 & 300 \\ 60 & -120 & -30 & 90 & 108 & 0 & -150 & -150 & 300 & 1608 \end{bmatrix}$$

$$D_1 = \frac{1}{1716}\begin{bmatrix} 1308 & 800 & 0 & -400 & -300 & 48 & 280 & 160 & -180 \\ 450 & 608 & 525 & 300 & 50 & -120 & -147 & -40 & 90 \\ 0 & 400 & 658 & 600 & 300 & -40 & -210 & -112 & 120 \\ -150 & 200 & 525 & 608 & 450 & 180 & -35 & -80 & 18 \\ -108 & 32 & 252 & 432 & 500 & 432 & 252 & 32 & -108 \\ 18 & -80 & -35 & 180 & 450 & 608 & 525 & 200 & -150 \\ 120 & -112 & -210 & -40 & 300 & 600 & 658 & 400 & 0 \\ 90 & -40 & -147 & -120 & 50 & 300 & 525 & 608 & 450 \\ -180 & 160 & 280 & 48 & -300 & -400 & 0 & 800 & 1308 \end{bmatrix}$$

$n = 10 \quad r = 4$ (continued.)

$$D_2 = \frac{1}{1716}\begin{bmatrix} 858 & 1050 & 525 & -175 & -525 & -357 & 70 & 270 \\ 450 & 658 & 525 & 225 & -25 & -105 & -42 & 30 \\ 150 & 350 & 483 & 475 & 325 & 105 & -70 & -102 \\ -42 & 126 & 399 & 575 & 525 & 273 & -14 & -126 \\ -126 & -14 & 273 & 525 & 575 & 399 & 126 & -42 \\ -102 & -70 & 105 & 325 & 475 & 483 & 350 & 150 \\ 30 & -42 & -105 & -25 & 225 & 525 & 658 & 450 \\ 270 & 70 & -357 & -525 & -175 & 525 & 1050 & 858 \end{bmatrix}$$

$$D_3 = \frac{1}{858}\begin{bmatrix} 204 & 400 & 400 & 200 & -50 & -176 & -120 \\ 150 & 304 & 325 & 200 & 25 & -80 & -66 \\ 96 & 208 & 250 & 200 & 100 & 16 & -12 \\ 42 & 112 & 175 & 200 & 175 & 112 & 42 \\ -12 & 16 & 100 & 200 & 250 & 208 & 96 \\ -66 & -80 & 25 & 200 & 325 & 304 & 150 \\ -120 & -176 & -50 & 200 & 400 & 400 & 204 \end{bmatrix}$$

$$D_4 = \frac{1}{286}\begin{bmatrix} 18 & 50 & 75 & 75 & 50 & 18 \\ & '' & & '' & & \\ & '' & & '' & & \\ & & & \text{6 rows.} & & \end{bmatrix}$$

$$D_5 = 0$$

$n = 10 \quad r = 5$

$$C = \frac{1}{2145}\begin{bmatrix} 2109 & 144 & -171 & -6 & 99 & 36 & -69 & -54 & 81 & -24 \\ 144 & 1549 & 774 & -111 & -356 & -69 & 246 & 151 & -264 & 81 \\ -171 & 774 & 919 & 624 & 204 & -104 & -174 & -24 & 151 & -54 \\ -6 & -111 & 624 & 999 & 744 & 156 & -264 & -174 & 246 & -69 \\ 99 & -356 & 204 & 744 & 859 & 576 & 156 & -104 & -69 & 36 \\ 36 & -69 & -104 & 156 & 576 & 859 & 744 & 204 & -356 & 99 \\ -69 & 246 & -174 & -264 & 156 & 744 & 999 & 624 & -111 & -6 \\ -54 & 151 & -24 & -174 & -104 & 204 & 624 & 919 & 774 & -171 \\ 81 & -264 & 151 & 246 & -69 & -356 & -111 & 774 & 1549 & 144 \\ -24 & 81 & -54 & -69 & 36 & 99 & -6 & -171 & 144 & 2109 \end{bmatrix}$$

$$D_1 = \frac{1}{2145}\begin{bmatrix} 1965 & 560 & -385 & -280 & 175 & 280 & -35 & -240 & 105 \\ 315 & 1090 & 945 & 210 & -350 & -315 & 105 & 280 & -135 \\ -165 & 720 & 1015 & 640 & 100 & -160 & -70 & 80 & -15 \\ -105 & 140 & 560 & 815 & 700 & 280 & -140 & -210 & 105 \\ 63 & -224 & 84 & 672 & 955 & 672 & 84 & -224 & 63 \\ 105 & -210 & -140 & 280 & 700 & 815 & 560 & 140 & -105 \\ -15 & 80 & -70 & -160 & 100 & 640 & 1015 & 720 & -165 \\ -135 & 280 & 105 & -315 & -350 & 210 & 945 & 1090 & 315 \\ 105 & -240 & -35 & 280 & 175 & -280 & -385 & 560 & 1965 \end{bmatrix}$$

$n = 10 \quad r = 5$ (continued.)

$$D_2 = \frac{1}{2145} \begin{bmatrix} 1650 & 1120 & -210 & -700 & -175 & 420 & 280 & -240 \\ 480 & 850 & 780 & 350 & -100 & -255 & -80 & 120 \\ -60 & 520 & 975 & 800 & 200 & -240 & -170 & 120 \\ -168 & 196 & 672 & 815 & 560 & 168 & -56 & -42 \\ -42 & -56 & 168 & 560 & 815 & 672 & 196 & -168 \\ 120 & -170 & -240 & 200 & 800 & 975 & 520 & -60 \\ 120 & -80 & -255 & -100 & 350 & 780 & 850 & 480 \\ -240 & 280 & 420 & -175 & -700 & -210 & 1120 & 1650 \end{bmatrix}$$

$$D_3 = \frac{1}{715} \begin{bmatrix} 390 & 480 & 150 & -200 & -225 & 0 & 120 \\ 180 & 290 & 225 & 75 & -25 & -30 & 0 \\ 36 & 144 & 245 & 240 & 120 & -16 & -54 \\ -42 & 42 & 210 & 295 & 210 & 42 & -42 \\ -54 & -16 & 120 & 240 & 245 & 144 & 36 \\ 0 & -30 & -25 & 75 & 225 & 290 & 180 \\ 120 & 0 & -225 & -200 & 150 & 480 & 390 \end{bmatrix}$$

$$D_4 = \frac{1}{715} \begin{bmatrix} 210 & 400 & 325 & 50 & -150 & -120 \\ 144 & 290 & 270 & 105 & -40 & -54 \\ 78 & 180 & 215 & 160 & 70 & 12 \\ 12 & 70 & 160 & 215 & 180 & 78 \\ -54 & -40 & 105 & 270 & 290 & 144 \\ -120 & -150 & 50 & 325 & 400 & 210 \end{bmatrix}$$

$$D_5 = \frac{1}{65} \begin{bmatrix} 6 & 16 & 21 & 16 & 6 \\ & '' & & '' & \\ & '' & & '' & \\ & & & 5 \text{ rows} & \end{bmatrix}$$

$$D_6 = 0$$

$n = 11 \quad r = 0$

$$C = \frac{1}{11} Q$$

$$D_1 = 0$$

$n = 11 \quad r = 1$

$$C = \frac{1}{110}\begin{bmatrix}
35 & 30 & 25 & 20 & 15 & 10 & 5 & 0 & -5 & -10 & -15 \\
30 & 26 & 22 & 18 & 14 & 10 & 6 & 2 & -2 & -6 & -10 \\
25 & 22 & 19 & 16 & 13 & 10 & 7 & 4 & 1 & -2 & -5 \\
20 & 18 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 & 0 \\
15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 \\
10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 \\
-5 & -2 & 1 & 4 & 7 & 10 & 13 & 16 & 19 & 22 & 25 \\
-10 & -6 & -2 & 2 & 6 & 10 & 14 & 18 & 22 & 26 & 30 \\
-15 & -10 & -5 & 0 & 5 & 10 & 15 & 20 & 25 & 30 & 35
\end{bmatrix}$$

$$D_1 = \frac{1}{110}\begin{bmatrix}
5 & 9 & 12 & 14 & 15 & 15 & 14 & 12 & 9 & 5 \\
 & " & & & " & & & " & & \\
 & " & & & " & & & " & & \\
 & & & & & & \text{10 rows.} & & &
\end{bmatrix}$$

$$D_2 = 0$$

$n = 11 \quad r = 2$

$$C = \frac{1}{2145}\begin{bmatrix}
1245 & 810 & 450 & 165 & -45 & -180 & -240 & -225 & -135 & 30 & 270 \\
810 & 597 & 414 & 261 & 138 & 45 & -18 & -51 & -54 & -27 & 30 \\
450 & 414 & 373 & 327 & 276 & 220 & 159 & 93 & 22 & -54 & -135 \\
165 & 261 & 327 & 363 & 369 & 345 & 291 & 207 & 93 & -51 & -225 \\
-45 & 138 & 276 & 369 & 417 & 420 & 378 & 291 & 159 & -18 & -240 \\
-180 & 45 & 220 & 345 & 420 & 445 & 420 & 345 & 220 & 45 & -180 \\
-240 & -18 & 159 & 291 & 378 & 420 & 417 & 369 & 276 & 138 & -45 \\
-225 & -51 & 93 & 207 & 291 & 345 & 369 & 363 & 327 & 261 & 165 \\
-135 & -54 & 22 & 93 & 159 & 220 & 276 & 327 & 373 & 414 & 450 \\
30 & -27 & -54 & -51 & -18 & 45 & 138 & 261 & 414 & 597 & 810 \\
270 & 30 & -135 & -225 & -240 & -180 & -45 & 165 & 450 & 810 & 1245
\end{bmatrix}$$

$$D_1 = \frac{1}{2145}\begin{bmatrix}
435 & 648 & 684 & 588 & 405 & 180 & -42 & -216 & -297 & -240 \\
360 & 543 & 584 & 518 & 380 & 205 & 28 & -116 & -192 & -165 \\
285 & 438 & 484 & 448 & 355 & 230 & 98 & -16 & -87 & -90 \\
210 & 333 & 384 & 378 & 330 & 255 & 168 & 84 & 18 & -15 \\
135 & 228 & 284 & 308 & 305 & 280 & 238 & 184 & 123 & 60 \\
60 & 123 & 184 & 238 & 280 & 305 & 308 & 284 & 228 & 135 \\
-15 & 18 & 84 & 168 & 255 & 330 & 378 & 384 & 333 & 210 \\
-90 & -87 & -16 & 98 & 230 & 355 & 448 & 484 & 438 & 285 \\
-165 & -192 & -116 & 28 & 205 & 380 & 518 & 584 & 543 & 360 \\
-240 & -297 & -216 & -42 & 180 & 405 & 588 & 684 & 648 & 435
\end{bmatrix}$$

$n = 11 \quad r = 2$ (continued.)

$$D_2 = \frac{1}{429}\begin{bmatrix} 15 & 36 & 56 & 70 & 75 & 70 & 56 & 36 & 15 \\ & & " & & & " & & & \\ & & " & & & " & & & \\ & & & & & \text{9 rows} & & & \end{bmatrix}$$

$$D_3 = 0$$

$n = 11 \quad r = 3$

$$C = \frac{1}{858}\begin{bmatrix} 678 & 288 & 48 & -72 & -102 & -72 & -12 & 48 & 78 & 48 & -72 \\ 288 & 246 & 192 & 132 & 72 & 18 & -24 & -48 & -48 & -18 & 48 \\ 48 & 192 & 246 & 232 & 172 & 88 & 2 & -64 & -88 & -48 & 78 \\ -72 & 132 & 232 & 251 & 212 & 138 & 52 & -23 & -64 & -48 & 48 \\ -102 & 72 & 172 & 212 & 206 & 168 & 112 & 52 & 2 & -24 & -12 \\ -72 & 18 & 88 & 138 & 168 & 178 & 168 & 138 & 88 & 18 & -72 \\ -12 & -24 & 2 & 52 & 112 & 168 & 206 & 212 & 172 & 72 & -102 \\ 48 & -48 & -64 & -23 & 52 & 138 & 212 & 251 & 232 & 132 & -72 \\ 78 & -48 & -88 & -64 & 2 & 88 & 172 & 232 & 246 & 192 & 48 \\ 48 & -18 & -48 & -48 & -24 & 18 & 72 & 132 & 192 & 246 & 288 \\ -72 & 48 & 78 & 48 & -12 & -72 & -102 & -72 & 48 & 288 & 678 \end{bmatrix}$$

$$D_1 = \frac{1}{858}\begin{bmatrix} 390 & 432 & 288 & 84 & -90 & -180 & -168 & -72 & 54 & 120 \\ 240 & 294 & 240 & 140 & 40 & -30 & -56 & -40 & 0 & 30 \\ 120 & 180 & 194 & 175 & 135 & 85 & 35 & -6 & -30 & -30 \\ 30 & 90 & 150 & 189 & 195 & 165 & 105 & 30 & -36 & -60 \\ -30 & 24 & 108 & 182 & 220 & 210 & 154 & 68 & -18 & -60 \\ -60 & -18 & 68 & 154 & 210 & 220 & 182 & 108 & 24 & -30 \\ -60 & -36 & 30 & 105 & 165 & 195 & 189 & 150 & 90 & 30 \\ -30 & -30 & -6 & 35 & 85 & 135 & 175 & 194 & 180 & 120 \\ 30 & 0 & -40 & -56 & -30 & 40 & 140 & 240 & 294 & 240 \\ 120 & 54 & -72 & -168 & -180 & -90 & 84 & 288 & 432 & 390 \end{bmatrix}$$

$$D_2 = \frac{1}{858}\begin{bmatrix} 150 & 288 & 336 & 280 & 150 & 0 & -112 & -144 & -90 \\ 120 & 234 & 280 & 245 & 150 & 35 & -56 & -90 & -60 \\ 90 & 180 & 224 & 210 & 150 & 70 & 0 & -36 & -30 \\ 60 & 126 & 168 & 175 & 150 & 105 & 56 & 18 & 0 \\ 30 & 72 & 112 & 140 & 150 & 140 & 112 & 72 & 30 \\ 0 & 18 & 56 & 105 & 150 & 175 & 168 & 126 & 60 \\ -30 & -36 & 0 & 70 & 150 & 210 & 224 & 180 & 90 \\ -60 & -90 & -56 & 35 & 150 & 245 & 280 & 234 & 120 \\ -90 & -144 & -112 & 0 & 150 & 280 & 336 & 288 & 150 \end{bmatrix}$$

$n = 11 \quad r = 3$ (continued)

$$D_3 = \frac{1}{858}\begin{bmatrix} 30 & 84 & 140 & 175 & 175 & 140 & 84 & 30 \\ & " & & " & & " & & \\ & " & & " & & " & & \\ & & & & & \text{8 rows} & & \\ \end{bmatrix}$$

$$D_4 = 0$$

$n = 11 \quad r = 4$

$$C = \frac{1}{429}\begin{bmatrix} 393 & 90 & -30 & -45 & -15 & 18 & 30 & 15 & -15 & -30 & 18 \\ 90 & 177 & 150 & 75 & 0 & -45 & -48 & -15 & 30 & 45 & -30 \\ -30 & 150 & 177 & 125 & 50 & -10 & -35 & -23 & 10 & 30 & -15 \\ -45 & 75 & 125 & 127 & 100 & 60 & 20 & -10 & -23 & -15 & 15 \\ -15 & 0 & 50 & 100 & 127 & 120 & 80 & 20 & -35 & -48 & 30 \\ 18 & -45 & -10 & 60 & 120 & 143 & 120 & 60 & -10 & -45 & 18 \\ 30 & -48 & -35 & 20 & 80 & 120 & 127 & 100 & 50 & 0 & -15 \\ 15 & -15 & -23 & -10 & 20 & 60 & 100 & 127 & 125 & 75 & -45 \\ -15 & 30 & 10 & -23 & -35 & -10 & 50 & 125 & 177 & 150 & -30 \\ -30 & 45 & 30 & -15 & -48 & -45 & 0 & 75 & 150 & 177 & 90 \\ 18 & -30 & -15 & 15 & 30 & 18 & -15 & -45 & -30 & 90 & 393 \end{bmatrix}$$

$$D_1 = \frac{1}{429}\begin{bmatrix} 303 & 216 & 36 & -84 & -99 & -36 & 42 & 72 & 27 & -48 \\ 120 & 147 & 120 & 70 & 20 & -15 & -28 & -20 & 0 & 15 \\ 15 & 90 & 142 & 140 & 90 & 20 & -35 & -48 & -15 & 30 \\ -30 & 45 & 120 & 147 & 120 & 60 & 0 & -30 & -18 & 15 \\ -33 & 12 & 72 & 112 & 119 & 96 & 56 & 16 & -9 & -12 \\ -12 & -9 & 16 & 56 & 96 & 119 & 112 & 72 & 12 & -33 \\ 15 & -18 & -30 & 0 & 60 & 120 & 147 & 120 & 45 & -30 \\ 30 & -15 & 48 & -35 & 20 & 90 & 140 & 142 & 90 & 15 \\ 15 & 0 & -20 & -28 & -15 & 20 & 70 & 120 & 147 & 15 \\ -48 & 27 & 72 & 42 & -36 & -99 & -84 & 36 & 216 & 303 \end{bmatrix}$$

$$D_2 = \frac{1}{429}\begin{bmatrix} 183 & 252 & 168 & 14 & -105 & -126 & -56 & 36 & 63 \\ 105 & 162 & 140 & 70 & 0 & -35 & -28 & 0 & 15 \\ 45 & 90 & 112 & 105 & 75 & 35 & 0 & -18 & -15 \\ 3 & 36 & 84 & 119 & 120 & 84 & 28 & -18 & -27 \\ -21 & 0 & 56 & 112 & 135 & 112 & 56 & 0 & -21 \\ -27 & -18 & 28 & 84 & 120 & 119 & 84 & 36 & 3 \\ -15 & -18 & 0 & 35 & 75 & 105 & 112 & 90 & 45 \\ 15 & 0 & -28 & -35 & 0 & 70 & 140 & 162 & 105 \\ 63 & 36 & -56 & -126 & -105 & 14 & 168 & 252 & 183 \end{bmatrix}$$

$n = 11 \quad r = 4$ (continued)

$$D_3 = \frac{1}{429}\begin{bmatrix} 78 & 168 & 196 & 140 & 35 & -56 & -84 & -48 \\ 60 & 132 & 160 & 125 & 50 & -20 & -48 & -30 \\ 42 & 96 & 124 & 110 & 65 & 16 & -12 & -12 \\ 24 & 60 & 88 & 95 & 80 & 52 & 24 & 6 \\ 6 & 24 & 52 & 80 & 95 & 88 & 60 & 24 \\ -12 & -12 & 16 & 65 & 110 & 124 & 96 & 42 \\ -30 & -48 & -20 & 50 & 125 & 160 & 132 & 60 \\ -48 & -84 & -56 & 35 & 140 & 196 & 168 & 78 \end{bmatrix}$$

$$D_4 = \frac{1}{143}\begin{bmatrix} 6 & 18 & 30 & 35 & 30 & 18 & 6 \\ & & \text{"} & & \text{"} & & \\ & & \text{"} & & \text{"} & & \\ & & & & \text{7 rows.} & & \end{bmatrix}$$

$$D_5 = 0$$

$n = 11 \quad r = 5$

$$C = \frac{1}{1716}\begin{bmatrix} 1671 & 162 & -153 & -48 & 72 & 72 & -12 & -72 & -27 & 78 & -27 \\ 162 & 1104 & 666 & 36 & -264 & -180 & 72 & 204 & 54 & -216 & 78 \\ -153 & 666 & 719 & 456 & 156 & -40 & -96 & -48 & 29 & 54 & -27 \\ -48 & 36 & 456 & 684 & 576 & 240 & -96 & -216 & -48 & 204 & -72 \\ 72 & -264 & 156 & 576 & 684 & 480 & 144 & -96 & -96 & 72 & -12 \\ 72 & -180 & -40 & 240 & 480 & 572 & 480 & 240 & -40 & -180 & 72 \\ -12 & 72 & -96 & -96 & 144 & 480 & 684 & 576 & 156 & -264 & 72 \\ -72 & 204 & -48 & -216 & -96 & 240 & 576 & 684 & 456 & 36 & -48 \\ -27 & 54 & 29 & -48 & -96 & -40 & 156 & 456 & 719 & 666 & -153 \\ 78 & -216 & 54 & 204 & 72 & -180 & -264 & 36 & 666 & 1104 & 162 \\ -27 & 78 & -27 & -72 & -12 & 72 & 72 & -48 & -153 & 162 & 1671 \end{bmatrix}$$

$$D_1 = \frac{1}{1716}\begin{bmatrix} 1509 & 567 & -252 & -336 & 0 & 252 & 168 & -108 & -189 & 105 \\ 315 & 753 & 700 & 280 & -140 & -280 & -112 & 140 & 165 & -105 \\ -105 & 525 & 788 & 560 & 140 & -140 & -140 & 28 & 105 & -45 \\ -120 & 180 & 480 & 588 & 480 & 240 & 0 & -120 & -72 & 60 \\ 0 & -84 & 112 & 448 & 652 & 560 & 224 & -112 & -168 & 84 \\ 84 & -168 & -112 & 224 & 560 & 652 & 448 & 112 & -84 & 0 \\ 60 & -72 & -120 & 0 & 240 & 480 & 588 & 480 & 180 & 120 \\ -45 & 105 & 28 & -140 & -140 & 140 & 560 & 788 & 525 & -105 \\ -105 & 165 & 140 & -112 & -280 & -140 & 280 & 700 & 753 & 315 \\ 105 & -189 & -108 & 168 & 252 & 0 & -336 & -252 & 567 & 1509 \end{bmatrix}$$

$n = 11 \quad r = 5$ (continued)

$$D_2 = \frac{1}{1716}\begin{bmatrix} 1194 & 1008 & 56 & -560 & -420 & 112 & 392 & 144 & -210 \\ 420 & 648 & 560 & 280 & 0 & -140 & -112 & 0 & 60 \\ 15 & 360 & 668 & 640 & 300 & -80 & -220 & -72 & 105 \\ -120 & 144 & 512 & 652 & 480 & 160 & -64 & -72 & 24 \\ -84 & 0 & 224 & 448 & 540 & 448 & 224 & 0 & -84 \\ 24 & -72 & -64 & 160 & 480 & 652 & 512 & 144 & -120 \\ 105 & -72 & -220 & -80 & 300 & 640 & 668 & 360 & 15 \\ 60 & 0 & -112 & -140 & 0 & 280 & 560 & 648 & 420 \\ -210 & 144 & 392 & 112 & -420 & -560 & 56 & 1008 & 1194 \end{bmatrix}$$

$$D_3 = \frac{1}{572}\begin{bmatrix} 258 & 378 & 210 & -70 & -210 & -126 & 42 & 90 \\ 135 & 231 & 195 & 75 & -25 & -45 & -9 & 15 \\ 45 & 117 & 169 & 165 & 105 & 25 & -27 & -27 \\ -12 & 36 & 132 & 200 & 180 & 84 & -12 & -36 \\ -36 & -12 & 84 & 180 & 200 & 132 & 36 & -12 \\ -27 & -27 & 25 & 105 & 165 & 169 & 117 & 45 \\ 15 & -9 & -45 & -25 & 75 & 195 & 231 & 135 \\ 90 & 42 & -126 & -210 & -70 & 210 & 378 & 258 \end{bmatrix}$$

$$D_4 = \frac{1}{572}\begin{bmatrix} 123 & 270 & 285 & 140 & -45 & -126 & -75 \\ 90 & 204 & 230 & 140 & 10 & -60 & -42 \\ 57 & 138 & 175 & 140 & 65 & 6 & -9 \\ 24 & 72 & 120 & 140 & 120 & 72 & 24 \\ -9 & 6 & 65 & 140 & 175 & 138 & 57 \\ -42 & -60 & 10 & 140 & 230 & 204 & 90 \\ -75 & -126 & -45 & 140. & 285 & 270 & 123 \end{bmatrix}$$

$$D_5 = \frac{1}{52}\begin{bmatrix} 3 & 9 & 14 & 14 & 9 & 3 \\ & " & & " & & \\ & " & & " & & \\ & & & \text{6 rows} & & \end{bmatrix}$$

$$D_6 = 0$$

$n = 12 \quad r = 0$

$$C = \frac{1}{12} Q$$

$$D_1 = 0$$

$n = 12 \quad r = 1$

$$C = \frac{1}{858}\begin{bmatrix} 253 & 220 & 187 & 154 & 121 & 88 & 55 & 22 & -11 & -44 & -77 & -110 \\ 220 & 193 & 166 & 139 & 112 & 85 & 58 & 31 & 4 & -23 & -50 & -77 \\ 187 & 166 & 145 & 124 & 103 & 82 & 61 & 40 & 19 & -2 & -23 & -44 \\ 154 & 139 & 124 & 109 & 94 & 79 & 64 & 49 & 34 & 19 & 4 & -11 \\ 121 & 112 & 103 & 94 & 85 & 76 & 67 & 58 & 49 & 40 & 31 & 22 \\ 88 & 85 & 82 & 79 & 76 & 73 & 70 & 67 & 64 & 61 & 58 & 55 \\ 55 & 58 & 61 & 64 & 67 & 70 & 73 & 76 & 79 & 82 & 85 & 88 \\ 22 & 31 & 40 & 49 & 58 & 67 & 76 & 85 & 94 & 103 & 112 & 121 \\ -11 & 4 & 19 & 34 & 49 & 64 & 79 & 94 & 109 & 124 & 139 & 154 \\ -44 & -23 & -2 & 19 & 40 & 61 & 82 & 103 & 124 & 145 & 166 & 187 \\ -77 & -50 & -23 & 4 & 31 & 58 & 85 & 112 & 139 & 166 & 193 & 220 \\ -110 & -77 & -44 & -11 & 22 & 55 & 88 & 121 & 154 & 187 & 220 & 253 \end{bmatrix}$$

$$D_1 = \frac{1}{286}\begin{bmatrix} 11 & 20 & 27 & 32 & 35 & 36 & 35 & 32 & 27 & 20 & 11 \\ & & '' & & & '' & & & '' & & \\ & & '' & & & '' & & & '' & & \\ & & & & & & \text{11 rows.} & & & & \end{bmatrix}$$

$$D_2 = 0 .$$

$n = 12 \quad r = 2$

$$C = \frac{1}{4004}\begin{bmatrix} 2189 & 1485 & 891 & 407 & 33 & -231 & -385 & -429 & -363 & -187 & 99 & 495 \\ 1485 & 1109 & 783 & 507 & 281 & 105 & -21 & -97 & -123 & -99 & -25 & 99 \\ 891 & 783 & 677 & 573 & 471 & 371 & 273 & 177 & 83 & -9 & -99 & -187 \\ 407 & 507 & 573 & 605 & 603 & 567 & 497 & 393 & 255 & 83 & -123 & -363 \\ 33 & 281 & 471 & 603 & 677 & 693 & 651 & 551 & 393 & 177 & -97 & -429 \\ -231 & 105 & 371 & 567 & 693 & 749 & 735 & 651 & 497 & 273 & -21 & -385 \\ -385 & -21 & 273 & 497 & 651 & 735 & 749 & 693 & 567 & 371 & 105 & -231 \\ -429 & -97 & 177 & 393 & 551 & 651 & 693 & 677 & 603 & 471 & 281 & 33 \\ -363 & -123 & 83 & 255 & 393 & 497 & 567 & 603 & 605 & 573 & 507 & 407 \\ -187 & -99 & -9 & 83 & 177 & 273 & 371 & 471 & 573 & 677 & 783 & 891 \\ 99 & -25 & -99 & -123 & -97 & -21 & 105 & 281 & 507 & 783 & 1109 & 1485 \\ 495 & 99 & -187 & -363 & -429 & -385 & -231 & 33 & 407 & 891 & 1485 & 2189 \end{bmatrix}$$

$n = 12 \quad r = 2$ (continued.)

$$D_1 = \frac{1}{2002}\begin{bmatrix}
352 & 540 & 594 & 544 & 420 & 252 & 70 & -96 & -216 & -260 & -198 \\
297 & 460 & 513 & 480 & 385 & 252 & 105 & -32 & -135 & -180 & -143 \\
242 & 380 & 432 & 416 & 350 & 252 & 140 & 32 & -54 & -100 & -88 \\
187 & 300 & 351 & 352 & 315 & 252 & 175 & 96 & 27 & -20 & -33 \\
132 & 220 & 270 & 288 & 280 & 252 & 210 & 160 & 108 & 60 & 22 \\
77 & 140 & 189 & 224 & 245 & 252 & 245 & 224 & 189 & 140 & 77 \\
22 & 60 & 108 & 160 & 210 & 252 & 280 & 288 & 270 & 220 & 132 \\
-33 & -20 & 27 & 96 & 175 & 252 & 315 & 352 & 351 & 300 & 187 \\
-88 & -100 & -54 & 32 & 140 & 252 & 350 & 416 & 432 & 380 & 242 \\
-143 & -180 & -135 & -32 & 105 & 252 & 385 & 480 & 513 & 460 & 297 \\
-198 & -260 & -216 & -96 & 70 & 252 & 420 & 544 & 594 & 540 & 352
\end{bmatrix}$$

$$D_2 = \frac{1}{2002}\begin{bmatrix}
55 & 135 & 216 & 280 & 315 & 315 & 280 & 216 & 135 & 55 \\
 & \text{"} & & & \text{"} & & & \text{"} & & \\
 & \text{"} & & & \text{"} & & & \text{"} & & \\
 & & & & & & & \text{10 rows.} & &
\end{bmatrix}$$

$$D_3 = 0$$

$n = 12 \quad r = 3$

$$C = \frac{1}{9009}\begin{bmatrix}
6831 & 3168 & 792 & -528 & -1023 & -924 & -462 & 132 & 627 & 792 & 396 & -792 \\
3168 & 2511 & 1872 & 1272 & 732 & 273 & -84 & -318 & -408 & -333 & -72 & 396 \\
792 & 1872 & 2295 & 2208 & 1758 & 1092 & 357 & -300 & -732 & -792 & -333 & 792 \\
-528 & 1272 & 2208 & 2455 & 2188 & 1582 & 812 & 53 & -520 & -732 & -408 & 627 \\
-1023 & 732 & 1758 & 2188 & 2155 & 1792 & 1232 & 608 & 53 & -300 & -318 & 132 \\
-924 & 273 & 1092 & 1582 & 1792 & 1771 & 1568 & 1232 & 812 & 357 & -84 & -462 \\
-462 & -84 & 357 & 812 & 1232 & 1568 & 1771 & 1792 & 1582 & 1092 & 273 & -924 \\
132 & -318 & -300 & 53 & 608 & 1232 & 1792 & 2155 & 2188 & 1758 & 732 & -1023 \\
627 & -408 & -732 & -520 & 53 & 812 & 1582 & 2188 & 2455 & 2208 & 1272 & -528 \\
792 & -333 & -792 & -732 & -300 & 357 & 1092 & 1758 & 2208 & 2295 & 1872 & 792 \\
396 & -72 & -333 & -408 & -318 & -84 & 273 & 732 & 1272 & 1872 & 2511 & 3168 \\
-792 & 396 & 792 & 627 & 132 & -462 & -924 & -1023 & -528 & 792 & 3168 & 6831
\end{bmatrix}$$

$n = 12 \quad r = 3$ (continued.)

$$D_1 = \frac{1}{9009}\begin{bmatrix}
3663 & 4320 & 3240 & 1440 & -315 & -1512 & -1890 & -1440 & -405 & 720 & 1188 \\
2376 & 3015 & 2592 & 1656 & 630 & -189 & -630 & -648 & -324 & 135 & 396 \\
1320 & 1920 & 2007 & 1760 & 1330 & 840 & 385 & 32 & -180 & -240 & -165 \\
495 & 1035 & 1485 & 1752 & 1785 & 1575 & 1155 & 600 & 27 & -405 & -495 \\
-99 & 360 & 1026 & 1632 & 1995 & 2016 & 1680 & 1056 & 297 & -360 & -594 \\
-462 & -105 & 630 & 1400 & 1960 & 2163 & 1960 & 1400 & 630 & -105 & -462 \\
-594 & -360 & 297 & 1056 & 1680 & 2016 & 1995 & 1632 & 1026 & 360 & -99 \\
-495 & -405 & 27 & 600 & 1155 & 1575 & 1785 & 1752 & 1485 & 1035 & 495 \\
-165 & -240 & -180 & 32 & 385 & 840 & 1330 & 1760 & 2007 & 1920 & 1320 \\
396 & 135 & -324 & -648 & -630 & -189 & 630 & 1656 & 2592 & 3015 & 2376 \\
1188 & 720 & -405 & -1440 & -1890 & -1512 & -315 & 1440 & 3240 & 4320 & 3663
\end{bmatrix}$$

$$D_2 = \frac{1}{9009}\begin{bmatrix}
1287 & 2592 & 3240 & 3024 & 2079 & 756 & -504 & -1296 & -1377 & -792 \\
1056 & 2151 & 2736 & 2632 & 1932 & 903 & -112 & -792 & -936 & -561 \\
825 & 1710 & 2232 & 2240 & 1785 & 1050 & 280 & -288 & -495 & -330 \\
594 & 1269 & 1728 & 1848 & 1638 & 1197 & 672 & 216 & -54 & -99 \\
363 & 828 & 1224 & 1456 & 1491 & 1344 & 1064 & 720 & 387 & 132 \\
132 & 387 & 720 & 1064 & 1344 & 1491 & 1456 & 1224 & 828 & 363 \\
-99 & -54 & 216 & 672 & 1197 & 1638 & 1848 & 1728 & 1269 & 594 \\
-330 & -495 & -288 & 280 & 1050 & 1785 & 2240 & 2232 & 1710 & 825 \\
-561 & -936 & -792 & -112 & 903 & 1932 & 2632 & 2736 & 2151 & 1056 \\
-792 & -1377 & -1296 & -504 & 756 & 2079 & 3024 & 3240 & 2592 & 1287
\end{bmatrix}$$

$$D_3 = \frac{1}{1287}\begin{bmatrix}
33 & 96 & 168 & 224 & 245 & 224 & 168 & 96 & 33 \\
 & '' & & & '' & & '' & & \\
 & '' & & & '' & & '' & & \\
 & & & & & & \text{9 rows} & &
\end{bmatrix}$$

$$D_4 = 0$$

$n = 12 \quad r = 4$

$$C = \frac{1}{10296}\begin{bmatrix}
9207 & 2475 & -495 & -1155 & -660 & 132 & 660 & 660 & 165 & -495 & -693 & 495 \\
2475 & 3807 & 3285 & 1905 & 420 & -660 & -1068 & -780 & -15 & 765 & 855 & -693 \\
-495 & 3285 & 4023 & 3075 & 1500 & 60 & -780 & -852 & -285 & 495 & 765 & -495 \\
-1155 & 1905 & 3075 & 3023 & 2300 & 1340 & 460 & -140 & -377 & -285 & -15 & 165 \\
-660 & 420 & 1500 & 2300 & 2648 & 2480 & 1840 & 880 & -140 & -852 & -780 & 660 \\
132 & -660 & 60 & 1340 & 2480 & 3032 & 2800 & 1840 & 460 & -780 & -1068 & 660 \\
660 & -1068 & -780 & 460 & 1840 & 2800 & 3032 & 2480 & 1340 & 60 & -660 & 132 \\
660 & -780 & -852 & -140 & 880 & 1840 & 2480 & 2648 & 2300 & 1500 & 420 & -660 \\
165 & -15 & -285 & -377 & -140 & 460 & 1340 & 2300 & 3023 & 3075 & 1905 & -1155 \\
-495 & 765 & 495 & -285 & -852 & -780 & 60 & 1500 & 3075 & 4023 & 3285 & -495 \\
-693 & 855 & 765 & -15 & -780 & -1068 & -660 & 420 & 1905 & 3285 & 3807 & 2475 \\
495 & -693 & -495 & 165 & 660 & 660 & 132 & -660 & -1155 & -495 & 2475 & 9207
\end{bmatrix}$$

$n = 12 \quad r = 4$ (continued)

$$D_1 = \frac{1}{10296}\begin{bmatrix}
6732 & 5400 & 1620 & -1440 & -2520 & -1728 & 0 & 1440 & 1620 & 360 & -1188 \\
2970 & 3492 & 2754 & 1584 & 504 & -216 & -504 & -432 & -162 & 108 & 198 \\
660 & 2040 & 2988 & 3040 & 2240 & 960 & -280 & -992 & -900 & -120 & 660 \\
-495 & 990 & 2565 & 3288 & 2940 & 1800 & 420 & -600 & -837 & -270 & 495 \\
-792 & 288 & 1728 & 2688 & 2856 & 2304 & 1344 & 384 & -216 & -288 & 0 \\
-528 & -120 & 720 & 1600 & 2240 & 2472 & 2240 & 1600 & 720 & -120 & -528 \\
0 & -288 & -216 & 384 & 1344 & 2304 & 2856 & 2688 & 1728 & 288 & -792 \\
495 & -270 & -837 & -600 & 420 & 1800 & 2940 & 3288 & 2565 & 990 & -495 \\
660 & -120 & -900 & -992 & -280 & 960 & 2240 & 3040 & 2988 & 2040 & 660 \\
198 & 108 & -162 & -432 & -504 & -216 & 504 & 1584 & 2754 & 3492 & 2970 \\
-1188 & 360 & 1620 & 1440 & 0 & -1728 & -2520 & -1440 & 1620 & 5400 & 6732
\end{bmatrix}$$

$$D_2 = \frac{1}{10296}\begin{bmatrix}
3762 & 5670 & 4536 & 1512 & -1512 & -3024 & -2520 & -648 & 1134 & 1386 \\
2310 & 3762 & 3528 & 2072 & 336 & -840 & -1064 & -504 & 234 & 462 \\
1155 & 2205 & 2628 & 2380 & 1680 & 840 & 140 & -252 & -315 & -165 \\
297 & 999 & 1836 & 2436 & 2520 & 2016 & 1092 & 108 & -513 & -495 \\
-264 & 144 & 1152 & 2240 & 2856 & 2688 & 1792 & 576 & -360 & -528 \\
-528 & -360 & 576 & 1792 & 2688 & 2856 & 2240 & 1152 & 144 & -264 \\
-495 & -513 & 108 & 1092 & 2016 & 2520 & 2436 & 1836 & 999 & 297 \\
-165 & -315 & -252 & 140 & 840 & 1680 & 2380 & 2628 & 2205 & 1155 \\
462 & 234 & -504 & -1064 & -840 & 336 & 2072 & 3528 & 3762 & 2310 \\
1386 & 1134 & -648 & -2520 & -3024 & -1512 & 1512 & 4536 & 5670 & 3762
\end{bmatrix}$$

$$D_3 = \frac{1}{10296}\begin{bmatrix}
1452 & 3360 & 4368 & 3808 & 1960 & -224 & -1680 & -1824 & -924 \\
1155 & 2712 & 3612 & 3304 & 1960 & 280 & -924 & -1176 & -627 \\
858 & 2064 & 2856 & 2800 & 1960 & 784 & -168 & -528 & -330 \\
561 & 1416 & 2100 & 2296 & 1960 & 1288 & 588 & 120 & -33 \\
264 & 768 & 1344 & 1792 & 1960 & 1792 & 1344 & 768 & 264 \\
-33 & 120 & 588 & 1288 & 1960 & 2296 & 2100 & 1416 & 561 \\
-330 & -528 & -168 & 784 & 1960 & 2800 & 2856 & 2064 & 858 \\
-627 & -1176 & -924 & 280 & 1960 & 3304 & 3612 & 2712 & 1155 \\
-924 & -1824 & -1680 & -224 & 1960 & 3808 & 4368 & 3360 & 1452
\end{bmatrix}$$

$$D_4 = \frac{1}{10296}\begin{bmatrix}
297 & 945 & 1701 & 2205 & 2205 & 1701 & 945 & 297 \\
 & \text{''} & & & \text{''} & & & \text{''} \\
 & \text{''} & & & \text{''} & & & \text{''} \\
 & & & & & \text{8 rows} & &
\end{bmatrix}$$

$$D_5 = 0$$

$$n = 12 \qquad r = 5$$

$$C = \frac{1}{9724}\begin{bmatrix}
9361 & 1188 & -891 & -506 & 264 & 528 & 220 & -264 & -429 & -44 & 495 & -198 \\
1188 & 5581 & 3834 & 789 & -1136 & -1320 & -312 & 796 & 996 & -9 & -1178 & 495 \\
-891 & 3834 & 4069 & 2532 & 852 & -200 & -480 & -240 & 103 & 198 & -9 & -44 \\
-506 & 789 & 2532 & 3369 & 2952 & 1620 & 80 & -912 & -870 & 103 & 996 & -429 \\
264 & -1136 & 852 & 2952 & 3684 & 2880 & 1200 & -352 & -912 & -240 & 796 & -264 \\
528 & -1320 & -200 & 1620 & 2880 & 3108 & 2400 & 1200 & 80 & -480 & -312 & 220 \\
220 & -312 & -480 & 80 & 1200 & 2400 & 3108 & 2880 & 1620 & -200 & -1320 & 528 \\
-264 & 796 & -240 & -912 & -352 & 1200 & 2880 & 3684 & 2952 & 852 & -1136 & 264 \\
-429 & 996 & 103 & -870 & -912 & 80 & 1620 & 2952 & 3369 & 2532 & 789 & -506 \\
-44 & -9 & 198 & 103 & -240 & -480 & -200 & 852 & 2532 & 4069 & 3834 & -891 \\
495 & -1178 & -9 & 996 & 796 & -312 & -1320 & -1136 & 789 & 3834 & 5581 & 1188 \\
-198 & 495 & -44 & -429 & -264 & 220 & 528 & 264 & -506 & -891 & 1188 & 9361
\end{bmatrix}$$

$$D_1 = \frac{1}{9724}\begin{bmatrix}
8173 & 3780 & -945 & -2240 & -840 & 1008 & 1540 & 480 & -945 & -980 & 693 \\
2079 & 3826 & 3591 & 1848 & -140 & -1260 & -1092 & -56 & 837 & 630 & -539 \\
-385 & 2660 & 4197 & 3360 & 1260 & -560 & -1120 & -448 & 525 & 620 & -385 \\
-770 & 1155 & 2835 & 3252 & 2520 & 1260 & 140 & -420 & -378 & -35 & 165 \\
-264 & -80 & 972 & 2304 & 3108 & 2880 & 1680 & 128 & -864 & -624 & 484 \\
308 & -700 & -420 & 1120 & 2800 & 3508 & 2800 & 1120 & -420 & -700 & 308 \\
484 & -624 & -864 & 128 & 1680 & 2880 & 3108 & 2304 & 972 & -80 & -264 \\
165 & -35 & -378 & -420 & 140 & 1260 & 2520 & 3252 & 2835 & 1155 & -770 \\
-385 & 620 & 525 & -448 & -1120 & -560 & 1260 & 3360 & 4197 & 2660 & -385 \\
-539 & 630 & 837 & -56 & -1092 & -1260 & -140 & 1848 & 3591 & 3826 & 2079 \\
693 & -980 & -945 & 480 & 1540 & 1008 & -840 & -2240 & -945 & -3780 & 8173
\end{bmatrix}$$

$$D_2 = \frac{1}{9724}\begin{bmatrix}
6094 & 6048 & 1512 & -2576 & -3276 & -1008 & 1624 & 2160 & 378 & -1232 \\
2464 & 3630 & 3024 & 1512 & 112 & -588 & -560 & -168 & 144 & 154 \\
385 & 1890 & 3252 & 3360 & 2100 & 280 & -980 & -1008 & -105 & 550 \\
-506 & 729 & 2592 & 3540 & 2952 & 1332 & -208 & -756 & -270 & 319 \\
-572 & 48 & 1440 & 2624 & 2932 & 2304 & 1184 & 192 & -252 & -176 \\
-176 & -252 & 192 & 1184 & 2304 & 2932 & 2624 & 1440 & 48 & -572 \\
319 & -270 & -756 & -208 & 1332 & 2952 & 3540 & 2592 & 729 & -506 \\
550 & -105 & -1008 & -980 & 280 & 2100 & 3360 & 3252 & 1890 & 385 \\
154 & 144 & -168 & -560 & -588 & 112 & 1512 & 3024 & 3630 & 2464 \\
-1232 & 378 & 2160 & 1624 & -1008 & 3276 & -2576 & 1512 & 6048 & 6094
\end{bmatrix}$$

$$D_3 = \frac{1}{9724}\begin{bmatrix}
3630 & 6048 & 4536 & 448 & -2940 & -3360 & -1176 & 1152 & 1386 \\
2079 & 3819 & 3591 & 1743 & -245 & -1113 & -693 & 147 & 396 \\
891 & 2052 & 2712 & 2532 & 1680 & 628 & -144 & -396 & -231 \\
66 & 747 & 1899 & 2815 & 2835 & 1863 & 471 & -477 & -495 \\
-396 & -96 & 1152 & 2592 & 3220 & 2592 & 1152 & -96 & -396 \\
-495 & -477 & 471 & 1863 & 2835 & 2815 & 1899 & 747 & 66 \\
-231 & -396 & -144 & 628 & 1680 & 2532 & 2712 & 2052 & 891 \\
396 & 147 & -693 & -1113 & -245 & 1743 & 3591 & 3819 & 2079 \\
1386 & 1152 & -1176 & -3360 & -2940 & 448 & 4536 & 6048 & 3630
\end{bmatrix}$$

$n = 12 \quad r = 5$ (continued.)

$$D_4 = \frac{1}{9724}\begin{bmatrix} 1551 & 3780 & 4725 & 3430 & 735 & -1512 & -1995 & 990 \\ 1188 & 2955 & 3834 & 3045 & 1120 & -621 & -1170 & -627 \\ 825 & 2130 & 2943 & 2660 & 1505 & 270 & -345 & -264 \\ 462 & 1305 & 2052 & 2275 & 1890 & 1161 & 480 & 99 \\ 99 & 480 & 1161 & 1890 & 2275 & 2052 & 1305 & 462 \\ -264 & -345 & 270 & 1505 & 2660 & 2943 & 2130 & 825 \\ -627 & -1170 & -621 & 1120 & 3045 & 3834 & 2955 & 1188 \\ 990 & -1995 & -1512 & 735 & 3430 & 4725 & 3780 & 1551 \end{bmatrix}$$

$$D_5 = \frac{1}{884}\begin{bmatrix} 33 & 108 & 189 & 224 & 189 & 108 & 33 \\ & '' & & '' & & '' & \\ & '' & & '' & & '' & \\ & & & & \text{7 rows} & & \end{bmatrix}$$

$$D_6 = 0$$

$n = 13 \quad r = 0$

$$C = \frac{1}{13} Q$$

$$D_1 = 0$$

$n = 13 \quad r = 1$

$$C = \frac{1}{182}\begin{bmatrix} 50 & 44 & 38 & 32 & 26 & 20 & 14 & 8 & 2 & -4 & -10 & -16 & -22 \\ 44 & 39 & 34 & 29 & 24 & 19 & 14 & 9 & 4 & -1 & -6 & -11 & -16 \\ 38 & 34 & 30 & 26 & 22 & 18 & 14 & 10 & 6 & 2 & -2 & -6 & -10 \\ 32 & 29 & 26 & 23 & 20 & 17 & 14 & 11 & 8 & 5 & 2 & -1 & -4 \\ 26 & 24 & 22 & 20 & 18 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 \\ 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 \\ 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 \\ 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 \\ -4 & -1 & 2 & 5 & 8 & 11 & 14 & 17 & 20 & 23 & 26 & 29 & 32 \\ -10 & -6 & -2 & 2 & 6 & 10 & 14 & 18 & 22 & 26 & 30 & 34 & 38 \\ -16 & -11 & -6 & -1 & 4 & 9 & 14 & 19 & 24 & 29 & 34 & 39 & 44 \\ -22 & -16 & -10 & -4 & 2 & 8 & 14 & 20 & 26 & 32 & 38 & 44 & 50 \end{bmatrix}$$

$$D_1 = \frac{1}{182}\begin{bmatrix} 6 & 11 & 15 & 18 & 20 & 21 & 21 & 20 & 18 & 15 & 11 & 6 \\ & '' & & & & '' & & & '' & & & \\ & '' & & & & '' & & & '' & & & \\ & & & & & & & \text{12 rows} & & & & \end{bmatrix}$$

$$D_2 = 0$$

$n = 13 \quad r = 2$

$$C = \frac{1}{1001}\begin{bmatrix}
517 & 363 & 231 & 121 & 33 & -33 & -77 & -99 & -99 & -77 & -33 & 33 & 121 \\
363 & 275 & 198 & 132 & 77 & 33 & 0 & -22 & -33 & -33 & -22 & 0 & 33 \\
231 & 198 & 167 & 138 & 111 & 86 & 63 & 42 & 23 & 6 & -9 & -22 & -33 \\
121 & 132 & 138 & 139 & 135 & 126 & 112 & 93 & 69 & 40 & 6 & -33 & -77 \\
33 & 77 & 111 & 135 & 149 & 153 & 147 & 131 & 105 & 69 & 23 & -33 & -99 \\
-33 & 33 & 86 & 126 & 153 & 167 & 168 & 156 & 131 & 93 & 42 & -22 & -99 \\
-77 & 0 & 63 & 112 & 147 & 168 & 175 & 168 & 147 & 112 & 63 & 0 & -77 \\
-99 & -22 & 42 & 93 & 131 & 156 & 168 & 167 & 153 & 126 & 86 & 33 & -33 \\
-99 & -33 & 23 & 69 & 105 & 131 & 147 & 153 & 149 & 135 & 111 & 77 & 33 \\
-77 & -33 & 6 & 40 & 69 & 93 & 112 & 126 & 135 & 139 & 138 & 132 & 121 \\
-33 & -22 & -9 & 6 & 23 & 42 & 63 & 86 & 111 & 138 & 167 & 198 & 231 \\
33 & 0 & -22 & -33 & -33 & -22 & 0 & 33 & 77 & 132 & 198 & 275 & 363 \\
121 & 33 & -33 & -77 & -99 & -99 & -77 & -33 & 33 & 121 & 231 & 363 & 517
\end{bmatrix}$$

$$D_1 = \frac{1}{1001}\begin{bmatrix}
154 & 242 & 275 & 264 & 220 & 154 & 77 & 0 & -66 & -110 & -121 & -88 \\
132 & 209 & 240 & 234 & 200 & 147 & 84 & 20 & -36 & -75 & -88 & -66 \\
110 & 176 & 205 & 204 & 180 & 140 & 91 & 40 & -6 & -40 & -55 & -44 \\
88 & 143 & 170 & 174 & 160 & 133 & 98 & 60 & 24 & -5 & -22 & -22 \\
66 & 110 & 135 & 144 & 140 & 126 & 105 & 80 & 54 & 30 & 11 & 0 \\
44 & 77 & 100 & 114 & 120 & 119 & 112 & 100 & 84 & 65 & 44 & 22 \\
22 & 44 & 65 & 84 & 100 & 112 & 119 & 120 & 114 & 100 & 77 & 44 \\
0 & 11 & 30 & 54 & 80 & 105 & 126 & 140 & 144 & 135 & 110 & 66 \\
-22 & -22 & -5 & 24 & 60 & 98 & 133 & 160 & 174 & 170 & 143 & 88 \\
-44 & -55 & -40 & -6 & 40 & 91 & 140 & 180 & 204 & 205 & 176 & 110 \\
-66 & -88 & -75 & -36 & 20 & 84 & 147 & 200 & 234 & 240 & 209 & 132 \\
-88 & -121 & -110 & -66 & 0 & 77 & 154 & 220 & 264 & 275 & 242 & 154
\end{bmatrix}$$

$$D_2 = \frac{1}{1001}\begin{bmatrix}
22 & 55 & 90 & 120 & 140 & 147 & 140 & 120 & 90 & 55 & 22 \\
 & \text{''} & & & & \text{''} & & & \text{''} & & \\
 & \text{''} & & & & \text{''} & & & \text{''} & & \\
 & & & & & \text{11 rows.} & & & & &
\end{bmatrix}$$

$$D_3 = 0$$

$n = 13 \quad r = 3$

$$C = \frac{1}{4004}\begin{bmatrix} 2915 & 1452 & 462 & -132 & -407 & -440 & -308 & -88 & 143 & 308 & 330 & 132 & -363 \\ 1452 & 1100 & 792 & 528 & 308 & 132 & 0 & -88 & -132 & -132 & -88 & 0 & 132 \\ 462 & 792 & 920 & 888 & 738 & 512 & 252 & 0 & -202 & -312 & -288 & -88 & 330 \\ -132 & 528 & 888 & 1004 & 932 & 728 & 448 & 148 & -116 & -288 & -312 & -132 & 308 \\ -407 & 308 & 738 & 932 & 939 & 808 & 588 & 328 & 77 & -116 & -202 & -132 & 143 \\ -440 & 132 & 512 & 728 & 808 & 780 & 672 & 512 & 328 & 148 & 0 & -88 & -88 \\ -308 & 0 & 252 & 448 & 588 & 672 & 700 & 672 & 588 & 448 & 252 & 0 & -308 \\ -88 & -88 & 0 & 148 & 328 & 512 & 672 & 780 & 808 & 728 & 512 & 132 & -440 \\ 143 & -132 & -202 & -116 & 77 & 328 & 588 & 808 & 939 & 932 & 738 & 308 & -407 \\ 308 & -132 & -312 & -288 & -116 & 148 & 448 & 728 & 932 & 1004 & 888 & 528 & -132 \\ 330 & -88 & -288 & -312 & -202 & 0 & 252 & 512 & 738 & 888 & 920 & 792 & 462 \\ 132 & 0 & -88 & -132 & -132 & -88 & 0 & 132 & 308 & 528 & 792 & 1100 & 1452 \\ -363 & 132 & 330 & 308 & 143 & -88 & -308 & -440 & -407 & -132 & 462 & 1452 & 2915 \end{bmatrix}$$

$$D_1 = \frac{1}{4004}\begin{bmatrix} 1463 & 1815 & 1485 & 825 & 110 & -462 & -770 & -770 & -495 & -55 & 363 & 495 \\ 990 & 1298 & 1170 & 810 & 380 & 0 & -252 & -340 & -270 & -90 & 110 & 198 \\ 594 & 858 & 890 & 774 & 580 & 364 & 168 & 20 & -66 & -90 & -66 & -22 \\ 275 & 495 & 645 & 717 & 710 & 630 & 490 & 310 & 117 & -55 & -165 & -165 \\ 33 & 209 & 435 & 639 & 770 & 798 & 714 & 530 & 279 & 15 & -187 & -231 \\ -132 & 0 & 260 & 540 & 760 & 868 & 840 & 680 & 420 & 120 & -132 & -220 \\ -220 & -132 & 120 & 420 & 680 & 840 & 868 & 760 & 540 & 260 & 0 & -132 \\ -231 & -187 & 15 & 279 & 530 & 714 & 798 & 770 & 639 & 435 & 209 & 33 \\ -165 & -165 & -55 & 117 & 310 & 490 & 630 & 710 & 717 & 645 & 495 & 275 \\ -22 & -66 & -90 & -66 & 20 & 168 & 364 & 580 & 774 & 890 & 858 & 594 \\ 198 & 110 & -90 & -270 & -340 & -252 & 0 & 380 & 810 & 1170 & 1298 & 990 \\ 495 & 363 & -55 & -495 & -770 & -770 & -462 & 110 & 825 & 1485 & 1815 & 1463 \end{bmatrix}$$

$$D_2 = \frac{1}{4004}\begin{bmatrix} 473 & 990 & 1305 & 1320 & 1050 & 588 & 70 & -360 & -585 & -550 & -297 \\ 396 & 836 & 1116 & 1152 & 952 & 588 & 168 & -192 & -396 & -396 & -220 \\ 319 & 682 & 927 & 984 & 854 & 588 & 266 & -24 & -207 & -242 & -143 \\ 242 & 528 & 738 & 816 & 756 & 588 & 364 & 144 & -18 & -88 & -66 \\ 165 & 374 & 549 & 648 & 658 & 588 & 462 & 312 & 171 & 66 & 11 \\ 88 & 220 & 360 & 480 & 560 & 588 & 560 & 480 & 360 & 220 & 88 \\ 11 & 66 & 171 & 312 & 462 & 588 & 658 & 648 & 549 & 374 & 165 \\ -66 & -88 & -18 & 144 & 364 & 588 & 756 & 816 & 738 & 528 & 242 \\ -143 & -242 & -207 & -24 & 266 & 588 & 854 & 984 & 927 & 682 & 319 \\ -220 & -396 & -396 & -192 & 168 & 588 & 952 & 1152 & 1116 & 836 & 396 \\ -297 & -550 & -585 & -360 & 70 & 588 & 1050 & 1320 & 1305 & 990 & 473 \end{bmatrix}$$

$$D_3 = \frac{1}{572}\begin{bmatrix} 11 & 33 & 60 & 84 & 98 & 98 & 84 & 60 & 33 & 11 \\ & \text{"} & & & \text{"} & & & \text{"} & & \\ & \text{"} & & & \text{"} & & & \text{"} & & \\ & & & & & & \text{10 rows.} & & & \end{bmatrix}$$

$$D_4 = 0$$

$n = 13 \quad r = 4.$

$$C = \frac{1}{34034}\begin{bmatrix} 29678 & 9075 & -825 & -3795 & -2915 & -572 & 1540 & 2420 & 1760 & -55 & -1947 & -2145 & 1815 \\ 9075 & 11528 & 9900 & 6270 & 2255 & -990 & -2772 & -2860 & -1485 & 660 & 2420 & 2178 & -2145 \\ -825 & 9900 & 12428 & 10140 & 5745 & 1280 & -1890 & -3072 & -2245 & -60 & 2160 & 2420 & -1947 \\ -3795 & 6270 & 10140 & 9992 & 7625 & 4460 & 1540 & -470 & -1283 & -990 & -60 & 660 & -55 \\ -2915 & 2255 & 5745 & 7625 & 8042 & 7220 & 5460 & 3140 & 715 & -1283 & -2245 & -1485 & 1760 \\ -572 & -990 & 1280 & 4460 & 7220 & 8678 & 8400 & 6400 & 3140 & -470 & -3072 & -2860 & 2420 \\ 1540 & -2772 & -1890 & 1540 & 5460 & 8400 & 9478 & 8400 & 5460 & 1540 & -1890 & -2772 & 1540 \\ 2420 & -2860 & -3072 & -470 & 3140 & 6400 & 8400 & 8678 & 7220 & 4460 & 1280 & -990 & -572 \\ 1760 & -1485 & 2245 & -1283 & 715 & 3140 & 5460 & 7220 & 8042 & 7625 & 5745 & 2255 & -2915 \\ -55 & 660 & -60 & -990 & -1283 & -470 & 1540 & 4460 & 7625 & 9992 & 10140 & 6270 & -3795 \\ -1947 & 2420 & 2160 & -60 & -2245 & -3072 & -1890 & 1280 & 5745 & 10140 & 12428 & 9900 & -825 \\ -2145 & 2178 & 2420 & 660 & -1485 & -2860 & -2772 & -990 & 2255 & 6270 & 9900 & 11528 & 9075 \\ 1815 & -2145 & -1947 & -55 & 1760 & 2420 & 1540 & -572 & -2915 & -3795 & -825 & 9075 & 29678 \end{bmatrix}$$

$$D_1 = \frac{1}{34034}\begin{bmatrix} 20603 & 18150 & 7425 & -2640 & -7810 & -7392 & -3080 & 2200 & 5445 & 4730 & 363 & -3960 \\ 9900 & 11528 & 9000 & 5130 & 1640 & -630 & -1512 & -1300 & -540 & 180 & 440 & 198 \\ 2970 & 6600 & 8888 & 9036 & 7156 & 3976 & 546 & -2056 & -3018 & -2088 & 132 & 1892 \\ -880 & 3135 & 7530 & 9897 & 9480 & 6720 & 2800 & -810 & -2808 & -2515 & -330 & 1815 \\ -2343 & 902 & 5367 & 8532 & 9354 & 7896 & 4956 & 1696 & -729 & -1542 & -715 & 660 \\ -2112 & -330 & 2840 & 5760 & 7520 & 7798 & 6720 & 4720 & 2400 & 390 & -792 & -880 \\ -880 & -792 & 390 & 2400 & 4720 & 6720 & 7798 & 7520 & 5760 & 2840 & -330 & -2112 \\ 660 & -715 & -1542 & -729 & 1696 & 4956 & 7896 & 9354 & 8532 & 5367 & 902 & -2343 \\ 1815 & -330 & -2515 & -2808 & -810 & 2800 & 6720 & 9480 & 9897 & 7530 & 3135 & -880 \\ 1892 & 132 & -2088 & -3018 & -2056 & 546 & 3976 & 7156 & 9036 & 8888 & 6600 & 2970 \\ 198 & 440 & 180 & -540 & -1300 & -1512 & -630 & 1640 & 5130 & 9000 & 11528 & 9900 \\ -3960 & 363 & 4730 & 5445 & 2200 & -3080 & -7392 & -7810 & -2640 & 7425 & 18150 & 20603 \end{bmatrix}$$

$$D_2 = \frac{1}{4862}\begin{bmatrix} 1529 & 2475 & 2250 & 1140 & -210 & -1176 & -1400 & -900 & -45 & 605 & 594 \\ 990 & 1694 & 1710 & 1152 & 364 & -294 & -588 & -480 & -126 & 198 & 242 \\ 550 & 1045 & 1239 & 1116 & 784 & 392 & 70 & -108 & -138 & -77 & -11 \\ 209 & 528 & 837 & 1032 & 1050 & 882 & 574 & 216 & -81 & -220 & -165 \\ -33 & 143 & 504 & 900 & 1162 & 1176 & 924 & 492 & 45 & -231 & -220 \\ -176 & -110 & 240 & 720 & 1120 & 1274 & 1120 & 720 & 240 & -110 & -176 \\ -220 & -231 & 45 & 492 & 924 & 1176 & 1162 & 900 & 504 & 143 & -33 \\ -165 & -220 & -81 & 216 & 574 & 882 & 1050 & 1032 & 837 & 528 & 209 \\ -11 & -77 & -138 & -108 & 70 & 392 & 784 & 1116 & 1239 & 1045 & 550 \\ 242 & 198 & -126 & -480 & -588 & -294 & 364 & 1152 & 1710 & 1694 & 990 \\ 594 & 605 & -45 & -900 & -1400 & -1176 & -210 & 1140 & 2250 & 2475 & 1529 \end{bmatrix}$$

$n = 13 \quad r = 4$ (continued.)

$$D_3 = \frac{1}{4862}\begin{bmatrix} 539 & 1320 & 1860 & 1848 & 1274 & 392 & -420 & -840 & -759 & -352 \\ 440 & 1089 & 1560 & 1596 & 1176 & 490 & -168 & -540 & -528 & -253 \\ 341 & 858 & 1260 & 1344 & 1078 & 588 & 84 & -240 & -297 & -154 \\ 242 & 627 & 960 & 1092 & 980 & 686 & 336 & 60 & -66 & -55 \\ 143 & 396 & 660 & 840 & 882 & 784 & 588 & 360 & 165 & 44 \\ 44 & 165 & 360 & 588 & 784 & 882 & 840 & 660 & 396 & 143 \\ -55 & -66 & 60 & 336 & 686 & 980 & 1092 & 960 & 627 & 242 \\ -154 & -297 & -240 & 84 & 588 & 1078 & 1344 & 1260 & 858 & 341 \\ -253 & -528 & -540 & -168 & 490 & 1176 & 1596 & 1560 & 1089 & 440 \\ -352 & -759 & -840 & -420 & 392 & 1274 & 1848 & 1860 & 1320 & 539 \end{bmatrix}$$

$$D_4 = \frac{1}{4862}\begin{bmatrix} 99 & 330 & 630 & 882 & 980 & 882 & 630 & 330 & 99 \\ & \text{''} & & & \text{''} & & & \text{''} & \\ & \text{''} & & & \text{''} & & & \text{''} & \\ & & & & & & \text{9 rows.} & & \end{bmatrix}$$

$$D_5 = 0$$

$n = 13 \quad r = 5$

$$C = \frac{1}{9724}\begin{bmatrix} 9240 & 1452 & -858 & -704 & 66 & 528 & 440 & 0 & -396 & -396 & 66 & 528 & -242 \\ 1452 & 5005 & 3762 & 1221 & -704 & -1320 & -792 & 220 & 924 & 759 & -242 & -1089 & 528 \\ -858 & 3762 & 4060 & 2586 & 906 & -200 & -540 & -312 & 94 & 294 & 108 & -242 & 66 \\ -704 & 1221 & 2586 & 3045 & 2628 & 1620 & 440 & -480 & -816 & -473 & 294 & 759 & -396 \\ 66 & -704 & 906 & 2628 & 3360 & 2880 & 1560 & 80 & -858 & -816 & 94 & 924 & -396 \\ 528 & -1320 & -200 & 1620 & 2880 & 3108 & 2400 & 1200 & 80 & -480 & -312 & 220 & 0 \\ 440 & -792 & -540 & 440 & 1560 & 2400 & 2708 & 2400 & 1560 & 440 & -540 & -792 & 440 \\ 0 & 220 & -312 & -480 & 80 & 1200 & 2400 & 3108 & 2880 & 1620 & -200 & -1320 & 528 \\ -396 & 924 & 94 & -816 & -858 & 80 & 1560 & 2880 & 3360 & 2628 & 906 & -704 & 66 \\ -396 & 759 & 294 & -473 & -816 & -480 & 440 & 1620 & 2628 & 3045 & 2586 & 1221 & -704 \\ 66 & -242 & 108 & 294 & 94 & -312 & -540 & -200 & 906 & 2586 & 4060 & 3762 & -858 \\ 528 & -1089 & -242 & 759 & 924 & 220 & -792 & -1320 & -704 & 1221 & 3762 & 5005 & 1452 \\ -242 & 528 & 66 & -396 & -396 & 0 & 440 & 528 & 66 & -704 & -858 & 1452 & 9240 \end{bmatrix}$$

$n = 13 \quad r = 5$ (continued)

$$D_1 = \frac{1}{9724}\begin{bmatrix} 7788 & 4235 & -385 & -2310 & -1540 & 308 & 1540 & 1320 & 0 & -1155 & -847 & 770 \\ 2310 & 3553 & 3255 & 1890 & 280 & -840 & -1092 & -560 & 270 & 735 & 385 & -462 \\ -154 & 2387 & 3861 & 3402 & 1680 & -140 & -1120 & -952 & -42 & 725 & 539 & -462 \\ -770 & 1155 & 2835 & 3252 & 2520 & 1260 & 140 & -420 & -378 & -35 & 165 & 0 \\ -462 & 154 & 1260 & 2268 & 2748 & 2520 & 1680 & 560 & -378 & -714 & -308 & 396 \\ 88 & -440 & -100 & 1080 & 2400 & 3108 & 2800 & 1600 & 120 & -800 & -572 & 440 \\ 440 & -572 & -800 & 120 & 1600 & 2800 & 3108 & 2400 & 1080 & -100 & -440 & 88 \\ 396 & -308 & -714 & -378 & 560 & 1680 & 2520 & 2748 & 2268 & 1260 & 154 & -462 \\ 0 & 165 & -35 & -378 & -420 & 140 & 1260 & 2520 & 3252 & 2835 & 1155 & -770 \\ -462 & 539 & 725 & -42 & -952 & -1120 & -140 & 1680 & 3402 & 3861 & 2387 & -154 \\ -462 & 385 & 735 & 270 & -560 & -1092 & -840 & 280 & 1890 & 3255 & 3553 & 2310 \\ 770 & -847 & -1155 & 0 & 1320 & 1540 & 308 & -1540 & -2310 & -385 & 4235 & 7788 \end{bmatrix}$$

$$D_2 = \frac{1}{9724}\begin{bmatrix} 5478 & 6160 & 2520 & -1680 & -3500 & -2352 & 280 & 2160 & 1890 & 0 & -1232 \\ 2464 & 3630 & 3024 & 1512 & 112 & -588 & -560 & -168 & 144 & 154 & 0 \\ 616 & 1848 & 2874 & 3024 & 2184 & 784 & -476 & -1008 & -672 & 88 & 462 \\ -308 & 693 & 2268 & 3252 & 3024 & 1764 & 224 & -756 & -756 & -77 & 396 \\ -550 & 44 & 1404 & 2592 & 2940 & 2352 & 1232 & 192 & -306 & -220 & 44 \\ -352 & -220 & 480 & 1440 & 2240 & 2548 & 2240 & 1440 & 480 & -220 & -352 \\ 44 & -220 & -306 & 192 & 1232 & 2352 & 2940 & 2592 & 1404 & 44 & -550 \\ 396 & -77 & -756 & -756 & 224 & 1764 & 3024 & 3252 & 2268 & 693 & -308 \\ 462 & 88 & -672 & -1008 & -476 & 784 & 2184 & 3024 & 2874 & 1848 & 616 \\ 0 & 154 & 144 & -168 & -560 & -588 & 112 & 1512 & 3024 & 3630 & 2464 \\ -1232 & 0 & 1890 & 2160 & 280 & -2352 & -3500 & -1680 & 2520 & 6160 & 5478 \end{bmatrix}$$

$$D_3 = \frac{1}{9724}\begin{bmatrix} 3014 & 5544 & 5040 & 1848 & -1764 & -3528 & -2688 & -360 & 1386 & 1232 \\ 1848 & 3630 & 3780 & 2268 & 196 & -1176 & -1260 & -420 & 396 & 462 \\ 924 & 2079 & 2685 & 2457 & 1617 & 637 & -63 & -315 & -231 & -66 \\ 242 & 891 & 1755 & 2415 & 2499 & 1911 & 903 & -45 & -495 & -352 \\ -198 & 66 & 990 & 2142 & 2842 & 2646 & 1638 & 390 & -396 & -396 \\ -396 & -396 & 390 & 1638 & 2646 & 2842 & 2142 & 990 & 66 & -198 \\ -352 & -495 & -45 & 903 & 1911 & 2499 & 2415 & 1755 & 891 & 242 \\ -66 & -231 & -315 & -63 & 637 & 1617 & 2457 & 2685 & 2079 & 924 \\ 462 & 396 & -420 & -1260 & -1176 & 196 & 2268 & 3780 & 3630 & 1848 \\ 1232 & 1386 & -360 & -2688 & -3528 & -1764 & 1848 & 5040 & 5544 & 3014 \end{bmatrix}$$

$$D_4 = \frac{1}{9724}\begin{bmatrix} 1166 & 3080 & 4340 & 3920 & 1960 & -392 & -1820 & -1760 & -770 \\ 924 & 2475 & 3570 & 3381 & 1960 & 147 & -1050 & -1155 & -528 \\ 682 & 1870 & 2800 & 2842 & 1960 & 686 & -280 & -550 & -286 \\ 440 & 1265 & 2030 & 2303 & 1960 & 1225 & 490 & 55 & -44 \\ 198 & 660 & 1260 & 1764 & 1960 & 1764 & 1260 & 660 & 198 \\ -44 & 55 & 490 & 1225 & 1960 & 2303 & 2030 & 1265 & 440 \\ -286 & -550 & -280 & 686 & 1960 & 2842 & 2800 & 1870 & 682 \\ -528 & -1155 & -1050 & 147 & 1960 & 3381 & 3570 & 2475 & 924 \\ -770 & -1760 & -1820 & -392 & 1960 & 3920 & 4340 & 3080 & 1166 \end{bmatrix}$$

$n = 13 \quad r = 5$ (continued.)

$$D_5 = \frac{1}{884}\begin{bmatrix} 22 & 77 & 147 & 196 & 196 & 147 & 77 & 22 \\ & \text{"} & & \text{"} & & \text{"} & & \\ & \text{"} & & \text{"} & & \text{"} & & \\ & & & & & \text{8 rows.} & & \end{bmatrix}$$

$$D_6 = 0 .$$

$n = 14 \quad r = 0$

$$C = \frac{1}{14} Q$$

$$D_1 = 0$$

$n = 14 \quad r = 1$

$$C = \frac{1}{455}\begin{bmatrix} 117 & 104 & 91 & 78 & 65 & 52 & 39 & 26 & 13 & 0 & -13 & -26 & -39 & -52 \\ 104 & 93 & 82 & 71 & 60 & 49 & 38 & 27 & 16 & 5 & -6 & -17 & -28 & -39 \\ 91 & 82 & 73 & 64 & 55 & 46 & 37 & 28 & 19 & 10 & 1 & -8 & -17 & -26 \\ 78 & 71 & 64 & 57 & 50 & 43 & 36 & 29 & 22 & 15 & 8 & 1 & -6 & -13 \\ 65 & 60 & 55 & 50 & 45 & 40 & 35 & 30 & 25 & 20 & 15 & 10 & 5 & 0 \\ 52 & 49 & 46 & 43 & 40 & 37 & 34 & 31 & 28 & 25 & 22 & 19 & 16 & 13 \\ 39 & 38 & 37 & 36 & 35 & 34 & 33 & 32 & 31 & 30 & 29 & 28 & 27 & 26 \\ 26 & 27 & 28 & 29 & 30 & 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 \\ 13 & 16 & 19 & 22 & 25 & 28 & 31 & 34 & 37 & 40 & 43 & 46 & 49 & 52 \\ 0 & 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 & 50 & 55 & 60 & 65 \\ -13 & -6 & 1 & 8 & 15 & 22 & 29 & 36 & 43 & 50 & 57 & 64 & 71 & 78 \\ -26 & -17 & -8 & 1 & 10 & 19 & 28 & 37 & 46 & 55 & 64 & 73 & 82 & 91 \\ -39 & -28 & -17 & -6 & 5 & 16 & 27 & 38 & 49 & 60 & 71 & 82 & 93 & 104 \\ -52 & -39 & -26 & -13 & 0 & 13 & 26 & 39 & 52 & 65 & 78 & 91 & 104 & 117 \end{bmatrix}$$

$$D_1 = \frac{1}{455}\begin{bmatrix} 13 & 24 & 33 & 40 & 45 & 48 & 49 & 48 & 45 & 40 & 33 & 24 & 13 \\ & & \text{"} & & & & \text{"} & & & \text{"} & & & \\ & & \text{"} & & & & \text{"} & & & \text{"} & & & \\ & & & & & & & & & \text{13 rows} & & & \end{bmatrix}$$

$$D_2 = 0$$

$n = 14 \quad r = 2.$

$$C = \frac{1}{3640}\begin{bmatrix}
1781 & 1287 & 858 & 494 & 195 & -39 & -208 & -312 & -351 & -325 & -234 & -78 & 143 & 429 \\
1287 & 989 & 726 & 498 & 305 & 147 & 24 & -64 & -117 & -135 & -118 & -66 & 21 & 143 \\
858 & 726 & 604 & 492 & 390 & 298 & 216 & 144 & 82 & 30 & -12 & -44 & -66 & -78 \\
494 & 498 & 492 & 476 & 450 & 414 & 368 & 312 & 246 & 170 & 84 & -12 & -118 & -234 \\
195 & 305 & 390 & 450 & 485 & 495 & 480 & 440 & 375 & 285 & 170 & 30 & -135 & -325 \\
-39 & 147 & 298 & 414 & 495 & 541 & 552 & 528 & 469 & 375 & 246 & 82 & -117 & -351 \\
-208 & 24 & 216 & 368 & 480 & 552 & 584 & 576 & 528 & 440 & 312 & 144 & -64 & -312 \\
-312 & -64 & 144 & 312 & 440 & 528 & 576 & 584 & 552 & 480 & 368 & 216 & 24 & -208 \\
-351 & -117 & 82 & 246 & 375 & 469 & 528 & 552 & 541 & 495 & 414 & 298 & 147 & -39 \\
-325 & -135 & 30 & 170 & 285 & 375 & 440 & 480 & 495 & 485 & 450 & 390 & 305 & 195 \\
-234 & -118 & -12 & 84 & 170 & 246 & 312 & 368 & 414 & 450 & 476 & 492 & 498 & 494 \\
-78 & -66 & -44 & -12 & 30 & 82 & 144 & 216 & 298 & 390 & 492 & 604 & 726 & 858 \\
143 & 21 & -66 & -118 & -135 & -117 & -64 & 24 & 147 & 305 & 498 & 726 & 989 & 1287 \\
429 & 143 & -78 & -234 & -325 & -351 & -312 & -208 & -39 & 195 & 494 & 858 & 1287 & 1781
\end{bmatrix}$$

$$D_1 = \frac{1}{3640}\begin{bmatrix}
494 & 792 & 924 & 920 & 810 & 624 & 392 & 144 & -90 & -280 & -396 & -408 & -286 \\
429 & 692 & 814 & 820 & 735 & 584 & 392 & 184 & -15 & -180 & -286 & -308 & -221 \\
364 & 592 & 704 & 720 & 660 & 544 & 392 & 224 & 60 & -80 & -176 & -208 & -156 \\
299 & 492 & 594 & 620 & 585 & 504 & 392 & 264 & 135 & 20 & -66 & -108 & -91 \\
234 & 392 & 484 & 520 & 510 & 464 & 392 & 304 & 210 & 120 & 44 & -8 & -26 \\
169 & 292 & 374 & 420 & 435 & 424 & 392 & 344 & 285 & 220 & 154 & 92 & 39 \\
104 & 192 & 264 & 320 & 360 & 384 & 392 & 384 & 360 & 320 & 264 & 192 & 104 \\
39 & 92 & 154 & 220 & 285 & 344 & 392 & 424 & 435 & 420 & 374 & 292 & 169 \\
-26 & -8 & 44 & 120 & 210 & 304 & 392 & 464 & 510 & 520 & 484 & 392 & 234 \\
-91 & -108 & -66 & 20 & 135 & 264 & 392 & 504 & 585 & 620 & 594 & 492 & 299 \\
-156 & -208 & -176 & -80 & 60 & 224 & 392 & 544 & 660 & 720 & 704 & 592 & 364 \\
-221 & -308 & -286 & -180 & -15 & 184 & 392 & 584 & 735 & 820 & 814 & 692 & 429 \\
-286 & -408 & -396 & -280 & -90 & 144 & 392 & 624 & 810 & 920 & 924 & 792 & 494
\end{bmatrix}$$

$$D_2 = \frac{1}{728}\begin{bmatrix}
13 & 33 & 55 & 75 & 90 & 98 & 98 & 90 & 75 & 55 & 33 & 13 \\
 & & '' & & & '' & & & '' & & & \\
 & & '' & & & '' & & & '' & & & \\
 & & & & & & & \text{12 rows.} & & & &
\end{bmatrix}$$

$$D_3 = 0$$

$n = 14 \quad r = 3$

$$C = \frac{1}{68068}\begin{bmatrix}
47619 & 25168 & 9438 & -572 & -5863 & -7436 & -6292 & -3432 & 143 & 3432 & 5434 & 5148 & 1573 & -6292 \\
25168 & 18579 & 13068 & 8558 & 4972 & 2233 & 264 & -1012 & -1672 & -1793 & -1452 & -726 & 308 & 1573 \\
9438 & 13068 & 14344 & 13728 & 11682 & 8668 & 5148 & 1584 & -1562 & -3828 & -4752 & -3872 & -726 & 5148 \\
-572 & 8558 & 13728 & 15624 & 14932 & 12338 & 8528 & 4188 & 4 & -3338 & -5152 & -4752 & -1452 & 5434 \\
-5863 & 4972 & 11682 & 14932 & 15387 & 13712 & 10572 & 6632 & 2557 & -988 & -3338 & -3828 & -1793 & 3432 \\
-7436 & 2233 & 8668 & 12338 & 13712 & 13259 & 11448 & 8748 & 5628 & 2557 & 4 & -1562 & -1672 & 143 \\
-6292 & 264 & 5148 & 8528 & 10572 & 11448 & 11324 & 10368 & 8748 & 6632 & 4188 & 1584 & -1012 & -3432 \\
-3432 & -1012 & 1584 & 4188 & 6632 & 8748 & 10368 & 11324 & 11448 & 10572 & 8528 & 5148 & 264 & -6292 \\
143 & -1672 & -1562 & 4 & 2557 & 5628 & 8748 & 11448 & 13259 & 13712 & 12338 & 8668 & 2233 & -7436 \\
3432 & -1793 & -3828 & -3338 & -988 & 2557 & 6632 & 10572 & 13712 & 15387 & 14932 & 11682 & 4972 & -5863 \\
5434 & -1452 & -4752 & -5152 & -3338 & 4 & 4188 & 8528 & 12338 & 14932 & 15624 & 13728 & 8558 & -572 \\
5148 & -726 & -3872 & -4752 & -3828 & -1562 & 1584 & 5148 & 8668 & 11682 & 13728 & 14344 & 13068 & 9438 \\
1573 & 308 & -726 & -1452 & -1793 & -1672 & -1012 & 264 & 2233 & 4972 & 8558 & 13068 & 18579 & 25168 \\
-6292 & 1573 & 5148 & 5434 & 3432 & 143 & -3432 & -6292 & -7436 & -5863 & -572 & 9438 & 25168 & 47619
\end{bmatrix}$$

$$D_1 = \frac{1}{68068}\begin{bmatrix}
22451 & 29040 & 25410 & 16280 & 5445 & -4224 & -10780 & -13200 & -11385 & -6160 & 726 & 6600 & 7865 \\
15730 & 21241 & 19965 & 14795 & 8085 & 1650 & -3234 & -5830 & -5940 & -3905 & -605 & 2541 & 3575 \\
10010 & 14520 & 15136 & 13240 & 9990 & 6320 & 2940 & 336 & -1230 & -1720 & -1320 & -440 & 286 \\
5291 & 8877 & 10923 & 11615 & 11160 & 9786 & 7742 & 5298 & 2745 & 395 & -1419 & -2343 & -2002 \\
1573 & 4312 & 7326 & 9920 & 11595 & 12048 & 11172 & 9056 & 5985 & 2440 & -902 & -3168 & -3289 \\
-1144 & 825 & 4345 & 8155 & 11295 & 13106 & 13230 & 11610 & 8490 & 4415 & 231 & -2915 & -3575 \\
-2860 & -1584 & 1980 & 6320 & 10260 & 12960 & 13912 & 12960 & 10260 & 6320 & 1980 & -1584 & -2860 \\
-3575 & -2915 & 231 & 4415 & 8490 & 11610 & 13230 & 13106 & 11295 & 8155 & 4345 & 825 & -1144 \\
-3289 & -3168 & -902 & 2440 & 5985 & 9056 & 11172 & 12048 & 11595 & 9920 & 7326 & 4312 & 1573 \\
-2002 & -2343 & -1419 & 395 & 2745 & 5298 & 7742 & 9786 & 11160 & 11615 & 10923 & 8877 & 5291 \\
286 & -440 & -1320 & -1720 & -1230 & 336 & 2940 & 6320 & 9990 & 13240 & 15136 & 14520 & 10010 \\
3575 & 2541 & -605 & -3905 & -5940 & -5830 & -3234 & 1650 & 8085 & 14795 & 19965 & 21241 & 15730 \\
7865 & 6600 & 726 & -6160 & -11385 & -13200 & -10780 & -4224 & 5445 & 16280 & 25410 & 29040 & 22451
\end{bmatrix}$$

$$D_2 = \frac{1}{68068}\begin{bmatrix}
6721 & 14520 & 19965 & 21450 & 18810 & 12936 & 5390 & -1980 & -7425 & -9680 & -8349 & -4290 \\
5720 & 12441 & 17270 & 18825 & 16920 & 12250 & 6076 & -90 & -4800 & -6985 & -6270 & -3289 \\
4719 & 10362 & 14575 & 16200 & 15030 & 11564 & 6762 & 1800 & -2175 & -4290 & -4191 & -2288 \\
3718 & 8283 & 11880 & 13575 & 13140 & 10878 & 7448 & 3690 & 450 & -1595 & -2112 & -1287 \\
2717 & 6204 & 9185 & 10950 & 11250 & 10192 & 8134 & 5580 & 3075 & 1100 & -33 & -286 \\
1716 & 4125 & 6490 & 8325 & 9360 & 9506 & 8820 & 7470 & 5700 & 3795 & 2046 & 715 \\
715 & 2046 & 3795 & 5700 & 7470 & 8820 & 9506 & 9360 & 8325 & 6490 & 4125 & 1716 \\
-286 & -33 & 1100 & 3075 & 5580 & 8134 & 10192 & 11250 & 10950 & 9185 & 6204 & 2717 \\
-1287 & -2112 & -1595 & 450 & 3690 & 7448 & 10878 & 13140 & 13575 & 11880 & 8283 & 3718 \\
-2288 & -4191 & -4290 & -2175 & 1800 & 6762 & 11564 & 15030 & 16200 & 14575 & 10362 & 4719 \\
-3289 & -6270 & -6985 & -4800 & -90 & 6076 & 12250 & 16920 & 18825 & 17270 & 12441 & 5720 \\
-4290 & -8349 & -9680 & -7425 & -1980 & 5390 & 12936 & 18810 & 21450 & 19965 & 14520 & 6721
\end{bmatrix}$$

$$D_3 = \frac{1}{9724}\begin{bmatrix}
143 & 440 & 825 & 1200 & 1470 & 1568 & 1470 & 1200 & 825 & 440 & 143 \\
 & \text{"} & & & & \text{"} & & & \text{"} & & \\
 & \text{"} & & & & \text{"} & & & \text{"} & & \\
 & & & & & & & & \text{11 rows.} & &
\end{bmatrix}$$

$D_4 = 0$

$n = 14 \quad r = 4$

$$C = \frac{1}{136136}\begin{bmatrix}
115687 & 39325 & 0 & -14300 & -13585 & -5863 & 2860 & 8580 & 9295 & 5005 & -2288 & -8580 & -7865 & 7865 \\
39325 & 43087 & 36300 & 24200 & 10945 & -385 & -7788 & -10340 & -8195 & -2585 & 4180 & 8712 & 6545 & -7865 \\
0 & 36300 & 46112 & 39600 & 25080 & 9020 & -3960 & -11088 & -11440 & -5940 & 2640 & 9680 & 8712 & -8580 \\
-14300 & 24200 & 39600 & 39712 & 31060 & 18880 & 7120 & -1560 & -5788 & -5480 & -1840 & 2640 & 4180 & -2288 \\
-13585 & 10945 & 25080 & 31060 & 30943 & 26605 & 19740 & 11860 & 4295 & -1807 & -5480 & -5940 & -2585 & 5005 \\
-5863 & -385 & 9020 & 18880 & 26605 & 30487 & 29700 & 24300 & 15225 & 4295 & -5788 & -11440 & -8195 & 9295 \\
2860 & -7788 & -3960 & 7120 & 19740 & 29700 & 34312 & 32400 & 24300 & 11860 & -1560 & -11088 & -10340 & 8580 \\
8580 & -10340 & -11088 & -1560 & 11860 & 24300 & 32400 & 34312 & 29700 & 19740 & 7120 & -3960 & -7788 & 2860 \\
9295 & -8195 & -11440 & -5788 & 4295 & 15225 & 24300 & 29700 & 30487 & 26605 & 18880 & 9020 & -385 & -5863 \\
5005 & -2585 & -5940 & -5480 & -1807 & 4295 & 11860 & 19740 & 26605 & 30943 & 31060 & 25080 & 10945 & -13585 \\
-2288 & 4180 & 2640 & -1840 & -5480 & -5788 & -1560 & 7120 & 18880 & 31060 & 39712 & 39600 & 24200 & -14300 \\
-8580 & 8712 & 9680 & 2640 & -5940 & -11440 & -11088 & -3960 & 9020 & 25080 & 39600 & 46112 & 36300 & 0 \\
-7865 & 6545 & 8712 & 4180 & -2585 & -8195 & -10340 & -7788 & -385 & 10945 & 24200 & 36300 & 43087 & 39325 \\
7865 & -7865 & -8580 & -2288 & 5005 & 9295 & 8580 & 2860 & -5863 & -13585 & -14300 & 0 & 39325 & 115687
\end{bmatrix}$$

$$D_1 = \frac{1}{136136}\begin{bmatrix}
76362 & 72600 & 36300 & -2200 & -26730 & -32208 & -21560 & -2640 & 14850 & 22440 & 15972 & -1320 & -15730 \\
39325 & 46112 & 36300 & 20900 & 6765 & -2640 & -6468 & -5720 & -2475 & 880 & 2420 & 1452 & -715 \\
14300 & 26400 & 32912 & 32800 & 26820 & 16960 & 5880 & -3648 & -9300 & -9760 & -5280 & 1760 & 6292 \\
-715 & 12540 & 27060 & 35712 & 35829 & 28104 & 15484 & 2064 & -8019 & -11692 & -8052 & 528 & 7293 \\
-7722 & 3608 & 19668 & 31848 & 36186 & 32304 & 22344 & 9904 & -1026 & -7128 & -6820 & -1320 & 4290 \\
-8723 & -1320 & 11660 & 23420 & 30285 & 31072 & 26460 & 18360 & 9285 & 1720 & -2508 & -2860 & -715 \\
-5720 & -3168 & 3960 & 12640 & 20520 & 25920 & 27832 & 25920 & 20520 & 12640 & 3960 & -3168 & -5720 \\
-715 & -2860 & -2508 & 1720 & 9285 & 18360 & 26460 & 31072 & 30285 & 23420 & 11660 & -1320 & -8723 \\
4290 & -1320 & -6820 & -7128 & -1026 & 9904 & 22344 & 32304 & 36186 & 31848 & 19668 & 3608 & -7722 \\
7293 & 528 & -8052 & -11692 & -8019 & 2064 & 15484 & 28104 & 35829 & 35712 & 27060 & 12540 & -715 \\
6292 & 1760 & -5280 & -9760 & -9300 & -3648 & 5880 & 16960 & 26820 & 32800 & 32912 & 26400 & 14300 \\
-715 & 1452 & 2420 & 880 & -2475 & -5720 & -6468 & -2640 & 6765 & 20900 & 36300 & 46112 & 39325 \\
-15730 & -1320 & 15972 & 22440 & 14850 & -2640 & -21560 & -32208 & -26730 & -2200 & 36300 & 72600 & 76362
\end{bmatrix}$$

$$D_2 = \frac{1}{19448}\begin{bmatrix}
5291 & 9075 & 9075 & 5775 & 990 & -3234 & -5390 & -4950 & -2475 & 605 & 2541 & 2145 \\
3575 & 6391 & 6875 & 5175 & 2310 & -490 & -2254 & -2550 & -1575 & -55 & 1045 & 1001 \\
2145 & 4125 & 4961 & 4545 & 3258 & 1666 & 294 & -522 & -705 & -429 & -33 & 143 \\
1001 & 2277 & 3333 & 3885 & 3834 & 3234 & 2254 & 1134 & 135 & -517 & -693 & -429 \\
143 & 847 & 1991 & 3195 & 4038 & 4214 & 3626 & 2418 & 945 & -319 & -935 & -715 \\
-429 & -165 & 935 & 2475 & 3870 & 4606 & 4410 & 3330 & 1725 & 165 & -759 & -715 \\
-715 & -759 & 165 & 1725 & 3330 & 4410 & 4606 & 3870 & 2475 & 935 & -165 & -429 \\
-715 & -935 & -319 & 945 & 2418 & 3626 & 4214 & 4038 & 3195 & 1991 & 847 & 143 \\
-429 & -693 & -517 & 135 & 1134 & 2254 & 3234 & 3834 & 3885 & 3333 & 2277 & 1001 \\
143 & -33 & -429 & -705 & -522 & 294 & 1666 & 3258 & 4545 & 4961 & 4125 & 2145 \\
1001 & 1045 & -55 & -1575 & -2550 & -2254 & -490 & 2310 & 5175 & 6875 & 6391 & 3575 \\
2145 & 2541 & 605 & -2475 & -4950 & -5390 & -3234 & 990 & 5775 & 9075 & 9075 & 5291
\end{bmatrix}$$

$n = 14 \quad r = 4$ (continued.)

$$
D_3 = \frac{1}{9724}\begin{bmatrix}
858 & 2200 & 3300 & 3600 & 2940 & 1568 & 0 & -1200 & -1650 & -1320 & -572 \\
715 & 1848 & 2805 & 3120 & 2646 & 1568 & 294 & -720 & -1155 & -968 & -429 \\
572 & 1496 & 2310 & 2640 & 2352 & 1568 & 588 & -240 & -660 & -616 & -286 \\
429 & 1144 & 1815 & 2160 & 2058 & 1568 & 882 & 240 & -165 & -264 & -143 \\
286 & 792 & 1320 & 1680 & 1764 & 1568 & 1176 & 720 & 330 & 88 & 0 \\
143 & 440 & 825 & 1200 & 1470 & 1568 & 1470 & 1200 & 825 & 440 & 143 \\
0 & 88 & 330 & 720 & 1176 & 1568 & 1764 & 1680 & 1320 & 792 & 286 \\
-143 & -264 & -165 & 240 & 882 & 1568 & 2058 & 2160 & 1815 & 1144 & 429 \\
-286 & -616 & -660 & -240 & 588 & 1568 & 2352 & 2640 & 2310 & 1496 & 572 \\
-429 & -968 & -1155 & -720 & 294 & 1568 & 2646 & 3120 & 2805 & 1848 & 715 \\
-572 & -1320 & -1650 & -1200 & 0 & 1568 & 2940 & 3600 & 3300 & 2200 & 858
\end{bmatrix}
$$

$$
D_4 = \frac{1}{9724}\begin{bmatrix}
143 & 495 & 990 & 1470 & 1764 & 1764 & 1470 & 990 & 495 & 143 \\
 & & '' & & & '' & & '' & & \\
 & & '' & & & '' & & '' & & \\
 & & & & & & \text{10 rows.} & & &
\end{bmatrix}
$$

$$D_5 = 0$$

$n = 14 \quad r = 5$

$$
C = \frac{1}{646646}\begin{bmatrix}
605748 & 113256 & -51909 & -56914 & -9867 & 29172 & 37180 & 17160 & -12870 & -30888 & -21879 & 11154 & 36179 & -18876 \\
113256 & 300828 & 240306 & 100551 & -19492 & -76395 & -67848 & -18260 & 35640 & 59202 & 34254 & -26499 & -65076 & 36179 \\
-51909 & 240306 & 266948 & 177936 & 68673 & -9790 & -40590 & -30888 & -1705 & 22242 & 22704 & -1936 & -26499 & 11154 \\
-56914 & 100551 & 177936 & 190788 & 158238 & 100845 & 38440 & -12030 & -38658 & -36733 & -10896 & 22704 & 34254 & -21879 \\
-9867 & -19492 & 68673 & 158238 & 200112 & 181800 & 116700 & 33400 & -35025 & -61716 & -36733 & 22242 & 59202 & -30888 \\
29172 & -76395 & -9790 & 100845 & 181800 & 202632 & 165000 & 91500 & 14500 & -35025 & -38658 & -1705 & 35640 & -12870 \\
37180 & -67848 & -40590 & 38440 & 116700 & 165000 & 172882 & 144000 & 91500 & 33400 & -12030 & -30888 & -18260 & 17160 \\
17160 & -18260 & -30888 & -12030 & 33400 & 91500 & 144000 & 172882 & 165000 & 116700 & 38440 & -40590 & -67848 & 37180 \\
-12870 & 35640 & -1705 & -38658 & -35025 & 14500 & 91500 & 165000 & 202632 & 181800 & 100845 & -9790 & -76395 & 29172 \\
-30888 & 59202 & 22242 & -36733 & -61716 & -35025 & 33400 & 116700 & 181800 & 200112 & 158238 & 68673 & -19492 & -9867 \\
-21879 & 34254 & 22704 & -10896 & -36733 & -38658 & -12030 & 38440 & 100845 & 158238 & 190788 & 177936 & 100551 & -56914 \\
11154 & -26499 & -1936 & 22704 & 22242 & -1705 & -30888 & -40590 & -9790 & 68673 & 177936 & 266948 & 240306 & -51909 \\
36179 & -65076 & -26499 & 34254 & 59202 & 35640 & -18260 & -67848 & -76395 & -19492 & 100551 & 240306 & 300828 & 113256 \\
-18876 & 36179 & 11154 & -21879 & -30888 & -12870 & 17160 & 37180 & 29172 & -9867 & -56914 & -51909 & 113256 & 605748
\end{bmatrix}
$$

$$
D_1 = \frac{1}{92378}\begin{bmatrix}
70356 & 43560 & 1815 & -20680 & -19305 & -4224 & 10780 & 15840 & 8910 & -3960 & -11979 & -6600 & 7865 \\
23595 & 32241 & 28435 & 17380 & 4785 & -4730 & -8624 & -6820 & -1485 & 3795 & 5445 & 1936 & -3575 \\
715 & 20680 & 33396 & 31560 & 18765 & 2960 & -8330 & -11024 & -5745 & 2680 & 7480 & 3960 & -4719 \\
-6721 & 10428 & 26037 & 30687 & 24705 & 13140 & 1960 & -4530 & -5049 & -1480 & 2211 & 2277 & -1287 \\
-5577 & 2552 & 13761 & 21960 & 24576 & 21600 & 14700 & 6400 & -675 & -4488 & -4213 & -792 & 2574 \\
-1144 & -2365 & 2035 & 10950 & 20250 & 25626 & 24500 & 17000 & 6000 & -3775 & -7579 & -3410 & 4290 \\
2860 & -4224 & -5610 & 1600 & 13500 & 24000 & 28126 & 24000 & 13500 & 1600 & -5610 & -4224 & 2860 \\
4290 & -3410 & -7579 & -3775 & 6000 & 17000 & 24500 & 25626 & 20250 & 10950 & 2035 & -2365 & -1144 \\
2574 & -792 & -4213 & -4488 & -675 & 6400 & 14700 & 21600 & 24576 & 21960 & 13761 & 2552 & -5577 \\
-1287 & 2277 & 2211 & -1480 & -5049 & -4530 & 1960 & 13140 & 24705 & 30687 & 26037 & 10428 & -6721 \\
-4719 & 3960 & 7480 & 2680 & -5745 & -11024 & -8330 & 2960 & 18765 & 31560 & 33396 & 20680 & 715 \\
-3575 & 1936 & 5445 & 3795 & -1485 & -6820 & -8624 & -4730 & 4785 & 17380 & 28435 & 32241 & 23595 \\
7865 & -6600 & -11979 & -3960 & 8910 & 15840 & 10780 & -4224 & -19305 & -20680 & 1815 & 43560 & 70356
\end{bmatrix}
$$

$n = 14 \quad r = 5$ (continued.)

$$D_2 = \frac{1}{92378}\begin{bmatrix} 46761 & 58080 & 31460 & -6600 & -30690 & -30184 & -10780 & 11880 & 22275 & 14520 & -2904 & -11440 \\ 22880 & 34441 & 29480 & 15300 & 1320 & -6370 & -6664 & -2460 & 1800 & 2915 & 880 & -1144 \\ 7436 & 17688 & 25047 & 25920 & 19980 & 9800 & -490 & -6984 & -7680 & -3520 & 1749 & 3432 \\ -1144 & 6732 & 19008 & 27735 & 27864 & 19404 & 6664 & -4266 & -8640 & -5632 & 792 & 3861 \\ -4433 & 484 & 12210 & 23220 & 27546 & 23520 & 13720 & 3120 & -3555 & -4268 & -902 & 1716 \\ -4004 & -2145 & 5500 & 14850 & 21600 & 23226 & 19600 & 12600 & 5100 & -275 & -2244 & -1430 \\ -1430 & -2244 & -275 & 5100 & 12600 & 19600 & 23226 & 21600 & 14850 & 5500 & -2145 & -4004 \\ 1716 & -902 & -4268 & -3555 & 3120 & 13720 & 23520 & 27546 & 23220 & 12210 & 484 & -4433 \\ 3861 & 792 & -5632 & -8640 & -4266 & 6664 & 19404 & 27864 & 27735 & 19008 & 6732 & -1144 \\ 3432 & 1749 & -3520 & -7680 & -6984 & -490 & 9800 & 19980 & 25920 & 25047 & 17688 & 7436 \\ -1144 & 880 & 2915 & 1800 & -2460 & -6664 & -6370 & 1320 & 15300 & 29480 & 34441 & 22880 \\ -11440 & -2904 & 14520 & 22275 & 11880 & -10780 & -30184 & -30690 & -6600 & 31460 & 58080 & 46761 \end{bmatrix}$$

$$D_3 = \frac{1}{92378}\begin{bmatrix} 23881 & 47520 & 49500 & 27600 & -4410 & -28224 & -32340 & -18000 & 2475 & 14080 & 10296 \\ 15444 & 32197 & 36630 & 26010 & 7350 & -8820 & -14994 & -10470 & -990 & 5445 & 4576 \\ 8580 & 19536 & 25575 & 23760 & 15876 & 6272 & -882 & -3600 & -2640 & -528 & 429 \\ 3289 & 9537 & 16335 & 20850 & 21168 & 17052 & 9996 & 2610 & -2475 & -3839 & -2145 \\ -429 & 2200 & 8910 & 17280 & 23226 & 23520 & 17640 & 8160 & -495 & -4488 & -3146 \\ -2574 & -2475 & 3300 & 13050 & 22050 & 25676 & 22050 & 13050 & 3300 & -2475 & -2574 \\ -3146 & -4488 & -495 & 8160 & 17640 & 23520 & 23226 & 17280 & 8910 & 2200 & -429 \\ -2145 & -3839 & -2475 & 2610 & 9996 & 17052 & 21168 & 20850 & 16335 & 9537 & 3289 \\ 429 & -528 & -2640 & -3600 & -882 & 6272 & 15876 & 23760 & 25575 & 19536 & 8580 \\ 4576 & 5445 & -990 & -10470 & -14994 & -8820 & 7350 & 26010 & 36630 & 32197 & 15444 \\ 10296 & 14080 & 2475 & -18000 & -32340 & -28224 & -4410 & 27600 & 49500 & 47520 & 23881 \end{bmatrix}$$

$$D_4 = \frac{1}{92378}\begin{bmatrix} 8437 & 23760 & 36630 & 38220 & 26460 & 7056 & -10290 & -17820 & -14355 & 5720 \\ 6864 & 19525 & 30580 & 32830 & 24304 & 9212 & -4900 & -11770 & -10120 & -4147 \\ 5291 & 15290 & 24530 & 27440 & 22148 & 11368 & 490 & -5720 & -5885 & -2574 \\ 3718 & 11055 & 18480 & 22050 & 19992 & 13524 & 5880 & 330 & -1650 & -1001 \\ 2145 & 6820 & 12430 & 16660 & 17836 & 15680 & 11270 & 6380 & 2585 & 572 \\ 572 & 2585 & 6380 & 11270 & 15680 & 17836 & 16660 & 12430 & 6820 & 2145 \\ -1001 & -1650 & 330 & 5880 & 13524 & 19992 & 22050 & 18480 & 11055 & 3718 \\ -2574 & -5885 & -5720 & 490 & 11368 & 22148 & 27440 & 24530 & 15290 & 5291 \\ -4147 & -10120 & -11770 & -4900 & 9212 & 24304 & 32830 & 30580 & 19525 & 6864 \\ 5720 & -14355 & -17820 & -10290 & 7056 & 26460 & 38220 & 36630 & 23760 & 8437 \end{bmatrix}$$

$$D_5 = \frac{1}{8398}\begin{bmatrix} 143 & 528 & 1078 & 1568 & 1764 & 1568 & 1078 & 528 & 143 \\ & " & & & " & & " & & \\ & " & & & " & & " & & \\ & & & & & & \text{9 rows.} & & \end{bmatrix}$$

$$D_6 = 0$$

$n = 15 \quad r = 0$

$$C = \frac{1}{15} Q$$

$$D_1 = 0$$

$n=15 \quad r = 1$

$$C = \frac{1}{840}\begin{bmatrix}
203 & 182 & 161 & 140 & 119 & 98 & 77 & 56 & 35 & 14 & -7 & -28 & -49 & -70 & -91 \\
182 & 164 & 146 & 128 & 110 & 92 & 74 & 56 & 38 & 20 & 2 & -16 & -34 & -52 & -70 \\
161 & 146 & 131 & 116 & 101 & 86 & 71 & 56 & 41 & 26 & 11 & -4 & -19 & -34 & -49 \\
140 & 128 & 116 & 104 & 92 & 80 & 68 & 56 & 44 & 32 & 20 & 8 & -4 & -16 & -28 \\
119 & 110 & 101 & 92 & 83 & 74 & 65 & 56 & 47 & 38 & 29 & 20 & 11 & 2 & -7 \\
98 & 92 & 86 & 80 & 74 & 68 & 62 & 56 & 50 & 44 & 38 & 32 & 26 & 20 & 14 \\
77 & 74 & 71 & 68 & 65 & 62 & 59 & 56 & 53 & 50 & 47 & 44 & 41 & 38 & 35 \\
56 & 56 & 56 & 56 & 56 & 56 & 56 & 56 & 56 & 56 & 56 & 56 & 56 & 56 & 56 \\
35 & 38 & 41 & 44 & 47 & 50 & 53 & 56 & 59 & 62 & 65 & 68 & 71 & 74 & 77 \\
14 & 20 & 26 & 32 & 38 & 44 & 50 & 56 & 62 & 68 & 74 & 80 & 86 & 92 & 98 \\
-7 & 2 & 11 & 20 & 29 & 38 & 47 & 56 & 65 & 74 & 83 & 92 & 101 & 110 & 119 \\
-28 & -16 & -4 & 8 & 20 & 32 & 44 & 56 & 68 & 80 & 92 & 104 & 116 & 128 & 140 \\
-49 & -34 & -19 & -4 & 11 & 26 & 41 & 56 & 71 & 86 & 101 & 116 & 131 & 146 & 161 \\
-70 & -52 & -34 & -16 & 2 & 20 & 38 & 56 & 74 & 92 & 110 & 128 & 146 & 164 & 182 \\
-91 & -70 & -49 & -28 & -7 & 14 & 35 & 56 & 77 & 98 & 119 & 140 & 161 & 182 & 203
\end{bmatrix}$$

$$D_1 = \frac{1}{280}\begin{bmatrix}
7 & 13 & 18 & 22 & 25 & 27 & 28 & 28 & 27 & 25 & 22 & 18 & 13 & 7 \\
 & '' & & & & & '' & & & & '' & & & \\
 & '' & & & & & '' & & & & '' & & & \\
 & & & & & & & & & \text{14 rows.} & & & &
\end{bmatrix}$$

$$D_2 = 0$$

$n = 15 \quad r = 2$

$$C = \frac{1}{30940}\begin{bmatrix}
14378 & 10647 & 7371 & 4550 & 2184 & 273 & -1183 & -2184 & -2730 & -2821 & -2457 & -1638 & -364 & 1365 & 3549 \\
10647 & 8294 & 6201 & 4368 & 2795 & 1482 & 429 & -364 & -897 & -1170 & -1183 & -936 & -429 & 338 & 1365 \\
7371 & 6201 & 5126 & 4146 & 3261 & 2471 & 1776 & 1176 & 671 & 261 & -54 & -274 & -399 & -429 & -364 \\
4550 & 4368 & 4146 & 3884 & 3582 & 3240 & 2858 & 2436 & 1974 & 1472 & 930 & 348 & -274 & -936 & -1638 \\
2184 & 2795 & 3261 & 3582 & 3758 & 3789 & 3675 & 3416 & 3012 & 2463 & 1769 & 930 & -54 & -1183 & -2457 \\
273 & 1482 & 2471 & 3240 & 3789 & 4118 & 4227 & 4116 & 3785 & 3234 & 2463 & 1472 & 261 & -1170 & -2821 \\
-1183 & 429 & 1776 & 2858 & 3675 & 4227 & 4514 & 4536 & 4293 & 3785 & 3012 & 1974 & 671 & -897 & -2730 \\
-2184 & -364 & 1176 & 2436 & 3416 & 4116 & 4536 & 4676 & 4536 & 4116 & 3416 & 2436 & 1176 & -364 & -2184 \\
-2730 & -897 & 671 & 1974 & 3012 & 3785 & 4293 & 4536 & 4514 & 4227 & 3675 & 2858 & 1776 & 429 & -1183 \\
-2821 & -1170 & 261 & 1472 & 2463 & 3234 & 3785 & 4116 & 4227 & 4118 & 3789 & 3240 & 2471 & 1482 & 273 \\
-2457 & -1183 & -54 & 930 & 1769 & 2463 & 3012 & 3416 & 3675 & 3789 & 3758 & 3582 & 3261 & 2795 & 2184 \\
-1638 & -936 & -274 & 348 & 930 & 1472 & 1974 & 2436 & 2858 & 3240 & 3582 & 3884 & 4146 & 4368 & 4550 \\
-364 & -429 & -399 & -274 & -54 & 261 & 671 & 1176 & 1776 & 2471 & 3261 & 4146 & 5126 & 6201 & 7371 \\
1365 & 338 & -429 & -936 & -1183 & -1170 & -897 & -364 & 429 & 1482 & 2795 & 4368 & 6201 & 8294 & 10647 \\
3549 & 1365 & -364 & -1638 & -2457 & -2821 & -2730 & -2184 & -1183 & 273 & 2184 & 4550 & 7371 & 10647 & 14378
\end{bmatrix}$$

$n = 15 \quad r = 2$ (continued)

$$D_1 = \frac{1}{30940}\begin{bmatrix}
3731 & 6084 & 7254 & 7436 & 6825 & 5616 & 4004 & 2184 & 351 & -1300 & -2574 & -3276 & -3211 & -2184 \\
3276 & 5369 & 6444 & 6666 & 6200 & 5211 & 3864 & 2324 & 756 & -675 & -1804 & -2466 & -2496 & -1729 \\
2821 & 4654 & 5634 & 5896 & 5575 & 4806 & 3724 & 2464 & 1161 & -50 & -1034 & -1656 & -1781 & -1274 \\
2366 & 3939 & 4824 & 5126 & 4950 & 4401 & 3584 & 2604 & 1566 & 575 & -264 & -846 & -1066 & -819 \\
1911 & 3224 & 4014 & 4356 & 4325 & 3996 & 3444 & 2744 & 1971 & 1200 & 506 & -36 & -351 & -364 \\
1456 & 2509 & 3204 & 3586 & 3700 & 3591 & 3304 & 2884 & 2376 & 1825 & 1276 & 774 & 364 & 91 \\
1001 & 1794 & 2394 & 2816 & 3075 & 3186 & 3164 & 3024 & 2781 & 2450 & 2046 & 1584 & 1079 & 546 \\
546 & 1079 & 1584 & 2046 & 2450 & 2781 & 3024 & 3164 & 3186 & 3075 & 2816 & 2394 & 1794 & 1001 \\
91 & 364 & 774 & 1276 & 1825 & 2376 & 2884 & 3304 & 3591 & 3700 & 3586 & 3204 & 2509 & 1456 \\
-364 & -351 & -36 & 506 & 1200 & 1971 & 2744 & 3444 & 3996 & 4325 & 4356 & 4014 & 3224 & 1911 \\
-819 & -1066 & -846 & -264 & 575 & 1566 & 2604 & 3584 & 4401 & 4950 & 5126 & 4824 & 3939 & 2366 \\
-1274 & -1781 & -1656 & -1034 & -50 & 1161 & 2464 & 3724 & 4806 & 5575 & 5896 & 5634 & 4654 & 2821 \\
-1729 & -2496 & -2466 & -1804 & -675 & 756 & 2324 & 3864 & 5211 & 6200 & 6666 & 6444 & 5369 & 3276 \\
-2184 & -3211 & -3276 & -2574 & -1300 & 351 & 2184 & 4004 & 5616 & 6825 & 7436 & 7254 & 6084 & 3731
\end{bmatrix}$$

$$D_2 = \frac{1}{6188}\begin{bmatrix}
91 & 234 & 396 & 550 & 675 & 756 & 784 & 756 & 675 & 550 & 396 & 234 & 91 \\
 & & \text{"} & & & & \text{"} & & & \text{"} & & & \\
 & & \text{"} & & & & \text{"} & & & \text{"} & & & \\
 & & & & & & & & \text{13 rows} & & & &
\end{bmatrix}$$

$$D_3 = 0$$

$n = 15 \quad r = 3$

$C = \frac{1}{278460}$ of

$$\begin{bmatrix}
187369 & 104104 & 44044 & 4004 & -19201 & -28756 & -27846 & -19656 & -7371 & 5824 & 16744 & 22204 & 19019 & 4004 & -26026 \\
104104 & 75829 & 52624 & 34034 & 19604 & 8879 & 1404 & -3276 & -5616 & -6071 & -5096 & -3146 & -676 & 1859 & 4004 \\
44044 & 52624 & 54709 & 51524 & 44294 & 34244 & 22599 & 10584 & -576 & -9656 & -15431 & -16676 & -12166 & -676 & 19019 \\
4004 & 34034 & 51524 & 58504 & 57004 & 49054 & 36684 & 21924 & 6804 & -6646 & -16396 & -20416 & -16676 & -3146 & 22204 \\
-19201 & 19604 & 44294 & 57004 & 59869 & 55024 & 44604 & 30744 & 15579 & 1244 & -10126 & -16396 & -15431 & -5096 & 16744 \\
-28756 & 8879 & 34244 & 49054 & 55024 & 53869 & 47304 & 37044 & 24804 & 12299 & 1244 & -6646 & -9656 & -6071 & 5824 \\
-27846 & 1404 & 22599 & 36684 & 44604 & 47304 & 45729 & 40824 & 33534 & 24804 & 15579 & 6804 & -576 & -5616 & -7371 \\
-19656 & -3276 & 10584 & 21924 & 30744 & 37044 & 40824 & 42084 & 40824 & 37044 & 30744 & 21924 & 10584 & -3276 & -19656 \\
-7371 & -5616 & -576 & 6804 & 15579 & 24804 & 33534 & 40824 & 45729 & 47304 & 44604 & 36684 & 22599 & 1404 & -27846 \\
5824 & -6071 & -9656 & -6646 & 1244 & 12299 & 24804 & 37044 & 47304 & 53869 & 55024 & 49054 & 34244 & 8879 & -28756 \\
16744 & -5096 & -15431 & -16396 & -10126 & 1244 & 15579 & 30744 & 44604 & 55024 & 59869 & 57004 & 44294 & 19604 & -19201 \\
22204 & -3146 & -16676 & -20416 & -16396 & -6646 & 6804 & 21924 & 36684 & 49054 & 57004 & 58504 & 51524 & 34034 & 4004 \\
19019 & -676 & -12166 & -16676 & -15431 & -9656 & -576 & 10584 & 22599 & 34244 & 44294 & 51524 & 54709 & 52624 & 44044 \\
4004 & 1859 & -676 & -3146 & -5096 & -6071 & -5616 & -3276 & 1404 & 8879 & 19604 & 34034 & 52624 & 75829 & 104104 \\
-26026 & 4004 & 19019 & 22204 & 16744 & 5824 & -7371 & -19656 & -27846 & -28756 & -19201 & 4004 & 44044 & 104104 & 187369
\end{bmatrix}$$

<u>$n = 15 \quad r = 3$</u> (continued.)

$$D_1 = \frac{1}{55692}\begin{bmatrix}
16653 & 22308 & 20592 & 14586 & 6825 & -702 & -6552 & -9828 & -10179 & -7800 & -3432 & 1638 & 5577 & 6006 \\
12012 & 16653 & 16236 & 12738 & 7800 & 2727 & -1512 & -4284 & -5292 & -4575 & -2508 & 198 & 2496 & 3003 \\
8008 & 11726 & 12363 & 10967 & 8425 & 5463 & 2646 & 378 & -1098 & -1700 & -1507 & -759 & 143 & 637 \\
4641 & 7527 & 8973 & 9273 & 8700 & 7506 & 5922 & 4158 & 2403 & 825 & -429 & -1233 & -1482 & -1092 \\
1911 & 4056 & 6066 & 7656 & 8625 & 8856 & 8316 & 7056 & 5211 & 3000 & 726 & -1224 & -2379 & -2184 \\
-182 & 1313 & 3642 & 6116 & 8200 & 9513 & 9828 & 9072 & 7326 & 4825 & 1958 & -732 & -2548 & -2639 \\
-1638 & -702 & 1701 & 4653 & 7425 & 9477 & 10458 & 10206 & 8748 & 6300 & 3267 & 243 & -1989 & -2457 \\
-2457 & -1989 & 243 & 3267 & 6300 & 8748 & 10206 & 10458 & 9477 & 7425 & 4653 & 1701 & -702 & -1638 \\
-2639 & -2548 & -732 & 1958 & 4825 & 7326 & 9072 & 9828 & 9513 & 8200 & 6116 & 3642 & 1313 & -182 \\
-2184 & -2379 & -1224 & 726 & 3000 & 5211 & 7056 & 8316 & 8856 & 8625 & 7656 & 6066 & 4056 & 1911 \\
-1092 & -1482 & -1233 & -429 & 825 & 2403 & 4158 & 5922 & 7506 & 8700 & 9273 & 8973 & 7527 & 4641 \\
637 & 143 & -759 & -1507 & -1700 & -1098 & 378 & 2646 & 5463 & 8425 & 10967 & 12363 & 11726 & 8008 \\
3003 & 2496 & 198 & -2508 & -4575 & -5292 & -4284 & -1512 & 2727 & 7800 & 12738 & 16236 & 16653 & 12012 \\
6006 & 5577 & 1638 & -3432 & -7800 & -10179 & -9828 & -6552 & -702 & 6825 & 14586 & 20592 & 22308 & 16653
\end{bmatrix}$$

$$D_2 = \frac{1}{55692}\begin{bmatrix}
4641 & 10296 & 14652 & 16500 & 15525 & 12096 & 7056 & 1512 & -3375 & -6600 & -7524 & -6084 & -3003 \\
4004 & 8931 & 12804 & 14575 & 13950 & 11214 & 7056 & 2394 & -1800 & -4675 & -5676 & -4719 & -2366 \\
3367 & 7566 & 10956 & 12650 & 12375 & 10332 & 7056 & 3276 & -225 & -2750 & -3828 & -3354 & -1729 \\
2730 & 6201 & 9108 & 10725 & 10800 & 9450 & 7056 & 4158 & 1350 & -825 & -1980 & -1989 & -1092 \\
2093 & 4836 & 7260 & 8800 & 9225 & 8568 & 7056 & 5040 & 2925 & 1100 & -132 & -624 & -455 \\
1456 & 3471 & 5412 & 6875 & 7650 & 7686 & 7056 & 5922 & 4500 & 3025 & 1716 & 741 & 182 \\
819 & 2106 & 3564 & 4950 & 6075 & 6804 & 7056 & 6804 & 6075 & 4950 & 3564 & 2106 & 819 \\
182 & 741 & 1716 & 3025 & 4500 & 5922 & 7056 & 7686 & 7650 & 6875 & 5412 & 3471 & 1456 \\
-455 & -624 & -132 & 1100 & 2925 & 5040 & 7056 & 8568 & 9225 & 8800 & 7260 & 4836 & 2093 \\
-1092 & -1989 & -1980 & -825 & 1350 & 4158 & 7056 & 9450 & 10800 & 10725 & 9108 & 6201 & 2730 \\
-1729 & -3354 & -3828 & -2750 & -225 & 3276 & 7056 & 10332 & 12375 & 12650 & 10956 & 7566 & 3367 \\
-2366 & -4719 & -5676 & -4675 & -1800 & 2394 & 7056 & 11214 & 13950 & 14575 & 12804 & 8931 & 4004 \\
-3003 & -6084 & -7524 & -6600 & -3375 & 1512 & 7056 & 12096 & 15525 & 16500 & 14652 & 10296 & 4641
\end{bmatrix}$$

$$D_3 = \frac{1}{7956}\begin{bmatrix}
91 & 286 & 550 & 825 & 1050 & 1176 & 1176 & 1050 & 825 & 550 & 286 & 91 \\
 & '' & & & '' & & & & '' & & & \\
 & '' & & & '' & & & & '' & & & \\
 & & & & & & & \text{12 rows.} & & & &
\end{bmatrix}$$

$$D_4 = 0$$

$n = 15 \quad r = 4.$

$$C = \frac{1}{11639628} \text{ of }$$

$$\begin{bmatrix}
9635626 & 3578575 & 275275 & -1101100 & -1251250 & -749749 & -45045 & 540540 & 810810 & 695695 & 251251 & -340340 & -770770 & -605605 & 715715 \\
3578575 & 3500926 & 2870725 & 1966250 & 1011725 & 177320 & -420849 & -720720 & -714285 & -447590 & -20735 & 412126 & 642785 & 408980 & -605605 \\
275275 & 2870725 & 3646126 & 3254900 & 2240975 & 1038785 & -26730 & -740124 & -995445 & -796235 & -255530 & 404140 & 850751 & 642785 & -770770 \\
-1101100 & 1966250 & 3254900 & 3337576 & 2698300 & 1732390 & 746460 & -41580 & -502524 & -595870 & -369820 & 38720 & 404140 & 412126 & -340340 \\
-1251250 & 1011725 & 2240975 & 2698300 & 2614126 & 2187505 & 1586115 & 946260 & 372870 & -60499 & -311665 & -369820 & -255530 & -20735 & 251251 \\
-749749 & 177320 & 1038785 & 1732390 & 2187505 & 2365126 & 2257875 & 1890000 & 1317375 & 627500 & -60499 & -595870 & -796235 & -447590 & 695695 \\
-45045 & -420849 & -26730 & 746460 & 1586115 & 2257875 & 2605626 & 2551500 & 2095875 & 1317375 & 372870 & -502524 & -995445 & -714285 & 810810 \\
540540 & -720720 & -740124 & -41580 & 946260 & 1890000 & 2551500 & 2787876 & 2551500 & 1890000 & 946260 & -41580 & -740124 & -720720 & 540540 \\
810810 & -714285 & -995445 & -502524 & 372870 & 1317375 & 2095875 & 2551500 & 2605626 & 2257875 & 1586115 & 746460 & -26730 & -420849 & -45045 \\
695695 & -447590 & -796235 & -595870 & -60499 & 627500 & 1317375 & 1890000 & 2257875 & 2365126 & 2187505 & 1732390 & 1038785 & 177320 & -749749 \\
251251 & -20735 & -255530 & -369820 & -311665 & -60499 & 372870 & 946260 & 1586115 & 2187505 & 2614126 & 2698300 & 2240975 & 1011725 & -1251250 \\
-340340 & 412126 & 404140 & 38720 & -369820 & -595870 & -502524 & -41580 & 746460 & 1732390 & 2698300 & 3337576 & 3254900 & 1966250 & -1101100 \\
-770770 & 642785 & 850751 & 404140 & -255530 & -796235 & -995445 & -740124 & -26730 & 1038785 & 2240975 & 3254900 & 3646126 & 2870725 & 275275 \\
-605605 & 408980 & 642785 & 412126 & -20735 & -447590 & -714285 & -720720 & -420849 & 177320 & 1011725 & 1966250 & 2870725 & 3500926 & 3578575 \\
715715 & -605605 & -770770 & -340340 & 251251 & 695695 & 810810 & 540540 & -45045 & -749749 & -1251250 & -1101100 & 275275 & 3578575 & 9635626
\end{bmatrix}$$

$$D_1 = \frac{1}{11639628} \text{ of }$$

$$\begin{bmatrix}
6057051 & 6134700 & 3539250 & 471900 & -1791075 & -2718144 & -2342340 & -1081080 & 444015 & 1587300 & 1859286 & 1106820 & -306735 & -1321320 \\
3303300 & 3933501 & 3158100 & 1869450 & 640200 & -221265 & -615384 & -595980 & -314820 & 33825 & 268620 & 276606 & 68640 & -165165 \\
1376375 & 2280850 & 2672076 & 2589400 & 2132075 & 1438470 & 665280 & -33264 & -526185 & -726550 & -612260 & -246840 & 199771 & 430430 \\
150150 & 1104675 & 2118600 & 2757876 & 2842050 & 2386935 & 1547280 & 559440 & -315954 & -851325 & -909480 & -500940 & 158730 & 591591 \\
-501501 & 332904 & 1535094 & 2501004 & 2927625 & 2750004 & 2078244 & 1134504 & 189999 & -498000 & -749166 & -523116 & 17589 & 444444 \\
-704704 & -106535 & 958980 & 1944910 & 2546300 & 2653551 & 2305800 & 1644300 & 865800 & 175925 & -257444 & -350790 & -151580 & 115115 \\
-585585 & -285714 & 427680 & 1215720 & 1855575 & 2223450 & 2277576 & 2041200 & 1585575 & 1012950 & 439560 & -21384 & -276705 & -270270 \\
-270270 & -276705 & -21384 & 439560 & 1012950 & 1585575 & 2041200 & 2277576 & 2223450 & 1855575 & 1215720 & 427680 & -285714 & -585585 \\
115115 & -151580 & -350790 & -257444 & 175925 & 865800 & 1644300 & 2305800 & 2653551 & 2546300 & 1944910 & 958980 & -106535 & -704704 \\
444444 & 17589 & -523116 & -749166 & -498000 & 189999 & 1134504 & 2078244 & 2750004 & 2927625 & 2501004 & 1535094 & 332904 & -501501 \\
591591 & 158730 & -500940 & -909480 & -851325 & -315954 & 559440 & 1547280 & 2386935 & 2842050 & 2757876 & 2118600 & 1104675 & 150150 \\
430430 & 199771 & -246840 & -612260 & -726550 & -526185 & -33264 & 665280 & 1438470 & 2132075 & 2589400 & 2672076 & 2280850 & 1376375 \\
-165165 & 68640 & 276606 & 268620 & 33825 & -314820 & -595980 & -615384 & -221265 & 640200 & 1869450 & 3158100 & 3933501 & 3303300 \\
-1321320 & -306735 & 1106820 & 1859286 & 1587300 & 444015 & -1081080 & -2342340 & -2718144 & -1791075 & 471900 & 3539250 & 6134700 & 6057051
\end{bmatrix}$$

$$D_2 = \frac{1}{1662804} \text{ of }$$

$$\begin{bmatrix}
393393 & 707850 & 762300 & 562650 & 215325 & -141372 & -388080 & -457380 & -348975 & -127050 & 100188 & 218790 & 165165 \\
275275 & 511368 & 580800 & 477950 & 264825 & 27720 & -155232 & -235620 & -205425 & -96800 & 29040 & 103818 & 85085 \\
175175 & 343200 & 422268 & 398200 & 296775 & 161280 & 35280 & -49392 & -79425 & -61600 & -19140 & 17160 & 23023 \\
93093 & 203346 & 286704 & 323400 & 311175 & 259308 & 183456 & 101304 & 29025 & -21450 & -44352 & -41184 & -21021 \\
29029 & 91806 & 174108 & 253550 & 308025 & 321804 & 289296 & 216468 & 119925 & 23650 & -46596 & -71214 & -47047 \\
-17017 & 8580 & 84480 & 188650 & 287325 & 348768 & 352800 & 296100 & 193275 & 73700 & -25872 & -72930 & -55055 \\
-45045 & -46332 & 17820 & 128700 & 249075 & 340200 & 373968 & 340200 & 249075 & 128700 & 17820 & -46332 & -45045 \\
-55055 & -72930 & -25872 & 73700 & 193275 & 296100 & 352800 & 348768 & 287325 & 188650 & 84480 & 8580 & -17017 \\
-47047 & -71214 & -46596 & 23650 & 119925 & 216468 & 289296 & 321804 & 308025 & 253550 & 174108 & 91806 & 29029 \\
-21021 & -41184 & -44352 & -21450 & 29025 & 101304 & 183456 & 259308 & 311175 & 323400 & 286704 & 203346 & 93093 \\
23023 & 17160 & -19140 & -61600 & -79425 & -49392 & 35280 & 161280 & 296775 & 398200 & 422268 & 343200 & 175175 \\
85085 & 103818 & 29040 & -96800 & -205425 & -235620 & -155232 & 27720 & 264825 & 477950 & 580800 & 511368 & 275275 \\
165165 & 218790 & 100188 & -127050 & -348975 & -457380 & -388080 & -141372 & 215325 & 562650 & 762300 & 707850 & 393393
\end{bmatrix}$$

$n=15 \quad r=4$ (continued.)

$$D_3 = \frac{1}{831402}\begin{bmatrix} 59059 & 157300 & 248050 & 290400 & 265650 & 181104 & 64680 & -46200 & -117975 & -133100 & -97526 & -40040 \\ 50050 & 134134 & 213400 & 253275 & 237300 & 170520 & 75264 & -17850 & -80850 & -98450 & -74360 & -31031 \\ 41041 & 110968 & 178750 & 216150 & 208950 & 159936 & 85848 & 10500 & -43725 & -63800 & -51194 & -22022 \\ 32032 & 87802 & 144100 & 179025 & 180600 & 149352 & 96432 & 38850 & -6600 & -29150 & -28028 & -13013 \\ 23023 & 64636 & 109450 & 141900 & 152250 & 138768 & 107016 & 67200 & 30525 & 5500 & -4862 & -4004 \\ 14014 & 41470 & 74800 & 104775 & 123900 & 128184 & 117600 & 95550 & 67650 & 40150 & 18304 & 5005 \\ 5005 & 18304 & 40150 & 67650 & 95550 & 117600 & 128184 & 123900 & 104775 & 74800 & 41470 & 14014 \\ -4004 & -4862 & 5500 & 30525 & 67200 & 107016 & 138768 & 152250 & 141900 & 109450 & 64636 & 23023 \\ -13013 & -28028 & -29150 & -6600 & 38850 & 96432 & 149352 & 180600 & 179025 & 144100 & 87802 & 32032 \\ -22022 & -51194 & -63800 & -43725 & 10500 & 85848 & 159936 & 208950 & 216150 & 178750 & 110968 & 41041 \\ -31031 & -74360 & -98450 & -80850 & -17850 & 75264 & 170520 & 237300 & 253275 & 213400 & 134134 & 50050 \\ -40040 & -97526 & -133100 & -117975 & -46200 & 64680 & 181104 & 265650 & 290400 & 248050 & 157300 & 59059 \end{bmatrix}$$

$$D_4 = \frac{1}{92378}\begin{bmatrix} 1001 & 3575 & 7425 & 11550 & 14700 & 15876 & 14700 & 11550 & 7425 & 3575 & 1001 \\ & & {''} & & & {''} & & & {''} & & \\ & & {''} & & & {''} & & & {''} & & \\ & & & & & & & \text{11 rows.} & & & \end{bmatrix}$$

$$D_5 = 0$$

$n=15 \quad r=5$

$$C = \frac{1}{1847560} \text{ of}$$

$$\begin{bmatrix} 1704417 & 368082 & -127413 & -182468 & -67353 & 55770 & 110825 & 85800 & 10725 & -64350 & -91377 & -46332 & 48763 & 103818 & -61347 \\ 368082 & 784212 & 651222 & 320892 & 10582 & -171600 & -201630 & -114400 & 21450 & 128700 & 146718 & 56628 & -93522 & -163592 & 103818 \\ -127413 & 651222 & 746097 & 524172 & 227337 & -6050 & -119625 & -117480 & -42625 & 44550 & 87813 & 56628 & -32307 & -93522 & 48763 \\ -182468 & 320892 & 524172 & 530112 & 422532 & 267300 & 113300 & -6600 & -74580 & -86900 & -52932 & 5808 & 56628 & 56628 & -46332 \\ -67353 & 10582 & 227337 & 422532 & 513417 & 478350 & 340275 & 150200 & -29325 & -140730 & -147947 & -52932 & 87813 & 146718 & -91377 \\ 55770 & -171600 & -6050 & 267300 & 478350 & 550020 & 476250 & 300000 & 91250 & -75000 & -140730 & -86900 & 44550 & 128700 & -64350 \\ 110825 & -201630 & -119625 & 113300 & 340275 & 476250 & 493145 & 405000 & 253125 & 91250 & -29325 & -74580 & -42625 & 21450 & 10725 \\ 85800 & -114400 & -117480 & -6600 & 150200 & 300000 & 405000 & 442520 & 405000 & 300000 & 150200 & -6600 & -117480 & -114400 & 85800 \\ 10725 & 21450 & -42625 & -74580 & -29325 & 91250 & 253125 & 405000 & 493145 & 476250 & 340275 & 113300 & -119625 & -201630 & 110825 \\ -64350 & 128700 & 44550 & -86900 & -140730 & -75000 & 91250 & 300000 & 476250 & 550020 & 478350 & 267300 & -6050 & -171600 & 55770 \\ -91377 & 146718 & 87813 & -52932 & -147947 & -140730 & -29325 & 150200 & 340275 & 478350 & 513417 & 422532 & 227337 & 10582 & -67353 \\ -46332 & 56628 & 56628 & 5808 & -52932 & -86900 & -74580 & -6600 & 113300 & 267300 & 422532 & 530112 & 524172 & 320892 & -182468 \\ 48763 & -93522 & -32307 & 56628 & 87813 & 44550 & -42625 & -117480 & -119625 & -6050 & 227337 & 524172 & 746097 & 651222 & -127413 \\ 103818 & -163592 & -93522 & 56628 & 146718 & 128700 & 21450 & -114400 & -201630 & -171600 & 10582 & 320892 & 651222 & 784212 & 368082 \\ -61347 & 103818 & 48763 & -46332 & -91377 & -64350 & 10725 & 85800 & 110825 & 55770 & -67353 & -182468 & -127413 & 368082 & 1704417 \end{bmatrix}$$

$n = 15 \quad r = 5$ (continued.)

$$D_1 = \frac{1}{1847560} \text{ of}$$

$$\begin{bmatrix}
1336335 & 920205 & 141570 & -361790 & -439725 & -212355 & 100100 & 300300 & 289575 & 96525 & -141570 & -244530 & -102245 & 165165 \\
495495 & 628485 & 533610 & 330330 & 113575 & -51975 & -133980 & -130900 & -66825 & 17325 & 76230 & 76230 & 15015 & -55055 \\
55065 & 385385 & 607310 & 601370 & 406175 & 132825 & -100100 & -210980 & -179025 & -47575 & 93170 & 143990 & 55055 & -95095 \\
-115115 & 195195 & 492030 & 599610 & 508725 & 297675 & 70700 & -86100 & -131355 & -77525 & 17490 & 76230 & 45045 & -45045 \\
-123123 & 59059 & 292446 & 447678 & 482745 & 411075 & 275100 & 125300 & 4725 & -61005 & -68222 & -34254 & 9009 & 27027 \\
-55055 & -25025 & 88550 & 242550 & 380625 & 454395 & 437500 & 332500 & 170625 & 4375 & -107030 & -119350 & -32175 & 75075 \\
25025 & -62205 & -64350 & 55550 & 245625 & 421875 & 510020 & 472500 & 320625 & 111875 & -67650 & -135630 & -60775 & 75075 \\
75075 & -60775 & -135630 & -67650 & 111875 & 320625 & 472500 & 510020 & 421875 & 245625 & 55550 & -64350 & -62205 & 25025 \\
75075 & -32175 & -119350 & -107030 & 4375 & 170625 & 332500 & 437500 & 454395 & 380625 & 242550 & 88550 & -25025 & -55055 \\
27027 & 9009 & -34254 & -68222 & -61005 & 4725 & 125300 & 275100 & 411075 & 482745 & 447678 & 292446 & 59059 & -123123 \\
-45045 & 45045 & 76230 & 17490 & -77525 & -131355 & -86100 & 70700 & 297675 & 508725 & 599610 & 492030 & 195195 & -115115 \\
-95095 & 55055 & 143990 & 93170 & -47575 & -179025 & -210980 & -100100 & 132825 & 406175 & 601370 & 607310 & 385385 & 55055 \\
-55055 & 15015 & 76230 & 76230 & 17325 & -66825 & -130900 & -133980 & -51975 & 113575 & 330330 & 533610 & 628485 & 495495 \\
165165 & -102245 & -244530 & -141570 & 96525 & 289575 & 300300 & 100100 & -212355 & -439725 & -361790 & 141570 & 920205 & 1336335
\end{bmatrix}$$

$$D_2 = \frac{1}{1847560} \text{ of}$$

$$\begin{bmatrix}
840840 & 1132560 & 740520 & 48400 & -504900 & -665280 & -431200 & 0 & 356400 & 435600 & 217800 & -102960 & -220220 \\
440440 & 683540 & 609840 & 338800 & 46200 & -138600 & -172480 & -92400 & 19800 & 84700 & 67760 & 0 & -40040 \\
170170 & 360360 & 475640 & 477400 & 374850 & 210000 & 39200 & -85680 & -133350 & -103400 & -27720 & 40040 & 50050 \\
8008 & 144144 & 343728 & 495660 & 521640 & 408240 & 203840 & -7560 & -143640 & -160160 & -74448 & 36036 & 72072 \\
-68068 & 16016 & 219912 & 425040 & 527160 & 483840 & 321440 & 114240 & -51660 & -117040 & -78232 & 6864 & 48048 \\
-80080 & -42900 & 110000 & 297000 & 432000 & 464520 & 392000 & 252000 & 102000 & -5500 & -44880 & -28600 & 0 \\
-50050 & -51480 & 19800 & 143000 & 276750 & 378000 & 415520 & 378000 & 276750 & 143000 & 19800 & -51480 & -50050 \\
0 & -28600 & -44880 & -5500 & 102000 & 252000 & 392000 & 464520 & 432000 & 297000 & 110000 & -42900 & -80080 \\
48048 & 6864 & -78232 & -117040 & -51660 & 114240 & 321440 & 483840 & 527160 & 425040 & 219912 & 16016 & -68068 \\
72072 & 36036 & -74448 & -160160 & -143640 & -7560 & 203840 & 408240 & 521640 & 495660 & 343728 & 144144 & 8008 \\
50050 & 40040 & -27720 & -103400 & -133350 & -85680 & 39200 & 210000 & 374850 & 477400 & 475640 & 360360 & 170170 \\
-40040 & 0 & 67760 & 84700 & 19800 & -92400 & -172480 & -138600 & 46200 & 338800 & 609840 & 683540 & 440440 \\
-220220 & -102960 & 217800 & 435600 & 356400 & 0 & -431200 & -665280 & -504900 & 48400 & 740520 & 1132560 & 840840
\end{bmatrix}$$

$$D_3 = \frac{1}{923780} \text{ of}$$

$$\begin{bmatrix}
200200 & 424710 & 490050 & 344850 & 69300 & -194040 & -323400 & -277200 & -108900 & 66550 & 141570 & 90090 \\
135135 & 296725 & 363825 & 294525 & 130200 & -44100 & -149940 & -153300 & -76725 & 17325 & 65065 & 45045 \\
81081 & 189189 & 255145 & 246015 & 172620 & 73500 & -8820 & -47880 & -42735 & -14355 & 9009 & 11011 \\
38038 & 102102 & 164010 & 199320 & 196560 & 158760 & 99960 & 39060 & -6930 & -28490 & -26598 & -12012 \\
6006 & 35464 & 90420 & 154440 & 202020 & 211680 & 176400 & 107520 & 30690 & -25080 & -41756 & -24024 \\
-15015 & -10725 & 34375 & 111375 & 189000 & 232260 & 220500 & 157500 & 70125 & -4125 & -36465 & -25025 \\
-25025 & -36465 & -4125 & 70125 & 157500 & 220500 & 232260 & 189000 & 111375 & 34375 & -10725 & -15015 \\
-24024 & -41756 & -25080 & 30690 & 107520 & 176400 & 211680 & 202020 & 154440 & 90420 & 35464 & 6006 \\
-12012 & -26598 & -28490 & -6930 & 39060 & 99960 & 158760 & 196560 & 199320 & 164010 & 102102 & 38038 \\
11011 & 9009 & -14355 & -42735 & -47880 & -8820 & 73500 & 172620 & 246015 & 255145 & 189189 & 81081 \\
45045 & 65065 & 17325 & -76725 & -153300 & -149940 & -44100 & 130200 & 294525 & 363825 & 296725 & 135135 \\
90090 & 141570 & 66550 & -108900 & -277200 & -323400 & -194040 & 69300 & 344850 & 490050 & 424710 & 200200
\end{bmatrix}$$

$n = 15 \quad r = 5$ (continued.)

$$D_4 = \frac{1}{923780} \text{ of}$$

$$\begin{bmatrix}
65065 & 193050 & 319275 & 369600 & 308700 & 158760 & -14700 & -138600 & -170775 & -121550 & -45045 \\
54054 & 161590 & 270275 & 318780 & 276360 & 158760 & 17640 & -87780 & -121770 & -90090 & -34034 \\
43043 & 130130 & 221265 & 267960 & 244020 & 158760 & 49980 & -36960 & -72765 & -58630 & -23023 \\
32032 & 98670 & 172260 & 217140 & 211680 & 158760 & 82320 & 13860 & -23760 & -27170 & -12012 \\
21021 & 67210 & 123255 & 166320 & 179340 & 158760 & 114660 & 64680 & 25245 & 4290 & -1001 \\
10010 & 35750 & 74250 & 115500 & 147000 & 158760 & 147000 & 115500 & 74250 & 35750 & 10010 \\
-1001 & 4290 & 25245 & 64680 & 114660 & 158760 & 179340 & 166320 & 123255 & 67210 & 21021 \\
-12012 & -27170 & -23760 & 13860 & 82320 & 158760 & 211680 & 217140 & 172260 & 98670 & 32032 \\
-23023 & -58630 & -72765 & -36960 & 49980 & 158760 & 244020 & 267960 & 221265 & 130130 & 43043 \\
-34034 & -90090 & -121770 & -87780 & 17640 & 158760 & 276360 & 318780 & 270270 & 161590 & 54054 \\
-45045 & -121550 & -170775 & -138600 & -14700 & 158760 & 308700 & 369600 & 319275 & 193050 & 65065
\end{bmatrix}$$

$$D_5 = \frac{1}{83980} \begin{bmatrix}
1001 & 3861 & 8316 & 12936 & 15876 & 15876 & 12936 & 8316 & 3861 & 1001 \\
 & " & & & " & & " & & & \\
 & " & & & " & & " & & & \\
 & & & & & & \text{10 rows.} & & &
\end{bmatrix}$$

$$D_6 = 0.$$

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