Oscillatory Singular Integrals with Variable Flat Phases, and Related Operators.

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Declaration

This thesis was composed by myself and has not been submitted for any other degree or professional qualification. All work not otherwise attributed is original.

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Abstract

We prove $L^p(\mathbb{R})$, $1 < p < \infty$, boundedness of oscillatory singular integral operators of the form

$$Tf(x) = p.v. \int_{-\infty}^{\infty} \frac{e^{iP(x)\gamma(x-y)}}{x-y} f(y)dy,$$

for P a real–valued polynomial, and $\gamma : \mathbb{R} \to \mathbb{R}$ a convex curve satisfying certain conditions that permit it to vanish to infinite order at the origin. The bounds are shown to be independent of the coefficients of the polynomial. This work allows us to conclude that, under the same conditions on γ, the Hilbert transform H, given by

$$Hf(x_1, x_2) = p.v. \int_{-\infty}^{\infty} f(x_1 - t, x_2 - P(x_1)\gamma(t)) \frac{dt}{t},$$

is bounded on $L^2(\mathbb{R}^2)$, with a bound that does not depend on the coefficients of P. We also obtain weak type 1–1 boundedness, and boundedness from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$ of the operator T, when P is linear, under similar conditions on γ.

In the final chapter we give necessary and sufficient conditions for a Calderón–Zygmund singular integral operator, of convolution type, to be injective on $L^1(\mathbb{R}^n)$. In addition, we show how our techniques allow us to reach similar conclusions for certain classes of oscillatory singular integrals.

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Chapter 1

Background

1.1 Introduction

In this thesis we shall be concerned with some questions that have arisen in the modern Calderón–Zygmund theory. Much of the thesis will be devoted to the study of singular integral operators whose kernels have an oscillating factor. The intimately related theory of singular and maximal Radon transforms, which has been largely responsible for the current far reaching perspective on Calderón–Zygmund theory, will form a complementary theme. We shall, therefore, begin with some well established preliminaries and a brief review of the theory of Calderón and Zygmund.

1.2 Preliminaries

The Fourier Transform.

For $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ the Fourier transform is defined by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{2\pi i x\cdot\xi} dx.$$

Plancherel's theorem states that $\mathcal{F}$ can be extended to a unitary operator on $L^2(\mathbb{R}^n)$. The Fourier transform is the central tool in the study of a variety of translation invariant operators. The simplest interesting example is the $L^2(\mathbb{R})$ boundedness of the Hilbert transform, which is defined a priori on a Schwarz function f by

$$Hf(x) = p.v.\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x-y)}{y} dy. \tag{1.1}$$

Taking the Fourier transform we see that $\widehat{Hf}(\xi) = -i\text{sign}(\xi)\widehat{f}(\xi)$. Given Plancherel's theorem, $L^2(\mathbb{R})$ boundedness of H now becomes obvious. Another classical example is the Hilbert transform along the parabola, defined a priori on a Schwarz function f by

$$H_{par}f(x_1,x_2) = p.v. \int_{-\infty}^{\infty} f(x_1 - t, x_2 - t^2)\frac{dt}{t}. \tag{1.2}$$

By taking the Fourier transform of (1.2) we obtain

$$\widehat{H_{par}f}(\xi_1,\xi_2) = \mathfrak{m}(\xi_1,\xi_2)\widehat{f}(\xi_1,\xi_2),$$

where

$$\mathfrak{m}(\xi_1,\xi_2) = p.v. \int_{-\infty}^{\infty} e^{2\pi i(\xi_1 t + \xi_2 t^2)}\frac{dt}{t}$$

is the Fourier multiplier corresponding to H_{par}.

Any translation invariant, L^2–bounded linear operator may be represented by a Fourier multiplier in this way; i.e. if T is such an operator and has Fourier multiplier $\mathfrak{m} : \mathbb{R}^n \to \mathbb{C}$, then $\widehat{Tf} = \mathfrak{m}\widehat{f}$. For $1 \le p \le \infty$, we say that $\mathfrak{m}$ is an L^p–multiplier (or $\mathfrak{m} \in \mathcal{M}_p(\mathbb{R}^n)$), if T is bounded on $L^p(\mathbb{R}^n)$. As we have observed, by Plancherel's Theorem, $L^\infty(\mathbb{R}^n) \subset \mathcal{M}_2(\mathbb{R}^n)$. In fact, one can easily see that there is equality here.

Interpolation of operators

An operator T is said to be bounded on $L^p(\mathbb{R}^n)$, or of strong type p–p, if there is a constant $A_p > 0$ for which

$$||Tf||_{L^p(\mathbb{R}^n)} \le A_p ||f||_{L^p(\mathbb{R}^n)} \tag{1.3}$$

for all $f \in L^p(\mathbb{R}^n)$. The smallest constant A_p for which (1.3) holds is called the $L^p(\mathbb{R}^n)$ operator norm of T, and is often denoted by $||T||_{p-p}$.

An operator T is said to be of weak type p–p if there is a constant $A_p > 0$ for which

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \alpha\}| \le \left(\frac{A_p ||f||_{L^p(\mathbb{R}^n)}}{\alpha}\right)^p \tag{1.4}$$

for all $f \in L^p(\mathbb{R}^n)$ and $\alpha > 0$. The smallest constant A_p for which (1.4) holds is called the weak type p–p operator bound of T. We observe that, by Chebychev's inequality, (1.3) implies (1.4).

On several occasions we will need to interpolate between operator norm estimates of a certain type. The following theorems will be sufficient. The reader is referred to [34] for the stronger forms.

Theorem 1 (Riesz–Thorin). *If a linear operator T is bounded on both $L^{p_0}(\mathbb{R}^n)$ and $L^{p_1}(\mathbb{R}^n)$, for some $1 \le p_0 < p_1 \le \infty$, then T is bounded on $L^{p_t}(\mathbb{R}^n)$ for $0 < t < 1$, where*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}.$$

Moreover,

$$\|T\|_{p_t-p_t} \leq \|T\|_{p_0-p_0}^{1-t}\|T\|_{p_1-p_1}^{t}.$$

Theorem 2 (Marcinkiewicz). *If a sublinear operator T is both of weak type p_0–p_0 and of weak type p_1–p_1 for some $1 \leq p_0 < p_1 \leq \infty$, then T is bounded on $L^p(\mathbb{R}^n)$ for $p_0 < p < p_1$.*

Weak–type estimates are generally thought of as 'end–point' results in the sense that they often hold in limiting cases where strong–type estimates fail. There are other types of end–point estimates that may be interpolated in a similar way. For example, for $H^1(\mathbb{R}^n)$ the real Hardy space defined at the end of this chapter, the following theorem is a special case of one proved in [15].

Theorem 3. *If a linear operator T is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, and is also bounded on $L^{p_0}(\mathbb{R}^n)$ for some $1 < p_0 \leq \infty$, then T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq p_0$.*

The Hardy–Littlewood Maximal Function, and Decompositions of $\mathbb{R}^n$.

For an appropriate function $f : \mathbb{R}^n \to \mathbb{C}$, its Hardy–Littlewood Maximal Function is defined to be

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x;r)|} \int_{B(x;r)} |f(y)|dy.$$

Using a covering lemma one can establish the weak type 1–1 inequality

$$|\{x : Mf(x) > \alpha\}| \leq 3^n \frac{\|f\|_1}{\alpha}.$$

This estimate can be interpolated with the trivial $L^\infty \to L^\infty$ estimate (via the Marcinkiewicz Interpolation Theorem), to give L^p boundedness of M for $1 < p \leq \infty$. The next important concept for us is that of a Whitney decomposition.

Theorem 4 ([35]). *Let F be a non-empty closed set in $\mathbb{R}^n$. There is a disjoint sequence of cubes $\{Q_k\}$, whose sides are parallel to the axes, and whose interiors are mutually disjoint, for which $\cup Q_k = F^c$, and*

$$diam(Q_k) \leq dist(Q_k, F) \leq 4diam(Q_k.)$$

By applying a Whitney decomposition to the set

$$F = \{x : Mf(x) \leq \alpha\},$$

and using the weak type 1–1 boundedness of M, one may arrive at the following theorem, which is a variant of the Calderón–Zygmund decomposition; see Stein [35] for further discussion.

Theorem 5. *Let f be a non-negative integrable function on $\mathbb{R}^n$, and let α be a positive constant. There exists a decomposition of $f = g + \sum b_j$, and a sequence of cubes $\{Q_j\}$ such that*

(i) $\|g\|_\infty \leq C\alpha$, $\|g\|_1 \leq \|f\|_1$,

(ii) b_j *is supported on* Q_j,

(iii) the Q_j*'s have pairwise disjoint interiors, and in addition,*

(iv) if $dist(Q_j, Q_k) \leq diam(Q_j)$, *then* $1 \leq |Q_j|/|Q_k| \leq 4$,

(v) $\int b_j = 0$ *and,*

(vi) there is a constant c *depending only on* n *for which*

$$\frac{1}{|Q_j|}\int_{Q_j} |b_j| \leq c\alpha$$

for all j.

The above theorem lies at the roots of Calderón–Zygmund theory, and is one of the main ideas in all of the weak–type estimates that we will discuss in this thesis.

In the following section, details of results not otherwise referenced can be found in Stein [35] and [36].

1.3 Calderón–Zygmund Theory.

The Calderón–Zygmund theory of singular integral operators largely evolved from a real variable understanding of the classical Hilbert transform by Besicovitch [2] and Titchmarsh [38] in the late 1920's. Prior to this, the Hilbert transform, given by (1.1), had long been understood to be a fundamental operator in Complex Analysis. To be precise, if f is an analytic function on $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ with boundary values given by $u+iv$, where $u, v : \mathbb{R} \to \mathbb{R}$, and $u \in L^p(\mathbb{R})$ for some $1 \leq p < \infty$, then H can be defined on u, and $v = Hu$. Before the work of Besicovitch and Titchmarsh (see for example, work of Plessner [30], and Kolmogorov [18]), all of the techniques involved were essentially complex analytic. Besicovitch and Titchmarsh gave real variable proofs of the weak type 1–1 boundedness of the Hilbert transform, and its almost everywhere existence on L^p for $1 \leq p < \infty$. As these techniques made no use of the special role of H in Complex Analysis, the way was paved for a general theory of singular integral operators.

The modern n–dimensional theory originates in Calderón and Zygmund [4], and a popular formulation of their ideas, due to Hörmander, is as follows.

Suppose $K : \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{C}$ satisfies

$$\int_{|x|\geq 2|y|} |K(x-y) - K(x)|dx \leq c \tag{1.5}$$

for all $y \neq 0$. Suppose T is bounded on $L^2(\mathbb{R}^n)$, commutes with translations and satisfies

$$Tf(x) = \int_{\mathbb{R}^n} K(y)f(x-y)dy \tag{1.6}$$

whenever $f \in \mathcal{S}(\mathbb{R}^n)$ with $x \notin \text{supp}(f)$, then such an operator is called a Calderón-Zygmund operator, with Calderón-Zygmund kernel K.

Theorem 6. *T, as defined above, satisfies the weak type 1–1 inequality*

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \alpha\}| \leq C\frac{\|f\|_{L^1(\mathbb{R}^n)}}{\alpha}, \tag{1.7}$$

and is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

In order to explain the relevance of the smoothness condition (1.5) in the definition of K, we shall outline the proof of Theorem 6.

Fix $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$. We decompose f as in Theorem 5. By the triangle inequality

$$\begin{aligned} |\{x : |Tf(x)| > \alpha\}| &\leq |\{x : |Tg(x)| > \alpha/2\}| \\ &\quad + \left|\left\{x : \left|T(\textstyle\sum b_j)(x)\right| > \alpha/2\right\}\right|, \end{aligned} \tag{1.8}$$

and so in order to prove (1.7) it is enough to dominate each of these two terms by $C\|f\|_1/\alpha$. By Chebyshev's inequality, and the $L^2(\mathbb{R})$ boundedness of T,

$$|\{x : |Tg(x)| > \alpha/2\}| \leq \left(\frac{2\|Tg\|_2}{\alpha}\right)^2 \leq C\frac{\|g\|_2^2}{\alpha^2}.$$

Using the trivial fact that $\|g\|_2^2 \leq \|g\|_\infty \|g\|_1$ and part (i) of Theorem 5, gives the required estimate for the first term of (1.8). We now turn to the second term.

Let Q_j^* be the concentric double of Q_j, y_j the centre of Q_j, and let $\Omega = (\cup Q_k^*)^c$. By part (vi) of Theorem 5, $|\Omega^c| \leq C\|f\|_1/\alpha$, so it suffices to show that

$$\left|\left\{x \in \Omega : |T(\textstyle\sum b_j)(x)| > \alpha/2\right\}\right| \leq C\frac{\|f\|_1}{\alpha}.$$

By Chebychev's inequality,

$$\left|\left\{x \in \Omega : |T(\sum b_j)(x)| > \alpha/2\right\}\right|$$
$$\leq \frac{2}{\alpha}\int_\Omega \left|\int_{\mathbb{R}^n} \sum_j b_j(y)K(x-y)dy\right| dx$$
$$= \frac{2}{\alpha}\int_\Omega \left|\sum_j \int_{\mathbb{R}^n} b_j(y)(K(x-y) - K(x-y_j))dy\right| dx$$
$$\text{(since } \int b_j = 0 \text{ for each } j)$$
$$\leq \frac{2}{\alpha}\sum_j \int_{Q_j} |b_j(y)| \int_{\mathbb{R}^n \setminus Q_j^*} |K(x-y) - K(x-y_j)| dx dy,$$

which by the smoothness condition on K, (1.5), is

$$\leq \frac{C}{\alpha}\sum_j \int_{Q_j} |b_j(y)| dy \leq C\frac{\|f\|_1}{\alpha}.$$

This completes the proof of (1.7).

The $L^p(\mathbb{R}^n)$ boundedness of T now follows from the Marcinkiewicz Interpolation Theorem, and a duality argument.

In Chapter 3 we consider the behaviour on L^1 of a class of singular integral operators for which the smoothness condition (1.5) fails.

Remarks

(i) In order to make the conditions on T more explicit, we remark that the hypothesis of $L^2(\mathbb{R}^n)$ boundedness may be replaced by the size condition

$$|K(x)| \leq \frac{c}{|x|^n},\ x \neq 0, \tag{1.9}$$

along with the cancellation condition

$$\sup_{0<\alpha\leq\beta} \left|\int_{\alpha\leq|x|\leq\beta} K(x)dx\right| < \infty. \tag{1.10}$$

(ii) *Injectivity of* T. Since T is bounded on $L^2(\mathbb{R}^n)$ and commutes with translations, it has a Fourier multiplier representation, i.e. there is a bounded function $\mathfrak{m}$ such that $\widehat{Tf} = \mathfrak{m}\hat{f}$ for all $f \in L^2(\mathbb{R}^n)$. Consequently T is injective on $L^2(\mathbb{R}^n)$ if and only if $\mathfrak{m} \neq 0$ almost everywhere. The question of injectivity on $L^1(\mathbb{R}^n)$ is much more subtle since it is not immediately clear how we should interpret $\widehat{Tf}$ for $f \in L^1(\mathbb{R}^n)$. In [1] we have recently overcome this problem under the additional size condition (1.9). In fact, the pointwise size condition (1.9) may be weakened at the expense of a strengthening of the smoothness condition (1.5). An appropriate setting for this more balanced result is in a class of operators which respect more general sets of dilations.

Calderón–Zygmund theory with general dilations.

If K is a convolution kernel giving rise to an L^p bounded operator T, and if $A \in GL(n;\mathbb{R})$, then the L^p operator norm of convolution with $\det A^{-1}K(A^{-1}x)$ is independent of A. However, the conditions imposed on the Calderón–Zygmund kernel in (1.5) and (1.9) do not hold uniformly under such actions by general $A \in GL(n;\mathbb{R})$. They are only invariant in this way under isotropic dilations, i.e. those given by $A = \lambda I$, for $\lambda \in \mathbb{R}$. A Calderón–Zygmund theory for kernels with a more general homogeneity has been developed in [6], see also [5]. It turns out that an appropriate condition to impose on the dilations is the so called Rivière condition. That is, we suppose that for each $t > 0$, $A(t) \in GL(n;\mathbb{R})$, and that

$$||A(s)^{-1}A(t)|| \leq C(t/s)^{\epsilon}, \tag{1.11}$$

for all $s \geq t$ and some $\epsilon > 0$.

Let B_0 be the unit ball in $\mathbb{R}^n$.

Theorem 7. *Suppose $Tf = f * K$ is an $L^2(\mathbb{R}^n)$-bounded operator. Suppose also that the distribution*

$$K = \sum_{j\in\mathbb{Z}} K_j,$$

with K_j supported in $A(2^{j+1})B_0$. Let $\widetilde{K}_j(x) = \det A(2^j)K_j(A(2^j)x)$. Suppose

$$\int |\widetilde{K}_j(x)|dx \leq C,$$

and

$$\int |\widetilde{K}_j(x-y) - \widetilde{K}_j(x)|dx \leq C|y|^{\epsilon} \tag{1.12}$$

for some $\epsilon > 0$. If $\{A(t)\}$ satisfies the Rivière condition (1.11), then T is of weak type 1–1, and bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

In Chapter 5 we give necessary and sufficient conditions for such an operator to be injective on $L^1(\mathbb{R}^n)$.

The analogue of the classical Hardy–Littlewood maximal function, where the averages are now taken over translates of the family of 'balls' $\{A(2^j)B_0\}_{j\in\mathbb{Z}}$, is also of weak type 1–1, and bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$. The reader is again referred to [6].

The Calderón–Zygmund theory for general dilations and its variants underpin the subtle theory of Singular and Maximal Radon Transforms; to which our discussion now turns.

1.4 Beyond the Calderón–Zygmund Theory

1.4.1 Singular and Maximal Radon Transforms

We begin by making some formal definitions in order to set the scene.

Let k be an integer strictly less than n. Let us asign to each point $x \in \mathbb{R}^n$, a "k–dimensional surface" given by $\{\Gamma(x,t) : t \in \mathbb{R}^k\}$, for some $\Gamma : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$. To this family of surfaces we associate the Maximal Radon Transform

$$Mf(x) = \sup_{h>0} \frac{1}{h^k} \left| \int_{|t|\leq h} f(\Gamma(x,t))dt \right|. \tag{1.13}$$

In addition, if K is a k–dimensional Calderón–Zygmund kernel, we may form the Singular Radon Transform

$$Tf(x) = \int_{\mathbb{R}^k} f(\Gamma(x,t))K(t)dt. \tag{1.14}$$

The question that we wish to address is the following:

Under what conditions on the family of surfaces, Γ, are M and T bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$?

Even though the above operators are much more singular than the standard Calderón–Zygmund operators, (i.e. the singularities of the kernels live on higher dimensional varieties), their $L^p(\mathbb{R}^n)$ boundedness can be seen partly as a consequence of the classical Calderón–Zygmund theory of the previous section. This often materialises in the form of Littlewood–Paley theory. In what follows we will discuss $L^2(\mathbb{R}^n)$ boundedness, and then, where possible, briefly describe the appropriate Calderón–Zygmund theory.

The translation invariant case.

In this case the surfaces involved are all translates of one fixed surface $\Gamma : \mathbb{R}^k \to \mathbb{R}^n$, i.e. $\Gamma(x,t) = x - \Gamma(t)$. Since the associated operators T, and M, are now translation invariant, one has the Fourier transform as a tool. For the singular integral, $L^p(\mathbb{R}^n)$ boundedness is equivalent, via the Fourier transform, to

$$m(\xi) = \int_{\mathbb{R}^k} e^{i\Gamma(t)\cdot\xi} K(t)dt \tag{1.15}$$

being an $L^p(\mathbb{R}^n)$ multiplier. For the maximal function, a further argument is required before we employ the Fourier transform.

For the sake of simplicity we shall describe the theory in the case of the parabola in $\mathbb{R}^2$, $\Gamma(t) = (t, t^2)$. In this case,

$$Mf(x) = \sup_{h>0} \frac{1}{2h} \left| \int_{|t|\leq h} f(x_1 - t, x_2 - t^2) dt \right| \tag{1.16}$$

and,

$$Tf(x) = \int_{-\infty}^{\infty} f(x_1 - t, x_2 - t^2) \frac{dt}{t}. \tag{1.17}$$

We shall begin with $L^2(\mathbb{R}^2)$ estimates for T. The nature of the measure dt/t suggests we write

$$T = \sum_{k\in\mathbb{Z}} T_k,$$

where

$$T_k f(x) = \int_{2^k<|t|\leq 2^{k+1}} f(x_1 - t, x_2 - t^2) \frac{dt}{t}.$$

Changing variables gives

$$\begin{aligned} T_k f(x) &= \int_{1<|t|\leq 2} f(x_1 - 2^k t, x_2 - 2^{2k} t^2) \frac{dt}{t} \\ &= \int_{1<|t|\leq 2} f(x - \delta_k \Gamma(t)) \frac{dt}{t}, \end{aligned} \tag{1.18}$$

where

$$\delta_k = \begin{pmatrix} 2^k & 0 \\ 0 & 2^{2k} \end{pmatrix}.$$

Taking the Fourier transform of (1.18) gives

$$\widehat{T_k f}(\xi) = m(\delta_k^{-1}\xi) \hat{f}(\xi), \tag{1.19}$$

where

$$m(\xi_1, \xi_2) = \int_{1<|t|\leq 2} e^{i(t\xi_1 + t^2\xi_2)} \frac{dt}{t}.$$

The curvature of the parabola ensures that the phase is not stationary to infinite order for any one $\xi \in \mathbb{R}^2$. This allows one to make the estimate $|m(\xi)| \leq c\min\{|\xi|, |\xi|^{-1/2}\}$, and conclude that

$$\sum_k m(\delta_k^{-1}\xi) \tag{1.20}$$

is bounded, and hence that T is bounded on $L^2(\mathbb{R}^2)$.

The $L^2(\mathbb{R}^2)$ estimates for the maximal function use the dilations $\{\delta_k\}$ in a more explicit way, which we now describe. We first remark that we may suppose the supremum in (1.16) is taken over h of the form 2^j, for $j \in \mathbb{Z}$. Define the averaging operators A_j by

$$A_j f(x) = \frac{1}{2^{j+1}} \int_{|t| \leq 2^j} f(x - \Gamma(t))dt.$$

We now wish to define some less singular averaging operators, S_j, which approximate A_j in some sense. Let $\psi \in C_c^\infty(\mathbb{R}^2)$ be non-negative and satisfy $\psi(0) = 1$. For $\psi_j(x) = \det \delta_j^{-1} \psi(\delta_j^{-1} x)$ define $S_j f = \psi_j * f$. Now

$$Mf(x) = \sup_j |A_j f(x)| \leq \sup_j |(A_j - S_j)f(x)| + \sup_j |S_j f(x)|.$$

By [3] (see also [6]), $f \mapsto \sup_j |S_j f(\cdot)|$ is bounded on $L^p(\mathbb{R}^2)$ for $1 < p \leq \infty$. Hence it suffices to control

$$f \mapsto \sup_j |(A_j - S_j)f(\cdot)|.$$

The idea now is to dominate the above by the square function

$$Gf(x) = \left(\sum_j |(A_j - S_j)f(x)|^2 \right)^{1/2},$$

which can be thought of as the l^2 norm of an l^2–valued singular integral operator, as described in [36]. Through this reasoning we see that, in principle, the analysis of M is very similar to that of T. By Plancherel's theorem, $L^2(\mathbb{R}^2)$ boundedness of G is equivalent to the boundedness of

$$\sum_j |m_j(\xi) - \widehat{\psi}_j(\xi)|^2, \tag{1.21}$$

where m_j is the Fourier multiplier corresponding to A_j. The boundedness of (1.21) may now be established in a similar way to that of (1.20).

As remarked, the $L^p(\mathbb{R}^2)$ estimates for $p \neq 2$ may be obtained by an appropriate variant of the Calderón–Zygmund theory, which we now sketch for T. Let K_j be the distributional convolution kernel of T_j. Next observe that $\{K_j\}$ and $\{A(2^j)\}$ satisfy the conditions of the Calderón–Zygmund theorem for general dilations (Theorem 7), with the exception of the smoothness estimate (1.12). This is not surprising since each K_j is singular with respect to Lebesgue measure on $\mathbb{R}^2$. However, by decomposing each K_j in ξ–space in an appropriate way, one may express K_j as the sum of kernels, each of which has enough smoothness to apply Theorem 7. An interpolation argument then gives $L^p(\mathbb{R}^2)$ boundedness for $1 < p < \infty$. For a fuller explanation of this argument see [5].

The $L^p(\mathbb{R}^2)$ estimates, $p \neq 2$, for T and M were originally obtained using Stein's Complex Interpolation. See [36].

As we alluded to in the above example, curvature has a decisive role to play in the wider theory of singular and maximal Radon transforms. In the 1970's Nagel, Rivière, Stein, and Wainger introduced the following notion of curvature for curves in $\mathbb{R}^n$. See [33] for further discussion.

Definition 8. *A curve $\Gamma : \mathbb{R} \to \mathbb{R}^n$ is well curved if for all t in a neighbourhood of the origin, $\Gamma(t)$ lies in the span of the vectors $\Gamma(0), \Gamma'(0), ..., \Gamma^{(j)}(0)$, for some fixed j.*

Theorem 9 (Nagel, Rivière, Stein, Wainger [33]). *If $\Gamma : \mathbb{R} \to \mathbb{R}^n$ well curved, and $\Gamma(0) = 0$, then the local Hilbert transform $\mathcal{H}_\Gamma$, and local maximal function $\mathcal{M}_\Gamma$, given by*

$$\mathcal{H}_\Gamma f(x) = p.v. \int_{-1}^{1} f(x - \Gamma(t)) \frac{dt}{t},$$

and

$$\mathcal{M}_\Gamma f(x) = \sup_{0<h<1} \frac{1}{h} \left| \int_0^h f(x - \Gamma(t)) dt \right|,$$

are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Following these satisfactory results in the well curved situation, it was observed that one could obtain positive results for curves under much weaker curvature conditions than those in the statement of Theorem 9. In fact certain curves that vanish to infinite order may be permitted. It became interesting to characterise the bounded operators in terms of geometrical properties of the curves. A great deal of this work has been focused on convex curves in the plane of the form

$$\Gamma(t) = (t, \gamma(t)), \tag{1.22}$$

where there has been much success. This was facilitated by the discovery of appropriate sets of dilations $\{\delta_k\}$ for certain classes of flat curves. Some of the main results are as follows.

Let Γ be a curve in $\mathbb{R}^2$ of the form (1.22), where

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R} \text{ is convex on } [0, \infty) \text{ and } \gamma(0) = \gamma'(0) = 0. \tag{1.23}$$

For $\gamma \in C^2(0, \infty)$, let

$$h(t) = t\gamma'(t) - \gamma(t), \ t > 0. \tag{1.24}$$

The Hilbert transform and maximal function along the curve Γ are defined by

$$H_\Gamma f(x) = p.v. \int_{-\infty}^{\infty} f(x - \Gamma(t)) \frac{dt}{t},$$

and

$$M_\Gamma f(x) = \sup_{h>0} \frac{1}{h} \left| \int_0^h f(x - \Gamma(t)) dt \right|.$$

Theorem 10 (Nagel, Vance, Wainger, and Weinberg [20]). *Suppose that Γ satisfies (1.22) and (1.23), and γ is an odd function of class $C^2(0,\infty)$. Suppose also that h satisfies the doubling property*

$$\exists C < \infty \text{ so that for each } t > 0,\ h(Ct) \geq 2h(t), \tag{1.25}$$

then both M_Γ and H_Γ are bounded on $L^2(\mathbb{R}^2)$. Moreover (1.25) is a necessary condition for the $L^2(\mathbb{R}^2)$ boundedness of H_Γ.

If γ satisfies (1.25) then we say that γ is h-doubling.

Theorem 11 (Carbery, Christ, Vance, Wainger, and Watson [6]). *Suppose that Γ satisfies (1.22) and (1.23), γ is of class $C^2(0,\infty)$, and γ is odd. Suppose also that*

$$\exists \epsilon > 0 \text{ so that for each } t > 0,\ h'(t) > \epsilon h(t)/t, \tag{1.26}$$

then both M_Γ and H_Γ are bounded on $L^p(\mathbb{R}^2)$, $1 < p < \infty$.

We refer to condition (1.26) as the infinitesimal doubling condition.

Theorem 12 ([11]). *Suppose that Γ satisfies (1.22) and (1.23), and γ is of class $C^2(0,\infty)$. If Γ is either even or odd and γ' satisfies the doubling property*

$$\exists C < \infty \text{ so that for each } t > 0,\ \gamma'(Ct) \geq 2\gamma'(t), \tag{1.27}$$

then M_Γ and H_Γ are bounded on $L^p(\mathbb{R}^2)$, $1 < p < \infty$. Moreover if γ is even, (1.27) is a necessary condition for $L^p(\mathbb{R}^2)$ boundedness of H_Γ, $1 < p < \infty$.

Carbery, Vance, Wainger, and Watson [7], later constructed appropriate dilations for convex curves in $\mathbb{R}^n$, and used them to extend Theorem 11. In [7] they also describe some different dilations which are particularly curious because of their connection with the theory of asymptotic stability of systems of ordinary differential equations.

For surfaces in $\mathbb{R}^n$ of dimension greater than or equal to two, natural dilations seem less apparent. Consequently, the operators associated to flat surfaces have been largely neglected; see however, [42] and [41]. In Chapter 3 we present a simple perspective on singular integrals and maximal functions associated to surfaces based on results for curves, such as Theorems 10, 11, and 12. Some very simple sufficient conditions for L^2 (and L^p, $1 < p < \infty$) boundedness of the operators are given which treat very many surfaces that vanish to infinite order at the origin.

The non translation invariant case

In the full non translation invariant case one might hope for a diffeomorphism invariant theory. In [13], Christ, Nagel, Stein, and Wainger achieve this under a certain local curvature condition on Γ. In the special case when the operators are translation invariant, they recover Theorem 9.

As yet there are no diffeomorphism invariant results for classes of curves or surfaces which allow the curvature condition in [13] to fail. To provide such a result is a major aim for the future of this theory. We refer the reader to [32] for partial results.

1.4.2 Oscillatory Singular Integrals.

In much of this thesis we shall be interested in singular integral operators whose kernels also have an oscillating factor. For an n–dimensional Calderón–Zygmund kernel K, a phase $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and $\lambda \in \mathbb{R}$, we formally define the singular oscillatory integral operator T_λ, by

$$T_\lambda f(x) = p.v. \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y)} K(x,y) f(y) dy. \tag{1.28}$$

Under certain conditions on Φ and K, we can make sense of this operator.

The study of the operators T_λ has been largely motivated by their intimate connection with the theory of singular integrals along curves [1]. For example, if $n = 2$, $K(x,y) = \frac{1}{x-y}$, $\Phi \in C(\mathbb{R}^2)$ and

$$Hf(x) = p.v. \int_{\mathbb{R}^2} f(x_1 - t, x_2 - \Phi(x_1, x_1 - t)) \frac{dt}{t},$$

then

$$\mathcal{F}_2 Hf(x_1, \lambda) = T_\lambda(\mathcal{F}_2 f(\cdot, \lambda))(x_1),$$

where $\mathcal{F}_2$ denotes the Fourier transform in the second variable. By applying Plancherel's theorem we see that

$$\sup_{\lambda \in \mathbb{R}} \|T_\lambda\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = \|H\|_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)}.$$

[1] Or singular Radon transforms

It should be remarked that in order to make sense of the above we need to suppose that H or the T_λ's make sense as principal values in an appropriate operator norm. All of the operators that we will discuss will exist as principal values in the strong operator norm. The operators T_λ also inherit uniform $L^p(\mathbb{R})$, $1 < p < \infty$, boundedness, via the following vector–valued version of deLeeuw's Theorem. For the standard scalar–valued version see [19].

Proposition 13. *Suppose $\Phi \in C(\mathbb{R}^2)$. If H, defined as a principal value in the strong operator topology, is bounded on $L^p(\mathbb{R}^2)$ for some $1 < p < \infty$, then T_λ is similarly well defined and*

$$\sup_{\lambda\in\mathbb{R}} ||T_\lambda||_{L^p(\mathbb{R})\to L^p(\mathbb{R})} \leq ||H||_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)}.$$

Proof. Let $\epsilon > 0$ and $g \in C_c^\infty(\mathbb{R})$. For some $\phi \in C_c^\infty(\mathbb{R})$ with $\phi(0) = 1$, define

$$f_\epsilon(x) = \epsilon^{1/p} g(x_1) e^{-i\lambda x_2} \phi(\epsilon x_2).$$

Now observe that

$$||Hf_\epsilon||_{L^p(\mathbb{R}^2)} \to ||T_\lambda g||_{L^p(\mathbb{R})} ||\phi||_{L^p(\mathbb{R})}$$

as $\epsilon \to 0$, and

$$||f_\epsilon||_{L^p(\mathbb{R}^2)} = ||g||_{L^p(\mathbb{R})} ||\phi||_{L^p(\mathbb{R})}$$

hence

$$\sup_{\lambda\in\mathbb{R}} ||T_\lambda||_{p-p} \leq ||H||_{p-p}.$$

□

A brief review of some known results.

(i) L^p Theory.

Oscillatory singular integrals were first described at this level of generality, and in this context, by Phong and Stein [29]. For a discussion of the history thereto the reader is referred to [29]. In their paper Phong and Stein show $L^p(\mathbb{R}^n)$ boundedness $(1 < p < \infty)$ of the operator

$$Tf(x) = \int_{\mathbb{R}^n} e^{i\langle Bx,y\rangle} K(x-y) f(y) dy,$$

where $\langle Bx, y\rangle$ is a real bilinear form and K is a Calderón–Zygmund kernel. The bound is shown to be independent of the matrix B. An important step was then made by Ricci and Stein in [31], where for a polynomial $P : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and a Calderón–Zygmund kernel K, they bound

$$Tf(x) = \int_{\mathbb{R}^n} e^{iP(x,y)} K(x-y) f(y) dy$$

on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. The bound is shown to be independent of the coefficients of P. In [21], Pan makes the natural extension of this to operators whose phases are smooth and of finite type. We say that $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is of finite type at point $w \in \mathbb{R}^n \times \mathbb{R}^n$ if for some $1 \leq j, k \leq n$,

$$\frac{\partial^2 \Phi}{\partial x_j \partial y_k}$$

does not vanish to infinite order at w. To be precise, Pan concludes that if $\varphi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and Φ is of finite type on

$$\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\} \cap \operatorname{supp}(\varphi),$$

then

$$Tf(x) = \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y)} K(x-y) \varphi(x,y) f(y) dy$$

is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, uniformly in λ.

We remark that these finite–type results can also be obtained (via Proposition 13) from the far reaching work of Christ, Nagel, Stein, and Wainger [13].

It is known that finite type conditions are not necessary for uniform L^p boundedness. For example, one can apply Proposition 13 to results about Hilbert transforms along curves in $\mathbb{R}^2$, such as Theorems 10, 11, and 12. This immediately gives positive results for the operators

$$T_\lambda f(x) = \int_{-\infty}^{\infty} \frac{e^{i\lambda\gamma(x-y)}}{x-y} f(y) dy, \tag{1.29}$$

under certain conditions which permit γ to vanish to infinite order at the origin. In particular, Theorem 12 implies the following.

Theorem 14. *Suppose $\gamma : \mathbb{R} \to \mathbb{R}$ is either even or odd, convex on $[0, \infty)$, and $\gamma(0) = \gamma'(0) = 0$.*

If γ' is doubling; i.e. (1.27) *holds, then the operators T_λ given by* (1.29) *are uniformly bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$.*

More recently, in [32], Seeger has generalised Theorem 14 to handle a class of phases that is diffeomorphism invariant. This clearly takes one out of the realm of translation invariant operators, but still permits the finite–type condition to fail. Previously, Carbery, Wainger, and Wright [9], by very different methods, concluded the following.

Theorem 15 (Carbery, Wainger, Wright). *Suppose γ is either even or odd, convex, $\gamma(0) = \gamma'(0) = 0$ and $t\gamma''(t)/\gamma'(t)$ is decreasing and bounded below. Then the Hilbert transform*

$$H_\gamma f(x) = p.v. \int_{-\infty}^{\infty} f(x_1 - t, x_2 - x_1\gamma(t))\frac{dt}{t}$$

is bounded on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$.

Again, using Proposition 13, one can deduce the uniform $L^p(\mathbb{R})$, $1 < p < \infty$, boundedness of the non translation invariant operators

$$T_\lambda f(x) = \int_{-\infty}^{\infty} \frac{e^{i\lambda x\gamma(x-y)}}{x-y} f(y)dy, \tag{1.30}$$

for γ satisfying the conditions of Theorem 15.

In Chapter 2 we consider generalisations of the operators (1.30), given by

$$Tf(x) = \int_{-\infty}^{\infty} \frac{e^{iP(x)\gamma(x-y)}}{x-y} f(y)dy,$$

where P is a polynomial.

(ii) L^1 Theory.

The question of weak type 1–1 boundedness of singular integrals and maximal functions along non-trivial curves has, so far, not been answered even in the simplest of cases. However there has been much success for the operators T_λ. Behind all of the known weak type 1–1 results is a certain $L^1 \to L^2$ estimate, the principle behind which first arose in a fundamental paper of C. Fefferman from 1970. See [16]. Using this principle, Chanillo and Christ [12] were able to obtain weak type 1–1 boundedness of T_λ when the phase Φ is a polynomial, with bounds depending only on the degree of the polynomial. Through work of Pan, this was extended to cover real–analytic, and later, finite type phases. See [26] and [28] respectively. However, in dimension greater than one, Pan makes the additional assumption that the phase is of the form $\Phi(x, y) = \phi(x - y)$; i.e. the associated operators are translation invariant.

Restricting himself to the translation invariant operators

$$T_\lambda f(x) = \int_{|x-y|\leq 1} \frac{e^{i\lambda\gamma(x-y)}}{x-y} f(y)dy,$$

Pan was able to obtain uniform weak type 1–1 boundedness for a class of phases which permit flatness at the singularity $x = y$.

In Chapter 3 we look for weak type 1–1 boundedness of the family of non translation invariant operators given by (1.30), under conditions on γ which also permit flatness at the origin.

(iii) The Real Hardy Space $H^1(\mathbb{R}^n)$

The n–dimensional analogues of the Hilbert transform on the line are the Riesz transforms R_j, given by convolution with the distributions (or Calderón–Zygmund kernels) $x_j/|x|^{n+1}$, for $1 \le j \le n$. Since $R_j f$, $1 \le j \le n$, are the boundary values of the conjugates of the harmonic extension of f to $\mathbb{R}^{n+1}_+$, then, by analogy with the one dimensional case we may define the real Hardy space $H^1(\mathbb{R}^n)$ as

$$H^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : R_j f \in L^1(\mathbb{R}^n), 1 \le j \le n\},$$

with norm given by

$$||f||_{H^1(\mathbb{R}^n)} = ||f||_{L^1(\mathbb{R}^n)} + \sum_{j=1}^{n} ||R_j f||_{L^1(\mathbb{R}^n)}.$$

This is one of several equivalent definitions of $H^1(\mathbb{R}^n)$, and can be found in [36].

Definition 16. *A function $a : \mathbb{R}^n \longrightarrow \mathbb{R}$ is an $H^1(\mathbb{R}^n)$ atom if*

(i) a is supported in a ball B,

(ii) $|a| \le |B|^{-1}$, and

(iii) $\int a(x)dx = 0$.

Theorem 17 (The Atomic Decomposition of $H^1(\mathbb{R}^n)$). *Given $f \in H^1(\mathbb{R}^n)$, there is a sequence of $H^1(\mathbb{R}^n)$ atoms $\{a_k\}$, and complex numbers $\{\lambda_k\}$ such that*

$$f = \sum_k \lambda_k a_k$$

in $H^1(\mathbb{R}^n)$ norm. Moreover,

$$\sum_k |\lambda_k| \le c||f||_{H^1(\mathbb{R}^n)}.$$

The Atomic Decomposition has the following very practical corollary.

Corollary 18. *If a linear operator T satisfies*

$$||Ta||_{L^1(\mathbb{R}^n)} \le c$$

uniformly over all $H^1(\mathbb{R}^n)$ atoms, then T is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

As we discussed earlier (see Theorem 3), one of the main motives for studying the behaviour of operators on H^1 is that H^1–L^1 estimates may be interpolated with L^p–L^p estimates, for $1 \le p \le \infty$.

The known H^1–L^1 boundedness results for the operators (1.28) run essentially parallel to the weak type 1–1 results discussed earlier. However, in certain cases, variants of H^1 are used that are tailored to the particular class of operators in question. For example, if $P : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a polynomial, we may alter the definition of an atom by replacing $\int a(x)dx = 0$ with

$$\int e^{iP(x_B,y)}a(y)dy = 0,$$

where x_B is the centre of B. For our purposes we will refer to such an a as a modified atom. The definition of the corresponding Hardy space H^1_E is as follows.

Definition 19. *A function f is said to be in $H^1_E(\mathbb{R}^n)$ if $f \in L^1(\mathbb{R}^n)$, and f can be written as*

$$f = \sum_j \beta_j a_j$$

for some $\{\beta_j\} \subset \mathbb{R}$ and modified atoms a_j. The $H^1_E(\mathbb{R}^n)$ norm of f is given by

$$\|f\|_{H^1_E(\mathbb{R}^n)} = \inf\left\{\sum_j |\beta_j| : f = \sum_j \beta_j a_j\right\}.$$

These variants of the classical Hardy space $H^1(\mathbb{R}^n)$ first appeared in work of Phong and Stein [29]. Phong and Stein also observe that H^1_E–L^1 estimates can be used for interpolation purposes just as in the standard case.

It was proved by Pan in [22] that

$$Tf(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x,y)}K(x-y)f(y)dy,$$

where K is a standard Calderón–Zygmund kernel, is bounded from $H^1_E(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, with a bound that does not depend on the coefficients of P. In [27] this is extended to cover real–analytic phases, but only in dimension 1. The natural extension from polynomial to finite–type phases has, so far, only been successful in the case where the operators are translation invariant; i.e. the phase is of the form $\Phi(x,y) = \phi(x-y)$, see [23].

In [25] Pan has shown uniform $H^1(\mathbb{R})$–$L^1(\mathbb{R})$ boundedness of a class of translation invariant operators whose phases may be flat. We refer the reader forward to Chapter 3 for a precise formulation of Pan's result. In Chapter 3 we extend what is known by showing that a class of non translation invariant operators, for which the finite–type condition may fail, is uniformly bounded from the standard $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$.

1.5 The structure of the thesis – a summary

Chapter 2

In this chapter we prove $L^p(\mathbb{R})$, $1 < p < \infty$, boundedness of oscillatory singular integrals of the form

$$Tf(x) = p.v. \int_{-\infty}^{\infty} \frac{e^{iP(x)\gamma(x-y)}}{x-y} f(y)dy,$$

for P a real valued polynomial, and $\gamma : \mathbb{R} \to \mathbb{R}$ a convex curve satisfying certain conditions that permit it to vanish to infinite order at the origin. The bounds are shown to be independent of the coefficients of the P.

Chapter 3

In this chapter we obtain weak type 1–1 boundedness and boundedness from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$ of the operators

$$Tf(x) = p.v. \int_{-\infty}^{\infty} \frac{e^{i\lambda x\gamma(x-y)}}{x-y} f(y)dy,$$

under conditions on γ similar to those in Chapter 2. The bounds we obtain are seen to be uniform in $\lambda \in \mathbb{R}$.

Chapter 4

In this chapter we describe a simple perspective on singular integrals and maximal functions associated to surfaces in $\mathbb{R}^n$. Our perspective allows us to formulate a variety of simple conditions that guarantee their $L^p(\mathbb{R}^n)$ boundedness. These conditions treat many surfaces that vanish to infinite order at the origin. Our results are consequences of the known theorems for Hilbert transforms and maximal functions along plane curves. We are also able to bound on $L^2(\mathbb{R}^n)$ some singular integrals associated to variable flat surfaces using the results of Chapter 2.

Chapter 5

The main purpose of this chapter is to characterise those Calderón–Zygmund operators, of convolution type, that are injective on $L^1(\mathbb{R}^n)$. We do this by proving a Fourier multiplier relation on $L^1(\mathbb{R}^n)$ which uses a generalised integral. Our techniques also allow us to come to a similar conclusion for a class of oscillatory singular integral operators.

Chapter 2

Some oscillatory singular integrals with variable flat phases; estimates on $L^p(\mathbb{R})$, $1 < p < \infty$.

This chapter is devoted to the study of the operators

$$Tf(x) = p.v. \int_{-\infty}^{\infty} \frac{e^{iP(x)\gamma(x-y)}}{x-y} f(y)dy,$$

for P a real–valued polynomial, and $\gamma : \mathbb{R} \to \mathbb{R}$ satisfying certain growth conditions. What is of prime interest to us is that these conditions will not exclude γ that vanish to infinite order at the origin. For example, γ may behave like $\exp(-t^{-2})$ for small t.

The following Theorem is a significant step forward from Theorem 15 of Carbery, Wainger, and Wright [9]. The proof we give uses many ideas from [9].

Theorem 20. *Let $P : \mathbb{R} \to \mathbb{R}$ be a real polynomial of degree n, and let $\gamma \in C^3(\mathbb{R})$ be either odd or even, convex, and satisfy*

(i) $\gamma(0) = \gamma'(0) = 0$,

(ii) $\lambda(t) = t\gamma''(t)/\gamma'(t)$ *is decreasing and bounded below on* $(0, \infty)$,

then

$$Tf(x) = p.v. \int_{-\infty}^{\infty} \frac{e^{iP(x)\gamma(x-y)}}{x-y} f(y)dy$$

is bounded on $L^p(\mathbb{R})$, $1 < p < \infty$, with bound independent of the coefficients of P.

An amusing feature of the proof of the above Theorem is that if we do not look for independence of the coefficients, we are unable to conclude that the operators are bounded. This is because our proof goes by induction, and the independence claim in the inductive hypothesis is crucial.

By the observation preceding Lemma 13, we can immediately conclude the following from Theorem 20.

Corollary 21. *If P is a polynomial, and γ satifsies the conditions of Theorem 20, then the Hilbert transform*

$$Hf(x_1,x_2) = p.v.\int_{-\infty}^{\infty} f(x_1 - t, x_2 - P(x_1)\gamma(t))\frac{dt}{t},$$

is bounded on $L^2(\mathbb{R}^2)$ with a bound that depends only on the degree of P.

Remarks

(i) The proof of Theorem 20 shows that $\|H\|_{2-2} \le C_n\lambda_0^{-1/2}$, where $\lambda_0 = \inf_{t>0}\lambda(t)$.

(ii) It is not possible to deduce $L^p(\mathbb{R}^2)$ boundedness, $1 < p < \infty$, of H from Theorem 20. Given the techniques developed in [9] and in this chapter, $L^p(\mathbb{R}^2)$ boundedness of H seems a viable proposition; this we hope to return to at a later date.

Prerequisites

We begin by establishing some simple properties of the curves γ.

Lemma 22. *γ' is doubling; i.e. there exists $C < \infty$ for which*

$$\gamma'(Ct) \ge 2\gamma'(t) \quad \textit{for all } t > 0. \tag{2.1}$$

Proof. If $C = e^{\frac{2}{\lambda_0}}$, then

$$\gamma'(Ct) = \int_0^{Ct}\gamma''(s)ds \ge \int_t^{Ct}\gamma''(s)ds \ge \lambda_0\int_t^{Ct}\frac{\gamma'(s)}{s}ds \ge \lambda_0\gamma'(t)\log C = 2\gamma'(t),$$

for all $t > 0$. □

Lemma 23. *If $g(s,t) = \frac{\gamma'(s)-\gamma'(t)}{\gamma(s)-\gamma(t)}$, then*

$$\frac{\partial g}{\partial s} \le 0 \quad \textit{for } s,t \ge 0,\ s \ne t. \tag{2.2}$$

Proof.

$$\begin{aligned}\frac{\partial g}{\partial s} =& \frac{(\gamma(s)-\gamma(t))\gamma''(s)-(\gamma'(s)-\gamma'(t))\gamma'(s)}{(\gamma(s)-\gamma(t))^2}\\ =& \frac{\gamma'(s)}{\gamma(s)-\gamma(t)}\left(\frac{\gamma''(s)}{\gamma'(s)}-\frac{\gamma'(s)-\gamma'(t)}{\gamma(s)-\gamma(t)}\right)\\ =& \frac{\gamma'(s)}{\gamma(s)-\gamma(t)}\left(\frac{\gamma''(s)}{\gamma'(s)}-\frac{\gamma''(\theta)}{\gamma'(\theta)}\right),\end{aligned}$$

for some θ between s and t, by the Generalised Mean Value Theorem. Since $\frac{\gamma''}{\gamma'}$ is decreasing, $\frac{\partial g}{\partial s} \leq 0$. □

Lemma 24. *There exists $c > 0$ for which*

$$\gamma(s)-\gamma(t) \geq c(s-t)\gamma(s), \tag{2.3}$$

and

$$\gamma'(s)-\gamma'(t) \geq c(s-t)\gamma'(s), \tag{2.4}$$

for all $1 \leq t \leq s \leq 2$.

Proof. We will prove (2.4); (2.3) is similar. We may suppose that $\gamma'(t) > \frac{1}{2}\gamma'(s)$, since on the other hand,

$$\gamma'(s)-\gamma'(t) \geq \tfrac{1}{2}\gamma'(s) \geq \tfrac{1}{2}(s-t)\gamma'(s),$$

for $1 \leq t \leq s \leq 2$.

Let $\lambda_0 = \inf_{t>0}\lambda(t)$. If $\gamma'(t) > \frac{1}{2}\gamma'(s)$, then

$$\gamma'(s)-\gamma'(t) = \int_t^s \gamma''(x)dx = \int_t^s \lambda(x)\frac{\gamma'(x)}{x}dx \geq \tfrac{\lambda_0}{2}(s-t)\gamma'(t) \geq \tfrac{\lambda_0}{4}(s-t)\gamma'(s),$$

for $1 \leq t \leq s \leq 2$. □

In what follows we shall need the following well–known lemma, which is a consequence of the Mean Value Theorem.

Lemma 25. *If P is a real monic polynomial of one variable, and of degree n, then, there is a constant C which depends only on n for which*

$$|\{x \in \mathbb{R} : |P(x)| \leq \delta\}| \leq C\delta^{\frac{1}{n}},$$

for all $\delta > 0$.

Lemma 26. *Suppose T is an $L^p(\mathbb{R}^n)$ bounded operator for some $1 \leq p \leq \infty$, and has integral kernel $K(x,y)$. If $\phi \in C_c^\infty(\mathbb{R}^n)$, then the operator T_ϕ with integral kernel $K(x,y)\phi(x-y)$ is bounded on $L^p(\mathbb{R}^n)$, and $\|T_\phi\|_{p-p} \leq \|\widehat{\phi}\|_1\|T\|_{p-p}$.*

Proof. By writing ϕ as the inverse Fourier transform of $\widehat{\phi}$, we have

$$\begin{aligned} T_\phi f(x) &= \int_{\mathbb{R}^n} \widehat{\phi}(\xi) \int_{\mathbb{R}^n} e^{-2\pi i \xi\cdot(x-y)} K(x,y) f(y) dy d\xi \\ &= \int_{\mathbb{R}^n} \widehat{\phi}(\xi) e^{2\pi i \xi\cdot x} \int_{\mathbb{R}^n} K(x,y) e^{-2\pi i \xi\cdot y} f(y) dy d\xi \\ &= \int_{\mathbb{R}^n} \widehat{\phi}(\xi) e^{-2\pi i \xi\cdot x} T f_\xi(x) d\xi, \end{aligned}$$

where $f_\xi(y) = e^{2\pi i \xi\cdot y} f(y)$. Since $\|f_\xi\|_p = \|f\|_p$ for all $\xi \in \mathbb{R}^n$, the conclusion of Lemma 26 follows by Minkowski's inequality for integrals. □

Before we begin the proof of Theorem 20 we need to introduce the notion of the 'Minkowski content' of a subset of $\mathbb{R}^2$. If $0 \leq d \leq 2$, a set $E \subset \mathbb{R}^2$ is said to have d–dimensional Minkowski content C if

$$|E_\delta| \sim C\delta^{2-d},$$

where

$$E_\delta = \{x \in \mathbb{R}^2 : \text{dist}(x, E) < \delta\}.$$

Example. Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is a monotone function and $\Gamma = \{(x, \varphi(x)) : x \in \mathbb{R}\}$. If $\Box$ is the interior of the unit square in $\mathbb{R}^2$, then the 1–dimensional Minkowski content of the set $\Gamma \cap \Box$ is bounded above by 9. In what follows, our conclusions concerning Minkowski content will be of this nature.

The proof of Theorem 20

The proof of Theorem 20 will proceed by induction on the degree of the polynomial.

When $n = 0$, the class of operators is reduced to $\{S_\lambda\}_{\lambda\in\mathbb{R}}$, where

$$S_\lambda f(x) = \int_{-\infty}^{\infty} \frac{e^{i\lambda\gamma(x-y)}}{x-y} f(y) dy.$$

By Proposition 13, uniform (in λ) $L^p(\mathbb{R})$ boundedness of S_λ is a consequence of the $L^p(\mathbb{R}^2)$ boundedness of

$$Hf(x) = \int_{-\infty}^{\infty} f(x_1 - t, x_2 - \gamma(t)) \frac{dt}{t},$$

which, by Lemma 22, follows from Theorem 12.

Suppose Theorem 20 is true for polynomials of degree $n - 1$.

Firstly we shall observe that it is enough to consider P monic, and γ satisfying $\gamma(1) = 1$. Suppose M is the coefficient of x^n in P, and that ω satisfies $M\omega^n\gamma(\omega) = 1$. Now,

$$Tf(\omega x) = \int_{-\infty}^{\infty} \frac{e^{iP(\omega x)\gamma(\omega(x-y))}}{x-y} f(\omega y)dy = \int_{-\infty}^{\infty} \frac{e^{i\widetilde{P}(x)\widetilde{\gamma}(x-y)}}{x-y} f(\omega y)dy,$$

where $\widetilde{P}(x) = \gamma(\omega)P(\omega x)$ and $\widetilde{\gamma}(x) = \gamma(\omega x)/\gamma(\omega)$. We now simply observe that $\widetilde{P}$ is monic, $\widetilde{\gamma}$ satisfies the conditions of Theorem 20, and $\widetilde{\gamma}(1) = 1$. Since the $L^p(\mathbb{R})$ operator norm of T is equal to that of

$$\widetilde{T}f(x) = \int_{-\infty}^{\infty} \frac{e^{i\widetilde{P}(x)\widetilde{\gamma}(x-y)}}{x-y} f(y)dy,$$

our claim follows. In what follows P will be monic, and γ will be 'normalised' in the sense that $\gamma(1) = 1$.

We now decompose

$$T = T^{(1)} + \sum_{k\geq 0} T_k,$$

where

$$T^{(1)}f(x) = \int_{|x-y|\leq 1} \frac{e^{iP(x)\gamma(x-y)}}{x-y} f(y)dy,$$

and

$$T_k f(x) = \int_{2^k\leq |x-y|\leq 2^{k+1}} \frac{e^{iP(x)\gamma(x-y)}}{x-y} f(y)dy.$$

2.1 The local part $T^{(1)}$.

Since the integral defining $T^{(1)}$ is restricted to $|x-y| \leq 1$, it suffices to consider $T^{(1)}$ acting on functions supported in balls of radius 1. Suppose $f \in L^p(\mathbb{R})$ is such a function, and has centre b. Let $Q_b(x) = P(x) - (x-b)^n$. Since Q_b is a polynomial of degree $n-1$, by the induction hypothesis

$$S_b f(x) = \int_{-\infty}^{\infty} \frac{e^{iQ_b(x)\gamma(x-y)}}{x-y} f(y)dy$$

is bounded on $L^p(\mathbb{R})$ with bound independent of b and the coefficients of P. Let $\phi \in C_c^\infty(\mathbb{R})$ be such that $\phi(t) = 1$ when $|t| \leq 1$. By Lemma 26

$$S_{b,\phi} f(x) = \int_{-\infty}^{\infty} \frac{e^{iQ_b(x)\gamma(x-y)}}{x-y} \phi(x-y) f(y)dy$$

is also bounded on $L^p(\mathbb{R})$ with bound independent of b and the coefficients of P.

We now define the operator S_b^l by

$$S_b^l f(x) = \int_{|x-y|\leq 1} \frac{e^{iQ_b(x)\gamma(x-y)}}{x-y} f(y)dy.$$

On observing that

$$|S_{b,\phi} f(x) - S_b^l f(x)| \leq |\phi| * |f|(x),$$

we conclude that S_b^l is bounded on $L^p(\mathbb{R})$ with bound independent of b and the coefficients of P.

Now

$$\begin{aligned}
|T^{(1)} f(x) - S_b^l f(x)| &= \left| \int_{|x-y|\leq 1} \frac{e^{iP(x)\gamma(x-y)} - e^{iQ_b(x)\gamma(x-y)}}{x-y} f(y)dy \right| \\
&\leq |x-b|^n \int_{|x-y|\leq 1} \left| \frac{\gamma(x-y)}{x-y} \right| |f(y)|dy \\
&\leq 2^n \int_{|x-y|\leq 1} |f(y)|dy \\
&\text{(since } \gamma \text{ is convex, } \gamma(0)=0, \text{ and } \gamma(1)=1) \\
&= 2^n Af(x),
\end{aligned}$$

where A is the averaging operator given by

$$Af(x) = \int_{|x-y|\leq 1} |f(y)|dy.$$

Since A is trivially bounded on $L^p(\mathbb{R})$, $T^{(1)}$ is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$, with a bound that is independent of the coefficients of P.

2.2 The global part.

Define the operator $\widetilde{T}_k$ by

$$\widetilde{T}_k f(x) = T_k(f(2^{-k}\cdot))(2^k x),$$

i.e.

$$\widetilde{T}_k f(x) = \int_{1\leq|x-y|\leq 2} \frac{e^{i2^{nk}\gamma(2^k)\widetilde{P}_k(x)\widetilde{\gamma}_k(x-y)}}{x-y} f(y)dy,$$

where $\widetilde{P}_k(x) = 2^{-nk}P(2^k x)$, and $\widetilde{\gamma}_k(x) = \gamma(2^k x)/\gamma(2^k)$. We should remark that this type of rescaling preserves the operator norm.

Since γ is either even or odd, we need only consider

$$\mathbb{T}_k f(x) = \int_{1\leq x-y\leq 2} \frac{e^{i2^{nk}\gamma(2^k)\widetilde{P}_k(x)\widetilde{\gamma}_k(x-y)}}{x-y} f(y)dy.$$

Since we are unable to bound the global part using the oscillation alone, we are forced to define some 'bad' sets E_k, on which we rely entirely on the size of the kernels of the operators $\mathbb{T}_k$. Let

$$E_k = \left\{ x \in \mathbb{R} : \left(\frac{\widetilde{P}_k'}{\widetilde{P}_k}\right)'(x) > 0 \text{ and } \left|\frac{\widetilde{P}_k'(x)}{\widetilde{P}_k(x)}\right| \geq \frac{\lambda_0}{4} \right\},$$

where

$$\lambda_0 = \inf_{t>0} \lambda(t).$$

We define the 'good' part of the operator $\mathbb{T}_k$ to be $\mathbb{T}_k^g$, where

$$\mathbb{T}_k^g f(x) = \chi_{E_k^c}(x) \mathbb{T}_k f(x),$$

and the 'bad' part to be

$$\mathbb{T}_k^b f(x) = \chi_{E_k}(x) \mathbb{T}_k f(x).$$

Lemma 27. *Let P be a monic polynomial, and γ be as in Theorem 20, with $\gamma(1) = 1$. If, for $\mu > 0$,*

$$R_\mu f(x) = \chi_{E^c}(x) \int_{1 \leq x-y \leq 2} \frac{e^{i\mu P(x)\gamma(x-y)}}{x-y} f(y)dy,$$

where

$$E = \left\{ x \in \mathbb{R} : \left(\frac{P'}{P}\right)'(x) > 0 \text{ and } \left|\frac{P'(x)}{P(x)}\right| \geq \frac{\lambda_0}{4} \right\},$$

then there is an $\epsilon > 0$ and a constant A, independent of the coefficients of P, for which

$$||R_\mu||_{2-2} \leq A\mu^{-\epsilon},$$

for all $\mu > 0$.

Lemma 28. *For any $\alpha > 0$,*

$$\sum_{k\geq 0} |E_k|^\alpha$$

is convergent, with a bound which depends only on α and the degree of the polynomial P.

We first show how Lemmas 27 and 28 imply Theorem 20.

By Lemma 27

$$||\mathbb{T}_k^g||_{2-2} \leq A(2^{nk}\gamma(2^k))^{-\epsilon},$$

for all $k \geq 0$. Since $\mathbb{T}_k^g$ is trivially bounded on L^1 and L^∞, uniformly in k, by interpolation we have

$$\sum_{k\geq 0} ||\mathbb{T}_k^g||_{p-p} \leq A \sum_{k\geq 0} (2^{nk}\gamma(2^k))^{-\epsilon'} < \infty,$$

with bound independent of the coefficients of P. Here we have used the fact that $\gamma(1) = 1$, and γ is increasing.

By interpolating between the trivial estimates

$$\|\mathbb{T}_k^b f\|_1 \leq |E_k| \|f\|_1,$$

and,

$$\|\mathbb{T}_k^b f\|_\infty \leq C \|f\|_\infty,$$

we obtain

$$\|\mathbb{T}_k^b\|_{p-p} \leq C|E_k|^{\frac{1}{p}},$$

and so by Lemma 28,

$$\sum_{k\geq 0} \|\mathbb{T}_k^b\|_{p-p} < \infty,$$

with bound independent of the coefficients of P.

We now turn to the proofs of Lemmas 27 and 28.

The proof of Lemma 27

In order to exploit the oscillation in R_μ, we will use the fact that $\|R_\mu\|_{2-2} = \|R_\mu^* R_\mu\|_{2-2}^{1/2}$.

Let $L_\mu(x, y)$ be the kernel of $R_\mu^* R_\mu$; i.e.

$$L_\mu(x, y) = \int_{1 \leq z-x, z-y \leq 2; z \in E^c} \frac{e^{i\mu P(z)(\gamma(z-x) - \gamma(z-y))}}{(z-x)(z-y)} dz.$$

Let

$$\psi(x, y, z) = P(z)(\gamma(z-x) - \gamma(z-y)),$$

and

$$\Delta = \{(z, x) \in \mathbb{R}^2 : 1 \leq z - y \leq z - x \leq 2;\ z \in E^c\}.$$

It suffices to consider the kernel

$$\mathbb{L}_\mu(x, y) = \int_{\{z:(z,x) \in \Delta\}} \frac{e^{i\mu\psi(x,y,z)}}{(z-x)(z-y)} dz,$$

since $L_\mu = \mathbb{L}_\mu + \mathbb{L}_\mu^*$.

Since the L^∞ operator norm of $R_\mu^* R_\mu$ is bounded uniformly in μ, it is enough, by interpolation, to obtain appropriate decay estimates for its L^1 operator norm. To this end we seek an estimate of the form

$$\sup_y \int |\mathbb{L}_\mu(x, y)| dx \leq c\mu^{-\epsilon}, \tag{2.5}$$

for some $\epsilon > 0$.

Throughout this chapter we will use the standard notation for partial derivatives where, for a twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, one writes f_i for $\frac{\partial f}{\partial x_i}$, and f_{ij} for $\frac{\partial^2 f}{\partial x_i \partial x_j}$.

In what follows $y \in \mathbb{R}$ will be fixed.

Lemma 29. *If $\frac{P'(z)}{P(z)} < 0$, and z is fixed, then there exists at most one value of $x < y$ for which $\psi_3(x, y, z) = 0$, and at most one for which $\psi_{31}(x, y, z) = 0$.*

Proof.

$$\psi_3(x, y, z) = P(z)(\gamma'(z-x) - \gamma'(z-y)) + P'(z)(\gamma(z-x) - \gamma(z-y)),$$

and

$$\psi_{31}(x, y, z) = -P(z)\gamma''(z-x) - P'(z)\gamma'(z-x),$$

and so

$$\frac{\partial}{\partial x}\left(\frac{\psi_{31}(x, y, z)}{\gamma'(z-x)}\right) = -P(z)\frac{\partial}{\partial x}\left(\frac{\lambda(z-x)}{z-x}\right),$$

which is of constant sign for fixed z. Hence $\frac{\psi_{31}(x,y,z)}{\gamma'(z-x)}$ is a monotone function of x, and so has at most one zero. Hence ψ_3 can have at most two zeros; one of which must be $x = y$. □

Lemma 30. *For fixed y, the zero sets of ψ_3 and ψ_{31} in Δ have bounded one-dimensional Minkowski content, with bound depending only on the degree of the polynomial P. In particular, the bound does not depend on y.*

Proof. Since $z \notin E$, either

(i)

$$\left|\frac{P'(z)}{P(z)}\right| \le \frac{\lambda_0}{4},$$

or (ii)

$$\left(\frac{P'}{P}\right)'(z) \le 0.$$

If (i), then by the Generalised Mean Value Theorem

$$\left|\frac{P'(z)(\gamma(z-x) - \gamma(z-y))}{P(z)(\gamma'(z-x) - \gamma'(z-y))}\right| = \left|\frac{P'(z)}{P(z)}\right| \frac{\gamma'(\theta)}{\gamma''(\theta)} = \left|\frac{P'(z)}{P(z)}\right| \frac{\theta}{\lambda(\theta)}, \tag{2.6}$$

for some $\theta \in (1, 2)$. Since $\lambda(\theta) \ge \lambda_0$, and $\theta \in (1, 2)$, (2.6) is less than

$$\left|\frac{P'(z)}{P(z)}\right| \frac{2}{\lambda_0} \le \frac{1}{2},$$

and consequently ψ_3 has no non–trivial zero as a function of x.

In case (ii), if ψ_3 has a non-trivial zero, say $\alpha_y(z)$, as a function of x, then it is defined implicitly by

$$P(z)[\gamma'(z-\alpha_y(z)) - \gamma'(z-y)] + P'(z)[\gamma(z-\alpha_y(z)) - \gamma(z-y)] = 0,$$

or

$$g(z-\alpha_y(z), z-y) = -\frac{P'(z)}{P(z)}, \tag{2.7}$$

where $g(s,t) = \frac{\gamma'(s)-\gamma'(t)}{\gamma(s)-\gamma(t)}$. Since ψ_{31} has at most one zero, and ψ_3 has at most two zeros (including $x = y$) as functions of x, then $\psi_{31}(\alpha_y(z), y, z) \neq 0$. This implies, by the Implicit Function Theorem, that α_y is defined on an open set U_y and is differentiable. Differentiating (2.7) with respect to z on U_y gives,

$$\left(1 - \frac{d\alpha_y}{dz}\right)\frac{\partial g}{\partial s}(z-\alpha_y(z), z-y) + \frac{\partial g}{\partial t}(z-\alpha_y(z), z-y) = -\left(\frac{P'}{P}\right)'(z) \geq 0.$$

By Lemma 23, $\frac{\partial g}{\partial s} \leq 0$ and $\frac{\partial g}{\partial t} \leq 0$, and so $\frac{d\alpha_y}{dz} \geq 1$. Hence $\{(z,x) \in \Delta : x = \alpha_y(z)\}$ has bounded one–dimensional Minkowski content.

We now turn to the zero set of ψ_{31}.

If ψ_{31} has a zero, say $\beta_y(z)$, as a function of x, then it is defined implicitly by

$$P(z)\gamma''(z-\beta_y(z)) + P'(z)\gamma'(z-\beta_y(z)) = 0.$$

Clearly β_y does not depend on y, so we simply write $\beta_y = \beta$. Since

$$\frac{\gamma''(z-\beta(z))}{\gamma'(z-\beta(z))} = -\frac{P'(z)}{P(z)},$$

and $\frac{\gamma''}{\gamma'}$ is strictly decreasing, $z - \beta(z)$ changes monotonicity exactly when $\frac{P'}{P}$ does; i.e. boundedly often. Hence $\{(z,x) \in \Delta : x = z - \beta(z)\}$ has bounded one–dimensional Minkowski content. By considering the shear $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\Phi(x,y) = (x, x-y)$ (a global diffeomorphism with Jacobian determinant equal to 1), one can deduce that $\{(z,x) \in \Delta : x = \beta(z)\}$ also has bounded one–dimensional Minkowski content.

At this point the following observation is appropriate. Since λ is decreasing and bounded below on $(0,\infty)$, $\frac{\gamma''(t)}{\gamma'(t)} \to \infty$ as $t \to 0$, and $\frac{\gamma''(t)}{\gamma'(t)} \to 0$ as $t \to \infty$. Consequently $\frac{\gamma''}{\gamma'} : (0,\infty) \to (0,\infty)$ is surjective, and so β is defined exactly on the set $\left\{z \in \mathbb{R} : \frac{P'(z)}{P(z)} < 0\right\}$. □

Let

$$F_1 = \{(z,x) \in \Delta : P(z) = 0\},$$

$$F_2 = \{(z,x) \in \Delta : P'(z) = 0\},$$

$$F_3 = \{(z,x) \in \Delta : P''(z) = 0\},$$

$$F_4 = \{(z,x) \in \Delta : x = y\},$$

$$F_5 = \{(z,x) \in \Delta : x = \alpha_y(z)\},$$

$$F_6 = \{(z,x) \in \Delta : x = \beta(z)\},$$

and

$$F_7 = \partial\Delta.$$

Our aim in what follows is to establish some lower bounds for ψ_3 on Δ. To do this we will divide Δ up into three pieces Δ_1, Δ_2, and Δ_3, and make estimates of a different type on each. For technical reasons we need to understand the nature of some of the boundaries between these regions; these are given by

$$F_8 = \{(z,x) \in \Delta : x = \beta_+(z)\},$$

and

$$F_9 = \{(z,x) \in \Delta : x = \beta_-(z)\},$$

where β_+ and β_- are given by

$$\frac{1}{2}\frac{\gamma''(z-\beta_+(z))}{\gamma'(z-\beta_+(z))} = -\frac{P'(z)}{P(z)},$$

and

$$2\frac{\gamma''(z-\beta_-(z))}{\gamma'(z-\beta_-(z))} = -\frac{P'(z)}{P(z)}.$$

As for $z-\beta(z)$, $z-\beta_+(z)$ and $z-\beta_-(z)$ change monotonicity boundedly often, and so F_8 and F_9 have bounded one–dimensional Minkowski content. As for β again, β_+ and β_- are defined exactly on $\left\{z \in \mathbb{R} : \frac{P'(z)}{P(z)} < 0\right\}$.

Let

$$F = \bigcup_{j=1}^{9} F_j.$$

By Lemma 30, F has one–dimensional Minkowski content bounded by a constant depending only on $n = \deg(P)$; i.e. not on y or the coefficients of P.

Decompose $\Delta \backslash F$ into a union of Whitney cubes $\{B_{l,m}\}_{l\geq 0,\, m\in\mathbb{N}}$ whose sides are parallel to the axes, and for which $\operatorname{diam}(B_{l,m}) = 2^{-l}$. Since F has bounded one–dimensional Minkowski content,

$$\#\{B \in \{B_{l,m}\} : \operatorname{diam}(B) = 2^{-l}\} \leq C2^l. \tag{2.8}$$

Next, we write

$$\mathbb{L}_\mu(x,y) = \int_{\{z:(z,x)\in\Delta\}} e^{i\mu\psi(x,y,z)} \frac{dz}{(z-x)(z-y)} = \sum_{l,m} \mathbb{L}_{l,m}(x,y),$$

where

$$\mathbb{L}_{l,m}(x,y) = \int_{\{z:(z,x)\in B_{l,m}\}} e^{i\mu\psi(x,y,z)} \frac{dz}{(z-x)(z-y)}. \tag{2.9}$$

We now claim that it is enough for us to obtain an estimate of the form

$$\sup_y \int |\mathbb{L}_{l,m}(x,y)|\, dx \leq C\frac{2^{Ml}}{\mu}, \tag{2.10}$$

for some $M > 0$ which is independent of l.

Assuming (2.10), and the trivial estimate,

$$\sup_y \int |\mathbb{L}_{l,m}(x,y)|\, dx \leq C2^{-2l}, \tag{2.11}$$

we obtain

$$\begin{aligned} \sup_y \int |\mathbb{L}_{\mu}(x,y)|\, dx &\leq \sum_{l,m} \sup_y \int |\mathbb{L}_{l,m}(x,y)|\, dx \\ &\leq C\sum_{l,m} \min\{2^{-2l}, 2^{Ml}/\mu\} \\ &\leq C\sum_{l} \min\{2^{-l}, 2^{(M+1)l}/\mu\} \quad \text{(by (2.8))} \\ &\leq C\mu^{-\frac{1}{M+2}}, \end{aligned} \tag{2.12}$$

as required. From here we will focus on finding an estimate of the form (2.10).

Before we integrate by parts in (2.9), we must establish some lower bounds for ψ_3 in terms of $\operatorname{dist}((z,x),F)$.

Let

$$\Delta_1 = \left\{(z,x) \in \Delta : \frac{P'(z)}{P(z)} \geq -\frac{1}{2}\frac{\gamma''(z-x)}{\gamma'(z-x)}\right\},$$

$$\Delta_2 = \left\{(z,x) \in \Delta : \frac{P'(z)}{P(z)} < -2\frac{\gamma''(z-x)}{\gamma'(z-x)}\right\},$$

$$\Delta_3 = \left\{(z,x) \in \Delta : -2\frac{\gamma''(z-x)}{\gamma'(z-x)} \leq \frac{P'(z)}{P(z)} \leq -\frac{1}{2}\frac{\gamma''(z-x)}{\gamma'(z-x)}\right\}.$$

Since λ is decreasing, $\{x : (z,x) \in \Delta_j\}$ is a line segment for each $z \in \mathbb{R}$ and $1 \leq j \leq 3$, and in fact

$$\Delta_1 = \left\{(z,x) \in \Delta : \frac{P'(z)}{P(z)} \geq 0, \text{ or } \frac{P'(z)}{P(z)} < 0 \text{ and } x \geq \beta_+(z)\right\},$$

$$\Delta_2 = \left\{(z,x) \in \Delta : \frac{P'(z)}{P(z)} < 0 \text{ and } x \leq \beta_-(z)\right\},$$

$$\Delta_3 = \left\{(z,x) \in \Delta : \frac{P'(z)}{P(z)} < 0 \text{ and } \beta_-(z) \leq x \leq \beta_+(z)\right\}.$$

Lemma 31. *On* Δ_1 *and* Δ_2,

$$|\psi_3(x,y,z)| \geq c \begin{cases} dist((z,x),F)|P(z)|\gamma'(z-x) \\ dist((z,x),F)|P'(z)|\gamma(z-x), \end{cases} \tag{2.13}$$

and on Δ_3,

$$|\psi_3(x,y,z)| \geq c \begin{cases} dist((z,x),F)^2|P(z)|\gamma'(z-x) \\ dist((z,x),F)^2|P'(z)|\gamma(z-x). \end{cases}$$

Before we begin the proof of Lemma 31 we remind the reader that

$$\alpha_y(z) \leq \beta(z) \leq y,$$

and

$$\beta_-(z) \leq \beta(z) \leq \beta_+(z),$$

on $\Delta_2 \cup \Delta_3$.

Proof. **Considering** Δ_1 **:**

If $\frac{P'(z)}{P(z)} \geq 0$, then by Lemma 24,

$$|\psi_3(x,y,z)| \geq c \begin{cases} |x-y||P(z)|\gamma'(z-x) \\ |x-y||P'(z)|\gamma(z-x), \end{cases} \tag{2.14}$$

which implies (2.13) since $|x-y| \geq \text{dist}((z,x),F)$. We will use Lemma 24 in this way several times in subsequent estimates.

If $-\frac{1}{2}\frac{\gamma''(z-x)}{\gamma'(z-x)} \leq \frac{P'(z)}{P(z)} < 0$, then by the Generalised Mean Value Theorem

$$\begin{aligned} \left|\frac{P'(z)(\gamma(z-x)-\gamma(z-y))}{P(z)(\gamma'(z-x)-\gamma'(z-y))}\right| &= \left|\frac{P'(z)}{P(z)}\right| \frac{\gamma'(\theta)}{\gamma''(\theta)} \\ &\quad \text{(for some } z-y<\theta<z-x\text{)} \\ &\leq \left|\frac{P'(z)}{P(z)}\right| \frac{\gamma'(z-x)}{\gamma''(z-x)} \leq \frac{1}{2}, \end{aligned}$$

since λ is decreasing. (2.13) now follows on Δ_1.

Considering Δ_2 **:**

$$\psi_{31}(x,y,z) = -P(z)\gamma''(z-x) - P'(z)\gamma'(z-x).$$

On Δ_2, $\left|\frac{P'(z)}{P(z)}\right| > 2\frac{\gamma''(z-x)}{\gamma'(z-x)}$, and so

$$\left|\frac{P'(z)\gamma'(z-x)}{P(z)\gamma''(z-x)}\right| \geq 2. \tag{2.15}$$

Consequently, ψ_{31} is of constant sign as a function of x on Δ_2, and so by (2.15),

$$|\psi_{31}(x,y,z)| \geq c\begin{cases}|P(z)|\gamma''(z-x)\\ |P'(z)|\gamma'(z-x),\end{cases} \tag{2.16}$$

on Δ_2. If $\alpha_y(z) < \beta_-(z)$, then by (2.16),

$$\begin{aligned}|\psi_3(x,y,z)| \geq& c|P(z)|\left|\int_x^{\alpha_y(z)} \gamma''(z-s)ds\right| \\ =& c|P(z)||\gamma'(z-x) - \gamma'(z-\alpha_y(z))| \\ \geq& c'|P(z)||x-\alpha_y(z)|\gamma'(z-x) \\ \geq& c'\mathrm{dist}((z,x),F)|P(z)|\gamma'(z-x).\end{aligned}$$

On the other hand, by (2.16)

$$\begin{aligned}|\psi_3(x,y,z)| \geq& c|P'(z)|\left|\int_x^{\alpha_y(z)} \gamma'(z-s)ds\right| \\ =& c|P'(z)||\gamma(z-x) - \gamma(z-\alpha_y(z))| \\ \geq& c'|P'(z)||x-\alpha_y(z)|\gamma(z-x) \\ \geq& c'\mathrm{dist}((z,x),F)|P'(z)|\gamma(z-x),\end{aligned}$$

If $\alpha_y(z) \geq \beta_-(z)$, then

$$\begin{aligned}|\psi_3(x,y,z)| \geq& c|P(z)|\left|\int_x^{\beta_-(z)} \gamma''(z-s)ds\right| \\ =& c|P(z)||\gamma'(z-x) - \gamma'(z-\beta_-(z))| \\ \geq& c'|P(z)||x-\beta_-(z)|\gamma'(z-x) \\ \geq& c'\mathrm{dist}((z,x),F)|P(z)|\gamma'(z-x),\end{aligned}$$

and,

$$|\psi_3(x,y,z)| \geq c'\mathrm{dist}((z,x),F)|P'(z)|\gamma(z-x).$$

Considering Δ_3 :

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\psi_{31}(x,y,z)}{\gamma'(z-x)}\right) &= -P(z)\frac{\partial}{\partial x}\left(\frac{\lambda(z-x)}{z-x}\right) \\ &= -P(z)\left(\frac{\lambda'(z-x)}{z-x} - \frac{\lambda(z-x)}{(z-x)^2}\right).\end{aligned}$$

Since λ is decreasing, $\frac{\psi_{31}(x,y,z)}{\gamma'(z-x)}$ is a monotone function of x, and since $1 \leq z-x \leq 2$,

$$\left|\frac{\partial}{\partial x}\left(\frac{\psi_{31}(x,y,z)}{\gamma'(z-x)}\right)\right| \geq \frac{1}{2}\frac{\gamma''(z-x)}{\gamma'(z-x)}|P(z)|. \tag{2.17}$$

On Δ_3,

$$|P(z)|\gamma''(z-x) \sim |P'(z)|\gamma'(z-x), \tag{2.18}$$

and so

$$\left|\frac{\partial}{\partial x}\left(\frac{\psi_{31}(x,y,z)}{\gamma'(z-x)}\right)\right| \geq c|P'(z)|. \tag{2.19}$$

Since $\frac{\psi_{31}(x,y,z)}{\gamma'(z-x)}$ is monotone as a function of x, and $(z,\beta(z)) \in \Delta_3$, (2.19) implies

$$\left|\frac{\psi_{31}(x,y,z)}{\gamma'(z-x)}\right| \geq c|P'(z)||x-\beta(z)|,$$

which implies, by (2.18),

$$|\psi_{31}(x,y,z)| \geq c\begin{cases} |P'(z)||x-\beta(z)|\gamma'(z-x) \\ |P(z)||x-\beta(z)|\gamma''(z-x). \end{cases}$$

The estimates we make now will depend on the location of x relative to the zeros of ψ_3, i.e. $\alpha_y(z)$ and y. Since $\alpha_y(z) \leq \beta(z) \leq y$ we consider three cases:

(i) $x \leq \alpha_y(z)$,

(ii) $\alpha_y(z) < x \leq \beta(z)$,

(iii) $x \geq \beta(z)$.

Since $\gamma \in C^3(\mathbb{R})$, $\psi_{31} \in C^1(\mathbb{R})$ as a function of x. Consequently, ψ_{31} is of constant sign in each of the regions (i)–(iii). This observation allows us to make the following estimates.

In case (i) we may suppose that $\alpha_y(z) \geq \beta_-(z)$, (or else (i) is vacuous), and so

$$\begin{aligned}
|\psi_3(x,y,z)| \geq& C|P(z)|\int_x^{\alpha_y(z)} |t-\beta(z)|\gamma''(z-t)dt \\
\geq& C|P(z)|\int_x^{\frac{\alpha_y(z)+x}{2}} |t-\beta(z)|\gamma''(z-t)dt \\
\geq& C|P(z)|\left|\frac{\alpha_y(z)+x}{2}-\beta(z)\right|\left|\gamma'\left(z-\tfrac{\alpha_y(z)+x}{2}\right)-\gamma'(z-x)\right| \\
\geq& C|P(z)|\left|\frac{\alpha_y(z)+x}{2}-\alpha_y(z)\right|\left|\frac{\alpha_y(z)+x}{2}-x\right|\gamma'(z-x) \\
=& C'|P(z)||x-\alpha_y(z)|^2\gamma'(z-x).
\end{aligned}$$

Similarly,

$$|\psi_3(x,y,z)| \geq C|P'(z)||x-\alpha_y(z)|^2\gamma(z-x).$$

In case (ii), if $\alpha_y(z) \geq \beta_-(z)$ then

$$\begin{aligned}|\psi_3(x,y,z)| \geq& C|P(z)| \int_{\alpha_y(z)}^{x} |t-\beta(z)|\gamma''(z-t)dt \\ \geq& C|P(z)||x-\beta(z)||\gamma'(z-x)-\gamma'(z-\alpha_y(z))| \\ \geq& C|P(z)||x-\beta(z)||x-\alpha_y(z)|\gamma'(z-x).\end{aligned}$$

Similarly,

$$|\psi_3(x,y,z)| \geq C|P'(z)||x-\beta(z)||x-\alpha_y(z)|\gamma(z-x).$$

If $\alpha_y(z) < \beta_-(z)$, we observe that $\beta_-(z) < x \leq \beta(z)$, and

$$\begin{aligned}|\psi_3(x,y,z)| \geq& C|P(z)| \int_{\beta_-(z)}^{x} |t-\beta(z)|\gamma''(z-t)dt \\ \geq& C|P(z)||x-\beta(z)||\gamma'(z-x)-\gamma'(z-\beta_-(z))| \\ \geq& C|P(z)||x-\beta(z)||x-\beta_-(z)|\gamma'(z-x).\end{aligned}$$

Similarly,

$$|\psi_3(x,y,z)| \geq C|P'(z)||x-\beta(z)||x-\beta_-(z)|\gamma(z-x).$$

In case (iii), if $y \leq \beta_+(z)$ then

$$\begin{aligned}|\psi_3(x,y,z)| \geq& C|P(z)| \int_{x}^{y} |t-\beta(z)|\gamma''(z-t)dt \\ \geq& C|P(z)||x-\beta(z)||\gamma'(z-x)-\gamma'(z-y)| \\ \geq& C|P(z)||x-\beta(z)||x-y|\gamma'(z-x).\end{aligned}$$

Similarly,

$$|\psi_3(x,y,z)| \geq C|P'(z)||x-\beta(z)||x-y|\gamma(z-x).$$

If $y > \beta_+(z)$ then we observe that $\beta(z) \leq x \leq \beta_+(z)$ and argue as before. □

Lemma 32. *Let P be a real monic polynomial of degree n and of one real variable. Let U be the union of the set of roots of P and of P' over $\mathbb{R}$. There exists $C > 0$, depending only on n, such that if dist$(x,U) > \epsilon$, then*

$$|P(x)| \geq C\epsilon^n,$$

for all $\epsilon > 0$.

Proof. Let $\epsilon > 0$, and suppose x is chosen so that $\text{dist}(x,U) > \epsilon$. Let $y_x \in U$ be such that $|x-y_x|$ is minimal; so $|x-y_x| > \epsilon$. Without loss of generality we may suppose that $y_x > x$. We observe that P is monotone on $[x, y_x]$. There are two cases to consider.

Case 1: $P(y_x) = 0$

By Lemma 25,

$$|\{z \in \mathbb{R} : |P(z)| < |P(x)|\}| \leq c|P(x)|^{\frac{1}{n}}, \tag{2.20}$$

with constant c independent of the coefficients of P. Since P is monotone on $[x, y_x]$, (2.20) implies that

$$|P(x)| \geq c'|x - y_x|^n \geq c'\epsilon^n,$$

as required.

Case 2: $P(y_x) \neq 0$

Since $y_x \in U$, $P'(y_x) = 0$. Let $y_x' \in U$ be such that $y_x' < x < y_x$ and is maximal in U.

If $P(y_x') = 0$, the argument in case 1 applies.

If $P(y_x') \neq 0$, then since y_x' was chosen maximally, P is single signed on $[y_x', y_x]$. Without loss of generality we may suppose that $|P(y_x)| < |P(y_x')|$, and hence $|P(x)| \geq |P(x) - P(y_x)|$. An application of the argument in case 1 to the polynomial $Q(z) = P(z) - P(y_x)$ completes the proof of Lemma 32. □

Remark

By applying Lemma 32 to the estimates in the statement of Lemma 31 we can conclude that on Δ,

$$|\psi_3(x, y, z)| \geq C \begin{cases} \text{dist}((z, x), F)^{n+2}\gamma'(z - x) \\ \text{dist}((z, x), F)^{n+1}\gamma(z - x). \end{cases} \tag{2.21}$$

We now show how Lemmas 31 and 32 finish the proof of Lemma 27.

Integrating by parts,

$$\begin{aligned}
\mathbb{L}_{l,m}(x, y) =& \int_{\{z:(z,x)\in B_{l,m}\}} e^{i\mu\psi(x,y,z)} \frac{dz}{(z-x)(z-y)} \\
=& \frac{1}{i\mu} \int_{\{z:(z,x)\in B_{l,m}\}} \frac{1}{\psi_3(x, y, z)} \frac{1}{(z-x)(z-y)} \frac{\partial}{\partial z} \left(e^{i\mu\psi(x,y,z)}\right) dz \\
=& \frac{1}{i\mu} \left[\frac{e^{i\mu\psi(x,y,z)}}{\psi_3(x, y, z)(z-x)(z-y)}\right]_{\{z:(z,x)\in\partial B_{l,m}\}} \\
&- \frac{1}{i\mu} \int_{\{z:(z,x)\in B_{l,m}\}} \frac{\partial}{\partial z} \left(\frac{1}{\psi_3(x, y, z)} \frac{1}{(z-x)(z-y)}\right) e^{i\mu\psi(x,y,z)} dz,
\end{aligned}$$

and so, for y fixed,

$$\begin{aligned}
\int |\mathbb{L}_{l,m}(x, y)|\, dx \leq& \frac{1}{\mu} \int_{\partial B_{l,m}} \left|\frac{1}{\psi_3(x, y, z)(z-x)(z-y)}\right| d\sigma_{l,m} \\
&+ \frac{1}{\mu} \int_{B_{l,m}} \left|\frac{\partial}{\partial z} \left(\frac{1}{\psi_3(x, y, z)} \frac{1}{(z-x)(z-y)}\right)\right| dzdx,
\end{aligned} \tag{2.22}$$

where $d\sigma_{l,m}$ is Lebesgue measure on $\partial B_{l,m}$.

By (2.21), the partially integrated term in (2.22) is bounded above by

$$\frac{1}{\mu}\int_{\partial B_{l,m}}\left|\frac{1}{\psi_3(x,y,z)}\right| d\sigma_{l,m} \leq \frac{C}{\mu}\left|\partial B_{l,m}\right|\frac{1}{2^{-(n+1)l}} \leq \frac{C2^{nl}}{\mu}, \tag{2.23}$$

uniformly in y.

The remaining term in (2.22) is bounded above by

$$\begin{aligned}&\frac{1}{\mu}\int_{B_{l,m}}\left|\frac{\psi_{33}(x,y,z)}{\psi_3(x,y,z)^2}\right| dzdx \\ &\quad+\frac{1}{\mu}\int_{B_{l,m}}\left|\frac{\partial}{\partial z}\left(\frac{1}{(z-x)(z-y)}\right)\right| dzdx \sup_{(z,x)\in B_{l,m}}\left|\frac{1}{\psi_3(x,y,z)}\right|.\end{aligned} \tag{2.24}$$

The second term in (2.24) is (by (2.21)) bounded above by $\frac{C2^{nl}}{\mu}$.

The remaining term in (2.24) is, by the triangle inequality, bounded by

$$\begin{aligned}&\frac{1}{\mu}\int_{B_{l,m}}\frac{|P(z)(\gamma''(z-x)-\gamma''(z-y))|}{\psi_3(x,y,z)^2}dzdx \\ &+\frac{1}{\mu}\int_{B_{l,m}}\frac{|P'(z)(\gamma'(z-x)-\gamma'(z-y))|}{\psi_3(x,y,z)^2}dzdx \\ &+\frac{2}{\mu}\int_{B_{l,m}}\frac{|P''(z)(\gamma(z-x)-\gamma(z-y))|}{\psi_3(x,y,z)^2}dzdx \\ &=I+II+III.\end{aligned}$$

By Lemmas 31 and 32,

$$\begin{aligned}I\leq&\frac{C}{\mu}\int_{B_{l,m}}\frac{|P(z)||\gamma''(z-x)-\gamma''(z-y)|}{P(z)^2 2^{-4l}\gamma'(z-x)^2}dzdx \\ \leq&\frac{C2^{4l}}{\mu}\int_{B_{l,m}}\frac{1}{|P(z)|}\left(\frac{\gamma''(z-x)}{\gamma'(z-x)^2}+\frac{\gamma''(z-y)}{\gamma'(z-y)^2}\right)dzdx \\ \leq&\frac{C2^{(n+4)l}}{\mu}\left|\int_{B_{l,m}}\frac{\partial}{\partial z}\left(\frac{1}{\gamma'(z-x)}+\frac{1}{\gamma'(z-y)}\right)dzdx\right| \\ \leq&\frac{C2^{(n+3)l}}{\mu},\end{aligned} \tag{2.25}$$

uniformly in y.

Again by Lemmas 31 and 32,

$$
\begin{aligned}
II \leq & \frac{C}{\mu}\int_{B_{l,m}} \frac{|P'(z)||\gamma'(z-x)-\gamma'(z-y)|}{P(z)^2 2^{-4l}\gamma'(z-x)^2} dzdx \\
\leq & \frac{C2^{4l}}{\mu}\int_{B_{l,m}} \left|\frac{\partial}{\partial z}\left(\frac{1}{P(z)}\right)\right| dzdx \\
\leq & \frac{C2^{4l}}{\mu}\left|\int_{B_{l,m}} \frac{\partial}{\partial z}\left(\frac{1}{P(z)}\right) dzdx\right| \\
& \text{(since } P \text{ is monotone on } B_{l,m}) \\
\leq & \frac{C2^{(n+3)l}}{\mu},
\end{aligned}
\tag{2.26}
$$

uniformly in y.

Similarly,

$$
\begin{aligned}
III \leq & \frac{C}{\mu}\int_{B_{l,m}} \frac{|P''(z)||\gamma(z-x)-\gamma(z-y)|}{P'(z)^2 2^{-4l}\gamma(z-x)^2} dzdx \\
\leq & \frac{C2^{4l}}{\mu}\int_{B_{l,m}} \left|\frac{\partial}{\partial z}\left(\frac{1}{P'(z)}\right)\right| dzdx \\
\leq & \frac{C2^{4l}}{\mu}\left|\int_{B_{l,m}} \frac{\partial}{\partial z}\left(\frac{1}{P'(z)}\right) dzdx\right| \\
& \text{(since } P' \text{ is monotone on } B_{l,m}) \\
\leq & \frac{C2^{(n+2)l}}{\mu},
\end{aligned}
\tag{2.27}
$$

uniformly in y.

Combining these estimates gives

$$
\sup_y \int |\mathbb{L}_{l,m}(x,y)|dx \leq \frac{C2^{(n+3)l}}{\mu},
$$

which is (2.10) with $M = n+3$.

This concludes the proof of Lemma 27.

Proof of Lemma 28

Suppose the roots of P are $\{\nu_j\}_{j=1}^m \subset \mathbb{R}$, and $\{\beta_j\}_{j=1}^{n'}, \{\bar{\beta}_j\}_{j=1}^{n'} \subset \mathbb{C}\backslash\mathbb{R}$, where $n' = \frac{n-m}{2}$ and $\beta_j = a_j + ib_j$. Now

$$
\widetilde{P}_k(x) = \Pi_{j=1}^m (x - 2^{-k}\nu_j)\Pi_{j=1}^{n'}(x - 2^{-k}\beta_j)(x - 2^{-k}\bar{\beta}_j),
$$

and so,

$$\begin{aligned}\frac{\widetilde{P}_k'(x)}{\widetilde{P}_k(x)} =&\sum_{j=1}^{m} \frac{1}{(x-2^{-k}\nu_j)} + \sum_{j=1}^{n'} \frac{1}{(x-2^{-k}\beta_j)} + \frac{1}{(x-2^{-k}\bar{\beta}_j)} \\ =&\sum_{j=1}^{m} \frac{1}{(x-2^{-k}\nu_j)} + 2\sum_{j=1}^{n'} \mathrm{Re}\left(\frac{1}{x-2^{-k}\beta_j}\right) \\ =&\sum_{j=1}^{m} \frac{1}{(x-2^{-k}\nu_j)} + 2\sum_{j=1}^{n'} \frac{x-2^{-k}a_j}{(x-2^{-k}a_j)^2+(2^{-k}b_j)^2},\end{aligned}$$

and,

$$\begin{aligned}\left(\frac{\widetilde{P}_k'}{\widetilde{P}_k}\right)'(x) =& -\sum_{j=1}^{m} \frac{1}{(x-2^{-k}\nu_j)^2} - 2\sum_{j=1}^{n'} \mathrm{Re}\left(\frac{1}{(x-2^{-k}\beta_j)^2}\right) \\ =& -\sum_{j=1}^{m} \frac{1}{(x-2^{-k}\nu_j)^2} - 2\sum_{j=1}^{n'} \frac{(x-2^{-k}a_j)^2-(2^{-k}b_j)^2}{((x-2^{-k}a_j)^2+(2^{-k}b_j)^2)^2},\end{aligned}$$

If $x \in E_k$, then, by definition,

$$\sum_{j=1}^{m} \frac{1}{(x-2^{-k}\nu_j)^2} + 2\sum_{j=1}^{n'} \frac{(x-2^{-k}a_j)^2-(2^{-k}b_j)^2}{((x-2^{-k}a_j)^2+(2^{-k}b_j)^2)^2} < 0, \tag{2.28}$$

and,

$$\left|\sum_{j=1}^{m} \frac{1}{(x-2^{-k}\nu_j)} + 2\sum_{j=1}^{n'} \frac{x-2^{-k}a_j}{(x-2^{-k}a_j)^2+(2^{-k}b_j)^2}\right| \geq \frac{\lambda_0}{4}. \tag{2.29}$$

By the triangle inequality, (2.29) implies that

$$\sum_{j=1}^{m} \frac{1}{|x-2^{-k}\nu_j|} + 2\sum_{j=1}^{n'} \frac{|x-2^{-k}a_j|}{(x-2^{-k}a_j)^2+(2^{-k}b_j)^2} \geq \frac{\lambda_0}{4},$$

and by the equivalence of the l^1 and l^2 norms on $\mathbb{R}^n$, the above implies

$$\sum_{j=1}^{m} \frac{1}{(x-2^{-k}\nu_j)^2} + 2\sum_{j=1}^{n'} \frac{(x-2^{-k}a_j)^2}{((x-2^{-k}a_j)^2+(2^{-k}b_j)^2)^2} \geq c\lambda_0^2, \tag{2.30}$$

for some constant c depending only on n. Combining (2.28) and (2.30) we obtain,

$$\sum_{j=1}^{n'} \frac{(2^{-k}b_j)^2}{((x-2^{-k}a_j)^2+(2^{-k}b_j)^2)^2} \geq c\lambda_0^2, \tag{2.31}$$

or, using the l^1 norm,

$$\sum_{j=1}^{n'} \frac{|2^{-k}b_j|}{(x-2^{-k}a_j)^2+(2^{-k}b_j)^2} \geq c'\lambda_0. \tag{2.32}$$

If x satisfies (2.32), then

$$\frac{|2^{-k}b_j|}{(x-2^{-k}a_j)^2+(2^{-k}b_j)^2} \geq \frac{c'\lambda_0}{n'}, \tag{2.33}$$

for some $1 \leq j \leq n'$. Hence

$$E_k \subset \bigcup_{j=1}^{n'} E_{jk}$$

where

$$E_{jk} = \left\{x \in \mathbb{R} : |2^{-k}b_j| \leq \lambda_0^{-1}, \text{ and } |x-2^{-k}a_j| \leq \lambda_0^{-1/2}|2^{-k}b_j|^{1/2}\right\}, \tag{2.34}$$

and so, for $0 < \alpha \leq 1$,

$$\sum_{k\geq 0} |E_k|^\alpha \leq \sum_{k\geq 0}\left(\sum_{j=1}^{n'}|E_{jk}|\right)^\alpha \leq \sum_{j=1}^{n'}\sum_{k\geq 0}|E_{jk}|^\alpha.$$

For $\alpha > 1$ the above holds with a constant factor depending only on n' and α.

Since $E_{jk} = \emptyset$ if $|2^{-k}b_j| > \lambda_0^{-1}$, and $|E_{jk}| \leq \lambda_0^{-1/2}|2^{-k}b_j|^{1/2}$ if $|2^{-k}b_j| \leq \lambda_0^{-1}$,

$$\sum_{k\geq 0}|E_{jk}|^\alpha \leq C\lambda_0^{-\alpha},$$

uniformly in j, for some constant C depending on α. Consequently,

$$\sum_{k\geq 0}|E_k|^\alpha \leq Cn'\lambda_0^{-\alpha}.$$

This completes the proof of Lemma 28, and hence the inductive step which leads to Theorem 20.

Chapter 3

Some oscillatory singular integrals with variable flat phases; estimates on $L^1(\mathbb{R})$ and $H^1(\mathbb{R})$.

Naturally we would like to look for weak type 1–1 boundedness of the operators studied in Chapter 2. The techniques used there seem too brutal for this much more subtle problem. Although we have been unable, as yet, to make the induction argument complete, we have been successful with certain non–trivial subclasses of the operators of Chapter 2. Our alternative techniques are also appropriate for obtaining boundedness from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$.

This chapter is mainly devoted to the study of the family of operators

$$T_\lambda f(x) = p.v. \int_{-\infty}^{\infty} \frac{e^{i\lambda x\gamma(x-y)}}{x-y} f(y)dy, \tag{3.1}$$

for $\lambda \in \mathbb{R}$, and $\gamma : \mathbb{R} \to \mathbb{R}$ satisfying certain growth conditions. What is, once again, of prime interest to us is that these conditions will not exclude γ that vanish to infinite order at the origin.

As described in Section 1.4.2 of the introductory chapter, operators such as T_λ arise when a partial Fourier transform is applied to certain Hilbert transforms along curves. The T_λ's are non translation invariant, and arise from semi translation invariant operators; the simplest of which is

$$Hf(x_1, x_2) = \int_{-\infty}^{\infty} f(x_1 - t, x_2 - x_1\gamma(t)) \frac{dt}{t},$$

whose L^p boundedness is studied in [9]. However, it is more insightful to view the operators T_λ as arising from certain translation invariant operators on the Heisenberg group. An appropriate operator (studied in [8]), is given by

$$Hf(x_1, x_2, x_3) = \int_{-\infty}^{\infty} f(x \cdot \Gamma^{-1}(t)) \frac{dt}{t},$$

where the group operation on $\mathbb{R}^3$ is given by

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + \tfrac{1}{2}(x_1 y_2 - y_1 x_2)),$$

and $\Gamma(t) = (t, \gamma(t), t\gamma(t))$ for certain $\gamma : \mathbb{R} \to \mathbb{R}$. In [8] it is observed that the operators T_λ are obtained (after a simple diffeomorphism of $\mathbb{R}^3$) by taking the Fourier transform in the second and third variables.

A far reaching theory of singular integrals and maximal functions along variable curves and surfaces has been developed in [13] through an understanding of general nilpotent Lie groups in this context. Unlike in [9] and [8], the curves considered there are required to satisfy a certain finite type condition.

Our aim is to study the behaviour of the operators T_λ on $L^1(\mathbb{R})$ and $H^1(\mathbb{R})$.

3.1 Weak type 1–1

Theorem 33. *Suppose γ is either even or odd and*

(i) $\gamma(0) = \gamma'(0) = 0$,

(ii) γ *and* γ' *are convex on* $(0, \infty)$,

(iii) $\gamma''(t)/\gamma'(t)$ *is decreasing on* $(0, \infty)$

then

$$T_\lambda f(x) = p.v. \int_{-\infty}^{\infty} \frac{e^{i\lambda x\gamma(x-y)}}{x-y} f(y)dy$$

satisfies the weak type 1-1 inequality

$$|\{x \in \mathbb{R} : |T_\lambda f(x)| \geq \alpha\}| \leq C \frac{\|f\|_1}{\alpha},$$

for any $\alpha > 0$, *uniformly in* $\lambda \in \mathbb{R}$.

Remark 1

We may also come to the conclusions of Theorem 33 for the local operators

$$T_\lambda^{loc} f(x) = p.v. \int_{|x-y|\leq 1} \frac{e^{i\lambda x\gamma(x-y)}}{x-y} f(y)dy,$$

with essentially no change in the analysis. The advantage of this observation is that we can treat curves that only satisfy the conditions of Theorem 33 locally.

Remark 2

The conditions imposed on γ in the above theorem differ from those in Theorem 20 of Chapter 2. In the above, we ask for $\gamma''(t)/\gamma'(t)$ to be decreasing, rather than $t\gamma''(t)/\gamma'(t)$ decreasing. However, in Theorem 20 we ask for $t\gamma''(t)/\gamma'(t)$ to be bounded below on $(0,\infty)$, whereas in the above the stronger condition, $\gamma''' \geq 0$ on $(0,\infty)$, is imposed. These differences are explained after the proof of Theorem 33.

As observed in Chapter 2, by rescaling it suffices to prove Theorem 33 in the case $\lambda = 1$, and γ satisfying $\gamma(1) = 1$.

Let

$$Tf(x) = p.v. \int_{-\infty}^{\infty} \frac{e^{ix\gamma(x-y)}}{x-y} f(y)dy,$$

where γ satisfies $\gamma(1) = 1$.

Before we continue it will be helpful to discuss the theorem from which our work grew.

Theorem 34 (Pan [24]). *Suppose $\gamma \in C^3([0,d])$ for some $d > 0$ and satisfies*

(i) γ is either even or odd,

(ii) $\gamma(0) = \gamma'(0) = 0$,

(iii) $\gamma'''(t) \geq 0$ on $[0,d]$,

then the operator S_λ given by

$$S_\lambda f(x) = \int_{|x-y|\leq 1} \frac{e^{i\lambda\gamma(x-y)}}{x-y} f(y)dy$$

is weak type 1–1 uniformly in λ.

In his proof, Pan writes S_λ as the sum of a local part S_λ^1, and a global part S_λ^2, where for ω satisfying $\lambda\gamma(\omega) = 1$,

$$S_\lambda^1 f(x) = \int_{|x-y|\leq\omega} \frac{e^{i\lambda\gamma(x-y)}}{x-y} f(y)dy.$$

With this choice of ω, the difference

$$S_\lambda^1 f(x) - \int_{|x-y|\leq\omega} \frac{f(y)}{x-y} dy$$

can be controlled by the Hardy–Littlewood maximal function. The uniform L^p, $1 < p < \infty$, and weak type 1–1 boundedness of the local Hilbert transform

$$H_\omega f(x) = \int_{|x-y|\leq\omega} \frac{f(y)}{x-y} dy,$$

and the Hardy–Littlewood maximal function then implies the same uniform bounds for S^1_λ.

In the case of our non translation invariant operators

$$Tf(x) = p.v. \int_{-\infty}^{\infty} \frac{e^{ix\gamma(x-y)}}{x-y} f(y)dy,$$

such an approximation by a local Hilbert transform is not possible, since the estimates will depend on where the function f is supported; i.e. if f is supported in a ball of radius 1 and centre b, then

$$\left| \int_{|x-y|\le 1} \frac{e^{ix\gamma(x-y)}}{x-y} f(y)dy - \int_{|x-y|\le 1} \frac{f(y)}{x-y} dy \right|$$
$$\le (|b|+2) \int_{|x-y|\le 1} \frac{\gamma(x-y)}{|x-y|} |f(y)|dy \le 2(|b|+1)Mf(x).$$

We will overcome this problem by allowing Pan's operators S_λ, to take the role of the local Hilbert transforms above. On this note we begin the proof of Theorem 33.

The local part.

We define the local part of the operator T_λ to be

$$T^1 f(x) = \int_{|x-y|\le 1} \frac{e^{ix\gamma(x-y)}}{x-y} f(y)dy.$$

Proposition 35. *Under the conditions of Theorem 34, T^1 is bounded on L^p, $1 < p < \infty$, and is weak type 1–1.*

Proof. For $\mu \in \mathbb{R}$, let

$$S_\mu f(x) = \int_{|x-y|\le 1} \frac{e^{i\mu\gamma(x-y)}}{x-y} f(y)dy.$$

Since the range of integration is localised to $|x-y| \le 1$, it suffices to check the claim for $f \in L^1$ supported in a ball of radius 1. Suppose this ball has centre b.

$$|T^1 f(x) - S_b f(x)| = \left| \int_{|x-y|\le 1} \left(\frac{e^{ix\gamma(x-y)} - e^{ib\gamma(x-y)}}{x-y} \right) f(y)dy \right|$$
$$\le \int_{|x-y|\le 1} |x-b| \left| \frac{\gamma(x-y)}{x-y} \right| |f(y)|dy$$
$$\le 2 \int_{|x-y|\le 1} |f(y)|dy$$

(since γ is convex, $\gamma(0) = 0$, and $\gamma(1) = 1$)

$$= 2Af(x),$$

where A is the averaging operator given by

$$Af(x) = \int_{|x-y|\leq 1} |f(y)|dy.$$

By Theorem 34, S_b is weak type 1–1 and $L^p(\mathbb{R})$ bounded for $1 < p < \infty$, with bound independent of b. Since A is similarly bounded, the proposition follows. □

We now turn to the remainder of the operator $T^2 = T - T^1$.

The global part.

For the classical Calderón–Zygmund singular integral operators, weak type 1–1 bounds can be obtained once L^2 boundedness has been established. For similar reasons, we shall first seek L^2 boundedness of the operator T^2. This was done in Chapter 2, but under different conditions on the curve γ.

Proposition 36. *Under the conditions of Theorem 33, T^2 is bounded on $L^2(\mathbb{R})$.*

Proposition 36 will follow from the following lemma, which will prove useful to us on a number of occasions.

Lemma 37. *Let θ_1 and θ_2 be positive, and let*

$$\psi(z) = z(\gamma(z-y) - \gamma(z-x)).$$

For $A = min\{x+\theta_1, y+\theta_2\}$, and $r \geq A$, let

$$J^r = \int_A^r e^{i\psi(z)}dz.$$

Under the conditions of Theorem 33,

$$|J^r| \leq c(|x-y|\gamma''(\theta))^{-1/2},$$

where $\theta = min\{\theta_1, \theta_2\}$.

Before we prove Lemma 37, we shall show how it implies Proposition 36. Since γ is either odd or even, it suffices to control

$$\mathbb{T}^2 f(x) = \int_{x-y\geq 1} \frac{e^{ix\gamma(x-y)}}{x-y} f(y)dy.$$

Since $||\mathbb{T}^2||_{2-2} = ||(\mathbb{T}^2)^*\mathbb{T}^2||_{2-2}^{1/2}$, it suffices to obtain L^2 boundedness of $(\mathbb{T}^2)^*\mathbb{T}^2$, whose kernel is given by

$$L(x,y) = \int_A^\infty \frac{e^{iz(\gamma(z-y)-\gamma(z-x))}}{(z-x)(z-y)}dz,$$

where $A = \max\{x+1, y+1\}$. Equivalently, we may write

$$L(x,y) = \int_A^\infty \frac{d}{dz}\left(\int_A^z e^{i\psi(s)}ds\right)\frac{dz}{(z-x)(z-y)},$$

which by integration by parts is equal to

$$\left[\frac{J^z}{(z-y)(z-x)}\right]_A^\infty - \int_A^\infty J^z \frac{d}{dz}\left(\frac{1}{(z-y)(z-x)}\right)dz. \tag{3.2}$$

For $z \in [A, \infty)$,

$$\frac{d}{dz}\left(\frac{1}{(z-y)(z-x)}\right) < 0,$$

and so

$$\left|\int_A^\infty J^z \frac{d}{dz}\left(\frac{1}{(z-y)(z-x)}\right)dz\right| \leq \sup_{z\geq A}|J^z|\left|\left[\frac{1}{(z-y)(z-x)}\right]_A^\infty\right|.$$

By applying this estimate to (3.2) we see that

$$|L(x,y)| \leq 2\sup_{z\geq A}|J^z|\left|\frac{1}{(A-y)(A-x)}\right| \leq 2\sup_{z\geq A}|J^z||x-y|^{-1}. \tag{3.3}$$

By Lemma 37, with $\theta_1 = \theta_2 = 1$,

$$|L(x,y)| \leq c(|x-y|^3\gamma''(1))^{-1/2}.$$

Since γ and γ' are convex, $\gamma''(1) \geq \gamma'(1) \geq \gamma(1)$, and so,

$$|L(x,y)| \leq c(|x-y|^3\gamma(1))^{-1/2},$$

and since $\gamma(1) = 1$, this reduces to

$$|L(x,y)| \leq c|x-y|^{-3/2}. \tag{3.4}$$

We also make the trivial estimate,

$$|L(x,y)| \leq \int_A^\infty \left|\frac{1}{(z-x)(z-y)}\right| dz \leq c. \tag{3.5}$$

We now use (3.4) and (3.5) to estimate the $L^1 \to L^1$ and $L^\infty \to L^\infty$ operator norms of $(\mathbb{T}^2)^*\mathbb{T}^2$.

$$\int_{\mathbb{R}} |L(x,y)|dx \leq \int_{|x-y|\leq 1} c\,dx + \int_{|x-y|>1} c|x-y|^{-3/2}dx \leq c' < \infty,$$

uniformly in y. By symmetry, the same estimate is true of $\int_{\mathbb{R}} |L(x,y)|dy$. By interpolation we conclude that $(\mathbb{T}^2)^*\mathbb{T}^2$ is bounded on $L^2(\mathbb{R})$, completing the proof of Proposition 36.

We now turn to the proof of Lemma 37.

The proof of Lemma 37.

The non translation invariance of the operators that we are considering prevents a direct application of the standard Van der Corput Lemma, as we shall see. Instead we shall argue from first principles.

Central to the proof of Lemma 37 is the following, which which will be our substitute for the monotonicity requirement in the first Van der Corput test.

Lemma 38. *If $\gamma''(t)/\gamma'(t)$ is decreasing on $(0,\infty)$, then for each fixed $a>0$,*

$$\Lambda_a(t) = \frac{\gamma'(a+t)-\gamma'(t)}{\gamma(a+t)-\gamma(t)}$$

is decreasing on $(0,\infty)$.

Proof. $\Lambda_a(t) = g(a+t,t)$, where $g(u,v) = \frac{\gamma'(u)-\gamma'(v)}{\gamma(u)-\gamma(v)}$, and so by Lemma 23,

$$\Lambda_a'(t) = \frac{\partial g}{\partial u}(a+t,t) + \frac{\partial g}{\partial v}(a+t,t) \leq 0.$$

□

Suppose $x>y$ are fixed, and that

$$\phi(z) = \gamma(z-y) - \gamma(z-x).$$

By Lemma 38,

$$\frac{\psi'(z)}{\phi'(z)} = z + \frac{\gamma(z-y)-\gamma(z-x)}{\gamma'(z-y)-\gamma'(z-x)} \tag{3.6}$$

is an increasing function of z, with derivative greater than or equal to 1. Hence $\frac{\psi'}{\phi'}$ can have at most one zero in the domain. Let this zero be $\mu = \mu(x,y)$. Consequently,

$$\left|\frac{\psi'(z)}{\phi'(z)}\right| \geq \left|\int_\mu^z ds\right| \geq |z-\mu|, \tag{3.7}$$

and so

$$|\psi'(z)| \geq |x-y||z-\mu|\gamma''(\theta).$$

Let $\delta>0$. Let $D=[A,r]\backslash B(\mu,\delta)$.

$$\int_D e^{i\psi(z)}dz = \int_D \left(\frac{d}{dz}\left(\frac{e^{i\psi(z)}}{\phi'(z)}\right) + \frac{\phi''(z)}{\phi'(z)^2}e^{i\psi(z)}\right)\frac{\phi'(z)}{i\psi'(z)}dz = J_1 + J_2.$$

$$J_1 = \int_D \frac{d}{dz}\left(\frac{e^{i\psi(z)}}{\phi'(z)}\right)\frac{\phi'(z)}{i\psi'(z)}dz = \left[\frac{e^{i\psi(z)}}{i\psi'(z)}\right]_{\partial D} - \int_D \frac{e^{i\psi(z)}}{\phi'(z)}\frac{d}{dz}\left(\frac{\phi'(z)}{i\psi'(z)}\right)dz.$$

The integrated term is less than $4(|x-y|\delta\gamma''(\theta))^{-1}$ in modulus, and

$$\left|\int_D \frac{e^{i\psi(z)}}{\phi'(z)}\frac{d}{dz}\left(\frac{\phi'(z)}{i\psi'(z)}\right)dz\right|$$
$$\leq (\inf_{z\in D}|\phi'(z)|)^{-1}\int_D\left|\frac{d}{dz}\left(\frac{\phi'(z)}{i\psi'(z)}\right)\right|dz,$$

and by monotonicity of $\frac{\phi'}{\psi'}$

$$\leq (|x-y|\gamma''(\theta))^{-1}\left|\int_D \frac{d}{dz}\left(\frac{\phi'(z)}{\psi'(z)}\right)dz\right|,$$

which by (5)

$$\leq (|x-y|\gamma''(\theta))^{-1}\left|\left[\frac{\phi'}{\psi'}\right]_{\partial D}\right| \leq 4(\delta|x-y|\gamma''(\theta))^{-1}.$$

$$|J_2| = \left|\int_D \frac{\phi''(z)}{\phi'(z)^2}e^{i\psi(z)}\frac{\phi'(z)}{\psi'(z)}dz\right| \leq \sup_{z\in D}\left|\frac{\phi'(z)}{\psi'(z)}\right|\int_D\left|\frac{d}{dz}\left(\frac{1}{\phi'(z)}\right)\right|dz \qquad (3.8)$$
$$\leq 4(\delta|x-y|\gamma''(\theta))^{-1},$$

by the monotonicity of ϕ', which in turn follows from the convexity of γ'. Trivially,

$$\left|\int_{B(\mu,\delta)} e^{i\psi(z)}dz\right| \leq 2\delta,$$

so,

$$|J^r| \leq c(\delta|x-y|\gamma''(\theta))^{-1} + 2\delta.$$

Setting $\delta = (|x-y|\gamma''(\theta))^{-1/2}$ gives

$$|J^r| \leq c(|x-y|\gamma''(\theta))^{-1/2}.$$

This completes the proof of Lemma 37.

We are now in a position to begin the proof of the weak type 1–1 boundedness of $\mathbb{T}^2$ (completing the proof of Theorem 33).

Let $\alpha > 0$. To a fixed non-negative $f \in L^1(\mathbb{R})$, we perform a Calderón–Zygmund decomposition,

$$f = g + \sum_j b_j,$$

as described by Lemma 5.

For each j, let $\omega_j = \max\{1, |I_j|\}$, and

$$T_j f(x) = \int_{x-y\geq\omega_j} \frac{e^{ix\gamma(x-y)}}{x-y}f(y)dy.$$

Let I_j^* be the concentric double of I_j, and $\Omega = \left(\bigcup_j I_j^*\right)^c$. By Lemma 5(vi),

$$|\Omega^c| \le C\|f\|_1/\alpha.$$

Hence it suffices to show that

$$\left|\{x \in \Omega : |\mathbb{T}^2 f(x)| > \alpha\}\right| \le C\|f\|_1/\alpha$$

for some absolute constant C. By the triangle inequality

$$\begin{aligned} \left|\{x \in \Omega : |\mathbb{T}^2 f(x)| > \alpha\}\right| &\le \left|\{x \in \Omega : |\mathbb{T}^2 g(x)| > \alpha/2\}\right| \\ &\quad + \left|\left\{x \in \Omega : \left|\mathbb{T}^2(\sum b_j)(x)\right| > \alpha/2\right\}\right|, \end{aligned}$$

and so it is enough to dominate each of these two terms by $C\|f\|_1/\alpha$. The first term may be dealt with by the L^2 boundedness of $\mathbb{T}^2$ in the standard way; see the proof of Theorem 6 in the introductory chapter.

We now turn to the second term. For $x \in \Omega$, we observe that

$$\mathbb{T}^2\left(\sum b_j\right)(x) = \sum \mathbb{T}^2 b_j(x) = \sum \mathbb{T}_j b_j(x).$$

Consequently we seek an estimate of the form

$$\left|\left\{x \in \Omega : \left|\sum \mathbb{T}_j b_j(x)\right| > \alpha\right\}\right| \le C\|f\|_1/\alpha.$$

This would follow from Chebychev's inequality if the following lemma were true.

Lemma 39. *There is a constant $C > 0$, not depending on α, for which*

$$\left\|\sum \mathbb{T}_j b_j\right\|_2^2 \le C\alpha\|f\|_1.$$

Before we prove Lemma 39, we make some reductions. Let $L_{i,j}$ be the kernel of $\mathbb{T}_i^*\mathbb{T}_j$. So

$$L_{i,j}(x,y) = \int_A^\infty \frac{e^{iz(\gamma(z-y)-\gamma(z-x))}}{(z-y)(z-x)}dz$$

where $A = \max(x+\omega_i, y+\omega_j)$. Since

$$\begin{aligned} \left\|\sum \mathbb{T}_j b_j\right\|_2^2 &\le 2\sum_{\omega_j \le \omega_i} |\langle \mathbb{T}_i b_i, \mathbb{T}_j b_j\rangle| \\ &\le 2\sum_{\omega_j \le \omega_i} |\langle b_i, \mathbb{T}_i^*\mathbb{T}_j b_j\rangle| \\ &\le 2\sum_i \Big\langle |b_i|, \sum_{j:\omega_j \le \omega_i} |\mathbb{T}_i^*\mathbb{T}_j b_j|\Big\rangle, \end{aligned}$$

Lemma 39 would follow if we could show that

$$\sup_{x\in I_i} \sum_{j:\omega_j\leq\omega_i} |\mathbb{T}_i^*\mathbb{T}_j b_j| \leq C\alpha,$$

i.e.

$$\sup_{x\in I_i} \sum_{j:\omega_j\leq\omega_i} \left| \int_{I_j} L_{i,j}(x,y)b_j(y)dy \right| \leq C\alpha \tag{3.9}$$

independently of i, and α. In order to achieve this we need to make some pointwise estimates on $L_{i,j}$ of a very specific nature. We point out that we were unable to obtain pointwise estimates of a similar type by the methods of Chapter 2.

Lemma 40.

$$|L_{i,j}(x,y)| \leq C\omega_i^{-1}(1+\log(\omega_i/\omega_j)), \tag{3.10}$$

and

$$|L_{i,j}(x,y)| \leq C|x-y|^{-3/2}. \tag{3.11}$$

Proof. We obtain (3.10) simply by taking the absolute values inside the integral. We will now prove (3.11). As in the proof of Proposition 36 we write

$$\begin{aligned} L_{i,j}(x,y) &= \int_A^\infty \frac{e^{i\psi(z)}}{(z-y)(z-x)}dz \\ &= \int_A^\infty \frac{d}{dz}\left(\int_A^z e^{i\psi(s)}ds\right)\frac{dz}{(z-y)(z-x)}. \end{aligned} \tag{3.12}$$

Integrating by parts in (3.12), and applying Lemma 37 with $\theta_1 = \omega_1$ and $\theta_2 = \omega_2$ gives (3.11). □

It remains now to prove estimate (3.9). This proceeds in the same way as in [24] for the translation invariant operators. Let

$$S_i = \{I_j : dist(I_j, I_i) \leq \omega_i, \omega_j \leq \omega_i\}$$

and,

$$F_i = \{I_j : dist(I_j, I_i) > \omega_i, \omega_j \leq \omega_i\}.$$

If $\omega_i = |I_i|$ and $I_j \in S_i$, then by Theorem 5, there is a constant c (which we can take to be equal to 4 here) such that $1 \leq |\omega_i/\omega_j| \leq c$; and so the number of elements of S_i is less than or equal to $2c$. Hence by (3.10) we have

$$\left| \sum_{I_j\in S_i} \int_{I_j} L_{i,j}(x,y)b_j(y)dy \right| \leq C\alpha.$$

If $\omega_i = 1$ and $I_j \in S_i$, then $\omega_j = \omega_i$. So

$$\left|\sum_{I_j \in S_i} \int_{I_j} L_{i,j}(x,y) b_j(y) dy\right| \leq C\alpha \sum_{I_j \in S_i} |I_j| \leq C\alpha.$$

If $I_j \in F_i$, $x \in I_i$, and $y \in I_j$, then $|x-y|$ is essentially constant, and so by (3.11) and Lemma 5(vi),

$$\begin{aligned}\left|\int_{I_j} L_{i,j}(x,y) b_j(y) dy\right| &\leq C \int_{I_j} \frac{|b_j(y)|}{|x-y|^{3/2}} dy \\ &\leq C\alpha \int_{I_j} |x-y|^{-3/2} dy,\end{aligned}$$

so that

$$\left|\sum_{I_j \in F_i} \int_{I_j} L_{i,j}(x,y) b_j(y) dy\right| \leq C\alpha \int_{|x-y|\geq 1} |x-y|^{-3/2} dy \leq C\alpha.$$

This completes the proof of (3.9), and hence Theorem 33.

Remark

We have now proved uniform $L^2(\mathbb{R})$ boundedness of operators of the form

$$T_\lambda f(x) = p.v. \int_{-\infty}^{\infty} \frac{e^{i\lambda x \gamma(x-y)}}{x-y} f(y) dy$$

in two different ways; once in Chapter 2, and once by Propositions 35 and 36 in this chapter. The most significant difference is that Theorem 20 of Chapter 2 requires $\frac{t\gamma''(t)}{\gamma'(t)}$ decreasing rather than just $\frac{\gamma''(t)}{\gamma'(t)}$ decreasing. This is essentially because, in Chapter 2, many of the 'decay' estimates are obtained from quantitative estimates on the derivative of $\frac{\gamma''(t)}{\gamma'(t)}$ on $[1,2]$; i.e.

$$\frac{d}{dt}\left(\frac{\gamma''(t)}{\gamma'(t)}\right) = \frac{d}{dt}\left(\frac{\lambda(t)}{t}\right) = \frac{\lambda'(t)}{t} - \frac{\lambda(t)}{t^2} \leq -\frac{\lambda(t)}{4},$$

on $[1,2]$. See the proof of Lemma 31 for more explanation. The oscillatory integral estimates in the proof of Proposition 36 rely on the presence of the factor 'x' in the phase, for the appropriate decay. We refer the reader back to (3.6) for details. On the other hand, Proposition 36 requires γ' to be convex, and Theorem 20 does not. However, by a further argument, this condition may be removed from Proposition 36 (and replaced by λ bounded below on $(0,\infty)$). As this chapter is concerned with weak type 1–1 boundedness, we leave this as a remark.

Remark

We can also prove uniform weak type 1–1 boundedness of the family of operators

$$T_\lambda f(x) = p.v. \int_{-\infty}^{\infty} \frac{e^{i\lambda x^2 \gamma(x-y)}}{x-y} f(y)dy,$$

under the conditions on γ given in Theorem 33. To prove this we use Theorem 33 to control the local part, and we observe that the global part presents no new obstacles.

3.2 Boundedness from H^1 to L^1

Theorem 41. *If γ satisfies the conditions of Theorem 33, then*

$$T_\lambda f(x) = p.v. \int_{-\infty}^{\infty} \frac{e^{i\lambda x \gamma(x-y)}}{x-y} f(y)dy$$

is bounded from H^1 to L^1, with bound independent of $\lambda \in \mathbb{R}$.

Based on our experience of the weak type estimates in the previous section, it comes as no surprise that we will need to use the following theorem of Pan [25].

Theorem 42 (Pan [25]). *If γ satisfies the conditions of Theorem 34, and $\phi \in C_c^\infty(\mathbb{R})$, then*

$$T_\lambda f(x) = \int_{\mathbb{R}} \frac{e^{i\lambda \gamma(x-y)}}{x-y} \phi(x-y) f(y)dy$$

is bounded from H^1 to L^1, with bound independent of $\lambda \in \mathbb{R}$.

A review of the proof of Theorem 42 shows that ϕ has no role to play. As such, we merely remark that Theorem 41 may also be localised in this way.

Let a be an $H^1(\mathbb{R})$ atom supported in an interval I. By the Atomic Decomposition of $H^1(\mathbb{R}^n)$ (Theorem 17), Theorem 41 will follow if we can show that for any such atom,

$$||T_\lambda a||_{L^1(\mathbb{R})} \leq C < \infty. \qquad (3.13)$$

When we considered the questions of uniform $L^p(\mathbb{R})$ boundedness and weak type 1–1 boundedness, we chose to rescale the operators so that the local–global cut–off was at $|x-y| = 1$. However, in proving Theorem 41, what we mean by the so called local and global estimates will be very different from their analogues in the weak type 1–1, and $L^p(\mathbb{R})$ boundedness proofs. With this different local–global notion a different rescaling is more natural, and so the parameter λ will persist. By making a change of variables in (3.13) one can see that it suffices to obtain uniform boundedness of (3.13) when the atom a is supported in an interval I of width 2.

Let y_I be the centre of I.

Lemma 43. *If η satisfies $\lambda\eta\gamma'(2\eta) = 1$, then*

$$\int_{|x-y_I|\leq\eta^*} |T_\lambda a(x)|dx \leq C,$$

where $\eta^* = max\{\eta, 2\}$.

Proof. Firstly, suppose $\eta^* = 2$. Using the Cauchy–Schwarz inequality and the uniform $L^2(\mathbb{R})$ boundedness of T_λ (Propositions 35 and 36),

$$\int_{|x-y_I|\leq 2} |T_\lambda a(x)|dx \leq 2\|T_\lambda\|_{2-2}\|a\|_2 \leq C.$$

Now suppose that $\eta^* = \eta$. By the above argument it suffices to control

$$\int_{2<|x-y_I|\leq\eta} |T_\lambda a(x)|dx,$$

uniformly in λ.

Let

$$S_{\lambda y_I} f(x) = \int_{\mathbb{R}} \frac{e^{i\lambda y_I \gamma(x-y)}}{x-y} f(y)dy.$$

By Theorem 42, $S_{\lambda y_I}$ is bounded from H^1 to L^1 with bound independent of λy_I. In particular $\int_{\mathbb{R}} |S_{\lambda y_I} a(x)|dx \leq C$. Now,

$$\begin{aligned}
&\int_{2<|x-y_I|\leq\eta} \left|T_\lambda a(x) - e^{i\lambda(x-y_I)\gamma(x-y_I)} S_{\lambda y_I} a(x)\right| dx \\
&\leq \int_{2<|x-y_I|\leq\eta} \left| \int_{\mathbb{R}} \frac{e^{i\lambda x\gamma(x-y)} - e^{i\lambda(y_I\gamma(x-y)+(x-y_I)\gamma(x-y_I))}}{x-y} a(y)dy \right| dx \\
&\leq C\lambda \int_{2<|x-y_I|\leq\eta} \int_{\mathbb{R}} |x-y_I| \left| \frac{\gamma(x-y)-\gamma(x-y_I)}{x-y} \right| |a(y)|dydx \\
&\leq C\lambda\eta \int_{\mathbb{R}} |a(y)| \int_{2<|x-y_I|\leq\eta} \left| \frac{\gamma(x-y)-\gamma(x-y_I)}{x-y} \right| dxdy \\
&\leq C\lambda\eta \int_{\mathbb{R}} |a(y)||y-y_I| \int_{2<|x-y_I|\leq\eta} \left| \frac{\gamma'(2(x-y))}{x-y} \right| dxdy \\
&\leq C\lambda\eta \int_{\mathbb{R}} |a(y)|dy \int_0^{2\eta} \frac{\gamma'(t)}{t} dt \\
&\leq C\lambda\eta \int_0^{2\eta} \gamma''(t)dt \leq C\lambda\eta\gamma'(2\eta) \leq C.
\end{aligned}$$

This completes the proof of Lemma 43. □

It now remains to show that

$$\int_{|x-y_I|\geq\eta^*} |T_\lambda a(x)|dx \leq C$$

uniformly in λ and I.

Merely for technical reasons, we make the trivial observation,

$$\int_{\eta^*\leq|x-y_I|\leq 5\eta^*} |T_\lambda a(x)|dx \leq \|a\|_{L^1(\mathbb{R})} \int_{\eta^*\leq|x-y_I|\leq 5\eta^*} \frac{1}{|x-y_I|}dx \leq C. \tag{3.14}$$

We will content ourselves with showing that

$$\int_{x-y_I\geq 5\eta^*} |T_\lambda a(x)|dx \leq C.$$

The integral over $x - y_I \leq -5\eta^*$ is similar since, by assumption, γ is either odd or even.

Let k^* be the smallest integer for which $2^{k^*} \geq 2\eta^*$, and let ψ be a smooth bump function satisfying $\psi(t) = 1$ for $|t| \leq 1$. For $k \geq k^*$ define

$$T^k f(x) = \int_{2^k\leq x-y\leq 2^{k+1}} e^{i\lambda x\gamma(x-y)}\psi(y-y_I)f(y)dy.$$

(Here we are supressing the dependence on λ and I.)

Since $|y - y_I| \leq 1$, and $x - y_I \geq 5\eta^* \geq 10$, $x - y_I$ and $x - y$ are comparable. With this in mind, we write

$$\begin{aligned}
&\int_{x-y_I\geq 5\eta^*} |T_\lambda a(x)|dx \\
&\quad\leq \int_{x-y_I\geq 5\eta^*} \left|\int_{\mathbb{R}} e^{i\lambda x\gamma(x-y)} \left(\frac{1}{x-y} - \frac{1}{x-y_I}\right) a(y)dy\right| dx \\
&\quad+ \int_{x-y_I\geq 5\eta^*} \left|\frac{1}{x-y_I}\int_{\mathbb{R}} e^{i\lambda x\gamma(x-y)} a(y)dy\right| dx.
\end{aligned}$$

Since $\eta^* \geq 2$, the first term in the above is dominated by

$$\int_{\mathbb{R}}\int_{x-y_I\geq 10} \left|\frac{y-y_I}{(x-y)(x-y_I)}\right| dx|a(y)|dy \leq C\int_{x\geq 9}\frac{1}{x^2}dx \leq C.$$

The second term can be expressed as

$$\begin{aligned}
&\sum_{k\geq k^*}\int_{x-y_I\geq 5\eta^*} \left|\frac{1}{x-y_I}\int_{x-y\sim 2^k} e^{i\lambda x\gamma(x-y)} a(y)dy\right| dx \\
&\qquad\leq C\sum_{k\geq k^*}\int_{2^{k-1}\leq x-y_I\leq 2^{k+2}} \frac{1}{x-y_I}|T^k a(x)|dx \\
&\qquad\leq C\sum_{k\geq k^*} 2^{-k/2}\|T^k a\|_2 \\
&\qquad\leq C\sum_{k\geq k^*} 2^{-k/2}\|(T^k)^*T^k\|_{2-2}^{1/2}.
\end{aligned}$$

Lemma 44.

$$\|(T^k)^*T^k\|_{2-2} \leq C2^k(\lambda 2^k\gamma'(2^k))^{-1/4}.$$

We first indicate how Lemma 44 finishes the proof of our theorem. Applying Lemma 44 to the expression immediately preceding it gives

$$C\sum_{k\geq k^*} 2^{-k/2}2^{k/2}(\lambda 2^k\gamma'(2^k))^{-1/8}$$

$$\leq (\lambda 2^{k^*}\gamma'(2^{k^*}))^{-1/8}\sum_{k\geq k^*}\left(\frac{2^{k^*}\gamma'(2^{k^*})}{2^k\gamma'(2^k)}\right)^{1/8} \leq C,$$

since the factor outside the sum is bounded by $(2\lambda\eta\gamma'(2\eta))^{-1/8} = 2^{-1/8}$.

We now turn to the proof of Lemma 44.

Proof. Let $L^k(x,y)$ be the kernel of $(T^k)^*T^k$, i.e.

$$L^k(x,y) = \psi(x-y_I)\psi(y-y_I)\int_{z-x,z-y\sim 2^k} e^{i\lambda z(\gamma(z-x)-\gamma(z-y))}dz.$$

By Lemma 37 of the previous section, with $\theta_1 = \theta_2 = 2^k$,

$$|L^k(x,y)| \leq C(\lambda|x-y|\gamma''(2^k))^{-1/2}\psi(x-y_I)\psi(y-y_I),$$

and trivially,

$$|L^k(x,y)| \leq 2^k.$$

Taking the geometric mean of these two estimates gives

$$|L^k(x,y)| \leq C(\lambda|x-y|\gamma''(2^k))^{-1/4}2^{k/2}\psi(x-y_I)^{1/2}\psi(y-y_I)^{1/2}.$$

Using this to estimate the $L^1 \to L^1$ and $L^\infty \to L^\infty$ norms of $(T^k)^*T^k$, proves the lemma. □

Chapter 4

Singular Integrals and Maximal Functions Associated to Flat Surfaces

4.1 Introduction

The aim of this chapter is to illustrate a method of deducing $L^p(\mathbb{R}^n)$ boundedness of maximal functions and singular integrals on surfaces in $\mathbb{R}^n$ from boundedness of their counterparts along curves in $\mathbb{R}^k$ (for some $k \leq n$). The modern proofs of the theorems for plane curves place emphasis on finding a set of dilations for a curve which localise the problem. This approach is appropriate even for flat curves. From this point of view, one of the main barriers to theorems for flat surfaces has been the problem of constructing dilations for surfaces which serve a similar purpose. See Section 1.4.1 of the introductory chapter for further discussion. Since the surfaces under consideration have an 'identified point', the origin, one possibility would be to consider non linear dilations inherited from the curves produced by restricting the surfaces to hyperplanes passing through the origin. An example of this simple idea is the following.

Let $\alpha : \mathbb{S}^1 \to (1, \infty)$. Define the surface $\Gamma : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\Gamma(t) = (t_1, t_2, |t|^{\alpha(t/|t|)}).$$

For each $\omega \in \mathbb{S}^1$, and $s > 0$, let

$$\delta_\omega(s) = \operatorname{diag}(s, s, s^{\alpha(\omega)}).$$

Now, if we write $x \in \mathbb{R}^3$ as $(x', x_3) \in \mathbb{R}^2 \times \mathbb{R}$, we can define the action of the non–linear dilations on x to be

$$\delta(s)x = \delta_{x'/|x'|}(s)x = (sx', s^{\alpha(x'/|x'|)}x_3).$$

We observe now that

$$\delta(s)\Gamma(t) = \Gamma(st).$$

Most of our results for surfaces will be consequences of known results for plane curves. On first sight this rather crude approach seems surprisingly effective, especially since our theorems cover surfaces that are not radial. We may explain this as follows.

We observe that for a plane curve $(t, \gamma(t))$, the corresponding operator norm is unchanged if γ is replaced by $a\gamma(b\cdot)$ for any $a, b \in \mathbb{R}$. The implications of this are that our conditions are invariant under 'star shaped dilations'; i.e. if $s : \mathbb{S}^{n-1} \to \mathbb{R}$, then the conditions of our theorems are unaltered if γ is replaced by $\gamma_s(t) = \gamma(s(t/|t|)t)$.

As we shall see, all of our results (which are technically quite trivial), rely heavily on the fact that the theorems for curves give bounds which are invariant under certain transformations.

Our approach is also appropriate for studying singular integrals and maximal functions associated to variable surfaces.

Spherical Polar Coordinates

The maximal functions

Surprisingly some interesting results come from the following simple majorisation.

Let $\Gamma : \mathbb{R}^k \longrightarrow \mathbb{R}^n$ for some $k \leq n$, and let

$$M_\Gamma f(x) = \sup_{h>0} \frac{c_k}{h^k} \int_{|t|\leq h} |f(x - \Gamma(t))| dt. \tag{4.1}$$

For each $\omega \in \mathbb{S}^{k-1}$, let

$$M_{\Gamma_\omega} f(x) = \sup_{h>0} \frac{1}{h} \int_0^h |f(x - \Gamma(r\omega))| dr.$$

Lemma 45. *If*

$$\int_{\mathbb{S}^{k-1}} ||M_{\Gamma_\omega}||_{p-p} \, d\sigma(\omega) < \infty$$

then $||M_\Gamma||_{p-p} < \infty$. *In particular, the conclusion follows if* $||M_{\Gamma_\omega}||_{p-p}$ *is uniformly bounded in* ω.

Proof Using spherical polar coordinates we see that

$$M_\Gamma f(x) = \sup_{h>0} \frac{c_k}{h^k} \int_{\mathbb{S}^{k-1}} \int_0^h |f(x - \Gamma(r\omega))| r^{k-1} dr d\sigma(\omega)$$
$$\leq \int_{\mathbb{S}^{k-1}} c_k \sup_{h>0} \frac{1}{h} \int_0^h |f(x - \Gamma(r\omega))| dr d\sigma(\omega)$$
$$\leq \int_{\mathbb{S}^{k-1}} c_k M_{\Gamma_\omega} f(x) d\sigma(\omega).$$

The conclusion now follows from Minkowski's integral inequality.

Remarks

(i) The above analysis can be equally well applied to the operator

$$f^*(x) = \sup_{h>0} \frac{c_k}{h^k} \int_{|t| \leq h} |f(x - \Gamma(t)) \Lambda(t)| dt$$

for any homogeneous of degree zero $\Lambda \in L^s(\mathbb{S}^{k-1})$, $1 \leq s \leq \infty$. In this case the condition in Lemma 1 becomes

$$\left(\int_{\mathbb{S}^{k-1}} \|M_{\Gamma_\omega}\|_{p-p}^{s'} d\sigma(\omega) \right)^{1/s'} < \infty,$$

where s' is the dual exponent of s.

(ii) Our techniques also apply to certain singular integrals associated to surfaces. We will discuss an example later in this chapter.

Some simple examples of this approach

Surprisingly, many of the previously known positive results can be proved using Lemma 45.

(1) Suppose $\Gamma : \mathbb{R}^k \longrightarrow \mathbb{R}^n$ is a polynomial, i.e.

$$\Gamma(t) = (P_1(t), ..., P_n(t)) \text{ for polynomials } P_j.$$

Now for fixed $\omega \in \mathbb{S}^{k-1}$, M_{Γ_ω} is just a maximal function along the polynomial curve $\Gamma_\omega(r) = \Gamma(r\omega)$. Since the bound of such an operator is dependent only on the degree of the polynomial we see that

$$\|M_{\Gamma_\omega}\|_{p-p} \leq A_p \; \forall \, \omega \in \mathbb{S}^{k-1}.$$

So by Lemma 1, M_Γ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$. It is appropriate to remark that the independence of the above estimates on the coefficients of the polynomials can be seen as a consequence of $GL(N, \mathbb{R})$ invariance of certain operator norms; see [36] for further discussion. This will be a common consideration.

(2) Suppose $\Gamma : \mathbb{R}^k \longrightarrow \mathbb{R}^n$ is an homogeneous surface with respect to a 1 parameter group of dilations $\{\delta(s)\}_{s>0}$. For fixed $\omega \in \mathbb{S}^{k-1}$, Γ_ω is an homogeneous curve with respect to the same 1 parameter dilation group $\{\delta(s)\}_{s>0}$. An application of the appropriate theorem for homogeneous curves, which can be found in [33], gives the desired uniform estimates for $\|M_{\Gamma_\omega}\|_{p-p}$, $1 < p \leq \infty$.

4.2 Surfaces of codimension 1 in $\mathbb{R}^n$

In this section we will use the theory developed for Hilbert transforms and maximal functions along convex curves in $\mathbb{R}^2$, (see for example Theorems 10, 11, 12), to obtain some simple theorems for surfaces of codimension 1 in $\mathbb{R}^n$. The theorems we obtain apply to many surfaces that vanish to infinite order at the origin.

Most of the theorems involving convex curves are formulated in terms of certain functionals acting on the graphing function γ. For example, Theorem 11 requires the functional F, given by

$$F(\gamma)(t) = \frac{th'(t)}{h(t)}, \tag{4.2}$$

where $h(t) = t\gamma'(t) - \gamma(t)$ to be bounded below on $(0, \infty)$.

In general the corresponding functional associated to the graphing function of the surface $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ is given by

$$\widetilde{F}(\gamma)(r\omega) = F(\gamma_\omega)(r),$$

where $\omega \in \mathbb{S}^{n-2}$ and $r \geq 0$.

The functionals for curves in $\mathbb{R}^2$ that we will encounter have a natural scale invariance, i.e.

$$F(\gamma(\lambda\cdot))(t) = F(\gamma)(\lambda t).$$

It is this homogeneity that is largely responsible for the natural appearance of the corresponding $\widetilde{F}$'s that we have encountered. For example, for F given by (4.2),

$$\widetilde{F}(\gamma)(t) = \frac{t \cdot \nabla H(t)}{H(t)},$$

where, $H(t) = t \cdot \nabla\gamma(t) - \gamma(t)$.

We come to the following conclusion based on the above example.

Theorem 46. *Suppose* $\Gamma : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^n$, *and*

$$\Gamma(t) = (t, \gamma(t)),$$

where $\gamma \in C^2(\mathbb{R}^{n-1})$. *If in addition,* *(i)* $\gamma(0) = 0$ *and* $\nabla\gamma(0) = 0$,

(ii) for each $\omega \in \mathbb{S}^{n-2}$, $\gamma_\omega(r) = |\gamma(r\omega)|$ *is convex on* $(0, \infty)$, *and*

(iii) $\exists \epsilon > 0$ *such that*

$$|t \cdot \nabla H(t)| \geq \epsilon |H(t)| \; \forall t \in \mathbb{R}^{n-1},$$

where $H(t) = t \cdot \nabla\gamma(t) - \gamma(t)$,

then M_Γ, *given by* (4.1), *is bounded on* $L^p(\mathbb{R}^n)$ *for* $1 < p \leq \infty$.

Proof. Let $\omega \in \mathbb{S}^{n-2}$ and suppose $\omega_j \neq 0 \; \forall j$. Let

$$\Gamma_\omega(r) = (r\omega_1, r\omega_2, ..., r\omega_{n-1}, \gamma_\omega(r))$$

and

$$D_\omega = \begin{pmatrix} \omega_1^{-1} & 0 & \cdot & \cdot & 0 & -\omega_{n-1}^{-1} & 0 \\ 0 & \omega_2^{-1} & 0 & \cdot & 0 & -\omega_{n-1}^{-1} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -\omega_{n-1}^{-1} & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix}$$

Clearly there are other matrices which serve the same purpose. Since $\det D_\omega = \omega_1 ... \omega_{n-1}$, $D_\omega \in GL(n, \mathbb{R})$ for a.e. $\omega \in \mathbb{S}^{n-2}$, and

$$D_\omega \Gamma_\omega(r) = (0, ..., 0, r, \gamma_\omega(r)).$$

So by Lemma 1 and the $GL(n, \mathbb{R})$ invariance of the operator norms, it suffices to bound the maximal function associated to the plane curve (r, γ_ω) with bound independent of ω. By Theorem 11, it is enough for γ_ω to satisfy the infinitesimal doubling property, (1.26), uniformly in $\mathbb{S}^{n-2}$. Let $h_\omega(r) = r\gamma_\omega'(r) - \gamma_\omega(r)$. By the chain rule,

$$\begin{aligned} h_\omega(r) &= |r\omega \cdot \nabla\gamma(r\omega) - \gamma(r\omega)| \\ &= |H(r\omega)|. \end{aligned}$$

So by hypothesis (iii),

$$\begin{aligned} rh_\omega'(r) &= |r\omega \cdot \nabla H(r\omega)| \\ &\geq \epsilon |H(r\omega)| \\ &= \epsilon h_\omega(r). \end{aligned}$$

Hence γ_ω satisfies the infinitesimal doubling condition uniformly in ω. □

Remark

In the statement of Theorem 46 we asked for the functional $\left|\frac{t\cdot\nabla H(t)}{H(t)}\right|$ to be bounded below by a positive constant. This condition may be weakened considerably, since by Lemma 45, we only require $\|M_{\Gamma_\omega}\|_{p-p}$ to be integrable over S^{n-2}. A possible way to formulate this weaker condition would be to observe how the bound in Theorem 11 depends on the constant ϵ. We will take this improved approach in Section 4.3 for a different class of operators.

Theorem 47. *Suppose* $\Gamma : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^n$, *and*

$$\Gamma(t) = (t, \gamma(t)),$$

where $\gamma \in C^2(\mathbb{R}^{n-1})$. *If in addition,*

(i) $\gamma(0) = 0$ *and* $\nabla\gamma(0) = 0$,

(ii) for each $\omega \in \mathbb{S}^{n-2}$, $\gamma_\omega(r) = |\gamma(r\omega)|$ *is convex on* $(0, \infty)$, *and*

(iii) $\exists C < \infty$ *such that*

$$|t\cdot\nabla\gamma(Ct)| \geq 2|t\cdot\nabla\gamma(t)|\ \forall t \in \mathbb{R}^{n-1},$$

then M_Γ *is bounded on* $L^p(\mathbb{R}^n)$ *for* $1 < p \leq \infty$.

The proof of Theorem 47 is very similar to that of Theorem 46. Obviously we use Theorem 12 instead of Theorem 11.

In a similar way an L^2 theorem corresponding to Theorem 10 can be formulated. Clearly we would require H to be doubling.

Some examples

(1) A natural application of Theorem 46 is the following.

Let $\rho : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ be homogeneous of degree 1. Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be convex, C^2, and satisfy $\phi(0) = \phi'(0) = 0$.

Let

$$h(s) = s\phi'(s) - \phi(s),$$

$$\gamma(t) = \phi(\rho(t)),\ \text{and}\ \Gamma(t) = (t, \gamma(t)).$$

If for some $\epsilon > 0$,

$$sh'(s) \geq \epsilon h(s)\ \forall s \in \mathbb{R}$$

then M_Γ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

(2) Theorems 46 and 47 apply to surfaces $\Gamma : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^n$ of the form

$$\Gamma(t) = (t, t^\alpha e^{-\frac{1}{t^\beta}})$$

where α and β are multi indices with each β_j even.

Remark

The main drawback of our approach is that we totally disregard any curvature of the surface that might exist in the angular variable ω. In fact, in certain circumstances, where the level curves of γ are of finite type, much more appropriate techniques have been developed by Wainger, Wright, and Ziesler (see [41]), which lead to much better results for the singular integrals [1] on L^2. In [41] they consider surfaces of the form

$$\Gamma(t) = (t, \phi(\psi(t))), \tag{4.3}$$

for ψ smooth, convex, and of finite type. Remarkably, as Wainger, Wright, and Ziesler rely on curvature in the angle variable, the only condition imposed on ϕ is that $\phi \in C^1(\mathbb{R})$, and $\phi(0) = 0$. This is in stark contrast to our approach since we exploit curvature of the surface along rays emmanating from the origin, and impose no conditions on the level sets of γ.

Wainger, Wright, and Ziesler are currently working on extending their techniques in order to handle surfaces not of the specific form (4.3).

4.3 Variable surfaces

Our aproach also applies to operators associated to variable hypersurfaces. We will give an example based on Corollary 21 of Chapter 2. Let K be a $n-1$–dimensional Calderón–Zygmund kernel satisfying

$$K(x) = \frac{\Omega(x)}{|x|^{n-1}}$$

where Ω is odd and homogeneous of degree 0. Suppose in addition that $\Omega \in L^s(\mathbb{S}^{n-2})$ for some $1 \leq s \leq \infty$.

The following notation will help to facilitate our discussion.
For $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, we let $x' = (x_1, x_2, ..., x_{n-1}) \in \mathbb{R}^{n-1}$, and $x'' = (x_1, x_2, ..., x_{n-2}) \in \mathbb{R}^{n-2}$.

Let $P : \mathbb{R}^{n-1} \to \mathbb{R}$ be a real polynomial, and let $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ be C^3 and satisfy

[1] So far the results for the maximal function are less satisfactory.

(i) $\gamma(0) = 0$, $\nabla\gamma(0) = 0$,

(ii) $|\gamma|$ is convex on rays emmanating from the origin, and

(iii) γ is odd or even on each ray passing through the origin.

Let

$$\Lambda(t) = \frac{t \cdot \nabla H(t)}{t \cdot \nabla\gamma(t)}.$$

Theorem 48. *If*

(i) $|\Lambda(t)|$ is decreasing on rays emmanating from the origin, and

(ii) $|\Lambda(t)| \geq \Lambda_0(t/|t|)$ on $\mathbb{R}^{n-1}$, where $\Lambda_0^{-1/2} \in L^{s'}(\mathbb{S}^{n-2})$, $\frac{1}{s} + \frac{1}{s'} = 1$,

then the singular Radon transform

$$T_\Gamma f(x) = \int_{\mathbb{R}^{n-1}} f(x' - t, x_n - P(x')\gamma(t))K(t)dt,$$

is bounded on $L^2(\mathbb{R}^n)$ with a bound that is independent of the coefficients of P.

Proof. Let

$$\Gamma_\omega(x, r) = \Gamma(x, r\omega) = (x_1 - r\omega_1, x_2 - r\omega_2, ..., x_{n-1} - P(x')\gamma_\omega(r)).$$

Let

$$\mathbb{S}_+ = \{\omega \in \mathbb{S}^{n-2} : \omega_1 \geq 0\}.$$

Using polar coordinates we can write

$$\begin{aligned} T_\Gamma f(x) &= \int_{\mathbb{S}_+} \Omega(\omega) \left(\int_{-\infty}^{\infty} f(\Gamma(x, r\omega)) \frac{dr}{r} \right) d\sigma(\omega) \\ &= \int_{\mathbb{S}_+} \Omega(\omega) H_{\Gamma_\omega} f(x) d\sigma(\omega), \end{aligned}$$

where,

$$H_{\Gamma_\omega} f(x) = \int_{-\infty}^{\infty} f(\Gamma(x, r\omega)) \frac{dr}{r}.$$

By $GL(n, \mathbb{R})$ invariance of the operator norms it suffices to consider the maximal function corresponding to the curve

$$\widetilde{\Gamma}_\omega(x, r) = (x_1, ..., x_{n-2}, x_{n-1} - r, x_n - P((M_\omega x)')\gamma_\omega(r)),$$

where $\widetilde{\Gamma}_\omega(x,r) = M_\omega^{-1}\Gamma_\omega(M_\omega x, r)$, and

$$M_\omega = \begin{pmatrix} \omega_1 & 0 & \cdot & \cdot & \cdot & 0 & \omega_1 & 0 \\ 0 & \omega_2 & 0 & \cdot & \cdot & 0 & \omega_2 & 0 \\ \cdot & & \cdot & & & & \cdot & \cdot \\ & & & \cdot & & & & \\ \cdot & & & & \cdot & & \cdot & \\ & & & & & \omega_{n-2} & \omega_{n-2} & \cdot \\ \cdot & & & & & 0 & \omega_{n-1} & 0 \\ 0 & \cdot & & \cdot & & \cdot & 0 & 1 \end{pmatrix}.$$

We observe that $\det(M_\omega) = \omega_1\omega_2 \ldots \omega_{n-1} \neq 0$ for almost every $\omega \in \mathbb{S}^{n-2}$, which is sufficient for our purposes.

Let $P_\omega(x') = P((M_\omega x)')$. For $\lambda \in \mathbb{R}^{n-2}$ let

$$H_{\lambda,\omega}g(y_1, y_2) = \int_{-\infty}^{\infty} g(y_1 - r, y_2 - P_\omega(\lambda, y_1)\gamma_\omega(r))\frac{dr}{r}.$$

For fixed $\lambda \in \mathbb{R}^{n-2}$, P_ω is a polynomial in y_1 of degree less than or equal to the degree of P. Since

$$\lambda_\omega(r) = \frac{r\gamma_\omega''(r)}{\gamma'(r)} = \Lambda(r\omega),$$

and γ_ω is odd or even, the remark following Corollary 21 of Chapter 2 implies that

$$||H_{\lambda,\omega}||_{2-2} \leq C\Lambda_0^{-1/2}(\omega),$$

uniformly in λ and ω. If $f_{x''}(x_{n-1}, x_n) = f(x)$, then we observe that

$$H_{\widetilde{\Gamma}_\omega} f(x) = H_{x'',\omega} f_{x''}(x_{n-1}, x_n),$$

and so

$$||H_{\widetilde{\Gamma}_\omega} f||_{L^2(dx_{n-1}dx_n)} \leq C\Lambda_0^{-1/2}(\omega)||f_{x''}||_2,$$

for all $x'' \in \mathbb{R}^{n-2}$. Taking the L^2 norm in x'' gives

$$||H_{\widetilde{\Gamma}_\omega} f||_{L^2(\mathbb{R}^n)} \leq C\Lambda_0^{-1/2}(\omega)||f||_{L^2(\mathbb{R}^n)},$$

which implies,

$$||T_\Gamma||_{2-2} \leq C\int_{\mathbb{S}_+} \Lambda_0^{-1/2}(\omega)\Omega(\omega)d\sigma(\omega) \leq C||\Lambda_0^{-1/2}||_{s'}||\Omega||_s < \infty.$$

□

Theorem 48 applies to many flat surfaces. For example, if β is a multi index with each β_j even, then

$$\gamma(t) = e^{-\frac{1}{t^\beta}},$$

satisfies the conditions of Theorem 48 for t in a certain neigbourhood of the origin in $\mathbb{R}^{n-1}$.

Surfaces of higher codimension in $\mathbb{R}^n$

Using our techniques one may also obtain $L^p(\mathbb{R}^n)$ boundedness of singular integrals and maximal functions associated to certain flat surfaces in $\mathbb{R}^n$ of codimension greater than one. Using the generalisation of Theorem 11 to curves in $\mathbb{R}^n$ for $n > 2$, (see Carbery, Vance, Wainger, and Watson [7]) one may generalise Theorem 46 to surfaces of any codimension. As we have illustrated the underlying principle several times, we leave this as a remark.

Chapter 5

A multiplier relation for Calderón-Zygmund operators on $L^1(\mathbb{R}^n)$

In this chapter we address a question raised in section 1.3 of the introductory chapter. As remarked there, injectivity on $L^2(\mathbb{R}^n)$ of a translation invariant Calderón–Zygmund operator is equivalent to its Fourier multiplier being almost everywhere non–zero. Our aim here is to come to a similar conclusion on $L^1(\mathbb{R}^n)$ for a wide class of these operators.

5.1 Introduction

The Hilbert transform, defined almost everywhere (a.e.) for $f \in L^p(\mathbb{R})$,$1 \leq p < \infty$, by

$$Hf(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|y| \geq \epsilon} \frac{f(x-y)}{y} dy$$

is well known to be bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$, and weak type 1–1. This is discussed in Chapter 1. For $f \in L^2(\mathbb{R})$, the action of H can also be described by a Fourier multiplier, $\widehat{(Hf)}(\xi) = -i\text{sign}(\xi)\widehat{f}(\xi)$. This multiplier relation also holds for all $f \in L^1(\mathbb{R})$ such that $Hf \in L^1(\mathbb{R})$. This may be seen as follows; the reader is referred back to section 1.4.2 of the introductory chapter for the relevant background. Recall that $\{f \in L^1(\mathbb{R}) : Hf \in L^1(\mathbb{R})\}$ is the real Hardy space $H^1(\mathbb{R})$, and H is bounded from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$. If $f \in H^1(\mathbb{R})$, by the atomic decomposition of $H^1(\mathbb{R})$ (Theorem 17), there exist non-negative constants $\{\lambda_k\}$ such that $\sum \lambda_k < \infty$, and $H^1(\mathbb{R})$ atoms $\{a_k\}$ such that $f = \sum \lambda_k a_k$ in the $H^1(\mathbb{R})$ norm. Since H is bounded from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$, $Hf = \sum \lambda_k Ha_k$ in $L^1(\mathbb{R})$. On taking the Fourier transform of this expression we get the desired result, since each atom is in $L^2(\mathbb{R})$, and hence satisfies the multiplier relation. Observe that this implies that H is injective on $L^1(\mathbb{R})$.

The above discussion has its roots in Zygmund [43], where the analogue for Fourier series is proved using the classical complex Hardy spaces. The analogue states that if f and its conjugate $\tilde{f}$ are in $L^1(\mathbb{T})$, then $c_k(\tilde{f}) = -i\text{sign}(k)c_k(f)$. Zygmund also describes a very different approach. He considers a generalised integral, refered to as integral B, with which the above multiplier relation for Fourier coefficients holds for all $f \in L^1(\mathbb{T})$.

The purpose of this chapter is to deduce analogous $L^1(\mathbb{R}^n)$ results for a wide class of Calderón-Zygmund operators for which Hardy space techniques are not necessarily appropriate. The main conclusion is the following, which is Corollary 65 of Section 5.4.

Theorem *Let the operator T satisfy the conditions* (5.2), (5.3) , *and* (5.4). *If $u \in L^1(\mathbb{R}^n)$ is such that $Tu \in L^1(\mathbb{R}^n)$, then*

$$\widehat{(Tu)}(\xi) = \mathfrak{m}(\xi)\widehat{u}(\xi)$$

for every $\xi \neq 0$, where $\mathfrak{m}$ is the Fourier multiplier corresponding to T.

The above will be achieved by obtaining a multiplier relation on $L^1(\mathbb{R}^n)$ using a generalised integral. This was done for the Hilbert transform by Toland in [39], following the alternative approach in Zygmund.

It is worth remarking that the previous observations about H suggest we might try to characterise those Calderón-Zygmund operators T for which $\{f \in L^1(\mathbb{R}) : Tf \in L^1(\mathbb{R})\} = H^1(\mathbb{R})$. For some related results see Janson [17], and Uchiyama [40].

5.2 The class of operators

As remarked in the introduction, we have some choice in how we define a so called Calderón-Zygmund singular integral operator. We are able to obtain positive results in a number of situations, however, we shall concern ourselves here with a class that is invariant under generalised dilations. For further discussion the reader is refered to Section 1.3 of the introductory chapter.

Suppose that for each $t > 0$, $A(t) \in GL(n; \mathbb{R})$, and that the Rivière condition holds; i.e.

$$||A(s)^{-1}A(t)|| \leq C(t/s)^{\delta}, \tag{5.1}$$

for all $s \geq t$, and some $\delta > 0$.

Let B_0 be the unit ball in $\mathbb{R}^n$.

Suppose $Tf = f * \mathbb{K}$ is an $L^2(\mathbb{R}^n)$ bounded operator. Suppose also that the distribution

$$\mathbb{K} = \sum_{j\in\mathbb{Z}} K_j, \tag{5.2}$$

with K_j supported in $A(2^{j+1})B_0$. Let $\widetilde{K}_j(x) = \det A(2^j)K_j(A(2^j)x)$. Suppose

$$\int_{\mathbb{R}^n} |\widetilde{K}_j(x)|dx \leq C, \tag{5.3}$$

and

$$\int_{\mathbb{R}^n} |\widetilde{K}_j(x-y) - \widetilde{K}_j(x)|dx \leq C|y|^{\epsilon_0} \tag{5.4}$$

for some $\epsilon_0 > 0$.

It will be convenient to denote by T^l and T^g, convolution with the distributions

$$\mathbb{K}^l = \sum_{j<0} K_j \quad \text{and} \quad \mathbb{K}^g = \sum_{j\geq 0} K_j$$

respectively.

Some useful properties of this class

(P1) For $\mu, \nu \in \mathbb{Z}$ with $\mu \leq \nu$,

$$\int_{\mathbb{R}^n} \sum_{j=\mu}^{\nu} K_j(x)dx$$

is bounded uniformly in μ and ν.

(P2) There is an $\mathfrak{m} \in L^\infty(\mathbb{R}^n)$ such that $\widehat{(Tf)}(\xi) = \mathfrak{m}(\xi)\widehat{f}(\xi)$ for $f \in L^2(\mathbb{R}^n)$.

(P3) T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and is weak type (1,1).

For (P1) and (P3) see [5], and for (P2) see [36].

It is of great importance for us to observe that the Fourier multiplier $\mathfrak{m}$ is continuous on $\mathbb{R}^n\backslash\{0\}$. This may be shown directly, but we prefer to give a more elegant proof based on the following lemma.

Lemma 49. *If a is an $H^1(\mathbb{R}^n)$ atom then $Ta \in L^1(\mathbb{R}^n)$.*

Proof. As T commutes with translations, we may suppose that the ball, B, associated to a is centred at the origin.

Firstly, by the Cauchy–Schwarz inequality and the $L^2(\mathbb{R}^n)$ boundedness of T,

$$\int_{|x|\leq 2|B|} |Ta(x)|dx \leq \|Ta\|_2 |2B|^{1/2} < \infty.$$

Secondly,

$$\begin{aligned}\int_{|x|>2|B|} |Ta(x)|dx &= \int_{(2B)^c} \left| \int_B \mathbb{K}(x-y)a(y)dy \right| dx \\ &= \int_{(2B)^c} \left| \int_B \left(\mathbb{K}(x-y) - \mathbb{K}(x)\right) a(y)dy \right| dx \\ &\left(\text{since } \int a = 0\right) \\ &\leq |B|^{-1} \int_B \left(\int_{(2B)^c} |\mathbb{K}(x-y) - \mathbb{K}(x)|dx \right) dy.\end{aligned}$$

If $y \in B$ and $x \in (2B)^c$ then $|x-y| \geq \frac{1}{2}\text{diam}(B)$. By the Rivière condition there is a $J \in \mathbb{Z}$ such that $\|A(2^{j+1})\| \leq \frac{1}{2}\text{diam}(B)$ for all $j \leq J$. Since K_j is supported in $A(2^{j+1})B_0$, where B_0 is the unit ball in $\mathbb{R}^n$,

$$\int_{(2B)^c} |K_j(x-y) - K_j(x)|dx = 0$$

for all $j \leq J$ and $y \in B$. Consequently,

$$\begin{aligned}\int_{|x|>2|B|} |Ta(x)|dx &\leq |B|^{-1} \int_B \sum_{j\geq J} \int_{\mathbb{R}^n} |K_j(x-y) - K_j(x)|dxdy \\ &\leq |B|^{-1} \int_B \sum_{j\geq J} \int_{\mathbb{R}^n} \left| \widetilde{K}_j(x - A(2^{j+1})^{-1}y) - \widetilde{K}_j(x) \right| dxdy \\ &\leq c|B|^{-1} \int_B \sum_{j\geq J} |A(2^{j+1})^{-1}y|^{\epsilon_0} dy < \infty,\end{aligned}$$

by the Rivière condition. □

Lemma 50. $\mathfrak{m}$ *is continuous on* $\mathbb{R}^n\backslash\{0\}$.

Proof. let a be a non zero $H^1(\mathbb{R}^n)$ atom. By Lemma 49, $Ta \in L^1(\mathbb{R}^n)$, and so $\widehat{a}$ and $\widehat{Ta}$ are continuous. Since $a \in L^2(\mathbb{R}^n)$, $\widehat{Ta} = \mathfrak{m}\widehat{a}$ a.e. Therefore $\mathfrak{m}$ is continuous at every point for which $\widehat{a} \neq 0$. Choose any $\xi \in \mathbb{R}^n\backslash\{0\}$. For some $\eta \in \mathbb{R}^n\backslash\{0\}$, $\widehat{a}(\eta) \neq 0$. Let λ be a non zero real number and ρ be an orthogonal matrix such that $\eta = \lambda\rho\xi$. Now $0 \neq \widehat{a}(\eta) = \int a(x)e^{2\pi i \lambda\rho\xi\cdot x}dx = \int a(x)e^{2\pi i\xi\cdot(\lambda\rho^{-1}x)}dx = \widehat{a_{\lambda,\rho}}(\xi)$, where $a_{\lambda,\rho}(x) = \lambda^{-n}a(\lambda^{-1}\rho x)$. Since $a_{\lambda,\rho}$ is an $H^1(\mathbb{R}^n)$ atom, $\mathfrak{m}$ is continuous at ξ and hence on $\mathbb{R}^n\backslash\{0\}$. □

We wish to thank F.Ricci for pointing out this alternative to the author's original argument. This proof is more appealing as it may be applied to any translation invariant $L^2(\mathbb{R}^n)$ bounded operator for which Lemma 49 holds.

Lemma 51. $\mathbb{K}^g \in L^{n/(n-\epsilon_0)}(\mathbb{R}^n)$.

Proof. By conditions (5.3) and (5.4),

$$\widetilde{K}_j \in \Lambda^{1,\infty}_{\epsilon_0}(\mathbb{R}^n) = \left\{ f : \|f\|_1 + \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x-y) - f(x)|}{|y|^{\epsilon_0}} dx < \infty \right\},$$

uniformly in j. By [37], $\Lambda^{1,\infty}_{\epsilon_0}(\mathbb{R}^n)$ is continuously embedded in $L^{n/(n-\epsilon_0)}(\mathbb{R}^n)$, and so $\widetilde{K}_j$ is in $L^{n/(n-\epsilon_0)}(\mathbb{R}^n)$ uniformly in j. Consequently,

$$\begin{aligned}
\|\mathbb{K}^g\|_p &\le \sum_{j\ge 0} \|K_j\|_p \\
&= \sum_{j\ge 0} \left(\int_{\mathbb{R}^n} |K_j(A(2^j x))|^p \det A(2^j) dx \right)^{\frac{1}{p}} \\
&= \sum_{j\ge 0} (\det A(2^j))^{\frac{1-p}{p}} \|\widetilde{K}_j\|_p \\
&\le c \sum_{j\ge 0} \|A(2^j)^{-1}\|^{\frac{n(p-1)}{p}} < \infty,
\end{aligned}$$

for $p = n/(n-\epsilon_0)$, by the Rivière condition (5.1). □

Realising the operators as principal values

Before we can make any progress, we must establish a workable relationship between the Fourier multiplier $\mathfrak{m}$, and the kernel $\mathbb{K}$. That is, we must describe a way to define the Fourier transform of $\mathbb{K}$ pointwise, and then compare it with $\mathfrak{m}$.

For $\mu, \nu \in \mathbb{Z}$ with $\mu \le \nu$, let

$$\mathbb{K}_{\mu,\nu} = \sum_{j=\mu}^{\nu} K_j.$$

It is well known (see [5]) that $\widehat{\mathbb{K}}_{\mu,\nu}$ is uniformly bounded in μ and ν. The following lemmas are refinements of this.

Lemma 52. *For $\xi \neq 0$, $\widehat{\mathbb{K}}_{\mu,\nu}(\xi)$ converges as $\nu \to \infty$, and*

$$\widehat{\mathbb{K}_\mu}(\xi) = \lim_{\nu\to\infty} \widehat{\mathbb{K}_{\mu,\nu}}(\xi)$$

is bounded independently of μ.

Proof. Fix $\xi \in \mathbb{R}^n \backslash \{0\}$, and let $\nu' \ge \nu \ge 0$. It suffices to show that

$$\widehat{\mathbb{K}}_{\nu,\nu'}(\xi) = \sum_{j=\nu}^{\nu'} \widehat{K}_j(\xi) \longrightarrow 0$$

as $\nu, \nu' \to \infty$. If $z = \frac{\xi}{2|\xi|^2}$, $z \cdot \xi = \frac{1}{2}$, and so

$$\begin{aligned}\left|\sum_{j=\nu}^{\nu'} \widehat{K}_j(\xi)\right| &= \frac{1}{2}\left|\sum_{j=\nu}^{\nu'} \int_{\mathbb{R}^n} (e^{2\pi i x\cdot\xi} - e^{2\pi i(x+z)\cdot\xi}) K_j(x)dx\right| \\ &= \frac{1}{2}\sum_{j=\nu}^{\nu'}\left|\int_{\mathbb{R}^n} (K_j(x) - K_j(x-z)) e^{2\pi i x\cdot\xi} dx\right| \\ &\leq \frac{1}{2}\sum_{j=\nu}^{\nu'} \int_{\mathbb{R}^n} \left|\widetilde{K}_j(x) - \widetilde{K}_j(x - A(2^j)^{-1}z)\right| dx \\ &\leq c\sum_{j=\nu}^{\nu'} |A(2^j)^{-1}z|^{\epsilon_0} \longrightarrow 0\end{aligned}$$

as $\nu, \nu' \to \infty$, by the Rivière condition (5.1). Hence $\widehat{\mathbb{K}_{\mu,\nu}}(\xi)$ converges to a bounded function as $\nu \to \infty$. □

Lemma 53. *There exists a decreasing sequence of integers $\{\mu_j\}$, for which $\{\widehat{\mathbb{K}_{\mu_j}}(\xi)\}$ converges everywhere on $\mathbb{R}^n \backslash \{0\}$ to a bounded function.*

Proof. Fix $\xi \neq 0$. $\{\widehat{\mathbb{K}}_\mu(\xi)\}_{\mu \leq 0}$ is a bounded sequence in $\mathbb{C}$, so there exists a subsequence $\{\mu_j\}$ such that $\{\widehat{\mathbb{K}}_{\mu_j}(\xi)\}_{j\geq 0}$ converges. Let $\xi' \in \mathbb{R}^n\backslash\{0\}$. We shall show that $\{\widehat{\mathbb{K}}_{\mu_j}(\xi')\}$ also converges.

$$\begin{aligned}&\left|\left(\widehat{\mathbb{K}_{\mu_j}}(\xi) - \widehat{\mathbb{K}_{\mu_j}}(\xi')\right) - \left(\widehat{\mathbb{K}_{\mu_l}}(\xi) - \widehat{\mathbb{K}_{\mu_l}}(\xi')\right)\right| \\ &\qquad = \left|\sum_{r=\mu_j}^{\mu_l} \widehat{K}_r(\xi) - \widehat{K}_r(\xi')\right| \\ &\qquad \leq \sum_{r=\mu_j}^{\mu_l}\left|\int_{A(2^{r+1})B_0} K_r(x)\left(e^{2\pi i x\cdot\xi} - e^{2\pi i x\cdot\xi'}\right) dx\right| \\ &\qquad \leq \sum_{r=\mu_j}^{\mu_l}\int_{A(2^{r+1})B_0} |\xi - \xi'||x||K_r(x)|dx \\ &\qquad \leq |\xi - \xi'|\sum_{r=\mu_j}^{\mu_l} \|A(2^{r+1})\| \int_{\mathbb{R}^n} |K_r(x)|dx \\ &\qquad \leq c|\xi - \xi'|\sum_{r=\mu_j}^{\mu_l} \|A(2^{r+1})\| \longrightarrow 0\end{aligned}$$

as $j, l \to \infty$, by the Rivière condition (5.1). So $\{\widehat{\mathbb{K}_{\mu_j}}(\xi) - \widehat{\mathbb{K}_{\mu_j}}(\xi')\}_j$ converges, and hence $\{\widehat{\mathbb{K}_{\mu_j}}(\xi')\}_j$ converges. □

Define $\widetilde{\mathrm{m}} \in L^\infty(\mathbb{R}^n)$ by $\widetilde{\mathrm{m}}(\xi) = \lim_{j\to\infty} \widehat{\mathbb{K}_{\mu_j}}(\xi)$, $\xi \neq 0$. We now make some observations.

(i) By the Dominated Convergence Theorem (D.C.T.) and Plancherel's theorem

$$\|\mathbb{K}_{\mu_j} * f - \mathcal{F}^{-1}(\widetilde{\mathfrak{m}}\widehat{f})\|_2 \longrightarrow 0 \text{ as } j \to \infty,$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

(ii) Fix $f \in \mathcal{S}(\mathbb{R}^n)$ and $x \notin supp(f)$. There is a $J \in \mathbb{N}$ such that

$$Tf(x) = \int_{\mathbb{R}^n} \mathbb{K}(x)f(x-y)dy = \int \mathbb{K}_{\mu_j}(y)f(x-y)dy = \mathbb{K}_{\mu_j} * f(x)$$

for $j \geq J$.

These observations allow us to define an operator $S : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ satisfying

(i) $\widehat{Sf} = \widetilde{\mathfrak{m}}\widehat{f}$, and

(ii) $Sf(x) = Tf(x)$ whenever $f \in \mathcal{S}(\mathbb{R}^n)$ and $x \notin supp(f)$.

The fact that $T - S$ is bounded on $L^2(\mathbb{R}^n)$ and commutes with translations allows one to show that $T - S = \lambda I$, for some $\lambda \in \mathbb{C}$. This is equivalent to $\mathfrak{m}(\xi) = \widetilde{\mathfrak{m}}(\xi) + \lambda$. For our purposes we may suppose that $\lambda = 0$, i.e. $S = T$. For further details of this argument, the reader is refered to [36], Chapter 1, Section 7.

5.3 A generalised integral

For a set $E \subset \mathbb{R}^n$, $|E|$ shall denote its Lebesgue measure.
As we intend our integral to be a type of principal value, it is appropriate to initially define it on functions of compact support.
Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ have compact support, $t \in [0,1]^n$, and $m \in \mathbb{Z}$. Let

$$I_m(f)(t) = \frac{1}{2^{nm}} \sum_{k \in \mathbb{Z}^n} f\left(t + \frac{k}{2^m}\right) \quad \text{(a finite sum)}$$

Definition 54. *For $I \in \mathbb{R}$, write $I = \#\int_{\mathbb{R}^n} f(x)dx$ (or more briefly $I = \#\int f$), if $I_m(f)(t) \to I$ in measure on $[0,1]^n$ as $m \to \infty$*

Observe that if $f \in C_c(\mathbb{R}^n)$, then $I_m(f)(t)$ is a Riemann partial sum. Hence $\#\int f = \int f$. From this we can deduce the following.

Lemma 55. *For $f \in L^1(\mathbb{R}^n)$ of compact support, $\#\int f = \int f$*

In order to prove this lemma, we shall define a simple extension of $\# \int$ to functions of non compact support. This is not an appropriate extension since it fails to acknowledge any global cancellation, however, we take this brief diversion because our techniques naturally encompass it.

Definition 56. *Define for some measurable $f : \mathbb{R}^n \longrightarrow \mathbb{R}$,*

$$\widetilde{I_m}(f)(t) \;=\; \frac{1}{2^{nm}} \sum_{k \in \mathbb{Z}^n} f\left(t + \frac{k}{2^m}\right) \quad t \in [0,1]^n$$

whenever the sum is absolutely convergent for a.e. $t \in [0,1]^n$. (So for $f \in L^1(\mathbb{R}^n)$ of compact support, $\widetilde{I_m} f = I_m f$.) Define $\widetilde{\#} \int f$ in analogy with $\# \int f$.

Lemma 57. *For $f \in L^1(\mathbb{R}^n)$, $\widetilde{\#} \int f = \int f$.*

Proof. We must first show that $I_m(f)$ is defined for $f \in L^1(\mathbb{R}^n)$. Let G be the set of lattice points in $[0, 2^m)^n$. Observe that

$$\sum_{k \in \mathbb{Z}^n} \frac{1}{2^{nm}} \int_{[0,1]^n} \left| f\left(t + \frac{k}{2^m}\right)\right| dt = \sum_{\gamma \in G} \sum_{k \in 2^m \mathbb{Z}^n + \{\gamma\}} \frac{1}{2^{nm}} \int_{[0,1]^n} \left| f\left(t + \frac{k}{2^m}\right)\right| dt$$
$$= \sum_{\gamma \in G} \frac{1}{2^{nm}} \|f\|_1 = \|f\|_1 < \infty.$$

So by the Monotone Convergence Theorem, $\sum_{k \in \mathbb{Z}^n} \left| f\left(t + \frac{k}{2^m}\right)\right| < \infty$ a.e. $t \in [0,1]^n$ as required. Observe that we also have,

$$\int_{[0,1]^n} \left|\widetilde{I_m}(f)(t)\right| dt \leq \|f\|_{L^1(\mathbb{R}^n)}. \tag{5.5}$$

Let $f \in L^1(\mathbb{R}^n)$, and $\alpha, \epsilon > 0$. Choose $f_1 \in C_c(\mathbb{R}^n)$, and $f_2 \in L^1(\mathbb{R}^n)$ such that $f = f_1 + f_2$ and $\|f_2\|_1 < \frac{\alpha}{4} \min(\epsilon, 1)$. By (5.5) and Chebychev's inequality,

$$\left|\left\{t \in [0,1]^n : \left|\widetilde{I_m}(f_2)(t)\right| \geq \frac{\alpha}{2}\right\}\right| \leq \frac{2\|f_2\|_1}{\alpha} < \frac{\epsilon}{2} \tag{5.6}$$

By the triangle inequality,

$$\left|\left\{t \in [0,1]^n : \left|\widetilde{I_m}(f)(t) - \int f\right| \geq \alpha\right\}\right|$$
$$\leq \left|\left\{t \in [0,1]^n : \left|\widetilde{I_m}(f_1)(t) - \int f_1\right| \geq \frac{\alpha}{4}\right\}\right| \tag{5.7}$$
$$+ \left|\left\{t \in [0,1]^n : \left|\widetilde{I_m}(f_2)(t)\right| \geq \frac{\alpha}{2}\right\}\right| \tag{5.8}$$
$$+ \left|\left\{t \in [0,1]^n : \left|\int f_2\right| \geq \frac{\alpha}{4}\right\}\right| \tag{5.9}$$

Since $\|f_2\|_1 < \frac{\alpha}{4}$, the term (5.9) is zero. By (5.6) the term (5.8) is less than $\frac{\epsilon}{2}$. Since $f_1 \in C_c(\mathbb{R}^n)$, the remark preceding Lemma 55 implies that the term (5.7) can be made less than $\frac{\epsilon}{2}$ for sufficiently large m. This concludes the proof. □

For our purposes it is more appropriate to extend $\#\int$ to functions of non-compact support by the following limiting process.
Let $\rho \in C_c^\infty(\mathbb{R}^n)$ satisfy

(i) $\rho(0) = 1$

(ii) $0 \le \rho(x) \le 1$

Let $\rho_N(x) = \rho(\frac{x}{N})$.

Definition 58. *For $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, we write $I = \#\int_{\mathbb{R}^n} f(x)dx$ (or $I = \#\int f$), if for every such ρ, $\#\int_{\mathbb{R}^n} \rho_N(x) f(x)dx$ converges to I as $N \to \infty$.*

By Lemma 55 and the Dominated Convergence Theorem, $\#\int f = \int f$ for every $f \in L^1(\mathbb{R}^n)$.

In order to exploit the translation invariance of T, we shall need the following Lemma.

Lemma 59. *Let $v \in C_c^1(\mathbb{R}^n)$, $u \in L^1(\mathbb{R}^n)$,*

$$S_v^l(u)(x) = (T^l vu)(x) - v(x)(T^l u)(x) \quad \textit{and,}$$

$$S_v^g(u)(x) = (T^g vu)(x) - v(x)(T^g u)(x).$$

S_v^l is bounded on $L^1(\mathbb{R}^n)$, and S_v^g is bounded from $L^1(\mathbb{R}^n)$ to $L^{n/(n-\epsilon)}(\mathbb{R}^n)$.

Proof. By Minkowski's inequality for integrals, it is sufficient to show that

$$\sup_{y \in \mathbb{R}^n} \|(v(y) - v(\cdot))\mathbb{K}^l(\cdot - y)\|_{L^1(\mathbb{R}^n)} < \infty \quad \text{and,}$$

$$\sup_{y \in \mathbb{R}^n} \|(v(y) - v(\cdot))\mathbb{K}^g(\cdot - y)\|_{L^{n/(n-\epsilon)}(\mathbb{R}^n)} < \infty.$$

Now,

$$\begin{aligned}\int_{\mathbb{R}^n} |(v(y) - v(x))\mathbb{K}^l(x-y)|dx &\le \|\nabla v\|_\infty \int_{\mathbb{R}^n} |x||\mathbb{K}^l(x)|dx \\ &\le \sum_{j<0} \int_{A(2^{j+1})B_0} |x||K_j(x)|dx \\ &\le c\sum_{j<0} \|A(2^{j+1})\|\|K_j\|_{L^1(\mathbb{R}^n)} < \infty\end{aligned}$$

and, by Lemma 51,

$$\left(\int_{\mathbb{R}^n} |(v(y) - v(x))\mathbb{K}^g(x-y)|^p dx\right)^{\frac{1}{p}} \le 2\|v\|_\infty \|\mathbb{K}^g\|_p < \infty,$$

for $p = n/(n - \epsilon_0)$. □

Remark. If we were to strengthen the Rivière condition (5.1) by requiring it to hold for all $s, t > 0$, then we would also have that $S_v^l : L^1(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, for $1 < p < \frac{n}{n-\epsilon_0}$. This would simplify the proof of the following lemma, since there would be no need to consider S_v^l and S_v^g separately. The said strengthening of the Rivière condition implies that there are constants c_1 and c_2 so that

$$c_1 s^\delta \leq \|A(s)\| \leq c_2 s^\delta$$

for all $s > 0$.

Our next lemma is at the heart of this chapter, since it allows us to approximate $L^1(\mathbb{R}^n)$ functions by smooth functions. In what follows we should think of the function u as an error of such an approximation.

Lemma 60. *Suppose $\phi \in C_c^1(\mathbb{R}^n)$, $\alpha > 0$, and $0 < \epsilon < 1$. There is a constant $\kappa = \kappa(\phi, n)$ such that for $u \in L^1(\mathbb{R}^n)$ with $\|u\|_1 \leq \kappa\alpha\epsilon$,*

$$|\{t \in [0,1]^n : |I_m(\phi T u)(t)| \geq \alpha\}| \leq \epsilon \text{ for all } m \in \mathbb{N}. \tag{5.10}$$

Proof. Let $t \in [0,1]^n$ and suppose N is chosen so that $supp(\phi) \in [-N, N]^n$. Let

$$A_{m,t} = \left\{ k \in \mathbb{Z}^n : t + \frac{k}{2^m} \in [-N, N]^n \right\}$$

We shall dominate $I_m(\phi T u)(t)$ by the sum of three terms, each of which will satisfy an expression of the form (5.10).

$$|I_m(\phi T u)| \leq \frac{1}{2^{nm}} \left| \sum_{k \in A_{m,t}} T(\phi u)\left(t + \frac{k}{2^m}\right) \right| + \frac{1}{2^{nm}} \left| \sum_{k \in A_{m,t}} S_\phi^l(u)\left(t + \frac{k}{2^m}\right) \right|$$
$$+ \frac{1}{2^{nm}} \left| \sum_{k \in A_{m,t}} S_\phi^g(u)\left(t + \frac{k}{2^m}\right) \right| \tag{5.11}$$

where S_ϕ^l and S_ϕ^g are defined in Lemma 59. Let

$$v_k(x) = \phi\left(x + \frac{k}{2^m}\right) u\left(x + \frac{k}{2^m}\right).$$

Since T is linear and commutes with translations,

$$\frac{1}{2^{nm}} \sum_{k \in A_{m,t}} T(\phi u)\left(t + \frac{k}{2^m}\right) = T\left(\frac{1}{2^{nm}} \sum_{k \in A_{m,t}} v_k\right)(t). \tag{5.12}$$

Observe that for each m, $A_{m,t}$ is constant, say A_m, on $(0,1)^n$. Using this, (5.12), and the fact that T is weak type 1–1, we get for some constant c,

$$\left|\left\{t \in [0,1]^n : \left|\frac{1}{2^{nm}} \sum_{k\in A_{m,t}} T(\phi u)\left(t+\frac{k}{2^m}\right)\right| \geq \alpha\right\}\right|$$
$$= \left|\left\{t \in (0,1)^n : \left|T\left(\frac{1}{2^{nm}} \sum_{k\in A_{m,t}} v_k\right)(t)\right| \geq \alpha\right\}\right|$$
$$\leq \frac{c}{\alpha}\left\|\frac{1}{2^{nm}}\sum_{k\in A_m} v_k\right\|_{L^1(\mathbb{R}^n)}$$
$$\leq c2^n N^n \|\phi\|_\infty \frac{\|u\|_{L^1(\mathbb{R}^n)}}{\alpha} < \epsilon$$

provided $\|u\|_{L^1(\mathbb{R}^n)} \leq \frac{\alpha\epsilon}{c2^nN^n\|\phi\|_\infty}$. This deals with the first term of (5.11) with $\kappa = \frac{1}{c2^nN^n\|\phi\|_\infty}$. We now turn to the remaining terms. Let

$$J_{m,\phi}(f)(t) = \frac{1}{2^{nm}} \sum_{k\in A_{m,t}} f\left(t+\frac{k}{2^m}\right) \quad \text{for } f \in L^p(\mathbb{R}^n),\ 1 \leq p \leq \infty.$$

By (5.5), $\|J_{m,\phi}(f)\|_{L^1([0,1]^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$, and by considering the number of elements of $A_{m,t}$, $\|J_{m,\phi}(f)\|_{L^\infty([0,1]^n)} \leq 2^n(N+1)^n\|f\|_{L^\infty(\mathbb{R}^n)}$. Therefore by the Riesz convexity theorem, $\|J_{m,\phi}(f)\|_{L^p([0,1]^n)} \leq (2^n(N+1)^n)^{\frac{1}{q}}\|f\|_{L^p(\mathbb{R}^n)}$ for $1 \leq p \leq \infty$. Here, as usual, $\frac{1}{p}+\frac{1}{q}=1$. By Lemma 59 and composition of $J_{m,\phi}$ with S^l_ϕ,

$$u \longmapsto \frac{1}{2^{nm}}\left|\sum_{k\in A_{m,t}} S^l_\phi(u)\left(t+\frac{k}{2^m}\right)\right|$$

is bounded on $L^1(\mathbb{R}^n)$ with bound independent of m. By Chebyshev's inequality, there is a constant $\kappa = \kappa(\phi, n)$ such that

$$\left|\left\{t \in [0,1]^n; \frac{1}{2^{nm}}\left|\sum_{k\in A_{m,t}} S^l_\phi(u)\left(t+\frac{k}{2^m}\right)\right| \geq \alpha\right\}\right| \leq \frac{\|u\|_{L^1(\mathbb{R}^n)}}{\kappa\alpha} < \epsilon$$

provided $\|u\|_{L^1(\mathbb{R}^n)} < \kappa\alpha\epsilon$.

Similarly, since

$$u \longmapsto \frac{1}{2^{nm}}\left|\sum_{k\in A_{m,t}} S^g_\phi(u)\left(t+\frac{k}{2^m}\right)\right|$$

is bounded from $L^1(\mathbb{R}^n)$ to $L^{n/(n-\epsilon_0)}(\mathbb{R}^n)$, with bound independent of m, there is a constant $\kappa' = \kappa'(\phi, n)$ such that

$$\left|\left\{t \in [0,1]^n; \frac{1}{2^{nm}}\left|\sum_{k\in A_{m,t}} S^g_\phi(u)\left(t+\frac{k}{2^m}\right)\right| \geq \alpha\right\}\right| \leq \left(\frac{\|u\|_{L^1(\mathbb{R}^n)}}{\kappa'\alpha}\right)^{n/(n-\epsilon_0)}$$
$$< \epsilon^{n/(n-\epsilon_0)} < \epsilon$$

provided $\|u\|_{L^1(\mathbb{R}^n)} < \kappa'\alpha\epsilon$. This deals with the second and third terms in (5.11).

$\square$

Lemma 61. *For $\phi \in C_c^1(\mathbb{R}^n)$, $T\phi \in L^\infty(\mathbb{R}^n)$.*

Proof. For any non–negative integer k,

$$\sum_{j\geq k} K_j * \phi(x) = \sum_{k\leq j<0} \int_{\mathbb{R}^n} K_j(x-y)(\phi(y)-\phi(x))dx$$
$$+ \phi(x) \sum_{k\leq j<0} \int_{\mathbb{R}^n} K_j(x-y)dy + \mathbb{K}^g * \phi(x) = I + II + III.$$

Now, as in the proof of Lemma 59,

$$|I| \leq c\|\nabla\phi\|_\infty \sup_j \|K_j\|_{L^1(\mathbb{R}^n)} \sum_{j<0} \|A(2^{j+1})\| < \infty.$$

By (P1),

$$|II| \leq \|\phi\|_\infty \sup_{\mu,\nu\in\mathbb{Z}} \left| \int_{\mathbb{R}^n} \mathbb{K}_{\mu,\nu}(x)dx \right| < \infty.$$

By Lemma 51, $\mathbb{K}^g \in L^{n/(n-\epsilon_0)}$, and so by Holder's inequality,

$$|III| = |\mathbb{K}^g * \phi(x)| \leq \|\mathbb{K}^g\|_{n/(n-\epsilon_0)}\|\phi\|_{n/\epsilon_0} < \infty.$$

Combining the above three estimates completes the proof of the lemma.

$\square$

Lemma 62. *If $\phi \in C_c^1(\mathbb{R}^n)$ and $u \in L^1(\mathbb{R}^n)$ then*

$$\# \int_{\mathbb{R}^n} \phi(x)\overline{(Tu)(x)}dx = \int_{\mathbb{R}^n} (T^*\phi)(x)\overline{u(x)}dx$$

where T^* is the L^2 adjoint of T, having Calderón-Zygmund kernel $\mathbb{K}^*(x) = \overline{\mathbb{K}(-x)}$. (Note that in general $Tu \notin L^1_{loc.}(\mathbb{R}^n)$.)

Proof. Let $u = v_j + w_j$ where $v_j \in C_c^1(\mathbb{R}^n)$ and $\|w_j\|_{L^1(\mathbb{R}^n)} \to 0$ as $j \to \infty$. Let $\alpha > 0$ and $0 < \epsilon < 1$. By the triangle inequality,

$$\left|\left\{t \in [0,1]^n : \left| I_m\left(\phi\overline{Tu}\right)(t) - \int (T^*\phi)(x)\overline{u(x)}dx \right| \geq \alpha\right\}\right|$$
$$\leq \left|\left\{t \in [0,1]^n : \left| I_m\left(\phi\overline{Tv_j}\right)(t) - \int (T^*\phi)(x)\overline{v_j(x)}dx \right| \geq \frac{\alpha}{3}\right\}\right| \quad (5.13)$$
$$+ \left|\left\{t \in [0,1]^n : \left| \int (T^*\phi)(x)\overline{w_j(x)}dx \right| \geq \frac{\alpha}{3}\right\}\right| \quad (5.14)$$
$$+ \left|\left\{t \in [0,1]^n : \left| I_m\left(\phi\overline{Tw_j}\right)(t)\right| \geq \frac{\alpha}{3}\right\}\right| \quad (5.15)$$

By Lemma 60, there is an integer J such that

$$\left|\left\{t \in [0,1]^n : \left|I_m\left(\phi\overline{Tw_j}\right)(t)\right| \geq \frac{\alpha}{3}\right\}\right| < \epsilon \quad \forall m \in \mathbb{Z}, j \geq J.$$

So the term (5.15) is less than ϵ for $j \geq J$. By Lemma 61, $T^*\phi \in L^\infty(\mathbb{R}^n)$, and hence

$$\int (T^*\phi)(x)\overline{w_j(x)}dx \to 0$$

as $j \to \infty$, so increasing J if necessary we may suppose that

$$\left|\int (T^*\phi)(x)\overline{w_j(x)}dx\right| < \frac{\alpha}{3} \quad \forall j \geq J.$$

So for $j \geq J$, the term (5.14) is zero. As $v_j, \phi \in L^2(\mathbb{R}^n)$,

$$\int (T^*\phi)(x)\overline{v_j(x)}dx = \int \phi(x)\overline{(Tv_j)(x)}dx,$$

so term (5.13) now becomes

$$\left|\left\{t \in [0,1]^n : \left|I_m\left(\phi\overline{Tv_j}\right)(t) - \int \phi(x)\overline{(Tv_j)(x)}dx\right| \geq \frac{\alpha}{3}\right\}\right|.$$

Fix $j \geq J$. $\phi\overline{Tv_j} \in L^1(\mathbb{R}^n)$, so by Lemma 55 this term (5.13) tends to zero as $m \to \infty$. □

5.4 The multiplier relation on $L^1(\mathbb{R}^n)$

Lemma 63. *If $\psi_N^{(\xi)}(y) = \rho_N(y)e^{2\pi i\xi\cdot y}$, $\xi \neq 0$, then*

$$T^*\psi_N^{(\xi)}(x) - \rho_N(x)\overline{\mathfrak{m}(-\xi)}e^{2\pi i\xi\cdot x} \longrightarrow 0$$

uniformly in x as $N \to \infty$.

Proof. Let $\mathbb{K}^*(x) = \overline{\mathbb{K}(-x)}$, and $\xi \neq 0$.

$$\begin{aligned}
&T^*\psi_N^{(\xi)}(x) - \rho_N(x)\overline{\mathfrak{m}(-\xi)}e^{2\pi i\xi\cdot x} \\
&= \lim_{j\to\infty}\lim_{\nu\to\infty} e^{2\pi i\xi\cdot x}\int_{\mathbb{R}^n} \mathbb{K}^*_{\mu_j,\nu}(y)(\rho_N(x-y) - \rho_N(x))e^{-2\pi i\xi\cdot y}dy.
\end{aligned}$$

By writing ρ as the inverse Fourier transform of $\widehat{\rho}$, and then by Fubini's theorem,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \mathbb{K}^*_{\mu_j,\nu}(y)(\rho_N(x-y) - \rho_N(x))e^{-2\pi i\xi\cdot y}dy\right| \\
&= \left|\int_{\mathbb{R}^n} \mathbb{K}^*_{\mu_j,\nu}(y)\int_{\mathbb{R}^n}\widehat{\rho}(s)\left(e^{-2\pi i\frac{(x-y)}{N}\cdot s} - e^{-2\pi i\frac{x}{N}\cdot s}\right)e^{-2\pi i\xi\cdot y}dsdy\right| \\
&= \left|\int_{\mathbb{R}^n}\widehat{\rho}(s)e^{2\pi ix\cdot\frac{s}{N}}\left(\overline{\widehat{\mathbb{K}_{\mu_j,\nu}}\left(\frac{s}{N}-\xi\right)} - \overline{\widehat{\mathbb{K}_{\mu_j,\nu}}(-\xi)}\right)ds\right| \\
&\leq \int_{\mathbb{R}^n}|\widehat{\rho}(s)|\left|\widehat{\mathbb{K}_{\mu_j,\nu}}\left(\frac{s}{N}-\xi\right) - \widehat{\mathbb{K}_{\mu_j,\nu}}(-\xi)\right|ds \\
&\longrightarrow \int_{\mathbb{R}^n}|\widehat{\rho}(s)|\left|\mathfrak{m}\left(\frac{s}{N}-\xi\right) - \mathfrak{m}(-\xi)\right|ds
\end{aligned}$$

as $\nu \to \infty$ and $j \to \infty$ by Lemmas 52, 53, and the D.C.T.. The last expression tends to zero uniformly in x as $N \to \infty$, by the continuity of $\mathfrak{m}$ on $\mathbb{R}^n \setminus \{0\}$, (Lemma 50), and the D.C.T.. □

Theorem 64. *Let T satisfy* (5.2), (5.3), *and* (5.4). *If $u \in L^1(\mathbb{R}^n)$ then,*

$$\# \int_{\mathbb{R}^n} (Tu)(x) e^{2\pi i \xi \cdot x} dx = \mathfrak{m}(\xi) \widehat{u}(\xi)$$

for every $\xi \neq 0$.

Proof. If $u \in L^1(\mathbb{R}^n)$, and $\xi \neq 0$ then

$$\begin{aligned} \# \int_{\mathbb{R}^n} & (Tu)(x) e^{2\pi i \xi \cdot x} \rho_N(x) dx \\ &= \# \int_{\mathbb{R}^n} (Tu)(x) \overline{e^{-2\pi i \xi \cdot x} \rho_N(x)} dx \\ &= \int_{\mathbb{R}^n} u(x) \overline{\left(T^* \psi_N^{(-\xi)} \right)(x)} dx \quad \text{(by Lemma 63)} \\ &\longrightarrow \int_{\mathbb{R}^n} u(x) e^{2\pi i \xi \cdot x} \mathfrak{m}(\xi) dx \end{aligned}$$

as $N \to \infty$ by Lemma 63 and the D.C.T.. Hence

$$\# \int_{\mathbb{R}^n} (Tu)(x) e^{2\pi i \xi \cdot x} dx = \mathfrak{m}(\xi) \widehat{u}(\xi).$$

□

Corollary 65. *Let T satisfy* (5.2), (5.3), *and* (5.4). *If $u \in L^1(\mathbb{R}^n)$ is such that $Tu \in L^1(\mathbb{R}^n)$, then*

$$\widehat{(Tu)}(\xi) = \mathfrak{m}(\xi) \widehat{u}(\xi), \quad \xi \neq 0.$$

Proof. Use Theorem 64 and the remark after Definition 3. □

Corollary 66. *If T satisfies* (5.2), (5.3), *and* (5.4), *then T is injective on $L^1(\mathbb{R}^n)$ if and only if*

$$E = \{\xi : \mathfrak{m}(\xi) = 0\}$$

has empty interior.

Proof. Suppose $u \in L^1(\mathbb{R}^n)$ is such that $Tu = 0$. By Theorem 64, $\mathfrak{m}(\xi)\widehat{u}(\xi) = 0$ for all $\xi \neq 0$. Hence $\widehat{u}(\xi)$ is supported in E. Since $\widehat{u}$ is continuous and E has empty interior, we conclude that $\widehat{u} = 0$. Since $u \in L^1(\mathbb{R}^n)$, $u = 0$.

Conversely, suppose T is injective on $L^1(\mathbb{R}^n)$. If E has non–empty interior, then there is a $\phi \in \mathcal{S}(\mathbb{R}^n)$, with $\phi \neq 0$, such that $\widehat{\phi}$ is supported in E. By the $L^2(\mathbb{R}^n)$ Fourier multiplier relation, $\widehat{T\phi}(\xi) = \mathfrak{m}(\xi)\widehat{\phi}(\xi) = 0$, for $\xi \neq 0$, contradicting the injectivity of T. □

Corollary 67. *Suppose* $\mathbb{K}$ *is homogeneous of degree* $-n$ *and* $f \in L^1(\mathbb{R}^n)$ *is non-negative. If* $f \not\equiv 0$ *then* $Tf \notin L^1(\mathbb{R}^n)$.

Proof. Use Corollary 65 and the fact that $\mathfrak{m}$ is homogeneous of degree 0. □

Remark

For $1 \leq p \leq 2$, it is easy to see that T is injective on $L^p(\mathbb{R}^n)$ if and only if

$$L^p_E = \{f \in L^p(\mathbb{R}^n) : \operatorname{supp}(\widehat{f}\,) \subset E\} = \{0\}.$$

Trivially $L^2_E = \{0\} \iff |E| = 0$, and by Corollary 66, $L^1_E = \{0\} \iff \operatorname{int}(E) = \varnothing$. For $1 < p < 2$, we have little qualitative information about $\widehat{f}$, for $f \in L^p(\mathbb{R}^n)$, other than that it is in $L^q(\mathbb{R}^n)$. Hence a simple characterisation of those E for which $L^p_E = \{0\}$ is less apparent.

5.5 Application to Oscillatory Singular Integrals

The techniques used in the proof of Theorem 64 can be applied to a much greater variety of translation invariant operators that are weak type 1–1. For example, one can handle some oscillatory singular integrals of the type described in the introductory chapter.

Theorem 68. *Suppose* $\phi : \mathbb{R} \to \mathbb{R}$ *satisfies*

(i) ϕ *is either even or odd,*

(ii) $\phi(0) = \phi'(0) = 0$,

(iii) $\phi''' \geq 0$ *on* $(0, \infty)$,

and define the operator T *by*

$$Tf(x) = p.v. \int_{-\infty}^{\infty} \frac{e^{i\phi(x-y)}}{x-y} f(y)dy.$$

If $\mathfrak{m}$ *is the Fourier multiplier corresponding to* T*, then the generalised multiplier relation*

$$\# \int_{-\infty}^{\infty} Tf(x) e^{2\pi i x \xi} dx = \mathfrak{m}(\xi)\widehat{f}(\xi),\ \xi \neq 0,$$

holds for all $f \in L^1(\mathbb{R})$.

In order to avoid repetition of many of our earlier arguments, we give a sketch of only the main points of the proof.

The proof of Theorem 68 essentially follows the same sequence of lemmas as that of Theorem 64. The appropriate version of Lemma 49 immediately follows from the H^1 boundedness of T (see Theorem 42). We are able to approximate the Fourier multiplier pointwise by a principal value integral (providing the analogues of Lemmas 52 and 53) by an integration by parts argument. The key calculation is the following.

For $0 < R \leq R' < \infty$,

$$\begin{aligned}\int_R^{R'} \frac{e^{i(\phi(x)+x\xi)}}{x} dx &= \int_R^{R'} \frac{1}{ix(\phi'(x)+\xi)} \frac{d}{dx}\left(e^{i(\phi(x)+x\xi)}\right) dx \\ &= \left[\frac{e^{i(\phi(x)+x\xi)}}{ix(\phi'(x)+\xi)}\right]_R^{R'} \\ &\quad - \int_R^{R'} e^{i(\phi(x)+x\xi)} \frac{d}{dx}\left(\frac{1}{ix(\phi'(x)+\xi)}\right) dx \\ &= I + II.\end{aligned}$$

Since $\phi''' \geq 0$, $\phi'(R) \to \infty$ as $R \to \infty$, and so

$$|I| \leq \frac{2}{R|\phi'(R)+\xi|} \longrightarrow 0,$$

as $R, R' \to \infty$.

$$|II| \leq \int_R^{R'} \left|\frac{d}{dx}\left(\frac{1}{x(\phi'(x)+\xi)}\right)\right| dx.$$

If $\psi(x) = x(\phi'(x)+\xi)$, then

$$\psi'(x) = x\phi''(x) + \phi'(x) + \xi,$$

and since ϕ is convex, there is an $R = R(\xi) > 0$ such that $\psi'(x) \geq 0$ for all $x \geq R$. Consequently,

$$\begin{aligned}|II| &\leq \left|\int_R^{R'} \frac{d}{dx}\left(\frac{1}{x(\phi'(x)+\xi)}\right) dx\right| \\ &\leq \frac{2}{R|\phi'(R)+\xi|} \longrightarrow 0\end{aligned}$$

as $R, R' \to \infty$, as in the estimate for $|I|$.

The remaining parts of the proof use the L^2 and weak type 1–1 boundedness of T, and the size of the absolute value of its kernel $\frac{e^{i\phi(x)}}{x}$; i.e. no new ideas are required.

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