Refuse or Reuse: Managing the Quality of Returns in Product Recovery Systems

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Sarah Elizabeth Marshall*)

“For with God nothing shall be impossible.”
Luke 1:37

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Abstract

Increasing legislative and societal pressures are forcing manufacturers to become environmentally-conscious and take responsibility for the fate of their goods after they have been used by consumers. As a result, some manufacturers operate hybrid systems which produce new goods and recover used goods. Product recovery describes the process by which used products are returned to their manufacturers or sent to a specialised facility for recovery, before being sold on the original or a secondary market. The quality of the returned goods is a significant issue in product recovery systems as it can affect both the type of recovery and costs associated with it. Quality in product recovery systems has not been adequately studied, with many authors either ignoring the possibility of receiving lower quality returns, or assuming they are disposed of rather than recovered. However, such assumptions ignore the possibility that the firm might be able to salvage value from lower quality returns by using them for parts or materials.

This thesis presents four models that investigate the importance of considering the quality of returns in the management of inventory in a product recovery system, by examining the cost-effectiveness of recovering both high quality and low quality returns.

The first model is a deterministic lot-sizing model of a product recovery system. It was found that performing both high and low quality recovery reduced the sensitivity of the optimal cost to operational restrictions on the choice of decision variables.

The second model is a discrete-time, periodic-review model formulated as a Markov decision process (MDP) and introduces uncertainty in demand, returns, and the quality of the returns. It was found that performing both types of recovery can lead to cost savings and better customer service for firms through an increased fill rate.

The third model addresses those industries where produced and recovered goods cannot be sold on the same market due to customers' perceptions and environmental legalisation. Using an MDP formulation, the model examines a product recovery system in which produced and recovered goods are sold on separate markets. The profitability of offering two-way substitution between these markets was investigated. It was found that offering substitution can allow firms to increase both their profits and fill rates.

The fourth model examines the issue of separate markets and substitution in the continuous time domain using a semi-Markov decision process. The continuous nature of the model allows more detailed examination of the substitution decision. It was found that offering substitution can allow firms to increase their profit and in some cases also increase their fill rate. In some cases, production is performed less frequently when downward substitution can be offered, and recovery is performed less often when upward substitution can be offered.

The findings of this thesis could be used to help a firm that is currently recovering high quality returns assess the cost-effectiveness of also recovering lower quality returns. Recovering low-quality items, rather than disposing of them, may allow a firm to increase the amount it recycles. The findings highlight the importance of considering the quality of returns when managing a product recovery system as they show that economic gains can be achieved by reusing rather than refusing low quality returns.

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Chapter 1

Introduction

These days the news is frequently filled with stories about global warming, pollution, and how we, as a society, are not doing enough to prevent or slow it. As a result of increasing governmental and societal pressures, companies and consumers alike have been encouraged to reduce, reuse and recycle in an effort to reduce the amount of waste that is sent to landfills. This has lead to dramatic increases in recycling in England with household recycling increasing from 11% to 40% between 2001 and 2011, and commercial and industrial recycling increasing from 42% to 52% between 2002/3 and 2009 (DEFRA, 2011c). Whilst these increases are impressive, further increases in recycling and further reductions in waste-generation are necessary in order to achieve the *zero-waste economy*, to which the current UK Government says it is committed to achieving (DEFRA, 2011a). One way in which further waste-reduction and recycling can be achieved is product recovery.

Product recovery describes the process by which used products are returned to their producer or are sent to a specialised facility, in order to undergo recovery. The recovered products are then sold either to the same market as their newly produced equivalents or to a separate, secondary market. For some used goods, if they do not undergo product recovery then they will be sent to a landfill. Whilst recycling and reuse are not new phenomena (e.g. reuse during wartime rationing), there seems to have been a recent resurgence in these activities. As an example of this resurgence, consider the many computer manufacturers and distributors that are now actively promoting their refurbishment facilities both to existing consumers (who may return used goods) and to new consumers (who may buy refurbished goods) (Apple, 2011; PCWorld, 2011; Fujitsu, 2011). An increase in product recovery, recycling and reuse in practice, has also led to an increase in related literature.

Three main arguments are cited in the literature as an explanation for the increase in and importance of product recovery: legislative, environmental, economic (Heisig and Fleischmann, 2001).

Legislative. Legislation requiring manufacturers in certain industries to take responsibility for the fate of their products has been introduced in many countries, with the aim of decreasing the quantity of harmful substances ending up in landfills. In particular, the European Commission has issued directives relating to waste electronic and electrical equipment (WEEE), end-of-life vehicles (ELV) and batteries, which regulate what happens to these types of products after they have been used by consumers (Environment Agency, 2011a). In the UK, producers of WEEE must sign-up to an approved take-back scheme, such as those offered by Valpak, REPIC and ECONO-WEEE (Environment Agency, 2011c) to ensure that their goods are recovered by an Approved Authorised Treatment Facility(AATF) (Environment Agency, 2011b). Whilst many firms appoint a specialised firm as their AATF, there are several well-known manufacturers who are themselves Approved Authorised Treatment Facilities, notably Fujitsu and Black & Decker. In addition to these European Commission directives, other measures, such as environment taxes and landfill bans on certain materials, may be extended in the UK to encourage recycling and waste-reduction (DEFRA, 2011a).

Environmental. Concern for the environment and the desire to be "environmentally friendly" have also led to an increase in product recovery. With increased publicity about issues such as global warming and pollution, consumers and manufacturers are becoming increasingly aware of the environmental impact of their actions. Consumers are being encouraged to make purchasing decisions based on environmental motivations (Recycle Now, 2011). However despite a growing trend in consumers'desire to be "green" there is still a stigma associated with "recycled goods" and view that they are inferior to new goods. The UK Government is aware of this and is striving to change people's perceptions of recycled goods by developing of End-of-Waste protocols, which will ensure that recovered goods meet a specified quality standard and will regulate how products with a "waste" component need to be identified (DEFRA, 2011a). Firms too are striving to be "green". Through corporate social responsibility, firms are developing "green" strategies for sustainable manufacturing and in some cases are using these to differentiate themselves from their competitors (e.g. Tesco, Sainsburys and Marks and Spencer). The "green" image associated with product recovery provides an incentive for firms to participate in product recovery and recycling schemes.

Economic. For some firms there is also an economic incentive to participate in product recovery and recycling. In some industries, significant cost-savings can be achieved by reusing, recycling or repairing used products or materials or by using renewable energy. However it is important that an increase in product recovery and other forms sustainable production does not harm or prevent economic growth. The need to "decouple" waste production and economic growth has been recognised as a necessary step on the path towards environmental sustainability (DEFRA, 2011a, 67). Environment Secretary Caroline Spelman, in reference to plans for the zero-waste economy, was recently reported as mentioning the importance of being able to "unlock the real value in the goods that people no longer want" (DEFRA, 2011b). The same applies to recovery. Recovery is about finding new and creative ways to salvage value contained in all parts of a used product, regardless of the condition that it is in.

Increased levels of interest in product recovery from governmental, societal and industrial spheres have also led to an increase in the product recovery literature. Ilgin and Gupta (2010) report that environmentally conscious manufacturing has seen "a surge in research activity" since 1999. One reason for this is that the management of a product recovery system is more complex than that of a regular production system, due to uncertainties relating to the timing, quantity and quality of returns (Fleischmann et al., 2002). Due to these uncertainties, production is often favoured over recovery, despite it generally being the more expensive option (Heisig and Fleischmann, 2001)

The aim of this thesis is to investigate how further waste reduction can be achieved through the use of product recovery by addressing the issue of the quality of the returns.

More specifically, we aim to investigate how the quality of returns could be taken into account in order to effectively use available resources. We compare the profitability of including a separate recovery channel for low quality returns and compare this with the alternative – the disposal of all low quality returns. In order to investigate this issue we begin by reviewing the relevant literature and methodology in Chapters 2 and 3 respectively, and then propose four models in Chapters 4–7. The modelling results will be discussed within each chapter and then conclusions will be drawn in Chapter 8.

The four models proposed in this thesis will address this issue of the profitability of low quality recovery, but will also fill gaps in their respective streams of literature. In Chapter 4, we begin our investigation with a *deterministic lot-sizing model* of a product recovery system. Uncertain demand, and uncertainty regarding the quality, quantity and timing of returns add challenges to the operation of a product recovery system, therefore we begin by assuming that all these values are known and constant. The model allows the quality of each return to determine which type of recovery is performed. The ability of the producer to only take-back high quality items is also incorporated. In Chapter 5, the second model extends the first by introducing uncertainty in the demand, returns and quality of the returns. A *discrete time periodic review model* is used, rather than a continuous time model, because it allows us to investigate the structure of the optimal policy. In both of these models it is assumed that recovered goods are "as good as new". In some cases this is a justified assumption, however in some industries even goods which may be functionally "as good as new" are not perceived by the consumer to be the same as a new item. This limitation is addressed in the third and fourth models by the introduction of separate markets for new and recovered goods.

In Chapter 6, the third model is presented. It is a *discrete time periodic review model* of a product recovery system with separate markets for newly produced and recovered goods. It is assumed that the produced and recovered goods have a similar functionality, therefore if there is insufficient inventory to meet demand for one of the goods, the firm may offer a substitute. The substitution is offered on a two-way basis and it is assumed that a consumer may accept or reject the substitution. In this discrete time model, demand, returns and substitution occur in batches each period. The "batched" framework may be appropriate for modelling demand and returns, however it does not allow substitution to be modelled in much detail. Therefore, the fourth and final model of this thesis examines a *continuous time continuous review model.* As in the discrete time model, this model has separate markets for new and recovered goods, however the continuous nature of the model means that leadtimes are also introduced.

In this thesis we find that performing both high and low quality recovery: reduces the sensitivity of the optimal cost to operational restrictions on the choice of decision variables; results in cost savings; and allows better customer service through increased fill rates. We also find that offering substitution between the markets for newly produced and recovered goods allows firms to increase both their profits and fill rates. The findings of this thesis could be used to help a firm assess the cost-effectiveness of recovering lower quality returns, in addition to high quality returns. Recovering low-quality items rather than disposing of them, may allow a firm to increase the amount it recycles. These findings highlight the importance of considering the quality of returns when managing a product recovery system as they show that economic gains can be achieved by reusing rather than refusing low quality returns.

Chapter 2

Literature Review

2.1 Introduction

Product recovery is performed in a large number of industries, from military to telecommunications and is becoming increasingly important due to economic, legislative and environmental pressures. From an operational perspective, product recovery systems pose additional operational challenges, compared with production or manufacturing systems, due to the increased uncertainty caused by the quality, quantity and timing of returns (Ilgin and Gupta, 2010). The quality of returns is widely acknowledged as being important, however comparatively little research has investigated this issue. In this chapter a review of the relevant product-recovery literature is presented, with a particular focus on the modelling of the quality of returns.

The quality of the returns is a significant issue in the operation of product recovery systems as it can affect the type of recovery which can be performed and the costs associated with that recovery (Fleischmann et al., 1997). Quality has been considered in a number of different ways. However, there are a number of limitations within the existing literature. These limitations will be discussed where relevant throughout this chapter and will be then summarised at the end of this chapter.

Before examining product recovery in more detail, it is important to understand where it fits within both an industrial and an academic context. This is important for 'putting product recovery on the map', so to speak. This background information will be discussed in Section 2.2. The remainder of this chapter will more or less follow the structure of this thesis. The first two models presented in this thesis assume that used goods are returned to their producer and are recovered to be "as good as new" and thus can be sold on the same market as newly produced goods. Therefore we begin, in Section 2.3, by reviewing literature relating to models which are produced and recovered by the same firm and are then sold on the same market. The "as good as new" assumption is relaxed in the third and fourth models presented in this thesis by allowing recovered and newly produced goods to be sold on separate markets, but also to be substitutes for each other. The literature relating to this type of system is discussed in Section 2.4. Some goods are recovered by specialised recovery firms, i.e., firms which recover used goods but do not produce new goods. Whilst none of the models in this thesis make this assumption, some of the literature in this field is relevant because of the way in which it addresses the issue of the quality of returns. Specialised recovery firms will be discussed in Section 2.4.3. The gaps in the literature that will be addressed by this thesis are summarised in Section 2.5.

2.2 Putting Product Recovery on the Map

2.2.1 Product Recovery in Industry

Product recovery describes the process by which used products are returned to their producer or are sent to a specialised facility, in order to undergo recovery. Organisations which perform production and recovery are sometimes referred to as *original equipment manufacturers (OEM)* or *hybrid manufacturing / remanufacturing systems* (Aras et al., 2004). Thierry et al. (1995) describe five types of recovery: repair, refurbishing, remanufacturing, cannibalisation and recycling. According to this definition, repair, refurbishing and remanufacturing involve reusing most of the existing product, whereas cannibalisation and recycling involve reusing only selected parts or materials from the existing product. Depending on the type of recovery that is performed, the returned product may be: used as is; recovered and used along with newly produced goods to satisfy consumer demand; used to satisfy demand in a secondary market; disassembled and used for parts or materials; or some combination of the above. However, since these classifications of the types of recovery do not appear to be standardised across the literature, in this thesis we use the term *recovery* to mean any activity which involves the reuse of part or all of a used product, thus *recovery* could include repair, refurbishing, remanufacturing, recycling or cannibalisation.

Product recovery is performed in a variety of contexts and for a variety of reasons, e.g, the buy-back of used goods from the consumer (Dobos and Richter, 2004), goods returned due to a warranty claim (Khawam et al., 2007), collection from consumer (Oh and Hwang, 2006), consumer return and legislative reasons (DeCroix and Zipkin, 2005). The reason for the return can influence the type of recovery that can be performed. A large number of industries and applications have used product recovery, for example military equipment (Schrady, 1967), single-use cameras (Toktay et al., 2000), leased office equipment (Aras et al., 2006), leased products (Inderfurth, 1997), Hitachi (Khawam et al., 2007), cargo hoists (Mahadevan et al., 2003), toner cartridges and personal computers DeCroix (2006), automotive industry, telecom industry, and steep scrap in China (Aras et al., 2006), reuseable packaging, electronics, car parts (Heisig and Fleischmann, 2001). However, a comprehensive review of applications of product recovery is necessary to bring the literature up-to-date with current technologies and trends.

From an operational perspective, product recovery systems are more complex than regular production systems. One reason for this is the increased uncertainty caused by the quality, quantity and timing of returns (Ilgin and Gupta, 2010). In a product recovery system there are additional factors that need to be considered, for instance, additional inventories to manage, two options for replenishing inventory (Fleischmann et al., 1997), salvage value of returns, quality requirements and sorting of returns (Ilgin and Gupta, 2010). The quality of returns, in particular, is an issue that is discussed widely in the literature as affecting industry practice because it can affect the type of recovery which can be performed and the costs associated with that recovery (Fleischmann et al., 1997).

These issues pose challenges for the product recovery industry. These challenges, together with the legislative changes, economic climate and growth in awareness about environmental issues, have contributed to the increased attention on product recovery in society, as well as in the literature. However, it is interesting to note that despite this recent attention in the literature, product recovery and reuse is not a new field; it has been studied in the literature for over 40 years, for instance Schrady (1967). The development of product recovery in the literature is discussed further in the following section.

2.2.2 Product Recovery in the Literature

Product recovery was first discussed in the literature at least 40 years ago (Schrady, 1967). Much of the early research in this area dealt with repairable inventory systems in a military context. Between 1960s and the mid-1990s there was not a great deal of research in the field of product recovery, however since the mid-1990s there has been a significant increase in the amount of research being conducted. This increase in research has encouraged numerous authors to publish reviews addressing product recovery and related subjects, (Fleischmann et al., 1997; Guide and Van Wassenhove, 2009; Pokharel and Mutha, 2009; Rubio et al., 2008; Ilgin and Gupta, 2010). There have also been numerous special issues dedicated to product recovery and related disciplines (Ferrer and Swaminathan, 2010). A list of some recent special issues is provided by Pokharel and Mutha (2009).

In the literature, product recovery falls under a number of broader fields, including *reverse logistics*, *environmentally conscious manufacturing* and *inventory control*. Managing inventory within reverse logistics and environmentally conscious manufacturing is closely related to the management of inventory in traditional forward-flow logistics situations as well as in repair, spare parts, production planning, disassembly and rework systems. These areas can give insight into aspects of modelling product recovery systems. The focus of this thesis is modelling the inventory of product recovery systems, therefore in this review we focus on issues relevant to inventory management models rather than to production planning or assembly system models.

Reverse logistics relates to the return of used products from the consumer to producer (Fleischmann et al., 1997). It involves all aspects of the return process, including the transportation of the returns from the customer to the producer, the storage of returns, and the type of recovery performed. Fleischmann et al. (1997)

classify the field into three subfields: distribution planning, inventory control and production planning.

In a recent review of environmentally conscious manufacturing and product recovery, Ilgin and Gupta (2010) classify the issues relating to product recovery as being "product design, reverse and close-loop supply chains, remanufacturing, disassembly". Under this structure, the management of inventory in product recovery system falls under the "remanufacturing" category. Some research has focussed on designing products so that they can be easily disassembled and recovered. The length of a product's life cycle is also an important issue (Kiesmüller and van der Laan, 2001; Angelus and Porteus, 2002).

Repairable inventory systems are closely related to product recovery systems. Repairable inventory models usually involve determining the number of spare parts to hold in stock, taking into account the ones which can be repaired. The failure of an item triggers demand for a replacement item and it also triggers the return of the failed item for repair. Thus, demand and returns are directly related and occur on a one for one basis. In contrast, in product recovery models demand and returns are usually assumed to be independent, and production and recovery occur in batches or lots, and decisions are made regarding their size and frequency (Inderfurth, 1997). Early papers dealing with the stocking of repairable inventory include (Phelps, 1962; Allen and D'Esopo, 1968; Simpson, 1978), many of which were motivated by the stocking of repairable spare parts by the military (Mabini et al., 1992), became the basis of product recovery research. However, these papers have also developed into a separate stream of research relating to repairable inventory. More recent papers in the field of repairable inventory include Wong et al. (2005a,b), who study single item repairable inventory models, and Wong et al. (2006), who discuss a two location multi-item repairable inventory system.

Production inventory and rework are also related to product recovery. In production inventory systems the items needing repair are defects from the production process, rather than used goods from consumers. This distinction is important as it has implications for the arrival rate of returns (defects are likely to arrive individually, whereas returns may arrive in batches) and the nature of the final product (reworked items are likely to be indistinguishable to the consumer, whereas recovered items may not be). Some other papers which consider the remanufacturing of defective units include (Buscher and Lindner, 2007; Chiu and Chiu, 2005, 2006a,b; Chiu et al., 2006, 2007a,b; Chiu, 2008; Sarker et al., 2008; Liu et al., 2009). The deterioration of products whilst waiting to be repaired has also been studied (Inderfurth et al., 2005, 2006).

Now that product recovery has be contextualised in terms of its place in industry and in the wider literature, we will focus on the literature which specifically addresses product recovery models in either a single market, or separate markets environment.

2.2.3 Examples of Current Practice

In this section examples of current product recovery practice will be discussed. The companies discussed below are by no means an exhaustive list, but are provided to give a flavour of the types of product recovery being performed across different industries. The examples discussed in this section motivate the models proposed in Chapters 4–7, therefore where appropriate, we will refer back to this section throughout this thesis.

Cartridge Recycling

The printer and photocopier company, Canon, is involved in recycling used toner cartridges. In particular, they reuse the charging roller, sleeve and magnetic roller (Canon, 2012h) (further details regarding the design of cartridges is available on Canon's website (Canon, 2012g)) Once the cartridges have been returned to Canon, they are sorted and then undergo either cannibalization or recycling. The cartridges that are selected for cannibalization undergo a crushing and selection processes in which materials are separated into aluminium, ferrous metals, mixed plastics and residue. These materials are then used in manufacture of other products (Canon, 2012a,g). The cartridges which are selected for recycling are disassembled and cleaned, before undergoing a quality inspection. These "recycled" cartridges are sold as new. Canon includes a label on *all* toner cartridges stating that they may contain "reconditioned and remoulded parts" (Canon, 2012h).

Printer Remanufacturing

Canon is also involved in remanufacturing of black and white multifunctional printers. End-of-lease products are returned to Canon's factory in Germany and are brought up to an "as good as new" or "better than new" condition (Canon, 2012c). Returned printers are "stripped down to the frame and thoroughly cleaned before being rebuilt" (Canon, 2012d). After undergoing this process, printers are made available for sale or lease. By remanufacturing used printers, Canon is able to re-use up to 91% of the parts (by weight) which yields economic and environmental benefits (Canon, 2012d).

Canon distinguishes between remanufacturing and refurbishment. As mentioned above, remanufacturing returns products to an "as good as new" condition. Refurbishment on the other hand, is a less intensive process (Canon, 2012e) that returns products to a "suitable standard for resale/lease" (Canon, 2012b). It may include "replacing or cleaning worn parts" (Canon, 2012e). While remanufacturing is carried out at the company's plant in Germany, refurbishment is typically performed by a local branch, i.e. printers used in the UK will be refurbished in the UK and then resold/leased in the UK.

Materials Recycling

A number of organisations are also involved in materials recycling. For example, in addition to the cartridge and print product recovery processes mentioned above, Canon is also involved in materials recycling. Waste materials generated by products and processes are re-used in the manufacture of new products (Canon, 2012f). In particular, they produce a range of calculators made from materials recycled from the production of camera lenses (Canon, 2012f).

The drinks manufacturer, Diageo, is also involved in materials recycling. They prefer to melt down used glass if possible, rather than produce new glass as it is more energy efficient (Diageo, 2011a). In Africa, they have implemented a scheme in which bottles are returned and then reused, after a thorough cleaning, on average 12-15 times (Diageo, 2011b).

Wooden Pallets

Pallet companies, such as Pallet World, make and sell new pallets to industry. Many of these companies also offer a buy-back service for used pallets. Used pallets are repaired, inspected to ensure they are "as good as new", and are then sold at a discounted price (PalletWorld, 2012). Pallet World states that saving of up to 40% can be achieved by buying used or repaired pallets rather than new ones (PalletWorld, 2012).

Whisky Barrels

Scotch whisky barrels are made out of American Oak and typically begin their life as a barrel for some other type of alcohol beverage (e.g. bourbon, sherry, wine etc). After being used for these other types of alcohol, the barrels may need to undergo some repairs (to the wooden slat or metal hoops) before being filled with whisky. Some barrels may require minor repairs, whereas others may have sustained so much damage they may be disassembled and used for parts. Cooperage's such as the Speyside Cooperage, perform these repairs (SpeysideCooperage, 2012b). As long as the barrels have been used for another form of alcohol, they can be reused as a whisky barrel repeatedly for up to 50 years (SpeysideCooperage, 2012a).

Refurbished Electronic Equipment

Many leading computer, satellite navigation and mobile phone manufacturers also offer consumers the choice of buying a refurbished product rather than a new one. Refurbished products typically have the same functionality as new products, but are sold for a discounted price. In some (but not all) cases, the products may have a cosmetic defect, however customers are assured that this does not affect the functionality and operation of the product/ Substantial quality testing is performed on refurbished items to ensure this. Examples of manufacturers who sell refurbished products include Apple (2011), Dell (2012), TomTom (2011) and Fujitsu (2011). In some cases, the sale of the refurbished product is performed by retailers rather than the original manufacturers, e.g. PC World stocks a range of new and refurbished computers. The nature of online shopping means that retailers could easily sell both new and recovered products, and, while they may be targeting different markets with each of these products, consumers can often see both, side-by-side on their computer screens.

Some companies, such as CCL North, provide specialised recycling and refurbishment services for electronic equipment such as computers, televisions and LCD monitors (CCLNorth, 2011b). Their computer recycling operations involve an initial visual inspection which, in addition to locating identifiable markers (e.g. asset tags), is also used to determine if the product could be resold. For example, computers which the beige casing popular in the 1990s have limited resale potential, compared with more modern casings. Even if a computer is not resaleable, it can be separated into different types of materials (e.g. plastics, ferrous metals) for further recycling (CCLNorth, 2011a).

Substitutable Products

In some industries, if a customer has requested a product that is no longer available, then they may be offered a substitute. For example, if you order groceries online and your chosen product is unavailable, then you may be offered a substitute (Asda, 2012). Policies regarding whether you pay for what you order or what you receive differ between different companies, for example, Asda states that "you'll always pay for the lowest priced product". Similarly, if you hire a rental car and your chosen model is unavailable, you may be offered an upgrade.

2.3 Single Market for Produced and Recovery Goods

In this section the literature relating to product recovery systems which produce (or procure) new goods and recover used goods and then sell them on the same market. It is assumed that recovered goods are "as good as new" goods. Figure 2.1 presents an example of a product recovery model in which items are supplied to the consumer from a store of serviceable inventory. After the consumer has finished with the items, some are returned and then stored while they await recovery. Recovered items are considered to be "as good as new" so are used to replenish serviceable inventory. Production is performed to ensure that there is sufficient serviceable inventory to meet demand. The majority of research in the field of product recovery studies this type of system.

The literature in this section is separated into three main sections based on the modelling assumptions applied to demand and returns. Section 2.3.1 discusses models which assume that demand and returns are deterministic. Models assuming stochastic demand and returns are then discussed, with discrete-time, periodic-review models being discussed in Section 2.3.2 and continuous-time, continuous-review models being discussed in Section 2.3.3.

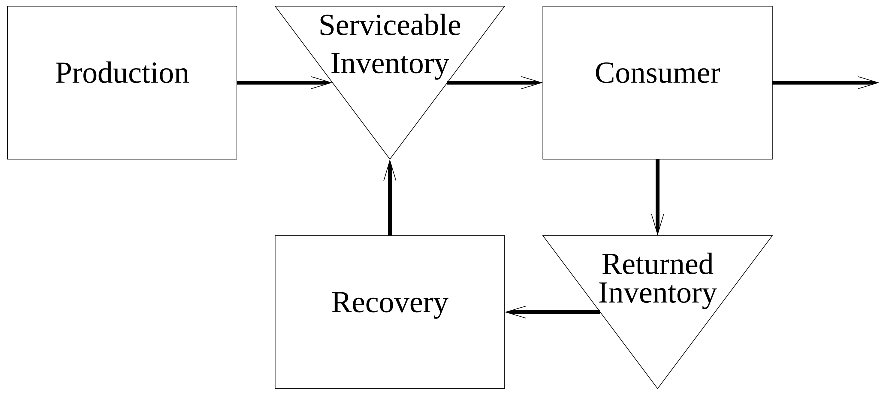


Figure 2.1: Structure of a simple product recovery model

2.3.1 Deterministic Models

Deterministic product recovery models can be broadly classified as being either static or dynamic. Dynamic models allow the problem parameters, such as the demand and return rate, to vary over time. Static models, on the other hand, do not include variation over time. In this review the focus is on static models. The motivation for this is that in this thesis, we want to begin by studying a very simple product recovery system in which demand returns are known and constant. To do this we will use a static, deterministic model. The second and subsequent models in this thesis will relax this assumption and allow for uncertain demand. Because of this we do not study, and thus do not review, deterministic dynamic product recovery models.

Static deterministic product recovery models typically have an economic order quantity (EOQ) structure and are concerned with determining the optimal lot size and the optimal number of lots per cycle. The main issues that are discussed in the literature are how to obtain values of the decision variables and whether or not to include a disposal option for returns. With regards to quality, as will be highlighted in this section, most research assumes that either all returns are recoverable or that low quality returns are disposed.

Most static deterministic product recovery literature stems from Schrady (1967), who uses an EOQ model to study a repairable inventory system, in which items are procured from an external source or are repaired. The procurement and repair rates are assumed to be infinite, which though may not be realistic, simplifies the derivation of the closed form expressions for optimal lot sizes of production and repair. This assumption was partially relaxed by Nahmias and Rivera (1979) who extend the model to incorporate a finite repair rate, with an infinite procurement rate. Both of these papers consider a single-item system with one production lot and multiple repair lots per cycle. This was extended by Mabini et al. (1992) to incorporate a multi-item system in which the recovery facility is shared by the items, however complexity of the resulting model meant that numerical methods were used. Additional details of the early research in this area can be found in a review by Fleischmann et al. (1997). In what follows we focus on more recent research, falling into two categories: recovery models and waste disposal models. Recovery models follow the typical product recovery structure discussed above, whereas waste disposal models include the waste disposal rate of returns as a decision variable. After these two streams have been discussed a brief discussion of other related literature is presented.

Recovery Models

Teunter (2001) extends the early work mentioned above by providing simple formulae for the cost-minimising lot sizes and number of lots per cycle for two policies: one production lot and multiple recovery lots, or one recovery lot and multiple production lots. Restricting the number of lots to one, for either production or recovery, greatly simplifies the derivation of the optimal lot sizes. However the motivation for doing this is not solely simplification, as Teunter (2001) also gives a proof which shows that the optimal number of lots for production and recovery can never both be even. It was also assumed that the both the lot sizes and the number of lots per cycle could take continuous, rather than discrete values. This assumption is addressed by Teunter (2004), who also extends his own previous paper by allowing both the production and recovery rates to be finite or infinite (rather than just the recovery rate). Cost-minimising lot sizes are given and then an approximation method is used at the final stage to obtain integer values for the optimal number of lots. However, this approximation method proposed is just that, an approximation. An exact approach for obtaining an integer number of production and recovery lots, using upper and lower bounds, was proposed by Konstantaras and Papachristos (2008b). In all of these papers, the quality of returns is either not mentioned, or it is assumed either that all goods returned are of sufficient quality to be recovered, or that any returns of 'low' quality are discarded. In Chapter 4, we propose a model that addresses this limitation and extends Teunter (2004); we apply the approach of Konstantaras and Papachristos (2008b) to our model.

A number of variations of this problem have been considered. For example Konstantaras and Papachristos (2006) allow backorders and Konstantaras and Skouri (2010) allow the number of lots per cycle to be variable and allow shortages of serviceable inventory to occur. The sequencing of the production and recovery lots within a cycle is studied by Choi et al. (2007) and Feng and Viswanathan (2011), of whom the latter also allows multiple production lots and multiple recovery lots. In Chapter 4, we also allow multiple production lots and multiple recovery lots.

Koh et al. (2002) study the relationship between the demand, production and recovery rates and propose a search procedure to obtain integer values for the optimal number of lots per cycle. They also include an additional decision variable which specifies the inventory level at which recovery should begin. Konstantaras and Papachristos (2008a) propose an alternative methodology for determining the optimal values of the decision variables by using upper and lower bounds (following an approach similar to Konstantaras and Papachristos (2008b)). Recently, Konstantaras et al. (2010) extended Koh et al. (2002) by introducing a secondary market to deal with items which are not of high enough quality level to be sold alongside newly produced items. These papers also assumed that all goods returned have sufficient quality to be recovered.

Waste Disposal Models

A second stream of literature that developed from Schrady (1967) focuses on the waste-disposal aspect of a product recovery system. This stream differs from Schrady (1967) in the way that returns are managed. Rather than assuming that all returns are recovered, the numbers of goods which are returned and recovered is managed either by the waste disposal rate or the buy-back rate. Much of the research in this stream has been carried out by either Richter, or both Dobos and Richter. The initial papers in this stream assume that returns arrive in batches, rather than at a constant rate (Richter, 1996a,b). The integer requirement for the number of lots per cycle is discussed briefly in these papers but no solution methodology is proposed. The integer requirement is addressed in more detail by Richter and Dobos (1999) and Dobos and Richter (2000). Jaber and El Saadany (2009) extend Richter (1996a,b) by assuming that recovered goods are not as good as new. This paper will be discussed further in Section 2.4 when models with separate markets for produced and recovered goods are reviewed.

Dobos and Richter (2004) consider a related model in which the number of production and recovery lots are both greater than one. The decision variables are the 'use' rate and the buyback rate of returns, as well as the number of production and recovery lots, the size of the production and recovery lots, and the length of the production and recovery intervals. This model is extended by El Saadany and Jaber (2010) who use the price paid for returns to control the quality and quantity of returns. As in the recovery models stream, the number of lots is also discussed: they consider policies with either one production and recovery lot, or multiple production and recovery lots.

Dobos and Richter (2006) consider a similar inventory model, which addresses the quality of the used items either by only buying back items of a satisfactory quality, or by buying back all items which are available and then performing a quality check. This essentially examines the decision about out-sourcing quality control. El Saadany and Jaber (2008) extend this stream of research by using switching costs instead of setup costs. While there are some papers in this stream which address the issue of the quality of returns, it is still assumed that any goods which are of insufficient quality are discarded. The possibility of salvaging value from these 'low quality' returns is not discussed.

Other Static Deterministic Models

Most research discussed thus far assumes that recovery is used to make a returned item 'as good as new' and thus to replenish the serviceable inventory, however other assumptions regarding the nature of the recovery process have also been studied. For example, Oh and Hwang (2006) study a product recovery model in which recovery is used to salvage raw materials or components from returned items (which are then used in the production process) rather than to replenish the serviceable inventory directly. A stochastic version of this model is studied by Mukhopadhyay and Ma (2009). Following an EOQ structure, Oh and Hwang (2006) present the optimal lot sizes for production and for orders of raw materials. The model presented in Chapter 4 will combine this type of 'components' recovery, with the 'as-good-as-new' recovery to provide two channels for recovery.

Disassembly and the sorting of returns have also been investigated. El Saadany and Jaber (2010) study a system in which returns are disassembled into "subassemblies" and then recovered to create items which are "as good as new". The decision variables are the ordering and recovery decisions for each subassembly.

Some product recovery models incorporate multiple items, for example Mandal and Roy (2006); Tang and Teunter (2006); Çorbacioğlu and van der Laan (2007); Teunter et al. (2008). Teunter et al. (2009) develop heuristics for a multi-item system which performs production and recovery by extending Tang and Teunter (2006). Çorbacioğlu and van der Laan (2007) investigate methods for setting the holding cost in a two item system in which returns can be used in two different recovery processes, one for each type of item. The type of recovery performed depends on quality of returns. However, their model assumes that the inventory of both types of goods can be replenished by production, as well as by recovery. Whilst the differentiation between the two products may be based on quality, it appears that this distinction is only used to differentiate between the two product types, and not between the type of recovery that can be performed.

Summary

Much of the deterministic product recovery literature assumes that all returns have sufficient quality for recovery or use a disposal option to eliminate low quality returns.

However some papers deal with the quality of returns more explicitly. El Saadany and Jaber (2010) use the price paid for the returns to control the quality of the returned goods. Dobos and Richter (2006) incorporate the quality of returns into their model by allowing the firm to choose either to buy back returns regardless of quality and then dispose of low quality items, or to only buy back high quality returns. These papers do not consider the possibility that the low-quality items may have some salvageable value. We address this limitation in this thesis. We also investigate whether or not performing low quality recovery (in addition to high quality recovery) is profitable.

2.3.2 Stochastic Discrete Time Models

Periodic review, discrete time, stochastic product recovery models allow demand and returns to be observed and decisions to be made on a periodic basis. Typically periodic models are studied over a discrete time horizon. The nature of the periodic review models mean that finding the structure of the optimal policy is sometimes possible, therefore much of the research has focussed on doing this (DeCroix, 2006; van der Laan and Teunter, 2006). However, a substantial body of literature has also investigated finding the optimal parameter values for various policy structures. Continuous time models, on the other hand, typically assume that the policy structure is fixed, and therefore focus on obtaining the policy parameter values (DeCroix, 2006). One reason for this is that it is easier to obtain an optimal policy structure for periodic review problems, than for continuous review (Inderfurth, 1997).

The literature in this section will be structured around these two main themes: finding the optimal policy structure, and finding the optimal parameter values. A three parameter policy, which specifies when to produce, recover and dispose of returns, has been discussed in the literature since at least Phelps (1962). However in today's environmentally conscious society, legislation may mean that disposal is not an option, i.e. firms must recover, in some form, everything that is returned to them. Much of the early research in this field had developed from an early paper by Simpson (1978), and a more recent paper by Inderfurth (1997).

Optimal Policy Structure

Proving the optimality of a particular policy structure has been the focus of one stream of research in this area. Early papers which examined the structure of the optimal policy did so in the context of managing inventories of repairable spare parts. For example, Phelps (1962) applies a three parameter (purchase, repair, dispose) policy in which unmet demand is lost, which Veinott (1966) extends by removing the disposal option and allowing unmet demand to be back-ordered, rather than lost (Simpson, 1978). A related problem was considered by Allen and D'Esopo (1968), who model the replenishment of serviceable inventory due to the repair of returns as a random variable (Simpson, 1978).

The first key paper in this stream is by Simpson (1978), who studied a product recovery model in which the stochastic demand is met by serviceable inventory. The serviceable inventory is replenished by the procurement of new goods and the repair of returned goods. Each period a decision is made about how many items to procure, how many items to repair and how many returned items to discard. There is no lead time for recovery or procurement and it is assumed that all returned inventory are able to be repaired. Unit costs are incurred for procurement and for repair, however there is no cost for disposing of excess returns and no fixed costs. While this assumption may have been realistic in 1978, with today's environmental legislation is unlikely that firms would be able to dispose of returns (or anything else for that matter) without incurring a cost. The key result from this paper is the proof of the optimality of a three parameter (recover-up-to, procure-up-to, dispose-down-to) policy for this finite horizon, periodic review problem.

Inderfurth (1997) extended the work of Simpson (1978) by including fixed, identical lead times for procurement and recovery. Variable costs, including a disposal cost are included, however fixed setup costs are not. This is one of the first periodic-review papers which positions itself as a product recovery or remanufacturing model, rather than as a repairable or spare parts inventory model. According to Inderfurth (1997), no research directly extended the work of Simpson (1978), until his own paper. He also states that, with the exception of one of his own working papers, the inclusion of lead times into a periodic review, product recovery model had not been previously considered in the literature. Inderfurth (1997) uses a dynamic programming formulation to prove the optimality of a three parameter (recover-up-to, procure-up-to, disposal-down-to) policy for a finite horizon, periodic review, stochastic product recovery system with identical lead times for procurement and recovery. This policy takes into account the combined level of serviceable and returned inventory. Demand and returns are modelled by continuous random variables and it is assumed that all returns are of sufficient quality to be recovered. Inderfurth (1997) mentions that, as for cash-balancing models, if setup costs are included in the model, a simple optimal policy will not exist. Three different lead time scenarios are considered for recovery and procurement: equal, recovery greater than procurement, and procurement greater than recovery. In all cases the lead times are fixed and known. Inderfurth (1997) finds that, as long as the difference between the lead times is no greater than one period, the structure of the optimal policy can be shown.

The relationship between the lead time for production and recovery is investigated further by Inderfurth and van der Laan (2001), who study the three lead time cases mentioned above using both periodic review and continuous review models. The periodic review model is used to investigate the structure of the optimal policy and they found that the structure tends to be complex if the production and recovery lead times differ.

A number of papers have studied product recovery systems by treating returns as "negative demands" and thus decomposing the inventory position so that the system is transformed into a model without returns. A traditional (s, S) (if inventory is less than s, order-up-to S) policy can then be applied to the system. Fleischmann and Kuik (2003) study such a system and prove the average cost optimality of a (s, S) policy. Fleischmann et al. (2003) adapt the findings of Fleischmann and Kuik (2003) and apply them to IBM's spare-parts system. Whilst their model does take into account alternative channels for recovery (a feature that we also include in all four of our models), the complexity of this real-life system means that analytical results are not available and thus that simulation must be used. The nature of the model may also be specific to the spare-parts system at IBM, and thus may not be applicable to other organisations.

Nakashima et al. (2002) study a product recovery model using a discrete time

Markov Chain. In their model there are two types of inventory: inventory in stock with the company and inventory with the consumer. The number of used goods which are returned to the firm each period depends on the number of goods in the "consumer inventory" and a known return rate. Each period used goods are also "disposed" directly from the consumers' inventory. The return and disposal rates which minimise the expected average cost per period are calculated exactly. This paper was extended by Nakashima et al. (2004) who consider a Markov decision process model. Like the earlier paper, the system is modelled in terms of two inventories: one for goods in-stock and one for goods with the consumer. The state space of the problem is the two inventories and the action chosen at each period is the number items to be produced. The optimal production policy for each state in the state space is presented. Neither of these papers consider the quality of the returns.

The inclusion of a disposal option in the policy has been an issue of contention in the literature. In many product recovery systems unwanted returns can be disposed of at each decision epoch (Inderfurth, 1997; Simpson, 1978), however some research has questioned the necessity of including this option. In considering this issue across a wide variety of scenarios, Teunter and Vlachos (2002) found that disposing of returns is only beneficial under certain conditions, namely, if there is a low demand rate, a high return rate, and the profitability of recovery is low. This result is one of the reasons why we do not explicitly include a disposal option in the models proposed in this thesis.

Policy Parameter Values

A related stream of research has investigated methods for finding parameter values in specified policies; in some cases optimal parameter for a given policy are sought, and in other cases heuristic methods are used.

Optimal parameter values for specific policies have been studied by Kiesmüller and van der Laan (2001); Kiesmüller and Scherer (2003); Wang et al. (2011). Kiesmüller and van der Laan (2001) find optimal parameter values for a two-parameter order-up-to level policy, which specifies the amount of inventory in stock at the start of the planning horizon, and the order-up-to level for orders placed at the start of each period. This paper differs from many other papers in this field of research as it assumes that there is a dependence between demand and returns. An uncertain proportion of goods are returned to the system, and of these returns, a stochastic proportion have sufficient quality to be recovered. The remainder are disposed. Kiesmüller and Scherer (2003) compute the optimal parameter values for the three parameter policy studied by Simpson (1978); Inderfurth (1997). They also present two approximation methods for computing the parameter values, which provide solutions more quickly. Wang et al. (2011) derive optimal parameter values for a two-parameter policy, which specifies the order size for production and the proportion of returns which should be recovered. In their model returns which are not recovered are disposed.

Another stream of research has focussed on using heuristic methods to find "good" parameter values for specified policies. This is particularly the case for more complicated models for which simple policies do not exist. Even the inclusion of fixed costs into an otherwise simple product recovery model can make it unlikely that the optimal policy will have a simple structure (Inderfurth, 1997). In Chapter 5 we investigate the structure of the optimal policy, but do not search for a simply-structured optimal policy. Instead, we use the insights gained about the optimal structure to develop simple characterisations and test their performance compared with the optimal policy.

Kiesmüller (2003) investigates a heuristic two-parameter (produce-up-to, recover-up-to) policy, in which two different definitions of the inventory positions are used in determining each of the actions in each period. Mahadevan et al. (2003) propose "push" policies for a periodic review product recovery system, which specify the review frequency and the order-up-to level. Using simulation they find the optimal order up to level for a fixed review period. Heuristics are developed to find the order-up-to level, and upper and lower bounds are calculated. The performance of the heuristics is tested using the results from the simulation. All returns are recoverable (hence no disposal). Khawam et al. (2007) analyses a periodic review stochastic warranty inventory system in which products are returned because of a warranty claim. Heuristics are proposed to find 'near optimal' policies. The quality is considered, in that not all returns can be recovered, however no alternative recovery option is available for low quality items. Ahiska and King (2010) use Markov decision processes and presents a simple characterisation of the optimal policy for the models by Inderfurth (1997) and Simpson

(1978). Their model includes fixed costs (unlike Inderfurth (1997)) and non-equal lead times. The policies they present are not optimal, but are simple and thus could be easily implemented. This paper highlights an important issue – even if an optimal policy can be found, it may be too complicated to implement in practice. This further motivates our search for "good", but simple heuristic policies in Chapter 5.

Other Stochastic Discrete Time Models

Another stream of the literature has examined multi-echelon inventory systems with returns. Much of the work in this area stems from Clark and Scarf (1960), who consider multi-echelon inventory systems without returns. The models in this stream typically to follow an "assembly system" structure, with a number of stages, in which returns may arrive at any stage. Some key papers in this stream include DeCroix and Zipkin (2005), DeCroix et al. (2005) and DeCroix (2006). Further details are not included here as, whilst this stream is related to the broader area of product recovery, they are not related to the models in this thesis.

Francas and Minner (2009) consider the network configuration of a one-period, multi-item, stochastic product recovery system. They investigate the efficiency of conducting production and recovery in common plants, compared with in separate plants. They also consider two cases: both products are sold on the same market, and products are sold on separate markets. They found the factors which contribute to the optimal network configuration include the network size, the costs, and the market structure. Throughout this thesis it is assumed that production and recovery require the use of some shared facility, i.e. a common plant.

Several approaches have been used to deal with the uncertainty of the returns. de Brito and van der Laan (2009) present four methods that could be used for forecasting the returns in future periods and find that the methods requiring more information do not necessarily lead to lower costs. Robust optimisation is used by Wei et al. (2010) to deal with the uncertainty of returns.

Summary

Much of the stochastic periodic-review literature has focussed on either finding the policy structure or the parameter values for a given policy structure. Some literature has focused on proving the optimality of the structure and/or parameter values, but the complexity of the models means that this is not always possible and thus heuristic methods must be used. In general, very little attention is paid to the quality of the returns. As was the case for the deterministic literature, it is assumed either that all returns can be recovered, or that "low quality" returns are discarded. This is somewhat surprising since it should be easier to model uncertain quality in stochastic models, compared with deterministic ones.

2.3.3 Stochastic Continuous Time Models

Another stream of research has examined continuous-time stochastic product recovery models, in which demand and returns are observed and decisions are made at any point across a continuous time horizon. In general, continuous time models assume a particular policy structure and focus on obtaining a cost function and the policy parameters. Finding the optimal policy structure is usually not possible for continuous time models. Much of the research in this area stems from the early papers by Heyman (1977) and Muckstadt and Isaac (1981). The key issues in this field of literature are the inclusion of an option for the disposal of returns, the definition of the inventory position, and the inclusion of leadtimes for production and recovery. Approaches for dealing with uncertain quality of returns and yield from recovery are also discussed by a number of authors.

In this thesis we study a continuous time model with separate markets (Chapter 7), however insights from this body of literature, particularly with respect to the quality of returns, are also useful for the single market discrete-time model (Chapter 5) and the separate markets discrete-time model (Chapter 6).

Policy Structures and Parameter Values

A large body of literature has focussed on finding parameter values for specific policy structures. Assuming a given policy structure (rather than trying to find an optimal one) allows more complicated features to be included in the models. Heyman (1977) was one of the first to consider a continuous time product recovery system with procurement and repair. Using a single server queue, Heyman (1977) finds an optimal policy which specifies the inventory levels for which disposal should be carried out, under the assumption that returns and demand are governed by Poisson processes. Returns could be either disposed of or repaired, however the number of returns accepted is limited by the capacity of the queue (Fleischmann et al., 2002). No fixed costs are incurred (Ilgin and Gupta, 2010). This model was extended by Muckstadt and Isaac (1981) by the inclusion of fixed costs and lead times, however the disposal of returns was not permitted (Ilgin and Gupta, 2010) and costs were not included for holding returned inventory (van der Laan et al., 1999a). A numerical procedure was used for determining the parameter values at each echelon under a (s, Q) policy structure (van der Laan et al., 1996b). The paper was extended by van der Laan et al. (1996a,b). The latter includes a disposal option and presents then tests two approximation methods for computing the parameter values of a (s, Q) policy. The former considers a more complex four parameter (s_p, Q_p, s_d, N) policy which specifies a procurement policy (if the inventory position is s_p or less, then an order of size Q_p is placed), and a disposal policy (if the inventory position is s_d or the number of returns in the system is N, then returns are disposed of when they arrive).

Push and pull policies for a continuous time product recovery system, with nonzero lead times are studied by van der Laan and Salomon (1997). Under a push strategy the returns are recovered as soon as possible, i.e. they are "pushed" through the system, and under a pull strategy the returns are only recovered when they are needed in order to satisfy demand. Exact expressions for the cost functions of these polices are provided and numerical experiments are used to show the advantage of incorporating "planned" disposal into the production planning system. van der Laan et al. (1999a) also examine push and pull strategies for managing production and recovery operations, however they focus on the impact of the length and variability of the lead-times of production and recovery. Their model does not include a disposal option for returns.

The managerial implications for firms which perform production and recovery and the added complexity that this presents (compared with performing just production) are discussed by van der Laan et al. (1999b). Inderfurth and van der Laan (2001) extend the work above and the discrete-time literature (discussed in Section 2.3.2) by examining the effect of the relationship between production and recovery lead times. They found that the structure of the optimal policies are complex if the production and recovery lead times differ, and in particular, that differing lead times mean that traditional definitions of the inventory position may not be appropriate.

Three policy structures and heuristics for obtaining the policy parameter values are studied by van der Laan and Teunter (2006). The first policy is a "push" policy which is defined as follows: if returns reach Q_r then recovery Q_r and if serviceable inventory falls to s_m, then produce Q_m. The second policy is a simple "pull" policy which is defined as follows: replenish if serviceable inventory falls to s by performing recovery if there are sufficient returns in stock Q_r, otherwise by production Q_m. The third is a more general "pull"policy which is defined as follows: if serviceable inventory drops to s_r and there sufficient returns in stock then recover Q_r, otherwise if serviceable inventory drops to $s_m < s_r$ produce Q_m. They use lot-sizing formulae from the equivalent deterministic models as the basis for their policy parameter estimates. We also follow this approach for obtaining parameter values for some of the heuristics presented in Chapter 5.

Zanoni et al. (2006) use simulation to compare the performance of three policy structures under different leadtime relationships. They consider some variations of the pull policy, including a dual sourcing policy which obtains part of the required replenishment batch from recovery and the remainder from production. The effect of leadtime on the performance of the system is also discussed by Zhou et al. (2006), who study a product recovery system which uses a Kanban policy.

Modelling Returns as "Negative" Demands

Another stream of literature has modelled product continuous time, stochastic product recovery systems by treating returns as "negative demands". In general these papers tend to have a simpler structure.

One such paper is by Fleischmann et al. (2002), who study a model in which returns are recovered instantaneously and in which there are independent Poisson demand and returns. The model is converted into a traditional inventory model by treating returns as ‘negative demands’ and by modelling the change in inventory with a continuous time Markov process. One simplification of this model is that it is assumed that returns undergo “instant” recovery so enter into serviceable inventory immediately after they arrive back into the system. All returns are recovered and there is no disposal. It is shown that the traditional (s, Q) policy remains optimal for the objective of minimising long-run expected average costs and a procedure for determining the optimal policy parameter values is presented. This is an important result because, though the model is quite simple, the optimal policy structure and optimal parameter values are determined.

Other Continuous-Time Stochastic Literature

Toktay et al. (2000) apply a queueing model to Kodak’s single-use camera remanufacturing system. Some parts of the queueing network are assumed to be “unobservable” in order to model the uncertainty associated with when the item is with the consumer. Toktay et al. (2000) assess the effect of the un-observability of different aspects of the queueing network, and use this to determine the “value of information”. This is an important issue and perhaps even more so now than in 2000, since stricter record-keeping requirements mean information may be available, but accessing it may incur some costs. Thus knowing the value of information is key for organisations.

A queueing model is also studied by Behret and Korugan (2009). In their model the returns are classified according to three different quality levels. The system has three separate stations for processing returns, with each station specialising in a particular classification of quality. All stations can remanufacture all quality levels, but processing time is quickest if done by a specialist station. Mitra (2009) study a stochastic two-echelon product recovery system. The production and recovery are performed by a depot which supplies the distributor, who in turn, supplies the consumer.

Many of the papers discussed above assumed that the demands and returns are governed by Poisson processes. de Brito and Dekker (2003) propose some frameworks in order to test this assumption using three different sets of data.

Souza and Ketzenberg (2002) study a ‘make-to-order’ product recovery model in which the objective is to maximize the long-run average profits, subject to service level constraints. In their model new products and recovered products undergo specific stages of production, but they also undergo a common stage of the production process. The returns that enter the recovery process do not all have sufficient quality to be recovered, thus some are disposed. Under production capacity restrictions, even if production is more profitable than recovery, it is still optimal to perform some recovery as this helps to satisfy the service level constraints. We find a similar result in this thesis – performing additional (low quality) recovery can improve the service level of a policy.

Aras et al. (2004) consider a continuous-time Markov chain model in which returns are categorised on arrival as being high or low quality. Recovery occurs in a batch of low quality items or a batch of high quality items, with the lead time of recovery dependent on the quality level of the batch. Production is also required to meet demand, however since the focus of this paper is on the recovery process and quality-categorisation, the production part of the process is not modelled. This is a signification limitation of this paper, as it is often the interaction between production and recovery that causes modelling complications. However despite this, a key result from this paper is that they demonstrate that cost savings are possible by categorising returns based on their quality. This provides further motivation for the models proposed in this thesis. The assumption that the quality can be determined on arrival may not be valid in some cases. For example, it may be the case that the quality of the return can not be determined until the recovery process has begun. We address this limitation by assuming that the quality of returns is determined during recovery.

Aras et al. (2006) also assume that the recovery lead time and cost depend on the quality of returns. There is a stocking point for raw materials, which are needed for production – the models in this thesis also follow this assumption and stock components. A three parameter pull policy which specifies the trigger level for ordering components, the trigger level for recovery and the returns' disposal level (if returns are greater than this level they are disposed). Disposal does not depend on the quality of the returns. It was found that when there is no coordination between recovery and production, it is cost effective to give priority to production if the return rate is below a certain level. However if there is coordination between the decision making, then there is not much difference between the prioritisation strategies.

The quality of returns has also been modelled by assuming that the yield of recovery is uncertain. Bayindir et al. (2006) study a product recovery model in which the success of the recovery process is uncertain and items which cannot be recovered are lost from the system. Mukhopadhyay and Ma (2009) study a model in which returns are recovered to obtain parts which then used in production. The yield rate of recovery is not perfect, so new components must also be bought.

Takahashi et al. (2007) discuss a remanufacturing system in which returns can be decomposed into parts, materials (or both), or are disposed of. Materials are used to make parts, and then parts are used to make serviceable goods. The flow of parts and materials from the decomposition process is stochastic, however no context is given for why there are different yields from different products. For instance, the quality of returns could explain the different types of recovery, however this is not mentioned in the paper.

Summary

The literature studying continuous-time models has either studied a simple model and then focused on finding the optimal policy, or studied more complicated models and used heuristic policies and simulation. The policy structures and methods for obtaining parameter values discussed above will be used to inform our development of policies in Chapter 5. More attention has been paid to the quality of the returns in this body of literature, compared with the discrete-time literature and a number of methods have been proposed for dealing with the quality of returns. Aras et al. (2004) classify returns as being high or low quality on arrival and find that doing so leads to cost-savings. The quality of returns has also been modelled by assuming that the yield of recovery is uncertain (Bayindir et al., 2006; Mukhopadhyay and Ma, 2009). Behret and Korugan (2009) classified returns according to three different quality levels. The system has three separate stations for processing returns which have different quality-dependent recovery lead-times. Takahashi et al. (2007) discuss a recovery system with multiple recovery channels, but they do not relate this to the quality of returns.

2.4 Separate Markets for Produced and Recovered Goods

The second key body of literature that is reviewed here relates to product recovery models in which produced and recovered goods are sold on separate markets. This literature relates to the models proposed in Chapters 6 and 7. Produced and recovered goods need to be sold on separate markets when recovered goods are not "as-good-as-new" ones. In some cases this assumption is valid, however this is not always the case. Particularly in the consumer goods industry, a consumer may not feel that a recovered good really is the same as a new one, even if it is functionally identical. Further, as discussed by Korugan and Gupta (2001), in some cases legislation prevents recovered goods being sold as new. In such cases, newly produced items and recovered items are sold in separate markets. The market for recovered items may be in developing countries or it may be a secondary market. Research in this area has considered factors such as network design, market structure, and substitution and pricing policies.

The management of inventory in production systems which sell to multiple markets has been widely discussed in the multi-item inventory literature, e.g. Federgruen et al. (1984). However in this review we focus on models which study separate markets within a product recovery context, i.e., in which used goods are recovered and then sold. We also discuss models of specialised recovery facilities, as in general, these papers assume that recovered goods are not as good as new. In cases where both goods are sold by the same firm, customers may accept the other type of good as a substitute, in the event of a stock out. In this section we first give a brief introduction to product substitution and then review the literature relating to product recovery models in which newly produced and recovered goods are sold on separate markets, before discussing specialised recovery models.

2.4.1 Substitution in Inventory Models

Substitution between goods has been studied in the context of product recovery, but also in the context of "regular" inventory models. For example, substitution between goods has been discussed in with respect to perishable goods (Deniz et al., 2004) and superior vs inferior semi-conductors (Hsu and Bassok, 1999). Substitution can be used, in the event of a stock out, in order to avoid shortage costs.

A key distinction with regards to substitution is consumer-driven versus firm-driven substitution (Hopp and Xu, 2008). *Consumer-driven substitution* is usually studied in the context of product assortment problems, in which the retailer chooses to stock at range of products in the hope that if the consumer is faced with a stock-out of their first-choice product, they may choose to buy an alternative product instead (Smith and Agrawal, 2000). The substitution decision in this case is made by the consumer, thus they must pay the full-cost of the product they purchase, even if it is more expensive than their first-choice product. Under *firm-driven substitution* the decision to offer substitution is made by the firm, therefore if there is a difference in the price of the two goods, the firm will cover the costs associated with this, e.g. by offering a discount and allowing customer to pay the price of the cheaper good. The consumer may still decide to accept or reject the substitution, however in initial decision is made by the firm. This type of substitution can be offered by the firm as a way of avoiding a lost sale. In this thesis we study firm-drive substitution.

Furthermore, there are two types of substitution that we distinguish between: downward substitution and upward substitution. Following Hsu and Bassok (1999), we define *downward substitution* as when a superior product is used to satisfy demand for an inferior product, and *upward substitution* as when an inferior products is used to satisfy demand for a superior product. This terminology is also used by Inderfurth (2004) and Robotis et al. (2005). In the context of the product recovery models, downward substitution occurs when there is a shortage of recovered products and upward substitution occurs when there is a shortage of produced goods. If both upward and downward substitution are permitted then this is sometimes referred to as two-way substitution.

2.4.2 Production and Recovery

Produced and recovered goods are sold on separate markets when recovered goods are considered to be "as-good-as-new". Many of the papers considering product recovery and separate markets study the problem over a single period. The key paper studying product recovery models with separate markets is by Inderfurth (2004). Many of these papers assume that all substitutions (if offered) are accepted. Bayindir et al. (2007) and

Kaya (2010) consider a partial acceptance, but these acceptance rates are deterministic and known.

Korugan and Gupta (2001) study a continuous-time product recovery system in which demand can be met by newly produced goods or by recovered goods. Either type of good can be used to meet the demand, however recovered goods are viewed as inferior so are sold for a lower price. While this paper does address the issue of separate prices for newly produced and recovered goods, it assumes that the customers are happy to receive either item, so strictly speaking, does not consider separate markets. They found that the decision about whether to use a new or recovered good to satisfy demand is governed by a switching function. They use Markov decision processes to obtain the optimal function and compare this to a number of heuristic functions.

Inderfurth (2004) studies a single period system in which produced and recovered items are sold on separate markets. Downward substitution may be offered in the event of a stock-out of recovered goods. Doing so prevents a shortage cost, however will result in a lower profit contribution as the produced good would be sold for the price of a recovered one. This paper has non-zero lead times for manufacturing and recovery, but no set up costs. They found that offering substitution leads to lower levels of recovered inventory. This is a key paper in the area of product recovery models with separate markets and substitution, and much of the recent research extends it. However, the quality of the returns and the possibility of upward substitution are not considered in this paper.

Li et al. (2006) study a dynamic deterministic multi-item product recovery model in which the firm must decide how much of each product to produce and recover each period, in order to minimise the total cost. The items stocked vary in quality. If there is insufficient stock to meet demand a downward substitution may be offered; substitution is always accepted by the customer. Li et al. (2007) study a similar problem, but over a two-period time horizon and allow emergency procurement orders.

Bayindir et al. (2007) study a single-period stochastic production system with production and recovery, in which downward substitution is offered. A model allowing two-way substitution is also presented, however they mostly address the one-way substitution case. The proportion of customers who accept a substitution is known and constant. A finite production capacity is included in the model; production and recovery 'compete' for this finite resource. The profitability of recovery is examined.

Jaber and El Saadany (2009) study a separate-markets deterministic model. Production and recovery can not occur at the same time and new and recovered goods can not both be held in stock at the same time. This means that during production, there are no recovered goods in stock, and thus all recovered sales are lost. Similarly, sales of produced goods are lost during periods of recovery. This paper also considers the case in which some unmet demand can be satisfied by substitution and briefly examines the relationship between acceptance of substitution and the level of compensation paid to the consumer.

Piñeyro and Viera (2010) study an extension of the economic lot sizing problem, which includes recovery and one-way substitution. The firm must decide how much to produce, recover and dispose of each period, in order to minimize costs, subject to demand constraints. They demonstrate that this deterministic dynamic problem is NP-hard and propose a tabu-search heuristic for obtaining a solution. There are separate markets for new and recovered goods, however in the event of a stock-out of recovered goods, a new good can be offered as a substitute. All customers who are offered a substitution will accept it.

Kaya (2010) studies a system produces goods from new materials, as well as recovers returned goods. In this model, an incentive is paid to the consumer in exchange for the returned goods with the level of the incentive affecting the quality of the returns. Three models are proposed: recovery only, single market for produced and recovered goods, and separate markets for produced and recovered goods. In this final model, there is some two-way substitution between produced and recovered goods if a stock out occurs. A known proportion of customers offered a substitution will accept it. This is one of the few product recovery models which considers two-way substitution.

Aras et al. (2011) study a profit maximising, finite horizon, periodic review system in which new products are leased, and then returned to the system, recovered and then sold. Recovered products cannot be leased and cannot be recovered more than once. There is no substitution between goods in the event of a stock out.

Francas and Minner (2009) study the network configuration of a multi-item stochastic product recovery system over a single time-period. They consider two market structure cases: when recovered goods can be sold as new (on the same market), and when they can be sold on a secondary market. When the goods are sold on the same market, new and recovered goods are considered to be perfect substitutes for others. When they are sold on separate markets there is no substitution between the two markets. Decisions are made about the capacity to be installed at each plant and the production decisions.

Debo et al. (2005) study the market segmentation for a product recovery firm which sells new and recovered goods on separate markets. The degree of recovery performed is a decision variable which determines the price for which the recovered good can be sold. The customers all have a different willingness to pay, thus different pricing strategies may influence the number of consumers wanting to buy each of the products. Both monopolistic and competitive market structures are considered.

Heese et al. (2005) use a duopoly game-theory model to analyse a manufacturing firm which is considering taking back products after use and recovering them. New and recovered goods are sold to separate markets. They analyse the conditions under which it is optimal for the producer to offer recovery and what effect this will have on the customers. In a related paper, Ferrer and Swaminathan (2010) use a game-theory approach to investigate pricing and production decisions in a monopolistic product recovery system. In the first period, only production occurs but in the second and subsequent periods both production and recovery may occur. The number of goods produced will affect the number of returns available for recovery in subsequent periods, thus the firm has to determine the prices for new and recovered goods, and also decide how many items to recover.

Summary

A range of issues have been studied in the context of separate markets for produced and recovered items. However across this body of research, the majority consider only downward substitution. This is a limitation of the existing literature as there may be some cases where a consumer would rather have a recovered good at a cheaper price, than nothing at all. Furthermore, the quality of returns is not addressed in this body of literature.

2.4.3 Specialised Recovery

Another stream of the literature has focused on specialised recovery firms, that is, firms which only recover used goods, rather than produce new items as well. The increase in product recovery in recent years has created demand for specialised recovery (or remanufacturing) firms, which has in turn, motivated research into such models. In general, the goods which are recovered by these firms are not assumed to be 'as good as new', thus are not sold on the same markets as newly produced ones. There is quite a large of body of literature which studies these specialised models, therefore in this section we mainly focus on the papers which incorporate the quality of returns into their models.

Inderfurth et al. (2001) study a periodic review, stochastic remanufacturing system, in which excess returns can be disposed off. Returns can be used to make multiple different types of products; each product is sold on a separate market. It addresses the problem of how the returned goods should be allocated to the different reuse options. This is paper relevant to this thesis as we too consider multiple reuse options.

Souza et al. (2002) study a capacitated remanufacturing system in which returns are graded based on three different conditions, i.e. each quality is treated as a different class in the queueing network. Processing stations specialise in dealing with one of the three grades of product. They can process other grades, however this costs more and takes longer. The system is analysed using queueing theory to investigate the constraints on the service level. Guide et al. (2008) also use a queueing system study a remanufacturing system, however their model focuses on a product with a short life cycle. The operational decision is whether to recovery the used product, or whether to sell it as is for a salvage price. Recovery will allow the item to be sold at a price higher than the salvage value, however the longer the lead-time associated with performing the manufacturing, the lower the sale price for the remanufactured good. The lead-time for recovery depends on the congestion at the facility. They found that if the recovery system is congested and the price decrease associated with the recovered good is steep, then profit can be increased by selling returns as-is.

Robotis et al. (2005) consider a dedicated remanufacturing system which receives returns from two suppliers, one of which provides high quality returns, the other of which provides low quality returns. Within each of these classes of returns, the returns vary in quality. Some are of sufficiently high quality to be sold on to their respective markets as is. Other returns require remanufacturing before they can be sold, otherwise they will be disposed of. The firm must decide whether to remanufacture or to dispose of returns that do not meet the required quality standard for each of the two classes of products. Downward substitution is available in the case that there is a stock out of the low quality goods. The problem is studied over a single period time horizon. They found that performing remanufacturing on the returns, rather than selling them as is, can allow the firm to achieve greater profits. Further, if remanufacturing is to be used, then the firm does not need to purchase as many used goods, since after remanufacturing the goods will meet the consumers required standard of quality. We find a similar result in this thesis - being able to offer substitution influences the decisions to replenish the inventories of produced and recovered goods.

Tang et al. (2007) study a make-to-order remanufacturing system in which returns have mixed quality. Returns with sufficient quality can be used in the assembly (remanufacturing) process and be combined with newly purchased components in order to meet a demand. Low quality components are disposed of, and an order is placed for a comparable component to be used in the assembly process. It was found that a newsboy problem could be used to obtain a solution.

Tagaras and Zikopoulos (2008) study an infinite horizon, multi-location remanufacturing problem in which there is a central remanufacturing facility and multiple collection sites. Three control policies are considered: no sorting of returns, sorting centrally, sorting at the collection sites. However, the sorting process can be inaccurate. The paper investigates the conditions under which it is optimal to perform each of the three sorting strategies and derives expressions specifying under what conditions each should be performed. Zikopoulos and Tagaras (2008) study a similar problem, in which sorting procedures are carried out at a single collection centre, and provide analytical expressions for the value of sorting. A single-period variation of this problem was considered by Zikopoulos and Tagaras (2007). The decision variables are the amount to collect from each site and the amount to remanufacture, in order to maximize profit. Gou et al. (2008) also study a product recovery model, in which returns are collected at various locations before being returned to a central recovery facility. This paper provides optimal batch sizes for shipments from the collection points to the recovery facility, and for the recovery of returns.

The issue of sorting, or grading returns based on quality was also considered by Ferguson et al. (2009). They consider the quality of returns in a firm which leases new goods and sells remanufactured goods to separate markets. In their finite horizon, periodic model they focus on the remanufacturing aspect of the operations and do not explicitly model the leasing aspect, thus the model can be considered a 'recovery-only' model. The decisions each period are the numbers of each quality level to remanufacture and to salvage, in order to maximize total expected profit. Returns are classified by multiple quality levels, with some returns being scrapped (and used for parts or materials) and the remainder enter the remanufacturing process. Returns which do not enter the remanufacturing process, i.e., the ones which are sold for a salvage value, leave the system. It was found that by sorting the returns, profits could be increased.

Denizel et al. (2010) study a periodic review, capacitated recovery system in which returns have differing qualities and lower quality items require more recovery effort. Demand and returns are deterministic, but the quality of the returns is stochastic so a stochastic programme formulation is used to specify the decision to be taken at each period. The decision variables are the number of returns to "grade", the number of returns to recovery at each level, and the amount of each type of inventory to keep in stock for future periods. They found that they relationship between the quality of the return and the cost of recovery has a significant impact on the firm's profit. The cost of sorting items and the salvage value of un-remanufactured returns also had an impact.

Kaya (2010) studies a production system which remanufactures returned goods. They consider three variations of this model: remanufacturing only, production and remanufacturing with a single market, production and remanufacturing with separate markets. Since this paper also considers production, it is discussed in Section 2.4.2.

Teunter and Flapper (2011) study a one-period, specialised recovery system in which the returns can be classified into multiple quality classes, where the cost of performing recovery is higher for items of lower quality. Models with deterministic and stochastic demand are presented. Although this paper uses the quality of returns determine the cost of recovery, it does not model the type of recovery that can be performed. This could be an oversimplification since it could be possible that returns of different quality levels not only have a different recovery cost, but also have to undergo different processes.

Summary

In general, more attention has been paid to the quality of returns in the specialised recovery systems, than in systems which also perform production. Teunter and Flapper (2011) was one that did consider the quality of returns, however they assumed it was only the cost recovery that was affected by the quality. As mentioned earlier this simplification may not be relevant in all situations. We address this issue in all models in this thesis by providing two channels for recovery, each with a different cost, but also a different process.

2.5 Literature Summary and Gaps

Product recovery is performed in a large number of industries, from military to telecommunications and is becoming increasingly important due to economic, legislative and environmental pressures. In recent years, the amount of research on product recovery management has increased dramatically. Product recovery systems pose additional operational challenges due to the increased uncertainty caused by the quality, quantity and timing of returns (Ilgin and Gupta, 2010). The quality of returns is widely acknowledged as being important, however comparatively little research has investigated this issue. In this section a brief summary of the literature reviewed in this chapter is provided and then an overview of the literature gaps addressed by this thesis is presented.

2.5.1 Literature Summary

Single Market for Produced and Recovered Goods

Production and recovery research can be classified as being deterministic, discrete-time stochastic, and continuous-time stochastic models.

Deterministic product recovery models typically have a EOQ structure and are concerned with determining the optimal lot size and the optimal number of lots per cycle. The main issues that are discussed are whether or not to include a disposal option for returns and obtaining values of the decision variables. With regards to quality, most research assumes that either all returns are recoverable or that low quality returns are disposed.

Discrete-time, periodic-review stochastic product recovery models allow demand and returns to be observed and decisions to be made on a periodic basis. Typically periodic models are studied over a discrete time horizon. The nature of the periodic review models mean that finding the structure of the optimal policy is sometimes possible, therefore much of the research has focussed on doing this. However, a substantial body of literature has also investigated finding the optimal parameter values for a variety of policy structures. A three parameter policy, which specifies when to produce, recover and dispose of returns, is and has been popular in the literature since it was introduced by Phelps (1962). However, in today's environmentally conscious society, legislation may mean that disposal is not an option, i.e. firms must recover, in some form, everything that is returned to them.

Continuous-time, continuous-review stochastic product recovery models allow demand and returns to be observed and decisions to be made at any point across a continuous time horizon. In general, continuous review models assume a particular policy structure and focus on obtaining a cost function and the policy parameters. Finding the optimal policy structure is often not possible for continuous time models. The key issues in this field of literature are the inclusion of a disposal option for returns, the definition of the inventory position, and the nature of the leadtimes for production and recovery. Approaches for dealing with uncertain quality of returns and yield from recovery have also been discussed by a number of authors.

Separate Markets for Produced and Recovered Goods

Another stream of literature dealing with product recovery management examines systems in which recovered goods are not sold on the same market as newly produced ones. Even when recovered goods are functionally "as good as new", customers may not perceive them to be so, and indeed in some countries, legislation prevents recovered goods being sold "as new". Key issues in this stream of literature include substitution, network configuration and market structure. Most of the papers which include substitution only consider one-way downward substitution, although some do consider two-way substitution. Downward substitution assumes that a high value (newly produced) item is used to satisfy demand for a lower value (recovered) item, rather than the reverse. This is a limitation of the literature as there may be cases where a customer would also be willing to receive a lower-value recovered item, rather than no item at all.

Some literature has considered specialised recovery systems, i.e., systems which only perform recovery, i.e., do not perform production or procurement of new goods. Some papers study consider multiple reuse options and investigate how returns should be allocated between them. Issues discussed in the literature include the network structure, the quality of returns and the sorting of returns. Of all of the product recovery literature, it is this area which discusses the quality of returns in most detail.

Quality of Returns

The quality of the returns is a significant issue in the operation of product recovery systems as it can affect the type of recovery which can be performed and the associated costs (Fleischmann et al., 1997). A summary of the key literature relating to the quality of returns is presented in Table 2.1. As shown in this table, quality has been considered in a number of different ways. Some papers assume that only goods which have sufficient quality are returned to the firm, others assume the firm can choose either to either buy back returns regardless of quality and then dispose of low quality items, or to only buy back high quality returns. A few papers address the issue of quality by introducing a sorting option for returns, for instance Ferguson et al. (2009) found that by sorting returns based on their quality, profits could be increased. The quality of returns has been modelled by introducing quality dependent costs and leadtimes for recovery (Aras et al., 2004, 2006). Controlling timing and quality of returns through buyback strategies has also been studied (Aras et al., 2006). Some of these papers assume returns which are disposed of are sold for scrap or used for parts/materials (Ferguson et al., 2009), however few actually include this in their models.

There are a number of limitations within the existing literature. Assuming that only high quality returns are returned is naive, and as suggested by various 'sorting' studies, can lead to underestimated costs. Further, assuming that low quality returns are disposed ignores the possibility that the firm may be able to salvage value from these returns, either by using them for parts or materials. If these parts can be used in the production of new items, then this could reduce the need for the purchase of new materials and thereby reducing the cost of producing new items, and thus helping to protect the environment.

Under current environmental recycling targets, firms are encouraged to recycle as much as possible, and can be penalised for items which they do not recover. Including a low quality recovery option could allow them to increase their recycling percentage and avoid such penalty costs. The profitability of performing low quality recovery is likely to be affected by the costs associated with production and recovery, but will also be affected by the penalty costs associated with not recycling.

2.5.2 Literature Gaps Addressed by this Thesis

Context

This thesis addresses the issue of the quality of the returns, by assuming that returns may be either high quality or low quality and that the quality of the returns determines the type of recovery that can be performed. High quality returns can be recovered (e.g. repair, refurbishing and remanufacturing) to become serviceable goods. Low quality returns can be recovered (e.g. cannibalisation and recycling) and used as components in the production of new items.

Table 2.1: Summary of papers in product recovery literature studying the quality of returns

Authors	Year	Type	Stream	Method for dealing with quality
Dobos and Richter	2006	Deterministic	Production/recycling	Low quality items are disposed
Çorbacioğlu and van der Laan	2007	Deterministic	Recovery	All returns are recovered in one of the two streams. No disposal
El Saadany and Jaber	2008	Deterministic	Waste disposal	Returns are recovered or disposed
El Saadany and Jaber	2010	Deterministic	Waste disposal	All returns entering recovery are useable. Items which are not are not accepted/disposed.
Rubio and Corominas	2008	Deterministic	Recovery	Returns are recovered or disposed
Koh et al.	2002	Deterministic	Recovery	All returns are recoverable, items are scrapped from consumer
Dobos and Richter	2004	Deterministic	Production/recycling	Returns are recovered or disposed
Simpson	1978	Stochastic Periodic	Optimal Policy Structure	All returns can be repaired.
Khawam et al.	2007	Stochastic Periodic	Heuristic Parameter Values	not all returns can be remanufactured
Wang et al.	2010	Stochastic Periodic	Parameter Values	Returns which are not recovered are disposed of
Fleischmann et al.	2002	Continuous	Negative Demands	All returns are recovered.
Aras et al.	2004	Continuous	Yield	Lead time and cost of recovery is quality dependent
Aras et al.	2006	Continuous	Yield	Quality is modelled by a uniform distribution
Souza and Ketzenberg	2002	Continuous	Yield	Returns are recovered or disposed depending on quality
Tang et al.	2007	Recovery Only	Recovery Only	Low quality components are disposed of, and an order is placed for a comparable component to be used in the assembly process.

Throughout this thesis we compare the profitability of including low quality recovery to the alternative of disposing of all low quality returns. The analyses that are conducted investigate whether or not it is possible for a firm to improve its performance by incorporating a low quality recovery option within their existing production recovery system. This thesis differs from the more complicated assembly system models which study use of parts, disassembly and assembly in a more detailed fashion, as the objective is not to study the production planning aspect, as it is in these studies. Rather we want to investigate whether or not it is profitable to incorporate this extra 'channel' of recovery into the model, under a inventory management context.

The results obtained in this thesis could be used to help a firm make a planning decision, using an inventory management framework, regarding the profitability of adding the additional type of recovery. Firms could compare the difference in costs to any one-off costs associated with the installation of the new recovery type.

Mathematical Models

In order to investigate the profitability of including a low quality recovery option into a product recovery model, we propose four models. These models will address this question, but will also fill gaps in their respective streams of literature.

Uncertain demand, and uncertainty regarding the quality, quantity and timing of returns add challenges to the operation of a product recovery system. Therefore in this research we begin our investigation with a *deterministic lot-sizing model* of a product recovery system. This model extends the deterministic product recovery literature (Section 2.3.1) by combining the models of Teunter (2004) and Oh and Hwang (2006). It allows the quality of each return to determine what type of recovery is performed. The ability of the producer to only take back high quality items, which was considered in Dobos and Richter (2006), is also incorporated into the current model.

The second model extends the first by introducing uncertainty in the demand, returns and quality of the returns. A *discrete time periodic review model* is used, rather than a continuous review model, because it allows us to more easily investigate the structure of the optimal policy. This model is positioned in the literature alongside the models of Inderfurth (1997) and Ahiska and King (2010). However it differs from these models as it includes the low quality recovery option.

The first two models of this thesis assume that recovered goods are "as good as new". In some cases this is a justified assumption, however in some industries even goods which may be functionally "as good as new" are not perceived by the consumer to be the same as a new item. This limitation is addressed in the third and fourth models by the introduction separate markets for new and recovered goods.

The third model is a *discrete time periodic review model* of a product recovery system with separate markets for new and recovered goods and batched arrivals of demand and returns. It is assumed that newly produced and recovered goods have a similar functionality, therefore if there is insufficient inventory to meet demand for one of the goods, the firm may offer the other as a substitute. Four substitution strategies are investigated: no substitution, downward, upward and two-way substitution. It is assumed that consumer may accept or reject the substitution. This model extends the 'separate markets' literature by allowing two-way substitution and an infinite horizon model, in particular Inderfurth (2004) and Kaya (2010). Piñeyro and Viera (2010) study a deterministic which has a similar modelling structure, but to the best of our knowledge this type of model has not been studied in a stochastic environment.

In this discrete time model, demand, returns and substitution occur in batches each period. This framework may be appropriate for modelling demand and returns, however it does not allow substitution to be modelled in much detail. Therefore, the fourth and final model of this thesis examines a *continuous time continuous review model*. As in the discrete time model, this model has separate markets for new and recovered goods, however the continuous nature of the model means that leadtimes are also introduced. This model extends the literature, in particular, Inderfurth (2004), Piñeyro and Viera (2010), and Kaya (2010).

Chapter 3

Methodology

A variety of solution methodologies have been used to study product recovery systems. The type of methodology which used depends on the structure of the problem under study, as some methodologies are more appropriate for certain types of problems. Methods which have been used to study product recovery systems include: simulation (Zanoni et al., 2006), queuing theory (Toktay et al., 2000; Souza et al., 2002), dynamic programming (DeCroix, 2006), Markov decision processes (Fleischmann and Kuik, 2003; Ahiska and King, 2010), Markov chains (Kiesmüller and van der Laan, 2001; Nakashima et al., 2002) mixed-integer non-linear programming (Ah kioon et al., 2009), lot-sizing models (Teunter, 2001, 2004) and heuristics (DeCroix and Zipkin, 2005). Three main methodologies are used in this thesis: lot-sizing models (Chapter 4), Markov decision processes (Chapters 5 and 6) and semi-Markov decision processes (Chapter 7). Simulation is also used in Chapters 5–7.

In this chapter a review is presented of key theory and methodological approaches used in this thesis. In this thesis we are particularly interested in the management of inventory in product recovery systems, therefore in Section 3.1 inventory models are discussed. This section will also discuss lot-sizing models. In Section 3.2 a review of Markov and semi-Markov decision processes, the value iteration algorithm and the implementation method used in this thesis is presented.

3.1 Inventory Models

The effective management of inventory is an issue which faces many organisations, and as a result it has become a widely-studied field within Operational Research and Management Science. In managing inventory systems a key concept relates to how much inventory is in stock. In inventory models there are two ways of quantifying stock: inventory level and inventory position. The *inventory level* refers the amount of inventory that is currently in stock whereas the *inventory position* refers the amount of inventory that is in stock, but also takes into account any outstanding orders (Axsäter, 2006). Inventory models typically aim to manage the stock levels in an inventory system in order to minimise the associated costs. They do this by answering two main questions: how much should be ordered, and how often should an order be placed Winston (2004).

There are two main trade-offs associated with inventory management (Beyer et al., 2010). The first being the trade-off between setup costs and holding costs (large infrequent orders vs small frequent orders), and the second being the trade-off between holding costs and shortage costs (high inventory costs, lower risk of shortages, vs low inventory holding costs, higher risk of shortages).

The costs associated with inventory systems typically include setup costs, unit ordering costs, holding costs and shortage costs. Setup costs are costs which do not depend on the size of an order and could include the cost of issuing an invoice, fixed overhead costs, or transportation costs (Axsäter, 2006). Unit-order costs are the variable costs which are incurred for each unit that is ordered or produced and could include labour costs, materials costs or overhead costs (Winston, 2004). Holding costs are the costs associated with storing inventory and not only includes the cost of physically storing inventory, but can also include the associated insurance costs, taxes, risk of theft, obsolescence or damage (Axsäter, 2006; Winston, 2004; Silver et al., 1998). The cost of holding inventory is related to the opportunity cost of having capital tied up in inventory, rather than some other investment, thus it is sometimes linked to the current interest rates, however this is not usually a direct relationship due to the potential risks of alternative investment opportunities (Axsäter, 2006). Finally, shortage costs are the costs associated with not being able to meet demand. Typically demand which cannot be met is either back-ordered and delivered at a later time, or is lost. Costs may be incurred for having back-ordered stock (perhaps an urgent delivery is required). If a sale is lost then this may result in lost goodwill, as customers may be disappointed at having not received the good they had wanted. However, these costs can often be very difficult to quantify (Axsäter, 2006)

The type of ordering policy that should be used depends on the nature of the product. For example, highly customised products are generally manufactured under a "make-to-order" policy, whereas a "make-to-stock" policy would be more appropriate for generic products (Slack et al., 2004). Another important distinction is whether the system operates as a "push" or "pull" strategy. Under a push strategy goods are produced as soon as possible, i.e. they are "pushed" through the system and then stored, whereas under a pull strategy goods are only produced when they are needed in order to satisfy demand (or needed for the next stage of the production process). Inventory is less likely to accumulate under a pull strategy (Slack et al., 2004). Many of the policies discussed henceforth assume that goods are produced in a "made-to-stock" environment.

3.1.1 Deterministic Inventory Models

A well-known deterministic inventory model is the *Economic Order Quantity* (EOQ) model. The following description is adapted from that given by Winston (2004). The basic economic order quantity Q^* is the total-cost-minimising order size, under assumptions of repeated (rather than one-off) ordering, constant deterministic demand of D items per year, zero lead time, no shortages, setup costs of k per order and holding costs of h per unit per year, and is given by:

$$Q^* = \sqrt{\frac{2kD}{h}}$$

The period of time from the moment an order arrives, to the moment in time before the next order arrives is often called a *cycle*. The EOQ is quite robust, so if the order size deviates slightly from the optimal quantity, then this will only result in a small increase in the total cost.

A number of variations of this basic model are possible. For example, if there is a nonzero lead time, then the optimal order quantity remains the same, but the order must be placed earlier, in order to allow sufficient time for the order to arrive. This order time is determined by the reorder point. If the lead time demand is less than the EOQ, then the reorder point is equal to the lead time demand. If the lead time demand exceeds the EOQ, then the reorder point is given by the remainder when lead time demand is divided by the EOQ (Winston, 2004, page 855). The EOQ can also be modified to take into account quanity discounts and back orders. For example, if a maximum of M items can be backordered, at a cost of s per unit per year, then the optimal order quantity is $Q^* = \text{EOQ}\sqrt{\frac{h+s}{s}}$ and the optimal shortage is $M^* = \text{EOQ}\sqrt{\frac{s}{h+s}}$.

In a production setting, items may be produced at a continual rate r, rather than all arriving instantly as may be the case with an order from an external supplier. In this situation, items are demanded whilst items are being produced, so the EOQ is no longer the optimal order quantity; rather the optimal number of goods which should be produced in a batch is $Q^* = \text{EOQ}\sqrt{\frac{r}{r-D}}$ This quantity is sometimes called the economic production quantity (EPQ) or the economic manufacturing quantity (EMQ) (Chiu, 2008; Darwish, 2008). As $r = \infty$ the EPQ approaches the EOQ.

3.1.2 Stochastic Inventory Models

In some situations demand may be uncertain or irregular, in which case a stochastic inventory model may be appropriate for studying the management of inventory. The inventory system may be studied using a single-period decision model in which the decision about how much to order is made only once, or it may be studied over multiple time periods.

The well-known newsvendor problem is an example of a single-period decision model. The following description of the news vendor problem is adapted from Winston (2004). The newsvendor problem studies the situation in which the 'news vendor' must decide how many newspapers to order, q. After this decision has been made, demand d is observed with probability $p(d)$ and then a cost $c(d, q)$ is incurred. There are costs associated with ordering too many c_o, or too few papers c_u. The optimal order quantity is the smallest q^* which satisfies $P(D \leq q^*) \geq \frac{c_u}{c_o+c_u}$. If the demand D is a continuous random variable, then this inequality can be solved to obtain q^*.

There are several inventory models in which decisions are made repeatedly. One such model adapts the EOQ to take into account uncertain demand. There are two cases which need to be considered: back-orders and lost sales. In each case, the optimal batch size Q^* and the re-order point r^* are sought. Let demand D be a random variable, with mean $E(D)$, the demand during the lead time be a random variable X and let c_b denote the per item shortage cost. In the back-ordered case, the optimal order quantity is $Q^* = \sqrt{\left(\frac{2KE(D)}{h}\right)}$ and the re-order point is given by $P(X \geq r^*) = \frac{hQ^*}{c_b E(D)}$. If $\frac{hQ^*}{c_b E(D)} > 1$ or $\frac{hQ^*}{c_b E(D)} < 0$ then the re-order point should be set "at the smallest acceptable level" (Winston, 2004). For the lost sales case the optimal order quantity Q^* is the same as under the back-ordered case and the re-order point is given by $P(X \geq r^*) = \frac{hQ^*}{hQ^* + c_l E(D)}$, where c_l is the per unit lost sales cost.

There are two common types of policies that can be applied to a stochastic continuous review inventory model: (r, q) and (s, S) policies. The (r, q) policy, sometimes called a *two-bin* policy, involves placing an order of size q if the inventory level falls to the reorder point r. If more than one item may be demanded at any point in time, then the (r, q) policy may not be appropriate, however the (s, S) policy could be used in such situations. Under this policy, if there is a zero lead time then an order, which replenishes inventory up to a level of S, is placed whenever the inventory level is less than or equal to s. This type of policy is sometimes referred to as an 'order-up-to' or 'base-stock' policy (Axsäter, 2006). As discussed by Silver et al. (1998), computing optimal values for s and S can be difficult. One approximation for these parameter values involves setting $s = r$ and $S = r + q$ (Winston, 2004).

In the examples discussed hitherto, the parameter values have been selected in order to minimise the total cost, however in some situations it may be difficult to quantify the cost of not meeting demand (and either incurring a back-order or lost sale). If a desired level of service is required, then the ordering policy can be determined by calculating the re-order point r required to achieve a specified level of service, rather than the re-order point which minimises costs. A variety of methods can be used to measure service. Two common measures include the "expected fraction ... of all demand that is met on time" or the "expected number of cycles per year during which a shortage occurs" (Winston, 2004, page 898). These measures are related to the fill rate, which is the "fraction of customer demand that is met" (Silver et al., 1998, page

245). For simpler models it is possible to calculate decision rules based on achieving specified measures of service (see Silver et al. (1998) for further details), however for more complicated models this is not always possible. Rather than using the fill rate to determine a policy, it could also be used as a measure of the performance of a policy. For a given policy, the fill rate (and other service measures) can be calculated using simulation.

Another well-known policy is the (R, S) periodic review policy. Under this policy, the inventory level is reviewed periodically, every R time units, and an order is placed which brings the inventory position up to S. If there is no back-ordering, i.e. all demand not met is lost, then for a given R, the S which minimises costs is given by $P(D_{L+R} \geq S) = \frac{Rh}{Rh+c_l}$ where D_{L+R} is the demand during the time interval $L+R$. The time between reviews R sometimes equals $EOQ/E(D) = \sqrt{\frac{2(K+J)E(D)}{h}}/E(D)$, where J is the cost of reviewing the inventory level (Winston, 2004).

The management of inventory, in both deterministic and stochastic settings, can also be studied using dynamic programming. In general, the dynamic programming is used for periodic review models in which the planning horizon is divided into set periods. The inventory level is reviewed once each period (at the beginning or end) and then an order is placed. If demand is known but can differ between periods (as in dynamic lot-sizing problems) then a traditional dynamic programming formulation becomes computationally burdensome, however the Wagner-Whitin Algorithm or Silver-Meal Heuristic can be used (Winston, 2004). If demand is uncertain, then a Markov decision process may be appropriate for studying the problem. Markov decision processes will be discussed in more detail in Section 3.2

3.1.3 Representing Inventory Policies

An inventory policy can be specified through a decision rule which details the condition under which certain actions are chosen. For instance if the ordering cost is less than the sum of the return and holding costs, then order x items (Archibald et al., 2007). For some problems, an inventory policy can be represented graphically by representing the state space (e.g. inventory levels) on a graph and then indicating for which regions certain actions should be performed. Simpson (1978) used graphs, with serviceable inventory on one axis and repairable inventory on the other axis, to show the structure of the optimal policy, i.e., at certain inventory levels particular actions are optimal. This idea will be used to represent the optimal policies obtain in Chapter 5.

3.2 Markov Decision Processes

A Markov decision process (MDP) is a stochastic process in which the progression of the process can be influenced by the actions of a decision maker. Markov processes are named as such due to their *Markovian property*: the future depends only on the present and not on the past. More specifically, the present state and decision depend not on the past history of states and decisions (Puterman, 1994). A Markov decision process can be described by the following: decision epochs, states, actions, rewards/costs, transition probabilities (Puterman, 1994). At each decision epoch, the state is observed by the decision maker, who then chooses an action. There is a cost or reward associated with the chosen action. The system will then move to the next state according to the associated transition probabilities.

A *policy* is another important concept in the study of Markov decision processes. Policies specify that decisions that should be taken in certain situations. Policies can be Markovian or history dependent, and deterministic or randomised, furthermore policies can be stationary or time-dependent. Stationary deterministic policies are the most specific type of policy and are therefore sometimes referred to as being *pure* (Puterman, 1994). Despite the many types of policies, in many cases there is exists a pure and stationary policy. Such policies can be described as "an assignment of an action to each state" (Archibald et al., 1993), in order words, they specify which action should be performed in each and every state. There is a cost or reward associated with a given policy, thus it is desirable to find a policy which either minimises the cost or maximises the reward. Often research which applies a Markov decision process model tries to find an associated optimal policy. However, as discussed by Ansell et al. (1998b) in the context of maintenance policies, policies obtained from Operational Research models, such as Markov decision models, may be seen as too complex to implement in practice, or conversely such models may be seen as too simple to represent real systems. In spite of this, optimal policies can still give insight into real-life applications and be interesting from a theoretical point of view.

There are several variations of Markov decision processes. When the decision epochs occur at regular, known times then discrete-time Markov decision process may be appropriate, whereas, when the decision epochs occur randomly, then a continuous time semi-Markov decision process may be appropriate (Tijms, 1994). Structuring discrete-time decision models as Markov decision models makes it possible to prove the nature of optimal policy (Ansell et al., 1998b) – this is one advantage of using Markov decision processes. The time-horizon over which a Markov decision process is studied can also vary and be finite or infinite horizon.

Markov decision processes fall into the category of decision models. It is generally Bellman (1957) and Howard (1960) who are credited with initiating modern research into Markov decision processes (Puterman, 1994). Since then Markov decision processes have been used to model a range of applications including ecological systems (Williams, 2009), profitability of credit cards (So and Thomas, 2011), inventory and production (Archibald et al., 2002; Thomas et al., 2003), repair and maintenance problems (Kim and Thomas, 2006; Ansell et al., 1998a), water industry (Ansell et al., 2004) and product recovery systems (Nakashima et al., 2004). Other applications include finance, queues, sports (White, 1993). Though it is now slightly out-of-date, the review of applications discussed by White (1993) gives a good indication of the scope of Markov Decision processes.

A common limitation of Markov decision processes is that for problems with a large state space, solution methodologies are computationally infeasible. In these cases heuristics may be used. In the context of inventory systems, heuristics have been proposed for a variety of problems, e.g. transshipments (Archibald et al., 2009).

In this thesis Markov decision processes will be used to model inventory decisions which, according to Puterman (1994), were one of their first applications. In its simplest form, a MDP representing an inventory system can be described as follows: the decision maker observes the inventory level (state), decides whether or not to place an order (action), and then demand is observed according to a certain probability (transition probability) and the inventory level is adjusted accordingly. Costs may be incurred as a result of the action or transition.

Two types of Markov decision processes will be used in this thesis to study the management of inventory in product recovery systems. In Chapters 5 and 6 discrete-time, infinite horizon Markov decision processes will be proposed, and in Chapter 7 a continuous-time, infinite horizon semi-Markov decision process will be proposed. In Chapter 5 the objective is to minimise the long run average cost and in Chapters 6 and 7 the objective is to maximise the long run average reward.

The remainder of this section is structured as follows. The elements of a Markov decision process are presented in Section 3.2.1 and then algorithms which can be used to solve MDPs are discussed in Section 3.2.2. Particular attention is paid to the value iteration algorithm, as it is the algorithm implemented in this thesis. The implementation of the value iteration algorithm is then discussed in Section 3.2.3. Finally, in Section 3.2.4 some theory relating to semi-Markov decision processes is presented.

3.2.1 Elements of a Discrete-Time MDP

In this section the elements of a discrete-time, infinite horizon Markov decision process under an average cost criterion are presented. The notation used in this section is adapted from Puterman (1994) and Tijms (1994) and will be used throughout this thesis.

Decision Epochs. Let T denote the set of decision epochs. In this thesis all Markov decision models are studied over an infinite time horizon, therefore

$$T = \{1, 2, \ldots, \infty\} = [0, \infty)$$

that is, T is an infinite, discrete set of the nonnegative real line. Each $t \in T$ is referred to as a *period*. Decisions are made at the start of each period t.

States. Let I denote the set of all possible states. In this thesis, each state $i \in I$ represents a vector of state variables. For example in Chapter 5, there are three state variables: serviceable inventory, returned inventory and component inventory. Letting these three state variables be denoted by i_s, i_r and i_c respectively, a state i can be defined as $i := (i_s, i_r, i_c)$. The set of all states I is assumed to be finite.

Actions. For each state $i \in I$ there is a set of actions $A(i)$ that can be chosen by the decision maker when the system is in state i. Across all states the set of all possible actions A is defined as $A = \bigcup_{i \in I} A(i)$. The set of all actions A is assumed to be finite.

Transition Probabilities. Suppose the system is in state i and action $a \in A(i)$ is chosen. The system will move to the next state $j \in I$ with a probability $p_{i,j}(a)$, where

$$\sum_{j \in I} p_{i,j}(a) = 1.$$

Costs. There is a cost $c(i, a)$ associated with the system being in state i and action $a \in A(i)$ being chosen. If the cost incurred depends on the transition to the next state j, then it may be denoted by $c(i, a, j)$. Using this notation, the expected cost incurred between the current and next decision epoch is then:

$$c(i, a) = \sum_{j \in S} c(i, a, j) p_{i,j}(a).$$

Policies. A policy is used to describe a particular set of actions that may be applied to the Markov decision model. Often a policy is sought which satisfies some objective, such as minimising the long run average cost. Other objectives include maximization of expected total reward, discounted expected total cost, or average reward (Puterman, 1994). In an infinite horizon problem, the total reward criterion may result in an infinite value, thus the average reward criterion may be more appropriate. In this thesis we will be concerned with finding a stationary policy, i.e., one which does not depend on time. A decision rule $d(i)$ specifies the action that should be chosen whenever the system is in state i. A policy π is a set of decision rules for all states $i \in I$, and thus a policy π is a mapping from the state space I to the action space A.

For a given policy π we are interested in the "value" of the policy, or in the context of this thesis, the long-run average cost or reward of the policy. Let $V_n(i,\pi)$ denote the "total expected costs over the first n decision epochs when the initial state is i and policy" π is implemented (Tijms, 1994). Then

$$V_n(i,\pi) = \sum_{t=0}^{n-1} \sum_{j \in I} p_{i,j}^{(t)}(\pi)\, c(j, \pi_j)$$

where $p_{i,j}^{(0)}(\pi) = 1$ for $j = i$, and $p_{i,j}^{(0)}(\pi) = 0$ for $j \neq i$, and where $p_{i,j}^{(n)}(\pi)$ are the n-step transition probabilities associated with the Markov chain $\{X_n\}$ such that $p_{i,j}^{(n)}(\pi) = P\{X_n = j | X_0 = i\}$ for $i, j \in I$ and $n = 1, 2, \ldots$. The average cost function $g_i(\pi)$ is

$$g_i(\pi) = \lim_{n\to\infty} \frac{1}{n} V_n(i,\pi), \quad i \in I$$

If it is assumed that the Markov chain corresponding to policy π has "no two disjoint sets", then the "long-run average expected cost per time unit is independent of the initial state i", and then "in the unichain case"

$$g_i(\pi) = g(\pi), \quad i \in I$$

At this point is it also worth mentioning the weak unichain assumption which states that "for each average cost optimal stationary policy the associated Markov chain $\{X_n\}$ has no two disjoint closed sets" (Tijms, 1994). These assumptions will be discussed further in Section 3.2.2.

3.2.2 Value Iteration Algorithm

Solving a Markov decision process involves finding a policy π which satisfies the objective, which in the case of this thesis means finding a policy that either minimises the long-run average cost or maximises the long-run average reward. There are several algorithms which can be used to solve Markov decision processes, namely the value iteration, policy iteration and linear programming algorithms. However both the policy iteration and linear programming algorithms involve solving a large system of linear equations, corresponding to the size of the state space (Tijms, 1994). The value iteration algorithm, on the other hand, uses a recursive approach. Therefore, for most problems, including the ones studied in this thesis, the value iteration algorithm is the only practical solution methodology.

The description of the value iteration algorithm presented below follows the structure used by Tijms (1994), with some notation adapted from Puterman (1994).

Step 0 Initialise.

- Choose a sufficiently small stopping condition $\epsilon > 0$
- Set $V_0(i)$ with $0 \leq V_0(i) \leq \min_a c(i,a)$ for all $i \in I$.
- Set $n := 1$

Step 1 For all states $i \in I$, compute $V_n(i)$

$$V_n(i) = \min_{a \in A(i)} \left\{ c(i,a) + \sum_{j \in I} p_{i,j}(a) V_{n-1}(j) \right\}$$

and determine the stationary policy $\pi_n(i)$ for all $i \in I$, where

$$\pi_n(i) = \arg\min_{a \in A(i)} \left\{ c(i,a) + \sum_{j \in I} p_{i,j}(a) V_{n-1}(j) \right\}$$

Step 2 Compute the bounds

$$m_n = \min_{j \in I} \{V_n(j) - V_{n-1}(j)\} \quad \text{and} \quad M_n = \max_{j \in I} \{V_n(j) - V_{n-1}(j)\}$$

Step 3 Stop the algorithm with policy π_n if:

$$0 \leq M_n - m_n \leq \epsilon$$

Otherwise, set $n := n + 1$ and return to Step 1.

If the weak unichain assumption, stated at the end of Section 3.2.1, is met and the Markov chain associated with each average cost optimal stationary policy is aperiodic, then the value iteration algorithm will converge and

$$g^* = \lim_{n \to \infty} M_n = \lim_{n \to \infty} m_n$$

is the minimum long-run average cost (Tijms, 1994). Some adjustments can be made to overcome the aperiodic requirement, see Tijms (1994, page 209) for further details.

For some parameter values and some problem types it is possible that there may be two or more disjoint closed sets, which would mean that the value iteration algorithm may not converge. We find that this is the case for some of the randomly generated problem scenarios used in Chapter 5.

3.2.3 Implementing the Value Iteration Algorithm

In this section we discuss the way in which the value iteration algorithm is implemented for this thesis.

Computational Requirements

The value iteration algorithm was presented in Section 3.2.2 however whilst this does contain the steps of the algorithm it does not represent how it is implement in java. For example, to calculate

$$V_n(i) = \min_{a \in A(i)} \left\{ c(i,a) + \sum_{j \in I} p_{i,j}(a) V_{n-1}(j) \right\}$$

for all $i \in I$, it is necessary to "loop" around all states $i \in I$, all actions $a \in A(i)$ and then again around all states $j \in I$. This sequence of loops must be performed at every iteration of the algorithm, until it converges. If the cost $c(i,a)$ is an expected cost, then some looping may be required to calculate this cost. Furthermore, depending on the problem, further loops may be required in order to obtain the transition probability $p_{i,j}(a)$. For example, using the problem discussed in Chapter 5, there are three random variables that affect the transition to the next state: demand, returns and the quality of returns. There are multiple ways that these variables could combine in order to reach a given state: for example if there are 2 high quality returns and a demand for 3 items, then the serviceable inventory level will increase by 2 and decrease by 3, resulting in an overall change in the inventory level of -1. Other combinations of demand and quality of returns could also result in an overall change of -1. Therefore for this problem, one way to obtain the probability $p_{i,j}(a)$, is to loop around all states j, all values of demand, all values of returns, all values of quality of returns and determine if each combination of the random variables could result in state j, and then keep track of the total probability of moving from state i to state j, given action a. If these probabilities are not stored, then this looping to find $p_{i,j}(a)$ would need to be performed for each iteration of the algorithm.

It is clear that this repeated looping and recalculation would quickly become computationally inefficient and burdensome as the number of states increases. During initial attempts at running this type of implementation of the value iteration algorithm it was found that the computational time was prohibitively long for all but the most trivial problems. In order to overcome this, two methods were used to reduce the computation time: precalculation and storage of values in an array structure and definition of the mid-state. These two methods will be discussed in turn.

The nature of the value iteration algorithm and of Markov decision processes means that there is a lot of repeated calculations performed during each iteration. For example the transition probabilities $p_{i,j}(a)$ are used in every iteration of the algorithm, and furthermore, many of these transition probabilities are zero. For larger problems it was found that storing some of this information in a series of arrays was computationally more efficient than calculating it repeatedly. One reason for this is that by storing only those with a non-zero transition probability, the number of states which need to be looped around was much lower. Therefore we implemented an array structure to store the transitions with non-zero transition probabilities associated with given states i and j and action a. Some additional information, such as the number of transition probabilities and the associated costs, was also stored.

A second method was also used to reduce the computational time. Using the notation from above, suppose the system is in state i, action a is chosen and the system moves to state j with probability $p_{i,j}(a)$. If the transition probability depends on multiple random variables, then it is possible that some of these random variables may depend on the action a, and some may not.

Let the mid-state m represent the state of the system after the action has been chosen and the state variables have been updated accordingly. The probability of moving from state i to mid-state m depends on the initial state i, the action a and any uncertainty associated with the action (in Chapter 5 the quality of returns). Let this probability be denoted by $p_{i,m}(a)$. The probability of moving from mid-state m to state j in the next period depends on the mid-state m, and on any uncertainty not associated with the action (in Chapter 5 the demand and the returns). Let this probability be denoted by $p_{m,j}$. Since the probability of moving from the mid-state to the next state does not depend on the action selected, the calculation of these transition probabilities and any associated costs could be calculated outside of the "action loop" in the algorithm. These values could then be calculated in advance, stored and then looked up during the value iteration algorithm. If the mid-state and array structure are used, then the value iteration algorithm could be described as follows.

Step 0 Initialise.

- Choose a sufficiently small stopping condition $\epsilon > 0$
- Set $V_0(i)$ with $0 \leq V_0(i) \leq \min_a c(i,a)$ for all $i \in I$.
- Set $n := 1$
- For $i, j, m \in I$ and $a \in A(i)$, calculate $p_{i,m}(a)$ and $p_{m,j}$, and:

  if $p_{i,m}(a) > 0$ then store i,m and a in set I_1

  if $p_{m,j}(a) > 0$ then store m and j in set I_2

Step 1 For all states $m \in I$, compute $W(m)$

$$W(m) = \sum_{j \in I_2} p_{m,j} V_{n-1}(j)$$

Step 2 For all states $i \in I$, compute $V_n(i)$:

$$V_n(i) = \min_{a \in A(i)} \left\{ c(i,a) + \sum_{m \in I_1} p_{i,m}(a) W(m) \right\}$$

and determine the stationary policy $\pi_n(i)$ for all $i \in I$, where

$$\pi_n(i) = \arg \min_{a \in A(i)} \left\{ c(i,a) + \sum_{m \in I_1} p_{i,m}(a) W(m) \right\}$$

Step 3 Compute the bounds

$$m_n = \min_{j \in I} \{V_n(j) - V_{n-1}(j)\} \quad \text{and} \quad M_n = \max_{j \in I} \{V_n(j) - V_{n-1}(j)\}$$

Step 4 Stop the algorithm with policy π_n if:

$$0 \leq M_n - m_n \leq \epsilon$$

Otherwise, set $n := n + 1$ and return to Step 1.

Note that the nature of the problem means that the sets I_1 and I_2 are substantially smaller than I, and this significantly reduces the "looping" The effectiveness of using the mid-state and the array structure to store the calculations was compared to the ‘naive’ approach of repeatedly calculating values at every iteration, for a variety of problem sizes of the stochastic product recovery model in Chapter 5. One of the primary determinants of the size of the problem is the state space, therefore the problem size was determined by varying the upper inventory capacities. (The action space and transition matrix also affect the size of the problem, but these are not considered here.) Table 3.1 shows the time required to obtain an optimal policy using arrays and using loops, for seven stochastic product recovery models with the following upper inventory capacities: $\{2, 5, 10, 15, 20, 25, 30\}$. The time stated for the arrays method includes the time taken to make the arrays and complete the value iteration algorithm. It was found that a computational saving of over 9 hours was attainable by using the array structure rather than repeated looping.

Inventory Capacity	Array-based approach	Naive, loop-based approach
2	00:00:00	00:00:01
5	00:00:01	00:00:05
10	00:00:05	00:02:25
15	00:00:30	00:14:03
20	00:02:15	01:03:09
25	00:08:29	03:41:40
30	00:28:42	10:13:11

Table 3.1: Comparison between computational time (in hh:mm:ss format) required for the value iteration algorithm with and without arrays for stochastic product recovery problems

Code Structure

The code design exploited the coding structures available in java (such as objects, classes and packages) and this allowed parts of the code to be used for multiple problems. This was done for practical reasons, but also to enable thorough error-checking to be performed. For example the value iteration code was contained in an independent file which meant that it could not only be used for the models in Chapters 5, 6 and 7 of this thesis, but also that it could also be used for MDP examples from Tijms (1994) and Puterman (1994). This was one method used to check the accuracy of the code. Further details about the measures taken to validate the code are described below.

Validation of Code

A variety of error-checking procedures were implemented within the java code. For instance, if a set of probabilities which are meant to sum to one, do not sum to one, or if insufficient parameters are entered, then an error message will be displayed and the run will be terminated.

Two versions of the value iteration code were developed, one which used the array structure and another which used repeated looping. During the initial code development, the value iteration algorithm was developed by using a number of examples from textbooks, including Example 3.1.1 from Tijms (1994, page 195) and Example 8.5.3 Puterman (1994, page 366). At the end of the code development process, these examples were once again used to test the code. In all cases, the policies obtained by both versions of the value iteration code were the same as those published in the textbooks. The cost/reward associated with the policies were not always provided in the textbooks, but where they were, they too were the same as what we obtained.

3.2.4 Semi-Markov Decision Processes

The structure of a semi-Markov decision process (SMDP) is similar to that of a Markov decision process (MDP), in that it too can be characterised by its decision epochs, states, actions, costs and transition probabilities. However a SMDP differs from a MDP because the time between decision epochs is uncertain, rather than being identical (Tijms, 1994). As discussed in Section 3.2.1, let I denote the state space, such that for any state i, $i \in I$ an action $a \in A(i)$ can be chosen, which will result in an expected cost $c(i, a)$ and transition probability $p_{i,j}(a)$. If the process is currently in state i and action a is chosen, then let the expected time to the next event be denoted by $\tau_i(a)$.

A semi-Markov decision process can be solved by converting the process to a discrete time model. This effectively means that the continuous time is split into a series of very small periods of length τ. The length of time τ is chosen such that the probability that more than one event could occur during that period is very small. The following description of the data transformation required to convert a semi-Markov decision process to a discrete-time model is adapted from Tijms (1994, page 222). Firstly, a value for τ must be selected, such that:

$$0 < \tau \leq \min_{i,a} \tau_i(a)$$

Secondly, the following transformations must be made:

$$\begin{aligned}
&\overline{I} = I \\
&\overline{A}(i) = A(i) \qquad i \in \overline{I} \\
&\overline{c}(i,a) = \tfrac{c(i,a)}{\tau_i(a)} \quad i \in \overline{I} \text{ and } a \in \overline{A}(i) \\
&\overline{p}_{i,j}(a) = \begin{cases} p_{i,j}(a)\tfrac{\tau}{\tau_i(a)} & \text{if } i \neq j,\ i \in \overline{I} \text{ and } a \in \overline{A}(i) \\ p_{i,j}(a)\tfrac{\tau}{\tau_i(a)} + \left(1 - \tfrac{\tau}{\tau_i(a)}\right) & \text{if } i = j,\ i \in \overline{I} \text{ and } a \in \overline{A}(i) \end{cases}
\end{aligned}$$

After making these adjustments to the expected costs and probabilities, the value iteration algorithm and implementation discussed in Section 3.2.2 can be used to solve the semi-Markov decision process. For the semi-Markov decision process, the value function calculated within the value iteration algorithm is:

$$V_n(i) = \min_{a \in \overline{A}(i)} \left\{ \overline{c}(i,a) + \sum_{j \in I} \overline{p}_{i,j}(a) V_{n-1}(j) \right\}$$

which, when the adjustments described above are taken into account, can be written as:

$$V_n(i) = \min_{a \in A(i)} \left\{ \frac{c(i,a)}{\tau_i(a)} + \frac{\tau}{\tau_i(a)} \sum_{j \in I} p_{i,j}(a) V_{n-1}(j) + \left(1 - \frac{\tau}{\tau_i(a)}\right) V_{n-1}(i) \right\}.$$

Chapter 4

Deterministic Lot-Sizing Product Recovery Model with a Single Market

4.1 Introduction

Product recovery describes the process by which used products are returned to their manufacturers or sent to a specialised facility for recovery, before being sold on the original or a secondary market. This chapter studies a product recovery system which in which returned products are recovered by the original manufacturer and are then sold on the original market. Thus in this chapter there is a single market for newly produced and recovered goods. A deterministic product recovery model using a EOQ-based lot-sizing framework is used to study this system.

This chapter addresses one of the gaps identified in the literature by differentiating between high quality and low quality returns and by providing two separate quality-dependent channels for the recovery of returned goods. Previous models either do not differentiate between high quality and low quality goods and thus provide one channel for recovery, or if they do differentiate then dispose of low quality goods and recover only the high quality ones. The current model combines the types of recovery used by Oh and Hwang (2006) and in traditional product recovery models, such as Schrady

(1967) and Teunter (2004). The producer may choose to take back all returns, or only those of a high quality, as in Dobos and Richter (2006).

This chapter begins in Section 4.2 with a description of the product recovery system being investigated. A model is proposed for analysing this system in Section 4.3. The inventory levels across a cycle are discussed and then the cost function is derived in Sections 4.4 and 4.5 respectively. The minimisation of the total cost function is not trivial as the resulting function is a non-differentiable function in one continuous variable and three integer variables. This will be discussed in Section 4.6. Properties of the model are investigated in Section 4.7 and finally a discussion of the results is presented in Section 4.8.

4.2 Problem Description

Suppose there is a firm which has the primary function of producing new goods. This firm accepts goods back after they have been used and, if they are above the required quality threshold, recovers them to the same quality standard as newly produced goods, before selling them as new. The decision to incorporate recovery into their business is threefold: to improve their 'green' image by taking back used products; to reduce their costs by reusing used goods and materials; and to comply with environmental legislation requiring the take-back of goods.

Produced and recovered items are both considered to be 'serviceable' and are viewed as identical by the consumer so are sold on the same market. For returns which are below the quality threshold for recovery, the firm has a choice: to dispose of them or to use them as components in the production of newly produced items. If an insufficient number of components are obtained from the recovery of low quality returns then additional components are bought from an external supplier.

The firm is a cost-minimising firm. Fixed and unit costs are incurred for production, recovery and buying components. Costs are also incurred for holding inventory and for lost sales. Demand for serviceable items, returns and the quality of returns are known with certainty. The firm must determine a production plan that specifies how much and how often to produce, recover and buy. The firm is also interested in the cost-effectiveness of the low-quality recovery channel.

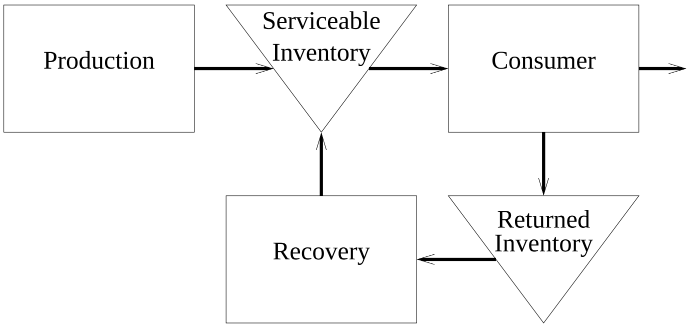
This system could describe Canon's remanufacturing processes for cartridges and printers, which were discussed in Section 2.2.3. Used printers and cartridges are returned to Canon to undergo remanufacturing before being sold as new. In both cases, returns which cannot be returned to this "as new" condition may be used as parts or materials in the production process. Rigorous quality standards allow Canon to sell these remanufactured products as new. This system could also describe the situation faced by a Cooperage with respect to whisky barrels. whisky barrels are used and then returned for repair. Some barrels will require only minor repairs, whereas some may require more substantial repairs or may only used for parts.

4.3 Model Description and Assumptions

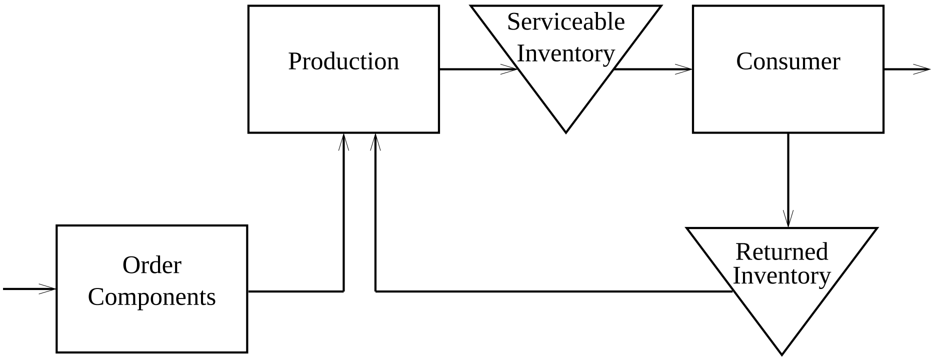
The model, modelling assumptions and the relationship between the current model and the existing models in the literature are discussed in this section.

4.3.1 Relationship with Existing Literature

The current model extends the literature by using quality to differentiate between returns and by providing two channels for recovery. The relevant literature was discussed in the literature review, therefore only the specific articles which are directly extended are discussed here, namely, Teunter (2004) and Oh and Hwang (2006). The structure of these two models is shown in Figures 4.1a and Oh and Hwang (2006) respectively. Teunter (2004) assumes that all returns are of sufficient quality to be recovered to be 'as good as new', and that the purchase of inputs for production happens outwith the model. Oh and Hwang (2006), on the other hand, does consider the purchase of inputs, however it assumes that returns are recovered to make these inputs, rather than to make completed goods. The types of recovery from each of these two papers are combined in the model proposed in this Chapter. Consequently, the model proposed in this chapter has two channels for recovery: recovery to become 'as good as new' and recovery to become input components. Figure 4.2 presents the model used to study the product recovery system in this chapter. The notation used in this model is summarised in Table 4.1. The modelling approach presented below follows the structure used by Teunter (2004).



(a) Product recovery model studied by Teunter (2004)



(b) Product recovery model studied by Oh and Hwang (2006)

Figure 4.1: Two product recovery models in the literature which are combined to form the current model

4.3.2 Model Parameters

In this model, consumer demand occurs constantly at a rate of d and is met by the stock of serviceable inventory. Serviceable inventory is replenished by either the production of new products or the recovery of high quality returned products. Production and

Table 4.1: Nomenclature used in this chapter

Symbol	Description	Conditions
Parameters		
d	demand rate	$d > 0$, $\alpha r \geq d$
f	return fraction	$0 < f < 1$ $f = \beta_H \zeta_H + \beta_L \zeta_L$
fd	return rate	$fd > 0$
r	recovery rate	$r > 0$, $r > d$
p	production rate	$p > 0$, $p > d$
b	arrival rate of orders	$b = \infty$
β_H	proportion of returns of high quality	$0 < \beta_H < 1$ $0 < \beta_H + \beta_L < 1$
β_L	proportion of returns of low quality	$0 < \beta_L < 1$ $0 < \beta_H + \beta_L < 1$
α	proportion of returns of high quality	$0 \leq \alpha \leq 1$ $\alpha = \frac{\beta_H}{\beta_H + \beta_L}$
Cost Parameters		
k_p, k_r, k_b	set up cost per production lot, recovery lot and ordering lot	$k_p, k_r > 0,\ k_b \geq 0$
h_s, h_r, h_c	holding cost per item per time unit for serviceable inventory, returned inventory and component inventory	$h_s, h_r > 0,\ h_c \geq 0$
c_p, c_h, c_l	unit processing cost for production, recovery of high quality items and recovery of low quality items	$c_p, c_h, c_l \geq 0$
c_r, c_b, c_d	unit processing cost for collecting returned products, ordering new components and disposing on unrecovered returns	$c_r, c_b, c_d \geq 0$
Variables		
T	cycle length	
N_p, N_r, N_b	number of production, recovery and ordering lots	$N_p, N_r, N_b \in \mathbb{Z}^+$
Q_p, Q_r, Q_b	size of a production, recovery and ordering lot	$Q_p, Q_r, Q_b \in \mathbb{R}^+$
Strategic Variables		
ζ_H	decision to recover high quality returns (unless stated assume $\zeta_H = 1$)	$\zeta_H + \zeta_L \geq 1$ $\zeta_H = \begin{cases} 1 & \text{high quality recovery} \\ 0 & \text{no high quality recovery} \end{cases}$
ζ_L	decision to recover low quality returns	$\zeta_H + \zeta_L \geq 1$ $\zeta_L = \begin{cases} 1 & \text{low quality recovery} \\ 0 & \text{no low quality recovery} \end{cases}$

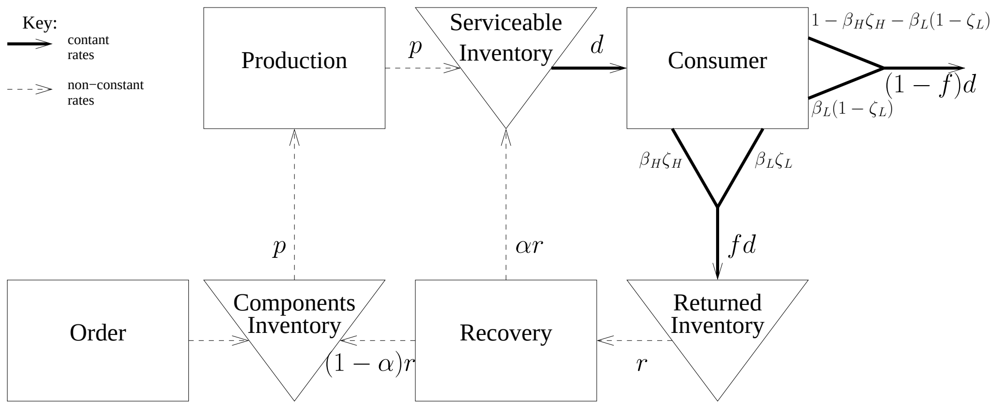


Figure 4.2: Product recovery model studied in this chapter

recovery occur in batches of a fixed size. It is assumed that they are performed on the same facility, so can not happen simultaneously. This is inline with assumptions made in the literature. Production, which occurs at rate p, uses raw materials from the component inventory to produce new products, thus during production, component inventory decreases at a rate of p and serviceable inventory increases at a rate of p. Following the literature, e.g., Dobos and Richter (2006), Teunter (2004), we assume that the production rate is greater than the demand rate $p > d$.

After serviceable goods have been used by the consumer, some are returned to the manufacturer for recovery. The recovery process has two channels, one which results in serviceable goods and one which results in components. It is assumed that with enough effort and expenditure, all returns could be recovered up to the same serviceable standard as newly produced goods. However it is also assumed that the firm has set a threshold which determines the quality level of returns for which it is considered worthwhile to recover to this serviceable standard. In the remainder of this chapter, returns which are above this quality threshold are referred to as 'high quality returns'. High quality returns undergo 'high quality recovery' which brings them up to a serviceable standard and are then sold alongside newly produced goods. Returns which fall below the quality threshold are termed 'low quality returns' and undergo 'low quality recovery' in order salvage components which can be used in production.

The proportion of used goods which are high quality (i.e. above the quality threshold) is β_H and the proportion of used goods which are low quality is β_L. The remaining proportion of used goods $1 - \beta_H - \beta_L$ are not returned, thus are lost from the system. It is assumed that there is always the need for production and recovery, therefore $0 < \beta_H + \beta_L < 1$.

It is assumed that the firm makes a strategic-level decision (outwith the model) about whether to recover low quality returns or dispose of them. Let ζ_H represent the firm's decision regarding the recovery of high quality items and ζ_L represent the firm's decision regarding the recovery of low quality items, where

$$\zeta_H = \begin{cases} 1 & \text{if high quality items are recovered} \\ 0 & \text{if high quality items are not recovered.} \end{cases}$$

$$\zeta_L = \begin{cases} 1 & \text{if low quality items are recovered} \\ 0 & \text{if low quality items are not recovered.} \end{cases}$$

Since recovery must occur, then the following condition must be satisfied:

$$\zeta_H + \zeta_L \geq 1$$

The main purpose of this model is to investigate the cost-effectiveness of recovering low quality items in addition to high quality items, therefore unless otherwise stated, it is assumed that high quality items are always recovered ($\zeta_H = 1$). The proportion of products demanded which are returned to the system is f, where:

$$f = \beta_H \zeta_H + \beta_L \zeta_L.$$

Returns arrive back in the system constantly at a rate of:

$$fd = (\beta_H \zeta_H + \beta_L \zeta_L)d$$

The definitions of β_H, β_L mean that $0 < f < 1$. All items which are returned to the system are recovered, thus if low quality recovery is not performed, then low quality items will not be returned. If the proportion of the returns that are high quality (i.e. above the quality threshold) is denoted by α, $0 \leq \alpha \leq 1$, then:

$$\alpha = \frac{\beta_H \zeta_H}{(\beta_H \zeta_H + \beta_L \zeta)}.$$

During recovery, high quality returns are recovered in order to replenish serviceable inventory. The remaining proportion of returns are low quality so undergo low-quality recovery in order to salvage components. These salvaged components are stored in component inventory before being used in the production of new products.

As mentioned above, the firm may choose to accept the return of high quality items, or both low and high quality items. If only high quality items are returned ($\zeta_H = 1$, $\zeta_L = 0$), then $f = \beta_H$ and $\alpha = 1$. If the recovery of low quality returns is not permitted ($\zeta_L = 0$) then the current model has a similar structure to existing models in the literature, such as Dobos and Richter (2004) and Teunter (2004). Though the case of low-quality only recovery is not explicitly considered here, it is worth mentioning that if the recovery of high quality items is not permitted ($\zeta_H = 0$) then the current model has a similar structure to the model in Oh and Hwang (2006).

While recovery is being performed, returned inventory is recovered at a rate of r, with serviceable inventory being replenished at a rate of αr and component inventory being replenished at a rate of $(1 - \alpha)r$. In literature it is assumed that the demand rate is less that the rate at which serviceable inventory is replenished during recovery, this assumption is made here as well, thus $d < \alpha r$ (Teunter, 2004; Dobos and Richter, 2006). Note that if there is no high quality recovery ($\zeta_H = 0$), then $\alpha r = 0$ and this condition does not hold, since the demand rate must be positive. However, since we assume that there is always high quality recovery $\beta_H = 1$, then throughout this Chapter it is assumed that $d < \alpha r$.

Components are required for the production of new products and can be bought from an external supplier or obtained from low quality recovery. If there are no low quality returns (i.e., $\zeta_L = 0$) then all components required for production must be bought. It is assumed that orders placed to buy components have a lead time of zero, thus arrive immediately after the order has been placed. Since it is assumed that the lead time for buying components is zero, orders only need to be placed the component inventory reaches zero; this will only happen during production. Component inventory decreases constantly during production at a rate of p. During recovery, if low quality recovery is performed, then the component inventory will increase at a rate of $(1 - \alpha)r$.

Referring to Figure 4.2, note that the rates relating to demand d, fd and $(1 - f)d$ are constant, whereas the rate p applies only during production, the rates r, αr, $(1-\alpha)r$ apply only during recovery, and the rate b applies only during ordering.

4.3.3 Model Variables

In this chapter, the product recovery model is studied over a period of time called a cycle. A cycle has a length of T time units and consists of a sequence of N_p production lots of size Q_p, followed by a sequence of N_r recovery lots of size Q_r. A *lot* is the term used to describe a batch or order of a fixed size. A cycle is defined in such a way that one cycle can be followed by another identical cycle, therefore the level of each type of inventory must be the same at the beginning and end of a cycle. For serviceable inventory we assume that this level is zero and for returned and component inventories we assume that this level is nonnegative. The inventory levels will be discussed and presented graphically in the next Section 4.4.

A cycle begins with a production lot. Subsequent production lots begin when serviceable inventory reaches zero. After N_p production lots have occurred, and serviceable inventory reaches zero, the first recovery lot begins. Subsequent recovery lots begin when serviceable inventory reaches zero. Components only need to be bought during production and when component inventory reaches zero. During the sequence of production lots, components are bought in N_b lots, each of size Q_b.

There are no backorders or lost sales in this model, therefore during a cycle the total number of products demanded must equal the number of products produced plus the number of high quality returned products which are recovered, hence:

$$dT = Q_p N_p + \alpha Q_r N_r \tag{4.1}$$

Once used goods have been returned to the system they cannot be discarded. This assumption, of not allowing the disposal of returns, is consistent with current governmental legislation discouraging disposal to landfills. It is also consistent with the work of Teunter and Vlachos (2002), who found that including a disposal option was only profitable under certain conditions. Therefore during a cycle, the total number of products returned must equal the total number of products recovered, hence:

$$Q_r N_r = f dT \tag{4.2}$$

The components used in production are sourced either by buying new components, or by recovering low quality returns. It is assumed that components cannot be backordered and that the number of components in stock at the beginning and the end of a cycle must be the same. One component is required for each good that is produced. Therefore, the total number of goods produced during a cycle, and thus the total number of components used, must equal the number bought plus the number obtained from the recovery of low quality returns, hence:

$$Q_p N_p = Q_b N_b + (1 - \alpha) Q_r N_r \tag{4.3}$$

These equalities can be used to express the total number of items produced during a cycle $Q_p N_p$, in terms of the total number of items recovered $Q_r N_r$ and the total number of components bought $Q_b N_b$. Using these equations Q_r can be defined in terms of Q_p, N_p, N_r, and similarly Q_b can be defined in terms of Q_p, N_p, N_b. After performing some algebraic manipulations (the details of which can be found in Appendix A.1.1), the following result is obtained:

$$Q_p N_p = \frac{Q_r N_r (1 - \alpha f)}{f} = \frac{Q_b N_b (1 - \alpha f)}{(1 - f)} \tag{4.4}$$

In a similar fashion the cycle length T can be defined in terms of Q_p and N_p, in terms of Q_r and N_r or in terms of Q_b and N_b. After performing some algebraic manipulations, the following result is obtained:

$$T = \frac{Q_p N_p}{d(1 - \alpha f)} = \frac{Q_r N_r}{f d} = \frac{Q_b N_b}{d(1 - f)} \tag{4.5}$$

There are seven variables in this model: the lot sizes and the number of lots for production Q_p, N_p, recovery Q_r, N_r, and buying components Q_b, N_b, respectively, and the cycle length T. However as shown by equations (4.4) and (4.5), these variables are not all independent of each other. In this model the production lot size Q_p and the number of lots per cycle for production, recovery and buying components (N_p, N_r, N_b) are selected as decision variables. The reason for selecting these variables to be the decision variables is twofold. Firstly, production is deemed to be the primary function of the firm, thus specifying the values of the variables in terms of the production lot size and the number of lots is most logical. Secondly, within the literature the lot size and the numbers of lots per cycle are often used as the decision variables.

In practice both the lot sizes and the numbers of lots should be positive integers. However, as in the previous literature, this restriction is only applied to the numbers of lots per cycle (Teunter, 2004). It is assumed that the lot sizes will be sufficiently large that rounding the values to the nearest integer will not greatly affect the model. However, the numbers of each type of lot, on the other hand, are likely to be comparatively small and therefore rounding the values to the nearest integer could have a significant effect on the total cost. Therefore, in this model it is assumed that Q_p, Q_r, Q_b, T are positive real numbers and that N_p, N_r, N_b are positive integers.

The nomenclature used in this model is summarised in Table 4.1 on page 71.

4.4 Inventory Levels Across a Cycle

Knowing the levels of the serviceable, returned and component inventories at different points in time across a cycle is important for two reasons. Firstly, the inventory levels are important for ensuring that there is sufficient stock to meet demand and secondly, because these levels determine the holding cost. In this section two cases are considered. Under the first case it is assumed that there is one of each type of lot per cycle, i.e. $N_p = N_r = N_b = 1$. Restricting the number of each type of lot to one, simplifies the model significantly. The second case relaxes this assumption and allows there to be at least one of each type of lot per cycle, i.e. $N_p, N_r, N_b \geq 1$. These two cases will be considered in Sections 4.4.1 and 4.4.2 respectively.

The area under the inventory level graphs, discussed in Section 4.4.2, is also important because it represents the amount of inventory that is held and how long it is held for, and will be used in the calculation of the holding costs in Section 4.5.

4.4.1 Case 1: $N_p = N_r = N_b = 1$

The levels of the serviceable, returned and component inventories across a cycle are shown in Figure 4.3 for a cycle with one production lot $N_p = 1$, one recovery lot $N_r = 1$ and one buying lot $N_b = 1$. The rates at which the inventory levels change are also shown in this Figure.

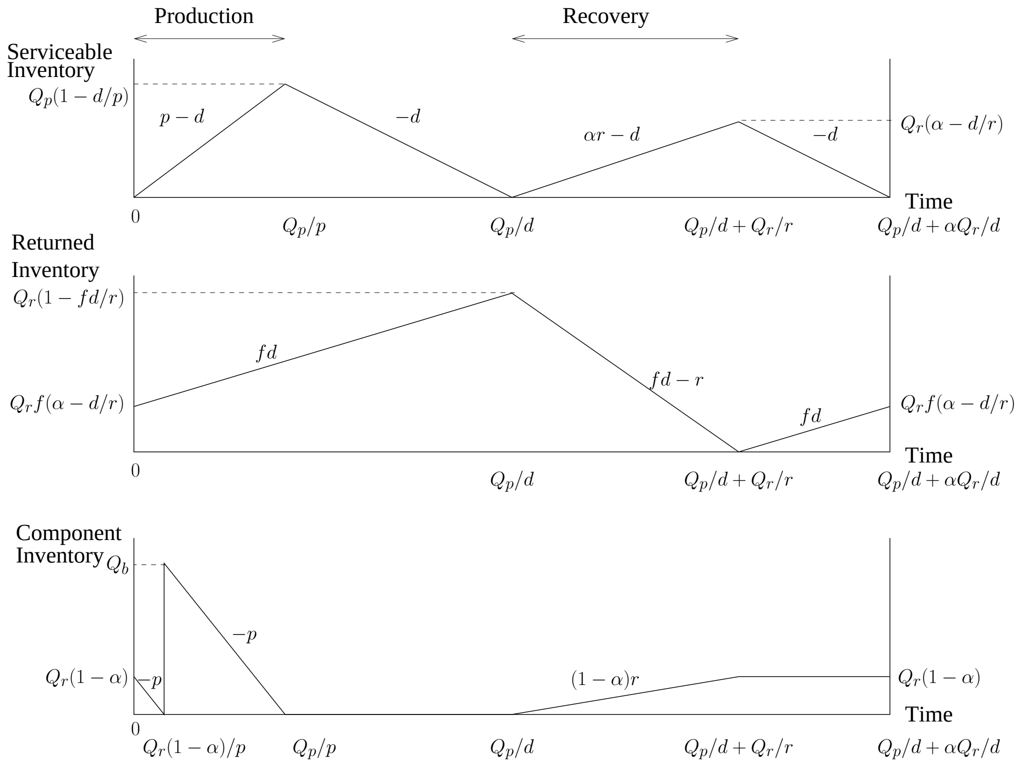


Figure 4.3: Inventory levels under a policy with $N_p = N_r = N_b = 1$

Serviceable Inventory

At the start of a cycle, the level of serviceable inventory is zero therefore a production lot begins immediately. During the production lot, which lasts for Q_p/p time units, the serviceable inventory level increases at a rate of $(p - d)$. At the completion of the production lot there are $Q_p(1 - d/p)$ units of serviceable inventory. The level of serviceable inventory then decreases according to the demand rate d until the inventory level reaches zero ($Q_p(1/d - 1/p)$ time units later), at which point recovery begins.

During recovery, which lasts for Q_r/r time units, the serviceable inventory level increases at a rate of $(\alpha r - d)$. At the completion of the recovery lot there are $Q_r(\alpha - d/r)$ units of serviceable inventory. The inventory level then decreases at a rate of d for $Q_r(\alpha/d - 1/r)$ time units until the inventory level reaches zero, at which point the cycle ends.

Returned Inventory

At the start of a cycle the level of returned inventory is $Q_r f(\alpha - d/r)$. This level is equal to the number of products returned during the $Q_r(\alpha/d - 1/r)$ time units between the end of the recovery lot and the end of the cycle. At the beginning of a cycle the returned inventory increases at a rate of fd for Q_p/d time units, reaching a level of $Q_r(1 - fd/r)$. When recovery begins, the returned inventory begins to decrease at a rate of $(fd - r)$. Recall that equation (4.2) requires that the number of items recovered during a cycle must equal the number of items returned during a cycle. Therefore, the returned inventory will reach zero at the completion of the recovery lot. At the completion of recovery, the level of returned inventory increases again at a rate of fd and reaches a level of $Q_r f(\alpha - d/r)$ by the end of the cycle.

Component Inventory

At the start of a cycle the level of component inventory is $Q_r(1-\alpha)$. This level is equal to the number low quality returns recovered during a cycle. The components which are in stock at the beginning of the cycle are used up before new components are bought. Each cycle begins with a production lot, therefore the level of component inventory initially decreases according to the production rate p, reaching zero at time $Q_r(1 - \alpha)/p$. At this point, Q_b components are bought and arrive instantly. The number of components required during a cycle, as defined in equation (4.3), is exactly equal to the number of components bought plus the number attained from low quality recovery. All components bought will be consumed during production, thus the component inventory will reach zero at the end of production. During recovery the level of component inventory increases at a rate of $(1 - \alpha)r$ up to a level of $Q_r(1 - \alpha)$, at which it remains until the end of the cycle. Note that if there is no low quality recovery ($\zeta_L = 0$), then $\alpha = 1$ and so components must be bought at the start of the cycle, i.e. at time $Q_r(1 - 1)/p = 0$.

4.4.2 Case 2: $N_p, N_r, N_b \geq 1$

In this section we discuss the levels of the serviceable, returned and component inventories across a cycle with more than one of each type of lot. The shape of the inventory level graphs for this general case, in particular the component inventory level, depend on the numbers of each type of lot and some parameter values. An example of the inventory levels for a system with $N_p = 4$ production lots, $N_r = 3$ recovery lots and $N_b = 2$ ordering lots per cycle is shown in Figure 4.4.

In this section the shape of and area under the inventory level graphs will be discussed. As mentioned above, the area under the graphs is important because it will be used in the calculation of the holding costs, in Section 4.5. Further details of the formulae presented in this section are available in Appendix A.2.

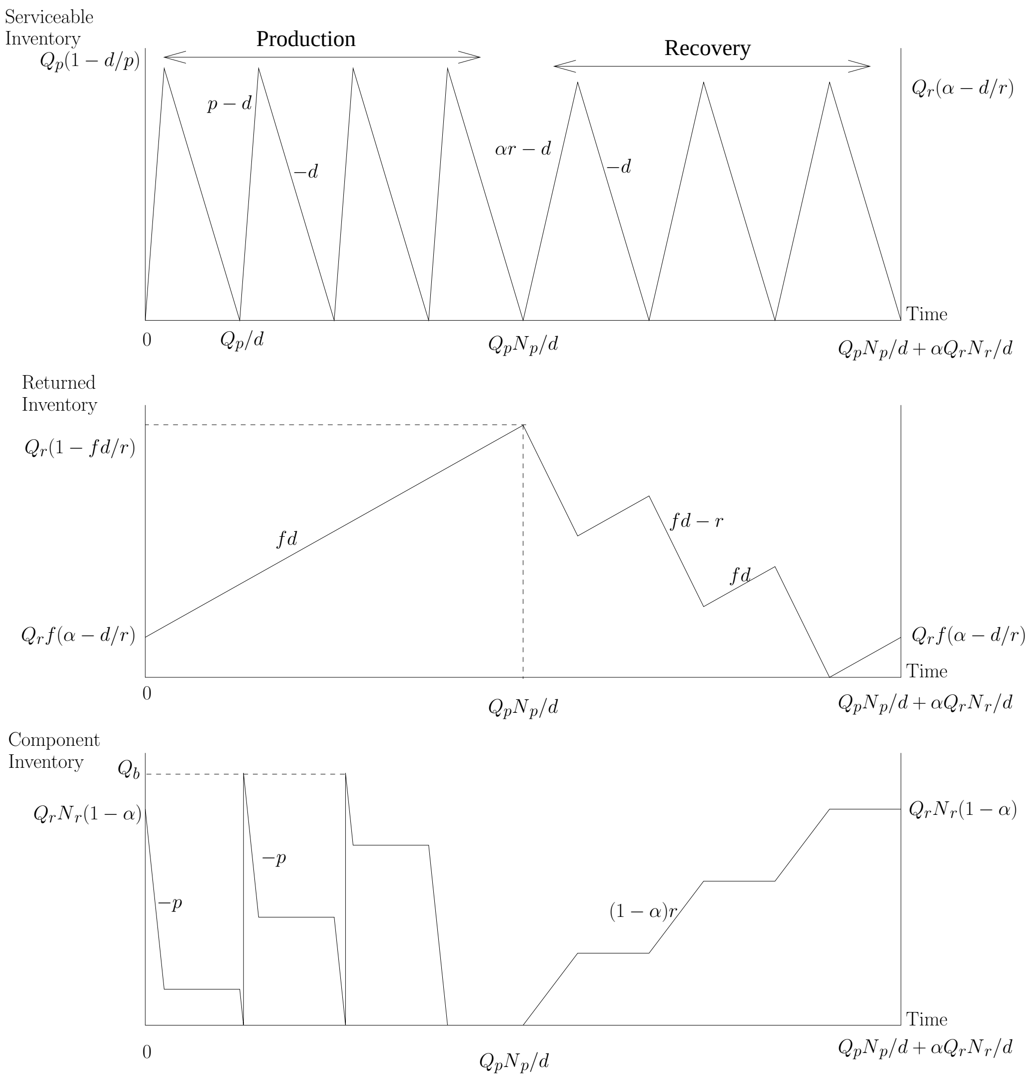


Figure 4.4: An example of the inventory levels under a policy with $N_p = 4, N_r = 3, N_b = 3$

Serviceable Inventory

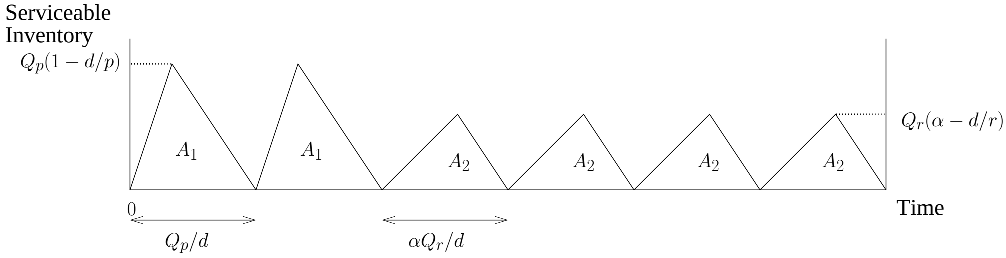


Figure 4.5: Serviceable inventory levels under a policy with $N_p = 2$ production lots, $N_r = 4$ recovery lots

Figure 4.5 shows the levels of serviceable inventory across a cycle with $N_p = 2$ production lots and $N_r = 4$ recovery lots. The number of buying lots N_b, is not mentioned here as it does not affect the shape of the serviceable inventory graph. The area under the graph is made up of N_p triangles A_1 and N_r triangles A_2, where:

Area of $A_1 = Q_p(1-d/p)(Q_p/d)(1/2)$

Area of $A_2 = Q_r(\alpha-d/r)(\alpha Q_r/d)(1/2)$

The total area under of the graph is:

$$A_s = N_pA_1 + N_rA_2 = \frac{Q_p^2N_p}{2}\left(\frac{1}{d}-\frac{1}{p}\right) + \frac{Q_r^2N_r\alpha}{2}\left(\frac{\alpha}{d}-\frac{1}{r}\right) \tag{4.6}$$

Using equations (4.4) and (4.5) and performing some simple algebra, the details of which can be found in Appendix A.2, the total area under the graph can be written in terms of Q_p and T:

$$A_s = \frac{TQ_p}{2}\left((1-\alpha f)\left(1-\frac{d}{p}\right) + \frac{N_p\alpha f^2}{N_r(1-\alpha f)}\left(\alpha-\frac{d}{r}\right)\right) \tag{4.7}$$

Returned Inventory

The level of returned inventory across a cycle with $N_r = 4$ recovery lots is presented in Figure 4.6. The number of production lots N_p and the number of ordering lots N_b are not mentioned here as they do not affect the shape of the returned inventory graph.

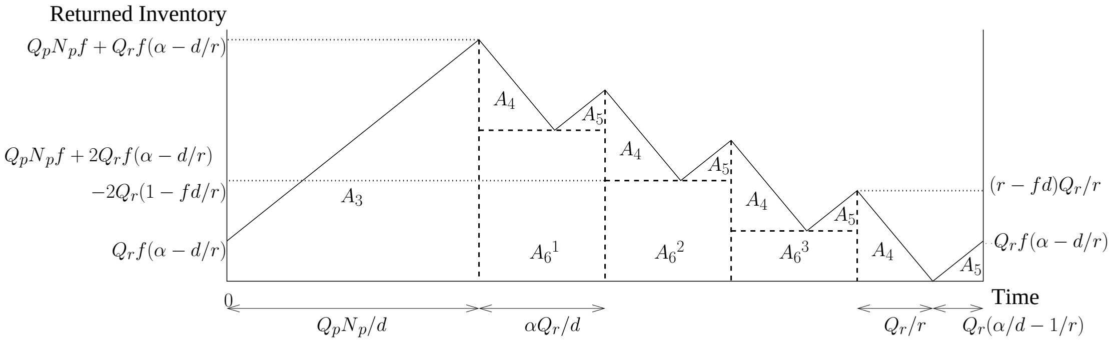


Figure 4.6: Returned inventory levels under a policy with $N_r = 4$ recovery lots

Following Choi et al. (2007), the area under the graph can be calculated by splitting it into a series of shapes.

The area under the graph consists of a trapezium A_3, N_r triangles A_4, N_r triangles A_5 and the rectangles A_6^i for $i = 1, \ldots N_r$. The area of the triangles A_3, A_4, A_5 are obvious from the graph. The height of rectangles is not quite as obvious. The height of rectangle A_6^i is the initial inventory plus the amount returned during production, less the amount recovered during i recovery lots plus the amount returned during $(i-1)$ recovery lots. Let A_6 denote the sum of the rectangles A_6^i for $i = 1, \ldots, N_r$. Note that the height of the N_r^{th} rectangle is 0, however the sum over N_r is still used as this aids the algebraic manipulations. The areas are defined as follows:

Area of $A_3 = (Q_pN_p/d)[Q_rf(\alpha - d/r) + Q_pN_pf + Q_rf(\alpha - d/r)](1/2)$

Area of $A_4 = (Q_r(r - fd)/r)(Q_r/r)(1/2)$

Area of $A_5 = fdQ_r(\alpha/d - 1/r)Q_r(\alpha/d - 1/r)(1/2)$

Area of $A_6^i = (\alpha Q_r/d)[Q_pN_pf + Q_rf(\alpha - d/r) + (i-1)Q_rf(\alpha - d/r) - i(r - fd)Q_r/r]$

Area of $A_6 = \sum_{i=1}^{N_r} A_6^i$

The total area under of the graph of the returned inventory is:

$$A_r = A_3 + N_rA_4 + N_rA_5 + A_6 \tag{4.8}$$

Using equations (4.4) and (4.5) the total area under the graph A_r can be represented in terms of Q_p and T. After some algebraic manipulations, the details of which can be found in Appendix A.2, the area under the graph can be expressed as follows:

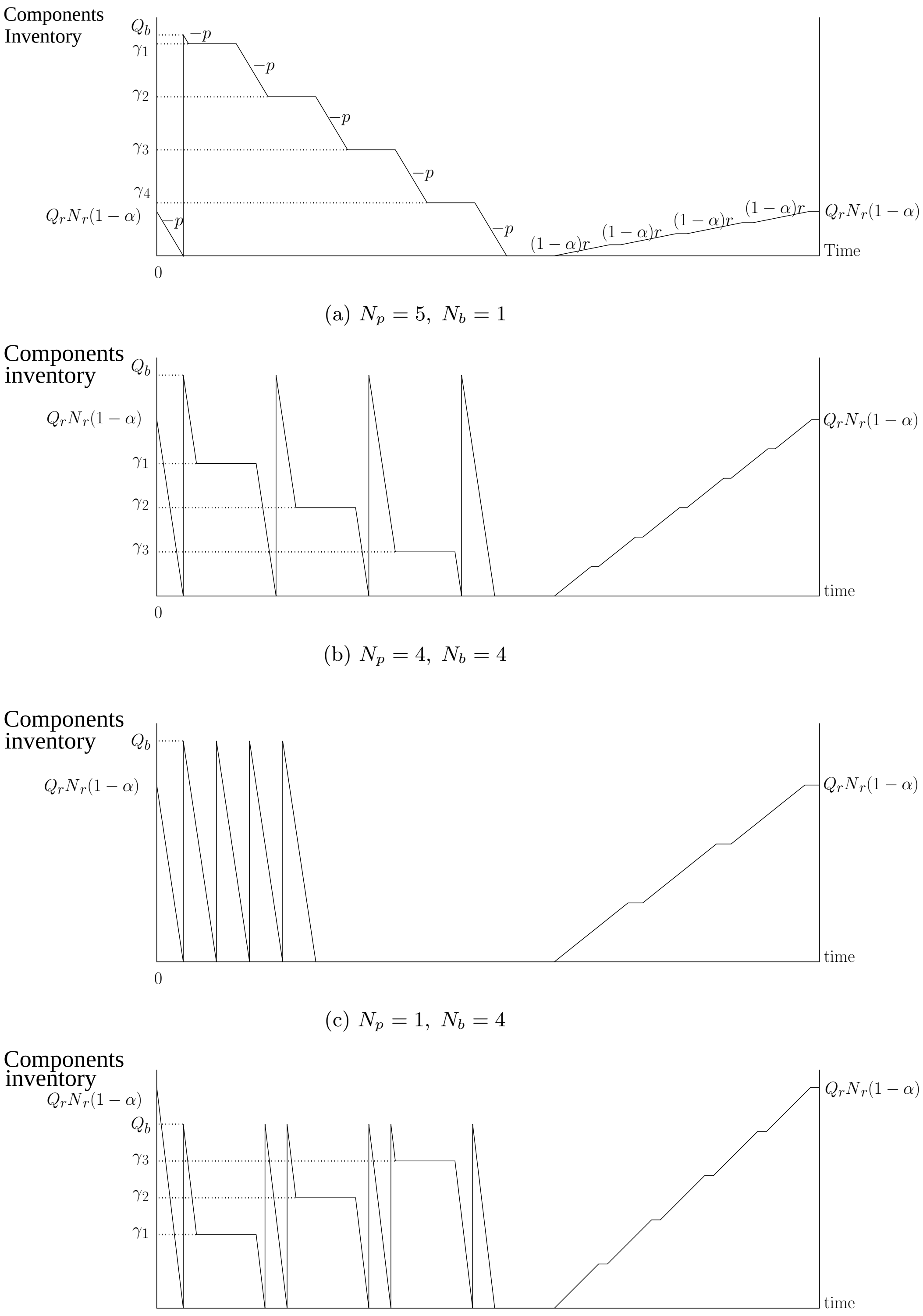
$$A_r = \frac{TQ_pN_pf^2}{2(1-\alpha f)}\left(\frac{1}{N_r}\left(\alpha - \frac{d}{r}\right) + \frac{1}{f} - \alpha\right) \tag{4.9}$$

Component Inventory

The shape of the component inventory level graph during production depends on the relationship between N_p, N_b and some of the parameters. In particular, for a cycle with $N_p, N_b \geq 1$, the shape of the graph depends on when components are bought, in relation to the production lots and how much inventory is still in stock when each of the production lots end.

Figure 4.7 shows four examples of the component inventory level graphs. Figure 4.7a shows the component inventory level for a system with $N_p = 5$ and $N_b = 1$. After each production lot there is a period when the serviceable inventory that has just been produced is used up. During this time there is no change in the component inventory level, i.e., the graph is horizontal. Compare the shape of this graph to the other graphs in Figure 4.7. Figure 4.7b shows the component inventory level for a system with $N_p = 4$ and $N_b = 4$. Figure 4.7c shows the component inventory level for a system with $N_p = 1$ and $N_b = 4$. Figure 4.7d shows the component inventory level for a system with $N_p = 4$ and $N_b = 6$. The shape of the inventory levels is important because it determines the inventory levels across a cycle, and thus the holding costs incurred.

In order to analyse the level of component inventory across a cycle, the cycle will be split into three parts: during the production lots, between the production lots and during recovery. During the first Q_pN_p/d time units of a cycle, there are times when production is being performed, and times when the goods that have been produced are being used up. The times during which production is being performed and serviceable inventory is increasing at a rate of $(p - d)$ and component inventory is decreasing at a rate of p, are referred to as being *during production lots*. The times during the first Q_pN_p/d time units of a cycle when production is not being performed and serviceable inventory is decreasing at a rate of d and component inventory is constant, are referred to as being *between production lots*.



(a) $N_p = 5,\ N_b = 1$

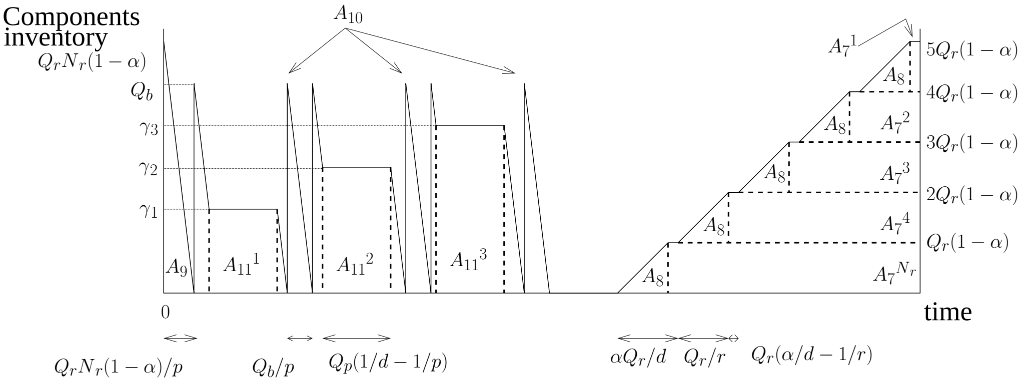
(b) $N_p = 4,\ N_b = 4$

(c) $N_p = 1,\ N_b = 4$

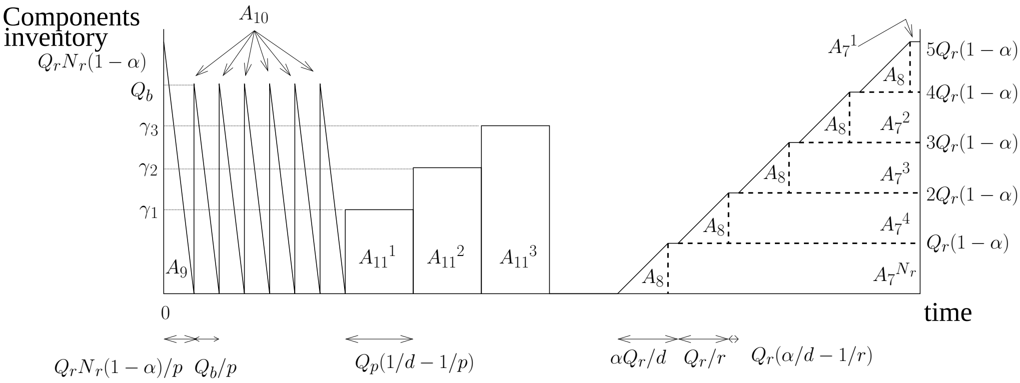
(d) $N_p = 4,\ N_b = 6$

Figure 4.7: Component inventory levels under policies with a variety of values for N_p and N_b

The remaining part of the cycle, from time Q_pN_p/d until the end of the cycle will be referred to as being *during recovery*. Figure 4.8 shows the component inventory levels across a cycle with $N_p = 4$ production lots, $N_r = 5$ recovery lots and $N_b = 6$ buying lots. The analyses which follow will refer to this figure, however they are applicable to the general case of any number of production and buying lots.



(a) Component inventory levels over a cycle



(b) Rearranged parts of the component inventory level over a cycle

Figure 4.8: Component inventory levels under a policy with $N_p = 4$, $N_r = 5$ and $N_b = 6$

During Production Lots. At the beginning of the cycle, the components from low quality recovery are used in production. These components are represented by the triangle A_9. After these components have been used, new components are bought. If low quality recovery is not performed ($\zeta_L = 0$) then all returns recovered are high quality ($\alpha = 1$) and thus the area of triangle A_9 is zero and new components are bought immediately (at time 0).

Components are bought in lots of size Q_b. While production is being performed, these components are used at a rate of p. For the system in Figure 4.8, the first three production lots end while the component inventory is greater than zero. During the period between the production lots, the component inventory level does not change, (represented by the rectangles A^i_{11}). When the next production lot starts, the inventory begins decreasing at a rate of p again. Ignoring for a moment, the period between production lots, the time taken to use up the components from one buying lot is Q_b/p. Therefore the area which corresponds to the inventory from one order can be represented by a triangle of height Q_b and width Q_b/p. For this system, however, these triangles have been split by some rectangular sections A^i_{11}. To help show this relationship, we have rearranged the pieces in Figure 4.8a to create the graph in Figure 4.8b. Therefore, the area under the graph during the production lots consists of triangle A_9 and N_b triangles A_{10}, as shown in these graphs:

Area of $A_9 = N_r Q_r (1-\alpha) N_r Q_r (1-\alpha)/p(1/2)$

Area of $A_{10} = Q_b Q_b/p(1/2)$

The total area under the graph during production A^{dp}_c is:

$$\begin{aligned} A^{dp}_c &= A_9 + N_b A_{10} \\ &= \frac{N_r^2 Q_r^2 (1-\alpha)^2}{2p} + \frac{N_b Q_b^2}{2p} \end{aligned}$$

Using equations (4.4) and (4.5) it can be shown that the total area under the graph during production is:

$$A^{dp}_c = T Q_p N_p \frac{d}{2p(1-\alpha f)} \left(f^2 (1-\alpha)^2 + \frac{(1-f)^2}{N_b} \right) \tag{4.10}$$

Between Production Lots. The period of time between production lots is represented in Figure 4.8a by the rectangles A^i_{11}, $i = 1, \ldots, N_b$. The width of these rectangles is $Q_p(1/d - 1/p)$. The height of rectangle i is the inventory level at the end of production lot i, which we denote by γ_i. Note that the model conditions ensure that $\gamma_{N_p} = 0$ (see equation (4.3)). The number of components needed for i production lots is $Q_p i$; of these $Q_r N_r (1-\alpha)$ are supplied from low quality recovery. This means that by the end of the i^{th} production lot, at least $Q_p i - Q_r N_r (1-\alpha)$ components must have been bought. Since components are bought in lots of Q_b, the number of lots required to buy the required number of components can be found by dividing the required number of components by the lot size Q_b and rounding up to the nearest integer, that is:

$$\left\lceil \frac{iQ_p - Q_r N_r(1-\alpha)}{Q_b} \right\rceil$$

where the operator $\lceil \frac{x}{y} \rceil$ rounds the ratio $\frac{x}{y}$ up to the nearest non-negative integer. Using equations (4.4) and (4.5), this expression can be written in terms of the numbers of production and buying lots N_p and N_b. Let $M(i, N_p, N_b)$ denote the number of buying lots to have occurred by the end of production lot i.

$$M(i, N_p, N_b) = \left\lceil \frac{iN_b(1-\alpha f)}{(1-f)N_p} - \frac{N_b f(1-\alpha)}{(1-f)} \right\rceil \tag{4.11}$$

If there is only one production lot $N_p = 1$, then all N_b buying lots must have occurred by the end of the first production lot, thus $M(1, 1, N_b) = N_b$ If there is only one buying lot $N_b = 1$, and that buying lot occurs during the first production lot (i.e. $Q_r N_r(1-\alpha) < Q_p$), then $M(i, N_p, 1) = 1$ for $(1 \leq i \leq N_p)$. This function $M(i, N_p, N_b)$ can be used to define the component inventory level at the end of production lot i. The component inventory level at the end of production lot i is equal to the number of components from low quality recovery plus the number of components ordered by the end of production lot i, less the number of goods produced by the end of production lot i. Let the component inventory level at the end of production lot i be denoted by γ_i, where:

$$\gamma_i = Q_r N_r(1-\alpha) + M(i, N_p, N_b)Q_b - iQ_p \tag{4.12}$$

As shown in Figures 4.8a and 4.8b, the area of rectangle A_{11}^i is:

$$\text{Area of } A_{11}^i = Q_p(1/d - 1/p)\gamma_i = Q_p(1/d - 1/p)[Q_r N_r(1-\alpha) + M(i, N_p, N_b)Q_b - iQ_p]$$

Using equations (4.4) and (4.5) it can be shown that the total area under the graph between production in terms of Q_p and T is:

$$\begin{aligned} A_c^{bp} &= \sum_{i=1}^{N_p} A_{11}^i \\ &= TQ_p N_p \frac{p-d}{p} \left(f(1-\alpha) + \frac{(1-f)}{N_b N_p} \sum_{i=1}^{N_p} M(i, N_p, N_b) - \frac{(1-\alpha f)}{2} - \frac{(1-\alpha f)}{2N_p} \right) \end{aligned} \tag{4.13}$$

Note that if $N_p = 1$, then this area A_c^{bp} reduces to zero.

During Recovery Lots. Referring again to Figure 4.8a, it can be seen that during recovery, the area under the graph is made up of rectangles A_7^i, for $i = 1, \ldots, N_r$, and N_r triangles A_8, where:

$$\begin{aligned}
\text{Area of } A_7^i &= Q_r(1-\alpha)[Q_r(\alpha/d - 1/r) + (i-1)Q_r\alpha/d] \\
\text{Area of } A_7 &= \sum_{i=1}^{N_r} A_7^i = Q_r(1-\alpha)\sum_{i=1}^{N_r}[Q_r\alpha/d - Q_r/r + iQ_r\alpha/d - Q_r\alpha/d] \\
\text{Area of } A_8 &= Q_r(1-\alpha)(Q_r/r)(1/2)
\end{aligned}$$

Using equations (4.4) and (4.5), it can be shown that the total area under the component inventory level graph during and between recovery lots, in terms of Q_p and T is:

$$A_c^{dr} = A_7 + N_r A_8 = TQ_pN_p\frac{f^2(1-\alpha)}{2(1-\alpha f)}\left(\frac{1}{N_r}\left(\alpha - \frac{d}{r}\right) + \alpha\right) \tag{4.14}$$

Across a cycle. The total area under the component inventory level graph can be expressed in terms of Q_p and T can be obtained by combining equations (4.10), (4.13) and (4.14). Further simplification of some parts of A_c is not performed at this stage as it will be helpful in the coming sections to be able to identify the origin of the negative terms in this expression.

$$\begin{aligned}
A_c = TQ_p\Bigg(&N_p\left[\left(1-\frac{d}{p}\right)\left(f(1-\alpha) - \frac{(1-\alpha f)}{2}\right)\right] + \frac{1}{N_b}\left[\frac{(p-d)(1-f)}{p}\sum_{i=1}^{N_p} M(i, N_p, N_b)\right] \\
&- \frac{(1-\alpha f)(p-d)}{2p} + N_p\left[\frac{f^2(1-\alpha)}{2(1-\alpha f)}\left(\frac{d\,(1-\alpha)}{p} + \alpha\right)\right] \\
&+ \frac{N_p}{N_b}\left[\frac{d(1-f)^2}{2p(1-\alpha f)}\right] + \frac{N_p}{N_r}\left[\frac{f^2(1-\alpha)(\alpha r - d)}{2r(1-\alpha f)}\right]\Bigg)
\end{aligned} \tag{4.15}$$

4.5 Cost Function

Fixed, unit and holding costs are incurred in this model. A summary of the cost parameters included in the model are presented in Table 4.1 (page 71). In this section the total cost function per time unit is derived (Section 4.5.1) and then the derivation is verified (Section 4.5.2).

4.5.1 Derivation of the Total Cost Function

In this section the fixed, unit and holding costs incurred in this model are discussed and the total cost per time unit is derived.

Setup Costs

Setup costs of k_p, k_r, and k_b are incurred for each production, recovery and buying lot respectively. The setup costs for production k_p and recovery k_r must be positive, and for buying components k_b must be nonnegative. The different requirements for k_b allow comparisons to be made with previous models in the literature which do not include the buying of components. The total setup costs per cycle are $k_pN_p + k_rN_r + k_bN_b$ and using equation (4.5) the total setup costs per time unit are

$$C_K(Q_p, N_p, N_r, N_b) = \frac{d(1-\alpha f)}{Q_pN_p}\Big(k_pN_p + k_rN_r + k_bN_b\Big) \tag{4.16}$$

Unit Processing Costs

Processing costs are incurred for each product which is produced or recovered. The nonnegative per item processing costs for producing new products is c_p, for recovering high quality returns is c_h and for recovering low quality returns is c_l. The purchase of new components incurs a nonnegative per item cost of c_b and the return of used products incurs a nonnegative per item cost of c_r. Any returns which are not recovered incur a per unit disposal cost of c_d. This disposal cost could be the actual cost of disposing of the item, or it could represent the cost of "taking responsibility" for what happens to the goods that they have produced, for example a penalty fee imposed by the government. The total processing costs per cycle are $c_pQ_pN_p + c_r\alpha Q_rN_r + c_l(1-\alpha)Q_rN_r + c_rQ_rN_r + c_bQ_bN_b + c_d(Q_pN_p + \alpha Q_rN_r)(1-f)$ and using equation (4.5) the total processing costs per time unit are:

$$C_P = c_p(1-\alpha f)d + c_h\alpha fd + c_l(1-\alpha)fd + c_rfd + c_bd(1-f) + c_d(1-f)d \tag{4.17}$$

Note that, when expressed per time unit, this cost is independent of the variables.

Holding Costs

Holding costs are incurred for each type of inventory on a per item, per time-unit basis. The holding costs can be calculated using the average inventory levels across a cycle. The average inventory level is equal to the area under the inventory level graphs which were calculated in Section 4.4. The per item, per time unit holding costs for serviceable inventory h_s and returned inventory h_r must be positive, and for component inventory h_c must be nonnegative. As was the case for the setup costs, the different requirements for h_c allow comparisons to be made with previous models in the literature.

Serviceable Inventory. The area under the serviceable inventory graph A_s is specified by equation (4.7). Using this area, the cost per time unit of holding serviceable inventory is:

$$H_s = h_s \frac{A_s}{T} = \frac{h_s Q_p}{2} \left((1-\alpha f) \left(1 - \frac{d}{p}\right) + \frac{N_p \alpha f^2}{N_r (1-\alpha f)} \left(\alpha - \frac{d}{r}\right) \right) \tag{4.18}$$

Returned Inventory. The area under the returned inventory graph A_r is specified by equation (4.9). Using this area, the cost per time unit of holding returned inventory is:

$$H_r = h_r \frac{A_r}{T} = \frac{h_r Q_p N_p f^2}{2(1-\alpha f)} \left(\frac{1}{N_r} \left(\alpha - \frac{d}{r}\right) + \frac{1}{f} - \alpha \right) \tag{4.19}$$

Component Inventory. The area under the component inventory graph A_c is specified by equation (4.15). Recall that $M(i, N_p, N_b)$ is the number of buying lots that will have occurred by the end of production lot i. Using the area A_c and this function $M(i, N_p, N_b)$ the cost per time unit of holding component inventory is:

$$\begin{aligned}
H_c &= h_c \frac{A_c}{T} \\
&= h_c Q_p \Bigg(N_p \left[\left(1 - \frac{d}{p}\right) \left(f(1-\alpha) - \frac{(1-\alpha f)}{2} \right) \right] \\
&+ \frac{1}{N_b} \left[\frac{(p-d)(1-f)}{p} \sum_{i=1}^{N_p} M(i, N_p, N_b) \right] \\
&\quad - \frac{(1-\alpha f)(p-d)}{2p} + N_p \left[\frac{f^2(1-\alpha)}{2(1-\alpha f)} \left(\frac{d(1-\alpha)}{p} + \alpha \right) \right] \\
&\quad + \frac{N_p}{N_b} \left[\frac{d(1-f)^2}{2p(1-\alpha f)} \right] + \frac{N_p}{N_r} \left[\frac{f^2(1-\alpha)(\alpha r - d)}{2r(1-\alpha f)} \right] \Bigg)
\end{aligned} \tag{4.20}$$

Total Cost

The total cost per time unit is the sum of equations (4.16), (4.17), (4.18), (4.19) and (4.20). The algebraic manipulations required to attain the total cost function in equation (4.21) are presented in available in Appendix A.3.1. The total cost per time unit is:

$$\begin{aligned}
TC = {} & c_r f d + c_b d(1-f) + c_p(1-\alpha f)d + c_h \alpha f d + c_l(1-\alpha) f d + c_d(1-f)d \\
& + \frac{d(1-\alpha f)}{Q_p N_p}(k_p N_p + k_r N_r + k_b N_b) + Q_p \Bigg(\frac{h_s(1-\alpha f)(p-d)}{2p} \\
& + N_p \left[\frac{h_r f}{2} + \frac{h_c f^2(1-\alpha)}{2(1-\alpha f)} \left(\frac{d\,(1-\alpha)}{p} + \alpha \right) \right] \\
& + \frac{N_p}{N_r} \left[\frac{f^2(\alpha r - d)}{2r(1-\alpha f)} \left(h_s \alpha + h_r + h_c(1-\alpha) \right) \right] \\
& + \frac{N_p}{N_b} \left[\frac{h_c d(1-f)^2}{2p(1-\alpha f)} \right] + N_p \left[h_c \left(1 - \frac{d}{p} \right) \left(f(1-\alpha) - \frac{(1-\alpha f)}{2} \right) \right] \\
& + \frac{1}{N_b} \left[\frac{h_c(p-d)(1-f)}{p} \sum_{i=1}^{N_p} M(i, N_p, N_b) \right] - \frac{h_c(1-\alpha f)(p-d)}{2p} \Bigg)
\end{aligned} \tag{4.21}$$

The total cost function is a function of the four decision variables Q_p, N_p, N_r, N_b. Following Konstantaras and Papachristos (2008b), the total cost function is rewritten by grouping constant terms together in order to more clearly see the relationship between the decision variables. Let:

$$C_P = c_p(1-\alpha f)d + c_h \alpha f d + c_l(1-\alpha) f d + c_r f d + c_b d(1-f) + c_d(1-f)d$$

$$\bar{K}(N_p, N_r, N_b) = \frac{d(1-\alpha f)}{N_p}(k_p N_p + k_r N_r + k_b N_b)$$

$$V = \frac{h_s(1-\alpha f)(p-d)}{2p}$$

$$W = \frac{h_r f}{2} + \frac{h_c f^2(1-\alpha)}{2(1-\alpha f)} \left(\frac{d\,(1-\alpha)}{p} + \alpha \right)$$

$$X = \frac{f^2(\alpha r - d)}{2r(1-\alpha f)}(h_s \alpha + h_r + h_c(1-\alpha))$$

$$Y = \frac{h_c d(1-f)^2}{2p(1-\alpha f)}$$

$$Z_1 = \frac{h_c(1-\alpha f)(p-d)}{2p}$$

$$Z_2 = h_c\left(1-\frac{d}{p}\right)\left(f(1-\alpha)-\frac{(1-\alpha f)}{2}\right)$$

$$Z_3 = \frac{h_c(p-d)(1-f)}{p}$$

$$M(i,N_p,N_b) = \left\lceil \frac{iN_b(1-\alpha f)}{(1-f)N_p} - \frac{N_b f(1-\alpha)}{(1-f)} \right\rceil$$

The total cost can now be rewritten as:

$$\begin{aligned} TC(Q_p,N_p,N_r,N_b) = C_P + \frac{\bar{K}(N_p,N_r,N_b)}{Q_p} + Q_p\Bigg(V + N_pW + \frac{N_p}{N_r}X \\ + \frac{N_p}{N_b}Y - Z_1 + N_pZ_2 + \frac{1}{N_b}Z_3\sum_{i=1}^{N_p} M(i,N_p,N_b)\Bigg) \end{aligned} \tag{4.22}$$

Before proceeding to the next section the sign of the functions C_P, $\bar{K}(N_p,N_r,N_b)$, V, W, X, Y, Z_1, Z_2, Z_3 will be discussed. First, recall from Section 4.3 the following requirements for the model parameters:

α	$0 \le \alpha \le 1,\ \ 0 \le (1-\alpha) \le 1$
f	$0 < f < 1,\ \ 0 < (1-f) < 1$
αf	$0 \le \alpha f < 1,\ \ 0 < (1-\alpha f) \le 1$
p, d	$p, d > 0,\ \ p > d$
$\alpha r - d$	$\alpha r \le d$
$c_p, c_h, c_l, c_d, c_r, c_b$	$c_p, c_h, c_l, c_d, c_r, c_b \ge 0$
k_p, k_r, k_b	$k_p, k_r > 0, k_b \ge 0$
h_s, h_r, h_c	$h_s, h_r > 0, h_c \ge 0$
N_p, N_r, N_b	$N_p, N_r, N_b > 0$

These conditions mean that:

$$
\begin{aligned}
C_P &\geq 0 \\
\bar{K}(N_p, N_r, N_b) &> 0 \\
V &> 0 \\
W &> 0 \\
X &> 0 \\
Y &\geq 0 \\
Z_1 &\geq 0 \\
Z_2 &\begin{cases} \geq 0 & \text{if } 2f - \alpha f - 1 \geq 0 \\ < 0 & \text{if } 2f - \alpha f - 1 < 0 \end{cases} \\
Z_3 &\geq 0
\end{aligned}
$$

It is not possible to determine the sign of Z_2 without knowing the values of the parameters α and f. However, observe that $-Z_1 + N_p Z_2 + \frac{1}{N_b} Z_3 \sum_{i=1}^{N_p} M(i, N_p, N_b)$ is the cost per time unit of holding components between production lots. This function is derived directly from the area under the component inventory graph (see equation (4.13)). Since this area can never be negative, it can be concluded that $-Z_1 + N_p Z_2 + \frac{1}{N_b} Z_3 \sum_{i=1}^{N_p} M(i, N_p, N_b) \geq 0$.

4.5.2 Validation of the Cost Function

The cost function calculated in the previous section and the areas under the inventory level graph derived in Section 4.4.2 have rather complicated formulae, and thus their derivations could be a potential source of algebraic errors. All of the algebraic manipulations were initially performed and checked numerous times by hand and then later verified by comparing the simplified formula to the unsimplified formula containing the areas under the graph, using two methods: MATLAB (R2009a) Symbolic Math package, and comparisons with the cost formulations by Teunter (2004).

The MATLAB (R2009a) Symbolic Math package allows formulae to be compared, without having to specify numeric values. It also performs simplification, expansion and factorisation of symbolic functions. Similar functionality is provided by software such as Maple. The simplified and unsimplified formulae were entered into Matlab for the individual components of the cost function. This allowed checks to be performed on individual parts, as well as, on the complete function. For all parts of the cost function and the entire cost function, the simplified version and the unsimplified version were found to be equivalent.

In addition to checking the symbolic representations of the formula, checks were also implemented in java, which was programming language used primarily in this research. The java class which calculates the total cost has two methods for calculating the cost, one which uses the simplified formula, and one which uses the unsimplified formula. Whenever the total cost is calculated, both methods are used, and then compared. If the java program found a difference between the costs calculated, then it would return an error message. This was tested extensively in the code development stage, and remained in place for all numerical experiments that were performed using this code.

The current model is an extension of the model described in Teunter (2004). These models have the same structure if only high quality items are recovered ($\zeta_H = 1$, $\zeta_L = 0$). These choices for ζ_H and ζ_L mean that the quality parameter $\alpha = 1$. Teunter (2004) does not include unit processing costs or costs for holding components ($c_r, c_b, c_p, c_r, c_d, h_c, k_b$). It is possible to show that with these cost parameters set to zero, the total cost in the current model is equivalent to equations (1) and (7) in Teunter's paper. Further details of this can be found in Appendix A.3.2 on page 369. This further validates the total cost formula. However obviously, since the current model uses non-zero values for these costs, some parts of the cost function can not be validated using this method.

4.6 Minimisation of the Total Cost Function

The total cost equation (4.21) is a function of four unknown, independent variables: Q_p, N_p, N_r, N_b. If each of these decision variables is assigned a value, then collectively that set of values is referred to as *policy*. We want to find a policy that will minimise the total cost function. In this section we seek the optimal policy for a given sequence of N_p product lots followed by N_r recovery lots. Any optimal policies found are only optimal within this class of policies. If the sequencing of the production and recovery lots was unrestricted, then the policies may no longer be optimal. However, the final three of these variables are required to be positive integers. This means that finding a policy that minimises the total cost is not a trivial task. The term $\sum_i M(i, N_p, N_b)$ in the total cost function also complicates this problem, because it is not differentiable.

The structure of this problem means that it can be formulated as a Mixed Integer Nonlinear Program (MINLP). The MINLP formulation is presented in Section 4.6.1. However, the small number of integer variables and preliminary numerical experiments suggest that MINLP methods may be unnecessarily computationally intensive. Therefore, alternative methods for determining an optimal solution are investigated.

In the literature, product recovery models (with one recovery channel) are often solved for the cases of $N_p = 1$ or $N_r = 1$ (Koh et al., 2002; Teunter, 2004). Not only does this assumption reduce the complexity of the problem (i.e. by having one rather than two integer variables), but it was shown by Teunter (2001) that for an even N_p and N_r, $TC(Q_p, N_p/2, N_r/2) < TC(Q_p, N_p, N_r)$. Following Teunter (2001) it may be possible to prove that $TC(Q_p, N_p/2, N_r/2, N_b) < TC(Q_p, N_p, N_r, N_b)$, or even that $TC(Q_p, N_p/2, N_r/2, N_b/2) < TC(Q_p, N_p, N_r, N_b)$. However, initial attempts found that this made non-trivial by the function $M(i, N_p, N_b)$.

Therefore, three cases are considered in this section. First the case in which $N_p = N_r = N_b = 1$ is considered in Section 4.6.2. This is the simplest case as it completely removes the integer requirement and the non-differentiable term becomes a constant. Then the case of $N_p = 1$, and $N_r, N_b \geq 1$ is considered in Section 4.6.3. When $N_p = 1$ the non-differentiable term is reduced to a constant (N_b), therefore the problem is simplified dramatically. Finally the most general case with $N_p, N_r, N_b \geq 1$ is considered in Section 4.6.4.

4.6.1 Mixed Integer Non Linear Programming Formulation

The product recovery model proposed in this chapter can be formulated as a *Mixed Integer Nonlinear Program* (MINLP). According to Floudas (1995), the general form of this type of problem is:

$$\begin{aligned}
\min_{x,y} \quad & f(x,y) & & \\
\text{subject to} \quad & h(x,y) & = 0 & \\
& g(x,y) & \leq 0 & \\
& x & \in \mathbf{X} & \subseteq \Re^n \\
& y & \in \mathbf{Y} & \text{ integer}
\end{aligned}$$

"where x is a vector of n continuous variables, y is a vector of integer variables, $h(x,y) = 0$ are m equality constraints, $g(x,y) \leq 0$ are p inequality constraints, and $f(x,y)$ is the objective function" (Floudas, 1995, page 4). For the current problem, the formulation is:

$$\begin{aligned}
\min_{Q_p,N_p,N_r,N_b} \quad & TC(Q_p, N_p, N_r, N_b) & & \\
\text{subject to} \quad & -N_p + 1 & \leq 0 & \\
& -N_r + 1 & \leq 0 & \\
& -N_b + 1 & \leq 0 & \\
& Q_p & & \subseteq \Re^n \\
& N_p, N_r, N_b & & \text{integer}
\end{aligned}$$

As mentioned above, while this product recovery model can be formulated as an MINLP, it will not be solved in this way.

4.6.2 Case 1: $N_p = N_r = N_b = 1$

In this section the case in which there is one of each type of lot ($N_p = N_r = N_b = 1$) is considered. This restriction on the number of lots dramatically simplifies the problem as it reduces the problem to an unconstrained minimization problem in one variable, Q_p, which can be solved using differentiation. When $N_p = N_r = N_b = 1$, the total cost function is:

$$TC(Q_p, 1, 1, 1) = C_P + \frac{d(1-\alpha f)(k_p + k_r + k_b)}{Q_p} + Q_p \left(V + W + X + Y - Z_1 + Z_2 + Z_3\right)$$

Differentiating with respect to Q_p, setting this derivative equal to zero, and then solving for Q_p gives:

$$Q_p = \sqrt{\frac{d(1-\alpha f)(k_p + k_r + k_b)}{V + W + X + Y - Z_1 + Z_2 + Z_3}}$$

The quantity under the square root sign is positive since d, $(1-\alpha f)$, $(k_p+k_r+k_b)$ and $(W+V+X+Y)$ must be greater than 0 and $(-Z_1+Z_2+Z_3)$ must be at least 0. The second derivative with respect to Q_p is:

$$\frac{d^2TC(Q_p,1,1,1)}{dQ_p^2} = \frac{d(1-\alpha f)(k_p+k_r+k_b)}{Q_p^3} > 0$$

Across the range of feasible values for Q_p, (i.e. $Q_p > 0$), this second derivative is always positive. This means that the stationary point found by the first derivative is a minimum. Therefore the production lot size that minimizes the total cost, when $N_p = N_r = N_b = 1$ is:

$$Q_p^*(1,1,1) = \sqrt{\frac{d(1-\alpha f)(k_p+k_r+k_b)}{V+W+X+Y-Z_1+Z_2+Z_3}} \tag{4.23}$$

This results in a total cost of:

$$TC(Q_p^*,1,1,1) = C_P + 2\sqrt{d(1-\alpha f)(k_p+k_r+k_b)(V+W+X+Y-Z_1+Z_2+Z_3)} \tag{4.24}$$

4.6.3 Case 2: $N_p = 1$, $N_r, N_b \geq 1$

In this section the restriction on the number of production lots is retained $N_p = 1$, however the restriction on the number of recovery and buying lots is relaxed to allow N_r and N_b to take integer values greater than or equal to one. The restriction on $N_p = 1$ means that the function $\sum_i M(i, N_p, N_b)$ becomes a constant and thus that the total cost function is differentiable. Despite being able to differentiate this function, the optimization of the function is still a reasonable complicated problem to solve, as it requires the minimization of a nonlinear function in one continuous variable and two integer variables.

In this section the convexity of the continuous relaxation of the total cost function is investigated. Though the convexity (or non-convexity) of the continuous relaxation does not, on its own, indicate that the same holds for the original integer function, it does give some insight into the nature of the function.

Convexity of the Continuous Relaxation

When $N_p = 1$, the total cost function simplifies to:

$$TC(Q_p, 1, N_r, N_b) = C_P + \frac{d(1-\alpha f)(k_p + N_r k_r + N_b k_b)}{Q_p} + Q_p\left(V + W + \frac{1}{N_r}X + \frac{1}{N_b}Y - Z_1 + Z_2 + Z_3\right)$$

Let $\tilde{N}_r$, $\tilde{N}_b$ denote the continuous relaxation of the variables N_r and N_b respectively. In this section the first and second partial derivatives with respect to Q_p, $\tilde{N}_r$, $\tilde{N}_b$ will be calculated and the Hessian matrix will be constructed. Using the Hessian matrix, the convexity of the continuous relaxation of this problem will be shown (Winston, 1987). The Hessian matrix of a function $f(x)$ is:

$$H(x_{i,j}) = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

For this problem the Hessian matrix has the following form:

$$H(Q_p, \tilde{N}_r, \tilde{N}_b) = \begin{bmatrix} \frac{\partial^2 TC}{\partial Q_p^2} & \frac{\partial^2 TC}{\partial Q_p \partial \tilde{N}_r} & \frac{\partial^2 TC}{\partial Q_p \partial \tilde{N}_b} \\ \frac{\partial^2 TC}{\partial \tilde{N}_r \partial Q_p} & \frac{\partial^2 TC}{\partial \tilde{N}_r{}^2} & \frac{\partial^2 TC}{\partial \tilde{N}_r \partial \tilde{N}_b} \\ \frac{\partial^2 TC}{\partial \tilde{N}_b \partial Q_p} & \frac{\partial^2 TC}{\partial \tilde{N}_b \partial \tilde{N}_r} & \frac{\partial^2 TC}{\partial \tilde{N}_b{}^2} \end{bmatrix}$$

where TC represents the total cost function $TC(Q_p, 1, \tilde{N}_r, \tilde{N}_b)$.

$$H(Q_p, \tilde{N}_r, \tilde{N}_b) = \begin{bmatrix} \frac{2d(1-\alpha f)(k_p + \tilde{N}_r k_r + \tilde{N}_b k_b)}{Q_p^3} & -\frac{d(1-\alpha f)k_r}{Q_p^2} - \frac{X}{\tilde{N}_r^2} & -\frac{d(1-\alpha f)k_b}{Q_p^2} - \frac{Y}{\tilde{N}_b^2} \\ -\frac{d(1-\alpha f)k_r}{Q_p^2} - \frac{X}{\tilde{N}_r{}^2} & \frac{2Q_p X}{\tilde{N}_r^3} & 0 \\ -\frac{d(1-\alpha f)k_b}{Q_p^2} - \frac{Y}{\tilde{N}_b{}^2} & 0 & \frac{2Q_p Y}{\tilde{N}_b^3} \end{bmatrix}$$

In order to show that the function $TC(Q_p, 1, \tilde{N}_r, \tilde{N}_b)$ is convex in Q_p, $\tilde{N}_r$ and $\tilde{N}_b$, it is must be shown that the first, second and third principle minors of $H(Q_p, \tilde{N}_r, \tilde{N}_b)$ are nonnegative (Winston, 1987). To calculate the principle minors, the determinants of several matrices will need to be calculated. In general, the determinant of an $m \times m$ matrix, A is:

$$\det(A) = (-1)^{i+1} a_{i,1}(\det A_{i,1}) + (-1)^{i+2} a_{i,2}(\det A_{i,2}) + \cdots + (-1)^{i+m} a_{i,m}(\det A_{i,m})$$

for any $i = 1, \ldots, m$, where $\det A_{i,j}$ is the determinant of the matrix obtained by deleting the i^{th} and j^{th} column (Winston, 1987).

All of the first principle minors are nonnegative. However for the second and third principle minors, it is not possible to determine the sign of the functions without further information about the relationship between the parameters. Since it is not possible to determine if the principle minors of the Hessian matrix are nonnegative, it is not possible to determine if the continuous relaxation of this mixed integer optimization problem is convex.

However, since $\frac{\partial^2 TC}{\partial Q_p^2} > 0$ it is possible to determine that for a fixed $\tilde{N}_r$ and $\tilde{N}_b$, the cost function is convex in Q_p, and thus that the cost minimizing production lot size is:

$$Q_p^* = \sqrt{\frac{d(1-\alpha f)(k_p + \tilde{N}_r k_r + \tilde{N}_b k_b)}{\left(V + W + \frac{1}{\tilde{N}_r}X + \frac{1}{\tilde{N}_b}Y - Z_1 + Z_2 + Z_3\right)}} \tag{4.25}$$

and the total cost is:

$$TC(Q_p^*, 1, N_r, N_b) = C_P + 2\sqrt{d(1-\alpha f)(k_p + \tilde{N}_r k_r + \tilde{N}_b k_b)\left(V + W + \frac{X}{\tilde{N}_r} + \frac{Y}{\tilde{N}_b} - Z_1 + Z_2 + Z_3\right)} \tag{4.26}$$

A method will now be proposed for solving this problem.

Upper and Lower Bounds

In this section we develop bounds for determining N_r and N_b when $N_p = 1$. This method extends Konstantaras and Papachristos (2008b), who applied this method to Teunter's 2004 model, to find N_r when $N_p = 1$ and N_p when $N_r = 1$. The minimum total cost for the system can be determined by finding integer values of N_r, N_b which minimise the total cost specified by equation (4.26).

Bounds for N_r. Following Konstantaras and Papachristos (2008b) it will be shown that for a fixed N_b, the optimal number of recovery lots N_r^* can be obtained by constructing an upper and lower bounds. Let us define a difference function, for values of $N_r \geq 2$:

$$\Delta TC(Q_p^*, N_r) = TC(Q_p^*, 1, N_r, N_b) - TC(Q_p^*, 1, N_r - 1, N_b)$$

after substituting equation (4.26) this function becomes:

$$\Delta TC(Q_p^*, N_r) = \left(C_P + 2\sqrt{d(1-\alpha f)(k_p + k_r N_r + k_b N_b)} \times \sqrt{\left(V + W + \frac{X}{N_r} + \frac{Y}{N_b} - Z_1 + Z_2 + Z_3\right)}\right) - \left(C_P + 2\sqrt{d(1-\alpha f)(k_p + k_r(N_r - 1) + k_b N_b)} \times \sqrt{\left(V + W + \frac{X}{(N_r - 1)} + \frac{Y}{N_b} - Z_1 + Z_2 + Z_3\right)}\right)$$

To aid with the simplification, let following temporary variables be defined:

$$c_1 = k_p + k_b N_b$$
$$c_2 = k_r$$
$$c_3 = V + W + \frac{Y}{N_b} - Z_1 + Z_2 + Z_3$$
$$c_4 = X$$

Note that $c_1, c_2, c_3, c_4 > 0$. Using this notation the total cost is:

$$TC(Q_p^*, 1, N_r, N_b) = C_P + 2\sqrt{d(1-\alpha f)\Big(c_1 + c_2 N_r\Big)\Big(c_3 + \frac{c_4}{N_r}\Big)}$$

and the difference function is:

$$\Delta TC(Q_p^*, N_r) = \frac{2\frac{d(1-\alpha f)}{N_p}\Big(c_2 c_3 - \frac{c_1 c_4}{N_r(N_r-1)}\Big)}{\sqrt{\frac{d(1-\alpha f)}{N_p}\Big(c_1 + c_2 N_r\Big)\Big(c_3 + \frac{c_4}{N_r}\Big)} + \sqrt{\frac{d(1-\alpha f)}{N_p}\Big(c_1 + c_2(N_r - 1)\Big)\Big(c_3 + \frac{c_4}{(N_r-1)}\Big)}} \tag{4.27}$$

Let N_r^* be the number of recovery lots required to minimise the total cost. If $\Delta TC(Q_p^*, N_r) \geq 0$, then this suggests that $N_r \geq N_r^*$, since the total cost function is convex. Whereas, if $\Delta TC(Q_p^*, N_r) < 0$, then this suggests that $N_r < N_r^*$. The difference function $\Delta TC(Q_p^*, N_r) \geq 0$ will have the same sign as $\big(c_2 c_3 - \frac{c_1 c_4}{N_r(N_r-1)}\big)$, since as all other terms are positive. The sign of the difference function will be used to determine whether $N_r^* = 1$ or $N_r^* > 1$, and if the latter, then it will also be used to create bounds on N_r^*. The conditions required for each case will be stated.

Case i: $N_r^* = 1$. Suppose that $N_r = 2$, then if $\Delta TC(Q_p^*, 2) \geq 0$, is must be the case that $N_r^* = 1$, due to the convexity of the cost function and the requirement that $N_r > 0$. As mentioned above, if $\Delta TC(Q_p^*, 2) \geq 0$, then $c_2 c_3 - \frac{c_1 c_4}{2} \geq 0$ and vice versa, since this term determines the sign of the difference function. This term can be rewritten as $\frac{c_1c_4}{c_2c_3} \leq 2$ to provide the following condition:

$$N_r^* = 1 \quad \text{if } \frac{c_1c_4}{c_2c_3} \leq 2 \quad \text{or equivalently, if } \frac{(k_p + k_bN_b)X}{k_r(V + W + \frac{Y}{N_b} - Z_1 + Z_2 + Z_3)} \leq 2$$

Case ii: $N_r^* > 1$. The condition required for Case i, implies that:

$$N_r^* > 1 \quad \text{if } \frac{c_1c_4}{c_2c_3} > 2 \quad \text{or equivalently, if } \frac{(k_p + k_bN_b)X}{k_r(V + W + \frac{Y}{N_b} - Z_1 + Z_2 + Z_3)} > 2$$

Since the total cost function is convex, if $\frac{c_1c_4}{c_2c_3} > 2$, then for the optimal number of recovery lots N_r^*, the following double inequality will be met:

$$\Delta TC(Q_p^*, N_r^*) < 0 \leq \Delta TC(Q_p^*, N_r^* + 1)$$

The sign of the difference function can be used to find N_r^*. Substituting the difference function from equation (4.27), and performing some algebraic manipulations gives:

$$N_r^*(N_r^* - 1) < \frac{c_1c_4}{c_2c_3} \leq N_r^*(N_r^* + 1)$$

Solving each part of this double inequality separately for N_r^* (using the quadratic formula and choosing the appropriate root in each case) and then rewriting the results as a double inequality gives:

$$\frac{1}{2}\Big(-1 + \sqrt{1 + 4\frac{c_1c_4}{c_2c_3}}\Big) \leq N_r^* < \frac{1}{2}\Big(1 + \sqrt{1 + 4\frac{c_1c_4}{c_2c_3}}\Big)$$

Substituting the expressions for c_1, c_2, c_3, c_4 into these expressions, gives:

$$\begin{aligned} &\frac{1}{2}\Big(-1 + \sqrt{1 + 4\frac{(k_pN_p + k_bN_b)N_pX}{k_r(V + N_pW + \frac{N_p}{N_b}Y + Z(N_p, N_b))}}\Big) \leq N_r^* < \\ &\qquad \frac{1}{2}\Big(1 + \sqrt{1 + 4\frac{(k_pN_p + k_bN_b)N_pX}{k_r(V + N_pW + \frac{N_p}{N_b}Y + Z(N_p, N_b))}}\Big) \end{aligned} \tag{4.28}$$

Therefore if $\frac{(k_p+k_bN_b)X}{k_r(V+W+\frac{Y}{N_b}-Z_1+Z_2+Z_3)} > 2$ then, for a fixed N_b, the optimal number of recovery lots N_r^* can be obtained by solving the double inequality give by equation (4.28).

Bounds for N_b^* Following Konstantaras and Papachristos (2008b) it will be shown that for $N_p = 1$ and a fixed N_r, upper and lower bounds on the optimal number of buying lots N_b^* can be constructed. Let us define a difference function, for values of $N_b \geq 2$:

$$\Delta TC(Q_p^*, N_b) = TC(Q_p^*, 1, N_r, N_b) - TC(Q_p^*, 1, N_r, N_b - 1)$$

after substituting equation (4.26) this becomes:

$$\begin{aligned}\Delta TC(Q_p^*, N_b) = \Bigg(& C_P + 2\sqrt{d(1-\alpha f)(k_p + N_r k_r + N_b k_b)} \times \\ & \sqrt{\left(V + W + \frac{X}{N_r} + \frac{Y}{N_b} - Z_1 + Z_2 + Z_3\right)}\Bigg) \\ - \Bigg(& C_P + 2\sqrt{d(1-\alpha f)(k_p + N_r k_r + (N_b - 1)k_b)} \times \\ & \sqrt{\left(V + W + \frac{X}{N_r} + \frac{Y}{N_b - 1} - Z_1 + Z_2 + Z_3\right)}\Bigg)\end{aligned}$$

To aid with the analysis, let the following temporary variables be defined:

$$\begin{aligned} c_5 &= k_p + k_r N_r \\ c_6 &= k_b \\ c_7 &= V + W + \frac{1}{N_r} X \\ c_8 &= Y \end{aligned}$$

Note that $c_5, c_6, c_7, c_8 > 0$. Using a similar procedure to the N_r^* case, there are two cases that need to be considered.

Case i: $N_b^* = 1$**.** Following the same logic as for the N_r^* case, the following condition is obtained:

$$N_b^* = 1 \quad \text{if } \frac{c_5 c_8}{c_6 c_7} \leq 2 \quad \text{or alternatively, if } \frac{(k_p + k_r N_r)Y}{k_b(V + W + \frac{X}{N_r})} \leq 2$$

Case ii: $N_b^* > 1$**.** The condition required for Case i implies that:

$$N_b^* > 1 \quad \text{if } \frac{c_5 c_8}{c_6 c_7} > 2 \quad \text{or alteratively, if } \frac{(k_p + k_r N_r)Y}{k_b(V + W + \frac{X}{N_r})} > 2$$

Since the total cost function is convex, if $\frac{c_5 c_8}{c_6 c_7} > 2$, then for the optimal number of buying lots N_b^*, the following double inequality will be met:

$$\Delta TC(Q_p^*, N_b^*) < 0 \leq \Delta TC(Q_p^*, N_b^* + 1)$$

The sign of the difference function can be used to find N_b^*. Substituting the difference function from equation (4.27), and performing some algebraic manipulations gives:

$$N_b^*(N_b^*-1) < \frac{c_5c_8}{c_6c_7} \le N_b^*(N_b^*+1)$$

Solving each part of this double inequality separately for N_b^* (using the quadratic formula and choosing the appropriate root in each case) and then rewriting the results as a double inequality gives:

$$\frac{1}{2}\Big(-1+\sqrt{1+4\frac{c_5c_8}{c_6c_7}}\Big) \le N_b^* < \frac{1}{2}\Big(1+\sqrt{1+4\frac{c_5c_8}{c_6c_7}}\Big)$$

Substituting back c_5, c_6, c_7, c_8 into these expressions, gives:

$$\frac{1}{2}\Big(-1+\sqrt{1+4\frac{(k_p+k_rN_r)Y}{k_b(V+W+\frac{X}{N_r})}}\Big) \le N_b^* < \frac{1}{2}\Big(1+\sqrt{1+4\frac{(k_p+k_rN_r)Y}{k_b(V+W+\frac{X}{N_r})}}\Big) \quad (4.29)$$

Therefore if $\frac{(k_p+k_rN_r)Y}{k_b(V+W+\frac{X}{N_r})} > 2$ then, for a fixed N_r, upper and lower bounds on the optimal number of buying lots N_b^* can be obtained by solving the double inequality in equation (4.29).

Combining bounds for N_r^* and N_b^* for a policy with $N_p = 1$. Assuming that $\frac{c_1c_4}{c_2c_3} > 2$, then for $N_p = 1$ and for a fixed N_b, the upper and lower bounds on the optimal number of recovery lots N_r^*, as given by equation (4.28) are:

$$N_r^* \ge \frac{1}{2}\left(-1+\sqrt{1+4\frac{(k_p+k_bN_b)X}{k_r\big(V+W+\frac{Y}{N_b}\big)}}\right) \quad (4.30)$$

$$N_r^* < \frac{1}{2}\left(1+\sqrt{1+4\frac{(k_p+k_bN_b)X}{k_r\big(V+W+\frac{Y}{N_b}\big)}}\right) \quad (4.31)$$

Assuming that $\frac{c_5c_8}{c_6c_7} > 2$, then for a $N_p = 1$ and a fixed N_r, the upper and lower bounds on the optimal number of buying lots N_b^*, as given by equation (4.29) are:

$$N_b^* \ge \frac{1}{2}\left(-1+\sqrt{1+4\frac{(k_p+k_rN_r)Y}{k_b\big(V+W+\frac{X}{N_r}\big)}}\right) \quad (4.32)$$

$$N_b^* < \frac{1}{2}\left(1+\sqrt{1+4\frac{(k_p+k_rN_r)Y}{k_b\big(V+W+\frac{X}{N_r}\big)}}\right) \quad (4.33)$$

Solving the equations (4.30), (4.31), (4.32) and (4.33) simultaneously to obtain values of N_r^* and N_b^* gives a convex region and allows the search area for N_r^* and N_b^* to be dramatically reduced. In our numerical experiments, these bounds provide a region containing only one integer point, thus providing a unique optimal solution to the minimization problem. However, future investigations are required to determine if this result holds in general. This is an area for future research.

4.6.4 Case 3: $N_p, N_r, N_b \geq 1$

In this section the restrictions on the numbers of lots are relaxed to allow N_p, N_r, N_b to take any positive integer values. This case is the most general case that we consider. Finding the values of N_p, N_r, N_b and Q_p which minimize the total cost function requires the minimization of a non-differentiable, nonlinear function with three integer variables.

The structure of the total cost function means that for a fixed N_p, the function is differentiable in Q_p. We begin by showing that the total cost function is convex in Q_p for a fixed N_p, N_r and N_b. Then we propose a search algorithm which can be used for finding the values of Q_p, N_p, N_r, and N_b that minimize the total cost function.

Convexity in Q_p

It will be shown that for a fixed N_p, N_r, N_b, the total cost function is convex in Q_p. (Further details are presented in Appendix A.4.2.) Convexity of a function can be determined by showing that the second derivative of that function is greater than or equal to zero (Winston, 1987). The first derivative of the total cost function, with respect to Q_p for a fixed N_p, N_r, N_b is:

$$\frac{dTC(Q_p, N_p, N_r, N_b)}{dQ_p} = -\frac{\bar{K}(N_p, N_r, N_b)}{Q_p^2} + \left(V + N_pW + \frac{N_p}{N_r}X + \frac{N_p}{N_b}Y \right.$$
$$\left. -Z_1 + N_pZ_2 + \frac{1}{N_b}Z_3 \sum_{i=1}^{N_p} M(i, N_p, N_b) \right) \tag{4.34}$$

The second derivative of the total cost function, with respect to Q_p for a fixed N_p, N_r, N_b is:

$$\frac{d^2TC(Q_p, N_p, N_r, N_b)}{dQ_p^2} = \frac{\bar{K}(N_p, N_r, N_b)}{Q_p^3} > 0$$

Across the range of feasible values for Q_p, (i.e. $Q_p > 0$), this second derivative is always positive. This means that the total cost function is convex with respect to Q_p and therefore that the stationary point found by the first derivative is a minimum. By setting the first derivative (equation (4.34)) equal to zero, and solving for Q_p, the value of Q_p which minimises the total cost can be found. The resultant equation contains a square root term, it is necessary to prove that the term under the square root is not negative. Indeed it is not negative since $\bar{K}(N_p, N_r, N_b) > 0$ and $W, V, X > 0$ and $Y, (-Z_1 + N_p Z_2 + \frac{1}{N_b} Z_3 \sum_{i=1}^{N_p} M(i, N_p, N_b)) \geq 0$. Thus, since the total cost function is convex in Q_p the minimum total cost is reached at:

$$Q_p^*(N_p, N_r, N_b) = \sqrt{\frac{\bar{K}(N_p, N_r, N_b)}{V + N_p W + \frac{N_p}{N_r} X + \frac{N_p}{N_b} Y - Z_1 + N_p Z_2 + \frac{1}{N_b} Z_3 \sum_{i=1}^{N_p} M(i, N_p, N_b)}}$$

which gives the corresponding total cost:

$$\begin{aligned} TC(Q_p^*, N_p, N_r, N_b) = C_P + 2\sqrt{\bar{K}\Big(N_p, N_r, N_b\Big)} \times \\ \sqrt{\Big(V + N_p W + \frac{N_p}{N_r} X + \frac{N_p}{N_b} Y - Z_1 + N_p Z_2 + \frac{1}{N_b} Z_3 \sum_{i=1}^{N_p} M(i, N_p, N_b)\Big)} \end{aligned} \tag{4.35}$$

The minimum total cost for the system can be determined by finding integer values of N_p, N_r, N_b which minimise equation (4.35).

Search Algorithm

In this section a basic search algorithm for determining the optimal values for N_p, N_r and N_b is proposed. More sophisticated algorithms could be designed using assumptions of convexity. Initial computational experiments and results in the literature (e.g. (Teunter, 2001, 2004; Konstantaras and Papachristos, 2008b) suggested that the optimal value of at least one of N_p and N_r is likely to be equal to 1. This means that the search area is likely to be relatively small, and therefore that the computational savings available by developing a more sophisticated search algorithm are likely to be minimal.

Initialise:

Set $TC^{MIN} = \infty$ and $N_p^{MIN} = N_r^{MIN} = N_b^{MIN} = \infty$

Choose suitably large $N_p^U = N_r^U = N_b^U$

For $N_p = 1$ to N_p^U

For $N_r = 1$ to N_r^U

For $N_b = 1$ to N_b^U

Calculate total cost $TC(Q_p^*, N_p, N_r, N_b)$ using equation (4.35)

If $TC(Q_p^*, N_p, N_r, N_b) < TC_{MIN}$

Set $TC^{MIN} = TC(Q_p^*, N_p, N_r, N_b)$

Set $N_p^{MIN} = N_p$, $N_r^{MIN} = N_r$, $N_b^{MIN} = N_b$

STOP when $N_p = N_p^U$, $N_r = N_r^U$, $N_b = N_b^U$

If $N_p^{MIN} = N_p^U$, $N_r^{MIN} = N_r^U$, or $N_b^{MIN} = N_b^U$

Increase the relevant upper bounds (N_p^U, N_r^U or N_b^U)

and repeat algorithm.

4.7 Properties of the Model

The properties of the product recovery model are explored in this section. Two main properties are investigated. Firstly, we investigate the effect on the optimal policy of the *restricting the lot sizes*. The reader is reminded that the optimal policy sought, is optimal within a class of policies with N_p production lots, followed by N_r recovery lots; it may not be optimal across all classes of policies. Recall that there are two recovery strategies: high quality recovery only ($\zeta_H = 1, \zeta_L = 0$) and high and low quality recovery ($\zeta_H = 1, \zeta_L = 1$). The second property that we investigate relates to these *recovery* strategies. We investigate the effect of the strategies on the optimal policy and the conditions under which each recovery strategy is optimal. These questions will be addressed in Sections 4.7.2 and 4.7.3 respectively. In investigating these properties a combination of analytical and numerical results are presented. The datasets used to obtain the numerical results are discussed in Section 4.7.1.

4.7.1 Datasets

In order to investigate the properties of the model a primary problem set is constructed. This problem set has been derived from dataset used by Konstantaras and Papachristos (2008b), however some additional parameters have been added in order to account for the two recovery channels and the component inventory. The relationships used to determine the values of these new parameters are discussed first, and then construction of the primary problem set is discussed. In addition to the primary problem set, two additional datasets are constructed.

Relationship between Cost Parameters

It is assumed that the cost of a newly produced serviceable item is greater than cost of high-quality recovered serviceable item, and greater than cost of a serviceable item produced with recovered components. These conditions are represented with the following inequalities:

$$c_p + c_b > c_p + c_r + c_l$$

$$c_p + c_b > c_r + c_h$$

The cost of carrying a unit of inventory for a year can be approximated by the variable per unit cost of inventory, multiplied by some annual holding charge (Silver et al., 1998, page 45). Following this formulation, it is assumed that the variable cost associated with serviceable inventory is c_p, with returned inventory is c_r, and with component inventory is c_b. Let γ_h denote the annual, per unit holding charge. Using this formulation, the costs of holding serviceable, returned and component inventory respectively are:

$$h_s = \gamma_h c_p$$

$$h_r = \gamma_h c_r$$

$$h_c = \gamma_h c_b$$

The rate γ_h is often based on the opportunity cost of using the money tied up in inventory in other ways (Silver et al., 1998, page 46).

In this model, high quality returns are required to undergo some recovery which transforms them into serviceable items. It is assumed that this process is similar to, but not as involved as, production. Because of this relationship, we relate the cost of high quality recovery to the cost of production through the multiplier γ_p.

$$c_h = \gamma_p c_p$$

Similarly, the use of low quality returns is assumed to be an alternative to buying components, therefore we relate the cost of low quality recovery to the cost of buying components through the multiplier γ_b.

$$c_l = \gamma_b c_b$$

The cost of disposing of returns that are not recovered could have numerous interpretations which are likely to depend on the type of good involved. For instance, the cost of disposing of a hazardous good, would be more than the cost of disposing of a non-hazardous good. The cost of disposal could also be related to the value of the good, e.g. the cost of manufacturing the good. We relate the disposal cost to the cost of producing a serviceable good with a new component, through the multiplier γ_d.

$$c_d = \gamma_d(c_p + c_b)$$

Primary Problem Set

The parameter values for the primary problem set are presented in Table 4.2. These problems are denoted by D followed by the problem number.

For the first base problem, which we denote by $D00$, the values of $d, p, r, k_p, k_r, h_s, h_r$ are exactly the same as those in the first example in Konstantaras and Papachristos (2008b). These parameter values are also used by Teunter (2004). The parameter f in their paper is equivalent to the parameter β_H in the current model, thus their values for f are used here as values for β_H. The values of the additional parameters $(\beta_L, k_b, h_c, c_p, c_h, c_l, c_b, c_r, c_d)$ are set to zero. Under the appropriate policy restrictions (e.g. 1 production lot and N_r recovery lots, or N_p production lots and 1 recovery lot), the optimal value for Q_p, the total cost and the lot sizes for $D00$ in the current model, are the same as those reported by Konstantaras and Papachristos (2008b). This problem instance was kept in the data set to assist with the continual model validation.

The next 13 base problems, denoted $D01$ to $D13$, use the same values as Konstantaras and Papachristos (2008b) for the parameters $d, p, r, k_p, k_r, h_s, h_r$. As for problem $D00$, β_H in the current model is set equal to their value for f. The values for β_L were chosen in order to represent a variety of high-low quality relationships across the problem set. The values for h_c and k_b were specified in the same way. The values for the remaining parameters $(c_p, c_h, c_l, c_b, c_r, c_d)$ were determined by the relationships discussed above and the following multipliers:

$$\gamma_h = 0.1$$

$$\gamma_p = 0.5$$

$$\gamma_b = 0.3$$

$$\gamma_d = 0.05$$

The final seven base problems, denoted $D14$ to $D20$, were not used by Konstantaras and Papachristos (2008b). For these problems the values for $d, p, r, k_p, k_r, k_b, h_s, h_r, h_c, \beta_H, \beta_L$, were selected in order to give a wider variety of cost scenarios. In particular, these additional problems explore the situation of having $r > p$ and $\beta_L > \beta_H$. The values for the remaining parameters $(c_p, c_h, c_l, c_b, c_r, c_d)$ were determined as for problems $D01$ to $D13$.

Additional Problem Sets

Two additional problem sets were constructed using the primary problem set D. These are named E and F. Problem set E is essentially the same as problem set D except that all of the processing unit costs are set to zero, i.e., $\{c_p = c_h = c_l = c_r = c_b = c_d = 0\}$. Problem set F was constructed from problem set D by varying one of the following parameters

$$\{d, r, p, k_p, k_r, k_b, h_s, h_r, h_c\}$$

Problem ID		$D00$	$D01$	$D02$	$D03$	$D04$	$D05$	$D06$	$D07$	$D08$	$D09$	$D10$	$D11$	$D12$	$D13$	$D14$	$D15$	$D16$	$D17$	$D18$	$D19$	$D20$
Proportions and Rates																						
High quality returns	β_H	0.8	0.8	0.2	0.3	0.3	0.5	0.7	0.8	0.8	0.2	0.5	0.5	0.5	0.5	0.8	0.4	0.1	0.3	0.2	0.2	0.5
Low quality returns	β_L	0	0.1	0.25	0.1	0.25	0.4	0.2	0.05	0.15	0.35	0.3	0.05	0.05	0.001	0.15	0.4	0.1	0.6	0.1	0.75	0.1
Demand	d	1000	1000	1000	500	500	800	1000	20	20	20	800	50	50	59	1000	1000	250	750	10	20	100
Production	p	5000	5000	4000	1000	2000	2400	3000	50	100	80	2300	70	70	70	5000	4000	1000	3000	100	80	2300
Recovery	r	3000	3000	2500	700	1000	1500	2000	35	50	60	1500	60	60	60	3000	2500	1500	2500	20	160	2300
Set up Costs																						
Production	k_p	20	20	10	10	20	20	20	30	30	50	28	12	12	120	20	100	1000	400	30	50	28
Recovery	k_r	5	5	5	10	12	8	20	20	25	30	8	2	2	10	5	50	1000	400	5	10	8
Buying	k_b	0	10	2	10	15	12	10	10	5	15	4	6	4	5	10	20	1000	200	15	50	4
Unit Costs																						
Production	c_p	0	100	60	80	120	150	100	60	70	200	170	20	70	70	100	60	100	100	70	200	150
High quality recovery	c_h	0	50	30	40	60	75	50	30	35	100	85	10	35	35	50	30	50	50	35	100	75
Low quality recovery	c_l	0	3	5	8	12	7	4	6	3	5	20	5	5	10	3	5	10	4	3	5	5
Buying	c_b	0	30	50	80	120	70	40	60	30	50	200	50	50	100	30	50	100	40	30	50	50
Acquiring returns	c_r	0	20	40	50	100	40	20	50	20	40	160	10	30	30	20	40	10	20	20	40	20
Disposal	c_d	0	6.5	5.5	8	12	11	7	6	5	12.5	18.5	3.5	6	8.5	6.5	5.5	10	7	5	12.5	10
Holding Costs																						
Serviceable	h_s	10	10	6	8	12	15	10	6	7	20	17	2	7	7	10	6	10	10	7	20	15
Returns	h_r	2	2	4	5	10	4	2	5	2	4	16	1	3	3	2	4	1	2	2	4	2
Components	h_c	0	3	5	8	12	7	4	6	3	5	20	5	5	10	3	5	10	4	3	5	5

Table 4.2: Parameters used in the primary problem set

using the multipliers

$$\{0, 0.1, 0.5, 0.9, 1.1, 1.5, 2.0, 5.0, 10.0\}$$

while holding all other parameters constant. Applying this method results in 9×9 problems created for each problem in problem set D. A total of 1701 problems were constructed in this way, however not all of the new problems were deemed to be feasible, e.g. some of the created problems had a demand rate equal to zero ($d = 0$). The following conditions were used to determine the feasibility of a problem:

$$\begin{array}{ll} 0 < \beta_H + \beta_L < 1 & 0 < \alpha < 1 \\ 0 < f < 1 & \alpha r < d \\ p \leq d & p > 0 \\ d > 0 & r > 0 \\ k_p > 0 & k_r > 0 \end{array}$$

Any problems which did not meet all of these conditions were deleted. In addition to the problems deemed infeasible due to these conditions, there were also some problems which were found to be infeasible in the solution process as the optimal values for N_p, N_r or N_b appeared to tend to infinity. These problems were also excluded from the data set. After these infeasible problems had been excluded, there were 1457 problems in problem set F.

4.7.2 Analysis of Lot Size Restrictions

In this section the effect of restricting the numbers of each type of lot is investigated. In the literature, research has focussed on searching for optimal policies within classes of policies with either 1 production lot and N_r recovery lots, or N_p production lots and 1 recovery lot (Teunter, 2001, 2004; Konstantaras and Papachristos, 2008b). Teunter (2001) shows that there is always a near-optimal policy within these classes by proving that for a fixed Q_p and Q_r, the "average total cost rate" of a policy with $\frac{N_p}{2}$ production lots and $\frac{N_r}{2}$ recovery lots is always less than the "average total cost rate" of a policy with N_p production lots and N_r recovery lots, when N_p and N_r are even numbers. Restricting the class of policies also simplifies the optimisation.

However, as mentioned by numerous authors including Teunter (2004), despite this proof, the optimal policy is not guaranteed to lie within these classes of policies. For some problems, the optimal policy may require the number of lots to be greater than one, and at least one of them to be not even. In the current model, it is also possible that with the addition of the component inventory and the buying of components, this result may no longer hold. Therefore it is interesting to investigate the effect of placing restrictions on the class of policies. To explore this further, the optimal policy will be examined under the following four policy restrictions: (N_p, N_r, N_b), $(1, N_r, N_b)$, $(N_p, 1, N_b)$, $(1, 1, 1)$. These four policy classes will also be compared under the two recovery strategies: high-quality recovery only and high and low quality recovery.

Tables 4.3 and 4.4 show the optimal policy for problem set D under each of the four policy restrictions for problems with $(\zeta_L = 1)$ and without $(\zeta_L = 0)$ low quality recovery respectively. These tables are presented to give some examples of optimal solution policies. However, as shown in Section 4.6, the optimal solution functions for Q_p, N_p, N_r and N_b depend not on the unit costs, but only on the holding costs and set up costs. The unit costs inflate the total cost and can make it more difficult to observe the differences caused by the policy restrictions. Therefore, in order to analyse the structure of these policies and their effect on the optimal solution, problem set E (with no acquisition or processing costs) is used.

Tables 4.5 and 4.6 show the optimal policy for problem set E under each of the four policy restrictions for problems with $(\zeta_L = 1)$ and without $(\zeta_L = 0)$ low quality recovery respectively. These tables are presented to give some examples of optimal solution policies. Examining Table 4.5, observe that for many of the problems it is optimal to produce, recover or buy components only per cycle, even under the unrestricted policy (N_p, N_r, N_b). Of the 21 base problems, $N_p > 1$ in no problems, $N_r > 1$ in 8 problems, and $N_b > 1$ in 2 problems. Overall, for 10 out of 21 problems is it optimal to deviate from the $(1, 1, 1)$ policy. In all cases the optimal policy is contained within one of the policy classes $(N_p, 1, N_b)$ or $(1, N_r, N_b)$. This means in these problems, it is never optimal for both N_r *and* N_p to be greater than one.

Table 4.6, which contains the optimal policy for the high-quality only recovery strategy, tells a similar story. In these problems, is never it optimal for both N_r and N_p to be greater than one. However there are more cases which deviate from the $(1, 1, 1)$

Table 4.3: Optimal lot sizes and numbers of lots for Problem Set D under various policy restrictions for the both high and low quality recovery strategy ($\zeta_L = 1$)

	N_p, N_r, N_b							$1, N_r, N_b$						
Problem	N_p	N_r	N_b	Q_p	Q_r	Q_b	TC	N_p	N_r	N_b	Q_p	Q_r	Q_b	TC
$D00$	1	6	1	51.75	34.50	51.75	386.44	1	6	1	51.75	34.50	51.75	386.44
$D01$	1	6	1	52.02	39.02	26.01	82411.35	1	6	1	52.02	39.02	26.01	82411.35
$D02$	1	1	1	65.25	36.70	44.86	104191.87	1	1	1	65.25	36.70	44.86	104191.87
$D03$	1	1	1	53.40	30.52	45.77	71193.23	1	1	1	53.40	30.52	45.77	71193.23
$D04$	1	1	1	47.38	37.23	30.46	110394.34	1	1	1	47.38	37.23	30.46	110394.34
$D05$	1	1	1	50.10	90.18	10.02	128158.73	1	1	1	50.10	90.18	10.02	128158.73
$D06$	1	2	1	65.17	97.76	21.72	89144.46	1	2	1	65.17	97.76	21.72	89144.46
$D07$	1	2	1	6.81	14.46	5.10	1868.03	1	2	1	6.81	14.46	5.10	1868.03
$D08$	1	2	1	7.17	17.02	1.79	1358.89	1	2	1	7.17	17.02	1.79	1358.89
$D09$	1	1	1	14.09	9.69	7.93	4853.24	1	1	1	14.09	9.69	7.93	4853.24
$D10$	1	1	1	32.58	52.12	13.03	245142.34	1	1	1	32.58	52.12	13.03	245142.34
$D11$	1	1	2	25.64	28.21	11.54	2291.94	1	1	2	25.64	28.21	11.54	2291.94
$D12$	1	1	1	13.67	15.04	12.31	4788.32	1	1	1	13.67	15.04	12.31	4788.32
$D13$	1	1	7	59.65	59.77	8.50	7342.42	1	1	7	59.65	59.77	8.50	7342.42
$D14$	1	5	1	46.04	43.74	11.51	81752.81	1	5	1	46.04	43.74	11.51	81752.81
$D15$	1	1	1	153.45	204.60	51.15	94429.41	1	1	1	153.45	204.60	51.15	94429.41
$D16$	2	1	2	330.79	147.02	294.03	49900.98	1	1	1	386.66	85.93	343.70	49991.42
$D17$	1	1	1	343.18	441.23	49.03	85634.66	1	1	1	343.18	441.23	49.03	85634.66
$D18$	1	1	1	11.50	4.31	10.06	1007.55	1	1	1	11.50	4.31	10.06	1007.55
$D19$	1	1	1	13.69	16.25	0.86	4754.70	1	1	1	13.69	16.25	0.86	4754.70
$D20$	1	2	1	18.99	11.40	15.19	15152.74	1	2	1	18.99	11.40	15.19	15152.74
	$N_p, 1, N_b$							$1, 1, 1$						
Problem	N_p	N_r	N_b	Q_p	Q_r	Q_b	TC	N_p	N_r	N_b	Q_p	Q_r	Q_b	TC
$D00$	1	1	1	18.63	74.54	18.63	536.66	1	1	1	18.63	74.54	18.63	536.66
$D01$	1	1	1	21.63	97.36	10.82	82597.10	1	1	1	21.63	97.36	10.82	82597.10
$D02$	1	1	1	65.25	36.70	44.86	104191.87	1	1	1	65.25	36.70	44.86	104191.87
$D03$	1	1	1	53.40	30.52	45.77	71193.23	1	1	1	53.40	30.52	45.77	71193.23
$D04$	1	1	1	47.38	37.23	30.46	110394.34	1	1	1	47.38	37.23	30.46	110394.34
$D05$	1	1	1	50.10	90.18	10.02	128158.73	1	1	1	50.10	90.18	10.02	128158.73
$D06$	1	1	1	46.48	139.43	15.49	89145.50	1	1	1	46.48	139.43	15.49	89145.50
$D07$	1	1	1	4.76	20.24	3.57	1874.79	1	1	1	4.76	20.24	3.57	1874.79
$D08$	1	1	1	4.72	22.40	1.18	1365.80	1	1	1	4.72	22.40	1.18	1365.80
$D09$	1	1	1	14.09	9.69	7.93	4853.24	1	1	1	14.09	9.69	7.93	4853.24
$D10$	1	1	1	32.58	52.12	13.03	245142.34	1	1	1	32.58	52.12	13.03	245142.34
$D11$	1	1	2	25.64	28.21	11.54	2291.94	1	1	1	19.25	21.17	17.32	2293.21
$D12$	1	1	1	13.67	15.04	12.31	4788.32	1	1	1	13.67	15.04	12.31	4788.32
$D13$	1	1	7	59.65	59.77	8.50	7342.42	1	1	1	35.46	35.53	35.39	7403.82
$D14$	1	1	1	21.46	101.96	5.37	81927.25	1	1	1	21.46	101.96	5.37	81927.25
$D15$	1	1	1	153.45	204.60	51.15	94429.41	1	1	1	153.45	204.60	51.15	94429.41
$D16$	2	1	2	330.79	147.02	294.03	49900.98	1	1	1	386.66	85.93	343.70	49991.42
$D17$	1	1	1	343.18	441.23	49.03	85634.66	1	1	1	343.18	441.23	49.03	85634.66
$D18$	1	1	1	11.50	4.31	10.06	1007.55	1	1	1	11.50	4.31	10.06	1007.55
$D19$	1	1	1	13.69	16.25	0.86	4754.70	1	1	1	13.69	16.25	0.86	4754.70
$D20$	1	1	1	15.05	18.06	12.04	15165.84	1	1	1	15.05	18.06	12.04	15165.84

Table 4.4: Optimal lot sizes and numbers of lots for Problem Set D under various policy restrictions for the high quality only recovery strategy ($\zeta_L = 0$)

	N_p, N_r, N_b							$1, N_r, N_b$						
Problem	N_p	N_r	N_b	Q_p	Q_r	Q_b	TC	N_p	N_r	N_b	Q_p	Q_r	Q_b	TC
$D00$	1	6	1	51.75	34.50	51.75	386.44	1	6	1	51.75	34.50	51.75	386.44
$D01$	1	7	1	61.04	34.88	61.04	83725.92	1	7	1	61.04	34.88	61.04	83725.92
$D02$	1	1	1	69.08	17.27	69.08	106793.75	1	1	1	69.08	17.27	69.08	106793.75
$D03$	1	1	1	52.64	22.56	52.64	72698.91	1	1	1	52.64	22.56	52.64	72698.91
$D04$	1	1	1	50.67	21.72	50.67	112849.30	1	1	1	50.67	21.72	50.67	112849.30
$D05$	1	2	1	60.81	30.41	60.81	139031.44	1	2	1	60.81	30.41	60.81	139031.44
$D06$	1	2	1	69.48	81.06	69.48	93704.48	1	2	1	69.48	81.06	69.48	93704.48
$D07$	1	2	1	7.09	14.17	7.09	1874.31	1	2	1	7.09	14.17	7.09	1874.31
$D08$	1	3	1	10.12	13.49	10.12	1386.99	1	3	1	10.12	13.49	10.12	1386.99
$D09$	2	1	2	12.57	6.29	12.57	4963.65	1	1	1	14.43	3.61	14.43	4970.68
$D10$	1	1	1	35.98	35.98	35.98	254289.44	1	1	1	35.98	35.98	35.98	254289.44
$D11$	1	1	2	25.96	25.96	12.98	2387.57	1	1	2	25.96	25.96	12.98	2387.57
$D12$	1	2	2	17.75	8.87	8.87	4842.61	1	2	2	17.75	8.87	8.87	4842.61
$D13$	1	1	7	59.66	59.66	8.52	7346.43	1	1	7	59.66	59.66	8.52	7346.43
$D14$	1	7	1	61.04	34.88	61.04	83725.92	1	7	1	61.04	34.88	61.04	83725.92
$D15$	1	1	1	175.15	116.77	175.15	98464.73	1	1	1	175.15	116.77	175.15	98464.73
$D16$	5	1	5	314.50	174.72	314.50	51897.88	1	1	1	383.03	42.56	383.03	52274.56
$D17$	2	1	2	300.38	257.46	300.38	95721.50	1	1	1	370.96	158.98	370.96	95755.46
$D18$	1	1	1	11.64	2.91	11.64	1018.73	1	1	1	11.64	2.91	11.64	1018.73
$D19$	1	1	1	15.40	3.85	15.40	4988.63	1	1	1	15.40	3.85	15.40	4988.63
$D20$	1	2	1	19.72	9.86	19.72	15493.45	1	2	1	19.72	9.86	19.72	15493.45

	$N_p, 1, N_b$							$1, 1, 1$						
Problem	N_p	N_r	N_b	Q_p	Q_r	Q_b	TC	N_p	N_r	N_b	Q_p	Q_r	Q_b	TC
$D00$	1	1	1	18.63	74.54	18.63	536.66	1	1	1	18.63	74.54	18.63	536.66
$D01$	1	1	1	22.00	88.01	22.00	83936.30	1	1	1	22.00	88.01	22.00	83936.30
$D02$	1	1	1	69.08	17.27	69.08	106793.75	1	1	1	69.08	17.27	69.08	106793.75
$D03$	1	1	1	52.64	22.56	52.64	72698.91	1	1	1	52.64	22.56	52.64	72698.91
$D04$	1	1	1	50.67	21.72	50.67	112849.30	1	1	1	50.67	21.72	50.67	112849.30
$D05$	1	1	1	50.40	50.40	50.40	139034.98	1	1	1	50.40	50.40	50.40	139034.98
$D06$	1	1	1	46.97	109.59	46.97	93738.75	1	1	1	46.97	109.59	46.97	93738.75
$D07$	1	1	1	4.86	19.46	4.86	1882.68	1	1	1	4.86	19.46	4.86	1882.68
$D08$	1	1	1	4.88	19.54	4.88	1398.27	1	1	1	4.88	19.54	4.88	1398.27
$D09$	2	1	2	12.57	6.29	12.57	4963.65	1	1	1	14.43	3.61	14.43	4970.68
$D10$	1	1	1	35.98	35.98	35.98	254289.44	1	1	1	35.98	35.98	35.98	254289.44
$D11$	1	1	2	25.96	25.96	12.98	2387.57	1	1	1	18.83	18.83	18.83	2390.62
$D12$	1	1	1	13.26	13.26	13.26	4842.88	1	1	1	13.26	13.26	13.26	4842.88
$D13$	1	1	7	59.66	59.66	8.52	7346.43	1	1	1	35.42	35.42	35.42	7408.10
$D14$	1	1	1	22.00	88.01	22.00	83936.30	1	1	1	22.00	88.01	22.00	83936.30
$D15$	1	1	1	175.15	116.77	175.15	98464.73	1	1	1	175.15	116.77	175.15	98464.73
$D16$	5	1	5	314.50	174.72	314.50	51897.88	1	1	1	383.03	42.56	383.03	52274.56
$D17$	2	1	2	300.38	257.46	300.38	95721.50	1	1	1	370.96	158.98	370.96	95755.46
$D18$	1	1	1	11.64	2.91	11.64	1018.73	1	1	1	11.64	2.91	11.64	1018.73
$D19$	1	1	1	15.40	3.85	15.40	4988.63	1	1	1	15.40	3.85	15.40	4988.63
$D20$	1	1	1	15.61	15.61	15.61	15506.23	1	1	1	15.61	15.61	15.61	15506.23

Table 4.5: Optimal lot sizes and numbers of lots for Problem Set E under various policy restrictions for the both high and low quality recovery strategy ($\zeta_L = 1$)

	N_p, N_r, N_b							$1, N_r, N_b$						
Problem	N_p	N_r	N_b	Q_p	Q_r	Q_b	TC	N_p	N_r	N_b	Q_p	Q_r	Q_b	TC
$E00$	1	6	1	51.755	34.503	51.755	386.437	1	6	1	51.755	34.503	51.755	386.437
$E01$	1	7	1	61.045	34.883	61.045	425.918	1	7	1	61.045	34.883	61.045	425.918
$E02$	1	1	1	67.898	16.975	67.898	400.600	1	1	1	67.898	16.975	67.898	400.600
$E03$	1	1	1	52.644	22.562	52.644	398.909	1	1	1	52.644	22.562	52.644	398.909
$E04$	1	1	1	51.377	22.019	51.377	640.371	1	1	1	51.377	22.019	51.377	640.371
$E05$	1	2	1	60.813	30.407	60.813	631.443	1	2	1	60.813	30.407	60.813	631.443
$E06$	1	2	1	69.481	81.061	69.481	604.483	1	2	1	69.481	81.061	69.481	604.483
$E07$	1	2	1	7.087	14.174	7.087	90.307	1	2	1	7.087	14.174	7.087	90.307
$E08$	1	3	1	10.116	13.488	10.116	86.994	1	3	1	10.116	13.488	10.116	86.994
$E09$	2	1	2	12.571	6.285	12.571	203.647	1	1	1	14.430	3.608	14.430	210.675
$E10$	1	1	1	35.978	35.978	35.978	889.436	1	1	1	35.978	35.978	35.978	889.436
$E11$	1	1	2	25.963	25.963	12.982	50.071	1	1	2	25.963	25.963	12.982	50.071
$E12$	1	2	2	17.748	8.874	8.874	67.612	1	2	2	17.748	8.874	8.874	67.612
$E13$	1	1	7	59.657	59.657	8.522	163.184	1	1	7	59.657	59.657	8.522	163.184
$E14$	1	2	1	74.023	37.012	74.023	540.370	1	2	1	74.023	37.012	74.023	540.370
$E15$	1	1	1	175.148	116.765	175.148	1164.732	1	1	1	175.148	116.765	175.148	1164.732
$E16$	2	1	2	313.209	69.602	313.209	2945.321	1	1	1	320.463	35.607	320.463	2948.856
$E17$	1	2	1	305.717	229.288	305.717	2747.635	1	2	1	305.717	229.288	305.717	2747.635
$E18$	1	1	1	11.640	2.910	11.640	68.731	1	1	1	11.640	2.910	11.640	68.731
$E19$	1	1	1	15.396	3.849	15.396	228.631	1	1	1	15.396	3.849	15.396	228.631
$E20$	1	2	1	20.298	10.149	20.298	236.474	1	2	1	20.298	10.149	20.298	236.474
	$N_p, 1, N_b$							$1, 1, 1$						
Problem	N_p	N_r	N_b	Q_p	Q_r	Q_b	TC	N_p	N_r	N_b	Q_p	Q_r	Q_b	TC
$E00$	1	1	1	18.634	74.536	18.634	536.656	1	1	1	18.634	74.536	18.634	536.656
$E01$	1	1	1	22.002	88.009	22.002	636.302	1	1	1	22.002	88.009	22.002	636.302
$E02$	1	1	1	67.898	16.975	67.898	400.600	1	1	1	67.898	16.975	67.898	400.600
$E03$	1	1	1	52.644	22.562	52.644	398.909	1	1	1	52.644	22.562	52.644	398.909
$E04$	1	1	1	51.377	22.019	51.377	640.371	1	1	1	51.377	22.019	51.377	640.371
$E05$	1	1	1	50.395	50.395	50.395	634.980	1	1	1	50.395	50.395	50.395	634.980
$E06$	1	1	1	46.967	109.589	46.967	638.749	1	1	1	46.967	109.589	46.967	638.749
$E07$	1	1	1	4.864	19.457	4.864	98.677	1	1	1	4.864	19.457	4.864	98.677
$E08$	1	1	1	4.884	19.537	4.884	98.273	1	1	1	4.884	19.537	4.884	98.273
$E09$	2	1	2	12.571	6.285	12.571	203.647	1	1	1	14.430	3.608	14.430	210.675
$E10$	1	1	1	35.978	35.978	35.978	889.436	1	1	1	35.978	35.978	35.978	889.436
$E11$	1	1	2	25.963	25.963	12.982	50.071	1	1	1	18.826	18.826	18.826	53.117
$E12$	1	1	1	13.260	13.260	13.260	67.876	1	1	1	13.260	13.260	13.260	67.876
$E13$	1	1	7	59.657	59.657	8.522	163.184	1	1	1	35.423	35.423	35.423	224.853
$E14$	1	1	1	61.347	61.347	61.347	570.526	1	1	1	61.347	61.347	61.347	570.526
$E15$	1	1	1	175.148	116.765	175.148	1164.732	1	1	1	175.148	116.765	175.148	1164.732
$E16$	2	1	2	313.209	69.602	313.209	2945.321	1	1	1	320.463	35.607	320.463	2948.856
$E17$	1	1	1	211.801	317.702	211.801	2832.843	1	1	1	211.801	317.702	211.801	2832.843
$E18$	1	1	1	11.640	2.910	11.640	68.731	1	1	1	11.640	2.910	11.640	68.731
$E19$	1	1	1	15.396	3.849	15.396	228.631	1	1	1	15.396	3.849	15.396	228.631
$E20$	1	1	1	16.303	16.303	16.303	245.357	1	1	1	16.303	16.303	16.303	245.357

Table 4.6: Optimal lot sizes and numbers of lots for Problem Set E under various policy restrictions for the high quality only recovery strategy ($\zeta_L = 0$)

	N_p, N_r, N_b							$1, N_r, N_b$						
Problem	N_p	N_r	N_b	Q_p	Q_r	Q_b	TC	N_p	N_r	N_b	Q_p	Q_r	Q_b	TC
$E00$	1	6	1	51.755	34.503	51.755	386.437	1	6	1	51.755	34.503	51.755	386.437
$E01$	1	6	1	52.022	39.016	26.011	461.346	1	6	1	52.022	39.016	26.011	461.346
$E02$	1	1	1	64.334	36.188	44.230	422.794	1	1	1	64.334	36.188	44.230	422.794
$E03$	1	1	1	53.404	30.517	45.775	393.228	1	1	1	53.404	30.517	45.775	393.228
$E04$	1	1	1	48.080	37.777	30.909	684.279	1	1	1	48.080	37.777	30.909	684.279
$E05$	1	1	1	50.099	90.179	10.020	638.732	1	1	1	50.099	90.179	10.020	638.732
$E06$	1	2	1	65.171	97.756	21.724	644.464	1	2	1	65.171	97.756	21.724	644.464
$E07$	1	2	1	6.806	14.463	5.105	94.035	1	2	1	6.806	14.463	5.105	94.035
$E08$	1	2	1	7.166	17.019	1.792	94.894	1	2	1	7.166	17.019	1.792	94.894
$E09$	1	1	1	14.091	9.687	7.926	215.744	1	1	1	14.091	9.687	7.926	215.744
$E10$	1	1	1	32.575	52.120	13.030	982.343	1	1	1	32.575	52.120	13.030	982.343
$E11$	1	1	2	25.644	28.209	11.540	50.693	1	1	2	25.644	28.209	11.540	50.693
$E12$	1	1	1	13.673	15.040	12.306	65.824	1	1	1	13.673	15.040	12.306	65.824
$E13$	1	1	7	59.646	59.766	8.504	163.212	1	1	7	59.646	59.766	8.504	163.212
$E14$	1	2	1	71.445	46.439	50.011	559.875	1	2	1	71.445	46.439	50.011	559.875
$E15$	1	1	1	155.435	194.293	64.764	1312.450	1	1	1	155.435	194.293	64.764	1312.450
$E16$	1	1	1	323.505	71.890	287.560	2921.130	1	1	1	323.505	71.890	287.560	2921.130
$E17$	1	2	1	292.467	255.909	219.350	2872.118	1	2	1	292.467	255.909	219.350	2872.118
$E18$	1	1	1	11.503	4.314	10.065	69.549	1	1	1	11.503	4.314	10.065	69.549
$E19$	1	1	1	13.686	16.252	0.855	257.202	1	1	1	13.686	16.252	0.855	257.202
$E20$	1	2	1	19.670	11.802	15.736	244.026	1	2	1	19.670	11.802	15.736	244.026

	$N_p, 1, N_b$							$1, 1, 1$						
Problem	N_p	N_r	N_b	Q_p	Q_r	Q_b	TC	N_p	N_r	N_b	Q_p	Q_r	Q_b	TC
$E00$	1	1	1	18.634	74.536	18.634	536.656	1	1	1	18.634	74.536	18.634	536.656
$E01$	1	1	1	21.635	97.357	10.818	647.101	1	1	1	21.635	97.357	10.818	647.101
$E02$	1	1	1	64.334	36.188	44.230	422.794	1	1	1	64.334	36.188	44.230	422.794
$E03$	1	1	1	53.404	30.517	45.775	393.228	1	1	1	53.404	30.517	45.775	393.228
$E04$	1	1	1	48.080	37.777	30.909	684.279	1	1	1	48.080	37.777	30.909	684.279
$E05$	1	1	1	50.099	90.179	10.020	638.732	1	1	1	50.099	90.179	10.020	638.732
$E06$	1	1	1	46.476	139.427	15.492	645.497	1	1	1	46.476	139.427	15.492	645.497
$E07$	1	1	1	4.763	20.241	3.572	100.788	1	1	1	4.763	20.241	3.572	100.788
$E08$	1	1	1	4.715	22.397	1.179	101.800	1	1	1	4.715	22.397	1.179	101.800
$E09$	1	1	1	14.091	9.687	7.926	215.744	1	1	1	14.091	9.687	7.926	215.744
$E10$	1	1	1	32.575	52.120	13.030	982.343	1	1	1	32.575	52.120	13.030	982.343
$E11$	1	1	2	25.644	28.209	11.540	50.693	1	1	1	19.245	21.170	17.321	51.962
$E12$	1	1	1	13.673	15.040	12.306	65.824	1	1	1	13.673	15.040	12.306	65.824
$E13$	1	1	7	59.646	59.766	8.504	163.212	1	1	1	35.461	35.532	35.390	224.614
$E14$	1	1	1	59.918	77.893	41.942	584.136	1	1	1	59.918	77.893	41.942	584.136
$E15$	1	1	1	155.435	194.293	64.764	1312.450	1	1	1	155.435	194.293	64.764	1312.450
$E16$	1	1	1	323.505	71.890	287.560	2921.130	1	1	1	323.505	71.890	287.560	2921.130
$E17$	1	1	1	204.191	357.334	153.143	2938.431	1	1	1	204.191	357.334	153.143	2938.431
$E18$	1	1	1	11.503	4.314	10.065	69.549	1	1	1	11.503	4.314	10.065	69.549
$E19$	1	1	1	13.686	16.252	0.855	257.202	1	1	1	13.686	16.252	0.855	257.202
$E20$	1	1	1	15.860	19.032	12.688	252.206	1	1	1	15.860	19.032	12.688	252.206

policy, 14 out of 21 cases. Of the 21 base problems, $N_p > 1$ in 2 problems, $N_r > 1$ in 10 problems, and $N_b > 1$ in 5 problems.

Relative Cost Error. As shown in these tables, the difference in the total cost under the unrestricted policy (N_p, N_r, N_b) and the other policies varies. This is shown more clearly in Table 4.7. This table shows the relative cost errors of the restricted policies compared with the unrestricted policies for problem sets D and E. For each of the restricted policies, the relative cost error is calculated as follows:

$$\text{RCE} = \frac{\text{restricted} - \text{unrestricted}}{\text{unrestricted}} \times 100\%$$

As shown in Table 4.7 RCEs vary depending on the structure of the unrestricted optimal policy. For some problems the RCE is zero, meaning that the optimal unrestricted policy is contained within that class, but for other problems the RCE is quite large. In general the RCE is lower for problem set D than problem set E. This is a reflection of the larger total costs associated with problem set D, which includes the processing unit costs.

Further Analysis. Problem set F was used to carry out further analysis on the effect of policy restrictions. This problem set was created by varying the values of individual parameters in problem set E and it contains 1457 problems. The unit processing costs are set to zero, as for problem set E. Across this larger dataset similar results were found. Of the 1457 problems in problem set F, there were 718 which had optimal policies contained within the $(1,1,1)$ policy class when $\zeta_L = 1$, and 403 which had optimal policies contained in this class when $\zeta_L = 0$.

The relative cost errors are summarised in Table 4.8. As shown in this table, restricting the number of lots per cycle had a greater impact on the relative cost error when only high-quality recovery is performed.

4.7.3 Analysis of Recovery Strategy

In this section the conditions determining the optimality of the recovery strategies are investigated. This analysis could help firms to decide under what conditions it is optimal

Table 4.7: Relative cost error (RCE) compared with the unrestricted (N_p, N_r, N_b) policy for both recovery strategies

(a) Problem Set D

	High and low quality recovery			High quality recovery only		
Problem	$1, N_r, N_b$	$N_p, 1, N_b$	$1, 1, 1$	$1, N_r, N_b$	$N_p, 1, N_b$	$1, 1, 1$
$D00$	0	38.873	38.873	0	38.873	38.873
$D01$	0	0.225	0.225	0	0.251	0.251
$D02$	0	0	0	0	0	0
$D03$	0	0	0	0	0	0
$D04$	0	0	0	0	0	0
$D05$	0	0	0	0	0.003	0.003
$D06$	0	0.001	0.001	0	0.037	0.037
$D07$	0	0.362	0.362	0	0.447	0.447
$D08$	0	0.508	0.508	0	0.813	0.813
$D09$	0	0	0	0.142	0	0.142
$D10$	0	0	0	0	0	0
$D11$	0	0	0.055	0	0	0.128
$D12$	0	0	0	0	0.005	0.005
$D13$	0	0	0.836	0	0	0.839
$D14$	0	0.213	0.213	0	0.251	0.251
$D15$	0	0	0	0	0	0
$D16$	0.181	0	0.181	0.726	0	0.726
$D17$	0	0	0	0.035	0	0.035
$D18$	0	0	0	0	0	0
$D19$	0	0	0	0	0	0
$D20$	0	0.086	0.086	0	0.082	0.082

(b) Problem Set E

	High and low quality recovery			High quality recovery only		
Problem	$1, N_r, N_b$	$N_p, 1, N_b$	$1, 1, 1$	$1, N_r, N_b$	$N_p, 1, N_b$	$1, 1, 1$
$E00$	0	38.873	38.873	0	38.873	38.873
$E01$	0	40.264	40.264	0	49.396	49.396
$E02$	0	0	0	0	0	0
$E03$	0	0	0	0	0	0
$E04$	0	0	0	0	0	0
$E05$	0	0	0	0	0.560	0.560
$E06$	0	0.160	0.160	0	5.669	5.669
$E07$	0	7.182	7.182	0	9.268	9.268
$E08$	0	7.277	7.277	0	12.965	12.965
$E09$	0	0	0	3.451	0	3.451
$E10$	0	0	0	0	0	0
$E11$	0	0	2.502	0	0	6.083
$E12$	0	0	0	0	0.390	0.390
$E13$	0	0	37.621	0	0	37.791
$E14$	0	4.333	4.333	0	5.581	5.581
$E15$	0	0	0	0	0	0
$E16$	0	0	0	0.120	0	0.120
$E17$	0	2.309	2.309	0	3.101	3.101
$E18$	0	0	0	0	0	0
$E19$	0	0	0	0	0	0
$E20$	0	3.352	3.352	0	3.756	3.756

	$\zeta_L = 1$			$\zeta_L = 0$		
	$1, N_r, N_b$	$N_p, 1, N_b$	$1,1,1$	$1, N_r, N_b$	$N_p, 1, N_b$	$1,1,1$
min	0	0	0	0	0	0
max	27.184	109.869	145.182	43.192	119.049	148.035
mean	0.264	5.764	7.834	0.579	7.187	9.947
sd	1.745	13.214	15.375	2.548	14.899	16.658
number of policies with $(1,1,1)$	772	1261	1457	578	1060	1457
number with optimal policies of $(1,1,1)$	718	718	718	403	403	403

Table 4.8: Summary of the relative cost errors (RCE) compared with the unrestricted (N_p, N_r, N_b) policy for both recovery strategies, $n = 1457$ problems

to perform both high quality recovery and low quality recovery. Firstly some general properties of the total cost function are discussed, and then two main properties are investigated: the effect of the *disposal cost* c_d and the effect of the *quality of the returns*. In this section we focus on the case with one of each type of lot, $(N_p = N_r = N_b = 1)$.

Recall from Section 4.6.2 that when there is one of each type of lot, the optimization problem is reduced to minimizing with respect to Q_p. The cost-minimising production lot size is:

$$Q_p^*(1,1,1) = \sqrt{\frac{d(1-\alpha f)(k_p + k_r + k_b)}{V + W + X + Y - Z_1 + Z_2 + Z_3}}$$

which results in a total cost of:

$$TC(Q_p^*, 1,1,1) = C_P + 2\sqrt{d(1-\alpha f)(k_p + k_r + k_b)(V + W + X + Y - Z_1 + Z_2 + Z_3)}$$

or replacing the substitutions of $C_P, V, W, X, Y, Z_1, Z_2, Z_3$:

$$\begin{aligned}
TC(Q_p^*, 1,1,1) = {} & c_r f d + c_b d(1-f) + c_p(1-\alpha f)d + c_h \alpha f d + c_l(1-\alpha) f d + c_d(1-f)d \\
& + 2\sqrt{d(1-\alpha f)(k_p+k_r+k_b)} \times \left(\frac{h_s(1-\alpha f)(p-d)}{2p} + \frac{h_r f}{2} \right. \\
& + \frac{h_c f^2 (1-\alpha)}{2(1-\alpha f)} \left(\frac{d\,(1-\alpha)}{p} + \alpha \right) + \frac{f^2(\alpha r - d)}{2r(1-\alpha f)} (h_s \alpha + h_r + h_c(1-\alpha)) \\
& + \frac{h_c d(1-f)^2}{2p(1-\alpha f)} - \frac{h_c(1-\alpha f)(p-d)}{2p} \\
& \left. + h_c \left(1 - \frac{d}{p}\right) \left(f(1-\alpha) - \frac{(1-\alpha f)}{2} \right) + \frac{h_c(p-d)(1-f)}{p} \right)^{0.5}
\end{aligned}$$

From this function it is obvious that the optimal total cost increases with c_r, c_b, c_p, c_h, c_l, c_d. The term which is derived from $(V + W + X + Y - Z_1 + Z_2 + Z_3)$ is always positive, as is $d(1 - \alpha f)$, therefore the optimal total cost also increases with k_p, k_r and k_b. The optimal production lot size Q_p^* is increasing in the set up costs k_p, k_r and k_b. Since the holding costs are represented by the term $(V + W + X + Y - Z_1 + Z_2 + Z_3)$ in the denominator, the optimal production lot size Q_p^* decreases as the holding costs h_s, h_r and h_c increase.

Disposal Cost

The term c_d denotes the cost of disposing of any returns which are not recovered. This cost could represent a production tax incurred by producers that do not take responsibility for the fate of their goods after the customer has used them. Under a high quality recovery strategy, this cost is incurred for more items, compared with under a both high and low quality recovery strategy. Thus, as this disposal cost c_d increases, it may become worthwhile to perform both low and high quality recovery.

Let $TC_0(Q_p^*, 1, 1, 1)$ and Q_{p0}^* denote the optimal total cost and production lot size under a high-quality only recovery strategy ($\zeta_L = 0$); and let $TC_1(Q_p^*, 1, 1, 1)$ and Q_{p1}^* denote the optimal total cost and production lot size when both high and low quality returns are covered ($\zeta_L = 1$). Let C_{P0} and C_{P1} denote the values of C_P under recovery strategies with $\zeta_L = 0$ and $\zeta_L = 1$ respectively. Let the similar notation apply to the other expressions $V, W, X, Y, Z_1, Z_2, Z_3$. Then under a high-quality only recovery strategy ($\zeta_L = 0$), the optimal total cost can be written as:

$$TC_0(Q_{p0}^*, 1, 1, 1) = C_{P0} + 2\sqrt{d(1 - \beta_H)(k_p + k_r + k_b)} \times \sqrt{(V_0 + W_0 + X_0 + Y_0 - Z_{10} + Z_{20} + Z_{30})}$$

and under a both high and low quality recovery strategy ($\zeta_L = 1$), the optimal total cost can be written as:

$$TC_1(Q_{p1}^*, 1, 1, 1) = C_{P1} + 2\sqrt{d(1 - \beta_H)(k_p + k_r + k_b)} \times \sqrt{(V_1 + W_1 + X_1 + Y_1 - Z_{11} + Z_{21} + Z_{31})}$$

It is profitable to perform low quality recovery when:

$$TC_1(Q_{p1}^*, 1, 1, 1) < TC_0(Q_{p0}^*, 1, 1, 1) \tag{4.36}$$

Using this relationship, it is possible to calculate the values of the disposal cost c_d, for which is it profitable to perform low quality recovery, given all other parameters remain unchanged.

Using the full expressions for C_{P0} and C_{P1} gives:

$$\begin{aligned}
C_{P0} &= c_p(1-\beta_H)d + c_h\beta_H d + c_l(0)\beta_H d + c_r\beta_H d + c_b d(1-\beta_H) + c_d(1-\beta_H)d \\
C_{P1} &= c_p(1-\beta_H)d + c_h\beta_H d + c_l\beta_L d + c_r(\beta_H+\beta_L)d + \\
&\qquad c_b d(1-(\beta_H+\beta_L)) + c_d(1-(\beta_H+\beta_L))d
\end{aligned}$$

Substituting these expressions into equation (4.36), and performing some simple algebra to solve for the disposal cost c_d yields the following result (further details available in Appendix A.5.1):

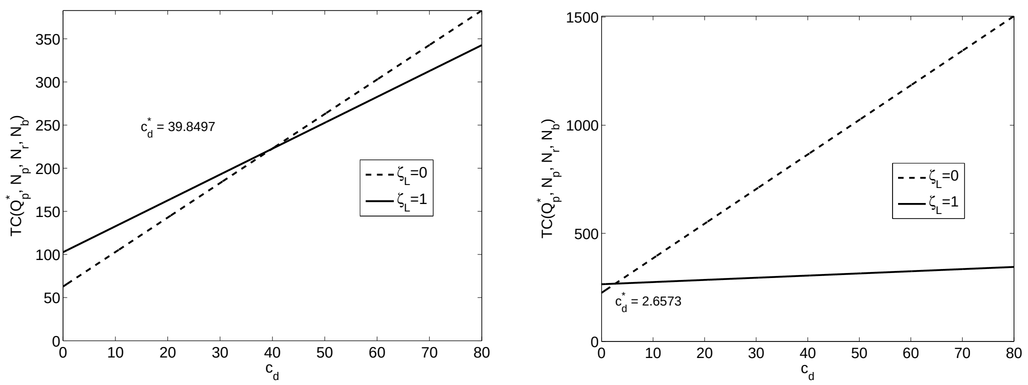
$$\begin{aligned}
c_d^* > \frac{1}{\beta_L d}\Bigg(& c_l\beta_L d + c_r\beta_L d - c_b d\beta_L \\
& + 2\sqrt{d(1-\beta_H)(k_p+k_r+k_b)(V_1+W_1+X_1+Y_1-Z_{11}+Z_{21}+Z_{31})} \\
& - 2\sqrt{d(1-\beta_H)(k_p+k_r+k_b)(V_0+W_0+X_0+Y_0-Z_{10}+Z_{20}+Z_{30})}\Bigg)
\end{aligned} \tag{4.37}$$

Thus if all cost parameters remain unchanged, then it will be profitable to perform low quality recovery if the disposal cost satisfies the condition given by equation (4.37). It may be possible to simplify this expression further, how we do not do this here due to the complicated nature of the function. Similar expressions are possible for the other unit cost parameters.

The use of this condition is applied to two problems from problem set E as an example. For problems $E07$ and $E19$, the values of c_d above which it becomes profitable to perform low quality recovery are 2.1108 and 1.9047 respectively. The relationship between the total cost and the value of the disposal cost is demonstrated for these two problems in Figures 4.9a and 4.9b respectively.

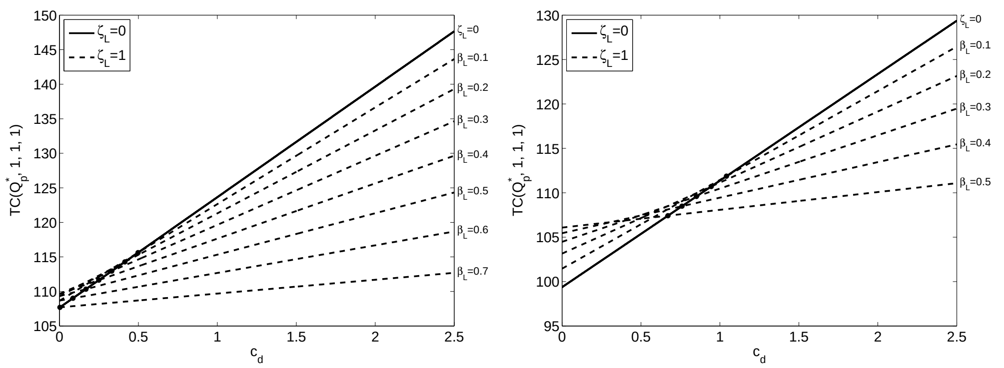
Quality of Returns

The quality of returns is likely to affect the performance of the system, and in particular, will influence the optimality of each recovery strategy. The restricted case of $(1, 1, 1)$ will be considered here in order to simplify the search for the optimal policy.

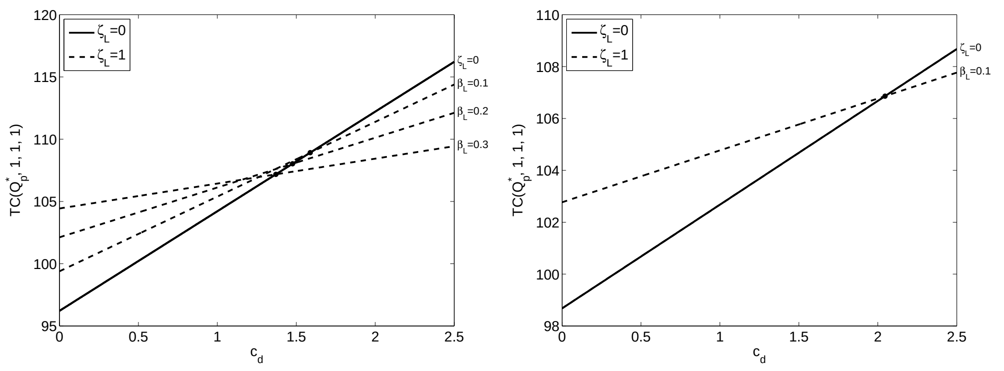


(a) Problem $E07$ (b) Problem $E19$

Figure 4.9: Values of the disposal cost c_d for which it is profitable to do low quality recovery



(a) $\beta_H = 0.2$ (b) $\beta_H = 0.4$



(c) $\beta_H = 0.6$ (d) $\beta_H = 0.8$

Figure 4.10: Optimal total cost across a range of disposal cost c_d for a variety of values of β_L under a policy with $N_p = N_r = N_b = 1$ for problem $E07$.

Recall that the proportion of returns that are high quality is denoted by β_H and the proportion of returns that are low quality is denoted by β_L. Figure 4.10 shows the effect on the total cost for a variety of values of β_H and β_L, across a range disposal costs for Problem $E07$. These graphs show that as the proportion of high quality returns β_H increases, the disposal cost has to be much higher in order for low quality recovery to be profitable. This effectively means that if there is a high proportion of high quality returns, the firm is not concerned about the low quality returns, unless the penalty cost associated with disposing of them is quite high. However for a fixed proportion of high quality returns β_H, the disposal cost for which low quality recovery is profitable decreases in β_L. These observations refer only to problem $E07$ displayed in Figure 4.10. Further investigations are required to determine if this relationship exists for all problems under a $(1, 1, 1)$ policy restriction, or a $N_p, N_r, N_b \geq 1$ policy.

4.8 Discussion

This chapter presents a product recovery model which uses the quality of returns to determine the type of recovery that is performed. High quality returns undergo high quality recovery, which is used to replenish the serviceable inventory; low quality returns undergo low quality recovery, which is used to replenish the component inventory. Components are required for the production of new products and can be bought, as well as salvaged from the low quality returns. The firm is interested in the strategic recovery decision of whether to recover both high quality and low quality returns, or only high quality returns.

An EOQ lot sizing model is used to study this problem. The total cost per time unit was derived in terms of the production lot size Q_p, and the number of production, recovery and buying lots per cycle N_p, N_r, N_b. The total cost function is a function in both integer and continuous variables and therefore the minimisation of this function is not trivial. In fact, the problem can be presented as a mixed integer non linear program (MINLP). The integer variables correspond to the number of lots per cycle, N_p, N_r, N_b, therefore we first consider the case in which $N_p = N_r = N_b = 1$. This is generalised to allow one production lot $N_p = 1$ and multiple recovery and buying lots $N_r, N_b \geq 1$.

Finally it is generalised further to allow multiple production, recovery and buying lots $N_p, N_r, N_b \geq 1$.

For the case of $N_p = N_r = N_b = 1$ the total cost function is convex in Q_p and closed form solution is derived for optimal lot size for production Q_p^*. For the case of $N_p = 1$ and $N_r, N_b \geq 1$, if N_r and N_b are fixed, then the total cost function is convex in Q_p. Following Konstantaras and Papachristos (2008b) a method was presented which allowed bounds to be developed to find values for N_r and N_b.

For the general case which allows multiple production, recovery and ordering lots per cycle, the convexity of the total cost function with respect to Q_p is proved, for a fixed N_p, N_r, N_b. An expression is provided for the optimal production lot size Q_p, for a fixed N_p, N_r, N_b and a search algorithm is presented to determine the optimal values of N_p, N_r, N_b.

The effect of restricting the number of lots per cycle is investigated by calculating the optimal policy under four policy scenarios, for both recovery strategies: high quality recovery only and both high and low quality recovery. It was found that restricting the number of lots per cycle resulted in a greater increase in the total cost when only high quality recovery was performed. This suggests that performing both high and low quality recovery reduces the sensitivity of the optimal solution to variations in the number of lots per cycle. This may be due to the fact that performing both types of recovery allows cost savings to be achieved, and these can counteract any increases resulting from a policy with a sub-optimal number of lots per cycle.

From a strategic planning point of view, it is interesting to investigate the conditions under which it is optimal to perform low quality recovery. It was found that if the disposal cost is greater than a certain level, say c_d^*, it will be optimal to perform both high and low quality recovery, otherwise only high quality recovery should be performed. For the $N_p = N_r = N_b = 1$ case, an expression was presented which allows this level c_d^* to be calculated. The effect of the quality of returns was investigated using numerical experiments. For the problems tested, it was found that if there is a high proportion of high quality returns, the firm should not be concerned about recovering the low quality returns, unless the penalty cost associated with disposing of them is quite high.

There are many possibilities for future research. Firstly, further results relating to the general case of multiple production, recovery and ordering lots per cycle may be possible, either theoretically or numerically. Secondly, the effect of rounding the lot size variables Q_p, Q_r, Q_b to the nearest integer could also be addressed. There are also several extensions to the model which could be studied, for example, it is possible that with enough expenditure all returns could be recovered into serviceable inventory. Therefore the model could be extended to allow the proportion of high quality returns to be a decision variable. A further extension could allow the arrival rate of a components order to be finite, i.e., the components would arrive gradually, rather than as a single batch.

The model in this chapter allows the profitability of having quality dependent recovery channels to be assessed in a deterministic context, however in reality, product recovery firms may not be operating in such a predictable world. Whilst it can be argued that in some manufacturing environments, contractual obligations remove much of the uncertainty around the rate of demand and returns, this does not apply to all firms. For many firms there are uncertainties associated with the quality, quantity and timing of returns, not to mention with the arrival of demand. This limitation will be addressed in Chapter 5 where a stochastic product recovery model is proposed.

Chapter 5

Discrete-Time Stochastic Product Recovery Model with a Single Market

5.1 Introduction

Product recovery systems operate in an uncertain environment. The demand for and return of goods can be uncertain, in terms of quantity and timing. The quality of returns can also be uncertain. These uncertainties add operational complications for firms in this industry.

This chapter presents a stochastic product recovery model in which goods no longer required by the consumer are returned to their producer for recovery. The returns have varying quality. If returns are of sufficiently high quality then they can be recovered to be 'as good as new', otherwise the components can be salvaged and used as inputs in the production of new goods. In this model demand, returns and the quality of returns are uncertain. This model extends the deterministic product recovery model presented in Chapter 4 by allowing the demand, returns and quality of returns to be stochastic. The introduction of uncertainty into the model addresses one of the limitations of the deterministic model in Chapter 4.

A discrete time Markov decision process is used to model this stochastic product recovery system. In each time period, the inventory levels are observed and decisions are made regarding production, recovery and buying components. A periodic discrete-time model is chosen to model this problem, rather than a continuous model because one of the aims of this research is to examine the structure of the optimal policy. As discussed by Inderfurth (1997) periodic models are generally used when the policy structure is to be analysed.

This chapter is structured as follows. The problem description is presented in Section 5.2. In Section 5.3, a model for analysing this problem is proposed and following from that a Markov decision process formulation is presented in Section 5.4. The implementation and validation of the Markov decision process model is discussed in Section 5.5. Properties of the optimal policy are discussed in Section 5.6 and some heuristic policies are proposed and tested in Section 5.7. The results of the chapter are discussed in Section 5.8.

5.2 Problem Description

Suppose there is a firm which has a primary function of producing new goods. This firm accepts these goods back after they have been used and, if they are above the required quality threshold, recovers them to the same quality standard as newly produced goods and then sells them. Produced and recovered items are both considered to be 'serviceable' and are viewed as identical by the consumer so are sold on the same market. For returns which are below the quality threshold for recovery, the firm has a choice: to dispose of them or to use them as components in the production of newly produced items. If insufficient components are obtained from the recovery of low quality returns then additional components are bought. High-quality recovered goods are considered to be "as good as new".

The firm is a cost-minimising firm. Fixed and unit costs are incurred for production, recovery and buying components. Costs are also incurred for holding inventory and for lost sales. Demand for serviceable items, returns and the quality of returns are uncertain. The firm must determine a production plan that specifies how much and how often to produce, recover and buy. The firm is also interested in the cost-effectiveness of recovering low quality returns.

As in Chapter 4, this system could describe Canon's remanufacturing processes for cartridges and printers, which were discussed in Section 2.2.3. Used printers and cartridges are returned to Canon to undergo remanufacturing before being sold as new. In both cases, returns which cannot be returned to this "as new" condition may be used as parts or materials in the production process. Rigorous quality standards allow Canon to sell these remanufactured products as new. This system could also describe the situation faced by a Cooperage with respect to whisky barrels. whisky barrels are used and then returned for repair. Some barrels will require only minor repairs, whereas some may require more substantial repairs or may only used for parts. In Chapter 4, it was assumed that there were no uncertainties, however it is likely that these companies will face uncertainties in demand and in the quantity and quality of returns.

5.3 Model Description and Assumptions

Figure 5.1 presents the product recovery system being modelled in this chapter. As shown in this diagram, consumer demand is met by the stock of serviceable goods. The model and the modelling assumptions will be discussed in this section.

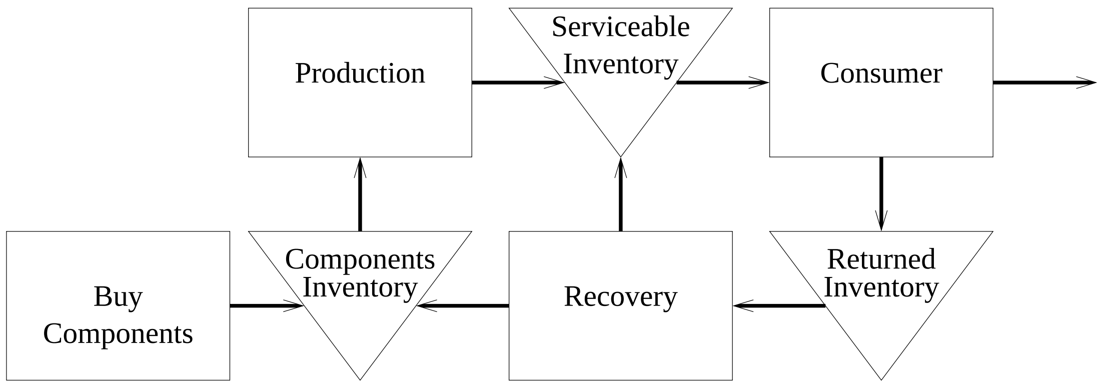


Figure 5.1: A product recovery system with dual-recovery channels

Inventory Levels. In this model there are three types of inventory: serviceable inventory (produced and recovered goods), returned inventory and component inventory.

It is assumed that there is a finite capacity available for storing each of the inventories. If this capacity is reached then a disposal cost may be incurred for each item which exceeds the maximum capacity. It is assumed that there is no backordering, therefore if there is insufficient inventory to meet demand, then the sale will be lost. Inventory levels must always be nonnegative.

Periodic Decision Making. In this model it is assumed that operational decisions are made periodically at set decision epochs. Nowadays many firms have the technology to monitor stock levels continuously, however despite this, many still only review this information and take action periodically, e.g. daily or weekly (Silver et al., 1998). It is assumed that the length of time between two subsequent decision epochs is one time unit.

At each decision epoch the inventory levels are reviewed and then a decision is made regarding, how much to produce and recover, and how many components to buy. The lead times for production, recovery and buying components are zero, therefore inventory levels are updated immediately to reflect the decision made. Demand and returns are then observed and the inventory levels are updated accordingly. This system is being studied over an infinite time horizon, which implies that the decisions regarding production, recovery and buying will not depend on time.

Production, Recovery and Buying. It is assumed that production and recovery require some shared facility, thus cannot both be selected at a given decision epoch. It is also assumed that components will only be bought at decision epochs at which production is also selected. This does not place any limitations on the model because it would be suboptimal to buy components in the same period as recovery, since any items bought would incur a holding cost until they were used. As mentioned above, the lead time for production, recovery and buying components is zero. At each decision epoch the amount that is produced or recovered and the number of components that are bought is non-negative and constrained by a finite upper limit. It is assumed that one component is required in the production of each new good.

The recovery process has two channels, one which results in serviceable goods and one which results in components. It is assumed that with enough effort and expenditure, all returns could be recovered up to the same serviceable standard as newly produced goods. However it is also assumed that the firm has set a threshold which determines the quality level of returns for which it is considered worthwhile to recover up to this serviceable standard. In the remainder of this chapter, returns which are above this quality threshold are referred to as 'high quality returns'. High quality returns undergo 'high quality recovery' which brings them up to a serviceable standard and are then sold alongside newly produced goods. Returns which fall below the quality threshold are termed 'low quality returns' and undergo 'low quality recovery' in order salvage components which can be used in production. It is assumed that the firm makes a strategic-level decision (outwith the model) about whether to recover low quality returns or to dispose of them. It is assumed that determining the quality of the returns requires some diagnostic testing which is performed as part of the recovery process. The quality of the returns and is modelled by a known probability distribution. If there is insufficient capacity in the serviceable inventory, then high quality returns will undergo low quality recovery and be used for components. Any recovered items which can not fit into either the serviceable inventory or the component inventory are discarded.

As mentioned above there is a finite capacity available for storing each of the inventories. It is assumed that decisions will never be made which could cause the inventories to exceed these maximum capacities with certainty. For instance, the amount produced must be less than or equal to the available capacity in serviceable inventory. Since the number of high quality returns resulting from a recovery lot is uncertain, the size of the recovery lot may exceed the available capacity of serviceable inventory.

Probabilistic Demand and Returns. It is assumed that the numbers of demand and returns observed each period are governed by known probability distributions and that these distributions are independent. It is assumed that the distributions governing the demand, returns and quality of returns is time-invariant, i.e., do not depend on time. This assumption is realistic for products which are established and are not seasonal and implies that the decisions regarding production, recovery and buying components will not depend on time, i.e. they will be stationary.

Costs. The costs incurred during a given time period depend on the initial inventory levels, the production, recovery and buying decisions, the quality of the returns, and the number of goods demanded and returned. The following types of costs are incurred: holding costs, setup costs, processing costs, lost sales costs, disposal costs. The objective of the firm is to minimise its long run average costs.

Setup costs are incurred each time a production, recovery or buying order is placed. This is a fixed cost which does not depend on the size of the order. It is assumed that a 'period' is a self-contained operational time, so that if an activity occurs, then a setup cost is incurred regardless of whether the activity was performed last period. Processing costs are incurred on a per unit basis for production, recovery and buying components. The cost of high-quality recovery is higher than for low quality recovery. Any inventory that is left in stock at the end of a period incurs a per unit holding cost.

If the demand exceeds the serviceable inventory, then a per unit lost sales cost is incurred. If the number of goods which undergo low quality recovery exceeds the available capacity in the component inventory, then a disposal cost is incurred. A disposal cost may also be incurred if returns exceed available capacity in returned inventory. These disposal, or penalty, costs may be positive or negative and may be different from each other. A negative disposal cost could represent a salvage value.

5.4 Markov Decision Process Formulation

The model described in the previous section is a discrete time stochastic decision problem, and the inventory levels in the next time period depend only on the inventory levels, the action taken and the demand and returns observed in the current time period. For these reasons an infinite-horizon Markov decision process is used to model this problem (Tijms, 1994). An overview of Markov decision processes is presented in Section 3.2. A Markov decision process formulation of this problem is presented in this section. A Markov decision process (MDP) is characterised by its decision epochs, states, actions, costs and transition probabilities. A summary of the notation used in this section is presented in Table 5.1.

Table 5.1: Summary of notation used in the MDP formulation of the product recovery model

State Space	
i_s, i_r, i_c	Level of serviceable, returned and component inventories in state i
I_s, I_r, I_c	State space for serviceable, returned and component inventories
W_s, W_r, W_c	Upper capacity limit on state space for serviceable, returned and component inventories
Action Space	
a_p, a_r, a_b	Number of items produced, recovered and bought for action $a(i)$
A_p, A_r, A_b	Action spaces for production, recovery and buying
L_p, L_r, L_b	Lower limit on action spaces for production, recovery and buying
U_p, U_r, U_b	Upper limit on action spaces for production, recovery and buying
Distributions	
X_d	Number of items demanded known distribution
X_r	Number of items returned known distribution
X_q	Number of high quality items in recovery batch a_r known distribution
α	Mean proportion of high quality items in recovery $0 < \alpha < 1$
λ_d	Mean demand
λ_r	Mean returns
Costs	
k_p, k_r, k_b	Set up cost per production lot, recovery lot and buying lot $k_p, k_r > 0,\ k_b \geq 0$
h_s, h_r, h_c	Holding cost per item per time unit for serviceable inventory, returned inventory and component inventory $h_s, h_r > 0,\ h_c \geq 0$
c_p, c_h, c_l	Unit processing cost for production, recovery of high quality items and recovery of low quality items $c_p, c_h, c_l \geq 0$
c_r, c_b, c_d	Unit acquisition cost for collecting returned products, buying new components and disposing on unrecovered returns $c_r, c_b, c_d \geq 0$
l_s, l_r	Unit penalty cost for lost sales and "lost" returns $l_r, l_r \geq 0$

5.4.1 Decision Epochs

Decision epochs occur at the beginning of each time period. At each decision epoch the state of the system is observed and then an action is chosen. The system then moves to the next state according to the transition probabilities. The time between decision epochs is one time unit.

5.4.2 States

The state of the system at a given time is determined by the levels of the serviceable, returned and component inventories. Let:

$$
\begin{aligned}
i_s &= \text{level of serviceable inventory}, & i_s &\in I_s = \{0,1,\ldots,W_s\}\\
i_r &= \text{level of returned inventory}, & i_r &\in I_r = \{0,1,\ldots,W_r\}\\
i_c &= \text{level of component inventory}, & i_c &\in I_c = \{0,1,\ldots,W_c\}
\end{aligned}
$$

where W_s, W_r, W_c are finite upper limits of inventory capacity and I_s, I_r, I_c are the sets of all possible inventory levels. Let i denote the state of the system at the beginning of a given time period, then:

$$
i = (i_s, i_r, i_c), \qquad i \in I = \{(i_s, i_r, i_c): \quad i_s \in I_s, \quad i_r \in I_r, \quad i_c \in I_c\}
$$

All state variables have finite upper limits of capacity, therefore the MDP can be described as having a finite state space.

5.4.3 Actions

At each decision epoch, the firm must decide how much to produce, how much to recover and how many components to buy. The number of items produced, recovered and bought are denoted by a_p, a_r and a_b. For a given state i, the values of these variables are chosen from sets of allowable actions: $A_p(i)$, $A_r(i)$, $A_b(i)$, for production, recovery and buying, respectively.

The minimum values that each of these actions can take are denoted by L_p, L_r, L_b for production, recovery and buying, respectively. Similarly, the maximum values that each action can take are denoted by U_p, U_r, U_b for production, recovery and buying, respectively. The upper and lower limits are nonnegative and finite. Using these upper and lower limits, a set of allowable actions can be defined for each action:

$$
a_p \in A_p(i) \subseteq \{0, L_p, \ldots, \times \min\{W_s - i_s, U_p\}\}
$$

$$
a_r \in A_r(i) \subseteq \begin{cases} \{0\} & \text{if } a_p > 0 \\ \{0, L_r, \ldots, \min\{i_r, U_r\}\} & \text{if } a_p = 0 \end{cases}
$$

$$
a_b \in A_b(i) \subseteq \begin{cases} \{0\} & \text{if } a_p = 0 \\ \{\min\{\max\{a_p - i_c, 0\}, L_b\}, \ldots, \max\{W_c - i_c, U_b\}\} & \text{if } a_p > 0 \end{cases}
$$

The restriction on the upper limit of $A_p(i)$ follows from the assumption that production will not result in the capacity of the serviceable inventory being exceeded. Therefore the amount of production must be less than the available capacity in serviceable inventory $W_s - i_s$, and than the upper limit on production U_p.

Recovery can only be performed if production is not performed ($a_p = 0$). If recovery can be performed, then the amount recovered must not exceed the number of returns in stock i_r or the upper limit on recovery U_r.

Components are only ordered when production is performed (as discussed in Section 5.3), therefore if production is not performed ($a_p = 0$) the number of components bought must equal 0. If production is performed, then there must be sufficient components for the entire production order. The lower limit on buying components $\max\{a_p - i_c, 0\}$ ensures that this is the case. The number of components which can be bought is restricted by the capacity available in the component inventory $W_c - i_c$ and by the upper limit on buying U_b.

For a given state i, an action $a(i)$ is chosen from the set of allowable actions $A(i)$, which is defined as:

$$a(i) \in A(i) = \{(a_p, a_r, a_b) : a_p \in A_p(i),\, a_r \in A_r(i),\, a_b \in A_b(i)\}.$$

5.4.4 Transition Probabilities

The transition from the current state to the next state depends on the action that is chosen, the quality of the returns recovered during the period, the number of goods which are returned, and the number of goods that are demanded.

Random Variables

The transition probabilities are determined by random variables X_d, X_r and X_q, representing the demand, the returns and the quality of the returns, respectively.

The random variable X_q represents the number of high quality returns and is governed by a probability distribution which depends the number of returns being recovered in that period (a_r) and the mean proportion of high quality returns (α). The random variable X_d represents the number of items demanded during a given period and the random variable X_r represents the number of items returned during a given period. It is assumed that these three random variables are independent of each other and that their distributions are known. Following the convention in probability, an observation of a random variable is denoted by the lower case equivalent, for instance, x_d is an observation of the random variable X_d.

System Dynamics

The system dynamics associated with this model are as follows. Suppose the system is in state $i = (i_s, i_r, i_c)$ at the start of the period and action $a(i) = (a_p, a_r, a_c)$ is chosen. In this model it is assumed that the state is updated to reflect the action before the demands and returns are observed. This is follows from the assumption that production, recovery and buying have a lead time of zero. However, alternative transitions are possible, for instance demand could be observed (and need to be met) before the replenishment orders have been completed.

The state transitions will be described according to three possible actions: production (including buying components), recovery, and neither (no replenishment).

Production. If production is chosen then the serviceable inventory is increased by the size of the production order a_p and decreased by the size of demand x_d. The inventory level $i_s + a_p$ will never exceed the capacity of serviceable inventory due to the assumptions made about the action set $A_p(i)$. If demand exceeds $i_s + a_p$ then the new inventory level will be zero. The new serviceable inventory level will be:

$$\max\{(i_s + a_p - x_d), 0\}$$

If production is performed, the returned inventory is affected only by the incoming returns, so the new level of returned inventory is $i_r + x_r$. If $i_r + x_r$ exceeds the maximum capacity for returned inventory W_r then the new inventory level is W_r and $i_r + x_r - W_r$ returns are discarded. The new returned inventory level will be:

$$\min\{(i_r + x_r), W_r\}$$

The component inventory is increased by the size of the order a_b, and then decreased by the size of the production lot a_p. Following from the assumptions regarding availability of components and the definition of $A_b(i)$, the number of components ordered a_b ensures that the component inventory remains nonnegative. The new component inventory level will be:

$$\min\{(i_c + a_b - a_p), W_c\}$$

Recovery. If recovery is chosen then the changes to the inventory levels depend on the quality of the returns observed during recovery. If a_r returns are recovered, then the number of high quality returns is x_q, where $x_q \leq a_r$. However the number of returns which actually undergo high quality recovery is restricted by the capacity of the serviceable inventory. Let the number of returns that undergo high quality recovery be denoted by $a_h(x_q)$, where:

$$a_h(x_q) = \min\{(W_s - i_s), x_q\}$$

This definition ensures that the capacity of serviceable inventory will not be exceeded. If the number of high quality returns is greater than the available capacity, then the excess high quality returns will undergo low quality recovery. Let the number of returns that undergo low quality recovery be denoted by $a_l(x_q)$, where:

$$a_l(x_q) = \min\{a_r - a_h(x_q), W_c - i_c\}$$

Let the number of returns that are discarded be denoted by $a_d(x_q)$, where:

$$a_d(x_q) = \max\{a_r - a_h(x_q) - a_l(x_q), 0\}$$

The serviceable inventory is increased by the number of returns undergoing high quality recovery $a_h(x_q)$ and decreased by the size of demand x_d. Therefore the new serviceable inventory level will be:

$$\max\{(i_s + a_h(x_q) - x_d), 0\}$$

The returned inventory will be decreased by the size of the recovery lot a_r and increased by the number of returns x_r, subject to the capacity limit W_r. The new returned inventory level will be:

$$\min\{(i_r - a_r + x_r), W_r\}$$

The component inventory will be increased by the number of low quality returns for which there is sufficient capacity to hold. The new component inventory will be:

$$\min\{(i_c + (a - (\min\{(W_s - i_s), x_q\}))), W_c\}$$

No Replenishment. If neither replenishment action is chosen (i.e. $a_r = a_p = 0$) then the new serviceable inventory will be:

$$\max\{(i_s - x_d), 0\}$$

The new returned inventory will be:

$$\min\{(i_r + x_r), W_r\}$$

Since components are only bought when production occurs, the component inventory will be unchanged and will remain at i_c.

Next State. Combining these equations, the next state $j = (j_s, j_r, j_c)$ can be related to state $i = (i_s, i_r, i_c)$ and action $a = (a_p, a_r, a_b)$ as follows:

$$\begin{aligned} j_s &= \max\left\{\Big(i_s + a_p + a_h(x_q) - x_d\Big), 0\right\} \\ j_r &= \min\left\{\Big(i_r - a_r + x_r\Big), W_r\right\} \\ j_c &= \min\left\{\Big(i_c + a_b - a_p + a_l(x_q)\Big), W_c\right\} \end{aligned} \tag{5.1}$$

5.4.5 Costs

In this model the following costs are incurred: holding costs, setup costs, processing costs, lost sales costs and disposal costs. The costs incurred in a given period depend on the current state, the action chosen, and the demand and returns and quality of returns observed.

Setup costs. Setup costs are incurred each time production and recovery are performed or components are bought. The costs for these actions are k_p, k_r, and k_b, respectively. Indicator functions can be used to define specify when each cost should be incurred. For instance $\min\{1, a_p\}$ will equal 1 if production occurs and will equal 0 otherwise. It is assumed that the setup costs for production k_p and recovery k_r are positive, and that the setup cost for buying components k_b is non-negative. The slightly relaxed restriction for k_b follows from the assumptions made in Chapter 4 which allowed comparisons to made with existing literature. This assumption is retained here.

The setup costs depend on the current state i and the action $a(i)$ that is chosen in that state. Let the function $C_K(i, a(i))$ denote the setup cost function: $C_K(i, a(i)) = k_p \min\{1, a_p\} + k_r \min\{1, a_r\} + k_b \min\{1, a_b\}$. Since this function does not depend on any random variables, the expected value of the setup cost function is equal to $C_K(i, a(i))$, that is:

$$E[C_K(i, a(i))] = C_K(i, a(i)) = k_p \min\{1, a_p\} + k_r \min\{1, a_r\} + k_b \min\{1, a_b\} \tag{5.2}$$

Processing costs. Processing costs are incurred on a per unit basis for production c_p, acquisition of returns c_r, buying components c_b, recovery of high quality returns c_h, recovery of low quality returns c_l, and disposal of excess low quality returns c_d. For a given state i, action $a(i)$ and number of observed high quality returns x_q, the actual processing costs $C_P(i, a(i), x_q)$ are:

$$C_P(i, a(i), x_q) = c_p a_p + c_r a_r + c_b a_b + c_h a_h(x_q) + c_l a_l(x_q) + c_d a_d(x_q)$$

The expected processing costs are:

$$E[C_P(i, a(i))] = c_p a_p + c_r a_r + c_b a_b + c_h E[a_h(X_q)] + c_l E[a_l(X_q)] + c_d E[a_d(X_q)] \tag{5.3}$$

Holding Costs. Holding costs are incurred for any stock which is carried between periods. The per period, per unit holdings costs are h_s, h_r and h_c for serviceable, returned and component inventories, respectively. The holding cost is charged based on the inventory which has been carried since the previous period, so it therefore calculated based on the inventory levels at the beginning of the period $i = (i_s, i_r, i_c)$. Alternative methods could be used for calculating the cost of holding inventory. If the system is in state $i = (i_s, i_r, i_c)$ at the beginning of a period, then the holding costs incurred for that period are:

$$C_H(i) = h_s i_s + h_r i_r + h_c i_c$$

Since the inventory levels are known with certainty at the beginning of a period, the expected holding cost is:

$$E[C_H(i)] = C_H(i) = h_s i_s + h_r i_r + h_c i_c \tag{5.4}$$

Lost Sales & Disposal Costs. If demand exceeds serviceable inventory, then a per unit lost sales cost of l_s is incurred. If the number of returns exceeds the available capacity of the returned inventory, a per unit disposal cost of l_r is incurred. A disposal cost is also incurred for the disposal of recovered goods which cannot be stored, however this has already been accounted for in the processing cost function $C_P(i, a(i), x_q)$. Suppose that the system is in state i, action $a(i)$ is chosen and there are x_q high quality returns (if $a_r = 0$ then $x_q = 0$), x_d units demanded and x_r units returned. The amount of unsatisfied demand is:

$$\max\Big\{x_d - (i_s + a_p + a_h(x_q)), 0\Big\}$$

and the number of returns which exceed the capacity is:

$$\max\Big\{x_r - (W_r - i_r + a_r), 0\Big\}$$

Therefore the lost sales and disposal cost is:

$$\begin{aligned} C_L(i, a(i), x_q, x_d, x_r) = l_s\Big(&\max\Big\{x_d - (i_s + a_p + a_h(x_q)), 0\Big\}\Big) \\ &+ l_r\Big(\max\Big\{x_r - (W_r - i_r + a_r), 0\Big\}\Big) \end{aligned}$$

and the expected lost sales and disposal cost is:

$$\begin{aligned} E[C_L(i, a(i))] = l_s\Big(&\max\Big\{E[X_d] - (i_s + a_p + E[a_h(X_q)]), 0\Big\}\Big) \\ &+ l_r\Big(\max\Big\{E[X_r] - (W_r - i_r + a_r), 0\Big\}\Big) \end{aligned}$$

Total costs. The total costs incurred during for one period, for a given state i, action $a(i)$, and observed quality x_q, demand x_d and returns x_r is:

$$C(i, a(i), x_q, x_d, x_r) = C_K(i, a(i)) + C_P(i, a(i), x_q) + C_H(i) + C_L(i, a(i), x_q, x_d, x_r) \quad (5.5)$$

and correspondingly the expected total costs for one period is:

$$E[C(i, a(i))] = E[C_K(i, a(i))] + E[C_P(i, a(i))] + E[C_H(i)] + E[C_L(i, a(i))] \quad (5.6)$$

5.5 Model Implementation and Validation

In order to explore the properties of the product recovery system under study, the Markov decision problem described in the previous section was implemented in java. Further details about the implementation of the model are discussed in the Methodology chapter in Section 3.2.3 and in Appendix B in Section B.1.1.

In addition to thorough error-checking and inspection of the output during the code development process, two forms of verification were used to validate the problem specific files. The calculation of the expected average rewards was checked using an Excel spreadsheet and the system was simulated using the optimal policy and the simulated cost was compared with the actual cost, as calculated by the MDP. To conduct these tests a set of test problems was constructed. For all test problems, the results of the tests were as expected. Further details regarding the validation of the java code can be found in Appendix B.

5.6 Properties of the Optimal Policy

The properties of the optimal policy are explored in this section. Three main properties are investigated in the chapter. Firstly, we investigate the *performance* of the optimal policy. Two performance measures are used: the long run average cost and the fill rate (a measure of service). Secondly, we investigate the structure of the optimal policy by examining the *actions* that are chosen in different states. Insights from the investigation into the structure of the optimal policy will be used in Section 5.7 to create heuristic policies. Throughout this section we investigate the effect of the *recovery* strategy on the performance and structure of the optimal policy.

5.6.1 Datasets

A dataset has been constructed in order to investigate the optimal policy across a range of scenarios. The parameters for this dataset are derived from the dataset used in Chapters 4, some of which were themselves derived from the dataset used by Konstantaras and Papachristos (2008b). This section explains how the current dataset extends those used to investigate the deterministic product recovery model in Chapter 4. Since the model in Chapter 4 is deterministic and the current model is stochastic, some modifications need to be made to the parameters used in the dataset. For instance, the lost sales cost l_s needs to be added and the constant demand, return and quality rates need to be converted to distributions.

State Variables

Each of the three state variables are constrained by an upper limit, as discussed in Section 5.3. The upper limits on the inventories are set to $W_s = W_r = W_c = 30$ and the lower limits are set to 0, therefore the state space for each of the inventories is limited to the values from zero to 30.

$$I_s = \{0, 1, \ldots, 30\} \qquad \text{Serviceable Inventory}$$

$$I_r = \{0, 1, \ldots, 30\} \qquad \text{Returned Inventory}$$

$$I_c = \{0, 1, \ldots, 30\} \qquad \text{Component Inventory}$$

Random Variables

Demands The Poisson distribution is selected to model demand and returns. The Poisson distribution is appropriate for modelling demand and returns because it is a discrete distribution, and gives the probability of a number of events (or arrival of demand) occurring within a fixed time interval (Ross, 1996). This fixed interval fits well with the notion of a 'period' which is used in this model. In addition, the Poisson distribution has the advantage of being able to be specified by only one parameter, the rate of arrivals.

For modelling purposes, it is desirable for the distributions to 'fit' almost completely within the state space, as this helps to minimise the effect of the limited the state space on the optimal policy. One way of doing this is to ensure that the probability that demand exceeds the upper limit for serviceable inventory W_s is very small. If the random variable governing demand X_d is governed by a Poisson distribution with mean $\lambda_d = 13$, then the probability that the demand will be greater than the upper limit on serviceable inventory $W_s = 30$ is very small, $P(X_d > 30) < 1 \times 10^{-4}$. This probability is sufficiently small, therefore an upper limit of 13 is placed on the parameter λ_d.

In the deterministic model it is assumed that there are no lost sales, therefore the production and recovery rates needed to be sufficiently large to prevent this from happening. More specifically, the production and recovery rates were required to be greater than the demand rate. The relationship between the demand rate and the production and recovery rates is significant and can used to model different types of products. For instance a demand rate of 100 and a production rate of 120 could be used to model demand for a fast moving item, whereas a model with the same demand rate, but a production rate of 1000 could be used to model a slow moving item. Thus, the size of the demand rate, relative to the production rate is particularly important.

In order to obtain values of λ_d for each problem in the dataset, the ratio of the demand rate over the production rate (from Chapter 4) was used. For all problems, this ratio gives values between 0 and 1, with more values being skewed towards 0. In order to study a wider range of demand rates, a cube-root transformation is used to spread the values more evenly between 0 and 1. These values are then multiplied by the upper limit for the demand parameter (i.e. 13) in order to distribute them between 0 and 13. These values were then rounded up to the nearest integer, this was done primarily for convenience. In summary, in order to obtain the mean demand λ_d for each problem in the dataset the following transformation is used:

$$\lambda_d = \left\lceil \sqrt[3]{\frac{d_d}{p_d}} \times 13 \right\rceil$$

where p_d and d_d are the production and demand rates from the deterministic dataset studied in Chapter 4, and the function $\lceil x \rceil$ rounds x up to the nearest integer.

Returns. In the literature, it is commonly assumed that returns are related to demand, but that across time periods, the specific quantities observed are independent.

Furthermore, in this model it is assumed that the goods being returned were previously produced by and then bought from the firm. Therefore in this dataset it is assumed that the mean demand rate λ_d is related to the mean return rate λ_r, but that the random variables are independent. In the deterministic model a fixed proportion f of goods demanded is returned and, if the system recovers both high quality and low quality returns, then f is the sum of the high quality proportion β_1 and low quality proportion β_2. In this chapter, let:

$$\gamma = \beta_1 + \beta_2$$

and then let the mean returns be:

$$\lambda_r = \lfloor \gamma \lambda_d \rfloor$$

where the function $\lfloor x \rfloor$ rounds x down to the nearest integer. It is assumed that the returns may be high quality or low quality, and that the quality of any one item is independent of the quality of any other item, therefore the binomial distribution is appropriate to model the quality of the returns. The number of trials used in the specification of the distribution is the number of goods being recovered a_r in a given period, and the probability that a return is high quality is given by the parameter β_1, thus:

$$X_q \sim \text{Bin}(a_r, \beta_1)$$

Costs

In the deterministic model the cost of c_d was equivalent to a penalty fee or surcharge incurred for all goods which were produced and demanded, but not recovered. However in this model, the demand and returns are uncertain and the number of returns which could have been recovered in a given period is unknown, therefore the cost of disposal c_d cannot be charged in the same way. In order to model this scenario in the current model, it is assumed that the cost of production includes a fee for the eventual disposal of that good but that this fee would be reimbursed if the product is later returned to the firm and recovered. This fee could be likened to a "tax", which is charged on production. This is incorporated into the model by adding a disposal fee to the cost of production, and subtracting the disposal fee from the cost of acquiring returns. If the good is returned and not recovered (due to capacity restrictions), then the fee is not reimbursed. Let c_r^d and c_p^d denote the returns acquisition and production costs for the deterministic model in Chapter 4, then the costs of acquiring returns and producing goods in this model are defined as:

$$c_r = c_r^d - c_d$$

$$c_p = c_p^d + c_d$$

where c_d are the disposal costs from Chapter 4. These changes were not specified in the model description in order to retain the generalisability of the model description with regard to alternative disposal scenarios. The values of c_p and c_r are altered accordingly based on this new relationship with the disposal cost.

All other cost parameters retain their values from Chapter 4, but in addition, two new cost parameters are introduced. A lost sales cost l_s is incurred for demand which is not met and a lost returns cost l_r is incurred for returns which cannot be accepted due to insufficient capacity. These costs are specified as follows. The lost sales cost is associated with a finished product, therefore this cost is derived from the cost of acquiring a new component and cost of production required to turn it into a serviceable good, that is:

$$l_s = \gamma_s(c_b + c_p)$$

In reality if the cost of production was prohibitively high, then it may not be profitable to continue operating. The firm would face a "continue" or "close" decision. However, in this model we assume that the firm is profitable and that the cost of producing a new item is less than the lost sales cost, therefore γ_s is set at 1.1.

A "lost" returns cost l_r is also incurred for returns which are returned, but can not be received due to insufficient capacity in the returns inventory. Since the current model already 'penalises' firms by withholding the disposal fee reimbursement for not recovering goods, this cost is set to zero $l_r = 0$.

Using these relationships a set of problems were constructed. The parameters for these problems are presented in Table 5.2. These problems are denoted by the prefix G, followed by the problem number 00 to 20. The transformations used above are not designed to make the current model directly comparable to the deterministic model from Chapter 4, but rather they are designed to provide a range of similar scenarios with which to test the model.

		$G00$	$G01$	$G02$	$G03$	$G04$	$G05$	$G06$	$G07$	$G08$	$G09$	$G10$	$G11$	$G12$	$G13$	$G14$	$G15$	$G16$	$G17$	$G18$	$G19$	$G20$
Distributions																						
Mean demand	λ_d	8	8	9	11	9	10	10	10	8	9	10	12	12	13	8	9	9	9	7	9	5
Mean returns	λ_r	6	7	4	4	4	9	9	8	7	4	8	6	6	6	5	6	1	6	2	8	3
High quality	α	0.8	0.8	0.2	0.3	0.3	0.5	0.7	0.8	0.8	0.2	0.5	0.5	0.5	0.5	0.5	0.4	0.1	0.6	0.2	0.2	0.5
Setup costs																						
production	k_p	20	20	10	10	20	20	20	30	30	50	28	12	12	120	20	100	1000	400	30	50	28
recovery	k_r	5	5	5	10	12	8	20	20	25	30	8	2	2	10	5	50	100	400	5	10	8
buying	k_b	0	10	2	10	15	12	10	10	5	15	4	6	4	5	10	20	1000	200	15	50	4
Processing costs																						
production	c_p	0	106.5	66	88	131	161	107	66	75	212.5	188.5	23.5	76	78.5	106.5	65.5	110	107	75	212.5	160
returns	c_r	0	13.5	34	42	89	29	13	44	15	27.5	141.5	6.5	24	21.5	13.5	34.5	0	13	15	27.5	10
buying	c_b	0	30	60	80	100	70	40	60	30	50	200	50	50	100	30	50	100	40	30	50	50
high quality recovery	c_h	0	50	30	40	60	75	50	30	35	100	85	10	35	35	50	30	50	50	35	100	75
low quality recovery	c_l	0	9	18	24	30	21	12	18	9	15	60	15	15	30	9	15	30	12	9	15	15
disposal	c_d	0	6.5	6	8	11	11	7	6	5	12.5	18.5	3.5	6	8.5	6.5	5.5	10	7	5	12.5	10
Holding costs																						
serviceable inventory	h_s	10	10	6	8	12	15	10	6	7	20	17	2	7	7	10	6	10	10	7	20	15
returned inventory	h_r	2	2	4	5	10	4	2	5	2	4	16	1	3	3	2	4	1	2	2	4	2
component inventory	h_c	0	3	6	8	10	7	4	6	3	5	20	5	5	10	3	5	10	4	3	5	5
Penalty costs																						
Lost sales	l_s	0	143	132	176	242	242	154	132	110	275	407	77	132	187	143	121	220	154	110	275	220
	l_r	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 5.2: Parameters for problem set G

5.6.2 Analysis of Performance

Costs

There is substantial variation between the long run average costs for each of the 21 problems. There is also variation between the two recovery strategies. When both high and low quality returns can be recovered, there is an additional source for obtaining serviceable goods and components. Due to the assumptions about the cost parameters, this additional source is also a cheaper source. It would therefore be expected that the long run average cost associated with a high-quality-only recovery strategy would be higher than with one in which both high and low quality returns could be recovered. Figure 5.2 compares the average cost for the two recovery strategies. As expected, the cost is higher under the high quality only recovery strategy. However, notice that for some problems the difference between the cost in the two strategies is not very large. The data for this Figure is presented in Tables 5.3 and 5.4. Further summary data relating to the simulation is presented in Appendix B, in Tables B.4 and B.5.

Fill rates

The fill rate measures the proportion of demand which is met by current stock and it can be used to assess the performance of a policy (Silver et al., 1998). The fill rate is:

$$\text{fill rate} = \frac{\text{number of met sales}}{\text{number of items demanded}}$$

The average fill rate can be determined using a simulation by calculating the fill rate each period and then by averaging this across the length of the simulation. A low fill rate indicates that there are high number of lost sales (and potentially a lot of dissatisfied customers!).

The fill rates under each of two recovery strategies for problem set G are shown in Figure 5.3, and Tables 5.3 and 5.4. In general, performing both high and low quality recovery leads to a higher fill rate. This implies that there are fewer lost sales and thus the firm offers a better level of service when it recovers both types of returns. When only high quality recovery is performed, 15 out of 21 have a fill rate above 60%. When

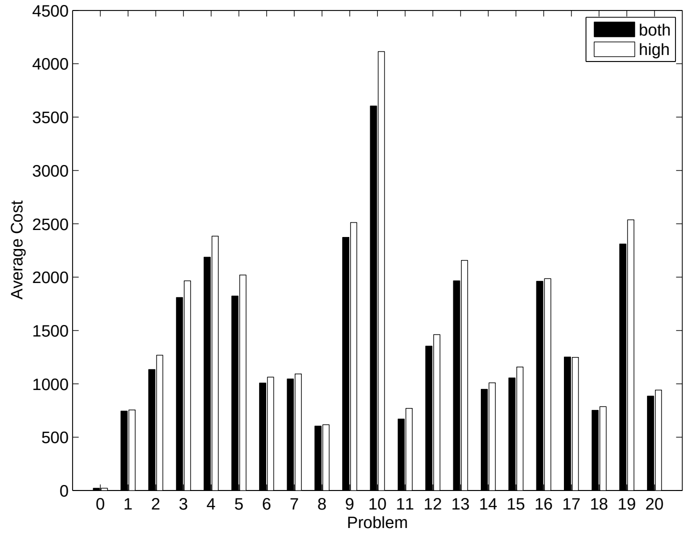


Figure 5.2: Average cost of the optimal policy calculated for two quality strategies

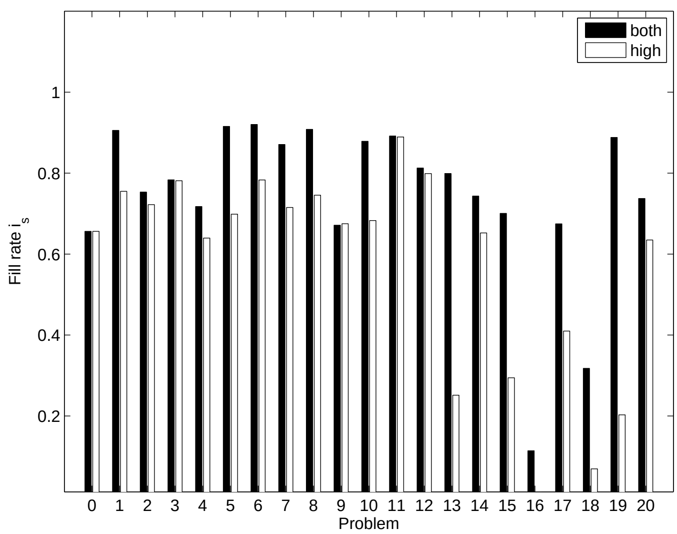


Figure 5.3: Fill rates under the optimal policy calculated for two quality strategies for a simulation of $T = 1,000,000$

Table 5.3: Summary of the performance of problem set G for a high quality only recovery strategy ($\zeta_L = 0$)

Problem	Optimal Cost	Simulation Cost	Simulation Fillrate
$G00$	21.841	21.846	0.6560
$G01$	755.204	755.235	0.7551
$G02$	1268.827	1268.832	0.7222
$G03$	1966.127	1965.960	0.7810
$G04$	2384.084	2384.078	0.6394
$G05$	2019.588	2019.461	0.6984
$G06$	1062.778	1062.727	0.7830
$G07$	1092.527	1092.534	0.7150
$G08$	617.019	617.032	0.7455
$G09$	2511.420	2511.438	0.6749
$G10$	4114.595	4114.478	0.6828
$G11$	770.370	770.294	0.8891
$G12$	1461.006	1460.908	0.7987
$G13$	2157.194	2157.338	0.2514
$G14$	1009.246	1009.116	0.6521
$G15$	1158.008	1157.836	0.2944
$G16$	1986.169	1986.273	0.0127
$G17$	1248.180	1248.236	0.4096
$G18$	785.778	785.746	0.0694
$G19$	2537.085	2537.080	0.2027
$G20$	941.318	941.201	0.6347

Table 5.4: Summary of the performance of problem set G for a both high and low quality recovery strategy ($\zeta_L = 1$)

Problem	Optimal Cost	Simulation Cost	Simulation Fillrate
$G00$	21.841	21.846	0.6560
$G01$	744.535	744.603	0.9055
$G02$	1134.757	1134.831	0.7530
$G03$	1809.169	1809.103	0.7833
$G04$	2187.034	2187.175	0.7174
$G05$	1823.303	1823.636	0.9154
$G06$	1007.910	1007.955	0.9201
$G07$	1045.545	1045.611	0.8708
$G08$	603.650	603.758	0.9079
$G09$	2372.582	2372.529	0.6712
$G10$	3604.191	3604.281	0.8785
$G11$	670.333	670.331	0.8914
$G12$	1353.736	1353.759	0.8126
$G13$	1965.953	1966.034	0.7986
$G14$	949.508	949.536	0.7433
$G15$	1055.832	1055.934	0.7005
$G16$	1961.925	1962.026	0.1140
$G17$	1251.555	1251.722	0.6747
$G18$	752.569	752.568	0.3177
$G19$	2310.692	2310.980	0.8882
$G20$	885.380	885.369	0.7372

both high quality and low quality recovery is performed 19 out of 21 have a fill rate of above 60%. For some problems, the recovery strategy does not appear to significantly affect the fill rate (e.g. $G03$, $G11$, $G12$).

Notice that some problems have a very low fill rate. Problem $G16$, in particular, has a very low fill rate. One reason for this could be that problem $G16$ has very high setup costs for production and buying components, compared with the other costs. This would serve to 'discourage' production and buying. The impact of this will be explored further when the structure of the optimal policy is investigated in Section 5.6.3.

Tables 5.5a and 5.5b show a summary of the inventory levels over the simulation for each recovery strategy. As shown in this table, under a high-quality-only strategy components are not held and the average levels of serviceable and returned inventory tend to be lower.

5.6.3 Analysis of Actions

In this section the structure of the optimal policy is analysed in order to investigate the relationship between the three replenishment actions (production, recovery, buying). Three main questions will be examined in this section. Firstly, across all states, what is the *frequency* with which each replenishment action chosen, and secondly what is the *size* of these actions. Finally, in which states are replenishment actions selected, i.e. what inventory levels *'trigger'* certain actions. Comparisons will be drawn between the two recovery strategies.

Action Size and Frequency

In this model there are a total of $31^3 = 29791$ states. This means that production, recovery and buying could each be chosen in a maximum of 29791 states. As well as choosing whether or not to replenish, a decision must also be made about the size of the replenishment order. The frequency with which each action is selected is presented for each of the recovery strategies in Figure 5.4. The data for these graphs is available in Appendix B, Tables B.6 and B.7. As shown in this Figure, the number of states in which each action is performed vary substantially across the 21 problems.

Table 5.5: Summary of inventory levels during the simulation

(a) High quality recovery

	Serviceable Inventory				Returned Inventory				Component Inventory			
Problem	mean	min	max	mode	mean	min	max	mode	mean	min	max	mode
$G00$	0.402	0	11	0	6.453	0	30	6	0	0	0	0
$G01$	0.754	0	12	0	7.989	0	30	7	0	0	0	0
$G02$	1.306	0	12	0	5.832	0	21	4	0	0	0	0
$G03$	1.509	0	13	0	6.374	0	25	5	0	0	0	0
$G04$	1.067	0	14	0	5.076	0	20	4	0	0	0	0
$G05$	0.538	0	16	0	10.359	0	30	9	0	0	0	0
$G06$	0.774	0	14	0	10.445	0	30	9	0	0	0	0
$G07$	0.738	0	24	0	8.126	0	30	8	0	0	0	0
$G08$	1.058	0	14	0	8.048	0	30	7	0	0	0	0
$G09$	0.685	0	10	0	6.334	0	21	5	0	0	0	0
$G10$	0.763	0	19	0	9.065	0	30	8	0	0	0	0
$G11$	2.833	0	21	0	8.725	0	29	6	0	0	0	0
$G12$	1.127	0	18	0	8.789	0	28	6	0	0	0	0
$G13$	0.006	0	8	0	6.328	0	25	6	0	0	0	0
$G14$	0.482	0	11	0	5.984	0	26	5	0	0	0	0
$G15$	0.099	0	12	0	8.789	0	27	6	0	0	0	0
$G16$	0.000	0	4	0	7.270	0	20	5	0	0	0	0
$G17$	1.873	0	24	0	16.075	0	30	6	0	0	0	0
$G18$	0.001	0	3	0	2.239	0	13	2	0	0	0	0
$G19$	0.006	0	7	0	8.024	0	26	7	0	0	0	0
$G20$	0.389	0	8	0	3.728	0	21	3	0	0	0	0

(b) Both high and low quality recovery

	Serviceable Inventory				Returned Inventory				Component Inventory			
Problem	mean	min	max	mode	mean	min	max	mode	mean	min	max	mode
$G00$	0.402	0	11	0	6.453	0	30	6	30.000	0	30	30
$G01$	2.142	0	14	0	12.040	0	30	8	3.147	0	16	0
$G02$	1.394	0	18	0	6.184	0	23	5	3.203	0	21	0
$G03$	1.431	0	16	0	6.593	0	25	5	2.801	0	20	0
$G04$	0.968	0	17	0	6.034	0	23	5	2.835	0	19	0
$G05$	2.797	0	20	0	15.712	0	30	10	4.908	0	29	0
$G06$	2.781	0	19	0	15.368	0	30	10	3.979	0	24	0
$G07$	2.432	0	29	0	10.153	0	30	8	2.820	0	20	0
$G08$	2.522	0	17	0	11.627	0	30	7	2.674	0	16	0
$G09$	0.677	0	12	0	6.599	0	24	5	3.249	0	21	0
$G10$	2.411	0	27	0	11.866	0	30	8	4.155	0	24	0
$G11$	2.489	0	20	0	9.394	0	30	6	3.004	0	22	0
$G12$	1.125	0	18	0	9.232	0	30	6	3.021	0	20	0
$G13$	2.223	0	19	0	10.039	0	30	6	3.070	0	25	0
$G14$	0.649	0	14	0	7.374	0	30	5	2.849	0	19	0
$G15$	1.465	0	26	0	11.236	0	30	6	3.850	0	28	0
$G16$	0.953	0	29	0	14.774	0	30	27	0.896	0	30	0
$G17$	3.547	0	29	0	15.994	0	30	30	6.770	0	30	0
$G18$	0.194	0	12	0	4.055	0	16	3	1.685	0	14	0
$G19$	2.970	0	18	0	14.569	0	30	8	8.197	0	30	0
$G20$	0.643	0	10	0	4.873	0	30	4	1.696	0	12	0

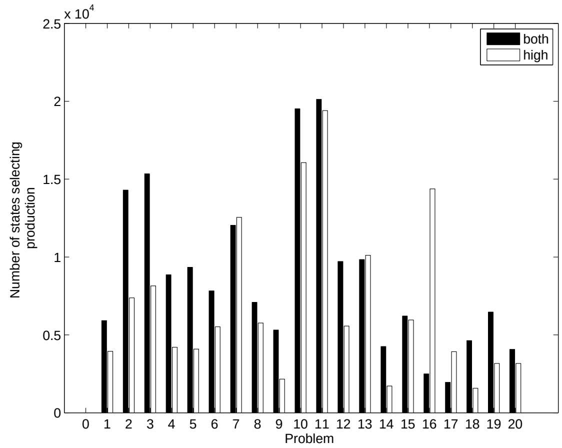
Figure 5.4a displays the frequency with which production is performed and shows that, in general, production is performed more frequently when both high and low quality returns are recovered. This could be because under this strategy, there are more components in the system (from low quality recovery), thus the costs associated with production are lower. Indeed under a high-quality recovery strategy, components are not held in stock at all and are only bought when needed for production (see Tables B.10 and B.11).

Figure 5.4b displays the frequency with which recovery is performed. Recovery is, in general, performed more frequently when a high-quality recovery strategy is implemented. Figure 5.4c displays the frequency with which components are bought. Components are bought, in general, more frequently when a high-quality recovery strategy is implemented. This is expected since under a high and low quality recovery strategy, components can also be obtained from recovery. However, for many problems there is not much difference between the two strategies.

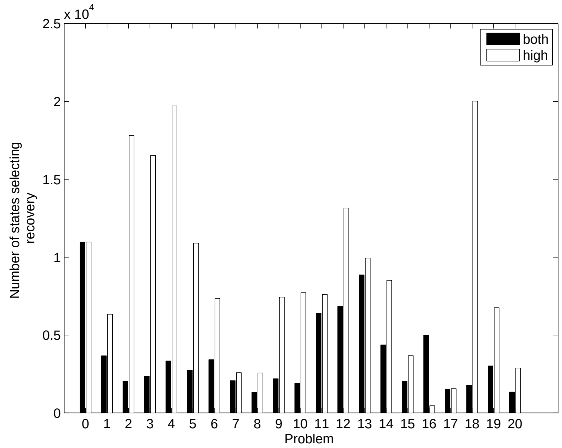
It is interesting to note that problem $G00$ never produces and that several problems never buy components (e.g. $G00$, $G16$, $G17$). Problem $G00$ has no unit processing costs and no lost sales costs. It was included in the dataset to allow for comparisons with previous literature relating to the deterministic model. The nature of the relationship between the costs is complicated so it is difficult to identify a single cause of this phenomenon, however the problems which never order components seem to have comparatively higher setup costs for buying components and for holding components. Both of these things would serve to 'discourage' the buying of components.

The size of the replenishment actions is also of interest. In this model, the numbers of goods which are produced, recovered or bought in any given period are also decision variables. Figure 5.5 shows the size of the production, recovery and buying lot sizes for problem $G01$. For all actions, the most common 'lot size' is 0, indicating that the action is not performed. When these states are included in the graph, the scale of the resultant graph makes it is difficult to observe patterns in the lot sizes when the actions are performed. Therefore, to allow the lot sizes to be examined in more detail, the states in each action is not performed (i.e., when the lot size is 0) are not included in this graph. First examining Figure 5.5a, which shows the lots sizes when high and low quality recovery can be performed when production is performed, the production quantity a_p is mostly distributed between 3 and 13. Compare this to the equivalent graph in Figure 5.5b (for high quality only recovery strategy), which has a similar but more dispersed shape. Similar trends can be observed for recovery and buying components.

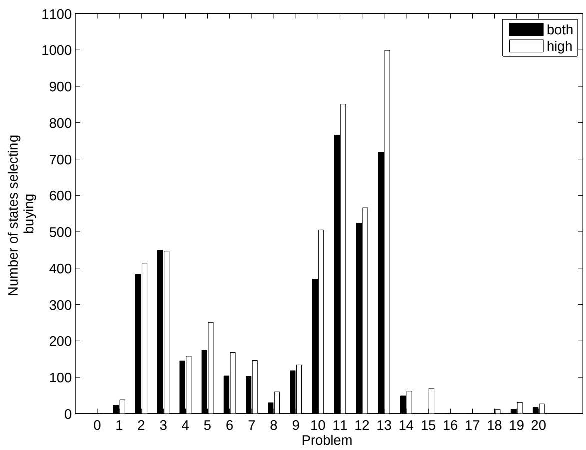
Figures 5.6 attempts to summarise the production lot sizes for all problems. For some problems the lot sizes fall within a narrow range (e.g. $G14$, $G20$), but some (e.g. $G02$) range across all possible lot sizes (0 to 30). Though there are some differences



(a) Production a_p

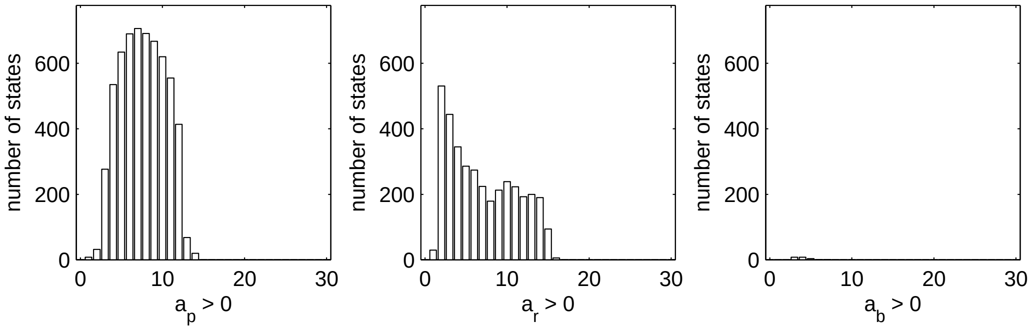


(b) Recovery a_r

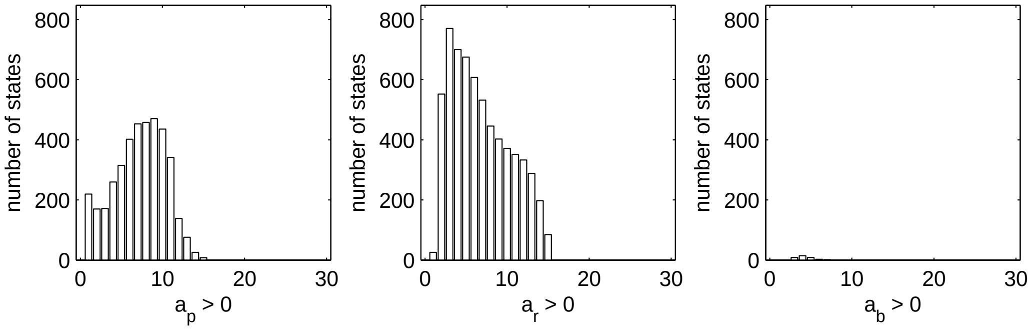


(c) Recovery a_b

Figure 5.4: Number of states with positive replenishment quantities for both recovery strategies, out of a total of $31^3 = 29791$ states



(a) High and low quality recovery



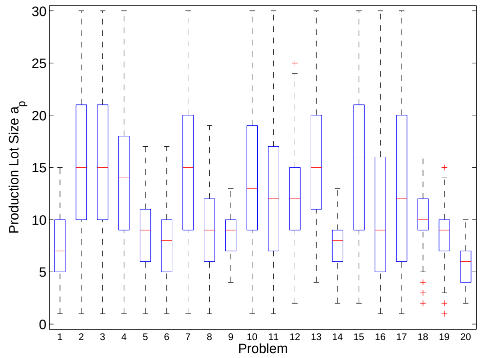
(b) High quality only

Figure 5.5: Optimal size of the replenishment lots for problem $G01$ under the two recovery strategies (excluding cases with $a_p = 0$ or $a_r = 0$)

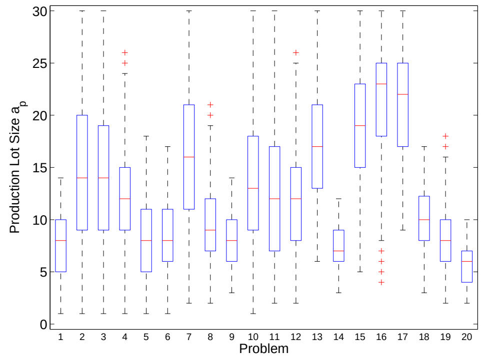
between the two recovery strategies, there are no obvious trends. The wide range of lot sizes suggests that an 'order-up-to' structured policy may be more appropriate than one with a fixed lot size. Figures 5.7 and 5.8 show the equivalent information for recovery and buying components. Once again, there is substantial variation between problems. Though not easy to identify from these graphs, the analysis found that under a high and low quality recovery strategy the range of lot sizes for recovery is smaller when there is a higher proportion of high quality returns α. This suggests that the proportion of high quality returns may influence the size of the recovery lot. A summary of the data displayed in these figures is available in Appendix B in Tables B.6 and B.7.

Trigger-States and Action

The inventory levels are taken into account when the action is selected at the beginning of each period, therefore it is interesting to investigate which inventory levels 'trigger' certain actions and whether or not these levels are affected by the recovery strategy.

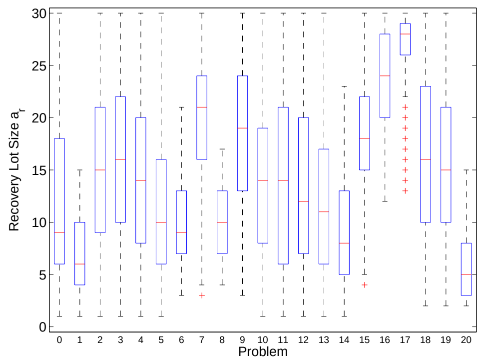


(a) High quality recovery

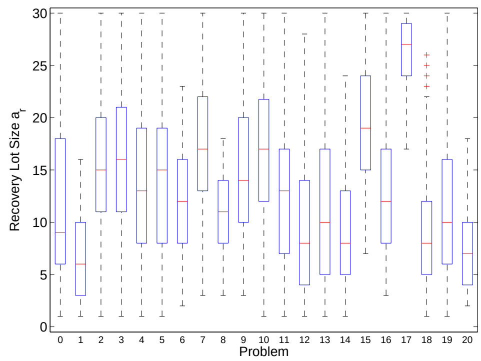


(b) High and low quality recovery

Figure 5.6: Graphs showing the size of the production lots for both recovery strategies

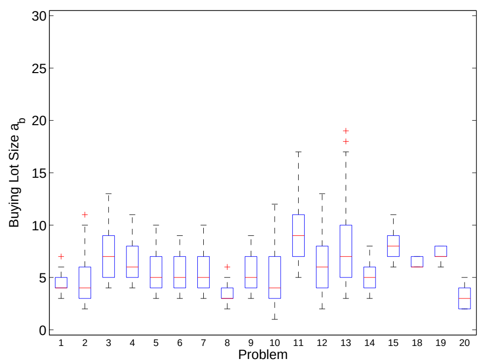


(a) High quality recovery

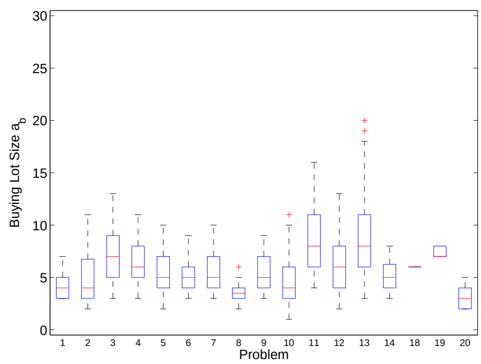


(b) High and low quality recovery

Figure 5.7: Graphs showing the size of the recovery lots for both recovery strategies



(a) High quality recovery



(b) High and low quality recovery

Figure 5.8: Graphs showing the size of the buying lots for both recovery strategies

The trigger states are introduced using *histograms* and then two types of analysis are performed. First the inventory levels which trigger each action will be examined using *box plots*. However, it is likely that the actions will not be affected by the level of a single type inventory, but rather by the relationship between the levels of all three types of inventory. These relationships will be examined using, what we term, *action plots*.

There are two inventory levels which are particularly of interest in the case of production. It is interesting to examine the level of serviceable inventory (i_s) when production is selected and the level of serviceable inventory ($i_s + a_p$) after production has been performed. For both these quantities we are only interested in the states for which the optimal action is to produce. In the case of recovery it is interesting to examine the level of returns in stock (i_r) when recovery is performed, and the level of serviceable inventory (i_s) when recovery is performed. For both these quantities we are only interested in the states for which the optimal action is to perform recovery. When components are bought it is interesting to examine the level of components in stock i_c and the level of serviceable inventory i_s. It would be expected that components are bought when the components inventory is low, and if components are primarily bought when production occurs, then it would be expected that the serviceable inventory would also be low.

Histograms. Figures 5.9a and 5.10a show the states in which each action is performed under each recovery strategy for problem $G01$. First examine Figure 5.9a. This figure shows the serviceable inventory level i_s and the initial state plus the action $i_s + a_p$ for serviceable inventory when production is chosen ($a_p > 0$) for problem $G01$ under a high-quality recovery strategy. Production is performed if serviceable inventory is less than approximately 8 and most commonly results in a new inventory level of between 9 and 13. This suggests that for this problem the 'trigger' level could be 8 and the 'produce-up-to' level could be 11. Figure 5.10a presents the same information, but for the high and low quality recovery strategy. This figure has a similar shape as under a high-quality only recovery strategy. Under this strategy it appears that the trigger or produce-up-to levels are slightly higher at 10 and 12 respectively.

Figures 5.9b and Figure 5.10b show level of returned inventory i_r and the level of serviceable inventory i_s when recovery is performed, under the two recovery strategies.

For this problem, recovery is performed across almost all levels of returned inventory. Notice that under the high-quality only recovery strategy, recovery is performed most frequently when returned inventory is high. This is expected. For both recovery strategies, recovery is performed if the level of serviceable inventory is less than approximately 10.

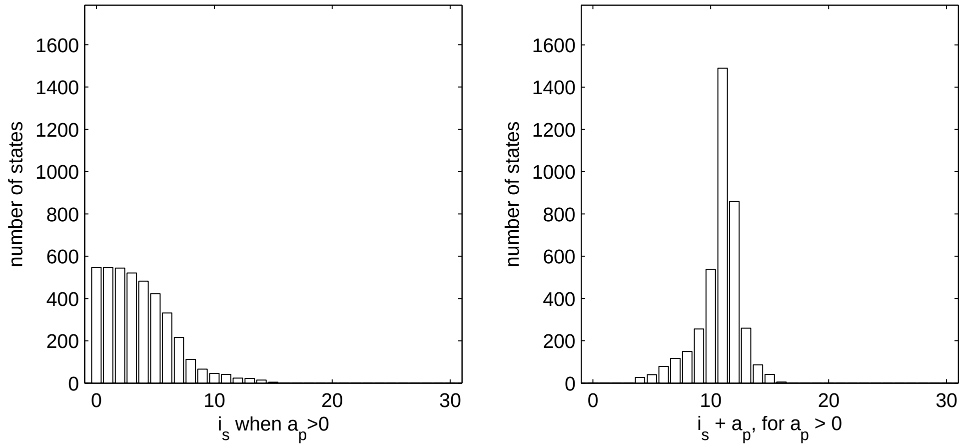
Figures 5.9c and Figure 5.10c show level of component inventory i_c and the level of serviceable inventory i_s when components are bought, under the two recovery strategies. For this problem, components are bought if the component inventory and the serviceable inventory are less than approximately 3.

Box Plots In order to summarise the trigger levels for all problems a series of box plots were created. Note that since production is not performed and components are not bought in all problems, the number of problems represented in each plot differs. Data summarising some of the values in the box plots examined in this section are available in Appendix B in Tables B.8, B.9, B.10 and B.11 .

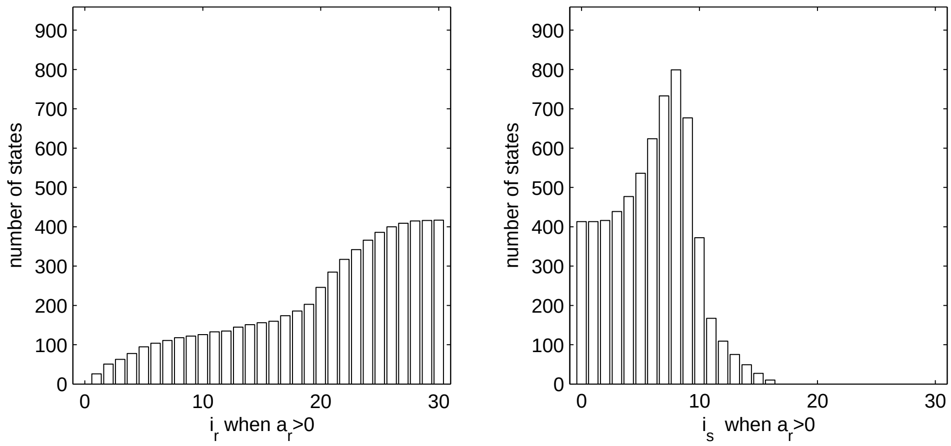
Figure 5.11 shows the level of serviceable inventory when production is chosen, for each of the recovery strategies. In general there appears to be little difference in the 'trigger' levels for production under each recovery strategy. Figure 5.12 shows the level of serviceable inventory after production has been performed for each of the recovery strategies. Though not easy to identify from these graphs, it was found that there was a moderate positive relationship between the the level of serviceable inventory after production $(i_s + a_p)$ and the average demand λ_d. This is not unexpected. Figure 5.13 shows the level of component inventory when components are bought. For all three graphs, there does not appear to be much difference between the two recovery strategies.

The inventory levels which trigger recovery will now be examined. The level of returned inventory when recovery is performed is of interest. Figure 5.14 shows this information for each of the recovery strategies. There are no obvious differences between the two recovery strategies in the level of returned inventory triggering recovery.

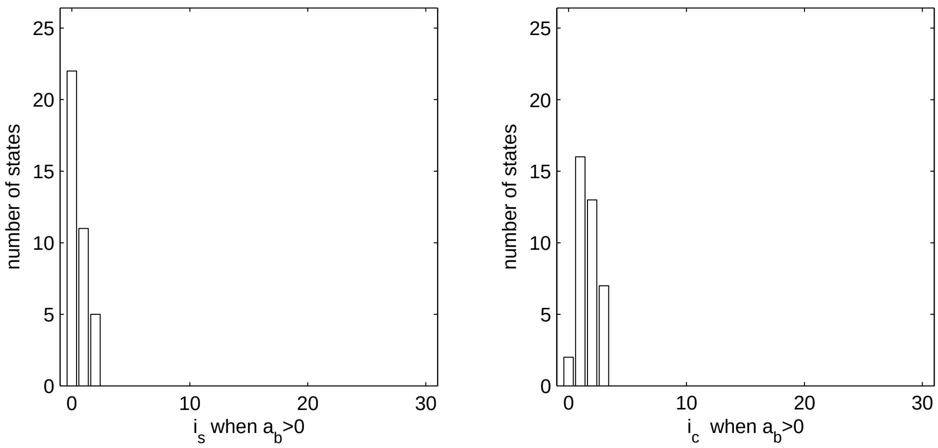
Figure 5.15 shows the level of serviceable inventory when recovery is chosen under each of the recovery strategies. It could be expected that recovery would be performed



(a) Distribution of states when production is selected

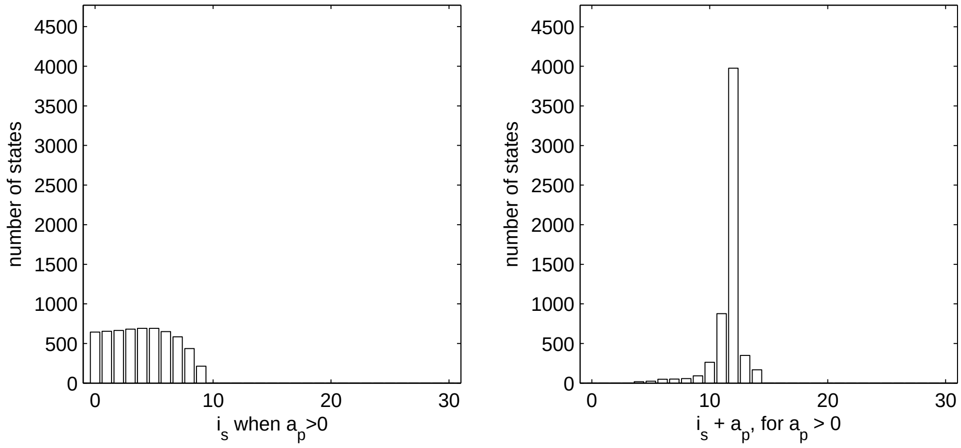


(b) Distribution of states when recovery is selected

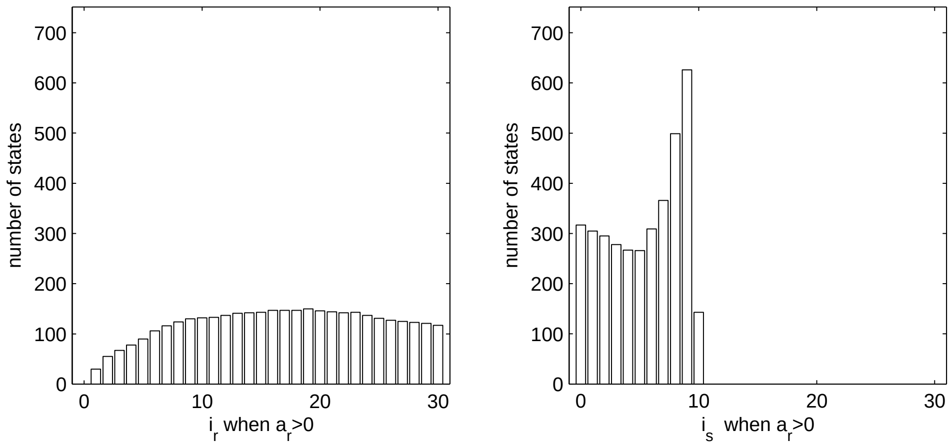


(c) Distribution of states when components are bought

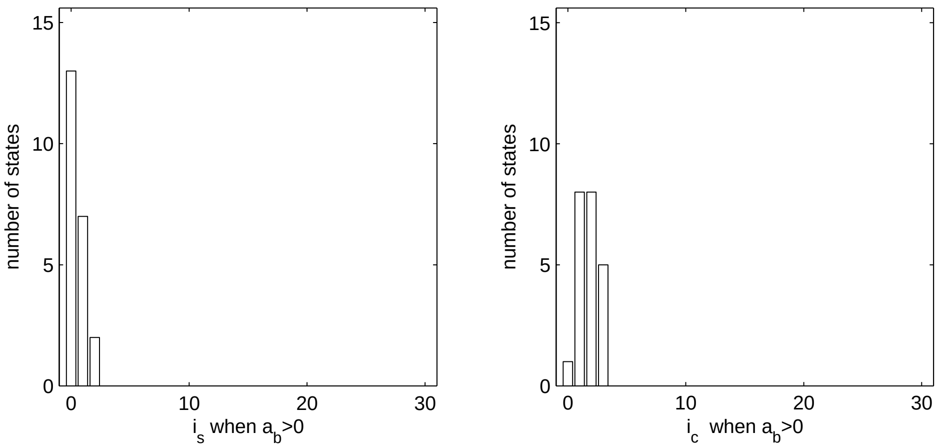
Figure 5.9: Histograms of showing states associated with positive replenishment actions under the optimal policy for test problem $G01$ under a high quality only recovery strategy



(a) Distribution of states when production is selected

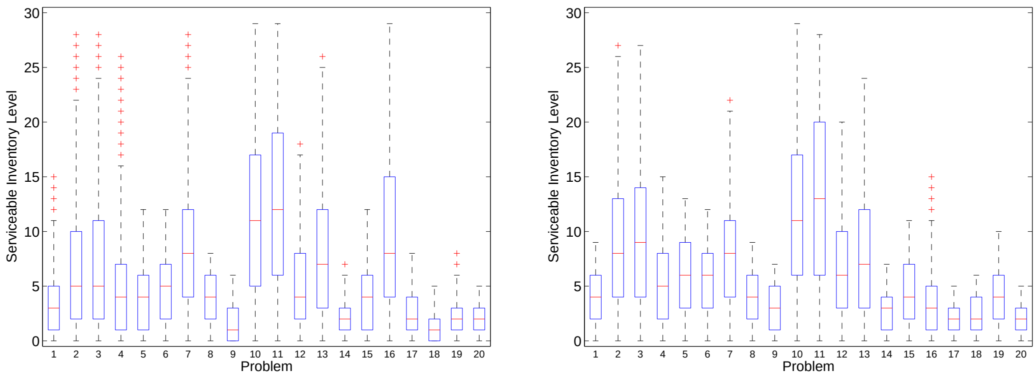


(b) Distribution of states when recovery is selected



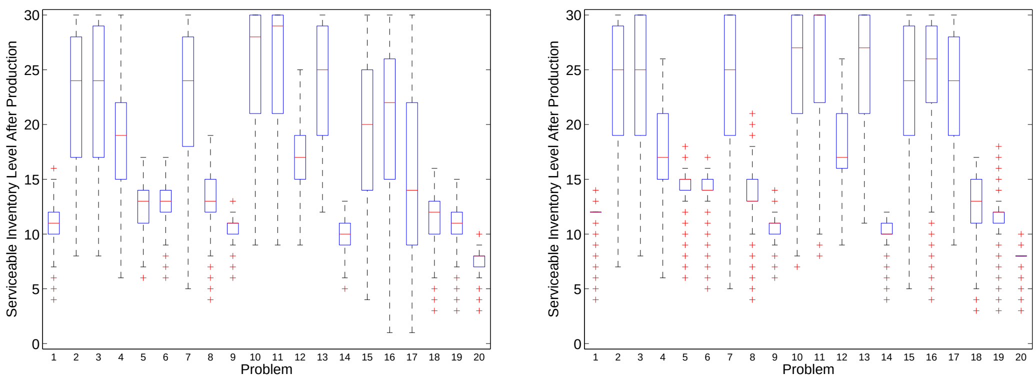
(c) Distribution of states when components are bought

Figure 5.10: Histograms of showing states associated with positive replenishment actions under the optimal policy for test problem $G01$ under a high and low quality recovery strategy



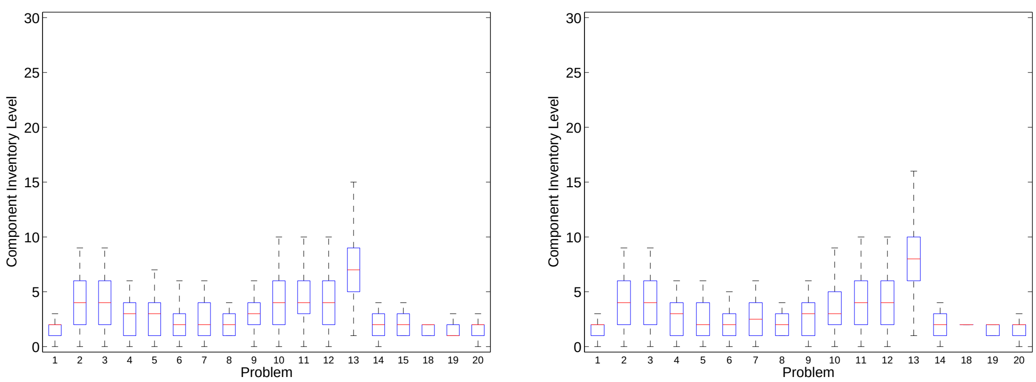
(a) High quality recovery only (b) Both high and low quality recovery

Figure 5.11: Graphs showing the level of serviceable inventory (trigger level) when production is performed (i_s), for two recovery strategies



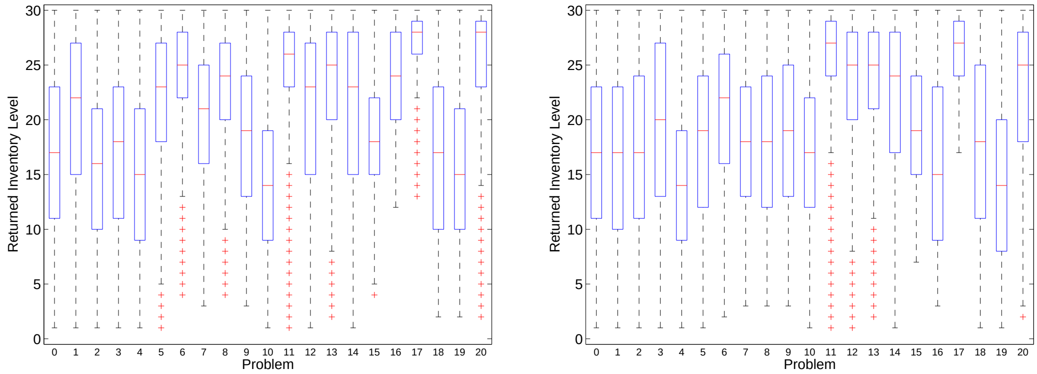
(a) High quality recovery only (b) Both high and low quality recovery

Figure 5.12: Graphs showing the level of serviceable inventory (trigger level) after production is performed ($i_s + a_p$), for two recovery strategies



(a) High quality recovery only (b) Both high and low quality recovery

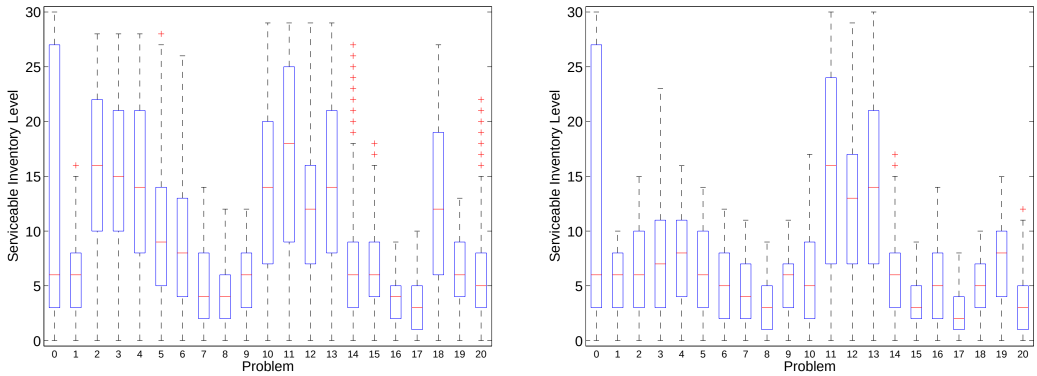
Figure 5.13: Graphs showing the level of component inventory (trigger level) when components are bought



(a) High quality recovery only

(b) Both high and low quality recovery

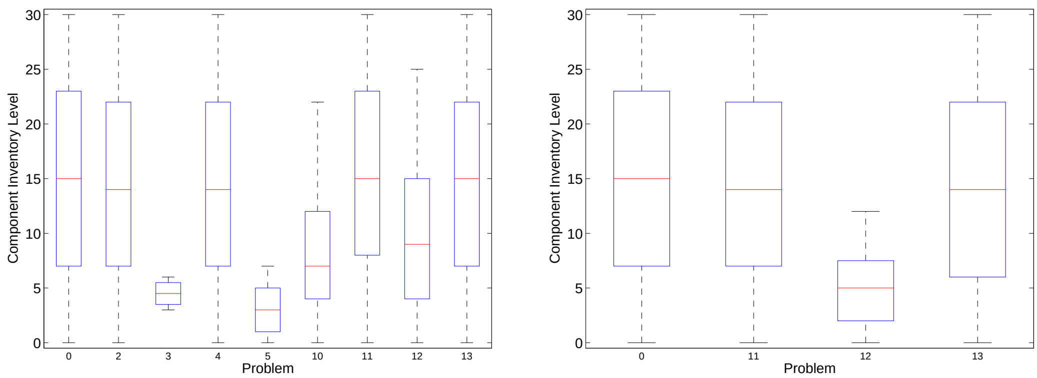
Figure 5.14: Graphs showing the level of returned inventory (trigger level) when recovery is performed



(a) High quality recovery only

(b) Both high and low quality recovery

Figure 5.15: Graphs showing the level of serviceable inventory (trigger level) when recovery is performed, for two recovery strategies .



(a) High quality recovery only

(b) Both high and low quality recovery

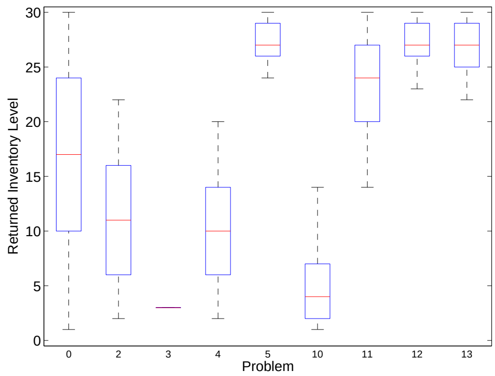
Figure 5.16: Graphs showing the level of component inventory (trigger level) when recovery is performed and serviceable inventory is at least than 28.

more often and for a wider range of inventory levels when there is a possibility of replenishing both the serviceable and component inventories (under a high and low quality recovery strategy), however this does not appear to be the case. The level of recovered inventory which triggers recovery is slightly higher and more variable for the high quality recovery strategy. Interestingly under both strategies, recovery is sometimes performed when serviceable inventory is near capacity. In these situations it seems unlikely that recovery is being performed in order to replenish the serviceable inventory. In order to investigate why recovery is performed in these situations two more box plots are created. In these box plots only the cases in which recovery is performed ($a_r > 0$) when serviceable inventory is near capacity ($i_s \geq 28$) are considered.

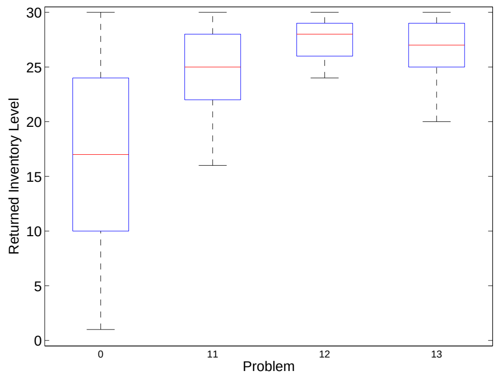
When both high and low quality returns can be recovered, recovery can also replenish the component inventory. If there is insufficient capacity in the serviceable inventory, then high quality returns will undergo low quality recovery and replenish the component inventory instead. Therefore if the serviceable inventory is high, recovery could be performed in order to replenish the stock of components. Figure 5.16 shows the level of component inventory when recovery is performed and the level of serviceable inventory is high. With the exception of problem $G12$, recovery is performed across a range levels of component inventory (from 0 to 30). This suggests that in these situations, replenishment of the component inventory may not always be the motivation for performing recovery.

Figure 5.17 shows the level of returned inventory when the serviceable inventory is near capacity and recovery is performed. For some problems the returned inventory is high, which suggests one reason for performing recovery is to use the returns. However under a high-quality recovery strategy there are some problems for which returned inventory is low (e.g. $G02$, $G03$, $G04$, $G10$). These problems seem to have comparatively higher holding cost h_r for returned inventory. This is consistent with the assumption of performing recovery in order to deplete the returned inventory.

In summary, there is evidence to suggest an ‘produce-up-to’ structured policy could be appropriate. The ‘produce-up-level’ seems to be positively associated with the demand rate. Recovery is performed over a wider range of states under a high-quality only recovery strategy. For problems with a particularly high holding cost for returns,



(a) High quality recovery only



(b) Both high and low quality recovery

Figure 5.17: Graphs showing the level of returned inventory (trigger level) when recovery is performed and serviceable inventory is at least than 28

recovery is sometimes performed when the serviceable and component inventories are near capacity, in order to dispose of returns.

Action plots. In order to examine the relationship between the three inventories and the action chosen for a given problem we construct what we call 'action plots'. An action plot is a plot of two inventory levels (e.g. serviceable inventory and returned inventory), and shows what action is taken for each combination of inventory level. A similar type of graph was used by Simpson (1978) to show which actions should be performed for different regions of the state space. These plots allow for the general structure of a policy, or lack thereof, to be observed at a glance. Insights into the structure of the optimal policy gained from these graphs, and the box plots above, will be used in the development of heuristic policies in Section 5.7. Ideally a well-performing heuristic policy with a simple structure is sought. In this section we focus on the optimal policy under recovery strategy which recovers high and low quality returns.

In these plots the symbol displayed at a given point on the graph represents the action that should be taken at the given inventory level:

- ◯ a circle is displayed when the action is to recover
- $+$ a plus is displayed when the action is to produce
- □ a square is displayed when the action is to produce and order components.

△ a triangle is displayed for states in which the action is to do nothing

Figure 5.18 shows the structure of a heuristic policy in which serviceable inventory is replenished if it less than 10 and the production or recovery lot size is $10 - i_s$.

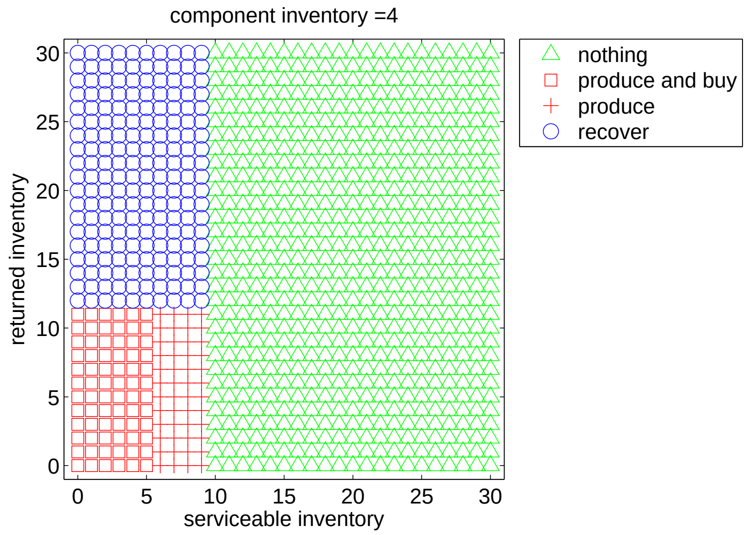


Figure 5.18: Action plot for a heuristic policy

In this plot, the serviceable inventory level is on the x-axis and the returned inventory is on the y-axis, thus, for a given level of serviceable and returned inventory, a symbol is used to represent the action that is chosen in that state. The type of replenishment is determined by the level of returned inventory. If there is "sufficient" returned inventory ($i_r \geq 12$), then recovery is performed, otherwise production is performed. This relationship between production and recovery is shown on the action plot by the rectangle in the lower left corner. If the serviceable inventory is 10 or more, then nothing is performed. This plot shows the action, assuming that there are 4 components in stock, so components must be bought if the production lot size $10 - i_s > 4$.

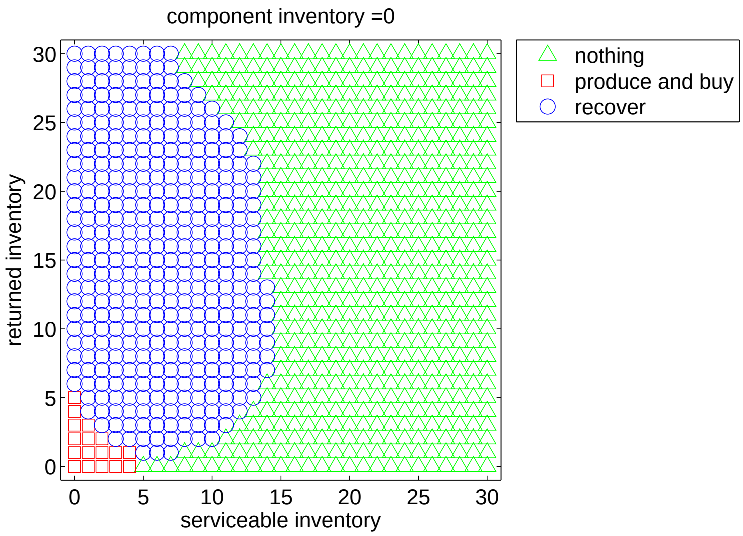
From the analysis conducted in the previous sections, it seems that the levels of serviceable inventory and returned inventory are the main drivers which determine the optimal action. Therefore action plots were created in order to examine the relationship between serviceable and returned inventory for the optimal policy.

Figure 5.19a presents the optimal policy for problem $G04$, when there are no components in stock ($i_c = 0$). As shown in this figure, if there are no components in stock and there are more than approximately 12 units of serviceable inventory in stock, then the optimal action is to do nothing (triangles). For low levels of serviceable and returned inventory the optimal action is to produce and buy components (squares). For low numbers of serviceable inventory and high numbers of returns, the optimal action is to do recovery (circles). This action plot shows the actions for only one level of component inventory.

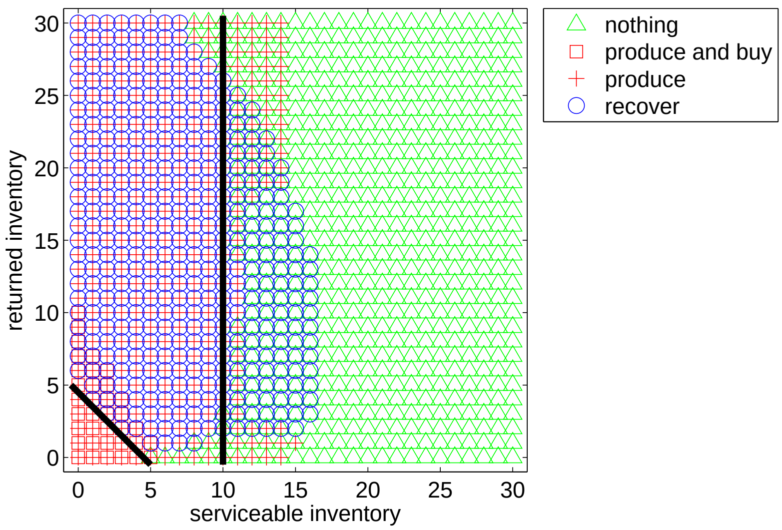
In order to represent the component inventory, it is possible to create 31 graphs, one for each inventory level. Whilst this may be useful, it detracts from the original purpose of the action plots, which was to be able to visualise the policy at a glance. Therefore, in order to represent the components inventory, we will overlay all of the components inventory levels onto one graph. The symbols used to represent the actions have been chosen so that no symbol can be hidden by another. Figure 5.19b shows the overlayed action plot for $G04$. If there are less than approximately 15 serviceable items and at least 2 returned items in stock, then depending on the number of components it is sometimes optimal to recover, produce, or do nothing.

Notice that this figure could be roughly divided into segments as shown by the thick black lines in Figure 5.19b. The small triangle in the bottom left represents only producing (and buying). The area above it represents recovery or production, and the area to the right of the vertical black line represents doing nothing.

Figures 5.20, 5.21 and 5.22 show the optimal policy actions for $G12$ and $G16$ in terms of different pairs of inventories. Observe that in general this action plots appear to have a more complicated structure than that of the action plot for $G04$, however there are still some similarities. For instance, in Figure 5.20 the bottom left corner in both plots indicates that for low levels of serviceable and returned inventory, production (and buying components) is performed. For high levels of serviceable inventory, no action is taken, and on the middle and upper left of the graph, recovery and production are performed. In Figure 5.20a notice that for problem $G12$ recovery is sometimes performed when the level of returned and serviceable inventory is high. For $G16$ notice that in the action plot for all values of serviceable and returned inventory, there is at least one level of components inventory for which it is optimal to do nothing, i.e., depending on the level of components, even if $i_s = 0$, it may be optimal to do nothing.



(a) Action plot for $G04$ with $i_c = 0$

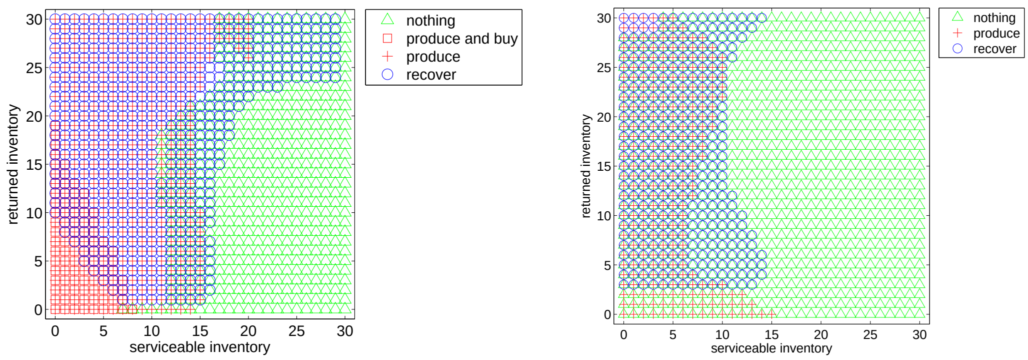


(b) Action plot for $G04$ for $i_c = \{0, \ldots, 30\}$

Figure 5.19: Action plot for $G04$ with $i_c = 0$ and $i_c = \{0, \ldots, 30\}$

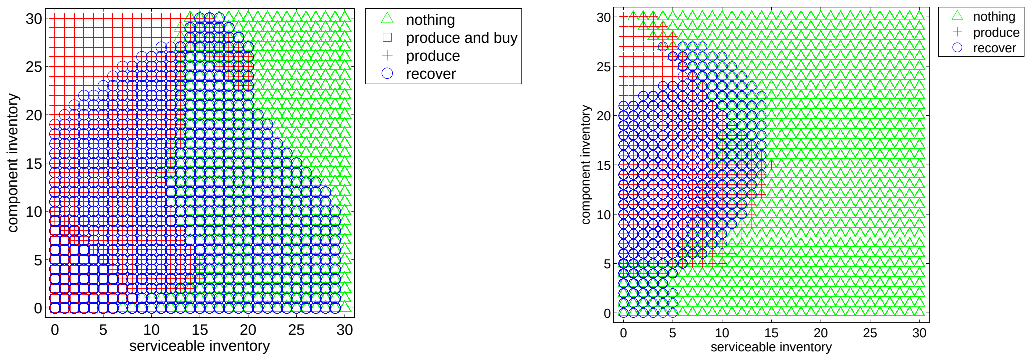
Presenting the action plots for all problems is impractical, therefore although all plots were examined, only a selection are presented here. Inspection of the action plots for all of problem set G revealed that many of the plots exhibit the 'triangular' pattern shown in Figures 5.19b, 5.20a and 5.20b. This pattern could be simplified to the structure shown in Figure 5.23.

The policy structure shown in Figure 5.23 could be described as follows: if the serviceable inventory level is below some level s_s, then produce or recover $(S_s - i_s)$, depending on the number of returns and components in stock; otherwise, do nothing.



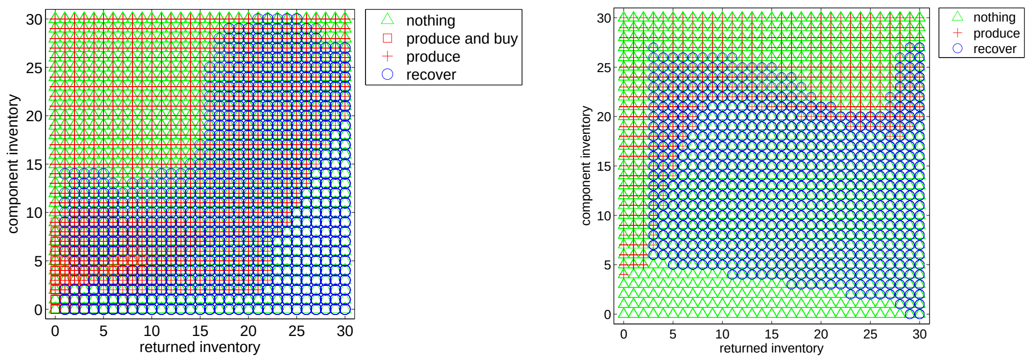
(a) Action plot for $G12$ (b) Action plot for $G16$

Figure 5.20: Serviceable vs Returned for all components for $i_c = \{0, \ldots, 30\}$



(a) Action plot for $G12$ (b) Action plot for $G16$

Figure 5.21: Serviceable vs Components for all returns for $i_r = \{0, \ldots, 30\}$



(a) Action plot for $G12$ (b) Action plot for $G16$

Figure 5.22: Returned vs Components for all serviceable for $i_s = \{0, \ldots, 30\}$

Note producing $S_s - i_s$ is the equivalent of producing up to S_s, however for recovery, since the quality of is uncertain, the 'recover-up-to' level will also be uncertain. The triangle in the lower left of the graph suggests that recovery is not performed if there are only a few returns in stock. Therefore, if there are 'sufficient' returns in stock, perform recovery, otherwise perform production. Figure 5.23 shows the action plot for this heuristic policy for the values of $s_s = S_s = 8$. Compare this figure to the action plot for the optimal policy of $G04$ (Figure 5.19b) and observe the similarities in the structure of the graphs.

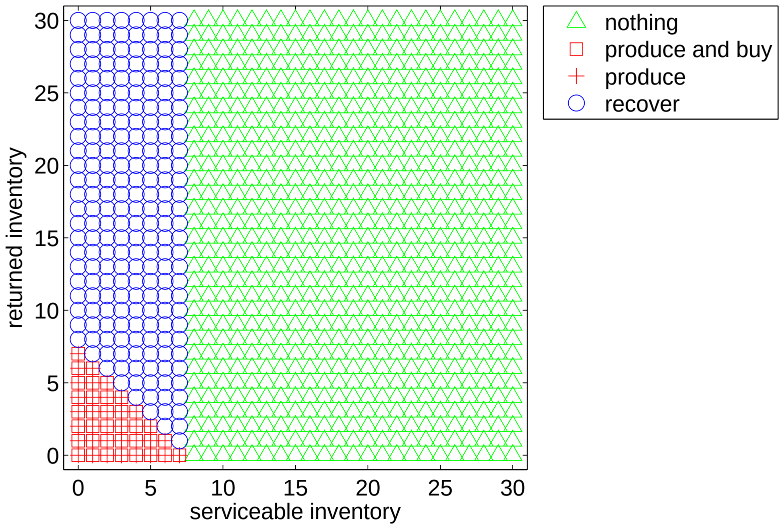


Figure 5.23: Action plot showing a 'triangular' heuristic policy

Insights into Policy Structure

The following insights have been gained from the analysis into the structure of the optimal policy.

The action plots for the base problems suggests that the optimal policy does not have a simple structure, therefore we will not try to find or prove the optimality of a fixed structure. Since the structure of the optimal policy is complex, it would be difficult to implement in practice. Therefore using some of the observations relating to when actions are performed in Section 5.7, we will develop some heuristics which do have a simple structure.

The 'produce-up-to' level seems to be related to the demand rate λ_d. The mean proportion of high quality returns α seems to affect the size of the recovery lots a_r.

High setup costs for buying components and high holding costs for components seem to ‘discourage’ buying components - and in some cases eliminate it completely.

5.7 Heuristic Policies

The analysis in Section 5.6 has demonstrated that the optimal policy for this model has a complicated structure. From a managerial point of view, the optimal policy is too complicated to implement in practice. Ideally a policy with a simple structure and near-optimal performance should be implemented. The aim of this section is to find such a policy. Insights gained about the structure of the optimal policy will be used in the development of the heuristic policies.

In the literature a number of policies structures have been used to study this type of system. For example van der Laan and Salomon (1997) study a push-pull policy and Kiesmüller (2009) discuss a ‘can order’ policy. Inderfurth (1997) mentions that if setup costs are included in the model, a simple optimal policy will not exist. This could be one reason for complicated policy structure observed in this chapter.

In Sections 5.7.1 and 5.7.2 two policy structures are proposed and tested on problem set G. Both policies extend a simple periodic review, order-up-to (R, s, S) policy (Silver et al., 1998, page 246). In the development of the heuristic policies, the *policy structure* and the *parameter selection* need to be considered. Note that problem $G00$ is excluded from the analysis as this problem does not have any unit costs. The type of policy that is appropriate for a problem with no unit costs is likely to be different to that which is appropriate for a problem with unit costs; thus trying to find a heuristic which performs well across both situations could detract from the purpose of this section. Similarly it is unlikely that the same heuristic policy will perform well on both recovery strategies. Therefore in this section we focus on systems which perform both high and low quality recovery, as this is the part of the model which extends the existing literature.

The proposed policies will be tested more extensively (on a set of randomly generated problems) in Section 5.7.3. The performance of the heuristic policies are compared to the optimal policy by calculating the ‘relative cost error’ (RCE):

$$\text{RCE} = \frac{\text{heuristic cost} - \text{optimal cost}}{\text{optimal cost}} \times 100\%$$

5.7.1 Policy $P1$: Periodic Review Order-up-to Policy

Policy Structure

Insights from the analysis of the optimal policy suggest that an 'order-up-to' structured policy may be appropriate for this problem. Therefore the first policy that is proposed is a simple periodic review order-up-to policy with two parameters. The structure of this policy is based on the traditional periodic review (s, S) 'order-up-to' policy (see Section 3.1 for a discussion of common inventory policies).

The structure of this policy is as follows. If the serviceable inventory level is less than some trigger level s, then inventory must be replenished up to a target level of S by either production or recovery. Ideally the serviceable inventory will be replenished by $S - i_s$. Recovery will be performed if there are sufficient returned items in stock to meet the target, otherwise production will be performed. Components will be bought if required in order to complete a production order.

If production is performed, then the production lot will be $a_p = S - i_s$. However, since the quality of the returns is not known until recovery has begun, it is not known how many of the items recovered will be added to serviceable inventory. For this policy it is assumed that if $i_r \geq (S - i_s)$, then recovery will be performed and the recovery lot will be $a_r = S - i_s$.

The policy can be described formally as follows:

If $i_s < s$

  If $i_r \geq (S - i_s)$

    recover $a_r = S - i_s$

  Else

    produce $a_p = S - i_s$

    order $a_c = \max\{0, S - i_s - i_c\}$

Else

  Do Nothing

An obvious limitation of this policy is that it does not take into account the uncertainty in the recovery yield. However at this point we choose to retain the simple structure of the policy; this limitation will be addressed by the second policy.

Parameter Selection

This policy has two parameters, the trigger level s and target level S. For the problems in set G, the trigger level s can take values $s = \{1, \ldots, 30\}$ and the target level S can take values $S = \{s, \ldots, 30\}$. This creates 465 possible parameter combinations for each problem. The first question that needs to be investigated is how well does the best performing combination of s and S perform compared with the optimal policy. Once this has been determined it can be used to gauge the performance of any heuristic parameter selection rules.

In order to investigate this question, the long run average cost was calculated for the 465 possible parameter combinations. The performance of these parameter combinations was compared to the optimal policy by calculating the 'relative cost error' (RCE).

Table 5.6 shows the RCE for the best and worst performing policies and the corresponding parameter values for s and S. The parameter combinations that perform the best will be referred to by s^* and S^*. Observe that for 17 of the 20 problems there is a $P1$ policy which results in a relative cost error of less than 4%, and for 19 of the problems there is a policy which has a relative cost error of less than 10%. Problem $G16$ has a much higher relative cost error (22.605%) than the other problems, however this problem has features that makes it quite different from the rest of the problems. In particular, problem $G16$ has very high setup costs for production and buying components, very low returns and quality of returns and the optimal policy never buys components. The action plot for this problem shows that the optimal policy appears to avoid ordering components, and instead uses ones which have been recovered.

In order to investigate how the cost of the heuristic policy $P1$ changes for different values of s and S, image plots were constructed using the MATLAB function `image()`. These plots show the values of s and S on the vertical and horizontal axes respectively, and display the relative cost error of the policy using the colour gradient. Figure 5.24 shows the figure for problems $G01$, $G09$, $G16$ and $G20$. The plots for all of the problems in set G were examined, however these four problems were selected in order to illustrate

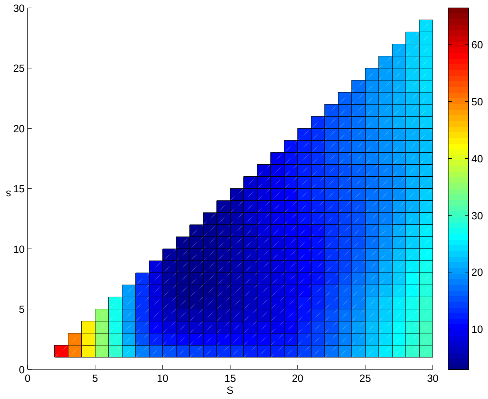
Problem	Best Combination s	S	RCE	Worst Combination s	S	RCE
$G01$	9	11	2.926	1	1	66.468
$G02$	11	12	1.577	2	3	36.113
$G03$	10	12	0.938	2	3	29.996
$G04$	10	11	1.675	2	3	32.927
$G05$	12	18	4.669	1	1	48.577
$G06$	11	15	3.585	1	1	66.910
$G07$	10	13	3.417	1	1	56.244
$G08$	7	12	3.509	1	1	67.482
$G09$	8	10	0.910	1	2	17.146
$G10$	13	17	4.890	1	1	42.827
$G11$	13	16	0.985	1	1	61.225
$G12$	13	14	1.080	1	1	33.297
$G13$	10	15	0.943	1	1	41.445
$G14$	8	10	1.513	1	1	33.674
$G15$	5	16	3.513	3	4	33.831
$G16$	1	1	22.345	29	30	89.180
$G17$	1	18	7.658	1	1	53.359
$G18$	1	7	1.416	1	1	23.376
$G19$	14	19	4.256	5	6	19.762
$G20$	4	7	1.596	1	1	42.379

Table 5.6: Best and worst performing parameter combinations for policy $P1$ for problem set G

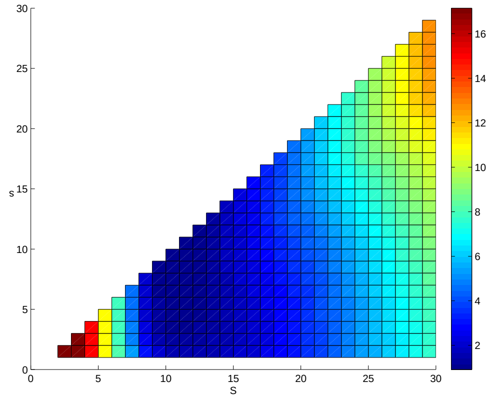
some common features across all problems. Note that the RCE represented by a given colour gradient differs across the problems.

The first feature of these graphs that is worth mentioning is that for all problems there are a range of parameters which have an RCE which is similar to the RCE associated with the 'best' parameter combination. However, there are some problems which perform very poorly. In general it appears that the RCE is not very sensitive to small changes in the parameter values, however for some problems, the RCE can be very sensitive to larger changes in the parameter values.

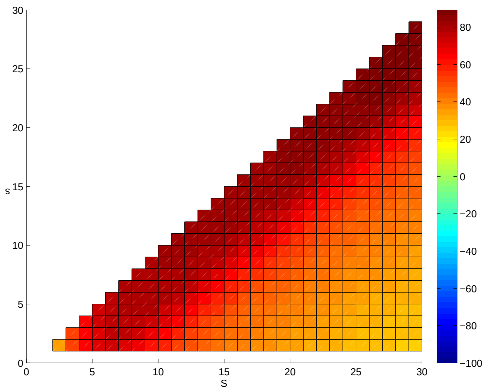
Using the insights gained from the image plots in Figure 5.24 and the analysis in Section 5.6.3, four methods for selecting parameter combinations for policy $P1$ are proposed. These methods will be denoted by $P1_A$, $P1_B$, $P1_C$ and $P1_D$; the parameters associated with each policy will also have the corresponding letter as a subscript (A, B, C, or D). However it is worth bearing in mind that the best performance that can be obtained by a policy of this structure are the shown in Table 5.6. These methods will be tested problem set G and then on a larger dataset in Section 5.7.3.



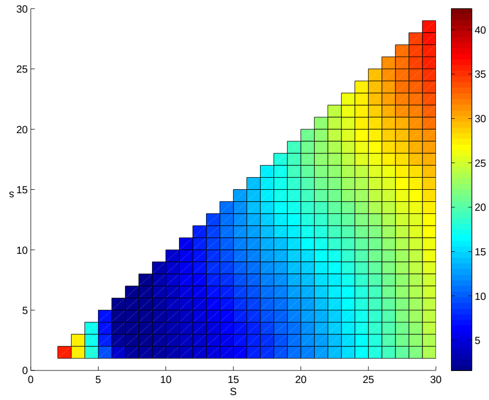
(a) Problem $G01$



(b) Problem $G09$



(c) Problem $G16$



(d) Problem $G20$

Figure 5.24: Relative cost error of a $P1$ policy for all parameters combinations (s, S) for selected problems from set G (s on the vertical axis and S on the horizontal axis).

Analysis of the optimal policy in Section 5.6.3 suggested that the 'order-up-to' level for production may be related to the mean demand λ_d. Therefore the first method for choosing the parameter values is based purely on the demand rates for each problem, such that:

$$s_A = \lambda_d$$
$$S_A = \lambda_d$$

However, while the analysis of the optimal policy did show a relationship between demand and the action sizes, other factors, such as the quality of the returns, also seemed to influence the structure of the optimal policy. In general the parameters which perform the best are greater than the expected value of demand λ_d. The second method adds an additional "boost" to the target level parameter S by taking into account the fact that not all of the returns recovered are likely to have sufficient quality to undergo high quality recovery and thus to replenish serviceable inventory. A "boost" factor of $0.1/\alpha$ was used. This ratio results in a larger increase for problems with a lower proportion of high quality returns (i.e. small α). It appears that for problems for which the demand rate λ_d and the best performing parameters (s^*, S^*) were of a similar magnitude, the target parameter S^* was only slightly greater than the demand rate. Therefore, a numerator of 0.1 was used to allow a slight increase in the target level. Thus the second method used to choose the policy parameter values is:

$$s_B = \lambda_d$$
$$S_B = \lambda_d \left(1 + \frac{0.1}{\alpha}\right)$$

The relationship between the setup costs and holding costs are also likely to affect the choice of the optimal action. The well-known EOQ formula uses the ratio of setup costs over holding costs $\sqrt{\frac{2\times\text{setup costs }\times\text{demand}}{\text{holding costs}}}$ to determine the optimal order size in a deterministic model (Silver et al., 1998). This ratio represents the tradeoff associated with placing frequent orders rather than holding stock. The third method for determining the parameter values takes this tradeoff into account.

In this method the trigger level s_C and target level S_C are chosen so that the resultant order quantity $S_C - s_C$ is approximately equal to an adjusted version of the economic order quantity (EOQ). The formula is adjusted according to costs associated with the production, thus the setup costs associated with production and buying components are included in the numerator. The sum of all three holding costs is used in the denominator of the adjusted EOQ formula to take into account the cost of holding all inventories. Thus the third method used to choose the policy parameter values is:

$$s_C = \max\left\{1, \left\lceil S_3 - \sqrt{2\lambda_d \frac{k_p + k_b}{(h_s + h_r + h_c)}} \right\rceil\right\}$$
$$S_C = \min\left\{30, \left\lceil \lambda_d + \sqrt{2\lambda_d \frac{k_p + k_b}{(h_s + h_r + h_c)}} \right\rceil\right\}$$

The relationship between the lost sales cost may also affect the choice of the optimal action. If the cost of losing a sale is very high, then a firm will try to avoid this by ensuring that there is sufficient inventory in stock. Using an EOQ-based formula again, a fourth method for determining the values of the parameters is developed. This method takes the cost of production and ordering into account as well as the lost sales and holding costs.

$$s_D = \max\left\{1, \left\lceil S_4 - \sqrt{2\lambda_d \frac{k_p + k_b}{(h_s + h_r + h_c)}} \right\rceil\right\}$$
$$S_D = \min\left\{30, \left\lceil \lambda_d(1 + \frac{0.1}{\alpha} + 2(\frac{c_p + c_b}{\lambda_d(l_s - h_s - h_c)})) \right\rceil\right\}$$

These methods were tested using problem set G. Table 5.7 shows the resultant policies and the relative cost errors (compared with the optimal policy) for problem set G. The column headed $P1_A$ shows the policy given by parameters s_A and S_A, and likewise for $P1_B$, $P1_C$, and $P1_D$. The policy $P1^*$ shows the parameters and RCE of $P1$ policy with the best performing parameter combination. Of the four parameter selection methods, method $P1_A$, which was based purely on the demand parameter λ_d performs the worst of the four methods. Policy $P1_C$ performs the best, with policy $P1_D$ a close second.

5.7.2 Policy $P2$: Periodic Review Order-up-to Policy with Yield Adjustment

Policy Structure

Policies $P1_B$ and $P1_D$ take into account the uncertain recovery yield by adjusting the target level S. However, as well as increasing the target level for recovery, this also increases the target for production. An alternative method of adjusting for an uncertain recovery yield is now proposed.

In Policy $P1$, recovery was chosen if $i_r \geq S - i_s$. In this policy, the number of returns required in order perform recovery and the recovery lot size are adjusted to account

	$P1_A$				$P1_B$				$P1_C$				$P1_D$				$P1^*$				Optimal Policy
Problem	s_A	S_A	Cost	RCE	s_B	S_B	Cost	RCE	s_C	S_C	Cost	RCE	s_D	S_D	Cost	RCE	s^*	S^*	Cost	RCE	Cost
G01	8	8	841.508	13.025	8	9	804.105	8.001	5	10	781.188	4.923	9	14	777.997	4.494	9	11	766.323	2.926	744.535
G02	9	9	1165.511	2.710	9	14	1155.567	1.834	12	15	1157.725	2.024	10	13	1153.532	1.655	11	12	1152.650	1.577	1134.757
G03	11	11	1827.235	0.999	11	15	1830.434	1.175	12	16	1833.554	1.348	12	16	1833.554	1.348	10	12	1826.141	0.938	1809.169
G04	9	9	2230.030	1.966	9	13	2231.105	2.015	9	13	2231.105	2.015	10	14	2236.832	2.277	10	11	2223.659	1.675	2187.034
G05	10	10	2200.504	20.688	10	12	2076.404	13.881	9	13	2026.063	11.120	11	15	1948.248	6.853	12	18	1908.425	4.669	1823.303
G06	10	10	1169.477	16.030	10	12	1094.574	8.598	6	12	1096.942	8.833	11	17	1048.271	4.004	11	15	1044.045	3.585	1007.910
G07	10	10	1172.859	12.177	10	12	1087.788	4.040	7	13	1082.766	3.560	11	17	1093.943	4.629	10	13	1081.269	3.417	1045.545
G08	8	8	704.759	16.750	8	9	667.752	10.619	4	10	650.325	7.732	9	15	635.323	5.247	7	12	624.833	3.509	603.650
G09	9	9	2399.821	1.148	9	14	2413.244	1.714	8	14	2412.359	1.677	10	16	2433.478	2.567	8	10	2394.164	0.910	2372.582
G10	10	10	4455.457	23.619	10	12	4115.351	14.182	10	13	3974.476	10.274	11	14	3875.396	7.525	13	17	3780.422	4.890	3604.191
G11	12	12	683.667	1.989	12	15	677.313	1.041	8	15	677.544	1.076	13	20	679.294	1.337	13	16	676.938	0.985	670.333
G12	12	12	1371.381	1.303	12	15	1369.349	1.153	10	15	1369.388	1.156	13	18	1378.014	1.793	13	14	1368.361	1.080	1353.736
G13	13	13	1988.182	1.131	13	16	1984.644	0.951	4	16	1987.536	1.098	14	26	2023.971	2.951	10	15	1984.497	0.943	1965.953
G14	8	8	990.867	4.356	8	10	963.870	1.513	5	10	964.585	1.588	9	14	979.790	3.189	8	10	963.870	1.513	949.508
G15	9	9	1199.991	13.654	9	12	1102.433	4.414	1	13	1131.726	7.188	9	21	1107.860	4.928	5	16	1092.919	3.513	1055.832
G16	9	9	3532.363	80.046	9	18	3143.350	60.218	1	28	2490.120	26.922	1	30	2470.942	25.945	1	1	2400.309	22.345	1961.925
G17	9	9	1504.347	20.198	9	11	1502.215	20.028	1	14	1371.300	9.568	5	30	1404.311	12.205	1	18	1347.401	7.658	1251.555
G18	7	7	764.635	1.603	7	11	774.169	2.870	5	12	773.918	2.837	8	15	788.376	4.758	1	7	763.228	1.416	752.569
G19	9	9	2662.195	15.212	9	14	2473.339	7.039	8	15	2455.652	6.273	10	17	2421.265	4.785	14	19	2409.042	4.256	2310.692
G20	5	5	945.512	6.792	5	6	903.329	2.027	4	7	899.510	1.596	6	9	912.704	3.086	4	7	899.510	1.596	885.380
Min				0.999				0.951				1.076				1.337				0.910	
Max				80.046				60.218				26.922				25.945				22.345	
Mean				12.770				8.366				5.640				5.279				3.670	
Median				9.484				3.455				3.198				4.249				2.301	
Std Dev				17.177				13.014				5.936				5.353				4.620	

Table 5.7: Performance of heuristic policy parameter selection methods for problem set G

for the fact that some returns may be low quality. The rationale for making this type of adjusted comes from insights gained during the analysis of the optimal policy in Section 5.6.3. Let y denote the 'yield factor' of recovery, and suppose that recovery is performed if $i_r \geq (S - i_s)/y$ and that the recovery lot size is $a_r = (S - i_s)/y$. If the yield factor $y = 1$, then the policy is the same as $P1$. If $y < 1$, then the number of returns recovered will be greater than the number required to meet the target $S - i_s$, and if $y > 1$, then the number of returns recovered will be less than the number required. This policy can be described as follows:

If $i_s < s$

    If $i_r \geq (S - i_s)/y$

        recover $a_r = (S - i_s)/y$

    Else

        produce $a_p = S - i_s$

        order $a_c = \max\{0, S - i_s - i_c\}$

Else

    Do Nothing

Parameter Selection

For this policy it is infeasible to completely enumerate across all possible values of the parameters s, S and y, therefore we need to use a different approach for selecting the parameters. Initial investigations suggested that if y is significantly different from 1, then the resultant policies performed dramatically worse than policies with no adjustment for the recovery yield. For instance, if $y = 0.5$ then twice as many returns are required in order to perform recovery. For some values of s and S, this means recovery is performed very infrequently, or never, as the returned inventory does not often reach the required level. Conversely, if y is higher, then recovery may be selected very frequently, meaning that production is rarely performed. Therefore, in this study we focus on values of y which are near to 1, $y = \{0.9, 1.0, 1.1, 1.2\}$. Note that if $y = 1.0$ then the policy is equivalent to policy $P1$.

The aim of the investigation in this section is to determine whether or not adjusting the policy for the recovery yield can improve the policies $P1_A$—$P1_D$, proposed in

Section 5.7.1. Therefore four variants of policy $P2$ are created, by selected the parameters as outlined by the four methods for policy $P1$. These four variants are denoted by $P2_A$, $P2_B$, $P2_C$ and $P2_D$. For each of these policies, four values of y are applied. This creates a total of 16 policies.

Table 5.8 presents a summary of the performance of the $P2$ policies. (The parameters, costs and RCEs associated with each of these policies is available in Appendix B in Tables B.12–B.15.) Observe that $P2_C$ has the lowest mean across all values of y, however the lowest median alternates between $P2_B$ and $P2_D$. The policy with the highest maximum performance is $P2_D$. For policies $P2_A$ and $P2_B$ the mean and median increase with y. For $P2_D$ the mean and median decrease with y. Policy $P2_C$ attains its lowest mean at $y = 1.0$ and the lowest median at $y = 0.9$. For $P2^*$, the parameters attained by complete enumeration of s and S, the lowest mean is attained at $y = 1.0$, and the lowest median at $y = 0.9$. For all the parameter selection methods $A - D$, the highest variability occurred for $y = 0.9$ and for $P2^*$ the highest variability occurred for $y = 1.2$.

Allowing flexibility in the yield adjustment y allows improvement in the RCE, however the size of y associated with the improvement, depends on the parameter selection method. Interestingly, it appears that allowing the yield adjustment affects the optimality of the $P2^*$ parameters. When $P2^*$ is used to determine the value of the parameters s and S, the median RCE is lowest for all values of y, but the mean is not.

5.7.3 Performance of Policies

In this section the policies proposed in previous sections are tested on a large set of randomly generated problems. In order to generate a set of test problems the following method is used:

1. Generate 14 uniform random variables $U_j \sim \text{Uniform}(0, 1),\ 1 \leq j \leq 14$ and calculate the parameters as follows:

y		$P2_A$	$P2_B$	$P2_C$	$P2_D$	$P2^*$
0.9	min	0.844	0.896	0.929	1.505	0.791
0.9	max	82.909	110.796	106.866	305.764	62.445
0.9	mean	13.806	12.164	9.702	20.427	7.291
0.9	median	5.101	3.209	3.375	4.576	2.780
0.9	std dev	23.508	26.135	23.060	65.782	15.215
1	min	0.999	0.951	1.076	1.337	0.910
1	max	80.046	90.371	89.317	271.661	61.812
1	mean	15.374	12.271	9.625	17.964	6.439
1	median	12.177	4.040	3.560	4.494	2.926
1	std dev	20.916	22.127	19.200	58.375	13.503
1.1	min	1.029	0.909	1.083	1.263	0.910
1.1	max	80.046	90.371	89.317	261.704	61.812
1.1	mean	15.422	12.744	10.091	17.551	6.567
1.1	median	12.177	5.035	3.942	4.219	3.342
1.1	std dev	20.883	22.092	19.208	56.204	13.480
1.2	min	1.126	0.905	1.126	1.243	0.970
1.2	max	79.325	73.845	74.061	235.979	125.507
1.2	mean	17.766	13.427	10.479	16.488	10.083
1.2	median	17.287	6.861	5.503	4.049	3.760
1.2	std dev	20.365	19.183	16.131	50.608	26.856

Table 5.8: Performance of $P2$ policy

Distributions		
mean demand	λ_d	$\lceil 13 * \sqrt{U_1} \rceil$
mean returns	λ_r	$\lceil \gamma\lambda_d \rceil$
returns ratio	γ	U_2
high quality	α	U_3
Setup costs		
maximum setup cost	M_k	100
production	k_p	$U_4 M_k$
recovery	k_r	$U_5 M_k$
buying	k_b	$U_6 M_k$
Cost ratios		
maximum processing cost ratio	M_h	0.3
maximum disposal ratio	M_d	0.3
holding cost ratio	γ_h	$U_7 M_h$
production ratio	γ_p	U_8
buying ratio	γ_b	U_9
disposal ratio	γ_d	$U_{10} M_d$

Processing costs		
production	c_p	hs/γ_h
returns	c_r	hr/γ_h
buying	c_b	hc/γ_h
high quality recovery	c_h	$\gamma_p c_p$
low quality recovery	c_l	$\gamma_b c_b$
disposal	c_d	$\gamma_d(c_p + c_b)$
Holding costs		
maximum holding costs	M_{hh}	100
serviceable inventory	h_s	$U_{11} M_{hh}$
returned inventory	h_r	$U_{12} M_{hh}$
component inventory	h_c	$U_{13} M_{hh}$
Penalty costs		
maximum lost sales ratio	M_l	0.3
lost sales ratio	γ_l	$U_{14} M_l$
lost sales	l_s	$(c_p + c_b - c_d)(1 + \gamma_l)$

2. If the following conditions are met, then accept the scenario.

$$\begin{aligned}
&hs > hr \\
&hs > hc \\
&cp + cb > cp + cr + cl \\
&cp + cb > cr + ch
\end{aligned}$$

3. Continue generating scenarios until the required number have been accepted.

This method was used to generate a set of 250 problems. These policies were used to test policy $P2_A$, $P2_B$, $P2_C$ and $P2_D$ for four values of $y = 0.9, 1.0, 1.1, 1.2$. Recall that policy $P2$ with $y = 1.0$ is equivalent to policy $P1$.

Since the problem parameters have been randomly generated it is possible that some combinations of parameters may result in a MDP for which the value iteration algorithm will not converge (see 3.2.2 for further details). Since there are a large number of problems to calculate, we do not want to waste time on problems which are not going to converge. After some initial experiments, it was determined that most problems which are going to converge should have done so by 1000 iterations. Therefore if a problem has not converged by 1000 iterations, the algorithm was terminated. For some problems it is possible that a few more iterations would have yielded the optimal solution, however, it is also possible that the problem scenario may have never converged. For this reason, some of the policies were tested on have fewer than 250 problems.

Table 5.9 presents a summary of the performance of the policies on this set of randomly generated problems and shows the number of policies for which an optimal policy was found. In general, the policies do not perform as well on these problems as they did on problem set G. However, all policies have a median relative cost error of less than 13%. Overall, the policies with $y = 0.9$ perform the best.

y		P2-A	P2-B	P2-C	P2-D
0.9	min	0.077	0.020	0.021	0.101
0.9	max	139.017	110.092	140.301	66.713
0.9	mean	16.126	12.698	12.969	12.916
0.9	median	9.494	8.815	8.794	9.748
0.9	var	363.831	177.963	202.144	128.787
0.9	n	240	240	240	240
1	min	0.068	0.035	0.035	0.080
1	max	157.897	136.340	145.055	75.505
1	mean	19.931	14.136	13.947	13.724
1	median	11.929	9.436	9.441	10.053
1	var	611.644	247.230	231.651	154.550
1	n	237	237	237	237
1.1	min	0.059	0.050	0.051	0.079
1.1	max	157.897	136.340	145.055	75.505
1.1	mean	20.476	14.549	14.322	14.017
1.1	median	11.929	9.766	10.097	10.485
1.1	var	620.603	251.683	234.921	160.105
1.1	n	237	237	237	237
1.2	min	0.056	0.096	0.097	0.079
1.2	max	157.897	136.340	145.055	85.776
1.2	mean	22.815	16.186	15.645	14.898
1.2	median	12.863	10.793	11.561	10.433
1.2	var	709.490	305.789	270.563	192.692
1.2	n	238	238	238	238

Table 5.9: Performance of the $P2$ policies on a randomly generated dataset

5.8 Discussion

This chapter extends the deterministic model from Chapter 4 by allowing demand, returns, and the quality of the returns to be uncertain. A Markov decision process formulation of this problem is presented and is implemented in java. Datasets derived from Konstantaras and Papachristos (2008b) and Chapter 4 are used to investigate some properties of the model, and the benefit of performing both high and low quality recovery. It was found that, in general, when both types of recovery are performed, the fill rate is higher and the average cost is lower, than when only high quality recovery is performed.

The structure of the optimal policy was investigated and it was found to be very complicated and complex, which would make it difficult to implement in practice. Insights from the structure of the policy were used to construct some simple heuristic policies. While the performance of the heuristics varied from problem to problem, on a large randomly generated problem set, one of the heuristics achieved an average cost of within 13% of the optimal cost. However, a limitation of using heuristic policies is that while they perform well on some, or even most problems, they may not perform well on all problems. On some of the problems tested, even some of the better performing heuristics, performed very poorly (eg relative cost error of over 100%).

From a managerial point of view, the results of this chapter are useful as they suggest that performing both types of recovery can not only lead to cost savings, but can also allow better customer service, through increased fill rates. These results support those obtained in the deterministic model in the previous chapter.

The complicated and complex structure of the optimal policy mean that further analysis could be performed to investigate this structure. Alternative policy structures and parameter selection methods could be investigated. For instance, a two-stage policy could be implemented, which first finds a 'produce-up-to' level and then for this parameter finds the best performing 'recover-up-to' level. Additional policies could allow the component inventory level to trigger production, although this would mean increasing the number of parameters in the heuristic policies.

In this model it was assumed that the distributions of returns and demands were independent and known. In some industries, this may not be a realistic assumption. Further research could investigate the effect of relaxing this assumption. It was assumed that the quality of returns could not be determined until the recovery process had begun. In some industries it may be possible to 'pre-sort' returns. This could be an interesting extension to the current model. Alternative methods for dealing with recovered goods which do not fit into the serviceable or component inventories could also be investigated.

One limitation of the current model is that it assumes that the customer perceives newly produced and recovered goods as identical. In some industries this may be the case, however in other industries, they may not be viewed as the same, and it may even be illegal for them to be sold as the same. In the next chapter this limitation is addressed by introducing separate markets for newly produced and recovery goods.

Chapter 6

Discrete-Time Stochastic Product Recovery Model with Separate Markets

6.1 Introduction

In some industries newly produced goods and recovered goods are not identical. Even when recovered goods are functionally "as good as new", customers may not perceive them to be so, and indeed in some countries, legislation prevents recovered goods being sold "as new". However, when the functionality of the two types of goods remains the same or similar, some consumers may be willing to substitute one good for the other, if their preferred good is out of stock. This chapter discusses a product recovery system in which newly produced goods and recovered goods are sold on separate markets but can act substitutes for each other.

This model extends the stochastic model in Chapter 5 by introducing separate markets and substitution. It also extends the 'separate markets' literature (Inderfurth, 2004; Kaya, 2010) by allowing two-way substitution (rather than one-way substitution) and by considering an infinite horizon model. Piñeyro and Viera (2010) study a deterministic model which has a similar modelling structure to the one considered in this chapter, but to the best of our knowledge this type of model has not been studied in a stochastic environment.

This chapter is structured as follows. In Section 6.2 the problem under study is described and then in Section 6.3 the model description and assumptions are presented. A Markov decision process formulation of this model is presented in Section 6.4 and the implementation of the model is discussed in Section 6.5. Properties of the optimal policy are explored in Section 6.6, including the effect of performing both high and low quality recovery. The results, limitations and areas for further research are discussed in Section 6.7.

6.2 Problem Description

Suppose there is a manufacturer which has a primary function of producing new goods. This manufacturer accepts these goods back after they have been used and, if they are of sufficient quality, recovers them and sells them to a secondary market. For returns which are below the quality threshold for recovery, the firm has a choice: to dispose of them or to use them as components in the production of newly produced items. If insufficient components are obtained from the recovery of low quality returns then additional components are bought. Produced and recovered items are viewed by the consumer as different so are sold on separate markets, however they are functionally similar so can act as substitutes. The firm may choose to offer substitution between these two types of goods if one of them sells out.

The firm is a profit-maximising firm which receives revenue for the sale of produced and recovered goods. Costs are incurred for holding inventory and for lost sales. Fixed and unit costs are incurred for production, recovery and buying components. Demand for produced items, demand for recovered items, returns and the quality of returns are uncertain.

The firm must determine a production plan that specifies how much and how often to produce, recover and buy. It must also determine a substitution policy which specifies if and when substitution will be offered to customers. If substitution is offered to, and then accepted by, a customer, then they will pay the price of the cheaper recovered item, regardless of the good they receive. Offering a substitution means that the firm will not incur a lost sales cost for that item, however there may be indirect costs associated with performing a substitution. For example, if a produced good is offered in place of a recovered good, then the firm will miss out on the revenue they would have received had they sold it for the full price. The firm may also incur a cost for "lost goodwill" for not being able to supply what the customer demanded, or an administration charge for offering a substitution rather than providing the product originally demanded.

This system could describe the refurbishment of electronic equipment, such as copiers (Canon, 2012e), computers (Apple, 2011; Dell, 2012) and satellite navigation systems (TomTom, 2011). Wooden pallets (PalletWorld, 2012) are another example of new and used products being sold side-by-side. In all of these cases, used products are brought up to a "suitable" standard before being resold or leased. They have the same functionality as a newly produced item, but are not recovered up to an "as new" standard so cannot be sold as such. Because the newly produced and recovered goods are functionally similar, they could act as substitutes. Substitution policies vary from industry to industry, and from company to company, and may also vary at different times of the year (e.g. peak seasons). However in all cases, the company could choose to offer these substitutions, if they were faced with a stock out of the customer's preferred good.

6.3 Model Description and Assumptions

A diagram of the product recovery system being modelled is presented in Figure 6.1. As shown in this diagram there are two consumer markets: the first demands newly produced goods and the second demands recovered goods. Substitution between the two markets is represented on the diagram by the dashed line. In this section the model will be described and the modelling assumptions will be discussed.

Periodic Decision Making. In this model it is assumed that operational decisions are made periodically. Firms which have the technological capabilities to monitor stock levels continuously, may still only review this information and take action periodically,

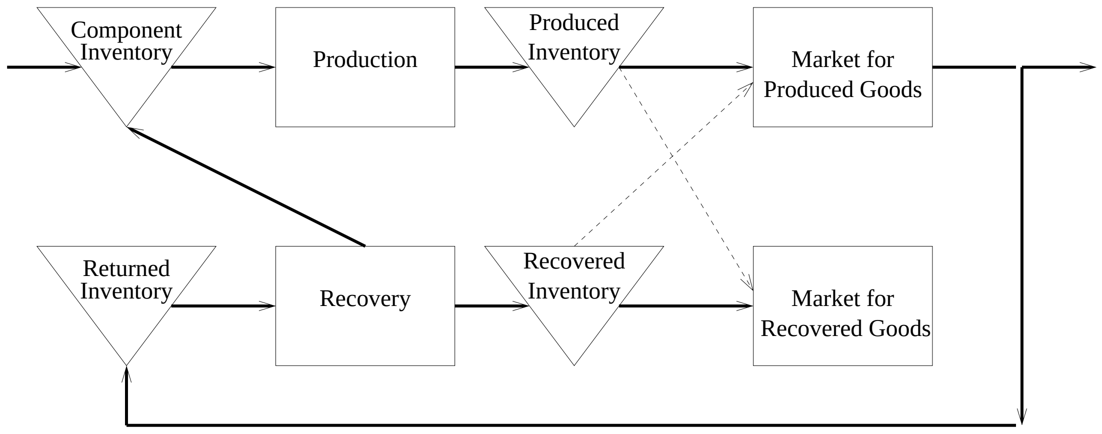


Figure 6.1: Product Recovery Model with Separate Markets and Substitution

e.g. daily or weekly (Silver et al., 1998). In this model, the inventory levels are observed at the start of each time period and then a decision is made regarding how much to produce or recover and how many components to buy. The lead times for production, recovery and buying are zero, therefore inventory levels are updated immediately to reflect the decision made. The rationale behind this is as follows. Suppose that the inventory levels are reviewed at the end of each day and then a production, recovery or buying order is placed. This order is completed or received before the firm opens for business in the morning, meaning that inventory levels will be updated before any customers arrive to demand goods.

After an action has been selected and inventory levels have been updated accordingly, the demands and returns are observed. The inventory levels cannot exceed the maximum limits at any point. Continuing the scenario above, if a production order is completed before the firm opens for business, then the inventory level must not exceed the maximum limit at any point during the night.

Production, Recovery and Buying. It is assumed that production and recovery require some shared facility so can not both happen in the same period. It is also assumed that within a given period, the amount that is produced or recovered and the number of components that are bought is non-negative and constrained by a finite upper limit. This system is being studied over an infinite time horizon, which implies that the decisions regarding production, recovery and buying will not depend on time.

With enough effort and expenditure, all returns could be recovered to a functional standard, similar to that of newly produced goods. However, it is assumed that the firm has a threshold which determines the quality level of returns for which it is considered worthwhile to recover to this functional standard. In the remainder of this chapter, returns which are above this quality threshold are referred to as 'high quality returns'. High quality returns undergo 'high quality recovery' and are then sold on a secondary market. Returns which fall below the quality threshold are termed 'low quality returns' and can undergo 'low quality recovery', in order to be used as components. It is assumed that the firm makes a strategic-level decision (outwith the model) about whether to recover low quality returns or to dispose of them. The quality of the returns is determined during recovery and is modelled by a known probability distribution.

If there is insufficient capacity in the recovered inventory, then high quality returns undergo low quality recovery and are used as components. Any recovered items which can not fit into either the recovered inventory or the component inventory are discarded.

It is assumed that there is a finite capacity available for storing each of the inventories. If this capacity is reached then a disposal cost may be incurred for each item which exceeds the inventory level. It is assumed that decisions will never be made which could cause the inventories to exceed the maximum capacities with certainty. For instance, the amount produced must be less than or equal to the available capacity in produced inventory. Since the number of high quality returns resulting from a recovery lot is uncertain, the size of the recovery lot may exceed the available capacity of recovered inventory.

Probabilistic Demand, Returns and Substitution. It is assumed that the size of demand for each market and the number of returns in each period are governed by known probability distributions. It is assumed that these three distributions are independent from each other and are time-invariant. It is assumed that there is no back ordering, however if there is insufficient inventory to meet demand, substitution of an alternative good may be offered, otherwise the sale will be lost. Inventory levels must always be non-negative.

The firm makes a strategic level decision about whether or not substitution should be offered in the event of a stock out of one or both types of goods. If the firm decides, at a strategic level to allow substitution then at an operational level, they need to decide when to offer substitution. The substitution is offered from a manufacturer's point of view rather than from a consumer-selection perspective. This means that substitution is offered by the firm as a way of avoiding a lost sale, rather than being chosen by the consumer when faced with a stock out of their preferred good. Because of this, the substitute good is offered to the consumer for the price of the lower-priced good (the recovered good).

If, after demand has been observed, there is a shortage of newly produced goods and a surplus of recovered goods, then the firm may choose to offer a recovered good in place of a produced good, i.e. a *upward substitution*. A customer may be willing to accept a upward substitution, as this allows them to purchase a product with the same functionality, for a lower price than they originally planned. On the other hand, if, after demand has been observed, there is a shortage of recovered goods and a surplus of produced goods, then the firm may choose to offer produced goods in place of recovered goods, i.e., a *downward substitution*.

It is assumed that customers can choose whether or not to accept the substitution. The proportions of customers from each market who accept the substitution are modelled by random variables with known distributions. The distributions governing the acceptance of substitution may be different for each type of substitution. For instance, it is possible that a consumer may be more willing to accept a newly produced good (downward substitution), than a recovered good (upward substitution).

Costs and Revenues. It is assumed that the objective of the firm is to maximise its long-run average reward. Each period, the reward is given by the revenues less costs. Revenue is received for the sale of produced and recovered goods. It is assumed that the price of a recovered good is always less than that of a produced good.

The costs incurred during a given time period depend on the initial inventory levels, the amount of production, recovery and buying performed, the quality of the returns, and the size of the demands and returns. The following types of costs are incurred:

holding costs, setup costs, processing costs, lost sales costs, disposal costs. It is assumed that the cost function is a linear function in the inventory levels.

Setup costs are incurred each time production, recovery or buying occurs. This is a fixed cost, and does not depend on the size of the action; it is incurred regardless of whether the activity was performed in the last period. Processing costs are incurred on a per unit basis for production, recovery and buying. Any inventory that is left in stock at the end of a period incurs a per unit, per time-unit holding cost. If demand exceeds the available inventory and a substitution is not offered, or offered but not accepted, then a per unit lost sales cost is incurred. If a substitution is offered and accepted then the firm receives revenue equal to the sale price of a recovered good. If returns exceed available capacity in returned inventory, then a disposal cost is incurred. This disposal cost may be positive or negative, representing the disposal cost or the salvage value.

6.4 Markov Decision Process Formulation

An infinite horizon Markov decision process is used to model this product recovery problem. A Markov decision process (MDP) is appropriate for studying this type of problem because the inventory levels in the next period depend only on the current inventory levels, the selected action (production/recovery/buying) and the uncertainty observed (demand, returns, substitution acceptance). An infinite horizon model is appropriate because it is assumed that demand and returns are time-invariant (Tijms, 1994). An overview of MDP theory was presented in Section 3.2. A MDP is characterised by its decision epochs, states, actions, rewards and transition probabilities. The MDP formulation of this product recovery model is presented in this section. A summary of the notation used in this section is presented in Table 6.1 on page 203.

6.4.1 Decision Epochs

At each decision epoch the state of the system is observed and then an action is chosen. The system then moves to the next state with a transition probability, which depends on the initial state observed and the action chosen. It is assumed that decision epochs occur at the beginning of each time period.

6.4.2 States

The state of the system is specified by the levels of the produced, recovered, returned and component inventories. Let:

$$\begin{aligned}
i_1 &= \text{inventory level of produced inventory,} & i_1 \in I_1 &= \{0, 1, \ldots, W_1\} \\
i_2 &= \text{inventory level of recovered inventory,} & i_2 \in I_2 &= \{0, 1, \ldots, W_2\} \\
i_r &= \text{inventory level of returned inventory,} & i_r \in I_r &= \{0, 1, \ldots, W_r\} \\
i_c &= \text{inventory level of component inventory,} & i_c \in I_c &= \{0, 1, \ldots, W_c\}
\end{aligned}$$

where W_1, W_2, W_r, W_c are finite upper limits of inventory capacity and I_1, I_2, I_r, I_c are the sets of all possible inventory levels. Since all state variables have finite upper limits of capacity, the MDP has a finite state space. Letting i denote the state of the system at the beginning of a given time period, then:

$$i \in I = \{(i_1, i_2, i_r, i_c) : \quad i_1 \in I_1, \quad i_2 \in I_2, \quad i_r \in I_r, \quad i_c \in I_c\}$$

6.4.3 Actions

At each decision epoch the firm must decide how much to produce, how much to recover, how many components to buy, and what substitution policy should used. The actions can be separated into replenishment actions (production, recovery and buying) and substitution actions (upward and downward).

Replenishment Actions

Three replenishment decisions must be made at each decision epoch: how much to produce, how much to recover and how many components to buy. The values of these actions are denoted by a_p, a_r and a_b and for a given state i are chosen from sets of allowable actions: $A_p(i)$, $A_r(i)$, $A_b(i)$ for production, recovery and buying, respectively.

A firm may not want to perform production (and incur setup costs) for a small number of items. Thus in practice, a firm may implement an operational 'minimum batch size' policy which specifies the minimum amount which should be produced, if production is to be performed. Similarly, for the recovery of returns and buying of components. To model this operational requirement, lower limits are placed on the size of the action variables. Let L_p, L_r, L_b denote the lower limit or 'minimum batch size' for production, recovering and buying respectively. If the firm did not wish to impose a 'minimum batch size policy' then these lower limits would be set to one.

The firm may be also restricted in the amount that is produced by the maximum capacity of their equipment or by other operational restrictions. To model these restrictions upper limits are placed on the size of the action variables. Let U_p, U_r, U_b denote upper limits for production, recovery, and buying respectively.

Using these upper and lower limits a set of allowable actions, i.e. the action space, can be defined for each type of action, for a given state i.

$$a_p \in A_p(i) \subseteq \{0, L_p, \ldots, \min\{W_1 - i_1, U_p\}\}$$

$$a_r \in A_r(i) \subseteq \begin{cases} \{0\} & \text{if } a_p > 0 \\ \{0, L_r, \ldots, \min\{i_r, U_r\}\} & \text{if } a_p = 0 \end{cases}$$

$$a_b \in A_b(i) \subseteq \begin{cases} \{0\} & \text{if } a_p = 0 \\ \{\min\{\max\{a_p - i_c, 0\}, L_b\}, \ldots, U_b\} & \text{if } a_p > 0 \end{cases}$$

The restriction on the upper limit of the production action set $A_p(i)$ follows from the assumption that production will not result in the maximum capacity of the produced inventory being exceeded. The amount produced must not be greater than the available capacity in produced inventory $W_1 - i_1$, or the upper limit on production U_p. Recovery can only occur if production does not, i.e., if $a_p = 0$. The amount recovered must not exceed the number of returns in stock i_r, or the upper limit on recovery U_r. Components are only bought if production is performed. The minimum restriction on the number of components bought ensures that there are sufficient components for the current production order.

Substitution Actions

Decisions about substitution are made at a strategic level and at an operational level. At a strategic level the firm determines whether or not substitution will be permitted ever, and at an operational level the firm decides whether or not to offer substitution in the current time-period. The strategic decisions, which are made outwith the model, are denoted by χ_1 and χ_2, which represent upward and downward substitution respectively. If the firm allows upward substitution, then $\chi_1 = 1$, otherwise $\chi_1 = 0$. Similarly, if the firm allows downward substitution, then $\chi_2 = 1$, otherwise $\chi_2 = 0$. This is summarised below:

Notation	Type of substitution	Description
χ_1	upward	Shortage of produced goods, substitute with a recovered good
χ_2	downward	Shortage of recovered goods, substitute with a produced good

If the firm permits substitution at a strategic level, then at each decision epoch the firm must make an operational decision about offering substitution during the coming period, if there is a shortage. In making this operational decision, the current inventory levels and replenishment actions are taken into account. The decision about offering substitution is made at the start of the period, before demand has been observed. This is obviously not an ideal way to represent the process of substitution, however, the periodic nature of the MDP means that all decisions must be made at the start of the period. A continuous time model would allow a more detailed treatment of substitution. However, the current periodic model is still useful as it provides a simple representation of the substitution process.

The sets of allowable substitution actions depend on the strategic variables χ_1 and χ_2. Let a_1 denote the upward substitution decision and let a_2 denote the downward substitution decision made in a given period, where

$$a_1 = \begin{cases} 1 & \text{if upward substitution is offered} \\ 0 & \text{otherwise} \end{cases}$$

$$a_2 = \begin{cases} 1 & \text{if downward substitution is offered} \\ 0 & \text{otherwise} \end{cases}$$

Let A_1 denote the set of allowable upward substitution actions and let A_2 denote the set of allowable downward substitution actions, where:

$$a_1 \in A_1 = \{0, \chi_1\}$$
$$a_2 \in A_2 = \{0, \chi_2\}$$

For a given state i, an action $a(i)$ is chosen from the set of allowable actions, denoted by $A(i)$, which can be defined as:

$$\begin{aligned} a(i) \in A(i) = \{(a_p, a_r, a_b, a_1, a_2) :\ & a_p \in A_p(i),\ a_r \in A_r(i),\ a_b \in A_b(i), \\ & a_1 \in A_1,\ a_2 \in A_2\}. \end{aligned} \tag{6.1}$$

6.4.4 Transition Probabilities

The transition from the current state to the next state depends on the action that is chosen, the quality of the returns recovered during the period, the number of goods which are returned, the number of goods that are demanded from each market, and, if offered, the acceptance of substitution.

Random Variables

There are six random variables which will be used to specify the transition probabilities: X_q, X_{d1}, X_{d2}, X_r, X_1, X_2. The random variable X_q represents the number of high quality returns and is governed by a probability distribution which depends on the number of returns being recovered (a_r) in that period. The random variable X_{d1} represents the demand for produced items, and the random variable X_{d2} represents the demand for recovered items during a period. The random variable X_r represents the number of items returned during a period. The random variable X_1 represents the number of customers whose demand for a produced good is unsatisfied and who would accept a recovered good in its place. This random variable is governed by a probability distribution which depends on the amount of unsatisfied demand for produced goods. The random variable X_2 represents the number of customers whose demand for recovered goods is unsatisfied and who would accept a produced good in its place. This random variable is governed by a probability distribution which depends on the amount of unsatisfied demand for recovered goods. The random variables X_q, X_{d1}, X_{d2}, X_r are independent of each other and have known distributions. The variables X_1 and X_2 are dependent on the demand observed during the period. Following the convention in probability, an observation of a random variable is denoted by the lower case equivalent, for instance, x_r is an observation of the random variable X_r.

System Dynamics

The system dynamics associated with this model are as follows. Suppose the system is in state $i = (i_1, i_2, i_r, i_c)$ at the start of the period and action $a(i) = (a_p, a_r, a_b, a_1, a_2)$ is chosen. The lead time for production, recovery and buying is zero, therefore the inventory levels are updated immediately after the action decision is made. After the state is updated for the action, the demands and returns are observed. The inventory transitions are presented in two stages, beginning with the replenishment actions, demand and returns, and then examining the effect of substitution.

Replenishment, Demands and Returns. The produced inventory is increased by the size of production a_p and decreased by the size of demand x_{d1}. The inventory level $(i_1 + a_p)$ will never exceed the capacity of serviceable inventory due to the assumptions made about the upper limit of the action set $A_p(i)$. If demand for produced items exceeds $(i_1 + a_p)$, then the produced inventory level will be zero. This gives a produced inventory level of:

$$\max\{i_1 + a_p - x_{d1}, 0\} \tag{6.2}$$

The recovered inventory is increased by the number of high quality returns that are recovered and decreased by the size of demand x_{d2}. If the available capacity in the recovered inventory is insufficient to hold the number of high quality returns, then some of the high quality returns will be recovered to be components. Recall that ζ_L represents the low quality recovery strategy. If low quality recovery is permitted then $\zeta_L = 1$, otherwise $\zeta_L = 0$. In this model, high quality recovery is always permitted so ζ_H is not included in this formulation. Let the number of returns that undergo high quality recovery be denoted by $a_h(x_q)$, where:

$$a_h(x_q) = \min\{W_2 - i_2, x_q\}$$

let the number of returns that undergo low quality recovered be denoted by $a_l(x_q)$, where:

$$a_l(x_q) = \zeta_L \min\{W_c - i_c, a_r - a_h(x_q)\}$$

and let the number of returns that are discarded be denoted by $a_d(x_q)$, where:

$$a_d(x_q) = \{a_r - a_h(x_q) - a_l(x_q), 0\}$$

Using this notation, the updated level of recovered inventory is:

$$\max\{i_2 + a_h(x_q) - x_{d2}, 0\} \tag{6.3}$$

The returned inventory will decrease by the size of the recovery lot a_r and increase by the number of returns x_r received, subject to the capacity limit for returned inventory W_r. Recall that the effect of the action is realised (and inventory levels updated) before the returns are observed. The updated returned inventory level will be:

$$\min\{i_r - a_r + x_r, W_r\} \tag{6.4}$$

The component inventory will increase by the number of components bought a_b, decrease by the production size a_p and increase by the size of low quality recovery $a_l(x_q)$. The new component inventory level will be:

$$i_c + a_b - a_p + a_l(x_q) \tag{6.5}$$

Substitution. The effect of substitution is now examined. Let u_1 and u_2 denote the amount of unsatisfied demand for produced and recovered items, respectively. Let n_1 denote the number of recovered goods offered for substitution, in the event of a shortage of produced goods; and let n_2 denote the number of produced goods offered for substitution, in the event of a shortage of recovered goods. Let s_1 denote the number of recovered goods which are accepted in place of produced goods; and let s_2 denote the number of produced goods which are accepted in place of the recovered goods. These quantities will now be defined. The amount of unsatisfied demand for produced items is

$$u_1 = \max\{0, x_{d1} - (i_1 + a_p)\}.$$

The amount of unsatisfied demand for recovered items is

$$u_2 = \max\{0, x_{d2} - i_2 - a_h(x_q)\}.$$

Taking into account the substitution decision a_1, the amount of recovered inventory available after demand for recovered goods has been met (equation 6.3), and the amount of unsatisfied demand, then the number of recovered goods offered for substitution is:

$$n_1 = a_1 \times \min\{u_1, \max\{0, i_2 + a_h(x_q) - x_{d2}\}\}$$

Taking into account the substitution decision a_2, the amount of produced inventory available after demand for produced goods has been met (equation 6.2), and the amount of unsatisfied demand, then the number of produced goods offered for substitution is:

$$n_2 = a_2 \times \min\{u_2, \max\{0, i_1 + a_p - x_{d1}\}\}$$

The number of customers with unsatisfied demand for produced goods, who would accept a substitution if offered is given by the random variable X_1. The variable X_1 has a general distribution with a known density. The number of recovered goods which are substituted for produced goods is:

$$s_1 = \min\{n_1, x_1\}$$

Therefore from s_1 and equation (6.3) the updated recovered inventory level is:

$$\max\{i_2 + a_h(x_q) - x_{d2}, 0\} - s_1 \tag{6.6}$$

Similarly, the number of customers who demanded recovered goods with unsatisfied demand, who would accept a substitution if offered, is given by the random variable X_2. The variable X_2 has a general distribution with a known density. The number of produced serviceable goods which are substituted for recovered serviceable goods is:

$$s_2 = \min\{n_2, x_2\}$$

Therefore, from s_2 and the equation (6.2), the updated produced inventory level is:

$$\max\{i_1 + a_p - x_{d1}, 0\} - s_2 \tag{6.7}$$

Next State. From equations (6.4) (6.5), (6.6) and (6.7), the next state $j = (j_1, j_2, j_r, j_c)$ can be related to state $i = (i_1, i_2, i_r, i_c)$ as follows:

$$\begin{aligned}
j_1 &= \max\left\{i_1 + a_p - x_{d1} - s_2, 0\right\} \\
j_2 &= \max\left\{i_2 + a_h(x_q) - x_{d2} - s_1, 0\right\} \\
j_r &= \min\left\{i_r - a_r + x_r, W_r\right\} \\
j_c &= i_c + a_b - a_p + a_l(x_q)
\end{aligned} \tag{6.8}$$

The probability that the system moves to state j in the next time period, given that it is currently in state i, depends on the joint probability function of the random variables X_q, X_{d1}, X_{d2}, X_r, X_1 and X_2.

6.4.5 Costs and Revenues

The objective of this Markov decision process is to maximise the long run average reward. The 'reward' received during a period is equal to the revenues less the costs, and the reward thereafter. In this model revenues are received for the sale of produced and recovered goods, and the following costs are incurred: holding costs, setup costs, processing costs, lost sales costs, substitution costs and disposal costs. The costs incurred in a given period depend on the current state, the action chosen, and the demand, returns, quality and substitutions that are observed. The revenues and costs will now be described in more detail.

Revenues. Sales revenues are received for the sale of produced and recovered goods. The sale of a produced good yields a revenue of p_1 and the sale of a recovered good yields a revenue of p_2. If a substitution occurs, then revenue equal to the price of the cheapest good is received p_2. For a given state i, action $a(i)$, observed quality x_q, demands x_{d1}, x_{d2} and substitutions x_1, x_2, the revenue that is received is:

$$\begin{aligned} R_S(i,a(i)) = p_1\Big(\min\{x_{d1},(i_1+a_p)\}\Big) \\ + p_2\Big(\min\{x_{d2}, i_2 + a_h(x_q)\} + s_1 + s_2\Big) \end{aligned}$$

Recall that s_1 and s_2 are the number of substitutions accepted by customers for produced and recovered goods respectively. The expected revenue is:

$$\begin{aligned} E[R_S(i,a(i),X_q,X_{d1},X_{d2},X_1,X_2] = p_1\Big(\min\{X_{d1}, i_1 + a_p\}\Big) \\ + p_2\Big(\min\{X_{d2}, i_2 + a_h(X_q)\} + s_1 + s_2\Big) \end{aligned} \tag{6.9}$$

Setup costs. Setup costs are incurred each time production, recovery and buying are initiated. These costs are denoted by k_p, k_r and k_b respectively. The following indicator variables specify when a particular action is performed and hence when a setup cost should be incurred.

$$\delta_p = \begin{cases} 1 & \text{if } a_p > 0 \quad \text{(production)} \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_r = \begin{cases} 1 & \text{if } a_r > 0 \quad \text{(recovery)} \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_b = \begin{cases} 1 & \text{if } a_b > 0 \quad \text{(buying)} \\ 0 & \text{otherwise} \end{cases}$$

Let the function $C_K(i, a(i))$ denote the setup cost function. This function does not depend on any random variables, therefore the expected value of the setup cost function is equal to $C_K(i, a(i))$, that is:

$$E[C_K(i, a(i))] = C_K(i, a(i)) = k_p\delta_p + k_r\delta_r + k_b\delta_b \tag{6.10}$$

Processing costs. Processing costs are incurred for production c_p, recovery of all returns c_r, recovery of high quality returns c_h, recovery of low quality returns c_l, and disposal of excess returns c_d. These costs occur on a per unit basis. For a given state i, action $a(i)$, observed number of high quality returns x_q, the actual processing costs $C_P(i, a(i), x_q)$ are:

$$C_P(i, a(i), x_q) = c_p a_p + c_r a_r + c_h a_h(x_q) + c_l a_l(x_q) + c_d a_d(x_q)$$

The expected processing costs are:

$$E[C_P(i, a(i))] = c_p a_p + c_r a_r + c_h E[a_h(X_q)] + c_l E[a_l(X_q)] + c_d E[a_d(X_q)] \tag{6.11}$$

Holding Costs. Holding costs are incurred for any goods that are in stock at the end of a period and are carried between periods. Each period, per unit holding costs of h_1, h_2, h_r and h_c are incurred for produced, recovered, returned and component inventories, respectively. The holding cost is calculated using the inventory levels at the beginning of each period. Since an infinite horizon MDP with an average reward criterion is used to model this problem, calculating the cost based on the inventory levels at the beginning or end of a period are equivalent. Alternative methods for calculating the holding cost could be used. Let $C_H(i)$ denote the holding costs incurred if the system is in state $i = (i_1, i_2, i_r, i_c)$ at the beginning of a period. Since the inventory levels are known with certainty at the beginning of a period, the expected holding cost is:

$$E[C_H(i)] = C_H(i) = h_1 i_1 + h_2 i_2 + h_r i_r + h_c i_c \tag{6.12}$$

Lost Sales & Substitution. Demand which is not met by inventory or by substitution incurs a lost sales cost. The per unit lost sales cost for produced goods is l_1 and for recovered goods is l_2. If the system is in state i, action $a(i)$ is chosen, there are x_q high quality returns (if $a_r = 0$ then $x_q = 0$), x_{d1} produced goods are demanded, x_{d2} recovered goods are demanded and x_r units returned, x_1 customers would accept upward substitution, and x_2 customers would accept downward substitution, then the lost sales, substitution and disposal costs will be:

$$\begin{aligned} C_L(i, a(i), x_q, x_{d1}, x_{d2}, x_1, x_2) = l_1 \Big(\max \Big\{ x_{d1} - (i_1 + a_p) - s_1(x_1), 0 \Big\} \Big) \\ + l_2 \Big(\max \Big\{ x_{d2} - (i_2 + a_h(x_q)) - s_2(x_2), 0 \Big\} \Big) \end{aligned}$$

and the expected lost sales costs are:

$$\begin{aligned} E[C_L(i, a(i), X_q, X_{d1}, X_{d2}, X_1, X_2)] = l_1 \Big(\max \Big\{ E[X_{d1}] - (i_1 + a_p) - E[s(X_1)], 0 \Big\} \Big) \\ + l_2 \Big(\max \Big\{ E[X_{d2}] - (i_2 + E[a_h(X_q)]) - E[s_2(X_2)], 0 \Big\} \Big) \end{aligned}$$

Disposal Costs. If the number of returns exceeds the available capacity of the returns inventory, then a per unit disposal cost of l_r is incurred. For a given state, action and number of returns, the disposal costs:

$$C_D(i, a(i), x_r) = l_r \Big(\max \Big\{ x_r - (W_r - i_r + a_r), 0 \Big\} \Big)$$

The expected disposal costs are:

$$E[C_D(i, a(i), X_r)] = l_r \Big(\max \Big\{ E[X_r] - (W_r - i_r + a_r), 0 \Big\} \Big)$$

Total costs. The total cost incurred during for one period, for a given state i, action $a(i)$, observed quality x_q, demands x_{d1}, x_{d2}, returns x_r and substitutions x_1, x_2 is:

$$\begin{aligned} C(i, a(i), x_q, x_{d1}, x_{d2}, x_r, x_1, x_2) = {} & C_K(i, a(i)) + C_P(i, a(i), x_q) + C_H(i) \\ & + C_L(i, a(i), x_q, x_{d1}, x_{d2}, x_1, x_2) + C_D(i, a(i), x_r) \end{aligned} \tag{6.13}$$

and correspondingly, the expected total costs for one period is:

$$\begin{aligned} E[C(i, a(i), X_q, X_{d1}, X_{d2}, X_r, X_1, X_2)] = {} & C_K(i, a(i)) + E[C_P(i, a(i))] + C_H(i) \\ & + E[C_L(i, a(i), X_q, X_{d1}, X_{d2}, X_1, X_2)] \\ & + E[C_D(i, a(i), X_r)] \end{aligned} \tag{6.14}$$

Total reward. The total rewards received during one period, depends on the state i, action $a(i)$, and observed quality X_q, demands X_{d1}, X_{d2}, returns X_r and substitutions X_1, X_2 is:

$$\begin{aligned} E[R(i, a(i), X_q, X_{d1}, X_{d2}, X_r, X_1, X_2)] = {} & E[R_S(X_q, X_{d1}, X_{d2}, X_1, X_2] \\ & - \Big(C_K(i, a(i)) + E[C_P(i, a(i))] + C_H(i) \\ & + E[C_L(i, a(i), X_q, X_{d1}, X_{d2}, X_1, X_2)] \\ & + E[C_D(i, a(i), X_r)] \Big) \end{aligned} \tag{6.15}$$

6.5 Model Implementation and Validation

The Markov decision problem described in the previous section was implemented in java in order to explore the properties of the model. Several algorithms can be used to find the optimal policy, however, as discussed in Section 3.2, the dimensionality of the problem under study means that the value iteration algorithm is the only practical option.

6.5.1 Dimensionality

This Markov decision problem has four state variables and five action variables. This makes the problem exponentially larger than the model in the previous chapter.

Table 6.1: Summary of notation used in MDP formulation

State Variables	
i_1	produced inventory
i_2	recovered inventory
i_r	returns
i_c	components
State Space Capacities	
W_1	new inventory
W_2	recovered inventory
W_r	returns
W_c	components
Action Variables	
a_p	production
a_r	recovery
a_b	buying of components
a_1	upward substitution
a_2	downward substitution
$a_h(x_q)$	high quality recovery
$a_l(x_q)$	low quality recovery
$a_d(x_q)$	disposed recovery
Action Quantities	(see section 6.5)
Q_p	production lot size
Q_r	recovery lot size
Action Space Capacities	
$U_p, U_r, U_b,$	upper limit on production, recovery, ordering
$L_p, L_r, L_b,$	lower limit on production, recovery, ordering
χ_1	upward substitution strategy
χ_2	downward substitution strategy
Random Variables	
X_{d1}	demand for new goods
X_{d2}	demand for recovered goods
X_r	returns
X_q	quality of returns
X_1	downward substitution acceptance
X_2	upward substitution acceptance
α	quality parameter
α_1	upward substitution acceptance parameter
α_2	downward substitution acceptance parameter

Unit costs	
c_p	production
c_r	recovery
c_1	upward substitution
c_2	downward substitution
c_b	buying components
c_h	high quality recovery
c_l	low quality recovery
c_d	disposal from recovery
c_a	acquisition of returns
Revenues	
p_1	produced inventory
p_2	recovered inventory
Setup Costs	
k_p	production
k_r	recovery
k_b	ordering components
δ_p	indicator production
δ_r	indicator recovery
δ_b	indicator buying
Holding Costs	
h_1	returns
h_2	produced inventory
h_r	recovered inventory
h_c	components
Lost Sales Costs	
l_r	excess returns
l_1	lost sales of produced goods
l_2	lost sales of recovered goods
Substitution quantities	
u_1, u_2	unmet demand
n_1, n_2	goods available for substitution
$s_1(x_1), s_2(x_2)$	number of substitutions made

Initial numerical experiments showed that dimensionality of the problem makes it computationally impractical to consider problems with upper capacity levels of about 5. This obviously places significant limitations on the numerical experiments that can be performed with this model. Alternative solution methods and heuristics could be investigated, however it would still be difficult to assess the performance of such methods. Instead, we propose a number of simplifications which allow the action space to be reduced and thus allow slightly larger problems to be investigated. These simplifications are as follows:

- Components are ordered only when they are needed for production. This means that the action variable a_b can be calculated and therefore does not need to be stored. In each period:

$$a_b = \max\{0, a_p - i_c\}.$$

- Production and recovery occur in fixed batch sizes of Q_p and Q_r respectively are used. It is assumed that these batches sizes are smaller than their respective upper limits for production U_p and recovery U_r. The action space for production becomes:

$$a_p \in A_p = \{0, \min\{W_1 - i_1, Q_p\}\}$$

  The action space for recovery becomes:

$$a_r \in A_r = \begin{cases} \{0\} & \text{if } a_p > 0 \text{ or } i_r < Q_r \\ \{0, Q_r\} & \text{if } a_p = 0 \end{cases}$$

  Recovery can only be performed if there are sufficient returns in stock, i.e., $i_r \geq Q_r$. This means that the decision within each period is restricted to whether or not to produce or recover, rather than how much to produce or recover.

Despite these adjustments, the dimensionality of this model is still an issue. For a given state $i = (i_1, i_2, i_r, i_c)$ and action $a = (a_p, a_r, a_b, a_1, a_2)$, the number of non-zero transitions is determined by the parameter values of the problem. Suppose the maximum inventory level was 10, ($i_1 = i_2 = i_r = i_c = 10$), the number of returns each period was 10 or less (with probability 0.999) and the batch size for returns was $Q_r = 10$ (meaning $X_q \sim \text{Bin}(Q_r, \alpha)$). Furthermore, suppose that demand for produced and recovered goods could vary between 1 and 10. The number of each type of good offered for substitution could also vary, therefore, between 1 and 10. In a given period all of the following events could occur: returns, production, recovery, demand for produced and recovered goods, and substitution of produced and recovered goods. Therefore, the maximum number of possible transitions is: num returns $\times$ num produced $\times$ num recovered $\times$ num demand produced $\times$ num demand recovered $\times$ num substitution produced $\times$ num substitution recovered $= 10 \times 1 \times 10^5 = 10^6$. Of course many of these transitions may lead to the same state j, but they could be reached through many differing combinations of the number of returns, high quality returns, demand and substitution. Because of this, the numerical experiments in this chapter will be limited to a maximum inventory level of 10.

6.5.2 Validation of the Code

Much of the programming code used in this chapter was discussed in Chapter 3 and was used Chapter 5, therefore this section refers only to the problem-specific code. In addition to thorough error-checking and inspection of output during the code development process, two forms of verification were used to validate the problem specific files. The calculation of the expected average rewards was checked using an Excel spreadsheet and using simulation. To conduct these tests a set of six example test problems was constructed. For all six problems, the results of the tests were as expected. Further details regarding the validation of the java code can be found in Appendix C.1.2.

6.6 Properties of the Optimal Policy

The properties of the optimal policy are explored in this section. Three main properties are investigated in the chapter. Firstly, we investigate the *performance* of the optimal policy under different substitution strategies. Two performance measures are used: the long run average reward and the fill rate (a measure of service). Secondly, we investigate the structure of the optimal policy by examining the *actions* that are chosen in different states. Finally, we investigate the effect of the *recovery* strategy on the performance and structure of the optimal policy.

As discussed in Section 6.5.1, to manage the dimensionality of the problem, the class of policies which are being considered has been restricted by introducing the fixed lot sizes Q_p and Q_r and by fixing the decision about buying components. The optimal policy within this class of policies is sought. This policy may not be optimal if the restrictions on the lot sizes and component buying decisions were omitted.

6.6.1 Datasets

Three datasets have been constructed in order to investigate the optimal policy across a range of scenarios. The parameters for these datasets are derived from the datasets used in Chapters 4 and 5, some of which were themselves derived from the dataset used by Konstantaras and Papachristos (2008b). This section explains how the current dataset extends those used to investigate the cost-minimising single-market stochastic product recovery model in Chapter 5 (Table 5.2, 146). Some additional parameters have been added to account for the additional inventory type, additional demand type and substitution. The model in this chapter uses the objective of maximising reward rather than minimising cost, thus price parameters also need to be included. The selection of these new parameters values is discussed in this section. Note that the first problem in the datasets in Chapters 4 and 5 (labelled 00) has been removed from the dataset as it did not contain any unit costs. It was included in the earlier models to enable comparisons to be drawn between models in the literature and the models proposed by this thesis. However, since the model in this chapter has a very different structure (e.g. two markets), such comparisons are not made here. The parameter values for the datasets used in the section are presented in Table 6.2.

State Variables

For computational reasons the state space for each of the four state variables must be constrained by an upper limit. Due to the dimensionality of the problem, the state space for each of the inventories is limited to the values from 0 to 10, thus:

$$I_1 = \{0, 1, \ldots, 10\}$$
$$I_2 = \{0, 1, \ldots, 10\}$$
$$I_r = \{0, 1, \ldots, 10\}$$
$$I_c = \{0, 1, \ldots, 10\}$$

Random Variables

The Poisson distribution is used to model demand and returns. The rates of the three Poisson random variables are chosen to be integers for convenience. Since the state space has been decreased, the demand and returns parameters need to be adjusted accordingly. Ideally, the capacity of the inventories should not affect the policy. The effect of the inventory capacity can be lessened by ensuring that the probability that demand exceeds the upper capacity is very small. If the random variable governing demand X_{d1} is governed by a Poisson distribution with mean $\lambda_{d1} = 3$, then $P(X_{d1} > 10) < 0.001$. This probability is sufficiently small, therefore an upper limit of 3 is placed on the parameter λ_d. Let p_d and d_d denote the production and demand rates from deterministic model in Chapter 4. The ratio of d_d to p_d results in values between 0 and 1 since the demand rate in the deterministic model is always less than the production rate. In order to spread the values between 0 and the upper limit 3, with a heavier weighting towards the upper end of the range, the square root of the ratio is taken. In summary, in order to obtain the mean demand λ_{d1} for each problem in the base dataset the following transformation is used:

$$\lambda_{d1} = \left\lceil \sqrt{\frac{d_d}{p_d}} \times 3 \right\rceil$$

where the function $\lceil x \rceil$ rounds x up to the nearest integer.

In this model an additional demand variable is included. In order to model a variety of relationships between the demand for the two products, the demand rate λ_{d2} was determined by randomly generating discrete values within the range of 1 to 3. This creates some scenarios with $\lambda_{d1} > \lambda_{d2}$, $\lambda_{d1} = \lambda_{d2}$ and $\lambda_{d1} < \lambda_{d2}$.

In Chapter 4, the parameters β_1 and β_2 specify the proportion of returns which are high quality and low quality respectively. These values, along with the demand for produced goods, are used to obtain the mean returns in the current model. Let:

$$\gamma = \beta_1 + \beta_2$$

and then let the mean returns be given by:

$$\lambda_r = \max\{1, \lfloor \gamma \lambda_{d1} \rfloor\}$$

where the function $\lfloor x \rfloor$ rounds x down to the nearest integer. Since maximum demand level is much lower than in Chapter 5, an additional restriction is added to ensure that the number of returns is at least 1.

Revenues and Costs

In order to model a range of scenarios where some products have small and some have large price mark-ups, the relationship

$$p_{d1} = (c_p + c_b)(1 + U)$$

is used to obtain the sale price of the produced serviceable goods, where U is a uniform $(0, 1)$ random variable. As mentioned in the problem description (§6.2), while the two goods are sold in separate markets, they are also assumed to have the same functionality and purpose. Consumers are likely to compare the prices of the two when making a purchase. Therefore, we relate the price of the recovered good to the price of the produced good, rather than to the cost of recovering a returned good. In practice the nature of the relationship between price of produced and recovered goods depends on the type of product and the associated consumer preferences. Therefore, in order to model a variety of price relationships we multiply the price of a produced good by a factor of between 0.5 and 1.0, such that:

$$p_{d2} = p_{d1} U$$

where U is a uniform $(0.5, 1.0)$ random variable.

The values for the holding cost of recovered inventory h_2 were selected in order to provide a range of values. The values for the other holding costs are the same as in

Chapter 5. The lost sales cost for unsatisfied demand for produced goods l_1 is related to the production cost of a produced item by the multiplier γ_h:

$$l_1 = \gamma_h(c_p + c_b - c_d)$$

and the lost sales cost for unsatisfied demand for recovered goods l_2 is related to the cost of a high quality recovery by the multiplier γ_h:

$$l_1 = \gamma_h(c_r + c_h - c_d)$$

For this investigation the multiplier $\gamma_h = 0.1$ is used to specify the lost sales cost. The lost sales costs are much lower than those used in Chapter 5. This is because in the previous model, the lost sales cost also represented part of the lost revenue from not meeting demand, however in this model revenue is included separately in the model through the price parameters. As discussed by Silver et al. (1998), the cost of losing a sale is hard to quantify, therefore the quantities used here serve to provide some form of disincentive against not meeting demand.

The setup cost for buying components is set to zero ($k_b = 0$) since a restriction was placed on this decision. Recall that a disposal cost c_d is applied to all goods which are not recovered. Since this is an open-loop system and goods are not monitored after they have been sold, the disposal cost is added as a surcharge to all goods which are produced and is then deducted from the recovery costs of goods which are recovered. Goods which are produced but not recovered will not have the fee reimbursed, thus are subject to a disposal charge.

The values for all other costs are the same as in Chapter 5.

Lot Sizes

The lot sizes for production Q_p and recovery Q_r are derived from the respective demand rates plus y_p or y_r standard deviations, respectively.

$$Q_p = \lceil \lambda_{d1} + y_p\sqrt{\lambda_{d1}} \rceil$$
$$Q_r = \lceil \lambda_{d2} + y_r\sqrt{\lambda_{d2}} \rceil$$

where the function $\lceil x \rceil$ rounds x up to the nearest integer. Three variations of the dataset are constructed by varying the number of standard deviations added to the demand rates. These datasets are labelled B, C and D:

Problem Set	y_p	y_r	
A	–	–	(Test Problems)
B	1	1	
C	2	1	
D	1	2	

Substitution Strategies

For problem sets B, C and D four substitution strategies are considered: no substitution ($\chi_1 = 0, \chi_2 = 0$), only downward substitution ($\chi_1 = 0, \chi_2 = 1$), only upward substitution ($\chi_1 = 1, \chi_2 = 0$), two-way substitution ($\chi_1 = 1, \chi_2 = 1$).

Notation	Type of substitution	Description
χ_1	upward	Shortage of produced goods, substitute with a recovered good
χ_2	downward	Shortage of recovered goods, substitute with a produced good

The proportion of customers for produced and recovered goods who would accept a substitution, if it was offered to them, is modelled by a Binomial distribution with the parameters α_1 and α_2 respectively.

The parameters for the problem sets are presented in Table 6.2.

6.6.2 Analysis of Performance

In this section we analyse the performance of the optimal policy by examining the average rewards and fill rate under different substitution strategies. Four substitution strategies are investigated: no substitution ($\chi_1 = 0, \chi_2 = 0$), upward substitution ($\chi_1 = 1, \chi_2 = 0$), downward substitution ($\chi_1 = 0, \chi_2 = 1$), two-way substitution ($\chi_1 = 1, \chi_2 = 1$). These substitution strategies do not force the systems to perform substitution in the case of a shortage, rather they allow substitution to be an option (i.e. an allowable action in the action space).

Table 6.2: Parameter values for the problem sets

	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18	19	20
Distributions																				
α	0.8	0.2	0.3	0.3	0.5	0.7	0.8	0.8	0.2	0.5	0.5	0.5	0.5	0.5	0.4	0.1	0.6	0.2	0.2	0.5
α_1	0.68	0.65	0.57	1	1	0.55	0.5	0.4	0.75	0.94	0.92	0.8	0.63	0.93	0.5	0.52	0.86	0.65	0.98	0.56
α_2	0.44	0.75	0.9	0.62	0.81	0.53	0.5	0.72	0.74	0.87	0.87	0.76	0.93	0.92	0.7	0.48	0.48	0.56	0.89	0.67
λ_{d1}	2	2	3	2	2	2	2	2	2	2	3	3	3	2	2	2	2	1	2	1
λ_{d2}	3	1	3	3	3	1	3	2	1	2	2	1	2	2	2	3	3	1	1	1
λ_r	2	1	1	1	2	2	2	2	1	2	2	2	2	1	2	1	1	1	2	1
Unit Costs																				
c_p	106.5	66	88	131	161	107	66	75	212.5	188.5	23.5	76	78.5	106.5	65.5	110	107	75	212.5	160
c_r	13.5	34	42	89	29	13	44	15	27.5	141.5	6.5	24	21.5	13.5	34.5	0	13	15	27.5	10
c_b	30	60	80	100	70	40	60	30	50	200	50	50	100	30	50	100	40	30	50	50
c_h	50	30	40	60	75	50	30	35	100	85	10	35	35	50	30	50	50	35	100	75
c_l	9	18	24	30	21	12	18	9	15	60	15	15	30	9	15	30	12	9	15	15
c_d	6.5	6	8	11	11	7	6	5	12.5	18.5	3.5	6	8.5	6.5	5.5	10	7	5	12.5	10
c_a	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Revenues																				
p_{d1}	209	169	269	316	434	269	242	112	378	563	146	214	277	231	186	330	212	140	263	279
p_{d2}	147	133	202	161	391	253	202	84	361	529	83	119	259	221	111	236	113	115	150	162
Set up Costs																				
k_p	20	10	10	20	20	20	30	30	50	28	12	12	120	20	100	1000	400	30	50	28
k_r	5	5	10	12	8	20	20	25	30	8	2	2	10	5	50	100	400	5	10	8
k_b	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Holding Costs																				
h_r	2	4	5	10	4	2	5	2	4	16	1	3	3	2	4	1	2	2	4	2
h_{s1}	10	6	8	12	15	10	6	7	20	17	2	7	7	10	6	10	10	7	20	15
h_{s2}	8	6	8	11	10	2	4	4	10	15	2	5	5	10	5	7	8	4	12	10
h_c	3	6	8	10	7	4	6	3	5	20	5	5	10	3	5	10	4	3	5	5
Penalty Costs																				
l_r	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
l_{s1}	13	12	16	22	22	14	12	10	25	37	7	12	17	13	11	20	14	10	25	20
l_{s2}	5.7	5.8	7.4	13.8	9.3	5.6	6.8	4.5	11.5	20.8	1.3	5.3	4.8	5.7	5.9	4	5.6	4.5	11.5	7.5
Order Sizes for B																				
Q_p	4	4	5	4	4	4	4	4	4	4	5	5	5	4	4	4	4	2	4	2
Q_r	5	2	5	5	5	2	5	4	2	4	4	2	4	4	4	5	5	2	2	2
Order Sizes for C																				
Q_p	5	5	7	5	5	5	5	5	5	5	7	7	7	5	5	5	5	3	5	3
Q_r	5	2	5	5	5	2	5	4	2	4	4	2	4	4	4	5	5	2	2	2
Order Sizes for D																				
Q_p	4	4	5	4	4	4	4	4	4	4	5	5	5	4	4	4	4	2	4	2
Q_r	7	3	7	7	7	3	7	5	3	5	5	3	5	5	5	7	7	3	3	3

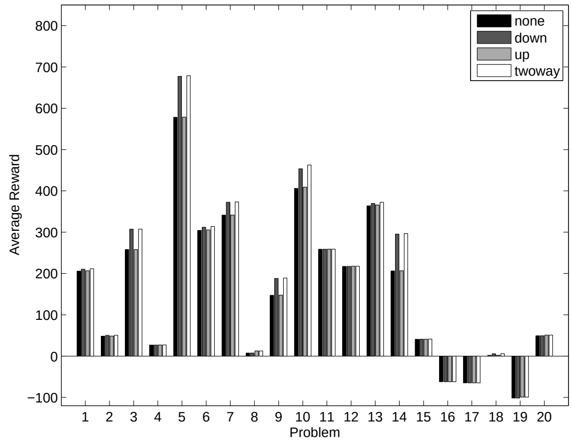
Average Reward

The average reward of the optimal policy varies across the three problem sets, with some being negative and some being positive. There is some variation between the substitution strategies. The average reward should be greatest when the two-way substitution strategy is used as it allows the widest range of policy actions. Under this strategy, substitution can and will be selected if it leads to the greatest immediate and future reward, whereas under the other substitution strategies the selection of substitution is limited to those permitted by the strategy. The average reward under each of the substitution strategies is presented for the three problem sets in Figure 6.2 and Table 6.3. As shown by Figure 6.2 the optimal reward is indeed highest under a two-way substitution strategy, however for some problems the highest reward is also attained by other substitution strategies.

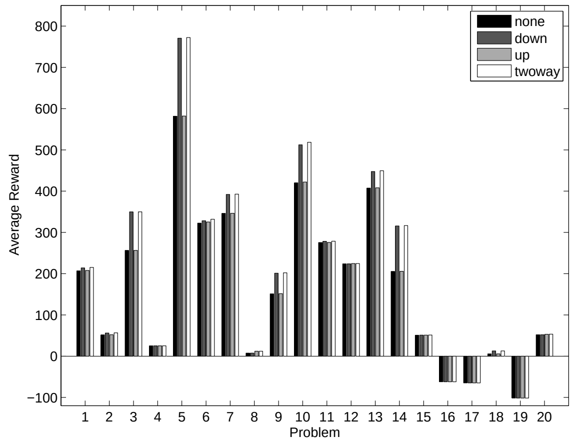
The question of interest is what additional revenue can be attained by introducing each of the substitution strategies. To investigate this, the relative increase in the reward which is attainable by allowing each type of the substitution is compared with not offering substitution. We refer to the attainable increase in reward as the *relative reward increase* (RRI). It is calculated as follows:

$$\text{RRI} = \frac{\text{Reward(substitution)} - \text{Reward(no substitution)}}{|\text{Reward(no substitution)}|} \times 100\%$$

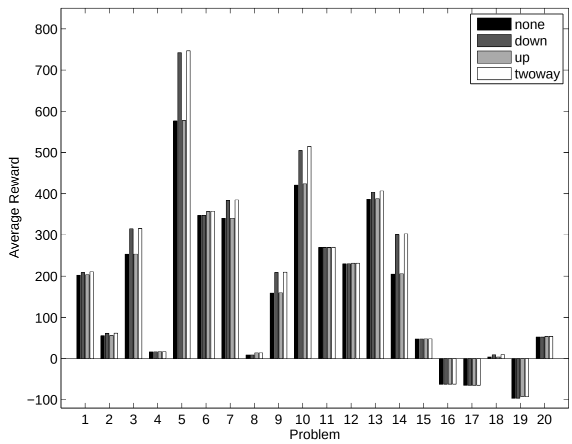
Since the long run average reward for some problems is negative, absolute value of the denominator needs to be taken. Note that the numerator will always be positive since the no-substitution strategy is the least flexible and thus has the lowest reward of all the substitution strategies. Figure 6.3 shows the relative reward increase attainable by offering substitution for problem sets B, C and D between problems. There is substantial variation in the percentage increase in the average reward attainable by offering substitution. For example, compare problems $B16$ and $B18$: for problem $B16$ offering any type of substitution has no effect on the average reward, whereas for problem $B18$ offering two-way substitution allows an increase in the average reward of almost 250% compared with not offering substitution. However it is important to note that while problem 18 has a large percentage increase, the magnitude of the average reward for this problem is much smaller than others. It is for this reason that there



(a) Problems B



(b) Problems C



(c) Problems D

Figure 6.2: Average reward of the optimal policy calculated for all substitution strategies

Table 6.3: Average reward of the optimal policy calculated for all substitution strategies

(a) Problems B

	Substitution Strategy			
Problem	None	Down	Up	Two-way
$B01$	205.6327	210.2666	206.6815	211.5259
$B02$	48.3231	50.1524	48.5907	50.4485
$B03$	258.0036	307.3568	258.0614	307.4512
$B04$	26.8350	26.8357	27.0030	27.0037
$B05$	578.1734	677.2367	578.6693	678.8897
$B06$	304.3658	311.9690	305.6469	313.7147
$B07$	341.2694	372.4779	341.5530	373.1980
$B08$	7.5144	7.5144	12.5345	12.5345
$B09$	147.2555	188.1642	147.7200	188.9560
$B10$	405.9674	453.4720	409.0333	462.6609
$B11$	258.7914	258.8317	258.9553	258.9960
$B12$	217.1785	217.1785	217.6327	217.6326
$B13$	363.8085	369.5037	365.8406	372.3621
$B14$	206.1956	295.4561	206.5465	296.6445
$B15$	40.8607	40.9289	41.0838	41.1527
$B16$	-61.9998	-61.9998	-61.9998	-61.9998
$B17$	-64.8000	-64.8000	-64.8000	-64.8000
$B18$	1.7273	5.5958	1.9205	5.8693
$B19$	-101.4996	-101.4995	-99.2472	-99.2472
$B20$	49.2246	49.2608	50.8572	50.8957

(b) Problems C

	Substitution Strategy			
Problem	None	Down	Up	Two-way
$C01$	206.9075	214.0137	207.7569	215.1768
$C02$	51.8501	56.2357	52.0333	56.4732
$C03$	256.5295	349.8017	256.5694	349.8866
$C04$	25.0915	25.0929	25.2221	25.2236
$C05$	581.6032	770.8180	582.0237	772.2306
$C06$	322.6048	328.3139	325.2434	331.8559
$C07$	346.2847	392.0980	346.4794	392.6603
$C08$	7.4771	7.4771	12.0837	12.0837
$C09$	151.1969	201.0447	151.6063	202.0693
$C10$	420.2436	512.3212	422.1603	518.6298
$C11$	275.6066	278.5170	275.7658	278.7292
$C12$	223.8000	223.8049	224.5272	224.5308
$C13$	407.5082	447.7006	408.1591	449.4213
$C14$	205.4133	315.6043	205.7440	316.5529
$C15$	50.7999	51.0689	50.9821	51.2546
$C16$	-61.9997	-61.9997	-61.9997	-61.9997
$C17$	-64.7993	-64.7994	-64.7993	-64.7994
$C18$	5.5319	12.6591	5.6659	12.8663
$C19$	-101.4995	-101.4996	-101.4995	-101.4995
$C20$	52.0044	52.0992	53.2187	53.3244

(c) Problems D

	Substitution Strategy			
Problem	None	Down	Up	Two-way
$D01$	202.1310	208.5794	203.4187	210.4709
$D02$	55.5843	61.2820	55.7953	61.5641
$D03$	253.6085	315.0881	253.6851	315.3195
$D04$	16.3997	16.4004	16.6168	16.6174
$D05$	576.7720	742.0547	577.5139	746.7642
$D06$	346.8426	347.6707	356.4312	357.5609
$D07$	340.3184	383.8900	340.6323	384.8664
$D08$	8.8406	8.8406	13.8216	13.8217
$D09$	159.0702	208.5078	159.5312	209.6617
$D10$	421.1900	504.6739	423.8750	514.6616
$D11$	269.2529	269.6960	269.5027	269.9624
$D12$	230.0933	230.0933	231.3473	231.3472
$D13$	386.4091	403.8067	387.8284	406.7766
$D14$	205.2108	300.8379	205.6899	302.5392
$D15$	47.5189	47.6618	47.7982	47.9296
$D16$	-61.9999	-61.9999	-61.9999	-61.9999
$D17$	-64.8000	-64.8000	-64.8000	-64.8000
$D18$	3.7045	9.2276	3.8825	9.4810
$D19$	-95.9167	-95.9167	-92.3012	-92.3012
$D20$	52.2228	52.2937	53.7832	53.8551

appears to be no difference between the substitution strategies for problem 18 in Figure 6.2.

It is expected that the greatest increase in the average reward would be attained by offering two-way substitution, compared with offering only downward or only upward substitution. This is confirmed by Figure 6.3. However, interestingly for most problems, the increase attainable is similar for the downward and two-way substitution strategies. This suggests that it is downward, rather than upward substitution that accounts for the increase attainable under a two-way substitution strategy. The main exception to this trend is problem 08, in which it appears that upward substitution accounts for the increase in the average reward attainable under a two-way substitution strategy.

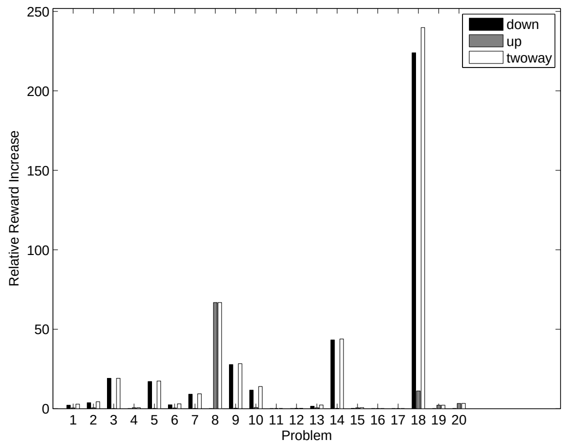
In general, the greatest increase in reward is attained by offering substitution strategy which includes downward substitution. Under downward substitution, produced goods are used to meet demand if there is a shortage of recovered goods. It is possible that downward substitution allows the manufacturer to mitigate the risks associated with the greater uncertainty in supply of the recovered goods, caused by the uncertainty in the arrival and quality of the returns.

Fill Rates

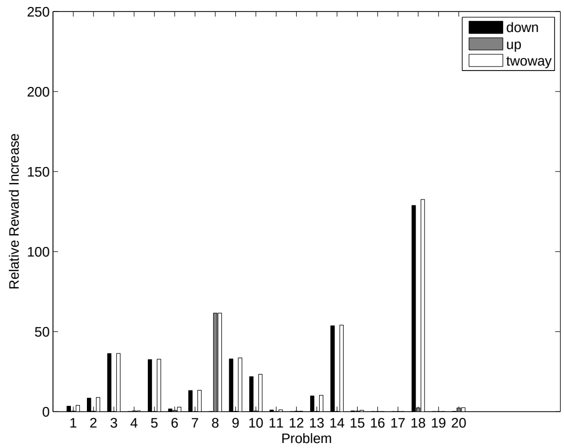
The fill rate is a measure of service which is determined by calculating the proportion of demand met by current stock (Silver et al., 1998). A low fill rate indicates that there has been a large number of lost sales, which could lead to higher levels of customer dissatisfaction. For each type of inventory that is demanded, the fill rate is:

$$\text{fill rate} = \frac{\text{number of met sales}}{\text{number of items demanded}}$$

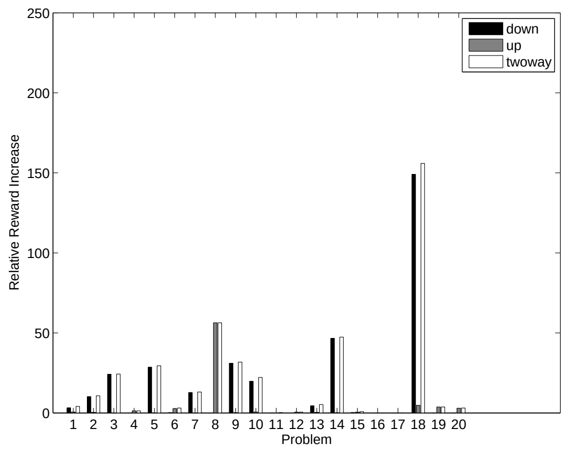
The average fill rate can be calculated using a simulation, by calculating the fill rate each period and then by averaging this across the length of the simulation. In situations where substitution is permitted, the 'number of met sales' could include or exclude the sales met by substitution, thus two versions of the fill rate can be calculated. It is interesting to compare the fill rates including and excluding substitution as it allows another way of measuring the effect of substitution. Let the *substitution-inclusive fill rate* refer to the fill rate calculation which includes sales met by substitution, and let



(a) Problems B



(b) Problems C



(c) Problems D

Figure 6.3: Relative reward increase attainable under each of the substitution strategies, compared with not offering substitution

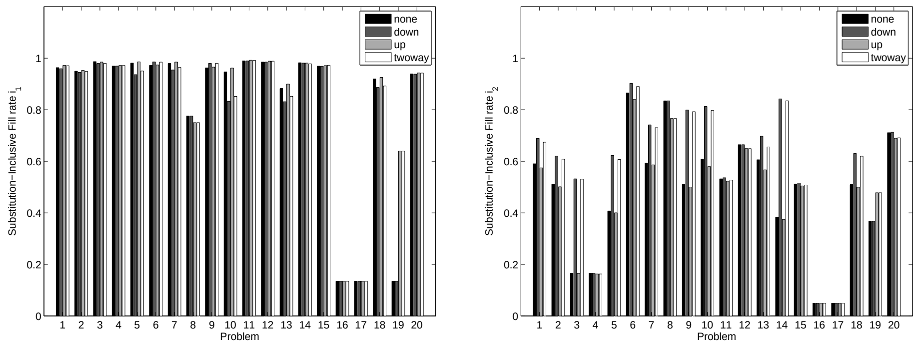
the *substitution-exclusive fill rate* refer to the fill rate calculation which excludes such sales. Similar approaches, for differentiating between demand met by substitution and demand met by current stock, were taken by Smith and Agrawal (2000) who considered substitutions in the context of a product assortment problem, and by Axsäter (2003) who considered them in the context of transshipments.

In this section two questions are investigated: firstly, how does the fill rate vary across the four substitution strategies, and secondly, how much demand is met by substitution under a two way substitution strategy. In order to study these questions, the fill rates were calculated from a simulation over $T = 1,000,000$ time units.

Effect of Substitution Strategy on Fill Rate. In order to consider the first question, graphs were constructed to compare the substitution-inclusive fill rate for produced and recovered inventory under the four substitution strategies. These graphs are presented in Figures 6.4, 6.5 and 6.6 for problem sets B, C, and D respectively. In each figure, graph (a) presents the fill rate of produced inventory and graph (b) presents the fill rate of recovered inventory. The data for these graphs is included in Appendix C in Tables C.4–C.6.

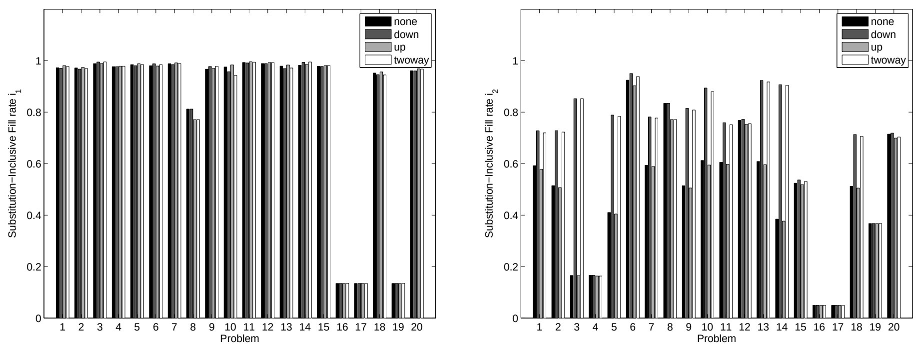
There is substantial amount of variation in the fill rates between the problems and, in general, the fill rates are greater for produced inventory, rather than recovered inventory. There is greater variability in the fill rates between the problems for the recovered inventory, than for the produced inventory. The substitution-inclusive fill rates for both types of inventory vary slightly across problem sets B, C and D, however for most problems are similar.

For produced inventory, the substitution-inclusive fill rates are generally similar across all substitution strategies. However, where there is a difference (e.g. $B09$) the downward or two-way substitution strategies tend to result in a lower fill rate. This may be due to the fact that under the downward or two-way substitution strategies produced goods can be used to meet the demand for recovered goods. In the long run, this potentially results in there being less produced inventory available to meet demand for produced goods.



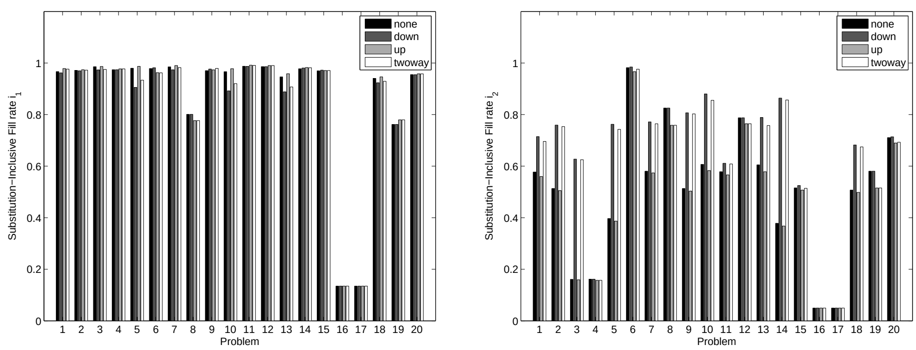
(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure 6.4: Fill rates for problem set B for all substitution strategies



(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure 6.5: Fill rates for problem set C for all substitution strategies



(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure 6.6: Fill rates for problem set D for all substitution strategies

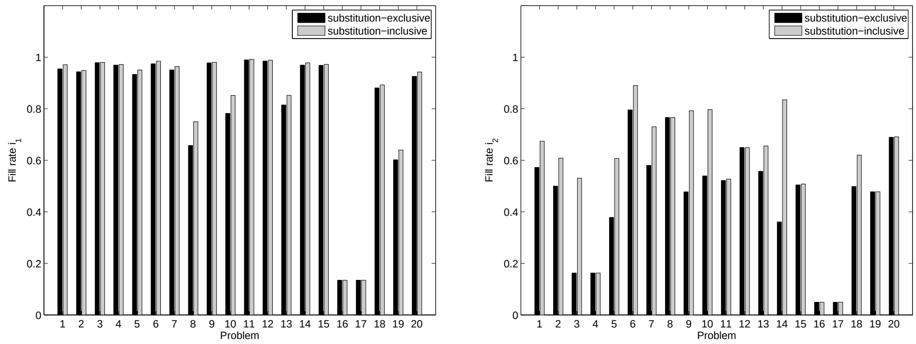
For recovered inventory, the substitution-inclusive fill rates vary significantly between the substitution strategies. In general, downward or two-way substitution strategies result in a higher fill rate. This is expected since the downward or two-way substitution strategies allow for substitution in the event of a shortage of recovered goods. This suggests that downward substitution helps to mitigate the uncertainty surrounding the arrival and quality of returns, and thus the replenishment of the recovered inventory.

Substitution Inclusive and Exclusive Fill Rate. The second question will now be considered. The replenishment and substitution policies may change under different substitution strategies, therefore while it is useful to compare the substitution-inclusive fill rates across substitution strategies, it does not give much information about how much demand is actually met by substitution. In order to examine how much demand is met by substitution, the substitution-inclusive and substitution-exclusive fill rates are examined in more detail. In this section, the performance of the model is considered under the two-way substitution strategy only ($\chi_1 = \chi_2 = 1$).

Figures 6.7, 6.8 and 6.9 show the fill rates including and excluding substitution for problems B, C and D respectively. In general the fill rates (both including and excluding substitution) are greater for produced inventory rather than recovered inventory. The tables showing the data for these graphs are presented in Appendix C in Tables C.4, C.5 and C.6.

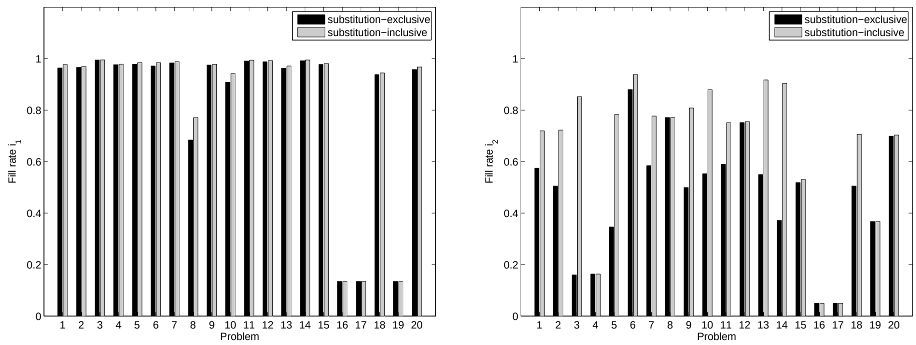
In summary, offering substitution has a greater effect on the fill rate of recovered inventory than produced inventory. This may be related to the greater levels of uncertainty surrounding the quality, quantity and timing of returns. The uncertainty in returns means that managing the levels of recovered inventory is difficult, and offering downward substitution may allow the firm to somewhat mitigate these risks.

In order to further investigate when substitution is offered, and when the replenishment actions are performed we analyse the optimal policy actions in the next section.



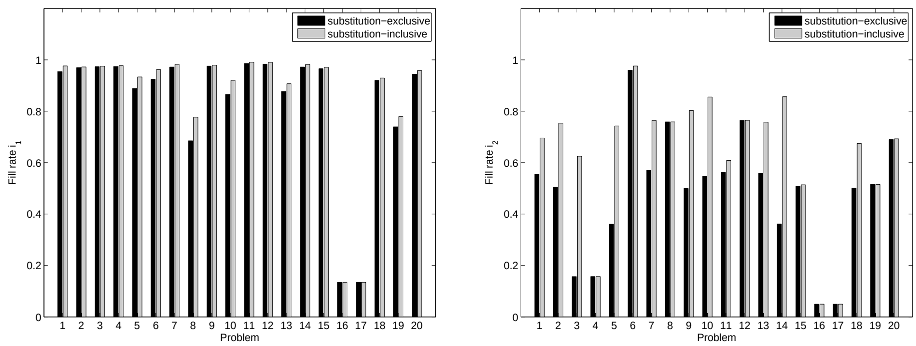
(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure 6.7: Fill rates for problem set B under a two-way substitution strategy



(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure 6.8: Fill rates for problem set C under a two-way substitution strategy



(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure 6.9: Fill rates for problem set D under a two-way substitution strategy

6.6.3 Analysis of Actions

The structure of the optimal policy is of interest as it can be used to investigate the relationship between replenishment actions (production and recovery) and substitution actions (upward and downward). Two main questions will be examined in this chapter. Firstly, across all states, with what *frequency* are replenishment and substitution actions chosen and how is this affected by the substitution strategy. Secondly, in which states are replenishment and substitution actions selected, i.e. what inventory levels *'trigger'* certain actions.

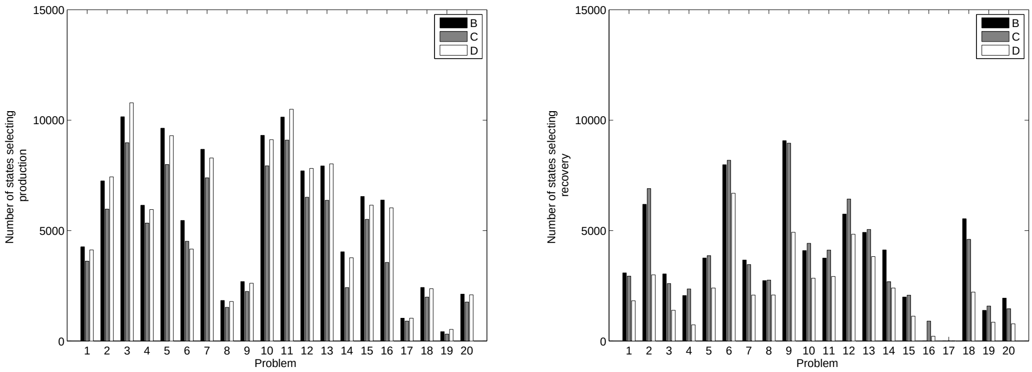
Action Size and Frequency

Replenishment Actions. The size of production and recovery lots are determined by the parameters Q_p and Q_r, therefore in this section we focus on the frequency with which the replenishment actions occur, rather than the size of the actions. In this model there is a total of 14641 states. Table 6.4 summarises the number of states in which each replenishment action is chosen under a two-way substitution strategy. (The other substitution strategies will be examined in the following section.) Figure 6.10 summarises this information graphically. Observe that for problems $B16$, $B17$, $C17$ and $D17$ it is never optimal to recover. This is likely to have a significant impact on the substitution policy for these problems, as demand for recovered goods has to be met by the substitution of produced goods, otherwise the sale will be lost. For problem 16 the comparatively low lost sales cost for recovered goods, and for problem 17 the high setup costs, may discourage recovery.

Comparisons can also be drawn between the problem sets. Problem set C has a greater production size Q_p than problem sets B and D. When the production size is larger (i.e. problem set C), the number of states in which production is performed is lower compared with when the production size is smaller (i.e. problem sets B, D). A similar result is observed for recovery. When the size of the recovery lot Q_r is larger (i.e. problem set D) the number of states in which it is optimal to recover is lower than when Q_r is smaller (i.e. problem sets B, C).

Table 6.4: Number of states in which replenishment is chosen under a two-way substitution strategy, out of a total of $11^4 = 14641$ states.

B			C			D		
Problem	a_p	a_r	Problem	a_p	a_r	Problem	a_p	a_r
$B01$	4266	3087	$C01$	3614	2936	$D01$	4125	1822
$B02$	7253	6190	$C02$	5971	6902	$D02$	7433	2997
$B03$	10156	3039	$C03$	8980	2602	$D03$	10787	1392
$B04$	6148	2059	$C04$	5339	2359	$D04$	5952	733
$B05$	9635	3761	$C05$	7991	3863	$D05$	9299	2401
$B06$	5460	7984	$C06$	4515	8185	$D06$	4165	6692
$B07$	8683	3667	$C07$	7391	3460	$D07$	8288	2083
$B08$	1836	2729	$C08$	1525	2760	$D08$	1790	2085
$B09$	2690	9073	$C09$	2244	8958	$D09$	2616	4922
$B10$	9314	4089	$C10$	7933	4424	$D10$	9117	2843
$B11$	10141	3755	$C11$	9096	4119	$D11$	10496	2921
$B12$	7705	5746	$C12$	6509	6427	$D12$	7816	4836
$B13$	7931	4919	$C13$	6371	5051	$D13$	8021	3825
$B14$	4040	4125	$C14$	2418	2686	$D14$	3771	2399
$B15$	6544	1994	$C15$	5510	2075	$D15$	6153	1128
$B16$	6380	0	$C16$	3558	902	$D16$	6031	217
$B17$	1034	0	$C17$	902	0	$D17$	1034	0
$B18$	2424	5536	$C18$	1990	4606	$D18$	2367	2216
$B19$	423	1389	$C19$	313	1586	$D19$	528	855
$B20$	2124	1942	$C20$	1766	1459	$D20$	2092	778



(a) Production a_p

(b) Recovery a_r

Figure 6.10: Number of states with positive replenishment quantities for three problem sets under a two-way substitution strategy, out of a total of $11^4 = 14641$ states.

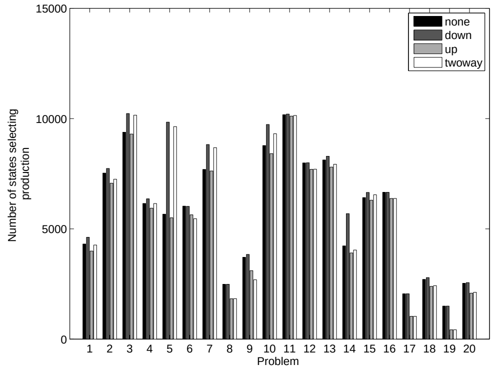
Frequency of Replenishment Under Substitution Strategies Offering substitution may influence whether or not production and recovery are chosen. For instance, it may be the case that it is better to not produce, and to allow substitution to cover any shortages. Figure 6.11 compares the number of states in which production and recovery are chosen under the four substitution strategies: no substitution, downward substitution, upward substitution, two-way substitution. The data for these graphs is available in Appendix C in Table C.7. These graphs show that for some problems (e.g. $B03, B05, B07$) the number of states which offer production is larger when downward or two-way substitution is permitted, which suggests that additional production is performed in order to meet the shortage of recovered goods. However this trend does not hold across all problems.

For some problems (e.g. $C9, C12$) upward or two-way substitution leads to an increase in the number of states in which recovery is performed. This suggests that recovered goods may be used to meet the shortage of produced goods. However this is not the case across all problems. For instance, for problems $B3, B14, B18$ downward or two-way substitution leads to an increase in the number of states which perform recovery.

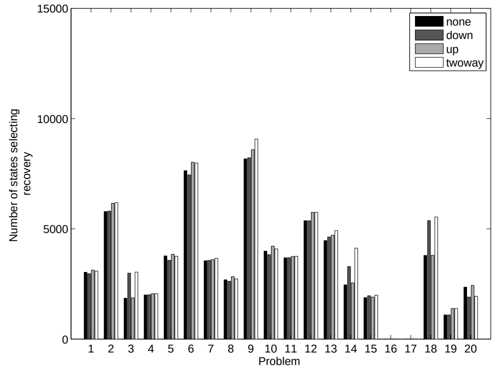
These graphs illustrate the complicated relationship between replenishment, substitution and the problem parameters. In some cases performing substitution leads to a less frequent replenishment, but in other cases in leads to more frequent replenishment. They also highlight that substitution can dramatically affect the optimal frequency with which production and recovery should be performed. It is important to bear this in mind when designing a substitution policy.

Substitution Actions. Figure 6.12 shows the number of states in which each of the substitution actions can be offered under each of the relevant substitution strategies. Upward substitution can only be chosen under a upward or two-way substitution strategy, therefore the number of states selecting upward substitution is only considered under these two strategies. Similarly, downward substitution can only be chosen under a downward or two-way substitution strategy, therefore the number of states selecting downward substitution is only considered under these two strategies. The data for these graphs is available in Appendix C in Table C.9.

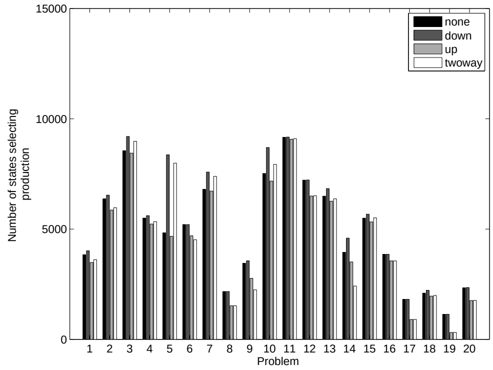
These graphs show that under the strategies which allow upward substitution, it is optimal to offer upward substitution in almost all of the 14641 states for all of the problems. The firm does not have much to lose by offering upward substitution as they receive the price of the good that they provide. The number of states in which upward



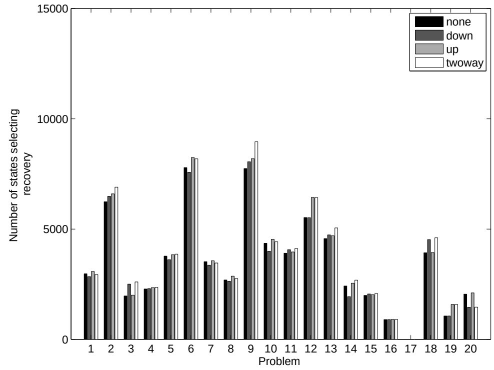
(a) Production for Problem Set B



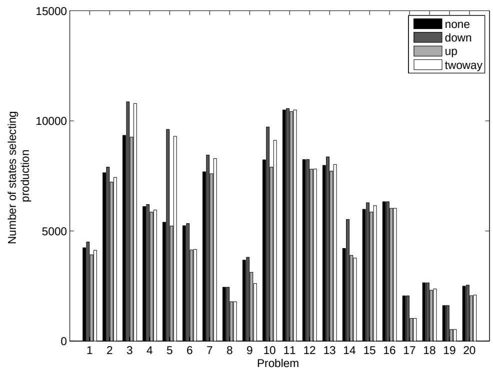
(b) Recovery for Problem Set B



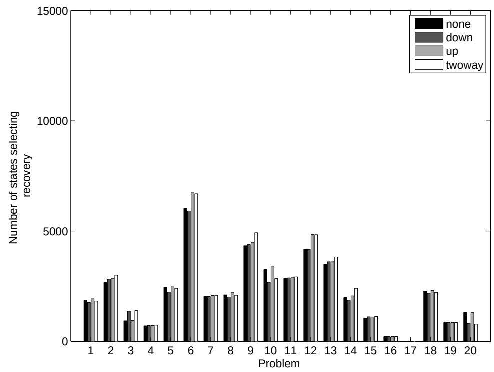
(c) Production for Problem Set C



(d) Recovery for Problem Set C



(e) Production for Problem Set D



(f) Recovery for Problem Set D

Figure 6.11: Graphs showing the number of states in which replenishment is chosen under the substitution strategies

substitution is offered remains reasonably constant across all problems and in the three datasets.

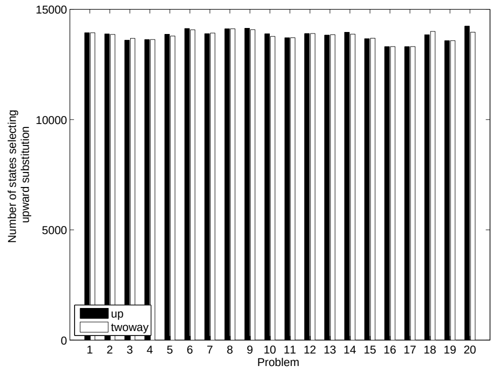
The number of states in which downward substitution, on the other hand, is offered varies considerably across the 20 problems. There is also more variation between the individual problems. This increased variability in downward substitution could be attributed to the firm not wanting to sell a produced good for a lower price. Notice that for the problems which never recover ($B16$, $B17$, $C17$, $D17$), it is never optimal to offer downward substitution.

Within each graph in Figure 6.12, notice the similarity between number of states selecting substitution under the two substitution strategies. Consider Figure 6.12a, for example, this graph shows the number of states in which upward substitution can be offered under two substitution strategies. Under the upward substitution strategy only upward substitution can be offered, whereas under the two-way substitution strategy both upward and downward substitution can be offered. For all problems, the number of states is similar across the two substitutions strategies. This suggests that offering downward substitution does not affect the number of states in which upward substitution is offered. This is also observed in the other graphs in Figure 6.12. This suggests that offering one type of substitution does not affect whether or not the other type of substitution is offered.

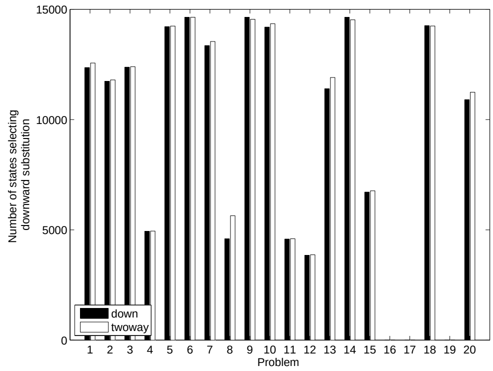
Trigger-States and Action

Replenishment Actions. The inventory levels are taken into account when the action is selected at the beginning of each period, therefore it is interesting to investigate which inventory levels 'trigger' certain actions. In the case of production, there are two inventory levels which are of particular interest: the level of produced inventory when production is selected (i_1) and the level of produced inventory after production has been performed ($i_1 + a_p$). For both these quantities we are only interested in the states for which the optimal action is to produce. These states can be identified as they have a positive value for their production action, i.e., $a_p > 0$.

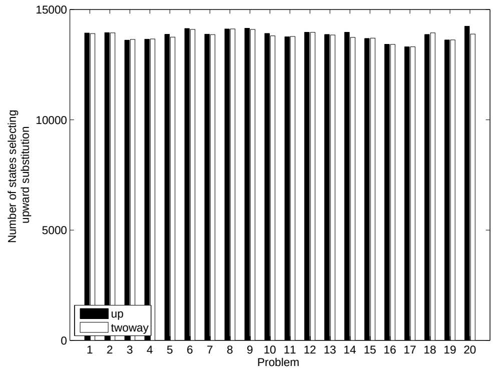
In the case of recovery it is interesting to examine the level of returns (i_r), and the level of recovered inventory (i_2) when recovery is performed. For both these quantities



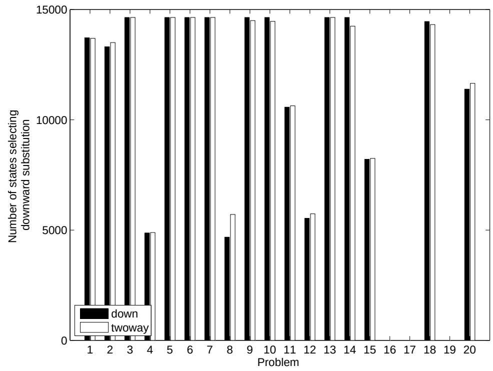
(a) Upward substitution for Problem Set B



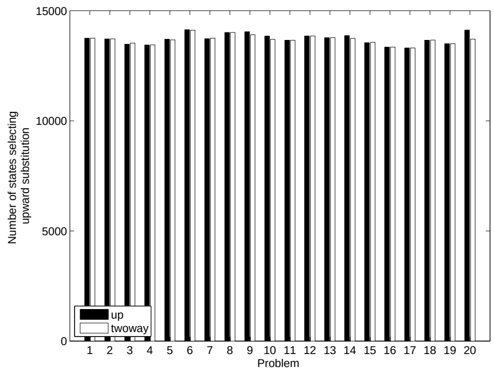
(b) Downward substitution for Problem Set B



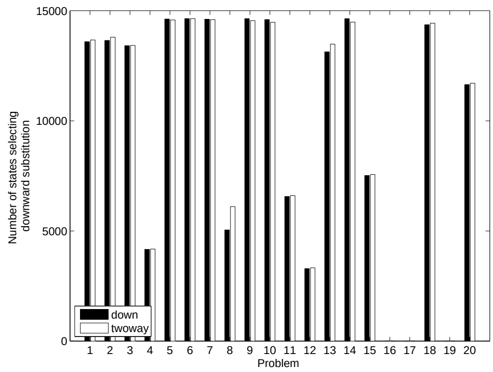
(c) Upward substitution for Problem Set C



(d) Downward substitution for Problem Set C



(e) Upward substitution for Problem Set D



(f) Downward substitution for Problem Set D

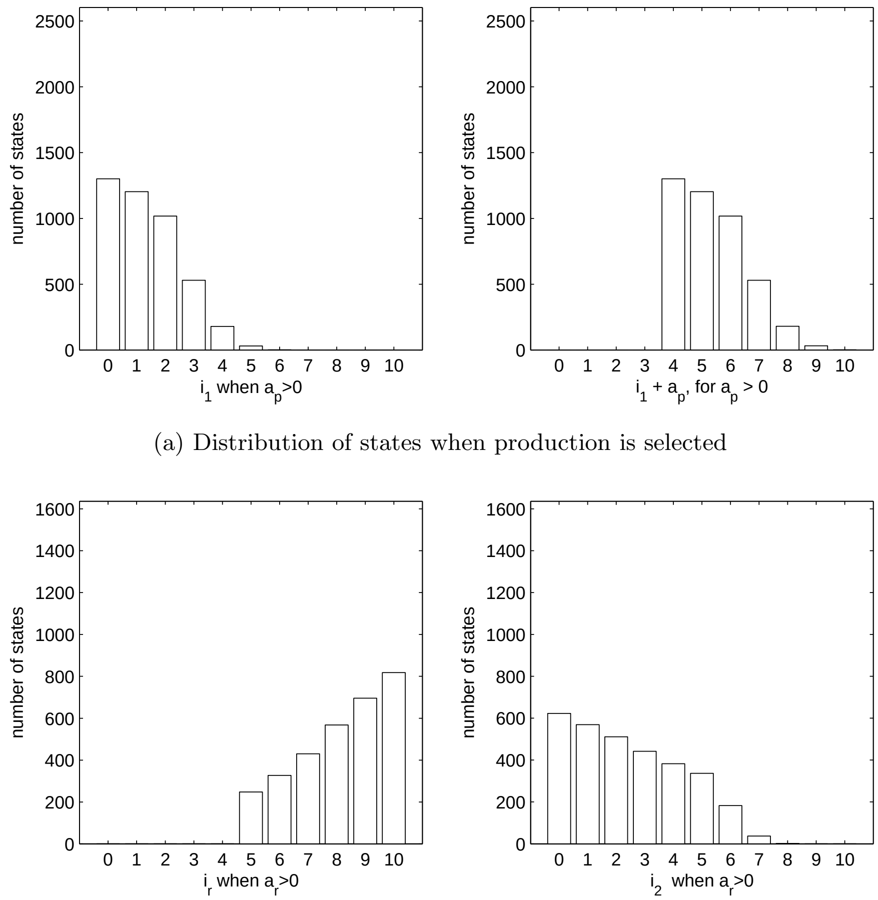
Figure 6.12: Graphs showing the number of states in which substitution is chosen under the substitution strategies

we are only interested in the states for which the optimal action is to perform recovery. These states can be identified as they have a positive value for their recovery action, i.e., $a_r > 0$.

Before considering all problems, we consider problem $B01$ as an example. Figure 6.13a shows the produced inventory level (i_1) and the initial state plus the action ($i_1 + a_p$) for produced inventory when production is chosen ($a_p > 0$) for problem $B01$, under a two-way substitution policy. Notice that production is performed if the produced inventory level is less than 6, and that after production has been performed the inventory level is at least 4. This minimum 'produce-up-to' level corresponds to the production batch size Q_p. These graphs could be used to obtain parameters for a 'produce-up-to' structured policy. Figure 6.13b shows the initial state of recovered inventory i_2 and the initial state of the returned inventory i_r, when recovery is chosen ($a_r > 0$). Notice that recovery is performed if the recovered inventory level is less than 8 and the returned inventory inventory level is at least 5. As expected, the minimum returned inventory level is at least as large as the recovery batch size Q_r.

In order to summarise the trigger levels for all problems, a series of boxplots were created. (It is impractical to include the individual graphs for all problems and substitution strategies as this would mean including about 1000 graphs!) For simplicity only the two-way substitution strategy is included here. Figure 6.14 shows the level of produced inventory when production is chosen, for problem sets B, C and D, under a two-way substitution strategy. The graphs for the other substitutions strategies are presented in Appendix C. As shown by Figure 6.14, the range of inventory levels for which production is performed varies from 0 to 9 depending on the problem. The boxplots appear to be skewed towards zero, this suggests a 'produce-up-to' policy may be appropriate. In general, it appears that the trigger levels of produced inventory for production are slightly lower for problem set C. This is expected since the lot size for production Q_p is greater for problem set C, than for B and D.

Figure 6.15 shows the level of recovery inventory when recovered is chosen for problem sets B, C and D, under a two-way substitution strategy. As shown by these graphs, the range of inventory levels for which recovery is performed varies from 0 to 10 depending on the problem. The boxplots appear to be somewhat skewed towards zero, however interestingly for some problems recovery is performed when $i_2 = 10$,



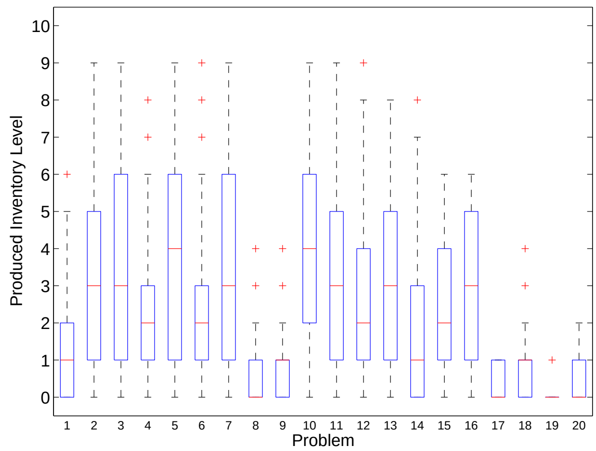
(a) Distribution of states when production is selected

(b) Distribution of states when recovery is selected

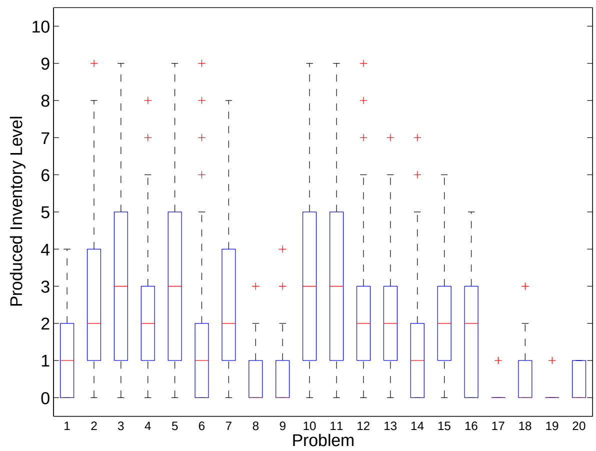
Figure 6.13: Histograms of showing states associated with positive replenishment actions under the optimal policy for test problem $B01$

i.e., when there is no space in the recovered inventory. In Chapter 5, some problems appeared to perform recovery when the serviceable and component inventories were almost at capacity, thus forcing the returns to be disposed. If this were the case, it would be expected that recovery is performed when the recovered inventory is at capacity ($i_2 = 10$), the component inventory would also be near capacity. However as shown by Figure 6.16 this is not the case. For many of the cases it appears that the component inventory levels are quite low. This suggests that recovery maybe performed, in these cases, in order to replenish the component inventory.

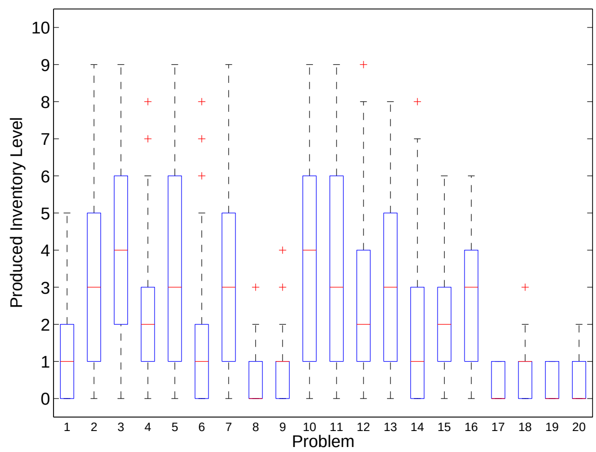
Considering Figure 6.15 again, observe that the trigger levels for recovery seem to be lower for problem set D. This is expected since problem set D has a greater recovery lot size Q_r. In problem set D there are fewer problems which perform recovery when recovered inventory is at maximum capacity, as shown by Figure 6.16c.



(a) Problem Set B

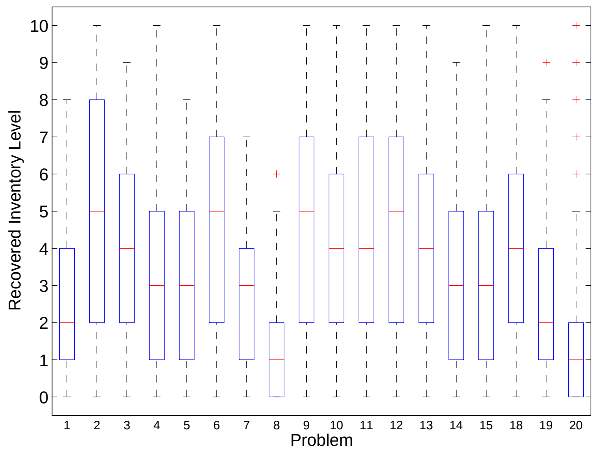


(b) Problem Set C

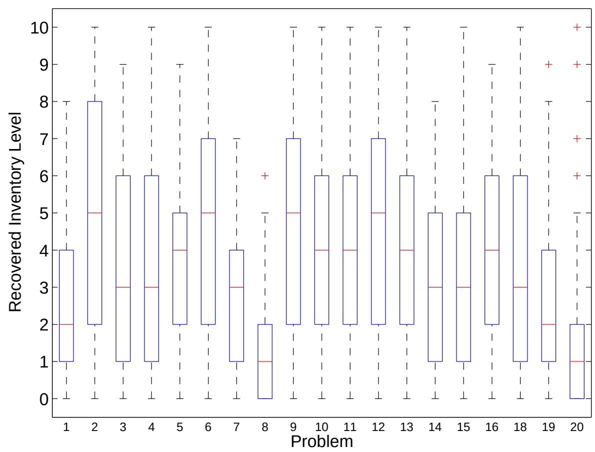


(c) Problem Set D

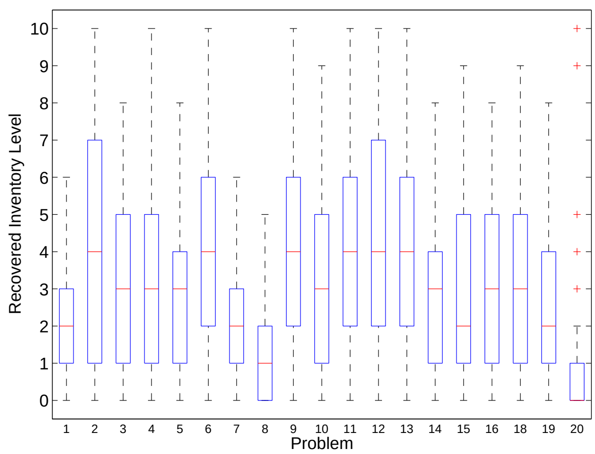
Figure 6.14: Graphs showing the level of produced inventory (trigger level) when production is performed under a two-way substitution policy



(a) Problem Set B

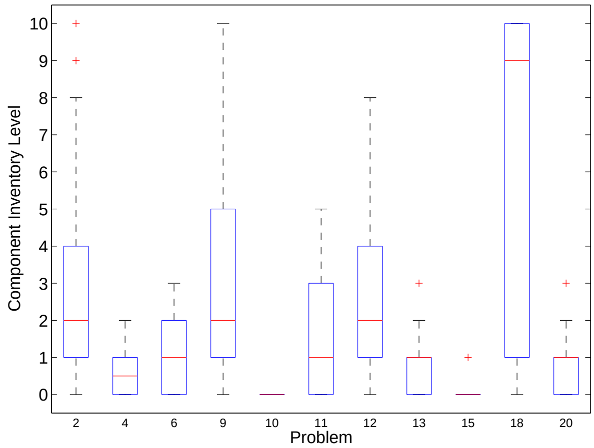


(b) Problem Set C

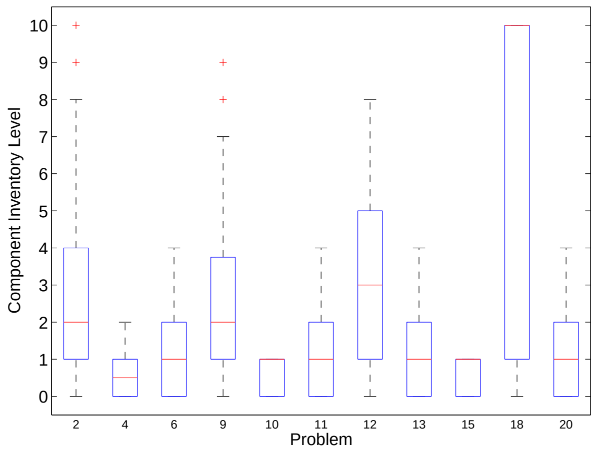


(c) Problem Set D

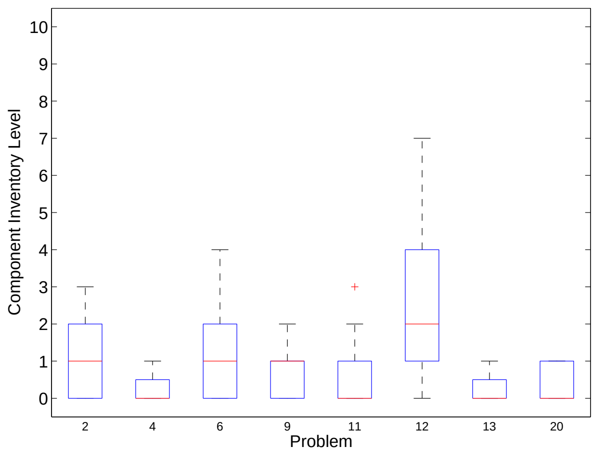
Figure 6.15: Graphs showing the level of recovered inventory (trigger level) when recovery is performed under a two-way substitution policy



(a) Problem Set B



(b) Problem Set C



(c) Problem Set D

Figure 6.16: Graphs showing the level of component inventory when recovery is performed and recovered inventory is at maximum capacity ($i_2 = 10$) under a two-way substitution policy

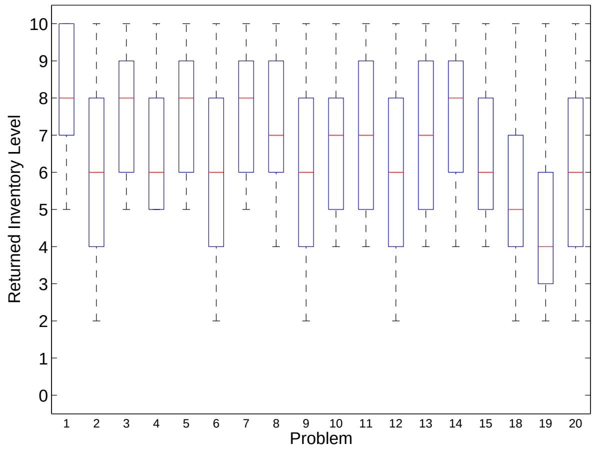
The level of returned inventory when recovery is performed is also of interest. The size of the recovery lot Q_r places a minimum bound on this level, as there must be at least Q_r returns in stock to be able to perform recovery. This is shown by minimum values in the boxplots in Figure 6.17. These boxplots are skewed towards the upper inventory limit of 10, which suggests that recovery is more frequently performed for large inventory levels. The trigger levels of returned inventory for recovery seem to be higher for problem set D. This is expected since problem set D has a greater recovery lot size Q_r and thus requires more returns to be in stock in order to perform recovery.

Substitution Actions. The levels of produced and recovered inventory will affect whether or not substitution is offered. It would be expected that substitution would be offered more often for higher inventory levels.

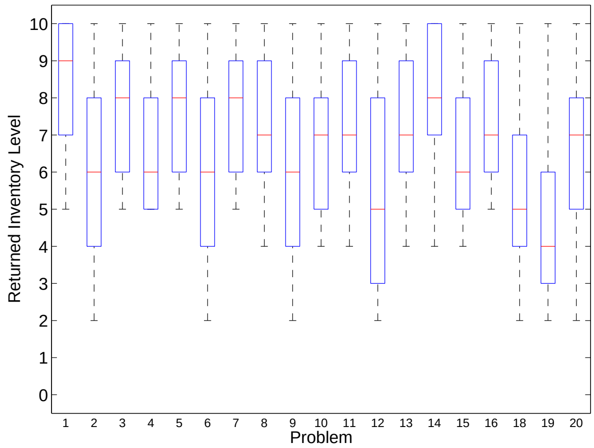
If there is a shortage of recovered goods then the level of produced inventory is important when considering whether or not a downward substitution should be offered. Production orders arrive before demand is observed and this newly arrived inventory can be used to satisfy demand within that period. The size of the production order is known with certainty, therefore the level of produced inventory that is of interest is the one after production has occurred, but before demand has been observed, i.e. $i_1 + a_p$. If the substitution action is selected, demand for produced inventory is always satisfied first, and if there are any produced goods left in stock and if there is a shortage of recovered goods, then a downward substitution is offered.

If there is a shortage of produced goods then the level of recovered inventory is important when considering whether or not an upward substitution should be offered. Since the yield of the recovery lot is not known with certainty, we examine the upward substitution behaviour for the recovered inventory level at the start of the period i_2 and also the recovered inventory level i_2 when recovery has been performed, i.e. $a_r > 0$. If the substitution action is selected, demand for recovered inventory is always satisfied first, and if there are any recovered goods left in stock and if there is a shortage of produced goods, then an upward substitution is offered.

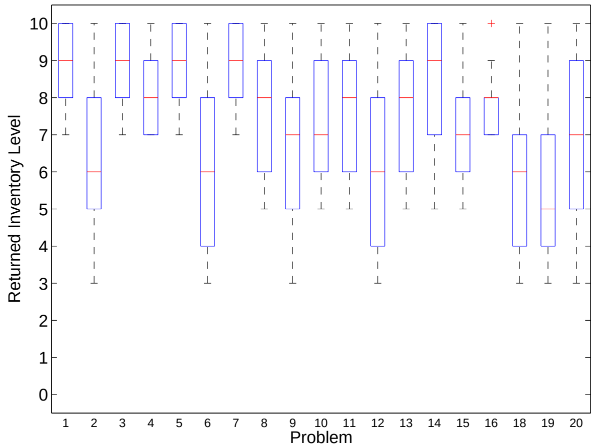
Figure 6.18 shows the inventory levels and the substitution decision for problem $B01$ under a two-way substitution strategy. These stacked bar charts show the number of



(a) Problem Set B



(b) Problem Set C



(c) Problem Set D

Figure 6.17: Graphs showing the level of returned inventory (trigger level) when recovery is performed under a two-way substitution policy.

states in which substitution is offered. The black part of the bars indicate the number of states in which substitution is not offered and the white part of the bars indicate the number of states in which substitution is offered. The first graph shows that downward substitution is always offered, if the level of produced inventory after ordering is 7 or more. It is sometimes offered if the inventory level is between 1 and 6.

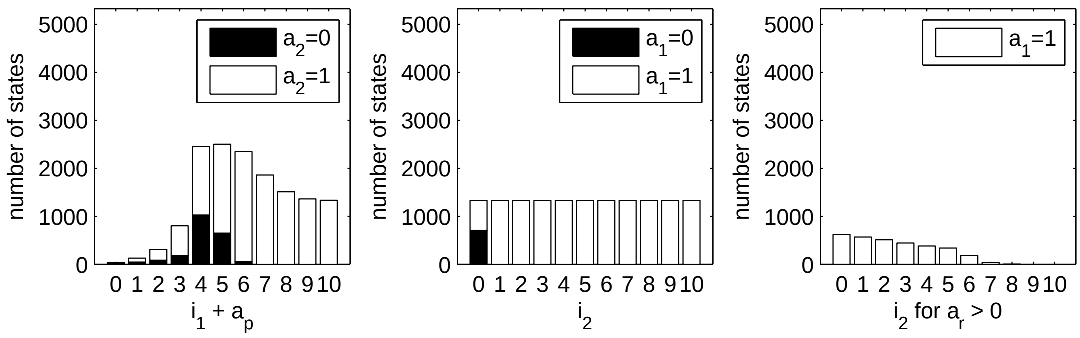


Figure 6.18: Histograms of the replenishment actions under the optimal policy for test problem $B01$

The second graph in Figure 6.18 shows that upward substitution is always offered if the inventory level of recovered inventory is 1 or more. If the inventory level is 0 then substitution is offered in approximately 600 states. It would be expected that recovery has been performed in these 600 states, otherwise there would be no recovered inventory to offer as a substitute. This is supported by the third graph which shows that when there is no recovered inventory, recovery is performed in approximately 600 states. The third graph also shows that if recovery is performed, upward substitution can always be offered.

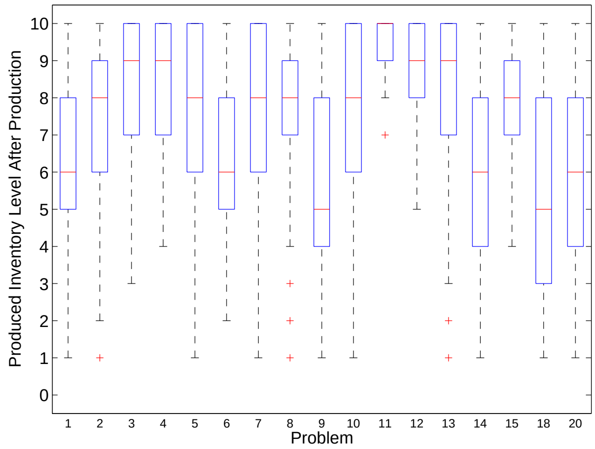
In order to summarise this information for all problems, a series of boxplots were created showing the inventory levels for which substitution was offered. Figure 6.19 shows the produced inventory levels after production has been completed, for the states in which downward substitution can be offered. As shown by these graphs the range of states over which substitution can be offered varies between the problems. However, as expected, the box plots do seem to be skewed towards 10 which is the upper limit of the produced inventory. The wide range of states in which substitution can be offered may be due to the fact that offering downward substitution does not carry much risk, since more goods can be produced before the next demand for produced goods. However under downward substitution, newly produced goods are sold for the (lower) price of recovered goods. Note, the number of states in which downward substitution is selected differs between problems, thus the number of states represented by each boxplot differs between problems. The number of states in which substitution is selected is presented in Appendix C (Table C.9).

Figure 6.20 shows the recovered inventory levels when upward substitution is selected. Since the increase to recovered inventory at the completion of recovery is uncertain, this post-recovery inventory level can not be represented on the graph. However, as shown on this graph, substitution can be offered across a range of all levels of recovered inventory. The exception is problems $B16$, $B17$, $C17$ and $D17$ in which it is never optimal to recover. This is inline with Figure 6.12 which shows that upward substitution is offered in almost all states.

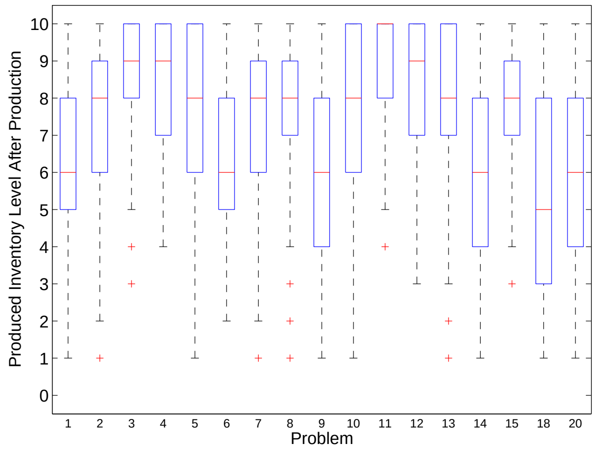
One reason for the optimality of upward substitution over most inventory levels could be because upward substitution involves selling a recovered good for its actual price, unlike downward substitution where a produced good is sold for a lower price. There is no cost-disincentive for upward substitution. There is little difference between the problem sets B, C and D. Note, the number of states in which upward substitution can be offered differs between problems, thus the number of states represented by each boxplot differs between problems. The number of states in which substitution is selected is presented in Appendix C (Table C.8).

6.6.4 Analysis of Recovery Strategy

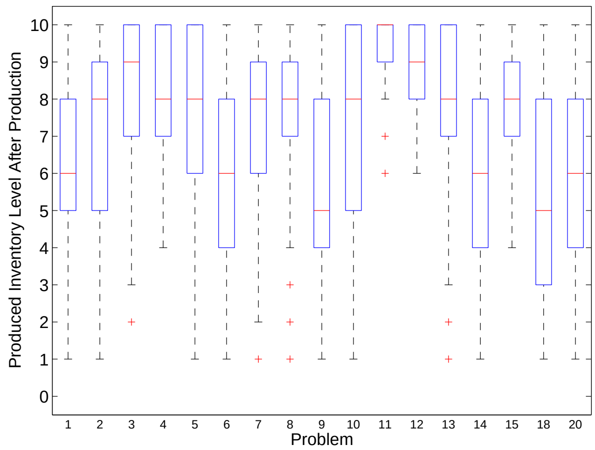
In this section two recovery strategies are compared: high quality returns only; and both low and high quality returns. In this section the focus is on the effect of the quality of returns, rather than the effect of substitution, therefore only a two-way substitution strategy is considered. The effect of the quality of returns is assessed with respect to four performance and policy features: the long run average reward, the fill rates, the frequency of replenishment and substitution, and states 'triggering' replenishment and substitution.



(a) Problem Set B

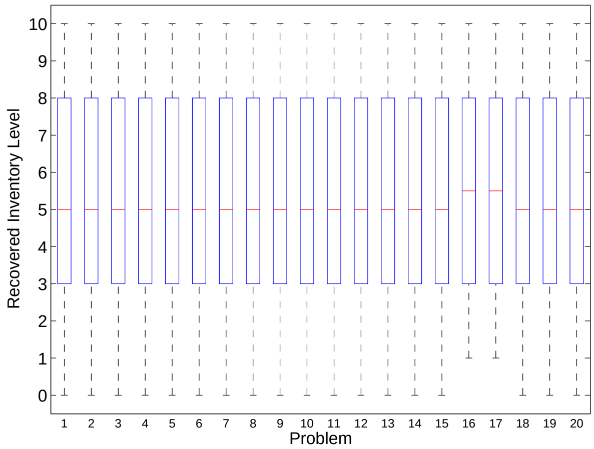


(b) Problem Set C

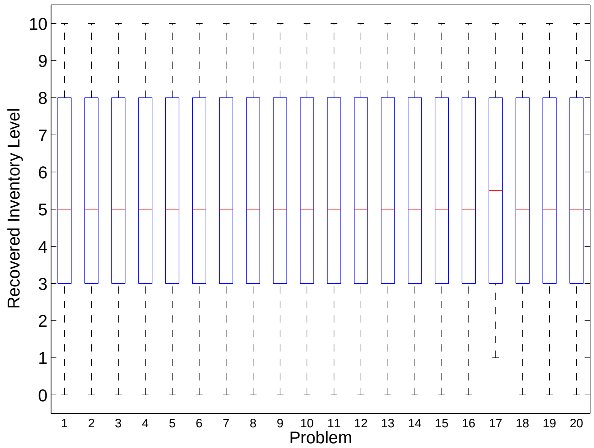


(c) Problem Set D

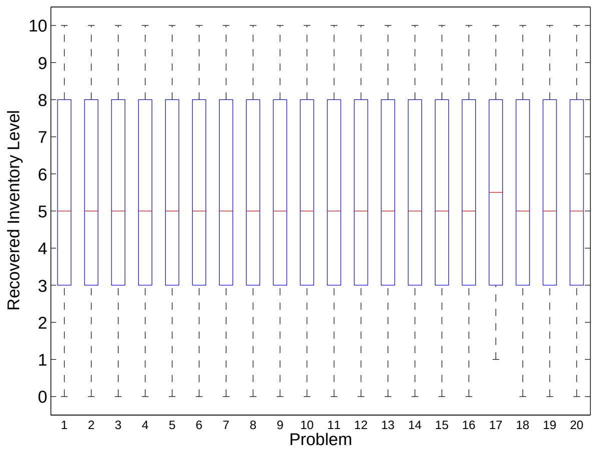
Figure 6.19: Graphs showing the levels of produced inventory after production has been completed ($i_1 + a_p$) in which downward substitution can be offered, under a two-way substitution strategy



(a) Problem Set B



(b) Problem Set C



(c) Problem Set D

Figure 6.20: Graphs showing the levels of recovered inventory (i_2) in which upward substitution can be offered, under a two-way substitution strategy.

Average Reward

Figure 6.21 shows the long run average reward of the optimal policy calculated for the two quality strategies under a two-way substitution strategy. As expected the reward is higher when both high and low quality returns are recovered. However for some problems, the inclusion of low quality recovery does not lead to a noticeable increase in the average reward. The data used to construct these tables is presented Table 6.5.

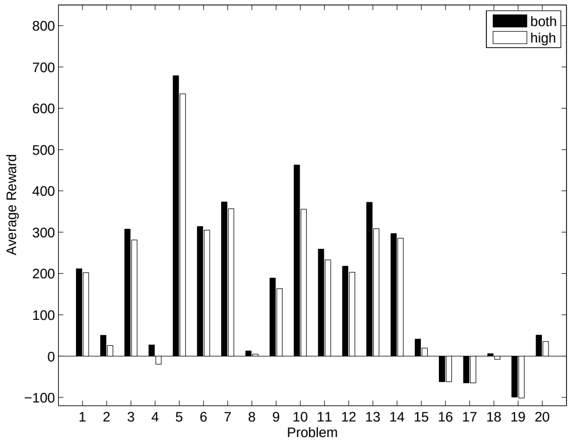
Table 6.5: Average reward of the optimal policy calculated for two quality strategies under a two-way substitution strategy

	B		C		D	
Problem	Only High	Low and High	Only High	Low and High	Only High	Low and High
01	202.1436	211.5259	205.9395	215.1768	210.4709	201.0826
02	25.6128	50.4485	25.5920	56.4732	61.5641	29.3206
03	281.3080	307.4512	312.8492	349.8866	315.3195	283.0035
04	-19.5735	27.0037	-20.4996	25.2236	16.6174	-20.1899
05	634.9699	678.8897	732.0661	772.2306	746.7642	695.2219
06	305.1127	313.7147	321.4910	331.8559	357.5609	342.9260
07	356.9213	373.1980	376.6520	392.6603	384.8664	368.4949
08	4.8093	12.5345	4.6477	12.0837	13.8217	5.8746
09	163.3835	188.9560	171.2241	202.0693	209.6617	177.9897
10	355.6379	462.6609	403.6609	518.6298	514.6616	383.9189
11	233.1060	258.9960	248.1966	278.7292	269.9624	240.1156
12	203.0774	217.6326	205.2325	224.5308	231.3472	209.7540
13	308.6027	372.3621	391.4959	449.4213	406.7766	340.0356
14	285.4041	296.6445	304.8582	316.5529	302.5392	290.6119
15	19.3797	41.1527	26.8810	51.2546	47.9296	19.3798
16	-61.9998	-61.9998	-61.9998	-61.9997	-61.9999	-60.2547
17	-64.8000	-64.8000	-64.7994	-64.7994	-64.8000	-64.8000
18	-7.8049	5.8693	-2.0649	12.8663	9.4810	-5.2455
19	-101.4998	-99.2472	-101.4997	-101.4995	-92.3012	-101.4998
20	35.2583	50.8957	38.3115	53.3244	53.8551	37.8018

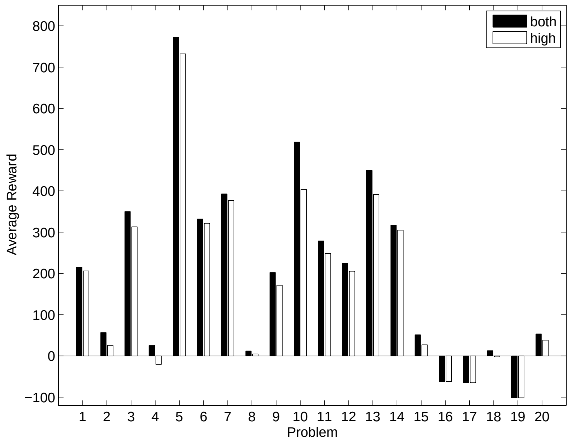
Fill Rates

As discussed in Section 6.6.2, a low fill rate indicates that there has been a large number of lost sales, which could lead to higher levels of customer dissatisfaction. The average fill rate can be calculated using a simulation, by calculating the fill rate each period and then averaging this across the length of the simulation. For each type of inventory that is demanded, the fill rate is:

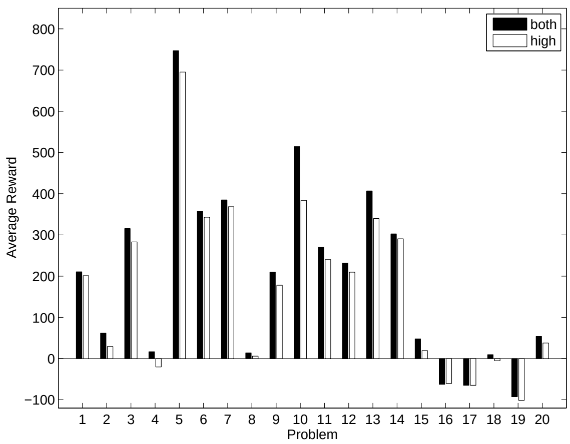
$$\text{fill rate} = \frac{\text{number of met sales}}{\text{number of items demanded}}$$



(a) Problem Set B



(b) Problem Set C



(c) Problem Set D

Figure 6.21: Average reward of the optimal policy calculated for two quality strategies under a two-way substitution strategy

As in Section 6.6.2, let the *substitution-inclusive fill rate* refer to the fill rate calculation which includes sales met by substitution, and let the *substitution-exclusive fill rate* refer to the fill rate calculation which excludes such sales.

In this section the following question is investigated: how does the fill rate vary between the two quality strategies. In order to study this questions, the fill rates were calculated from a simulation over $T = 1,000,000$ time units.

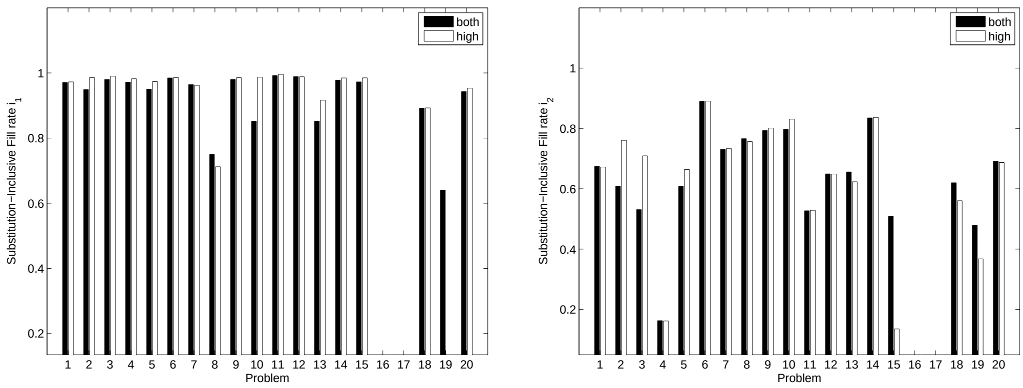
Effect of Recovery Strategy on Fill Rate. In order to consider this question, graphs were constructed to compare the substitution-inclusive fill rate for produced and recovered inventory under the four substitution strategies. These graphs are presented in Figures 6.22, 6.23 and 6.24 for problem sets B, C, and D respectively. The data for these figures is presented in Appendix C in Tables C.10 and C.11.

As in Section 6.6.2, there is substantial amount of variation in the fill rates between the problems and in general the fill rates are greater for produced inventory rather than recovered inventory. For the recovered inventory there is greater variability in the fill rates between the problems, than for the produced inventory. The substitution-inclusive fill rates for both types of inventory vary somewhat across problem sets B, C and D, however for most problems are similar.

In general, the fill rate for produced inventory is greater than the fill rate for recovered inventory and the fill rates are similar for the two recovery strategies. However for some problems performing both types of recovery leads to a higher fill rate, and for other problems a lower fill rate. This highlights the complexity of the relationship between the substitution and recovery strategies.

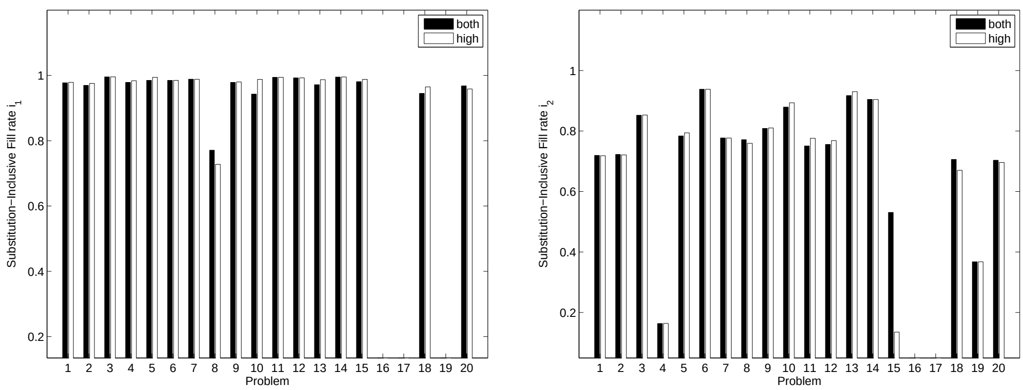
Action Size and Frequency

Replenishment Actions. The size of production and recovery lots are determined by the parameters Q_p and Q_r, therefore as in Section 6.6.3, in this section we focus on the frequency with which the replenishment actions occur, rather than the size of the actions. Table 6.6 summarises the number of states for which each replenishment action is chosen under a two-way substitution, high-quality only recovery strategy.



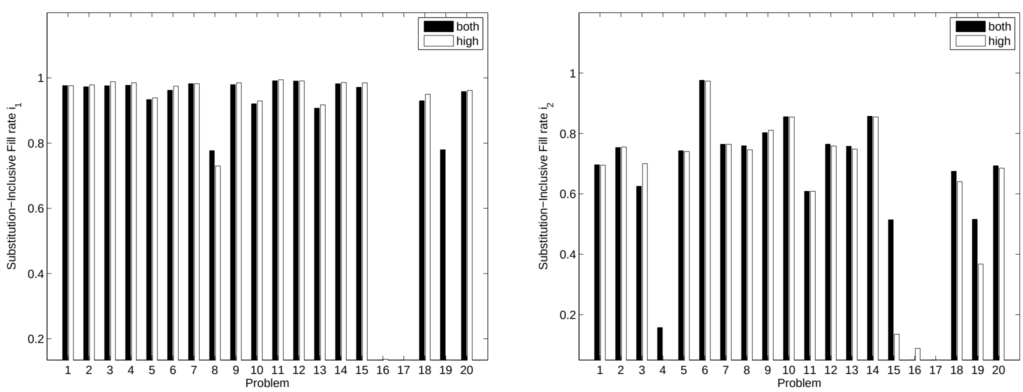
(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure 6.22: Fill rates for problem set B for both quality strategies



(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure 6.23: Fill rates for problem set C for both quality strategies



(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure 6.24: Fill rates for problem set D for both quality strategies

The equivalent data for both high and low quality recovery was presented in Table 6.4. Figure 6.25 summarises this information graphically. In this model there is a total of 14641 states. All of the problems for which it was not optimal to perform recovery under a high and low quality recovery strategy, it is also not optimal to perform recovery under a high-quality-only recovery strategy. Furthermore there are six additional problems for which it is not optimal to perform recovery under a high-quality-only recovery strategy. This suggests that for these additional six problems, it is the recovery of low quality items which makes recovery cost-efficient. However interestingly, for some problems (e.g. $D02$, $D09$ and $D18$) performing only high-quality recovery increases the frequency with which recovery is performed.

Table 6.6: Number of states in which replenishment is chosen under a two-way substitution, high-quality only strategy, out of a total of $11^4 = 14641$ states

B			C			D		
Problem	a_p	a_r	Problem	a_p	a_r	Problem	a_p	a_r
$B01$	4206	3131	$C01$	3601	3014	$D01$	4076	1838
$B02$	6927	4785	$C02$	5598	7668	$D02$	6171	6421
$B03$	9903	3131	$C03$	8429	3473	$D03$	10091	1996
$B04$	5445	4016	$C04$	5278	4077	$D04$	5358	72
$B05$	9156	4169	$C05$	7356	4066	$D05$	8719	2820
$B06$	5308	7777	$C06$	4468	7702	$D06$	4055	6355
$B07$	8540	3737	$C07$	7341	3474	$D07$	8208	2092
$B08$	1747	2992	$C08$	1521	2961	$D08$	1688	2371
$B09$	2516	9151	$C09$	2129	9714	$D09$	2407	7874
$B10$	9092	4070	$C10$	7835	4283	$D10$	8676	3404
$B11$	10242	3447	$C11$	9158	3837	$D11$	10475	2717
$B12$	7107	4419	$C12$	6610	4425	$D12$	7133	3426
$B13$	8533	4197	$C13$	6401	4695	$D13$	8225	3541
$B14$	3510	4796	$C14$	2173	3376	$D14$	3190	3156
$B15$	5676	0	$C15$	5126	0	$D15$	5676	46
$B16$	6380	0	$C16$	4961	0	$D16$	5972	2045
$B17$	1034	0	$C17$	902	0	$D17$	1034	0
$B18$	1974	10028	$C18$	1713	10047	$D18$	1849	8161
$B19$	330	0	$C19$	297	0	$D19$	330	0
$B20$	1991	2893	$C20$	1597	2352	$D20$	1958	1396

Referring to the parameters in Table 6.2, appears that the largest differences in the frequency of recovery are observed for problems for which mean proportion of high quality items α is less that approximately 0.5. This suggests that the effect of the recovery strategy is more dramatic when there is a high proportion low quality returns. This is not unexpected, since the effect of performing low quality recovery as well as high quality recovery is likely to have more impact when the number of low quality returns is greater. However interestingly, performing low quality recovery in addition to high quality recovery results in more frequent recovery for some problems (e.g. $B02$, $B12$), less frequent recovery for some problems (e.g. $B04$, $B18$, $C04$, $C18$, $D02$, $D18$), and does not affect the recovery frequency of some problems (e.g. $B06 - B10$).

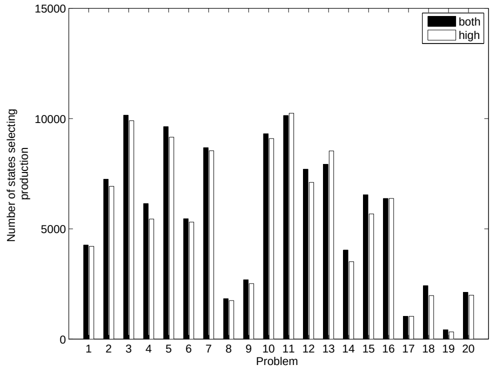
There appears to be some variation between the problem sets. Consider problem 02: compared with high-quality only recovery, under a high and low quality recovery strategy $B02$ has more frequent recovery, $C02$ similar recovery levels, and $D02$ has less frequent recovery. This is interesting as it shows that the order size and the recovery strategy interact to effect the frequency with which recovery is performed.

Substitution Actions. Figure 6.26 shows the number of states in which each of the substitution actions is chosen for all problems sets under the two recovery strategies. The number of states in which substitution is selected is presented in Table 6.7. As shown in these graphs, for both recovery strategies it is optimal to offer upward substitution in almost all of the 14641 states across all problems. The firm does not have much to lose by offering upward substitution as they receive the price of the good that they provide, and it appears that this is not affected by the recovery strategy.

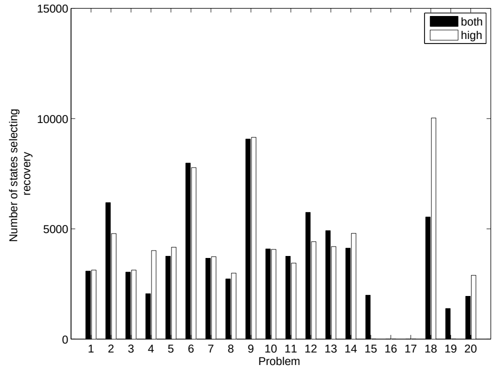
The number of states in which downward substitution is offered varies considerably across the 20 problems. Compared with a high-quality only recovery strategy, the number of states in which downward substitution is offered under a low and high quality recovery strategy is higher for some problems (e.g. 04 and 08) and lower for some problems (e.g. $B02$, $B03$, $B12$, $C12$, $D3$). However, for most problems, there is little variation between the two recovery strategies.

Trigger-States and Action

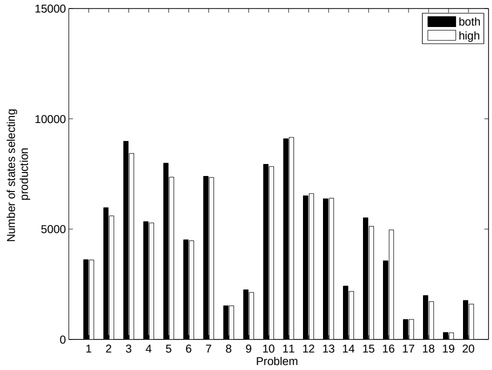
Replenishment Actions. The inventory levels are taken into account when the action is selected at the beginning of each period, therefore it is interesting to investigate which inventory levels 'trigger' certain actions and whether or not these levels are affected by the recovery strategy. As in Section 6.6.3 there are two inventory levels which are particularly of interest in the case of production. It is interesting to examine



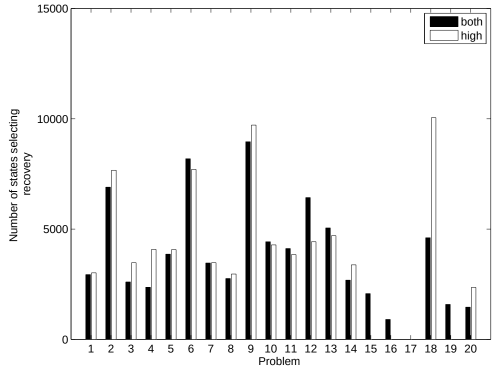
(a) Production a_p for Problem Set B



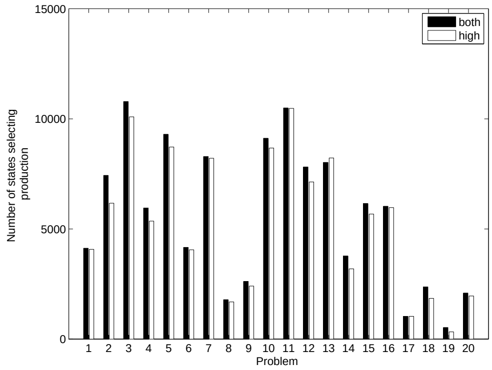
(b) Recovery a_r for Problem Set B



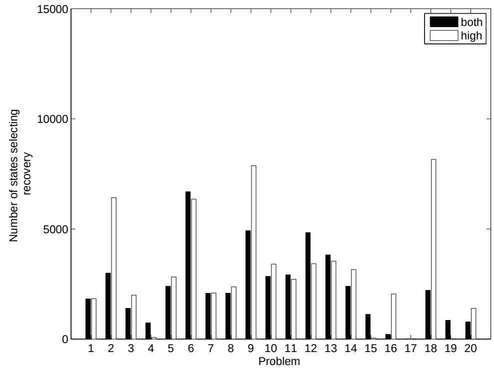
(c) Production a_p for Problem Set C



(d) Recovery a_r for Problem Set C

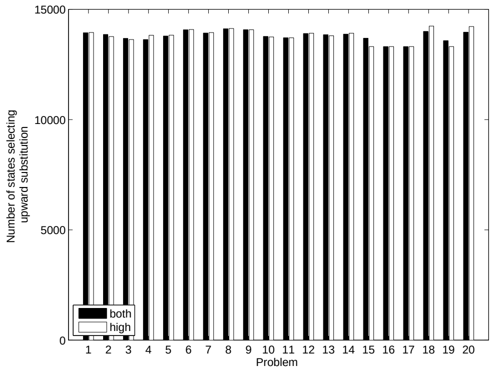


(e) Production a_p for Problem Set D

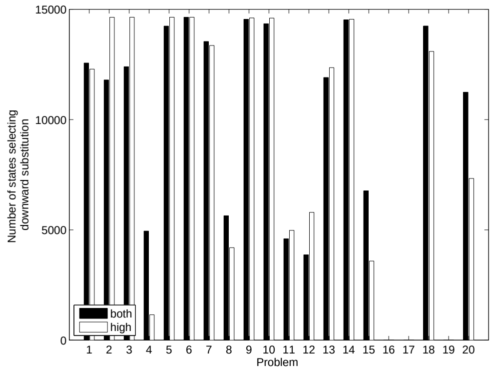


(f) Recovery a_r for Problem Set D

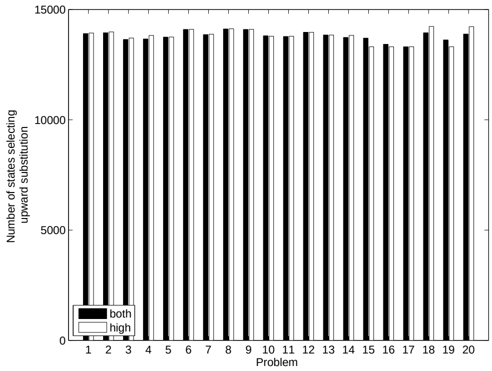
Figure 6.25: Number of states with positive replenishment quantities under a two-way substitution strategy for each recovery strategy, out of a total of $11^4 = 14641$ states



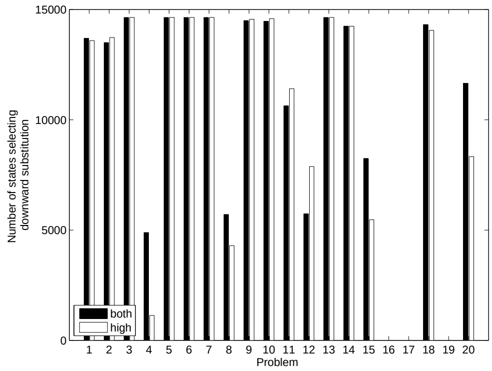
(a) Upward substitution for Problem Set B



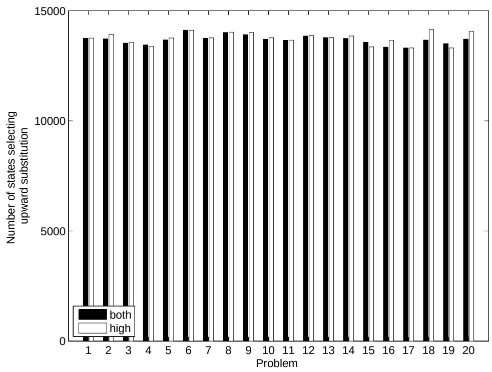
(b) Downward substitution for Problem Set B



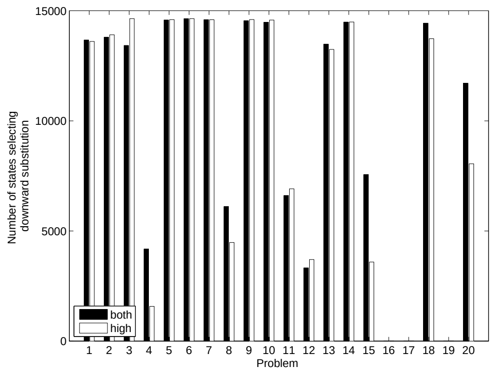
(c) Upward substitution for Problem Set C



(d) Downward substitution for Problem Set C



(e) Upward substitution for Problem Set D



(f) Downward substitution for Problem Set D

Figure 6.26: Graphs showing the number of states in which substitution is chosen under the recovery strategies

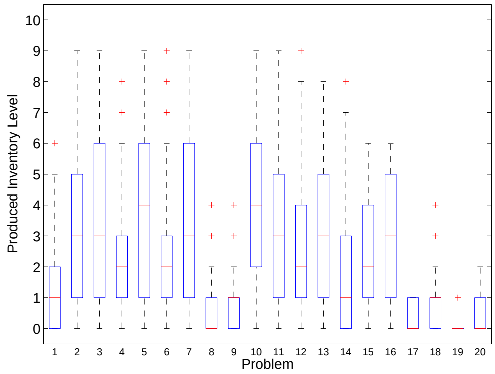
Table 6.7: Number of states in which substitution can be offered under a two-way substitution, high-quality only strategy, out of a total of $11^4 = 14641$ states

B			C			D		
Problem	a_{s1}	a_{s2}	Problem	a_{s1}	a_{s2}	Problem	a_{s1}	a_{s2}
$B01$	13948	12287	$C01$	13932	13589	$D01$	13757	13604
$B02$	13763	14638	$C02$	13984	13733	$D02$	13917	13907
$B03$	13628	14641	$C03$	13708	14641	$D03$	13561	14641
$B04$	13825	1150	$C04$	13825	1131	$D04$	13382	1578
$B05$	13827	14641	$C05$	13753	14641	$D05$	13760	14595
$B06$	14084	14641	$C06$	14101	14641	$D06$	14110	14641
$B07$	13946	13359	$C07$	13881	14641	$D07$	13767	14591
$B08$	14128	4193	$C08$	14125	4292	$D08$	14026	4478
$B09$	14073	14609	$C09$	14099	14558	$D09$	14007	14596
$B10$	13747	14601	$C10$	13789	14582	$D10$	13769	14576
$B11$	13708	4978	$C11$	13788	11409	$D11$	13664	6915
$B12$	13916	5793	$C12$	13965	7879	$D12$	13865	3702
$B13$	13799	12355	$C13$	13841	14641	$D13$	13775	13242
$B14$	13917	14547	$C14$	13826	14241	$D14$	13852	14487
$B15$	13310	3586	$C15$	13310	5467	$D15$	13356	3590
$B16$	13310	0	$C16$	13310	0	$D16$	13662	0
$B17$	13310	0	$C17$	13310	0	$D17$	13310	0
$B18$	14239	13088	$C18$	14231	14054	$D18$	14148	13727
$B19$	13310	0	$C19$	13310	0	$D19$	13310	0
$B20$	14217	7329	$C20$	14222	8330	$D20$	14063	8050

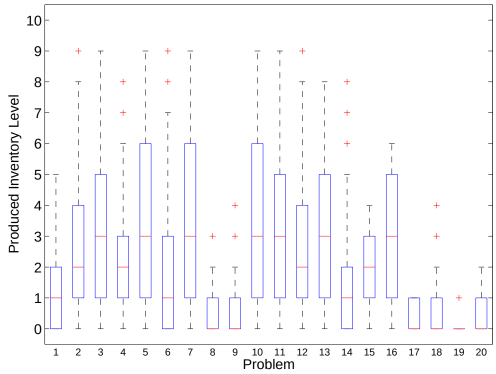
a.) the level of produced inventory (i_1) when production is selected and b.) the level of produced inventory ($i_1 + a_p$) after production has been performed. For both these quantities we are only interested in the states for which the optimal action is to produce. These states can be identified as they have a positive value for their production action, i.e., $a_p > 0$.

In the case of recovery it is interesting to examine the level of returns in stock (i_r) when recovery is performed, and the level of recovered inventory (i_2) when recovery is performed. For both these quantities we are only interested in the states for which the optimal action is to perform recovery, i.e. when $a_r > 0$.

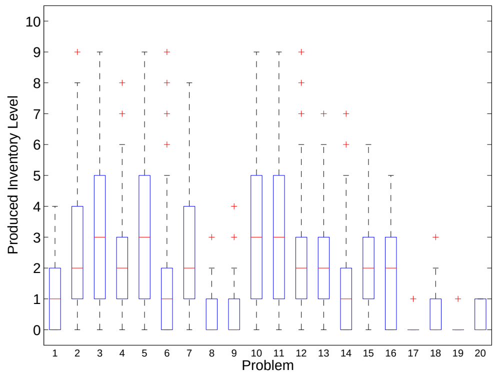
In order to summarise the trigger levels for all problems a series of boxplots were created. Figure 6.27 shows the level of produced inventory when production is chosen for problem sets B, C and D, under a two-way substitution strategy for each of the recovery strategies. In general there appears to be little difference between the 'trigger' levels for production.



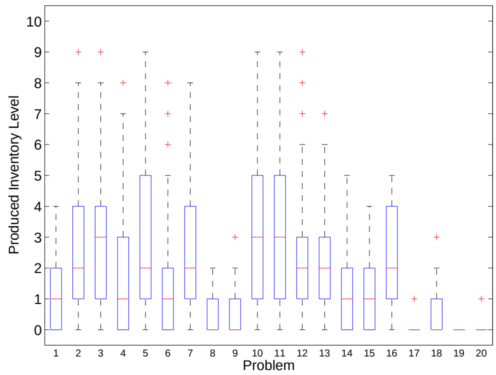
(a) Problem Set B – Both high and low quality recovery



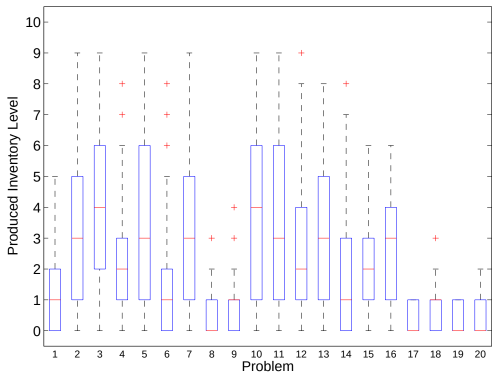
(b) Problem Set B – High quality recovery only



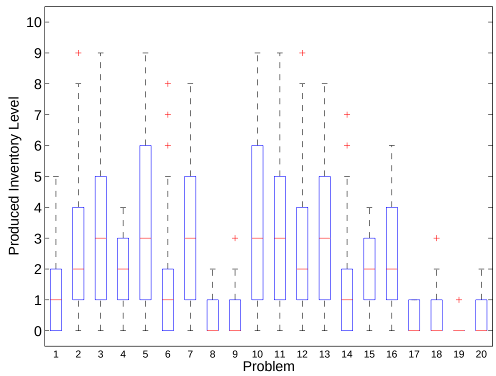
(c) Problem Set C– Both high and low quality recovery



(d) Problem Set C– High quality recovery only



(e) Problem Set D– Both high and low quality recovery



(f) Problem Set D– High quality recovery only

Figure 6.27: Graphs showing the level of produced inventory (trigger level) when production is performed under a two-way substitution policy for two recovery strategies.

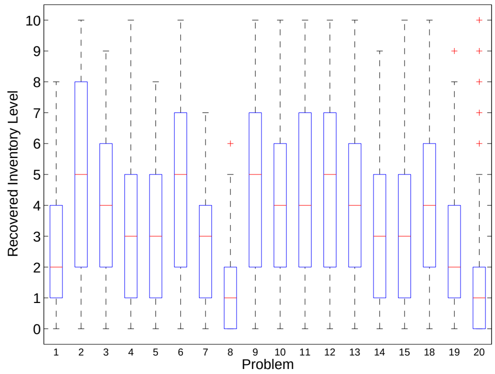
Figure 6.28 shows the level of recovered inventory when recovery is chosen for problem sets B, C and D, under a two-way substitution strategy for the two recovery strategies. In general the level of recovered inventory which triggers recovery is lower for the high quality recovery strategy. There are certainly fewer problems for which is it optimal to recovery when recovered inventory is at its capacity $i_2 = 10$, under a high-quality only recovery strategy. This is expected because when both high and low quality returns are recovered, if the recovered inventory is at capacity, then returns can be recovered to be components. Under a high-quality recovery strategy on the other hand, if the recovered inventory is at capacity, then returns are discarded.

The level of returned inventory when recovery is performed is also of interest. The size of the recovery lot Q_r places a minimum bound on this level, as there must be at least Q_r returns in stock to be able to perform recovery. This is shown by minimum values in the box plots in Figure 6.29. Note that since recovery is not performed in all problems, the number of problems represented in each plot differs. There are no obvious differences between the two recovery strategies in the level of returned inventory triggering recovery, except that as noted above, there are fewer problems which perform recovery under a high quality recovery strategy.

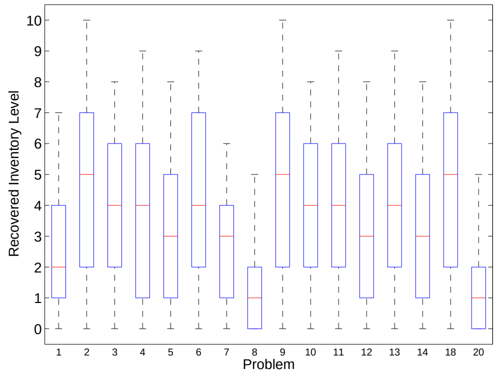
Substitution Actions. The inventory levels of produced and recovered goods will affect whether or not substitution is offered. It would be expected that substitution would be offered more often for higher inventory levels.

If there is a shortage of recovered goods then the level of produced inventory is important in considering whether or not a substitution would be offered if needed. As in Section 6.6.3, the produced inventory after production occurred, but before demand has been observed is $i_1 + a_p$ is used to examine when downward substitution is offered.

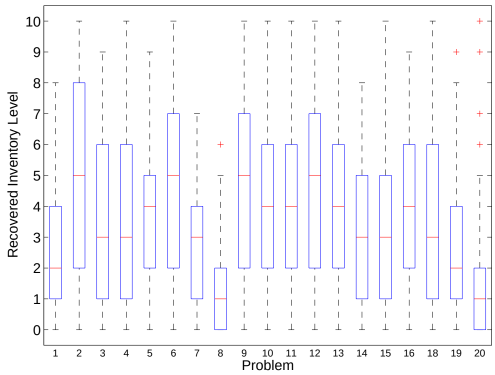
In order to summarise this information for all problems a series of box plots were created showing the inventory levels for which substitution was offered. Figure 6.30 shows the produced inventory levels after production has been completed, for the states in which downward substitution can be offered. As shown by these graphs the range over which substitution is able to be offered varies between the problems, however there appears to be little variation between the two substitution strategies.



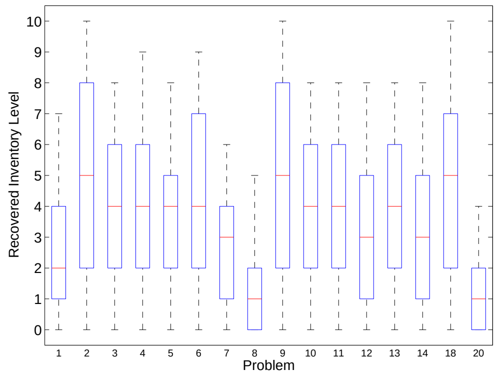
(a) Problem Set B – Both high and low quality recovery



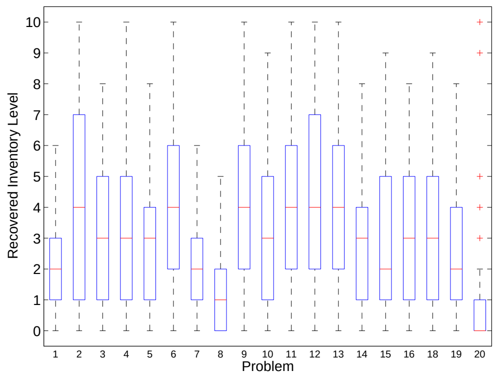
(b) Problem Set B – High quality recovery only



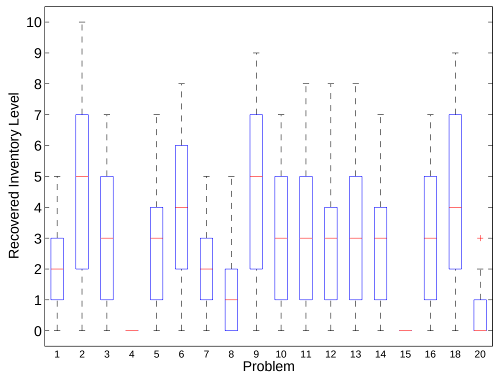
(c) Problem Set C– Both high and low quality recovery



(d) Problem Set C– High quality recovery only

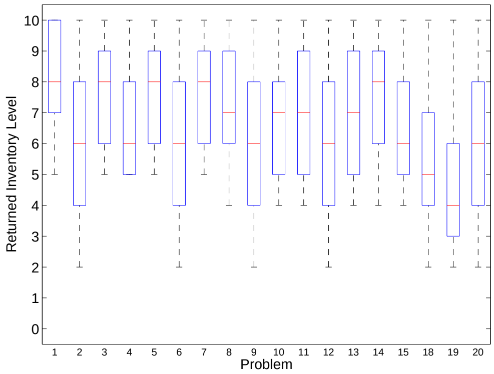


(e) Problem Set D– Both high and low quality recovery

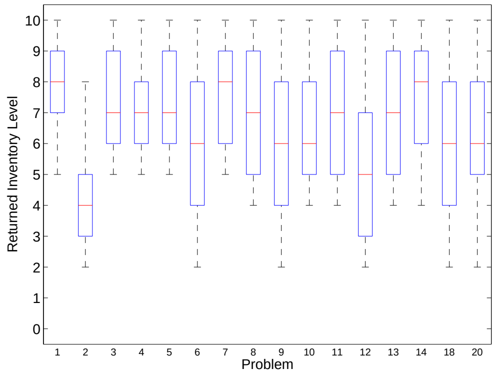


(f) Problem Set D– High quality recovery only

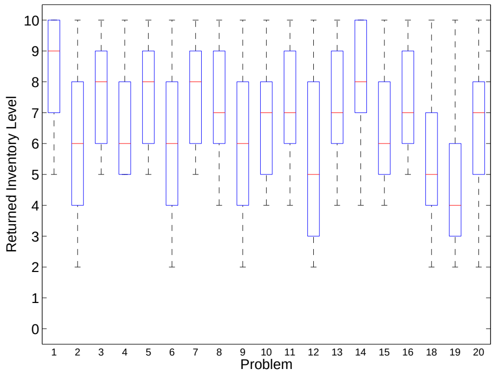
Figure 6.28: Graphs showing the level of recovered inventory (trigger level) when recovery is performed under a two-way substitution policy



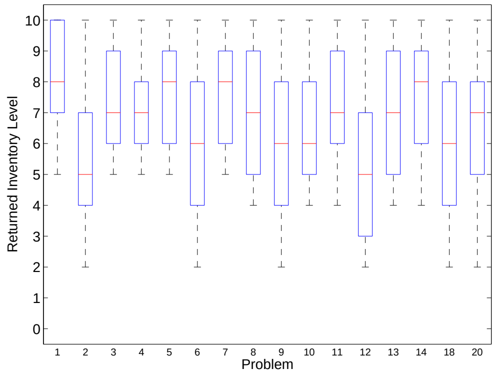
(a) Problem Set B – Both high and low quality recovery



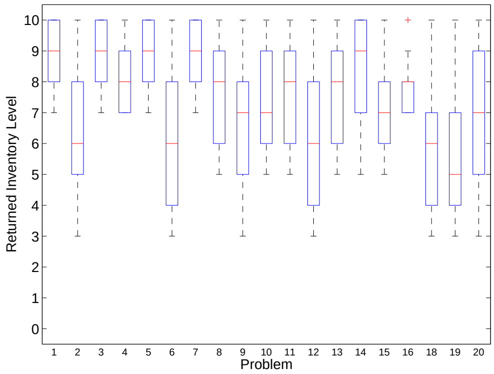
(b) Problem Set B – High quality recovery only



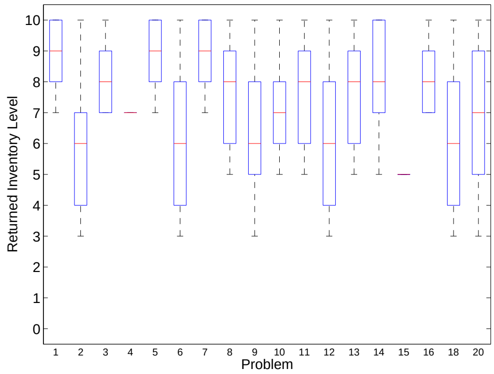
(c) Problem Set C– Both high and low quality recovery



(d) Problem Set C– High quality recovery only

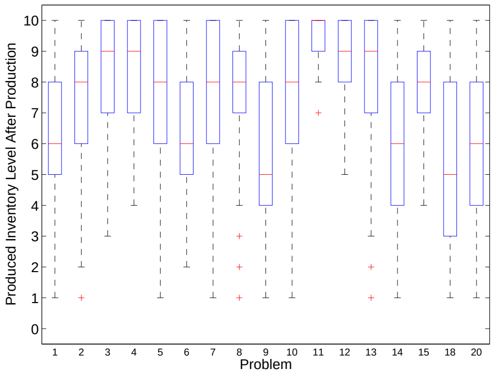


(e) Problem Set D– Both high and low quality recovery

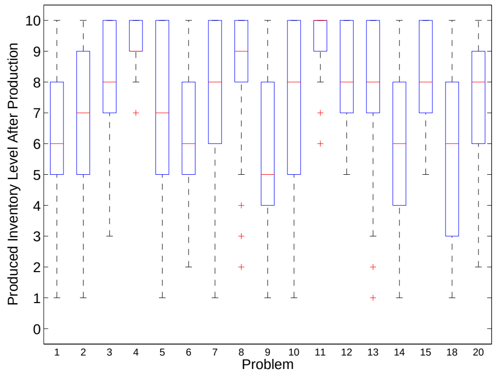


(f) Problem Set D– High quality recovery only

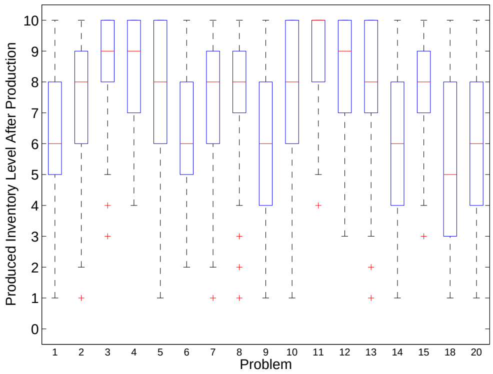
Figure 6.29: Graphs showing the level of returned inventory (trigger level) when recovery is performed under a two-way substitution policy



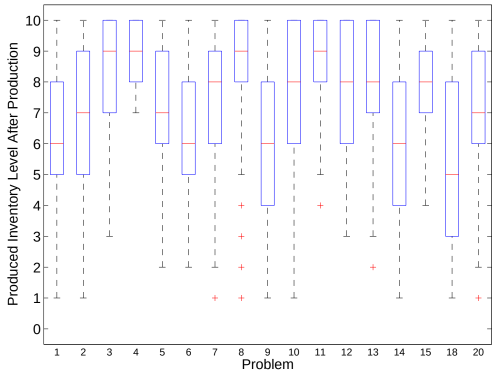
(a) Problem Set B – Both high and low quality recovery



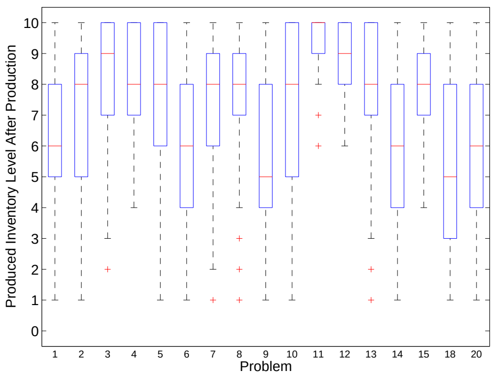
(b) Problem Set B – High quality recovery only



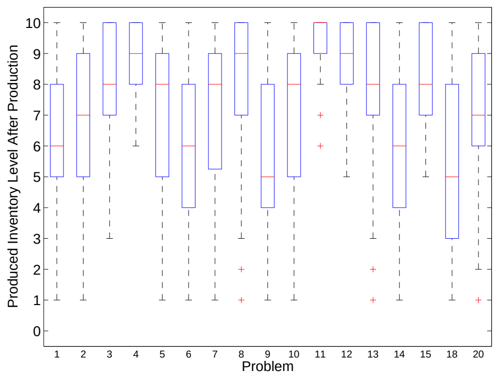
(c) Problem Set C– Both high and low quality recovery



(d) Problem Set C– High quality recovery only



(e) Problem Set D– Both high and low quality recovery



(f) Problem Set D– High quality recovery only

Figure 6.30: Graphs showing the levels of produced inventory after production has been completed $(i_1 + a_p)$ in which substitution can be offered, under a two-way substitution strategy

Figure 6.31 shows the recovered inventory levels when upward substitution is selected. There appears to be little variation between the two recovery strategies.

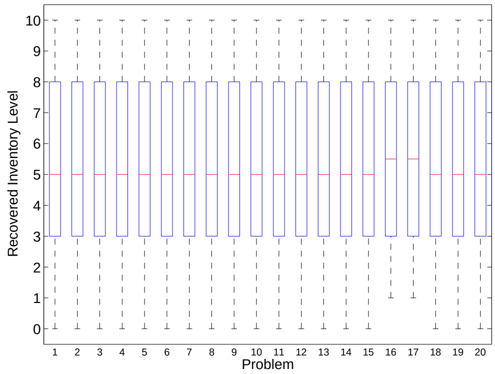
6.7 Discussion

This chapter has presented a discrete-time product recovery model with separate markets and substitution.

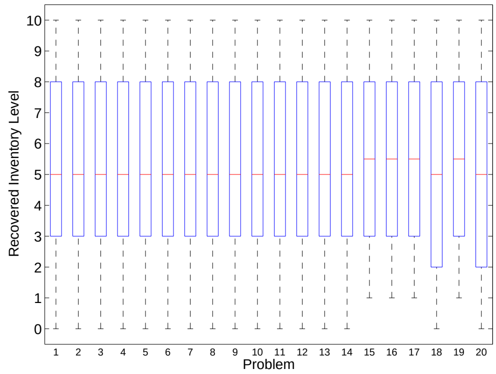
In this model, if there is a shortage of recovered goods, then the firm may offer a downward substitution. In downward substitution, the firm offers the customer a produced good for a reduced price and may also incur an opportunity cost associated with no longer being able to sell the produced good for the full price. However the firm does receive some revenue, rather than none at all, as would have been the case if the sale had been lost. If there is a shortage of produced goods, then the firm may offer an upward substitution. In the case of upward substitution, the firm offers the customer a recovered good and charges the price of the lower item, thereby missing out on the revenue it would have received had it had produced goods in stock and been able to meet the demand with produced goods. However, if the substitution is accepted, then the good that is does sell is sold for its full value. For both types of substitution, if a substitution is not offered (or offered and not accepted), then a lost sales cost is incurred, no revenue is received and holding costs are incurred. The firm must weigh up these costs and benefits when deciding to offer substitution or not.

The analyses in the section have shown that for some problems offering substitution can allow firms to increase their profits and increase the proportion of met sales (fill rates). However the magnitude of the increase depends heavily on the parameters associated with a given problem. Offering substitution has a greater effect on the fill rate of recovered inventory than that of produced inventory. This may be because offering downward substitution allows the firm to mitigate the risks associated with the uncertainty of the quality, quantity and timing of returns.

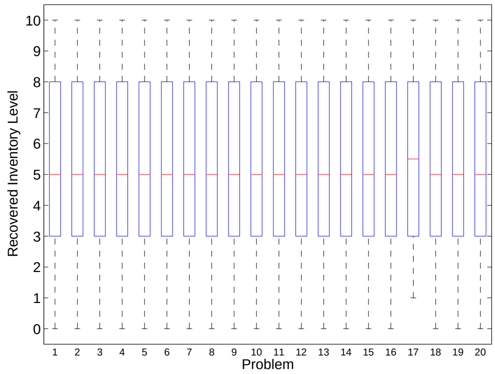
The frequency and occurrence of replenishment was also examined. It was found that in some cases performing substitution leads to a less frequent replenishment, but in other cases it leads to more frequent replenishment. This analysis highlights the



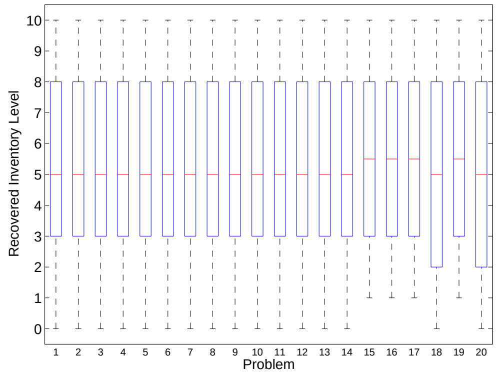
(a) Problem Set B – Both high and low quality recovery



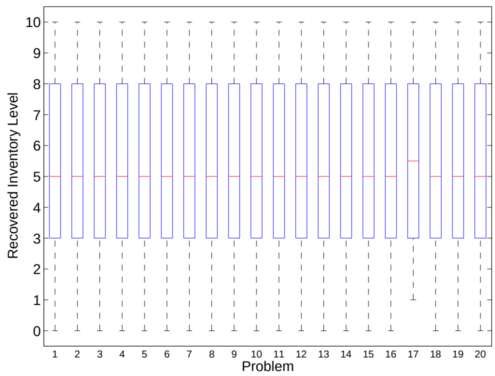
(b) Problem Set B – High quality recovery only



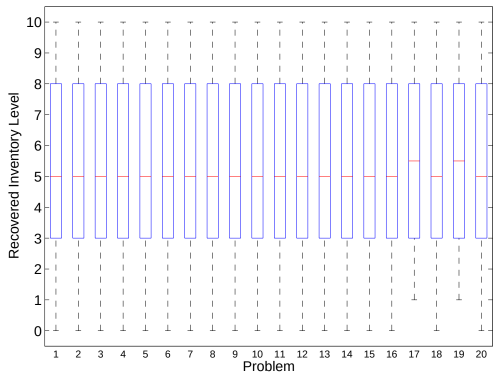
(c) Problem Set C– Both high and low quality recovery



(d) Problem Set C– High quality recovery only



(e) Problem Set D– Both high and low quality recovery



(f) Problem Set D– High quality recovery only

Figure 6.31: Graphs showing the levels of recovered inventory (i_2) in which substitution can be offered, under a two-way substitution strategy

complicated relationship between substitution and replenishment, and the fact that substitution can affect the frequency with which production and recovery should be performed. It is important for managers to be aware of the impact of this when designing a substitution and replenishment policy.

In reality, offering a substitution carries the risk associated with offering a good as a substitute and then not being to offer that good in the future when demand for it arrives. However, the nature of this model means that inventory replenishment happens at the beginning of each period, subject to the shared resource constraint. Thus, offering a substitution does not carry the same trade off as in other substitution problems where there may not be an opportunity to replenish the stock before the next demand occurs. However, there is still some uncertainty surrounding the size and timing of the next replenishment due to the shared resource constraint (only one of production or recovery can be performed each period) and due to the quality uncertainty. For the most part, the tradeoff to be considered when choosing to offer a substitution is the difference between no revenue, a holding cost, a lost sales cost, and some revenue, no holding cost, no lost sales cost, the lost opportunity to sell the good for a higher price at a later date.

The risk associated with downward and upward substitution differs. There is a lead time of zero in this model, which means that if required, the inventory of produced goods can always be replenished at the start of the next period. However, the inventory of recovered goods can only be replenished if there are sufficient returns in stock, and furthermore, the size of the replenishment is uncertain due to the quality of the returns. This means that offering upward substitution could be viewed as more risky than downward substitution. On the other hand, upward substitution sells a good for its actual price, whereas downward substitution sells a produced good for the price of a recovered one. However there may be risks associated with upward substitution that are not modelled here, but which may affect the substitution decision. For example if customers purchase these goods repeatedly, then by introducing a "produced" customer to a "recovered" good, the firm may encourage the "produced" customer to switch to the cheaper "recovered" good for future purchases. This would add an additional risk to upward substitution.

The effect of the recovery strategy was also examined. As expected the reward is at least as high when both high and low quality returns are recovered, compared with when only high quality returns are recovered. The fill rates for produced inventory and recovered inventory are similar for the two recovery strategies.

Under a high-quality-only recovery strategy there are more problems for which it is not optimal to ever perform recovery. This suggests that for these problems, it is the recovery of low quality items which makes recovery cost-effective. The recovery strategy has a greatest effect on the frequency of replenishment when there is a high proportion of low quality returns. This is not unexpected, since the effect of performing low quality recovery as well as high quality recovery is likely to have more impact when the number of low quality returns is greater. The recovery strategy does not have a noticeable impact on the frequency with which substitution is offered. There appears to be little difference between the 'trigger' levels for production under the two recovery strategies. However, in general the level of recovered inventory which triggers recovery is lower for the high quality recovery strategy.

A major limitation of the model presented in this section is that the dimensionality of the problem means that the state space used in the computational experiments is limited to a maximum of 10. While this still does allow us to investigate some properties of the model, the applicability of such properties for larger, more realistic problems is uncertain.

The problem of offering substitution between produced and recovered goods is similar to the transshipment problem in a multi-location inventory system and to a substitution problem in a multi-item inventory system. However, it differs from these types of problems because of the returns element and the uncertainty surrounding the yield of the recovery process. It is possible that heuristics from this field could be applied to this model, thereby allowing problems with a larger and more realistic state space to be studied.

Another imitation of the discrete time model presented here is that decisions about substitution are made in 'bulk' and after *all* demand for the period has been observed. While discrete modelling may be appropriate for production and recovery decisions, which may only happen periodically (e.g. daily), in reality the decisions regarding substitution are likely to happen continuously throughout a day, as and when demand arrives. Modelling the problem using a continuous time framework would allow a finer-grained treatment of substitution and would allow substitution to be addressed as and when each demand instance arrives. This limitation will be addressed in Chapter 7 by extending the model to the continuous time domain.

Chapter 7

Continuous-Time Stochastic Product Recovery Model with Separate Markets

7.1 Introduction

Consumers usually differentiate between newly produced and recovered used goods. Even when recovered goods are functionally “as good as new”, customers’ perceptions, and indeed in some countries, legislation prevent recovered goods being sold “as new”. However, when the functionality of the two types of goods remains the same or similar, some consumers may be willing to substitute one good for the other, if their preferred good is out of stock.

This chapter discusses a product recovery system in which newly produced goods and recovered goods are sold on separate markets, but can act substitutes for each other. It extends Chapter 5 by introducing separate markets and substitution and extends Chapter 6 by incorporating a continuous time element into the problem. The continuous time domain allows greater flexibility in modelling the substitution aspect of the problem, compared with the discrete time domain used in Chapter 6. A semi-Markov decision process (SMDP) is used to model this problem.

This chapter also extends the 'separate markets' literature (Inderfurth, 2004; Kaya, 2010) by allowing two-way substitution rather than one-way substitution, and by considering an infinite horizon model. A deterministic model with a similar modelling structure to the one considered in this chapter was studied by Piñeyro and Viera (2010), but to the best of our knowledge this type of model has not been studied in a stochastic environment.

This chapter is structured as follows. The problem description and modelling assumptions are presented in Sections 7.2 and 7.3 respectively. The formulation of the problem as an SMDP is presented in Section 7.4 and the implementation and validation of the model are discussed in Section 7.5. Properties of the optimal policy are explored in Section 7.6; in particular, the performance and structure of the optimal policy are analysed under different substitution and recovery strategies. In Section 7.7 the results, limitations and directions for future research are discussed.

7.2 Problem Description

In this chapter a new modelling approach is applied to the problem that was considered in Chapter 6. Since the underlying problem in this chapter and Chapter 6 is the same, the problem description is also the same.

Suppose there is a firm which has a primary function of producing new goods. The firm accepts these goods back after they have been used and, if they are of sufficient quality, recovers them and sells them to a secondary market. For returns which are below the quality threshold for recovery, the firm has a choice: to dispose of them or to use them as components in the production of newly produced items. If insufficient components are obtained from the recovery of low quality returns then additional components are bought. Produced and recovered items are viewed by the consumer as different so are sold on separate markets, however they are functionally similar so can act as substitutes. The firm may choose to offer substitution between these two types of goods if one of them sells out.

The firm is a profit-maximising firm which receives revenue for the sale of produced and recovered goods. Costs are incurred for holding inventory and for lost sales. Fixed and unit costs are incurred for production, recovery and buying components. Demand for produced items, demand for recovered items, returns and the quality of returns are uncertain.

The firm must determine a production plan that specifies how much and how often to produce, recover and buy. It must also determine a substitution policy which specifies if and when substitution will be offered to customers. If substitution is offered to and then accepted by a customer, then they will pay the price of the cheaper recovered item, regardless of the good they receive. Offering a substitution means that the firm will not incur a lost sales cost for that item, however there may be indirect costs associated with performing a substitution. For example if a produced good is offered in place of a recovered good, then the firm will miss out on the revenue they would have received had they sold it for the full price. The firm may also incur an administration charge for offering a substitute rather than the product originally demanded, or a cost for "lost goodwill" for not being able to supply what the customer demanded.

As discussed in Chapter 6, this system could describe the refurbishment of electronic equipment, such as copiers (Canon, 2012e), computers (Apple, 2011; Dell, 2012) and satellite navigation systems (TomTom, 2011). Wooden pallets (PalletWorld, 2012) are another example of new and used products being sold side-by-side. In all of these cases, used products are brought up to a "suitable" standard before being resold or leased. They have the same functionality as a newly produced item, but are not recovered up to an "as new" standard so cannot be sold as such. Because the newly produced and recovered goods are functionally similar, they could act as substitutes. Substitution policies vary from industry to industry, and from company to company, and may also vary at different times of the year (e.g. peak seasons). However in all cases, the company could choose to offer these substitutions, if they were faced with a stock out of the customer's preferred good.

7.3 Model Description and Assumptions

Figure 7.1 presents the product recovery system being modelled in this chapter. As shown in this diagram, there are two consumer markets: the first demands newly produced goods and the second demands recovered goods. Substitution between the two markets is represented on the diagram by the dashed line. The model and the modelling assumptions will be discussed in this section.

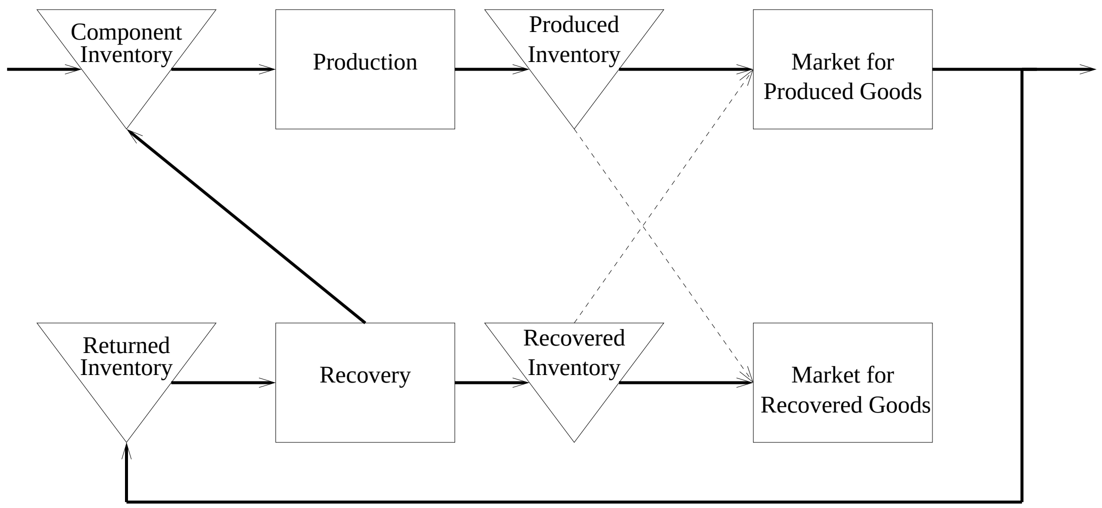


Figure 7.1: Product Recovery Model with Separate Markets and Substitution

Inventory Levels. There are four types of inventory in this model: produced goods, recovered goods, returns and components. It is assumed that there is a finite capacity available for storing each of the inventories. If this capacity is reached then a disposal cost may be incurred for each item which is in excess of the inventory capacity. If there are insufficient items in stock to meet demand, an alternative good may be offered as a substitute, otherwise the sale will be lost. It is assumed that there is no backordering and that the inventory levels must always be nonnegative.

Continuous Decision Making. This problem is modelled in continuous time and inventory levels are reviewed continually. Decisions regarding inventory replenishment and substitution are made after the following situations: demand occurs, returns arrive, or a replenishment order is completed. The points in time when decisions are made are called "decision epochs". At each decision epoch the current inventory levels and status of outstanding orders are examined, and two types of decisions are made: replenishment (production and recovery) and substitution. The time between decision epochs varies and will depend on the inventory levels, order status and the decision that is made.

Production, Recovery and Buying. At each decision epoch a decision is made about whether to replenish the produced inventory or the recovered inventory. It is assumed that production and recovery require a shared resource at the beginning of the process, therefore cannot both be selected at the same decision epoch. The amount of time required by this shared resource is minimal, so does not restrict the selection of a replenishment action at the next decision epoch. Replenishment orders which have been placed, but have not yet arrived are said to be "outstanding". Only one of each type of order may be outstanding at any time. Production and recovery have stochastic lead times, governed by separate stochastic processes with known rates. It is assumed that the inventory levels are updated to reflect the increase in produced or recovered inventory at the end of the process, e.g. produced goods "enter" the produced inventory when the entire production order has been completed.

The size of each production and recovery lot is non-negative and constrained by a finite upper limit. The yield from a production order is certain, but the yield from a recovery order, on the other hand, depends on the quality of returns used in recovery. With sufficient effort and expenditure, all returns could be recovered to have the functionality of newly produced goods. It is assumed, however that there is a quality threshold that determines which returns are worthwhile recovering up to this functional standard. For the remainder of this chapter, returns which are above this quality threshold are referred to as 'high quality returns'. High quality returns undergo 'high quality recovery', which brings them up to the same functional standard as newly produced goods, and are then sold on a secondary market. Returns which fall below the quality threshold are termed 'low quality returns'. It is assumed that the firm makes a strategic-level decision (outwith the model) about whether to recover low quality returns or to dispose of them. If low quality returns are to be recovered, then they undergo 'low quality recovery' and are used as components. The quality of the returns is determined during recovery and is modelled by a known probability distribution. The firm can set the quality threshold by specifying the relevant parameters of the quality distribution.

It is assumed that replenishment decisions will not cause the upper inventory level capacities to be exceeded with certainty. For instance, the amount produced must be less than or equal to the available capacity in produced inventory. However, since the quality returns is uncertain the size of the recovery order *can* exceed the available capacity of recovered inventory.

Components are required for production. They may be bought from an external supplier or sourced from low quality recovery. It is assumed that components are purchased as they are needed and arrive immediately, therefore the lead time for buying components is zero. The number of components that will need to be bought for a particular production order is uncertain as the components inventory may be replenished through low quality recovery while the production order is in progress. Therefore, the number of components that are bought for a production order is calculated at the end of the production process. Effectively this means that if components are required for production, then they are "bought" at the end of the production process. This is in line with the assumption above, which states that inventory levels are updated at the end of the process.

The model could be easily adapted for alternative modelling assumptions. For example, components could be bought at the beginning of the production process and then held in stock until the completion of a production lot. However, this approach may result in an over-ordering of components. Another alternative approach would be to allow production to occur, only if there were sufficient components in stock at the beginning of the order, however this may be overly strict and discourage production.

Probabilistic Demand, Returns and Substitution. The arrival of demand for produced goods, demand for recovered goods and returns are assumed to be uncertain and are modelled by stochastic processes with a known intensities. It is assumed that one customer arrives at a time, and each customer demands only one item, whether it be a produced or recovered one. It is assumed that the returns are collected from multiple locations before being returned to the firm in a batch, therefore the arrival of returns is unlikely to be related to the demand for a produced item. For this reason it is assumed that the stochastic processes governing returns and demands are independent of each other and that they are time-homogeneous.

The size of each batch of returns is uncertain, however, it is assumed that the firm has information about the availability returns which allows them to know the distribution of the batch sizes. Therefore the batch size of returns is modelled by a random variable with a known distribution. There is a high degree of uncertainty surrounding the return of used goods, and in some cases there may be a problem with the delivery of an entire batch of returns. To model these cases could be a non-zero probability that the batch size is 0.

Decisions about substitution are made at both strategic and operational levels. At a strategic level the firm determines whether or not substitution will be permitted ever, and at an operational level the firm decided whether or not to offer substitution in the current time period. If substitution is permitted at a strategic level, and if there are insufficient items in stock to meet demand, then a substitution may be offered to the customer. Upward substitution is offered when there is a shortage of produced goods and results in the customer being offered a functionally similar recovered good. Downward substitution is offered when there is a shortage of recovered serviceable goods and results in the customer being offered a produced good. Customers who are offered a substitution may choose to accept or reject it. The probability that a customer accepts the substitution is governed by random variable with a known distribution and depends on the type of substitution (upward or downward) offered.

Costs and Revenues. It is assumed that the objective of the firm is to maximise its long run average reward, where the reward is the revenue less the costs.

The following costs are incurred: holding costs, setup costs, processing costs, lost sales costs, disposal costs. It is assumed that the cost function is a linear function in the inventory levels. Inventory that is in stock between decision epochs incurs a per unit, per time unit holding cost. Setup costs are incurred each time a production or recovery order is placed. This is a fixed cost and does not depend on the size of the order. Processing costs are incurred for production and recovery, on a per unit basis. If demand exceeds the available inventory and a substitution is not offered, or offered and not accepted, then a lost sales cost is incurred. A disposal cost is incurred for any returns which do not fit into the returned inventory. This disposal cost may be negative representing a salvage value, or positive representing a cost.

Revenues are received for the sale of produced and recovered goods. It is assumed that the sale price of a recovered good is less than that of a produced good. If a substitution is offered and accepted then revenue equal to the price of the recovered good is received.

The costs incurred and revenues received by the firm between the current and next decision epoch depend three factors: the current inventory levels, the time to the next decision epoch, and the reason for the next decision epoch.

7.4 Semi-Markov Decision Process Formulation

The problem studied in this chapter is a continuous time decision problem and can be modelled by a semi-Markov decision process (SMDP). The structure of a semi-Markov decision process is similar to that of a Markov decision process (MDP), as discussed in Section 3.2.4. A semi-Markov decision process is characterised by its decision epochs, states, actions, costs and transition probabilities. These elements will be described in the following sections. The notation used in the coming sections is summarised in Table 7.1 (page 277).

7.4.1 Decision Epochs

A decision epoch is a point in time when a decision is made. In this model decision epochs can be triggered by one of five possible events: arrival of a production order, arrival of a recovery order, demand for produced goods, demand for recovered goods, or arrival of returns batch. Once the state has been updated to reflect this event, the state of the system is examined and a decision is made about what action to perform. The state of the system does not change between decision epochs.

7.4.2 States

The state of the system is characterised by six state variables: returned inventory level i_r, produced inventory level i_1, recovered inventory level i_2, component inventory i_c, outstanding production order indicator i_{op}, and outstanding recovery order indicator i_{or}. The inventory state variables are defined as follows:

$$
\begin{aligned}
i_r &= \text{ level of returned inventory,} && i_r \in I_r = \{0, 1, \ldots, W_r\} \\
i_1 &= \text{ level of produced inventory,} && i_1 \in I_1 = \{0, 1, \ldots, W_1\} \\
i_2 &= \text{ level of recovered inventory,} && i_2 \in I_2 = \{0, 1, \ldots, W_2\} \\
i_c &= \text{ level of component inventory,} && i_c \in I_c = \{0, 1, \ldots, W_c\}
\end{aligned}
$$

where W_r, W_1, W_2, W_c are finite upper limits of inventory capacity and I_r, I_1, I_2, I_c are the sets of all possible inventory levels.

The state of the system also indicates whether or not there are any outstanding production or recovery orders. The outstanding order variables are equal to zero if no order has been placed, and are equal to the size of the outstanding order if an order has been placed and the firm is still awaiting its arrival. The maximum order sizes for production and recovery are denoted by U_p and U_r respectively. The outstanding order state variables are defined as follows:

$$
\begin{aligned}
i_{op} &\in I_{op} \subseteq \{0, \ldots, U_p\} \\
i_{or} &\in I_{or} \subseteq \{0, \ldots, U_r\}
\end{aligned}
$$

Let i denote the state of the system at a given point in time. The state space can then be defined as:

$$i \in I = \{(i_r, i_1, i_2, i_c, i_{op}, i_{or}) : \, i_1 \in I_1, \, i_2 \in I_2, \, i_r \in I_r, \, i_c \in I_c, \, i_{op} \in I_{op}, \, i_{or} \in I_{or}\}$$

All state variables have finite upper and lower limits of capacity, therefore the SMDP has a finite state space.

7.4.3 Actions

At each decision epoch the firm must make decisions regarding replenishment (how much to produce or recover), and substitution (what substitution policy should be used until the next decision epoch).

Replenishment Actions. The firm makes replenishment decisions at a strategic and at an operational level. At a strategic level the firm must decide whether to recover low quality returns or to dispose of them. Let ζ_L denote the firms low-quality recovery strategy, such that:

$$\zeta_L = \begin{cases} 1 & \text{if low quality recovery is performed} \\ 0 & \text{otherwise} \end{cases}$$

This decision is made outwith the model. It is assumed that high quality recovery will always be performed, therefore $\zeta_H = 1$.

At an operational level the firm must decide whether to produce, recover or neither for a given state. Production can only be performed if there is not already an outstanding production order. Similarly, recovery can only be performed if there is not already an outstanding recovery order. Furthermore, at each decision epoch the firm must choose to either produce or recover. As discussed in Section 7.3, it is assumed that a shared resource is required during the setup phase of both replenishment actions, meaning that production and recovery cannot both be selected at the same decision epoch. This setup phase is not long enough to prevent the other type of order from being placed at the next decision epoch.

The size of the production and recovery orders are referred to as the action quantities for production and recovery, and are denoted by a_p and a_r respectively. These quantities are constrained by nonnegative finite lower and upper limits. The lower limits are denoted by L_p, L_r and the upper limits are denoted by U_p, U_r, for production and recovery respectively. Using these upper and lower limits a set of allowable actions can be defined for each type of action, for a given state i. These sets are denoted by $A_p(i)$ for production and $A_r(i)$ for recovery, where:

$$a_p \in A_p(i) \subseteq \begin{cases} \{0, L_p, \ldots, \min\{W_1 - i_1, U_p\}\} & \text{if } i_{op} = 0 \text{ and } a_r = 0 \\ \{0\} & \text{otherwise} \end{cases}$$

$$a_r \in A_r(i) \subseteq \begin{cases} \{0, L_r, \ldots, \min\{i_r, U_r\}\} & \text{if } i_{or} = 0 \text{ and } a_p = 0 \\ \{0\} & \text{otherwise} \end{cases}$$

The upper limit on the production set $A_p(i)$ is required to ensure that the produced inventory capacity W_1 and that the upper limit on production U_p will not be exceeded. The upper limit on the recovery set $A_r(i)$ is required to ensure that the amount recovered does not exceed the number of returns in stock i_r, or the upper limit on recovery U_r. Note that 0 is always part of these sets as the decision may be to not perform the replenishment action.

Substitution Actions. Decisions regarding substitution are made both at strategic and operational levels, as discussed above. The firm makes a strategic decision outwith the model about whether or not they will offer substitution ever. The variables χ_1 and χ_2 are used to represent the firm's strategic decisions about offering upward and downward substitution respectively. If the firm allows upward substitution, i.e. substitution due to a shortage of produced goods, then $\chi_1 = 1$. If upward substitution is not allowed then $\chi_1 = 0$. Similarly, if the firm allows downward substitution, i.e. substitution due to a shortage of recovered goods, then $\chi_2 = 1$, otherwise $\chi_2 = 0$.

In addition to the strategic level decisions regarding substitution, the firm makes an operational decision about whether or not to offer substitution in a particular situation. This operational substitution decision depends on the current inventory levels. The customers, who each demand one item, can choose to accept or reject a substitution if it is offered to them.

Let a_1 denote the upward substitution action and a_2 denote the downward substitution action chosen at a particular decision epoch, where

$$a_k = \begin{cases} 1 & \text{if substitution is offered} \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } k = 1, 2$$

Since it is assumed that each customer demands only one item, the variables a_1, a_2 also denote the number of goods offered for substitution (either 1 or 0). The sets of allowable substitution actions depend on the strategic variables (χ_1, χ_2), and the current inventory levels. For any state i, let $A_1(i)$ denote the set of allowable upward substitution actions and let $A_2(i)$ denote the set of allowable downward substitution actions. For a given state i,

$$a_1 \in A_1(i) = \begin{cases} \{0, \chi_1\} & \text{if } i_1 = 0 \text{ and } i_2 > 0 \\ \{0\} & \text{otherwise} \end{cases}$$

$$a_2 \in A_2(i) = \begin{cases} \{0, \chi_2\} & \text{if } i_2 = 0 \text{ and } i_1 > 0 \\ \{0\} & \text{otherwise} \end{cases}$$

The set of allowable actions for a given state i is denoted by $A(i)$ and can be defined as:

$$a \in A(i) = \{(a_p, a_r, a_1, a_2) : a_p \in A_p(i),\ a_r \in A_r(i),\ a_1 \in A_1(i),\ a_2 \in A_2(i)\}.$$

7.4.4 Transition Probabilities

If the system is in state i and action a has been chosen, then the probability that the system will be in state j at the next decision epoch is denoted by the transition probability $p_{i,j}(a)$. There are five events which could trigger the occurrence of a decision epoch: arrival of returns, demand for produced goods, demand for recovered goods, arrival of a production order and arrival of a recovery order. It is assumed that all five events have exponential inter-arrival times. The arrival of the events are modelled by independent Poisson processes with rates λ_r, λ_{d1}, λ_{d2}, λ_p and λ_{rec} respectively. Since returns arrive in batches, a compound Poisson process is used to model the arrival of returns. The size of the batch is governed by the random variable X_r with a known distribution and mean μ_r. It is assumed that the arrival of a batch of returns and the size of the batch are independent.

The next event may be the arrival of a production order, if there is currently an outstanding production order ($i_{op} > 0$) or if the action chosen is to produce ($a_p > 0$). If, on the other hand, there is not an outstanding production order ($i_{op} = 0$) and production is not chosen ($a_p = 0$), then the probability that the next event is the arrival of a production order is zero. Similar statements can be made regarding the arrival of a recovery order. Using this logic, the following function is defined:

$$\lambda(i_{op}, i_{or}, a) = \lambda_r + \lambda_{d1} + \lambda_{d2} + \min\{\max\{i_{op}, a_p\}, 1\}\lambda_p + \min\{\max\{i_{or}, a_r\}, 1\}\lambda_{rec} \quad (7.1)$$

Using this function, the expected time until the next decision epoch, denoted by $\tau_i(a)$, can be defined as:

$$\tau_i(a) = \frac{1}{\lambda(i_{op}, i_{or}, a)} \tag{7.2}$$

The transition probabilities described in this section are summarised in Table 7.2. In this section the convention of describing an instance of a random variable X by the lower case equivalent x is used. The transitions associated with each the five events are now described.

Returns. If the next event is the arrival of a batch of returns of size x_r, then the returned inventory will be updated to reflect the size of the batch, therefore the next state will be:

$$j = (\min\{i_r + r, W_r\},\, i_1,\, i_2,\, i_c,\, \max\{i_{op}, a_p\},\, \max\{i_{or}, a_r\})\,.$$

The arrival of the returns is governed by a compound Poisson process, therefore the probability of the next event being the arrival of returns is given by the exponential probability $\frac{\lambda_r}{\lambda(i_{op}, i_{or}, a)}$. The probability of the batch being of size x_r is given by the probability $P(X_r = x_r)$. Since the arrival of the event and the size of the batch are assumed to be independent, the probability of moving from state i to state j with action a is:

$$\frac{\lambda_r P(X_r = x_r)}{\lambda(i_{op}, i_{or}, a)}$$

Demand for Produced Inventory. Customers arrive to demand a produced good according to a Poisson process with rate λ_{d1}, therefore the probability that the next event is a demand for a produced good is given by $\frac{\lambda_{d1}}{\lambda(i_{op}, i_{or}, a)}$. After the occurrence of a demand for produced goods, the next state depends on the current state, the replenishment actions and the upward substitution action a_1. Two cases need to be considered: upward substitution is not offered or not required ($a_1 = 0$) and upward substitution is required and is offered ($a_1 = 1$).

1. If $a_1 = 0$, then either there is sufficient stock to meet demand or upward substitution is not offered. The next state will be:

$$j = (i_r,\, \max\{0, i_1 - 1\},\, i_2,\, i_c,\, \max\{i_{op}, a_p\},\, \max\{i_{or}, a_r\})$$

2. If $a_1 = 1$, then there is insufficient stock to meet demand and upward substitution is offered. The next state depends on whether or not the consumer accepts the upward substitution. Customers will accept an upward substitution with a fixed probability of α_1. Let the random variable Y_1 equal 1 if upward substitution is accepted and 0 otherwise, where $P(Y_1 = 1) = \alpha_1$.

   (a) If the consumer rejects the upward substitution then the next state is:

   $$j = (i_r,\, 0,\, i_2,\, i_c,\, \max\{i_{op}, a_p\},\, \max\{i_{or}, a_r\})$$

   and this occurs with probability:

   $$\frac{(1-\alpha_1)\,\lambda_{d1}}{\lambda(i_{op},\, i_{or},\, a)}.$$

   (b) If the consumer accepts the upward substitution then the next state is:

   $$j = (i_r,\, 0,\, i_2 - 1,\, i_c,\, \max\{i_{op}, a_p\},\, \max\{i_{or}, a_r\})$$

   and this occurs with probability:

   $$\frac{\alpha_1\,\lambda_{d1}}{\lambda(i_{op},\, i_{or},\, a)}.$$

Demand for Recovered Inventory. Customers arrive to demand a recovered good according to a Poisson process with rate λ_{d2}, therefore the probability that the next event is a demand is $\frac{\lambda_{d2}}{\lambda(i_{op}, i_{op}, a)}$. After the occurrence of a demand for recovered goods, the next state depends on the current state, the replenishment decision and the downward substitution action a_2. Two cases are considered: downward substitution is not offered or not required ($a_2 = 0$) and downward substitution is required and offered ($a_2 = 1$).

1. If $a_2 = 0$, then either there is sufficient stock to meet demand, or a downward substitution is not offered. The next state will be:

   $$j = (i_r, i_1, \max\{0, i_2 - 1\}, i_c, \max\{i_{op}, a_p\}, \max\{i_{or}, a_r\})$$

2. If $a_2 = 1$, then there is insufficient stock to meet demand and downward substitution is offered. The next state depends on whether or not the consumer accepts the offered downward substitution. Customers will accept a downward substitution with a fixed probability of α_2. Let the random variable Y_2 equal 1 if downward substitution is accepted and 0 otherwise.

(a) If the consumer rejects the downward substitution then the next state is:

$$j = (i_r,\, i_1,\, 0,\, i_c,\, \max\{i_{op}, a_p\},\, \max\{i_{or}, a_r\})$$

and this occurs with probability

$$\frac{(1-\alpha_2)\lambda_{d2}}{\lambda(i_{op}, i_{or}, a)}$$

(b) If the consumer accepts the downward substitution then the next state is:

$$j = (i_r,\, i_1 - 1,\, 0,\, i_c,\, \max\{i_{op}, a_p\},\, \max\{i_{or}, a_r\})$$

and this occurs with probability

$$\frac{\alpha_2\lambda_{d2}}{\lambda(i_{op}, i_{or}, a)}$$

Production. If there is an outstanding production order ($i_{op} > 0$) or production is selected at the current state ($a_p > 0$), then the next event may be the arrival of a production order with probability $\frac{\lambda_p}{\lambda(i_{op}, i_{or}, a)}$. The size of the order will be $\max\{i_{op}, a_p\}$. As mentioned in Section 7.3, the number of components required is for a production order is calculated when that order arrives. If a production order arrives, the number of components that will be bought is:

$$a_b = \max\{0, \max\{i_{op}, a_p\} - i_c\}$$

and the components inventory will be increased by the size of this order and decreased by the size of the production lot, i.e. $i_c + a_b - \max\{i_{op}, a_p\}$. If the next event is the arrival of a production order then the next state will be:

$$j = (i_r,\, i_1 + \max\{i_{op},\, a_p\},\, i_2,\, i_c + a_b - \max\{i_{op}, a_p\},\, 0,\, \max\{i_{or}, a_r\})$$

Recovery. If there is an outstanding recovery order ($i_{or} = 1$) or the action chosen at the current state is recovery ($a_r > 0$), then the next event may be the arrival of a recovery order, with probability $\frac{\lambda_{rec}}{\lambda(i_{op}, i_{or}, a)}$. If the next event is the arrival of a recovery order then the next state will depend on the quality of the returns in the recovery order. The number of high quality returns is modelled by a Binomial random variable X_q, where $X_q \sim \text{Bin}(\alpha, a_r)$. Recall that ζ_L represents the low quality recovery strategy.

If low quality recovery is permitted then $\zeta_L = 1$, and if it is not permitted then $\zeta_L = 0$. It is assumed that high quality recovery is always performed therefore ζ_H is not included in this formulation. Suppose there are x_q high quality returns in the recovery order then:

$$
\begin{aligned}
a_h(x_q) &= \max\{W_2 - i_2, x_q\} && \text{amount of high quality recovery} \\
a_l(x_q) &= \zeta_L \min\{W_c - i_c, a_r - a_h(x_q)\} && \text{amount of low quality recovery} \\
a_d(x_q) &= a_r - a_h(x_q) - a_l(x_q) && \text{amount of disposal}
\end{aligned}
$$

If the next event is an arrival of a recovery order then the next state will be:

$$j = (i_r, i_1, i_2 + a_h(x_q), i_c + a_l(x_q), \max\{i_{op}, \min\{1, a_p\}\}, 0)$$

The probability of this occurring is:

$$\frac{\lambda_{rec}}{\lambda(i_{op}, i_{or}, a)} \binom{a_r}{x_q} \alpha^{x_q}(1-\alpha)^{a_r - x_q}$$

These transitions are summarised in Table 7.2.

7.4.5 Costs and Revenues

The objective of this semi-Markov decision process is to maximise the long run average reward. The 'reward' is equal to the revenues less the costs. In this model revenues are received for the sale of produced and recovered goods, and the following costs are incurred: holding costs, setup costs, processing costs, lost sales costs, substitution costs, disposal costs. The reward evaluated at each decision epoch is the expected reward that will be received until the next decision epoch.

Costs and revenues could be related to a specific *state and action* or to an *event*. The holding and setup costs are incurred at every decision epoch and depend on the current state and the action chosen. On the other hand, the revenues and the processing, lost sales, substitution and disposal costs depend on the type of event that occurs at a particular decision epoch, as well as the current state and the action chosen. Therefore, in what follows, the costs are classified as being related to a specific *state and action* or to a specific *event*.

State and Action Costs

The following costs are associated with a specific state and action.

Holding Costs. If the system is state i, then the cost of holding stock per time unit is $h_r i_r + h_1 i_1 + h_2 i_2 + h_c i_c$. The expected time to until the next decision is $\tau_i(a) = \frac{1}{\lambda(i_{op}, i_{or}, a)}$ time units, therefore the expected cost of holding inventory until the next decision epoch is:

$$E[R_h(i,a)] = -(h_r i_r + h_1 i_1 + h_2 i_2 + h_c i_c)\tau_i(a)$$

Setup costs. Setup costs are incurred each time production or recovery occurs. These costs are denoted by k_p and k_r, respectively. Components are bought on an "as-needed" basis, therefore there is no setup cost for ordering components. The following indicator variables δ_p, δ_r are defined in order to specify when a particular action is performed and hence when a setup cost should be incurred:

$$\delta_p = \begin{cases} 1 & \text{if } a_p > 0 \quad \text{(production)} \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_r = \begin{cases} 1 & \text{if } a_r > 0 \quad \text{(recovery)} \\ 0 & \text{otherwise} \end{cases}$$

The setup costs incurred are $k_p\delta_p + k_r\delta_r$. The expected setup costs incurred until the next decision epoch is:

$$E[R_k(i,a)] = -(k_p\delta_p + k_r\delta_r) \tag{7.3}$$

Event Costs

The following costs and revenues are associated with the occurrence of a particular event.

Returns. If the next event is the arrival of a batch of returns, then returns will be added to the returns inventory. A cost of c_a is incurred for acquiring the returns and a cost of l_r is incurred for disposing of any returns which do not fit within the returned inventory. If the system is in state i and x_r returns are received, then the cost incurred is $l_r \max\{0, (x_r - W_r + i_r)\} + c_a x_r$, therefore the expected cost incurred until the next decision epoch is:

$$E[R_r(i, X_r)] = -l_r \max\{0, (E[X_r] - W_r + i_r)\} + c_a E[X_r] \tag{7.4}$$

Demand for Produced Inventory. If the next event is the arrival of demand for a produced good, the revenue received and costs incurred depend on the current state and whether or not demand is met by produced inventory or by an upward substitution, or is lost. Using the same structure as for the transition probabilities, two main cases are considered: upward substitution is not required or not offered ($a_1 = 0$), and upward substitution is required and offered ($a_1 = 1$).

1. If $a_1 = 0$, then either (a) there is sufficient stock to meet demand or (b) upward substitution is not offered. Under this case there are two subcases which could occur, depending on the inventory level i_1.

    (a) There is sufficient produced inventory in stock ($i_1 > 0$) and a revenue of p_1 is received.

    (b) There is insufficient produced inventory in stock ($i_1 = 0$) and upward substitution is not offered ($a_1 = 0$), meaning that the sale is lost and a lost sales cost of l_1 is incurred.

2. If $a_1 = 1$ then there is insufficient stock to meet demand and upward substitution is offered. Under this case there are two sub-cases which could occur depending on whether the substitution is (a) accepted or (b) not accepted.

    (a) The consumer rejects the upward substitution ($y_1 = 0$) and a lost sales cost of l_1 is incurred.

    (b) The consumer accepts the upward substitution ($y_1 = 1$) and a revenue of a recovered good p_2 is received.

Indicator functions can be defined to specify when these cases, and thus when each of these costs and revenues should be included in the reward function. The derivation of these functions is provided in Appendix D.1.1. Combining these indicator functions gives the following expression for the expected reward received in the event of demand for a produced good:

$$E[R_{d1}(i,a,Y_1)] = p_1 \min\{i_1,1\} + p_2 \min\{E[Y_1],a_1\} - l_1(\max\{0,1-i_1\} - \min\{E[Y_1],a_1\}) \tag{7.5}$$

Demand for Recovered Inventory. If the next event is the arrival of demand for a recovered good, then the revenue received or cost incurred depends on the current state and whether or not demand is met by recovered inventory or a downward substitution, or is lost. Using the same structure as for the transition probabilities, two main cases are considered: downward substitution is not required or not offered ($a_2 = 0$) and downward substitution is required and offered ($a_2 = 1$).

1. If $a_2 = 0$, then either (a) there is sufficient stock to meet demand or (b) downward substitution is not offered, thus there are two sub-cases which could occur, depending on the recovered inventory level i_2.

   (a) There is sufficient recovered inventory in stock ($i_2 > 0$) and a revenue of p_2 is received.

   (b) There is insufficient recovered inventory in stock ($i_2 = 0$) and downward substitution is not offered ($a_2 = 0$), meaning that the sale is lost and a lost sales cost of l_2 is incurred.

2. If $a_2 = 1$ then there is insufficient stock to meet demand and downward substitution is offered, thus there are two sub-cases which could occur which depend on whether substitution is (a) accepted or (b) not accepted.

   (a) The consumer rejects the downward substitution ($y_2 = 0$) and a lost sales cost of l_2 is incurred.

   (b) The consumer accepts the downward substitution ($y_2 = 1$) and a revenue of a recovered good p_2 is received. (The customer is charged for the cheaper of the two goods, i.e. the recovered good).

Indicator functions can be defined which specify when each of these costs and revenues should be included in the reward function. The derivation of these functions is provided in Appendix D.1.1. Combining these indicator functions gives the following expression for the reward received in the event of demand for a recovered good:

$$E[R_{d2}(i,a,Y_2)] = p_2 \min\{i_2,1\} + p_2 \min\{E[Y_2],a_2\} - l_2(\max\{0,1-i_2\} - \min\{E[Y_2],a_2\}) \tag{7.6}$$

Production. If the next event is the arrival of a production order then costs are incurred for performing production and for purchasing components. The size of a production order is $\max\{i_{op},a_p\}$, therefore the number of components that need to be bought is: $a_b = \max\{0, \max\{i_{op},a_p\} - i_c\}$. Costs are incurred on a per unit basis for each good that is produced and bought. If the next event is the arrival of a production, then the cost incurred is: $\max\{i_{op},a_p\}c_p + a_b c_b$. and therefore the expected reward is:

$$E[R_p(i,a)] = -(\max\{i_{op},a_p\}c_p + a_b c_b) \tag{7.7}$$

Recovery. If the next event is the arrival of a recovery order, then costs are incurred for the processing of the recovery lot, the high quality recovery, the low quality recovery, and the disposal of items which are not recovered. If there are x_q high quality returns then the cost incurred is: $Q_r c_r + a_h(x_q)c_h + a_l(x_q)c_l + a_d(x_q)c_d$. Therefore the expected reward is:

$$E[R_{rec}(i,a,X_q)] = -\left(Q_r c_r + E[a_h(X_q)]c_h + E[a_l(X_q)]c_l + E[a_d(X_q)]c_d\right) \tag{7.8}$$

Total rewards

The expected total reward, for a given state i and action a, is:

$$\begin{aligned} E[R(i,a)] = R_h(i,a) + R_k(i,a) + \frac{\lambda_r}{\lambda(i_{op},i_{op},a)}E[R_r(i,X_r)] + \\ \frac{\lambda_{d1}}{\lambda(i_{op},i_{op},a)}E[R_{d1}(i,a,Y_1)] + \frac{\lambda_{d2}}{\lambda(i_{op},i_{op},a)}E[R_{d2}(i,a,Y_2)] + \\ \frac{\lambda_p}{\lambda(i_{op},i_{op},a)}R_p(i,a) + \frac{\lambda_{rec}}{\lambda(i_{op},i_{op},a)}E[R_{rec}(i,a,X_q)] \end{aligned} \tag{7.9}$$

Table 7.1: Summary of notation used in the SMDP formulation

State Variables	
i_1	produced inventory
i_2	recovered inventory
i_r	returns
i_c	components
i_{op}	outstanding production order
i_{or}	outstanding recovery order
Inventory State Space Capacities	
W_1	produced inventory
W_2	recovered inventory
W_r	returns
W_c	components
Action Variables	
a_p	production
a_r	recovery
a_b	buying of Components
a_1	downward substitution
a_2	upward substitution
$a_h(x_q)$	high quality recovery
$a_l(x_q)$	low quality recovery
$a_d(x_q)$	disposal from recovery
Action Quantities	
Q_p	production lot size
Q_r	recovery lot size
Action Space Capacities	
$U_p, U_r, U_b,$	upper limit on production, recovery, ordering
$L_p, L_r, L_b,$	lower limit on production, recovery, ordering
Random Variables and Distribution Parameters	
X_r	size of returns batch
X_q	quality of returns
Y_1	upward substitution acceptance
Y_2	downward substitution acceptance
α_1	upward substitution acceptance parameter
α_2	downward substitution acceptance parameter
α	quality parameter

Setup Costs	
k_p	production
k_r	recovery
δ_p	production indicator
δ_r	recovery indicator
Holding Costs	
h_1	produced
h_2	recovered
h_r	returns
h_c	components
Unit costs	
c_p	production
c_r	recovery
c_b	buying components
c_h	high quality recovery
c_l	low quality recovery
c_d	disposal from recovery
c_a	acquisition of returns
Revenues	
p_{d1}	revenue from produced goods
p_{d2}	revenue from recovered goods
Lost Sales Costs	
l_r	excess returns
l_1	lost sales of produced goods
l_2	lost sales of recovered
Arrival Rates	
λ_r	Arrival of returns
λ_{d1}	demand for new goods
λ_{d2}	demand for recovered goods
λ_p	arrival of production lot
λ_{rec}	arrival of recovery lot
Strategic Variables	
ζ_L	low quality recovery strategy
ζ_H	high quality recovery strategy
χ_1	upward substitution strategy
χ_2	downward substitution strategy

Table 7.2: Summary of transition probabilities, costs and functions in the SMDP formulation

Event type	Case	Next State	Probability	Costs
Returns		$(\min\{i_r + r, U_r\}, i_1, i_2, i_c, j_{op}, j_{or})$	$\frac{\lambda_r}{\lambda(i_{op}, i_{op}, a)} P(X_r = x_r)$	$-l_r \max\{0, (x_r - U_r + i_r)$
Demand 1	$a_1 = 0$	$(i_r, \max\{0, i_1 - 1\}, i_2, i_c, j_{op}, j_{or})$	$\frac{\lambda_{d1}}{\lambda(i_{op}, i_{op}, a)}$	$p_1 \min\{i_1, 1\}$
(Produced)	$a_1 = 1, y_1 = 0$	$(i_r, 0, i_2, i_c, j_{op}, j_{or})$	$\frac{\lambda_{d1}}{\lambda(i_{op}, i_{op}, a)}(1 - \alpha_1)$	$-l_1(\max\{0, 1 - i_1\} - \min\{y_1, a_1\})$
	$a_1 = 1, y_1 = 1$	$(i_r, 0, i_2 - 1, i_c, j_{op}, j_{or})$	$\frac{\lambda_{d1}}{\lambda(i_{op}, i_{op}, a)}\alpha_1$	$p_2 \min\{y_1, a_1\}$
Demand 2	$a_2 = 0$	$(i_r, i_1, \max\{0, i_2 - 1\}, i_c, j_{op}, j_{or})$	$\frac{\lambda_{d2}}{\lambda(i_{op}, i_{op}, a)}$	$p_2 \min\{i_2, 1\}$
(Recovered)	$a_2 = 1, y_2 = 0$	$(i_r, i_1, 0, i_c, j_{op}, j_{or})$	$\frac{\lambda_{d2}}{\lambda(i_{op}, i_{op}, a)}(1 - \alpha_2)$	$-l_2(\max\{0, 1 - i_2\} - \min\{y_2, a_2\})$
	$a_2 = 1, y_2 = 1$	$(i_r, i_1 - 1, 0, i_c, j_{op}, j_{or})$	$\frac{\lambda_{d2}}{\lambda(i_{op}, i_{op}, a)}\alpha_2$	$p_2 \min\{y_2, a_2\}$
Production		$(i_r, i_1 + \max\{i_{op}, a_p\}, i_2, i_c + a_b - \max\{i_{op}, a_p\}, 0, j_{or})$	$\frac{\lambda_p}{\lambda(i_{op}, i_{or}, a)}$	$-(\max\{i_{op}, a_p\}c_p + a_b c_b)$
Recovery		$(i_r, i_1, i_2 + a_h(x_q), i_c + a_l(x_q), \max\{i_{op}, \min\{1, a_p\}\}, 0)$	$\frac{\lambda_{rec}}{\lambda(i_{op}, i_{or}, a)} P(X_q = x_q)$	$-\left(Q_r c_r + a_h(x_q)c_h + a_l(x_q) + a_d(x_q)\right)$

7.5 Model Implementation and Validation

In this section the implementation and validation of the semi-Markov decision process is discussed.

7.5.1 Continuous Time to Discrete Time

A semi-Markov decision process can be solved by converting the process to a discrete time model. Further details are provided in Section 3.2.4. If the system is in state i at time t, then the probability that the system is in state j at time $t + \tau$ is the probability that one of the five events will occur during the next τ time units. In this implementation, the length of time τ is:

$$\tau = \frac{1}{\lambda_r + \lambda_{d1} + \lambda_{d2} + \lambda_p + \lambda_{rec}}$$

The expected time to the next event is denoted by $\tau_i(a)$ and was specified in equation (7.2). In the discrete adaptation of the continuous time model, the probability of moving to state j given that the system is currently in state i, and action a has been chosen, is denoted by $\overline{p}_{i,j}(a)$ where:

$$\overline{p}_{i,j}(a) = \begin{cases} p_{i,j}\frac{\tau}{\tau_i(a)} & \text{if } i \neq j \\ p_{i,j}\frac{\tau}{\tau_i(a)} + \left(1 - \frac{\tau}{\tau_i(a)}\right) & \text{if } i = j \end{cases}$$

The reward associated with a transition from i to j, given action a, is the expected reward received from the current time until the next event. To obtain the expected reward per time unit, this reward needs to be divided by $\tau_i(a)$, and then to get the reward per period, it needs to be multiplied by τ. The reward in the discrete adaptation of the continuous time model is denoted by $\overline{R}(i,a)$, where:

$$\overline{R}(i,a) = \frac{\tau}{\tau_i(a)} R(i,a)$$

After making these adjustments to the expected rewards and probabilities, the value iteration algorithm can be used to solve the continuous time problem.

7.5.2 Dimensionality

This semi-Markov decision problem has six state variables and four action variables. This makes the state space and action space of this problem exponentially larger than those in the Chapters 5 and 6.

However, the structure of this continuous time model means that the dimensionality is not as burdensome at in the previous chapter. In this model, each decision epoch is associated with a specific event, thus only one of production, recovery, returns, demand for produced goods, demand for recovered goods, can occur at each decision epoch. Furthermore, in this model customers arrive one at a time, and only demand a single good. Because only one event can occur at each decision epoch, the maximum number of transitions is equal to the sum of the maximum number of outcomes for each event. In Chapter 6, any and all of these events could occur during each period and multiple customers could arrive during a period. The continuous time structure used in the current chapter leads to a substantial reduction in the number of non-zero transitions for a given state i and action a, compared with Chapter 6. For instance, suppose that the maximum inventory level was 10 ($i_1 = i_2 = i_r = i_c = 10$), the number of returns each period was 10 or less (with probability 0.999) and the batch size for returns was $Q_r = 10$ (meaning $X_q \sim \text{Bin}(Q_r, \alpha)$). At most, one good will be demanded at each decision epoch. If a substitution is offered, there are two alternatives - it is accepted or rejected by the customer. Therefore, at a given decision epoch, the number of possible transitions would be no more than: $\text{num returns} + \text{num produced} + \text{num recovered} + \text{num demand/substitution produced} + \text{num demand/substitution recovered} = 10 + 1 + 10 + 2 + 2 = 25$. Compare this to 10^6, the number of possible transitions calculated in Section 6.5.1. For this reason, we are able to use a higher maximum inventory level than in Chapter 6.

Despite the fact that the number of possible transitions is substantially lower than in the previous Chapter, the larger state space means that the dimensionality of the model is still an issue. Initial numerical experiments suggested that dimensionality of the problem makes it computationally impractical to consider problems with upper inventory capacity levels of more than approximately $20 - 25$; the computational time associated with solving one problem with upper capacity levels of 25 can range between

1 to 5 hours. This obviously places significant limitations on the numerical experiments that can be performed with this model. To aid with the computational investigations a simplification is proposed in order to reduce the computational burden per problem and thus allow a larger number of problem instances to be studied. This simplification, similar to previous chapters, involves the introduction of fixed order sizes of Q_p for production and of Q_r for recovery. The action space for production then becomes:

$$a_p \in A_p(i) = \{0, \min\{W_1 - i_1, Q_p\}\}$$

and for recovery becomes:

$$a_r \in A_r(i) = \begin{cases} \{0, Q_r\} & \text{if } i_r \geq Q_r \\ \{0\} & \text{if } i_r < Q_r \end{cases}$$

Recall that the components buying decisions has already been simplified. This means that at each decision epoch, the decision is restricted to whether or not to produce or recover, rather than how much to produce or recover.

7.5.3 Implementation of the Model

The semi-Markov decision process described in the previous sections can be solved as a discrete time Markov decision process, by making the adjustments described Section 7.5.1. Some of the java code used in Chapters 5 and 6 can also be used for the model in this chapter.

7.5.4 Validation of the Code

Much of the code used for implementing this model has been used and validated in previous chapters, therefore in this section we focus on the validation of the code created specifically for this model. In addition to thorough checks during the code development process, an Excel spreadsheet was developed to calculate the expected reward for a given state and action. To conduct these tests, a set of six test problems was developed. For a random selection of 30 states, the costs calculated by the java programme and those calculated by the Excel spreadsheet were checked for the optimal policy action and a heuristic policy and were found to be the same. Further details of this are provided in Appendix D.2.2.

The code can also be validated by comparing the reward from the value iteration algorithm and the reward obtained from a simulation of the problem. Two versions of the simulation code were developed for this model: a discrete time simulation and a continuous time simulation. In order to validate both versions of the simulation code, repeated simulations were performed and the long run average reward was recorded. Over a large number of trials, repeated sampling from the distribution of the long run average reward should yield an approximately normally distributed curve, centred around the actual optimal reward, as calculated by the value iteration algorithm. The set of six test problems used above were also used here. It was found that all test problems performed as expected. Further details are provided in Appendix D.

7.6 Properties of the Optimal Policies

The properties of the optimal policy are explored in this section. Three main properties will be investigated. Firstly, we investigate the *performance* of the optimal policy under different substitution strategies. Two performance measures are used: the long run average reward and the fill rate (a measure of service). Secondly, we investigate the structure of the optimal policy by examining the *actions* that are chosen in different states. Finally, we investigate the effect of the *recovery* strategy on the performance and structure of the optimal policy.

The class of policies being considered has been restricted by introducing the fixed lot sizes Q_p and Q_r and by fixing the decision about buying components, as discussed in Section 7.5.2. The optimal policy within this class of policies is sought. This policy may not be optimal if these restrictions were not imposed.

This section is structured as follows. The datasets used to explore the properties of the optimal policy are described in Section 7.6.1 and the performance, actions, and recovery of the optimal policy are investigated in Sections 7.6.2, 7.6.3 and 7.6.4, respectively.

7.6.1 Datasets

A set of problems has been constructed to investigate the properties of the optimal policy. The parameters values for these problems are derived from the datasets used in Chapters 4, 5 and 6, some of which were themselves derived from the dataset used by Konstantaras and Papachristos (2008b). This section explains how the dataset used in this chapter extends the ones used in previous chapters. Some additional parameters have been added to account for the additional state variables and random variables. Note that the first problem in the datasets in Chapters 4 and 5 (labelled 00) is not used in this chapter as it does not contain any unit costs. It was included in the earlier models to enable comparisons to be drawn between models in the literature and the models proposed by this thesis. However, since the model in this chapter has a different structure (e.g. two markets), such comparisons are not made here. The parameter values for the datasets used in this chapter are presented in Table 7.3.

State Variables

The state space for each of the inventories is limited to the discrete values from 0 to 20. The upper limit was selected due to the dimensionality and computational requirements associated with this problem. The state space for each of the inventory variables is defined as follows:

$$I_1 = \{0, 1, \ldots, 20\} \qquad \text{produced inventory}$$

$$I_2 = \{0, 1, \ldots, 20\} \qquad \text{recovered inventory}$$

$$I_r = \{0, 1, \ldots, 20\} \qquad \text{returned inventory}$$

$$I_c = \{0, 1, \ldots, 20\} \qquad \text{component inventory}$$

The outstanding order variables indicate the number of goods that have been ordered (through production or recovery), but have not yet arrived. The state space for the outstanding order variables is:

$$I_{op} \subseteq \{0, \ldots, 20\} \qquad \text{outstanding production}$$

$$I_{or} \subseteq \{0, \ldots, 20\} \qquad \text{outstanding recovery}$$

Table 7.3: Datasets used in the analysis of the optimal policy

Problem ID	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18	19	20
Arrival Rates																				
λ_r	5	3	3	3	6	6	6	6	3	6	5	5	5	4	5	1	4	2	6	2
λ_{d1}	6	6	8	6	7	7	7	6	6	7	9	9	9	6	6	6	6	5	6	4
λ_{d2}	5	1	8	5	6	2	7	4	1	4	6	2	5	2	4	6	5	2	2	2
λ_p	30	24	16	24	21	21	18	30	24	21	13	13	11	30	24	24	24	50	24	80
λ_{rec}	18	15	12	12	14	14	13	15	18	14	11	11	10	18	15	36	32	10	48	20
Distributions																				
Returns Size	U(2,7)	U(0,5)	U(0,5)	U(0,5)	U(3,8)	U(3,8)	U(3,8)	U(3,8)	U(0,5)	U(3,8)	U(2,7)	U(2,7)	U(2,7)	U(1,6)	U(2,7)	U(1,1)	U(1,6)	U(2,2)	U(3,8)	U(2,2)
α	0.8	0.2	0.3	0.3	0.5	0.7	0.8	0.8	0.2	0.5	0.5	0.5	0.5	0.5	0.4	0.1	0.6	0.2	0.2	0.5
α_1	0.68	0.65	0.57	1	1	0.55	0.5	0.4	0.75	0.94	0.92	0.8	0.63	0.93	0.5	0.52	0.86	0.65	0.98	0.56
α_2	0.44	0.75	0.9	0.62	0.81	0.53	0.5	0.72	0.74	0.87	0.87	0.76	0.93	0.92	0.7	0.48	0.48	0.56	0.89	0.67
Unit Costs																				
c_p	106.5	66	88	131	161	107	66	75	212.5	188.5	23.5	76	78.5	106.5	65.5	110	107	75	212.5	160
c_r	13.5	34	42	89	29	13	44	15	27.5	141.5	6.5	24	21.5	13.5	34.5	0	13	15	27.5	10
c_b	30	60	80	100	70	40	60	30	50	200	50	50	100	30	50	100	40	30	50	50
c_h	50	30	40	60	75	50	30	35	100	85	10	35	35	50	30	50	50	35	100	75
c_l	9	18	24	30	21	12	18	9	15	60	15	15	30	9	15	30	12	9	15	15
c_d	6.5	6	8	11	11	7	6	5	12.5	18.5	3.5	6	8.5	6.5	5.5	10	7	5	12.5	10
Revenues																				
p_{d1}	209	169	269	316	434	269	242	112	378	563	146	214	277	231	186	330	212	140	263	279
p_{d2}	147	133	202	161	391	253	202	84	361	529	83	119	259	221	111	236	113	115	150	162
Set up Costs																				
k_p	20	10	10	20	20	20	30	30	50	28	12	12	120	20	100	1000	400	30	50	28
k_r	5	5	10	12	8	20	20	25	30	8	2	2	10	5	50	100	400	5	10	8
k_b	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Holding Costs																				
h_r	2	4	5	10	4	2	5	2	4	16	1	3	3	2	4	1	2	2	4	2
h_{s1}	10	6	8	12	15	10	6	7	20	17	2	7	7	10	6	10	10	7	20	15
h_{s2}	8	6	8	11	10	2	4	4	10	15	2	5	5	10	5	7	8	4	12	10
h_c	3	6	8	10	7	4	6	3	5	20	5	5	10	3	5	10	4	3	5	5
Penalty Costs																				
l_r	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
l_{s1}	13	12	16	22	22	14	12	10	25	37	7	12	17	13	11	20	14	10	25	20
l_{s2}	5.7	5.8	7.4	13.8	9.3	5.6	6.8	4.5	11.5	20.8	1.3	5.3	4.8	5.7	5.9	4	5.6	4.5	11.5	7.5
Order Sizes B																				
Q_p	9	9	11	9	10	10	10	9	9	10	12	12	12	9	9	9	9	8	9	6
Q_r	8	2	11	8	9	4	10	6	2	6	9	4	8	4	6	9	8	4	4	4
Order Sizes C																				
Q_p	11	11	14	11	13	13	13	11	11	13	15	15	15	11	11	11	11	10	11	8
Q_r	8	2	11	8	9	4	10	6	2	6	9	4	8	4	6	9	8	4	4	4
Order Sizes D																				
Q_p	9	9	11	9	10	10	10	9	9	10	12	12	12	9	9	9	9	8	9	6
Q_r	10	3	14	10	11	5	13	8	3	8	11	5	10	5	8	11	10	5	5	5

Random Variables

There are five types of event which could occur or "arrive": returns, demand for produced goods, demand for recovered goods, production lots and recovery lots. All five are governed by independent Poisson processes. If the rate of a Poisson process is λ then the expected number of events to occur in t time units is λt, and the expected time between events is $1/\lambda$ (Ross, 1996). Following the convention in stochastic processes, let $N(t)$ denote the number of events to have occurred by time t, then $N(t)$ is a Poisson distributed random variable with mean λt.

Demand. The arrival of demand for produced and recovered goods are governed by Poisson processes with rates of λ_{d1} and λ_{d2}, respectively. At each demand instance one good is demanded. The rates of these Poisson processes are chosen to be integers for convenience.

In Chapters 5 and 6 the demands were governed by Poisson distributions, where the rates indicated the expected number events to be demanded in one time period. Since one good is demanded at each demand instance, if it is assumed that each time period has a length of one time unit (as was the case in Chapters 5 and 6), then the rates from these previous chapters have a similar meaning to the rates used in this chapter, i.e., the rates λ_{d1} and λ_{d2} specify the expected number of events to occur within 1 time unit.

To retain some comparability between the previous models and the current model, the expected demands used in this chapter are derived from the demand and production rates in Chapter 4, as they have been Chapters 5 and 6. Ideally, the capacity of the inventories should not affect the policy, therefore the maximum state space is also taken into account when selecting the rates. In Chapters 5 and 6 this was done by ensuring that the probability that demand exceeds this upper capacity is very low. In this chapter, since only one item is demanded at each demand instance, ensuring that expected demand per time unit does not often exceed the maximum inventory capacity has a different effect compared with previous models. However to retain some consistency, the same method is used in this chapter. Suppose $\lambda_{d1} = 11$, then the probability that the number of goods demanded during one time unit will exceed the maximum inventory capacity of 20 is very small ($P(N_{d1}(t) > 20) < 1 \times 10^{-4}$). This probability is sufficiently small, therefore an upper limit of 11 is placed on the parameter λ_{d1}.

Let p_d and d_d denote the production and demand rates from the deterministic model in Chapter 4. The ratio of p_d and d_d is used to determine the demand parameter for produced goods. However, a transformation needs to be applied to this ratio in order to scale it to the state space of the model, and to provide a range of values spread across the state space. In summary, in order to obtain the values for λ_{d1} (i.e. number of demand events per one time unit) for each problem in the dataset the following transformation is used:

$$\lambda_{d1} = \left\lceil \sqrt[3]{\frac{d_d}{p_p}} \times 11 \right\rceil$$

where the function $\lceil x \rceil$ rounds x up to the nearest integer. Parameters are rounded to integer values for convenience.

In order to model a variety of relationships between the demand for produced and recovered goods, the demand rate λ_{d2} was determined by randomly generating values within the range of 1 to 11 and rounding to the nearest integer. This creates some scenarios with $\lambda_{d1} > \lambda_{d2}$, some with $\lambda_{d1} = \lambda_{d2}$, and some with $\lambda_{d1} < \lambda_{d2}$.

Returns. The arrival of returns is governed by a compound Poisson process and the size of each batch of returns is modelled by the uniform distribution. The uniform distribution is used in order to model the situation in which the size of a returns batch is known to fall within a certain range. For most problems the variance of the uniform distribution was chosen such that $b - a = 5$, except in cases where the mean is less than 3. In those cases the distributions were selected such that $a = b = \mu_r$, this could correspond to a supply contract in which the batch size is known with certainty. In some of the problems, the lower limit is equal to 0. This represents a non-zero probability that there may be a problem with the arrival of a batch. This models additional uncertainty associated with receiving returns. The arrival rate and batch sizes are chosen to model the situation in which returns arrive in (relatively) small batches, but arrive often.

Production and Recovery. The arrival rates of production and recovery orders are derived from the production and recovery rates in the dataset from Chapter 4. However since the demand rate has been transformed from the values used in Chapter 4, a transformation also needs to be applied here in order to maintain the relationship between the demand, production and recovery rates. Let d_d, p_d, r_d denote the demand, production and recovery rates used in the deterministic model in Chapter 4. Then the arrival rate of the production order for this model is given by:

$$\lambda_p = \left\lfloor \frac{\lambda_{d1} p_d}{d_d} \right\rfloor$$

and the arrival rate of the recovery order for this model is given by:

$$\lambda_{rec} = \left\lfloor \frac{\lambda_{d1} r_d}{d_d} \right\rfloor$$

where the function $\lfloor x \rfloor$ rounds the value x down to the nearest integer.

Revenues and Costs

The revenues and costs used in this chapter are the same as those used Chapter 6.

Lot Sizes

Using the same method as in Chapter 6, the lot sizes for production Q_p and recovery Q_r are derived from the respective demand rates plus y_p or y_r standard deviations respectively, such that:

$$Q_p = \lceil \lambda_{d1} + y_p \sqrt{\lambda_{d1}} \rceil, \qquad Q_r = \lceil \lambda_{d2} + y_r \sqrt{\lambda_{d2}} \rceil$$

where $\lceil x \rceil$ rounds x up to the nearest integer. Three variations of the dataset are constructed by varying the number of standard deviations added to the demand rates. These datasets are labelled B, C and D such that:

Problem Set	y_p	y_r	
A	–	–	Test Problems
B	1	1	
C	2	1	
D	1	2	

Substitution Strategies

Four substitution strategies are considered: no substitution ($\chi_1 = 0, \chi_2 = 0$), only downward substitution ($\chi_1 = 0, \chi_2 = 1$), only upward substitution ($\chi_1 = 1, \chi_2 = 0$), two-way substitution ($\chi_1 = 1, \chi_2 = 1$).

Notation	Type of substitution	Description
χ_1	upward	Shortage of produced goods, substitute with a recovered good
χ_2	downward	Shortage of recovered goods, substitute with a produced good

The proportion of customers for produced and recovered goods who would accept a substitution, if it was offered to them, is modelled by a Binomial distribution with the parameters α_1 and α_2 respectively.

The parameters of the base problems are presented in Table 7.3. Many of these parameters are the same as the ones used in Chapter 6.

7.6.2 Analysis of Performance

In this section the performance of the optimal policy is analysed by examining the average reward and fill rates under the four substitution strategies: no substitution ($\chi_1 = 0, \chi_2 = 0$), upward substitution ($\chi_1 = 1, \chi_2 = 0$), downward substitution ($\chi_1 = 0, \chi_2 = 1$), two-way substitution ($\chi_1 = 1, \chi_2 = 1$). Upward substitution relates to the shortage of produced goods and downward relates to the shortage of recovered goods. These substitution strategies do not force the system to perform substitution in the case of a shortage, rather they allow substitution to be an option (i.e. an allowable action in the action space).

Simulation is used to calculate the fill rates of the optimal policy. Either of the discrete-time or continuous-time simulations could have been used, however the discrete time simulation was used, as it more closely aligned to the discretization of the Markov decision process implementation.

Rewards

The value of the average reward varies across the problems. There is variation between problems in the three sets and across the four substitution strategies. Some problems have a negative average reward, which suggests that some problems are not financially viable. The average rewards for the three problems sets, under the four substitution strategies are presented in Figure 7.2 and Table 7.4.

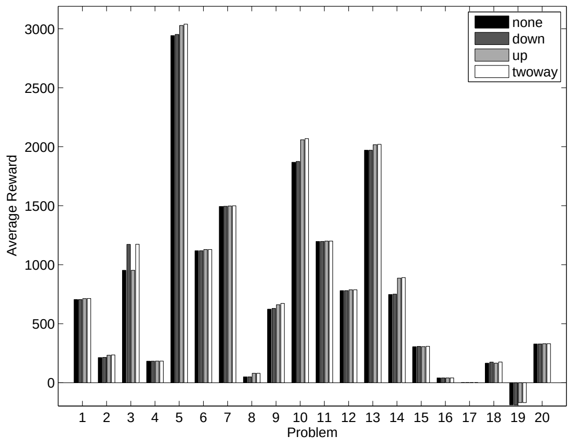
The reward associated with the two-way substitution strategy should be the highest as it offers the greatest freedom in choosing the policy actions. As shown by Figure 7.2 and Table 7.4 this indeed is the case; the optimal reward is highest under a two-way substitution strategy, however for some problems the highest reward is also attained by other substitution strategies.

The question of interest here is what additional reward can be achieved by allowing substitution. To investigate this question the relative reward increase (RRI) attainable by allowing each of the substitution strategies is compared with not allowing substitution. The relative reward increase is calculated as follows:

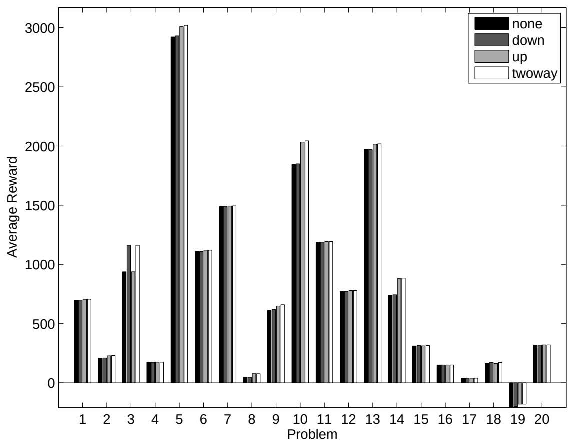
$$\text{RRI} = \frac{\text{Reward(no substitution)} - \text{Reward(other substitution)}}{\text{Reward(no substitution)}} \times 100\%$$

Figure 7.3 shows the relative reward increase for problem sets B, C and D.

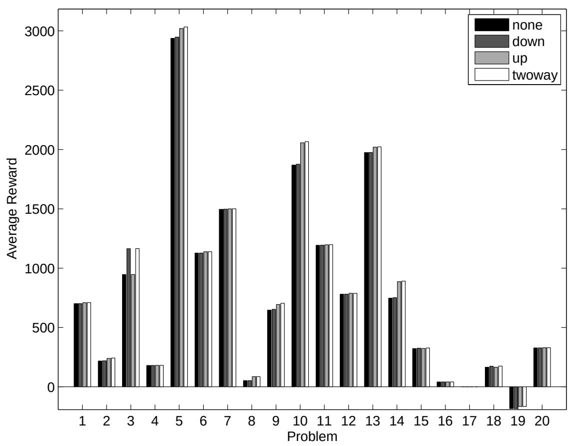
Comparing the three graphs, it is apparent that the savings available for each problem are similar across the three data sets, however vary considerably between the 20 problems. The RRI attainable by allowing substitution varies from 0 (no benefit from substitution) to approximately 60% (substantial benefit from substitution). For some problems (e.g. 08) the increase in reward available by allowing upward substitution is greater than for downward substitution, however for some problems (e.g. 03) the reverse can be observed. This variation suggests that the benefits available from allowing substitution depends heavily on the problem parameters. In general, the greatest increase in reward is attained by allowing a substitution strategy which includes upward substitution. Under a upward substitution recovered goods are used to meet demand if there is a shortage of produced goods. There is little risk associated with upward substitution as recovered goods are sold for their regular price.



(a) Problems B



(b) Problems C



(c) Problems D

Figure 7.2: Average reward of the optimal policy for all substitution strategies

Table 7.4: Average reward of the optimal policy for all substitution strategies

(a) Problems B

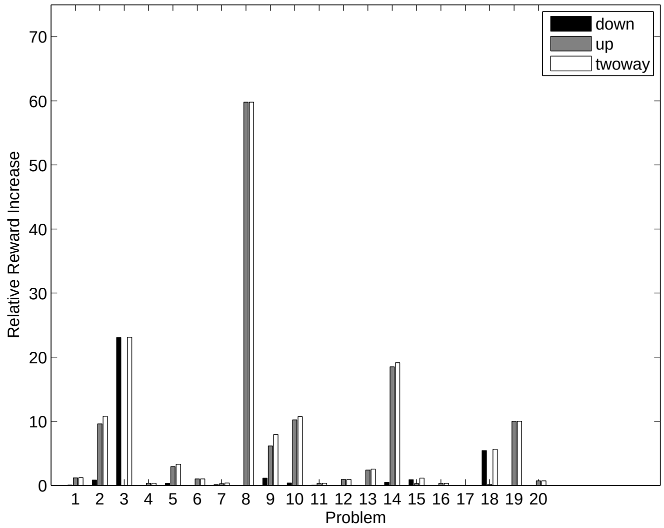
	Substitution Strategy			
Problem	None	Down	Up	Two-way
B01	705.3792	705.6431	713.4715	713.7553
B02	213.0802	214.8265	233.5067	236.0227
B03	953.1586	1172.9329	953.2487	1173.3080
B04	183.0032	183.0032	183.5772	183.5772
B05	2942.3782	2951.2749	3027.9538	3038.9410
B06	1118.5293	1118.5293	1129.6040	1129.6040
B07	1494.1489	1495.6817	1497.8446	1499.3618
B08	50.0060	50.0060	79.9090	79.9090
B09	623.2927	630.2194	661.5581	672.6597
B10	1868.2836	1875.0448	2059.0831	2068.4609
B11	1197.1239	1197.4120	1200.6414	1200.9714
B12	780.8096	780.8191	787.9061	787.9194
B13	1970.5853	1970.7320	2017.3735	2020.4322
B14	747.8507	751.3398	886.1845	890.8854
B15	305.4939	308.1406	306.2887	308.9396
B16	41.0997	41.0997	41.2261	41.2261
B17	1.0815	1.0815	1.0814	1.0814
B18	166.0895	175.0832	166.3100	175.4455
B19	-187.2994	-187.2987	-168.5815	-168.5815
B20	328.8674	328.8674	331.1385	331.1385

(b) Problems C

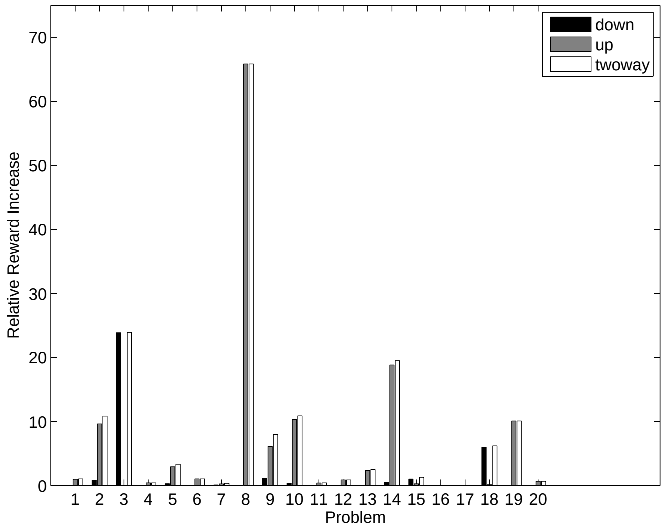
	Substitution Strategy			
Problem	None	Down	Up	Two-way
C01	698.7423	699.0217	705.6471	705.9478
C02	207.7474	209.4980	227.7293	230.3007
C03	937.8397	1161.7678	937.9336	1162.1803
C04	173.4298	173.4298	174.1442	174.1442
C05	2921.9751	2930.7146	3007.9105	3019.1330
C06	1108.6435	1108.6435	1120.0610	1120.0609
C07	1489.1721	1490.7274	1492.9538	1494.4392
C08	46.4153	46.4153	76.9781	76.9781
C09	610.7875	617.9158	648.1387	659.4672
C10	1843.2418	1849.7324	2033.3397	2043.9817
C11	1188.2421	1188.5112	1192.9287	1193.2640
C12	772.7961	772.8147	779.5477	779.5721
C13	1969.0867	1969.2407	2015.4702	2018.2048
C14	740.0975	743.7213	879.4855	884.4565
C15	311.4163	314.5791	312.2569	315.4457
C16	150.0331	150.0331	150.0708	150.0708
C17	39.4667	39.4667	39.4667	39.4667
C18	161.9989	171.6935	162.2076	172.0579
C19	-199.8629	-199.8629	-179.7160	-179.7160
C20	318.0211	318.0211	320.1212	320.1212

(c) Problems D

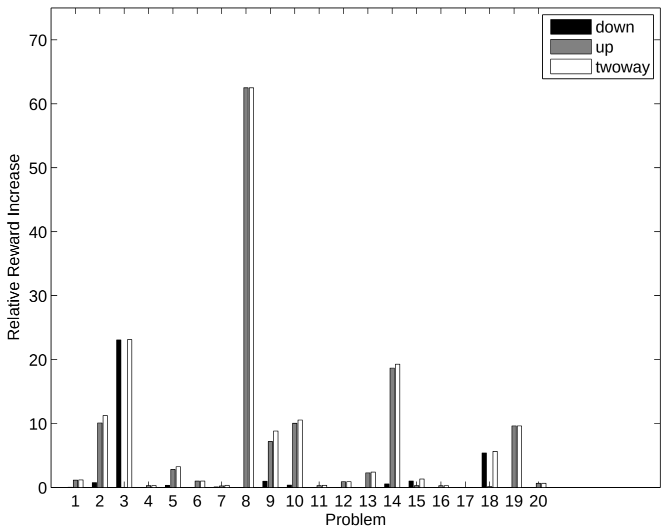
	Substitution Strategy			
Problem	None	Down	Up	Two-way
D01	701.0779	701.2895	709.2029	709.4329
D02	217.5816	219.2315	239.5535	242.0744
D03	946.0521	1164.4283	946.1486	1164.7638
D04	180.4413	180.4413	180.9843	180.9843
D05	2937.2535	2947.3478	3019.9809	3032.5782
D06	1127.3137	1127.3137	1138.8094	1138.8094
D07	1495.7627	1497.1643	1499.4542	1500.9091
D08	52.6219	52.6219	85.5068	85.5068
D09	646.9946	653.3268	693.4792	704.2378
D10	1869.0423	1875.9195	2056.7591	2066.4963
D11	1193.3866	1193.7250	1197.0492	1197.4479
D12	780.7896	780.8060	787.9121	787.9270
D13	1974.6202	1974.8150	2020.0231	2022.3762
D14	746.9873	751.0783	886.5235	891.1434
D15	322.4409	325.7021	323.3985	326.7173
D16	40.7580	40.7580	40.8775	40.8775
D17	1.0815	1.0815	1.0814	1.0814
D18	164.9608	173.8717	165.1927	174.2624
D19	-183.4333	-183.4331	-165.7716	-165.7716
D20	327.4696	327.4696	329.5992	329.5992



(a) Problems B



(b) Problems C



(c) Problems D

Figure 7.3: Relative reward increase for all substitution strategies

Table 7.5: Summary of inventory levels during the simulation under a two-way substitution strategy

(a) Problems B

	Produced Inventory				Recovered Inventory				Returned Inventory				Component Inventory			
Problem	mean	min	max	mode	mean	min	max	mode	mean	min	max	mode	mean	min	max	mode
B01	4.891	0	9	2	4.533	0	9	2	17.974	0	20	20	1.083	0	15	0
B02	5.356	0	19	0	2.547	0	11	1	9.905	0	20	20	1.819	0	15	0
B03	8.405	0	20	7	0.739	0	13	0	6.504	0	20	9	1.795	0	20	0
B04	5.888	0	11	8	1.016	0	15	0	8.079	0	20	20	2.883	0	18	0
B05	5.032	0	12	0	5.352	0	17	4	16.048	0	20	20	5.528	0	20	0
B06	5.822	0	11	4	18.635	0	20	20	17.812	0	20	20	2.040	0	12	0
B07	6.202	0	17	4	6.664	0	16	7	17.649	0	20	20	0.937	0	13	0
B08	1.409	0	9	0	3.995	0	10	4	18.513	0	20	20	2.356	0	12	0
B09	4.599	0	10	8	1.992	0	7	1	9.749	0	20	20	3.326	0	20	0
B10	4.542	0	19	0	5.132	0	18	4	16.892	0	20	20	1.994	0	19	0
B11	10.780	0	20	10	14.046	0	20	16	12.735	0	20	20	3.607	0	20	0
B12	7.037	0	16	6	2.993	0	7	3	18.877	0	20	20	1.094	0	20	0
B13	5.351	0	20	0	8.964	0	20	9	13.917	0	20	20	3.134	0	20	0
B14	3.221	0	10	0	2.537	0	7	2	15.598	0	20	20	4.051	0	20	0
B15	6.034	0	12	3	3.406	0	15	2	16.658	0	20	20	2.175	0	16	0
B16	4.848	0	9	5	0.024	0	5	0	4.863	0	16	8	0.036	0	9	0
B17	4.859	0	9	7	0	0	0	0	19.999	0	20	20	0	0	0	0
B18	4.410	0	9	3	0.716	0	12	0	2.616	0	20	2	2.238	0	12	0
B19	4.321	0	9	0	0.449	0	5	0	18.503	0	20	20	1.164	0	11	0
B20	3.427	0	6	2	2.054	0	6	2	9.360	0	20	2	1.164	0	16	0

(b) Problems C

	Produced Inventory				Recovered Inventory				Returned Inventory				Component Inventory			
Problem	mean	min	max	mode	mean	min	max	mode	mean	min	max	mode	mean	min	max	mode
C01	5.885	0	12	3	4.524	0	9	2	17.989	0	20	20	1.264	0	15	0
C02	6.124	0	20	0	2.595	0	12	1	9.892	0	20	20	2.112	0	18	0
C03	9.503	0	20	7	0.752	0	13	0	6.752	0	20	9	2.255	0	20	0
C04	6.705	0	13	7	1.005	0	14	0	8.607	0	20	20	2.509	0	18	0
C05	6.202	0	15	0	5.357	0	17	4	15.992	0	20	20	6.067	0	20	0
C06	6.654	0	13	6	18.205	0	20	20	17.599	0	20	20	2.638	0	15	0
C07	7.624	0	20	3	6.888	0	17	7	17.641	0	20	20	1.183	0	15	0
C08	1.736	0	11	0	4.031	0	11	4	18.490	0	20	20	2.739	0	14	0
C09	5.483	0	12	0	1.940	0	7	1	9.653	0	20	20	3.590	0	20	0
C10	5.602	0	20	0	5.283	0	20	4	16.773	0	20	20	2.414	0	19	0
C11	10.631	0	20	14	14.081	0	20	17	13.044	0	20	20	4.800	0	20	0
C12	8.417	0	19	10	2.927	0	7	3	18.856	0	20	20	1.308	0	20	0
C13	6.645	0	20	0	10.071	0	20	8	13.854	0	20	20	3.858	0	20	0
C14	3.855	0	12	0	2.635	0	7	2	15.357	0	20	20	4.376	0	20	0
C15	6.903	0	13	3	3.585	0	14	2	16.483	0	20	20	2.303	0	19	0
C16	6.015	0	12	9	0.023	0	4	0	4.868	0	17	8	0.037	0	9	0
C17	5.863	0	11	3	0	0	0	0	19.999	0	20	20	0	0	0	0
C18	5.407	0	10	2	0.704	0	12	0	2.982	0	20	2	2.427	0	13	0
C19	5.120	0	11	0	0.399	0	5	0	18.428	0	20	20	1.255	0	13	0
C20	4.381	0	8	4	2.047	0	7	2	9.016	0	20	2	1.355	0	20	0

(c) Problems D

	Produced Inventory				Recovered Inventory				Returned Inventory				Component Inventory			
Problem	mean	min	max	mode	mean	min	max	mode	mean	min	max	mode	mean	min	max	mode
D01	4.881	0	9	1	5.328	0	11	3	17.734	0	20	20	1.084	0	16	0
D02	5.287	0	19	9	2.534	0	12	1	9.972	0	20	20	1.677	0	19	0
D03	8.824	0	20	11	0.860	0	14	0	7.657	0	20	6	1.549	0	20	0
D04	5.849	0	11	8	1.089	0	13	0	9.353	0	20	20	2.187	0	18	0
D05	5.016	0	13	0	5.619	0	18	5	16.087	0	20	20	5.972	0	20	0
D06	5.789	0	11	2	18.530	0	20	20	17.695	0	20	20	2.124	0	13	0
D07	6.226	0	19	9	7.710	0	18	7	17.387	0	20	20	0.965	0	14	0
D08	1.376	0	9	0	4.582	0	10	4	18.376	0	20	20	2.432	0	12	0
D09	4.609	0	10	9	2.000	0	8	1	9.652	0	20	20	3.309	0	20	0
D10	4.520	0	20	0	5.318	0	17	4	16.819	0	20	20	2.186	0	20	0
D11	10.895	0	20	11	14.089	0	20	18	12.858	0	20	20	4.128	0	20	0
D12	7.011	0	18	12	3.029	0	8	3	18.801	0	20	20	0.907	0	19	0
D13	5.592	0	20	0	10.716	0	20	9	13.370	0	20	20	2.840	0	20	0
D14	3.239	0	10	0	2.629	0	8	2	15.363	0	20	20	4.060	0	20	0
D15	6.027	0	12	5	3.786	0	15	3	16.425	0	20	20	2.255	0	16	0
D16	4.855	0	9	7	0.024	0	5	0	5.887	0	17	10	0.176	0	11	0
D17	4.859	0	9	7	0	0	0	0	19.999	0	20	20	0	0	0	0
D18	4.402	0	9	2	0.743	0	12	0	3.716	0	20	4	2.150	0	12	0
D19	4.395	0	9	0	0.540	0	5	0	18.454	0	20	20	1.303	0	12	0
D20	3.443	0	6	2	2.261	0	7	2	9.660	0	20	20	1.212	0	17	0

Fill Rates

The fill rate measures the proportion of demand which is met by current stock; it can be used to assess the performance of a policy. For each type of good the fill rate is:

$$\text{fill rate} = \frac{\text{number of met sales}}{\text{number of items demanded}}$$

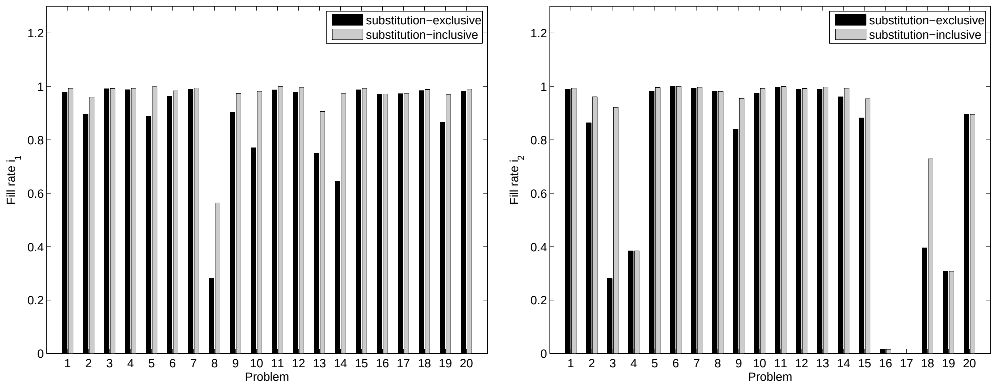
The average fill rate can be calculated by using a simulation to determine the fill rate each period and then by averaging this across the length of the simulation. As discussed in Chapter 6, in situations where substitution is permitted the 'number of met sales' could include or exclude the sales met by substitution. Thus two versions of the fill rate can be calculated. Let the *substitution-inclusive fill rate* refer to the fill rate calculation which includes sales met by substitution, and let the *substitution-exclusive fill rate* refer to the fill rate calculation which excludes such sales.

In this section two questions are investigated: firstly, for a given substitution strategy how much demand is met by substitution, and secondly, how does the fill rate vary across the four substitution strategies. In order to study these questions, the fill rates were calculated from a simulation over $T = \tau \times 1000000$ time units. Table 7.5 summarises the inventory levels reached during the simulation.

The substitution-inclusive and exclusive fill rates for the produced and recovered inventories are available for problem sets B, C and D in Appendix D in Tables D.5, D.6 and D.7 for all substitution strategies.

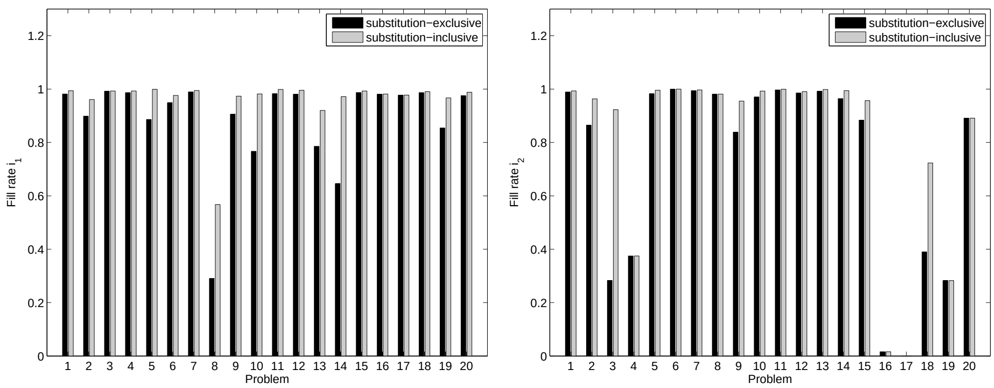
Substitution Inclusive and Exclusive Fill Rate. In order to investigate how much demand is met by substitution, the substitution-inclusive and substitution-exclusive fill rates are examined. In this initial analysis of the fill rates, only the two-way substitution strategy is examined. Figures 7.4, 7.5 and 7.6 show the fill rates under the two-way substitution strategy for problem sets B, C and D respectively. The graphs for the other substitution strategies are available in Appendix D.

The fill rate is higher when demand met by substitution is included the fill rate, as expected. For some problems substitution leads to little increase in the fill rate when demand met by substitution is included, indicating that most demand is met by the



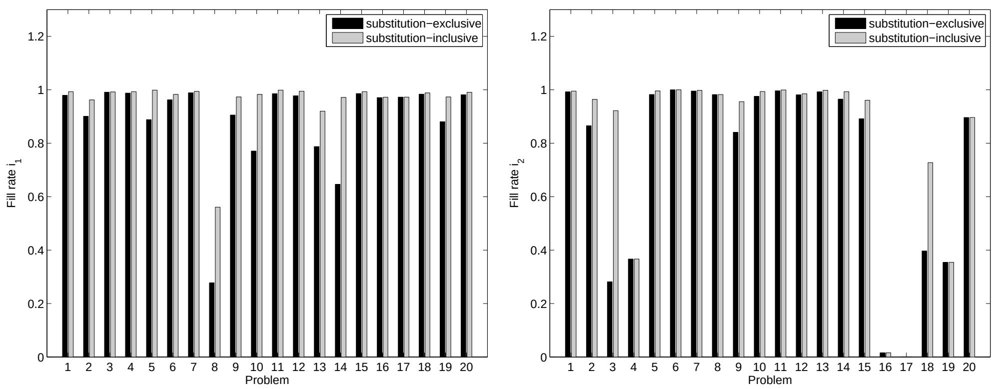
(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure 7.4: Fill rates for problem set B under a two-way substitution strategy



(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure 7.5: Fill rates for problem set C under a two-way substitution strategy



(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure 7.6: Fill rates for problem set D under a two-way substitution strategy

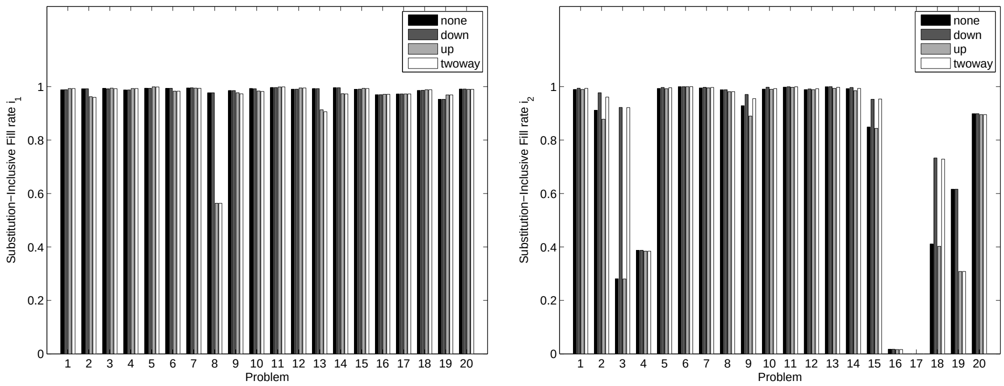
demanded good rather than by substitution. However, for some problems the increase is very large indicating a large amount of demand satisfied by substitution. For example consider problem $B03$, the fill rate for recovered inventory more than triples when sales met by substitution are included in the fill rate. This means that for this problem a large proportion of demand for recovered items is met by produced items.

Notice that for recovered inventory, the fill rates including and excluding substitution for problems 16 and 17 (across all problems sets) are close or equal to zero. This means that most demand for recovered goods is lost. This suggests that for these problems the average level of recovered inventory across the simulation is very low and that substitution is not offered. The average inventory levels during simulation, shown in Table 7.5, confirm that this is indeed the case. Problem 17 has a mean recovered inventory level of 0 across the three problem sets.

Though the models in Chapter 6 and this chapter are not directly comparably, we make the following observation. It appears that in the current model, substitution has a greater influence on the fill rates, especially for produced inventory, than it does for the model in Chapter 6 (c.f. Figure 6.7, page 220).

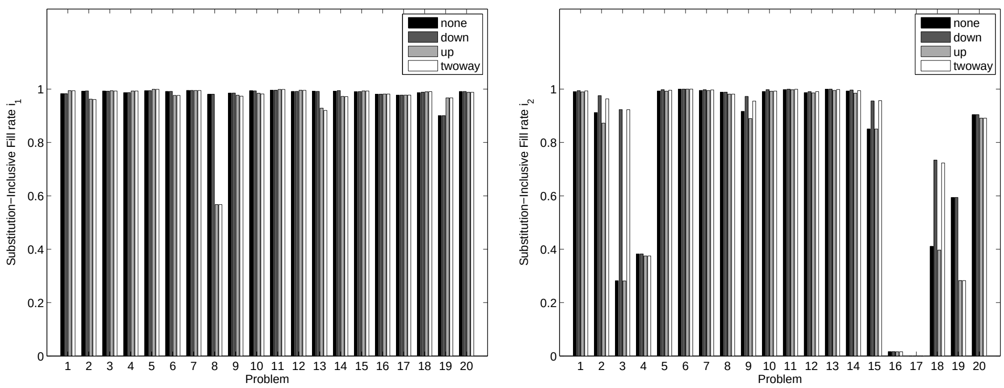
Effect of Substitution Strategy on Fill Rate. The fill rates including substitution for each of the substitution strategies are displayed for each of the problem sets in Figures 7.7, 7.8 and 7.9. In general, the fill rates are similar across all three datasets. This suggests that changes in the order sizes for production Q_p and recovery Q_r do not significantly affect the fill rates.

There is some difference in the fill rates across the four substitution strategies, however for most problems the fill rates are similar. It could be expected that the produced inventory fill rate is higher when substitution in the event of a shortage of produced inventory is permitted. However, for some problems (e.g. $B08$) this appears not to be the case. The fill rate for this problem is lower when upward substitution is permitted (upward and two-way strategies). For problem $B08$ this may be related to the low acceptance rate of upward substitution for this problem $\alpha_1 = 0.4$, however it may also be related to the replenishment policy under the various substitution strategies. The relationship between substitution strategies and the replenishment policy is investigated in Section 7.6.3.



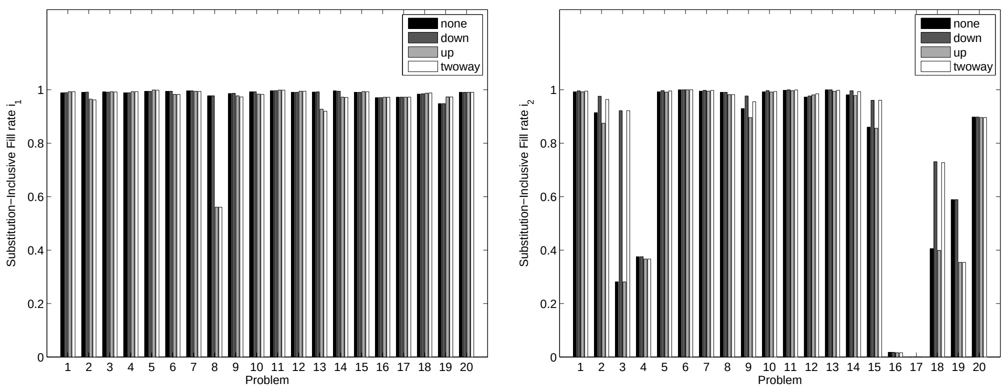
(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure 7.7: Fill rates for problem set B for all substitution strategies



(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure 7.8: Fill rates for problem set C for all substitution strategies



(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure 7.9: Fill rates for problem set D for all substitution strategies

In general the fill rates appear to be higher for produced inventory than for recovered inventory. The fill rates for recovered inventory were, on the whole, more variable across the substitution strategies, than those for produced inventory. In general, it appears that downward substitution (downward and two-way strategies) lead to higher fill rates for recovered inventory. This is expected since downward substitution is offered when there is a shortage of recovered goods. It is possible that the uncertainty surrounding the supply of recovered goods, means that downward substitution is offered more than upward substitution. This could also contribute to the difference in fill rates under the four substitution strategies.

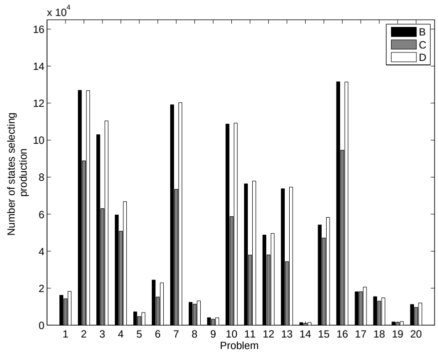
In order to further investigate when substitution is offered and when the replenishment actions are performed, we analyse the optimal policy actions in the next section.

7.6.3 Analysis of Actions

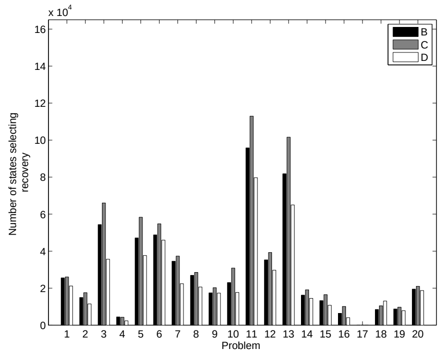
In this section the structure of the optimal policy is analysed in order to investigate the relationship between replenishment actions (production and recovery) and substitution actions (upward and downward). Two main questions will be examined in this section. Firstly, across all states, with what *frequency* are replenishment and substitution actions chosen and how is this affected by the substitution strategy. Secondly, in which states are replenishment and substitution actions selected, i.e. what inventory levels *'trigger'* certain actions. Comparisons will be drawn between the three datasets and the four substitution strategies.

Action Size and Frequency

Replenishment Actions. In this model, production can be chosen as long as there is not already a production order outstanding. This means that production could be chosen in a maximum of $21^4 \times 1 \times 2 = 388,962$ states. Recovery can be chosen as long as there is not already a recovery order outstanding and there are at least Q_r returns in stock. This means that recovery could be chosen in a maximum of $21^3 \times (W_r - Q_r + 1) \times 2 \times 1$ states. The size of the production and recovery actions is determined by the parameters Q_p and Q_r respectively. The action size for production takes values of 0 or Q_p and for recovery takes values 0 or Q_r. In this initial examination of the action frequency we focus on the two-way substitution strategy only. Table 7.6 and Figure 7.10 show the number of states in which each replenishment action is chosen under a two way substitution strategy. The data for the other substitution strategies is presented in Appendix D, in Tables D.8, D.9 and D.10.

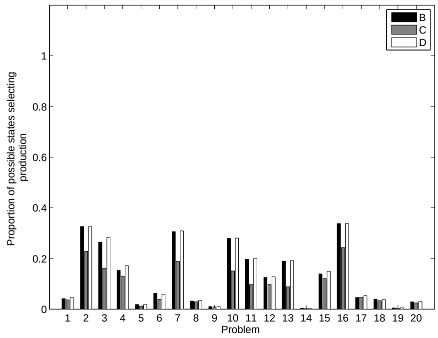


(a) Production a_p

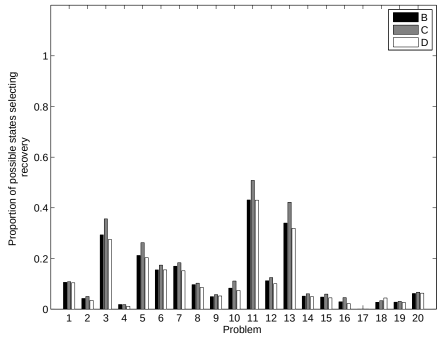


(b) Recovery a_r

Figure 7.10: Number of states with positive replenishment quantities for three problem sets, under a two-way substitution strategy



(a) Production a_p



(b) Recovery a_r

Figure 7.11: Proportion of possible states with positive replenishment quantities for three problem sets, under a two-way substitution strategy

Recall that Problem set C has a greater production size Q_p than problem sets B and D and that Problem set D has a greater recovery size Q_r than problem sets B and C. Referring to Figure 7.10 observe that when the production size is higher (C), the number of states in which production is performed is, in general, lower than when the production size is lower (B, D). A similar result is observed for recovery, although there is more variation across all three problems sets. When the size of the recovery lot Q_r is larger (D) the number of states for which it is optimal to recover is lower than

Table 7.6: Number of states in which replenishment is chosen under a two-way substitution strategy

B			C			D		
Problem	a_p	a_r	Problem	a_p	a_r	Problem	a_p	a_r
$B01$	16187	25560	$C01$	14316	26067	$D01$	18333	21189
$B02$	126923	14961	$C02$	88805	17592	$D02$	126727	11480
$B03$	102950	54345	$C03$	63006	66028	$D03$	110394	35639
$B04$	59596	4466	$C04$	50794	4331	$D04$	66785	2363
$B05$	7242	47174	$C05$	4688	58335	$D05$	6853	37631
$B06$	24413	48818	$C06$	15225	54771	$D06$	22883	45963
$B07$	119131	34566	$C07$	73435	37319	$D07$	120232	22438
$B08$	12378	26938	$C08$	11363	28528	$D08$	13153	20624
$B09$	4079	17451	$C09$	3241	20270	$D09$	4037	17341
$B10$	108695	23067	$C10$	58715	30873	$D10$	109182	17718
$B11$	76423	95796	$C11$	37912	112947	$D11$	77909	79672
$B12$	48724	35315	$C12$	38018	39276	$D12$	49608	29736
$B13$	73772	81840	$C13$	34351	101585	$D13$	74617	64958
$B14$	1422	16245	$C14$	1046	19097	$D14$	1456	14504
$B15$	54197	13305	$C15$	47131	16516	$D15$	58276	10789
$B16$	131510	6440	$C16$	94574	10126	$D16$	131430	4070
$B17$	18081	0	$C17$	18144	0	$D17$	20580	0
$B18$	15434	8493	$C18$	13027	10467	$D18$	14828	13074
$B19$	1787	8744	$C19$	1516	9733	$D19$	2000	7856
$B20$	11209	19514	$C20$	9613	21035	$D20$	12007	18702

when Q_r is lower (B, C). However, unlike the production case, the number of states in which recovery is performed differs between B and C for most problems, despite the fact that the recovery size is the same. Recovery is performed in a larger number of states in problem set C than B. This suggests that the production lot size (which differs between B and C) may affect the frequency of recovery.

However, these graphs could be slightly misleading as the number of states in which recovery could be performed differs for each problem. In order to take this into account, the proportion of possible states in which recovery is performed is examined. This proportion is calculated as the number of states performing recovery out of the states in which it is possible to perform recovery. Figure 7.11 shows these proportions for production and recovery. The data for these graphs is available in Appendix D, Table D.12. Referring to Figure 7.11, observe that Figure 7.11a, which shows the proportion of states in which production is selected, has the same shape as Figure 7.10 - this is expected as the number of states in which production can be chosen is the same across all problems. Figure 7.11b shows the proportion of states in which it is optimal to perform recovery. When the number of states in which recovery can be performed is taken into account, the frequency of recovery appears similar for problem sets B and D. Furthermore, when Q_p is larger (problem set C), recovery is performed more often.

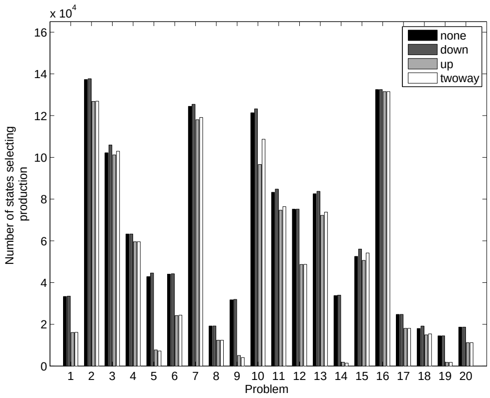
This could be due to the fact that production is performed less frequently for problem set C (as observed in Figure 7.10), therefore there are more states in which recovery could be performed. Recall that production and recovery cannot both be performed at the same decision epoch. This highlights the complicated nature of the relationship between production and recovery under the optimal policy.

Note that for problems $B17$, $C17$ and $D17$ it is never optimal to recover. This could be due to values of some of the parameters used in these problems (see Table 7.3). Firstly, the setup cost of recovery in problem 17 is the same as the setup cost of production, making recovery relatively more expensive than in the other problems. Secondly, the sale price of a recovered item is about half the price of a produced item and thirdly, the holding cost of recovered items is quite high compared to that of produced items. The combination of these factors make recovery less desirable compared with production. As observed in Section 7.6.2, the recovered-inventory fill rate is zero for problem 17. This lack of recovery is the likely cause. The states which trigger certain replenishment or substitution actions will be examined later in this section.

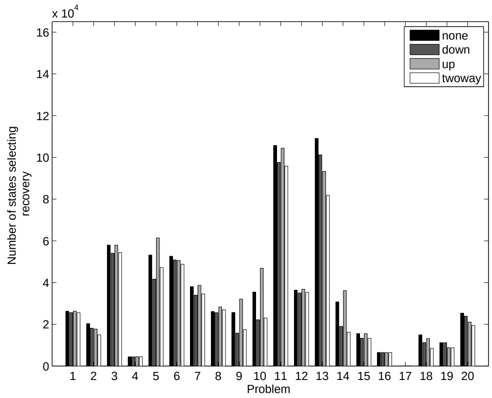
Frequency of Production and Recovery Under Substitution Strategies. Thus far, the frequency of replenishment has been considered for the two-way substitution strategy only. However, the ability to offer substitution may influence whether or not production and recovery are chosen. For instance, it may be the case that in some situations it is better to not produce and to allow upward substitution of recovered goods to cover any shortages.

Figure 7.12 shows the numbers of states in which production and recovery are chosen under the four substitution strategies for datasets B, C and D. The number of states in which recovery can be performed is not affected by the substitution strategy, therefore the number, rather than proportion, of states is used in these Figures. The data for these graphs is available in Appendix D, Tables D.8, D.9 and D.10.

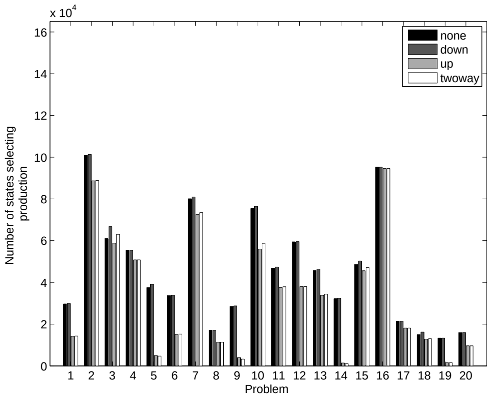
Upward substitution is offered in the event of a shortage of produced goods. When upward substitution can be offered (under the upward and two-way strategies), in general, the number of states in which production is performed is lower, than when upward substitution cannot be offered. This trend is observed across datasets B, C,



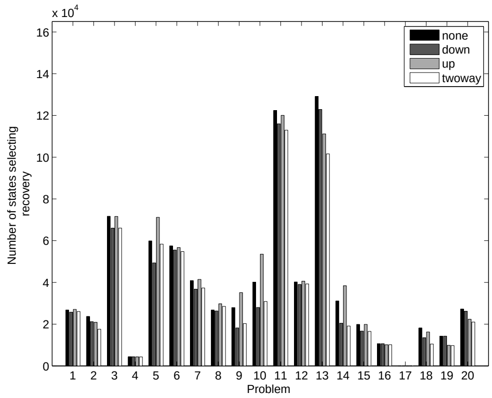
(a) Production for Problem Set B



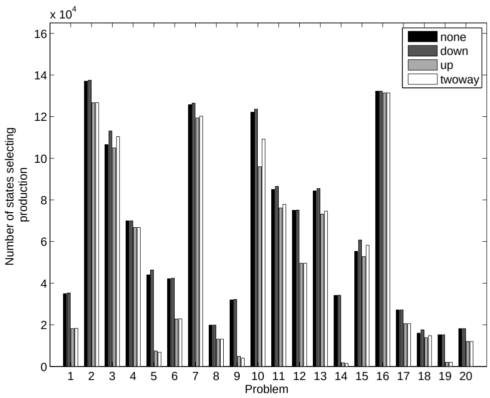
(b) Recovery for Problem Set B



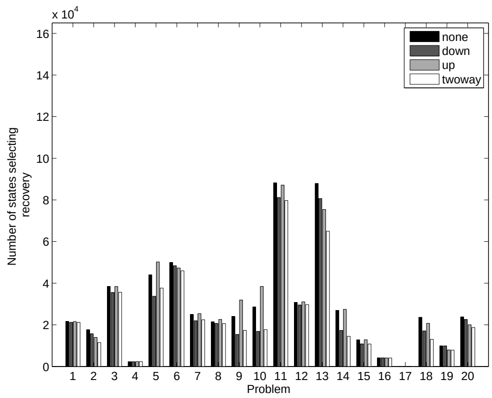
(c) Production for Problem Set C



(d) Recovery for Problem Set C



(e) Production for Problem Set D



(f) Recovery for Problem Set D

Figure 7.12: The number of states in which replenishment is chosen under the substitution strategies

and D. This suggests that, when substitution is available, the firm produces less as it is willing to let some demand for produced goods be met by recovered goods.

Recall that in the analysis of the policy performance (§7.6.2) it was observed that the fill rate for problem 08 was lower when upward substitution could be offered (under the upward and two-way strategies). Examining Figure 7.12, notice that for problem 08, the number of states in which production is performed is lower for the upward and two-way strategies. This suggests that one reason for the lower fill rate could be that when substitution is offered, production can be performed less frequently.

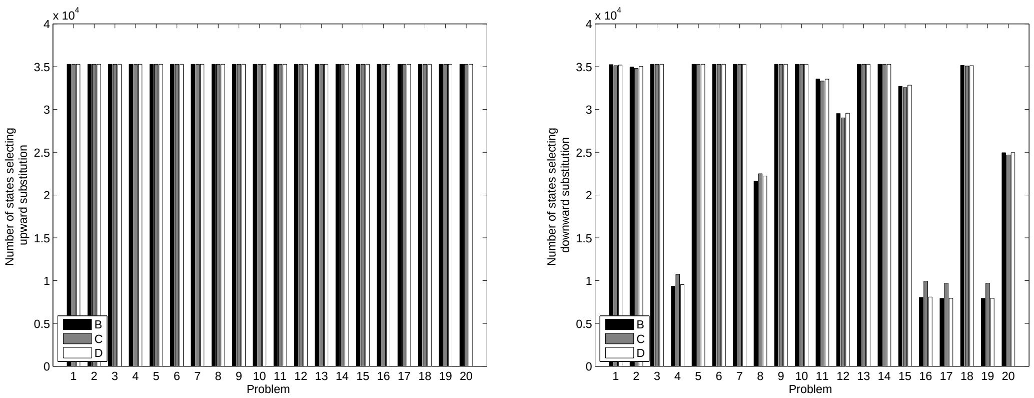
Downward substitution is offered in the event of a shortage of recovered goods. When downward substitution can be offered (under the downward and two-way strategies) then the number of states performing recovery is lower, in general, than when substitution cannot be offered. However, this trend does not seem to hold across all problems.

Substitution Actions. Upward substitution can only be offered when $i_1 = 0$ and $i_2 > 0$, and downward substitution can only be offered when $i_2 = 0$ and $i_1 > 0$. Therefore, each type of substitution can be offered in a maximum of $1 \times 20 \times 21^2 \times 4 = 35280$ states. In the following analysis we focus on the two-way substitution strategy because under the other three strategies, one or both of the substitution actions will always take a value of 0. Table 7.7 shows the number of states for which it is optimal to offer each type of substitution under a two-way substitution strategy. Data for the other substitution strategies is available in Appendix D, in Tables D.13, D.14 and D.15.

Figure 7.13 shows the number of states for which it is optimal to offer each type of substitution, under a two-way substitution strategy. Observe that it is optimal to offer upward substitution in all of the possible 35280 states. For downward substitution on the other hand, there are some problems which always offer substitution, and some which sometimes offer substitution. Across the 20 base problems the number of states offering downward substitution is quite variable, however for specific problems, there is not much difference between the data sets B, C and D. This suggests that for these problems the order size for production and recovery does not have a major influence on the substitution policy.

Table 7.7: Number of states in which substitution is offered for a two-way substitution strategy, out of a possible 35280 states

B			C			D		
Problem	a_{s1}	a_{s2}	Problem	a_{s1}	a_{s2}	Problem	a_{s1}	a_{s2}
$B01$	35280	35246	$C01$	35280	35124	$D01$	35280	35184
$B02$	35280	34961	$C02$	35280	34812	$D02$	35280	35028
$B03$	35280	35280	$C03$	35280	35280	$D03$	35280	35280
$B04$	35280	9366	$C04$	35280	10737	$D04$	35280	9531
$B05$	35280	35280	$C05$	35280	35280	$D05$	35280	35280
$B06$	35280	35280	$C06$	35280	35280	$D06$	35280	35280
$B07$	35280	35280	$C07$	35280	35280	$D07$	35280	35280
$B08$	35280	21623	$C08$	35280	22489	$D08$	35280	22220
$B09$	35280	35280	$C09$	35280	35280	$D09$	35280	35280
$B10$	35280	35280	$C10$	35280	35278	$D10$	35280	35280
$B11$	35280	33563	$C11$	35280	33325	$D11$	35280	33546
$B12$	35280	29526	$C12$	35280	29014	$D12$	35280	29553
$B13$	35280	35280	$C13$	35280	35280	$D13$	35280	35280
$B14$	35280	35280	$C14$	35280	35280	$D14$	35280	35280
$B15$	35280	32709	$C15$	35280	32561	$D15$	35280	32838
$B16$	35280	8043	$C16$	35280	9951	$D16$	35280	8085
$B17$	35280	7938	$C17$	35280	9702	$D17$	35280	7938
$B18$	35280	35162	$C18$	35280	35074	$D18$	35280	35124
$B19$	35280	7938	$C19$	35280	9702	$D19$	35280	7938
$B20$	35280	24951	$C20$	35280	24683	$D20$	35280	24959



(a) Upward substitution a_1

(b) Downward substitution a_2

Figure 7.13: Number of states which offer substitution for three problem sets under a two-way substitution strategy

Interestingly, for problem 16 and 17 it is optimal to offer downward substitution in approximately a quarter of possible states. Given the low fill rates, it is surprising that this figure is so high. One might have expected that it was never optimal to offer substitution. However it is possible that during the course of the simulation the system rarely moves to the states in which it is optimal to offer substitution.

Trigger-States and Action

In this section the relationships between the states and selected actions are investigated. Throughout this section it is assumed that the two-way substitution strategy is used.

Replenishment Actions. The inventory levels are taken into account when the action is selected, therefore it is interesting to investigate which inventory levels 'trigger' certain actions. In the case of production, the level of produced inventory (i_1) when production is selected is of particular interest. The states in which production is selected as the optimal action can be identified as they have a positive value for their production action, i.e., $a_p > 0$. In the case of recovery it is interesting to examine the level of returns in stock (i_r) when recovery is performed, and the level of recovered inventory (i_2) when recovery is performed. For both these inventories we are only interested in the states for which the optimal action is to perform recovery. These states can be identified as they has a positive value for their recovery action, i.e., $a_r > 0$. These states will be referred to as "trigger states".

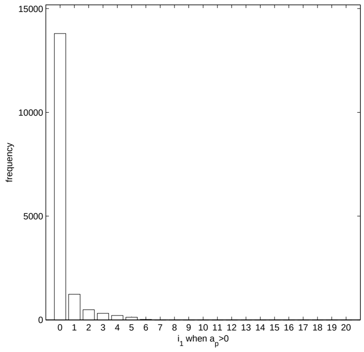
The trigger states for problem $B01$ are discussed as an example, and then the trigger levels for all problems will be examined using boxplots. Figure 7.14a shows the frequency with which the production was performed for different levels of produced inventory. Notice that production is most commonly performed if the produced inventory level is 0, however it is performed in some states if the inventory is less than 6. These graphs could be used to obtain parameters for an 'order-up-to' structured policy.

Figures 7.14b and 7.14c show the initial state of recovered inventory i_2 and the initial state of the returned inventory i_r when recovery is chosen ($a_r > 0$). Notice that recovery is performed if the recovered inventory level is 1 or less, but also when recovered inventory is 18 or more. The purpose of recovery is primarily to replenish the recovered and component inventories. When recovery is performed at these high levels of i_2, the intention cannot be to replenish the recovered inventory, instead it may be to replenish the components inventory. However since recovered goods which do not fit into the recovered or component inventories are discarded, recovery performed at high levels of recovered inventory could done in order to dispose of excess returns.

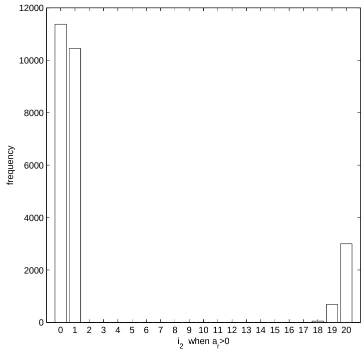
Figures 7.15a and 7.15b show the levels of the returned and component inventories when there are high levels of recovered inventory and recovery is performed for problem $B01$. If recovery is being performed in order to dispose of returns, then it would be expected that if recovered inventory is near capacity ($i_2 = 19$ or $i_2 = 20$), the component inventory level would also be near capacity. However, as shown by these Figures, while recovery is performed for high levels of returned inventory, the component inventory is comparatively low. In fact, the maximum component inventory level for which recovery is performed is 12, which is equivalent to the required inventory capacity needed if the entire recovery batch were to enter the component inventory ($U_c - Q_r = 20 - 8$). This suggests that for this problem, recovery is performed at high levels of recovered inventory in order to replenish the component inventory, not to dispose of returns.

Figure 7.16 shows the level of produced inventory when production is selected (i.e. the level of produced inventory which triggers production), under two-way and no substitution strategies. This Figure summarises the information in Figure 7.14a for all problems. First consider the trigger states under the two-way substitution strategy. Notice that for about half of the problems, production is most commonly performed when produced inventory is equal to zero ($i_1 = 0$) and is occasionally performed at higher inventory levels. For the remainder of the problems the trigger state is distributed over a range of states, approximately between 0–12. It could be possible that the problems with the more distributed trigger states have a higher relative lost sales cost and sale price, than the problems with trigger states of $i_1 = 0$.

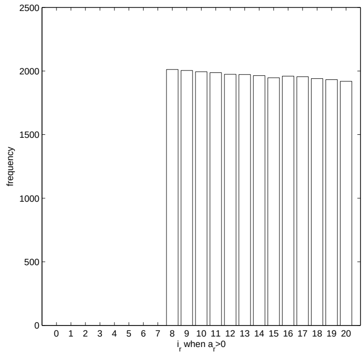
For the problems which mostly order when the inventory level is 0, this suggests that the firm should wait until the inventory has run out before placing an order. With this in mind, compare the trigger levels under the two-way and no substitution strategies. The trigger levels are higher when substitution is not possible. This suggests that the optimal policy is willing to wait until inventory reaches zero, because substitution can be used to meet demand which arrives during the production lead time. This is not possible under the no-substitution strategy, therefore the order must be placed sooner.



(a) Produced Inventory i_1 when $a_p > 0$

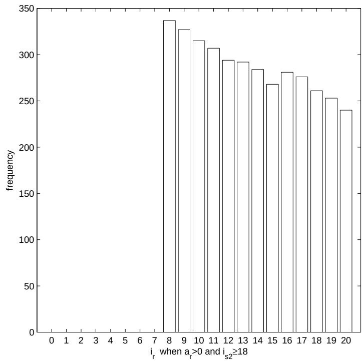


(b) Recovered Inventory i_2 when $a_r > 0$

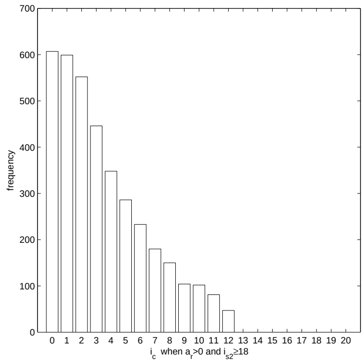


(c) Returns Inventory i_r when $a_r > 0$

Figure 7.14: Histograms showing trigger states associated with positive replenishment actions under the optimal policy for problem $B01$ for a two way substitution policy



(a) Returns Inventory i_r when $a_r > 0$ and $i_2 \geq 18$



(b) Component Inventory i_c when $a_r > 0$ and $i_2 \geq 18$

Figure 7.15: Histograms showing trigger states associated with recovery when the level of recovered inventory is high, under the optimal policy for problem $B01$ for a two way substitution policy

Recall that problem set C has a greater production order size Q_p than B and D. Trigger levels are, in general, lower for problems set C, than for B and D. This suggests that because the order sizes are larger, the optimal policy waits until the inventory levels are lower before placing an order.

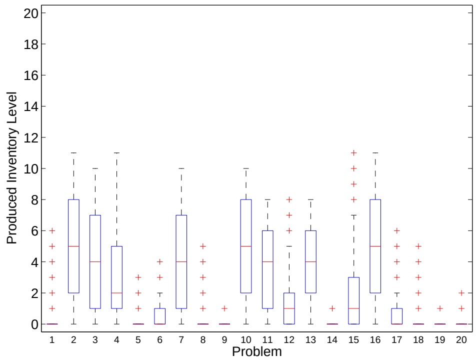
The states which trigger recovery will now be considered. Figures 7.17 and 7.18 show the level of recovered and returned inventory respectively, when recovery is performed, under a two-way recovery strategy. First observe that because recovery is not performed in problem 17, there is no box on the graph for this problem.

Considering first Figure 7.17, notice that the box plots are skewed towards 20, indicating that recovery is performed more often for higher levels of returned inventory. There must be at least Q_r returns in stock in order to perform recovery – the lower limit of these box plots reflects this.

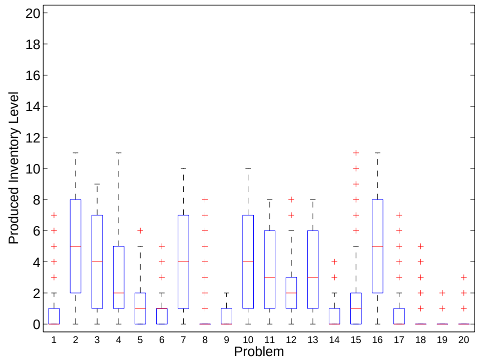
Now consider Figure 7.18. This figure shows that the levels of recovered inventory which trigger recovery are distributed across a wide range of states, but are skewed towards zero, indicating that recovery is performed more often when the level of recovered inventory is low. However, for some problems recovery is performed when recovered inventory is very high. This suggests that the objective of performing recovery in these situations is not to replenish the recovered inventory.

Recovery could be performed in order to replenish the component inventory or to dispose of excess returns. Figure 7.19 shows the level of component inventory when recovery is performed and the level of recovered inventory is high. Figure 7.20 shows the level of returned inventory when recovery is performed and the level of recovered inventory is high. These figures shows that the component inventory level is not near capacity, and that the returned inventory is distributed across a range of states, which indicates that recovery is performed in order replenish the component inventory rather than dispose of returns.

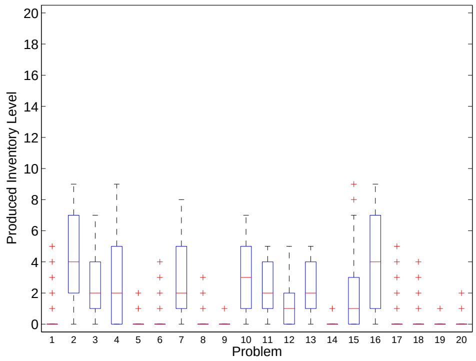
Substitution Actions. The substitution actions can take values of 0 or 1 where for a given state, a value 1 indicates that substitution should be offered and a value of 0 indicates that substitution should not be offered. In this section we investigate in which states substitution is offered, bearing in mind that substitution can only be offered in



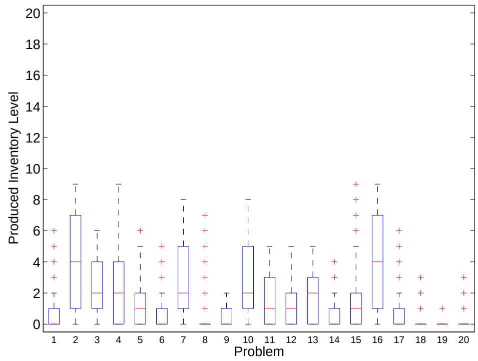
(a) Problem Set B - two-way substitution



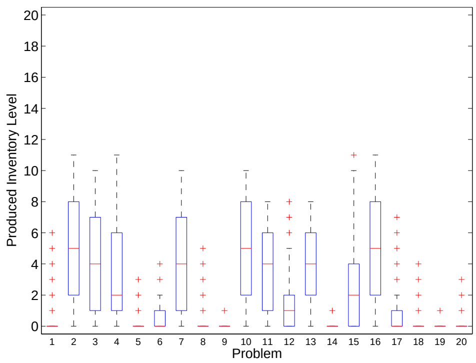
(b) Problem Set B - no substitution



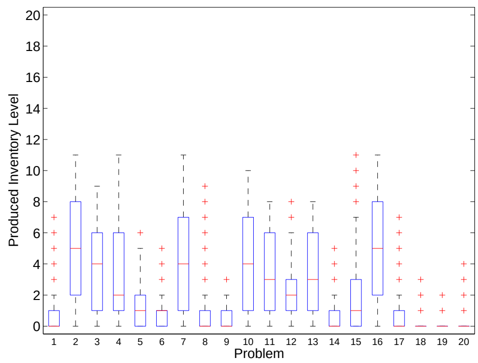
(c) Problem Set C - two-way substitution



(d) Problem Set C - no substitution

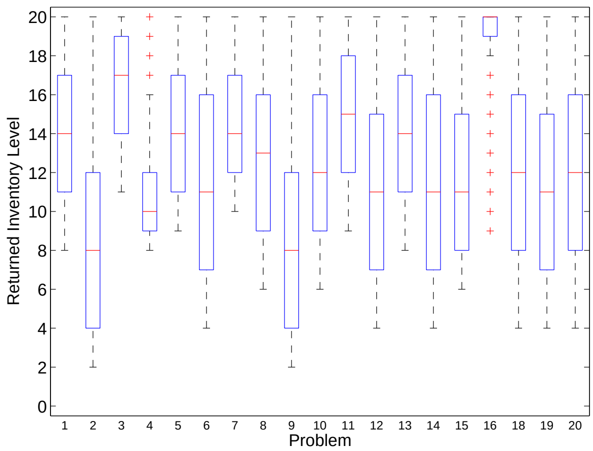


(e) Problem Set D - two-way substitution

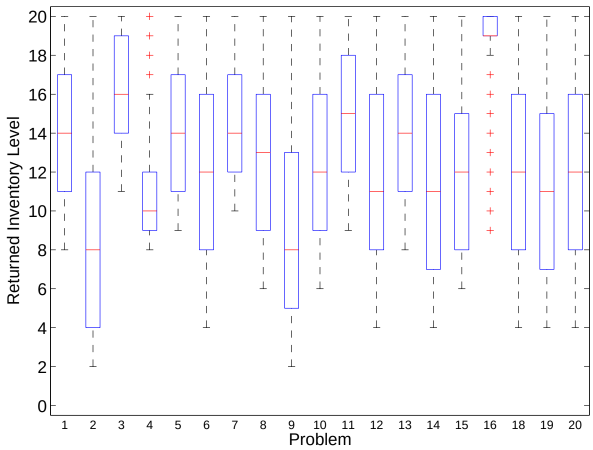


(f) Problem Set D - no substitution

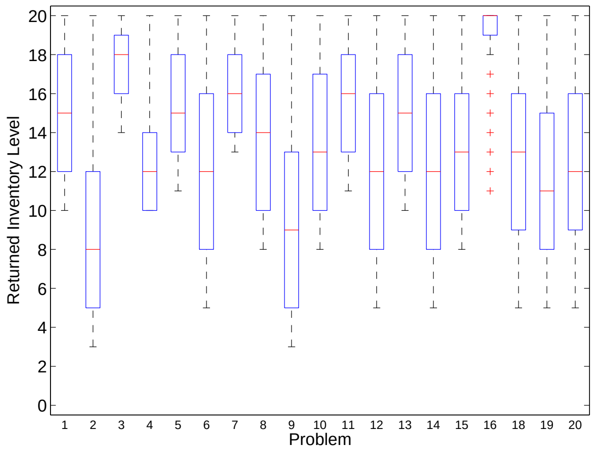
Figure 7.16: Graphs showing the level of produced inventory (trigger level) when production is performed under a two-way and no substitution policies.



(a) Problem Set B

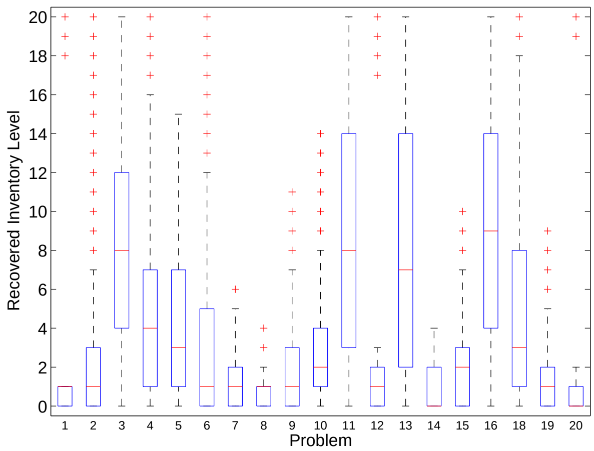


(b) Problem Set C

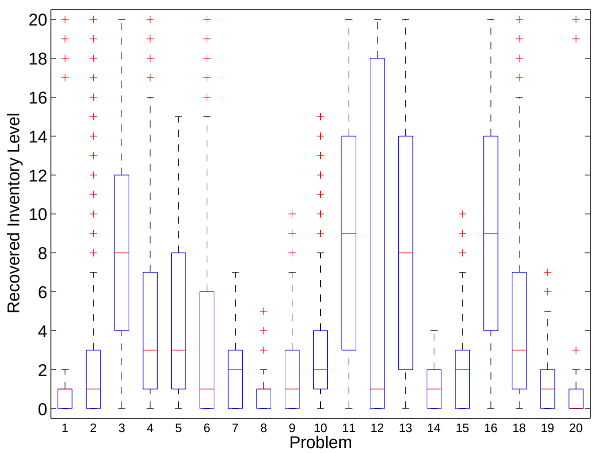


(c) Problem Set D

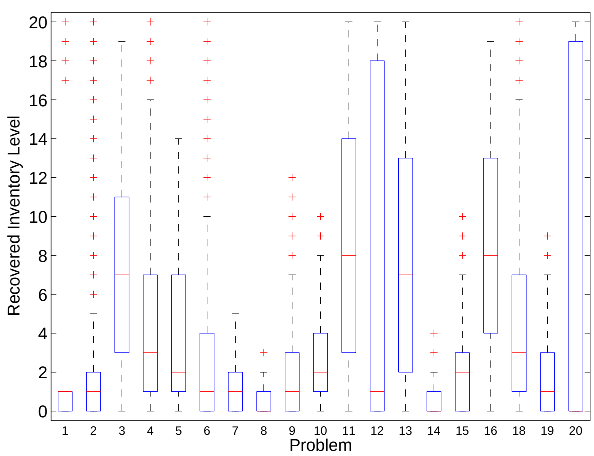
Figure 7.17: Graphs showing the level of returned inventory (trigger level) when recovery is performed under a two-way substitution policy



(a) Problem Set B

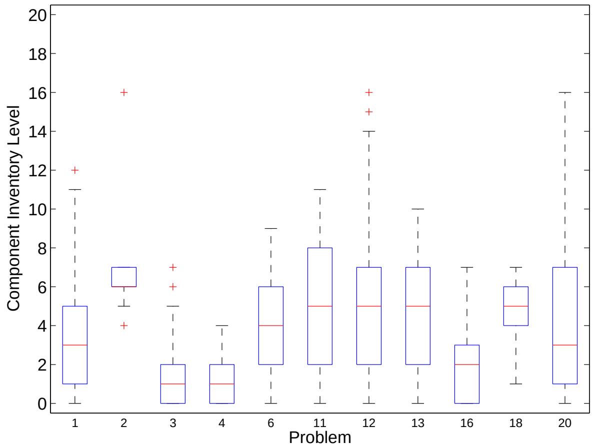


(b) Problem Set C

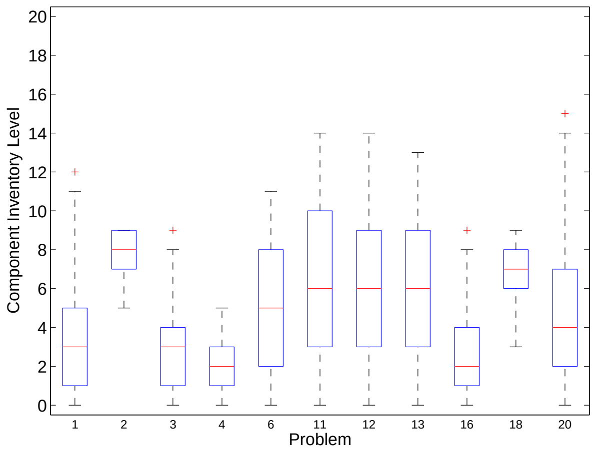


(c) Problem Set D

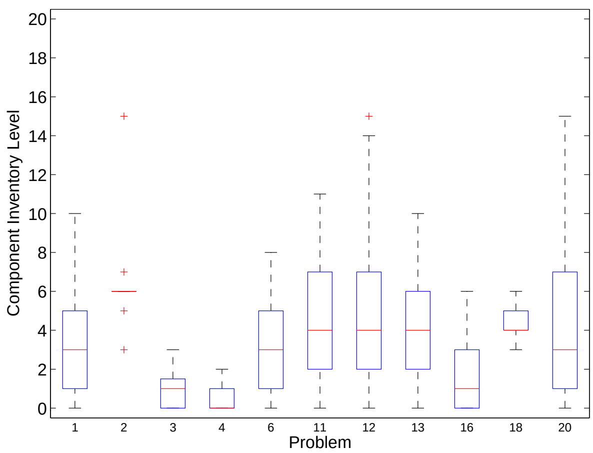
Figure 7.18: Graphs showing the level of recovered inventory (trigger level) when recovery is performed under a two-way substitution policy



(a) Problem Set B

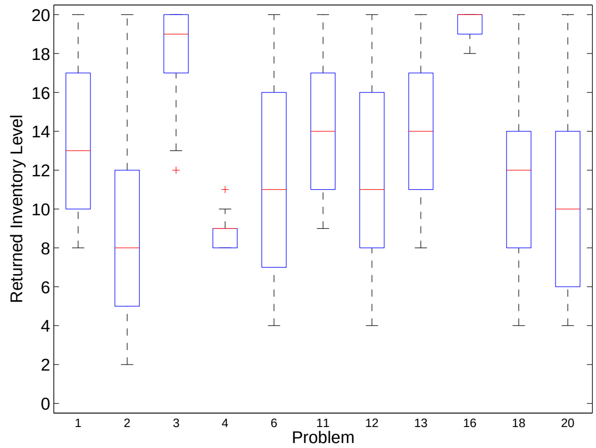


(b) Problem Set C

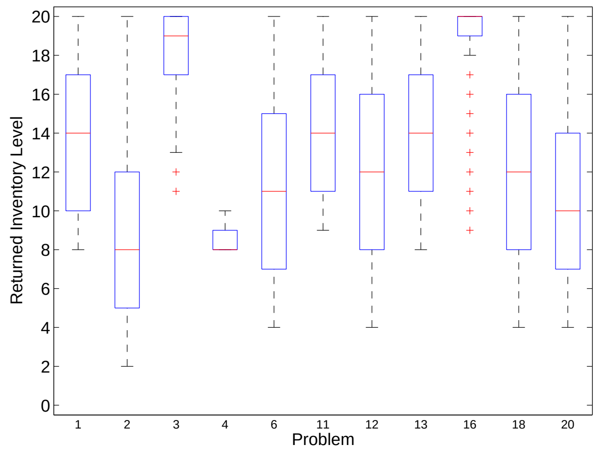


(c) Problem Set D

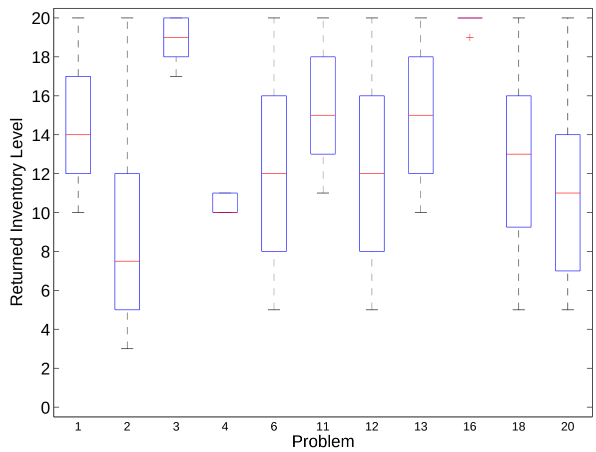
Figure 7.19: Graphs showing the level of component inventory when recovery is performed and recovered inventory is at near maximum capacity ($i_2 > 18$) under a two-way substitution policy



(a) Problem Set B



(b) Problem Set C



(c) Problem Set D

Figure 7.20: Graphs showing the level of returned inventory when recovery is performed and recovered inventory is at near maximum capacity ($i_2 > 18$) under a two-way substitution policy.

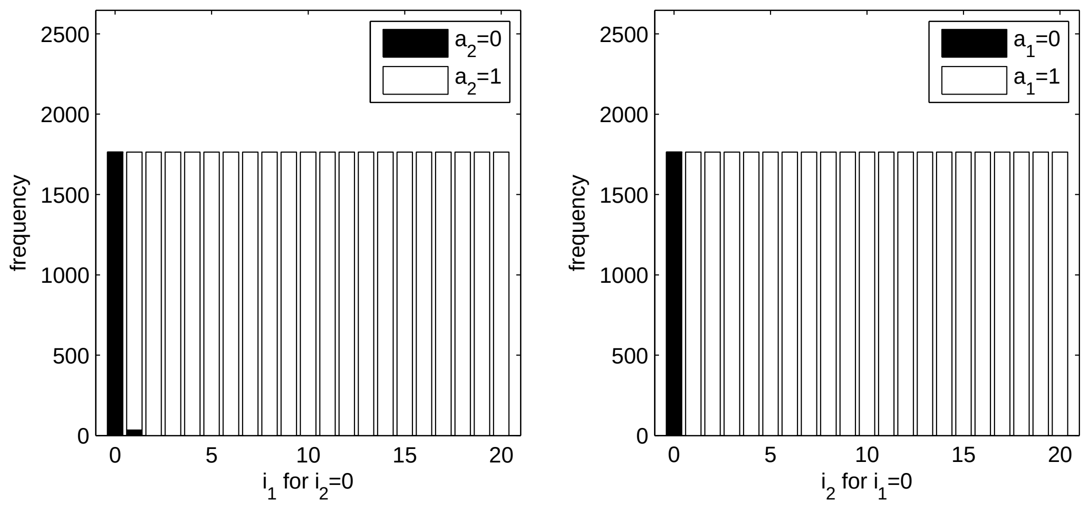


Figure 7.21: Histogram showing states in which substitution is offered for problem $B01$ under a two-way substitution strategy

some states. Upward substitution can only be offered if the produced inventory level is zero ($i_1 = 0$) and the recovered inventory level is greater than zero ($i_2 > 0$). These restrictions on the inventory levels mean that for each level of recovered inventory, there are a total of $1 \times 1 \times 21^2 \times 4 = 1764$ states in which upward substitution could be offered. Correspondingly, downward substitution can only be offered if the recovered inventory is zero ($i_2 = 0$) and the produced inventory level is greater than zero ($i_1 > 0$). For each level of produced inventory there are a total of 1764 states in which downward substitution could be offered.

Figure 7.21 shows the inventory levels and the substitution decision for problem $B01$. Notice that for this problem upward substitution is offered whenever recovered inventory is greater than 0. Downward substitution is always offered if the produced inventory level is greater than 1 and sometimes offered if the inventory is equal to 1.

Figure 7.22 summarises the level of produced inventory in stock when downward substitution is offered. Notice that for many of the problems it is optimal to offer downward substitution in states where the produced inventory is greater than one. However, for problems $4, 8, 12, 15, 16, 17, 19$ the level of produced inventory must be greater than one in order to offer substitution. The produced inventory level for which downward substitution is offered is lower for dataset C, than for B and D. This suggests that an increased production size (as in C) allows a more flexible substitution policy.

Recall that problem 17 had a fill rate of 0 for recovered inventory, despite it being optimal to offer downward substitution in approximately a quarter of possible states. Referring to Figure 7.22, notice that it is *not* optimal to offer substitution if the produced inventory level is less than 12 for problems $B17$ and $D17$, and less than 10 for problem $C17$. Referring to Table 7.5, notice that the maximum level of produced inventory over the length of the simulation 9 for problems $B17$ and $D17$, and 11 for problem $C17$. This suggests that for this problem, the system never (or very rarely) has sufficient produced inventory in stock for it to be optimal to offer downward substitution. This helps to explain why the recovered inventory fill rate for problem 17 is zero.

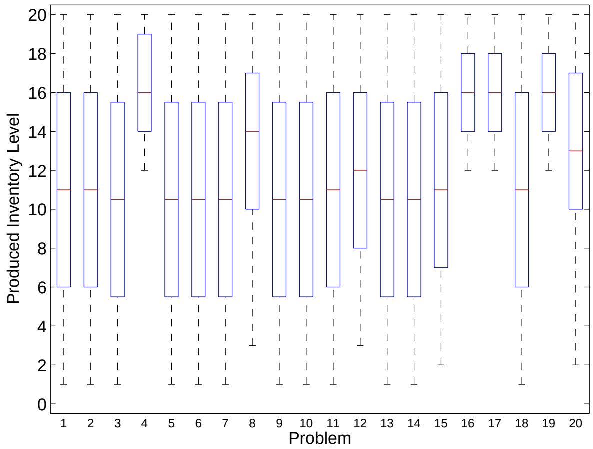
Figure 7.23 shows the level of recovered inventory when upward substitution can be offered. Observe that upward substitution can be offered for all states. This is in line with the frequency analysis which showed that substitution is offered in all allowable states. The is no cost-disincentive in offering upward substitution, since the recovered goods are sold for the same price, regardless of whether they are sold as a substitution or as a regular sale.

7.6.4 Analysis of Recovery Strategy

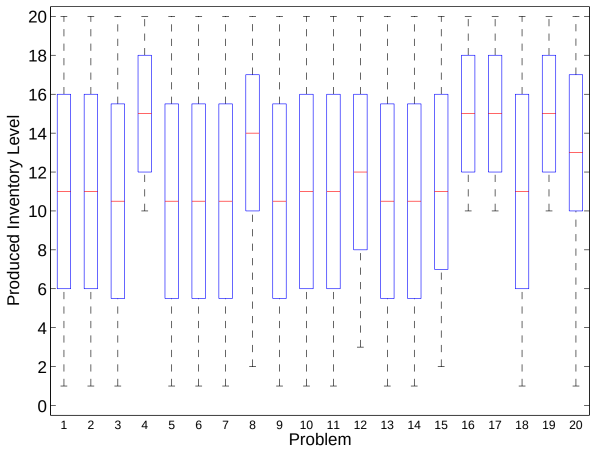
In this section we compare the performance of the system under two recovery quality strategies: recovery of high quality returns only and recovery of both high and low quality returns. In this section the focus is on the effect of the quality of returns, rather than the effect of substitution, therefore only a two-way substitution strategy is considered. The effect of the quality of returns is assessed with respect to performance (the long run average reward, the fill rates) and the policy structure (the frequency of replenishment and substitution, and states 'triggering' replenishment and substitution).

Average Reward

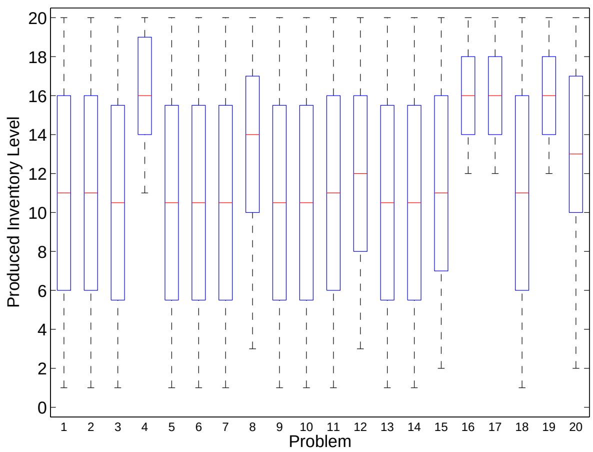
Figure 7.24 shows the long run average reward of the optimal policy calculated for the two quality strategies under a two-way substitution strategy. As expected the reward is higher when both high and low quality returns are recovered. However for some problems, the inclusion of low quality recovery does not lead to a noticeable increase in the average reward. The data used to construct these Figures is presented Table 7.8.



(a) Problem Set B

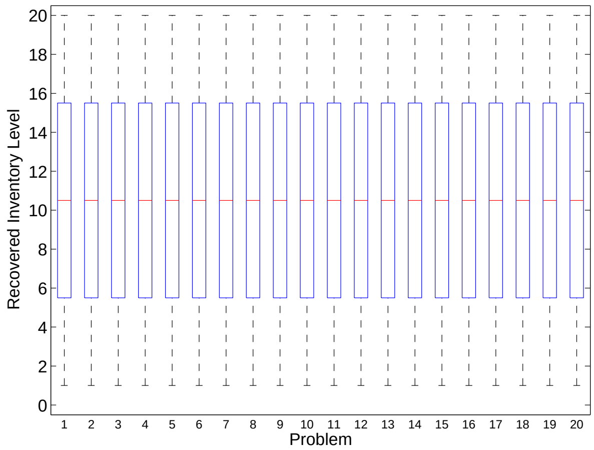


(b) Problem Set C

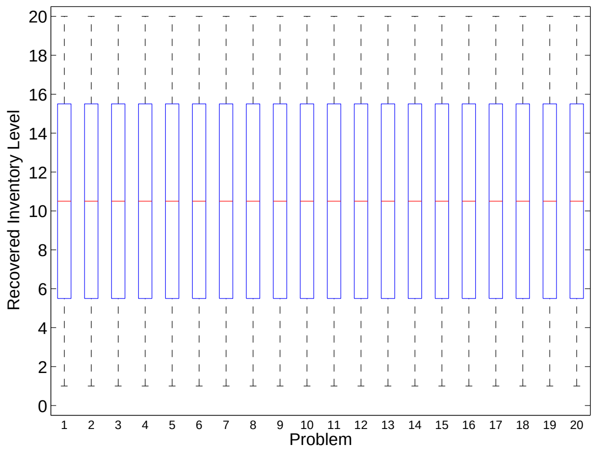


(c) Problem Set D

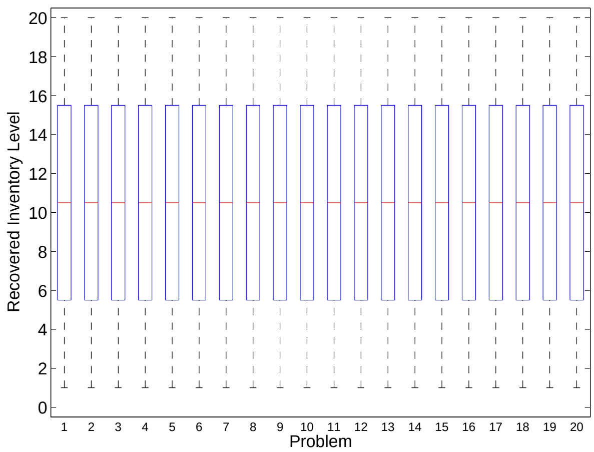
Figure 7.22: Graphs showing the levels of produced inventory for which downward substitution can be offered, under a two-way substitution strategy



(a) Problem Set B

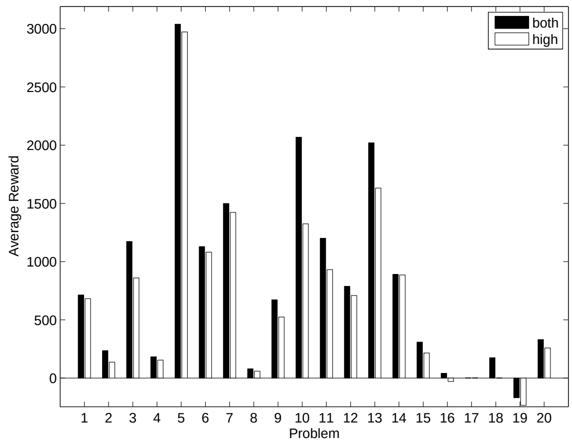


(b) Problem Set C

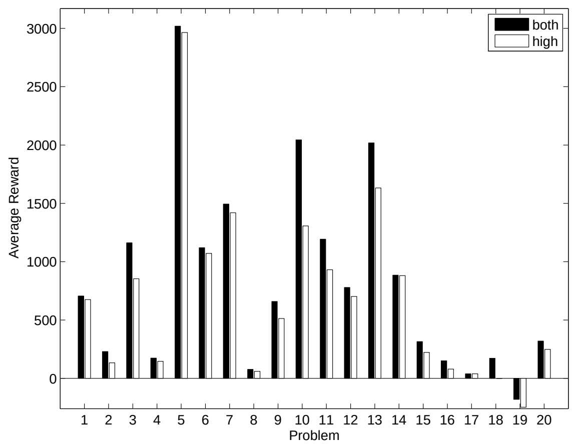


(c) Problem Set D

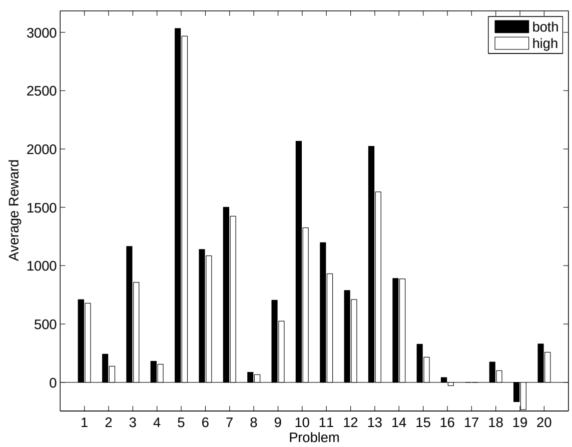
Figure 7.23: Graphs showing the levels of recovered inventory for which upward substitution can be offered, under a two-way substitution strategy



(a) Problem Set B



(b) Problem Set C



(c) Problem Set D

Figure 7.24: Average reward of the optimal policy calculated for two quality strategies under a two-way substitution strategy

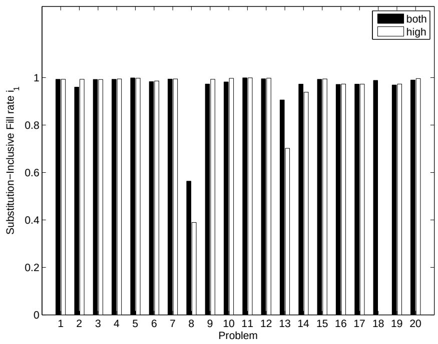
Table 7.8: Average reward of the optimal policy calculated for two quality strategies under a two-way substitution strategy

	B		C		D	
Problem	Only High	Low and High	Only High	Low and High	Only High	Low and High
01	682.3279	713.7553	675.2347	705.9478	677.9475	709.4329
02	137.5913	236.0227	133.3471	230.3007	137.5913	242.0744
03	860.8656	1173.3080	853.6164	1162.1803	856.3637	1164.7638
04	154.9781	183.5772	146.0045	174.1442	154.9781	180.9843
05	2972.9250	3038.9410	2964.3916	3019.1330	2967.1857	3032.5782
06	1081.6905	1129.6040	1071.4471	1120.0609	1084.3413	1138.8094
07	1422.6796	1499.3618	1419.5938	1494.4392	1424.2572	1500.9091
08	60.1156	79.9090	60.1156	76.9781	66.5772	85.5068
09	524.5998	672.6597	513.0015	659.4672	524.5998	704.2378
10	1324.6929	2068.4609	1306.6230	2043.9817	1324.5207	2066.4963
11	931.4246	1200.9714	930.6251	1193.2640	930.6130	1197.4479
12	709.6151	787.9194	702.0750	779.5721	709.6124	787.9270
13	1632.1267	2020.4322	1631.6483	2018.2048	1632.6060	2022.3762
14	885.7947	890.8854	880.6552	884.4565	886.8221	891.1434
15	215.7922	308.9396	222.2733	315.4457	215.7922	326.7173
16	-29.2792	41.2261	80.0500	150.0708	-28.2704	40.8775
17	1.0814	1.0814	39.4667	39.4667	1.0814	1.0814
18	0.0000	175.4455	0.0000	172.0579	100.3376	174.2624
19	-233.8644	-168.5815	-247.3999	-179.7160	-233.8644	-165.7716
20	258.7136	331.1385	248.7530	320.1212	257.7139	329.5992

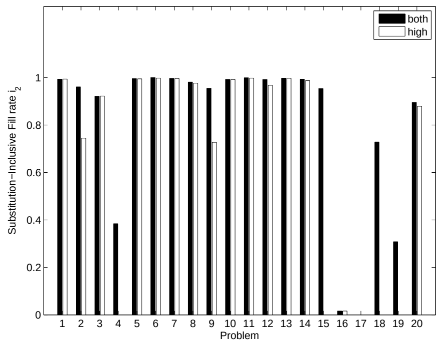
Fill Rates

Effect of Recovery Strategy on Fill Rates. The fill rates under the two recovery strategies are compared in Figure 7.25. The data for these graphs is presented in Appendix D, in Tables D.16 and D.17.

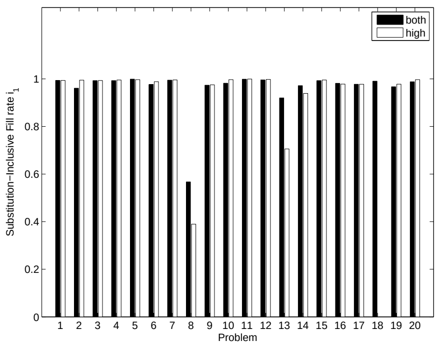
The fill rates appear to be similar across the three problem sets B, C and D. In general the fill rates are higher under the recovery strategy in which both high and low quality returns are recovered, however this is not the case for all problems (e.g. problem $B02$). For produced inventory, several problems has a substantially lower fill rate under a high-quality only recovery strategy (e.g. $B08$, $B18$). For recovered inventory, under a high-quality only recovery strategy there are more problems which have a zero fill rate for recovered inventory. This suggests that for these problems, it may become unprofitable to sell recovered goods when low quality recovery is not performed. This will be investigated further when the policy structure is examined.



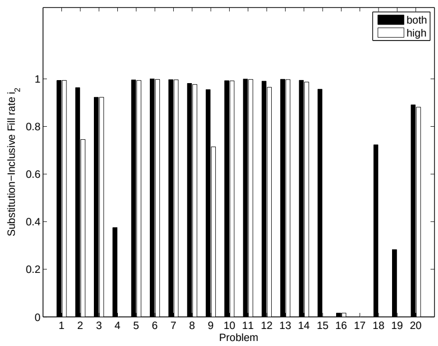
(a) Problems B - Produced Inventory i_1



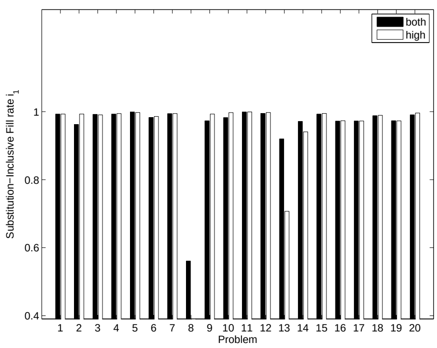
(b) Problems B - Recovered Inventory i_2



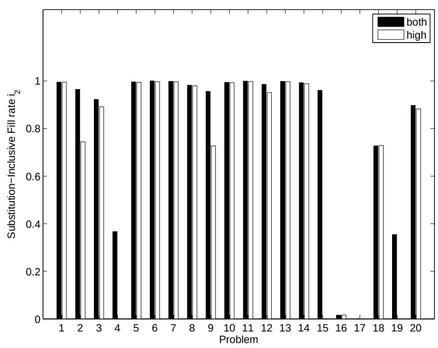
(c) Problems C - Produced Inventory i_1



(d) Problems C - Recovered Inventory i_2



(e) Problems D - Produced Inventory i_1



(f) Problems D - Recovered Inventory i_2

Figure 7.25: Substitution-inclusive fill rates for all problems sets for both quality strategies

Action Size and Frequency

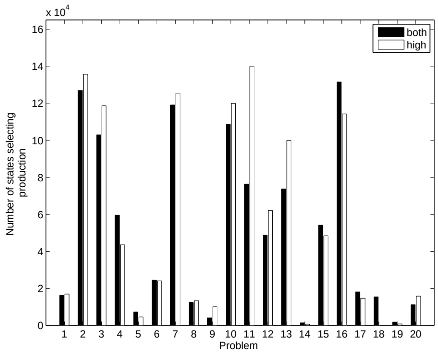
Replenishment Actions. Table 7.10 and Figure 7.26 shows the number of states in which replenishment is selected for each of the recovery strategies, under a two-way substitution strategy. The recovery strategy does not affect the maximum number of states in which each action could be chosen. Therefore within each graph, the number of states in which each action can be selected is the same. (See Table D.11 for the maximum number of states in which replenishment could be selected.) In general, production is performed more under the high-quality recovery strategy than under the both high and low recovery strategy. There are some problems for which it is optimal to never produce, or to produce very rarely ($B18$, $C18$). Recovery, on the other hand in general, is performed less frequently under a high-quality recovery strategy. The exception to this is problems ($B16$,$C16$, $D16$, $D18$). There are more problems which choose to recover infrequently (or not at all) under the high-quality only recovery strategy.

Substitution Actions. Table 7.11 and Figure 7.27 show the number of states in which substitution is offered for both recovery strategies, under a two-way substitution strategy.

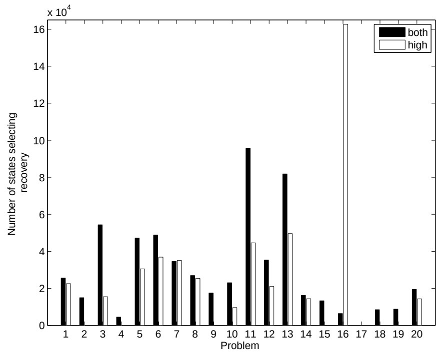
With the exception of problems $B18$ and $C18$, there is no difference in the number of states for which upward substitution is offered under each of the recovery strategies. Similarly, with the exception of problems $B18$ and $C18$, there is very little difference in the number of states for which downward substitution is offered under each of the recovery strategies. For problems $B18$ and $C18$ it is not optimal to perform either type of substitution under the high-quality only recovery strategy. As observed above it is not optimal to produce or recovery for this problem, and as such the inventory levels of produced and recovered inventory remain at zero through out the simulation (see Table 7.9).

Trigger-states and Actions

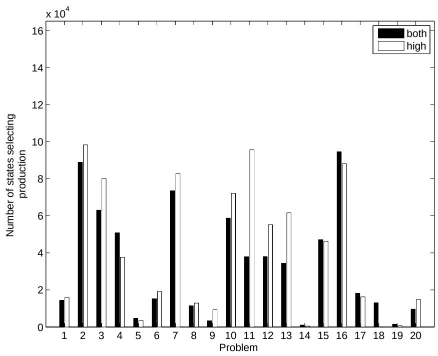
In this section we compare when replenishment and substitution performed under each of the recovery strategies.



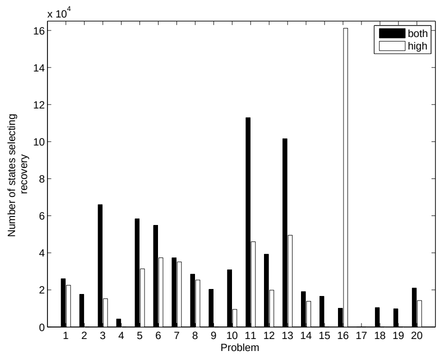
(a) Production a_p for Problem Set B



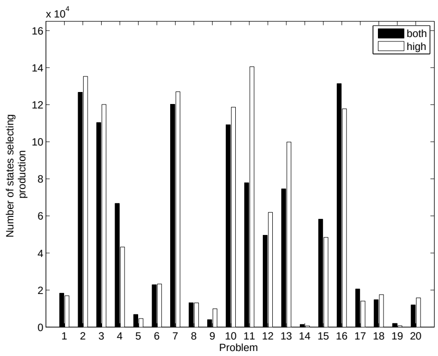
(b) Recovery a_r for Problem Set B



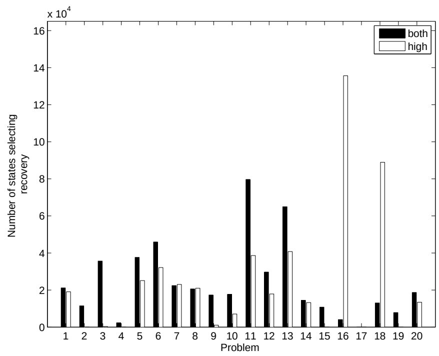
(c) Production a_p for Problem Set C



(d) Recovery a_r for Problem Set C

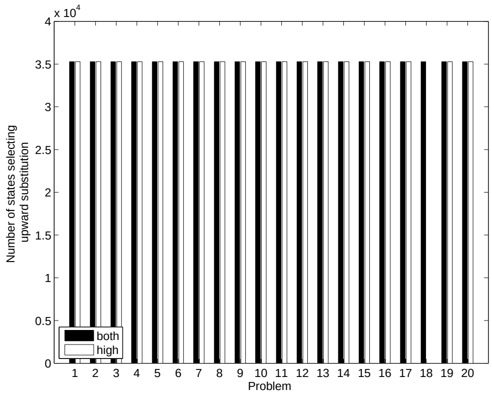


(e) Production a_p for Problem Set D

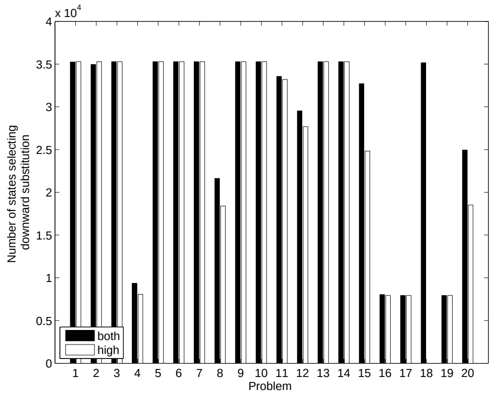


(f) Recovery a_r for Problem Set D

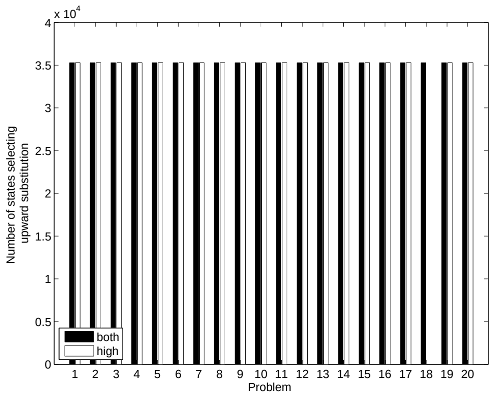
Figure 7.26: Number of states with positive replenishment quantities under a two-way substitution strategy for both quality strategies.



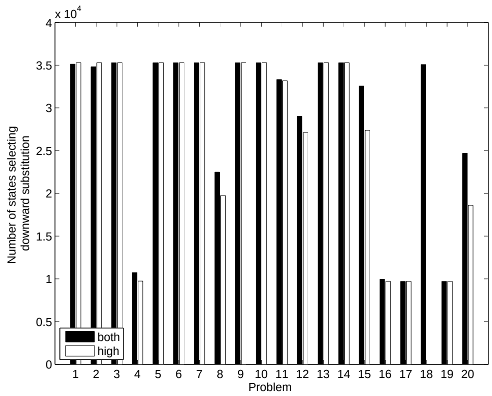
(a) Upward substitution for Problem Set B



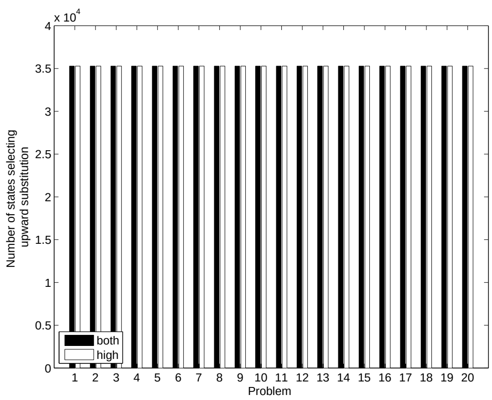
(b) Downward substitution for Problem Set B



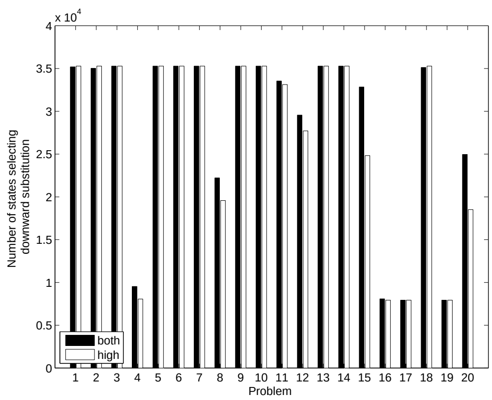
(c) Upward substitution for Problem Set C



(d) Downward substitution for Problem Set C



(e) Upward substitution for Problem Set D



(f) Downward substitution for Problem Set D

Figure 7.27: Graphs showing the number of states in which substitution is chosen under the recovery strategies

Replenishment Actions. Figure 7.28 shows the levels of produced inventory which trigger production. Notice that the trigger levels are lower under a high quality only recovery strategy. This could be because under a high-quality only strategy, components are bought when needed and are not stored (and thus the level of components is zero), therefore production is performed purely to replenish stock, rather than use up stored components (see Table 7.9).

Figure 7.29 shows the level of recovered inventory when recovery is performed. Notice that fewer problems perform recovery under a high-quality only recovery strategy, compared with under the both high and low recovery strategy. Under the high-quality strategy no problems perform recovery when recovered inventory is at capacity. This suggests that under the both high and low recovery strategy, recovery is performed in order to replenish the components inventory rather than the recovered inventory.

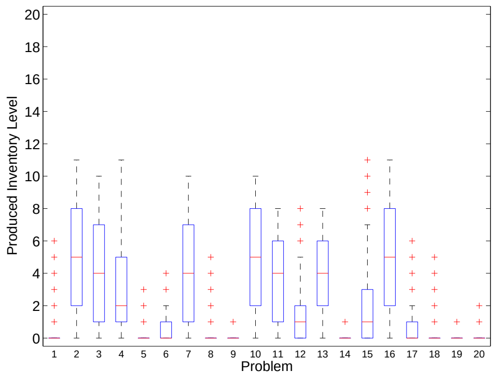
Figure 7.30 shows the level of returned inventory when recovery is performed. Apart of the fact that fewer problems perform recovery under the high-quality strategy, the graphs appear similar. Thus, the level of returns for which recovery is performed is not affected by the recovery strategy.

Substitution Actions. Figures 7.31 show the levels of produced inventory for which substitution is offered. There does not appear to be any difference between the two recovery strategies. This is perhaps to be expected since the number of states and costs are the same under both recovery strategies.

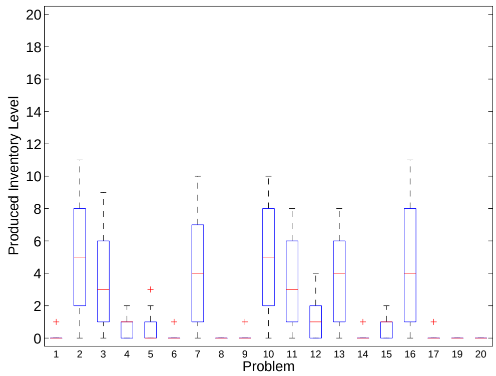
7.7 Discussion

This chapter has presented a continuous time product recovery model with separate markets and substitution.

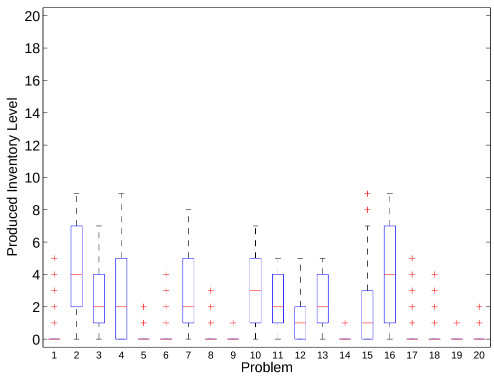
In this model, if there is a shortage of recovered goods then the firm may offer the customer a produced good instead (downward substitution), and if there is a shortage of produced goods then the firm may offer the customer a recovered good instead (upward



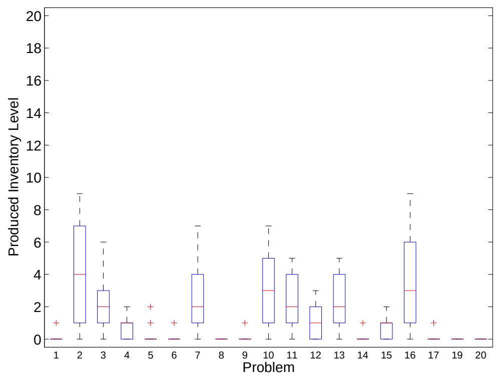
(a) Problem Set B – Both high and low quality recovery



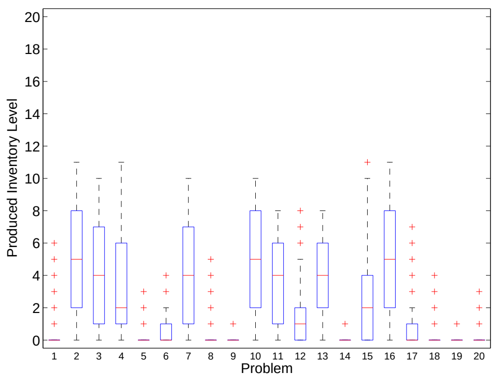
(b) Problem Set B – High quality recovery only



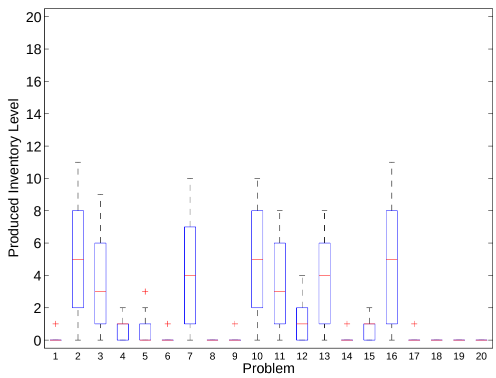
(c) Problem Set C– Both high and low quality recovery



(d) Problem Set C– High quality recovery only

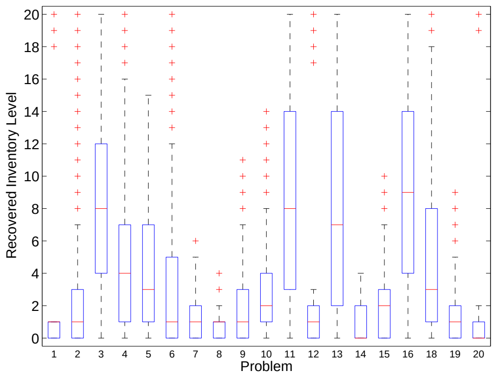


(e) Problem Set D– Both high and low quality recovery

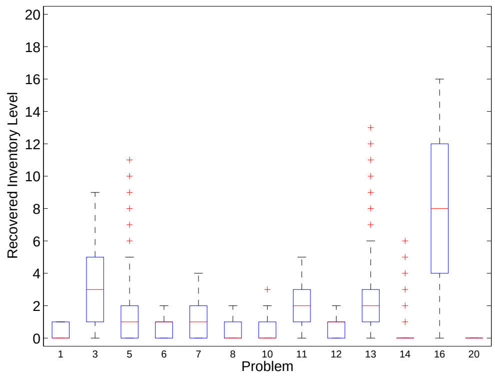


(f) Problem Set D– High quality recovery only

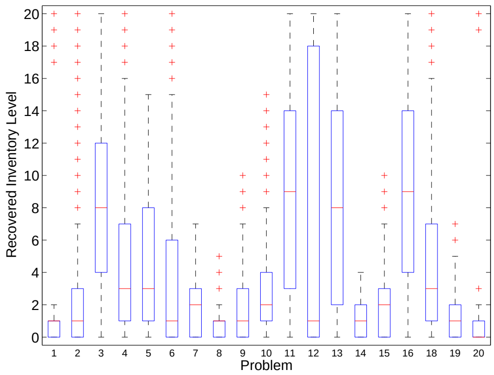
Figure 7.28: Graphs showing the level of produced inventory (trigger level) when production is performed under a two-way substitution policy for two recovery strategies.



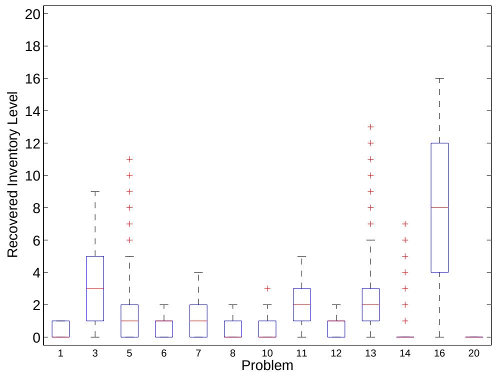
(a) Problem Set B – Both high and low quality recovery



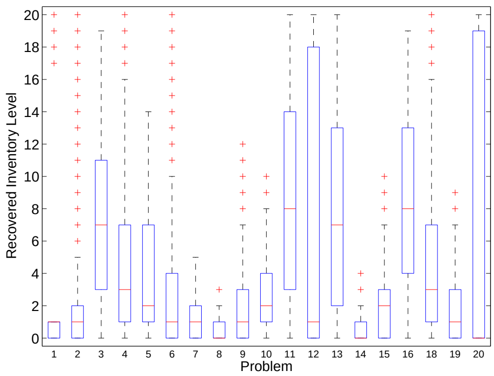
(b) Problem Set B – High quality recovery only



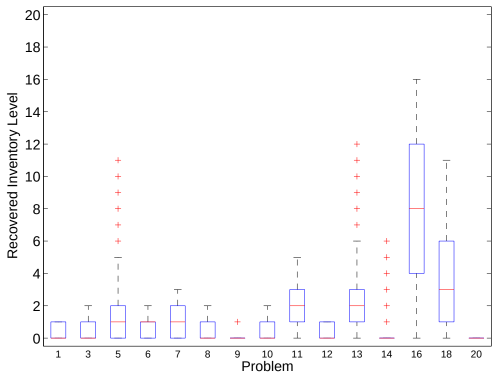
(c) Problem Set C– Both high and low quality recovery



(d) Problem Set C– High quality recovery only

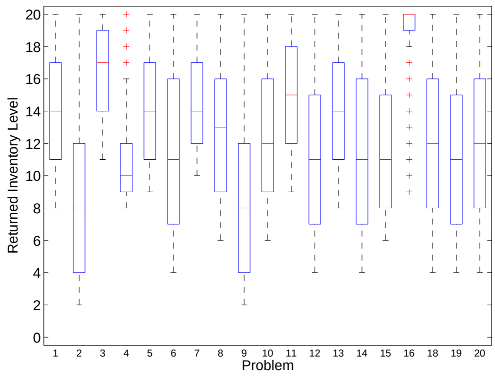


(e) Problem Set D– Both high and low quality recovery

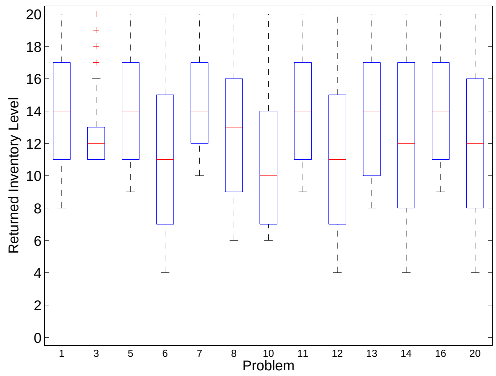


(f) Problem Set D– High quality recovery only

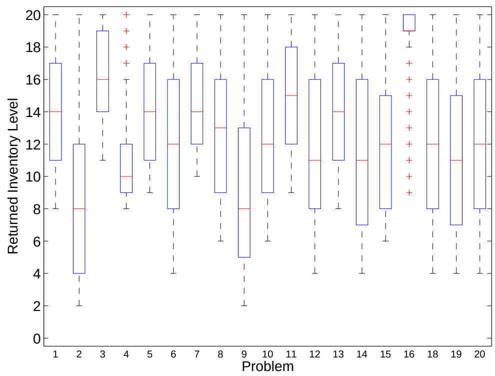
Figure 7.29: Graphs showing the level of recovered inventory (trigger level) when recovery is performed under a two-way substitution policy.



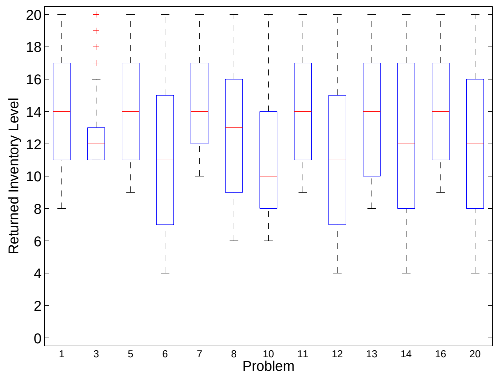
(a) Problem Set B – Both high and low quality recovery



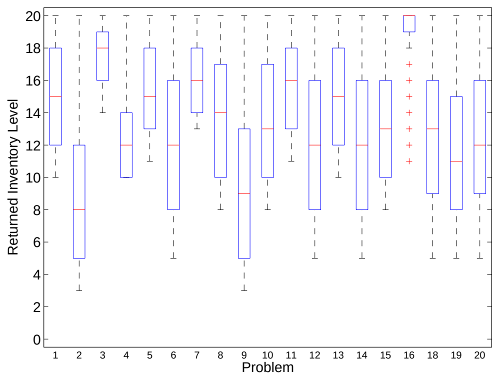
(b) Problem Set B – High quality recovery only



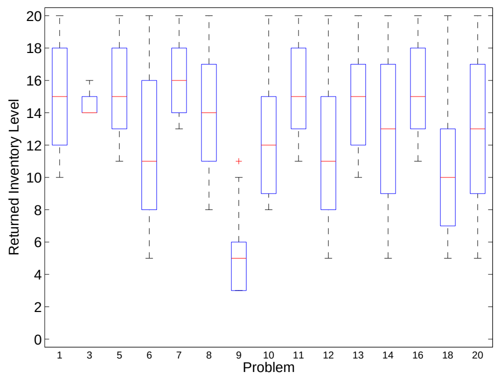
(c) Problem Set C– Both high and low quality recovery



(d) Problem Set C– High quality recovery only

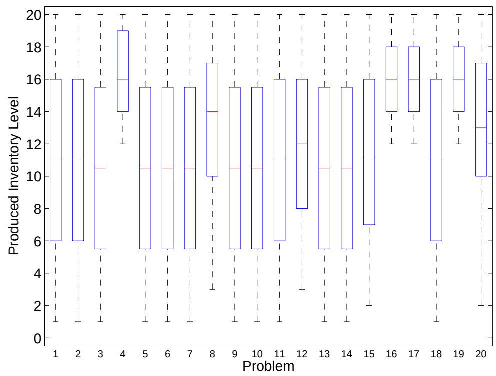


(e) Problem Set D– Both high and low quality recovery

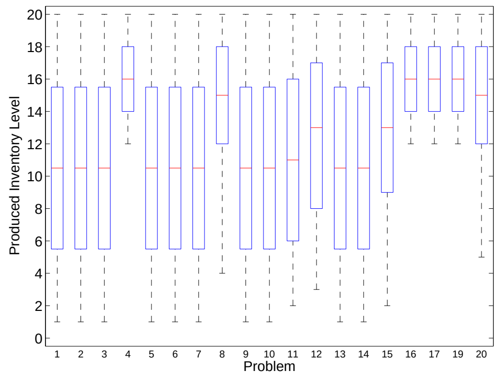


(f) Problem Set D– High quality recovery only

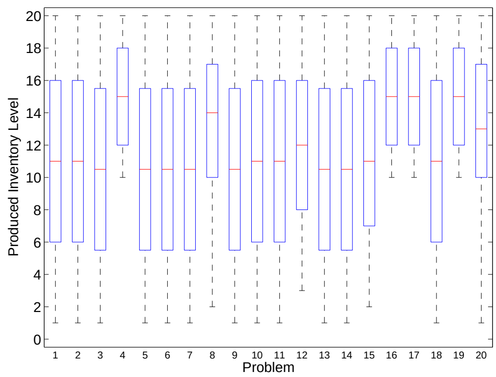
Figure 7.30: Graphs showing the level of returned inventory (trigger level) when recovery is performed under a two-way substitution policy.



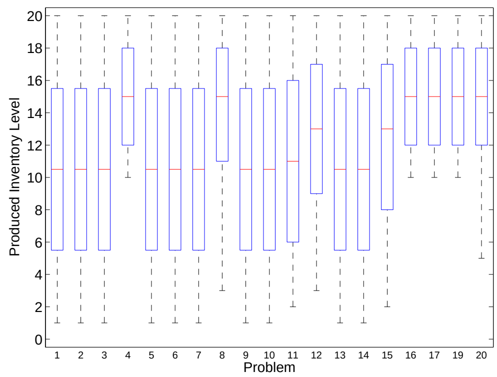
(a) Problem Set B – Both high and low quality recovery



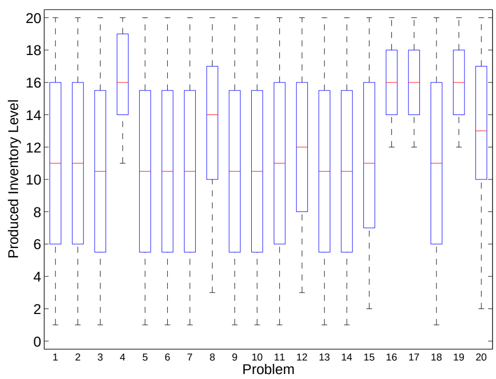
(b) Problem Set B – High quality recovery only



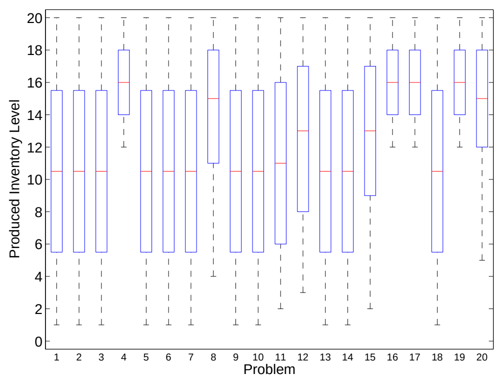
(c) Problem Set C– Both high and low quality recovery



(d) Problem Set C– High quality recovery only



(e) Problem Set D– Both high and low quality recovery



(f) Problem Set D– High quality recovery only

Figure 7.31: Graphs showing the levels of produced inventory for which substitution can be offered, under a two-way substitution strategy.

substitution). The rationale for offering substitution is that doing so allows the firm to potentially gain some revenue, rather than losing the sale and also some goodwill. The customer can choose to accept or reject the offered substitution, therefore offering substitution does not remove all uncertainty. In downward substitution, the firm offers a produced good for a reduced price and may also incur an opportunity cost associated with no longer being able to sell the produced good for the full price. In the case of upward substitution, the firm charges the price of the lower item so it misses out on the revenue it would have received had it been able to meet the demand with produced goods.

Table 7.9: Summary of inventory levels during the simulation under a two-way substitution strategy and a high quality recovery strategy

(a) Problems B

	Produced Inventory				Recovered Inventory				Returned Inventory				Component Inventory			
Problem	mean	min	max	mode	mean	min	max	mode	mean	min	max	mode	mean	min	max	mode
B01	4.889	0	9	1	4.534	0	9	2	17.979	0	20	20	0	0	0	0
B02	5.744	0	10	5	0	0	0	0	19.999	0	20	20	0	0	0	0
B03	7.880	0	14	8	0.775	0	14	0	5.659	0	20	10	0	0	0	0
B04	5.770	0	10	6	0	0	0	0	19.999	0	20	20	0	0	0	0
B05	0.593	0	11	0	10.736	0	20	12	11.446	0	20	20	0	0	0	0
B06	5.317	0	10	1	3.373	0	6	3	19.455	0	20	20	0	0	0	0
B07	6.155	0	12	4	6.228	0	14	5	17.690	0	20	20	0	0	0	0
B08	0	0	0	0	3.757	0	8	3	18.315	0	20	20	0	0	0	0
B09	5.740	0	10	2	0	0	0	0	19.999	0	20	20	0	0	0	0
B10	5.403	0	11	10	2.593	0	8	2	18.107	0	20	20	0	0	0	0
B11	7.887	0	16	4	5.868	0	13	5	14.647	0	20	20	0	0	0	0
B12	7.852	0	14	12	2.267	0	5	2	18.927	0	20	20	0	0	0	0
B13	1.299	0	15	0	11.048	0	20	14	9.129	0	20	20	0	0	0	0
B14	1.104	0	9	0	4.157	0	10	4	10.046	0	20	16	0	0	0	0
B15	5.778	0	10	9	0	0	0	0	20.000	0	20	20	0	0	0	0
B16	4.868	0	9	7	0.023	0	5	0	4.021	0	11	4	0	0	0	0
B17	4.859	0	9	7	0	0	0	0	19.999	0	20	20	0	0	0	0
B18	0	0	0	0	0	0	0	0	19.998	0	20	20	0	0	0	0
B19	4.859	0	9	3	0	0	0	0	20.000	0	20	20	0	0	0	0
B20	3.467	0	6	2	1.543	0	4	1	10.237	0	20	2	0	0	0	0

(b) Problems C

	Produced Inventory				Recovered Inventory				Returned Inventory				Component Inventory			
Problem	mean	min	max	mode	mean	min	max	mode	mean	min	max	mode	mean	min	max	mode
C01	5.888	0	11	3	4.529	0	9	3	17.996	0	20	20	0	0	0	0
C02	6.738	0	12	8	0	0	0	0	19.999	0	20	20	0	0	0	0
C03	9.324	0	17	12	0.768	0	13	0	5.642	0	20	8	0	0	0	0
C04	6.762	0	12	10	0	0	0	0	19.999	0	20	20	0	0	0	0
C05	0.753	0	14	0	10.880	0	20	12	11.304	0	20	20	0	0	0	0
C06	6.828	0	13	13	3.292	0	6	3	19.466	0	20	20	0	0	0	0
C07	7.664	0	15	10	6.229	0	14	5	17.680	0	20	20	0	0	0	0
C08	0	0	0	0	3.757	0	8	3	18.315	0	20	20	0	0	0	0
C09	5.846	0	11	3	0	0	0	0	19.999	0	20	20	0	0	0	0
C10	6.817	0	14	1	2.545	0	8	2	18.131	0	20	20	0	0	0	0
C11	9.388	0	19	14	5.824	0	14	5	14.650	0	20	20	0	0	0	0
C12	9.221	0	17	5	2.205	0	5	2	18.926	0	20	20	0	0	0	0
C13	1.664	0	18	0	10.981	0	20	14	9.202	0	20	20	0	0	0	0
C14	1.333	0	11	0	4.419	0	11	4	10.395	0	20	16	0	0	0	0
C15	6.771	0	12	10	0	0	0	0	20.000	0	20	20	0	0	0	0
C16	5.876	0	11	9	0.023	0	5	0	4.021	0	11	4	0	0	0	0
C17	5.863	0	11	3	0	0	0	0	19.999	0	20	20	0	0	0	0
C18	0	0	0	0	0	0	0	0	19.998	0	20	20	0	0	0	0
C19	5.862	0	11	4	0	0	0	0	20.000	0	20	20	0	0	0	0
C20	4.465	0	8	5	1.548	0	4	1	10.284	0	20	20	0	0	0	0

(c) Problems D

	Produced Inventory				Recovered Inventory				Returned Inventory				Component Inventory			
Problem	mean	min	max	mode	mean	min	max	mode	mean	min	max	mode	mean	min	max	mode
D01	4.882	0	9	1	5.326	0	11	3	17.732	0	20	20	0	0	0	0
D02	5.744	0	10	5	0	0	0	0	19.999	0	20	20	0	0	0	0
D03	8.091	0	14	11	0	0	0	0	19.999	0	20	20	0	0	0	0
D04	5.770	0	10	6	0	0	0	0	19.999	0	20	20	0	0	0	0
D05	0.646	0	11	0	11.036	0	20	12	11.622	0	20	20	0	0	0	0
D06	5.318	0	10	1	3.295	0	7	3	19.431	0	20	20	0	0	0	0
D07	6.143	0	12	3	7.308	0	16	6	17.394	0	20	20	0	0	0	0
D08	0	0	0	0	4.455	0	10	4	18.142	0	20	20	0	0	0	0
D09	5.740	0	10	2	0.000	0	1	0	19.997	0	20	20	0	0	0	0
D10	5.376	0	11	10	3.037	0	10	2	17.925	0	20	20	0	0	0	0
D11	7.909	0	16	7	6.639	0	15	5	14.683	0	20	20	0	0	0	0
D12	7.845	0	14	3	2.150	0	6	2	18.854	0	20	20	0	0	0	0
D13	1.380	0	15	0	10.939	0	20	13	9.548	0	20	20	0	0	0	0
D14	1.144	0	9	0	4.410	0	11	4	10.130	0	20	15	0	0	0	0
D15	5.778	0	10	9	0	0	0	0	20.000	0	20	20	0	0	0	0
D16	4.868	0	9	7	0.024	0	6	0	5.052	0	13	10	0	0	0	0
D17	4.859	0	9	7	0	0	0	0	19.999	0	20	20	0	0	0	0
D18	4.443	0	8	7	0.793	0	10	0	2.398	0	16	4	0	0	0	0
D19	4.859	0	9	3	0	0	0	0	20.000	0	20	20	0	0	0	0
D20	3.467	0	6	2	1.759	0	5	1	10.395	0	20	20	0	0	0	0

Table 7.10: Number of states in which replenishment is chosen under a two-way substitution, high-quality only strategy.

B			C			D		
Problem	a_p	a_r	Problem	a_p	a_r	Problem	a_p	a_r
$B01$	16918	22527	$C01$	15910	22541	$D01$	16930	19074
$B02$	135639	0	$C02$	98259	0	$D02$	135282	0
$B03$	118636	15488	$C03$	80127	15289	$D03$	120116	346
$B04$	43596	0	$C04$	37569	0	$D04$	43239	0
$B05$	4529	30496	$C05$	3596	31372	$D05$	4625	25110
$B06$	24068	36880	$C06$	19128	37321	$D06$	23278	32139
$B07$	125420	35055	$C07$	82813	35104	$D07$	127012	23040
$B08$	13365	25445	$C08$	12904	25357	$D08$	13115	21024
$B09$	10164	0	$C09$	9303	0	$D09$	9910	1055
$B10$	119886	9623	$C10$	72086	9512	$D10$	118702	7081
$B11$	139952	44589	$C11$	95656	46013	$D11$	140515	38640
$B12$	62047	21007	$C12$	55177	19871	$D12$	61925	17913
$B13$	99966	49612	$C13$	61615	49501	$D13$	99878	40780
$B14$	644	14386	$C14$	482	13884	$D14$	636	13254
$B15$	48426	0	$C15$	46284	0	$D15$	48405	0
$B16$	114226	162661	$C16$	88088	161194	$D16$	117798	135610
$B17$	14658	0	$C17$	16275	0	$D17$	14091	0
$B18$	0	0	$C18$	0	0	$D18$	17532	88945
$B19$	777	0	$C19$	630	0	$D19$	714	0
$B20$	15795	14306	$C20$	14848	14252	$D20$	15785	13451

Table 7.11: Number of states in which substitution is offered under a two-way substitution.

B			C			D		
Problem	a_{s1}	a_{s2}	Problem	a_{s1}	a_{s2}	Problem	a_{s1}	a_{s2}
$B01$	35280	35280	$C01$	35280	35124	$D01$	35280	35280
$B02$	35280	35280	$C02$	35280	34812	$D02$	35280	35280
$B03$	35280	35280	$C03$	35280	35280	$D03$	35280	35280
$B04$	35280	8064	$C04$	35280	10737	$D04$	35280	8064
$B05$	35280	35280	$C05$	35280	35280	$D05$	35280	35280
$B06$	35280	35280	$C06$	35280	35280	$D06$	35280	35280
$B07$	35280	35280	$C07$	35280	35280	$D07$	35280	35280
$B08$	35280	18396	$C08$	35280	22489	$D08$	35280	19572
$B09$	35280	35280	$C09$	35280	35280	$D09$	35280	35280
$B10$	35280	35280	$C10$	35280	35278	$D10$	35280	35280
$B11$	35280	33202	$C11$	35280	33325	$D11$	35280	33128
$B12$	35280	27688	$C12$	35280	29014	$D12$	35280	27713
$B13$	35280	35280	$C13$	35280	35280	$D13$	35280	35280
$B14$	35280	35280	$C14$	35280	35280	$D14$	35280	35280
$B15$	35280	24822	$C15$	35280	32561	$D15$	35280	24822
$B16$	35280	7938	$C16$	35280	9951	$D16$	35280	7938
$B17$	35280	7938	$C17$	35280	9702	$D17$	35280	7938
$B18$	0	0	$C18$	35280	35074	$D18$	35280	35280
$B19$	35280	7938	$C19$	35280	9702	$D19$	35280	7938
$B20$	35280	18518	$C20$	35280	24683	$D20$	35280	18515

The uncertainty associated with the supply of the recovered inventory, in terms of quality, quantity and timing, means that planning and ensuring that there is sufficient inventory in stock is much harder. Offering downward substitution allows the firm to somewhat mitigate these risks. In this model, there is also uncertainty associated with the lead time of the production and recovery orders. On the one hand, offering substitution allows the firm to mitigate the risks associated with the arrival of replenishment orders, however on the other hand, offering substitution carries the risk that the inventory (of the offered good) may run out before the next replenishment arrives. The firm must weigh up these costs and benefits when deciding to offer substitution.

The analysis conducted in this chapter has shown that allowing both upward and downward substitution can enable firms to receive an increased average reward. In general, the greatest increase in reward is attained by allowing a substitution strategy which includes upward substitution. In this model, there is little risk associated with upward substitution as recovered goods are sold for their regular price.

When sales met by substitution are included in the fill rate the fill rate is higher, however the magnitude of the increase depends on the amount of substitution performed and on the associated costs and revenues. The fill rates appear, in general, to be higher and more stable for different problems for produced inventory than for recovered inventory. The uncertainty surrounding the supply of recovered goods could be one reason for this.

Four substitution strategies were investigated. The substitution strategies which permit downward substitution (downward and two-way strategies) led to higher substitution-inclusive fill rates for recovered inventory. This is expected since downward substitution is offered when there is a shortage of recovered goods. A comparison between substitution strategies reveals that allowing substitution, in general, leads to increased fill rates. However, in some cases the substitution-inclusive fill rate is actually lower when substitution is allowed. The reason for this is that the SMDP is maximizing the reward, not the fill rate. However it does highlight the complicated nature of the relationship between the substitution and replenishment decisions in this model. Future research could investigate how the policy changes if the objective is to maximize the fill rate, rather than to maximize the long run average reward.

Production was performed less frequently when the size of the production order was larger. A similar trend was found for recovery in some cases. Offering substitution can change the optimal replenishment plan. For some problems, production is performed less frequently when upward substitution can be offered, and recovery is performed less frequently when downward substitution can be offered.

In general, the number of states in which production is performed is lower when upward substitution cannot be offered. This suggests that, when substitution is available the firm produces less as it is willing to let some demand for produced goods be met by recovered goods. When downward substitution can be offered (under the downward and two-way strategies) then the number of states performing recovery is lower, in general, than when substitution cannot be offered. However, this trend does not seem to hold across all problems.

The use of the shared resource in the initial stages of production and recovery means that there is a complicated relationship between the size of the replenishment orders and the frequency of replenishment. For instance, when the production size is larger, recovery is performed more often. This could be because larger production sizes result in production being performed less frequently, meaning that there are more states in which recovery could be performed.

In some problems, it was observed that it was never optimal to recover. Factors such as the setup cost of recovery, the sale price and holding cost of recovered items compared with the equivalent production-related costs contribute to making recovery less desirable than production.

In this analysis it was always optimal to offer upward substitution and sometimes optimal to offer downward substitution. The frequency with which substitution is offered does not seems to be affected by the replenishment order sizes. When substitution is available, the optimal policy waits until the levels of produced inventory (trigger levels) are lower before placing a production order. Increased order sizes for production also lead to lower trigger levels for production. There is no cost-disincentive in offering upward substitution, since the recovered goods are sold for the same price, regardless of whether they are sold as a substitution or as a regular sale.

Recovery is performed more often when the level of returned inventory is high. In general recovery is triggered by low levels of recovered inventory, however in some cases recovery is performed when recovered inventory is very high. In these cases the objective of performing recovery is not to replenish the recovered inventory, but rather to replenish the component inventory.

As expected, the long run average reward is higher when both high and low quality returns are recovered. In general, the fill rates are higher when both high and low quality returns are recovered. Under a high-quality only recovery strategy there are more problems which have a zero fill rate for recovered inventory, compared with a high-and-low quality recovery strategy.

In general, production is performed more under the high-quality recovery strategy than under the both high and low recovery strategy. Recovery, on the other hand is, in general, performed less frequently under a high-quality recovery strategy. This suggests that for these problems, it may become unprofitable to sell recovered goods when low quality recovery is not performed. There is little difference in the optimal substitution under the two recovery strategies.

The analysis conducted here is limited by the number of problems considered. The computational time required to obtain the optimal policy for each problems restricted the number of problems that could be investigated.

It was assumed that production and recovery require a shared resource at the beginning of the process, meaning that production and recovery could not be chosen at the same time. Some of the results suggested that if it was not for this requirement, recovery may be chosen more often. Future research could relax this assumption and investigate the effect of this and other operational restrictions.

In this analysis upward substitution was always offered if there was a stock out of produced inventory, providing that there was at least one unit of recovered inventory in stock. Under upward substitution the firm receives the same revenue for the recovered good as it would have if it had sold the good to a consumer originally demanding a recovered good. There is no disincentive for offering upward substitution. However, if the goods in question were purchased repeatedly, then offering a substitution to a "produced" customer could encourage them to switch to the recovered product for their next purchase. The recovered product is cheaper, thus there may be some costs associated with losing "produced" customers to the recovered market. Future research could investigate this by modelling demand using dynamic functions which change if customers are introduced to an alternative product.

The model in this chapter extended the model in Chapter 6 by incorporating the continuous time element into the problem. This allows the substitution aspect of the problem to be studied in more detail by taking into account the risk of uncertain replenishment lead time. The continuous nature of this model means that production and recovery decisions are monitored continuously. However, in reality these decisions may only be acted on periodically, e.g. once per day or once per week. Future research could investigate a hybrid model with periodic decision making for production and recovery and continuous arrivals of demand and decision making for substitution.

As mentioned in Chapter 6, there are some similarities between offering substitution between produced and recovered goods, transhippments in a multi-location inventory system and substitution in a multi-item inventory system. Future research could investigate whether insights may be gained from multi-item inventory substitution and multi-location transshipment problems, in particular with regards to the dimensionality and computational requirements of the current model.

This research has highlighted some properties of product recovery with substitution that could be useful to firms who are currently in or who are considering entering this industry. Future research could investigate simple, yet effective, policies for use by managers in charge of such systems.

Chapter 8

Conclusion

This thesis investigates the importance of considering the quality of returns in the management of inventory in a product recovery system, by examining the cost-effectiveness of recovering both high quality and low quality returns. Quality in product recovery systems has not been sufficiently studied, with many authors either ignoring the possibility of receiving lower quality returns, or assuming they are disposed of rather than recovered. However, such assumptions ignore the possibility that the firm might be able to salvage value from lower quality returns by using them for parts or materials. These assumptions also ignore the impact of governmental and societal pressures on producers to reduce waste by recycling as much as possible.

The four models presented in this thesis were used to investigate the cost-effectiveness of performing low quality recovery, but also fill gaps in their respective streams of literature. The main findings, obtained through the analysis of these four models, relate to the following strategic issues: the cost-effectiveness of low quality recovery and the profitability of substitution between newly produced and recovered goods. With regard to the cost-effectiveness of low quality recovery, it was found that performing both high and low quality recovery resulted in cost savings; reduced the sensitivity of the optimal cost to operational restrictions on the choice of decision variables; and allowed better customer service through increased fill rates. The nature of the cost savings is problem dependent. The findings of this thesis could be used therefore to help a firm assess the cost-effectiveness of recovering low quality returns, in addition to high quality returns.

Offering substitution between produced and recovered goods ameliorates the uncertainty associated with quality, quantity, and timing of returns. This uncertainty makes it difficult to ensure that there is sufficient stock to meet demand, however by offering substitution the firm is able to mitigate this and therefore gain some revenue rather than losing sales and goodwill. However, there are some trade-offs associated with substitution. In offering downward (upward) substitution, the firm faces the risk that the produced (recovered) inventory may also run out before the next replenishment arrives. However, these risks are not symmetric. With regard to downward substitution, the firm must therefore consider the following trade-off: not offering substitution and losing sales of recovered goods; or offering substitution, selling produced goods for a lower price and *potentially* losing future sales of produced goods. Upward substitution, on the other hand, involves a trade-off only with respect to potential for future lost sales, not with respect to the revenue received. This is because recovered goods are sold for the same price regardless of whether they are sold to "recovered" customers, or as substitutes to "produced" customers. Notwithstanding these trade-offs, the findings of this thesis show that substitution between the markets for newly produced and recovered goods allows firms to increase both their profits and fill rates. The findings also highlight the need for an integrated decision-making process as the substitution policy can affect the frequency with which replenishment orders are placed.

The two types of substitution may have different affects in the long and short term. For instance, upward substitution has a short-term benefit of receiving the full price for the good that is sold. However in the longer-term, there may a be risk that "produced" customers who are offered a recovered good as a substitute, may choose to switch to recovered goods for future purchases. The opposite affect could be observed for downward substitution. In the short-term, the firm may lose some revenue by selling a "produced" good for a reduced price, however, "recovered" customers will generally feel positive about the experience of being offered this type of substitution, so in the long-term may be more loyal to the firm in the future.

While we have shown that there is potential for improvement in managing uncertainty in product recovery systems by performing low quality recovery, and by offering substitution between produced and recovered goods, there are some limitations in our approach. One limitation of this research is that the models presented here do not relate to a specific company or industry, and because of this many assumptions had to be made regarding the nature of the model. Despite the fact that these assumptions were based on research conducted using a range of sources (literature, industry, media and a personal visit to a recycling company), there is still a risk that the assumptions may not be broadly applicable. However, on the other hand, not working with a specific company or industry has allowed us to develop more general models, which may have a wider range of applications.

Another risk associated with not working with a specific company or industry is that the current prevalence of hybrid product recovery firms and thus practical relevance of this thesis, is uncertain. While the research conducted for this thesis does suggest that there are some companies who currently perform both production and recovery, many of the examples of hybrid product recovery cited in the literature are now several years old. The product recovery industry has grown substantially in the past decade, and with more stringent legislation it is possible that it is now a trend towards recovery being performed by specialist firms, rather than by the original producers. A current review of product recovery industry is required to provide more details on this point.

Another limitation of this thesis was caused by the aptly-named "curse of dimensionality". The dimensionality associated with the three Markov decision process models (Chapters 5–7) meant that the capacity of each of the inventories had to be restricted to unrealistically low levels. Larger, more realistic problems could not be solved in any reasonable time using the computing power currently available. Alternative solution methodologies, such as heuristics, would need to be considered in order to solve problems of realistic size. The application of heuristic methods to the models in this thesis is one area for future work.

The limitations of this thesis have highlighted avenues for future research. For instance directions for future research are could be found by altering some of the modelling assumptions. Assumptions which could be altered include the independence of demands and returns; the nature of the recovery process (sorting of returns within recovery and the use of shared resource in production and recovery); ways for dealing with recovered goods which do not fit into the serviceable or component inventories; multiple quality classes; and multi-objective models which jointly maximise profit and recovery rates.

The relationship between substitution policies and future demand is also an area for future research. Assuming that customers purchase goods repeatedly, then it is possible that if “produced” customers are introduced to recovered goods through a substitution, their demand preferences may change for future purchases. If this were a possibility then the firm could end up losing revenue in the long-run, since recovered goods are sold for less than newly produced goods. It would therefore also present an additional risk associated with upward substitution. This problem could be investigated with a finite horizon model and a dynamic demand function. However this would require further analysis of consumer behaviour. For example Ansell et al. (2007) discuss cross-selling opportunities and highlight that customer behaviour may differ for different types of customers and in particular, mention that excessive cross selling can damage the future relationship with the client. The same could apply in our model – some customers may be offended at being offered a substitute.

This thesis has demonstrated the importance of considering the quality of returns when managing a product recovery system and of understanding the trade-offs associated with substitution between produced and recovered goods. Moreover, it has highlighted numerous avenues for future research in topics relating to product recovery. Product recovery can enable businesses to reduce the amount of waste sent to landfills, which is a key issue in the context of increasing governmental and societal pressures to achieve a zero-waste economy. This thesis sheds light on how this could be achieved by examining how businesses can reuse and salvage value from what would otherwise be refuse.

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Appendix A

Appendix for Deterministic Lot-Sizing Model with a Single Market

Introduction

This section presents additional information related to the first model in this thesis – the deterministic product recovery model, which was presented in Chapter 4.

A.1 Model Description

A.1.1 Model Variables

This section relates to Section 4.3.3 and provides further details about the derivation of the model conditions contained in equations (4.4) and (4.5). The derivations are presented through equations (A.4), (A.5), (A.6) and (A.8). Recall that this product recovery model is studied over a period of time called a cycle. A cycle is of length T and consists of a sequence of N_p production lots of size Q_p, followed by a sequence of

N_r recovery lots of size Q_r. The model conditions in equations (4.1), (4.2) and (4.3) are restated here for convenience in equations (A.1), (A.2) and (A.3) respectively:

$$dT = Q_p N_p + \alpha Q_r N_r \tag{A.1}$$

$$Q_r N_r = f dT \tag{A.2}$$

$$Q_p N_p = Q_b N_b + (1-\alpha) Q_r N_r \tag{A.3}$$

Equations (A.1), (A.2) and (A.3) can be used to express the total number of products produced, Q_pN_p, in terms of the total number of products recovered, Q_rN_r, and the total number of components bought, Q_bN_b. Using equations (A.1) and (A.2), Q_r can be defined in terms of Q_p, N_p, N_r. Beginning with equation (A.1), substituting equation (A.2) and carrying out some simple algebraic manipulation produces the following result:

$$\begin{aligned}
dT &= Q_p N_p + \alpha Q_r N_r \\
Q_p N_p &= dT - \alpha Q_r N_r \\
Q_p N_p &= d\frac{Q_r N_r}{fd} - \alpha Q_r N_r \\
Q_p N_p &= \frac{Q_r N_r}{f} - \alpha Q_r N_r \\
Q_p N_p &= Q_r N_r \Big(\frac{1}{f} - \alpha\Big) \\
Q_p N_p &= Q_r N_r \frac{(1-\alpha f)}{f} \\
Q_r &= \frac{Q_p N_p}{N_r} \frac{f}{(1-\alpha f)}
\end{aligned} \tag{A.4}$$

Similarly, using equations (A.3) and (A.4), Q_b can be defined as follows:

$$\begin{aligned}
Q_p N_p &= (1-\alpha) Q_r N_r + Q_b N_b \\
Q_p N_p &= (1-\alpha) Q_p N_p \frac{f}{(1-\alpha f)} + Q_b N_b \\
Q_b N_b &= Q_p N_p \Big(1 - \frac{f(1-\alpha)}{(1-\alpha f)}\Big) \\
Q_b N_b &= Q_p N_p \Big(\frac{1-\alpha f - f + \alpha f}{(1-\alpha f)}\Big) \\
Q_b N_b &= Q_p N_p \Big(\frac{(1-f)}{(1-\alpha f)}\Big) \\
Q_b &= Q_p \frac{N_p}{N_b} \frac{1-f}{1-\alpha f}
\end{aligned} \tag{A.5}$$

In a similar fashion the cycle length, T, can be defined in terms of Q_p, N_p. Once again using equations (A.1) and (A.2) the following result is obtained:

$$\begin{aligned}
dT &= Q_pN_p + \alpha Q_rN_r \\
dT &= Q_pN_p + \alpha f dT \\
dT - \alpha f dT &= Q_pN_p \\
dT(1-\alpha f) &= Q_pN_p \\
T &= \frac{Q_pN_p}{d(1-\alpha f)} \qquad \text{(A.6)}
\end{aligned}$$

The cycle length T can also be expressed in terms of Q_r and in terms of Q_b. From equation (A.2)

$$T = \frac{Q_rN_r}{fd} \qquad \text{(A.7)}$$

and from equations (A.3), (A.6) and (A.7)

$$\begin{aligned}
T &= \frac{Q_pN_p}{d(1-\alpha f)} \\
T &= \frac{Q_bN_b(1-\alpha f)}{(1-f)}\frac{1}{d(1-\alpha f)} \\
T &= \frac{Q_bN_b}{d(1-f)} \qquad \text{(A.8)}
\end{aligned}$$

Thus in summary, as presented in equations (4.4) and (4.5):

$$Q_pN_p = \frac{Q_rN_r(1-\alpha f)}{f} = \frac{Q_bN_b(1-\alpha f)}{(1-f)} \qquad \text{(A.9)}$$

$$T = \frac{Q_pN_p}{d(1-\alpha f)} = \frac{Q_rN_r}{fd} = \frac{Q_bN_b}{d(1-f)} \qquad \text{(A.10)}$$

A.2 Inventory Levels Across a Cycle

This section presents additional information to support Section 4.4.2, in particular it provides further details about the derivation of the area under the graphs. Some information presented in Section 4.4.2 is repeated here for clarity.

A.2.1 Case 2: $N_p, N_r, N_b \geq 1$

Serviceable Inventory

Figure A.1 shows a graph of the serviceable inventory level across a cycle with $N_p = 2$ production lots and $N_r = 4$ recovery lots. The number of ordering lots, N_b, is not

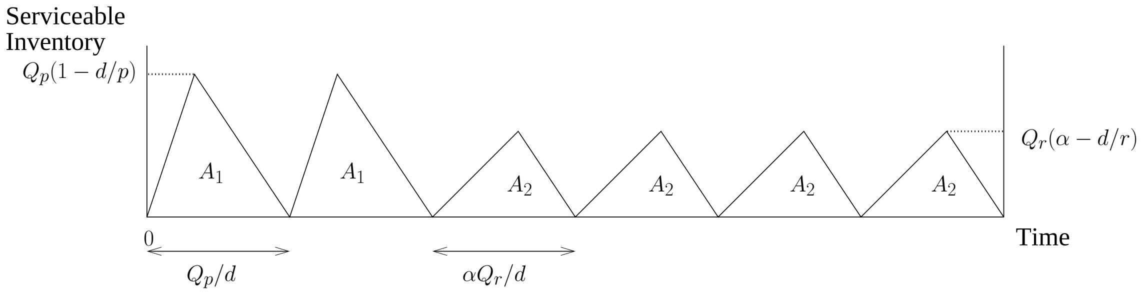


Figure A.1: Serviceable inventory levels for a system with $N_p = 2$, $N_r = 4$.

stated here as it does not affect the shape of the serviceable inventory graph. The area under the graph is made up of N_p triangles A_1 and N_r triangles A_2, where:

$$\text{Area of } A_1 = Q_p(1-d/p)(Q_p/d)(1/2)$$

$$\text{Area of } A_2 = Q_r(\alpha-d/r)(\alpha Q_r/d)(1/2)$$

The total area under of the graph is:

$$\begin{aligned}
A_s &= N_pA_1 + N_rA_2 \\
&= N_pQ_p(1-d/p)(Q_p/d)(1/2) + N_rQ_r(\alpha-d/r)(\alpha Q_r/d)(1/2) \\
&= \frac{Q_p^2N_p}{2}\left(\frac{1}{d}-\frac{1}{p}\right) + \frac{Q_r^2N_r\alpha}{2}\left(\frac{\alpha}{d}-\frac{1}{r}\right) \qquad \text{(A.11)}
\end{aligned}$$

From equations (A.4) and (A.10) we know that:

$$Q_r = \frac{Q_pN_pf}{N_r(1-\alpha f)} \qquad \text{and } quad \quad T = \frac{Q_pN_p}{d(1-\alpha f)}$$

Using these equations, it can be shown that the total area under the graph in terms of Q_p and T is:

$$\begin{aligned}
A_s &= \frac{Q_p^2N_p}{2}\left(\frac{1}{d}-\frac{1}{p}\right) + \frac{Q_r^2N_r\alpha}{2}\left(\frac{\alpha}{d}-\frac{1}{r}\right) \\
&= \frac{Q_p^2N_p}{2}\left(\frac{1}{d}-\frac{1}{p}\right) + \left(\frac{Q_pN_pf}{N_r(1-\alpha f)}\right)^2\frac{N_r\alpha}{2}\left(\frac{\alpha}{d}-\frac{1}{r}\right) \\
&= \frac{Q_p^2N_p}{2}\left(\frac{1}{d}-\frac{1}{p}\right) + \frac{Q_p^2N_p^2\alpha f^2}{2N_r(1-\alpha f)^2}\left(\frac{\alpha}{d}-\frac{1}{r}\right) \\
&= \frac{Q_p^2N_p}{2}\frac{d(1-\alpha f)}{d(1-\alpha f)}\left(\left(\frac{1}{d}-\frac{1}{p}\right) + \frac{N_p\alpha f^2}{N_r(1-\alpha f)^2}\left(\frac{\alpha}{d}-\frac{1}{r}\right)\right) \\
&= \frac{TQ_pd(1-\alpha f)}{2}\left(\left(\frac{1}{d}-\frac{1}{p}\right) + \frac{N_p\alpha f^2}{N_r(1-\alpha f)^2}\left(\frac{\alpha}{d}-\frac{1}{r}\right)\right) \\
&= \frac{TQ_p}{2}\left((1-\alpha f)\left(1-\frac{d}{p}\right) + \frac{N_p\alpha f^2}{N_r(1-\alpha f)}\left(\alpha-\frac{d}{r}\right)\right) \qquad \text{(A.12)}
\end{aligned}$$

Returned Inventory

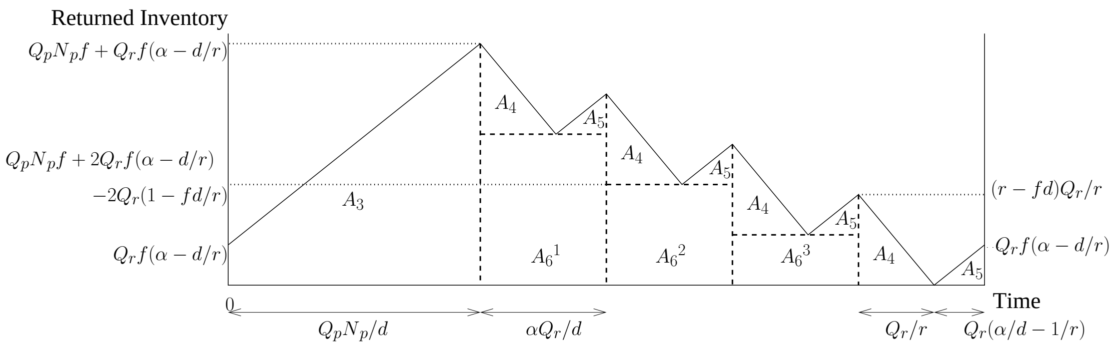


Figure A.2: Returned inventory levels under a policy with $N_r = 4$ recovery lots.

Figure A.2 shows the returned inventory levels across a cycle with $N_r = 4$ recovery lots. The number of production lots N_p and the number of ordering lots N_b are not stated here as they do affect the shape of the returned inventory graph. The area under the graph consists of trapezium A_3, N_r triangles A_4, N_r triangles A_5 and the rectangles A_6^i for $i = 1, \ldots N_r$. The area of the triangles A_3, A_4, A_5 are obvious from the graph. The height of rectangles is not quite as obvious. The height of rectangle A_6^i is the initial inventory plus amount returned during production less the amount recovered during i recovery lots plus the amount returned during $i-1$ recovery lots. The areas are defined as follows:

$$\text{Area of } A_3 = (Q_pN_p/d)[Q_rf(\alpha - d/r) + Q_pN_pf + Q_rf(\alpha - d/r)](1/2)$$

$$\text{Area of } A_4 = (Q_r(r - fd)/r)(Q_r/r)(1/2)$$

$$\text{Area of } A_5 = fdQ_r(\alpha/d - 1/r)Q_r(\alpha/d - 1/r)(1/2)$$

$$\text{Area of } A_6^i = (\alpha Q_r/d)[Q_pN_pf + Q_rf(\alpha - d/r) + (i-1)Q_rf(\alpha - d/r) - i(r - fd)Q_r/r]$$

$$\begin{aligned}
\text{Area of } A_6 &= \frac{Q_r\alpha}{d}\sum_{i=1}^{N_r}[Q_pN_pf + iQ_rf(\alpha - d/r) - iQ_r(r - fd)/r] \\
&= \frac{Q_r\alpha}{d}\left(Q_pN_pN_rf + \sum_{i=1}^{N_r} iQ_r\left(f(\alpha - d/r) - (r - fd)/r\right)\right) \\
&= \frac{Q_r\alpha}{d}\left(Q_pN_pN_rf + Q_r\left(f(\alpha - d/r) - (r - fd)/r\right)\sum_{i=1}^{N_r} i\right) \\
&= \frac{Q_r\alpha}{d}\left(Q_pN_pN_rf + Q_r\left(f(\alpha - d/r) - (r - fd)/r\right)\frac{N_r(N_r + 1)}{2}\right)
\end{aligned}$$

The total area under of the graph of the returned inventory is:

$$\begin{aligned}
A_r &= A_3 + N_r A_4 + N_r A_5 + A_6 \\
&= \frac{Q_p N_p}{2d}\left(2Q_r f\left(\alpha - \frac{d}{r}\right) + Q_p N_p f\right) + \quad \frac{Q_r^2 N_r (r - fd)}{2r^2} + \\
& \frac{Q_r^2 N_r f d}{2}\left(\frac{\alpha}{d} - \frac{1}{r}\right)^2 + \quad \frac{Q_r \alpha}{d}\left(Q_p N_p N_r f + Q_r\left(f\left(\alpha - \frac{d}{r}\right) - \frac{(r - fd)}{r}\right)\frac{N_r(N_r + 1)}{2}\right)
\end{aligned} \tag{A.13}$$

The total area under the graph A_r can be represented in terms of Q_p and T. In order retain clarity each of the areas A_3, A_4, A_5 and A_6 will be algebraically manipulated separately.

$$\begin{aligned}
A_3 &= \frac{N_p Q_p}{2d}\left(2Q_r f\left(\alpha - \frac{d}{r}\right) + N_p Q_p f\right) \\
&= \frac{N_p Q_p}{2d}\left(\frac{2Q_p N_p f^2}{N_r(1 - \alpha f)}\left(\alpha - \frac{d}{r}\right) + N_p Q_p f\right) \\
&= \frac{Q_p^2 N_p^2 f^2}{2d}\left(\frac{2}{N_r(1 - \alpha f)}\left(\alpha - \frac{d}{r}\right) + \frac{1}{f}\right) \\
&= \frac{Q_p^2 N_p^2 f^2}{2d(1 - \alpha f)^2}\left(\frac{2(1 - \alpha f)}{N_r}\left(\alpha - \frac{d}{r}\right) + \frac{(1 - \alpha f)^2}{f}\right) \\
&= \frac{T Q_p N_p f^2}{2(1 - \alpha f)}\left(\frac{2(1 - \alpha f)}{N_r}\left(\alpha - \frac{d}{r}\right) + \frac{(1 - \alpha f)^2}{f}\right)
\end{aligned}$$

$$\begin{aligned}
N_r A_4 &= \frac{N_r Q_r^2 (r - fd)}{2r^2} \\
&= \frac{Q_p^2 N_p^2 f^2}{N_r^2 d(1 - \alpha f)^2}\frac{N_r (r - fd) d}{2r^2} \\
&= \frac{T Q_p N_p f^2}{2(1 - \alpha f)}\frac{d(r - fd)}{N_r r^2}
\end{aligned}$$

$$\begin{aligned}
N_r A_5 &= \frac{N_r Q_r^2 f d}{2}\left(\frac{\alpha}{d} - \frac{1}{r}\right)^2 \\
&= \frac{Q_p^2 N_p^2 f^2}{N_r^2 d(1 - \alpha f)^2}\frac{N_r f d^2}{2}\left(\frac{\alpha}{d} - \frac{1}{r}\right)^2 \\
&= \frac{T Q_p N_p f^2}{2(1 - \alpha f)}\frac{f d^2}{N_r}\left(\frac{\alpha}{d} - \frac{1}{r}\right)^2
\end{aligned}$$

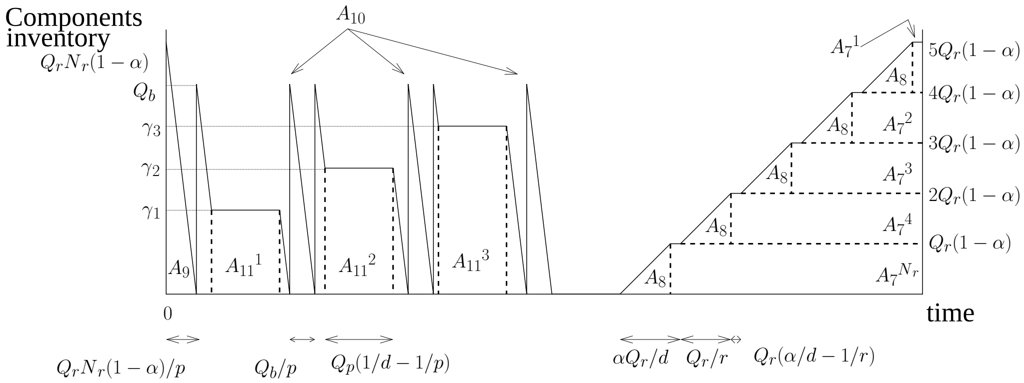
$$
\begin{aligned}
A_6 &= \frac{Q_r N_r \alpha}{d}\left(Q_p N_p f + \frac{Q_r(N_r+1)}{2}\left(f\left(\alpha - \frac{d}{r}\right) - \frac{(r-fd)}{r}\right)\right)\\
&= \frac{Q_p N_p f}{N_r(1-\alpha f)}\frac{N_r \alpha}{d}\left(Q_p N_p f + \frac{Q_p N_p f}{N_r(1-\alpha f)}\frac{(N_r+1)}{2}\left(\alpha f - \frac{df}{r} - 1 + \frac{fd}{r}\right)\right)\\
&= \frac{Q_p^2 N_p^2 f^2}{(1-\alpha f)}\frac{\alpha}{d}\left(1 + \frac{(N_r+1)}{2N_r(1-\alpha f)}(\alpha f - 1)\right)\\
&= \frac{Q_p^2 N_p^2 f^2}{2d(1-\alpha f)^2}\left(2\alpha(1-\alpha f) - \frac{\alpha(N_r+1)}{N_r}(1-\alpha f)\right)\\
&= \frac{TQ_p N_p f^2}{2(1-\alpha f)}\left(2\alpha(1-\alpha f) - \alpha\left(1+\frac{1}{N_r}\right)(1-\alpha f)\right)\\
&= \frac{TQ_p N_p f^2}{2(1-\alpha f)}\left(2\alpha - 2\alpha^2 f - \alpha\left(1 - \alpha f + \frac{1}{N_r}(1-\alpha f)\right)\right)\\
&= \frac{TQ_p N_p f^2}{2(1-\alpha f)}\left(2\alpha - 2\alpha^2 f - \alpha + \alpha^2 f - \frac{\alpha}{N_r}(1-\alpha f)\right)\\
&= \frac{TQ_p N_p f^2}{2(1-\alpha f)}\left(\alpha(1-\alpha f) - \frac{\alpha}{N_r}(1-\alpha f)\right)
\end{aligned}
$$

Combining these expressions gives:

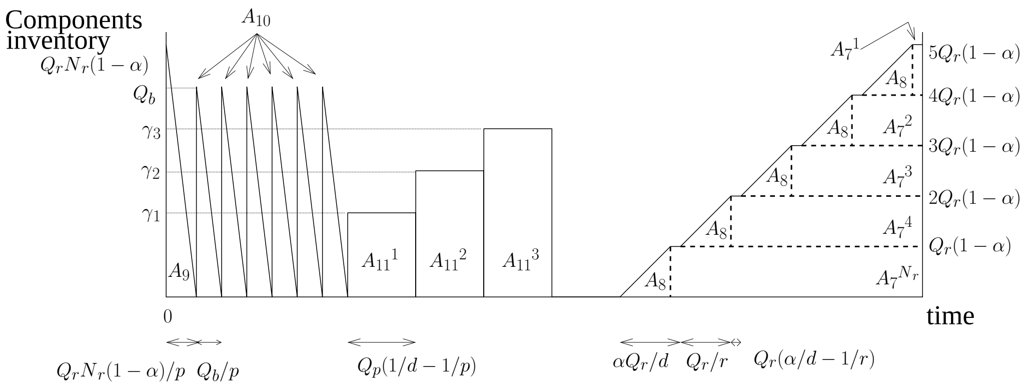
$$
\begin{aligned}
A_r &= A_3 + N_r A_4 + N_r A_5 + A_6\\
&= \frac{TQ_p N_p f^2}{2(1-\alpha f)}\Bigg(\frac{2(1-\alpha f)}{N_r}\left(\alpha - \frac{d}{r}\right) + \frac{(1-\alpha f)^2}{f} + \frac{d(r-fd)}{N_r r^2} + \frac{fd^2}{N_r}\left(\frac{\alpha}{d} - \frac{1}{r}\right)^2\\
&\qquad\qquad + \alpha(1-\alpha f) - \frac{\alpha}{N_r}(1-\alpha f)\Bigg)\\
&= \frac{TQ_p N_p f^2}{2(1-\alpha f)}\Bigg(\frac{1}{N_r}\left(2(1-\alpha f)\left(\alpha - \frac{d}{r}\right) + \frac{dr - fd^2}{r^2} + fd^2\left(\frac{\alpha}{d} - \frac{1}{r}\right)^2 - \alpha(1-\alpha f)\right)\\
&\qquad\qquad + \frac{1 - 2\alpha f + \alpha^2 f^2}{f} + \alpha - \alpha^2 f\Bigg)\\
&= \frac{TQ_p N_p f^2}{2(1-\alpha f)}\Bigg(\frac{1}{N_r}\left(2\alpha - 2\alpha^2 f - \frac{2d}{r} + \frac{2\alpha f d}{r} + \frac{d}{r} - \frac{fd^2}{r^2} + \frac{fd^2\alpha^2}{d^2} - \frac{2fd^2\alpha}{dr} + \frac{fd^2}{r^2} - \alpha + \alpha^2 f\right)\\
&\qquad\qquad + \frac{1}{f} - 2\alpha + \alpha^2 f + \alpha - \alpha^2 f\Bigg)\\
&= \frac{TQ_p N_p f^2}{2(1-\alpha f)}\left(\frac{1}{N_r}\left(\alpha - \frac{d}{r}\right) + \frac{1}{f} - \alpha\right) \qquad (A.14)
\end{aligned}
$$

Components Inventory

The shape of the component inventory level graph during production depends on the relationship between N_p, N_b and some of the parameters.



(a) Components inventory levels over a cycle



(b) Rearranged parts of a the components inventory level

Figure A.3: Components inventory levels under a policy with $N_p = 4$, $N_r = 5$ and $N_b = 6$

In order to analyse the shape of the components inventory level graph, the cycle will be considered in three parts: during the production lots, between the production lots and during recovery.

During Production Lots. The area under the graph during the production lots consists of triangle A_9, N_b triangles A_{10}. As shown in these graphs,

Area of $A_9 = N_rQ_r(1-\alpha)N_rQ_r(1-\alpha)/p(1/2)$

Area of $A_{10} = Q_bQ_b/p(1/2)$

The total area under the graph during production A_c^{dp} is:

$$\begin{aligned} A_c^{dp} &= A_9 + N_bA_{10} \\ &= \frac{N_r^2Q_r^2\,(1-\alpha)^2}{2p} + \frac{N_bQ_b^2}{2p} \end{aligned} \tag{A.15}$$

Using equations (A.9) and (A.10) it can be shown that the total area under the graph during production in terms of Q_p and T is:

$$\begin{aligned}
A_c^{dp} &= \frac{N_r^2 Q_r^2 (1-\alpha)^2}{2p} + \frac{N_b Q_b^2}{2p} \\
&= \frac{Q_p^2 N_p^2 f^2}{(1-\alpha f)^2} \frac{(1-\alpha)^2}{2p} + \frac{Q_p^2 N_p^2 (1-f)^2}{N_b^2 (1-\alpha f)^2} \frac{N_b}{2p} \\
&= \frac{Q_p^2 N_p^2}{(1-\alpha f)^2} \frac{d}{2pd} \left(f^2 (1-\alpha)^2 + \frac{(1-f)^2}{N_b} \right) \\
&= T Q_p N_p \frac{d}{2p(1-\alpha f)} \left(f^2 (1-\alpha)^2 + \frac{(1-f)^2}{N_b} \right) \qquad \text{(A.16)}
\end{aligned}$$

Between Production Lots. The period of time between production lots is represented in Figure A.3a by the rectangles A_{11}^i, $i = 1, \ldots, N_b$. The width of these rectangles is $Q_p(1/d - 1/p)$. The height of rectangle i is the inventory level at the end of production lot i, which we denote by γ_i. Note that the model conditions ensure that $\gamma_{N_p} = 0$ (see equation (A.3)). Let $M(i, N_p, N_b)$ denote the number of buying lots to have occurred by the end of production lot i.

$$M(i, N_p, N_b) = \left\lceil \frac{i N_b (1-\alpha f)}{(1-f) N_p} - \frac{N_b f (1-\alpha)}{(1-f)} \right\rceil \qquad \text{(A.17)}$$

Let the component inventory level at the end of production lot i be denoted by γ_i, where:

$$\gamma_i = Q_r N_r (1-\alpha) + M(i, N_p, N_b) Q_b - i Q_p \qquad \text{(A.18)}$$

As shown in Figures A.3 and A.3b, the area of rectangle A_{11}^i is:

Area of $A_{11}^i = Q_p(1/d - 1/p)\gamma_i = Q_p(1/d - 1/p)[Q_r N_r (1-\alpha) + M(i, N_p, N_b) Q_b - i Q_p]$

The total area under the graph between production lots is:

$$\begin{aligned}
A_c^{bp} &= A_{11} = \sum_{i=1}^{N_p} A_{11}^i \\
&= Q_p \left(\frac{1}{d} - \frac{1}{p} \right) \sum_{i=1}^{N_p} (Q_r N_r (1-\alpha) + M(i, N_p, N_b) Q_b - i Q_p) \\
&= Q_p \left(\frac{1}{d} - \frac{1}{p} \right) \left(N_p Q_r N_r (1-\alpha) + Q_b \sum_{i=1}^{N_p} M(i, N_p, N_b) - Q_p \frac{N_p (N_p + 1)}{2} \right) \qquad \text{(A.19)}
\end{aligned}$$

Using equations (A.9) and (A.10) it can be shown that the total area under the graph between production in terms of Q_p and T is:

$$
\begin{aligned}
A_c^{bp} &= Q_p\left(\frac{1}{d}-\frac{1}{p}\right)\left(\frac{Q_pN_pf}{(1-\alpha f)}N_p(1-\alpha)+\frac{Q_pN_p(1-f)}{N_b(1-\alpha f)}\sum_{i=1}^{N_p}M(i,N_p,N_b)-Q_p\frac{N_p(N_p+1)}{2}\right)\\
&= \frac{Q_p^2N_p^2}{(1-\alpha f)}\frac{d}{d}\left(\frac{1}{d}-\frac{1}{p}\right)\left(f(1-\alpha)+\frac{(1-f)}{N_bN_p}\sum_{i=1}^{N_p}M(i,N_p,N_b)-\frac{(N_p+1)(1-\alpha f)}{2N_p}\right)\\
&= TQ_pN_p\frac{p-d}{p}\left(f(1-\alpha)+\frac{(1-f)}{N_bN_p}\sum_{i=1}^{N_p}M(i,N_p,N_b)-\frac{(1-\alpha f)}{2}-\frac{(1-\alpha f)}{2N_p}\right)
\end{aligned}
\tag{A.20}
$$

Note that if $N_p = 1$, then this area A_c^{bp} reduces to zero.

During Recovery Lots. During recovery the area under the graph is made up of rectangles A_7^i, for $i = 1, \ldots, N_r$, and N_r triangles A_8 where:

Area of $A_7^i = Q_r(1-\alpha)[Q_r(\alpha/d - 1/r) + (i-1)Q_r\alpha/d]$

Area of $A_7 = \sum_{i=1}^{N_r} A_7^i = Q_r(1-\alpha)\sum_{i=1}^{N_r}[Q_r\alpha/d - Q_r/r + iQ_r\alpha/d - Q_r\alpha/d]$

Area of $A_8 = Q_r(1-\alpha)(Q_r/r)(1/2)$

The total area under the graph during and between recovery lots is:

$$
\begin{aligned}
A_c^{dr} &= A_7 + N_rA_8\\
&= Q_r(1-\alpha)\sum_{i=1}^{N_r}\left(\frac{Q_r\alpha}{d}-\frac{Q_r}{r}+\frac{iQ_r\alpha}{d}-\frac{Q_r\alpha}{d}\right)+\frac{N_rQ_r^2(1-\alpha)}{2r}\\
&= Q_r^2N_r\left((1-\alpha)\left(-\frac{1}{r}+\frac{\alpha}{N_rd}\sum_{i=1}^{N_r}i\right)+\frac{(1-\alpha)}{2r}\right)\\
&= Q_r^2N_r\left(-\frac{(1-\alpha)}{r}+\frac{\alpha(1-\alpha)}{N_rd}\frac{N_r(N_r+1)}{2}+\frac{(1-\alpha)}{2r}\right)\\
&= Q_r^2N_r\left((1-\alpha)\left(-\frac{2}{2r}+\frac{1}{2r}\right)+\frac{\alpha(1-\alpha)(N_r+1)}{2d}\right)\\
&= Q_r^2N_r(1-\alpha)\left(-\frac{1}{2r}+\frac{\alpha(N_r+1)}{2d}\right)
\end{aligned}
$$

Using equations (A.9) and (A.10) it can be shown that the total area under the graph during and between recovery lots, in terms of Q_p and T is:

$$\begin{aligned}
A_c^{dr} &= A_7 + N_r A_8 \\
&= Q_r^2 N_r (1-\alpha)\left(-\frac{1}{2r} + \frac{\alpha(N_r+1)}{2d}\right) \\
&= \frac{Q_p^2 N_p^2 f^2}{N_r^2 (1-\alpha f)^2} N_r (1-\alpha)\frac{d}{d}\left(-\frac{1}{2r} + \frac{\alpha(N_r+1)}{2d}\right) \\
&= \frac{T Q_p N_p}{(1-\alpha f)} \frac{f^2(1-\alpha)}{2N_r}\left(-\frac{d}{r} + \alpha(N_r+1)\right) \\
&= T Q_p N_p \frac{f^2(1-\alpha)}{2(1-\alpha f)}\left(-\frac{d}{N_r r} + \frac{\alpha N_r}{N_r} + \frac{\alpha}{N_r}\right) \\
&= T Q_p N_p \frac{f^2(1-\alpha)}{2(1-\alpha f)}\left(\frac{1}{N_r}\left(\alpha - \frac{d}{r}\right) + \alpha\right)
\end{aligned} \tag{A.21}$$

Across a cycle. Using equations (A.16), (A.20) and (A.21) the total area under the components inventory level graph can be expressed in terms of Q_p and T. Further simplification of some parts of A_c is not performed at this stage as it will be necessary in the coming sections to be able to identify the origin of the negative terms in this expression. The simplified version of this expression is presented in equation (A.22). The derivation of this expression is presented on the coming pages. For notation convenience, the expression $M(i, N_p, N_b)$ is replaced with m_i on the coming pages.

$$\begin{aligned}
A_c = T Q_p \Bigg(& N_p \left[\left(1 - \frac{d}{p}\right)\left(f(1-\alpha) - \frac{(1-\alpha f)}{2}\right)\right] + \frac{1}{N_b}\left[\frac{(p-d)(1-f)}{p} \sum_{i=1}^{N_p} M(i, N_p, N_b)\right] \\
& - \frac{(1-\alpha f)(p-d)}{2p} + N_p \left[\frac{f^2(1-\alpha)}{2(1-\alpha f)}\left(\frac{d\,(1-\alpha)}{p} + \alpha\right)\right] \\
& + \frac{N_p}{N_b}\left[\frac{d(1-f)^2}{2p(1-\alpha f)}\right] + \frac{N_p}{N_r}\left[\frac{f^2(1-\alpha)(\alpha r - d)}{2r(1-\alpha f)}\right]\Bigg)
\end{aligned} \tag{A.22}$$

$$
\begin{aligned}
A_c &= A_c^{dp} + A_c^{dr} + A_c^{dr} \\
&= TQ_pN_p \frac{d}{2p(1-\alpha f)} \left(f^2 (1-\alpha)^2 + \frac{(1-f)^2}{N_b} \right) + TQ_pN_p \left(1 - \frac{d}{p}\right) \left(f(1-\alpha) + \frac{(1-f)}{N_bN_p} \sum_{i=1}^{N_p} M(i, N_p, N_b) - \frac{(1-\alpha f)}{2} - \frac{(1-\alpha f)}{2N_p} \right) + TQ_pN_p \frac{f^2(1-\alpha)}{2(1-\alpha f)} \left(\frac{1}{N_r} \left(\alpha - \frac{d}{r} \right) + \alpha \right) \\
&= TQ_pN_p \left(\frac{d}{2p(1-\alpha f)} \left(f^2 (1-\alpha)^2 + \frac{(1-f)^2}{N_b} \right) + \left(1 - \frac{d}{p}\right) \left(f(1-\alpha) + \frac{(1-f)}{N_bN_p} \sum_{i=1}^{N_p} M(i, N_p, N_b) - \frac{(1-\alpha f)}{2} - \frac{(1-\alpha f)}{2N_p} \right) + \frac{f^2(1-\alpha)}{2(1-\alpha f)} \left(\frac{1}{N_r} \left(\alpha - \frac{d}{r} \right) + \alpha \right) \right) \\
&= TQ_pN_p \left(\frac{df^2 (1-\alpha)^2}{2p(1-\alpha f)} + \frac{d(1-f)^2}{2p(1-\alpha f)N_b} + \left(1 - \frac{d}{p}\right) f(1-\alpha) + \left(1 - \frac{d}{p}\right) \frac{(1-f)}{N_bN_p} \sum_{i=1}^{N_p} M(i, N_p, N_b) - \frac{(1-\alpha f)}{2} \left(1 - \frac{d}{p}\right) - \frac{(1-\alpha f)}{2N_p} \left(1 - \frac{d}{p}\right) + \frac{f^2(1-\alpha)}{2(1-\alpha f)N_r} \left(\alpha - \frac{d}{r} \right) + \frac{\alpha f^2(1-\alpha)}{2(1-\alpha f)} \right) \\
&= TQ_pN_p \left(\frac{df^2 (1-\alpha)^2}{2p(1-\alpha f)} + \frac{f(1-\alpha)(p-d)}{p} - \frac{(1-\alpha f)(p-d)}{2p} + \frac{\alpha f^2(1-\alpha)}{2(1-\alpha f)} + \frac{1}{N_b} \frac{d(1-f)^2}{2p(1-\alpha f)} + \frac{1}{N_bN_p} \frac{(p-d)(1-f)}{p} \sum_{i=1}^{N_p} M(i, N_p, N_b) - \frac{1}{N_p} \frac{(1-\alpha f)(p-d)}{2p} + \frac{1}{N_r} \frac{f^2(1-\alpha)(\alpha r - d)}{2(1-\alpha f)r} \right) \\
&= TQ_pN_p \left(\frac{1}{2p(1-\alpha f)} \left(df^2 (1-\alpha)^2 + 2f(1-\alpha)(1-\alpha f)(p-d) - (1-\alpha f)^2(p-d) + p\alpha f^2(1-\alpha) \right) + \frac{1}{N_b} \frac{d(1-f)^2}{2p(1-\alpha f)} + \frac{1}{N_bN_p} \frac{(p-d)(1-f)}{p} \sum_{i=1}^{N_p} M(i, N_p, N_b) - \frac{1}{N_p} \frac{(1-\alpha f)(p-d)}{2p} + \frac{1}{N_r} \frac{f^2(1-\alpha)(\alpha r - d)}{2(1-\alpha f)r} \right) \\
&= TQ_pN_p \left(\frac{1}{2p(1-\alpha f)} \left(p\alpha f^2 - p\alpha^2 f^2 + df^2(1 - 2\alpha + \alpha^2) + 2f(1 - \alpha f - \alpha + \alpha^2 f)(p-d) - (1 - 2\alpha f + \alpha^2 f^2)(p-d) \right) + \frac{1}{N_b} \frac{d(1-f)^2}{2p(1-\alpha f)} + \frac{1}{N_bN_p} \frac{(p-d)(1-f)}{p} \sum_{i=1}^{N_p} M(i, N_p, N_b) - \frac{1}{N_p} \frac{(1-\alpha f)(p-d)}{2p} + \frac{1}{N_r} \frac{f^2(1-\alpha)(\alpha r - d)}{2(1-\alpha f)r} \right) \\
&= TQ_pN_p \left(\frac{1}{2p(1-\alpha f)} \left(df^2 - df^2\alpha(p-d)(\alpha f^2 - \alpha^2 f^2) + (p-d)(2f - 2\alpha f^2 - 2\alpha f + 2\alpha^2 f^2 - 1 + 2\alpha f - \alpha^2 f^2) \right) + \frac{1}{N_b} \frac{d(1-f)^2}{2p(1-\alpha f)} + \frac{1}{N_bN_p} \frac{(p-d)(1-f)}{p} \sum_{i=1}^{N_p} M(i, N_p, N_b) - \frac{1}{N_p} \frac{(1-\alpha f)(p-d)}{2p} + \frac{1}{N_r} \frac{f^2(1-\alpha)(\alpha r - d)}{2(1-\alpha f)r} \right) \\
&= TQ_pN_p \left(\frac{1}{2p(1-\alpha f)} \left(df^2(1-\alpha) + (p-d)(\alpha f^2 - \alpha^2 f^2 + 2f - 2\alpha f^2 - 2\alpha f + 2\alpha^2 f^2 - 1 + 2\alpha f - \alpha^2 f^2) \right) + \frac{1}{N_b} \frac{d(1-f)^2}{2p(1-\alpha f)} + \frac{1}{N_bN_p} \frac{(p-d)(1-f)}{p} \sum_{i=1}^{N_p} M(i, N_p, N_b) - \frac{1}{N_p} \frac{(1-\alpha f)(p-d)}{2p} + \frac{1}{N_r} \frac{f^2(1-\alpha)(\alpha r - d)}{2(1-\alpha f)r} \right) \\
&= TQ_pN_p \left(\frac{1}{2p(1-\alpha f)} \left(df^2(1-\alpha) + (p-d)(2f - \alpha f^2 - 1) \right) + \frac{1}{N_b} \frac{d(1-f)^2}{2p(1-\alpha f)} + \frac{1}{N_bN_p} \frac{(p-d)(1-f)}{p} \sum_{i=1}^{N_p} M(i, N_p, N_b) - \frac{1}{N_p} \frac{(1-\alpha f)(p-d)}{2p} + \frac{1}{N_r} \frac{f^2(1-\alpha)(\alpha r - d)}{2(1-\alpha f)r} \right) \\
&= TQ_pN_p \frac{d}{2p(1-\alpha f)} \left(f^2 (1-\alpha)^2 + \frac{(1-f)^2}{N_b} \right) + TQ_pN_p \left(1 - \frac{d}{p}\right) \left(f(1-\alpha) + \frac{(1-f)}{N_bN_p} \sum_{i=1}^{N_p} M(i, N_p, N_b) - \frac{(1-\alpha f)}{2} - \frac{(1-\alpha f)}{2N_p} \right) + TQ_pN_p \frac{f^2(1-\alpha)}{2(1-\alpha f)} \left(\frac{1}{N_r} \left(\alpha - \frac{d}{r} \right) + \alpha \right) \\
&= TQ_pN_p \left(\frac{d}{2p(1-\alpha f)} \left(f^2 (1-\alpha)^2 + \frac{(1-f)^2}{N_b} \right) + \left(1 - \frac{d}{p}\right) \left(f(1-\alpha) + \frac{(1-f)}{N_bN_p} \sum_{i=1}^{N_p} M(i, N_p, N_b) - \frac{(1-\alpha f)}{2} - \frac{(1-\alpha f)}{2N_p} \right) + \frac{f^2(1-\alpha)}{2(1-\alpha f)} \left(\frac{1}{N_r} \left(\alpha - \frac{d}{r} \right) + \alpha \right) \right) \\
&= TQ_pN_p \left(\left(1 - \frac{d}{p}\right) \left(f(1-\alpha) - \frac{(1-\alpha f)}{2} + \frac{1}{N_bN_p} \left((1-f) \sum_{i=1}^{N_p} M(i, N_p, N_b) \right) - \frac{1}{N_p} \frac{(1-\alpha f)}{2} \right) + \frac{df^2 (1-\alpha)^2}{2p(1-\alpha f)} + \frac{1}{N_b} \frac{d(1-f)^2}{2p(1-\alpha f)} + \frac{1}{N_r} \frac{f^2(1-\alpha)(\alpha r - d)}{2r(1-\alpha f)} + \frac{f^2(1-\alpha)\alpha}{2(1-\alpha f)} \right) \\
&= TQ_p \left(N_p \left[\left(1 - \frac{d}{p}\right) \left(f(1-\alpha) - \frac{(1-\alpha f)}{2} \right) \right] + \frac{1}{N_b} \left[\frac{(p-d)(1-f)}{p} \sum_{i=1}^{N_p} M(i, N_p, N_b) \right] - \frac{(1-\alpha f)(p-d)}{2p} + N_p \left[\frac{f^2(1-\alpha)}{2(1-\alpha f)} \left(\frac{d\,(1-\alpha)}{p} + \alpha \right) \right] + \frac{N_p}{N_b} \left[\frac{d(1-f)^2}{2p(1-\alpha f)} \right] + \frac{N_p}{N_r} \left[\frac{f^2(1-\alpha)(\alpha r - d)}{2r(1-\alpha f)} \right] \right)
\end{aligned}
$$

A.3 Cost Function

A.3.1 Derivation of the Total Cost Function

Total Cost

The total cost per time unit is the sum of equations (4.17), (4.16), (4.18), (4.19) and (4.20). Some algebraic manipulations on the sum of these equations are performed on the following pages. Equation (A.23) presents a simplified version of the total cost per time unit; the derivation of this expression is shown on the coming pages.

The total cost per time unit is:

$$
\begin{aligned}
TC = {} & c_r f d + c_b d(1-f) + c_p(1-\alpha f)d + c_h \alpha f d + c_l(1-\alpha)fd + c_d(1-f)d + \\
& \frac{d(1-\alpha f)}{Q_p N_p}(k_p N_p + k_r N_r + k_b N_b) + \\
& Q_p \left(\frac{h_s(1-\alpha f)(p-d)}{2p} + N_p \left[\frac{h_r f}{2} + \frac{h_c f^2(1-\alpha)}{2(1-\alpha f)} \left(\frac{d(1-\alpha)}{p} + \alpha \right) \right] \right. \\
& + \frac{N_p}{N_r} \left[\frac{f^2(\alpha r - d)}{2r(1-\alpha f)} (h_s \alpha + h_r + h_c(1-\alpha)) \right] \\
& + \frac{N_p}{N_b} \left[\frac{h_c d(1-f)^2}{2p(1-\alpha f)} \right] + N_p \left[h_c \left(1 - \frac{d}{p} \right) \left(f(1-\alpha) - \frac{(1-\alpha f)}{2} \right) \right] \\
& \left. + \frac{1}{N_b} \left[\frac{h_c(p-d)(1-f)}{p} \sum_{i=1}^{N_p} M(i, N_p, N_b) \right] - \frac{h_c(1-\alpha f)(p-d)}{2p} \right)
\end{aligned}
$$

$$
\begin{aligned}
TC &= C_P + K + H_s + H_r + H_c \\
&= c_r f d + c_b d(1-f) + c_p(1-\alpha f)d + c_h \alpha f d + c_l(1-\alpha) f d + c_d(1-f)d + \frac{d(1-\alpha f)}{Q_p N_p}(k_p N_p + k_r N_r + k_b N_b) + \\
&\quad + \frac{h_s Q_p}{2}\left((1-\alpha f)\left(1-\frac{d}{p}\right) + \frac{N_p \alpha f^2}{N_r(1-\alpha f)}\left(\alpha - \frac{d}{r}\right)\right) + \frac{h_r Q_p N_p f^2}{2(1-\alpha f)}\left(\frac{1}{N_r}\left(\alpha - \frac{d}{r}\right) + \frac{1}{f} - \alpha\right) \\
&\quad + h_c Q_p \Big(N_p \left[\left(1-\frac{d}{p}\right)\left(f(1-\alpha) - \frac{(1-\alpha f)}{2}\right)\right] + \frac{1}{N_b}\left[\frac{(p-d)(1-f)}{p}\sum_{i=1}^{N_p} M(i, N_p, N_b)\right] - \frac{(1-\alpha f)(p-d)}{2p} \\
&\quad + N_p\left[\frac{f^2(1-\alpha)}{2(1-\alpha f)}\left(\frac{d\,(1-\alpha)}{p} + \alpha\right)\right] + \frac{N_p}{N_b}\left[\frac{d(1-f)^2}{2p(1-\alpha f)}\right] + \frac{N_p}{N_r}\left[\frac{f^2(1-\alpha)(\alpha r - d)}{2r(1-\alpha f)}\right]\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= c_r f d + c_b d(1-f) + c_p(1-\alpha f)d + c_h \alpha f d + c_l(1-\alpha) f d + c_d(1-f)d + \frac{d(1-\alpha f)}{Q_p N_p}(k_p N_p + k_r N_r + k_b N_b) + \\
&\quad Q_p\Bigg(\frac{h_s}{2}(1-\alpha f)\left(1-\frac{d}{p}\right) + \frac{N_p}{N_r}\left[\frac{h_s \alpha f^2(\alpha r - d)}{2r(1-\alpha f)}\right] + \frac{N_p}{N_r}\left[\frac{h_r f^2(\alpha r - d)}{2r(1-\alpha f)}\right] + N_p\left[\frac{h_r f^2(1-\alpha f)}{2f(1-\alpha f)}\right] + N_p\left[h_c\left(1-\frac{d}{p}\right)\left(f(1-\alpha) - \frac{(1-\alpha f)}{2}\right)\right] \\
&\quad + \frac{1}{N_b}\left[\frac{h_c(p-d)(1-f)}{p}\sum_{i=1}^{N_p} M(i, N_p, N_b)\right] - \frac{h_c(1-\alpha f)(p-d)}{2p} + N_p\left[\frac{h_c f^2(1-\alpha)}{2(1-\alpha f)}\left(\frac{d\,(1-\alpha)}{p} + \alpha\right)\right] + \frac{N_p}{N_b}\left[\frac{h_c d(1-f)^2}{2p(1-\alpha f)}\right] + \frac{N_p}{N_r}\left[\frac{h_c f^2(1-\alpha)(\alpha r - d)}{2r(1-\alpha f)}\right]\Bigg) \\
&= c_r f d + c_b d(1-f) + c_p(1-\alpha f)d + c_h \alpha f d + c_l(1-\alpha) f d + c_d(1-f)d + \frac{d(1-\alpha f)}{Q_p N_p}(k_p N_p + k_r N_r + k_b N_b) + \\
&\quad Q_p\Bigg(\frac{h_s}{2}(1-\alpha f)\left(1-\frac{d}{p}\right) + N_p\left[\frac{h_r f}{2} + \frac{h_c f^2(1-\alpha)}{2(1-\alpha f)}\left(\frac{d\,(1-\alpha)}{p} + \alpha\right)\right] + \frac{N_p}{N_r}\left[\frac{h_s f^2 \alpha(\alpha r - d)}{2r(1-\alpha f)} + \frac{h_r f^2(\alpha r - d)}{2r(1-\alpha f)} + \frac{h_c f^2(1-\alpha)(\alpha r - d)}{2r(1-\alpha f)}\right] \\
&\quad + \frac{N_p}{N_b}\left[\frac{h_c d(1-f)^2}{2p(1-\alpha f)}\right] + N_p\left[h_c\left(1-\frac{d}{p}\right)\left(f(1-\alpha) - \frac{(1-\alpha f)}{2}\right)\right] + \frac{1}{N_b}\left[\frac{h_c(p-d)(1-f)}{p}\sum_{i=1}^{N_p} M(i, N_p, N_b)\right] - \frac{h_c(1-\alpha f)(p-d)}{2p}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= c_r f d + c_b d(1-f) + c_p(1-\alpha f)d + c_h \alpha f d + c_l(1-\alpha) f d + c_d(1-f)d + \frac{d(1-\alpha f)}{Q_p N_p}(k_p N_p + k_r N_r + k_b N_b) + \\
&\quad Q_p\Bigg(\frac{h_s(1-\alpha f)(p-d)}{2p} + N_p\left[\frac{h_r f}{2} + \frac{h_c f^2(1-\alpha)}{2(1-\alpha f)}\left(\frac{d\,(1-\alpha)}{p} + \alpha\right)\right] + \frac{N_p}{N_r}\left[\frac{f^2(\alpha r - d)}{2r(1-\alpha f)}\left(h_s \alpha + h_r + h_c(1-\alpha)\right)\right] \\
&\quad + \frac{N_p}{N_b}\left[\frac{h_c d(1-f)^2}{2p(1-\alpha f)}\right] + N_p\left[h_c\left(1-\frac{d}{p}\right)\left(f(1-\alpha) - \frac{(1-\alpha f)}{2}\right)\right] + \frac{1}{N_b}\left[\frac{h_c(p-d)(1-f)}{p}\sum_{i=1}^{N_p} M(i, N_p, N_b)\right] - \frac{h_c(1-\alpha f)(p-d)}{2p}\Bigg)
\end{aligned}
$$

A.3.2 Validation of the Cost Function

In this section, further details about the comparison of the total cost function with Teunter (2004) are discussed.

Comparison to Teunter (2004)

The current model is an extension of the model described in Teunter (2004). These models have the same structure if only high quality items are recovered, ($zeta_H = 1$, $\zeta_L = 0$). These choices for ζ_H and ζ_L mean that the quality parameter $\alpha = 1$. This parameter is not replaced until the end of the simplification below, in order to retain comparability with the current mode. Teunter (2004) does not processing costs or costs for holding components ($c_r, c_b, c_p, c_r, c_d, h_c, k_b$). We will show that with these cost parameters set to zero, the total cost in the current model is equivalent to the equations (1) and (7) in Teunter (2004). Equation (1) is the total cost under a policy with $N_p = 1$ production lots and N_r recovery lots and equation (7) is the total cost under a policy with N_p production lots and $N_r = 1$ recovery lots. This further validates the part of the total cost formula. However, obviously, since the current model uses nonzero costs for these costs, some parts of the cost function can not be validated using this method. The effect of taking the total cost function in equation (A.23) and setting these cost parameters to zero is now presented.

$$
\begin{aligned}
TC = {} & c_r f d + c_b d(1-f) + c_p(1-\alpha f)d + c_h \alpha f d + c_l(1-\alpha) f d + c_d(1-f)d + \frac{d(1-\alpha f)}{Q_p N_p}(k_p N_p + k_r N_r + k_b N_b) + Q_p \Bigg(\frac{h_s(1-\alpha f)(p-d)}{2p} + \\
& N_p \left[\frac{h_r f}{2} + \frac{h_c f^2 (1-\alpha)}{2(1-\alpha f)} \left(\frac{d\,(1-\alpha)}{p} + \alpha \right) \right] + \frac{N_p}{N_r} \left[\frac{f^2(\alpha r - d)}{2r(1-\alpha f)} \left(h_s \alpha + h_r + h_c(1-\alpha) \right) \right] + \frac{N_p}{N_b} \left[\frac{h_c d (1-f)^2}{2p(1-\alpha f)} \right] + N_p \left[h_c \left(1 - \frac{d}{p} \right) \left(f(1-\alpha) - \frac{(1-\alpha f)}{2} \right) \right] \\
& + \frac{1}{N_b} \left[\frac{h_c(p-d)(1-f)}{p} \sum_{i=1}^{N_p} M(i, N_p, N_b) \right] - \frac{h_c(1-\alpha f)(p-d)}{2p} \Bigg)
\end{aligned}
$$

$$
= \frac{d(1-\alpha f)}{Q_p N_p}(k_p N_p + k_r N_r) + \frac{h_s Q_p}{2} \left[(1-\alpha f) \left(1 - \frac{d}{p} \right) + \frac{N_p}{N_r} \frac{\alpha f^2}{(1-\alpha f)} \left(\alpha - \frac{d}{r} \right) \right] + \frac{h_r Q_p N_p f}{2 N_r (1-\alpha f)} \left[N_r - \frac{f d}{r} - \alpha f (N_r - 1) \right]
$$

$$
= \frac{d(1-\alpha f)}{Q_p N_p} k_p N_p + \frac{d(1-\alpha f)}{Q_p N_p} k_r N_r + \frac{h_s}{2} \left[Q_p (1-\alpha f) \left(1 - \frac{d}{p} \right) + \frac{Q_p N_p}{N_r} \frac{\alpha f^2}{(1-\alpha f)} \left(\alpha - \frac{d}{r} \right) \right] + \frac{h_r f}{2} \frac{Q_r}{f} \left[N_r - \frac{f d}{r} - \alpha f (N_r - 1) \right]
$$

$$
= \frac{k_p d(1-\alpha f)}{Q_p} + \frac{k_r f d (1-\alpha f)}{Q_r N_r (1-\alpha f)} N_r + \frac{h_s}{2} \left[Q_p (1-\alpha f) \left(1 - \frac{d}{p} \right) + \frac{Q_r N_r (1-\alpha f)}{f N_r} \frac{\alpha f^2}{(1-\alpha f)} \left(\alpha - \frac{d}{r} \right) \right] + \frac{h_r f}{2} \left[\frac{Q_r}{f} N_r - \frac{Q_r}{f} \frac{f d}{r} - \frac{Q_r}{f} \alpha f (N_r - 1) \right]
$$

$$
= \frac{k_p d(1-\alpha f)}{Q_p} + \frac{k_r f d}{Q_r} + \frac{h_s}{2} \left[Q_p (1-\alpha f) \left(1 - \frac{d}{p} \right) + Q_r \alpha f \left(\alpha - \frac{d}{r} \right) \right] + \frac{h_r f}{2} \left[\frac{Q_r N_r}{f} - \frac{Q_r d}{r} - Q_r N_r \alpha + Q_r \alpha \right]
$$

$$
= \frac{k_p d(1-\alpha f)}{Q_p} + \frac{k_r f d}{Q_r} + \frac{h_s}{2} \left[Q_p (1-\alpha f) \left(1 - \frac{d}{p} \right) + Q_r \alpha f \left(\alpha - \frac{d}{r} \right) \right] + \frac{h_r f}{2} \left[Q_r N_r \Big(\frac{1}{f} - \alpha \Big) + Q_r \Big(\alpha - \frac{d}{r} \Big) \right]
$$

$$
= \frac{k_p d(1-\alpha f)}{Q_p} + \frac{k_r f d}{Q_r} + \frac{h_s}{2} \left[Q_p (1-\alpha f) \left(1 - \frac{d}{p} \right) + Q_r \alpha f \left(\alpha - \frac{d}{r} \right) \right] + \frac{h_r f}{2} \left[Q_r N_r \Big(\frac{(1-\alpha f)}{f} \Big) + Q_r \Big(\alpha - \frac{d}{r} \Big) \right]
$$

$$
= \frac{k_p d(1-\alpha f)}{Q_p} + \frac{k_r f d}{Q_r} + \frac{h_s}{2} \left[Q_p (1-\alpha f) \left(1 - \frac{d}{p} \right) + Q_r \alpha f \left(\alpha - \frac{d}{r} \right) \right] + \frac{h_r f}{2} \left[Q_p N_p + Q_r \Big(\alpha - \frac{d}{r} \Big) \right]
$$

Now setting $\alpha = 1$ and $N_p = 1$ gives:

$$TC = \frac{k_p d(1-f)}{Q_p} + \frac{k_r f d}{Q_r} + \frac{h_s}{2}\left[Q_p(1-f)\left(1-\frac{d}{p}\right) + Q_r f\left(1-\frac{d}{r}\right)\right] + \frac{h_r f}{2}\left[Q_p + Q_r\left(1-\frac{d}{r}\right)\right]$$

which is equivalent to equation (1) from Teunter (2004). Alternatively, setting $\alpha = 1$ and $N_r = 1$:

$$\begin{aligned}
&= \frac{k_p d(1-f)}{Q_p} + \frac{k_r f d}{Q_r} + \frac{h_s}{2}\left[Q_p(1-f)\left(1-\frac{d}{p}\right) + Q_r f\left(1-\frac{d}{r}\right)\right] \\
&\quad + \frac{h_r f}{2}\left[\frac{Q_r(1-f)}{f} + Q_r\left(\alpha - \frac{d}{r}\right)\right] \\
&= \frac{k_p d(1-f)}{Q_p} + \frac{k_r f d}{Q_r} + \frac{h_s}{2}\left[Q_p(1-f)\left(1-\frac{d}{p}\right) + Q_r\left(1-\frac{d}{r}\right)\right] + \frac{h_r f}{2}\left[Q_r\left(1-f+f-\frac{df}{r}\right)\right] \\
&= \frac{k_p d(1-f)}{Q_p} + \frac{k_r f d}{Q_r} + \frac{h_s}{2}\left[Q_p(1-f)\left(1-\frac{d}{p}\right) + Q_r\left(1-\frac{d}{r}\right)\right] + \frac{h_r f}{2}\left[Q_r\left(1-\frac{df}{r}\right)\right]
\end{aligned}$$

which is equivalent to equation (7) from Teunter (2004). These results further validate the cost function of the current model.

A.4 Minimisation of the Total Cost Function

The section contains information to support Section 4.6. Details relating to the case where $N_p = 1$, and $N_r, N_b \geq 1$ are presented in Section A.4.1. The most general case with $N_p, N_r, N_b \geq 1$ is considered in Section A.4.2.

A.4.1 Case 2: $N_p = 1$, $N_r, N_b \geq 1$

Convexity of the Continuous Relaxation

When $N_p = 1$, the total cost function simplifies to:

$$TC(Q_p, 1, N_r, N_b) = C_P + \frac{d(1-\alpha f)(k_p + N_r k_r + N_b k_b)}{Q_p} + Q_p\left(V + W + \frac{1}{N_r}X + \frac{1}{N_b}Y - Z_1 + Z_2 + Z_3\right)$$

Let $\tilde{N}_r$, $\tilde{N}_b$ denote the continuous relaxation of the variables N_r and N_b respectively. In this section the first and second partial derivatives with respect to Q_p, $\tilde{N}_r$, $\tilde{N}_b$ will be calculated and the Hessian matrix will be constructed. The partial derivatives of the total cost function are presented below:

$$
\begin{aligned}
\frac{\partial TC(Q_p, 1, \tilde{N}_r, \tilde{N}_b)}{\partial Q_p} &= -\frac{d(1-\alpha f)(k_p + \tilde{N}_r k_r + \tilde{N}_b k_b)}{Q_p^2} + \\
&\quad \left(V + W + \frac{1}{\tilde{N}_r} X + \frac{1}{\tilde{N}_b} Y - Z_1 + Z_2 + Z_3\right) \\
\frac{\partial TC(Q_p, 1, \tilde{N}_r, \tilde{N}_b)}{\partial \tilde{N}_r} &= \frac{d(1-\alpha f) k_r}{Q_p} - \frac{Q_p}{N_r^2} X \\
\frac{\partial TC(Q_p, 1, \tilde{N}_r, \tilde{N}_b)}{\partial \tilde{N}_b} &= \frac{d(1-\alpha f) k_b}{Q_p} - \frac{Q_p}{N_b^2} Y
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 TC(Q_p, 1, \tilde{N}_r, \tilde{N}_b)}{\partial Q_p^2} &= \frac{2d(1-\alpha f)(k_p + \tilde{N}_r k_r + \tilde{N}_b k_b)}{Q_p^3} \\
\frac{\partial^2 TC(Q_p, 1, \tilde{N}_r, \tilde{N}_b)}{\partial Q_p \partial \tilde{N}_r} &= -\frac{d(1-\alpha f) k_r}{Q_p^2} - \frac{X}{\tilde{N}_r^2} \\
\frac{\partial^2 TC(Q_p, 1, \tilde{N}_r, \tilde{N}_b)}{\partial Q_p \partial \tilde{N}_b} &= -\frac{d(1-\alpha f) k_b}{Q_p^2} - \frac{Y}{\tilde{N}_b^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 TC(Q_p, 1, \tilde{N}_r, \tilde{N}_b)}{\partial \tilde{N}_r \partial Q_p} &= -\frac{d(1-\alpha f) k_r}{Q_p^2} - \frac{X}{N_r^2} \\
\frac{\partial^2 TC(Q_p, 1, \tilde{N}_r, \tilde{N}_b)}{\partial \tilde{N}_r^{\,2}} &= \frac{2 Q_p X}{\tilde{N}_r^{\,3}} \\
\frac{\partial^2 TC(Q_p, 1, \tilde{N}_r, \tilde{N}_b)}{\partial \tilde{N}_r \partial \tilde{N}_b} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 TC(Q_p, 1, \tilde{N}_r, \tilde{N}_b)}{\partial \tilde{N}_b \partial Q_p} &= -\frac{d(1-\alpha f) k_b}{Q_p^2} - \frac{Y}{\tilde{N}_b^{\,2}} \\
\frac{\partial^2 TC(Q_p, 1, \tilde{N}_r, \tilde{N}_b)}{\partial \tilde{N}_b \partial \tilde{N}_r} &= 0 \\
\frac{\partial^2 TC(Q_p, 1, \tilde{N}_r, \tilde{N}_b)}{\partial \tilde{N}_b^{\,2}} &= \frac{2 Q_p Y}{\tilde{N}_b^{\,3}}
\end{aligned}
$$

Using the Hessian matrix, the convexity of the continuous relaxation of this problem will be shown (Winston, 1987). The Hessian matrix of a function $f(x)$ is:

$$H(x_{i,j}) = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

For this problem the Hessian matrix has the following form:

$$H(Q_p, \tilde{N}_r, \tilde{N}_b) = \begin{bmatrix} \frac{\partial^2 TC}{\partial Q_p^2} & \frac{\partial^2 TC}{\partial Q_p \partial \tilde{N}_r} & \frac{\partial^2 TC}{\partial Q_p \partial \tilde{N}_b} \\ \frac{\partial^2 TC}{\partial \tilde{N}_r \partial Q_p} & \frac{\partial^2 TC}{\partial \tilde{N}_r{}^2} & \frac{\partial^2 TC}{\partial \tilde{N}_r \partial \tilde{N}_b} \\ \frac{\partial^2 TC}{\partial \tilde{N}_b \partial Q_p} & \frac{\partial^2 TC}{\partial \tilde{N}_b \partial \tilde{N}_r} & \frac{\partial^2 TC}{\partial \tilde{N}_b{}^2} \end{bmatrix}$$

where TC represents the total cost function $TC(Q_p, 1, \tilde{N}_r, \tilde{N}_b)$.

$$H(Q_p, \tilde{N}_r, \tilde{N}_b) = \begin{bmatrix} \frac{2d(1-\alpha f)(k_p+\tilde{N}_r k_r+\tilde{N}_b k_b)}{Q_p^3} & -\frac{d(1-\alpha f)k_r}{Q_p^2} - \frac{X}{\tilde{N}_r^2} & -\frac{d(1-\alpha f)k_b}{Q_p^2} - \frac{Y}{\tilde{N}_b^2} \\ -\frac{d(1-\alpha f)k_r}{Q_p^2} - \frac{X}{\tilde{N}_r{}^2} & \frac{2Q_p X}{\tilde{N}_r^3} & 0 \\ -\frac{d(1-\alpha f)k_b}{Q_p^2} - \frac{Y}{\tilde{N}_b{}^2} & 0 & \frac{2Q_p Y}{\tilde{N}_b^3} \end{bmatrix}$$

In order to show that the function $TC(Q_p, 1, \tilde{N}_r, \tilde{N}_b)$ is convex in Q_p, $\tilde{N}_r$ and $\tilde{N}_b$, it is must be shown that the first, second and third principle minors of $H(Q_p, \tilde{N}_r, \tilde{N}_b)$ are nonnegative (Winston, 1987). To calculate the principle minors, the determinants of several matrices will need to be calculated. In general, the determinant of an $m \times m$ matrix, A is:

$$\det(A) = (-1)^{i+1} a_{i,1}(\det A_{i,1}) + (-1)^{i+2} a_{i,2}(\det A_{i,2}) + \cdots + (-1)^{i+m} a_{i,m}(\det A_{i,m}) \tag{A.23}$$

for any $i = 1, \ldots, m$, where $\det A_{i,j}$ is the determinant of the matrix obtained by deleting the i^{th} and j^{th} column (Winston, 1987).

For a 2×2 matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, equation (A.23) is equivalent to using the following formula to calculate the determinant: $ad - bc$. Similarly, for a 3×3 matrix $\left(\begin{smallmatrix} a & b & c \\ d & e & f \\ g & h & i \end{smallmatrix}\right)$ equation (A.23) is equivalent to: $aei - afh - bdi + bfg + cdh - cge$.

The first principle minors are the elements on the main diagonal,

$$\frac{2d(1-\alpha f)(k_p + \tilde{N}_r k_r + \tilde{N}_b k_b)}{Q_p^3}, \quad \frac{2Q_p X}{\tilde{N}_r^3}, \quad \text{and} \quad \frac{2Q_p Y}{\tilde{N}_b^3}$$

Since d, $(1-\alpha f)$, k_p, k_r, Q_p, $\tilde{N}_r$, $\tilde{N}_b$, X and Y are all greater than zero, and k_b is greater than or equal to 0, all first principle minors are nonnegative.

The second principle minors in a 3×3 matrix are obtained by deleting the i^{th} row and i^{th} column and calculating the determinant of the remaining 2×2 matrix. Deleting the 1^{st} row and column leaves:

$$H(Q_p, \tilde{N}_r, \tilde{N}_b) = \left[\begin{array}{c|cc} & & \\ \hline & \frac{2Q_pX}{\tilde{N}_r^3} & 0 \\ & 0 & \frac{2Q_pY}{\tilde{N}_b^3} \end{array}\right]$$

therefore the determinant is:

$$\frac{2Q_pX}{\tilde{N}_r^{\,3}}\frac{2Q_pY}{\tilde{N}_b^{\,3}} - \left(0\right)\left(0\right) = \frac{4Q_p^2XY}{\tilde{N}_r^{\,3}\tilde{N}_b^{\,3}} > 0$$

Deleting the 2^{nd} row and column leaves:

$$H(Q_p, \tilde{N}_r, \tilde{N}_b) = \left[\begin{array}{c|c} \frac{2d(1-\alpha f)(k_p+\tilde{N}_r k_r+\tilde{N}_b k_b)}{Q_p^3} & -\frac{d(1-\alpha f)k_b}{Q_p^2} - \frac{Y}{\tilde{N}_b^2} \\ \hline -\frac{d(1-\alpha f)k_b}{Q_p^2} - \frac{Y}{\tilde{N}_b^{\,2}} & \frac{2Q_pY}{\tilde{N}_b^3} \end{array}\right]$$

and therefore the determinant is:

$$\frac{2d(1-\alpha f)(k_p+\tilde{N}_r k_r+\tilde{N}_b k_b)}{Q_p^3}\frac{2Q_pY}{\tilde{N}_b^{\,3}} - \left(-\frac{d(1-\alpha f)k_b}{Q_p^2} - \frac{Y}{\tilde{N}_b^{\,2}}\right)\left(-\frac{d(1-\alpha f)k_b}{Q_p^2} - \frac{Y}{\tilde{N}_b^{\,2}}\right)$$

It is not immediately obvious whether this function is non-negative. As this function is quite complicated, Maple 14 was used to investigate its sign. The expression Y was replaced with its actual expression and appropriate sign restrictions for the parameters were included. However, it was found that it is not possible to determine the sign of this function without further information about the relationship between the parameters. This means that the continuous relaxation of the total cost function may not always be convex. Nevertheless, we shall continue the procedure for determining convexity.

Deleting the 3rd row and column leaves:

$$H(Q_p, \tilde{N}_r, \tilde{N}_b) = \left[\begin{array}{cc|c} \frac{2d(1-\alpha f)(k_p+\tilde{N}_r k_r+\tilde{N}_b k_b)}{Q_p^3} & -\frac{d(1-\alpha f)k_r}{Q_p^2} - \frac{X}{\tilde{N}_r^2} & \\ -\frac{d(1-\alpha f)k_r}{Q_p^2} - \frac{X}{\tilde{N_r}^2} & \frac{2Q_pX}{\tilde{N}_r^3} & \\ \hline & & \end{array}\right]$$

and therefore the determinant is:

$$\frac{2d(1-\alpha f)(k_p+\tilde{N}_r k_r+\tilde{N}_b k_b)}{Q_p^3}\frac{2Q_pX}{\tilde{N_r}^3} - \left(-\frac{d(1-\alpha f)k_r}{Q_p^2} - \frac{X}{\tilde{N_r}^2}\right)\left(-\frac{d(1-\alpha f)k_r}{Q_p^2} - \frac{X}{\tilde{N_r}^2}\right)$$

As was the case above, the sign of this determinant is not immediately obvious. Using Maple 14 once again, the same result was found: without further information about the relationships between the variables it is not possible to determine the sign of the function.

The third principle minor of a 3×3 matrix is the determinant of the matrix.

$$\begin{aligned}
\det H(Q_p, \tilde{N}_r, \tilde{N}_b) = {} & \frac{2d(1-\alpha f)(k_p+\tilde{N}_r k_r+\tilde{N}_b k_b)}{Q_p^3}\frac{2Q_pX}{\tilde{N_r}^3}\frac{2Q_pY}{\tilde{N_b}^3} \\
& - \left(\frac{2d(1-\alpha f)(k_p+\tilde{N}_r k_r+\tilde{N}_b k_b)}{Q_p^3}\right)\left(0\right)\left(0\right) \\
& - \left(\frac{-d(1-\alpha f)k_r}{Q_p^2} - \frac{X}{N_r^2}\right)\left(-\frac{d(1-\alpha f)k_r}{Q_p^2} - \frac{X}{\tilde{N_r}^2}\right)\left(\frac{2Q_pY}{\tilde{N_b}^3}\right) \\
& + \left(\frac{-d(1-\alpha f)k_r}{Q_p^2} - \frac{X}{\tilde{N_r}^2}\right)\left(0\right)\left(-\frac{d(1-\alpha f)k_b}{Q_p^2} - \frac{Y}{\tilde{N_b}^2}\right) \\
& + \left(\frac{-d(1-\alpha f)k_b}{Q_p^2} - \frac{Y}{\tilde{N_b}^2}\right)\left(-\frac{d(1-\alpha f)k_r}{Q_p^2} - \frac{X}{\tilde{N_r}^2}\right)\left(0\right) \\
& - \left(\frac{-d(1-\alpha f)k_b}{Q_p^2} - \frac{Y}{\tilde{N_b}^2}\right)\left(-\frac{d(1-\alpha f)k_b}{Q_p^2} - \frac{Y}{\tilde{N_b}^2}\right)\left(\frac{2Q_pX}{\tilde{N_r}^3}\right) \\
= {} & \frac{2d(1-\alpha f)(k_p+\tilde{N}_r k_r+\tilde{N}_b k_b)}{Q_p}\frac{4XY}{\tilde{N_r}^3\tilde{N_b}^3} \\
& - \left(\frac{-d(1-\alpha f)k_r}{Q_p^2} - \frac{X}{\tilde{N_r}^2}\right)^2\left(\frac{2Q_pY}{\tilde{N_b}^3}\right) \\
& - \left(\frac{-d(1-\alpha f)k_b}{Q_p^2} - \frac{Y}{\tilde{N_b}^2}\right)^2\left(\frac{2Q_pX}{\tilde{N_r}^3}\right)
\end{aligned}$$

As for the second principle minors, the it is not possible to determine the sign of this determinant without more information about the relationship between the parameters. Since it is not possible to determine if the principle minors of the Hessian matrix are nonnegative, it is not possible to determine if the continuous relaxation of this mixed integer optimization problem is convex.

A.4.2 Case 3: $N_p, N_r, N_b \geq 1$

Convexity in Q_p

For fixed N_p, N_r, N_b, ζ, the total cost function is convex in Q_p. The first derivative of the total cost function, with respect to Q_p, given a fixed N_p, N_r, N_b is:

$$\frac{dTC(Q_p, N_p, N_r, N_b)}{dQ_p} = -\frac{\bar{K}(N_p, N_r, N_b)}{Q_p^2} + \left(V + N_pW + \frac{N_p}{N_r}X + \frac{N_p}{N_b}Y - Z_1 + N_pZ_2 + \frac{1}{N_b}Z_3\sum_{i=1}^{N_p} M(i, N_p, N_b)\right) \tag{A.24}$$

The second derivative of the total cost function, with respect to Q_p, given a fixed N_p, N_r, N_b is:

$$\frac{d^2TC(Q_p, N_p, N_r, N_b)}{dQ_p^2} = \frac{\bar{K}(N_p, N_r, N_b)}{Q_p^3} > 0$$

Across the range of feasible values for Q_p, (i.e. $Q_p > 0$) this second derivative is always positive. This means that the total cost function is convex with respect to Q_p and therefore that the stationary point found by the first derivative is minimum. By setting the first derivative (equation (A.24)) equal to zero, and solving for Q_p, the value of Q_p which minimises the total cost can be found:

$$0 = -\frac{\bar{K}(N_p, N_r, N_b)}{Q_p^2} + \left(V + N_pW + \frac{N_p}{N_r}X + \frac{N_p}{N_b}Y - Z_1 + N_pZ_2 + \frac{1}{N_b}Z_3\sum_{i=1}^{N_p} M(i, N_p, N_b)\right)$$

$$\frac{\bar{K}(N_p, N_r, N_b)}{Q_p^2} = \left(V + N_pW + \frac{N_p}{N_r}X + \frac{N_p}{N_b}Y - Z_1 + N_pZ_2 + \frac{1}{N_b}Z_3\sum_{i=1}^{N_p} M(i, N_p, N_b)\right)$$

$$Q_p^2 = \frac{\bar{K}(N_p, N_r, N_b)}{\left(V + N_pW + \frac{N_p}{N_r}X + \frac{N_p}{N_b}Y - Z_1 + N_pZ_2 + \frac{1}{N_b}Z_3\sum_{i=1}^{N_p} M(i, N_p, N_b)\right)}$$

$$Q_p = \sqrt{\frac{\bar{K}(N_p, N_r, N_b)}{\left(V + N_pW + \frac{N_p}{N_r}X + \frac{N_p}{N_b}Y - Z_1 + N_pZ_2 + \frac{1}{N_b}Z_3\sum_{i=1}^{N_p} M(i, N_p, N_b)\right)}}$$

Since this equation contains a square root term, it is necessary to prove that the term being square rooted is not negative. This condition holds since $\bar{K}(N_p, N_r, N_b) > 0$ and

$W, V, X > 0$ and $Y, (-Z_1 + N_p Z_2 + \frac{1}{N_b} Z_3 \sum_{i=1}^{N_p} M(i, N_p, N_b)) \geq 0$. Since the total cost function is convex in Q_p the minimum total cost is reached at

$$Q_p^*(N_p, N_r, N_b) = \sqrt{\frac{\bar{K}(N_p, N_r, N_b)}{V + N_p W + \frac{N_p}{N_r} X + \frac{N_p}{N_b} Y - Z_1 + N_p Z_2 + \frac{1}{N_b} Z_3 \sum_{i=1}^{N_p} M(i, N_p, N_b)}}$$

which gives the corresponding total cost

$$TC(Q_p^*, N_p, N_r, N_b) = C_P + 2\sqrt{\bar{K}\Big(N_p, N_r, N_b\Big)} \times$$
$$\sqrt{\Big(V + N_p W + \frac{N_p}{N_r} X + \frac{N_p}{N_b} Y - Z_1 + N_p Z_2 + \frac{1}{N_b} Z_3 \sum_{i=1}^{N_p} M(i, N_p, N_b)\Big)}$$

The minimum total cost for the system can be determined by finding integer values of N_p, N_r, N_b which minimise equation (A.25).

A.5 Properties of the Model

A.5.1 Analysis of Recovery Strategy

In this Section the conditions under which each of the recovery strategies is optimal is analysed. This analysis could help firms to decide under what conditions it is optimal to perform high quality recovery and low quality recovery. Before conducting some numerical investigations, some theoretical results are provided for the case with one of each type of lot, i.e. $N_p = N_r = N_b = 1$.

Disposal Cost

Let $TC_0(Q_p^*, 1, 1, 1)$ and Q_{p0}^* denote the optimal total cost and production lot size under a high-quality only recovery strategy ($\zeta_L = 0$); and let $TC_1(Q_p^*, 1, 1, 1)$ and Q_{p1}^* denote the optimal total cost and production lot size when both high and low quality returns are covered ($\zeta_L = 1$). Let C_{P0} and C_{P1} denote the values of C_P under recovery strategies with $\zeta_L = 0$ and $\zeta_L = 1$ respectively. Let the similar notation apply to the other expressions $V, W, X, Y, Z_1, Z_2, Z_3$. Then under a high-quality only recovery strategy ($\zeta_L = 0$), the optimal total cost can be written as:

$$TC_0(Q_{p0}^*, 1, 1, 1) = C_{P0} + 2\sqrt{d(1 - \beta_H)(k_p + k_r + k_b)} \times$$
$$\sqrt{(V_0 + W_0 + X_0 + Y_0 - Z_{10} + Z_{20} + Z_{30})}$$

and under a both high and low quality recovery strategy ($\zeta_L = 1$), the optimal total cost can be written as:

$$TC_1(Q^*_{p1}, 1, 1, 1) = C_{P1} + 2\sqrt{d(1-\beta_H)(k_p + k_r + k_b)} \times \sqrt{(V_1 + W_1 + X_1 + Y_1 - Z_{11} + Z_{21} + Z_{31})}$$

It is profitable to perform low quality recovery when:

$$TC_1(Q^*_{p1}, 1, 1, 1) < TC_0(Q^*_{p0}, 1, 1, 1) \tag{A.25}$$

Using this relationship, it is possible to calculate the values of the disposal cost c_d, for which is it profitable to perform low quality recovery, given all other parameters remain unchanged.

Using the full expressions for C_{P0} and C_{P1} gives:

$$\begin{aligned}
C_{P0} &= c_p(1-\beta_H)d + c_h\beta_H d + c_l(0)\beta_H d + c_r\beta_H d + c_b d(1-\beta_H) + c_d(1-\beta_H)d \\
C_{P1} &= c_p(1-\beta_H)d + c_h\beta_H d + c_l\beta_L d + c_r(\beta_H + \beta_L)d + \\
&\quad c_b d(1-(\beta_H+\beta_L)) + c_d(1-(\beta_H+\beta_L))d
\end{aligned}$$

Substituting these expressions into equation (A.25), and performing some simple algebra to solve for the disposal cost c_d yields the expression given by equation (4.37) in Chapter 4. The details of the derivation of this expression are presented below.

For the disposal cost, it is profitable to do low quality recovery if the following condition is meet:

$$
\begin{aligned}
& TC_1(Q^*_{p1},1,1,1) < TC_0(Q^*_{p0},1,1,1) \\
& 0 > TC_1(Q^*_{p1},1,1,1) - TC_0(Q^*_{p0},1,1,1) \\
& 0 > c_p(1-\beta_H)d + c_h\beta_H d + c_l\left(1-\frac{\beta_H}{(\beta_H+\beta_L)}\right)(\beta_H+\beta_L)d + c_r(\beta_H+\beta_L)d + \\
& \qquad c_b d(1-(\beta_H+\beta_L)) + c_d(1-(\beta_H+\beta_L))d \\
& \qquad + 2\sqrt{d(1-\beta_H)(k_p+k_r+k_b)(V_1+W_1+X_1+Y_1-Z_{11}+Z_{21}+Z_{31})} \\
& \qquad - \Bigg(c_p(1-\beta_H)d + c_h\beta_H d + c_l(0)\beta_H d + c_r\beta_H d + c_b d(1-\beta_H) + c_d(1-\beta_H)d \\
& \qquad + 2\sqrt{d(1-\beta_H)(k_p+k_r+k_b)(V_0+W_0+X_0+Y_0-Z_{10}+Z_{20}+Z_{30})}\Bigg) \\
& 0 > c_l\beta_L d + c_r\beta_L d - c_b d\beta_L - c_d\beta_L d \\
& \qquad + 2\sqrt{d(1-\beta_H)(k_p+k_r+k_b)(V_1+W_1+X_1+Y_1-Z_{11}+Z_{21}+Z_{31})} \\
& \qquad - 2\sqrt{d(1-\beta_H)(k_p+k_r+k_b)(V_0+W_0+X_0+Y_0-Z_{10}+Z_{20}+Z_{30})} \\
& c_d > \frac{1}{\beta_L d}\Bigg(c_l\beta_L d + c_r\beta_L d - c_b d\beta_L \\
& \qquad + 2\sqrt{d(1-\beta_H)(k_p+k_r+k_b)(V_1+W_1+X_1+Y_1-Z_{11}+Z_{21}+Z_{31})} \\
& \qquad - 2\sqrt{d(1-\beta_H)(k_p+k_r+k_b)(V_0+W_0+X_0+Y_0-Z_{10}+Z_{20}+Z_{30})}\Bigg)
\end{aligned}
$$

Appendix B

Appendix for Stochastic Model with a Single Market

Introduction

This section presents additional information related to the second model in this thesis – the stochastic product recovery model, which was presented in Chapter 5.

B.1 Model Implementation and Validation

In this section information to support Section 5.5 is presented. Details relating to the implementation of the MDP in java are discussed in Section B.1.1 and in Section B.1.2 the validation of the programming code used is discussed.

B.1.1 Implementation of the MDP and Mid-State

The MDP described in Section 5.4 has three state variables. If the upper capacity was set to 30 for each inventory, then there would be a total of $31^3 = 29791$ states. For each of these states there are numerous actions which could be chosen, and for each action numerous transitions could occur. Thus the problem we are studying is very large and indeed does suffer from the curse of dimensionality. Several algorithms could be used to find the optimal policy, as discussed in Section 3.2. However due to the dimensionality of the problem under study, the value iteration algorithm is the only practical option here. Other algorithms, such as the policy iteration and linear programming, require solving a large set of linear equations (Tijms, 1994, page 182).

However as shown in the Methodology Chapter (Section 3.2), even the value iteration algorithm can be extremely computationally intensive for larger problems. Therefore in order to implement the model in Chapter 5, we use the concept of the mid-state that was described in 3.2, which allows us to calculate some values in advance and store them, rather than calculating them for every iteration of the algorithm. Referring to MDP formulation in Section 5.4, it can be observed that the effect of demand and returns on the next state does not depend on the decisions taken during the period. This means that it is possible to separate the transitions associated with demand and returns from the transitions associated with the action. This has the effect of reducing the number of transitions which need to occur within the *action loop* in the value iteration algorithm. The definition of the mid-state m for the model in Chapter 5 will now be presented.

Let the mid-state $m = (m_s, m_r, m_c)$ denote the value of the state variables after the action has been chosen and the state has been updated accordingly (but before demand and returns have been observed). The mid-state m is related to the state at the beginning of the period $i = (i_s, i_r, i_c)$ in the following way:

$$m_s = (i_s + a_p\delta + a_h(x_q)(1-\delta))$$
$$m_r = (i_r - a_r(1-\delta))$$
$$m_c = \min\{(i_c - a_p\delta + a_b\delta + a_l(x_q)(1-\delta)), W_c\}$$

and the mid-state m is related to the state at the beginning of the next period $j = (j_s, j_r, j_c)$ in the following way:

$$j_s = \max\{m_s - x_d, 0\}$$
$$j_r = \min\{m_r + x_r, W_r\}$$
$$j_c = m_c$$

The probability of moving from state i to mid-state m depends on the initial state i, the action $a(i)$ and the quality variable x_q. The probabilistic part of the transition is only applicable when recovery occurs. If recovery does not occur then the mid-state will be known with certainty.

The probability of moving from mid-state m to state j in the next period depends on the mid-state m, the demand x_d and the returns x_r. Observe that the probability of moving from the mid-state to the next state does not depend on the action selected. This means that these transition probabilities could be calculated in advance, outside of the action loop. The introduction of the mid-state means that some calculations could be performed in advance and stored. Then, during the value iteration algorithm, the values need to be looked up, rather than calculated. Furthermore, since the values are calculated in the advance, the non-zero probability can be identified and thus only these probabilities are looked up during the algorithm. This significantly reduces the computational time required to perform the value iteration algorithm.

B.1.2 Validation of Code

In addition to thorough error-checking and inspection of output during the code development process, two forms of verification were used to validate the problem specific files. The calculation of the expected average costs was checked using an Excel spreadsheet and the system was simulated using the optimal policy and the simulated cost was compared with the actual cost, as calculated by the MDP.

To conduct these tests a set of five example problems are used. The parameters for these problems are shown in Table B.1. These five test problems were chosen because they represent a range of different scenarios. One of the test problems uses the uniform distribution to model demand and returns, the other four use the Poisson distribution. All five test problems allow the recovery of high and low quality returns. The quality parameter α varies to represent there being a high probability of the returns being low quality, mixed quality, or high quality. Note that test problem $E03$ represents the case of having all high quality returns.

In order to test the calculation of the costs within the java code, an Excel spreadsheet was constructed to calculate the expected costs independently from the

Table B.1: Test problems used in the validation of the code

		$E01$	$E02$	$E03$	$E04$	$E05$
State Space						
serviceable inventory	U_s	20	20	20	20	20
returned inventory	U_r	20	20	20	20	20
component inventory	U_c	20	20	20	20	20
Distributions						
demand	X_d	Pois(8)	Pois(8)	Pois(8)	Pois(8)	$\text{Uni}(0, 16)$
returns	X_r	Pois(6)	Pois(6)	Pois(6)	Pois(6)	$\text{Uni}(0, 12)$
quality	X_q	$\text{Bin}(a, \alpha)$	$\text{Bin}(a, \alpha)$	$\text{Bin}(a, \alpha)$	$\text{Bin}(a, \alpha)$	$\text{Bin}(a, \alpha)$
recovery strategy	ζ_L	1	1	1	1	1
mean quality	α	0.6	0.6	1	0.2	0.6
Setup costs						
production	k_p	4	1	1	20	20
recovery	k_r	2	1	1	10	10
buying	k_c	3	1	1	20	10
Processing costs						
production	c_b	0	0	1	10	5
returns	c_p	0	0	1	2	10
buying	c_r	0	0	1	0.1	1
high quality recovery	c_h	0	0	2	1	5
low quality recovery	c_l	0	0	0.5	3	2
disposal	c_d	0	0	1	1	1
Holding costs						
serviceable inventory	h_s	2	1	1	4	5
returned inventory	h_r	0.1	1	1	1	1
component inventory	h_c	1	1	1	3	1
Penalty costs						
Lost sales	l_s	10	10	10	100	10
Lost returns	l_r	0	1	1	1	1

calculations used in the java programme. For a given state the expected cost associated with the optimal policy action and a heuristic policy action were calculated in Excel, by completely enumerating across possible values of the random variables X_q, X_d, X_r. The heuristic policy requires that if serviceable inventory i_s is greater than or equal to $S = 10$, then do nothing. If i_s is below S and returned inventory i_r is greater than $S - i_s$, then recover $\min\{i_r, S - i_s\}$, otherwise produce $S - i_s$ and order components if required to complete production. Note that since the quality of the returns is uncertain, if recovery is chosen, the new serviceable inventory level could be less than S.

Table B.2 shows the long run average cost associated with the optimal and heuristic policies for all five test problems. Since the optimal policy minimises the long run average cost, the costs associated with the optimal policy should be lower than the heuristic policy. The results in this table confirm that this is indeed the case. Note also that the heuristic policy performs particularly poorly for problem $E04$. It is likely that this is caused by the large cost incurred for lost sales. Calculation of the difference between the heuristic and optimal policies was performed during the numerical experiments detailed later in this chapter as well. If a heuristic policy returned a cost which was less than that of the optimal policy, an error message was displayed and the run was terminated.

Test Problem	Optimal Policy	Heuristic Policy
$E01$	17.3326	30.0075
$E02$	20.0219	36.9989
$E03$	37.2657	40.4818
$E04$	133.2897	451.7227
$E05$	106.0037	121.0157

Table B.2: Test problems cost of optimal policy and heuristic policy

The sheer number of states and actions in the problem make it impractical to compare the costs for every state, therefore a selection of 30 states for the first four test problems were used. The fifth test problem was not used as in this problem the uniform distribution is used to govern returns and demand, rather than the Poisson distribution as in the first four problems. The repeated probabilities in the uniform distribution means that it is not as effective at highlighting problems with the code. Ten states were chosen because they provide a range of states, including some which have inventory levels equal to the minimum or maximum capacity levels. The remaining 20 states were selected randomly using the Excel function `rand()`.

The states and actions for the optimal and heuristic policy are shown in Table B.3. The states marked with an asterisk * were part of the set of 10 states which were selected. For each of the 30 states, the costs calculated by the java programme and those calculated by the Excel spreadsheet were checked for the optimal policy action and for a heuristic policy action. Note that the heuristic policy actions are the same for all of the problems because the heuristic rule speficied is not affected by any of the problem parameters. For all states and actions that were examined, the costs calculated by the java code were the same as the costs calculated by the excel spreadsheet. This provides evidence that the methods used to calculate the costs in the java code are correct.

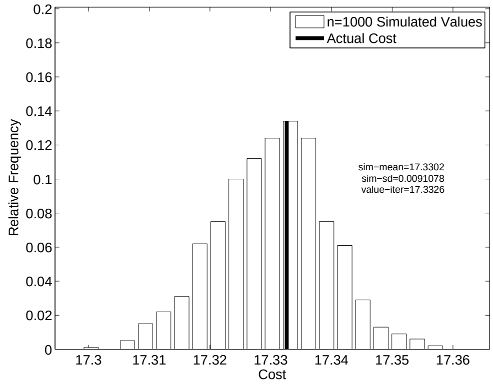
Simulation can also be used to calculate the average cost of a policy and hence can be used to validate the calculation of the average total cost in the value iteration algorithm.

Table B.3: Selection of states for which the costs were calculated using an Excel spreadsheet

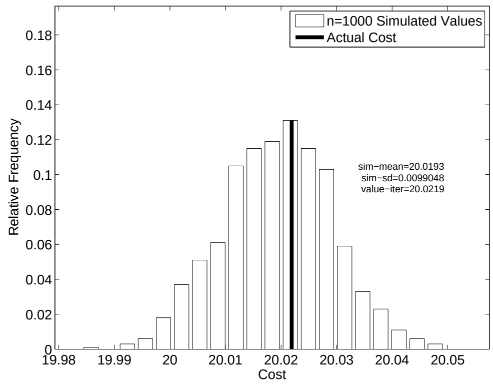
					$E01$			$E02$			$E03$			$E04$					
		State Values			Optimal Action			Optimal Action			Optimal Action			Optimal Action			Heuristic Policy Action		
	State	i_s	i_r	i_c	a_p	a_r	a_b	a_p	a_r	a_b	a_p	a_r	a_b	aa_p	a_r	a_b	a_p	a_r	a_b
1*	0	0	0	0	11	0	11	13	0	13	12	0	12	19	0	19	10	0	10
2*	5	0	0	5	11	0	6	13	0	8	12	0	7	19	0	14	10	0	5
3	303	0	14	9	9	0	0	10	0	1	0	14	0	17	0	8	0	10	0
4	383	0	18	5	11	0	6	0	18	0	0	18	0	17	0	12	0	10	0
5*	396	0	18	18	12	0	0	18	0	0	18	0	0	18	0	0	0	10	0
6*	399	0	19	0	0	19	0	0	19	0	0	19	0	17	0	17	0	10	0
7	1054	2	8	4	9	0	5	10	0	6	0	8	0	16	0	12	0	8	0
8	1097	2	10	5	9	0	4	9	0	4	0	10	0	15	0	10	0	8	0
9	1116	2	11	3	9	0	6	0	11	0	0	11	0	15	0	12	0	8	0
10	1120	2	11	7	7	0	0	9	0	2	0	11	0	15	0	8	0	8	0
11	1211	2	15	14	14	0	0	14	0	0	14	0	0	14	0	0	0	8	0
12	2541	5	16	0	0	11	0	0	16	0	0	15	0	12	0	12	0	5	0
13	3049	6	19	4	4	0	0	4	0	0	4	0	0	0	19	0	0	4	0
14	3618	8	4	6	6	0	0	6	0	0	6	0	0	6	0	0	0	2	0
15*	3906	8	18	0	0	4	0	0	18	0	0	0	0	0	18	0	0	2	0
16	4609	10	9	10	0	0	0	10	0	0	10	0	0	10	0	0	0	0	0
17*	4630	10	10	10	0	0	0	10	0	0	10	0	0	10	0	0	0	0	0
18	4661	10	11	20	0	0	0	0	11	0	10	0	0	7	0	0	0	0	0
19	4903	11	2	10	0	0	0	9	0	0	9	0	0	9	0	0	0	0	0
20	5283	11	20	12	0	0	0	9	0	0	9	0	0	9	0	0	0	0	0
21*	5292	12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	5552	12	12	8	0	0	0	8	0	0	8	0	0	8	0	0	0	0	0
23	5798	13	3	2	0	0	0	2	0	0	2	0	0	0	0	0	0	0	0
24*	6237	14	3	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0	0
25*	7152	16	4	12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26	7562	17	3	2	0	0	0	2	0	0	2	0	0	0	0	0	0	0	0
27	7643	17	6	20	0	0	0	0	6	0	0	0	0	0	0	0	0	0	0
28	8874	20	2	12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
29	9075	20	12	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30*	9260	20	20	20	0	20	0	0	20	0	0	20	0	0	20	0	0	0	0

As part of the development of this code, the initial state, the action, the random variables, mid-state and next state were printed and were inspected and compared with the equivalent manual calculations. In addition to this, the distributions of the three random variables X_q, X_d, X_r were investigated. Across a simulation of $T = 49999$ time units, the observed values of the random variables were stored, and then compared with their specified distributions. For the five test problems in Table B.1, a graphical comparison between the simulated distribution and the theoretical distribution suggested that the simulated distributions were governed by the specified distributions.

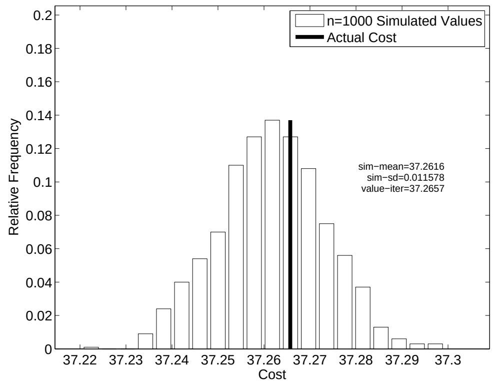
Figure B.1 shows a histogram of the simulated costs for the test problems across the 1000 runs of the simulation code. The mean and standard deviation of the simulated values are also shown on the graphs. For all test problems the value-iteration cost and the mean simulated cost were approximately the same, and as shown by these graphs, the values produced for the 'average total cost' appear to be normally distributed and centred around the cost from the value iteration algorithm. This provide further evidence to support the accuracy of the stochastic product recovery code and also the simulation code. This offers further validation of java code.



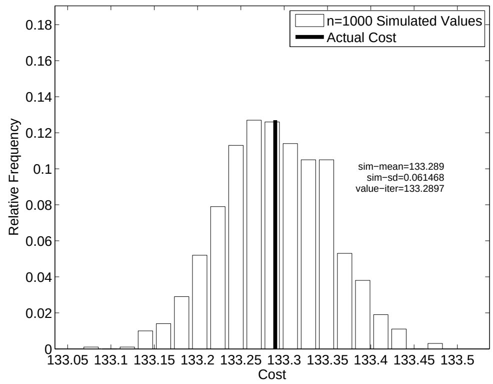
(a) $E01$



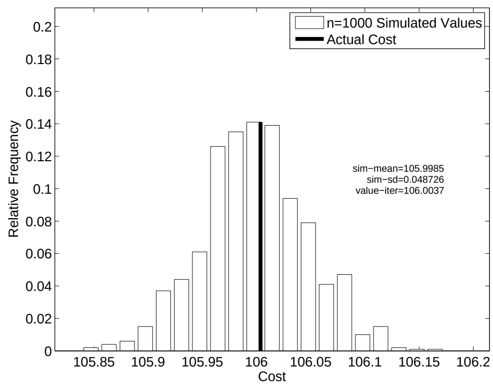
(b) $E02$



(c) $E03$



(d) $E04$



(e) $E05$

Figure B.1: Histogram of average total cost from 1000 simulations for test problems

B.2 Properties of the Optimal Policy

In this section information to support the analysis of the optimal policy in Section 5.6 is presented.

B.2.1 Analysis of Performance

This section relates to Section 5.6.2 and provides additional information about the performance of the optimal policy in terms of the average cost and the fill rate, under the two recovery strategies.

Tables B.4 and B.5 contain some additional information relating to the simulation of the optimal policy over $T = 1,000,000$ time units. The mean observed demand and returns are also shown in the Tables; these values are equal to λ_d and λ_r to 3 decimal places. The same seed was used in all runs. As expected the average amount recovered per period is at least as high when both high quality and low quality recovery is performed.

Table B.4: Summary of the performance and simulation data for problem set G under a high only quality recovery strategy ($\zeta_L = 0$)

$\zeta = 0$		Simulation over $T = 1000000$						
Problem	Optimal Cost	$\bar{x_d}$	$\bar{x_r}$	Fill rate %	$\bar{a_h}$	$\bar{a_l}$	$\bar{a_d}$	Mean recovery
$G00$	21.841	8.000	6.000	65.599	4.800	0	1.200	4.800
$G01$	755.204	8.000	7.000	75.508	5.600	0	1.400	5.600
$G02$	1268.827	9.000	4.000	72.220	0.800	0	3.200	0.800
$G03$	1966.127	11.000	4.000	78.102	1.201	0	2.799	1.201
$G04$	2384.084	9.000	4.000	63.938	1.200	0	2.800	1.200
$G05$	2019.588	10.000	9.001	69.843	4.501	0	4.500	4.501
$G06$	1062.778	10.000	9.001	78.302	6.300	0	2.700	6.300
$G07$	1092.527	10.000	8.000	71.496	6.400	0	1.600	6.400
$G08$	617.019	8.000	7.000	74.545	5.599	0	1.400	5.599
$G09$	2511.420	9.000	4.000	67.494	0.800	0	3.200	0.800
$G10$	4114.595	10.000	8.000	68.279	4.000	0	4.000	4.000
$G11$	770.370	12.000	6.000	88.911	3.001	0	2.999	3.001
$G12$	1461.006	12.000	6.000	79.866	3.001	0	3.000	3.001
$G13$	2157.194	13.000	6.000	25.138	3.000	0	3.001	3.000
$G14$	1009.246	8.000	5.000	65.206	2.501	0	2.499	2.501
$G15$	1158.008	9.000	6.000	29.441	2.402	0	3.598	2.402
$G16$	1986.169	9.000	1.000	1.275	0.100	0	0.900	0.100
$G17$	1248.180	9.000	6.000	40.963	3.573	0	2.382	3.573
$G18$	785.778	7.000	2.000	6.936	0.400	0	1.600	0.400
$G19$	2537.085	9.000	8.000	20.265	1.600	0	6.400	1.600
$G20$	941.318	5.000	3.000	63.473	1.501	0	1.499	1.501

Table B.5: Summary of the performance and simulation data for problem set G under a both high and low quality recovery strategy ($\zeta_L = 1$)

$\zeta = 1$		Simulation over $T = 1000000$						
Problem	Optimal Cost	$\bar{x_d}$	$\bar{x_r}$	Fill rate %	$\bar{a_h}$	$\bar{a_l}$	$\bar{a_d}$	Mean recovery
$G00$	21.841	8.000	6.000	65.599	4.800	0.000	1.200	4.800
$G01$	744.535	8.000	7.000	90.549	5.572	1.393	0	6.965
$G02$	1134.757	9.000	4.000	75.297	0.800	3.201	0	4.000
$G03$	1809.169	11.000	4.000	78.325	1.200	2.800	0	4.000
$G04$	2187.034	9.000	4.000	71.738	1.199	2.801	0	4.000
$G05$	1823.303	10.000	9.001	91.535	4.431	4.434	0	8.865
$G06$	1007.910	10.000	9.001	92.013	6.233	2.671	0	8.904
$G07$	1045.545	10.000	8.000	87.077	6.396	1.599	0	7.995
$G08$	603.650	8.000	7.000	90.792	5.570	1.393	0	6.962
$G09$	2372.582	9.000	4.000	67.117	0.801	3.199	0	4.000
$G10$	3604.191	10.000	8.000	87.852	3.993	3.993	0	7.986
$G11$	670.333	12.000	6.000	89.135	3.000	3.001	0	6.000
$G12$	1353.736	12.000	6.000	81.263	2.999	3.001	0	6.000
$G13$	1965.953	13.000	6.000	79.865	3.000	3.000	0	6.000
$G14$	949.508	8.000	5.000	74.326	2.499	2.501	0	5.000
$G15$	1055.832	9.000	6.000	70.055	2.398	3.601	0	5.999
$G16$	1961.925	9.000	1.000	11.402	0.099	0.896	0	0.995
$G17$	1251.555	9.000	6.000	67.467	3.548	2.368	0.001	5.916
$G18$	752.569	7.000	2.000	31.769	0.399	1.601	0	2.000
$G19$	2310.692	9.000	8.000	88.817	1.554	6.222	0.000	7.777
$G20$	885.380	5.000	3.000	73.719	1.500	1.500	0	3.000

B.2.2 Analysis of Actions

This section relates to Section 5.6.3 and provides additional information about the structure of the optimal policy under the two recovery strategies.

Action Size and Frequency

Tables B.6 and B.7 summarise the size of the production, recovery and buying lots. In these tables n refers to the number of states in which the action is chosen. A '-' in the table indicates that there were no observations so the statistic could not be calculated. For these problems there are $(31 \times 31 \times 31) = 29791$ states .

Trigger-States and Action

Tables B.8 and B.9 show the value of the inventory levels when production and recovery were chosen. Tables B.10 and B.11 summarise the mid-states after each of the actions

Table B.6: Summary of the optimal policy action quantities for problem set G under a high only quality recovery strategy ($\zeta_L = 0$)

$\zeta_L = 0$		Proportion of states	a_p for $a_p > 0$					a_r for $a_r > 0$					a_b for $a_b > 0$				
	Num. states	$a_p = a_r = a_b = 0$	Num. states	mean	median	min	max	Num. states	mean	median	min	max	Num. states	mean	median	min	max
$G00$	29791	0.632	0	-	-	-	-	10974	11.921	9	1	30	0	-	-	-	-
$G01$	29791	0.655	3946	7.202	7	1	15	6336	6.869	6	1	15	38	4.316	4	3	7
$G02$	29791	0.154	7378	15.486	15	1	30	17816	14.785	15	1	30	414	4.896	4	2	11
$G03$	29791	0.171	8149	15.048	15	1	30	16534	16.170	16	1	30	447	7.065	7	4	13
$G04$	29791	0.197	4207	13.842	14	1	30	19712	14.100	14	1	30	158	6.601	6	4	11
$G05$	29791	0.497	4089	8.529	9	1	17	10901	11.559	10	1	30	251	5.394	5	3	10
$G06$	29791	0.568	5520	7.717	8	1	17	7357	10.283	9	3	21	168	5.321	5	3	9
$G07$	29791	0.492	12547	14.501	15	1	30	2589	19.500	21	3	30	146	5.527	5	3	10
$G08$	29791	0.721	5763	9.247	9	1	19	2563	10.319	10	4	17	60	3.583	3	2	6
$G09$	29791	0.678	2159	8.793	9	4	13	7439	18.404	19	3	30	134	5.552	5	3	9
$G10$	29791	0.202	16063	13.734	13	1	30	7721	13.861	14	1	30	505	4.891	4	1	12
$G11$	29791	0.093	19401	12.699	12	1	30	7610	13.796	14	1	30	851	9.246	9	5	17
$G12$	29791	0.372	5567	11.834	12	2	25	13155	13.329	12	1	30	566	6.235	6	2	13
$G13$	29791	0.327	10111	15.673	15	4	30	9947	12.508	11	1	30	999	7.784	7	3	19
$G14$	29791	0.657	1716	7.553	8	2	13	8514	9.333	8	1	23	62	5.177	5	3	8
$G15$	29791	0.677	5957	15.249	16	2	30	3676	18.522	18	4	30	70	8.214	8	6	11
$G16$	29791	0.502	14375	10.693	9	1	30	465	23.630	24	12	30	0	-	-	-	-
$G17$	29791	0.816	3926	12.973	12	1	30	1556	26.965	28	13	30	0	-	-	-	-
$G18$	29791	0.275	1574	10.071	10	2	16	20024	16.593	16	2	30	11	6.364	6	6	7
$G19$	29791	0.667	3166	8.349	9	1	15	6759	15.381	15	2	30	31	7.097	7	6	8
$G20$	29791	0.797	3167	5.650	6	2	10	2883	5.940	5	2	15	27	3.000	3	2	5

Table B.7: Summary of the optimal policy action quantities for problem set G under a both high and low quality recovery strategy ($\zeta_L = 1$)

$\zeta_L = 1$		Proportion of states	a_p for $a_p > 0$					a_r for $a_r > 0$					a_b for $a_b > 0$				
	Num. states	$a_p = a_r = a_b = 0$	Num. states	mean	median	min	max	Num. states	mean	median	min	max	Num. states	mean	median	min	max
$G00$	29791	0.632	0	-	-	-	-	10974	11.921	9	1	30	0	-	-	-	-
$G01$	29791	0.678	5917	7.650	8	1	14	3671	7.006	6	1	16	22	4.045	4	3	7
$G02$	29791	0.452	14298	14.364	14	1	30	2038	15.245	15	1	30	383	4.914	4	2	11
$G03$	29791	0.405	15345	14.434	14	1	30	2371	16.080	16	1	30	448	6.946	7	3	13
$G04$	29791	0.590	8860	12.330	12	1	26	3340	14.027	13	1	30	145	6.324	6	3	11
$G05$	29791	0.595	9339	8.390	8	1	18	2738	14.135	15	1	30	175	5.343	5	2	10
$G06$	29791	0.623	7826	8.332	8	1	17	3420	12.361	12	2	23	104	5.115	5	3	9
$G07$	29791	0.526	12038	15.817	16	2	30	2074	17.186	17	3	30	102	5.588	5	3	10
$G08$	29791	0.717	7092	9.625	9	2	21	1340	10.880	11	3	18	30	3.633	3.5	2	6
$G09$	29791	0.748	5313	7.836	8	3	14	2196	15.049	14	3	30	118	5.703	5	3	9
$G10$	29791	0.281	19522	13.387	13	1	30	1895	16.619	17	1	30	370	4.662	4	1	11
$G11$	29791	0.110	20125	12.699	12	2	30	6398	12.375	13	1	30	766	8.785	8	4	16
$G12$	29791	0.444	9716	11.810	12	2	26	6835	9.640	8	1	28	524	6.284	6	2	13
$G13$	29791	0.372	9836	17.332	17	6	30	8860	11.465	10	1	30	719	8.570	8	3	20
$G14$	29791	0.711	4257	7.456	7	3	12	4367	9.552	8	1	24	49	5.367	5	3	8
$G15$	29791	0.723	6213	19.000	19	5	30	2050	19.290	19	7	30	0	-	-	-	-
$G16$	29791	0.748	2499	21.138	23	4	30	4997	12.798	12	3	30	0	-	-	-	-
$G17$	29791	0.883	1957	20.946	22	9	30	1519	26.269	27	17	30	0	-	-	-	-
$G18$	29791	0.785	4633	10.359	10	3	17	1786	9.283	8	1	26	1	6.000	6	6	6
$G19$	29791	0.681	6468	8.271	8	2	18	3022	11.498	10	1	30	11	7.455	7	7	8
$G20$	29791	0.818	4073	5.643	6	2	10	1347	7.522	7	2	18	18	3.000	3	2	5

have occurred. This mid-state serviceable inventory level might be useful in determining a 'produce-up-to' level in a heuristic policy. The level of components inventory after ordering has occurred and production has taken place, (i.e. at the end of the period), is also of interest. The components in stock at the end of a period indicate the willingness to hold components. Observe that the mid-state of the components inventory is always zero. This suggests that components are only ordered when needed and are used up during production.

B.3 Heuristic Policies

B.3.1 Policy $P2$: Periodic Review Order-up-to Policy with Yield Adjustment

This section contains the parameter values, costs and relative cost error associated with the $P2$ policies. This information is presented in Tables B.12–B.15.

Table B.8: Summary of the optimal policy initial states for problem set G under a high only quality recovery strategy ($\zeta_L = 0$) (n= Number of states)

$\zeta = 0$	i_s for $a_p > 0$					i_s for $a_r > 0$					i_r for $a_r > 0$					i_c for $a_b > 0$				
	n	mean	median	min	max	n	mean	median	min	max	n	mean	median	min	max	n	mean	median	min	max
$G00$	0	-	-	-	-	10974	13.853	6	0	30	10974	16.802	17	1	30	0	-	-	-	-
$G01$	3946	3.554	3	0	15	6336	5.935	6	0	16	6336	20.481	22	1	30	38	1.658	2	0	3
$G02$	7378	6.774	5	0	28	17816	15.683	16	0	28	17816	15.328	16	1	30	414	4.196	4	0	9
$G03$	8149	7.359	5	0	28	16534	15.153	15	0	28	16534	17.172	18	1	30	447	4.004	4	0	9
$G04$	4207	4.723	4	0	26	19712	14.470	14	0	28	19712	14.882	15	1	30	158	2.797	3	0	6
$G05$	4089	3.960	4	0	12	10901	10.152	9	0	28	10901	21.867	23	1	30	251	2.717	3	0	7
$G06$	5520	4.803	5	0	12	7357	8.809	8	0	26	7357	24.207	25	4	30	168	2.315	2	0	6
$G07$	12547	8.359	8	0	28	2589	4.888	4	0	14	2589	20.207	21	3	30	146	2.541	2	0	6
$G08$	5763	3.860	4	0	8	2563	4.010	4	0	12	2563	22.918	24	4	30	60	2.000	2	1	4
$G09$	2159	1.703	1	0	6	7439	5.641	6	0	12	7439	18.404	19	3	30	134	2.881	3	0	6
$G10$	16063	11.309	11	0	29	7721	13.729	14	0	29	7721	14.374	14	1	30	505	3.958	4	0	10
$G11$	19401	12.602	12	0	29	7610	16.551	18	0	29	7610	24.676	26	1	30	851	4.343	4	0	10
$G12$	5567	5.059	4	0	18	13155	12.062	12	0	29	13155	20.684	23	1	30	566	4.366	4	0	10
$G13$	10111	8.082	7	0	26	9947	14.386	14	0	29	9947	23.196	25	2	30	999	7.135	7	1	15
$G14$	1716	2.004	2	0	7	8514	6.944	6	0	27	8514	20.897	23	1	30	62	2.065	2	0	4
$G15$	5957	4.038	4	0	12	3676	6.409	6	0	18	3676	18.522	18	4	30	70	2.157	2	1	4
$G16$	14375	9.687	8	0	29	465	3.852	4	0	9	465	23.630	24	12	30	0	-	-	-	-
$G17$	3926	2.416	2	0	8	1556	3.246	3	0	10	1556	26.970	28	13	30	0	-	-	-	-
$G18$	1574	1.424	1	0	5	20024	12.705	12	0	27	20024	16.804	17	2	30	11	1.545	2	1	2
$G19$	3166	2.099	2	0	8	6759	6.271	6	0	13	6759	15.381	15	2	30	31	1.516	1	1	3
$G20$	3167	1.716	2	0	5	2883	5.627	5	0	22	2883	25.068	28	2	30	27	1.667	2	0	3

Table B.9: Summary of the optimal policy initial states for problem set G under a both high and low quality recovery strategy ($\zeta_L = 1$) (n= Number of states)

$\zeta = 1$	i_s for $a_p > 0$					i_s for $a_r > 0$					i_r for $a_r > 0$					i_c for $a_b > 0$				
	n	mean	median	min	max	n	mean	median	min	max	n	mean	median	min	max	n	mean	median	min	max
G00	0	-	-	-	-	10974	13.85310734	6	0	30	10974	16.802	17	1	30	0	-	-	-	-
G01	5917	4.002	4	0	9	3671	5.338872242	6	0	10	3671	16.756	17	1	30	22	1.773	2	0	3
G02	14298	8.830	8	0	27	2038	6.448478901	6	0	15	2038	17.138	17	1	30	383	4.063	4	0	9
G03	15345	9.371	9	0	27	2371	7.28553353	7	0	23	2371	19.417	20	1	30	448	3.982	4	0	9
G04	8860	5.296	5	0	15	3340	7.755988024	8	0	16	3340	14.190	14	1	30	145	2.786	3	0	6
G05	9339	5.762	6	0	13	2738	6.058802045	6	0	14	2738	17.938	19	1	30	175	2.343	2	0	6
G06	7826	5.512	6	0	12	3420	5.147368421	5	0	12	3420	20.815	22	2	30	104	2.077	2	0	5
G07	12038	7.660	8	0	22	2074	4.175024108	4	0	11	2074	18.094	18	3	30	102	2.598	2.5	0	6
G08	7092	4.128	4	0	9	1340	2.946268657	3	0	9	1340	18.049	18	3	30	30	2.133	2	1	4
G09	5313	2.916	3	0	7	2196	5.154371585	6	0	11	2196	18.867	19	3	30	118	2.746	3	0	6
G10	19522	11.383	11	0	29	1895	5.615831135	5	0	17	1895	16.779	17	1	30	370	3.541	3	0	9
G11	20125	12.929	13	0	28	6398	15.5140669	16	0	30	6398	25.278	27	1	30	766	4.223	4	0	10
G12	9716	6.236	6	0	20	6835	12.49144111	13	0	29	6835	22.906	25	1	30	524	4.260	4	0	10
G13	9836	7.878	7	0	24	8860	13.87968397	14	0	30	8860	23.605	25	2	30	719	7.613	8	1	16
G14	4257	2.867	3	0	7	4367	5.541103733	6	0	17	4367	21.882	24	1	30	49	1.939	2	0	4
G15	6213	4.450	4	0	11	2050	3.592195122	3	0	9	2050	19.331	19	7	30	0	-	-	-	-
G16	2499	3.383	3	0	15	4997	5.048028817	5	0	14	4997	16.120	15	3	30	0	-	-	-	-
G17	1957	2.058	2	0	5	1519	2.354180382	2	0	8	1519	26.269	27	17	30	0	-	-	-	-
G18	4633	2.473	2	0	6	1786	4.660134378	5	0	10	1786	17.593	18	1	30	1	2.000	2	2	2
G19	6468	3.624	4	0	10	3022	7.11780278	8	0	15	3022	14.546	14	1	30	11	1.545	2	1	2
G20	4073	2.161	2	0	5	1347	3.317743133	3	0	12	1347	22.625	25	2	30	18	1.611	2	0	3

Table B.10: Summary of the optimal policy mid-states for problem set G under a high only quality recovery strategy ($\zeta_L = 0$) (n= Number of states)

$\zeta = 0$	$i_s + a_p$ for $a_p > 0$					$i_r - a_r$ for $a_r > 0$					$i_c + a_b$ for $a_b > 0$					$i_c + a_b - a_p$ for $a_b > 0, a_p > 0$				
	n	mean	median	min	max	n	mean	median	min	max	n	mean	median	min	max	n	mean	median	min	max
$G00$	0	-	-	-	-	10974	4.881	0	0	21	0	-	-	-	-	0	-	-	-	-
$G01$	3946	10.756	11	4	16	6336	13.612	14	0	28	38	5.974	6	4	7	38	0	0	0	0
$G02$	7378	22.260	24	8	30	17816	0.543	0	0	18	414	9.092	9	4	11	414	0	0	0	0
$G03$	8149	22.407	24	8	30	16534	1.001	0	0	17	447	11.069	11	5	13	447	0	0	0	0
$G04$	4207	18.565	19	6	30	19712	0.783	0	0	17	158	9.399	10	5	11	158	0	0	0	0
$G05$	4089	12.489	13	6	17	10901	10.308	11	0	28	251	8.112	8	4	10	251	0	0	0	0
$G06$	5520	12.519	13	6	17	7357	13.924	16	0	26	168	7.637	8	4	9	168	0	0	0	0
$G07$	12547	22.860	24	5	30	2589	0.708	0	0	13	146	8.068	8	5	10	146	0	0	0	0
$G08$	5763	13.107	13	4	19	2563	12.599	13	0	25	60	5.583	6	3	7	60	0	0	0	0
$G09$	2159	10.497	11	6	13	7439	0.000	0	0	0	134	8.433	9	7	9	134	0	0	0	0
$G10$	16063	25.043	28	9	30	7721	0.514	0	0	13	505	8.850	9	3	12	505	0	0	0	0
$G11$	19401	25.301	29	9	30	7610	10.880	12	0	28	851	13.589	14	8	17	851	0	0	0	0
$G12$	5567	16.893	17	9	25	13155	7.355	1	0	29	566	10.601	11	4	13	566	0	0	0	0
$G13$	10111	23.755	25	12	30	9947	10.688	12	0	29	999	14.919	15	9	20	999	0	0	0	0
$G14$	1716	9.557	10	5	13	8514	11.564	11	0	27	62	7.242	7	6	8	62	0	0	0	0
$G15$	5957	19.288	20	4	30	3676	0.000	0	0	0	70	10.371	10	9	12	70	0	0	0	0
$G16$	14375	20.381	22	1	30	465	0.000	0	0	0	0	-	-	-	-	0	-	-	-	-
$G17$	3926	15.389	14	1	30	1556	0.006	0	0	1	0	-	-	-	-	0	-	-	-	-
$G18$	1574	11.495	12	3	16	20024	0.211	0	0	13	11	7.909	8	7	8	11	0	0	0	0
$G19$	3166	10.447	11	3	15	6759	0.000	0	0	0	31	8.613	9	7	9	31	0	0	0	0
$G20$	3167	7.366	8	3	10	2883	19.128	20	0	27	27	4.667	5	4	5	27	0	0	0	0

Table B.11: Summary of the optimal policy mid-states for problem set G under a both high and low quality recovery strategy ($\zeta_L = 1$), (n= Number of states)

$\zeta = 1$	$i_s + a_p$ for $a_p > 0$					$i_r - a_r$ for $a_r > 0$					$i_c + a_b$ for $a_b > 0$					$i_c + a_b - a_p$ for $a_b > 0, a_p > 0$				
	n	mean	median	min	max	n	mean	median	min	max	n	mean	median	min	max	n	mean	median	min	max
$G00$	0	-	-	-	-	10974	4.881	0	0	21	0	-	-	-	-	0	-	-	-	-
$G01$	5917	11.651	12	4	14	3671	9.750	9	0	28	22	5.818	6	4	7	22	0	0	0	0
$G02$	14298	23.194	25	7	30	2038	1.893	0	0	15	383	8.977	9	4	11	383	0	0	0	0
$G03$	15345	23.805	25	8	30	2371	3.337	0	0	26	448	10.929	11	5	13	448	0	0	0	0
$G04$	8860	17.626	17	6	26	3340	0.163	0	0	11	145	9.110	9	5	11	145	0	0	0	0
$G05$	9339	14.152	15	6	18	2738	3.803	0	0	28	175	7.686	8	4	10	175	0	0	0	0
$G06$	7826	13.844	14	5	17	3420	8.454	8	0	26	104	7.192	7	4	9	104	0	0	0	0
$G07$	12038	23.476	25	5	30	2074	0.907	0	0	26	102	8.186	8	5	10	102	0	0	0	0
$G08$	7092	13.753	13	4	21	1340	7.169	7	0	25	30	5.767	6	4	7	30	0	0	0	0
$G09$	5313	10.753	11	6	14	2196	3.817	0	0	24	118	8.449	9	7	9	118	0	0	0	0
$G10$	19522	24.771	27	7	30	1895	0.160	0	0	8	370	8.203	8	3	11	370	0	0	0	0
$G11$	20125	25.628	30	8	30	6398	12.903	14	0	29	766	13.008	13	8	16	766	0	0	0	0
$G12$	9716	18.046	17	9	26	6835	13.267	15	0	29	524	10.544	11	4	13	524	0	0	0	0
$G13$	9836	25.211	27	11	30	8860	12.140	15	0	29	719	16.184	16	11	21	719	0	0	0	0
$G14$	4257	10.322	10	4	12	4367	12.330	12	0	27	49	7.306	7	6	8	49	0	0	0	0
$G15$	6213	23.450	24	5	30	2050	0.041	0	0	3	0	-	-	-	-	0	-	-	-	-
$G16$	2499	24.520	26	4	30	4997	3.322	0	0	27	0	-	-	-	-	0	-	-	-	-
$G17$	1957	23.004	24	9	30	1519	0.000	0	0	0	0	-	-	-	-	0	-	-	-	-
$G18$	4633	12.832	13	3	17	1786	8.310	7	0	26	1	8.000	8	8	8	1	0	0	0	0
$G19$	6468	11.895	12	3	18	3022	3.048	0	0	28	11	9.000	9	9	9	11	0	0	0	0
$G20$	4073	7.804	8	3	10	1347	15.103	16	0	28	18	4.611	5	4	5	18	0	0	0	0

Table B.12: Performance of heuristic policy $P2$ with yield adjustment $y = 0.9$ for problem set G

	$P2_A$				$P2_B$				$P2_C$				$P2_D$				$P1^*$				Optimal Policy
Problem	s_A	S_A	Cost	RCE	s_B	S_B	Cost	RCE	s_C	S_C	Cost	RCE	s_D	S_D	Cost	RCE	s^*	S^*	Cost	RCE	Cost
$G01$	8	8	807.632	8.475	8	9	778.077	4.505	5	10	769.665	3.375	9	14	785.387	5.487	9	11	765.230	2.780	744.535
$G02$	9	9	1157.137	1.972	9	14	1156.878	1.949	12	15	1159.543	2.184	10	13	1154.348	1.726	11	12	1152.772	1.588	1134.757
$G03$	11	11	1826.775	0.973	11	15	1832.903	1.312	12	16	1836.594	1.516	12	16	1836.594	1.516	10	12	1826.761	0.972	1809.169
$G04$	9	9	2223.605	1.672	9	13	2236.251	2.250	9	13	2236.251	2.250	10	14	2242.693	2.545	10	11	2226.566	1.808	2187.034
$G05$	10	10	2100.795	15.219	10	12	2000.556	9.722	9	13	1959.398	7.464	11	15	1908.937	4.697	12	18	1904.755	4.467	1823.303
$G06$	10	10	1112.973	10.424	10	12	1055.960	4.767	6	12	1060.288	5.197	11	17	1054.029	4.576	11	15	1041.369	3.320	1007.910
$G07$	10	10	1098.875	5.101	10	12	1078.708	3.172	7	13	1081.792	3.467	11	17	1099.042	5.117	10	13	1079.989	3.294	1045.545
$G08$	8	8	671.311	11.209	8	9	640.928	6.176	4	10	637.093	5.540	9	15	639.482	5.936	7	12	625.658	3.646	603.650
$G09$	9	9	2392.612	0.844	9	14	2417.186	1.880	8	14	2415.978	1.829	10	16	2439.760	2.831	8	10	2391.347	0.791	2372.582
$G10$	10	10	4217.294	17.011	10	12	3919.344	8.744	10	13	3832.888	6.345	11	14	3787.875	5.096	13	17	3767.960	4.544	3604.191
$G11$	12	12	679.317	1.340	12	15	676.337	0.896	8	15	676.562	0.929	13	20	680.422	1.505	13	16	676.471	0.916	670.333
$G12$	12	12	1367.455	1.013	12	15	1369.768	1.184	10	15	1369.806	1.187	13	18	1380.788	1.998	13	14	1367.798	1.039	1353.736
$G13$	13	13	1987.866	1.115	13	16	1987.824	1.113	4	16	1990.333	1.240	14	26	2039.919	3.762	10	15	1986.755	1.058	1965.953
$G14$	8	8	970.620	2.223	8	10	962.713	1.391	5	10	962.889	1.409	9	14	984.552	3.691	8	10	962.713	1.391	949.508
$G15$	9	9	1164.166	10.260	9	12	1095.836	3.789	1	13	1115.625	5.663	9	21	1113.470	5.459	5	16	1094.171	3.631	1055.832
$G16$	9	9	3561.595	81.536	9	18	3158.271	60.978	1	28	2535.058	29.213	1	30	2567.675	30.875	1	1	2760.976	40.728	1961.925
$G17$	9	9	1493.895	19.363	9	11	1513.127	20.900	1	14	1357.948	8.501	5	30	1548.677	23.740	1	18	1347.233	7.645	1251.555
$G18$	7	7	765.849	1.765	7	11	776.722	3.209	5	12	776.012	3.115	8	15	791.273	5.143	1	7	763.466	1.448	752.569
$G19$	9	9	2615.591	13.195	9	14	2434.980	5.379	8	15	2422.726	4.848	10	17	2400.934	3.905	14	19	2403.187	4.003	2310.692
$G20$	5	5	905.807	2.307	5	6	897.239	1.339	4	7	899.616	1.608	6	9	917.190	3.593	4	7	899.616	1.608	885.380
Min				0.844				0.896				0.929				1.505				0.791	
Max				81.536				60.978				29.213				30.875				40.728	
Mean				10.351				7.233				4.844				6.160				4.534	
Median				3.704				3.191				3.245				4.241				2.294	
Std Dev				17.376				13.126				6.011				7.269				8.473	

Table B.13: Performance of heuristic policy $P2$ with yield adjustment $y = 1.0$ for problem set G

	$P1_A$				$P1_B$				$P1_C$				$P1_D$				$P1^*$				Optimal Policy
Problem	s_A	S_A	Cost	RCE	s_B	S_B	Cost	RCE	s_C	S_C	Cost	RCE	s_D	S_D	Cost	RCE	s^*	S^*	Cost	RCE	Cost
$G01$	8	8	841.508	13.025	8	9	804.105	8.001	5	10	781.188	4.923	9	14	777.997	4.494	9	11	766.323	2.926	744.535
$G02$	9	9	1165.511	2.710	9	14	1155.567	1.834	12	15	1157.725	2.024	10	13	1153.532	1.655	11	12	1152.650	1.577	1134.757
$G03$	11	11	1827.235	0.999	11	15	1830.434	1.175	12	16	1833.554	1.348	12	16	1833.554	1.348	10	12	1826.141	0.938	1809.169
$G04$	9	9	2230.030	1.966	9	13	2231.105	2.015	9	13	2231.105	2.015	10	14	2236.832	2.277	10	11	2223.659	1.675	2187.034
$G05$	10	10	2200.504	20.688	10	12	2076.404	13.881	9	13	2026.063	11.120	11	15	1948.248	6.853	12	18	1908.425	4.669	1823.303
$G06$	10	10	1169.477	16.030	10	12	1094.574	8.598	6	12	1096.942	8.833	11	17	1048.271	4.004	11	15	1044.045	3.585	1007.910
$G07$	10	10	1172.859	12.177	10	12	1087.788	4.040	7	13	1082.766	3.560	11	17	1093.943	4.629	10	13	1081.269	3.417	1045.545
$G08$	8	8	704.759	16.750	8	9	667.752	10.619	4	10	650.325	7.732	9	15	635.323	5.247	7	12	624.833	3.509	603.650
$G09$	9	9	2399.821	1.148	9	14	2413.244	1.714	8	14	2412.359	1.677	10	16	2433.478	2.567	8	10	2394.164	0.910	2372.582
$G10$	10	10	4455.457	23.619	10	12	4115.351	14.182	10	13	3974.476	10.274	11	14	3875.396	7.525	13	17	3780.422	4.890	3604.191
$G11$	12	12	683.667	1.989	12	15	677.313	1.041	8	15	677.544	1.076	13	20	679.294	1.337	13	16	676.938	0.985	670.333
$G12$	12	12	1371.381	1.303	12	15	1369.349	1.153	10	15	1369.388	1.156	13	18	1378.014	1.793	13	14	1368.361	1.080	1353.736
$G13$	13	13	1988.182	1.131	13	16	1984.644	0.951	4	16	1987.536	1.098	14	26	2023.971	2.951	10	15	1984.497	0.943	1965.953
$G14$	8	8	990.867	4.356	8	10	963.870	1.513	5	10	964.585	1.588	9	14	979.790	3.189	8	10	963.870	1.513	949.508
$G15$	9	9	1199.991	13.654	9	12	1102.433	4.414	1	13	1131.726	7.188	9	21	1107.860	4.928	5	16	1092.919	3.513	1055.832
$G16$	9	9	3532.363	80.046	9	18	3143.350	60.218	1	28	2490.120	26.922	1	30	2470.942	25.945	1	1	2400.309	22.345	1961.925
$G17$	9	9	1504.347	20.198	9	11	1502.215	20.028	1	14	1371.300	9.568	5	30	1404.311	12.205	1	18	1347.401	7.658	1251.555
$G18$	7	7	764.635	1.603	7	11	774.169	2.870	5	12	773.918	2.837	8	15	788.376	4.758	1	7	763.228	1.416	752.569
$G19$	9	9	2662.195	15.212	9	14	2473.339	7.039	8	15	2455.652	6.273	10	17	2421.265	4.785	14	19	2409.042	4.256	2310.692
$G20$	5	5	945.512	6.792	5	6	903.329	2.027	4	7	899.510	1.596	6	9	912.704	3.086	4	7	899.510	1.596	885.380
Min				0.999				0.951				1.076				1.337				0.910	
Max				80.046				60.218				26.922				25.945				22.345	
Mean				12.770				8.366				5.640				5.279				3.670	
Median				9.484				3.455				3.198				4.249				2.301	
Std Dev				17.177				13.014				5.936				5.353				4.620	

Table B.14: Performance of heuristic policy $P2$ with yield adjustment $y = 1.1$ for problem set G

	$P2_A$				$P2_B$				$P2_C$				$P2_D$				$P1^*$				Optimal Policy
Problem	s_A	S_A	Cost	RCE	s_B	S_B	Cost	RCE	s_C	S_C	Cost	RCE	s_D	S_D	Cost	RCE	s^*	S^*	Cost	RCE	Cost
$G01$	8	8	841.508	13.025	8	9	804.105	8.001	5	10	781.188	4.923	9	14	775.947	4.219	9	11	769.415	3.342	744.535
$G02$	9	9	1165.511	2.710	9	14	1154.617	1.750	12	15	1156.561	1.922	10	13	1152.717	1.583	11	12	1152.267	1.543	1134.757
$G03$	11	11	1827.794	1.029	11	15	1829.078	1.100	12	16	1832.022	1.263	12	16	1832.022	1.263	10	12	1825.727	0.915	1809.169
$G04$	9	9	2230.030	1.966	9	13	2228.187	1.882	9	13	2228.187	1.882	10	14	2233.651	2.132	10	11	2222.331	1.614	2187.034
$G05$	10	10	2200.504	20.688	10	12	2121.628	16.362	9	13	2064.495	13.228	11	15	1974.690	8.303	12	18	1915.400	5.051	1823.303
$G06$	10	10	1169.477	16.030	10	12	1119.149	11.037	6	12	1121.146	11.235	11	17	1047.474	3.925	11	15	1050.100	4.186	1007.910
$G07$	10	10	1172.859	12.177	10	12	1099.977	5.206	7	13	1086.761	3.942	11	17	1091.703	4.415	10	13	1084.828	3.757	1045.545
$G08$	8	8	704.759	16.750	8	9	667.752	10.619	4	10	650.325	7.732	9	15	633.739	4.985	7	12	625.781	3.666	603.650
$G09$	9	9	2399.821	1.148	9	14	2412.011	1.662	8	14	2411.217	1.628	10	16	2431.608	2.488	8	10	2394.164	0.910	2372.582
$G10$	10	10	4455.457	23.619	10	12	4225.680	17.244	10	13	4066.171	12.818	11	14	3940.634	9.335	13	17	3792.957	5.237	3604.191
$G11$	12	12	687.728	2.595	12	15	678.123	1.162	8	15	678.417	1.206	13	20	679.067	1.303	13	16	677.445	1.061	670.333
$G12$	12	12	1375.266	1.590	12	15	1369.498	1.164	10	15	1369.548	1.168	13	18	1377.048	1.722	13	14	1369.106	1.135	1353.736
$G13$	13	13	1990.040	1.225	13	16	1983.817	0.909	4	16	1987.246	1.083	14	26	2016.754	2.584	10	15	1984.191	0.928	1965.953
$G14$	8	8	990.867	4.356	8	10	963.870	1.513	5	10	964.585	1.588	9	14	978.597	3.064	8	10	963.870	1.513	949.508
$G15$	9	9	1199.991	13.654	9	12	1108.988	5.035	1	13	1145.200	8.464	9	21	1105.914	4.743	5	16	1093.603	3.577	1055.832
$G16$	9	9	3532.363	80.046	9	18	3140.071	60.051	1	28	2488.323	26.831	1	30	2467.641	25.777	1	1	2400.309	22.345	1961.925
$G17$	9	9	1504.347	20.198	9	11	1499.286	19.794	1	14	1380.242	10.282	5	30	1401.298	11.965	1	18	1349.275	7.808	1251.555
$G18$	7	7	764.635	1.603	7	11	774.067	2.857	5	12	773.449	2.774	8	15	787.732	4.672	1	7	763.228	1.416	752.569
$G19$	9	9	2662.195	15.212	9	14	2492.809	7.881	8	15	2473.199	7.033	10	17	2433.436	5.312	14	19	2414.325	4.485	2310.692
$G20$	5	5	945.512	6.792	5	6	903.329	2.027	4	7	899.510	1.596	6	9	912.704	3.086	4	7	899.510	1.596	885.380
Min				1.029				0.909				1.083				1.263				0.910	
Max				80.046				60.051				26.831				25.777				22.345	
Mean				12.821				8.863				6.130				5.344				3.804	
Median				9.484				3.946				3.358				4.072				2.478	
Std Dev				17.145				13.103				6.279				5.421				4.635	

Table B.15: Performance of heuristic policy $P2$ with yield adjustment $y = 1.2$ for problem set G

	$P2_A$				$P2_B$				$P2_C$				$P2_D$				$P1^*$				Optimal Policy
Problem	s_A	S_A	Cost	RCE	s_B	S_B	Cost	RCE	s_C	S_C	Cost	RCE	s_D	S_D	Cost	RCE	s^*	S^*	Cost	RCE	Cost
$G01$	8	8	890.046	19.544	8	9	838.412	12.609	5	10	803.930	7.978	9	14	772.373	3.739	9	11	776.446	4.286	744.535
$G02$	9	9	1181.252	4.097	9	14	1155.188	1.801	12	15	1156.673	1.931	10	13	1153.815	1.679	11	12	1154.507	1.740	1134.757
$G03$	11	11	1829.547	1.126	11	15	1829.048	1.099	12	16	1831.657	1.243	12	16	1831.657	1.243	10	12	1827.662	1.022	1809.169
$G04$	9	9	2244.008	2.605	9	13	2227.764	1.862	9	13	2227.764	1.862	10	14	2232.533	2.080	10	11	2224.954	1.734	2187.034
$G05$	10	10	2281.304	25.119	10	12	2187.218	19.959	9	13	2118.249	16.176	11	15	2012.925	10.400	12	18	1936.382	6.202	1823.303
$G06$	10	10	1221.107	21.152	10	12	1157.362	14.828	6	12	1158.517	14.942	11	17	1048.721	4.049	11	15	1063.595	5.525	1007.910
$G07$	10	10	1251.720	19.719	10	12	1141.369	9.165	7	13	1103.083	5.503	11	17	1090.266	4.277	10	13	1100.827	5.287	1045.545
$G08$	8	8	748.773	24.041	8	9	701.706	16.244	4	10	672.195	11.355	9	15	631.601	4.630	7	12	632.088	4.711	603.650
$G09$	9	9	2413.323	1.717	9	14	2410.636	1.604	8	14	2410.038	1.579	10	16	2428.576	2.360	8	10	2399.342	1.128	2372.582
$G10$	10	10	4639.105	28.714	10	12	4386.778	21.713	10	13	4206.783	16.719	11	14	4053.000	12.452	13	17	3821.391	6.026	3604.191
$G11$	12	12	697.381	4.035	12	15	679.834	1.417	8	15	680.185	1.470	13	20	678.962	1.287	13	16	678.588	1.231	670.333
$G12$	12	12	1385.358	2.336	12	15	1370.813	1.261	10	15	1370.877	1.266	13	18	1376.036	1.647	13	14	1371.593	1.319	1353.736
$G13$	13	13	1995.650	1.511	13	16	1983.753	0.905	4	16	1988.083	1.126	14	26	2010.220	2.252	10	15	1985.032	0.970	1965.953
$G14$	8	8	1022.773	7.716	8	10	968.794	2.031	5	10	970.256	2.185	9	14	976.235	2.815	8	10	968.794	2.031	949.508
$G15$	9	9	1238.352	17.287	9	12	1128.276	6.861	1	13	1163.190	10.168	9	21	1102.590	4.428	5	16	1095.536	3.760	1055.832
$G16$	9	9	3518.231	79.325	9	18	3129.142	59.493	1	28	2485.437	26.684	1	30	2462.935	25.537	1	1	2400.309	22.345	1961.925
$G17$	9	9	1539.131	22.977	9	11	1497.204	19.627	1	14	1391.371	11.171	5	30	1402.164	12.034	1	18	1357.674	8.479	1251.555
$G18$	7	7	763.917	1.508	7	11	772.387	2.633	5	12	771.875	2.565	8	15	785.637	4.394	1	7	762.799	1.359	752.569
$G19$	9	9	2725.806	17.965	9	14	2522.501	9.166	8	15	2498.461	8.126	10	17	2451.595	6.098	14	19	2428.969	5.119	2310.692
$G20$	5	5	945.512	6.792	5	6	919.408	3.843	4	7	902.666	1.952	6	9	910.691	2.859	4	7	902.666	1.952	885.380
Min				1.126				0.905				1.126				1.243				0.970	
Max				79.325				59.493				26.684				25.537				22.345	
Mean				15.464				10.406				7.300				5.513				4.311	
Median				12.501				5.352				4.034				3.894				2.896	
Std Dev				17.422				13.280				6.925				5.649				4.669	

Appendix C

Appendix for Discrete-Time with Separate Markets

Introduction

This section presents additional information related to the third model in this thesis – the discrete-time stochastic product recovery model with separate markets and substitution.

C.1 Model Implementation and Validation

The implementation of the model and validation of the java programming code are described in this section. Some of the java code was used in Chapter 5 and described in Section 3.2.3 therefore only the code which is specific to this model is described and validated here.

C.1.1 Implementation of the Stochastic Product Recovery Model

As in Chapter 5 the mid-state is used to reduce the computational burden of the problem. The introduction of the mid-state means that some calculations can be performed in advance and stored, so that during the value iteration algorithm the values need to be looked up, rather than calculated. However in this model, the substitution action needs to be taken into account when addressing the effect of demand, therefore the mid-state does not offer the same computational savings for this model, compared with the second model (Chapter 5).

Definition of the Mid State. Let $m = (m_1, m_2, m_r, m_c)$ denote the value of the state variables after the action has been chosen, and demand and substitutions have been observed. Recall that s_1 is the number of recovered goods used to meet demand of produced goods, and s_2 is the number of produced goods used to meet demand of recovered goods. The mid-state m is then related to the state at the beginning of the period i in the following way:

$$
\begin{aligned}
m_1 &= \max\{\big(i_1 + a_p - x_{d1} - s_2(x_{d2})\big), 0\} \\
m_2 &= \max\{\big(i_2 + a_h(x_q) - x_{d2} - s_1(x_{d1})\big), 0\} \\
m_r &= i_r - a_r \\
m_c&(i_c + a_b - a_p + a_l(x_q))
\end{aligned}
$$

The expected reward associated with the transition from the initial state to the mid-state is:

$$
\begin{aligned}
E[R_{i,m}(i, a(i), X_q, X_{d1}, X_{d2}, X_r, X_1, X_2)] = E[S(X_q, X_{d1}, X_{d2}, X_1, X_2] \\
- \Big(E[K(i, a(i))] + E[P(i, a(i))] + E[H(i)] \\
+ E[L(i, a(i), X_q, X_{d1}, X_{d2}, X_1, X_2)]\Big)
\end{aligned}
\tag{C.1}
$$

The mid-state is related to the state at the beginning of the the next period j in the following way:

$$
\begin{aligned}
j_1 &= m_1 \\
j_2 &= m_2 \\
j_r &= \min\{m_r + x_r, W_r\} \\
j_c &= m_c
\end{aligned}
$$

The expected reward associated with the transition from the mid-state to the next-state is:

$$
E[R_{m,j}(X_r)] = l_r \max\Big\{E[X_r] - (W_r - m_r), 0\Big\} \tag{C.2}
$$

The probability of moving from state i to mid-state m depends on the initial state i, the action $a(i)$, the quality variable X_q, the demand variables X_{d1}, X_{d2} and the substitution variables X_1, X_2. The probability of moving from mid-state m to state j in the next period depends on the mid-state m and the returns variable X_r.

C.1.2 Validation of the Code

The code structure used for this model is the same as in Chapter 5 and was described in Section 3.2.3.

In addition to thorough error-checking and inspection of output during the code development process, two forms of verification were used to validate the problem specific files. The calculation of the expected average rewards was checked using an Excel spreadsheet and the system was simulated using the optimal policy and the simulated reward was compared with the actual reward, as calculated by the MDP.

To conduct these tests a set of six example test problems were constructed. The parameters of these problems are shown in Table C.1. These test problems were chosen because they represent a range of different scenarios. The test problems are labelled $A01$–$A06$. The uniform distribution is used to model returns for problem $A05$ and demand and returns for problem $A06$. For the other problems demand and returns are modelled by the Poisson distribution. The quality of returns and the acceptance of upward and downward substitutions are modelled by the Binomial distribution with parameters $\alpha, \alpha_1, \alpha_2$, respectively. These test problems have a limited upper capacity on each inventory level of $W_1 = W_2 = W_r = W_c = 5$. This level is lower than the one used in the previous chapter due to the increased state and action spaces and hence increased computational time of the current model.

Several variations of these test problems were created. All four substitution strategies were considered under a low and high quality recovery strategy. The high-quality recovery strategy was investigated under a two-way substitution strategy only. All problems were studied under the optimal policy and a heuristic policy. This created a test set of 60 problems. A heuristic policy was included to ensure that the code was not always choosing the action with the highest reward. The heuristic policy used the following decision rule:

If $W_1 - i_1 > Q_p$ then produce Q_p

else if $i_r > Q_r$ then recover Q_r.

Always offer substitution is the strategy allows.

Table C.1: Test Problems used in Model Validation

	$A01$	$A02$	$A03$	$A04$	$A05$	$A06$
Distributions						
α	0.9	0.6	1	0.5	0.2	0.6
α_1	0.8	0.2	0.5	1	0.8	0.5
α_2	0.3	0.8	0.5	1	0.8	0.5
λ_1	1	1	1	1	1	U(0,5)
λ_2	1	1	1	1	1	U(0,5)
λ_r	2	2	2	2	U(2,6)	U(0,4)
Costs and Revenues						
c_p	4	4	15	15	5	10
c_r	1	2	5	5	1	1
c_1	0	0	0	0	0	0
c_2	0	0	0	0	0	0
c_b	3	3	5	1	10	2
c_h	2	3	1	1	1	1
c_l	1	1	3	3	5	3
c_d	1.1	1.1	2	2	2	1
c_a	0	0	0	0	0	0
p_r	0	0	0	0	0	0
p_1	21	21	60	48	45	36
p_2	13	9	55	16	34	10
k_p	3	2	10	2	5	1
k_r	1	1	5	1	3	1
k_b	0	0	0	0	0	0
h_1	1	0.1	1	1	1	3
h_2	3	2	5	2	4	5
h_r	2	1	2	1	4	4
h_c	1	1	0	1	1	2
l_r	0.1	0.1	0.1	0.1	0.1	0.1
l_1	0.7	0.7	2	1.6	1.5	1.2
l_2	0.3	0.5	0.6	0.6	0.2	0.2
Order Size						
Q_p	4	3	4	4	4	4
Q_r	4	3	4	4	4	4

Reward Calculations. In order to test the calculation of the costs and rewards within the java code an Excel spreadsheet was constructed to calculate the expected rewards for a given state and action independently of the calculations used in the java code. The expected reward associated with a given state and action (equation 6.15) was calculated, by completely enumerating across possible values of the random variables X_q, X_{d1}, X_{d2}, X_r, X_1, X_2. The result of this equation should be the same when calculated using Excel and when calculated using the java code.

The java code creates a `.csv` file containing the policy and rewards associated with each state, so these values are easily attainable.

The number of states and actions in this problem make it infeasible to manually compare the rewards for every state using the Excel spreadsheet, therefore for the test problems the rewards associated with 30 states were checked. These states are listed in Table C.2. The first and last states were chosen because they have inventory levels equal to the minimum or maximum capacity levels. The remaining 28 states were selected randomly using the Excel function `rand()`. For each of the 30 states, the rewards calculated by the java programme and those calculated by the Excel spreadsheet were checked for the optimal policy action and for a heuristic policy action, under the four substitution strategies. They were also checked for the high-quality recovery, under a two-way substitution strategy. For all 60 test problems and all states tested, the rewards calculated by the java code were the same as those calculated by the Excel spreadsheet.

Optimal Policies. The optimal and heuristic policy actions for all the test problems were also examined. Histograms showing the frequency of the size of each action variable were examined for all test problems. The histograms for problem $A01$ are presented in Figure C.1 and are discussed here as an example.

Figure C.1a shows that the maximum value for the production action a_p is 4, which corresponds to Q_p, the maximum lot size for production (see Table C.1). The action size can be distributed across the values 0 to 4, although for this problem it only takes values 0 and 4. The size of a recovery lot can take two values 0 and Q_r and this is represented by the histogram of a_r.

Figure C.1b shows that substitution variables take two values: 0 (do not offer substitution) or 1 (offer substitution). For this problem, substitution is offered in

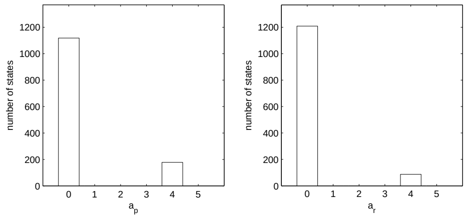
Table C.2: Selection of states for which the rewards were calculated using an Excel spreadsheet

State	i_r	i_{s1}	i_{s2}	i_c
0	0	0	0	0
20	0	0	3	2
52	0	1	2	4
64	0	1	4	4
85	0	2	2	1
253	1	1	0	1
292	1	2	0	4
324	1	3	0	0
371	1	4	1	5
434	2	0	0	2
577	2	4	0	1
623	2	5	1	5
678	3	0	5	0
689	3	1	0	5
722	3	2	0	2
739	3	2	3	1
746	3	2	4	2
762	3	3	1	0
789	3	3	5	3
892	4	0	4	4
961	4	2	4	1
991	4	3	3	1
1000	4	3	4	4
1002	4	3	5	0
1016	4	4	1	2
1033	4	4	4	1
1095	5	0	2	3
1112	5	0	5	2
1235	5	4	1	5
1295	5	5	5	5

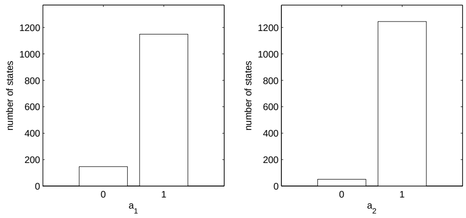
most states. Upward substitution (a_1) is offered slightly less frequently than downward substitution (a_2).

Figure C.1c and C.1d show the inventory levels when recovery and production, respectively, are chosen. Figure C.1e show the inventory levels when each type of substitution is chosen.

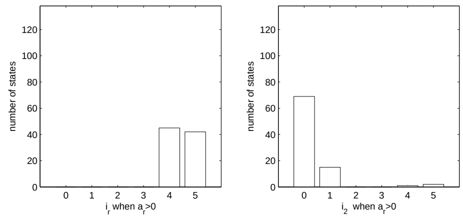
Figure C.2 shows the equivalent graphs but for the heuristic policy. These histograms are typical of the ones for the other 60 test problems. All graphs showed the expected results.



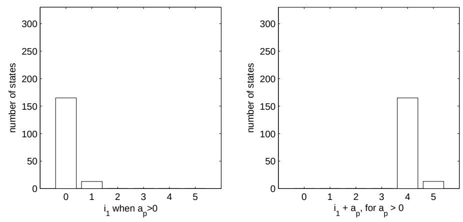
(a) Action Quantities for Production a_p and Recovery a_r



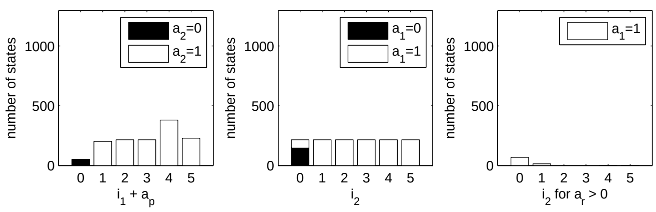
(b) Action Quantities for Upward Substitution a_1 and Downward Substitution a_2



(c) Level of Returned Inventory i_r and Recovered Inventory i_2 when recovery is performed

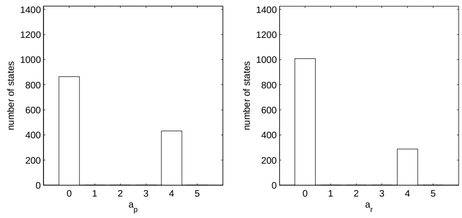


(d) Level of Produced Inventory i_1 when production is performed (before and after order is received)

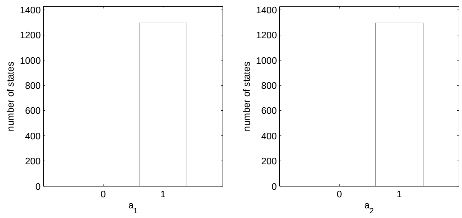


(e) Inventory levels when substitution is performed

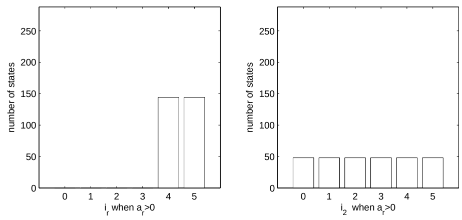
Figure C.1: Histograms showing the frequency of action sizes for the optimal policy for test problem $A01$ under a two-way substitution strategy



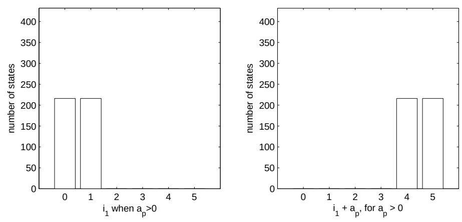
(a) Action Quantities for Production a_p and Recovery a_r



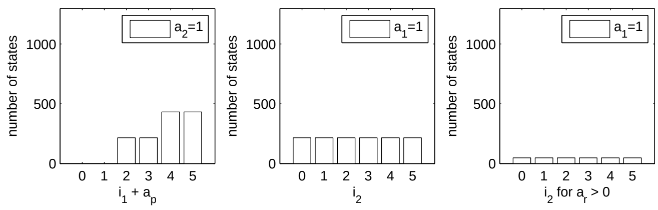
(b) Action Quantities for Upward Substitution a_1 and Downward Substitution a_2



(c) Level of Returned Inventory i_r and Recovered Inventory i_2 when recovery is performed



(d) Level of Produced Inventory i_1 when production is performed (before and after order is received)



(e) Inventory levels when substitution is performed

Figure C.2: Histograms showing the frequency of action sizes for the heuristic policy for test problem $A01$ under a two-way substitution strategy

Simulation Code. Simulation can also be used to calculate the average reward of a policy and hence can be used to validate the calculation of the average total reward in the value iteration file. Across a large number of simulations, the distribution of the rewards should be centred around the actual average total reward obtained from the value iteration algorithm. For each of the test problems, 1000 runs of $T = 1,000,000$ time units were completed. The long run average reward was calculated for each simulation and the distribution of these rewards obtained over the 1000 runs was examined using histograms. The histograms were calculated for the optimal policy for the 30 variations of the test problems. This histograms for the 30 problems were examined and appeared to be approximately normally distributed and centred around the reward from the value iteration algorithm. As an example, the histograms of simulated costs for test problem $A01$–$A05$ under a two-way substitution strategy with high and low quality recovery is shown in Figure C.3. The mean and standard deviation of the simulated values are printed on the graph. This provides further evidence to support the accuracy of the stochastic product recovery code and also the simulation code.

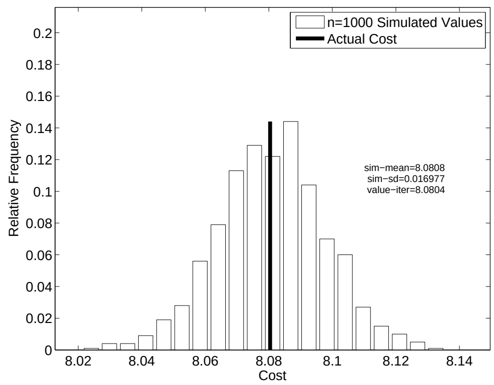
Table C.3 contains the value iteration reward and simulation statistics relating to the test problems under a two-way substitution strategy with high and low quality recovery, under a optimal and heuristic policy. The values in these tables were calculated using a single simulation over $T = 1,000,000$ time units. As shown in these tables, the mean values for produced demand, recovered demand, returns and the fill rates are close to the parameter values for all problems. The simulated and optimal profits are also similar for all test problems.

C.2 Properties of the Optimal Policy

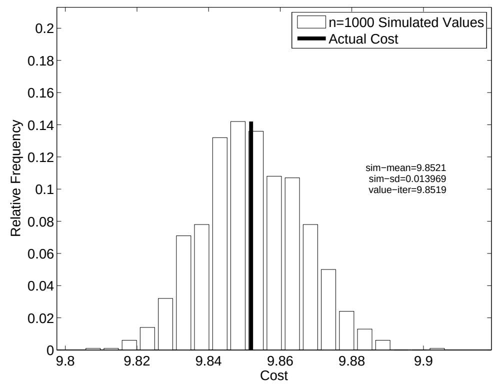
In this section information to support the analysis of the optimal policy in Section 6.6 is presented.

C.2.1 Analysis of Performance

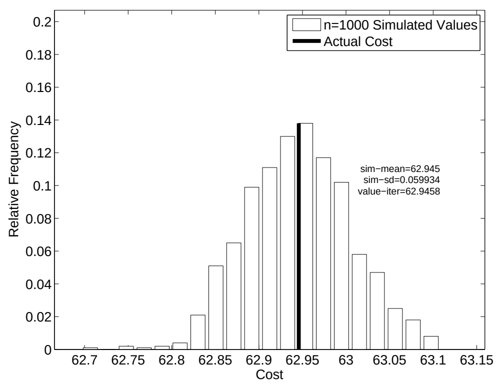
This section relates to Section 6.6.2 and provides additional information about the fill rate of the optimal policy, under the four substitution strategies: no substitution



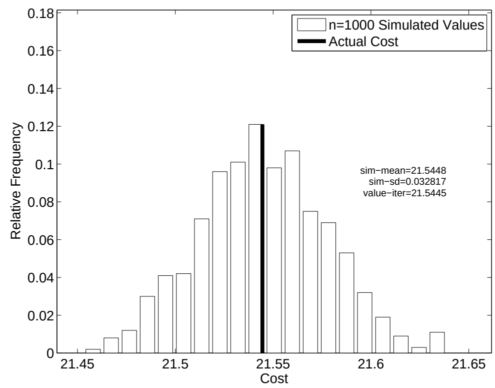
(a) $A01$



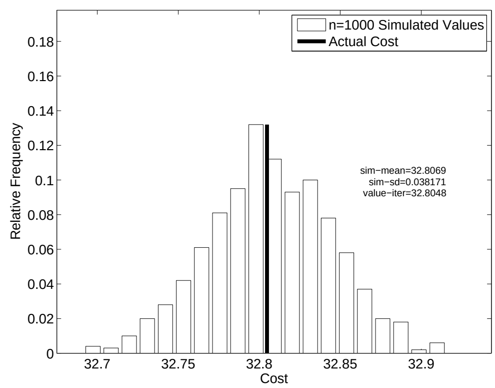
(b) $A02$



(c) $A03$



(d) $A04$



(e) $A05$

Figure C.3: Histogram of average total reward from 1000 simulations for test problem under a two-way substitution strategy with high and low quality recovery.

Table C.3: Policy and Simulation Results for Test Problems under a two-way substitution strategy with high and low quality recovery

(a) Optimal Policy

$\chi_1 = 1$, $\chi_2 = 1$		Simulation over $T = 1,000,000$							
Problem	Optimal Profit	Simulation Profit	$\bar{x}_{d1}$	$\bar{x}_{d2}$	$\bar{x}_r$	Fill rate i_1	Fill rate i_1 including substitution	Fill rate i_2	Fill rate i_2 including substitution
$A01$	8.0804	8.0607	0.9988	0.9994	1.9994	0.8345	0.9521	0.9316	0.9478
$A02$	9.8519	9.8562	1.0001	0.9993	1.9998	0.9857	0.9871	0.5983	0.8792
$A03$	62.9458	63.0551	1.0007	1.0020	1.9985	0.9317	0.9637	0.9762	0.9863
$A04$	21.5445	21.4869	0.9982	0.9999	1.9994	0.9929	0.9967	0.6923	0.9801
$A05$	32.8048	32.8246	0.9999	0.9999	3.9980	0.9811	0.9868	0.5860	0.8810
$A06$	36.0285	36.0667	2.5019	2.4986	1.9978	0.9716	0.9752	0.4115	0.4638

(b) Heuristic Policy

$\chi_1 = 1$, $\chi_2 = 1$		Simulation over $T = 1,000,000$							
Problem	Heuristic Policy Profit	Simulation Profit	$\bar{x}_{d1}$	$\bar{x}_{d2}$	$\bar{x}_r$	Fill rate i_1	Fill rate i_1 including substitution	Fill rate i_2	Fill rate i_2 including substitution
$A01$	3.6565	3.6495	1.0000	0.9995	1.9993	0.9911	0.9974	0.9554	0.9675
$A02$	6.0577	6.0352	1.0000	0.9984	1.9984	0.9978	0.9981	0.8643	0.9667
$A03$	56.8159	56.8230	0.9998	1.0001	2.0003	0.9911	0.9951	0.9628	0.9798
$A04$	19.2974	19.2958	0.9995	1.0005	1.9991	0.9911	0.9969	0.8239	0.9826
$A05$	30.4580	30.5071	1.0000	1.0011	4.0000	0.9913	0.9940	0.6835	0.9089
$A06$	34.3514	34.3680	2.5000	2.5000	2.0012	0.9333	0.9408	0.3912	0.5486

$(\chi_1 = 0, \chi_2 = 0)$, upward substitution $(\chi_1 = 1, \chi_2 = 0)$, downward substitution $(\chi_1 = 0, \chi_2 = 1)$, two-way substitution $(\chi_1 = 1, \chi_2 = 1)$.

Fill Rates

Substitution Inclusive and Exclusive Fill Rate. Tables C.4, C.5 and C.6 show the fill rates including and excluding substitution for problems B, C and D respectively. They provide the data used for the graphs in Figures 6.4, 6.5, 6.6, 6.7, 6.8 and 6.9.

Table C.4: Summary of the optimal policy fill rates for substitution strategies for problem set B

(a) Produced inventory i_1

	Fill rate i_1 excluding substitution					Fill rate i_1 including substitution			
Problem	None	Down	Up	Two-way	Problem	None	Down	Up	Two-way
$B01$	0.9635	0.9592	0.9566	0.9540	$B01$	0.9635	0.9592	0.9718	0.9708
$B02$	0.9494	0.9452	0.9481	0.9435	$B02$	0.9494	0.9452	0.9525	0.9485
$B03$	0.9868	0.9792	0.9842	0.9791	$B03$	0.9868	0.9792	0.9848	0.9799
$B04$	0.9694	0.9694	0.9693	0.9693	$B04$	0.9694	0.9694	0.9717	0.9717
$B05$	0.9811	0.9361	0.9809	0.9327	$B05$	0.9811	0.9361	0.9857	0.9502
$B06$	0.9716	0.9851	0.9599	0.9745	$B06$	0.9716	0.9851	0.9735	0.9846
$B07$	0.9798	0.9539	0.9793	0.9502	$B07$	0.9798	0.9539	0.9851	0.9639
$B08$	0.7755	0.7755	0.6568	0.6568	$B08$	0.7755	0.7755	0.7498	0.7498
$B09$	0.9621	0.9800	0.9619	0.9777	$B09$	0.9621	0.9800	0.9654	0.9801
$B10$	0.9468	0.8327	0.9391	0.7821	$B10$	0.9468	0.8327	0.9616	0.8519
$B11$	0.9892	0.9891	0.9890	0.9889	$B11$	0.9892	0.9891	0.9919	0.9919
$B12$	0.9850	0.9850	0.9848	0.9848	$B12$	0.9850	0.9850	0.9885	0.9885
$B13$	0.8823	0.8312	0.8737	0.8150	$B13$	0.8823	0.8312	0.8994	0.8521
$B14$	0.9823	0.9811	0.9764	0.9695	$B14$	0.9823	0.9811	0.9810	0.9781
$B15$	0.9690	0.9687	0.9669	0.9682	$B15$	0.9690	0.9687	0.9713	0.9725
$B16$	0.1347	0.1347	0.1347	0.1347	$B16$	0.1347	0.1347	0.1347	0.1347
$B17$	0.1347	0.1347	0.1347	0.1347	$B17$	0.1347	0.1347	0.1347	0.1347
$B18$	0.9192	0.8862	0.9179	0.8809	$B18$	0.9192	0.8862	0.9255	0.8920
$B19$	0.1347	0.1347	0.6019	0.6019	$B19$	0.1347	0.1347	0.6397	0.6397
$B20$	0.9386	0.9382	0.9255	0.9252	$B20$	0.9386	0.9382	0.9427	0.9424

(b) Recovered inventory i_2

	Fill rate i_2 excluding substitution					Fill rate i_2 including substitution			
Problem	None	Down	Up	Two-way	Problem	None	Down	Up	Two-way
$B01$	0.5903	0.5900	0.5745	0.5724	$B01$	0.5903	0.6890	0.5745	0.6740
$B02$	0.5117	0.5116	0.5011	0.5000	$B02$	0.5117	0.6201	0.5011	0.6083
$B03$	0.1656	0.1644	0.1645	0.1628	$B03$	0.1656	0.5319	0.1645	0.5308
$B04$	0.1659	0.1659	0.1627	0.1627	$B04$	0.1659	0.1659	0.1627	0.1627
$B05$	0.4073	0.3993	0.4002	0.3785	$B05$	0.4073	0.6228	0.4002	0.6072
$B06$	0.8648	0.8175	0.8394	0.7955	$B06$	0.8648	0.9025	0.8394	0.8901
$B07$	0.5934	0.5947	0.5862	0.5803	$B07$	0.5934	0.7413	0.5862	0.7303
$B08$	0.8344	0.8344	0.7659	0.7659	$B08$	0.8344	0.8344	0.7659	0.7659
$B09$	0.5102	0.4843	0.5001	0.4777	$B09$	0.5102	0.7988	0.5001	0.7926
$B10$	0.6089	0.6097	0.5792	0.5396	$B10$	0.6089	0.8126	0.5792	0.7971
$B11$	0.5318	0.5311	0.5228	0.5220	$B11$	0.5318	0.5360	0.5228	0.5270
$B12$	0.6643	0.6643	0.6492	0.6492	$B12$	0.6643	0.6643	0.6492	0.6492
$B13$	0.6058	0.6109	0.5668	0.5575	$B13$	0.6058	0.6974	0.5668	0.6557
$B14$	0.3838	0.3645	0.3742	0.3609	$B14$	0.3838	0.8421	0.3742	0.8348
$B15$	0.5121	0.5124	0.5045	0.5045	$B15$	0.5121	0.5158	0.5045	0.5079
$B16$	0.0495	0.0495	0.0495	0.0495	$B16$	0.0495	0.0495	0.0495	0.0495
$B17$	0.0495	0.0495	0.0495	0.0495	$B17$	0.0495	0.0495	0.0495	0.0495
$B18$	0.5092	0.5117	0.4994	0.4984	$B18$	0.5092	0.6300	0.4994	0.6198
$B19$	0.3675	0.3675	0.4780	0.4780	$B19$	0.3675	0.3675	0.4780	0.4780
$B20$	0.7111	0.7110	0.6894	0.6893	$B20$	0.7111	0.7126	0.6894	0.6910

Table C.5: Summary of the optimal policy fill rates for substitution strategies for problem set C

(a) Produced inventory i_1

	Fill rate i_1 excluding substitution					Fill rate i_1 including substitution			
Problem	None	Down	Up	Two-way	Problem	None	Down	Up	Two-way
$C01$	0.9725	0.9704	0.9705	0.9638	$C01$	0.9725	0.9704	0.9808	0.9773
$C02$	0.9720	0.9673	0.9715	0.9662	$C02$	0.9720	0.9673	0.9741	0.9693
$C03$	0.9886	0.9949	0.9886	0.9949	$C03$	0.9886	0.9949	0.9890	0.9952
$C04$	0.9768	0.9768	0.9768	0.9768	$C04$	0.9768	0.9768	0.9786	0.9786
$C05$	0.9848	0.9805	0.9840	0.9784	$C05$	0.9848	0.9805	0.9880	0.9846
$C06$	0.9803	0.9878	0.9649	0.9716	$C06$	0.9803	0.9878	0.9793	0.9844
$C07$	0.9883	0.9849	0.9881	0.9835	$C07$	0.9883	0.9849	0.9915	0.9883
$C08$	0.8120	0.8120	0.6839	0.6839	$C08$	0.8120	0.8120	0.7710	0.7710
$C09$	0.9670	0.9776	0.9669	0.9754	$C09$	0.9670	0.9776	0.9700	0.9784
$C10$	0.9756	0.9569	0.9731	0.9086	$C10$	0.9756	0.9569	0.9836	0.9427
$C11$	0.9932	0.9912	0.9930	0.9909	$C11$	0.9932	0.9912	0.9956	0.9942
$C12$	0.9889	0.9889	0.9880	0.9881	$C12$	0.9889	0.9889	0.9925	0.9925
$C13$	0.9789	0.9694	0.9778	0.9630	$C13$	0.9789	0.9694	0.9832	0.9716
$C14$	0.9822	0.9935	0.9816	0.9920	$C14$	0.9822	0.9935	0.9852	0.9950
$C15$	0.9782	0.9778	0.9780	0.9777	$C15$	0.9782	0.9778	0.9811	0.9809
$C16$	0.1347	0.1347	0.1347	0.1347	$C16$	0.1347	0.1347	0.1347	0.1347
$C17$	0.1347	0.1347	0.1347	0.1347	$C17$	0.1347	0.1347	0.1347	0.1347
$C18$	0.9521	0.9458	0.9516	0.9383	$C18$	0.9521	0.9458	0.9564	0.9447
$C19$	0.1347	0.1347	0.1347	0.1347	$C19$	0.1347	0.1347	0.1347	0.1347
$C20$	0.9608	0.9604	0.9587	0.9581	$C20$	0.9608	0.9604	0.9683	0.9678

(b) Recovered inventory i_2

	Fill rate i_2 excluding substitution					Fill rate i_2 including substitution			
Problem	None	Down	Up	Two-way	Problem	None	Down	Up	Two-way
$C01$	0.5920	0.5906	0.5785	0.5748	$C01$	0.5920	0.7276	0.5785	0.7196
$C02$	0.5144	0.5134	0.5073	0.5051	$C02$	0.5144	0.7280	0.5073	0.7226
$C03$	0.1657	0.1607	0.1648	0.1600	$C03$	0.1657	0.8525	0.1648	0.8523
$C04$	0.1658	0.1658	0.1633	0.1633	$C04$	0.1658	0.1659	0.1633	0.1634
$C05$	0.4103	0.3549	0.4043	0.3463	$C05$	0.4103	0.7886	0.4043	0.7836
$C06$	0.9245	0.9028	0.9022	0.8803	$C06$	0.9245	0.9504	0.9022	0.9384
$C07$	0.5939	0.5911	0.5890	0.5846	$C07$	0.5939	0.7814	0.5890	0.7772
$C08$	0.8344	0.8344	0.7715	0.7715	$C08$	0.8344	0.8344	0.7715	0.7715
$C09$	0.5142	0.5082	0.5055	0.4995	$C09$	0.5142	0.8151	0.5055	0.8085
$C10$	0.6126	0.5742	0.5945	0.5530	$C10$	0.6126	0.8938	0.5945	0.8794
$C11$	0.6053	0.5992	0.5978	0.5897	$C11$	0.6053	0.7586	0.5978	0.7509
$C12$	0.7682	0.7673	0.7523	0.7518	$C12$	0.7682	0.7725	0.7523	0.7554
$C13$	0.6086	0.5686	0.5960	0.5501	$C13$	0.6086	0.9234	0.5960	0.9174
$C14$	0.3844	0.3786	0.3768	0.3717	$C14$	0.3844	0.9067	0.3768	0.9044
$C15$	0.5243	0.5251	0.5182	0.5189	$C15$	0.5243	0.5367	0.5182	0.5308
$C16$	0.0495	0.0495	0.0495	0.0495	$C16$	0.0495	0.0495	0.0495	0.0495
$C17$	0.0495	0.0495	0.0495	0.0495	$C17$	0.0495	0.0495	0.0495	0.0495
$C18$	0.5121	0.5139	0.5053	0.5052	$C18$	0.5121	0.7133	0.5053	0.7061
$C19$	0.3675	0.3675	0.3675	0.3675	$C19$	0.3675	0.3675	0.3675	0.3675
$C20$	0.7147	0.7145	0.6993	0.6990	$C20$	0.7147	0.7187	0.6993	0.7035

Table C.6: Summary of the optimal policy fill rates for substitution strategies for problem set D

(a) Produced inventory i_1

	Fill rate i_1 excluding substitution					Fill rate i_1 including substitution			
Problem	None	Down	Up	Two-way	Problem	None	Down	Up	Two-way
$D01$	0.9666	0.9620	0.9616	0.9541	$D01$	0.9666	0.9620	0.9787	0.9763
$D02$	0.9717	0.9700	0.9714	0.9693	$D02$	0.9717	0.9700	0.9743	0.9726
$D03$	0.9858	0.9735	0.9858	0.9734	$D03$	0.9858	0.9735	0.9865	0.9753
$D04$	0.9742	0.9742	0.9742	0.9742	$D04$	0.9742	0.9742	0.9772	0.9772
$D05$	0.9797	0.9056	0.9796	0.8887	$D05$	0.9797	0.9056	0.9872	0.9334
$D06$	0.9788	0.9819	0.9264	0.9249	$D06$	0.9788	0.9819	0.9629	0.9619
$D07$	0.9853	0.9739	0.9848	0.9721	$D07$	0.9853	0.9739	0.9903	0.9825
$D08$	0.8011	0.8011	0.6852	0.6852	$D08$	0.8011	0.8011	0.7770	0.7770
$D09$	0.9706	0.9768	0.9701	0.9760	$D09$	0.9706	0.9768	0.9735	0.9793
$D10$	0.9664	0.8921	0.9618	0.8657	$D10$	0.9664	0.8921	0.9781	0.9201
$D11$	0.9876	0.9868	0.9872	0.9861	$D11$	0.9876	0.9868	0.9918	0.9910
$D12$	0.9860	0.9860	0.9835	0.9835	$D12$	0.9860	0.9860	0.9905	0.9905
$D13$	0.9462	0.8885	0.9441	0.8771	$D13$	0.9462	0.8885	0.9585	0.9072
$D14$	0.9779	0.9807	0.9776	0.9723	$D14$	0.9779	0.9807	0.9827	0.9819
$D15$	0.9700	0.9725	0.9658	0.9656	$D15$	0.9700	0.9725	0.9712	0.9710
$D16$	0.1347	0.1347	0.1347	0.1347	$D16$	0.1347	0.1347	0.1347	0.1347
$D17$	0.1347	0.1347	0.1347	0.1347	$D17$	0.1347	0.1347	0.1347	0.1347
$D18$	0.9403	0.9237	0.9397	0.9205	$D18$	0.9403	0.9237	0.9463	0.9294
$D19$	0.7620	0.7620	0.7394	0.7394	$D19$	0.7620	0.7620	0.7798	0.7798
$D20$	0.9553	0.9550	0.9443	0.9439	$D20$	0.9553	0.9550	0.9584	0.9581

(b) Recovered inventory i_2

	Fill rate i_2 excluding substitution					Fill rate i_2 including substitution			
Problem	None	Down	Up	Two-way	Problem	None	Down	Up	Two-way
$D01$	0.5770	0.5779	0.5595	0.5554	$D01$	0.5770	0.7146	0.5595	0.6961
$D02$	0.5129	0.5131	0.5051	0.5045	$D02$	0.5129	0.7592	0.5051	0.7536
$D03$	0.1608	0.1601	0.1592	0.1568	$D03$	0.1608	0.6273	0.1592	0.6251
$D04$	0.1612	0.1612	0.1572	0.1572	$D04$	0.1612	0.1612	0.1572	0.1572
$D05$	0.3967	0.3975	0.3867	0.3608	$D05$	0.3967	0.7624	0.3867	0.7428
$D06$	0.9823	0.9726	0.9668	0.9598	$D06$	0.9823	0.9847	0.9668	0.9762
$D07$	0.5803	0.5812	0.5738	0.5710	$D07$	0.5803	0.7717	0.5738	0.7645
$D08$	0.8255	0.8255	0.7588	0.7588	$D08$	0.8255	0.8255	0.7588	0.7588
$D09$	0.5128	0.5091	0.5033	0.4995	$D09$	0.5128	0.8071	0.5033	0.8028
$D10$	0.6068	0.6080	0.5827	0.5482	$D10$	0.6068	0.8800	0.5827	0.8554
$D11$	0.5785	0.5761	0.5661	0.5617	$D11$	0.5785	0.6110	0.5661	0.6085
$D12$	0.7876	0.7876	0.7648	0.7648	$D12$	0.7876	0.7876	0.7648	0.7648
$D13$	0.6050	0.6070	0.5783	0.5585	$D13$	0.6050	0.7887	0.5783	0.7576
$D14$	0.3782	0.3734	0.3678	0.3619	$D14$	0.3782	0.8640	0.3678	0.8566
$D15$	0.5157	0.5174	0.5071	0.5078	$D15$	0.5157	0.5247	0.5071	0.5140
$D16$	0.0495	0.0495	0.0495	0.0495	$D16$	0.0495	0.0495	0.0495	0.0495
$D17$	0.0495	0.0495	0.0495	0.0495	$D17$	0.0495	0.0495	0.0495	0.0495
$D18$	0.5072	0.5128	0.4983	0.5013	$D18$	0.5072	0.6821	0.4983	0.6745
$D19$	0.5804	0.5804	0.5153	0.5153	$D19$	0.5804	0.5804	0.5153	0.5153
$D20$	0.7106	0.7103	0.6898	0.6897	$D20$	0.7106	0.7137	0.6898	0.6928

C.2.2 Analysis of Actions

This section relates to Section 6.6.3 and provides additional information about when and how often the replenishment (production and recovery) and substitution actions are chosen.

Action Size and Frequency

Replenishment Actions. The size of the replenishment actions is determined by the parameters Q_p and Q_r respectively. This means that some features of the graph are not particular interesting, for instance a graph of the sizes of the recovery action a_r can take only values of 0 or Q_r. However, it is still useful to examine these graphs as they allow us to check that the model is performing in the expected way. The graphs for problems $B01$, $C12$ and $D16$ are presented in Figures C.4, C.5 and C.6 respectively to illustrate some common properties of the graphs. For problem $B01$ and $D16$ the production the action size a_p takes values of 0 and Q_p. For problem $C12$ the production action size a_p takes values between 0 and 7. The action size for recovery takes only two values 0 or Q_r for all problems.

Frequency of Replenishment Under Substitution Strategies Offering substitution may influence whether or not production and recovery are chosen. For instance, it may be the case that it is better to not produce, and to allow substitution to cover any shortages. In Chapter 6, Figure 6.11 compared the number of states in which production and recovery are chosen under the four substitution strategies: no substitution, downward substitution, upward substitution, two-way substitution. The data from which these graphs were constructed is presented in Table C.7.

Substitution Actions The substitution actions take value of 0 or 1. However, for some of the substitution strategies, the one or both of the substitution actions must take values of 0. The graphs for problems $B01$, $C12$ and $D16$ are presented in Figures C.7, C.8 and C.9 respectively to illustrate some common properties of the graphs. These graphs were summarised for all problems in Chapter 6 in Figure 6.12.

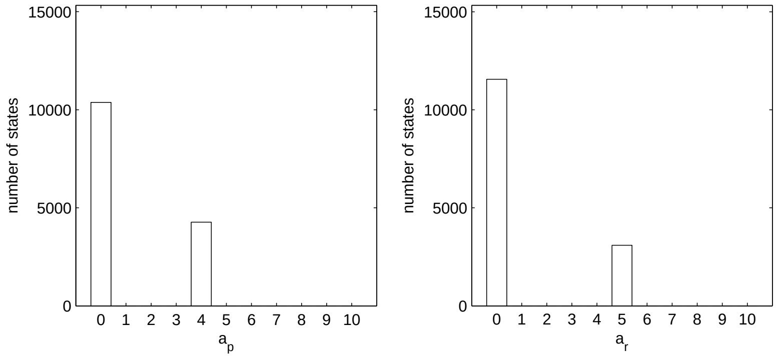


Figure C.4: Histograms of the replenishment actions under the optimal policy for test problem $B01$ under a two-way substitution strategy

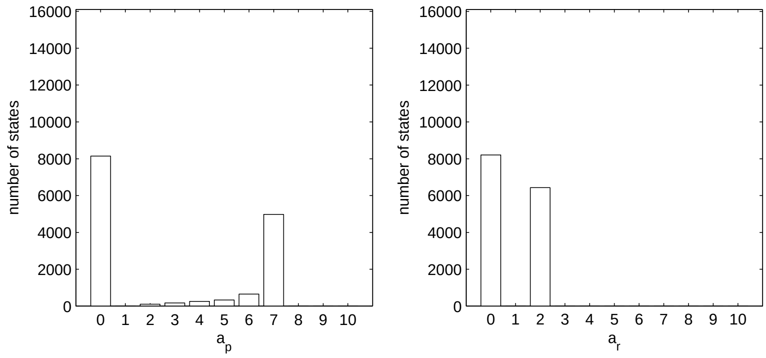


Figure C.5: Histograms of the replenishment actions under the optimal policy for test problem $C12$ under a downward substitution strategy

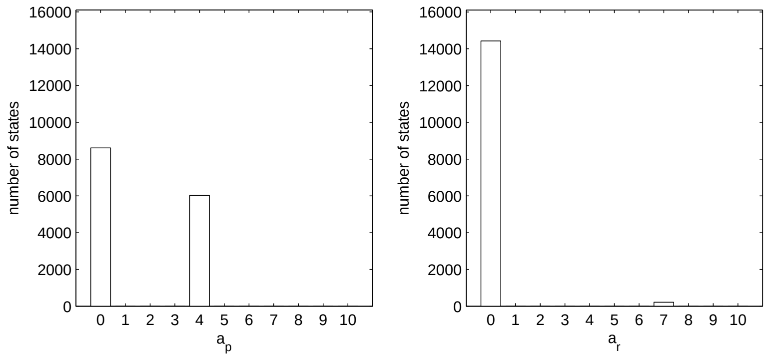


Figure C.6: Histograms of the replenishment actions under the optimal policy for test problem $D16$ under a two-way substitution strategy

Table C.7: Replenishment actions under optimal policy for substitution strategies for problem sets

(a) Problems B

	None		Down		Up		Two-way	
Problem	a_p	a_r	a_p	a_r	a_p	a_r	a_p	a_r
$B01$	4315	3034	4619	2963	3995	3131	4266	3087
$B02$	7529	5787	7738	5802	7071	6160	7253	6190
$B03$	9379	1859	10228	2995	9299	1872	10156	3039
$B04$	6153	2004	6361	2014	5939	2060	6148	2059
$B05$	5664	3773	9838	3572	5502	3845	9635	3761
$B06$	6032	7638	6022	7447	5636	8021	5460	7984
$B07$	7696	3554	8824	3572	7623	3603	8683	3667
$B08$	2486	2690	2487	2624	1834	2837	1836	2729
$B09$	3709	8176	3842	8221	3106	8594	2690	9073
$B10$	8776	3990	9733	3826	8413	4218	9314	4089
$B11$	10177	3688	10205	3693	10112	3748	10141	3755
$B12$	7990	5370	7996	5364	7699	5749	7705	5746
$B13$	8123	4471	8292	4634	7801	4716	7931	4919
$B14$	4227	2461	5690	3291	3910	2551	4040	4125
$B15$	6419	1882	6653	1963	6300	1908	6544	1994
$B16$	6655	0	6655	0	6380	0	6380	0
$B17$	2057	0	2057	0	1034	0	1034	0
$B18$	2711	3797	2790	5375	2395	3799	2424	5536
$B19$	1500	1099	1500	1099	423	1389	423	1389
$B20$	2533	2357	2563	1909	2085	2437	2124	1942

(b) Problems C

	None		Down		Up		Two-way	
Problem	a_p	a_r	a_p	a_r	a_p	a_r	a_p	a_r
$C01$	3837	2969	4014	2845	3485	3078	3614	2936
$C02$	6371	6241	6537	6480	5864	6595	5971	6902
$C03$	8553	1968	9199	2505	8445	2003	8980	2602
$C04$	5499	2283	5610	2301	5229	2345	5339	2359
$C05$	4832	3773	8367	3611	4670	3834	7991	3863
$C06$	5208	7786	5204	7576	4695	8246	4515	8185
$C07$	6801	3521	7586	3363	6723	3561	7391	3460
$C08$	2169	2692	2169	2632	1523	2867	1525	2760
$C09$	3451	7745	3562	8051	2765	8191	2244	8958
$C10$	7521	4352	8698	3995	7176	4532	7933	4424
$C11$	9157	3904	9172	4066	9074	3963	9096	4119
$C12$	7219	5522	7228	5515	6501	6434	6509	6427
$C13$	6492	4568	6835	4735	6266	4695	6371	5051
$C14$	3948	2420	4589	1937	3512	2546	2418	2686
$C15$	5496	1992	5676	2055	5325	2025	5510	2075
$C16$	3857	897	3857	897	3558	902	3558	902
$C17$	1815	0	1815	0	902	0	902	0
$C18$	2094	3928	2224	4522	1962	3936	1990	4606
$C19$	1138	1061	1138	1061	313	1586	313	1586
$C20$	2333	2051	2350	1453	1762	2106	1766	1459

(c) Problems D

	None		Down		Up		Two-way	
Problem	a_p	a_r	a_p	a_r	a_p	a_r	a_p	a_r
$D01$	4238	1863	4498	1760	3918	1928	4125	1822
$D02$	7645	2672	7898	2819	7220	2839	7433	2997
$D03$	9340	931	10865	1365	9264	947	10787	1392
$D04$	6105	705	6201	719	5857	720	5952	733
$D05$	5399	2449	9611	2231	5226	2508	9299	2401
$D06$	5240	6037	5337	5906	4139	6738	4165	6692
$D07$	7687	2044	8449	2032	7601	2077	8288	2083
$D08$	2445	2107	2445	2010	1789	2224	1790	2085
$D09$	3677	4333	3807	4385	3124	4488	2616	4922
$D10$	8232	3254	9720	2677	7898	3409	9117	2843
$D11$	10495	2858	10560	2876	10424	2909	10496	2921
$D12$	8237	4176	8246	4173	7806	4840	7816	4836
$D13$	7978	3504	8366	3606	7717	3634	8021	3825
$D14$	4209	1983	5524	1870	3895	2059	3771	2399
$D15$	5986	1052	6283	1111	5859	1066	6153	1128
$D16$	6329	210	6329	210	6031	217	6031	217
$D17$	2057	0	2057	0	1034	0	1034	0
$D18$	2649	2277	2647	2182	2311	2310	2367	2216
$D19$	1616	854	1616	854	528	855	528	855
$D20$	2495	1309	2542	813	2065	1302	2092	778

Table C.9 shows the number of states in which substitution is chosen under each of the substitution strategies. This data counts the number of states for which $a_1 = 1$ and $a_2 = 1$. It does not count the number of times substitution is actually offered - this is only possible to examine using simulation. This information is summarised for the two-way substitution strategy in Table C.8

Trigger-States and Action

Replenishment Actions The model takes the inventory levels into account when selecting the optimal action, therefore it is interesting to investigate which inventory levels 'trigger' certain actions. Many of the problems seem to exhibit an 'order-up-to' policy structure for production and recovery. Figures C.10a and C.10b show this structure for problem $B01$. Figure C.10a shows the initial state i_1 and the initial state plus the action $i_1 + a_p$ for produced inventory when production is chosen ($a_p > 0$). Notice that production is performed if the produced inventory level is less than 5 and that after production has been performed the inventory level is at least 4. (Actually production is performed at $i_1 = 6$, however the number of states in which this occurs is too small to appear of this graph.) This minimum 'produce-up-to' level corresponds to the production batch size Q_p. These graphs could be used to obtain parameters for an 'order-up-to' structured policy.

Figure C.10b shows the initial state of recovered inventory i_2 and the initial state of the returned inventory i_r when recovery is chosen ($a_r > 0$). Notice that recovery is performed if the recovered inventory level is less than 7 and the returned inventory inventory level is at least 4. This minimum returned inventory level corresponds to the recovery batch size Q_r. The graphs are also presented for problem $C12$ under a downward substitution strategy and $D16$ under a two-way substitution strategy in Figures C.11a and C.12a respectively.

The trigger states for replenishment all substitution strategies are shown in Figures C.13, C.14 and C.15 for problem set B, in Figures C.16, C.17 and C.18 for problem set C and in Figures C.19, C.20 and C.21 for problem set D.

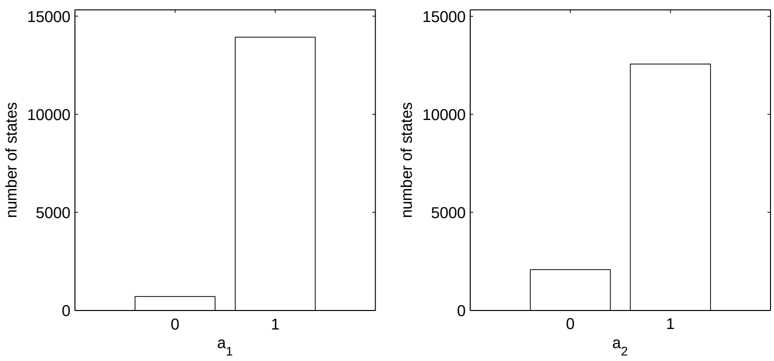


Figure C.7: Histograms of the substitution actions under the optimal policy for test problem $B01$ under a two-way substitution strategy

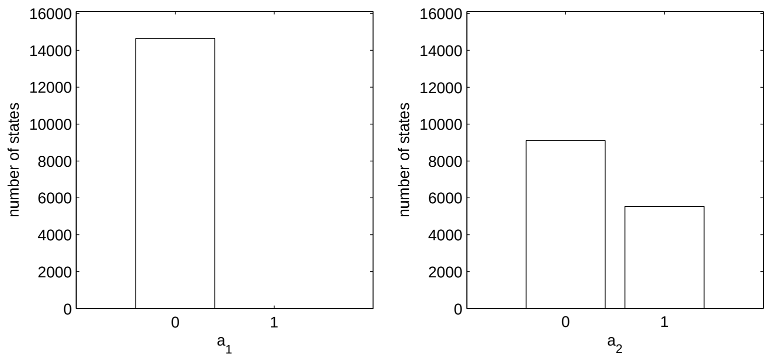


Figure C.8: Histograms of the substitution actions under the optimal policy for test problem $C12$ under a downward substitution strategy

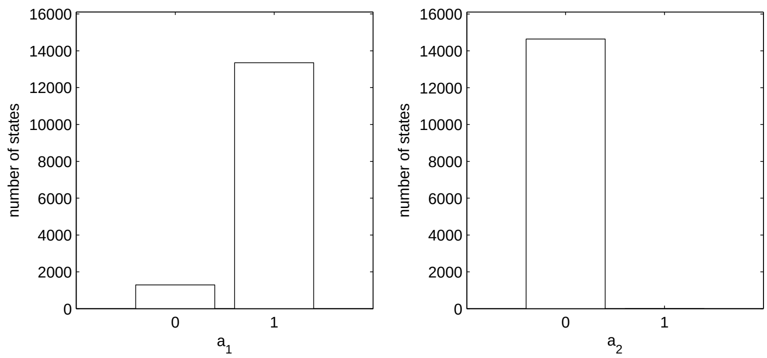
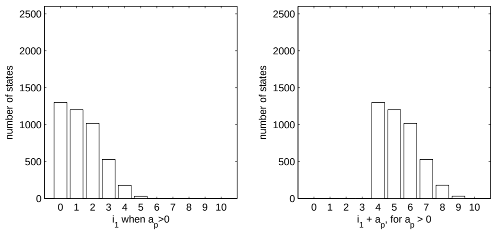
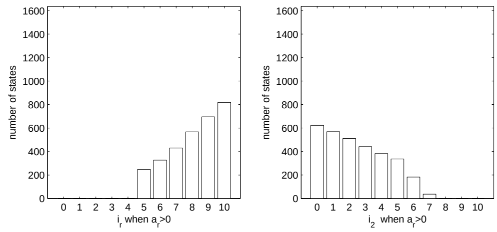


Figure C.9: Histograms of the substitution actions under the optimal policy for test problem $D16$ under a two-way substitution strategy

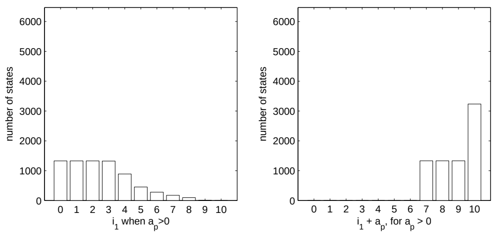


(a) Distribution of states when production is selected

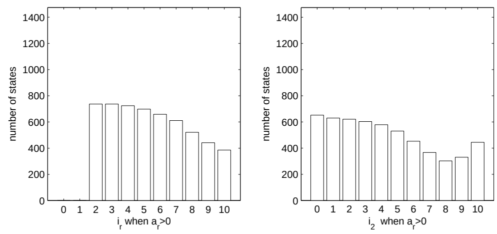


(b) Distribution of states when recovery is selected

Figure C.10: Histograms of showing states associated with positive replenishment actions under the optimal policy for test problem $B01$

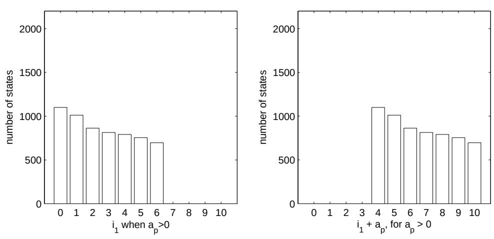


(a) Distribution of states when production is selected

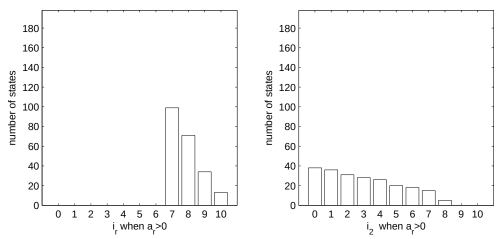


(b) Distribution of states when recovery is selected

Figure C.11: Histograms of showing states associated with positive replenishment actions under the optimal policy for test problem $C12$ under a downward substitutions strategy

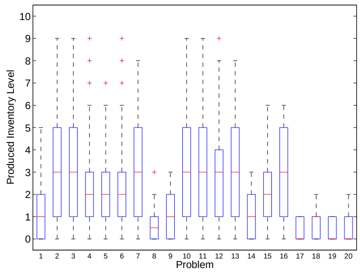


(a) Distribution of states when production is selected

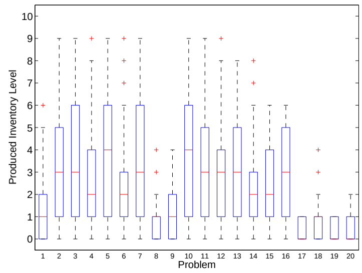


(b) Distribution of states when recovery is selected

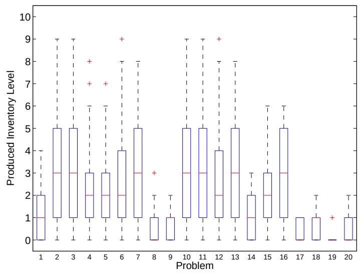
Figure C.12: Histograms of showing states associated with positive replenishment actions under the optimal policy for test problem $D16$ under a two-way substitutions strategy



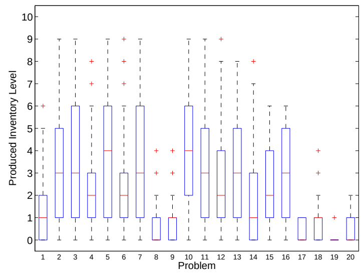
(a) No substitution



(b) Downward substitution

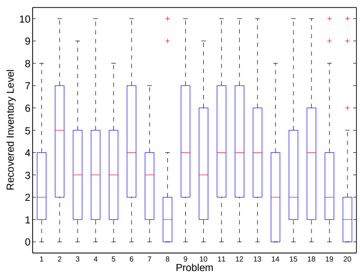


(c) Upward substitution

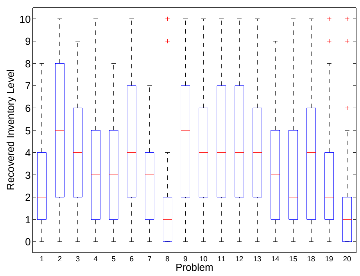


(d) Two-way substitution

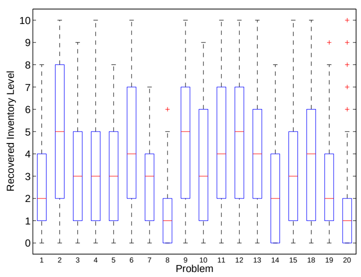
Figure C.13: Graphs showing the level of produced inventory (trigger level) when production is performed for problem set B.



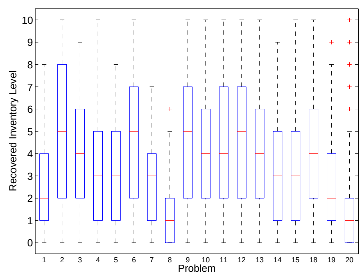
(a) No substitution



(b) Downward substitution

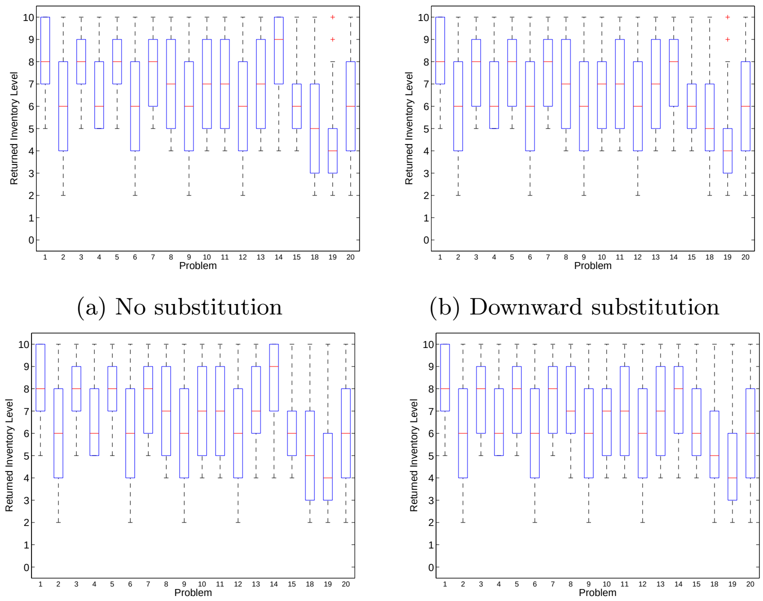


(c) Upward substitution



(d) Two-way substitution

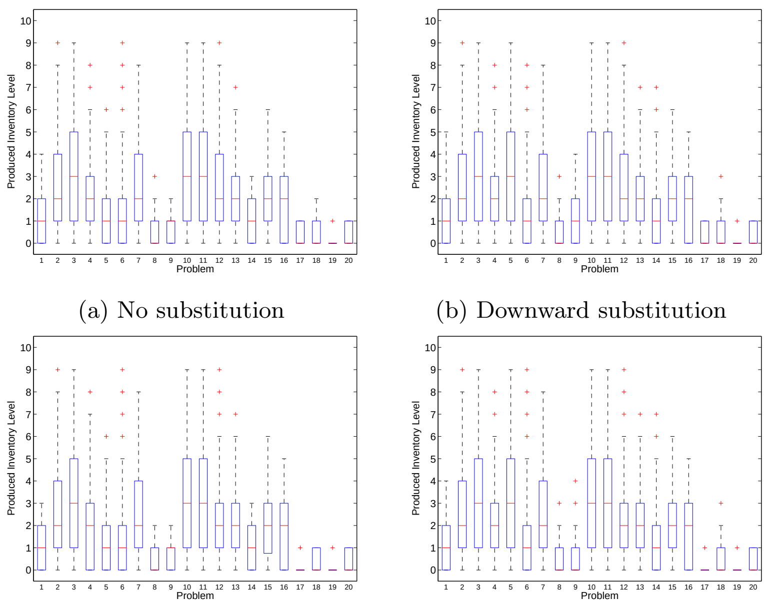
Figure C.14: Graphs showing the level of recovered inventory (trigger level) when recovery is performed for problem set B.



(a) No substitution (b) Downward substitution

(c) Upward substitution (d) Two-way substitution

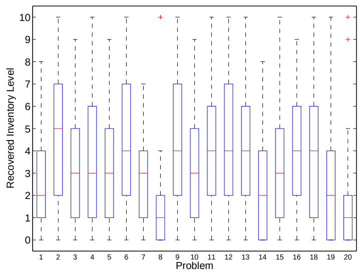
Figure C.15: Graphs showing the level of returned inventory (trigger level) when recovery is performed for problem set B.



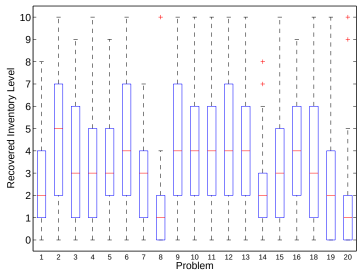
(a) No substitution (b) Downward substitution

(c) Upward substitution (d) Two-way substitution

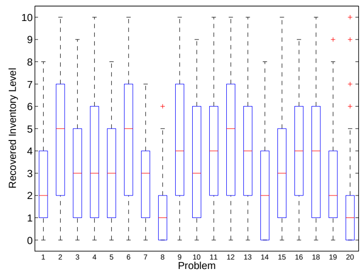
Figure C.16: Graphs showing the level of produced inventory (trigger level) when production is performed for problem set C.



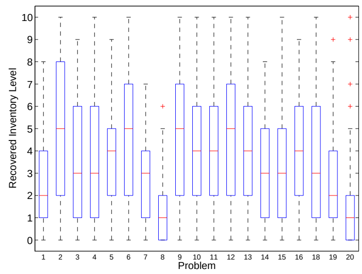
(a) No substitution



(b) Downward substitution

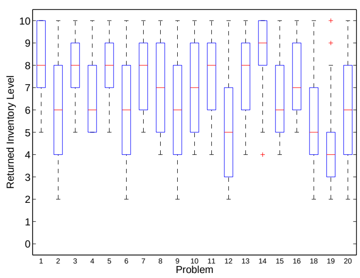


(c) Upward substitution

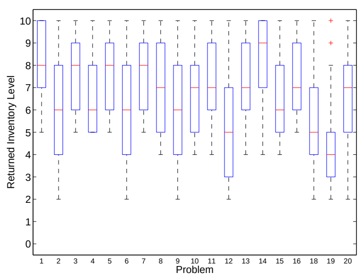


(d) Two-way substitution

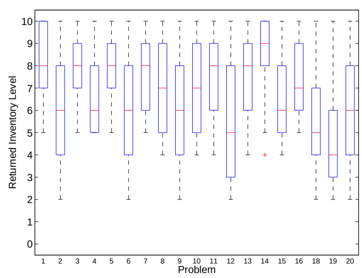
Figure C.17: Graphs showing the level of recovered inventory (trigger level) when recovery is performed for problem set C.



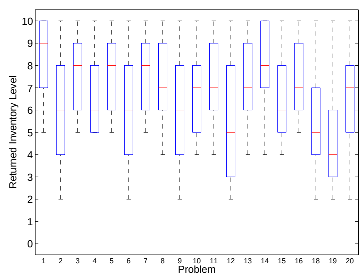
(a) No substitution



(b) Downward substitution

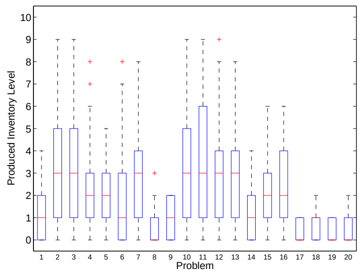


(c) Upward substitution

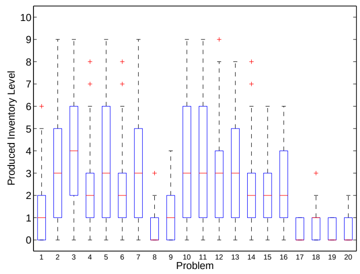


(d) Two-way substitution

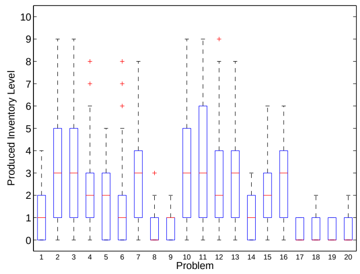
Figure C.18: Graphs showing the level of returned inventory (trigger level) when recovery is performed for problem set C.



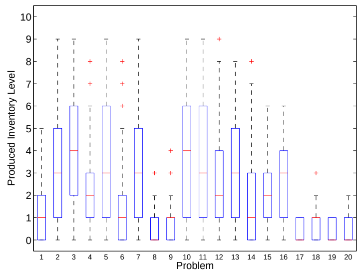
(a) No substitution



(b) Downward substitution

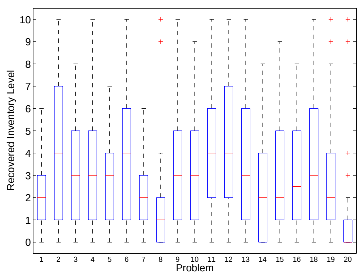


(c) Upward substitution

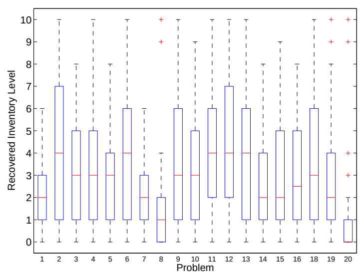


(d) Two-way substitution

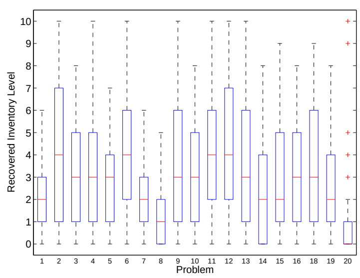
Figure C.19: Graphs showing the level of produced inventory (trigger level) when production is performed for problem set D.



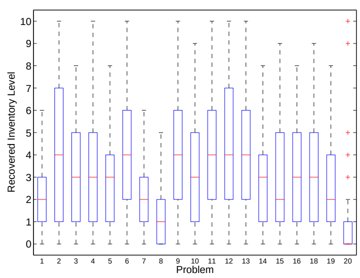
(a) No substitution



(b) Downward substitution

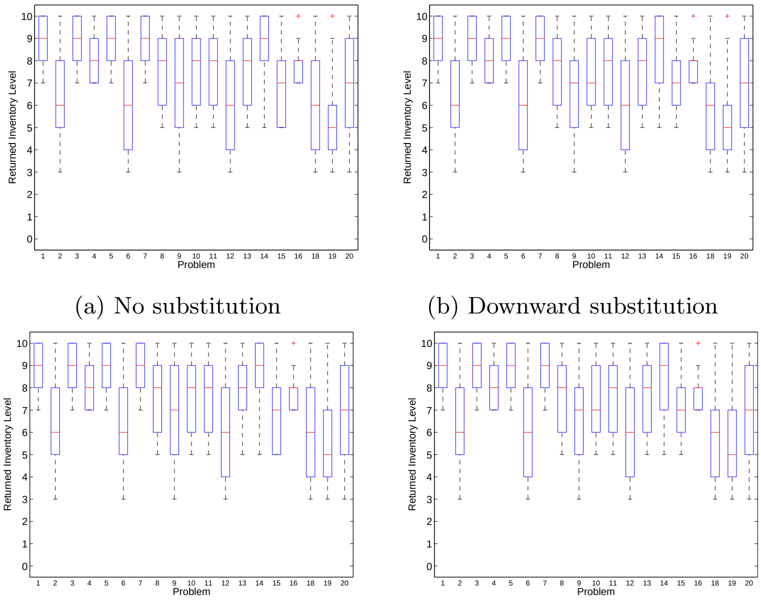


(c) Upward substitution



(d) Two-way substitution

Figure C.20: Graphs showing the level of recovered inventory (trigger level) when recovery is performed for problem set D.

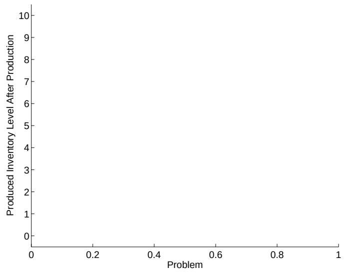


(a) No substitution (b) Downward substitution

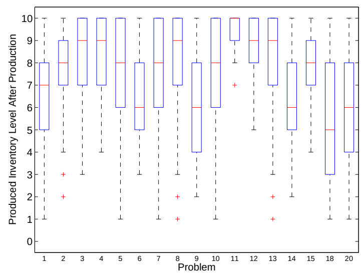
(c) Upward substitution (d) Two-way substitution

Figure C.21: Graphs showing the level of returned inventory (trigger level) when recovery is performed for problem set D.

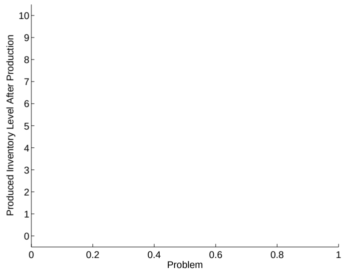
Substitution Actions. The levels of produced inventory after production has been completed $(i_1 + a_p)$ in which substitution is selected, under all substitution strategies presented in Figures C.22, C.24 and C.26. The levels of recovered inventory (i_2) in which substitution is selected, under all substitution strategies are presented in Figures C.23, C.25 and C.27. These figures correspond to Figures 6.19 and 6.20 in Chapter 6, which present only the two-way substitution strategies.



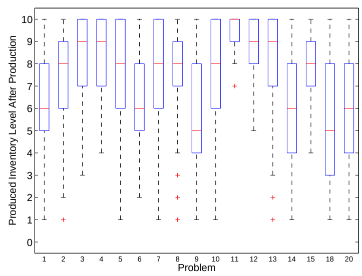
(a) No substitution



(b) Downward substitution

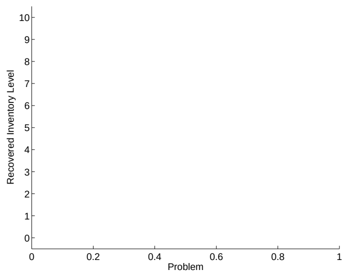


(c) Upward substitution

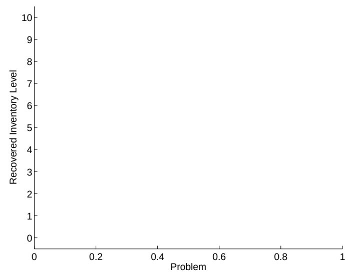


(d) Two-way substitution

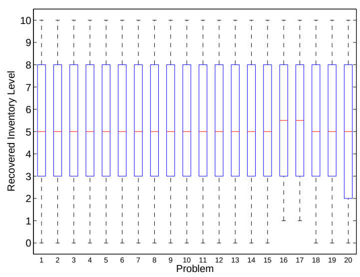
Figure C.22: Graphs showing the level of produced inventory (trigger level) when downward substitution is selected for problem set B.



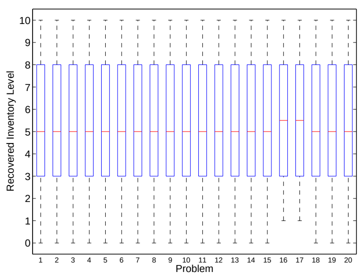
(a) No substitution



(b) Downward substitution

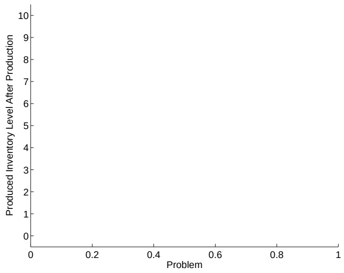


(c) Upward substitution

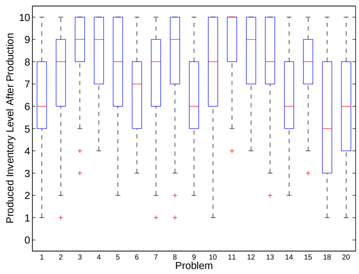


(d) Two-way substitution

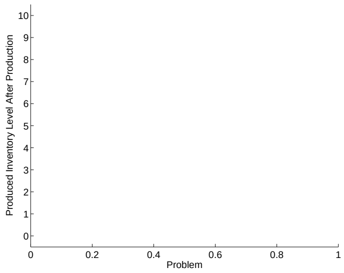
Figure C.23: Graphs showing the level of recovered inventory (trigger level) when upward substitution is selected for problem set B.



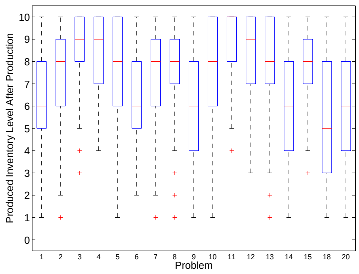
(a) No substitution



(b) Downward substitution

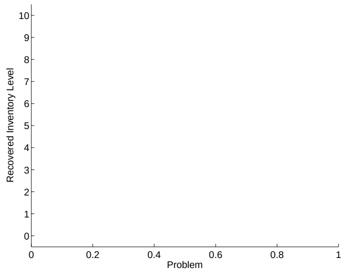


(c) Upward substitution

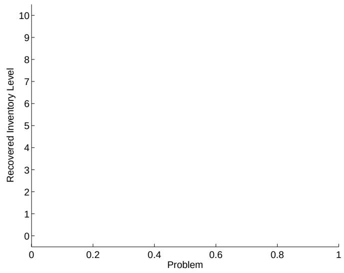


(d) Two-way substitution

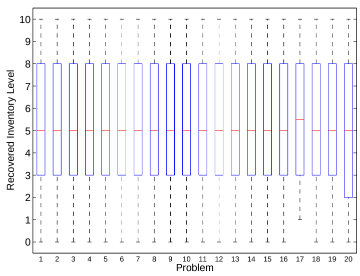
Figure C.24: Graphs showing the level of produced inventory (trigger level) when downward substitution is selected for problem set C.



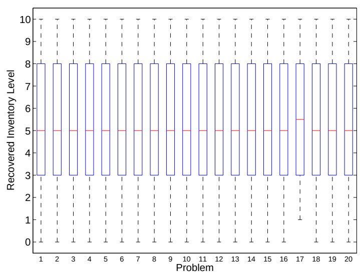
(a) No substitution



(b) Downward substitution

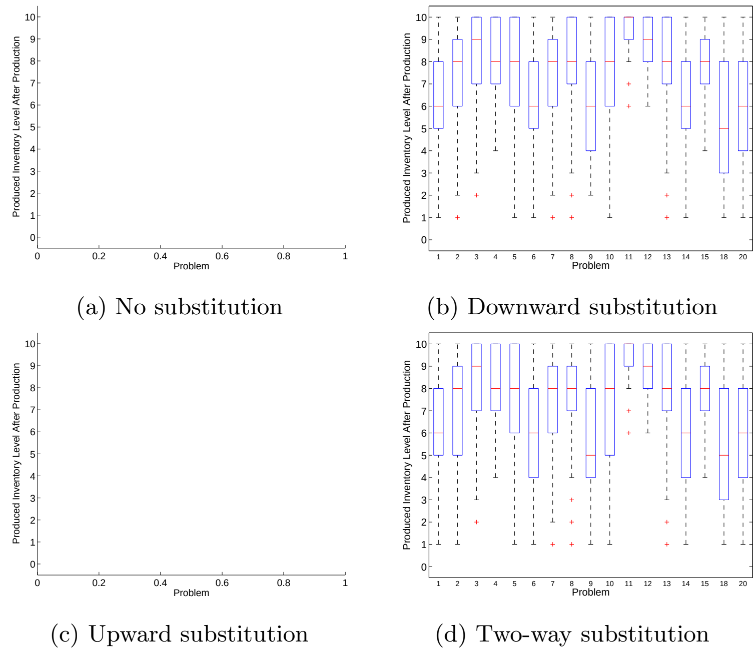


(c) Upward substitution



(d) Two-way substitution

Figure C.25: Graphs showing the level of recovered inventory (trigger level) when upward substitution is selected for problem set C.



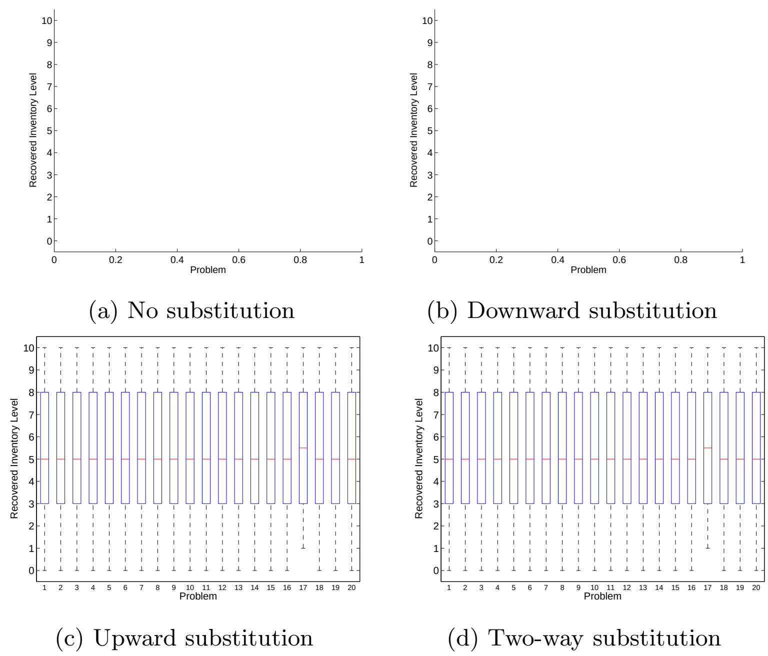
(a) No substitution

(b) Downward substitution

(c) Upward substitution

(d) Two-way substitution

Figure C.26: Graphs showing the level of produced inventory (trigger level) when downward substitution is selected for problem set D.



(a) No substitution

(b) Downward substitution

(c) Upward substitution

(d) Two-way substitution

Figure C.27: Graphs showing the level of recovered inventory (trigger level) when upward substitution is selected for problem set D.

Table C.8: Number of states selecting substitution actions under two-way substitution (total number of states = $11^4 = 14641$)

B			C			D		
Problem	a_{s1}	a_{s2}	Problem	a_{s1}	a_{s2}	Problem	a_{s1}	a_{s2}
$B01$	13933	12563	$C01$	13910	13694	$D01$	13754	13671
$B02$	13860	11797	$C02$	13940	13502	$D02$	13719	13800
$B03$	13681	12398	$C03$	13644	14641	$D03$	13529	13421
$B04$	13628	4944	$C04$	13664	4887	$D04$	13447	4179
$B05$	13787	14238	$C05$	13749	14641	$D05$	13676	14579
$B06$	14068	14641	$C06$	14096	14641	$D06$	14113	14641
$B07$	13923	13538	$C07$	13862	14641	$D07$	13752	14594
$B08$	14115	5641	$C08$	14117	5709	$D08$	14012	6105
$B09$	14077	14549	$C09$	14096	14497	$D09$	13909	14549
$B10$	13772	14345	$C10$	13807	14464	$D10$	13700	14475
$B11$	13711	4597	$C11$	13776	10632	$D11$	13658	6604
$B12$	13902	3869	$C12$	13967	5735	$D12$	13852	3327
$B13$	13848	11908	$C13$	13845	14639	$D13$	13778	13482
$B14$	13875	14522	$C14$	13736	14246	$D14$	13741	14483
$B15$	13690	6770	$C15$	13702	8249	$D15$	13566	7560
$B16$	13310	0	$C16$	13419	0	$D16$	13348	0
$B17$	13310	0	$C17$	13310	0	$D17$	13310	0
$B18$	13998	14243	$C18$	13942	14317	$D18$	13668	14433
$B19$	13578	0	$C19$	13620	0	$D19$	13499	0
$B20$	13962	11242	$C20$	13888	11652	$D20$	13711	11708

Table C.9: Number of states in which the substitution actions are selected under the optimal policy for each substitution strategy (total number of states = $11^4 = 14641$)

(a) Problems B

	None		Down		Up		Two-way	
Problem	a_{s1}	a_{s2}	a_{s1}	a_{s2}	a_{s1}	a_{s2}	a_{s1}	a_{s2}
$B01$	0	0	0	12359	13936	0	13933	12563
$B02$	0	0	0	11735	13883	0	13860	11797
$B03$	0	0	0	12376	13601	0	13681	12398
$B04$	0	0	0	4934	13627	0	13628	4944
$B05$	0	0	0	14215	13866	0	13787	14238
$B06$	0	0	0	14641	14128	0	14068	14641
$B07$	0	0	0	13354	13896	0	13923	13538
$B08$	0	0	0	4599	14115	0	14115	5641
$B09$	0	0	0	14641	14139	0	14077	14549
$B10$	0	0	0	14193	13887	0	13772	14345
$B11$	0	0	0	4583	13710	0	13711	4597
$B12$	0	0	0	3844	13903	0	13902	3869
$B13$	0	0	0	11400	13831	0	13848	11908
$B14$	0	0	0	14641	13955	0	13875	14522
$B15$	0	0	0	6711	13668	0	13690	6770
$B16$	0	0	0	0	13310	0	13310	0
$B17$	0	0	0	0	13310	0	13310	0
$B18$	0	0	0	14259	13845	0	13998	14243
$B19$	0	0	0	0	13578	0	13578	0
$B20$	0	0	0	10904	14238	0	13962	11242

(b) Problems C

	None		Down		Up		Two-way	
Problem	a_{s1}	a_{s2}	a_{s1}	a_{s2}	a_{s1}	a_{s2}	a_{s1}	a_{s2}
$C01$	0	0	0	13717	13933	0	13910	13694
$C02$	0	0	0	13312	13953	0	13940	13502
$C03$	0	0	0	14641	13612	0	13644	14641
$C04$	0	0	0	4872	13653	0	13664	4887
$C05$	0	0	0	14641	13872	0	13749	14641
$C06$	0	0	0	14641	14141	0	14096	14641
$C07$	0	0	0	14641	13879	0	13862	14641
$C08$	0	0	0	4679	14117	0	14117	5709
$C09$	0	0	0	14641	14151	0	14096	14497
$C10$	0	0	0	14640	13916	0	13807	14464
$C11$	0	0	0	10573	13760	0	13776	10632
$C12$	0	0	0	5535	13968	0	13967	5735
$C13$	0	0	0	14641	13869	0	13845	14639
$C14$	0	0	0	14641	13968	0	13736	14246
$C15$	0	0	0	8208	13688	0	13702	8249
$C16$	0	0	0	0	13419	0	13419	0
$C17$	0	0	0	0	13310	0	13310	0
$C18$	0	0	0	14456	13868	0	13942	14317
$C19$	0	0	0	0	13620	0	13620	0
$C20$	0	0	0	11390	14242	0	13888	11652

(c) Problems D

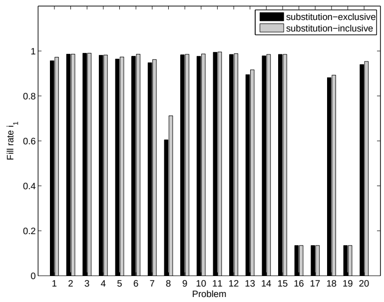
	None		Down		Up		Two-way	
Problem	a_{s1}	a_{s2}	a_{s1}	a_{s2}	a_{s1}	a_{s2}	a_{s1}	a_{s2}
$D01$	0	0	0	13586	13744	0	13754	13671
$D02$	0	0	0	13649	13718	0	13719	13800
$D03$	0	0	0	13404	13476	0	13529	13421
$D04$	0	0	0	4166	13443	0	13447	4179
$D05$	0	0	0	14622	13708	0	13676	14579
$D06$	0	0	0	14641	14135	0	14113	14641
$D07$	0	0	0	14613	13729	0	13752	14594
$D08$	0	0	0	5048	14012	0	14012	6105
$D09$	0	0	0	14641	14045	0	13909	14549
$D10$	0	0	0	14599	13849	0	13700	14475
$D11$	0	0	0	6560	13657	0	13658	6604
$D12$	0	0	0	3291	13853	0	13852	3327
$D13$	0	0	0	13132	13774	0	13778	13482
$D14$	0	0	0	14641	13868	0	13741	14483
$D15$	0	0	0	7517	13545	0	13566	7560
$D16$	0	0	0	0	13348	0	13348	0
$D17$	0	0	0	0	13310	0	13310	0
$D18$	0	0	0	14360	13662	0	13668	14433
$D19$	0	0	0	0	13499	0	13499	0
$D20$	0	0	0	11646	14120	0	13711	11708

C.2.3 Analysis of Recovery Strategy

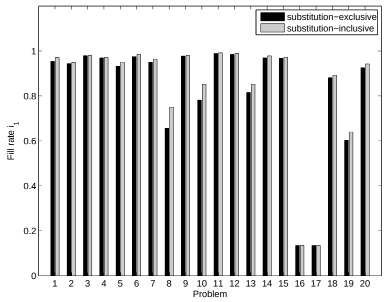
Fill Rates

Substitution Inclusive and Exclusive Fill Rates It is interesting to examine the effect that substitution has on the fill rates under the two quality strategies. Figures C.28, C.29 and C.30 show the fill rates including and excluding sales met from substitution for problem sets B, C and D respectively. These figures show that in general, substitution has a greater affect on the fill rate of recovered inventory than produced inventory, however there is also substantial variation between problems. There are some small differences in the fill rates between the quality strategies, however in general, the substitution inclusive and exclusive fill rates are similar under both quality strategies.

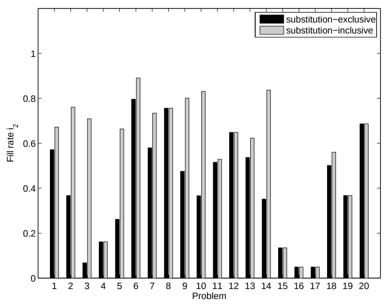
Tables C.10 and C.11 provide the data used for the graphs in Figures C.28, C.29 and C.30. They show the fill rates including and excluding substitution for problems B, C and D respectively.



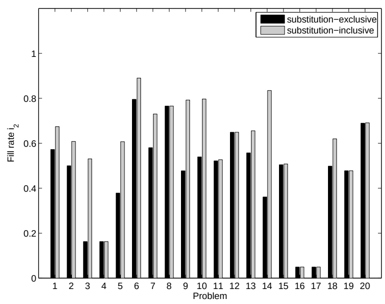
(a) Produced Inventory i_1 - High Quality Recovery



(b) Produced Inventory i_1 - High & Low Quality Recovery

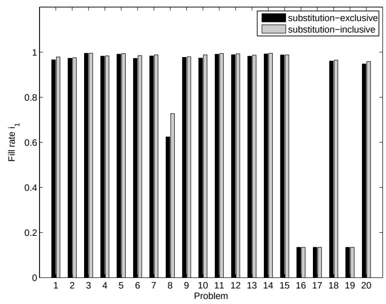


(c) Recovered Inventory i_2 - High Quality Recovery

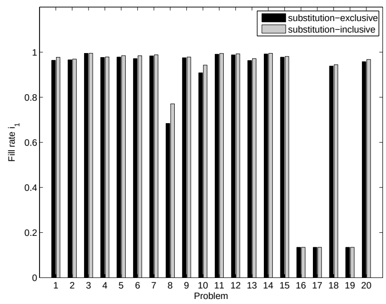


(d) Recovered Inventory i_2 - High & Low Quality Recovery

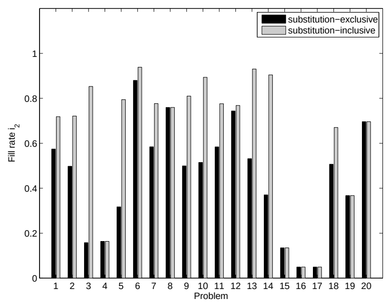
Figure C.28: Fill rates for problem set B under a two-way substitution strategy for both quality strategies



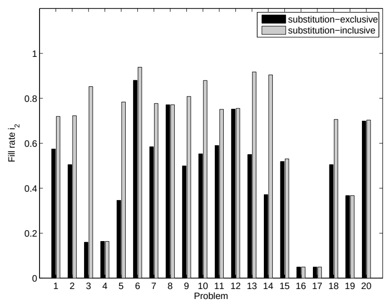
(a) Produced Inventory i_1 - High Quality Recovery



(b) Produced Inventory i_1 - High & Low Quality Recovery

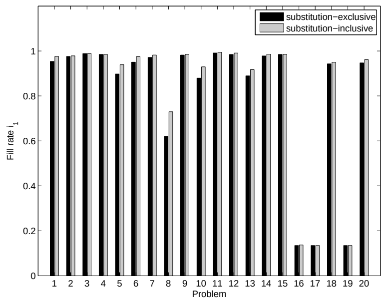


(c) Recovered Inventory i_2 - High Quality Recovery

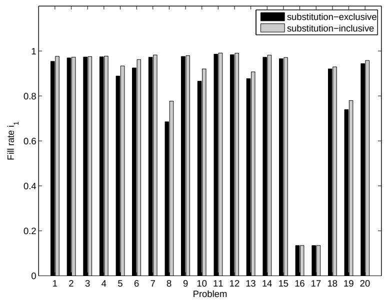


(d) Recovered Inventory i_2 - High & Low Quality Recovery

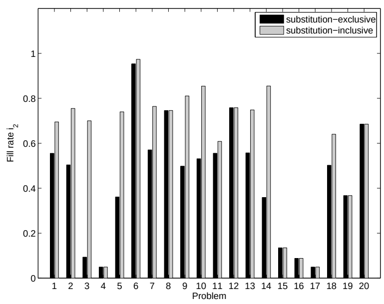
Figure C.29: Fill rates for problem set C under a two-way substitution strategy for both quality strategies



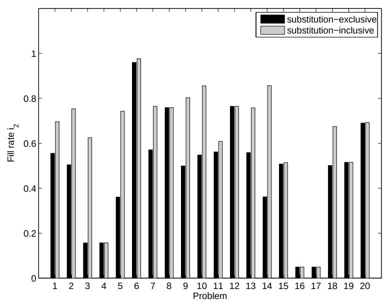
(a) Produced Inventory i_1 - High Quality Recovery



(b) Produced Inventory i_1 - High & Low Quality Recovery



(c) Recovered Inventory i_2 - High Quality Recovery



(d) Recovered Inventory i_2 - High & Low Quality Recovery

Figure C.30: Fill rates for problem set D under a two-way substitution strategy for both quality strategies

Table C.10: Summary of the optimal policy produced-inventory-fill-rates under a two-way substitution strategies under both quality strategies

(a) Problem set B

	Fill rate i_1 excluding substitution		Fill rate i_1 including substitution	
Problem	Only High	Low and High	Only High	Low and High
$B01$	0.9566	0.9540	0.9725	0.9708
$B02$	0.9860	0.9435	0.9860	0.9485
$B03$	0.9902	0.9791	0.9902	0.9799
$B04$	0.9809	0.9693	0.9821	0.9717
$B05$	0.9643	0.9327	0.9734	0.9502
$B06$	0.9766	0.9745	0.9858	0.9846
$B07$	0.9479	0.9502	0.9619	0.9639
$B08$	0.6049	0.6568	0.7119	0.7498
$B09$	0.9833	0.9777	0.9853	0.9801
$B10$	0.9764	0.7821	0.9871	0.8519
$B11$	0.9942	0.9889	0.9957	0.9919
$B12$	0.9846	0.9848	0.9883	0.9885
$B13$	0.8946	0.8150	0.9163	0.8521
$B14$	0.9784	0.9695	0.9847	0.9781
$B15$	0.9849	0.9682	0.9849	0.9725
$B16$	0.1347	0.1347	0.1347	0.1347
$B17$	0.1347	0.1347	0.1347	0.1347
$B18$	0.8814	0.8809	0.8926	0.8920
$B19$	0.1347	0.6019	0.1347	0.6397
$B20$	0.9399	0.9252	0.9534	0.9424

(b) Problem set C

	Fill rate i_1 excluding substitution		Fill rate i_1 including substitution	
Problem	Only High	Low and High	Only High	Low and High
$C01$	0.9663	0.9638	0.9788	0.9773
$C02$	0.9732	0.9662	0.9755	0.9693
$C03$	0.9953	0.9949	0.9955	0.9952
$C04$	0.9825	0.9768	0.9836	0.9786
$C05$	0.9913	0.9784	0.9941	0.9846
$C06$	0.9724	0.9716	0.9850	0.9844
$C07$	0.9833	0.9835	0.9882	0.9883
$C08$	0.6244	0.6839	0.7276	0.7710
$C09$	0.9771	0.9754	0.9799	0.9784
$C10$	0.9740	0.9086	0.9880	0.9427
$C11$	0.9910	0.9909	0.9943	0.9942
$C12$	0.9887	0.9881	0.9926	0.9925
$C13$	0.9822	0.9630	0.9870	0.9716
$C14$	0.9924	0.9920	0.9953	0.9950
$C15$	0.9878	0.9777	0.9878	0.9809
$C16$	0.1347	0.1347	0.1347	0.1347
$C17$	0.1347	0.1347	0.1347	0.1347
$C18$	0.9607	0.9383	0.9648	0.9447
$C19$	0.1347	0.1347	0.1347	0.1347
$C20$	0.9477	0.9581	0.9585	0.9678

(c) Problem set D

	Fill rate i_1 excluding substitution		Fill rate i_1 including substitution	
Problem	Only High	Low and High	Only High	Low and High
$D01$	0.9537	0.9541	0.9760	0.9763
$D02$	0.9759	0.9693	0.9785	0.9726
$D03$	0.9883	0.9734	0.9885	0.9753
$D04$	0.9849	0.9742	0.9849	0.9772
$D05$	0.8978	0.8887	0.9389	0.9334
$D06$	0.9505	0.9249	0.9748	0.9619
$D07$	0.9716	0.9721	0.9822	0.9825
$D08$	0.6198	0.6852	0.7298	0.7770
$D09$	0.9820	0.9760	0.9848	0.9793
$D10$	0.8797	0.8657	0.9295	0.9201
$D11$	0.9913	0.9861	0.9942	0.9910
$D12$	0.9846	0.9835	0.9908	0.9905
$D13$	0.8895	0.8771	0.9172	0.9072
$D14$	0.9782	0.9723	0.9858	0.9819
$D15$	0.9849	0.9656	0.9849	0.9710
$D16$	0.1347	0.1347	0.1370	0.1347
$D17$	0.1347	0.1347	0.1347	0.1347
$D18$	0.9429	0.9205	0.9497	0.9294
$D19$	0.1347	0.7394	0.1347	0.7798
$D20$	0.9473	0.9439	0.9614	0.9581

Table C.11: Summary of the optimal policy recovered-inventory-fill-rates under a two-way substitution strategies under both quality strategies

(a) Problem set B

	Fill rate i_2 excluding substitution		Fill rate i_2 including substitution	
Problem	Only High	Low and High	Only High	Low and High
B01	0.5715	0.5724	0.6719	0.6740
B02	0.3675	0.5000	0.7606	0.6083
B03	0.0684	0.1628	0.7090	0.5308
B04	0.1616	0.1627	0.1616	0.1627
B05	0.2619	0.3785	0.6639	0.6072
B06	0.7963	0.7955	0.8907	0.8901
B07	0.5799	0.5803	0.7340	0.7303
B08	0.7561	0.7659	0.7561	0.7659
B09	0.4753	0.4777	0.8011	0.7926
B10	0.3668	0.5396	0.8309	0.7971
B11	0.5158	0.5220	0.5287	0.5270
B12	0.6486	0.6492	0.6486	0.6492
B13	0.5366	0.5575	0.6231	0.6557
B14	0.3518	0.3609	0.8365	0.8348
B15	0.1351	0.5045	0.1351	0.5079
B16	0.0495	0.0495	0.0495	0.0495
B17	0.0495	0.0495	0.0495	0.0495
B18	0.5012	0.4984	0.5602	0.6198
B19	0.3675	0.4780	0.3675	0.4780
B20	0.6867	0.6893	0.6867	0.6910

(b) Problem set C

	Fill rate i_2 excluding substitution		Fill rate i_2 including substitution	
Problem	Only High	Low and High	Only High	Low and High
C01	0.5743	0.5748	0.7184	0.7196
C02	0.4979	0.5051	0.7211	0.7226
C03	0.1580	0.1600	0.8529	0.8523
C04	0.1636	0.1633	0.1636	0.1634
C05	0.3172	0.3463	0.7941	0.7836
C06	0.8798	0.8803	0.9384	0.9384
C07	0.5845	0.5846	0.7769	0.7772
C08	0.7596	0.7715	0.7596	0.7715
C09	0.4994	0.4995	0.8102	0.8085
C10	0.5149	0.5530	0.8935	0.8794
C11	0.5840	0.5897	0.7761	0.7509
C12	0.7441	0.7518	0.7684	0.7554
C13	0.5316	0.5501	0.9303	0.9174
C14	0.3705	0.3717	0.9043	0.9044
C15	0.1351	0.5189	0.1351	0.5308
C16	0.0495	0.0495	0.0495	0.0495
C17	0.0495	0.0495	0.0495	0.0495
C18	0.5069	0.5052	0.6703	0.7061
C19	0.3675	0.3675	0.3675	0.3675
C20	0.6961	0.6990	0.6961	0.7035

(c) Problem set D

	Fill rate i_2 excluding substitution		Fill rate i_2 including substitution	
Problem	Only High	Low and High	Only High	Low and High
D01	0.5553	0.5554	0.6950	0.6961
D02	0.5041	0.5045	0.7551	0.7536
D03	0.0936	0.1568	0.7001	0.6251
D04	0.0495	0.1572	0.0495	0.1572
D05	0.3613	0.3608	0.7400	0.7428
D06	0.9536	0.9598	0.9735	0.9762
D07	0.5708	0.5710	0.7642	0.7645
D08	0.7459	0.7588	0.7459	0.7588
D09	0.4984	0.4995	0.8107	0.8028
D10	0.5314	0.5482	0.8544	0.8554
D11	0.5558	0.5617	0.6086	0.6085
D12	0.7585	0.7648	0.7585	0.7648
D13	0.5573	0.5585	0.7484	0.7576
D14	0.3591	0.3619	0.8549	0.8566
D15	0.1351	0.5078	0.1351	0.5140
D16	0.0882	0.0495	0.0882	0.0495
D17	0.0495	0.0495	0.0495	0.0495
D18	0.5024	0.5013	0.6403	0.6745
D19	0.3675	0.5153	0.3675	0.5153
D20	0.6851	0.6897	0.6851	0.6928

Appendix D

Appendix for Continuous-Time Model with Separate Markets

Introduction

This section presents additional information related to the fourth model in this thesis – the continuous-time stochastic product recovery model with separate markets and substitution.

D.1 Semi-Markov Decision Process Formulation

D.1.1 Costs and Revenues

This section provides additional details to support Section 7.4.5.

Event Costs

Demand for Produced Inventory. If the next event is the arrival of demand for a produced good the revenues received and costs incurred depend on the current state and whether or not demand is met by produced inventory or by an upward substitution, or is lost.

1. If $a_1 = 0$, then either (a) there is sufficient stock to meet demand or (b) upward substitution is not offered. Under this case there are two sub-cases which could occur, depending on the inventory level i_1.

   (a) There is sufficient produced inventory in stock $i_1 > 0$ and a revenue of p_1 is received.

   (b) There is insufficient produced inventory in stock $i_1 = 0$ and upward substitution is not offered $a_1 = 0$, meaning that the sale is lost and a lost sales cost of l_1 is incurred.

2. If $a_1 = 1$ then there is insufficient stock to meet demand and upward substitution is offered. Under this case there are two sub-cases which could occur depending on whether the substitution is (a) accepted or (b) not accepted.

   (a) The consumer rejects the upward substitution ($y_1 = 0$) and a lost sales cost of l_1 is incurred.

   (b) The consumer accepts the upward substitution ($y_1 = 1$) and a revenue of a recovered good p_2 is received.

Indicator functions can be defined which specify when each of these costs and revenues should be included in the reward function. These are summarised and explained below.

The function $\min\{i_1, 1\}$ will be equal to 1 when demand is met by produced inventory and revenue of p_1 is received, and 0 otherwise.

Recall that Y_1 is the random variable representing the acceptance of upward substitution. If substitution is offered and accepted, then $\min\{y_1, a_1\}$ will be 1 and 0 otherwise. If the produced inventory is greater than zero $i_1 > 0$ then both y_1 and a_1 are equal to zero, making $\min\{y_1, a_1\} = 0$.

If $i_1 > 0$, then $\max\{0, 1 - i_1\}$ will be 0 and a substitution is not required (and not offered). Therefore $\max\{0, 1 - i_1\} - \min\{y_1, a_1\} = 0 - 0 = 0$. Thus if substitution is not required then $\max\{0, 1 - i_1\} - \min\{y_1, a_1\} = 0$. If $i_1 = 0$ then $\max\{0, 1 - i_1\}$ will be 1 and a substitution is required. If substitution is offered and accepted, then

$\max\{0, 1 - i_1\} - \min\{y_1, a_1\} = 1 - 1 = 0$. If substitution is not offered, then $a_1 = 0$ so $\max\{0, 1 - i_1\} - \min\{y_1, a_1\} = 1 - 0 = 1$. If substitution is offered and not accepted, then $y_1 = 0$ and $a_1 = 1$ so $\max\{0, 1 - i_1\} - \min\{y_1, a_1\} = 1 - 0 = 1$. This is summarised in the following table:

Case	Cost	i_1	a_1	y_1	$\max\{0, 1 - i_1\}$	$\min\{i_1, 1\}$	$\min\{y_1, a_1\}$	$\max\{0, 1 - i_1\} - \min\{y_1, a_1\}$
1a	p_1	> 0	0	0	0	1	0	0
1b	l_1	0	0	0	1	0	0	1
2a	l_1	0	1	0	1	0	0	1
2b	p_2	0	1	1	1	0	1	0

Combining these expressions gives the following expression for the expected reward received in the event of demand for a produced good:

$$E[R_{d1}(i, a, Y_1)] = p_1 \min\{i_1, 1\} + p_2 \min\{E[Y_1], a_1\} - l_1(\max\{0, 1 - i_1\} - \min\{E[Y_1], a_1\}) \tag{D.1}$$

Demand for Recovered Inventory. If the next event is the arrival of demand for a recovered good the revenue received or cost incurred depends on the current state and whether or not demand is met by recovered inventory or a downward substitution, or is lost.

1. If $a_2 = 0$, then either (a) there is sufficient stock to meet demand or (b) downward substitution is not offered. Under this case there are two sub-cases which could occur, depending on the recovered inventory level i_2.
   (a) There is sufficient recovered inventory in stock $i_2 > 0$ and a revenue of p_2 is received.
   (b) There is insufficient recovered inventory in stock $i_2 = 0$ and downward substitution is not offered $a_2 = 0$, meaning that the sale is lost and a lost sales cost of l_2 is incurred.
2. If $a_2 = 1$ then there is insufficient stock to meet demand and downward substitution is offered. Under this case there are two sub-cases which could occur which depend on whether substitution is (a) accepted or (b) not accepted.

(a) The consumer rejects the downward substitution ($y_2 = 0$) and a lost sales cost of l_2 is incurred.

(b) The consumer accepts the downward substitution ($y_2 = 1$) and a revenue of a recovered good p_2 is received. (The customer is charged for the cheaper of the two goods, i.e. the recovered good).

Indicator functions can be defined which specify when each of these costs and revenues should be included in the reward function.

The function $\min\{i_2, 1\}$ will be equal to 1 when demand is met by recovered inventory and revenue of p_2 is received, and 0 otherwise.

Recall that Y_2 is the random variable representing the acceptance of downward substitution. If substitution is offered and accepted, then $\min\{y_2, a_2\}$ will be 1 and 0 otherwise. If the recovered inventory is greater than zero $i_2 > 0$ then both y_2 and a_2 are equal to zero, making $\min\{y_2, a_2\} = 0$.

If $i_2 > 0$ then $\max\{0, 1 - i_2\}$ will be 0 and a substitution is not required (and not offered). Therefore $\max\{0, 1 - i_2\} - \min\{y_2, a_2\} = 0 - 0 = 0$. Thus if substitution is not required then $\max\{0, 1 - i_2\} - \min\{y_2, a_2\} = 0$. If $i_2 = 0$ then $\max\{0, 1 - i_2\}$ will be 1 and a substitution is required. If substitution is offered and accepted, then $\max\{0, 1 - i_2\} - \min\{y_2, a_2\} = 1 - 1 = 0$. If substitution is not offered, then $a_2 = 0$ so $\max\{0, 1 - i_2\} - \min\{y_2, a_2\} = 1 - 0 = 1$. If substitution is offered and not accepted, then $y_2 = 0$ and $a_2 = 1$ so $\max\{0, 1 - i_2\} - \min\{y_2, a_2\} = 1 - 0 = 1$. This is summarised in the following table:

Case	Cost	i_2	a_2	y_2	$\max\{0, 1 - i_2\}$	$\min\{i_2, 1\}$	$\min\{y_2, a_2\}$	$\max\{0, 1 - i_2\} - \min\{y_2, a_2\}$
1a	p_2	> 0	0	0	0	1	0	0
1b	l_2	0	0	0	1	0	0	1
2a	l_2	0	1	0	1	0	0	1
2b	p_1	0	1	1	1	0	1	0

Combining these expressions gives the following expression for the reward received in the event of demand for a recovered good:

$$E[R_{d2}(i, a, Y_2)] = p_2 \min\{i_2, 1\} + p_2 \min\{E[Y_2], a_2\} - l_2(\max\{0, 1 - i_2\} - \min\{E[Y_2], a_2\}) \tag{D.2}$$

D.2 Model Implementation and Validation

D.2.1 Implementation of the Stochastic Product Recovery Model

This section relates to Section 7.5.3. Further details about semi-Markov decision processes were provided in the Methodology chapter (in Section 3.2). As in the previous chapters, several forms of validation are used in order to check the accuracy of the code. These are discussed in the following sections.

In the previous models the mid-state has been used to reduce the computational burden of the transitions between states. In this model, however, all parts of the transition from the current state i to the next state depend on the action chosen, therefore the mid-state does not offer any computational savings in this model.

D.2.2 Validation of the Code

In addition to thorough error-checking and inspection of output during the code development process, two forms of verification were used to validate the problem specific files. The calculation of the expected average rewards was checked using an Excel spreadsheet and by simulation for the optimal and a heuristic policy.

To conduct these tests, a set of six test problems was developed. The parameters for these problems are shown in Table D.1. These six test problems were chosen because they represent a range of different scenarios. The test problems are labelled $A01$–$A06$. The test problems have a limited upper capacity on each inventory level of 5. In all six problems Poisson processes are used to model the arrival of the events: returns, demand for produced goods, demand for recovered goods, production, recovery. The number of returns received is governed by the Poisson distribution in problems $A01$–$A04$ and the Uniform distribution in problems $A05$ and $A06$. The size of the demand for produced and recovered goods is always 1.

All six test problems allow the recovery of high and low quality returns, and allow upward and downward substitution. The quality of returns and the acceptance of substitution are modelled by the Binomial distribution. The quality parameter α varies to represent there being a low or high probability of receiving high quality returns. The acceptance parameters α_1 and α_2 represent the probability of acceptance of upward and downward substitution respectively.

Table D.1: Parameters used in the test problems used in model validation

Problem ID	*A*01	*A*02	*A*03	*A*04	*A*05	*A*06
Arrival Rates						
λ_r	1	2	1	8	1	1
λ_{d1}	8	10	10	8	1	3
λ_{d2}	6	5	10	8	1	3
λ_p	1	1	0.2	3	1	100
λ_{rec}	1	0.5	0.4	2	1	100
Distributions						
X_r	Pois(2)	Pois(2)	Pois(2)	Pois(2)	U(2,6)	U(0,4)
α	0.9	0.6	1	0.5	0.2	0.6
α_1	0.8	0.2	0.5	1	0.8	0.5
α_2	0.3	0.8	0.5	1	0.8	0.5
Unit Costs						
c_p	4	4	15	15	5	10
c_r	1	2	5	5	1	1
c_b	3	3	5	1	10	2
c_h	2	3	1	1	1	1
c_l	1	1	3	3	5	3
c_d	1.1	1.1	2	2	2	1
Revenues						
p_r	0	0	0	0	0	0
p_{d1}	21	21	60	48	45	36
p_{d2}	13	9	55	16	34	10
Setup Costs						
k_p	3	2	10	2	5	1
k_r	1	1	5	1	3	1
k_b	0	0	0	0	0	0
Holding Costs						
h_r	1	0.1	1	1	1	3
h_{s1}	3	2	5	2	4	5
h_{s2}	2	1	2	1	4	4
h_c	1	1	0	1	1	2
Penalty Costs						
l_r	0.1	0.1	0.1	0.1	0.1	0.1
l_{s1}	0.7	0.7	2	1.6	1.5	1.2
l_{s2}	0.3	0.5	0.6	0.6	0.2	0.2
Order Sizes A						
Q_p	4	3	4	4	4	4
Q_r	4	3	4	4	4	4
Strategic Variables						
χ_1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1
ζ	1	1	1	1	1	1

Reward Calculations. In order to test the calculation of the costs and rewards within the java code, an Excel spreadsheet was constructed to calculate the expected rewards independently of the calculations used in the java code. The expected reward associated with a given state and action (as defined by equation (7.9)) was calculated by completely enumerating across possible transitions for a given state, i.e., all possible events and size of events. The result of this equation should be the same when calculated using Excel and when calculated using the java code. The expected reward associated with the optimal action for each state is printed in the optimal policy file.

The number of states and actions in the problem makes it infeasible to manually compare the rewards for every state. Therefore, for the six test problems the rewards associated with 30 states were checked. The first and last states (marked with a *) were chosen because they have inventory levels equal to the minimum or maximum capacity levels. The remaining 28 states were selected randomly using the Excel function `rand()`. The 30 states are displayed in Table D.2.

For each of the 30 states, the rewards calculated by the java programme and those calculated by the Excel spreadsheet were checked for the optimal policy action and for a heuristic policy action. A heuristic policy was included to ensure that the code was not always choosing the action with the highest reward. The heuristic policy used the following decision rule:

If $i_1 \leq s_p$ and $i_{op} = 0$ then produce Q_p

Else if $i_2 \leq s_r$ and $i_{or} = 0$ and $i_r > Q_r$ then recover Q_r.

If $i_1 = 0$ and $i_2 \geq ss_2$ and $\chi_1 = 1$ and $i_{op} = 1$ then offer upward substitution.

If $i_2 = 0$ and $i_1 \geq ss_1$ and $\chi_2 = 1$ and $i_{or} = 1$ then offer downward substitution.

For this validation, the $s_p = s_r = 4$ and $ss_1 = ss_2 = 1$. For all six problems, the rewards calculated by the java code were the same as the rewards calculated by the excel spreadsheet for all states.

Table D.2: Selection of states for which the costs were calculated using an Excel spreadsheet

		State Values					
	State	i_r	i_{s1}	i_{s2}	i_c	i_{op}	i_{or}
1*	0	0	0	0	0	0	0
2	58	0	0	2	2	1	0
3	84	0	0	3	3	0	0
4	269	0	1	5	1	0	1
5	479	0	3	1	5	1	1
6	482	0	3	2	0	1	0
7	516	0	3	3	3	0	0
8	756	0	5	1	3	0	0
9	794	0	5	3	0	1	0
10	1151	1	1	5	5	1	1
11	1340	1	3	1	5	0	0
12	1399	1	3	4	1	1	1
13	1453	1	4	0	3	0	1
14	1758	2	0	1	1	1	0
15	2271	2	3	4	3	1	1
16	2357	2	4	2	1	0	1
17	2375	2	4	2	5	1	1
18	2552	2	5	4	2	0	0
19	3249	3	4	3	2	0	1
20	3734	4	1	5	3	1	0
21	4055	4	4	0	5	1	1
22	4185	4	5	0	2	0	1
23	4572	5	1	4	3	0	0
24	4728	5	2	5	0	0	0
25	4769	5	3	0	4	0	1
26	4810	5	3	2	2	1	0
27	4914	5	4	0	4	1	0
28	4949	5	4	2	1	0	1
29	5041	5	5	0	0	0	1
30*	5183	5	5	5	5	1	1

Optimal Policies. The optimal and heuristic policy actions for all the test problems were also examined. Histograms showing the frequency of the size of each action variable were examined for all test problems. The histograms for problem $A01$ are presented in Figure D.1 and are discussed here as an example.

Figure D.1a shows that the action a_p can take two values 0 and Q_p. For recovery, the action a_r can take two values 0 and Q_r and this is represented by the histogram of a_r. Figure D.1b shows that the substitution variables can take two values 0 (do not offer substitution) or 1 (offer substitution). For this problem, both upward substitution (a_1) and downward substitution (a_2) are offered in some states.

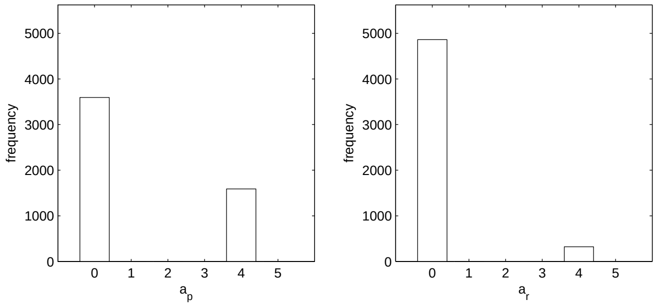
Figure D.2 show the inventory levels when production and recovery are chosen. Figure D.2a shows that production is selected when the produced inventory level is 3 or less. Recovery is performed when the recovered inventory level is less than 3 and when the returned inventory level is 4 or more, as shown in Figures D.2b and D.2c respectively.

Figure D.3 show the inventory levels when each type of substitution is chosen. Downward substitution should only be offered when $i_2 = 0$ and $i_1 > 0$ and upward substitution can only be offered when $i_1 = 0$ and $i_2 > 0$. As expected downward substitution is only offered when $i_2 = 0$ and $i_1 > 0$ and upward substitution is only offered when $i_1 = 0$ and $i_2 > 0$. For this problem, downward substitution is only offered in some states when if $i_1 = 5$ and upward substitution is offered whenever $i_2 > 0$.

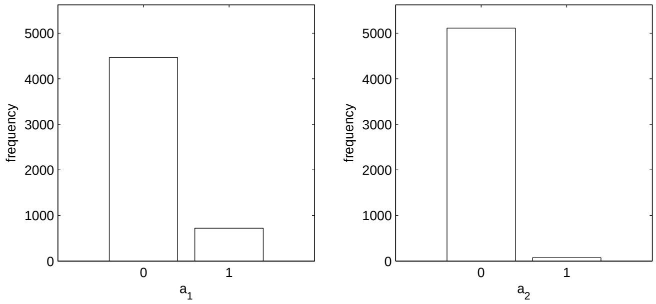
These histograms are typical of the ones for the other test problems. All graphs were examined and showed the expected results. This gives further validation to the accuracy of the code.

Simulation Code

Simulation can also be used to calculate the average reward of a policy and hence can be used to validate the calculation of the average total reward in the value iteration file. Two simulation codes were developed for this model: a discrete time simulation and a continuous time simulation. Repeated simulations were performed using both

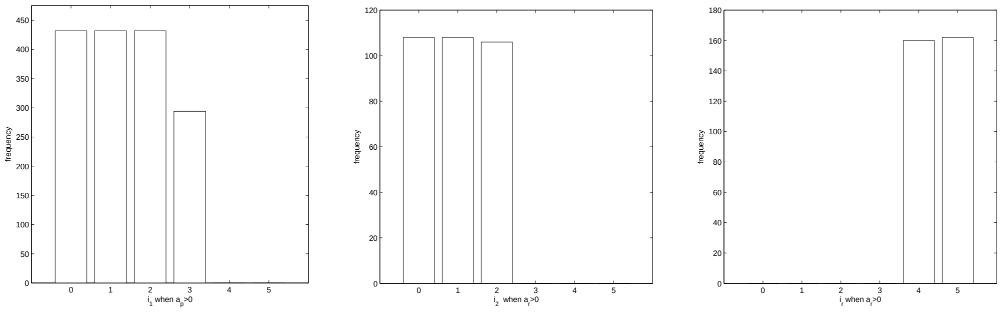


(a) Replenishment



(b) Substitution

Figure D.1: Action quantities for production and recovery for test problem $A01$



(a) Production (b) Recovery (c) Recovery

Figure D.2: Trigger states for production and recovery for test problem $A01$

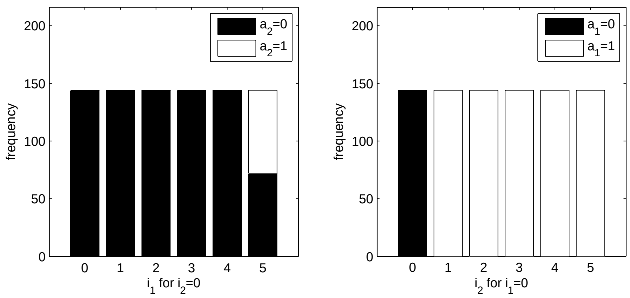


Figure D.3: States in which substitution is performed for test problem $A01$

codes and the long run average reward was recorded. Over a large number of trials, repeated sampling from the distribution of the long run average reward should yield an approximately normally distributed curve, centred around the actual optimal reward calculated by the value iteration algorithm. For each test problem 1000 simulations were performed.

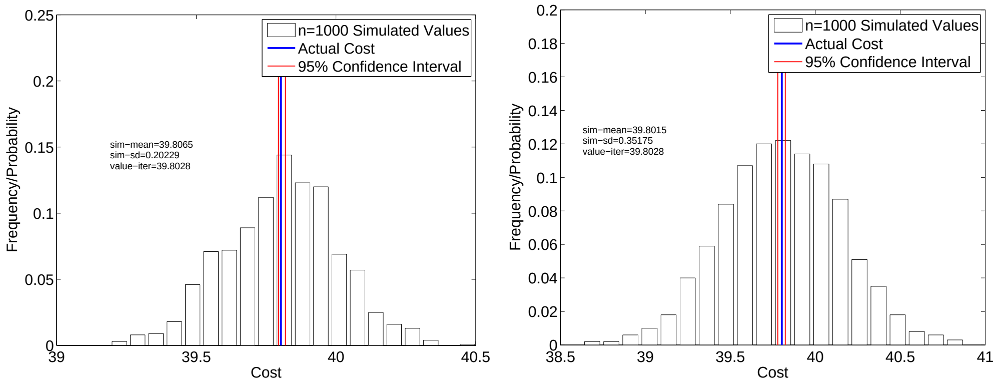
The discrete time simulation code models the problem by using a structure similar to the discretization of the SMDP, in which each period has a length τ. The length of each period τ corresponds to the value used in the conversion of the SMDP to a discrete problem. The value of τ for each problem is presented in Table D.4. The probability of an event happening in each period is given by the probabilities $\overline{p}_{i,j}(a)$ (see Table 7.2). The discrete simulation the simulations had a length of $T = 1000000$ periods, which corresponds to $\tau \times 1000000$ time units.

The continuous time simulation code models a continuous time version of the problem in which the time to the next event is determined by generating a exponential random variable for each event and selecting the one with the earliest arrival time. The continuous simulation had a length of $T = 10000$ time units.

Histograms showing the distribution of the simulation costs were constructed from the discrete and continuous simulations for each of the test problems. The histograms for the first test problem $A01$ are presented in Figure D.4 as an example. The histograms were visually inspected and all appeared to be approximately normally distributed with the data centred around the value iteration reward value, which is also shown on the graphs.

A summary of the optimal policy reward and simulation results for the test problems are presented in Tables D.3 and D.4. Tables D.3a and D.3b show the optimal policy reward and the fill rates calculated by a discrete and continuous simulation respectively. For both types of simulation, the simulation costs are approximately equal to the optimal policy reward from the value iteration algorithm. The fill rates vary considerably across the six test problems and between the two types of inventory. It was not optimal to perform downward substitution in problems $A01$–$A04$.

The average values of the random variables across the simulations are presented in Table D.4. These can be compared to the parameters of the problems in Table D.1.



(a) Discrete Time Simulation (b) Continuous Time Simulation

Figure D.4: Histogram of average total reward from 1000 simulations for problem $A01$.

The values for the $\bar{x}_{d1}, \bar{x}_{d2}, \bar{x}_r$ are the mean number of arrivals per time unit. The similarity between these values further validates the programming code.

D.3 Properties of the Optimal Policy

This section contains Figures and Tables to support the analysis in Section 7.6.

D.3.1 Analysis of Performance

This section relates to Section 7.6.2 and provides additional information about the performance of the optimal policy in terms of the average reward and the fill rate, under the four substitution strategies: no substitution ($\chi_1 = 0, \chi_2 = 0$), upward substitution ($\chi_1 = 1, \chi_2 = 0$), downward substitution ($\chi_1 = 0, \chi_2 = 1$), two-way substitution ($\chi_1 = 1, \chi_2 = 1$). Upward substitution relates to the shortage of produced goods and downward relates to the shortage of recovered goods.

Rewards

In Section 7.6.2 the relative reward increase associated with choosing substitution, compared with no substitution was presented in Chapter 7, in Figure 7.3. Figures D.5

Table D.3: Performance of optimal policy for test problems

(a) Discrete Time Simulation

		Discrete Time Simulation over $T = \tau \times 1,000,000$						
Problem	Optimal Profit	τ	Simulation Profit	Fill rate i_1	Fill rate i_1 including substitution	Fill rate i_2	Fill rate i_2 including substitution
$A01$	39.8028	0.0588	39.9178	0.3820	0.4347	0.1228	0.1228
$A02$	29.0319	0.0541	29.0825	0.2795	0.2942	0.1071	0.1071
$A03$	42.1640	0.0463	42.1436	0.0771	0.1044	0.0603	0.0603
$A04$	164.9202	0.0345	164.8589	0.7255	0.7819	0.2679	0.2679
$A05$	25.8380	0.2000	25.7434	0.8037	0.8353	0.2788	0.7305
$A06$	56.0325	0.0048	55.7303	0.9922	0.9935	0.3484	0.4136

(b) Continuous Time Simulation

		Continuous Time Simulation over $T = 10000$ time units					
Problem	Optimal Profit	τ	Simulation Profit	Fill rate i_1	Fill rate i_1 including substitution	Fill rate i_2	Fill rate i_2 including substitution
$A01$	39.8028	0.0588	38.9342	0.3708	0.4232	0.1214	0.1214
$A02$	29.0319	0.0541	28.5734	0.2751	0.2896	0.1075	0.1075
$A03$	42.1640	0.0463	41.5402	0.0754	0.1017	0.0610	0.0610
$A04$	164.9202	0.0345	164.5277	0.7225	0.7796	0.2691	0.2691
$A05$	25.8380	0.2000	25.8591	0.7997	0.8350	0.2833	0.7359
$A06$	56.0325	0.0048	56.7560	0.9908	0.9926	0.3515	0.4153

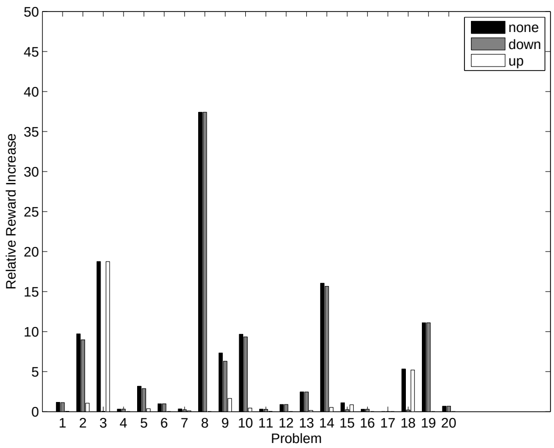
show the relative reward decrease associated with not choosing two-way substitution. The relative reward decrease (RRD) is calculated as follows:

$$\mathrm{RRD} = \frac{\text{Reward(two-way substitution)} - \text{Reward(other substitution)}}{\text{Reward(two-way substitution)}} \times 100\%$$

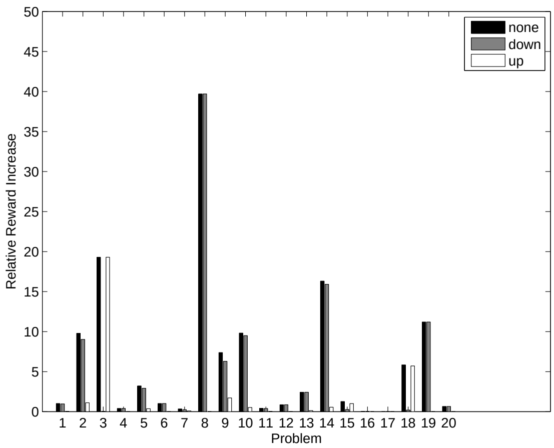
Fill rates

Tables D.5, D.6 and D.7 show the substitution inclusive and exclusive fill rates for produced and recovered goods, for problem sets B, C and D respectively. These fill rates were calculated from a simulation over $T = \tau \times 1000000$ time units.

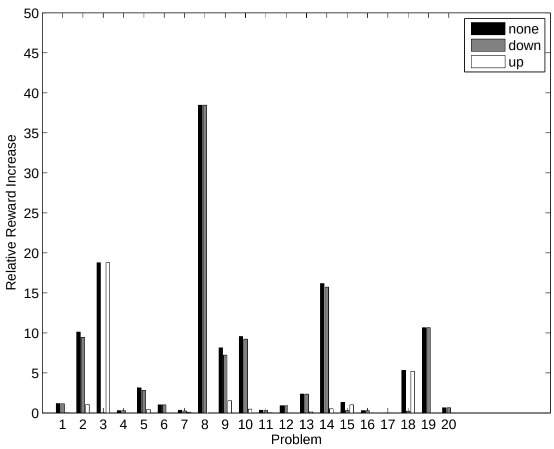
Notice that under the strategies allowing upward substitution (upward and two-way), the fill rates for produced goods (i_1) including and excluding substitution are the same. This is expected as no substitution is offered for produced goods under this strategy. Similar observations can be made regarding the fill rates for recovered goods (i_2) when a strategy allowing downward substitution is applied (downward and two-way).



(a) Problems B



(b) Problems C



(c) Problems D

Figure D.5: Relative reward increase for substitution strategies, compared with two-way substitution

Table D.4: Average value of random variables in simulation of the test problems

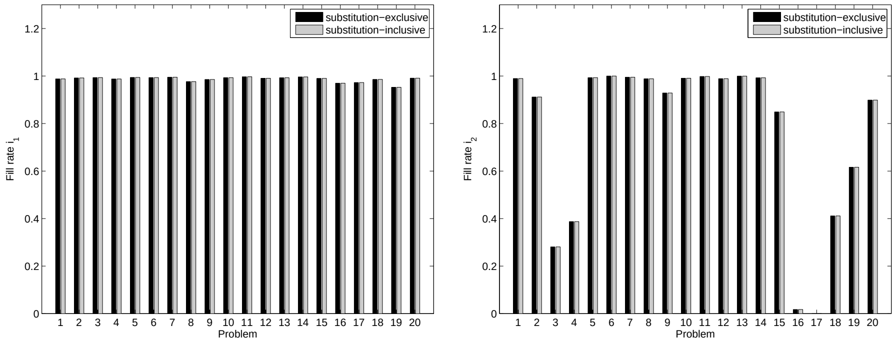
(a) Discrete Time Simulation

	Discrete Time Simulation over $T = \tau \times 1,000,000$						
Problem	$\bar{x}_{d1}$	$\bar{x}_{d2}$	$\bar{x}_r$	$\bar{x}_r$ size	$\bar{x}_q$	$\bar{x}_{s1}$	$\bar{x}_{s2}$
A01	7.9929	5.9988	1.0032	1.9973	0.9006	0.7957	0.0000
A02	9.9975	4.9992	1.9998	1.9967	0.5959	0.1988	0.0000
A03	9.9938	9.9992	1.0030	1.9931	1.0000	0.5020	0.0000
A04	7.9923	8.0043	8.0052	2.0000	0.5010	1.0000	0.0000
A05	0.9955	1.0023	1.0020	4.0000	0.2002	0.8015	0.7988
A06	2.9825	3.0106	1.0149	1.9984	0.5953	0.5263	0.5080

(b) Continuous Time Simulation

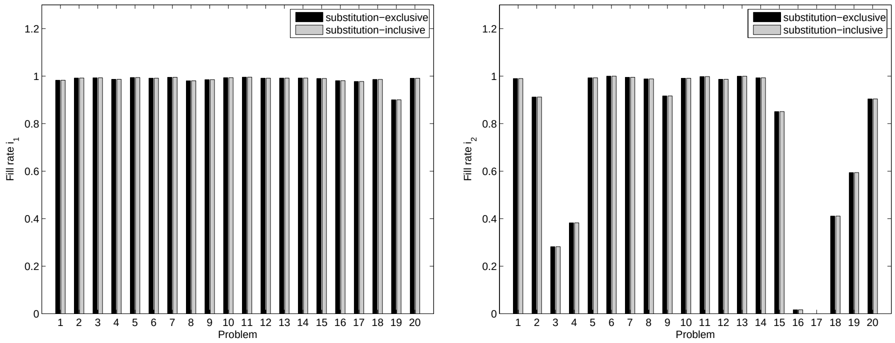
	Continuous Time Simulation over $T = 10000$ time units						
Problem	$\bar{x}_{d1}$	$\bar{x}_{d2}$	$\bar{x}_r$	$\bar{x}_r$ size	$\bar{x}_q$	$\bar{x}_{s1}$	$\bar{x}_{s2}$
A01	8.0419	6.0342	1.0141	2.0119	0.8986	0.7969	0.0000
A02	10.0279	5.0460	2.0167	2.0082	0.6014	0.1944	0.0000
A03	10.0267	10.0724	1.0130	2.0359	1.0000	0.4969	0.0000
A04	8.0083	8.0039	8.0353	2.0019	0.4996	1.0000	0.0000
A05	1.0036	1.0001	1.0152	4.0097	0.2004	0.8101	0.8001
A06	3.0359	3.0068	1.0158	2.0209	0.5959	0.5340	0.5088

Figures 7.4, 7.5 and 7.6 show the substitution inclusive and exclusive fill rates for the two-way substitution strategy. Graphs for the other substitution strategies are shown below in Figures D.6, D.9 and D.12.



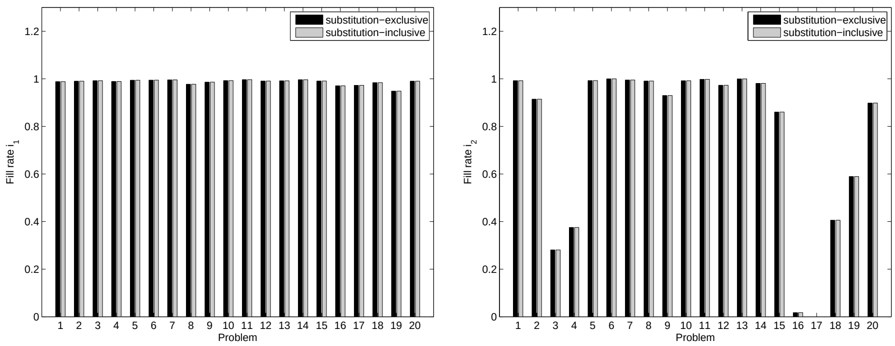
(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure D.6: Fill rates for problem set B under a no substitution strategy



(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure D.7: Fill rates for problem set C under a no substitution strategy



(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure D.8: Fill rates for problem set D under a no substitution strategy

Table D.5: Summary of the optimal policy fill rates for substitution strategies for problem set B

(a) Produced inventory i_1

	Fill rate i_{is1} excluding substitution					Fill rate i_{is1} including substitution			
Problem	None	Down	Up	Two-way	Problem	None	Down	Up	Two-way
B01	0.9880	0.9887	0.9788	0.9783	B01	0.9880	0.9887	0.9929	0.9928
B02	0.9921	0.9919	0.9020	0.8961	B02	0.9921	0.9919	0.9625	0.9600
B03	0.9935	0.9917	0.9937	0.9912	B03	0.9935	0.9917	0.9944	0.9922
B04	0.9878	0.9878	0.9874	0.9874	B04	0.9878	0.9878	0.9929	0.9929
B05	0.9941	0.9940	0.8897	0.8874	B05	0.9941	0.9940	0.9991	0.9989
B06	0.9936	0.9936	0.9631	0.9631	B06	0.9936	0.9936	0.9832	0.9832
B07	0.9951	0.9956	0.9889	0.9882	B07	0.9951	0.9956	0.9944	0.9938
B08	0.9766	0.9766	0.2820	0.2820	B08	0.9766	0.9766	0.5635	0.5635
B09	0.9856	0.9851	0.9155	0.9038	B09	0.9856	0.9851	0.9769	0.9729
B10	0.9931	0.9920	0.7798	0.7703	B10	0.9931	0.9920	0.9836	0.9819
B11	0.9969	0.9965	0.9869	0.9869	B11	0.9969	0.9965	0.9989	0.9990
B12	0.9906	0.9905	0.9789	0.9786	B12	0.9906	0.9905	0.9952	0.9951
B13	0.9929	0.9927	0.7681	0.7496	B13	0.9929	0.9927	0.9132	0.9058
B14	0.9964	0.9959	0.6534	0.6454	B14	0.9964	0.9959	0.9734	0.9726
B15	0.9901	0.9907	0.9880	0.9872	B15	0.9901	0.9907	0.9934	0.9929
B16	0.9700	0.9700	0.9700	0.9700	B16	0.9700	0.9700	0.9713	0.9713
B17	0.9725	0.9725	0.9725	0.9725	B17	0.9725	0.9725	0.9725	0.9725
B18	0.9858	0.9862	0.9844	0.9840	B18	0.9858	0.9862	0.9884	0.9884
B19	0.9525	0.9525	0.8643	0.8643	B19	0.9525	0.9525	0.9690	0.9690
B20	0.9910	0.9910	0.9803	0.9803	B20	0.9910	0.9910	0.9900	0.9900

(b) Recovered inventory i_2

	Fill rate i_{is2} excluding substitution					Fill rate i_{is2} including substitution			
Problem	None	Down	Up	Two-way	Problem	None	Down	Up	Two-way
B01	0.9894	0.9895	0.9894	0.9893	B01	0.9894	0.9942	0.9894	0.9937
B02	0.9116	0.9124	0.8783	0.8638	B02	0.9116	0.9769	0.8783	0.9609
B03	0.2810	0.2810	0.2803	0.2806	B03	0.2810	0.9222	0.2803	0.9217
B04	0.3875	0.3875	0.3842	0.3842	B04	0.3875	0.3875	0.3842	0.3842
B05	0.9932	0.9858	0.9923	0.9823	B05	0.9932	0.9970	0.9923	0.9957
B06	1.0000	1.0000	1.0000	1.0000	B06	1.0000	1.0000	1.0000	1.0000
B07	0.9951	0.9944	0.9953	0.9939	B07	0.9951	0.9972	0.9953	0.9970
B08	0.9883	0.9883	0.9812	0.9812	B08	0.9883	0.9883	0.9812	0.9812
B09	0.9286	0.8973	0.8898	0.8402	B09	0.9286	0.9707	0.8898	0.9552
B10	0.9909	0.9850	0.9902	0.9747	B10	0.9909	0.9979	0.9902	0.9926
B11	0.9983	0.9977	0.9975	0.9969	B11	0.9983	0.9996	0.9975	0.9993
B12	0.9891	0.9890	0.9887	0.9884	B12	0.9891	0.9915	0.9887	0.9921
B13	0.9998	0.9997	0.9934	0.9898	B13	0.9998	1.0000	0.9934	0.9976
B14	0.9926	0.9617	0.9848	0.9609	B14	0.9926	0.9969	0.9848	0.9931
B15	0.8489	0.8819	0.8439	0.8819	B15	0.8489	0.9526	0.8439	0.9537
B16	0.0175	0.0175	0.0161	0.0161	B16	0.0175	0.0175	0.0161	0.0161
B17	0.0000	0.0000	0.0000	0.0000	B17	0.0000	0.0000	0.0000	0.0000
B18	0.4113	0.4088	0.4023	0.3956	B18	0.4113	0.7329	0.4023	0.7288
B19	0.6163	0.6163	0.3082	0.3082	B19	0.6163	0.6163	0.3082	0.3082
B20	0.8987	0.8987	0.8953	0.8953	B20	0.8987	0.8987	0.8953	0.8953

Table D.6: Summary of the optimal policy fill rates for substitution strategies for problem set C

(a) Produced inventory i_1

	Fill rate i_{is1} excluding substitution					Fill rate i_{is1} including substitution			
Problem	None	Down	Up	Two-way	Problem	None	Down	Up	Two-way
C01	0.9827	0.9826	0.9824	0.9816	C01	0.9827	0.9826	0.9942	0.9938
C02	0.9918	0.9931	0.9018	0.8981	C02	0.9918	0.9931	0.9625	0.9610
C03	0.9929	0.9921	0.9930	0.9917	C03	0.9929	0.9921	0.9939	0.9928
C04	0.9867	0.9867	0.9864	0.9864	C04	0.9867	0.9867	0.9928	0.9928
C05	0.9940	0.9940	0.8864	0.8862	C05	0.9940	0.9940	0.9990	0.9989
C06	0.9914	0.9914	0.9486	0.9486	C06	0.9914	0.9914	0.9764	0.9764
C07	0.9951	0.9951	0.9900	0.9896	C07	0.9951	0.9951	0.9948	0.9947
C08	0.9802	0.9802	0.2908	0.2908	C08	0.9802	0.9802	0.5673	0.5673
C09	0.9853	0.9853	0.9165	0.9054	C09	0.9853	0.9853	0.9771	0.9737
C10	0.9934	0.9932	0.7812	0.7668	C10	0.9934	0.9932	0.9844	0.9819
C11	0.9958	0.9954	0.9843	0.9828	C11	0.9958	0.9954	0.9987	0.9986
C12	0.9912	0.9914	0.9804	0.9804	C12	0.9912	0.9914	0.9956	0.9955
C13	0.9916	0.9914	0.8095	0.7858	C13	0.9916	0.9914	0.9285	0.9200
C14	0.9919	0.9944	0.6509	0.6455	C14	0.9919	0.9944	0.9726	0.9717
C15	0.9901	0.9907	0.9869	0.9870	C15	0.9901	0.9907	0.9933	0.9929
C16	0.9810	0.9810	0.9810	0.9810	C16	0.9810	0.9810	0.9814	0.9814
C17	0.9776	0.9776	0.9776	0.9776	C17	0.9776	0.9776	0.9776	0.9776
C18	0.9862	0.9884	0.9857	0.9865	C18	0.9862	0.9884	0.9899	0.9904
C19	0.9002	0.9002	0.8544	0.8544	C19	0.9002	0.9002	0.9670	0.9670
C20	0.9911	0.9911	0.9747	0.9747	C20	0.9911	0.9911	0.9879	0.9879

(b) Recovered inventory i_2

	Fill rate i_{is2} excluding substitution					Fill rate i_{is2} including substitution			
Problem	None	Down	Up	Two-way	Problem	None	Down	Up	Two-way
C01	0.9897	0.9897	0.9896	0.9892	C01	0.9897	0.9942	0.9896	0.9932
C02	0.9121	0.9091	0.8726	0.8645	C02	0.9121	0.9753	0.8726	0.9633
C03	0.2823	0.2821	0.2812	0.2827	C03	0.2823	0.9230	0.2812	0.9227
C04	0.3826	0.3826	0.3747	0.3747	C04	0.3826	0.3826	0.3747	0.3747
C05	0.9930	0.9857	0.9922	0.9823	C05	0.9930	0.9974	0.9922	0.9957
C06	1.0000	1.0000	1.0000	1.0000	C06	1.0000	1.0000	1.0000	1.0000
C07	0.9951	0.9944	0.9949	0.9942	C07	0.9951	0.9971	0.9949	0.9970
C08	0.9884	0.9884	0.9814	0.9814	C08	0.9884	0.9884	0.9814	0.9814
C09	0.9167	0.9021	0.8897	0.8385	C09	0.9167	0.9727	0.8897	0.9551
C10	0.9911	0.9838	0.9916	0.9707	C10	0.9911	0.9979	0.9916	0.9926
C11	0.9982	0.9976	0.9977	0.9968	C11	0.9982	0.9996	0.9977	0.9993
C12	0.9867	0.9864	0.9858	0.9855	C12	0.9867	0.9905	0.9858	0.9904
C13	0.9995	0.9995	0.9949	0.9916	C13	0.9995	1.0000	0.9949	0.9984
C14	0.9929	0.9613	0.9846	0.9642	C14	0.9929	0.9965	0.9846	0.9941
C15	0.8508	0.8835	0.8504	0.8833	C15	0.8508	0.9556	0.8504	0.9565
C16	0.0163	0.0163	0.0159	0.0159	C16	0.0163	0.0163	0.0159	0.0159
C17	0.0000	0.0000	0.0000	0.0000	C17	0.0000	0.0000	0.0000	0.0000
C18	0.4109	0.4082	0.3967	0.3902	C18	0.4109	0.7340	0.3967	0.7233
C19	0.5940	0.5940	0.2825	0.2825	C19	0.5940	0.5940	0.2825	0.2825
C20	0.9039	0.9039	0.8911	0.8911	C20	0.9039	0.9039	0.8911	0.8911

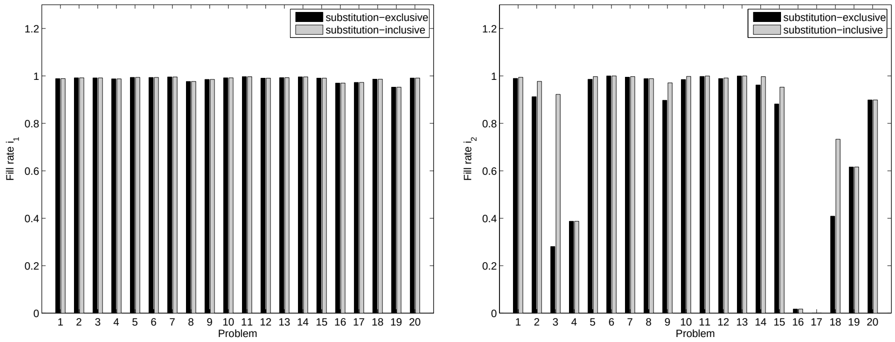
Table D.7: Summary of the optimal policy fill rates for substitution strategies for problem set D

(a) Produced inventory i_1

	Fill rate i_{is1} excluding substitution					Fill rate i_{is1} including substitution			
Problem	None	Down	Up	Two-way	Problem	None	Down	Up	Two-way
$D01$	0.9883	0.9892	0.9788	0.9789	$D01$	0.9883	0.9892	0.9928	0.9929
$D02$	0.9902	0.9914	0.9066	0.9011	$D02$	0.9902	0.9914	0.9651	0.9623
$D03$	0.9923	0.9913	0.9920	0.9912	$D03$	0.9923	0.9913	0.9930	0.9921
$D04$	0.9888	0.9888	0.9877	0.9877	$D04$	0.9888	0.9888	0.9929	0.9929
$D05$	0.9942	0.9941	0.8922	0.8884	$D05$	0.9942	0.9941	0.9993	0.9988
$D06$	0.9945	0.9945	0.9621	0.9621	$D06$	0.9945	0.9945	0.9828	0.9828
$D07$	0.9956	0.9958	0.9888	0.9889	$D07$	0.9956	0.9958	0.9943	0.9941
$D08$	0.9772	0.9772	0.2773	0.2773	$D08$	0.9772	0.9772	0.5608	0.5608
$D09$	0.9861	0.9865	0.9158	0.9055	$D09$	0.9861	0.9865	0.9772	0.9731
$D10$	0.9927	0.9927	0.7825	0.7709	$D10$	0.9927	0.9927	0.9841	0.9826
$D11$	0.9966	0.9969	0.9865	0.9854	$D11$	0.9966	0.9969	0.9989	0.9988
$D12$	0.9908	0.9908	0.9773	0.9774	$D12$	0.9908	0.9908	0.9949	0.9949
$D13$	0.9917	0.9921	0.8063	0.7874	$D13$	0.9917	0.9921	0.9273	0.9201
$D14$	0.9964	0.9948	0.6515	0.6463	$D14$	0.9964	0.9948	0.9726	0.9714
$D15$	0.9908	0.9906	0.9878	0.9858	$D15$	0.9908	0.9906	0.9935	0.9928
$D16$	0.9709	0.9709	0.9709	0.9709	$D16$	0.9709	0.9709	0.9723	0.9723
$D17$	0.9725	0.9725	0.9725	0.9725	$D17$	0.9725	0.9725	0.9725	0.9725
$D18$	0.9840	0.9850	0.9835	0.9837	$D18$	0.9840	0.9850	0.9875	0.9882
$D19$	0.9483	0.9483	0.8805	0.8805	$D19$	0.9483	0.9483	0.9733	0.9733
$D20$	0.9900	0.9900	0.9815	0.9815	$D20$	0.9900	0.9900	0.9907	0.9907

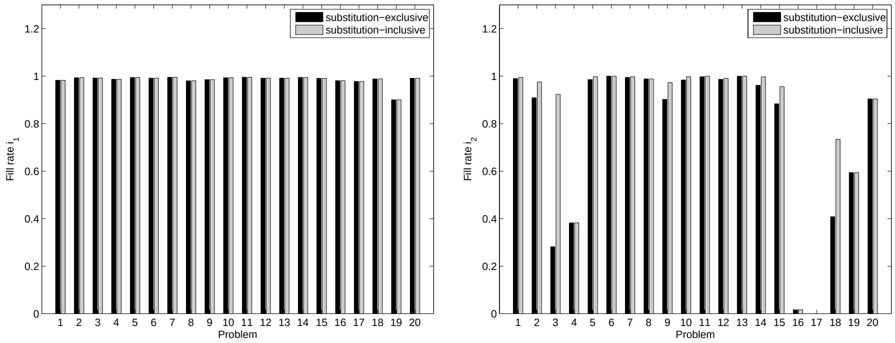
(b) Recovered inventory i_2

	Fill rate i_{is2} excluding substitution					Fill rate i_{is2} including substitution			
Problem	None	Down	Up	Two-way	Problem	None	Down	Up	Two-way
$D01$	0.9924	0.9924	0.9922	0.9920	$D01$	0.9924	0.9956	0.9922	0.9950
$D02$	0.9144	0.9145	0.8751	0.8656	$D02$	0.9144	0.9759	0.8751	0.9640
$D03$	0.2811	0.2816	0.2808	0.2812	$D03$	0.2811	0.9220	0.2808	0.9218
$D04$	0.3753	0.3753	0.3669	0.3669	$D04$	0.3753	0.3753	0.3669	0.3669
$D05$	0.9927	0.9850	0.9913	0.9825	$D05$	0.9927	0.9972	0.9913	0.9958
$D06$	1.0000	1.0000	1.0000	1.0000	$D06$	1.0000	1.0000	1.0000	1.0000
$D07$	0.9950	0.9951	0.9950	0.9952	$D07$	0.9950	0.9977	0.9950	0.9978
$D08$	0.9907	0.9907	0.9821	0.9821	$D08$	0.9907	0.9907	0.9821	0.9821
$D09$	0.9297	0.9165	0.8963	0.8410	$D09$	0.9297	0.9769	0.8963	0.9554
$D10$	0.9919	0.9793	0.9918	0.9755	$D10$	0.9919	0.9971	0.9918	0.9935
$D11$	0.9978	0.9972	0.9969	0.9964	$D11$	0.9978	0.9996	0.9969	0.9994
$D12$	0.9728	0.9727	0.9809	0.9811	$D12$	0.9728	0.9770	0.9809	0.9849
$D13$	0.9996	0.9993	0.9947	0.9922	$D13$	0.9996	1.0000	0.9947	0.9981
$D14$	0.9809	0.9663	0.9788	0.9651	$D14$	0.9809	0.9966	0.9788	0.9928
$D15$	0.8604	0.8911	0.8564	0.8917	$D15$	0.8604	0.9605	0.8564	0.9609
$D16$	0.0173	0.0173	0.0159	0.0159	$D16$	0.0173	0.0173	0.0159	0.0159
$D17$	0.0000	0.0000	0.0000	0.0000	$D17$	0.0000	0.0000	0.0000	0.0000
$D18$	0.4059	0.4100	0.3990	0.3968	$D18$	0.4059	0.7312	0.3990	0.7274
$D19$	0.5891	0.5891	0.3543	0.3543	$D19$	0.5891	0.5891	0.3543	0.3543
$D20$	0.8981	0.8981	0.8964	0.8964	$D20$	0.8981	0.8981	0.8964	0.8964



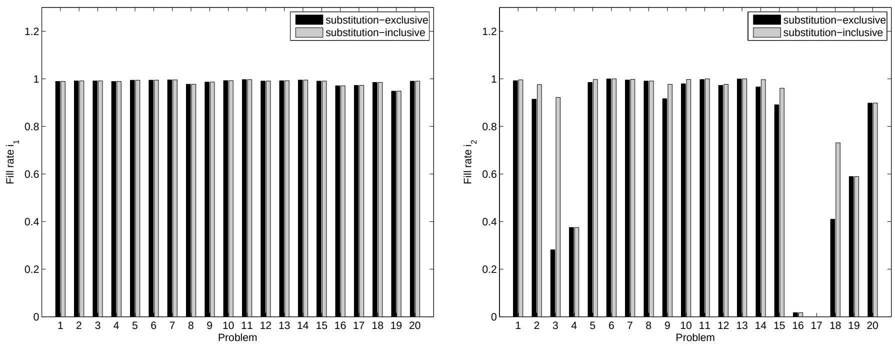
(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure D.9: Fill rates for problem set B under a downward substitution strategy



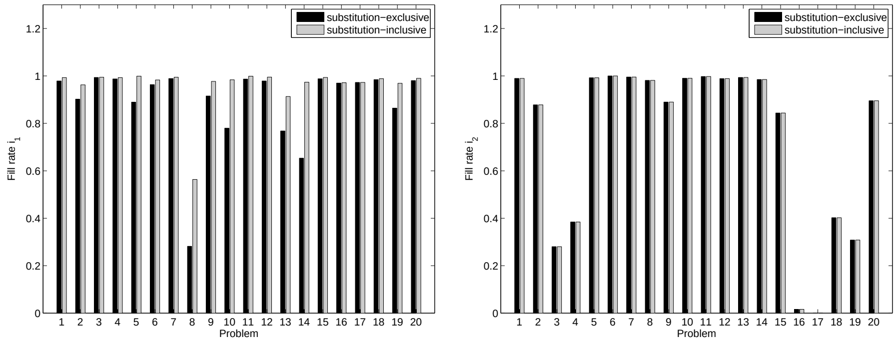
(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure D.10: Fill rates for problem set C under a downward substitution strategy



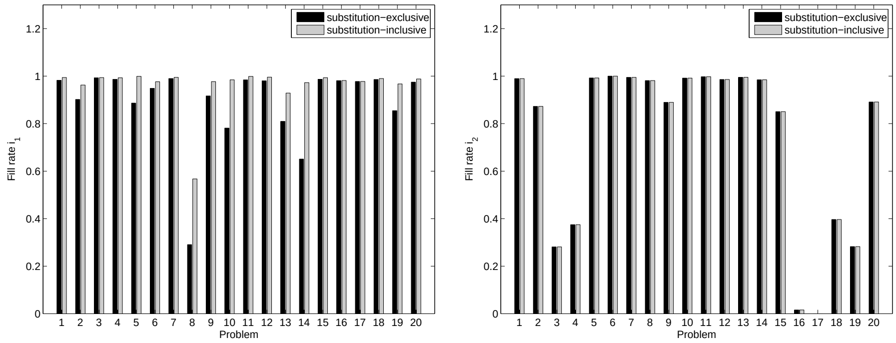
(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure D.11: Fill rates for problem set D under a downward substitution strategy



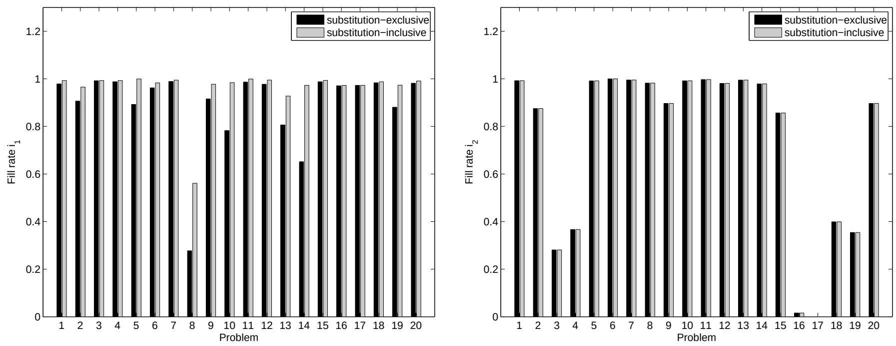
(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure D.12: Fill rates for problem set B under a upward substitution strategy



(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure D.13: Fill rates for problem set C under a upward substitution strategy



(a) Produced Inventory i_1 (b) Recovered Inventory i_2

Figure D.14: Fill rates for problem set D under a upward substitution strategy

D.3.2 Analysis of Actions

Action size and frequency

Replenishment Actions. Figure D.15 shows the size of the production and recovery actions for problem $B01$. The production variable a_p can take values of 0 and Q_p and the recovery variable can take values of 0 and Q_r. The fixed lot sizes mean that some features of the graph are not particularly interesting, however it is still useful to examine these graphs as they allow us to check that the model is performing in the expected way. Figures D.16 and D.17 show the size of the production and recovery actions for problems $C17$ and $D18$ respectively.

Tables D.8, D.9 and D.10 summarise the number of states for which each replenishment action is chosen and for each substitution strategy for datasets B, C and D respectively. The number of states in which each action can be performed in shown in Table D.11. Table D.12 shows the proportion of these states in which replenishment is performed, taking into account the number of states in which each action can be performed. The data in these Tables relate to Figures 7.10 and 7.12.

Substitution Actions. The substitution actions a_1 and a_2 can each take values of 0 or 1. Figure D.18 the frequency with which each substitution actions is chosen for problem $B01$. Figure D.19 shows this frequency for problem $C17$ and Figure D.20 shows this frequency for problem $D18$. Tables D.13, D.14 and D.15 show the number of states in which each substitution action is chosen for each of the substitution strategies. These Tables relate to Figures 7.13.

Trigger-states and actions

Replenishment. Figures D.21, D.22 and D.23 show the states in which replenishment is selected for problems $B01$, $C17$ and $D18$. These graphs relate to Figures 7.16, 7.18 and 7.17, which summarise the states in which replenishment is chosen using boxplots. In Chapter 7, the boxplots were presented for two-way and no substitution strategies. The boxplots for all substitution strategies are presented for Problem set B in Figures D.24, D.25 and D.26, Problem set C in Figures D.27, D.28 and D.29, and Problem set D in Figures D.30, D.31 and D.32.

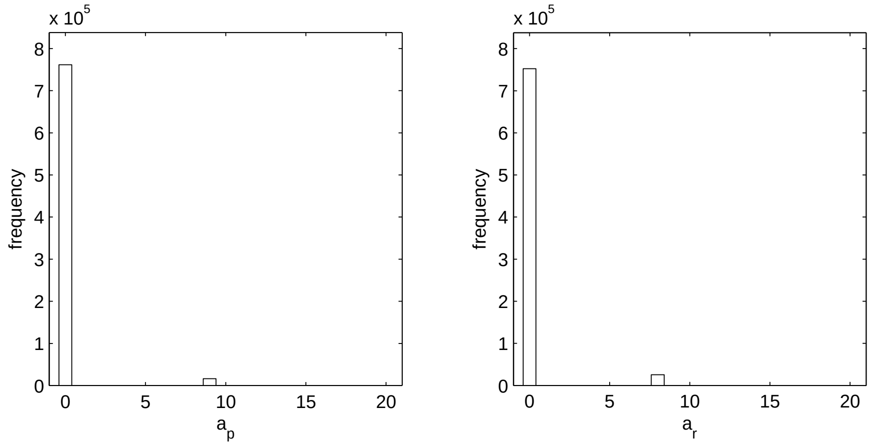


Figure D.15: Histograms showing the size of the replenishment actions under the optimal policy for test problem $B01$ under a two-way substitution strategy

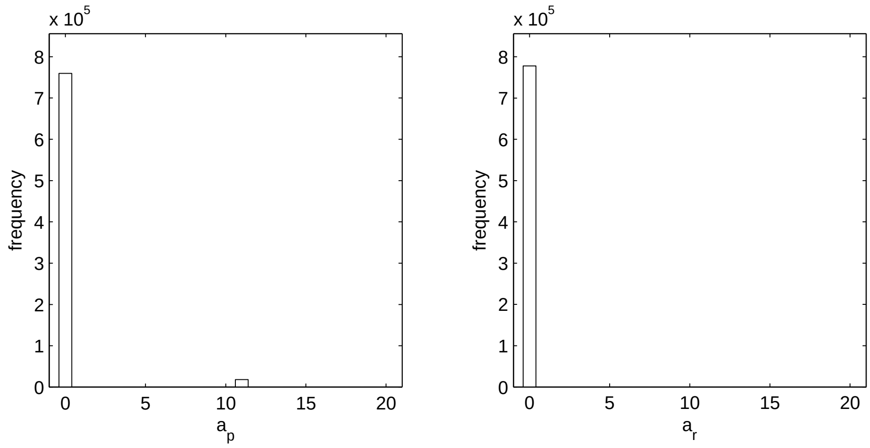


Figure D.16: Histograms showing the size of the replenishment actions under the optimal policy for test problem $C17$ under a two-way substitution strategy

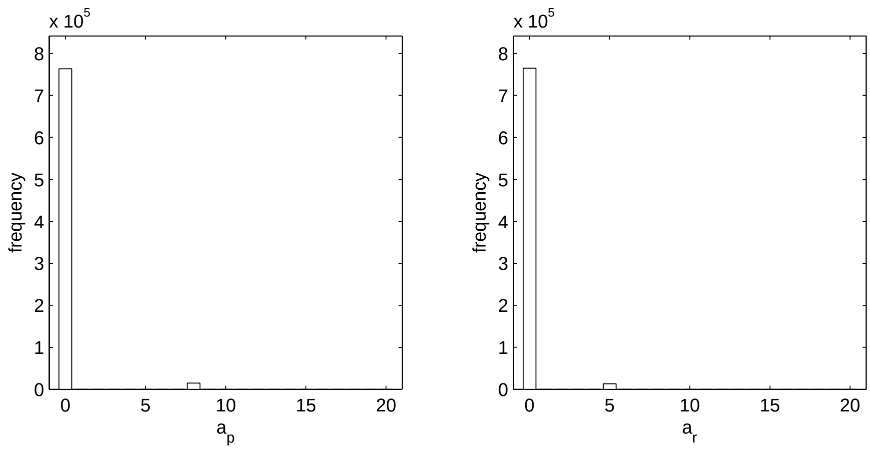


Figure D.17: Histograms showing the size of the replenishment actions under the optimal policy for test problem $D18$ under a two-way substitution strategy

Table D.8: Number of states in which each replenishment action is chosen for each substitution strategy for dataset B

	None		Down		Up		Two-way	
Problem	a_p	a_r	a_p	a_r	a_p	a_r	a_p	a_r
$B01$	33360	26278	33504	25657	16093	26304	16187	25560
$B02$	137313	20307	137712	18129	126796	17794	126923	14961
$B03$	102212	57941	105931	54091	101201	57946	102950	54345
$B04$	63300	4426	63300	4426	59596	4466	59596	4466
$B05$	42833	53241	44589	41678	7752	61398	7242	47174
$B06$	44053	52628	44261	50889	24229	50611	24413	48818
$B07$	124451	38064	125437	33973	118073	38695	119131	34566
$B08$	19234	26176	19234	25592	12378	28341	12378	26938
$B09$	31765	25706	31948	15824	5103	32122	4079	17451
$B10$	121397	35450	123268	22194	96602	46908	108695	23067
$B11$	83289	105674	84818	97598	74698	104390	76423	95796
$B12$	75194	36374	75199	35040	48711	36823	48724	35315
$B13$	82586	109096	83766	101239	72226	93247	73772	81840
$B14$	33787	30772	33970	19045	1838	36150	1422	16245
$B15$	52541	15607	56091	13353	50646	15542	54197	13305
$B16$	132503	6519	132503	6510	131510	6449	131510	6440
$B17$	24780	0	24780	0	18081	0	18081	0
$B18$	18016	14980	19200	11264	14961	13214	15434	8493
$B19$	14514	11176	14514	11169	1787	8821	1787	8744
$B20$	18665	25337	18665	23892	11209	21095	11209	19514

Table D.9: Number of states in which each replenishment action is chosen for each substitution strategy for dataset C

	None		Down		Up		Two-way	
Problem	a_p	a_r	a_p	a_r	a_p	a_r	a_p	a_r
$C01$	29691	26786	29908	25754	14239	27058	14316	26067
$C02$	100829	23714	101232	21179	88682	20879	88805	17592
$C03$	60902	71689	66708	65990	58816	71628	63006	66028
$C04$	55422	4299	55422	4299	50794	4331	50794	4331
$C05$	37526	59892	39083	49371	4939	71164	4688	58335
$C06$	33641	57534	33881	55545	15036	56682	15225	54771
$C07$	80086	40772	80939	36746	72576	41444	73435	37319
$C08$	17114	26794	17113	26343	11363	29734	11363	28528
$C09$	28529	27939	28730	18200	3979	35113	3241	20270
$C10$	75400	40078	76436	27999	55964	53490	58715	30873
$C11$	46794	122390	47354	115994	37482	120037	37912	112947
$C12$	59415	40223	59491	39063	38003	40564	38018	39276
$C13$	45743	129114	46392	122845	33805	111167	34351	101585
$C14$	32166	31133	32412	20430	1371	38378	1046	19097
$C15$	48568	19905	50259	16677	45582	19887	47131	16516
$C16$	95237	10611	95237	10611	94574	10126	94574	10126
$C17$	21462	0	21462	0	18144	0	18144	0
$C18$	15003	18229	16215	13498	12814	16274	13027	10467
$C19$	13323	14282	13323	14282	1516	9838	1516	9733
$C20$	15965	27282	15965	26123	9613	22358	9613	21035

Table D.10: Number of states in which each replenishment action is chosen for each substitution strategy for dataset D

	None		Down		Up		Two-way	
Problem	a_p	a_r	a_p	a_r	a_p	a_r	a_p	a_r
$D01$	35006	21678	35254	21198	18215	21562	18333	21189
$D02$	137021	17684	137426	15760	126675	13954	126727	11480
$D03$	106539	38401	113142	35499	105008	38390	110394	35639
$D04$	69966	2314	69966	2314	66785	2363	66785	2363
$D05$	44011	44036	46330	33730	7494	50225	6853	37631
$D06$	42191	49961	42404	48419	22777	47291	22883	45963
$D07$	125724	24960	126489	21998	119396	25381	120232	22438
$D08$	19875	21482	19875	20714	13153	22591	13153	20624
$D09$	31986	24041	32231	15354	4909	31909	4037	17341
$D10$	122090	28618	123595	16893	95943	38438	109182	17718
$D11$	85072	88232	86510	81146	76127	87124	77909	79672
$D12$	75057	30715	75081	29476	49588	31065	49608	29736
$D13$	84358	87927	85427	80629	73182	75385	74617	64958
$D14$	34087	26962	34228	17339	1761	27461	1456	14504
$D15$	55320	12821	60757	10807	52812	12847	58276	10789
$D16$	132227	4118	132227	4091	131430	4097	131430	4070
$D17$	27174	0	27174	0	20580	0	20580	0
$D18$	16018	23626	17632	17041	13882	20758	14828	13074
$D19$	15252	9883	15252	9806	2000	7912	2000	7856
$D20$	18195	23786	18196	22561	12005	20028	12007	18702

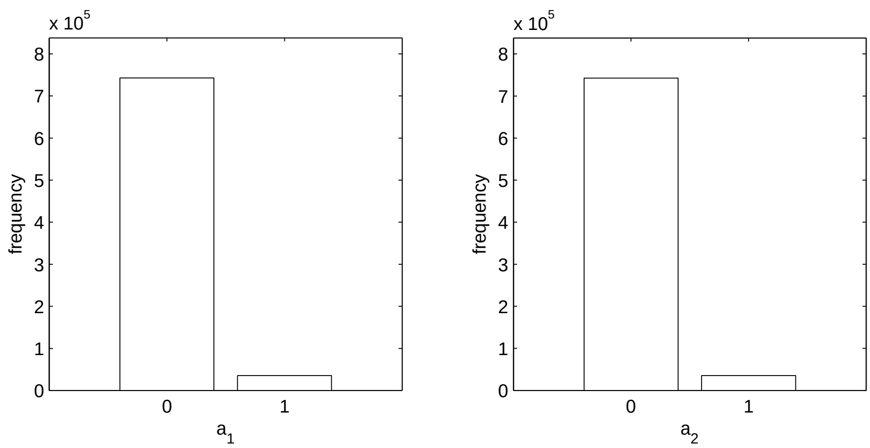


Figure D.18: Histograms of the substitution actions under the optimal policy for test problem $B01$

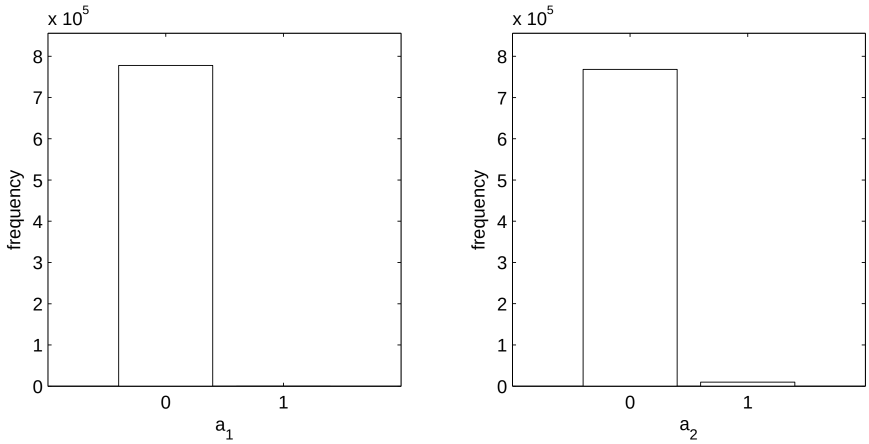


Figure D.19: Histograms of the substitution actions under the optimal policy for test problem $C17$ under a downward substitution strategy

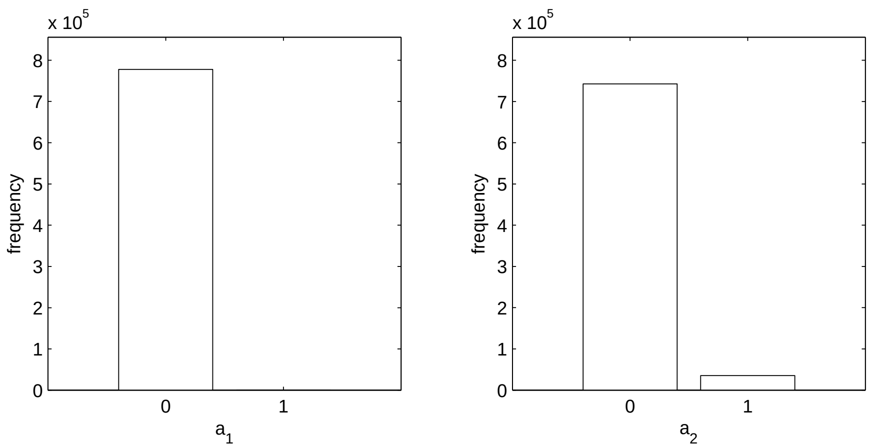


Figure D.20: Histograms of the substitution actions under the optimal policy for test problem $D18$ under an upward substitution strategy

Table D.11: Number of states in which replenishment can be chosen under a two-way substitution strategy

B			C			D		
Problem	a_p	a_r	Problem	a_p	a_r	Problem	a_p	a_r
$B01$	388962	240786	$C01$	388962	240786	$D01$	388962	203742
$B02$	388962	351918	$C02$	388962	351918	$D02$	388962	333396
$B03$	388962	185220	$C03$	388962	185220	$D03$	388962	129654
$B04$	388962	240786	$C04$	388962	240786	$D04$	388962	203742
$B05$	388962	222264	$C05$	388962	222264	$D05$	388962	185220
$B06$	388962	314874	$C06$	388962	314874	$D06$	388962	296352
$B07$	388962	203742	$C07$	388962	203742	$D07$	388962	148176
$B08$	388962	277830	$C08$	388962	277830	$D08$	388962	240786
$B09$	388962	351918	$C09$	388962	351918	$D09$	388962	333396
$B10$	388962	277830	$C10$	388962	277830	$D10$	388962	240786
$B11$	388962	222264	$C11$	388962	222264	$D11$	388962	185220
$B12$	388962	314874	$C12$	388962	314874	$D12$	388962	296352
$B13$	388962	240786	$C13$	388962	240786	$D13$	388962	203742
$B14$	388962	314874	$C14$	388962	314874	$D14$	388962	296352
$B15$	388962	277830	$C15$	388962	277830	$D15$	388962	240786
$B16$	388962	222264	$C16$	388962	222264	$D16$	388962	185220
$B17$	388962	388962	$C17$	388962	388962	$D17$	388962	388962
$B18$	388962	314874	$C18$	388962	314874	$D18$	388962	296352
$B19$	388962	314874	$C19$	388962	314874	$D19$	388962	296352
$B20$	388962	314874	$C20$	388962	314874	$D20$	388962	296352

Table D.12: Proportion of states in which replenishment is chosen under a two-way substitution strategy

B			C			D		
Problem	a_p	a_r	Problem	a_p	a_r	Problem	a_p	a_r
$B01$	0.0416	0.1062	$C01$	0.0368	0.1083	$D01$	0.0471	0.1040
$B02$	0.3263	0.0425	$C02$	0.2283	0.0500	$D02$	0.3258	0.0344
$B03$	0.2647	0.2934	$C03$	0.1620	0.3565	$D03$	0.2838	0.2749
$B04$	0.1532	0.0185	$C04$	0.1306	0.0180	$D04$	0.1717	0.0116
$B05$	0.0186	0.2122	$C05$	0.0121	0.2625	$D05$	0.0176	0.2032
$B06$	0.0628	0.1550	$C06$	0.0391	0.1739	$D06$	0.0588	0.1551
$B07$	0.3063	0.1697	$C07$	0.1888	0.1832	$D07$	0.3091	0.1514
$B08$	0.0318	0.0970	$C08$	0.0292	0.1027	$D08$	0.0338	0.0857
$B09$	0.0105	0.0496	$C09$	0.0083	0.0576	$D09$	0.0104	0.0520
$B10$	0.2794	0.0830	$C10$	0.1510	0.1111	$D10$	0.2807	0.0736
$B11$	0.1965	0.4310	$C11$	0.0975	0.5082	$D11$	0.2003	0.4301
$B12$	0.1253	0.1122	$C12$	0.0977	0.1247	$D12$	0.1275	0.1003
$B13$	0.1897	0.3399	$C13$	0.0883	0.4219	$D13$	0.1918	0.3188
$B14$	0.0037	0.0516	$C14$	0.0027	0.0606	$D14$	0.0037	0.0489
$B15$	0.1393	0.0479	$C15$	0.1212	0.0594	$D15$	0.1498	0.0448
$B16$	0.3381	0.0290	$C16$	0.2431	0.0456	$D16$	0.3379	0.0220
$B17$	0.0465	0.0000	$C17$	0.0466	0.0000	$D17$	0.0529	0.0000
$B18$	0.0397	0.0270	$C18$	0.0335	0.0332	$D18$	0.0381	0.0441
$B19$	0.0046	0.0278	$C19$	0.0039	0.0309	$D19$	0.0051	0.0265
$B20$	0.0288	0.0620	$C20$	0.0247	0.0668	$D20$	0.0309	0.0631

Table D.13: Number of states in which each substitution action is chosen for each substitution strategy for dataset B

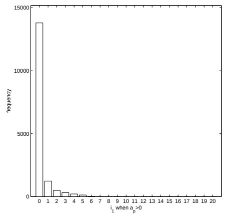
Problem	None		Down		Up		Two-way	
	a_{s1}	a_{s2}	a_{s1}	a_{s2}	a_{s1}	a_{s2}	a_{s1}	a_{s2}
$B01$	0	0	0	35280	35280	0	35280	35246
$B02$	0	0	0	34932	35280	0	35280	34961
$B03$	0	0	0	35280	35280	0	35280	35280
$B04$	0	0	0	9356	35280	0	35280	9366
$B05$	0	0	0	35280	35280	0	35280	35280
$B06$	0	0	0	35280	35280	0	35280	35280
$B07$	0	0	0	35280	35280	0	35280	35280
$B08$	0	0	0	13468	35280	0	35280	21623
$B09$	0	0	0	35280	35280	0	35280	35280
$B10$	0	0	0	35280	35280	0	35280	35280
$B11$	0	0	0	33168	35280	0	35280	33563
$B12$	0	0	0	28370	35280	0	35280	29526
$B13$	0	0	0	35280	35280	0	35280	35280
$B14$	0	0	0	35280	35280	0	35280	35280
$B15$	0	0	0	32669	35280	0	35280	32709
$B16$	0	0	0	8043	35280	0	35280	8043
$B17$	0	0	0	7938	35280	0	35280	7938
$B18$	0	0	0	35163	35280	0	35280	35162
$B19$	0	0	0	7938	35280	0	35280	7938
$B20$	0	0	0	24748	35280	0	35280	24951

Table D.14: Number of states in which each substitution action is chosen for each substitution strategy for dataset C

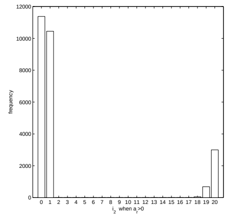
Problem	None		Down		Up		Two-way	
	a_{s1}	a_{s2}	a_{s1}	a_{s2}	a_{s1}	a_{s2}	a_{s1}	a_{s2}
$C01$	0	0	0	35280	35280	0	35280	35124
$C02$	0	0	0	34764	35280	0	35280	34812
$C03$	0	0	0	35280	35280	0	35280	35280
$C04$	0	0	0	10689	35280	0	35280	10737
$C05$	0	0	0	35280	35280	0	35280	35280
$C06$	0	0	0	35280	35280	0	35280	35280
$C07$	0	0	0	35280	35280	0	35280	35280
$C08$	0	0	0	14183	35280	0	35280	22489
$C09$	0	0	0	35280	35280	0	35280	35280
$C10$	0	0	0	35280	35280	0	35280	35278
$C11$	0	0	0	32980	35280	0	35280	33325
$C12$	0	0	0	27724	35280	0	35280	29014
$C13$	0	0	0	35280	35280	0	35280	35280
$C14$	0	0	0	35280	35280	0	35280	35280
$C15$	0	0	0	32508	35280	0	35280	32561
$C16$	0	0	0	9951	35280	0	35280	9951
$C17$	0	0	0	9702	35280	0	35280	9702
$C18$	0	0	0	35059	35280	0	35280	35074
$C19$	0	0	0	9702	35280	0	35280	9702
$C20$	0	0	0	24489	35280	0	35280	24683

Table D.15: Number of states in which each substitution action is chosen for each substitution strategy for dataset D

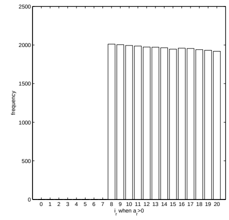
	None		Down		Up		Two-way	
Problem	a_{s1}	a_{s2}	a_{s1}	a_{s2}	a_{s1}	a_{s2}	a_{s1}	a_{s2}
D01	0	0	0	35280	35280	0	35280	35184
D02	0	0	0	34951	35280	0	35280	35028
D03	0	0	0	35280	35280	0	35280	35280
D04	0	0	0	9539	35280	0	35280	9531
D05	0	0	0	35280	35280	0	35280	35280
D06	0	0	0	35280	35280	0	35280	35280
D07	0	0	0	35280	35280	0	35280	35280
D08	0	0	0	13482	35280	0	35280	22220
D09	0	0	0	35280	35280	0	35280	35280
D10	0	0	0	35280	35280	0	35280	35280
D11	0	0	0	33214	35280	0	35280	33546
D12	0	0	0	28486	35280	0	35280	29553
D13	0	0	0	35280	35280	0	35280	35280
D14	0	0	0	35280	35280	0	35280	35280
D15	0	0	0	32807	35280	0	35280	32838
D16	0	0	0	8085	35280	0	35280	8085
D17	0	0	0	7938	35280	0	35280	7938
D18	0	0	0	35122	35280	0	35280	35124
D19	0	0	0	7938	35280	0	35280	7938
D20	0	0	0	24795	35280	0	35280	24959



(a) Produced inventory i_1 when production is selected $a_p > 0$

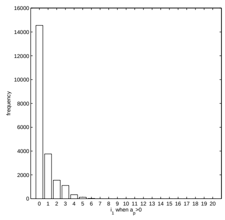


(b) Recovered inventory i_2 when recovery is selected $a_r > 0$

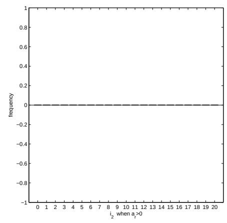


(c) Returned inventory i_r when recovery is selected $a_r > 0$

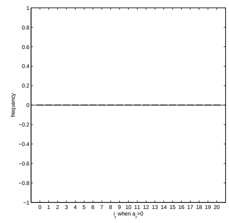
Figure D.21: Histograms showing states associated with positive replenishment actions under the optimal policy for test problem $B01$ under a two-way substitutions strategy



(a) Produced inventory i_1 when production is selected $a_p > 0$

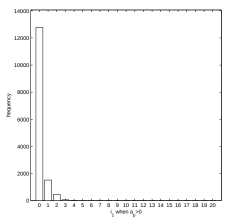


(b) Recovered inventory i_2 when recovery is selected $a_r > 0$

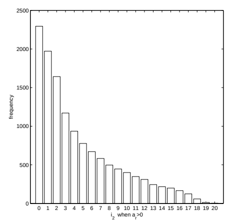


(c) Returned inventory i_r when recovery is selected $a_r > 0$

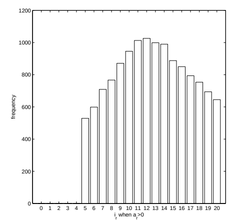
Figure D.22: Histograms showing states associated with positive replenishment actions under the optimal policy for test problem $C17$ under a downward substitution strategy



(a) Produced inventory i_1 when production is selected $a_p > 0$

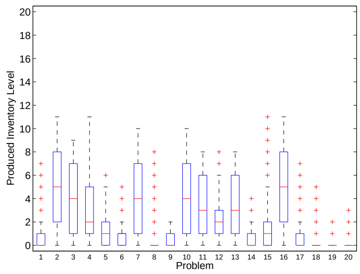


(b) Recovered inventory i_2 when recovery is selected $a_r > 0$

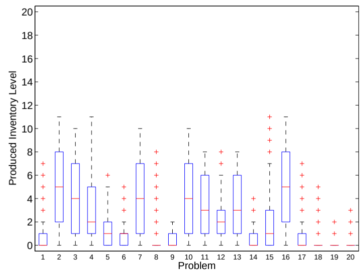


(c) Returned inventory i_r when recovery is selected $a_r > 0$

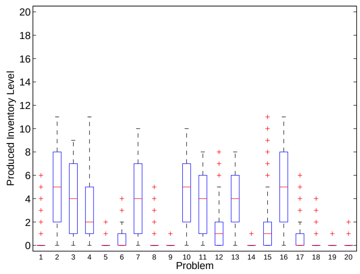
Figure D.23: Histograms showing states associated with positive replenishment actions under the optimal policy for test problem $D18$ under an upward substitution strategy



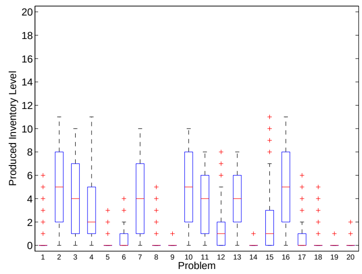
(a) No substitution



(b) Downward substitution

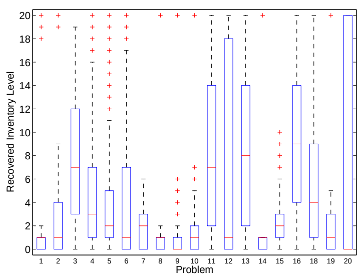


(c) Upward substitution

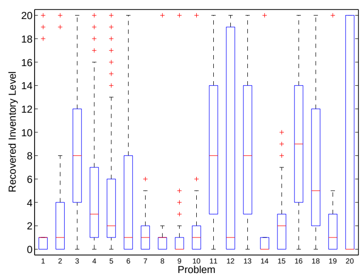


(d) Two-way substitution

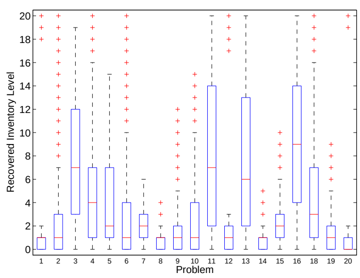
Figure D.24: Graphs showing the level of produced inventory (trigger level) when production is performed for problem set B.



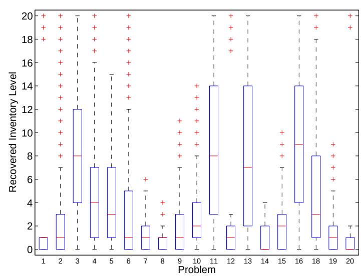
(a) No substitution



(b) Downward substitution

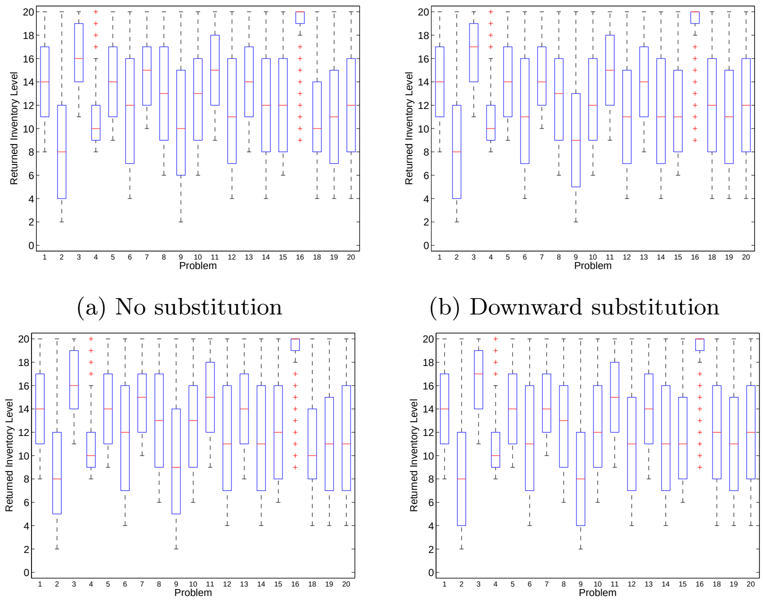


(c) Upward substitution



(d) Two-way substitution

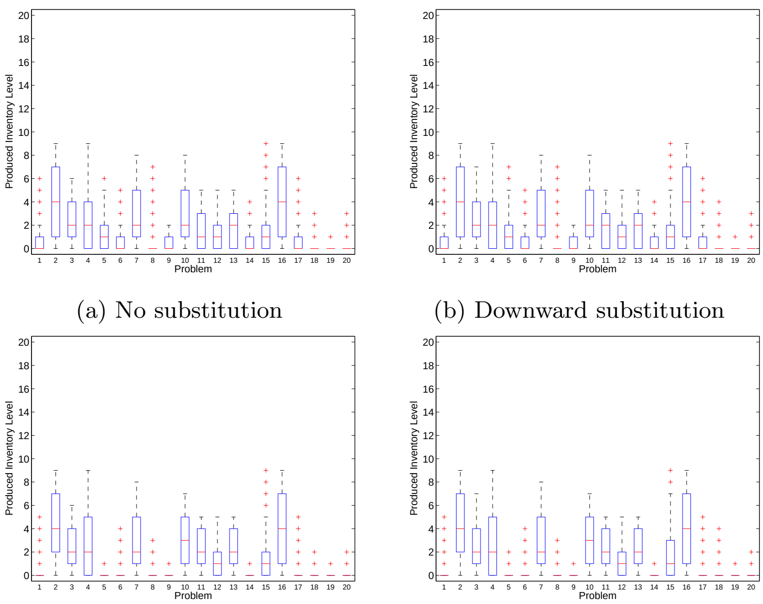
Figure D.25: Graphs showing the level of recovered inventory (trigger level) when recovery is performed for problem set B.



(a) No substitution (b) Downward substitution

(c) Upward substitution (d) Two-way substitution

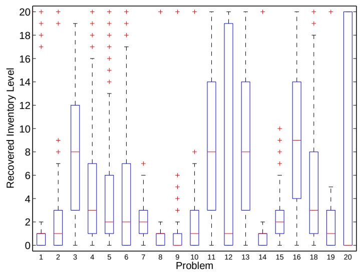
Figure D.26: Graphs showing the level of returned inventory (trigger level) when recovery is performed for problem set B.



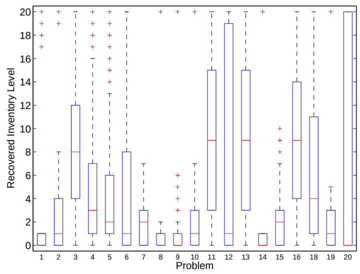
(a) No substitution (b) Downward substitution

(c) Upward substitution (d) Two-way substitution

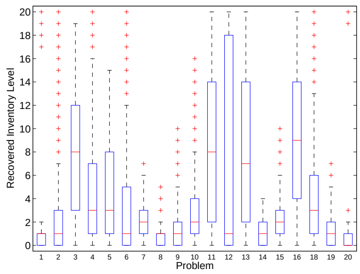
Figure D.27: Graphs showing the level of produced inventory (trigger level) when production is performed for problem set C.



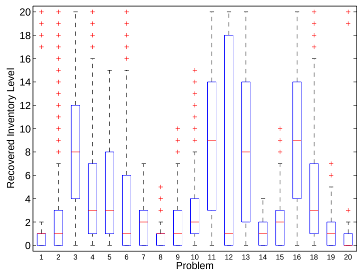
(a) No substitution



(b) Downward substitution

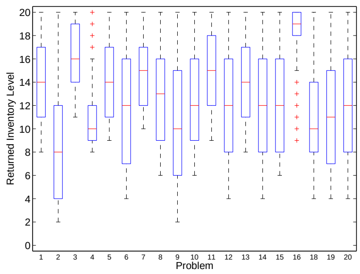


(c) Upward substitution

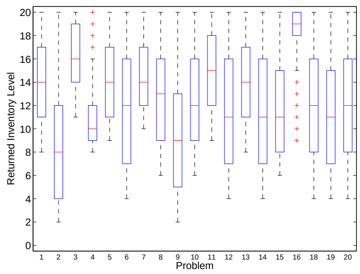


(d) Two-way substitution

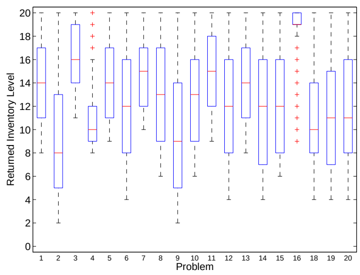
Figure D.28: Graphs showing the level of recovered inventory (trigger level) when recovery is performed for problem set C.



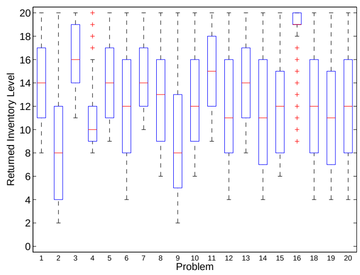
(a) No substitution



(b) Downward substitution

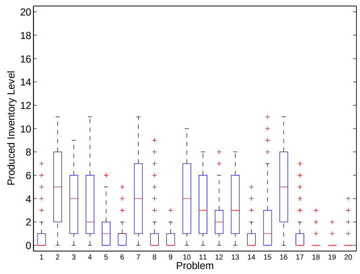


(c) Upward substitution

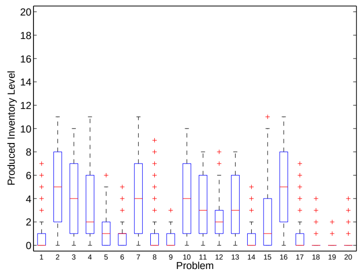


(d) Two-way substitution

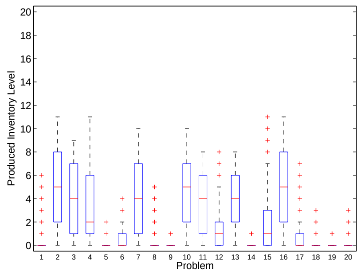
Figure D.29: Graphs showing the level of returned inventory (trigger level) when recovery is performed for problem set C.



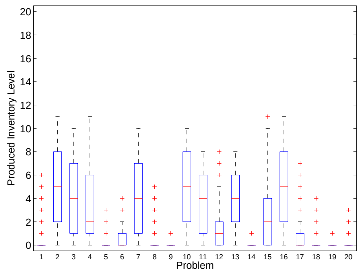
(a) No substitution



(b) Downward substitution

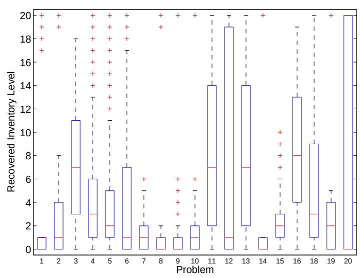


(c) Upward substitution

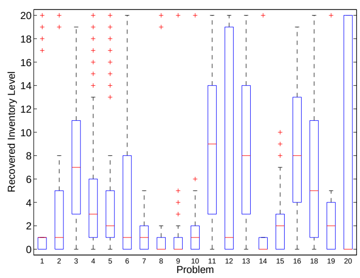


(d) Two-way substitution

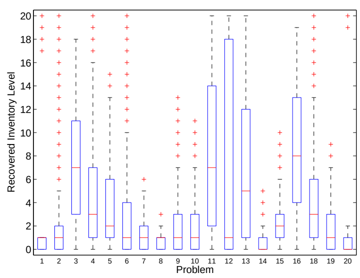
Figure D.30: Graphs showing the level of produced inventory (trigger level) when production is performed for problem set D.



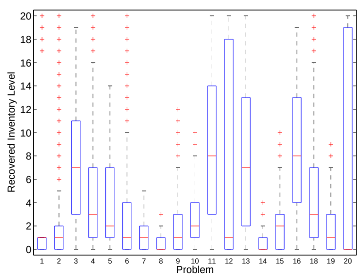
(a) No substitution



(b) Downward substitution

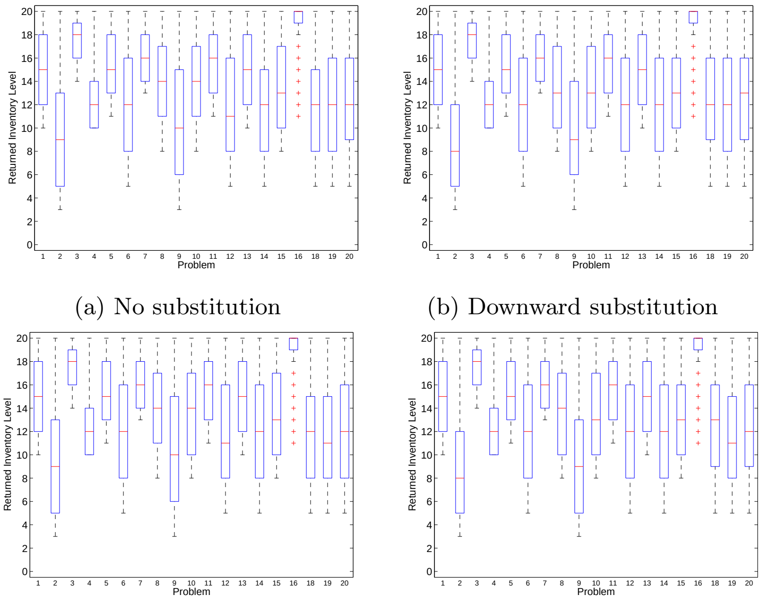


(c) Upward substitution



(d) Two-way substitution

Figure D.31: Graphs showing the level of recovered inventory (trigger level) when recovery is performed for problem set D.



(a) No substitution (b) Downward substitution

(c) Upward substitution (d) Two-way substitution

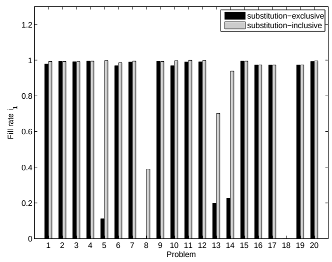
Figure D.32: Graphs showing the level of returned inventory (trigger level) when recovery is performed for problem set D.

D.3.3 Analysis of Recovery Strategy

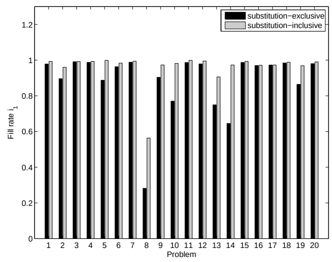
In this section additional figures and tables relating to Section 7.6.4, the analysis of the recovery strategy, are presented.

Fill Rates

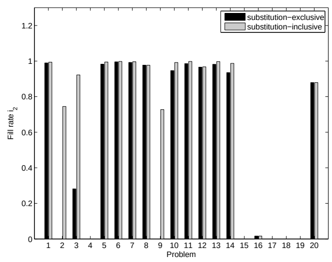
Effect of Recovery Strategy on Fill Rates. Figures D.33, D.34 and D.35 show the substitution inclusive and exclusive fill rates for the two recovery strategies for problems B, C and D respectively. Tables D.16 and D.17 show the substitution inclusive and exclusive fill rates for produced and recovered inventory respectively.



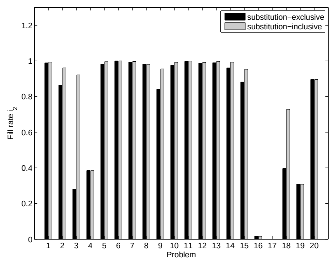
(a) Produced Inventory i_1 - High Quality Recovery



(b) Produced Inventory i_1 - High & Low Quality Recovery

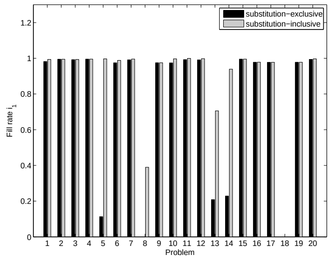


(c) Recovered Inventory i_2 - High Quality Recovery

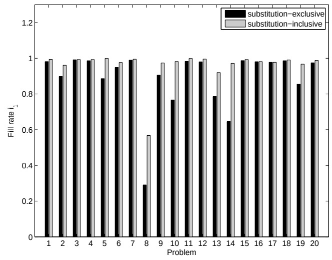


(d) Recovered Inventory i_2 - High & Low Quality Recovery

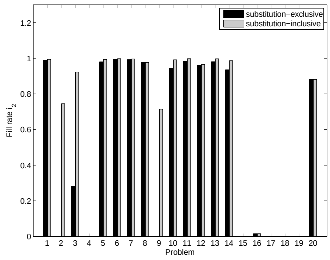
Figure D.33: Fill rates for problem set B under a two-way substitution strategy for both quality strategies



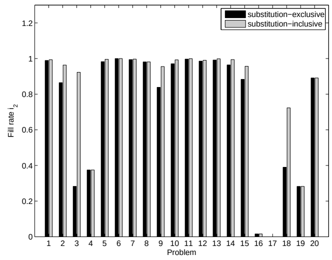
(a) Produced Inventory i_1 - High Quality Recovery



(b) Produced Inventory i_1 - High & Low Quality Recovery

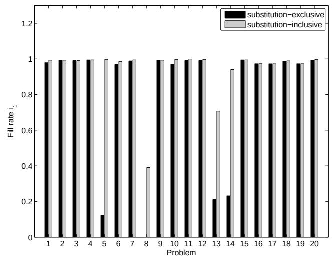


(c) Recovered Inventory i_2 - High Quality Recovery

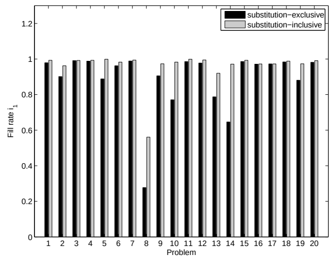


(d) Recovered Inventory i_2 - High & Low Quality Recovery

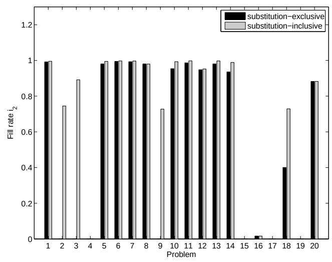
Figure D.34: Fill rates for problem set C under a two-way substitution strategy for both quality strategies



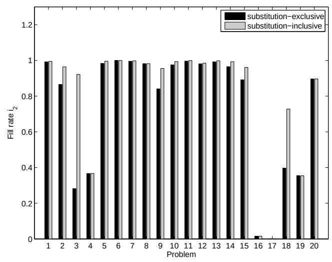
(a) Produced Inventory i_1 - High Quality Recovery



(b) Produced Inventory i_1 - High & Low Quality Recovery



(c) Recovered Inventory i_2 - High Quality Recovery



(d) Recovered Inventory i_2 - High & Low Quality Recovery

Figure D.35: Fill rates for problem set D under a two-way substitution strategy for both quality strategies

Table D.16: Summary of the optimal policy produced inventory fill rates under a two-way substitution strategies under both quality strategies

(a) Problem set B

	Fill rate i_1 excluding substitution		Fill rate i_1 including substitution	
Problem	Only High	Low and High	Only High	Low and High
$B01$	0.9783	0.9783	0.9928	0.9928
$B02$	0.9930	0.8961	0.9930	0.9600
$B03$	0.9910	0.9912	0.9920	0.9922
$B04$	0.9944	0.9874	0.9944	0.9929
$B05$	0.1111	0.8874	0.9976	0.9989
$B06$	0.9688	0.9631	0.9860	0.9832
$B07$	0.9894	0.9882	0.9945	0.9938
$B08$	0.0000	0.2820	0.3896	0.5635
$B09$	0.9928	0.9038	0.9928	0.9729
$B10$	0.9690	0.7703	0.9970	0.9819
$B11$	0.9903	0.9869	0.9991	0.9990
$B12$	0.9908	0.9786	0.9978	0.9951
$B13$	0.1990	0.7496	0.7021	0.9058
$B14$	0.2269	0.6454	0.9384	0.9726
$B15$	0.9946	0.9872	0.9946	0.9929
$B16$	0.9729	0.9700	0.9730	0.9713
$B17$	0.9725	0.9725	0.9725	0.9725
$B18$	0.0000	0.9840	0.0000	0.9884
$B19$	0.9730	0.8643	0.9730	0.9690
$B20$	0.9923	0.9803	0.9958	0.9900

(b) Problem set C

	Fill rate i_1 excluding substitution		Fill rate i_1 including substitution	
Problem	Only High	Low and High	Only High	Low and High
$C01$	0.9818	0.9816	0.9936	0.9938
$C02$	0.9947	0.8981	0.9947	0.9610
$C03$	0.9922	0.9917	0.9933	0.9928
$C04$	0.9953	0.9864	0.9953	0.9928
$C05$	0.1124	0.8862	0.9968	0.9989
$C06$	0.9749	0.9486	0.9884	0.9764
$C07$	0.9914	0.9896	0.9957	0.9947
$C08$	0.0000	0.2908	0.3896	0.5673
$C09$	0.9751	0.9054	0.9751	0.9737
$C10$	0.9740	0.7668	0.9969	0.9819
$C11$	0.9925	0.9828	0.9993	0.9986
$C12$	0.9912	0.9804	0.9978	0.9955
$C13$	0.2082	0.7858	0.7056	0.9200
$C14$	0.2288	0.6455	0.9392	0.9717
$C15$	0.9953	0.9870	0.9953	0.9929
$C16$	0.9779	0.9810	0.9780	0.9814
$C17$	0.9776	0.9776	0.9776	0.9776
$C18$	0.0000	0.9865	0.0000	0.9904
$C19$	0.9779	0.8544	0.9779	0.9670
$C20$	0.9937	0.9747	0.9968	0.9879

(c) Problem set D

	Fill rate i_1 excluding substitution		Fill rate i_1 including substitution	
Problem	Only High	Low and High	Only High	Low and High
$D01$	0.9789	0.9789	0.9929	0.9929
$D02$	0.9930	0.9011	0.9930	0.9623
$D03$	0.9906	0.9912	0.9906	0.9921
$D04$	0.9944	0.9877	0.9944	0.9929
$D05$	0.1216	0.8884	0.9974	0.9988
$D06$	0.9688	0.9621	0.9857	0.9828
$D07$	0.9891	0.9889	0.9945	0.9941
$D08$	0.0000	0.2773	0.3906	0.5608
$D09$	0.9929	0.9055	0.9929	0.9731
$D10$	0.9693	0.7709	0.9972	0.9826
$D11$	0.9908	0.9854	0.9992	0.9988
$D12$	0.9908	0.9774	0.9973	0.9949
$D13$	0.2106	0.7874	0.7069	0.9201
$D14$	0.2318	0.6463	0.9405	0.9714
$D15$	0.9946	0.9858	0.9946	0.9928
$D16$	0.9729	0.9709	0.9732	0.9723
$D17$	0.9725	0.9725	0.9725	0.9725
$D18$	0.9858	0.9837	0.9892	0.9882
$D19$	0.9730	0.8805	0.9730	0.9733
$D20$	0.9923	0.9815	0.9959	0.9907

Table D.17: Summary of the optimal policy recovered inventory fill rates under a two-way substitution strategies under both quality strategies

(a) Problem set B

	Fill rate i_2 excluding substitution		Fill rate i_2 including substitution	
Problem	Only High	Low and High	Only High	Low and High
$B01$	0.9895	0.9893	0.9939	0.9937
$B02$	0.0000	0.8638	0.7446	0.9609
$B03$	0.2815	0.2806	0.9220	0.9217
$B04$	0.0000	0.3842	0.0000	0.3842
$B05$	0.9824	0.9823	0.9948	0.9957
$B06$	0.9958	1.0000	0.9980	1.0000
$B07$	0.9924	0.9939	0.9963	0.9970
$B08$	0.9767	0.9812	0.9767	0.9812
$B09$	0.0000	0.8402	0.7272	0.9552
$B10$	0.9462	0.9747	0.9921	0.9926
$B11$	0.9852	0.9969	0.9977	0.9993
$B12$	0.9654	0.9884	0.9676	0.9921
$B13$	0.9818	0.9898	0.9973	0.9976
$B14$	0.9344	0.9609	0.9871	0.9931
$B15$	0.0000	0.8819	0.0000	0.9537
$B16$	0.0160	0.0161	0.0160	0.0161
$B17$	0.0000	0.0000	0.0000	0.0000
$B18$	0.0000	0.3956	0.0000	0.7288
$B19$	0.0000	0.3082	0.0000	0.3082
$B20$	0.8792	0.8953	0.8792	0.8953

(b) Problem set C

	Fill rate i_2 excluding substitution		Fill rate i_2 including substitution	
Problem	Only High	Low and High	Only High	Low and High
$C01$	0.9892	0.9892	0.9939	0.9932
$C02$	0.0000	0.8645	0.7453	0.9633
$C03$	0.2818	0.2827	0.9227	0.9227
$C04$	0.0000	0.3747	0.0000	0.3747
$C05$	0.9803	0.9823	0.9934	0.9957
$C06$	0.9957	1.0000	0.9979	1.0000
$C07$	0.9929	0.9942	0.9964	0.9970
$C08$	0.9767	0.9814	0.9767	0.9814
$C09$	0.0000	0.8385	0.7147	0.9551
$C10$	0.9429	0.9707	0.9916	0.9926
$C11$	0.9845	0.9968	0.9977	0.9993
$C12$	0.9608	0.9855	0.9650	0.9904
$C13$	0.9809	0.9916	0.9971	0.9984
$C14$	0.9358	0.9642	0.9866	0.9941
$C15$	0.0000	0.8833	0.0000	0.9565
$C16$	0.0160	0.0159	0.0160	0.0159
$C17$	0.0000	0.0000	0.0000	0.0000
$C18$	0.0000	0.3902	0.0000	0.7233
$C19$	0.0000	0.2825	0.0000	0.2825
$C20$	0.8814	0.8911	0.8814	0.8911

(c) Problem set D

	Fill rate i_2 excluding substitution		Fill rate i_2 including substitution	
Problem	Only High	Low and High	Only High	Low and High
$D01$	0.9921	0.9920	0.9955	0.9950
$D02$	0.0000	0.8656	0.7446	0.9640
$D03$	0.0000	0.2812	0.8918	0.9218
$D04$	0.0000	0.3669	0.0000	0.3669
$D05$	0.9801	0.9825	0.9943	0.9958
$D06$	0.9946	1.0000	0.9971	1.0000
$D07$	0.9930	0.9952	0.9967	0.9978
$D08$	0.9798	0.9821	0.9798	0.9821
$D09$	0.0002	0.8410	0.7273	0.9554
$D10$	0.9540	0.9755	0.9932	0.9935
$D11$	0.9866	0.9964	0.9980	0.9994
$D12$	0.9479	0.9811	0.9521	0.9849
$D13$	0.9802	0.9922	0.9971	0.9981
$D14$	0.9355	0.9651	0.9887	0.9928
$D15$	0.0000	0.8917	0.0000	0.9609
$D16$	0.0163	0.0159	0.0163	0.0159
$D17$	0.0000	0.0000	0.0000	0.0000
$D18$	0.4005	0.3968	0.7290	0.7274
$D19$	0.0000	0.3543	0.0000	0.3543
$D20$	0.8826	0.8964	0.8826	0.8964