Function Theoretic Methods in Partial Differential Equations

by

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ABSTRACT OF THESIS

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ABSTRACT

Part I of my thesis gives a brief survey of the area of function theoretic methods in partial differential equations. My principal contribution is my two monographs Partial Differential Equations in the Complex Domain and Solution of Boundary Value Problems by the Method of Integral Operators, both published by Pitman Press. These monographs are contained in Part II of this submission, while in Part III I am presenting a supplementary presentation of 41 papers. Each of these last two parts is accompanied by a brief synopsis.

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STATEMENT

None of the following work has been submitted for any other degree or diploma. All of the following work is my own, except in the case of joint papers, where I and the co-author have made equal contributions.

Signature ....

The motivation for developing an analytic theory of partial differential equations lies in the classical results concerning the analyticity of solutions to elliptic and parabolic equations with analytic coefficients (c.f. [46]). This basic motivation is further strengthened by such properties of solutions to elliptic and parabolic equations as maximum principles and Phragmén-Lindelöf theorems ([55]). The analytic theory of partial differential equations seeks to more fully develop this close parallel between the behaviour of solutions to partial differential equations and the behaviour of analytic functions of a complex variable and to apply these results to solve boundary value, initial-boundary value, and inverse problems arising in various areas of mathematical physics. There have been two main directions in this development. The first of these is the theory of integral operators as developed by Bergman ([44]), Vekua ([56]), Gilbert ([49], [50]) and Colton ([1], [2]), and the second direction is the area of generalized analytic function theory as created by Bers ([45]), Vekua ([57]), and Haack ([51]). Roughly speaking, the theory of integral operators treats the case of partial differential equations with analytic coefficients, and the theory of generalized analytic functions extends this theory to the case of non-analytic coefficients. By imposing the stronger assumption of analyticity on the coefficients it is of course possible to develop a more complete theory, and for this reason the description "integral operator methods in partial differential equations" and "function theoretic methods in partial differential equations" are often used interchangeably. The basic idea of the theory of integral operators is to construct an integral operator which maps analytic functions onto solutions of partial differential equations with analytic coefficients, and to use this relationship to develop an analytic theory for the class of equations under investigation. It is, of course, of paramount importance that the mapping between analytic functions and solutions of the partial differential equation be one to one and onto, since otherwise it is not possible to obtain results concerning solutions in general, but only for certain (usually rather artificially defined) subclasses of solutions.

The theory of integral operators for elliptic partial differential equations in two independent variables was created by Bergman ([44]) and Vekua ([56]) in the late 1930's and early 1940's. Their main contribution was to construct an integral operator mapping analytic functions of a single complex variable onto real valued solutions of the elliptic equation

$$u_{xx} + u_{yy} + a(x,y)\, u_x + b(x,y)\, u_y + c(x,y)\, u = 0 \tag{1}$$

where the coefficients in (1) are analytic in some polydisc in the space of two complex variables. Vekua's operator is of the form

$$u(x,y) = \operatorname{Re}\left[H_o(z)\, \phi(z) + \int_o^z H(z,t)\, \phi(t)dt\right] \tag{2}$$

where $z = x + iy$, $\phi(z)$ is an arbitrary analytic function, and the kernels $H_o(z)$ and $H(z,t)$ can be expressed in terms of Vekua's complex Riemann function. Bergman's operator is of the form

$$u(x,y) = \operatorname{Re}\int_{-1}^{1} E(z,\bar{z},t)\, f(\tfrac{1}{2}z(1-t^2))\, \frac{dt}{\sqrt{1-t^2}} \tag{3}$$

where $f(z)$ is any analytic function and $E(z,\bar{z},t)$ is the Bergman generating function. Both Bergman and Vekua used the above integral operators to give constructive methods for solving boundary value problems associated with equation (1) defined in a bounded simply connected domain. Bergman accomplished this by using his operator to construct a complete family of solutions, orthonomalized this set, and then represented the solution of the given boundary value problem by means of a generalized Fourier series. Vekua, on the other hand, represented the analytic function $\phi(z)$ by means of Cauchy integral with unknown density $\mu(t)$, substituted this into his representation (2), interchanged orders of integration, and, after applying the given boundary conditions, arrived at an invertible singular integral equation for the unknown density $\mu(t)$.

The initial work of Bergman and Vekua was continued by R. P. Gilbert, beginning with a series of papers appearing in the early 1960's. Gilbert's first main contribution was the discovery and use of his "envelope method" in conjunction with the theory of integral operators to investigate the analytic behaviour of solutions to Laplace's equation in three and four variables and certain classes of singular elliptic equations in two independent variables ([49]). His second main contribution was his development (along with other mathematicians) of a function theoretic approach to partial differential equations with variable coefficients in more than two independent variables. Of particular note here is his "method of ascent" which maps solutions $h(\underset{\sim}{x})$ of the n dimensional Laplace equation onto solutions of

$$\Delta_n u + B(r^2)\, u = 0 \tag{4}$$

by means of the transformation

$$u(\underset{\sim}{x}) = h(\underset{\sim}{x}) + \int_0^1 \sigma^{n-1}\, G(r,1-\sigma^2)\, h(\underset{\sim}{x}\sigma^2)d\sigma, \tag{5}$$

where $G(r,\tau)$ denotes Gilbert's "G-function" and $B(r^2)$ need only be continuously differentiable. For a full discussion of Gilbert's work in this area, in particular for applications to the solution of boundary value problems, the reader is referred to [50].

My main contribution to the area of function theoretic methods in partial differential equations was to develop an integral operator approach for the study of pseudoparabolic and parabolic equations. Of particular concern has been the application of this approach to the solution of initial-boundary value problems, inverse problems, and the unique continuation of solutions. Since a full description of this work can be found in my monographs ([1], [2]) only a brief outline shall now be given. The simplest example of a pseudoparabolic equation is

$$u_{xxt} + \gamma u_t + \eta u_{xx} = 0 \tag{6}$$

where γ and η are constants. By means of the Riemann function for pseudoparabolic equations $v(\xi,\tau;x,t)$ ([1]) the solution of (6) satisfying

$$\begin{aligned} u(o,t) &= f(t) \\ u_x(o,t) &= g(t) \\ u(x,o) &= h(x) \end{aligned} \tag{7}$$

can be expressed in the form

$$\begin{aligned} u(x,t) = h(x) - \eta \int_o^x h'(\xi)v_\xi\,(\xi,o;x,t)d\xi \\ + \int_o^t \Big[g'(\tau)v_\xi(o,\tau;x,t) - f'(\tau)\, v_{\xi\tau}\,(o,\tau;x,t) \\ + \eta\, g(\tau)\, v_\tau\,(0,\tau;x,t) + \eta f'(\tau)v_\xi(o,\tau,x,t)\Big]\, d\,\tau. \end{aligned} \tag{8}$$

If $\gamma < o$ the solution of the first initial-boundary value problem for (6) in a rectangle can now be obtained by using (8) to derive a Volterra integral equation of the second kind for the unknown initial data $g(t)$. Such results can be extended to equations with variable coefficients and to equations in more than two independent variables ([28], [29], [30]). Turning now to parabolic equations one first writes the general linear second order parabolic equation in one space variable in the form

$$u_{xx} + q(x,t)u = u_t. \tag{9}$$

If $h(x,t)$ denotes a solution of the heat equation

$$h_{xx} = h_t \tag{10}$$

then I have shown that every solution of (9) can be expressed as

$$u(x,t) = h(x,t) + \int_{-x}^{x} E(s,x,t)\ h(s,t)ds \tag{11}$$

for some $h(x,t)$, where $E(s,x,t)$ is a known function depending only on $q(x,t)$([2]). The operator (11) can now be used to obtain reflection principles for parabolic equations with variable coefficients and, through the use of this reflection principle, to construct approximate solutions to initial-boundary value problems for parabolic equations defined in domains with moving boundaries ([2]). These results can be extended to the case of parabolic equations in more than one space variable ([2], [37]), and provide the analogue for parabolic equations of the work of Bergman and Vekua for elliptic equations.

In closing I should like to emphasize that I have only rather quickly highlighted the main developments in the area of function theoretic methods in partial differential equations and have not discussed the many other contributions the above mathematicians have made, in particular Gilbert's and my work on improperly posed initial-value problems ([49], [50], [1], [2]), my work on elliptic equations in three independent variables and its application to the solution of boundary value problems arising in scattering theory ([1], [2]), Bergman's use of integral operators in fluid dynamics ([53]), etc. Furthermore, the contributions of many other researchers have not been mentioned, notably Henrici ([52]), Garabedian ([47], [48]) and Lewy ([54]). The reason for these omissions is that it is impossible in the space of a few pages to discuss the entire area of function theoretic methods in partial differential equations, and I have instead tried, albeit in a rather subjective manner, to highlight the main developments in the theory for the benefit of the reader who is not necessarily an expert in this rapidly growing and diverse area of mathematics.

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PART II: SYNOPSIS

As mentioned in Part I, my main submission is my monographs [1] and [2] (numbers refer to references in Part I). The first of these monographs represents a general survey of the area of function theoretic methods in partial differential equations, with particular emphasis on improperly posed initial value problems and the analytic continuation of solutions to partial differential equations. Contained here is a description of some of my work on improperly posed initial value problems, integral operators for elliptic equations in three independent variables, pseudoparabolic equations, and inverse problems in scattering theory. In addition a survey is given of some of the work of Bergman, Vekua, Gilbert, Garabedian and Lewy. The second monograph is designed as a companion volume to the first set, with particular emphasis on the use of integral operators in the solution of boundary and initial-boundary value problems. Included here is a more complete discussion of the work of Bergman and Vekua, by work on parabolic equations (including a discussion of the inverse Stefan problem) and Gilbert's and my work on the "method of ascent", in particular my study of integral operators and their application to scattering theory (a more up to date survey can be found in [43]).

Research Notes in Mathematics 4

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Contents

Introduction

The subject matter of these lectures can conveniently be introduced by quoting a paragraph from Methods of Mathematical Physics, Vol. II by Courant and Hilbert: "The stipulation about existence, uniqueness, and stability of solutions dominate classical mathematical physics. They are deeply inherent in the ideal of a unique, complete and stable determination of physical events by appropriate conditions at the boundaries, at infinity, at time $t = 0$, or in the past. Laplace's vision of the possibility of calculating the whole future of the physical world from complete data of the present state is an extreme expression of this attitude. However, this rational ideal of causal-mathematical determination was gradually eroded by confrontation with physical reality. Nonlinear phenomena, quantum theory, and the advent of powerful numerical methods have shown that 'properly posed' problems are by far not the only ones which appropriately reflect real phenomena. So far, unfortunately, little mathematical progress has been made in the important task of solving or even identifying and formulating such problems which are not 'properly posed' but still are important and motivated by realistic situations". In one sense these notes are an introduction to the use of function theoretic methods in the investigation of one important class of physically motivated "improperly posed" problems, that is improperly posed initial value problems. However, such a study extends far beyond the immediate physical situation in which these problems arise. The following example serves as an illustration: the uniqueness of a solution to the "improperly posed" elliptic Cauchy problem is equivalent to the Runge approximation property. In order to exploit such a property one is led to construct an integral operator which maps analytic functions onto solutions of the elliptic equation under investigation, thus giving a practical method of constructing both a complete family of solutions and analytic approximations to the elliptic Cauchy problem. However in order to construct this integral operator it is necessary (in the case of three independent variables) to again examine an "improperly posed" problem, this time an exterior characteristic initial value problem for a hyperbolic equation. Having now constructed the desired integral operator it can in turn be used not only to construct a complete family of solutions but also to analytically continue solutions of the elliptic equation from a knowledge of the domain of regularity of their Cauchy or (complex) Goursat data along prescribed analytic surfaces. Proceeding in this manner and considering selected "improperly posed" initial value problems it is possible to systematically develop an analytic theory of partial differential equations based on the analytic theory of functions of a complex variable, and in a broad sense it is to this general theory that these notes are devoted.

We now briefly outline the content of the lectures. Chapter one consists of statements of basic results on the existence, uniqueness, and regularity of solutions to initial and boundary value problems in partial differential equations which will be needed in future chapters. In Chapter two we consider "improperly posed" initial value problems, give examples of their appearance in physics, and obtain results on the analytic continuation of solutions to elliptic and parabolic equations. In particular, the result mentioned above on the equivalence of the uniqueness to the elliptic Cauchy problem and the Runge approximation property is proved. In Chapter three we construct integral operators which map analytic functions of one and several complex variables onto real valued solutions of elliptic equations in two and three independent variables. Since this particular topic has been the subject matter of three different books ([1], [21], [39]), we concentrate on newer results, in particular integral operators for elliptic equations in three independent variables. Nevertheless for the sake of completeness we construct (in section 8) Bergman's integral operators for elliptic equations in two independent variables and derive (in section 11) Vekua's representation of solutions to this class of equation. Chapter four is concerned with the use of integral operators in the analytic continuation of solutions to elliptic equations. Lewy's reflection principle and Gilbert's envelope method are derived and as an example of their use are applied to the problem of analytically continuing solutions of the axially symmetric Laplace and Helmholtz equations. Included in this chapter is a discussion of radiation conditions and asymptotic expansions of solutions to the axially symmetric Helmholtz equation. In Chapter five we introduce a class of third order equations which have been the object of recent study in various areas of fluid dynamics and derive the basic analytic properties of solutions to such equations. It is shown that in terms of their analytic behaviour the solutions of the third order equations considered here occupy a position somewhere in between that of parabolic and elliptic equations. "Improperly posed" problems associated with these equations can be found in [5].

None of the material presented in the last four chapters of these notes has appeared in previous research monographs on the subject, except for section 4 (which can be found in Chapter 16 of [20]), section 8 (which is taken from [2] and can also be found in [1] and [21]) and section 12 (which is taken from [21]). This fact is reflected in the bibliography, where only those results which are directly referred to or used in these notes are referenced. For other work in this area the reader is directed to the bibliographies contained in the books by Bergman ([1]), Garabedian ([20]), Gilbert ([21]), and Vekua ([39]).

The following course of lectures was given in the spring semester of 1971 at Indiana University and again in the fall of that year at the University of Glasgow where the author was a visiting research fellow participating in the North British

Symposium on Partial Differential Equations and Their Applications. Gratitude is expressed to the Science Research Council the National Science Foundation, and the Air Force Office of Scientific Research for their financial support, and to Professor Ian Sneddon and the University of Glasgow for their hospitality during the academic year 1971-1972 when the first draft of these notes was typed and circulated.

<u>Note</u>: Since the time these notes were written a considerable amount of new research has been completed in areas closely related to the subject matter of these lectures. For some of these new results the reader is referred to

R.P.Gilbert, <u>Constructive Methods for Elliptic Partial Differential Equations</u>, Springer Verlag Lecture Notes Series Berlin, 1974.

I Preliminary results

Preliminaries

Let $\underset{\sim}{x} = (x_1,\ldots,x_n)$, $\underset{\sim}{\xi} = (\xi_1,\ldots,\xi_n)$ and let $u(\underset{\sim}{x})$ satisfy the equation

$$L[u] = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{n} b_i(\underset{\sim}{x}) \frac{\partial u}{\partial x_i} + c(\underset{\sim}{x})u = 0$$

where $b_i(\underset{\sim}{x})$, $i = 1,\ldots, n$, and $c(\underset{\sim}{x})$ are analytic functions of their independent variables in a domain $\tilde{D}$.

1. Fundamental Solutions and Analyticity

A fundamental solution $S = S(\underset{\sim}{x};\underset{\sim}{\xi})$ is a solution of the equation

$$L[S] = \delta(|\underset{\sim}{x} - \underset{\sim}{\xi}|)$$

where δ denotes the Dirac delta function. S has the form

$$S = \frac{U(\underset{\sim}{x};\underset{\sim}{\xi})}{r^{n-2}} + V(\underset{\sim}{x};\underset{\sim}{\xi}) \log r + W(\underset{\sim}{x};\underset{\sim}{\xi}); \; n > 2 \tag{1.1}$$

$$S = A(x_1,x_2;\xi_1,\xi_2) \log \frac{1}{r} + B(x_1,x_2,\xi_1,\xi_2); \; n = 2 \tag{1.2}$$

where $r = |\underset{\sim}{x} - \underset{\sim}{\xi}|$. For n odd $V \equiv 0$, $L[W] = 0$. In (1.2) set $z = x_1 + ix_2$, $z^* = x_1 - ix_2$, $\zeta = \xi_1 + i\xi_2$, $\zeta^* = \xi_1 - i\xi_2$.

Then

$$\underset{\sim}{L}[A] = \frac{\partial^2 A}{\partial z \partial z^*} + \alpha\frac{\partial A}{\partial z} + \beta\frac{\partial A}{\partial z^*} + \gamma A = 0 \tag{1.3}$$

where $\alpha = \frac{1}{4}(b_1+ib_2)$, $\beta = \frac{1}{4}(b_1-ib_2)$, $\gamma = \frac{1}{4}c$, and

$$\left[\frac{\partial}{\partial z} + \beta(z,\zeta^*)\right] A(z,\zeta^*;\zeta,\zeta^*) = 0 \tag{1.4}$$

$$\left[\frac{\partial}{\partial z^*} + \alpha(\zeta, z^*)\right] \quad A(\zeta, z^*; \zeta, \zeta^*) = 0 \tag{1.5}$$

$$A(\zeta, \zeta^*, \zeta, \zeta^*) = 1. \tag{1.6}$$

$A(z, z^*, \zeta, \zeta^*)$ is the Riemann function for $L[A] = 0$ (See Exercise 3.1 of these notes, [20], [21], [24], [39]). Now let D be a simply connected domain with smooth boundary ∂D. Then

$$u(\underset{\sim}{x}) = -\int_{\partial D} B\,[u(\underset{\sim}{\xi}),\ S_M(\underset{\sim}{\xi};\underset{\sim}{x})] \tag{1.7}$$

$$B[u,v] = \sum_{i=1}^{n} (-1)^i \left[b_i uv + v\frac{\partial u}{\partial x_i} - u\frac{\partial v}{\partial x_i}\right] dx,\ldots,\widetilde{dx_i},\ldots,dx_n \tag{1.8}$$

where S_M is fundamental solution of the adjoint equation $M[u] = 0$. Since for $\underset{\sim}{x} \neq \underset{\sim}{\xi}$, S_M is an analytic function of $\underset{\sim}{x}$ and $\underset{\sim}{\xi}$ for $\underset{\sim}{x}$, $\underset{\sim}{\xi} \in D$ (for $n = 2$ S_M is an analytic function of z, z^*, ζ, ζ^* for z, $\zeta \in D$, z^*, $\zeta^* \in D^*$, where $D^* = \{z^* \mid \bar{z}^* \in D\}$), provided the coefficients of L have the same property, we have

<u>Theorem A</u> ([20]): If $b_i(\underset{\sim}{x})$, $c(\underset{\sim}{x})$ are analytic functions of $x_1,\ldots,x_n$ in D and $u(\underset{\sim}{x}) \in \tilde{C}^2(D)$, then $u(\underset{\sim}{x})$ is an analytic function of $x_1,\ldots,x_n$ in D.

<u>Theorem B</u> ([39]): Let $n = 2$. If $\alpha(z, z^*)$, $\beta(z, z^*)$, $\gamma(z, z^*)$ are analytic in $D \times D^*$ and $u(x_1, x_2) \in C^2(D)$, then $U(z, z^*) = u\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right)$ is analytic in $D \times D^*$.

2. <u>Existence and Uniqueness of Solutions to the Dirichlet and Cauchy Problems</u>.

Let f, $g \in L_2(D)$. $(f,g) = \iint_D fg$.

Definition: u is a weak solution of $Lu = f$ if $(M\phi, u) = (\phi, f)$ for every $\phi \in C_0^\infty(D)$.

Theorem C ([3]): Let $f(\underset{\sim}{x})$ be analytic in D. Then if u is a weak solution of Lu = f, u is analytic in D.

Theorem D ([3]): Let $\phi \in C^o(\partial D)$. Then the boundary value problem

$$\begin{aligned} Lu &= f \text{ in } D \\ u &= \phi \text{ on } \partial D \end{aligned} \tag{2.1}$$

where $f \in L_2(D)$, has a (weak) solution if and only if

$$\iint_D f\omega = \int_{\partial D} \phi \frac{\partial \omega}{\partial \nu} \tag{2.2}$$

for all solutions ω of

$$\begin{aligned} M\omega &= 0 \quad \text{in } D \\ \omega &= 0 \quad \text{on } \partial D. \end{aligned} \tag{2.3}$$

The set of solutions of (2.3) is finite dimensional.

Corollary ([3]): The orthogonal complement on ∂D of the space of all boundary values of all solutions of Lu = 0 is the finite dimensional space spanned by $\frac{\partial \omega}{\partial \nu}$, ω a solution of (2.3).

Theorem E ([20]): Let $L[u] = 0$, $u \in C^2(D) \cap C^o(\bar{D})$, $c(\underset{\sim}{x}) \leq 0$. Then $u(\underset{\sim}{x})$ achieves its maximum and minimum on ∂D.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$; $\alpha_j \geq 0$, $|\alpha| = \sum_{j=1}^{n} \alpha_j$,

$D_j = \frac{\partial}{\partial z_j}$, $D^\alpha = D_1^{\alpha_1}, \ldots, D_n^{\alpha_n}$, $z^\alpha = z_1^{\alpha_1}, \ldots, z_n^{\alpha_n}$.

Theorem F ([28]): Consider

$$D^\beta u = \sum_{|\alpha| \leq |\beta|} a_\alpha D^\alpha u + f \tag{2.4}$$

where f, a_α are analytic functions of $z = (z_1, \ldots, z_n)$ in a neighbourhood of the origin in C^n and prescribe

$$D_j^k\,(u-\phi) = 0 \text{ when } z_j = 0 \text{ if } 0 \le k < \beta_j;\ j = 1,\dots,n, \quad (2.5)$$

where ϕ is analytic in a neighbourhood of the origin. Let A be the set of multi-indices in the sum on the right hand side of (2.4) such that $a_\alpha \not\equiv 0$, and assume that β does not belong to the convex hull of A considered as a subset of R^n. Then there exists a unique analytic solution of (2.4), (2.5) analytic in a neighbourhood Ω of the origin. u depends continuousl on the initial data $D_j^k\,\phi$ in $\Omega' \cap \{z_j = 0\}$ where $\Omega' \subset \Omega$ and the size of Ω' depends only on $\sum_{|\alpha| \le \beta} |a_\alpha|$.

Remark: It follows from the results of J. Persson, J. Persson, Linear Goursat problems for entire functions when the coefficients are variable, Ann. Scoula Norm. Sup. Pisa (3) 23 (1969), 87-98, that if a_β are constant and ϕ is entire then u is also entire.

II Improperly posed initial value problems

3. Cauchy's Problem in Two Independent Variables

The elliptic Cauchy problem is of interest for the following reasons:

1. analytic continuation

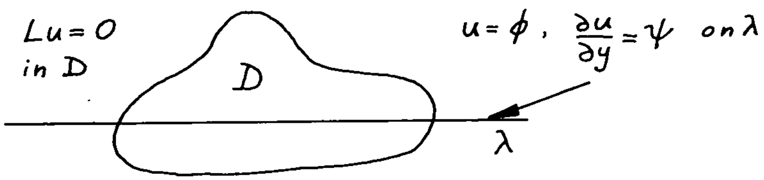


figure 3.1

What is the domain of regularity of u in terms of the domain of regularity of ϕ, ψ and the coefficients of L? Theorem F does not answer this question.

2. inverse boundary value problems

Consider the flow of an incompressible, irrotational fluid about a curved plate B such that behind B there is a "dead water" region Ω.

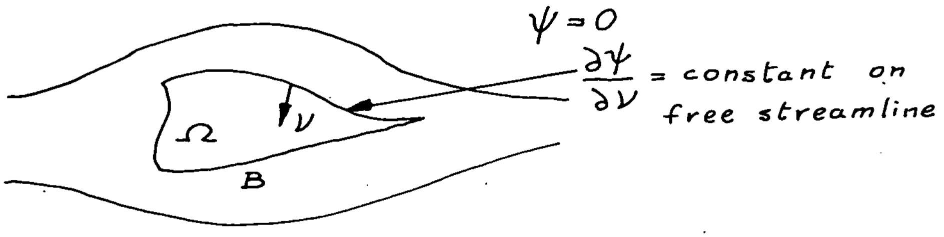


figure 3.2

Let ψ be the stream function. Along the free streamline $\psi=0$. Since the pressure is the same on both sides of the streamline, we have by Bernoulli's equation that $\frac{\partial\psi}{\partial\nu}$ = constant on the streamline. The inverse problem is to find the shape of B given the Cauchy data on the streamline. Analogous problems arise in semilinear and quasilinear equations (see [20]).

3. <u>boundary value problems with incomplete data</u>:

Suppose we have a clamped membrane vibrating with the frequency ω and the slope of deflection is measured on a portion of the boundary.

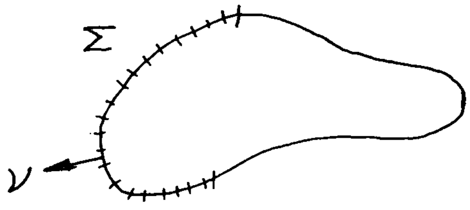


figure 3.3

Then we have the following Cauchy problem:

$$\Delta_2 u + \frac{\omega^2}{c^2} u = 0$$

$$u = 0 \text{ on } \Sigma$$

$$\frac{\partial u}{\partial \nu} = f \text{ on } \Sigma$$

where c = velocity of sound. For a discussion on problems of this type and their solution see [34].

In (2) and (3) the problem is to construct an approximation to the elliptic Cauchy problem, including error estimates. The following example shows why difficulties arise when an attempt is made to do this.

<u>Example 3.1</u> (Hadamard). Let $u(x,y)$ satisfy the equation

$$\Delta_2 u = 0 \tag{3.1}$$

and the Cauchy data

$$u(x,0) = 0 \qquad u_y(x,0) = \frac{1}{n} \sin nx. \tag{3.2}$$

Then

$$u(x,y) = \frac{1}{n^2} \sin nx \sinh ny. \tag{3.3}$$

is the unique solution to (3.1), (3.2). However as $n \to \infty$ the Cauchy data tends to zero but the solution does not tend to the corresponding zero solution, i.e. Cauchy's problem for elliptic equations does not depend continuously on the initial data.

We now consider the following elliptic Cauchy problem:

$$u_{xx} + u_{yy} = g(x,y,u,u_x,u_y) \tag{3.4}$$

$$u(x,y) = \Phi(x + iy), \quad x + iy \in C$$

$$\frac{\partial u(x,y)}{\partial \nu} = \Omega(x + iy), \quad x + iy \in C \tag{3.5}$$

where C is a givenanalytic arc. (Regularity conditions on g, Φ, Ω will be prescribed shortly). By the use of a conformal mapping we can assume without loss of generality, that C is a segment of the x-axis containing the origin, i.e. $y = 0$ in (3.5). Setting

$$\begin{aligned} z &= x + iy \\ z^* &= x - iy \end{aligned} \tag{3.6}$$

(3.4), (3.5) becomes

$$U_{zz^*} = f(z,z^*,U,U_z,U_{z^*}) \tag{3.7}$$

where

$$u\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right) = U(z,z^*),$$

and $U(z,z^*) = \Phi(z)$ on $z = z^*$

$$\frac{\partial U(z,z^*)}{\partial z} - \frac{\partial U(z,z^*)}{\partial z^*} = -i\Omega(z) \text{ on } z = z^*. \tag{3.8}$$

Assume i) $f(z,z^*,\xi_1,\xi_2,\xi_3)$ is holomorphic in $G \times G^* \times B_3$ where $G^* = \{z | \bar{z} \in G\}$ and B_3 is a ball containing the origin in ξ_1,ξ_2,ξ_3 space.

ii) G is a disk containing the origin, in particular $G = G^*$.

iii) $\Phi(z)$, $\Omega(z)$ are holomorphic for all $z \in G$.

Now let $s(z,z^*) = U_{zz^*}(z,z^*)$. Then

$$U(z,z^*) = \int_0^z\int_0^{z^*} s(\xi,\xi^*)d\xi^* d\xi + \int_0^z \phi(\xi)d\xi + \int_0^{z^*}\psi(\xi^*)d\xi^* + U(0,0) \tag{3.9}$$

$$U_z(z,z^*) = \int_0^{z^*} s(z,\xi^*)d\xi^* + \phi(z) \tag{3.10}$$

$$U_{z^*}(z,z^*) = \int_0^z s(\xi,z^*)d\xi + \psi(z^*) \tag{3.11}$$

where $\phi(z) = U_z(z,0)$, $\psi(z^*) = U_{z^*}(0,z^*)$.

The initial conditions (3.8) become

$$\int_0^z\int_0^z s\,(\xi,\xi^*)d\xi^* d\xi + \int_0^z \phi(\xi)d\xi + \int_0^z \psi(\xi^*)d\xi^* + U(0,0) = \Phi(z),$$

or differentiating with respect to z

$$\int_0^z s(z,\xi^*)d\xi^* + \int_0^z s(\xi,z)d\xi + \phi(z) + \psi(z) = \Phi'(z) \tag{3.12}$$

and

$$\int_0^z s(z,\xi^*)d\xi^* + \phi(z) - \int_0^z s(\xi,z)d\xi - \psi(z) = -i\Omega(z). \tag{3.13}$$

Equations (3.12), (3.13) imply that

$$\phi(z) = \tfrac{1}{2}[\Phi'(z) - i\Omega(z)] - \int_0^z s(z,\xi^*)d\xi^* \tag{3.14}$$

$$\psi(z) = \tfrac{1}{2}[\Phi'(z) + i\Omega(z)] - \int_0^z s(\xi,z)d\xi. \tag{3.15}$$

Hence, defining the operators B_i, $i = 1,2,3$, by the right hand side of (3.9), (3.10), (3.11) respectively, where $\phi(z)$, $\psi(z)$ are defined by (3.14), and (3.15) (note that $U(0,0) = \Phi(0)$)

leads to the problem of finding $s(z,z^*)$ satisfying

$$s(z,z^*) = f(z,z^*,B_1[s(z,z^*)],B_2[s(z,z^*)],B_3[s(z,z^*)]).$$

Let HB be the Banach space of functions of two complex variables which are holomorphic and bounded in $G \times G^*$ with norm

$$\| s \|_\lambda = \sup_{(z,z^*)\in G\times G^*} \{e^{-\lambda(|z|+|z^*|)}|s(z,z^*)|\}. \tag{3.17}$$

where $\lambda > 0$ is a fixed constant.

Theorem 3.1 ([6]): The operator T defined by

$$Ts = f(z,z,\ B_1 s,\ B_2 s,\ B_3 s)$$

maps a closed ball of HB into itself and is a contraction mapping.

Proof: Since f is holomorphic in a compact subset of the space of five complex variables, by Schwarz's lemma (c.f.[21]) a Lipschitz condition holds there with respect to the last three arguments, i.e.

$$\begin{aligned}&|f(z,z^*,\xi_1,\xi_2,\xi_3) - f(z,z^*,\xi_1^0,\xi_2^0,\xi_3^0)|\\ &\le C_0\{|\xi_1-\xi_1^0| + |\xi_2-\xi_2^0| + |\xi_3-\xi_3^0|\}\end{aligned} \tag{3.18}$$

where C_0 is a positive constant. Hence for $s_1,s_2\in HB$ and G sufficiently small

$$\begin{aligned}\| Ts_1-Ts_2\|_\lambda \le C_0\{ &\|B_1s_1-B_1s_2\|_\lambda + \|B_2s_1-B_2s_2\|_\lambda\\ &+ \|B_3s_1-B_3s_2\|_\lambda\}.\end{aligned} \tag{3.19}$$

From estimates of the form

$$\left|\int_0^z s(\xi,z^*)d\xi\right| \le \int_0^{|z|} \|s\|_\lambda e^{\lambda|\xi|+\lambda|z^*|}|d\xi|$$
$$\le \frac{1}{\lambda} e^{\lambda|z|+\lambda|z^*|} \|s\|_\lambda \tag{3.20}$$

i.e.

$$\left\|\int_0^z s(\xi,z^*)d\xi\right\|_\lambda \le \frac{\|s\|_\lambda}{\lambda} \tag{3.21}$$

we have

$$\|B_i s_1 - B_i s_2\|_\lambda \le \frac{N_i}{\lambda}\|s_1-s_2\|_\lambda;\ i = 1,2,3 \tag{3.22}$$

where N_i are positive constants independent of λ. Hence there exists a positive constant M such that

$$\|Ts_1 - Ts_2\|_\lambda \le \frac{M}{\lambda}\|s_1-s_2\|_\lambda \tag{3.23}$$

and for every $s \in HB$

$$\|Ts\|_\lambda \le \frac{M}{\lambda}\|s\|_\lambda + \|T_0\|_\lambda$$
$$\le \frac{M}{\lambda}\|s\|_\lambda + \frac{M_0}{2} \tag{3.24}$$

where M_0 is a positive constant. Hence for $\|s\|_\lambda \le M_0$ and λ sufficiently large, $\|Ts\|_\lambda \le M_0$ i.e. T takes a closed ball in HB into itself. Equation (3.23) shows that for λ sufficiently large T is a contraction mapping.

Corollary ([6]): There exists a stable iterative procedure for solving the semilinear elliptic Cauchy problem in two independent variables. When (3.4) is linear global solutions are obtained; if the Cauchy data is analytic in G and the coefficients are analytic functions of z and z* in G x G*, then the solution is an analytic function of z and z* in G x G*.

Exercise 3.1. Use the above method of exponential majorization to construct the complex Riemann function (see Section 1). Compare to [20], pp. 139-141, and [39].

Remark: For related results see [20], pp. 625-631 and [24].

4. Cauchy's Problem for Quasilinear Systems.

Consider a quasilinear system of m partial differential equations of order one in n + 1 independent variables $x,\dots,x_n,t$

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{n} A_j \frac{\partial u}{\partial x_j} + B \tag{4.1}$$

where

$$u = u(x,t) = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

and $A_j = A_j(x,t,u)$, $j = 1,\dots,n$ are $m \times m$ matrices which are analytic functions of $x = (x,\dots,x_n)$, t and u, and $B = B(x,t,u)$ is a column vector of analytic function of x,t and u. (Any system of partial differential equations can be written as (4.1) if $t = 0$ is not characteristic: c.f. [20], pp. 6-12). Pose the Cauchy problem for (4.1) by prescribing the initial condition

$$u(x,0) = f(x) \tag{4.2}$$

where $f(x)$ is analytic. By the Cauchy-Kowalewski Theorem (c.f. [20]) there exists locally an analytic solution u of (4.1), (4.2).

Now keep t real and replace x_j by $z_j = x_j + iy_j$ (x_j, y_j are real). Then

$$\frac{\partial U}{\partial z_j} = \frac{1}{2}\left(\frac{\partial u}{\partial x_j} - i\frac{\partial u}{\partial y_j}\right) \tag{4.3}$$

where $U(z,\bar{z},t) = u(x,t)$, $z = (z_1,\dots,z_n)$, $\bar{z} = (\bar{z}_1,\dots,\bar{z}_n)$.

Furthermore, since U is an analytic function of z_j, $j = 1, \ldots, n$, the n Cauchy-Riemann equations are satisfied:

$$\frac{\partial U}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial u}{\partial x_j} + i\frac{\partial u}{\partial y_j}\right) = 0. \qquad (4.4)$$

Equations (4.1), (4.2) become

$$\frac{\partial U}{\partial t} = \sum_{j=1}^{n} A_j \frac{\partial U}{\partial z_j} + B \qquad (4.5)$$

$$U(z,0) = f(z). \qquad (4.6)$$

Let A_j^* be the transpose of the complex conjugate of A_j. Multiplying (4.4) by A_j^* and adding it to (4.5) gives (see [20])

$$\frac{\partial U}{\partial t} = \sum_{j=1}^{n} A_j \frac{\partial U}{\partial z_j} + \sum_{j=1}^{n} A_j^* \frac{\partial U}{\partial \bar{z}_j} + B$$

or by (4.3) and (4.4)

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{n} \frac{A_j + A_j^*}{2} \frac{\partial u}{\partial x_j} + \sum_{j=1}^{n} \frac{A_j - A_j^*}{2i} \frac{\partial u}{\partial y_j} + B. \qquad (4.7)$$

Recall that a system of the form (4.1) is symmetric hyperbolic if and only if the A_j are symmetric, i.e. for arbitrary but fixed λ_j the roots λ of the polynomial

$$\det\left| \sum_{j=1}^{n} \lambda_j A_j - \lambda I \right| = 0 \qquad (4.8)$$

are real since they are the eigenvalues of a symmetric matrix. A characteristic of the system (4.1) is any level surface $\phi(x,t) = \text{constant}$ where $\phi(x,t)$ satisfies

$$\det\left| \sum_{j=1}^{n} \phi_{x_j} A_j - \phi_t \right| = 0 \qquad (4.9)$$

Since the coefficients of (4.7) are Hermitian matrices, the system (4.7) can be written as a symmetric hyperbolic system of 2m real equations in 2n + 1 real independent variables, independent of the type of the system (4.1). Such problems are well posed (c.f. [20] pp. 434-448).

The characteristic surfaces of (4.7) are real manifolds of dimension 2n and define the domain of dependence in the initial hyperplane t = 0 where data must be known if the value of the solution at a given point is to be determined ([20], pp. 614-621).

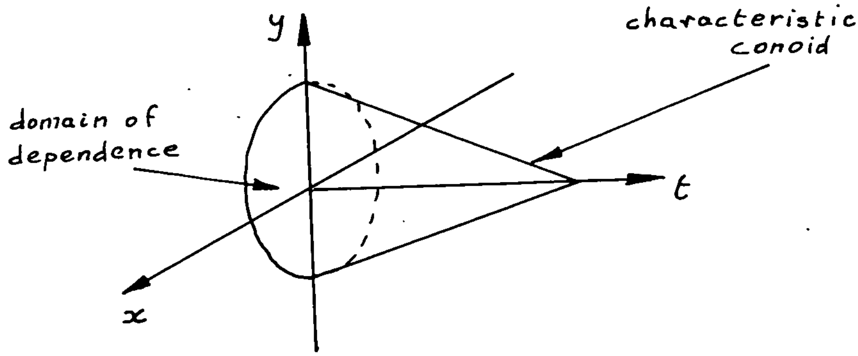


figure 4.1

5. Uniqueness of Solutions to Cauchy's Problem and the Runge Approximation Property

Let L be a second order elliptic operator with analytic coefficients and Laplacian as its principal part. Let M be the adjoint operator.

Definition 5.1: Solutions of an equation Lu = 0 are said to have the Runge approximation property if, whenever D_1 and D_2 are two bounded simply connected domains, D_1 a subset of D_2, any solution in D_1 can be approximated uniformly in compact subsets of D_1 by a sequence of solutions which can be extended as solutions to D_2.

Theorem 5.1 ([30], [32]): Solutions of Lu = 0 have the Runge approximation property if and only if solutions of Mu = 0 are uniquely determined throughout their domain of existence by their Cauchy data along any smooth hypersurface.

Remarks: From a result of Friedrichs ([18]) the L_2 norm of a solution u over a domain bounds the maximum norm of u over any compact subset of this domain. Hence in order to show that a solution over D_1 can be approximated uniformly over any compact subset by solutions in D_2, it is sufficient to show that it can be approximated in the L_2 sense over any subdomain whose closure lies in D_1. Let D_0 be such a subdomain; denote by S_1 the restriction of solutions in D_1 to D_0, by S_2 the restriction of solutions in D_2 to D_0. S_2 is a subspace of S_1 (our aim in the first part of the theorem is to show that it is a dense subspace in the L_2 topology). By a classical criterion, S_2 is dense in S_1 if and only if every function v_0 in L_2 over D_0 orthogonal to S_2 is also orthogonal to S_1. Finally we write Green's formula in the form

$$\int_D\int uMw - wLu = \int_C u\frac{\partial w}{\partial \nu} - w\frac{\partial u}{\partial \nu} + cuw \tag{5.1}$$

where c is some function of the coefficients of L(see equation (1.8)).

Proof of Theorem: (uniqueness of solution to Cauchy problem implies Runge approximation property): Let $v_0 \in L_2(D_0)$, $v_0 \perp S_2$. We will show $v_0 \perp S_1$. Let w_0 be a solution of

$$Mw_0 = \begin{cases} v_0 \text{ in } D_0 \\ 0 \text{ in } D_2 - D_0 \end{cases} \tag{5.2}$$

$$w_0 = 0 \text{ on } C_2 = \partial D_2.$$

By Theorem D, w_0 exists since $v_0 \perp S_2$. Let $u \in S_2$, $w = w_0$ in (5.1). Since $v_0 \perp S_2$ this shows that $\frac{\partial w_0}{\partial \nu}$ is orthogonal to the boundary values of functions in S_2. By the corollary to Theorem D, $\frac{\partial w_0}{\partial \nu} = \frac{\partial w}{\partial \nu}$, where w satisfies the homogeneous equation $Mw = 0$ in D_2, $w = 0$ on C_2, which implies by Theorem D that (5.2) has a solution w_0 such that $\frac{\partial w_0}{\partial \nu} = 0$ on C_2 (e.g. $w - w_0$ satisfies this). By the uniqueness of the solution to the

Cauchy problem this implies that $w_0 = 0$ in $D_2 - D_0$.

Now applying (5.1) to $w = w_0$ and $u \in S_1$ over a domain D such that $D_1 \supset D \supset D_0$, we conclude that $v_0 \perp S_1$.

(Runge approximation property implies uniqueness of the solution to the Cauchy problem): Let w_0 be a solution of $Mw_0 = 0$ with zero Cauchy data on a piece of a surface C. We will show $w_0 = 0$ wherever it is defined. First we assume that C is a closed surface and w_0 is defined in a boundary strip of the domain D bounded by C (see figure 5.1).

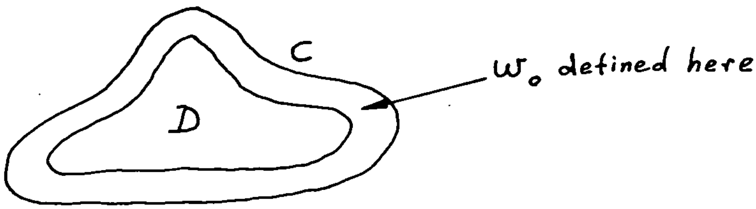


figure 5.1

Since uniqueness is a local problem, without loss of generality choose D so small that L and M are positive definite over D i.e. $Lu = 0$ or $Mu = 0$ in $D_0 \subset D$, $u = 0$ on C, has only the trivial solution for any subdomain D_0.

First we extend w_0 to the whole interior of C (not necessarily as a solution of $Mw_0 = 0$ but such that $Mw_0 \in L_2(D)$). Equation (5.1) with $w = w_0$ and u a solution of $Lu = 0$ shows that $Mw_0 \perp u$ for every such u. By the Runge approximation property this implies that Mw_0 is orthogonal to any solution of $Lu = 0$ in a domain $\tilde{D} \subset D$ which contains the support of $v_0 = Mw_0$. Let $\tilde{C}$ be the (smooth) boundary of $\tilde{D}$.

Now define $\tilde{w}_0$ as the solution of

$$\begin{aligned} M\tilde{w}_0 &= 0 \quad \text{in } \tilde{D} \\ \tilde{w}_0 &= w_0 \quad \text{on } \tilde{C} \end{aligned} \tag{5.3}$$

$\tilde{w}_0$ exists by Theorem D and the fact that L is positive definite. In (5.1) set $w = w_0 - \tilde{w}_0$ and let u be any solution of $Lu = 0$

over $\tilde{D}$ to get

$$\iint_{\tilde{D}} u \, Mw_0 = \int_{\tilde{C}} u \frac{\partial(w_0-\tilde{w}_0)}{\partial\nu} \quad . \tag{5.4}$$

Since $Mw_0 \perp u$, the left-hand side of (5.4) equals zero. Since the boundary values of u on $\tilde{C}$ are arbitrary, we conclude $\frac{\partial(w-\tilde{w}_0)}{\partial\nu} = 0$ on $\tilde{C}$. Now define w_1 by

$$w_1 = \begin{cases} w_0 \text{ in } D - \tilde{D} \\ \tilde{w}_0 \text{ in } \tilde{D} \end{cases} \tag{5.5}$$

Note that w_1 satisfies $Mw_1 = 0$ in both domains and has continuous first derivatives across $\tilde{C}$.

Exercise 5.1. Show that w_1 is a weak solution of $Mw_1 = 0$ and hence a strong (or genuine) solution.

Exercise 5.1 shows that w_1 is an extension of w_0 to the whole interior of C as a solution of $Mw_1 = 0$. Since M is positive definite over D and $w_1 = 0$ on C, $w_1 \equiv 0$ in D which implies $w_0 = 0$ in $D - \tilde{D}$. Since the only restriction on $\tilde{D}$ was that $\tilde{C}$ should be contained in the boundary strip in which w_0 was originally defined, we conclude that $w_0 = 0$ in the whole boundary strip.

We now remove the restriction that C be a closed surface. Let F be a sufficiently small piece of a surface. (F = AB in figure 5.2 below).

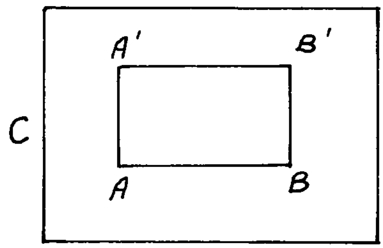


figure 5.2

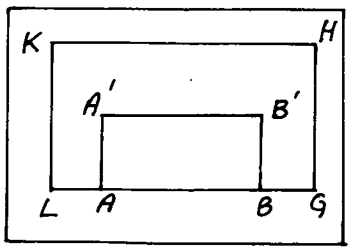


figure 5.3

Let w_0 be a solution of $Mw_0 = 0$ in ABB'A' of figure 5.1 which has zero Cauchy data on F. Let C be the boundary of the whole rectangle in figure 5.2. Define $w_0 = 0$ outside ABB'A' and consider w_0 in the boundary strip consisting of those points inside C which lie outside the octagon AA'B'BGHKL offigure 5.3. Since we have already shown that solutions with zero Cauchy data on a closed surface are identically zero, $w_0 = 0$ in ABB'A' and the proof is complete . (Note that w_0 has continuous first derivatives across AB which implies that w_0 is a weak solution of $Mw_0 = 0$ and hence w_0 is a strong solution of $Mw_0 = 0$ in the region under consideration).

Exercise 5.2. Let $L[u] = 0$ be a linear second order elliptic equation with analytic coefficients and Laplacian as its principal part. Suppose u has zero Cauchy data on a smooth hypersurface T. Show that $u \equiv 0$ in its domain of definition and conclude that solutions of $L[u] = 0$ have the Runge Approximation Property.

6. The Non-Characteristic Cauchy Problem for Parabolic Equations.

The Stefan problem for the heat equation is defined as follows:

Find $u(x,t)$ and $s(t)$ satisfying

$$u_{xx} - u_t = 0 , \qquad 0 < x < s(t) , \qquad 0 < t ,$$

$$u(s(t),t) = 0 , \quad u_x(s(t),t) = -\dot{s}(t), \quad 0 < t ,$$

$$u(x,0) = \phi(x) \geq 0, \quad 0 \leq x \leq s(0) = b$$

with either $u(0,t) \geq 0$ or $u_x(0,t) \leq 0$ also given. u may be thought of as the temperature distribution in the water component of a one-dimensional ice and water system. The free boundary s(t) represents the interface between the ice and water. The initial temperature ϕ and interface position b are given, along with either the temperature u or its gradient u_x at $x = 0$.

Conversely, we can also consider the <u>inverse</u> Stefan problem, i.e. assume $x = s(t)$ is known and find u, i.e. how must one heat the water in order to melt the ice along a prescribed curve? This is a non-characteristic Cauchy problem for the heat equation. The following example shows that this problem is "improperly posed".

<u>Example 6.1 ([35])</u>:
Let

$$u_n(x,t) = \frac{1}{n^{2k}}\left[e^{nx}\sin(2n^2t+nx) + e^{-nx}\sin(2n^2t+nx)\right].$$

$u_n(x,t)$ satisfies the heat equation and Cauchy data

$$u_n(0,t) = \frac{2}{n^{2k}}\sin 2n^2t,\quad u_{nx}(0,t) = 0.$$

$u_n(0,t)$ and its derivatives up to order $k-1$ tend to zero as $n \to \infty$ while for $|x| \geq \delta > 0$ $u_n(x,t)$ assumes arbitrarily large values as $n \to \infty$, i.e. the inverse Stefan problem is improperly posed. Now consider the Cauchy problem

$$\mathcal{L}[u] \equiv u_{xx} + a(x)u_x + b(x)u - c(x)u_t = F(x,t) \tag{6.1}$$

$$\begin{aligned} u(s(t),t) &= f(t) \\ u_x(s(t),t) &= g(t) \end{aligned} \tag{6.2}$$

where a, b, c, F and s are analytic in a sufficiently large neighbourhood of the origin and $x = s(t)$ is non-characteristic. We will first construct a fundamental solution S to equation (6.1) which has an essential singularity at $t = \tau$ (as opposed to the usual multi-valued fundamental solution) and use this to solve (6.1), (6.2). The results which follow are due to Hill ([26]).

<u>Example 6.2</u>. When $\mathcal{L}[u] \equiv u_{xx} - u_t = 0$ the fundamental solution S that we will construct is defined by

$$S(x,t;\xi,\tau) = \frac{\sqrt{\pi}(x-\xi)}{2(\tau-t)}\, E\left\{-\frac{(\xi-x)^2}{4(\tau-t)}\right\}$$

where

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+3/2)} ,$$

as opposed to the usual fundamental solution

$$w(x,t;\xi,\tau) = \frac{1}{2\sqrt{\pi(\tau-t)}} \exp\left\{-\frac{(\xi-x)^2}{4(\tau-t)}\right\}.$$

The fundamental solution $S(x,t;\xi,\tau)$ of equation (6.1) is a solution of the adjoint equation

$$\begin{aligned} \mathfrak{M}[v] &\equiv v_{xx} - (av)_x + bv + cv_t \\ &\equiv M[v] + cv_t = 0 \end{aligned} \tag{6.3}$$

with initial conditions

$$S(\xi,t;\xi,\tau) = 0, \quad S_x(\xi,t;\xi,\tau) = \frac{-1}{t-\tau}. \tag{6.4}$$

Let

$$S(x,t;\xi,\tau) = \sum_{j=0}^{\infty} S_j(x,\xi) \frac{j!}{(t-\tau)^{j+1}} . \tag{6.5}$$

Equations (6.3) and (6.4) imply that

$$\begin{aligned} &S_0(\xi,\xi) = 0, \quad S_{0x}(\xi,\xi) = -1 \\ &S_j(\xi,\xi) = S_{jx}(\xi,\xi) = 0, \quad j = 1,2,\ldots . \end{aligned} \tag{6.6}$$

Inserting (6.5) into (6.3) gives

$$\mathfrak{M}[S] = \frac{M[S_0]}{(t-\tau)} + \sum_{j=1}^{\infty} \{M[S_j] - cS_{j-1}\} \frac{j!}{(t-\tau)^{j+1}}$$

which implies that

$$\begin{aligned} &M[S_0] = 0 \\ &M[S_j] = cS_{j-1}, \ j = 1,2,\ldots , \end{aligned} \tag{6.7}$$

Equations (6.6) and (6.7) determine the S_j's uniquely. We must now show that (6.5) converges. By Duhamel's principle

$$S_j(x,\xi) = \int_\xi^x R(x,\eta)c(\eta)S_{j-1}(\eta,\xi)d\eta \; ; \; j = 1,2,\ldots \tag{6.8}$$

where $M[R] = 0$

$$R(\eta,\eta) = 0 \qquad R_x(\eta,\eta) = 1.$$

On any compact interval $|\xi|$, $|\eta|$, $|x| \le h$ from the analytic theory of ordinary differential equations there exist constants M_0, K, C such that

$$|S_{0x}(x,\xi)| \le M_0, \qquad |S_0(x,\xi)| \le M_0|x - \xi|$$

$$|R_x(x,\eta)| \le K, \qquad |R(x,\eta)| \le K\,|x - \eta|$$

$$|c(\eta)| \le C.$$

From the observation that

$$\left| \int_\xi^x |x-\eta| \frac{|\eta-\xi|^{2j-1}}{(2j-1)!} d\eta \right| = \frac{|x-\xi|^{2j+1}}{(2j+1)!}$$

we have by induction on (6.8) that

$$\begin{aligned} |S_j(x,\xi)| &\le M_0 M^j \frac{|x-\xi|^{2j+1}}{(2j+1)!} \\ |S_{jx}(x,\xi)| &\le M_0 M^j \frac{|x-\xi|^{2j}}{(2j)!} \end{aligned} \tag{6.9}$$

where $M = KC$. Equation (6.9) implies that the series (6.5) converges absolutely and uniformly and can be differentiated termwise.

Now consider the identity

$$\begin{aligned} \int\int_D \{v\mathcal{L}[u] - u\mathcal{M}[v]\}dxdt \\ = \int_{\partial D} \{(vu_x - uv_x + auv)dt + cuvdx\} \end{aligned} \tag{6.10}$$

where D is some two dimensional chain in the region of analyticity of the integrand and ∂D is its one dimensional boundary.

In equation (6.10) let $v = S$, u be a solution of $\mathcal{L}[u] = F$, and let D be the lateral surface of a cylinder that wraps around $t = \tau$ and has γ_0 and γ as its two rims, where γ_0 is a loop about $t = \tau$ in the plane $x = \xi$ and γ is some other loop about $t = \tau$ (see figure 6.1 below).

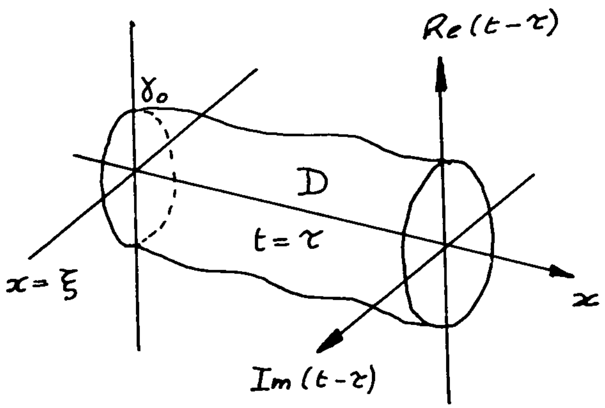


figure 6.1

Because of (6.4) we have

$$\oint_{\gamma 0} \{(Su_x - uS_x + auS)dt + cuSdx\} = \oint_{\gamma_0} \frac{u(\xi,t)}{t-\tau}dt$$
$$= 2\pi i\ u(\xi,t)$$

and hence (6.10) becomes

$$u(\xi,\tau) = \frac{1}{2\pi i} \oint_{\gamma} (Su_x - uS_x + auS)dt + cuSdx + \frac{1}{2\pi i} \int_D\int SFdxdt$$

Placing the cycle γ on the two dimensional manifold $x = s(t)$ (t complex) where the Cauchy data (6.2) is prescribed gives

$$u(\xi,\tau) = \frac{1}{2\pi i} \oint_{\gamma} \left\{(Su_x - uS_x + auS + cuS\dot{s} + \oint_{\xi}^{s(t)} SFdx\right\}dt, \tag{6.11}$$

the desired solution of the Cauchy problem (6.1), (6.2).

Suppose now $F \equiv 0$ and $s(t) =$ constant $= x_0$. Then (6.11) becomes

$$u(\xi,\tau) = \operatorname*{Res}_{t=\tau} \left\{ Su_x - uS_x + auS + cuS\dot{s} \right\}$$

$$= \sum_{j=0}^{\infty} \frac{\partial^j}{\partial\tau^j} \{S_j(x_0,\xi)u_x(x_0,\tau) \qquad (6.1$$

$$+ [a(x_0)S_j(x_0,\xi) - S_{jx}(x_0,\xi)]u(x_0,\tau)\}.$$

Suppose the Cauchy data, as a function of τ, is analytic for $|\tau| \leq \rho$. Then by Cauchy's inequality there exists a constant A such that

$$\left|\frac{\partial^j}{\partial\tau^j} u(x_0,\tau)\right| \leq A\frac{j!}{\rho^j} \qquad (6.1$$

$$\left|\frac{\partial^j}{\partial\tau^j} u_x(x_0,\tau)\right| \leq A\frac{j!}{\rho^j} \quad .$$

The estimates (6.9) and (6.13) imply that the series (6.12) is dominated (up to multiplication by a constant) by

$$\sum_{j=0}^{\infty} \frac{1}{j!} \left\{ \frac{M|x_0-\xi|^2}{\rho} \right\}^j = \exp \left\{\frac{M|x_0-\xi|^2}{\rho}\right\}, \qquad (6.1$$

which implies the following theorem:

<u>Theorem 6.1</u> ([26]): Assume the coefficients of $\mathcal{L}[u] = 0$ ar entire functions of x and let u be a solution of $\mathcal{L}[u] = 0$ which is a real analytic function of x and t in the circle $x^2 + t^2 < \rho^2$. Then u can be continued as an analytic function of x and t into the strip $-\rho < t < \rho$, $-\infty < x < \infty$

<u>Remarks</u>: For related results for parabolic equations in two space variables see [27].

7. <u>Improperly Posed Initial-Value Problems for Hyperbolic Equations</u>.

Consider the equation

$$u_{x_1x_1} = u_{x_2x_2} + u_{x_3x_3} + q(x_1,x_2,x_3)u - f(x_1,x_2,x_3) \qquad (7.1)$$

and assume q and f are entire functions of their independent (complex) variables. The Cauchy problem along a space-like surface is well posed:

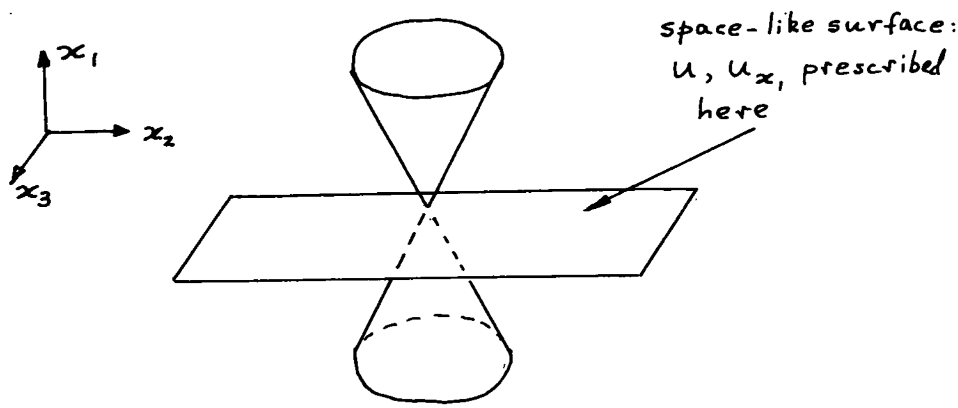


figure 7.1

However the Cauchy problem along a time-like surface is improperly posed (c.f. [20], p. 176):

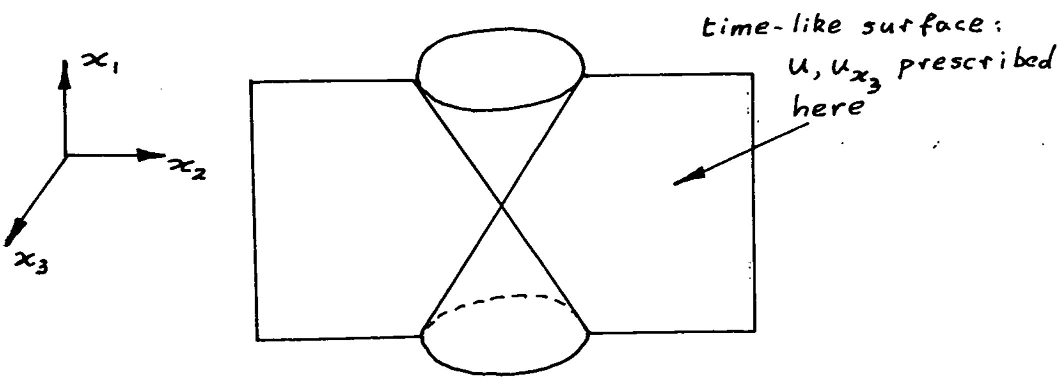


figure 7.2

Note that the distinction between time-like and space-like surfaces is not important in one space dimension.

Example 7.1:

$$u = \frac{1}{n^2} \sin hnx_2 \sin nx_3$$

is a solution of

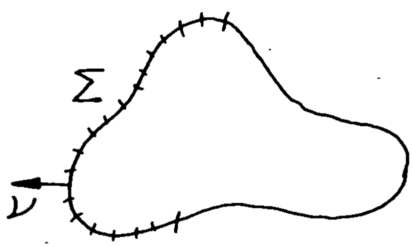
$$u_{x_1x_1} = u_{x_2x_2} + u_{x_3x_3}$$

satisfying

$$u(x_1,0,x_3) = 0 \quad u_{x_2}(x_1,0,x_3) = \frac{1}{n} \sin nx_3.$$

As $n \to \infty$ the initial data tends to zero but the solution does not, i.e. the Cauchy problem for hyperbolic equations along a time-like surface is improperly posed.

Example 7.2: Suppose we have a clamped, vibrating membrane and the slope of the deflection is measured on portion Σ of the boundary.



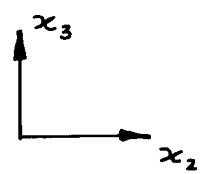


figure 7.3

We ask the following question: What must the initial displacement and velocity be to produce a prescribed slope of deflection (as a function of time) on Σ? This leads us to ask for a solution of the time-like Cauchy problem.

$$\frac{1}{c^2} u_{x_1x_1} = u_{x_2x_2} + u_{x_3x_3}$$

$$u = 0 \text{ on } \Sigma$$

$$\frac{\partial u}{\partial \nu} = g \text{ on } \Sigma$$

(where c = velocity of sound) and to then evaluate u and $\frac{\partial u}{\partial x_1}$ at $x_1 = 0$. Suppose the plane $x_2 = 0$ in figure 7.2 is bent 45^o on each side of the x_3 axis to form two intersecting planes tangent to a genertor of each nape of the characteristic cone. The exterior characteristic initial value problem for equation (7.1) is to find a solution u of (7.1) in the quarter space bounded by these intersecting planes such that u assumes prescribed values on each of the two planes. This problem is also improperly posed ([17]).

Example 7.3 ([17]): Suppose u is a solution of the three dimensional wave equation in spherical coordinates

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} \tag{7.2}$$

that is defined in $r \geq a > 0$ and vanishes for $t \leq r$. (This represents an outgoing wave produced by sources in $r < a$). It can be shown that if $t - r = \tau$ is bounded then

$$\lim_{r\to\infty} \{ru(r, \theta, \phi, r + \tau)\} = f(\theta, \phi, \tau)$$

exists i.e. $ru \sim f(\theta, \phi, t - r)$ for large r. The inverse problem is, given the "radiation field" f, to determine, u. Set

$$\begin{aligned} ru &= v \\ t - r &= \tau \\ t + r &= \frac{1}{\sigma} \end{aligned}$$

Then (7.2) becomes

$$\frac{\partial^2 v}{\partial\tau\partial\sigma} + \frac{1}{(1-\sigma\tau)^2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2}\right\} = 0 \tag{7.3}$$

and the data for the inverse problem is

$$v\Big|_{\tau = 0} = 0, \quad 0 \leq \sigma \leq \frac{1}{2a}$$

$$v\Big|_{\sigma = 0} = f(\theta, \phi, \tau), \quad \tau \geq 0 \qquad (7.4$$

$$f(\theta, \phi, 0) = 0.$$

Equations (7.3) and (7.4) constitute an exterior characteristi initial value problem.

We will need the following definition:

<u>Definition 7.1</u>: A function $g(x_1,x_2)$ of two real variables x_1 and x_2 will be said to be partially analytic with respect to x_1 for $x_1 = a$ in the interval $\alpha \leq x_2 \leq \beta$ provided it can be represented by a series of the form

$$g(x_1,x_2) = b_0(x_2) + b_1(x_2)(x_1 - a) + b_2(x_2)(x_1 - a)^2 + \dots \qquad (7.5$$

whose coefficients are continuous functions of x_2 in the inter val $\alpha \leq x_2 \leq \beta$ and provided that the series (7.5) converges absolutely and uniformly for $\alpha \leq x_2 \leq \beta$, $|x_1 - a| \leq \gamma$. The region $\alpha \leq x_2 \leq \beta$, $|x_1 - a| \leq \gamma$ is known as the region of partial analyticity. The extension to more variables is eviden First consider the time-like Cauchy problem. Let

$$x = x_3 - x_1$$
$$y = x_1 + x_3 \qquad (7.6$$
$$z = x_2.$$

Then (7.1) becomes

$$L[u] \equiv u_{zz} + 4u_{xy} + Q(x,y,z)u = F(x,y,z) \qquad (7.7$$

where $F(x,y,z) = f(x_1,x_2,x_3)$, $Q(x,y,z) = q(x_1,x_2,x_3)$. Let u and v be "well behaved" functions to be prescribed shortly and integrate the identity

$$vL[u] - uL[v] = (2vu_y - 2v_y u)_x + (2vu_x - 2v_x u)_y$$
$$+ (vu_z - uv_z)_z \tag{7.8}$$

over the torus $D \times \Omega$ where $\Omega = \Omega(\zeta)$: $|z - \zeta| = \delta > 0$ is a circle in the complex z plane and $D = D(\zeta) \subset \mathbb{R}^2$ ($\mathbb{R}^2$ is the Euclidean plane) is as in figure (7.4) below.

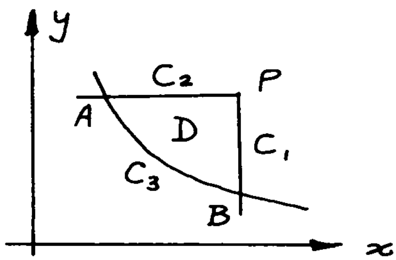


figure 7.4

$C_3 = \mathbf{C}_3(\zeta)$ is on the complex extension with respect to ζ of the intersection of the plane $z = \zeta$ with the smooth convex initial surface on which the Cauchy data is prescribed. It is assumed that the normal to this surface is never parallel to the z-axis and that the initial surface is partially analytic with respect to z. Note that in the special case when C_3 is independent of ζ the cylinder C_3 is time-like in Euclidean three space $\mathbb{R}^3$, but the cylinder C_3' in figure (7.5) below is space like.

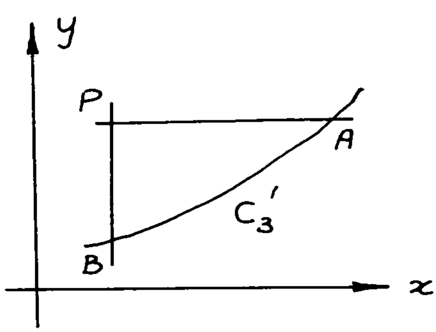


figure 7.5

Integrating (7.8) over $D \times \Omega$ gives

$$\iiint_{D\times\Omega} (vL[u] - uL[v])\,dxdydz$$

$$+ \int_{\Omega} [2v(A,z)u(A,z) + 2v(B,z)u(B,z) + 4v(P,z)u(P,z)]dz$$

$$+ 4\int\int_{C_1\times\Omega} uv_y dydz - 4\int\int_{C_2\times\Omega} uv_x dxdz \tag{7.9}$$

$$+ 2\int\int_{C_3\times\Omega} [(uv_y - vu_y)dydz - (uv_x - vu_x)dxdz] = 0$$

(Note that $dxdy = 0$ on $\partial D \times \Omega$). In equation (7.9)$v(A,z)=v(x,y$
where $A = (x,y)$, etc.. Now, let

1) u be a C^2 solution of $L[u] = F$ such that u and its derivatives of order less than or equal to two are partiall analytic with respect to z in some neighbourhood of a smooth (time-like) convex surface, where $C_3 = C_3(\zeta)$ lies on the complex extension of the intersection of this surface with the plane $z = \zeta$.

2) v be a fundamental solution of $L[u] = 0$ such that

$$v_y = 0 \qquad \text{on } C_1 \times \Omega \tag{7.10}$$

$$v_x = 0 \qquad \text{on } C_2 \times \Omega \tag{7.11}$$

and at the point $(P,z) = (\xi,\eta,z)$

$$v(P,z) = \frac{1}{8\pi i(z-\zeta)} + \text{analytic function of } (z-\zeta) \tag{7.12}$$

We must now construct v. A fundamental solution S of $L[v] = 0$ is of the (normalized) form

$$S = \frac{1}{8\pi i R} + \sum_{\ell=1}^{\infty} U_\ell R^{2\ell-1} + W \tag{7.13}$$

where $R = \sqrt{(z-\zeta)^2 + (x-\xi)(y-\eta)}$. The $U_\ell \equiv U_\ell(x,y,z;\ \xi,\eta,\zeta)$ can be computed recursively and are entire functions of their independent variables (since Q is - c.f. [20]). W is a regular solution of $L[w] = 0$. Let W satisfy the boundary conditions

$$W = -\sum_{=1}^{\infty} U_\ell (z-\zeta)^{2\ell-1} \quad \text{on } x = \xi \tag{7.14}$$

$$W = -\sum_{=1}^{\infty} U_\ell (z-\zeta)^{2\ell-1} \quad \text{on } y = \eta. \tag{7.15}$$

W exists from Theorem F and is entire since Q is. Then S defined by (7.13) satisfies equations (7.10)-(7.12) and we can set $v = S$ in (7.9) provided $|z - \zeta|^2 > |(x-\xi)(y-\eta)|$ i.e. z lies outside the cut in the complex z plane along a line parallel to the imaginary axis between $\zeta \pm i\sqrt{(x-\xi)(y-\eta)}$.

Note that in view of equations (7.10)-(7.12) W can actually be chosen in a variety of different ways, e.g. when $q = \text{constant} = \lambda^2$ a possible choice for $v = S_\lambda$ is

$$S_\lambda = \frac{\cos \lambda R}{8\pi\, iR} \quad . \tag{7.16}$$

From (7.12) we have

$$4 \int_\Omega v(P,z)u(P,z)dz = u(\xi,\eta,\zeta) \tag{7.17}$$

and hence (7.9) becomes (setting $v = S$)

$$\begin{aligned} u(\xi,\eta,\zeta) = {} & -2\int_{\Omega(\zeta)} [S(A,z;\xi,\eta,\zeta)u(A,z) + S(B,z;\xi,\eta,\zeta)u(B,z)]dz \\ & +2\int\int_{C_3(\zeta)\times\Omega(\zeta)} [u(x,y,z)S_x(x,y,z;\xi,\eta,\zeta)-S(x,y,z;\xi,\eta,\zeta)u_x(x,y,z)]dxdz \\ & -2\int\int_{C_3(\zeta)\times\Omega(\zeta)} [u(x,y,z)S_y(x,y,z;\xi,\eta,\zeta)-S(x,y,z;\xi,\eta,\zeta)u_y(x,y,z)]dydz \\ & -\iiint_{D(\zeta)\times\Omega(\zeta)} S(x,y,z;\xi,\eta,\zeta)F(x,y,z)dxdydz. \end{aligned} \tag{7.18}$$

Equation (7.18) gives the solution of the time-like Cauchy problem along a smooth convex surface.

Note that

1) Partial analyticity of the Cauchy data and its derivatives of order less than or equal to two along $y = y(x)$ implies that u and its derivative of order less than or equal to two are partially analytic in $D \times \Omega$.

2) Recall that S is an analytic function of z outside the cut between $\zeta \pm i\sqrt{(x-\xi)(y-\eta)}$. Let $G(x,y)$ be an arbitrarily small neighbourhood of this cut. Then (by deforming Ω) equation (7.18) shows that at the point (ξ,η,ζ) u depends continuously on its Cauchy data in $C_3 \times G$, where $G \supset G(x,y)$ for all points $(x,y) \in C_3$.

By deforming the curve C_3 onto the characteristics $C_4 = AT$, $C_5 = TB$ (see figure 7.6 below), and integrating by parts to eliminate the partial derivatives of u along these characteristics, we arrive at the solution of the exterior characteristic initial-value problem ([7]):

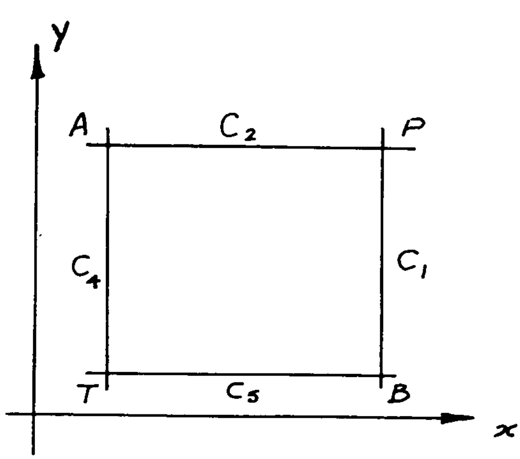


figure 7.6

$$u(\xi,\eta,\zeta) = -2\int_{\Omega} [S(A,z;\xi,\eta,\zeta)u(A,z) + S(B,z;\xi,\eta,\zeta)u(B,z) - S(T,z;\xi,\eta,\zeta)u(T,z)]dz$$

$$-4\int\int_{C_4\times\Omega} S_y(x,y,z;\xi,\eta,\zeta)u(x,y,z)dydz \tag{7.19}$$

$$+4\int\int_{C_5\times\Omega} S_x(x,y,z;\xi,\eta,\zeta)u(x,y,z)dxdz$$

$$-\int\int_{D\times\Omega}\int S(x,y,z;\xi,\eta,\zeta)F(x,y,z)dxdydz.$$

Note that the regions of integration are now independent of ζ.

Remark: Similar integral representations can be obtained for the Cauchy and Goursat problems for the elliptic equation

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} + q(x_1,x_2,x_3)u = f(x_1,x_2,x_3) \tag{7.20}$$

with data along a convex analytic surface. To see this set

$$X = x_1$$

$$Z = x_2 + ix_3$$

$$Z^* = x_2 - ix_3.$$

Then (7.20) becomes

$$U_{XX} + 4U_{ZZ^*} + Q(X,Z,Z^*)U = F(X,Z,Z^*)$$

which is of the same form as (7.7). Repeating the previous analysis now leads to the representations (7.18) and (7.19) (with z replaced by X, x replaced by Z, and y replaced by Z*). for the solution of the Cauchy and complex Goursat problem respectively.

III Integral operators for elliptic equations

8. Integral Operators in Two Independent Variables

In one sense the integral representations obtained in sections 6 and 7 can be viewed as integral operators for parabolic and hyperbolic equations. Here we obtain integral operators for elliptic equations such that the kernel of the operator is an entire function of its independent variables.

Consider the self-adjoint equation (see, however, the remark at the end of this section).

$$\Delta_2 u - q(x,y)u = 0 \tag{8.1}$$

Where $u(x,y) \in C^2(D)$, D is simply connected with C^2 boundary ∂D, $q(x,y)$ is a real valued (for x, y real) entire function of the (complex) variables x,y, (with minor modifications we could have considered q to be analytic only in some polycylinder). We want to generalize the following example:

Example 8.1: Suppose $\Delta u = 0$ in D. Then $u = \text{Re } f(z)$ where $f(z)$ is analytic. But by Runge's theorem $\{z^n\}$ is a complete family of analytic functions in D. Hence $\{\text{Re } z^n\} = \{r^n \cos n\theta\}$ and $\{\text{Im } z^n\} = \{r^n \sin n\theta\}$ together form a complete family of solutions for $\Delta u = 0$ in D. To approximate solutions of $\Delta u = 0$ in D, $u = f$ on ∂D, set $u_N = \sum_{m=0}^{N} a_m r^n \cos n\theta + b_m r^n \sin n\theta$ and minimise $|u_N - f|$ on ∂D. By Theorem E this minimizes $|u_N - u|$ in D. A complete family of solutions can also be used to approximate solutions to Cauchy's problem:

Example 8.2: Suppose $\Delta_n u + q(\underset{\sim}{x})u = 0$ in D and $u = f, \frac{\partial u}{\partial \nu} = g$ on $\Sigma \subset \partial D$. Then if $|u| \leq 2M$ in D then there exist constants

$C_1 = C_1(M)$, $C_2 = C_2(M)$ such that

$$\max_{\underset{\sim}{x}\in\bar{D}} |u|^{2/\delta} \le C_1 \int_\Sigma |u|^2 ds + C_2 \int_\Sigma \left|\frac{\partial u}{\partial \nu}\right| ds$$

where $0 < \delta < 1$. (c.f. [34]).

Let $\{\phi_k\}$ be a complete family of solutions to $\Delta_n u + q(\underset{\sim}{x})u = 0$

$$u_N = \sum_{k=1}^{N} a_k \phi_k(\underset{\sim}{x}) .$$

To approximate u (under the assumption $|u| \le M$ in D) use the Rayleigh Ritz procedure to minimize

$$C_1 \int_\Sigma (f-u_N)^2 ds + C_2 \int_\Sigma \left(g - \frac{\partial u_N}{\partial \nu}\right)^2 ds$$

subject to the constraint $|u_N| \le M$ in D.

Now let

$$\begin{aligned} z &= x + iy \\ z^* &= x - iy \end{aligned} \tag{8.2}$$

be a mapping of $C_2 \to C_2$ (C_2 denotes the space of two complex variables). Equation (8.1) becomes

$$L(U) = \frac{\partial^2 U}{\partial z \partial z^*} + Q(z,z^*)U = 0 \tag{8.3}$$

$$\text{where } Q(z,z^*) = -\frac{1}{4} q\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right)$$

$$U(z,z^*) = u\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right). \tag{8.4}$$

<u>Theorem 8.1 ([2])</u>: Let $E(z,z^*,t)$ be an analytic function of t, z, z^* for $|t| \le 1$ and z, z^* in some neighbourhood of the origin which satisfies the partial differential equation

$$-(1 - t^2)E_{z^*t} + \frac{1}{t} E_{z^*} - 2tzL(E) = 0 \tag{8.5}$$

and is such that E_{z^*}/zt is continuous at $z = 0$, $t = 0$. Then if $f(z)$ is an analytic function of z in a neighbourhood of $z = 0$,

$$\mathcal{U}(z,z^*) = \underset{\sim}{P}_2\{f\} = \int_{-1}^{1} E(z,z^*,t) f(\tfrac{z}{2}(1-t^2)) \frac{dt}{(1-t^2)^{\frac{1}{2}}} \tag{8.6}$$

will be a solution of (8.3) in a sufficiently small neighbourhood of $z = 0$, $z^* = 0$.

Proof:

$$\mathcal{U}_{zz^*} = \int_{-1}^{1}\left(E_{zz^*} f(\tfrac{z}{2}(1-t^2)) + E_{z^*}\frac{\partial\, f(\frac{z}{2}(1-t^2))}{\partial z}\right)\frac{dt}{\sqrt{1-t^2}}$$

But $f_z = -f_t(1-t^2)/2zt$ which implies that

$$\mathcal{U}_{zz^*} = \int_{-1}^{1}\left(E_{zz^*}f - E_{z^*}(1-t^2)(2zt)^{-1}f_t\right)\frac{dt}{\sqrt{1-t^2}} .$$

Integrating the second term by parts gives

$$\mathcal{U}_{zz^*} = \int_{-1}^{1} E_{zz^*}\, f\frac{dt}{\sqrt{1-t^2}} - \left(\frac{E_{z^*}\sqrt{1-t^2}}{2zt}\, f\left(\frac{z}{2}(1-t^2)\right)\right)_{t=-1}^{t=+1}$$

$$+\int_{-1}^{1}\left(\frac{E_{z^*}\sqrt{1-t^2}}{2zt}\right)_t f\,dt = \int_{-1}^{1}\left(\frac{E_{zz^*}}{\sqrt{1-t^2}} + \left(\frac{E_{z^*}\sqrt{1-t^2}}{2zt}\right)_t\right) f\left(\frac{z}{2}(1-t^2)\right) dt.$$

Equations (8.3), (8.5) now imply the theorem.

Theorem 8.2 ([2]): There exists a function $E(z,z^*,t)$ satisfying all the conditions of Theorem 8.1 and also

$$E(0,z^*,t) = E(z,0,t) = 1. \tag{8.7}$$

If $q(x,y)$ is entire then $E(z,z^*,t)$ is an entire function of z and z^*.

Proof: Let

$$E(z,z^*,t) = 1 + \sum_{n=1}^{\infty} t^{2n}\, z^n \int_0^{z^*} P^{(2n)}(z,z^*)dz^*. \tag{8.8}$$

Substitute (8.8) into (8.5) and compare powers of t.

This yields

$$P^{(2)}(z,z^*) = -2\,Q(z,z^*) \tag{8.9}$$

$$(2n+1)P^{(2n+2)}(z,z^*) = -2\left(P_z^{(2n)} + Q(z,z^*)\int_0^{z^*} P^{(2n)}(z,z^*)\,ds^*\right);$$

$n = 1,2,\ldots$

Thus the $P^{(2n)}$ are uniquely determined. We must now show (8.8) converges.

<u>Definition 8.1</u>: Let $S = \sum_{m,n=0}^{\infty} a_{mn} z^m z^{*n}$, $\tilde{S} = \sum_{m,n=0}^{\infty} \tilde{a}_{mn} z^m z^{*n}$

where $\tilde{a}_{mn} \geq 0$. Then we say the series $\tilde{S}$ dominates the series S if $|a_{mn}| \leq \tilde{a}_{mn}$ $(m,n = 0,1,\ldots)$, and write $S << \tilde{S}$. Note that if $S << \tilde{S}$ then

1) $\dfrac{\partial S}{\partial z} << \dfrac{\partial \tilde{S}}{\partial z}$

2) $\int_0^{z^*} S(z,z^*)\,dz^* << \int_0^{z^*} \tilde{S}(z,z^*)\,dz^*$

3) $S << \dfrac{\tilde{S}}{1-az}$, $a \geq 0$.

Since $Q(z,z^*)$ is entire we have $Q(z,z^*) = \sum_{m,n=0}^{\infty} a_{mn} z^m z^{*n}$ converges uniformly and absolutely for $|z| \leq r$, $|z^*| \leq r$ for every $r > 0$. Hence there exists an $M > 0$ such that $|a_{mn} r^m r^n| < M$ $(m,n = 0, 1, 2,\ldots)$, i.e.

$$Q(z,z^*) << M\left(1 - \frac{z}{r}\right)^{-1}\left(1 - \frac{z^*}{r}\right)^{-1} \equiv \tilde{Q}(z,z^*). \tag{8.10}$$

Now define $\tilde{P}^{(2n)}(z,z^*)$ $(n = 1, 2, \ldots)$ by

$$\tilde{P}^{(2)}(z,z^*) = 2\,\tilde{Q}(z,z^*)$$

$$(2n + 1)\tilde{P}^{(2n+2)}(z,z^*) = 2\left(\tilde{P}_z^{(2n)}(z,z^*)\left(1 - \frac{z^*}{r}\right)^{-1} + \right.$$

$$+ \tilde{Q}(z,z^*)\int_0^{z^*} \tilde{P}^{(2n)}(z,z^*)\left(1 - \frac{z^*}{r}\right)^{-1} dz^* +$$

$$+ c^{(2n)} r M n^{-1} \left(1 - \frac{z}{r}\right)^{-n-1}\left(1 - \frac{z^*}{r}\right)^{-n-1}\Big), \quad (n = 1,2,\ldots) \tag{8.11}$$

where

$$c^{(2)} = 2M$$

$$c^{(2m+2)} = c^{(2n)}\left(\frac{2n}{2n+1}\,\frac{1}{r} + \frac{2Mr}{n(2n+1)}\right), \quad (n=1,2,\ldots) \tag{8.12}$$

Note that the $\tilde{P}^{(2n)}$ are uniquely determined and $P^{(2n)} \ll \tilde{P}^{(2n)}$.

<u>Exercise 8.1</u>: Show that

$$\tilde{P}^{(2n)}(z,z^*) = c^{(2n)}\left(1 - \frac{z}{r}\right)^{-n}\left(1 - \frac{z^*}{r}\right)^{-n}, \quad (n = 1,2 \ldots)$$

where for each $\epsilon > 0$

$$c^{(2n)} \leq N\left(\frac{1+\epsilon}{r}\right)^{n}, \quad (n = 1,2,\ldots) \text{ for some } N = N(\epsilon)$$

Exercise 8.1 implies that

$$\tilde{P}^{(2n)} \ll \frac{N(1+\epsilon)^n}{r^n\left(1 - \frac{z}{r}\right)^n\left(1 - \frac{z^*}{r}\right)^n}, \quad (n = 1,\ 2,\ \ldots) \tag{8.13}$$

Hence

$$1 + N\int_0^{z^*} \sum_{n=1}^{\infty} \frac{z^n(1+\epsilon)^n}{r^n\left(1 - \frac{z}{r}\right)^n\left(1 - \frac{z^*}{r}\right)^n}\, dz^* \tag{8.14}$$

is a dominant for (8.8) with $|t| \leq 1$. Since ϵ is arbitrary (8.14) will converge uniformly and absolutely provided that

$$\left|\frac{z}{r\left(1 - \frac{|z|}{r}\right)\left(1 - \frac{|z^*|}{r}\right)}\right| \leq \eta < 1. \tag{8.15}$$

Since Q is entire, r can be arbitrarily large which implies that the series (8.8) converges to an entire function of z and

z^* for $|t| \leq 1$. Q.E.D.

Since $q(x,y)$ is real valued for x,y real, $\operatorname{Re} \underset{\sim}{P}_2\{f\}$ is a solution of $\Delta_2 u - q(x,y)u = 0$. For x, y real we have

$$\operatorname{Re} \underset{\sim}{P}_2\{f\} = \frac{1}{2}\left[\int_{-1}^{1} E(z, \bar{z}, t) f(\tfrac{z}{2}(1 - t^2)) \frac{dt}{\sqrt{1-t^2}} + \right. \left. + \int_{-1}^{1} \bar{E}(\bar{z}, z, t)\, \bar{f}(\tfrac{\bar{z}}{2}(1-t^2)) \frac{dt}{\sqrt{1-t^2}}\right] \tag{8.16}$$

where $\bar{f}(z) = \overline{f(\bar{z})}$, $\bar{E}(z,\bar{z},t) = \overline{E(\bar{z}, z, t)}$. Now extend x and y into the complex plane, i.e. set $\bar{z} = z^*$ in (8.16). From (8.7) we have

$$\operatorname{Re} \underset{\sim}{P}_2\{f\} \Big|_{z^*=0} = \frac{1}{2}\left[\pi \bar{f}(0) + \int_{-1}^{1} f(\tfrac{z}{2}(1-t^2)) \frac{dt}{\sqrt{1-t^2}}\right] \tag{8.17}$$

Now let $u(x,y)$ be a real valued C^2 solution of $\Delta_2 u - q(x,y)u = 0$ in D. Then we can (locally) expand $U(z,z^*) = u(\frac{z+z^*}{2}, \frac{z-z^*}{2i})$ as

$$U(z,z^*) = \sum_{m,n=0}^{\infty} a_{mn}\, z^m z^{*n} \tag{8.18}$$

Since $U(z,z^*)$ is real valued for x, y real, $a_{mn} = \overline{a_{nm}}$

<u>Exercise 8.2</u>: Show from equation (8.3) and the fact that $a_{mn} = \overline{a_{nm}}$ that $U(z,z^*)$ is uniquely determined by $U(z,0)$.

Exercise 8.2 shows that if we choose $f(z)$ in equation (8.17) such that $\operatorname{Re} \underset{\sim}{P}_2\{f\}\Big|_{z^*=0} = U(z,0)$, then we have

$$u(x,y) = \operatorname{Re} \underset{\sim}{P}_2\{f\} \tag{8.19}$$

i.e. <u>every</u> real valued C^2 solution of $\Delta_2 u - q(x,y)u = 0$ can be expressed in the form of equation (8.19) for some analytic function $f(z)$.

<u>Theorem 8.3 ([2])</u>: The functions $\{\operatorname{Re} \underset{\sim}{P}_2\{z^n\}\}_{n=0}^{\infty}$ and

$\{\text{Im}\ \underset{\sim}{P}_2\{z^n\}\}_{n=0}^{\infty}$ together form a complete family of solutions for $\Delta_2 u - q(x,y)u = 0$ in any simply connected domain D.

Proof: The proof follows from (8.19) Theorem B, and Runge's Theorem for analytic functions of a single complex variable.

Remark: A similar analysis (c.f. [1]) as in this section yields integral operators for the equation

$$\Delta_2 u + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0 \tag{8.20}$$

Alternatively, integral operators for equation (8.20) can be obtained via the Riemann function, c.f. Section 1, exercise 3.1 and equation (11.7). Equation (11.7) defines an operator mapping ordered pairs of analytic functions onto complex valued solutions of equation (8.20). Taking the real parts of both sides of equation (11.7) yields an operator mapping a single analytic function onto real valued solutions of equation (8.20) (c.f. [39]).

9. Integral Operators for Self Adjoint Equations in Three Independent Variables

Consider the partial differential equation

$$\Delta_3 u - q(x,y,z)u = 0 \tag{9.1}$$

where $q(x,y,z)$ is a real valued (for x, y, z real) entire function of the (complex) variables x, y, z.

Theorem 9.1 ([8]):

Let

$$\begin{aligned} X &= x \\ Z &= \frac{1}{2}(y + iz) \\ Z^* &= \frac{1}{2}(-y + iz) \end{aligned} \tag{9.2}$$

and let $u(x,y,z)$ be a real valued C^2 solution of (9.1) in a neighbourhood of the origin. Then $\mathcal{U}(X,Z,Z^*) = u(x,y,z)$ is an analytic function of X, Z, Z* in some neighbourhood of the origin in C_3 and is uniquely determined by the function $F(X,Z^*) = \mathcal{U}(X,0,Z^*)$.

<u>Proof</u>: $u(x,y,z) \in C^2$ implies that $\mathcal{U}(X,Z,Z^*)$ is analytic. Hence locally

$$\mathcal{U}(X,Z,Z^*) = \sum a_{mn\ell} X^{\ell} Z^n Z^{*m} \tag{9.3}$$

$$\mathcal{U}(X,0,Z^*) = \sum a_{mo\ell} X^{\ell} Z^{*m} \tag{9.4}$$

$$\mathcal{U}(X,Z,0) = \sum a_{on\ell} X^{\ell} Z^n .$$

$u(x,y,z)$ real valued implies that for x, y, z real

$$\mathcal{U}(X,Z,Z^*) = \mathcal{U}\overline{(X,Z,Z^*)} \tag{9.5}$$

and hence for x, y, z real

$$\sum a_{mn\ell}\, X^{\ell}\, Z^n Z^{*m} = \sum \overline{a_{mn\ell}}\quad X^{\ell}(-\, Z^*)^n(-Z)^m, \tag{9.6}$$

i.e.

$$a_{mn\ell} = (-\,1)^{n+m}\ \overline{a_{nm\ell}}\ . \tag{9.7}$$

Equations (9.7) and (9.4) imply that $\mathcal{U}(X,Z,0)$ is uniquely determined from $\mathcal{U}(X,0,Z^*)$. But in X, Z, Z* coordinates (9.1) becomes

$$\mathcal{U}_{XX} - \mathcal{U}_{ZZ^*} - Q(X,Z,Z^*)\mathcal{U} = 0 \tag{9.8}$$

(where for x,y, z real $Q(X,Z,Z^*) = q(x,y,z)$). Hence from Theorem F (see also section 7) $\mathcal{U}(X,Z,Z^*)$ is uniquely determined from $\mathcal{U}(X,Z,0)$ and $\mathcal{U}(X,0,Z^*)$ i.e. from $\mathcal{U}(X,0,Z^*)$ alone.

Now define

$$\xi_1 = 2\zeta Z \tag{9.9}$$

$$\xi_2 = X + 2\zeta Z$$

$$\xi_3 = X + 2\zeta^{-1}Z^* \tag{9.9}$$

$$\mu = \tfrac{1}{2}(\xi_2 + \xi_3) = X + \zeta Z + \zeta^{-1}Z^* \tag{9.10}$$

where

$$1 - \epsilon < |\zeta| < 1 + \epsilon,\ 0 < \epsilon < \tfrac{1}{2}.$$

Theorem 9.2 ([8]): Let D be a neighbourhood of the origin in the μ plane, $B = \{\zeta : 1 - \epsilon < |\zeta| < 1 + \epsilon\}$, G a neighbourhood of the origin in the ξ_1, ξ_2, ξ_3 space, and $T = \{t : |t| \leq 1\}$. Let $f(\mu,\zeta)$ be an analytic function of two complex variables in $D \times B$ and let $E^*(\xi_1,\xi_2,\xi_3,\zeta,t) \equiv E(X,Z,Z^*,\zeta,t)$ be a regular solution in $G \times B \times T$ of the partial differential equation

$$\mu t(4E^*_{13} + 2E^*_{23} - E^*_{22} - E^*_{33} + Q^*E^*) + (1-t^2)E^*_{1t} - \frac{1}{t}E^*_1 = 0 \tag{9.11}$$

where

$$Q^*(\xi_1,\xi_2,\xi_3,\zeta) \equiv Q(X,Z,Z^*) \text{ and } E^*_i = \frac{\partial E^*}{\partial \xi_i}.$$

Then

$$\mathcal{U}(X,Z,Z^*) = \underset{\sim}{P}_3\{f\} =$$

$$= \frac{1}{2\pi i}\int_{|\zeta|=1}\int_{-1}^{+1} E(X,Z,Z^*,\zeta,t)f(\mu(1-t^2),\zeta)\frac{dt}{\sqrt{1-t^2}}\,\frac{d\zeta}{\zeta} \tag{9.12}$$

is a (complex valued) solution of (9.1) which is regular in a neighbourhood of the origin in X, Z, Z^* space.

Proof: The Jacobian of the transformation (9.9) is -4 which implies that $\mathcal{U}(X,Z,Z^*) = \underset{\sim}{P}_3\{f\}$ is regular in a neighbourhood of the origin. Differentiating and integrating by parts in (9.12) (using $\frac{\partial f}{\partial w} = \frac{-1}{2\mu t}\frac{\partial f}{\partial t}$ where $w = \mu(1-t^2)$) leads to

$$U_{zz^*} - U_{xx} + QU =$$

$$= \frac{1}{2\pi i} \int_{|\zeta|=1} \int_{-1}^{+1} \frac{f(\mu(1-t^2),\zeta)}{\mu t} \{\mu t(4E_{13}^* + 2E_{23}^* - E_{33}^* + Q^*E^* - E_{22}^*)$$

$$+ (1-t^2)E_{1t}^* - \frac{1}{t} E_1^*\} \frac{dt}{\sqrt{1-t^2}} \frac{d\zeta}{\zeta}$$

which implies the theorem.

Theorem 9.3([8]): Let $D_r = \{(\xi_1, \xi_2, \xi_3) + |\xi_i| < r, i = 1,2,3\}$ where r is an arbitrary positive number, and $B_{2\epsilon} = \{\zeta : |\zeta - \zeta_0| < 2\epsilon\}$, $0 < \epsilon < \frac{1}{2}$, where ζ_0 is arbitrary with $|\zeta_0| = 1$. Then for every n, n = 0, 1, 2, ... there exists a unique function $p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta)$ regular in $\overline{D}_r \times \overline{B}_{2\epsilon}$ and satisfying

$$p_1^{(n+1)} = \frac{1}{2n+1}\left[p_{22}^{(n)} + p_{33}^{(n)} - 4p_{13}^{(n)} - 2p_{23}^{(n)} - Q^*p^{(n)}\right]$$

$$p^{(0)}(\xi_1, \xi_2, \xi_3, \zeta) \equiv 1, \quad p^{(n+1)}(0, \xi_2, \xi_3, \zeta) = 0;$$

$$n = 0,1,2,... \tag{9.13}$$

where $p_i^{(n)} = \frac{\partial p^{(n)}}{\partial \xi_i}$. Furthermore the function

$$E^*(\xi_1, \xi_2, \xi_3, \zeta, t) = 1 + \sum_{n=1}^{\infty} t^{2n}\mu^n p^{(n)}(\xi_1,\xi_2,\xi_3,\zeta) \tag{9.14}$$

is a solution of (9.11) which is regular in $G_R \times B \times T$ where R is an arbitrary positive number and

$$G_R = \{(\xi_1,\xi_2,\xi_3) + |\xi_i| < R, i = 1,2,3\}$$

$$B = \{\zeta : 1 - \epsilon < |\zeta| < 1 + \epsilon\}, \quad 0 < \epsilon < \tfrac{1}{2}$$

$$T = \{t : |t| \leq 1\}.$$

The function defined in (9.14) satisfies

$$E^*(0, \xi_2 \;\; \xi_3 \;\; \zeta, t) = 1 \tag{9.15}$$

<u>Proof</u>: $p^{(1)}(\xi_1,\xi_2,\xi_3,\zeta) = -\int_0^{\xi} Q^*(\xi_1{}',\xi_2\ \xi_3\ \zeta)d\xi_1{}'$ is uniquely determined and is regular in $\overline{D}_r \times \overline{B}_{2\epsilon}$. By induction all the $p^{(n)}$ are uniquely determined. Substituting (9.14) into (9.11) shows that E* formally satisfies (9.11). We must now show the series converges uniformly in $G_R \times B \times T$. Since $\overline{B}$ is compact there exists $\zeta_j, |\zeta_j| = 1$, $j = 1, \ldots, N$ such that B is covered by $\bigcup_{j=1}^{N} N_j$ where $N_j = \{\zeta - \zeta_j| < \frac{3}{2}\epsilon\}$. Hence it is sufficient to show that the series converges in $\overline{G}_R \times \overline{N}_j \times T$. Since Q is entire, in $\overline{D}_r \times \overline{B}_{2\epsilon}$ we have

$$Q^*(\xi_1,\xi_2,\xi_3,\zeta) << C\left(1-\frac{\xi_1}{r}\right)^{-1}\left(1-\frac{\xi_2}{r}\right)^{-1}\left(1-\frac{\xi_3}{r}\right)^{-1}\left(1-\frac{\zeta-\zeta_0}{2\epsilon}\right)^{-1}$$

for some $C > 0$ where "<<" means "is dominated by".

<u>Exercise 9.1</u>: Show by induction that in $\overline{D}_r \times \overline{B}_{2\epsilon}$

$$p_1^{(n)} << M(8+\delta)^n(2n-1)^{-1}\left(1-\frac{\xi_1}{r}\right)^{-(2n-1)}\left(1-\frac{\xi_2}{r}\right)^{-(2n-1)}$$

$$\left(1-\frac{\xi_3}{r}\right)^{-(2n-1)}\left(1-\frac{\zeta-\zeta_0}{2\epsilon}\right)^{n} r^{-n}$$

where M and δ are positive constants independent of n.

Exercise 9.1 implies that

$$p^{(n)} << M(8+\delta)^n(2n)^{-1}(2n-1)^{-1}\left(1-\frac{\xi_1}{r}\right)^{-2n}\left(1-\frac{\xi_2}{r}\right)^{-(2n-1)}$$

$$\left(1-\frac{\xi_3}{r}\right)^{-(2n-1)}\left(1-\frac{\zeta-\zeta_0}{2\epsilon}\right)^{-n} r^{-n+1}$$

and hence in $\overline{D}_r \times \overline{N}_j \times T$ we have

$$|p^{(n)}(\xi_1,\xi_2,\xi_3,\zeta)| \leq M(8+\delta)^n(2n)^{-1}(2n-1)^{-2}\left(1-\frac{|\xi_1|}{r}\right)^{-2n}$$

$$\left(1-\frac{|\xi_2|}{r}\right)^{-(2n-1)}\left(1-\frac{|\xi_3|}{r}\right)^{-(2n-1)}\left(1-\frac{|\zeta-\zeta_j|}{2\epsilon}\right)^{-n} r^{-n+1}.$$

Now consider $|t^{2n}\ \mu^n\ p^{(n)}(\xi_1,\xi_2,\xi_3,\zeta)|$ in $\overline{D}_{\alpha r} \times \overline{N}_j \times T$ where

$$D_{\alpha r} = \{(\xi_1,\xi_2,\xi_3) : |\xi_i| < \frac{r}{\alpha} ;\ \alpha > 1,\ i = 1,2,3\}$$

In $\overline{D}_{\alpha r} \times \overline{N}_j \times T$ we have

$$1 - \frac{|\xi_i|}{r} \geq \frac{\alpha-1}{\alpha},\ i = 1,\ 2,\ 3$$

$$1 - \frac{|\zeta-\zeta_j|}{2\epsilon} \geq \frac{1}{4}$$

$$|\mu| = \tfrac{1}{2}|\xi_2 + \xi_3| \leq \frac{r}{\alpha}$$

$$|t| \leq 1$$

which implies

$$|t^{2n}\mu^n\ p^{(n)}(\xi_1,\xi_2,\xi_3,\zeta)| \leq Mr(\alpha-1)^2\alpha^{-2}(2n)^{-1}(2n-1)^{-1}$$
$$\{4\ \alpha^5(\delta+\delta)(\alpha-1)^{-6}\}^n$$

and hence for α sufficiently large the series (9.14) converges absolutely and uniformly in $\overline{D}_{\alpha r} \times \overline{N}_j \times T$. Setting $r = \alpha R$ shows that E^* is regular in $\overline{G}_R \times \overline{N}_j \times T$ and hence is regular in $G_R \times B \times T$.

Exercise 9.2: Show that if $Q(X,Z,Z^*) = \lambda =$ constant, then

$$E(X,Z,Z^*,\zeta,t) = \cos\sqrt{4\lambda(\zeta XZ + \zeta^2 Z^2 + ZZ^*)}\ t.$$

Exercise 9.3: Show that $E(X,Z,Z^*,\zeta,t)$ is an entire function of its five independent complex variables if $Q(X,Z,Z^*)$ is entire.

Theorem 9.4 ([8]): Let $u(x,y,z)$ $(= U(X,Z,Z^*))$ be a real valued C^2 solution of (9.1) in some neighbourhood of the origin in $\mathbb{R}^3$. Then there exists an analytic function of two complex variables $f(\mu,\zeta)$ which is regular for $|\zeta| < 1 + \epsilon, \epsilon > 0$, and μ in some neighbourhood of the origin such that locally $u(x,y,z) = \mathrm{Re}\ \underset{\sim}{P}_3\{f\}$.

In particular

$$f(\mu,\zeta) = -\frac{1}{2\pi}\int_\gamma g(\mu(1-t^2),\zeta)\,\frac{dt}{t^2}$$

where

$$g(\mu,\zeta) = 2\,\frac{\partial}{\partial\mu}\left[\mu\int_0^1 \mathbf{U}(t\mu,\ 0,\ (1-t)\mu\zeta)dt\right] - \mathbf{U}(\mu,0,0) \tag{9.16}$$

and γ is a rectifiable arc joining the points $t = -1$ and $t = +1$ and not passing through the origin.

<u>Proof</u>: $u(x,y,z)\in C^2$ implies that $u(x,y,z)$ is analytic and $q(x,y,z)$ real implies that Re $\underset{\sim}{P}_3\{f\}$ (where x,y,z are real) is a real valued solution of (9.1) for every analytic function $f(\mu,\zeta)$. Now suppose locally that

$$g(\mu,\zeta) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{nm}\,\mu^n\,\zeta^m$$

$$f(\mu,\zeta) = -\frac{1}{2\pi}\int_\gamma g(\mu(1-t^2),\zeta)\,\frac{dt}{t^2} \tag{9.17}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{nm}\,\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})}\,\mu^n\zeta^m$$

$$Q(X,Z,Z^*) = \sum_{\ell,m,n=0}^{\infty} b_{mn\ell}\,X^\ell Z^n Z^{*m}.$$

<u>Exercise 9.4</u>: Show that

$$g(\mu,\zeta) = \int_{-1}^{1} f(\mu(1-t^2),\zeta)\,\frac{dt}{\sqrt{1-t^2}}\ .$$

Define

$$\overline{Q}(X,Z,Z^*) = \sum_{\ell,m,n=0}^{\infty} \overline{b_{mn\ell}}\,X^\ell Z^n Z^{*m}$$

$$\overline{f}(\mu,\zeta) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \overline{a_{nm}}\,\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})}\,\mu^n\,\zeta^m. \tag{9.18}$$

Let $\overline{E}\,(X,Z,Z^*,\zeta,t)$ be the generating function corresponding to the equation $U_{xx} - U_{zz^*} - \overline{Q}U = 0$. Then for x,y,z real

$$\mathrm{Re}\ \underset{\sim}{P}_3\{f\} = \frac{1}{4\pi i}\int_{|\zeta|=1}\int_{-1}^{1} E(X,Z,Z^*,\zeta,t)f(\mu(1-t^2),\zeta)\frac{dt}{\sqrt{1-t^2}}\frac{d\zeta}{\zeta}$$

$$+ \frac{1}{4\pi i}\int_{|\zeta|=1}\int_{-1}^{1} \overline{E}(X,-Z^*,-Z,\zeta,t)\overline{f}(\overline{\mu}(1-t^2),\zeta)\frac{dt}{\sqrt{1-t^2}}\frac{d\zeta}{\zeta} \tag{9.19}$$

where $\overline{\mu} = X - \zeta Z^* - \zeta^{-1}Z$. From Theorem 9.1 $U(X,Z,Z^*)$ is uniquely determined by $U(X,0,Z^*)$ and hence we try and determine $f(\mu,\zeta)$ from the integral equation

$$U(X,0,Z^*) = \frac{1}{4\pi i}\int_{|\zeta|=1}\int_{-1}^{1} f(\mu_1(1-t^2),\zeta)\frac{dt}{\sqrt{1-t^2}}\frac{d\zeta}{\zeta}$$

$$+ \frac{1}{4\pi i}\int_{|\zeta|=1}\int_{-1}^{1}\overline{E}(X\ -Z^*,0,\zeta,t)\overline{f}(\mu_2(1-t^2),\zeta)\frac{dt}{\sqrt{1-t^2}}\frac{d\zeta}{\zeta} \tag{9.20}$$

where $\mu_1 = X + \zeta^{-1}Z^*$, $\mu_2 = X - \zeta Z^*$. But

$$\overline{E}^*\,(\xi_1,\xi_2,\xi_3,\zeta,\ t) = 1 + \sum_{n=1}^{\infty} t^{2n}\mu^n\,\overline{p}^{(n)}(\xi_1,\xi_2,\xi_3,\zeta) \tag{9.21}$$

where $\overline{p}^{(1)} = -\int_0^{\xi_1}\overline{Q}^*(\xi_1',\xi_2,\xi_3,\zeta)d\xi_1'$

$$\overline{p}_1^{(n+1)} = \frac{1}{2n+1}\{\overline{p}_{22}^{(n)} + \overline{p}_{33}^{(n)} - 4\overline{p}_{13}^{(n)} - 2\overline{p}_{23}^{(n)} - \overline{Q}^*\overline{p}^{(n)}\}$$

$$\overline{p}^{(n+1)}(0,\xi_2,\xi_3,\zeta) = 0;\ n = 0,\ 1,\ 2,\ \ldots \tag{9.22}$$

Equations (9.9), (9.22) imply that

$$\overline{p}^{(1)} = -2\zeta\int_0^{Z}\overline{Q}(X + 2\zeta Z - 2\zeta\tau,\ \tau,\ \tfrac{\zeta}{2}(2\zeta^{-1}Z^* - 2\zeta Z + 2\zeta\tau))d\tau$$

i.e. $\overline{p}^{(1)}$ is an entire function of X, Z, Z* and ζ, and vanishe for $\zeta = 0$. A similar calculation using (9.22) shows that the same can be said for $\overline{p}^{(n)}$ $n = 1, 2, \ldots$.

Substituting (9.21) into (9.20) and integrating termwise (this is possible due to the absolute and uniform convergence of the series (9.21)) gives

$$\begin{aligned} U(X,0,Z^*) &= \frac{1}{4\pi i}\int_{|\zeta|=1}\int_{-1}^{1} f(\mu_1(1-t^2),\zeta)\,\frac{dt}{\sqrt{1-t^2}}\,\frac{d\zeta}{\zeta} \\ &+ \frac{1}{4\pi i}\int_{|\zeta|=1}\int_{-1}^{1} \overline{f}(\mu_2(1-t^2),\zeta)\,\frac{dt}{\sqrt{1-t^2}}\,\frac{d\zeta}{\zeta} \\ &= \frac{1}{4\pi i}\int_{|\zeta|=1} g(\mu_1,\zeta)\,\frac{d\zeta}{\zeta} \\ &+ \frac{1}{4\pi i}\int_{|\zeta|=1} \overline{g}(\mu_2,\zeta)\,\frac{d\zeta}{\zeta} \end{aligned} \tag{9.23}$$

where

$$\overline{g}(\mu,\zeta) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \overline{a_{nm}}\ \mu^n\ \zeta^m.$$

We will now show that (9.16) gives the solution of (9.23). Let

$$U(X,0,Z^*) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\ X^n Z^{*m} \tag{9.24}$$

Since $u(x,y,z)$ is real valued we have that c_{no}, $n = 0,1,2,\ldots$ are all real. Equating coefficients of $X^n\ Z^{*m}$ in (9.23) gives

$$\begin{aligned} 2n!m!\ c_{nm} &= (n+m)!\ a_{n+m,m}\ ;\ n \geq 0,\ m > 0 \\ 2c_{no} &= a_{no} + \overline{a}_{no} \end{aligned} \tag{9.25}$$

Without loss of generality assume that a_{no}, $n = 0,1,2,\ldots$

re real.   Equations (9.24), (9.25) imply that

$$\begin{aligned}
U(X,0,Z^*) &= \frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(n+m+1)}{\Gamma(n+1)\Gamma(m+1)}\, a_{n+m,m}X^n Z^{*m} \\
&\quad + \frac{1}{2}\sum_{n=0} c_{no}X^n \\
&= \frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n-m+1)\Gamma(m+1)}\, a_{nm}X^{n-m}Z^{*m} \\
&\quad + \frac{1}{2}\sum_{n=0}^{\infty} c_{no}\, X^n \qquad (9.26)\\
&= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\Gamma(n+1)}{\Gamma(n-m+1)\Gamma(m+1)}\, a_{nm}\, X^{n-m}Z^{*m} \\
&\quad + \frac{1}{2}\sum_{n=0}^{\infty} c_{no}X^n.
\end{aligned}$$

rom the definition of the Beta function $B(x,y)$

$$B(x,y) \equiv \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

e have

$$\int_0^1 U(tX,0,(1-t)Z^*)dt = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{a_{nm}}{n+1}\, X^{n-m}Z^{*m} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{c_{no}}{n+1}\, X^n$$

r

$$\begin{aligned}
&\frac{\partial}{\partial\mu}\left[\mu\int_0^1 U(t\mu,0,(1-t)\mu\zeta)dt\right] = \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{nm}\,\mu^n\zeta^m + \frac{1}{2}\sum_{n=0}^{\infty} c_{no}\mu^n \\
&= \frac{1}{2}g(\mu,\zeta) + \frac{1}{2}\,U(\mu,0,0)
\end{aligned}$$

which implies the theorem.

<u>Theorem 9.5 ([8])</u>: Let G be a bounded, simply connected domain in $\mathbb{R}^3$, and define

$$u_{2n,m}(x,y,z) = \operatorname{Re} \underset{\sim}{P}_3\{\mu^n\zeta^m\};\ 0 \leq n < \infty,\ m=0,\ 1,\ldots\ n.$$
$$u_{2n+1,m}(x,y,z) = \operatorname{Im} \underset{\sim}{P}_3\{\mu^n\zeta^m\};\ 0 \leq n < \infty,\ m=0,\ 1,\ldots,n. \tag{9.27}$$

Then the set $\{u_{nm}\}$ is a complete family of solutions for equation (9.1) in the space of real valued C^2 solutions of (9.1) defined in G.

<u>Proof</u>: Let $u(x,y,z)\in C^2$ be a solution of (9.1) in G and let $\overline{G}_1 \subset G$. By the Runge approximation property for every $\epsilon > 0$ there exists a solution $u_1(x,y,z)$ of (9.1) which is regular in a sphere S, $S \supset G$, such that

$$\max_{(x,y,z)\in\overline{G}_1} |u - u_1| < \frac{\epsilon}{3}\ . \tag{9.28}$$

From section 4 we can conclude that the Cauchy data for u_1 must be regular in some <u>convex</u> region B in C^2 and u_1 depends continuously on this data in S. Since convex domains are Runge domains of the first kind ([19] p.229), on compact subsets of B we can approximate the Cauchy data for u_1 by polynomials and construct a (real valued) solution u_2 of (9.1) with polynomial Cauchy data. By Theorem F $u_2(x,y,z)$ is an entire function of its independent (complex) variables. Furthermore there exists a domain G_2, $G \subset \overline{G}_2 \subset S$, such that

$$\max_{(x,y,z)\in\overline{G}_2} |u_1 - u_2| < \frac{\epsilon}{3}\ . \tag{9.29}$$

The fact that u_2 is entire implies that $\mathbf{U}_2(X,Z,Z^*)$ $(=u_2(x,y,z)$ for x,y,z real) is an entire function of X, Z,Z*, which implies that $\mathbf{U}_2(X,0,Z^*)$ is regular in $\{|X| \leq R\} \times \{|Z^*| \leq R\}$ for R arbitrarily large. Since product domains are Runge domains of the first kind ([19], p.49), we can approximate $\mathbf{U}_2(X,0,Z^*)$

by a polynomial in $\{|x| \le R\} \times \{|Z^*| \le R\}$ and use Theorems 9.1 and F to construct a (real valued) entire solution $u_3(x,y,z)$ of equation (9.1) (with polynomial Goursat data in the X, Z, Z* variables) such that

$$\max_{(x,y,z)\in\overline{G}_2} |u_2 - u_3| < \frac{\epsilon}{3} . \tag{9.30}$$

Theorem 9.4 implies that there exists a polynomial $h_N(\mu,\zeta)$ such that $u_3(x,y,z) = \mathrm{Re}\ \underset{\sim}{P}_3\ \{h_N\}$. Equations (9.28) - (9.30) now imply that

$$\max_{(x,y,z)\in\overline{G}_1} |u - \mathrm{Re}\ \underset{\sim}{P}_3\ \{h_N\}| < \epsilon$$

and the theorem follows.

Example 9.1: When $q \equiv 0$ we have

$$u_{2n,m}(x,y,z) = \frac{n!}{(n+m)!} r^n P_n^m (\cos\theta)\mathrm{Re}(i^m e^{im\phi})$$

$$u_{2n+1,m}(x,y,z) = \frac{n!}{(n+m)!} r^n P_n^m (\cos\theta)\mathrm{Im}(i^m e^{im\phi})$$

where r, θ, ϕ are spherical coordinates and P_n^m denote the associated Legendre polynomials.

Remarks: The results in this section first appeared in [8] and [9]. Prior to this paper partial results in this direction had been obtained by Bergman [1], Tjong [38] and Gilbert and Lo [23]. For the extension of the results in this section to elliptic equations in four independent variables see [10]. For recent results in this area see the book by Gilbert referred to in the introduction and the Indiana University Ph.D. thesis of D. Kukral and M. Stecher.

10. Integral Operators for Non Self Adjoint Equations in Three Independent Variables.

The result just presented for equation (9.1) can also be obtained for the more general equation

$$\Delta_3 u + a(x,y,z)u_x + b(x,y,z)u_y + c(x,y,z)u_z + d(x,y,z)u = 0 \tag{10.1}$$

where a, b, c, d are real valued entire functions of the (complex variables) x, y, z(see [9]). In complex form (10.1) becomes

$$\mathcal{U}_{XX} - \mathcal{U}_{ZZ^*} + A(X,Z,Z^*)\mathcal{U}_X + B(X,Z,Z^*)\mathcal{U}_Z + C(X,Z,Z^*)\mathcal{U}_{Z^*} + D(X,Z,Z^*)\mathcal{U} = 0 \tag{10.2}$$

where $A = a$, $B = \frac{1}{2}(b + ic)$, $C = \frac{1}{2}(-b + ic)$, $D = d$. Substitutin

$$V(X,Z,Z^*) = \mathcal{U}(X,Z,Z^*)e^{-\int_0^Z C(X,Z',Z^*)dZ'} \tag{10.3}$$

yields the following equation satisfied by $V(X,Z,Z^*)$:

$$V_{XX} - V_{ZZ^*} + \tilde{A}(X,Z,Z^*)V_X + \tilde{B}(X,Z,Z^*)V_Z + \tilde{D}(X,Z,Z^*)V = 0 \tag{10.4}$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{D}$ can be expressed in terms of A, B, C. An integral operator mapping analytic functions $f(\mu,\zeta)$ onto solutions $V(X,Z,Z^*)$ of (10.4) is given by

$$\begin{aligned} V(X,Z,Z^*) &\equiv \underset{\sim}{C}_3'\{f\} \\ &= \frac{1}{2\pi i}\int_{|\zeta|=1}\int_{-1}^{1} E(X,Z,Z^*,\zeta,t)f(\mu(1-t^2),\zeta)\frac{dt}{\sqrt{1-t^2}}\frac{d\zeta}{\zeta} \end{aligned} \tag{10.5}$$

where in the ξ_1, ξ_2, ξ_3 variables $E^* = E$ satisfies

$$\begin{aligned} &\mu t(4E^*_{13} + 2E^*_{23} - E^*_{22} - E^*_{33} - \tilde{D}^*E^*) + (1-t^2)E^*_{1t} \\ &- \frac{1}{t}E^*_1 - \tilde{A}^*[(E^*_2 + E^*_3)\mu t + \frac{1}{2}(1-t^2)E^*_t - \frac{1}{2t}E^*] \\ &- \tilde{B}^*\zeta[(2E^*_1 + 2E^*_2)\mu t + \frac{1}{2}(1-t^2)E^*_t - \frac{1}{2t}E^*] = 0 \end{aligned} \tag{10.6}$$

where $\tilde{A}^*$, $\tilde{B}^*$, $\tilde{D}^*$ are $\tilde{A}$, $\tilde{B}$, $\tilde{D}$ in the ξ_i, i = 1, 2, 3, variables.

It can be shown as in Theorem 9.3 that

$$E^*(\xi_1,\xi_2,\xi_3,\zeta,t) = \sum_{n=1}^{\infty} t^{2n} \mu^n p^{(n)}(\xi_1,\xi_2,\xi_3,\zeta) \qquad (10.7)$$

is a regular solution of equation (10.6) in $G_R \times B \times T$, where the $p^{(n)}$ are given recursively by

$$p_1^{(n+1)} - \frac{1}{2}(\tilde{A}^* + \tilde{B}^*\zeta)p^{(n+1)} = \frac{1}{2n+1}\{p_{22}^{(n)} + p_{33}^{(n)} - 4p_{13}^{(n)}$$

$$- 2p_{23}^{(n)} + (\tilde{A}^* + 2\tilde{B}^*\zeta)p_2^{(n)} + \tilde{A}^* p_3^{(n)} + 2\tilde{B}^*\zeta\, p_1^{(n)} + \tilde{D}^* p^{(n)}\},$$

$$p^{(1)}(\xi_1,\xi_2,\xi_3,\zeta) = e^{\frac{1}{2}\int_0^{\xi_1}(\tilde{A}^* + \tilde{B}^*\zeta)d\xi_1},$$

$$p^{(n+1)}(0,\xi_2,\xi_3,\zeta) = 0;\ n = 1,\ 2,\ \ldots \qquad (10.8)$$

(In the present case it is not possible to have $E^*(0,\xi_2,\xi_3,\zeta,t)=1$ as in Theorem 9.3 since in this case equation $(10.\underset{\sim}{6})$ cannot be satisfied due to the appearance of the term $\frac{1}{2t}E^*(\tilde{A}^* + \tilde{B}^*\zeta)$).

Proceeding now as in Theorem 9.4 we can show that every real valued solution of equation (10.3) can be represented locally in the form

$$U(X,Z,Z^*) = \bar{U}(0,0,0)U_0(X,Z,Z^*) + \operatorname{Re} \underset{\sim}{C}_3\{f\} \qquad (10.9)$$

where

$$\underset{\sim}{C}_3\{f\} = \frac{1}{2\pi i} \int_{|\zeta|=1} \int_{-1}^{+1} e^{\int_0^{Z} C(X,Z',Z^*)dZ'} E(X,Z,Z^*,\zeta,t) \times$$

$$f(\mu(1-t^2),\zeta) \frac{dt}{\sqrt{1-t^2}} \frac{d\zeta}{\zeta}, \qquad (10.10)$$

$U_0(X,Z,Z^*)$ is the unique solution of equation (10.2) satisfying $U_0(X,0,Z^*) = U_0(X,Z,0) = 1$ (which can be constructed by iteration - see Theorem F and [28]) and $f(\mu,\zeta)$ is constructed from the formulas

$$f(\mu,\zeta) = \frac{3}{2\pi}\int_\gamma g(\mu(1-t^2),\zeta)\,\frac{(1-t^2)}{t^4}\,dt$$

$$g(\mu,\zeta) = \sum_{n=0}^{\infty}\sum_{m=0}^{n+1} a_{nm}\,\mu^n\zeta^m$$

$$a_{n+m-1,m} = \frac{2n!m!}{(n+m)!}\,\gamma_{nm} - \sum_{k=0}^{n-1}\frac{n!}{(n+m)!k!}\,\delta_{km}\gamma_{n-k,0} \qquad (10.11)$$

where

$$\mathbf{U}(X,0,Z^*) - \mathbf{U}(0,0,0) = \sum_{n=0}^{\infty}\sum_{\substack{m=0\\ n+m\neq 0}}^{\infty}\gamma_{nm}X^nZ^{*m}$$

$$\delta_{km} = \left(\frac{\partial^{k+m}}{\partial X^k\partial Z^{*m}}\,e^{\int_0^{-Z^*}\overline{C}(X,Z',0)dZ'}\right)_{X=Z^*=0}$$

Remark: It was in order to achieve this inversion formula that equation (10.2) was reduced to the form of equation (10.4

Following the analysis of Theorem 9.5 it can now be shown ([9]) that the set

$$u_0(x,y,z) = \mathbf{U}_0(X,Z,Z^*)$$

$$u_{2n,m} = \mathrm{Re}\ \underset{\sim}{C}_3\{\mu^n\zeta^m\};\ 0 \le n < \infty,\ m = 0,1,\ \ldots,\ n+1$$

(10.12

$$u_{2n+1,m} = \mathrm{Im}\ \underset{\sim}{C}_3\{\mu^n\zeta^m\};\ 0 \le n < \infty,\ m = 0,1,\ \ldots,\ n+1$$

is a complete family of solutions for equation (10.1).

IV Analytic continuation

11. Lewy's Reflection Principle and Vekua's Integral Operators.

Let u satisfy the partial differential equation

$$u_{xx} + u_{yy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0 \qquad (11.1)$$

where $u(x,y) \in C^2(D) \cap C^1(\bar{D})$, D being a simply connected domain of the x,y plane whose boundary contains a segment σ of the x-axis with the origin as the interior point and such that D contains the portion $y < 0$ of a neighbourhood of each point of σ.

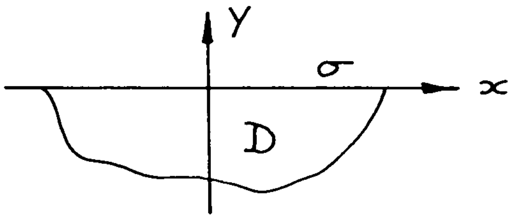


figure 11.1

In complex form (11.1) becomes (c.f. section 1)

$$L[U] = \frac{\partial^2 U}{\partial z \partial z^*} + A(z,z^*)\frac{\partial U}{\partial z} + B(z,z^*)\frac{\partial U}{\partial z^*} + C(z,z^*)U = 0. \quad (11.2)$$

Assume A, B, C are regular for z, z* in $D \cup \sigma \cup D^* \times D \cup \sigma \cup D^*$. Now let $R(z,z^*,\zeta,\zeta^*)$ be the Riemann function of the adjoint equation (see Section 1 and Exercise 3.1)

$$M[v] = \frac{\partial^2 v}{\partial z \partial z^*} - \frac{\partial (Av)}{\partial z} - \frac{\partial (Bv)}{\partial z^*} + Cv = 0. \qquad (11.3)$$

We have the identity for analytic functions $U(z,z^*)$, $v(z,z^*)$,

$$vL[\mathcal{U}] - \mathcal{U}M[v] = (v\mathcal{U})_{zz^*} - (\mathcal{U}[v_{z^*} - Av])_z - - (\mathcal{U}[v_z - Bv])_{z^*}. \tag{11.4}$$

Setting v = R, letting $\mathcal{U}$ be a solution of $L[\mathcal{U}] = 0$ and usin Green's theorem over a plane region S in C^2 bounded by a smoot curve ∂S gives

$$0 = \oint_{\partial S} (\mathcal{U}R)_z dz - \oint_{\partial S} \mathcal{U}(R_z - BR)dz + \oint_{\partial S} \mathcal{U}(R_{z^*} - AR)dz^* \tag{11.5}$$

Letting S be the triangle with corners $(\zeta,\overline{\zeta})$, (ζ,ζ^*) and $(\overline{\zeta}^*,\zeta$ gives (setting, at the end, $\zeta = z$, $\zeta^* = z^*$)

$$\begin{aligned}\mathcal{U}(z,z^*) &= \mathcal{U}(\overline{z}^*,z^*)R(\overline{z}^*,z^*,z,z^*) \\ &+ \int_d \mathcal{U}(t,\overline{t})(R_{\overline{t}}(t,\overline{t},z,z^*) - A(t,\overline{t})R)d\overline{t} \\ &+ \int_d [(\mathcal{U}(t,\overline{t})R(t,\overline{t},z,z^*))_t - \mathcal{U}(R_t - BR)]dt\end{aligned} \tag{11.6}$$

where d is the diagonal from $(\overline{z}^*,z^*)$ to $(z,\overline{z})$ and use has been made of the boundary conditions satisfied by R. Similarly, letting S be the quadrilateral $(0,0)$, $(\zeta,0)$, $(\zeta,\overline{\zeta})$, $(0,\overline{\zeta})$, we have

$$\begin{aligned}\mathcal{U}(z,\overline{z}) &= -\mathcal{U}(0,0)R(0,0,z,\overline{z}) + \mathcal{U}(z,0)R(z,0,z,\overline{z}) \\ &+ \mathcal{U}(0,\overline{z})R(0,\overline{z},z,\overline{z}) \\ &- \int_0^z \mathcal{U}(t,0)(R_t(t,0,z,\overline{z}) - B(t,0)R)dt \\ &- \int_0^{\overline{z}} \mathcal{U}(0,t)(R_{\overline{t}}(0,\overline{t},z,\overline{z}) - A(0,\overline{t})R)d\overline{t}.\end{aligned} \tag{11.7}$$

(Equation (11.7) is Vekua's operator ([39]) mapping analytic functions onto solutions of equation (11.1)).

Now suppose on σ we have

$$u(x,0) = \mathcal{U}(x,x) = \delta(x) \tag{11.8}$$

here $\delta(z)$ is regular for z in $D \cup \sigma \cup D^*$. In equation (11.7) et $\mathcal{U}(z,0) = f(z)$, $\mathcal{U}(0,z) = g(z)$, and obtain for z on σ

$$\begin{aligned}\delta(z) &= -\delta(0)R(0,0,z,z) + f(z)R(z,0,z,z) + g(z)R(0,z,z,z) \\ &- \int_0^z f(t)(R_t(t,0,z,z) - B(t,0)R)dt \\ &- \int_0^z g(t)(R_t(0,t,z,z) - B(0,t)R)dt\end{aligned} \tag{11.9}$$

ote that the boundary conditions satisfied by R imply that $(z,0,z,z)$ and $R(0,z,z,z)$ do not vanish for z in $D \cup \sigma \cup D^*$. rom equation (11.6) (setting $z^* = 0$) we see that $f(z)$ is known or z in $D \cup \sigma$ in terms of the given solution $u(x,y) = \mathcal{U}(z,\bar{z})$ nd (setting $z = 0$) so in $g(z)$ for z in $D^* \cup \sigma$.

Now for z in $D \cup \sigma$ (11.9) is a Volterra integral equation or $g(z)$ since $f(z)$ and $\delta(z)$ are known in $D \cup \sigma$. Since the ernel and terms not involving $g(z)$ are analytic in D, coninuous in $D \cup \sigma$, so must the solution $g(z)$. But $g(z)$ is already nown to be analytic in D^* and continuous in $D^* \cup \sigma$, which mplies that the above construction of $g(z)$ furnishes the nalytic continuation of $g(z)$ into $D \cup \sigma \cup D^*$ (see [37], p.157).

Similarly $f(z)$ can be continued into $D \cup \sigma \cup D^*$. Equation 11.7) now gives $\mathcal{U}(z,\bar{z})$ for arbitrary z in $D \cup \sigma \cup D^*$ i.e. $\mathcal{U}(z,\bar{z})$ has been extended to z in $D \cup \sigma \cup D^*$.

<u>Theorem 11.1</u> ([31]): Let $u(x,y) = \mathcal{U}(z,\bar{z}) \in C^2(D) \cap C^1(\bar{D})$ satisfy $L[\mathcal{U}] = 0$ in D and suppose $\delta(z) = \mathcal{U}(z,z)$ is regular in $D \cup \sigma \cup D^*$. Then $u(x,y)$ can be analytically continued into all of $D \cup \sigma \cup D^*$ where D^* is the mirror image of D reflected across σ.

12. The Envelope Method and Analytic Continuation

We have already seen one method of analytic continuation, that is in sections 3 and 4 where regularity of Cauchy data determines regularity of solution. The following theorem shows how integral operators can also be used for this purpose.

Theorem 12.1. (The Envelope Method [21]): Let $F(z) \equiv F(z_1, z_2, \ldots, z_n)$ be defined by the integral representation

$$F(z) = \int_{\mathscr{L}} K(z;\zeta)d\zeta \tag{12.1}$$

where $K(z;\zeta)$ is a holomorphic function of $(n+1)$ complex variables for $(z;\zeta)$ contained in, save for certain singularities indicated below, C^{n+1}. Furthermore, let the integration path $\mathscr{L}$ be a closed rectifiable contour, and let all of the singula[r] points of $K(z;\zeta)$ be contained on the analytic set $\mathfrak{S}_0 = \{z | S(z,\zeta) = 0; \zeta \in C^1\}$. Then $F(z)$ is regular for all point[s]

$$z \notin \mathfrak{S} \equiv \mathfrak{S}_0 \cap \mathfrak{S}_1$$

where

$$\mathfrak{S}_1 = \{z | \frac{\partial S(z,\zeta)}{\partial \zeta} = 0; \ \zeta \in C^1\} \tag{12.2}$$

Proof: Let $F(z)$ be regular at $z = z_0$ and hence in a neighbou[r]hood $N(z^0)$ of z^0. Now analytically continue $F(z)$ along a pat[h] γ with one endpoint in $N(z^0)$. This can be done as long as no point of γ corresponds to a singularity of the integrand on $\mathscr{L}$. Even when this happens we can keep on continuing $F(z)$ along γ by deforming the path of integration to avoid the singularity $\zeta = \alpha(z)$ threatening to cross it. In particular suppose we have continued $F(z)$ along γ to a point $z = z_1$ and at that poi[nt] there exists a singularity $\zeta = \alpha$ on $\mathscr{L}$. Suppose however $S(z_1,\zeta)$ has a simple zero at $\zeta = \alpha$, i.e. in a sufficiently small neigh[bo]bourhood $N(\alpha) = \{\zeta| \ |\zeta - \alpha| < \epsilon\}$ we have

$$S(z_1,\zeta) \approx (\zeta - \alpha) \frac{\partial S(z_1;\alpha)}{\partial \zeta}$$

where $\dfrac{\partial S(z_1,\alpha)}{\partial \zeta} \neq 0$. Then we can deform $\mathscr{L}$ about the point $\zeta = \alpha$ by letting it follow a portion of the circle $|\zeta - \alpha| = \epsilon/$ which implies that $F(z)$ is regular at z_1. Q.E.D.

Corollary 12.1 (Hadamard): Let $n = 1$ and suppose $f(z)$ is singular at $\alpha_1, \alpha_2, \ldots$ and $g(z)$ is singular at $\beta_1, \beta_2, \ldots$. Then

$$F(z) = \frac{1}{2\pi i}\int_{\mathcal{L}} f(\zeta)g(z/\zeta)^{d\zeta}/\zeta \tag{12.3}$$

is regular for $z \neq \alpha m\ \beta_n$, $m,n = 1, 2, \ldots$

Proof: Without loss of generality suppose that α_m and β_n are the only finite singularities of $f(z)$ and $g(z)$ respectively. Then

$$S(z,\zeta) = (\zeta - \alpha_m)(z - \beta_n\zeta)$$

and

$$\frac{\partial S(z,\zeta)}{\partial\zeta} = z - 2\beta_n\zeta + \alpha_m\beta_n$$

setting $S(z,\zeta) = \frac{\partial S(z,\zeta)}{\partial\zeta} = 0$ and eliminating ζ

gives $(z - \alpha_m\beta_n)^2 = 0$, which implies the corollary.

Remark 1: If $\mathcal{L}$ is not a closed contour but an open contour between two fixed points ζ_1 and ζ_2, then we cannot deform $\mathcal{L}$ away from these points, and hence $F(z)$ may be singular on the set

$$\mathcal{G}_3 = \{z\,|\,S(z,\zeta) = 0;\quad \zeta = \zeta_1 \text{ and } \zeta = \zeta_2\}. \tag{12.4}$$

Such singularities are called endpoint-pinch singularities. In summary the possible singularities are those points which we are unable to list as regular points by the Hadamard method or envelope method (taking into account possible endpoint-pinch singularities).

Remark 2: For extensions of Theorem 12.1 and Corollary 12.1 to multiple integrals see Chapter 1 of [21]. Theorem 12.1 is often mistakenly credited to Landau, Bjorken and/or Polkinghorne and Screaton. It was actually first proved by R.P. Gilbert in his 1958 thesis. For a historical discussion of the origins of Theorem 12.1 see the introduction in [21].

The envelope method can be applied whenever an integral representation of the solution is available, e.g. the representation derived in sections 6-10. Here we apply Theorem 12.1 to the axially symmetric potential equation, i.e. the equation

$$\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0 \qquad (12.5)$$

where $(r.z.\phi)$ are cylindrical coordinates and u is assumed to be independent of ϕ.

Exercise 12.1: Show that if $u(z,r)$ is an analytic solution of (12.5) in some neighbourhood of the origin, then $u(z,r)$ is an even function of r and is uniquely determined by $u(z,0) = f(z)$

Exercise 12.2: Using Theorem 9.2 show that for every analytic function $f(z)$

$$u(z,r) = \underset{\sim}{A}\{f\} \equiv \frac{1}{2\pi i}\int_{\mathcal{L}} f(\sigma)\frac{d\zeta}{\zeta} \qquad (12.6)$$

where

$$\mathcal{L} = \{\zeta | \zeta = e^{i\phi} ; 0 \le \phi \le 2\pi\} \qquad (12.7)$$

$$\sigma = z + i\frac{r}{2}(\zeta + \zeta^{-1}) \qquad (12.8)$$

defines a regular solution of (12.5) in some neighbourhood of the origin such that $u(z,0) = f(z)$.

Theorem 12.2: If the only finite singularities of $f(\sigma)$ are at $\sigma = \alpha$ then the only possible singularities of $u(z,r) = \mathcal{U}(\eta,\bar{\eta})$ ($\eta = z+ir$, $\bar{\eta} = z-ir$) on its first Riemann sheet are at $\eta = \alpha$ and $\eta = \bar{\alpha}$.

Proof: We represent $u(z,r)$ by the operator $\underset{\sim}{A}$ and apply the enevelope method. "Envelope" singularities:

$$\mathcal{G}_0 = \{(z,r) \mid S(z,r;\zeta) = (z-\alpha)\zeta + \frac{ir}{2}(\zeta^2+1) = 0\}$$

$$\mathfrak{S}_1 = \{(z,r) \mid \frac{\partial S(z,r;\zeta)}{\partial \zeta} = (z - \alpha) + ir\zeta = 0\}.$$

Eliminating ζ gives

$$\mathfrak{S} = \mathfrak{S}_0 \cap \mathfrak{S}_1 = \{(z,r) \mid (z - \alpha)^2 + r^2 = 0\}$$
$$= \{\eta \mid (\eta - \alpha)(\eta - \bar{\alpha}) = 0\},$$

where $\eta = z + ir$.

"Hadamard" singularities:

$$\mathfrak{S}_0 \cap \{\zeta = 0\} = \{(z,r) \mid r = 0\}.$$

But it can easily be verified by termwise integration in equation 12.6 that $\underset{\sim}{A}\{f\}$ is regular about origin if $f(\sigma)$ is. But (12.5) is invariant under translations along z axis and hence $\underset{\sim}{A}\{f\}$ is regular at points on the z axis provided they correspond to regular points of $f(\sigma)$.

Remarks: Theorem 12.2 has a long history: [15], [25], [22] and [11]. For further applications of Theorem 12.1 to the analytic continuation of solutions of partial differential equations see [21].

13. The Axially Symmetric Helmholtz Equation

Consider

$$\Delta_3 u + u = 0 \tag{13.1}$$

defined in the exterior of a bounded domain. Assume that in cylindrical coordinates (r,z,ϕ) u is independent of ϕ. Then (13.1) becomes

$$L[u] \equiv u_{zz} + u_{rr} + \frac{1}{r} u_r + u = 0. \tag{13.2}$$

Now let D be a bounded, simply connected domain in the (r,z) plane which is symmetric with respect to the axis r = 0 and has smooth boundary ∂D. Let f(r,z) be a continuous function defined on $\partial D^+ = \partial D \cap \{(r,z)\,|\,r \geq 0\} \ni$

$$u\Big|_{\partial D^+} = f \tag{13.}$$

Example 13.1: $\frac{\sin R}{R}$ (where $z = R\cos\theta$, $r = R\sin\theta$) is a solution of (13.2) which vanishes on the circle $R = \pi$, i.e. uniqueness does not in general hold for the boundary value problem (13.2), (13.3).

Theorem 13.1 ([40]): There is at most one solution of (13.2), (13.3) which is regular in the exterior of D and satisfies

$$\lim_{R\to\infty} R\left(\frac{\partial u}{\partial R} - iu\right) = 0 \tag{13.4}$$

uniformly for $\theta \in [0,\pi]$.

Remark 1: (13.4) is known as the Sommerfeld radiation condition.

Remark 2: It is easily shown (c.f. Exercise 12.1) that the regularity of u in the exterior of D implies that u is an even function of r.

Proof of Theorem: On the circle of radius R (where the radius of D is less than R) expand u in a Legendre series

$$u(R,\theta) = \sum_{n=0}^{\infty} a_n(R)P_n(\cos\theta); \quad R > R_0 \tag{13.5}$$

where

$$a_n(R) = \int_0^{\pi} u(R,\theta)P_n(\cos\theta)\sin\theta\, d\theta. \tag{13.6}$$

From (13.2) and (13.6) it can be verified that

$$a_n(R) = a_n R^{-\frac{1}{2}}H^{(1)}_{n+\frac{1}{2}}(R) + b_n R^{-\frac{1}{2}}H^{(2)}_{n+\frac{1}{2}}(R) \tag{13.7}$$

where $H^{(i)}_{n+\frac{1}{2}}$ denotes the Hankel function of the i^{th} kind.

But (c.f. [16])

$$H^{(1)}_{n+\frac{1}{2}}(R) = \sqrt{\frac{2}{\pi R}}\, e^{i(R - \frac{1}{2}n\pi - \frac{1}{2}\pi)} + O\left(R^{-3/2}\right); \quad R \to \infty$$

$$H^{(2)}_{n+\frac{1}{2}}(R) = \sqrt{\frac{2}{\pi R}}\, e^{-i(R - \frac{1}{2}n\pi - \frac{1}{2}\pi)} + O\left(R^{-3/2}\right); \quad R \to \infty \tag{13.8}$$

which implies (by the Sommerfeld radiation condition) that $b_n = 0$. Hence

$$u(R,\theta) = R^{-\frac{1}{2}} \sum_{n=0}^{\infty} a_n H^{(1)}_{n+\frac{1}{2}}(R) P_n(\cos\theta); \quad R > R_0. \tag{13.9}$$

Now assume $u = 0$ on ∂D^+ and apply Green's formula in region B below to u and $\bar{u}$:

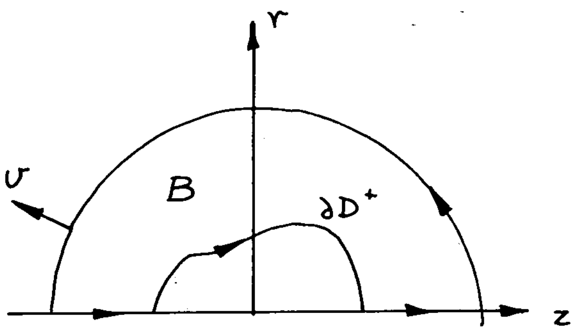


figure 13.1

$$\int_{\partial B} r\left(u \frac{\partial \bar{u}}{\partial v} - \bar{u} \frac{\partial u}{\partial v}\right) dS = \iint_B r(uM(\bar{u}) - \bar{u}M(u))\,dV \tag{13.10}$$

where $M(u) = u_{zz} + u_{rr} + \frac{1}{r} u_r$, the axially symmetric harmonic equation. But (13.10) implies (since $L(u) = L(\bar{u}) = 0$) that

$$\int_0^{\pi} R^2 \left(u \frac{\partial \bar{u}}{\partial R} - \bar{u} \frac{\partial u}{\partial R}\right) \sin\, d\theta = 0. \tag{13.11}$$

From the relations (see [16])

$$\lim_{n\to\infty} \frac{(R/_2)^{n+\frac{1}{2}}}{\Gamma(n+\frac{1}{2})} \quad H^{(1)}_{n+\frac{1}{2}} \ (R) = - \frac{i}{\pi} \tag{13.12}$$

$$\max_{\theta\in[0,\pi]} |P_n \ (\cos\theta)| \leq 1 \ ,$$

it is seen that the series (13.9) can be differentiated term-wise. Furthermore from Abel's formula for the Wronskian we h (using equation (13.8) to evaluate the constant $-\frac{4i}{\pi}$)

$$H^{(1)}_{n+\frac{1}{2}} \ (R) \ \frac{d}{dR} \ H^{(2)}_{n+\frac{1}{2}}(R) \ - \ H^{(2)}_{n+\frac{1}{2}}(R) \ \frac{d}{dR} \ H^{(1)}_{n+\frac{1}{2}} \ (R) = \frac{-4i}{\pi R} \tag{13.13}$$

Since $\overline{H^{(1)}_{n+\frac{1}{2}}} = H^{(2)}_{n+\frac{1}{2}}$, substitution of (13.9) into (13.11) and making use of the orthogonality property of the $P_n(\cos\theta)$, giv

$$0 = \int_0^\pi R^2(u\frac{\partial\bar{u}}{\partial R} - \bar{u} \ \frac{\partial u}{\partial R}) \ \sin\theta \ d\theta = - \ \frac{4i}{\pi} \ \Sigma \ |a_n|^2. \tag{13.14}$$

Hence $a_n = 0$ for every n, which implies that $u \equiv 0$.

Theorem 13.2 ([29],[40]): Let $u(R,\theta)$ be a regular solution of (13.2) for $R > C$ satisfying the Sommerfeld radiation condition Then $u \ (R,\theta)$ has the representation

$$u(R,\theta) = R^{-\frac{1}{2}}H_{\frac{1}{2}}{}^{(1)}(R) \sum_{n=0}^{\infty} \frac{F_n(\cos\theta)}{R^n} + R^{-\frac{1}{2}}H^{(1)}_{3/_2}(R) \sum_{n=0}^{\infty} \frac{G_n(\cos\theta)}{R^n} \tag{13.15}$$

where the series converge uniformly and absolutely for $R \geq C' > C$, $\theta\in [0,\pi]$. $F_0 \ (\cos\theta)$ and $G_0(\cos\theta)$ determine $u(R,\theta)$ uniquely.

Proof: If the recursion formula

$$H^{(1)}_{v-1}(R) + H^{(1)}_{v+1} \ (R) = \frac{2v}{R} \ H^{(1)}_v(R); \tag{13.16}$$

is used repeatedly one gets

$$H^{(1)}_{n+\frac{1}{2}}(R) = H^{(1)}_{\frac{1}{2}}(R)\, R_{n,\frac{1}{2}}(R) - H_{-\frac{1}{2}}(R)\, R_{n-1,3/2}(R) \quad (13.17)$$

here $R_{n,\nu}(R)$ are Lommel polynomials defined as

$$R_{n,\nu}(z) = \sum_{k=0}^{[n/2]} \frac{(-1)^n (n-k)!\,\Gamma(\nu+n-k)}{k!\,(n-2k)!\,\Gamma(\nu+k)} \left(\frac{z}{2}\right)^{-n+2k} \qquad (13.18)$$

onsider now the series ($z = R$, $\xi = \cos\theta$)

$$E(z,\xi) = \sum_{n=0}^{\infty} a_n R_{n,\frac{1}{2}}(z) P_n(\xi)$$

$$Q(z,\xi) = \sum_{n=0}^{\infty} a_n R_{n-1,3/2}(z) P_n(\xi) \qquad (13.19)$$

hich result when (13.17) is substituted into the series rep-
esentation (13.9) of $u(R,\theta)$. Now set $z = C'e^{i\alpha}$, $0 \leq \alpha \leq 2\pi$.
Ve can rewrite $E(z,\xi)$ as

$$E(C'e^{i\alpha},\xi) = \sum_{n=0}^{\infty} \frac{a_n P_n(\xi)\ \frac{1}{\Gamma(n+\frac{1}{2})}\left(\frac{C'}{2}e^{i\alpha}\right)^{n+\frac{1}{2}} H^{(1)}_{n+\frac{1}{2}}(C')}{\frac{1}{\Gamma(n+\frac{1}{2})}\left(\frac{C'}{2}e^{i\alpha}\right)^{n+\frac{1}{2}} H^{(1)}_{n+\frac{1}{2}}(C')} R_{n,\frac{1}{2}}(C'e^{i\alpha}) \qquad (13.20)$$

From [16], p.35 we have

$$\lim_{n\to\infty} \frac{(z/2)^{n+\frac{1}{2}} R_{n,\frac{1}{2}}(z)}{\Gamma(n+\frac{1}{2})} = \left(\frac{z}{2}\right) J_{\frac{1}{2}}(z). \qquad (13.21)$$

Now note that (c.f. equation (13.12))

$$\sum_{n=0}^{\infty} a_n H^{(1)}_{n+\frac{1}{2}}(C') P_n(\xi) \qquad (13.22)$$

is absolutely and uniformly convergent for $\xi\in[-1,+1]$. Equation (13.12), (13.20) and (13.21) imply that $E(z,\xi)$ is uniformly and absolutely convergent for $|z| = C'$, $\xi\in[-1, +1]$. But the series defining $E(z,\xi)$ is a series of polynomials in $1/_z$ which converges uniformly and absolutely on the circle $|1/_z| = 1/_{C'}$ and hence is analytic for $|1/_z| < \frac{1}{C'}$, $\xi\in[-1, +1]$. A similar result holds for $Q(z,\xi)$. Hence

$$u(R,\theta) = R^{-\frac{1}{2}}H_{\frac{1}{2}}^{(1)}(R)E(\tfrac{1}{R}, \cos\theta) + R^{-\frac{1}{2}}H_{3/2}^{(1)}(R)Q(\tfrac{1}{R}, \cos\theta) \tag{13.23}$$

where E and Q are analytic functions of $\frac{1}{R}$ and are regular in the interior of the circle $|\frac{1}{R}| = \frac{1}{C'}$ in the complex $\frac{1}{R}$ plane. Equation (13.15) follows from this statement. If F_0 and G_0 are known, F_n and G_n can be computed recursively by substituting into equation (13.2). It is easily verified that

$$F_0(\cos\theta) = \sum_{n=0}^{\infty} a_{2n}\ (-1)^n\ P_{2n}(\cos\theta)$$

$$G_0(\cos\theta) = \sum_{n=0}^{\infty} a_{2n+1}(-1)^n P_{2n+1}(\cos\theta). \tag{13.24}$$

<u>Corollary</u> ([29], [40]): Let $u(R,\theta)$ be a solution of (13.2) for $R > R_0$, $\theta\in[0,\pi]$ satisfying the Sommerfeld radiation condition. Then

$$\lim_{R\to\infty} e^{-iR}\ Ru(R,\theta) = f(\cos\theta) = -G_0(\cos\theta)\ -iF_0(\cos\theta)$$

exists uniformly for $\theta\in[0,\pi]$. If $u(R,\theta)$ has the expansion (for $R > R_0$, $\theta\in[0,\pi]$)

$$u(R,\theta) = \sqrt{\frac{\pi}{2}}\ R^{-\frac{1}{2}} \sum_{n=0}^{\infty} a_n i^{n+1}\ H_{n+\frac{1}{2}}^{(1)}(R)P_n(\cos\theta) \tag{13.25}$$

then

$$f(\cos\theta) = \sum_{n=0}^{\infty} a_n P_n(\cos\theta) \tag{13.26}$$

Proof: The corollary follows from equations(13.8), (13.15) and (13.24).

We now consider the inverse scattering problem associated with equation (13.2), i.e. given the radiation pattern $f(\cos\theta)$, to determine $u(R,\theta)$ and its domain of regularity. In particular we want to analytically continue $u(R,\theta)$, from its initial domain of definition in a neighbourhood of infinity.

14. Analytic Continuation of Solutions to the Axially Symmetric Helmholtz Equation

Let u be a regular solution of

$$u_{zz} + u_{rr} + \frac{1}{r}u_r + u = 0 \tag{14.1}$$

in the exterior of a bounded domain D, and suppose

$$\lim_{R\to\infty} R(\frac{\partial u}{\partial R} - iu) = 0; \quad R = \sqrt{r^2 + z^2} \tag{14.2}$$

From section 13 we have

$$u \sim \frac{e^{iR}}{R} f(\cos\theta); \; R \to \infty . \tag{14.3}$$

where $f(\cos\theta)$ is known as the radiation pattern of u.

Theorem 14.1 ([33]): A necessary and sufficient condition for a function $f(\cos\theta)$ to be a radiation pattern is that there exists an (axially symmetric) harmonic function $h(z,r) = \tilde{h}(R,\theta)$ which is regular in the entire space such that $h(1,\theta) = f(\cos\theta)$ and furthermore has the property that

$$\int_0^{\pi} |\tilde{h}(R,\theta)|^2 \sin\theta \, d\theta \tag{14.4}$$

is an entire function of R of order one and finite type C. When this condition holds there exists a unique function $u(z,r) = \tilde{u}(R,\theta)$ which satisfies the Sommerfeld radiation condition (14.2) and is a regular solution of the (axially symmetric) Helmholtz equation for $R > C$ such that

$$\tilde{u}(R,\theta) \sim \frac{e^{iR}}{R} f(\cos\theta) + O\left(\frac{1}{R^2}\right); \quad R \to \infty. \tag{14.5}$$

Proof: Suppose $f(\cos\theta)$ is a radiation pattern. Then

$$f(\cos\theta) = \sum_{n=0}^{\infty} a_n P_n(\cos\theta) \tag{14.6}$$

where the series

$$\sum_{n=0}^{\infty} a_n \; i^{(n+1)} H^{(1)}_{n+\frac{1}{2}}(C') P_n(\cos\theta) \tag{14.7}$$

converges absolutely and uniformly for $C' > C$ for some $C > 0$. From equation (13.12) this implies that

$$|a_n| \Gamma(n + \tfrac{1}{2}) ({}^2/C')^{n+\frac{1}{2}} \tag{14.8}$$

is bounded, i.e. (using Stirling's formula)

$$\overline{\lim_{n\to\infty}} \; n|a_n|^{1/n} = \tfrac{1}{2} eC'. \tag{14.9}$$

But $\tilde{h}(R,\theta) = \sum_{n=0}^{\infty} a_n R^n P_n(\cos\theta)$ and (14.9) implies that

$\int_0^{\pi} |\tilde{h}(R,\theta)|^2 \sin\theta d\theta = \sum_{n=0}^{\infty} \frac{2}{2n+1} |a_n|^2 R^{2n}$ is an entire function of order 1 and exponential type C.

Suppose $\int_0^{\pi} |\tilde{h}(R,\theta)|^2 \sin\theta d\theta = \sum_{n=0}^{\infty} \frac{2}{2n+1} |a_n|^2 R^{2n}$ is an entire function of order 1 and exponential type C. Then from equations (14.8) and (14.9) the series (14.7) converges for each $C' > C$.

From equation (13.12), the series

$$\tilde{u}(R,\theta) = \sqrt{\frac{\pi}{2}} \; R^{-\frac{1}{2}} \sum_{n=0}^{\infty} a_n i^{n+1} H^{(1)}_{n+\frac{1}{2}}(R) P_n(\cos\theta) \tag{14.10}$$

can be differentiated termwise and defines a solution of (14.1) for $R > C$ which satisfies the Sommerfeld radiation condition. From the corollary to Theorem 13.2 $u(r,\theta)$ has $f(\cos\theta)$ as its radiation pattern.

We now want to analytically continue $\tilde{u}(R,\theta)$ pastthe circle $R=C$. From section 12 $h(z,r)$ is uniquely determined by the function

$$h(z,0) = \sum_{n=0}^{\infty} a_n z^n = \tfrac{1}{2}\int_{-1}^{+1} f(\xi)\,\frac{1-z^2}{(1-2\xi z+z^2)^{3/2}}d\xi.$$

Equation (14.9) implies that $h(z,0)$ is an entire function of order 1 and type ${}^{C}/2$. Let $f(z)$ be the Borel transform of $h(2iz,0)$, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n 2^n i^n n!\; z^{-n-1}. \tag{14.11}$$

Before we can state our first lemma we will need to introduce the concept of the indicator diagram of an entire function of exponential type. Suppose $g(z)$ is an entire function of exponential type. Then the indicator function of $g(z)$ is defined as

$$k(\theta) = \overline{\lim_{R\to\infty}}\, R^{-1}\log|g(Re^{i\theta})|.$$

It can be shown that $k(\theta)$ is the supporting function of a convex set, called the indicator diagram of $g(z)$.

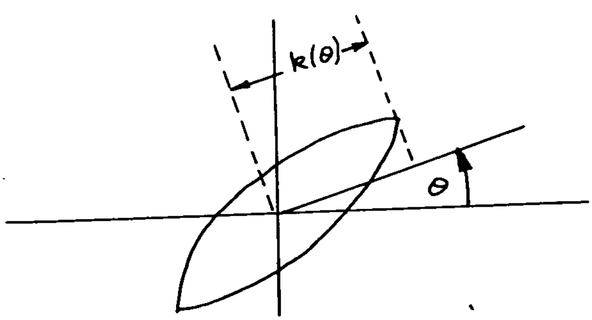


figure 14.1

Lemma 14.1 (Polya): $f(z)$ is regular in the exterior of the co-jugate indicator diagram $\mathfrak{G}$ of $h(2iz,0)$.

Proof: [4], p.75.

Remark: $\mathfrak{G}$ is a closed convex set contained in $|z| \leq C$.

Now define

$$g(z) = C'^{-\frac{1}{2}} \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} a_n i^{n+1} H^{(1)}_{n+\frac{1}{2}}(C') \left(\frac{z}{C'}\right)^{-n-1} \qquad (14.1$$

where $C' > C$.

Lemma 14.2 ([12]): $g(z)$ is regular in the exterior of $\mathfrak{G}$.

Proof: From Lemma 14.1 and Hadamard's multiplication of singularities theorem (Corollary 12.1; [37], p. 157) it suffices to show that the singularities of

$$G(z) = \sum_{n=0}^{\infty} \frac{H^{(1)}_{n+\frac{1}{2}}(C')}{n!\ 2^n} \left(\frac{z}{C'}\right)^{-n} \qquad (14.1$$

lie on the closed interval [0,1]. But from [16] p.78, 100 we can actually sum (14.13) to give

$$G(z) = -i \sqrt{\frac{2}{\pi C'}}\ \left(1 - \frac{1}{z}\right)^{-\frac{1}{2}} e^{iC'(1 - {}^1/z)^{\frac{1}{2}}} \qquad (14.14)$$

i.e. the only singularities of $G(z)$ are branch points at $z=0$ and $z = +1$.

We now construct the axially symmetric harmonic function $v(z,r)$ such that $v(z,0) = g(z)$:

$$v(z,r) = \tilde{v}(R,\theta) = \sqrt{\frac{\pi}{2}}\ C'^{-\frac{1}{2}} \sum_{n=0}^{\infty} a_n i^{n+1} H^{(1)}_{n+\frac{1}{2}}(C') \left(\frac{R}{C'}\right)^{-n-1} P_n(\cos\theta)$$

$$(14.15)$$

Note that $\frac{1}{R}\tilde{v}(\frac{1}{R},\theta)$ is an axially symmetric harmonic function in a neighbourhood of the origin. Applying Theorem 12.2 to $\frac{1}{R}\tilde{v}(\frac{1}{R},\theta)$ and using Lemma 14.2 we have that $v(z,r) = \tilde{v}(R,\theta)$ is regular in the exterior of $\mathfrak{G} \cup \overline{\mathfrak{G}}$ (where $\overline{G}$ denotes the image of G under conjugation). By the law of permanence of functional equations $v(z,r)$ is harmonic in $G \cup \overline{G}$. By construction we have that $v(C',\theta) = \tilde{u}(C',\theta)$.

<u>Lemma 14.3 ([12])</u>: Let $A = \{(z,r) | \sqrt{z^2 + r^2} = C'\}$. Then for $(z,r) \in A$, $v(z,r)$ $(= u(z,r))$ is an analytic function of $\eta = z + ir$ and can be continued analytically as a function of η into the exterior of $G \cup \overline{G} \cup \{(r,z) | r = 0\}$.

<u>Proof</u>: Let ϕ conformally map the exterior of the circle A onto the upper plane in the complex $\zeta = x + iy$ plane such that ϕ maps the line $r = 0$ onto the line $x = 0$. Under such a mapping the exterior of $G \cup \overline{G}$ is taken into a region Ω containing the upper half plane in its interior. The equation for $v(z,r)$ is transformed into an elliptic equation for a function $w(x,y)$ with coefficients analytic in $\Omega' = \Omega - \{(x,y) | x = 0\}$. Since $v(z,r)$ is regular in the exterior of $G \cup \overline{G}$, by Theorem B $w(x,y) = W(\zeta,\overline{\zeta})$ is an analytic function of ζ and $\overline{\zeta}$ for $(\zeta,\overline{\zeta}) \in \Omega' \times \overline{\Omega}'$ where $\overline{\Omega}'$is the image of Ω' under conjugation. Hence we can conclude that

$$f(\zeta) = W(\zeta,\zeta) \tag{14.16}$$

is regular for ζ in $\Omega' \cap \overline{\Omega}'$. Using the inverse conformal mapping now shows that $v(z,r)\Big|_{(z,r)\in A}$ can be continued to an analytic function of η in the inverse image of $\Omega' \cap \overline{\Omega}'$. From Theorem 11.1 we can now conclude that $u(z,r)$ is regular in the exterior of $G \cup \overline{G} \cup \{(r,z) | r = 0\}$.

<u>Remark</u>: A similar analysis shows that $\left.\frac{\partial u}{\partial R}\right|_{R=C'}$ can be continued to an analytic function of η in the exterior of $G \cup \overline{G} \cup \{r = 0\}$. Hence the continuation of $u(z,r)$ across the circle $R = C'$ can be accomplished by referring to the results of section 4 instead of making use of Lewy's reflection principle.

We now collect our results in the following theorem:

Theorem 14.2 ([12]): Let $f(\cos\theta)$ be the radiation pattern of a solution $\tilde{u}(R,\theta)$ of the three dimensional axially symmetric Helmholtz equation where (R,θ) are polar coordinates and let

$$F(z) = \tfrac{1}{2}\int_{-1}^{+1} f(\xi)\ \frac{(1 + 4z^2)d\xi}{(1 - 4iz\xi - 4z^2)^{3/2}}.$$

Then $F(z)$, $z = Re^{i\theta}$, is an entire function of order one and finite exponential type C. If G is the conjugate indicator diagram of $F(z)$, then $\tilde{u}(R,\theta)$ is regular in the exterior of $G \cup \overline{G} \cup \{(R,\theta)\,|\,\theta = 0,\pi, R \leq C\}$ (see figure 14.2 below).

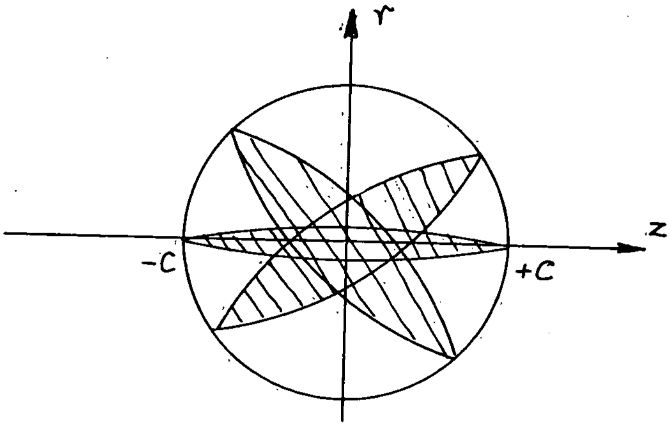


figure 14.2

Remark: The results of this section have recently been extended by B.D. Sleeman in

B.D. Sleeman, The three-dimensional inverse scattering problem for the Helmholtz equation, Proc. Camb. Phil. Soc. 73 (1973), 477-488.

V Pseudoparabolic equations

15. Pseudoparabolic Equations in One Space Variable

Consider the pseudoparabolic equation (c.f. [36])

$$\mathscr{L}[u] \equiv u_{xtx} + d(x,t)u_t + \eta u_{xx} + a(x)u_x + b(x)u = q(x,t) \tag{15.1}$$

defined in $D(H,T) = \{(x,t)\,|\,0 < x < H,\ 0 < t < T\}$. Assume $d(x,t) \in C^1\ (\bar{D}(H,T))$ ($\bar{D}(H,T)$ denotes the closure of $D(H,T)$), $q(x,t) \in C^0\ (\bar{D}(H,T))$, $a(x) \in C^1[0,H]$ and $b(x) \in C^0[0,H]$. η is a constant. (References for the appearance of equation (15.1) in physics can be found in [5] and [36]).

Define the adjoint equation by

$$\mathcal{M}[v] = v_{xtx} + d(x,t)v_t - \eta v_{xx} + (av)_x - bv = 0. \tag{15.2}$$

Now let $(\xi,\tau) \in D(H,T)$ and integrate

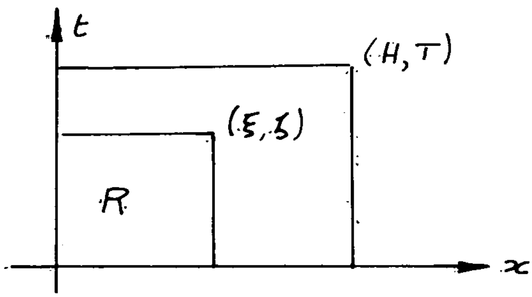


figure 15.1

the identity

$$v_t\mathscr{L}[u] - u_t\mathcal{M}[v] = \frac{\partial}{\partial x}[u_{xt}v_t - u_tv_{xt} - au_tv + \eta u_xv_t + \eta u_tv_x] + \frac{\partial}{\partial t}[au_xv + buv - \eta u_xv_x]$$

over the rectangle R in figure 15.1. An application of Green's formula gives

$$\int_0^\tau \int_0^\xi (v_t \mathscr{L}[u] - u_t \mathcal{M}[v])\,dx\,dt = \tag{15.3}$$

$$= \int_{\partial R} (u_{xt}v_t - u_t v_{xt} - au_t v + \eta u_x v_t + \eta u_t v_x)\,dt - (au_x v + buv - \eta u_x v_x)\,dx.$$

Suppose there exists a function $v(x,t;\xi,\tau)$ such that

$$\mathcal{M}[v] = 0 \tag{15.4a}$$

$$v_x(\xi,t;\xi,\tau) = \frac{1}{\eta}[1 - e^{\eta(t-\tau)}] \tag{15.4b}$$

$$v(\xi,t;\xi,\tau) = 0 \tag{15.4c}$$

$$v(x,\tau;\xi,\tau) = 0. \tag{15.4d}$$

If $\eta = 0$, (15.4b) is interpreted in its limiting form as $n \to 0$. Then if there exists a function $u(x,t)$ such that

$$\mathscr{L}[u] = q \tag{15.5a}$$

$$u(0,t) = f(t) \tag{15.5b}$$

$$u_x(0,t) = g(t) \tag{15.5c}$$

$$u(x,0) = h(x) \tag{15.5d}$$

where $f(t)$, $g(t) \in C^1[0,T]$, $h(x) \in C^2[0,H]$, then equation (15.3) implies that

$$\begin{aligned} u(\xi,\tau) = h(\xi) &+ \int_0^\xi [a(x)h'(x)v(x,0;\xi,\tau) - \eta h'(x)v_x(x,0;\xi,\tau) \\ &\qquad + b(x)h(x)v(x,0;\xi,\tau)]dx \\ &+ \int_0^\tau [g'(t)v_t(0,t;\xi,\tau) - f'(t)v_{xt}(0,t;\xi,\tau) \\ &\qquad -a(0)f'(t)v(0,t;\xi,\tau) + \eta g(t)v_t(0,t;\xi,\tau) \\ &\qquad + \eta f'(t)v_x(0,t;\xi,\tau)]dt \\ &+ \int_0^\tau \int_0^\xi q(x,t)v_t(x,t;\xi,\tau)\,dx\,dt. \end{aligned} \tag{15.6}$$

Equation (15.6) gives the solution of the Goursat problem (15.5) in terms of the <u>Riemann function</u> $v(x,t;\xi,\tau)$.

<u>Exercise 15.1</u>: Show $v(x,t;\xi,\tau)$ exists by using the methods of section 3. Let

$$||.||_\lambda = \max_{(x,t)\in R} \{e^{-\lambda[(\xi-x)+(\tau-t)]}|.|\}.$$

Show that as a function of ξ and τ, $\mathcal{L}[v] = 0$.

<u>Exercise 15.2</u>: Suppose the coefficients a, b and d are entire functions of x and q = 0. Show that $v(x,t;\xi,\tau)$ is an entire function of ξ. Conclude from equation (15.6) that if $h(x) = 0$ and $u(x,t)$ is a solution of $\mathcal{L}[u] = 0$ which is analytic in a neighbourhood of the origin, then $u(x,t)$ can be analytically continued into a strip of the form $|t| < t_0$, $-\infty < x < \infty$. Compare this result to the behaviour of solutions to parabolic and elliptic equations in two independent variables.

We now want so solve the first initial boundary value problem for equation (15.1) i.e. find a solution of $\mathcal{L}[u] = q$ in $D(H,T)$, continuously differentiable in $\overline{D}(H,T)$, such that

$$u(0,t) = f(t) \tag{15.7a}$$

$$u(x,0) = h(x) \tag{15.7b}$$

$$u(H,t) = \phi(t) \tag{15.7c}$$

where $f(t)$, $\phi(t)\in C^1[0,T]$ $h(x)\in C^2[0,H]$. To find u set $\xi = H$ in (15.6) and integrate by parts to arrive at

$$\gamma(\tau) = g(\tau)v_t(0,\tau;H,\tau) + \int_0^\tau [\eta v_t(0,t;H,\tau) - v_{tt}(0,t;H,\tau) - \eta v_{xt}(0,t;H,\tau)]g(t)dt \tag{15.8}$$

where

$$\begin{aligned}\gamma(\tau) = \phi(\tau) - h(H) &- \int_0^H [h'(x)(a(x)v(x,0;H,\tau) - \eta v_x(x,0;H,\tau))\\ &+ h(x)b(x)v(x,0;H,\tau)]dx\\ &+ h'(0)[v_t(0,0;H,\tau) + \eta v_x(0,0;H,\tau)]\\ &+ \int_0^\tau f'(t)[v_{xt}(0,t;H,\tau) - a(0)v(0,t;H,\tau)\\ &+ \eta v_x(0,t;H,\tau)]dt\\ &- \int_0^\tau\int_0^H q(x,t)v_t(x,t;H,\tau)dxdt\end{aligned} \tag{15.9}$$

<u>Exercise 15.3</u>: Use exercise 15.1 and the fact that $d(x,t)\in C^1(\overline{D}(H,T))$ to show that $\gamma(\tau)$ and the kernel of the integral equation (15.8) is continuously differentiable with respect to τ for $0 \leq \tau \leq T$, $0 \leq t \leq \tau$; see also [20] pp. 116-117. Conclude that if a solution $g(\tau)$ exists then $g(\tau)\in C^1[0,T]$.

To show that a solution $g(\tau)$ of (15.8) exists, it suffices to show $v_t(0,\tau;H,\tau) \neq 0$ for $\tau\in[0,T]$. To this end consider

$$\mu(x) = v_t(x,\tau;H,\tau) \tag{15.10}$$

for arbitrary (but fixed) τ in $[0,T]$. The differential equation (15.4a) and the boundary condition (15.4d) imply that

$$\mu_{xx} + d(x,\tau)\mu = 0 \tag{15.11}$$

Suppose $d(x,t) \leq 0$ for $(x,t)\in\overline{D}(H;T)$. Then if $\mu(0) = 0$, $\mu(x) \equiv 0$, since (15.4c) implies that $\mu(H) = 0$. But $\mu(x) \equiv 0$ implies that $u_x(H) = v_{xt}(H,\tau;H,\tau)=0$ which contradicts (15.4b). ((15.4b) implies that $v_{xt}(H,\tau;H,\tau) = -1$). Hence if $d(x,t) \leq 0$ we can solve (15.8) for $g(\tau)$ and substitute into (15.6) to give the (unique) solution of the first initial boundary value problem for $\mathcal{L}[u] = q$.

Theorem 15.1 ([13], [36]): Let $d(x,t)$ be continuously differentiable and nonpositive in $\overline{D}(H,T)$, $q(x,t)$ continuous in $D(H,T)$, and assume $a(x)\in C^{1}[0,H]$, $b(x)\in C[0,H]$. Let $f(t)$, $g(t)\in C^{1}[0,T]$ and $h(x)\in C^{2}[0,H]$. Then there exists a unique solution to $\mathcal{L}[u] = q(x,t)$ satisfying the initial - boundary data (15.7).

Example 15.1: In general $d(x,t) \leq 0$ in $\overline{D}(H,T)$ is necessary. For example $u(x,t) = t \sin kx$ is a solution of

$$u_{xtx} + k^2 u_t = 0 \tag{15.12}$$

for $(x,t)\in D(\frac{\pi}{k},T)$, T arbitrary. $u\in C^{1}[\overline{D}(\frac{\pi}{k},T)]$. But $u(0,t) = u(\frac{\pi}{k},t) = u(x,0) = 0$, i.e. the solution of the first initial boundary value problem is not unique.

A result similar to that of Theorem 15.1 can be obtained for pseudoparabolic equations in an arbitrary number of space dimensions. Rather than pursuing this investigation we will now turn our attention to the analytic properties of pseudoparabolic equations in two space variables.

16. Pseudoparabolic Equations in Two Space Variables.

Consider

$$\underset{\sim}{M}\left[\frac{\partial \underset{\sim}{u}}{\partial t}\right] + \gamma \underset{\sim}{L}\ [\underset{\sim}{u}] = 0 \tag{16.1}$$

where $\underset{\sim}{M} \equiv \Delta + \underset{\sim}{d}(x,y)$, $\underset{\sim}{L} = \Delta + \underset{\sim}{a}(x,y)\frac{\partial}{\partial x} + \underset{\sim}{b}(x,y)\frac{\partial}{\partial y} + \underset{\sim}{c}(x,y)$, and γ is a constant. Let $\underset{\sim}{u} = e^{-\gamma t}u$. Then (16.1) becomes

$$\mathcal{L}[u] \equiv M\left[\frac{\partial u}{\partial t}\right] + L[u] = 0 \tag{16.2}$$

where

$$M = \Delta + d(x,y)$$

$$L = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} + c(x,y).$$

Assumption: As a function of $z = x + iy$ and $z^* = x - iy$, a,b,c,d are analytic in $D \times D^*$ where $D^* = \{z^* | \overline{z^*} \in D\}$ and D is a bounded simply connected domain in $\mathbb{R}^2$.

Define the adjoint of $\mathcal{L}[u] = 0$ to be

$$\mathcal{M}[v] \equiv M[v_t] - L^*[v] = 0 \tag{16.3}$$

where $L^*[v] = -(av)_x - (bv)_y + cv$.

Definition 16.1: A function S of the form

$$S(x,y,t;\xi,\eta,\tau) = A(x,y,t;\xi,\eta,\tau)\log\frac{1}{r} + B(x,y,t;\xi,\eta,\tau) \tag{16.4}$$

where $r = [(x - \xi)^2 + (y - \eta)^2]^{\frac{1}{2}}$ will be called a fundamental solution if

1) $\mathcal{M}[S] = 0$ for $r \neq 0$.

2) A and B are analytic functions of their independent variables.

3) $A_t = 1$ at $x = \xi$, $y = \eta$, $t = \tau$. $A = B = 0$ at $t = \tau$.

We will now construct S. Let

$$\begin{aligned} z &= x + iy & \zeta &= \xi + i\eta \\ z^* &= x - iy & \zeta^* &= \xi - i\eta \end{aligned} \tag{16.5}$$

Then (16.3) can be written as

$$\begin{aligned} \mathcal{M}[V] &= M[V_t] - L^*[V] \\ &= V_{zz^*t} + \delta V_t + (\alpha V)_z + (\beta V)_{z^*} - \gamma V = 0 \end{aligned} \tag{16.6}$$

where $V(z,z^*,t) = v(x,y,t)$, $\alpha = \frac{1}{4}(a + ib)$, $\beta = \frac{1}{4}(a - ib)$, $\gamma = {}^c/_4$, $\delta = {}^d/_4$. Note that $r^2 = (z - \zeta)(z^* - \zeta^*)$.

Substituting (16.4) into (16.6) gives

$$\mathcal{M}[S] = \mathcal{M}[A]\log\frac{1}{r} - \frac{A_{zt} + \beta A}{2(z^* - \zeta^*)} - \frac{A_{z^*t} - \alpha A}{2(z - \zeta)} + \mathcal{M}[B] = 0$$

which implies that

$$\mathcal{M}[A] = 0 \tag{16.7}$$

$$\left[\frac{\partial^2}{\partial z\,\partial t} + \beta(z,\zeta^*)\right] A(z,\zeta^*,t;\zeta,\zeta^*,\tau) = 0 \tag{16.8}$$

$$\left[\frac{\partial^2}{\partial z^*\partial t} + \alpha(\zeta,z^*)\right] A(\zeta,z^*,t;\zeta,\zeta^*,\tau) = 0 \tag{16.9}$$

Once we have found A, B is any solution of

$$\mathcal{M}[B] = \frac{A_{zt} + \beta A}{2(z^*-\zeta^*)} + \frac{A_{z^*t} + \alpha A}{2(z-\zeta)} . \tag{16.10}$$

Now let

$$A = \sum_{j=1}^{\infty} A_j(z,z^*;\zeta,\zeta^*)\,\frac{(t-\tau)^j}{j!} \tag{16.11}$$

Substituting (16.11) into equations (16.7) - (16.9) gives

$$\begin{aligned}
&\frac{\partial A_1}{\partial z} = 0 \text{ on } z^* = \zeta^* \\
&\frac{\partial A_1}{\partial z^*} = 0 \text{ on } z = \zeta \\
&\frac{\partial A_{j+1}}{\partial z} + \beta A_j = 0 \text{ on } z^* = \zeta^*;\ j = 0,1,2,\ldots \\
&\frac{\partial A_{j+1}}{\partial z^*} + \alpha A_j = 0 \text{ on } z = \zeta;\ j = 0,1,2,\ldots
\end{aligned} \tag{16.12}$$

and

$$\begin{aligned}
M[A_1] &= 0 \\
M[A_{j+1}] &= L^*[A_j]
\end{aligned} \tag{16.13}$$

Condition (3) satisfied by S implies that

$$\begin{aligned}
&A_1(\zeta,\zeta^*;\zeta,\zeta^*) = 1 \\
&A_j(\zeta,\zeta^*;\zeta,\zeta^*) = 0,\ j = 2,3,\ldots
\end{aligned} \tag{16.14}$$

and hence

$$A_1 = 1 \quad \text{on } z^* = \zeta^*$$
$$A_1 = 1 \quad \text{on } z = \zeta \tag{16.15}$$

and

$$A_{j+1}(z,\zeta^*;\zeta,\zeta^*) = -\int_{\zeta}^{z} \beta(\sigma,\zeta^*)A_j(\sigma,\zeta^*;\zeta,\zeta^*)d\sigma; \; j=1,2,...$$
$$A_{j+1}(\zeta,z^*;\zeta,\zeta^*) = -\int_{\zeta^*}^{z^*} \alpha(\zeta,\rho)A_j(\zeta,\rho;\zeta,\zeta^*)d\rho; \; j = 1,2,... \tag{16.16}$$

Note that equations (16.13) and (16.15) imply that A_1 is the Riemann function for $M[u] = 0$.

Lemma 16.1 ([14]): $A(z,z^*,t;\zeta,\zeta^*,\tau)$ is an analytic function of its six independent variables for all (complex) t, τ and z, $\zeta \in D$, z^*, $\zeta^* \in D^*$.

Proof: Integrating the identity (11.4) of section 11 with $U = A_{j+1}$, $V = A_1$, over the quadrilateral (ζ,ζ^*), (z,ζ^*), (z,z^*), (ζ,z^*) gives (after an integration by parts)

$$\begin{aligned} A_{j+1}(z,z^*;\zeta,\zeta^*) = &-\int_{\zeta}^{z} A_1(\sigma,z^*;z,z^*)\beta(\sigma,z^*)A_j(\sigma,z^*;\zeta,\zeta^*)d\sigma \\ &-\int_{\zeta^*}^{z^*} A_1(z,\rho,z,z^*)\alpha(z,\rho)A_j(z,\rho;\zeta,\zeta^*)d\rho \\ &+\int_{\zeta}^{z}\int_{\zeta^*}^{z^*} A_1(\sigma,\rho;z,z^*)\gamma(\sigma,\rho) + \frac{\partial}{\partial\sigma}A_1(\sigma,\rho;z,z^*)\alpha(\sigma,\rho) \\ &+ \frac{\partial}{\partial\rho}A_1(\sigma,\rho;z,z^*)\beta(\sigma,\rho)A_j(\sigma,\rho;\zeta,\zeta^*)d\rho d\sigma. \end{aligned} \tag{16.17}$$

By induction A_j is analytic for z, $\zeta \in D$, z^*, $\zeta^* \in D^*$ (since the Riemann function A_1 is - see [20], p.141, and [39]). Let k be an upper bound on $|A_1\beta|$, $|A_1\alpha|$ and $|A_1\gamma + A_{1_z}\alpha + A_{1_{z^*}}\beta|$ for z, $\zeta \in \bar{\Omega} \subset D$, z^*, $\zeta^* \in \bar{\Omega}^* \subset D^*$, where $\bar{\Omega}$ and $\bar{\Omega}^*$ are arbitrary compact subsets of D and D^* respectively. Let ℓ be an upper bound on the legnth of the paths of integration in (16.17) and let

$A_1| \leq C$ for $z, \zeta \in \bar{\Omega} \subset D$, $z^*, \zeta^* \in \bar{\Omega}^* \subset D^*$. Then by induction (16.17) implies that

$$|A_j| \leq C\, k^j \ell^j (2+\ell)^j; \quad z, \zeta \in \Omega, \; z^*, \zeta^* \in \Omega^* \tag{16.18}$$

and hence the series (16.11) converges in $|t - \tau| < T_1$, z, $\zeta \in \bar{\Omega}$, z^*, $\zeta^* \in \bar{\Omega}^*$, where T_1 is arbitrarily large. The lemma follows from this last statement.

Now look at the function B. Set

$$B = \sum_{j=1}^{\infty} B_j(z, z^*; \zeta, \zeta^*) \frac{(t-\tau)^j}{j!} \tag{16.19}$$

Substituting (16.19) into (16.10) gives

$$M[B_{j+1}] = L^*[B_j] + \frac{\frac{\partial}{\partial z} A_{j+1} + \beta A_j}{2(z^* - \zeta^*)} + \frac{\frac{\partial}{\partial z^*} A_{j+1} + \alpha A_j}{2(z - \zeta)}; \; j=1,2,\ldots$$

$$M[B_1] = \frac{\partial A_1}{\partial z} \Big/ 2(z^* - \zeta^*) + \frac{\partial A_1}{\partial z^*} \Big/ 2(z - \zeta) \tag{16.20}$$

Since B is an arbitrary solution of (16.10), without loss of generality we impose the boundary conditions

$$B_j(z, \zeta^*; \zeta, \zeta^*) = B_j(\zeta, z^*; \zeta, \zeta^*) = 0; \; j=1,2,3,\ldots \tag{16.21}$$

Imitating the proof of lemma 16.1 now gives

<u>Lemma 16.2 ([14])</u>: $B(z, z^*, t; \zeta, \zeta^*, \tau)$ is an analytic function of its six independent variables for all (complex) t, τ, and z, $\zeta \in D$, z^*, $\zeta^* \in D^*$.

Now let

$$T = \{t \,|\, 0 \leq t < T_0\}$$

$$G(D \times T) = \{u(x,y,t) \,|\, u, u_x, u_y \in C^1(D \times T); u_{xyt}, u_{xxt}, u_{yyt} \in C^0(D \times T)\}$$

<u>Theorem 16.1 ([14])</u>: Let $u(x,y,t) \in G(D \times T)$ be a solution of (16.2) in $D \times T$ and assume $\mathcal{U}(z, z^*, 0) = u(x,y,0)$ is analytic in $D \times D^*$. Then for each fixed $t \in T$, $\mathcal{U}(z, z^*, t) = u(x,y,t)$ is

an analytic function of z and z^* in $D\times D^*$.

Proof: We first show that without loss of generality we can assume $\mathcal{U}(z,z^*,0) = 0$. Let $f(z,z^*) = L[\mathcal{U}(z,z^*,0)]$ and define

$$C(z,z^*,\zeta,\zeta^*,t) = \sum_{j=1}^{\infty} C_j(z,z^*,\zeta,\zeta^*) \frac{t^j}{j!} \tag{16.22}$$

where

$$\begin{gathered} M[C_1] = f(z,z^*) \\ M[C_{j+1}] = -L[C_j];\ j = 1,\ 2,\ \ldots \\ C_j(z,\zeta^*;\zeta,\zeta^*) = C_j(\zeta,z^*,\zeta,\zeta^*) = 0;\ j = 1,\ 2,\ \ldots \end{gathered} \tag{16.23}$$

Using the analysis of lemma 16.1, it is easy to show that C exists, is analytic for all complex, t, $z,\zeta\in D, z^*,\zeta^*\in D^*$, and satisfies

$$\mathcal{L}[C] = f \tag{16.24}$$

$$C(z,z^*;\zeta,\zeta^*,0) = 0. \tag{16.25}$$

Hence $V(z,z^*,t) = \mathcal{U}(z,z^*,t) - \mathcal{U}(z,z^*,0) + C(z,z^*,\zeta,\zeta^*,t)$ satisfies $\mathcal{L}[V] = 0$, $V(z,z^*,0) = 0$, and to prove the theorem it suffices to show that V is analytic in $D\times D^*$ for each fixed t. Without loss of generality assume that $u(x,y,t)\in G(\overline{D}\times T)$ and that D has a smooth boundary., Integrating the identity

$$\begin{aligned} v_t\mathcal{L}[u] - u_t\mathcal{M}[v] &= \frac{\partial}{\partial x}[u_{xt}v_t - u_tv_{xt} - au_tv] \\ &+ \frac{\partial}{\partial y}[u_{yt}v_t - u_tv_{yt} - bu_tv] + \frac{\partial}{\partial t}[cuv + au_xv + bu_yv] \end{aligned} \tag{16.26}$$

over $D\times T$ gives

$$\begin{aligned} &\iiint_{D\times T}(v_t\mathcal{L}[u] - u_t\mathcal{M}[v])dxdydt = \\ &\iint_{\partial(D\times T)} (u_{xt}v_t-u_tv_{xt}-au_tv)dydt - (u_{yt}v_t-u_tv_{yt}-bu_tv)dxdt \\ &\qquad + (cuv + au_xv + bu_yv)dxdy. \end{aligned} \tag{16.27}$$

Now let $T_0 = \tau$, $v = S$ and $u = V$ and replace $D\times T$ in (16.27) by $D\times T - \Omega\times T$ where Ω is a thin cylinder surrounding the singular line $r = 0$. Note that $V = 0$ on $t = 0$, $S = 0$ on $t = \tau$. Computing the residue as Ω shrinks onto $r = 0$ gives

$$0 = 2\pi \int_0^\tau V_t(\xi,\eta,t)dt + \int_0^t\int_{\partial D}[(V_{xt}S_t - V_tS_{xt} - aV_tS)dy$$
$$- (V_{yt}S_t - V_tS_{yt} - bV_tS)dx]dt$$

or

$$V(\xi,\eta,\tau) = -\frac{1}{2\pi}\int_0^\tau\int_{\partial D}\Big[(V_{xt}S_t - V_tS_{xt} - aV_tS)dy$$
$$- (V_{yt}S_t - V_tS_{yt} - bV_tS)dx\Big]dt, \tag{16.28}$$

and the theorem now follows from lemmas 16.1 and 16.2

<u>Example 16.1</u>: In Theorem 16.1 it is not possible to remove the assumption that $\mathcal{U}(z,z^*,0)$ is analytic. Consider the special case of equation (16.1) when $\underset{\sim}{M} = \underset{\sim}{L}$ and $\gamma = 1$. Then $\underset{\sim}{u}(x,y,t) = e^{-t}\underset{\sim}{u}(x,y,0)$ is a solution of (16.1) and is not analytic unless $u(x,y,0)$ is.

We now turn our attention to constructing a reflection principle for solutions of equation (16.2).

Integrate the identity

$$W_t\mathcal{L}[\mathcal{U}] - \mathcal{U}_t\mathcal{M}[W] = \frac{\partial}{\partial w}(\mathcal{U}_{tz^*}W_t - \alpha\mathcal{U}_tW)$$
$$= \frac{\partial}{\partial z^*}(\mathcal{U}_tW_{tz} + \beta\mathcal{U}_tW) + \frac{\partial}{\partial t}(\alpha\mathcal{U}_zW + B\mathcal{U}_{z^*}W + \gamma\mathcal{U}W) \tag{16.29}$$

over a three dimensional cell $G \subseteq D\times D^*\times T$, set $W = A$, and let $\bar{\mathcal{L}}[\mathcal{U}] = 0$. By lemma 16.1 and Theorem 16.1 the derivatives in (16.29) are well defined. We arrive at

$$0 = \iint_{\partial G}(\mathcal{U}_tA_t)_{z^*}dz^*dt - \iint_{\partial G}\mathcal{U}_t(A_{tz^*} + \alpha A)dz^*dt$$
$$+ \iint_{\partial G}\mathcal{U}_t(A_{tz} + \beta A)dzdt + \iint_{\partial G}(\alpha\mathcal{U}_zA + \beta\mathcal{U}_{z^*}A + \gamma\mathcal{U}A)dzdz^*. \tag{16.30}$$

Now (paying attention to the boundary conditions (16.8) and (16.9) satisfied by A) first let G be the parallelpiped with base $(\zeta_0, \zeta_0^*, 0)$, $(\zeta, \zeta_0^*, 0)$, $(\zeta, \zeta^*, 0)$ and $(\zeta_0, \zeta^*, 0)$ and height τ, and then let G be the wedge with base $(\zeta, \overline{\zeta}, 0)$, $(\overline{\zeta}^*, \zeta^*, 0)$, $(\zeta, \zeta^*, 0)$ and height τ. This yields two equations analogous to equations (11.6) and (11.7) of section 11, and following the analysis of section (11) we arrive at the following analogue of Lewy's reflection principle for elliptic equations

Theorem 16.2 ([14]): Let $D \times T$ be a simply connected cylindrica[illegible] domain in the half space $y < 0$ whose boundary contains a portion σ of the plane $y = 0$. Let $u(x,y,t) \in G(D \times T) \cap C^2(\overline{D} \times T)$ be a solution of $\mathcal{L}[u] = 0$ in $D \times T$, and on σ suppose $u(x,0,t) = \rho(x,$[illegible] where $\rho(z,t) \in C^1(D \cup \sigma \cup D^* \times T)$ and for each fixed $t \in T$ is an analyti[illegible] function of z [illegible] z^* in $D \cup \sigma \cup D^*$ [illegible]. Then $u(x,y,t)$ can be uniquely continued as a solution of $\mathcal{L}[u] = 0$ in class $G(D \cup \sigma \cup D^* \times T)$ into all of $D \cup \sigma \cup D^* \times T$ provided $U(z, z^*, 0)$ $(= u(x,y,0))$ is analytic in $D \cup \sigma \cup D^* \times D \cup \sigma \cup D^*$.

Remark: For more recent results on the analytic theory of pseudoparabolic equations see the University of Glasgow Ph.D. thesis of W. Rundell and the Indiana University Ph.D. thesis of S. Bhatnagar.

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Note that since each $\phi_n(x,t)$ is a solution of (1), the maximum error (in absolute value) occurs on the base or vertical sides of the rectangle $-1 \leq x \leq 1$, $0 \leq t \leq 1$; in this case at the points $(x,t)=(\pm 1,1)$, where the relative error is 8.4473×10^{-8} in absolute value.

The computation time to construction $u^*(x,t)$ (i.e. to find the Taylor coefficients of $\phi_n(x,t)$, the coefficients a_n, and to evaluate $u^*(x,t)$ at selected grid points) using the CDC 6600 computer was approximately six seconds.

TABLE I

x	t	Approximate solution	Relative error
0	0	1.00000	-6.9580×10^{-9}
0.2	0	0.98020	-3.3762×10^{-9}
0.4	0	0.92312	4.2696×10^{-9}
0.6	0	0.83527	8.0803×10^{-9}
0.8	0	0.72615	-4.3613×10^{-10}
1.0	0	0.60653	-2.2466×10^{-8}
0	0.2	0.81873	4.0571×10^{-10}
0.2	0.2	0.80252	9.3730×10^{-10}
0.4	0.2	0.75578	2.7202×10^{-9}
0.6	0.2	0.68386	6.1830×10^{-9}
0.8	0.2	0.59452	1.1356×10^{-8}
1.0	0.2	0.49659	1.5536×10^{-8}
0	0.4	0.67032	2.2209×10^{-9}
0.2	0.4	0.65705	1.7415×10^{-9}
0.4	0.4	0.61878	3.2910×10^{-11}
0.6	0.4	0.55990	-3.6697×10^{-9}
0.8	0.4	0.48675	-1.0332×10^{-8}
1.0	0.4	0.40657	-2.0325×10^{-8}
0.	0.6	0.54881	-1.1541×10^{-9}
0.2	0.6	0.53794	-8.6797×10^{-10}
0.4	0.6	0.50662	3.4421×10^{-10}
0.6	0.6	0.45841	3.6095×10^{-9}
0.8	0.6	0.39852	1.0898×10^{-8}
1.0	0.6	0.33287	2.4115×10^{-8}
0	0.8	0.44933	2.7676×10^{-9}
0.2	0.8	0.44043	2.5721×10^{-9}
0.4	0.8	0.41478	1.5339×10^{-9}
0.6	0.8	0.37531	-1.8415×10^{-9}
0.8	0.8	0.32628	-1.0323×10^{-8}
1.0	0.8	0.27253	-2.7649×10^{-8}
0	1.0	0.36788	-7.3333×10^{-11}
0.2	1.0	0.36059	4.9411×10^{-10}
0.4	1.0	0.33960	2.3005×10^{-9}
0.6	1.0	0.30728	4.1011×10^{-9}
0.8	1.0	0.26714	-5.4443×10^{-9}
1.0	1.0	0.22313	-8.4473×10^{-8}

Since $u(x,t)$ and $u^*(x,t)$ are even functions of x, values of the approximate solution and relative error are only given for $0 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant 1$.

entire rectangle $-1 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant 1$. Such an advantage is of course particularly important in higher dimensional problems. We finally note that since the solution of (1), (2) is an even function of x, the odd coefficients a_1, a_3, ..., a_{13} in (14), (15) all turn out to be identically zero.

The exact solution of (1), (2) is

$$u(x,t) = e^{-\frac{1}{2}x^2-t} \quad . \tag{16}$$

In Table I below we give the values of $u^*(x,t)$ at selected grid points and also the relative error defined by

$$\text{relative error} = \frac{u^*(x,t)-u(x,t)}{u(x,t)} \quad . \tag{17}$$

TABLE I /

This is done for n=0,1,2, ... , 14. The integration in (11) is exact, i.e. the polynomials $P_{10}(s,x)$ and $h_n(s,t)$ are multiplied together, integrated, and added to $h_n(x,t)$. The set $\{u_n(x,t)\}_{n=0}^{14}$ is now orthonormalized over the base and vertical sides of the rectangle $-1 \leq x \leq 1$, $0 \leq t \leq 1$ by means of the Gram-Schmidt process to obtain the set $\{\phi_n(x,t)\}_{n=0}^{14}$. This is done using double precision arithmetic. The inner product used is

$$(\phi,\psi) = \int_0^1 \phi(-1,t)\psi(-1,t)dt + \int_{-1}^1 \phi(x,0)\psi(x,0)dx + \int_0^1 \phi(1,t)\psi(1,t)dt \quad . \tag{13}$$

The integrations performed in the Gram-Schmidt process are again exact, i.e. polynomials are multiplied together and integrated. The solution to the initial-boundary value problem (1),(2) is now approximated by the sum

$$u^*(x,t) = \sum_{n=0}^{14} a_n\phi_n(x,t) \tag{14}$$

where

$$a_n = (u,\phi_n) \quad . \tag{15}$$

Note that the coefficients a_n, n=0,1, ... , 14, can be computed solely from a knowledge of the functions $\phi_n(x,t)$, n=0,1, ..., 14, and the initial-boundary data (2). The coefficients a_n are computed by truncating the Taylor series for the functions $e^{-\frac{1}{2}t}$ and $e^{-\frac{1}{2}x^2}$ to the same order of accuracy as (10), and then computing (15) exactly by multiplying the appropriate polynomials together and integrating. Note that although the initial-boundary value problem (1), (2) is two dimensional, due to the fact we are approximating by means of a complete family of solutions only one dimensional integrals need be computed. This is an advantage of the present approach over other methods, where integrations must be performed over the

$$P(x,x) = \frac{1}{2}\int_0^x s^2 ds = \frac{x^3}{6} \tag{6a}$$

$$P(-x,x) = 0 \tag{6b}$$

and $h(x,t)$ is a solution of

$$h_{xx} = h_t \quad . \tag{7}$$

Note that since the coefficients of (1) are independent of t, so is the kernel $P(s,x)$. The initial value problem (5),(6) satisfied by $P(s,x)$ follows from the initial value problems satisfied by $K(s,x)$ and $M(s,x)$ (c.f. (2.1.30), (2.1.31), (2.1.33), (2.1.34) and the facts that

$$\begin{aligned} K(s,x) &= -K(-s,x) \\ M(s,x) &= M(-s,x). \quad) \end{aligned} \tag{8}$$

From (5), (6) we have that $\tilde{P}(\xi,\eta) = P(\xi-\eta,\ \xi+\eta)$ can be constructed by the iterative scheme

$$\begin{aligned} \tilde{P}(\xi,\eta) &= \lim_{n\to\infty} \tilde{P}_n(\xi,\eta) \\ \tilde{P}_1(\xi,\eta) &= \frac{\xi^3}{6} \\ \tilde{P}_{n+1}(\xi,\eta) &= \frac{\xi^3}{6} + \int_0^\eta \int_0^\xi (\xi+\eta)^2 \tilde{P}_n(\xi,\eta)\, d\xi d\eta \end{aligned} \tag{9}$$

for $n=1,2,\ldots$. As an approximation to the kernel $P(s,x)$ we use $P_{10}(s,x)$ as defined by (9). A short calculation using (9) shows that

$$\max_{\substack{-1\leq x\leq 1 \\ 1\leq s\leq 1}} |P(s,x) - P_{10}(s,x)| \leq 1.6 \times 10^{-20}. \tag{10}$$

We now construct the (approximate) complete family of solutions

$$u_n(x,t) = h_n(x,t) + \int_{-x}^{x} P_{10}(s,x) h_n(s,t)\, ds \tag{11}$$

where

$$h_n(x,t) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{x^{n-2k} t^k}{(n-2k)!k!} \quad . \tag{12}$$

spirit to the above steps for solving boundary value problems, but of course the details vary depending on the type of inverse problem being investigated. The method for obtaining an analytic solution to various inverse problems has been given in these lectures (c.f. sections 1.5, 2.3, 3.3 and 4.3) and each of these is ameniable to numerical computations. In addition to approximating kernels of integral operators and the numerical integration of certain integrals, one must in some cases (e.g. in inverse problems in subsonic fluid flow and the inverse Stephan problem for the heat equation in two space variables) construct approximations to certain conformal mappings.

To illustrate the general approach for using integral operators to obtain numerical solutions to boundary or initial-boundary value problems we consider the following simple example due to Y.F. Chang of the Department of Computer Science, University of Nebraska (see also [13]). We want to use a complete family of solutions to construct an approximate solution to the initial-boundary value problem

$$u_{xx} - x^2u = u_t \quad ; \quad -1 < x < 1,\ 0 < t < 1 \tag{1}$$

$$\begin{aligned} u(-1,t) = e^{-\frac{1}{2}-t} \quad , \quad u(1,t) = e^{-\frac{1}{2}-t} \quad ; \quad 0 \leqslant t \leqslant 1 \\ u(x,0) = e^{-\frac{1}{2}x^2} \quad ; \ -1 \leqslant x \leqslant 1. \end{aligned} \tag{2}$$

To construct a complete family of solutions we use the operator $\underset{\sim}{T}_3$ of section 2.1:

$$u(x,t) = \underset{\sim}{T}_3\{h\} = h(x,t) + \int_{-x}^{x} P(s,x)h(s,t)ds \tag{3}$$

where

$$P(s,x) = \frac{1}{2}\left[K(s,x) + M(s,x)\right] \tag{4}$$

is the (unique) solution of the initial value problem

$$P_{xx} - P_{ss} - x^2P = 0 \tag{5}$$

3) Orthonormalization of the complete family by use of the Gram-Schmidt process, using numerical integration if necessary.

4) Computation of the Fourier coefficients. If the boundary of the domain is reasonably simple this (as well as step 3) can again be reduced to the "problem" of integrating the product of two polynomials (where we have assumed that the boundary or initial-boundary data has been approximated by polynomials).

5) Construction of the approximate solution and error estimates. Once one has completed step 4) the approximate solution of course follows immediately. Error estimates can be found by means of a priori estimates or (more simply) the maximum principle.

If one uses the integral operator in conjunction with double or single layer potentials to solve the desired boundary or initial-boundary value value problem then of course steps 2)-5) are replaced by

6) Solution of the integral equation. Since the resulting integral equations are of Volterra or Fredholm type, a numerical solution can be obtained by any one of a variety of known methods (c.f. [1],[22]).

7) Construction of the approximate solution. This is obtained by substituting the approximate density obtained from step 6) back into the double or single layer representation for the solution to the heat or Laplace or Helmholtz equation which the operator is operating on, constructing this solution to the heat or Laplace or Helmholtz equation, and then using the integral operator (and step 1)) to obtain the desired approximate solution to the original equation.

8) Error estimate. These are obtained from step 6), assuming the kernel of the integral operator has been approximated to a known degree of accuracy. In the case of inverse problems, the numberical approach is similar in

Appendix

A Numerical Example .

Throughout these lectures we have given reference in the literature where numerical examples of the use of integral operator methods to solve boundary and initial-boundary value problems can be found (In particular see [1] ,[3] , [4], [5], [13], [24], [33], [34], [47]). The numerical procedure consists basically in the following steps (in the case of the solution of boundary or initial-boundary value problems by means of a complete family of solutions).

1) Approximation of the kernel of the integral operator. Since the kernel is given by either a recursion or iteration scheme, approximations can be obtained by truncating the iteration (or recursion) process after a finite number of steps. Error estimates can be obtained by either using the estimates used to show the series for the kernel converges, or by qualitatively observing that the contributions to the kernel become negligible after a finite number of iteration (or recursion)steps. The qualitative approach is reasonably safe since in practice the coefficients of the differential equation are polynomials and the kernel of the integral operator converges at a steady rate (in general geometrically).
2) Construction of a complete family of solutions. Since the approximation to the kernel of the integral operator is a polynomial (if the coefficients of the differential equation are polynomials) and so is the function operated on (i.e. z^n, $z^n t^m$ or the heat polynomial $h_n(x,t)$) this is merely a question of multiplying one polynomial by another and then performing the integration indicated in the definition of the integral operator.

$$= \int_{1/a}^{a} f(s)\left[s^{2n+2} - s^{2n+4}\right]ds \tag{4.3.24}$$

$$= \frac{1}{2}\int_{1/a^2}^{a^2} f(s^{\frac{1}{2}})\left[s^{\frac{1}{2}} - s^{\frac{3}{2}}\right]s^n ds$$

for $n=0,1,2,\ldots$. Since the set $\{r^n\}_{n=0}^{\infty}$ is complete in $L^2[\frac{1}{a},a]$, we have from (4.3.24) that

$$f(r^{1/2})(r^{1/2} - r^{3/2})=0 \tag{4.3.25}$$

for $r\varepsilon[\frac{1}{a},a^2]$, and hence $f(r)=0$ for $r\varepsilon[1,a]$.

The theorem is now proved.

The uniqueness of the function $B(r)$ follows immediately from the above theorem. Furthermore the function $B(r)$ can be approximated in $L^2[1,a]$ be orthonormalizing the set $\{P_n(r)\}_{n=0}^{\infty}$ over the interval $[1,a]$ to obtain the orthonormal set $\{\phi_n(r)\}_{n=0}^{\infty}$ and then approximating $B(r)$ in $L^2[1,a]$ by the function

$$B_N(r) = \sum_{n=0}^{N} b_n\phi_n(r) \tag{4.3.26}$$

where

$$b_n = \int_1^a \phi_n(s)B(s)ds \quad . \tag{4.3.27}$$

The coefficients b_n can be found by using (4.3.19), (4.3.20). If it is assumed that $B(r)\varepsilon C^1[1,a]$, then it can be concluded that $B_N(r)$ approximates $B(r)$ pointwise almost everywhere on $[1,a]$ (c.f.[43]).

have that a_{n1} is purely imaginary, and hence μ_n is real for each $n, n=0,1,2,\ldots$

We will now assume the existence of a continuously differentiable function $B(r)$ such that (4.3.19), (4.3.20) is valid, and address ourselves to the problems of uniqueness and approximation in $L^2[1,a]$. We restrict ourselves solely to the problem of uniqueness and approximation, since it is assumed a priori that the sequence μ_n (or a_{n1}) is a (generalized) moment sequence for some function $B(r)$ to be determined and hence the existence of $B(r)$ is not in question. The basic problems of uniqueness and approximation can be settled by appealing to the following theorem:

Theorem 4.3.1 ([19]): The functions

$$P_n(r) = r^{2n+2} + r^{-2n} - 2r \quad ,$$

$n=0,1,2,\ldots$, are complete in $L^2[1,a]$.

Proof: Let $f(r)$ be a continuous function on the interval $[1,a]$. Since the space of continuous functions on $[1,a]$ is dense in $L^2[1,a]$, to prove the theorem it suffices to show that if

$$\int_1^a f(s)P_n(s)ds = 0 \tag{4.3.21}$$

for $n=0,1,2,\ldots$, then $f(r)=0$ for $r\varepsilon[1,a]$. For $r\varepsilon\ [\frac{1}{a},1]$ define $f(r)$ by

$$f(r) = r^{-4}f(\tfrac{1}{r}) \quad ; \quad r\varepsilon\ [\tfrac{1}{a},1] \ . \tag{4.3.22}$$

Then

$$\int_1^a f(s)s^{-2n}ds = \int_{1/a}^1 f(s)s^{2n+2}ds \tag{4.3.23}$$

and hence from (4.3.21) we have

$$0 = \int_1^a f(s)\left[P_n(s) - P_{n+1}(s)\right]ds$$

$$= \int_1^a f(s)\left[s^{2n+2} + s^{-2n} - s^{2n+4} - s^{-2n-2}\right]ds$$

$$- \frac{(\frac{1}{2})^{n+2}}{\Gamma(n+\frac{5}{2})} + \int_1^\infty N_0(\log 1, \log s) \frac{(\frac{1}{2})^n s^{n-\frac{1}{2}}}{\Gamma(n+\frac{3}{2})} ds$$

$$= a_{n0}\, i\, \frac{(-1)^n (\frac{1}{2})^{-n+1}}{\Gamma(-n+\frac{3}{2})} + a_{n1}\, i\, \frac{(-1)^{n+1}(\frac{1}{2})^{-n-1}}{\Gamma(-n+\frac{1}{2})} \tag{4.3.17}$$

$$+ a_{n0}\, i \int_1^\infty N_0(\log 1, \log s) \frac{(-1)^{n+1}(\frac{1}{2})^{-n-1} s^{-n-\frac{3}{2}}}{\Gamma(-n+\frac{1}{2})} ds \quad .$$

The equation corresponding to (4.3.17) for n=0 is exactly the same except that the term $c_{oo}/\Gamma(\frac{3}{2})$ is added to the right hand side. Note that the coefficient a_{no} is independent of B(r). From (4.2.43) and (4.2.35) we have

$$\int_1^\infty N_0(\log 1, \log s)\, s^m ds = -\frac{1}{2}\int_1^{a^2}\int_{s^{\frac{1}{2}}}^{a} \xi B(\xi) s^m d\xi ds$$

$$= -\frac{1}{2}\int_1^{a}\int_1^{\xi^2} \xi B(\xi) s^m ds d\xi \tag{4.3.18}$$

$$= -\frac{1}{2(m+1)}\int_1^a \xi^{2m+3} B(\xi)\, d\xi + \frac{1}{2(m+1)}\int_1^a \xi B(\xi) d\xi,$$

and hence using (4.3.16) and (4.3.18) we can rewrite (4.3.17) as

$$\mu_n = \int_1^a B(s)\left[s^{2n+2} + s^{-2n} - 2s\right] ds \tag{4.3.19}$$

where for n > 0

$$\mu_n = -(2n+1)\left[\frac{-(2n+1)}{(2n+3)(1-2n)} + a_{n1}\, i\, \frac{(-1)^{n+1}(\frac{1}{2})^{-2n-1}\Gamma(n+\frac{3}{2})}{\Gamma(-n+\frac{1}{2})}\right]. \tag{4.3.20}$$

For n=0, μ_o is the same as defined above except that the term c_{oo} is subtracted from the right hand side. The μ_n are known from the far field pattern, and hence the problem of determining the function B(r) has now been reduced to solving the generalized moment problem (4.3.19), (4.3.20). Note that if we assume that B(r) is real valued, then from (4.3.16), (4.3.17) we

Recall once again that although the far field pattern $f(\theta;\lambda)$ is assumed to be known, the functions $j_{n+\frac{1}{2}}(r)$ and $h_{n+\frac{1}{2}}(r)$ are unknown since $B(r)$ is as of yet unknown. However if we expand $f(\theta,\lambda)$ in a Legendre series

$$f(\theta;\lambda) = \sum_{n=0}^{\infty} \tilde{a}_n(\lambda)P_n(\cos\theta) , \tag{4.3.12}$$

then from (4.3.11) and (4.3.13) we have

$$\begin{aligned}\frac{j_{n+\frac{1}{2}}(1)}{h_{n+\frac{1}{2}}(1)} &= a_n(\lambda) \\ &= \lambda^{2n+1}(a_{n0} + a_{n1}\lambda^2 + \dots) \\ &\qquad +\lambda^{4n+2}(c_{n0} + c_{n1}\lambda^2 + \dots)\end{aligned} \tag{4.3.13}$$

where

$$a_n(\lambda) = \frac{\lambda\tilde{a}_n(\lambda)}{i(2n+1)} \tag{4.3.14}$$

are known analytic functions of λ. The fact that $a_n(\lambda)$ has a zero of order $2n+1$ at the origin follows from (4.2.44), (4.3.6) and the series representations (c.f. [25] p.4)

$$(\lambda r)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\lambda r) = \sqrt{\frac{1}{2}} \sum_{m=0}^{\infty} (-1)^m \frac{(\lambda r/2)^{2m+n}}{m!\,\Gamma(m+n+\frac{3}{2})} \tag{4.3.15}$$

$$(\lambda r)^{-\frac{1}{2}}H^{(1)}_{n+\frac{1}{2}}(\lambda r) = \sqrt{\frac{1}{2}} \sum_{m=0}^{\infty} (-1)^m \left[\frac{(\lambda r/2)^{2m+n}}{m!\,\Gamma(m+n+\frac{3}{2})} + i\,\frac{(-1)^{n+1}(\lambda r/2)^{2m-n-1}}{m!\,\Gamma(m-n+\frac{1}{2})}\right].$$

Equating the coefficients of λ^{2n+1} and λ^{2n+3} respectively in (4.3.13) we have for $n \geqslant 0$ (using (4.2.44))

$$a_{n0} = i\,\frac{(-1)^n\Gamma(-n+\frac{1}{2})(\frac{1}{2})^{2n+1}}{\Gamma(n+\frac{3}{2})} \tag{4.3.16}$$

and, for $n > 0$,

$$f(\theta,\phi;\lambda) = \lim_{r\to\infty} re^{-i\lambda r}u(\underset{\sim}{x}) \tag{4.3.5}$$

we want to determine the function $B(r)$. We will solve this problem by using the operator $\underset{\sim}{I}+\underset{\sim}{K}$ constructed in the previous section (c.f. [19]).

Let $J_{n+\frac{1}{2}}(\lambda r)$ and $H^{(1)}_{n+\frac{1}{2}}(\lambda r)$ denote respectively a Bessel function and Hankel function of the first kind, and define $j_{n+\frac{1}{2}}(r)$ and $h_{n+\frac{1}{2}}(r)$ by

$$\begin{aligned} j_{n+\frac{1}{2}}(r) &= (\underset{\sim}{I}+\underset{\sim}{K})((\lambda r)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\lambda r)) \\ h_{n+\frac{1}{2}}(r) &= (\underset{\sim}{I}+\underset{\sim}{K})((\lambda r)^{-\frac{1}{2}}H^{(1)}_{n+\frac{1}{2}}(\lambda r)) \quad . \end{aligned} \tag{4.3.6}$$

Then from the representation (c.f. [25], p.64)

$$e^{i\lambda z} = \sqrt{\frac{\pi}{2\lambda r}} \sum_{n=0}^{\infty} (2n+1)i^n \, J_{n+\frac{1}{2}}(\lambda r)P_n(\cos\theta) \tag{4.3.7}$$

where $P_n(\cos\theta)$ denotes Legendre's polynomial, it is easily verified using (4.2.46) that the solution of (4.3.2)-(4.3.4) is given by

$$u(\underset{\sim}{x}) = u(r,\theta) = -\sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{(2n+1)i^n j_{n+\frac{1}{2}}(1)}{h_{n+\frac{1}{2}}(1)} h_{n+\frac{1}{2}}(r)P_n(\cos\theta) \; . \tag{4.3.8}$$

Note that from Theorem 4.2.2 we can conclude that $h_{n+\frac{1}{2}}(1) \neq 0$, and the convergence of the series (4.3.8) for $1 \leqslant r < \infty$, $0 \leqslant \theta \leqslant \pi$ follows from (4.3.6) and standard estimates for Bessel functions and Legendre polynomials for large values of n (c.f. [25] p.22-23 and p.205). From the fact that

$$h_{n+\frac{1}{2}}(r) = (\lambda r)^{-\frac{1}{2}}H^{(1)}_{n+\frac{1}{2}}(\lambda r); \; r \geqslant a \tag{4.3.9}$$

and the asymptotic estimate ([25] p.85)

$$H^{(1)}_{n+\frac{1}{2}}(\lambda r) = (-i)^{n+1} \sqrt{\frac{2}{\pi\lambda r}} \, e^{i\lambda r}\left[1+O\left(\frac{1}{\lambda r}\right)\right] \tag{4.3.10}$$

we can conclude that the far field pattern $f(\theta,\phi;\lambda) = f(\theta;\lambda)$ is given by (c.f. [8])

$$f(\theta;\lambda) = \sum_{n=0}^{\infty} \frac{i(2n+1)j_{n+\frac{1}{2}}(1)}{\lambda \, h_{n+\frac{1}{2}}(1)} P_n(\cos\theta) \quad . \tag{4.3.11}$$

and the references contained in this paper. A discussion of the use of integral operators in the investigation of certain inverse problems in scattering theory can also be found in [8] and [30].

4.3 The Inverse Scattering Problem.

The inverse problem we will consider in this section has its origins in the following problem connected with the scattering of acoustic waves in a nonhomogeneous medium (c.f. section 4.2). Let an incoming plane acoustic wave of frequency ω moving in the direction of the z axis be scattered off a "soft" sphere Ω of radius one which is surrounded by a pocket of rarefied or condensed air in which the local speed of sound is given by $c(r)$ where $r=|\underset{\sim}{x}|$ for $\underset{\sim}{x} \in \mathbb{R}^3$. Let $u_s(\underset{\sim}{x})e^{i\omega t}$ be the velocity potential of the scattered wave and let r,θ,ϕ be spherical coordinates in $\mathbb{R}^3$. Then from a knowledge of the far field pattern

$$f(\theta,\phi;\lambda) = \lim_{r\to\infty} re^{-i\lambda r}u_s(\underset{\sim}{x}) \tag{4.3.1}$$

for $\lambda = \frac{\omega}{c_o}$ (where $c(r) = c_o$ = constant for $r \geqslant a > 1$) contained in some finite interval $[\lambda_0,\lambda_1]$, we would like to determine the unknown function $c(r)$. Under the assumption that $|\nabla c(r)|$ is small compared with $\lambda c(r)$, we can formulate this problem mathematically as follows (c.f. [19], [20]): Let $B(r) = (\frac{c_o}{c(r)})^2-1$ and set $u_s(\underset{\sim}{x}) = v(\underset{\sim}{x}) + u(\underset{\sim}{x})$ where $u(\underset{\sim}{x})$ satisfies

$$\Delta_3 u + \lambda^2(1+B(r))u = 0 \text{ in } \mathbb{R}^3\backslash\Omega \tag{4.3.2}$$

$$u(\underset{\sim}{x}) = -(e^{i\lambda z} + v(\underset{\sim}{x})) \text{ on } \partial\Omega \tag{4.3.3}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - i\lambda u\right) = 0 \tag{4.3.4}$$

and $v(\underset{\sim}{x})$ is such that $e^{i\lambda z} + v(\underset{\sim}{x})$ is a solution of (4.3.2) in $\mathbb{R}^3\backslash\Omega$ where $v(\underset{\sim}{x}) = 0$ for $r \geqslant a$. Then given conclude that $\psi(\underset{\sim}{x})=0$ and it follows from the Fredholm alternative that $(\underset{\sim}{I}-\underset{\sim}{T}_0(\lambda))^{-1}$ exists.

From the previously described work of Jones we now have the following Corollary:

Corollary 4.2.1: Let M be such that $\lambda < \lambda_{M+2}$ where λ_j denotes the j^{th} eigenvalue for the interior Dirichlet problem for $\Delta_3 h+\lambda^2 h=0$ in D. Then $(\underset{\sim}{I}-\underset{\sim}{T}(\lambda))^{-1}$ exists.

Remark: A similar approach to that described above can be used to solve the Dirichlet, Neumann, and Robin problems for solutions of (4.2.9) defined in the exterior of D for all $n \geq 2$.

In the next section we will discuss an inverse problem associated with (4.2.9), i.e. the problem of determining the unknown function B(r) when the behaviour of $u(\underset{\sim}{x})$ at infinity is known, as well as the shape of the scattering body D and the boundary conditions on ∂D. It should be noted that other inverse problems can also be considered, for example that of determining the scattering body D given the function B(r), the behaviour of $u(\underset{\sim}{x})$ at infinity, and the boundary conditions on ∂D(c.f. [8]). Such inverse problems are in general improperly posed in the sense that the solution does not depend continuously on the behaviour of $u(\underset{\sim}{x})$ at infinity and a solution will not exist for arbitrarily prescribed "far field" data. We will not discuss the regularization of such problems, but instead consider only the case when the "far field" pattern is known exactly and is such that a solution is known to exist. For further discussion of inverse scattering problems for acoustic waves we refer the reader to

D. Colton, A reflection principle for solutions to the Helmholtz equation and an application to the inverse scattering problem, to appear in Glasgow Math. J.

inverting an integral equation of Volterra type (which implies that $h(\underset{\sim}{x})$ has the same smoothness properties that $u(\underset{\sim}{x})$ does), we can conclude that $h(\underset{\sim}{x}) \in C^2(\mathbb{R}^3\backslash\bar{D}) \cap C^1(\mathbb{R}^3\backslash D)$ and $h(\underset{\sim}{x})=0$ for $r > a$. Hence, since twice continuously differentiable solutions of $\Delta_3 h+\lambda^2 h=0$ are analytic functions of their independent variables (c.f. [21], [29]), we can conclude that $h(\underset{\sim}{x})=0$ for $\underset{\sim}{x} \in \mathbb{R}^3\backslash D$, and hence $u(\underset{\sim}{x})=(\underset{\sim}{I}+\underset{\sim}{K})h=0$ for $\underset{\sim}{x} \in \mathbb{R}^3\backslash D$.

We can now establish the following result on the invertibility of the Fredholm operator $\underset{\sim}{I}-\underset{\sim}{T}(\lambda)$:

<u>Theorem 4.2.3</u> ([20]): Let $\lambda > 0$ and define the operator $\underset{\sim}{T}_o(\lambda)$ by

$$\underset{\sim}{T}_o(\lambda)\psi = \frac{1}{2\pi} \int_{\partial D} \psi(\underset{\sim}{\xi}) \frac{\partial}{\partial \nu_{\underset{\sim}{x}}} \Gamma(\underset{\sim}{\xi},\underset{\sim}{x};\lambda)\, d\omega_{\underset{\sim}{\xi}} \quad ; \quad \underset{\sim}{x} \in \partial D.$$

Then $(\underset{\sim}{I}-\underset{\sim}{T}(\lambda))^{-1}$ exists if and only if $(\underset{\sim}{I}-\underset{\sim}{T}_o(\lambda))^{-1}$ exists (where all mappings are understood to be in the space C^o, the space of continuous functions over ∂D with the maximum norm).

<u>Proof</u>: Since $\underset{\sim}{T}(\lambda)$ and $\underset{\sim}{T}_o(\lambda)$ are integral operators with weakly singular kernels, the Fredholm alternative is valid. Now let ψ be a solution of $(\underset{\sim}{I}-\underset{\sim}{T}(\lambda))\psi = 0$. Then the potential defined by (4.2.47) generates by (4.2.51) a solution of (4.2.6) in the exterior of D such that $u(\underset{\sim}{x})$ satisfies the Sommerfeld radiation condition, and, since $(\underset{\sim}{I}-\underset{\sim}{T}(\lambda))\psi=0$, we have $\frac{\partial u}{\partial \nu} = 0$ for $\underset{\sim}{x} \in \partial D$. From Theorem 4.2.2 we can now conclude that $u(\underset{\sim}{x})=0$ in the exterior of D. By inverting the Volterra equation (4.2.51) we can conclude that $h(\underset{\sim}{x})=0$ in the exterior of D and hence $(\underset{\sim}{I}-\underset{\sim}{T}_o(\lambda))\psi=0$ for $\underset{\sim}{x} \in \partial D$. If $(\underset{\sim}{I}-\underset{\sim}{T}_o(\lambda))^{-1}$ exists then we can conclude that $\psi(\underset{\sim}{x})=0$, and hence by the Fredholm alternative $(\underset{\sim}{I}-\underset{\sim}{T}(\lambda))^{-1}$ exists.

Conversely, if ψ is a solution of $(\underset{\sim}{I}-\underset{\sim}{T}_o(\lambda))\psi=0$, then $h(\underset{\sim}{x})$ as defined by (4.2.47) is zero for $\underset{\sim}{x} \in \mathbb{R}^3\backslash D$ and hence from (4.2.51) $u(\underset{\sim}{x})=0$ for $\underset{\sim}{x} \in \mathbb{R}^3\backslash D$. Then $\frac{\partial u}{\partial \nu} = 0$ for $\underset{\sim}{x} \in \partial D$ and $(\underset{\sim}{I}-\underset{\sim}{T}(\lambda))\psi=0$. Hence if $(\underset{\sim}{I}-\underset{\sim}{T}(\lambda))^{-1}$ exists we can

Theorem 4.2.2 ([20]): Let $\lambda > 0$ and let $u(\underset{\sim}{x}) \in C^2(\mathbb{R}^3\setminus\bar{D}) \cap C^1(\mathbb{R}^3\setminus D)$ be a solution of (4.2.6) in the exterior of D satisfying the Sommerfeld radiation condition (4.2.8) at infinity and the boundary condition $\frac{\partial u}{\partial \nu} = 0$ (or $u=0$) on ∂D. Then $u(\underset{\sim}{x}) \equiv 0$ for $\underset{\sim}{x} \in \mathbb{R}^3\setminus D$.

Proof: Let Ω be a ball of radius $r > a$ (recalling that $B(r)=0$ for $r \geqslant a$). Then from Green's formula we have

$$\iint_{\Omega\setminus D} (u\Delta\bar{u}-\bar{u}\Delta u)\,dV = \int_{\partial D} \left(\bar{u}\,\frac{\partial u}{\partial \nu} - u\,\frac{\partial \bar{u}}{\partial \nu}\right) d\omega - \int_{\partial\Omega} \left(\bar{u}\,\frac{\partial u}{\partial r} - u\,\frac{\partial \bar{u}}{\partial r}\right) d\omega \qquad (4.2.54)$$

where dV denotes an element of volume and $d\omega$ an element of surface area. Since λ and $B(r)$ are real and $\frac{\partial u}{\partial \nu} = \frac{\partial \bar{u}}{\partial \nu} = 0$ (or $u=\bar{u}=0$) on ∂D, we have from (4.2.54) that

$$\int_{\partial D} \left(\bar{u}\,\frac{\partial u}{\partial r} - u\,\frac{\partial \bar{u}}{\partial r}\right) d\omega = 0. \qquad (4.2.55)$$

But, for $r > a$, $u(\underset{\sim}{x})$ is a solution of $\Delta_3 h + \lambda^2 h = 0$ satisfying the Sommerfeld radiation condition (4.2.3), and hence for $r > a$

$$u(\underset{\sim}{x}) = \sum_{m=o}^{\infty} \sum_{n=-m}^{m} a_{mn} h_m^{(1)}(\lambda|\underset{\sim}{x}|) S_{mn}\left(\frac{\underset{\sim}{x}}{|\underset{\sim}{x}|}\right) \qquad (4.2.56)$$

where the series converges absolutely and uniformly for $|\underset{\sim}{x}| \geqslant a+\varepsilon$, $\varepsilon > 0$ (c.f. [49]). By the orthogonality of the functions $S_{mn}\left(\frac{\underset{\sim}{x}}{|\underset{\sim}{x}|}\right)$ over the unit sphere and the formula

$$\overline{h_m^{(1)}(\lambda r)}\,\frac{d}{dr}\,h_m^{(1)}(\lambda r) - h_m^{(1)}(\lambda r)\,\frac{d}{dr}\,\overline{h_m^{(1)}(\lambda r)} = \frac{4i}{\pi\lambda^2 r^2} \qquad (4.2.57)$$

we have from (4.2.55) and (4.2.56) that

$$\sum_{m=0}^{\infty} \sum_{n=-m}^{m} |a_{mn}|^2 = 0 , \qquad (4.2.58)$$

which implies that $u(\underset{\sim}{x})=0$ for $r > a$. Let $u(\underset{\sim}{x}) = (\underset{\sim}{I}+\underset{\sim}{K})h$ for $\underset{\sim}{x} \in \mathbb{R}^3\setminus\bar{D}$. Then from (4.2.12) and the fact that $h(\underset{\sim}{x})$ can be determined from $u(\underset{\sim}{x})$ by of integration gives

$$u(\underset{\sim}{x}) = \int_{\partial D} \psi(\underset{\sim}{\xi})\Gamma(\underset{\sim}{\xi},\underset{\sim}{x};\lambda)d\omega_{\underset{\sim}{\xi}} \tag{4.2.51}$$

$$+ \int_{\partial D} \psi(\underset{\sim}{\xi}) \left\{ \int_{|\underset{\sim}{x}|}^{\infty} K(|\underset{\sim}{x}|,s;\lambda)\Gamma(\underset{\sim}{\xi}, s\frac{\underset{\sim}{x}}{|\underset{\sim}{x}|};\lambda)ds \right\} d\omega_{\underset{\sim}{\xi}} .$$

As in section 4.1 one can show that for $\underset{\sim}{x}$, $\underset{\sim}{\xi}$ on ∂D

$$\left| \frac{\partial}{\partial\nu_{\underset{\sim}{x}}} \left\{ \int_{|\underset{\sim}{x}|}^{\infty} K(|\underset{\sim}{x}|,s;\lambda)\Gamma(\underset{\sim}{\xi},s\frac{\underset{\sim}{x}}{|\underset{\sim}{x}|};\lambda)ds \right\} \right| \leq \frac{\text{constant}}{|\underset{\sim}{x}-\underset{\sim}{\xi}|} \tag{4.2.52}$$

where $\frac{\partial}{\partial\nu_{\underset{\sim}{x}}}$ denotes differentiation with respect to $\underset{\sim}{x}$ in the direction of the outward normal at $\underset{\sim}{x}$. Now let $\underset{\sim}{x}\varepsilon\partial D$, evaluate (4.2.51) at $\underset{\sim}{x}^1\varepsilon\, \mathbb{R}^3\setminus\bar{D}$, and apply the operator $\nu_{\underset{\sim}{x}}\cdot\nabla$ to both sides of (4.2.51). Letting $\underset{\sim}{x}^1$ tend to $\underset{\sim}{x}$, and using (4.2.52) and the discontinuity properties of the derivatives of single layer potentials (c.f. [21],[29]), we arrive at the following integral equation for $\psi(\underset{\sim}{x})$:

$$-\frac{1}{2\pi} f(\underset{\sim}{x}) = \psi(\underset{\sim}{x}) - \frac{1}{2\pi}\int_{\partial D}\psi(\underset{\sim}{\xi})\frac{\partial}{\partial\nu_{\underset{\sim}{x}}}\Gamma(\underset{\sim}{\xi},\underset{\sim}{x};\lambda)d\omega_{\underset{\sim}{\xi}}$$

$$-\frac{1}{2\pi}\int_{\partial D}\psi(\underset{\sim}{\xi})\frac{\partial}{\partial\nu_{\underset{\sim}{x}}}\left\{ \int_{|\underset{\sim}{x}|}^{\infty} K(|\underset{\sim}{x}|,s;\lambda)\Gamma(\underset{\sim}{\xi},s\frac{\underset{\sim}{x}}{|\underset{\sim}{x}|};\lambda)ds \right\} d\omega_{\underset{\sim}{\xi}} \tag{4.2.53}$$

$$= (\underset{\sim}{I}-\underset{\sim}{T}(\lambda))\psi .$$

A contstructive method for determining the desired function $u(\underset{\sim}{x})$ can now be obtained if we can show that the Fredholm integral equation (4.2.53) with weakly singular kernel can be uniquely solved for the unknown density $\psi(\underset{\sim}{x})$, i.e. that the operator $\underset{\sim}{I}-\underset{\sim}{T}(\lambda)$ is invertible. We will accomplish this by proving two theorems. The first theorem below proceeds along classical lines (c.f.[49])except for the conclusion, where we make use of the operator $\underset{\sim}{I}+\underset{\sim}{K}$.

$$h(\underset{\sim}{x}) = \int_{\partial D} \psi(\underset{\sim}{\xi})\Gamma(\underset{\sim}{\xi},\underset{\sim}{x};\lambda)\,d\omega_{\underset{\sim}{\xi}} \quad . \tag{4.2.47}$$

In (4.2.47)

$$\Gamma(\underset{\sim}{\xi},\underset{\sim}{x};\lambda) = \frac{e^{i\lambda R}}{R} + \sum_{m=0}^{M} \sum_{n=-m}^{m} b_{mn}\psi_{mn}(\underset{\sim}{x})\psi_{mn}(\underset{\sim}{\xi}) \;, \tag{4.2.48}$$

$R=|\underset{\sim}{x}-\underset{\sim}{\xi}|$, $d\omega_{\underset{\sim}{\xi}}$ is an element of surface area at the point $\underset{\sim}{\xi}\varepsilon\partial D$, the b_{mn} are nonzero real constants (arbitrary, but fixed), and

$$\psi_{mn}(\underset{\sim}{x}) = h_m^{(1)}(\lambda|\underset{\sim}{x}|)S_{mn}\left(\frac{\underset{\sim}{x}}{|\underset{\sim}{x}|}\right) \tag{4.2.49}$$

where $h_m^{(1)}$ denotes a spherical Hankel function and S_{mn} a spherical harmonic. Note that if the finite sum in (4.2.48) is not present, then in general it is not possible to represent $h(\underset{\sim}{x})$ in the form of the single layer potential (4.2.47) (c.f. [49]). Jones has also shown that for a given λ a suitable value of M can be chosen as follows: Let $\mu_1,\ldots,\mu_j,\ldots$ be the eigenvalues of the interior Dirichlet problem for (4.2.11) (for n=3) in the unit sphere (which can be computed from a knowledge of the zeros of the spherical Bessel functions - for a table of these zeros see [44]) and let r_o be the radius of the smallest sphere contained in D and r_1 the radius of the largest sphere containing D. Then

$$\frac{\mu_j}{r_o} \geqslant \lambda_j \geqslant \frac{\mu_j}{r_1} \quad . \tag{4.2.50}$$

In order to construct a solution $u(\underset{\sim}{x})$ of (4.2.6)-(4.2.8) we will look for a solution in the form

$$u(\underset{\sim}{x}) = (\underset{\sim}{I}+\underset{\sim}{K})h \tag{4.2.51}$$

where $h(\underset{\sim}{x})$ is a solution of (4.2.11) (for n=3) having the representation (4.2.47) in terms of an unknown continuous density $\psi(\underset{\sim}{x})$ to be determined. Note that $h(\underset{\sim}{x})$, and hence $u(\underset{\sim}{x})$, satisfies the Sommerfeld radiation condition (4.2.8). Substituting (4.2.47) into (4.2.51) and interchanging the orders

$u(\underset{\sim}{x})$ of (4.2.9) defined in the exterior of D there exists a solution $h(\underset{\sim}{x})$ of (4.2.11) defined in the exterior of D such that $u(\underset{\sim}{x})=(\underset{\sim}{I}+\underset{\sim}{K})h$. We summerize our results in the following theorem:

<u>Theorem 4.2.1</u> ([20]): Let $u(\underset{\sim}{x})$ be a twice continuously differentiable solution of (4.2.9) in the exterior of D where D is strictly starlike with respect to the origin. Then $u(\underset{\sim}{x})$ can be represented in the form $u(\underset{\sim}{x})=(\underset{\sim}{I}+\underset{\sim}{K})h$ where $h(\underset{\sim}{x})$ is a twice continuously differentiable solution of (4.2.11) in the exterior of D. Conversely if $h(\underset{\sim}{x})$ is a solution of (4.2.11) in the exterior of D, then $u(\underset{\sim}{x})=(\underset{\sim}{I}+\underset{\sim}{K})h$ is a solution of (4.2.9) in the exterior of D. $u(\underset{\sim}{x})$ satisfies the Sommerfeld radiation condition (4.2.13) if and only if $h(\underset{\sim}{x})$ satisfies this condition.

We now want to use the integral operator $\underset{\sim}{I}+\underset{\sim}{K}$ (for n=3) to construct the functions $v(\underset{\sim}{x})$ and $u(\underset{\sim}{x})$ described in the introduction to this section. Since $e^{i\lambda z}$ is a solution of (4.2.11) we have that

$$w(\underset{\sim}{x}) = (\underset{\sim}{I}+\underset{\sim}{K})e^{i\lambda z} \tag{4.2.45}$$

is a solution of (4.2.9),and from (4.2.12) it is seen that we can choose $v(\underset{\sim}{x})$ to be

$$v(\underset{\sim}{x}) = \underset{\sim}{K}\, e^{i\lambda z} \quad . \tag{4.2.46}$$

To construct a solution $u(\underset{\sim}{x})$ of (4.2.6)-(4.2.8) we will use the operator $\underset{\sim}{I}+\underset{\sim}{K}$ in conjunction with the work of D.S.Jones on the exterior Neumann problem for the Helmholtz equation (4.2.11) (c.f. [38]). To describe the work of Jones, let $\lambda_1,\lambda_2,\ldots,\lambda_j,\ldots$ be the eigenvalues of the interior Dirichlet problem for (4.2.11) in D (for n=3). Then Jones has shown that if $\lambda < \lambda_{M+2}$ and $h(\underset{\sim}{x})$ is a solution of (4.2.11) (for n=3) satisfying prescribed Neumann data on ∂D and the Sommerfeld radiation condition (4.2.8) at infinity, there exists a continuous density $\psi(\underset{\sim}{x})$ such that $h(\underset{\sim}{x})$ can be represented in the form

$\frac{1}{2}(\xi+\eta) \leqslant \tau$, which implies $\eta \leqslant 2\tau - \xi$, and hence $\eta-\xi \leqslant 2(\tau-\xi)$. Therefore from (4.2.38) we have

$$|M_1(\xi,\eta;\lambda)| \leqslant 2C^2 \int_{\xi}^{\log a} (\log a-\tau)(\tau-\xi)d\tau \; . \tag{4.2.39}$$

But for $j \geqslant 0$ we have

$$\frac{1}{(2j+1)!} \int_{\xi}^{\log a} (\log a-\tau)^{2j+1}(\tau-\xi)d\tau = \frac{(\log a-\xi)^{2j+3}}{(2j+3)!} \tag{4.2.40}$$

and hence

$$|M_1(\xi,\eta;\lambda)| \leqslant \frac{2C^2}{3!} \; (\log a-\xi)^3. \tag{4.2.41}$$

By induction we have

$$|M_j(\xi,\eta;\lambda)| \leqslant \frac{2C^{j+1}}{(2j+1)!} \; (\log a-\xi)^{2j+1} \tag{4.2.42}$$

$$\leqslant \frac{2C^{j+1}}{(2j+1)!} \; (\log a+\xi_o)^{2j+1}$$

for $j \geqslant 0$, and hence the series (4.2.34) is absolutely and uniformly convergent for $\eta \geqslant \xi \geqslant -\xi_o$. This establishes the existence of the function $M(\xi,\eta;\lambda)$ and hence the kernel $K(r,s;\lambda)$. It is easily seen that since $B(r)$ is continuously differentiable, $K(r,s;\lambda)$ is twice continuously differentiable for $s \geqslant r > 0$. We note that $M(\xi,\eta;\lambda)$ is an entire function of λ and that

$$\lambda^{-2-2j}M_j(\xi,\eta;\lambda) = N_j(\xi,\eta) \tag{4.2.43}$$

is independent of λ. In particular $s^{n-3}K(r,s;\lambda)$ has the Taylor expansion

$$s^{n-3}K(r,s;\lambda) = s^{(n-4)/2}r^{(2-n)/2} \sum_{j=0}^{\infty} \lambda^{2j+2} \; N_j(\log r, \; \log s) \tag{4.2.44}$$

which is uniformly convergent for all complex values of λ. Note also that since $\underset{\sim}{K}$ is a Volterra operator, $(\underset{\sim}{I}+\underset{\sim}{K})^{-1}$ exists, in particular we can conclude (using (4.2.12), (4.2.14) and (4.2.15)) that for every solution

We now want to solve (4.2.33) through the method of successive approximations. We look for a solution of (4.2.33) in the form

$$M(\xi,\eta;\lambda) = \sum_{j=0}^{\infty} M_j(\xi,\eta;\lambda) \tag{4.2.34}$$

where

$$\begin{aligned} M_o(\xi,\eta;\lambda) &= -\frac{1}{2}\lambda^2 \int_{\frac{1}{2}(\xi+\eta)}^{\log a} e^{2\tau}B(e^{\tau})d\tau \\ M_j(\xi,\eta;\lambda) &= -\frac{1}{2}\lambda^2 \int_{\xi}^{\frac{1}{2}(\xi+\eta)} \int_{\eta+\xi-\tau}^{\eta+\tau-\xi} F(\tau,\mu)M_{j-1}(\tau,\mu;\lambda)d\mu d\tau \\ &\quad -\frac{1}{2}\lambda^2 \int_{\frac{1}{2}(\xi+\eta)}^{\log a} \int_{\tau}^{\eta+\tau-\xi} F(\tau,\mu)M_{j-1}(\tau,\mu;\lambda)d\mu d\tau \end{aligned} \tag{4.2.35}$$

for $j \geqslant 1$. Note that the region of integration in (4.2.35) is only in the half-space $\frac{1}{2}(\xi+\eta) \leqslant \log a$ since $M_o(\xi,\eta;\lambda) = 0$ for $\frac{1}{2}(\xi+\eta) \geqslant \log a$ and this implies that for $\frac{1}{2}(\xi+\eta) \geqslant \log a$, $M_j(\xi,\eta;\lambda) = 0$ for each j. Assume $\eta \geqslant \xi \geqslant -\xi_o$ where ξ_o is a positive constant, and let

$$C = \frac{1}{2}|\lambda|^2 \max_{\substack{-\xi_o\leqslant\xi\leqslant\log a \\ -\xi_o\leqslant\eta\leqslant\xi_o+\log a}} \{ e^{2\xi}|B(e^{\xi})|, |F(\xi,\eta)| \}. \tag{4.2.36}$$

Then for $\eta \geqslant \xi \geqslant -\xi_o$, $\frac{1}{2}(\xi+\eta) \leqslant \log a$, we have

$$\begin{aligned} |M_o(\xi,\eta;\lambda)| &\leqslant C(\log a - \frac{1}{2}(\xi+\eta)) \\ &\leqslant C(\log a - \xi) \end{aligned} \tag{4.2.37}$$

and

$$\begin{aligned} |M_1(\xi,\eta;\lambda)| &\leqslant 2C^2 \int_{\xi}^{\frac{1}{2}(\xi+\eta)} (\log a-\tau)(\tau-\xi)d\tau \\ &\quad +C^2 \int_{\frac{1}{2}(\xi+\eta)}^{\log a} (\log a-\tau)(\tau-\xi)d\tau \quad . \end{aligned} \tag{4.2.38}$$

But in the second integral on the right hand side of (4.2.38) we have of the integral equation

$$\tilde{M}(x,y;\lambda) = -\frac{1}{2}\lambda^2 \int_x^\infty e^{2\tau}B(e^\tau)d\tau \tag{4.2.30}$$
$$- \lambda^2 \int_y^\infty \int_x^\infty F(\alpha+\beta,\alpha-\beta)\tilde{M}(\alpha,\beta;\lambda)d\alpha d\beta.$$

Note that (4.2.28) implies that the solution of the integral equation (4.2.30) satisfies the initial condition (4.2.26), and (4.2.27), (4.2.28), and the fact that B(r) has compact support guarnatee the existence of the integrals appearing in (4.2.30). Now in (4.2.30) make the change of variables

$$\alpha = \frac{1}{2}(\tau+\mu)$$
$$\beta = \frac{1}{2}(\tau-\mu) \quad . \tag{4.2.31}$$

Then (4.2.30) becomes

$$M(\xi,\eta;\lambda) = -\frac{1}{2}\lambda^2 \int_{\frac{1}{2}(\xi+\eta)}^\infty e^{2\tau}B(e^\tau)d\tau \tag{4.2.32}$$
$$- \frac{1}{2}\lambda^2 \int_\xi^\infty \int_{\eta+\xi-\tau}^{\eta+\tau-\xi} F(\tau,\mu)M(\tau,\mu;\lambda)d\mu d\tau.$$

Now note that in (4.2.32) if $\eta+\xi-\tau > \tau$, then $\mu > \tau$, and hence $M(\tau,\mu;\lambda)$ is not identically zero. On the other hand if $\eta+\xi-\tau < \tau$, then μ may be less than τ, and in such cases $M(\tau,\mu;\lambda) = 0$. Taking these facts into consideration we have that , for $\eta > \xi$, $M(\xi,\eta;\lambda)$ is the solution of the integral equation

$$M(\xi,\eta;\lambda) = -\frac{1}{2}\lambda^2 \int_{\frac{1}{2}(\xi+\eta)}^\infty e^{2\tau}B(e^\tau)d\tau$$
$$- \frac{1}{2}\lambda^2 \int_\xi^{\frac{1}{2}(\xi+\eta)} \int_{\eta+\xi-\tau}^{\eta+\tau-\xi} F(\tau,\mu)M(\tau,\mu;\lambda)d\mu d\tau \tag{4.2.33}$$
$$- \frac{1}{2}\lambda^2 \int_{\frac{1}{2}(\xi+\eta)}^\infty \int_\tau^{\eta+\tau-\xi} F(\tau,\mu)M(\tau,\mu;\lambda)d\mu d\tau \quad .$$

i.e.

$$K(r,s;\lambda) = (rs)^{-\left(\frac{n-2}{2}\right)} M(\log r, \log s;\lambda) . \tag{4.2.18}$$

Then $M(\xi,\eta;\lambda)$ satisfies the differential equation

$$M_{\xi\xi}-M_{\eta\eta} + \lambda^2(e^{2\xi}-e^{2\eta}+e^{2\xi}B(e^{\xi}))M = 0 \tag{4.2.19}$$

for $\eta > \xi$ and the auxilliary conditions

$$M(\xi,\xi;\lambda) = -\frac{1}{2}\lambda^2 \int_{\xi}^{\infty} e^{2\tau}B(e^{\tau})d\tau \tag{4.2.20}$$

$$M(\xi,\eta;\lambda) = 0 \quad \text{for } \tfrac{1}{2}(\xi+\eta) \geqslant \log a. \tag{4.2.21}$$

We assume that in addition to (4.2.19)-(4.2.21),

$$M(\xi,\eta;\lambda) = 0 \quad \text{for } \xi > \eta \quad . \tag{4.2.22}$$

Note that $M(\xi,\eta;\lambda)$, if it exists, is independent of the dimension n, and in this sense the operator (4.2.10) can be described as a "method of ascent".

We now proceed to construct a solution of (4.2.19)-(4.2.22). Our approach resemble that of section 2.1 for the operator $\underset{\sim}{A}_1$. Let

$$\begin{aligned} x &= \tfrac{1}{2}(\xi+\eta) \\ y &= \tfrac{1}{2}(\xi-\eta) \end{aligned} \tag{4.2.23}$$

and define $\tilde{M}(x,y;\lambda)$ by

$$\tilde{M}(x,y;\lambda) = M(x+y, x-y;\lambda) \quad . \tag{4.2.24}$$

The $\tilde{M}(x,y;\lambda)$ satisfies

$$\tilde{M}_{xy} - \lambda^2 F(x+y,x-y)\tilde{M} = 0 \quad ; \quad y < 0 \tag{4.2.25}$$

$$\tilde{M}(x,0;\lambda) = -\frac{1}{2}\lambda^2 \int_{x}^{\infty} e^{2\tau}B(e^{\tau})d\tau \tag{4.2.26}$$

$$\tilde{M}(x,y;\lambda) = 0 \quad \text{for } x \geqslant \log a \tag{4.2.27}$$

$$\tilde{M}(x,y;\lambda) = 0 \quad \text{for } y > 0 \quad , \tag{4.2.28}$$

where in (4.2.25)

$$F(\xi,\eta) = -\left[e^{2\xi}-e^{2\eta}+e^{2\xi}B(e^{\xi})\right] \quad . \tag{4.2.29}$$

For $y \leqslant 0$, (4.2.25)-(4.2.27) imply that $\tilde{M}(x,y;\lambda)$ is the solution in place of (4.2.6) and later on set n=3.

We now look for a twice continuously differentiable solution $u(\underset{\sim}{x})$ of (4.2.9) defined in the exterior of D in the form

$$
\begin{aligned}
u(r,\theta) &= (\underset{\sim}{I}+\underset{\sim}{K})h \\
&= h(r,\theta) + \int_r^\infty s^{n-3}K(r,s;\lambda)h(s,\theta)ds
\end{aligned}
\tag{4.2.10}
$$

where $(r,\theta) = (r,\theta_1,\ldots\theta_{n-1})$ are spherical coordinates, $h(r,\theta)$ is a twice continuously differentiable solution of

$$\Delta_n h + \lambda^2 h = 0 \tag{4.2.11}$$

in the exterior of D, and $K(r,s;\lambda)$ is a function to be determined. We assume

$$K(r,s;\lambda) = 0 \quad \text{for} \quad rs \geqslant a^2 \tag{4.2.12}$$

and note that if $h(r,\theta)$ satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{(n-1)/2}\left(\frac{\partial u}{\partial r} - i\lambda u\right) = 0 \quad , \tag{4.2.13}$$

then by (4.2.12) so will $u(r,\theta)$. We now substitute (4.2.10) into (4.2.9) and integrate by parts using (4.2.12). The result of this calculation is that (4.2.10) will be a solution of (4.2.9) provided $K(r,s;\lambda)$ is a twice continuously differentiable solution of

$$r^2\left[K_{rr} + \frac{n-1}{r}K_r + \lambda^2(1+B(r))K\right] = s^2\left[K_{ss} + \frac{n-1}{s}K_s + \lambda^2 K\right] \tag{4.2.14}$$

for $s > r$ satisfying (4.2.12) and the initial condition

$$K(r,r;\lambda) = -\frac{1}{2}\lambda^2 r^{2-n}\int_r^\infty sB(s)ds \quad . \tag{4.2.15}$$

Now let

$$
\begin{aligned}
&\xi = \log r \\
&\eta = \log s
\end{aligned}
\tag{4.2.16}
$$

and define $M(\xi,\eta;\lambda)$ by

$$M(\xi,\eta;\lambda) = \exp\left[\left(\frac{n-2}{2}\right)(\xi+\eta)\right]K(e^\xi,e^\eta;\lambda), \tag{4.2.17}$$

Then, assuming $|\nabla c(r)|$ is small compared with $\lambda c(r)$ where $\lambda = \frac{\omega}{c_o}$, we are led to the following boundary value problem, where $u_s(\underset{\sim}{x})$ is the velocity potential of the scattered wave and ν denotes the outward normal to ∂D:

$$U(\underset{\sim}{x}) = e^{i\lambda z} + u_s(\underset{\sim}{x}) \tag{4.2.1}$$

$$\Delta_3 U + \lambda^2(1+B(r))U = 0 \text{ in } \mathbb{R}^3\backslash D \tag{4.2.2}$$

$$\frac{\partial U}{\partial \nu} = 0 \text{ on } \partial D \tag{4.2.3}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u_s}{\partial r} - i\lambda u_s\right) = 0 \tag{4.2.4}$$

where the Sommerfeld radiation condition (4.2.4) is assumed to hold uniformly in all directions. Now let

$$u_s(\underset{\sim}{x}) = v(\underset{\sim}{x}) + u(\underset{\sim}{x}) \tag{4.2.5}$$

where $v(\underset{\sim}{x}) \varepsilon C^2(\mathbb{R}^3\backslash\bar{D}) \cap C^1(\mathbb{R}^3\backslash D)$ is such that $e^{i\lambda z} + v(\underset{\sim}{x})$ is a solution of (4.2.2) in $\mathbb{R}^3\backslash\bar{D}$ and $v(\underset{\sim}{x})$ satisfies (4.2.4). If such a function $v(\underset{\sim}{x})$ can be found, then the boundary value problem (4.2.1)-(4.2.4) for $U(\underset{\sim}{x})$ can be reduced to the following boundary value problem for $u(\underset{\sim}{x})$:

$$\Delta_3 u + \lambda^2(1+B(r))u = 0 \text{ in } \mathbb{R}^3\backslash\bar{D} \tag{4.2.6}$$

$$\frac{\partial u}{\partial \nu} = f(\underset{\sim}{x}) \text{ on } \partial D \tag{4.2.7}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - i\lambda u\right) = 0 \tag{4.2.8}$$

where $f(\underset{\sim}{x}) = -\frac{\partial}{\partial \nu}(e^{i\lambda z} + v(\underset{\sim}{x}))$. We will now show how the functions $v(\underset{\sim}{x})$ and $u(\underset{\sim}{x})$ can be constructed by means of a "method of ascent". We make the assumption that $B(r)$ is a real valued continuously differentiable function of r for $r > 0$ with compact support contained in the interval $[0,a]$ where $a > 0$, and that D is bounded and strictly starlike with respect to the origin. In order to establish a "method of ascent" we consider the equation

$$\Delta_n u + \lambda^2(1+B(r))u = 0 \tag{4.2.9}$$

maximum principle for elliptic equations that $u(\underset{\sim}{x})\equiv 0$ in D. This implies from our previous discussion that $h(\underset{\sim}{x})$ as defined by (4.1.33) is identically zero in D, and hence letting $\underset{\sim}{x}$ tend to ∂D and using the discontinuity properties of double layer potentials we have

$$0=\psi(\underset{\sim}{x})+\frac{1}{2\pi}\int_{\partial D}\psi(\underset{\sim}{\xi})\frac{\partial}{\partial\nu}\left(\frac{1}{R}\right)d\omega_{\xi}\ ;\ \underset{\sim}{x}\in\partial D \tag{4.1.49}$$

But from classical results in potential theory (c.f. [29]) (4.1.49) implies that $\psi(\underset{\sim}{\xi})=0$ for $\underset{\sim}{\xi}\in\partial D$ and hence $(\underset{\sim}{I}+\underset{\sim}{T})^{-1}$ exists.

4.2 Exterior Domains.

We now want to obtain a "method of ascent" for solutions of equations of the form (4.1.1) which are defined in exterior domains. In the case when $B(r^2)$ decays sufficiently rapidly at infinity, a "method of ascent" for equations defined in exterior domains can be obtained by simply applying a Kelvin transformation (c.f. [29]) and then using the results of section 4.1. However if we are interested in problems which arise from scattering theory, then $B(r^2)$ does not tend to zero as r tends to infinity, and the above approach can no longer be used. It is this type of problem which we will be interested in for the remainder of this chapter. In particular the mathematical problems which we will consider in the present section have their origin in the following problem connected with the scattering of acoustic waves in a non homogeneous medium. Let an incoming plane acoustic wave of frequency ω moving in the direction of the z axis be scattered off a bounded rigid obstacle D which is surrounded by a pocket of rarefied or condensed air in which the local speed of sound is given by $c(r)$ where $r=|\underset{\sim}{x}|$ for $\underset{\sim}{x}\in\mathbb{R}^3$. Assume that this pocket of air is contained in a ball of radius a and that for $r\geq a$ we have $c(r)=c_o=\text{constant}$. Let $U(\underset{\sim}{x})$ be the velocity potential (factoring out a term of the form $e^{i\omega t}$) and set $B(r)=\left(\frac{c_o}{c(r)}\right)^2-1$.

$$|(\underset{\sim}{x}-\underset{\sim}{\xi})\cdot\underset{\sim}{x}| \leq \alpha|\underset{\sim}{x}-\underset{\sim}{\xi}||\underset{\sim}{x}| \tag{4.1.45}$$

uniformly for all $\underset{\sim}{x},\underset{\sim}{\xi}\varepsilon\partial D$ such that $\underset{\sim}{x}.\underset{\sim}{\xi} \geq 0$. Hence from (4.1.41) we have

$$|\underset{\sim}{x}\rho_o-\underset{\sim}{\xi}|^2 \geq (1-\alpha^2)|\underset{\sim}{x}-\underset{\sim}{\xi}|^2 \tag{4.1.46}$$

uniformly for all $\underset{\sim}{x},\underset{\sim}{\xi}\varepsilon\partial D$ such that $\underset{\sim}{x}.\underset{\sim}{\xi} \geq 0$ and from (4.1.41)

$$\begin{aligned} |\underset{\sim}{x}\rho-\underset{\sim}{\xi}|^2 &= |\underset{\sim}{x}\rho_o-\underset{\sim}{\xi}|^2 + (\rho-\rho_o)^2 \\ &\geq (1-\alpha^2)|\underset{\sim}{x}-\underset{\sim}{\xi}|^2+(\rho-\rho_o)^2 . \end{aligned} \tag{4.1.47}$$

Therefore for $\underset{\sim}{x}.\underset{\sim}{\xi} \geq 0$ and the case (4.1.41) we have

$$\begin{aligned} \int_o^1 |\underset{\sim}{x}\rho-\underset{\sim}{\xi}|^{-2}d\rho &\leq \frac{1}{(1-\alpha^2)}\int_o^1 \frac{d\rho}{|\underset{\sim}{x}-\underset{\sim}{\xi}|^2+(\rho-\rho_o)^2} \\ &= \frac{1}{(1-\alpha^2)|\underset{\sim}{x}-\underset{\sim}{\xi}|}\arctan\left(\frac{\rho-\rho_o}{|\underset{\sim}{x}-\underset{\sim}{\xi}|}\right)\Bigg|_{\rho=0}^{\rho=1} \\ &\leq \frac{4\pi}{(1-\alpha^2)|\underset{\sim}{x}-\underset{\sim}{\xi}|} , \end{aligned} \tag{4.1.48}$$

and from a consideration of the remaining (trivial) cases we can now conclude from (4.1.37) that (4.1.35) is valid.

To complete our discussion of the Dirichlet problem for (4.1.1) we now show that $(\underset{\sim}{I}+\underset{\sim}{T})^{-1}$ exists where $\underset{\sim}{T}$ is defined in (4.1.36). Since we have already shown that $\underset{\sim}{T}$ has a weakly singular kernel, if $(\underset{\sim}{I}+\underset{\sim}{T})^{-1}$ exists then there are a variety of constructive methods for obtaining the unknown density $\psi(\underset{\sim}{\xi})$ and hence the solution of the Dirichlet problem for (4.1.1) (c.f. [1],[22]).

To show that $(\underset{\sim}{I}+\underset{\sim}{T})^{-1}$ exists, by the Fredholm alternative it suffices to show that if $(\underset{\sim}{I}+\underset{\sim}{T})\psi = 0$ then $\psi=0$. Suppose $(\underset{\sim}{I}+\underset{\sim}{T})\psi = 0$. Then the potential defined by (4.1.33) generates by (4.1.31)a solution $u(\underset{\sim}{x})$ of (4.1.1) such that $u(\underset{\sim}{x})=0$ for $\underset{\sim}{x}\varepsilon\partial D$. Since $B(r^2) \leq 0$ for $\underset{\sim}{x}\varepsilon D$, we can conclude from the

We now examine the function $|\boldsymbol{x}\rho-\boldsymbol{\xi}|^{-2}$ for $\boldsymbol{x}$ and $\boldsymbol{\xi}$ on ∂D. Without loss of generality we can restrict our attention to values of $\boldsymbol{x}$ and $\boldsymbol{\xi}$ such that $\boldsymbol{x}.\boldsymbol{\xi} \geqslant 0$. This follows from the fact that if $\boldsymbol{x}.\boldsymbol{\xi} < 0$ then

$$\begin{aligned} |\boldsymbol{x}\rho-\boldsymbol{\xi}|^2 &= \rho^2|\boldsymbol{x}|^2+|\boldsymbol{\xi}|^2 - 2\rho\boldsymbol{x}.\boldsymbol{\xi} \\ &\geqslant \rho^2|\boldsymbol{x}|^2+|\boldsymbol{\xi}|^2 , \end{aligned} \tag{4.1.38}$$

and hence for such values of $\boldsymbol{x}$ and $\boldsymbol{\xi}$ the integral on the right hand side of (4.1.37) can be bounded by a constant independent of $\boldsymbol{x}$ and $\boldsymbol{\xi}$ (since D contains the origin). This in turn implies that (4.1.35) is valid. Hence we now assume that $\boldsymbol{x}.\boldsymbol{\xi} \geqslant 0$ and observe that either

$$|\boldsymbol{x}\rho-\boldsymbol{\xi}| \geqslant |\boldsymbol{x}-\boldsymbol{\xi}| \tag{4.1.39}$$

for $0 \leqslant \rho \leqslant 1$, or

$$|\boldsymbol{x}\rho-\boldsymbol{\xi}| \geqslant |\boldsymbol{\xi}| \tag{4.1.40}$$

for $0 \leqslant \rho \leqslant 1$, or there exists a ρ_o, $0 < \rho_o < 1$, such that

$$|\boldsymbol{x}\rho-\boldsymbol{\xi}| \geqslant |\boldsymbol{x}\rho_o-\boldsymbol{\xi}| \tag{4.1.41}$$

for $0 \leqslant \rho \leqslant 1$, where

$$(\boldsymbol{x}\rho_o-\boldsymbol{\xi}).\boldsymbol{x} = 0 \quad . \tag{4.1.42}$$

In the first two cases we can immediately conclude from (4.1.3~~5~~7) that an estimate of the form (4.1.37) is valid. Hence we now consider the third case. From (4.1.42) we have that $\rho_o|\boldsymbol{x}|^2 = \boldsymbol{\xi}.\boldsymbol{x}$ and hence 15

$$\begin{aligned} \boldsymbol{x}\rho_o-\boldsymbol{\xi} &= \boldsymbol{x}\,\frac{\boldsymbol{\xi}.\boldsymbol{x}}{|\boldsymbol{x}|^2} - \boldsymbol{\xi} \\ &= \boldsymbol{x}-\boldsymbol{\xi}-\boldsymbol{x}\,\frac{(\boldsymbol{x}-\boldsymbol{\xi}).\boldsymbol{x}}{|\boldsymbol{x}|^2} \quad . \end{aligned} \tag{4.1.43}$$

Therefore

$$|\boldsymbol{x}\rho_o-\boldsymbol{\xi}|^2 = |\boldsymbol{x}-\boldsymbol{\xi}|^2 - \frac{((\boldsymbol{x}-\boldsymbol{\xi}).\boldsymbol{x})^2}{|\boldsymbol{x}|^2} \quad . \tag{4.1.44}$$

Since D is strictly starlike, there exists a positive constant $\alpha < 1$ which is independent of $\boldsymbol{x}$ and $\boldsymbol{\xi}$ such that

(This is similar to the approach used in section 1.3 for elliptic equations in two independent variables). In (4.1.33) $d\omega_{\underset{\sim}{\xi}}$ denotes an element of surface area on ∂D, $R = |\underset{\sim}{x}-\underset{\sim}{\xi}|$, ν is the inward normal to ∂D at the point $\underset{\sim}{\xi}$, and ψ is a continuous density to be determined. Substituting (4.1.33) into (4.1.) and interchanging the orders of integration gives

$$u(\underset{\sim}{x}) = \frac{1}{2\pi} \int_{\partial D} \psi(\underset{\sim}{\xi}) \frac{\partial}{\partial\nu} \left(\frac{1}{R} \right) d\omega_{\underset{\sim}{\xi}}$$

$$+ \frac{1}{2\pi} \int_{\partial D} \psi(\underset{\sim}{\xi}) \left\{ \int_0^1 \sigma^2 G(r^2, 1-\sigma^2) \frac{\partial}{\partial\nu} \left(\frac{1}{|\underset{\sim}{x}\sigma^2-\underset{\sim}{\xi}|} \right) d\sigma \right\} d\omega_{\underset{\sim}{\xi}} \quad . \qquad (4.1.34)$$

We will show shortly that for $\underset{\sim}{\xi}$, $\underset{\sim}{x}$ on ∂D

$$\left| \int_0^1 \sigma^2 G(r^2, 1-\sigma^2) \frac{\partial}{\partial\nu} \left(\frac{1}{|\underset{\sim}{x}\sigma^2-\underset{\sim}{\xi}|} \right) d\sigma \right| \leq \frac{\text{constant}}{|\underset{\sim}{x}-\underset{\sim}{\xi}|} \quad . \qquad (4.1.35)$$

Assuming this fact for the time being, we let $\underset{\sim}{x}$ tend to ∂D, and, using the discontinuity properties of double and single layer potentials (c.f. [21], [29]), we arrive at the following integral equation for $\psi(\underset{\sim}{\xi})$:

$$\begin{aligned} f(\underset{\sim}{x}) &= \psi(\underset{\sim}{x}) + \frac{1}{2\pi} \int_{\partial D} \psi(\underset{\sim}{\xi}) \frac{\partial}{\partial\nu} \left(\frac{1}{R} \right) d\omega_{\underset{\sim}{\xi}} \\ &\quad + \frac{1}{2\pi} \int_{\partial D} \psi(\underset{\sim}{\xi}) \left\{ \int_0^1 \sigma^2 G(r^2, 1-\sigma^2) \frac{\partial}{\partial\nu} \left(\frac{1}{|\underset{\sim}{x}\sigma^2-\underset{\sim}{\xi}|} \right) d\sigma \right\} d\omega_{\underset{\sim}{\xi}} \\ &= (\underset{\sim}{I}+\underset{\sim}{T})\psi \qquad ; \qquad \underset{\sim}{x} \in \partial D \ . \end{aligned} \qquad (4.1.36)$$

Before discussing the invertibility of the operator $\underset{\sim}{I}+\underset{\sim}{T}$ we prove the extimate (4.1.35). Since $G(r^2, 1-\sigma^2)$ is continuous, there exists a positive constant C such that for $\underset{\sim}{\xi}$, $\underset{\sim}{x}$ on ∂D

$$\begin{aligned} &\left| \int_0^1 \sigma^2 G(r^2, 1-\sigma^2) \frac{\partial}{\partial\nu} \left(\frac{1}{|\underset{\sim}{x}\sigma^2-\underset{\sim}{\xi}|} \right) d\sigma \right| \\ &\qquad \leq C \int_0^1 \frac{1}{|\underset{\sim}{x}\rho-\underset{\sim}{\xi}|^2} d\rho \ . \end{aligned} \qquad (4.1.37)$$

Remark 1: The assumption that $B(r^2)$ is an entire function can be considerably weakened. This follows from the fact that it can be shown ([31]) that

$$G(r,1-\sigma^2) = -2r\, R_3(r,r;r\sigma^2,0) \tag{4.1.30}$$

where $R(x,y;\xi,\eta)$ is the Riemann function for the hyperbolic equation

$$u_{xy} + \tfrac{1}{4}B(xy)u = 0 \quad , \tag{4.1.31}$$

and the subscript denotes differentiation with respect to ξ. Hence if $\tilde{B}(r) = B(r^2)$ is continuously differentiable we can conclude that $G(r,1-\sigma^2)$ exists and is twice continuously differentiable.

Remark 2: An alternate approach to constructing the operator $\underset{\sim}{I}+\underset{\sim}{G}$ has been outlined by M. Eichler in [23]. This approach is somewhat similar to that which we will use in section 4.2 to obtain a "method of ascent" for equations of the form (4.1.1) definied in exterior domains.

We will now show how the integral operator $\underset{\sim}{I}+\underset{\sim}{G}$ can be used to solve (interior) boundary value problems for (4.1.1). To be specific we will consider the interior Dirichlet problem for (4.1.1) in the case n=3 under the assumption that $B(r^2) \leqslant 0$ in D; the same approach can be used to treat the Dirichlet, Neumann, and Robin problems for $n \geqslant 2$. We want to construct a solution $u(\underset{\sim}{x}) \in C^2(D) \cap C^0(\bar{D})$ of (4.1.1) in D such that $u(\underset{\sim}{x}) = f(\underset{\sim}{x})$ on ∂D where $f(\underset{\sim}{x})$ is a known continuous function defined on ∂D. We look for a solution in the form

$$u(\underset{\sim}{x}) = (\underset{\sim}{I}+\underset{\sim}{G})h \tag{4.1.32}$$

where $h(\underset{\sim}{x})$ is represented in terms of the double layer potential

$$h(\underset{\sim}{x}) = \frac{1}{2\pi} \int_{\partial D} \psi(\underset{\sim}{\xi}) \frac{\partial}{\partial \nu} \left(\frac{1}{R} \right) d\omega_{\underset{\sim}{\xi}}.$$

$$\Phi(r;\theta;\phi) = r^{(n-2)/2}u(r;\theta;\phi)$$

$$\psi(r;\theta;\phi) = r^{(n-2)/2}h(r;\theta;\phi) \qquad (4.1.27)$$

$$K(r,\rho) = \frac{1}{2r}\, G(r^2,\ 1-(\ \frac{\rho}{r}\ ))$$

and $(r;\theta;\phi)$ are spherical coordinates. From the recursion formula (4.1.22) it is seen that each $c^{(k)}(r^2;n)$ is of the form

$$c^{(k)}(r^2;n) = r^{2k}\tilde{c}(r^2;n) \qquad (4.1.28)$$

where $\tilde{c}^{(k)}(r^2;n)$ is an entire function of r^2. This follows from the fact that the differential operator $(2k-3)(\frac{d}{dr}) - r(\frac{d^2}{dr^2})$ annihilates r^{2k-2}.

Hence the function $K(r,\rho)$ defined in (4.1.27) is an entire function of r and ρ. Since (4.1.26) is a Volterra integral equation of the second kind it is now clear that there exists a unique solution $\psi(r;\theta;\phi)$ of (4.1.26). From the fact that

$$0 = \Delta_n u + B(r^2)u = \Delta h + \int_0^1 \sigma^{n-1}G(r^2,1-\sigma^2)\Delta h(\underset{\sim}{x}\sigma^2)d\sigma \qquad (4.1.29)$$

it can easily be seen that $r^{-(n-2)/2}\psi(r;\theta;\phi) = h(r,\theta;\phi)$ is a harmonic function in D (rewrite (4.1.29) in the form (4.1.26) where Φ now equals zero, and appeal to the uniqueness of solutions to Volterra integral equations of the second kind). We can now conclude that the operator $\underset{\sim}{I}+\underset{\sim}{G}$ is invertible.

We summarize our results in the following theorem:

<u>Theorem 4.1.1</u> ([32]): Let $u(\underset{\sim}{x})$ be a real valued twice continuously differentiable solution of (4.1.1) in D where D is strictly starlike with respect to the origin. Then $u(\underset{\sim}{x})$ can be represented in the form $u(\underset{\sim}{x}) = (\underset{\sim}{I}+\underset{\sim}{G})h$ where $h(\underset{\sim}{x})$ is a real valued harmonic function in D. Conversely, if $h(\underset{\sim}{x})$ is harmonic in D, then $u(\underset{\sim}{x}) = (\underset{\sim}{I}+\underset{\sim}{G})h$ is a solution of (4.1.1) in D.

$$c_r^{(1)} = -rB \tag{4.1.22}$$

$$2(k-1)c_r^{(k)} = (2k-3)c_r^{(k-1)} - rc_{rr}^{(k-1)} - rBc^{(k-1)} \quad ; \quad k \geq 2$$

and the initial conditions

$$c^{(k)}(0;n) = 0 \quad ; \quad k \geq 1 \ . \tag{4.1.23}$$

(4.1.22) and (4.1.23) imply that the $c^{(k)}(r^2;n)$ are in fact independent of n. Since we know the series (4.1.18) is convergent when n=2, we can now conclude from (4.1.21) and the fact that the $c^{(k)}(r^2;n)$ are independent of n that the series (4.1.18) converges absolutely and uniformly for r and t arbitrarily large (but bounded). This establishes the existence of the operator defined by (4.1.16) and (4.1.18). If in this operator we now set

$$h(\underset{\sim}{x}) = \int_0^1 t^{n-2}H(\underset{\sim}{x}(1-t^2))\frac{dt}{\sqrt{1-t^2}} \tag{4.1.24}$$

we arrive at the following integral operator which maps real valued harmonic functions defined in D into the class of real valued solutions of (4.1.1) defined in D:

$$u(\underset{\sim}{x}) = (\underset{\sim}{I}+\underset{\sim}{G})h = h(\underset{\sim}{x}) + \int_0^1 \sigma^{n-1}G(r^2,1-\sigma^2)h(\underset{\sim}{x},\sigma^2)d\sigma \tag{4.1.25}$$

where $G(r^2,\rho)$ is defined by (4.1.15) and is independent of n. This last fact is the basis for referring to the approach used in this section as a "method of ascent".

We now want to show that the operator $\underset{\sim}{I}+\underset{\sim}{G}$ is invertible, i.e. for every solution $u(\underset{\sim}{x})$ of (4.1.1) in D there exists a harmonic function $h(\underset{\sim}{x})$ in D such that (4.1.25) is valid. To this end we rewrite (4.1.25) as the Volterra integral equation

$$\Phi(r;\theta;\phi) = \psi(r;\theta;\phi) + \int_0^r K(r,\rho)\psi(\rho;\theta;\phi)d\rho \tag{4.1.26}$$

where

by integrating by parts that $E(r^2,t;n)$ must satisfy the singular partial differential equation

$$(1-t^2)E_{rt} + \frac{n-3}{t} E_r + rs\left[E_{rr} + \frac{1}{r} E_r + BE\right] = 0 \quad . \tag{4.1.17}$$

We now look for a solution of (4.1.17) in the form

$$E(r^2,t;n) = 1 + \sum_{k=1}^{\infty} t^{2k} e^{(k)}(r^2;n) \quad . \tag{4.1.18}$$

Substituting (4.1.18) into (4.1.17) yields the following recursion formulas for the determination of the $e^{(k)}(r^2;n)$:

$$\begin{aligned} &(n-1)e_r^{(1)} = -rB \\ &(2k+n-3)e_r^{(k)} = (2k-3)e_r^{(k-1)} - re_{rr}^{(k-1)} - rBe^{(k-1)}; \\ &\qquad\qquad\qquad\qquad\qquad\qquad k \geqslant 2. \end{aligned} \tag{4.1.19}$$

From the initial condition $E(0,t;n) = 0$ we have the initial conditions

$$e^{(k)}(0;n) = 0 \quad ; \quad k = 1,2,\ldots \quad . \tag{4.1.20}$$

Hence, each of the $e^{(k)}(r^2;n)$ in (4.1.18) is uniquely determined. We must now show that the series (4.1.18) converges for t and r arbitrarily large (but bounded). We first note that for n=2 the $e^{(k)}(r^2;2)$ are identical with the functions $e^{(k)}(r^2)$ defined by (4.1.12). This follows from the facts that the form of the series expansion for $\tilde{E}(r^2,t)$ and $E(r^2,t;2)$ are the same and these functions satisfy the same differential equation and initial condition. Hence, the series (4.1.18) converges when n=2. Now define functions $c^{(k)}(r^2;n)$ by the formula

$$c^{(k)}(r^2;n) = \frac{2e^{(k)}(r^2;n)\Gamma(k+\frac{n}{2}-\frac{1}{2})}{\Gamma(\frac{n}{2}-\frac{1}{2})\Gamma(k)} \quad ; \quad k \geqslant 1 \quad . \tag{4.1.21}$$

Then from (4.1.19) and (4.1.20) it is seen that the $c^{(k)}(r^2;n)$ satisfy the recursion formula bounded). Now define the harmonic function $h(x,y)$ by

$$h(x,y) = \int_{-1}^{1} H(x(1-t^2),\ y(1-t^2)) \frac{dt}{\sqrt{1-t^2}} \ . \qquad (4.1.13)$$

Then (4.1.8) can be rewritten as

$$u(x,y) = h(x,y) + \int_0^1 \sigma G(r^2,1-\sigma^2)h(x\sigma^2,y\sigma^2)d\sigma \qquad (4.1.14)$$

where

$$G(r^2,\rho) = \sum_{k=1}^{\infty} \frac{2e^{(k)}(r^2)\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(k)} \rho^{k-1} \ . \qquad (4.1.15)$$

These last two equations follow immediately from expanding $h(x,y)$ and $H(x,y)$ in a series of harmonic polynomials, integrating (4.1.8) termwise using (4.1.12), and using the elementary properties of the Beta function. From section 1.3 it is clear that (4.1.14) defines a mapping of the class of real valued harmonic functions defined in D onto the class of real valued solutions of (4.1.1) (for n=2) defined in D.

We now want to generalize the representation (4.1.14) from n=2 to general n. To this end we first look for real valued twice continuously differentiable solutions of (4.1.1) in the form

$$u(\underset{\sim}{x}) = \int_0^1 t^{n-2}E(r^2,t;n)H(\underset{\sim}{x}(1-t^2)) \frac{dt}{\sqrt{1-t^2}} \qquad (4.1.16)$$

where $\underset{\sim}{x} = (x_1,,,.,x_n)$ and $H(\underset{\sim}{x})$ is a real valued harmonic function in D (which is now of course a domain in $\mathbb{R}^n$). We require that $E(r^2,t;n)$ be an entire function of t and r^2 and satisfy the initial condition $E(0,t;n)=1$. We now temporarily replace the path of integration from zero to one by a loop starting from s=+1, passing counterclockwise around the origin and onto the second sheet of the Riemann surface of the integrand, and then back up to t=+1, and substitute the resulting expression into the differential equation (4.1.1). If $u(\underset{\sim}{x})$ is to be a solution of (4.1.1), it is then easily verified

$$Q^{(2k)}(z,0) = 0 \tag{4.1.6c}$$

for $k=1,2,\ldots$. (4.1.6a) can be rewritten in the form

$$\frac{\partial Q^{(2)}}{\partial(r^2)} + 2B(r^2) = 0, \tag{4.1.7a}$$

and if we require $Q^{(2)}(0) = 0$ it is seen that $Q^{(2)}$ depends only on r^2 and satisfies (4.1.6c). Now assume that $Q^{(2k)}$ depends only on r^2. Then (4.1.6b) and (4.1.6c) will be satisfied if $Q^{(2k+2)}$ is a solution of

$$(2k+1)\frac{\partial Q^{(2k+2)}}{\partial(r^2)} + 2\left[\frac{\partial\left(r^2 \frac{\partial Q^{(2k)}}{\partial(r^2)}\right)}{\partial(r^2)} + B(r^2)Q^{(2k)} - k\frac{\partial Q^{(2k)}}{\partial(r^2)}\right] = 0 \tag{4.1.7b}$$

such that $Q^{(2k+2)}(0) = 0$. From (4.1.7b) we see that $Q^{(2k+2)}$ is a function only of r^2, and the lemma now follows by induction. The fact that $\tilde{E}(r^2,t)$ is entire follows from the fact that $E(z,z^*,t)$ is entire (section 1.2).

From lemma 4.1 we can write (4.1.2) in the form

$$u(x,y) = \int_{-1}^{1} \tilde{E}(r^2,t)H(x(1-t^2),y(1-t^2))\frac{dt}{\sqrt{1-t^2}} \tag{4.1.8}$$

where

$$H(x,y) = \operatorname{Re} f\left(\frac{z}{2}\right) \tag{4.1.9}$$

is a harmonic function. From section 1.3 and lemma 4.1 it can be shown that $\tilde{E}(r^2,t)$ satisfies the partial differential equation

$$(1-t^2)\tilde{E}_{rt} - \frac{1}{t}\tilde{E}_r + rt\left[\tilde{E}_{rr} + \frac{1}{r}\tilde{E}_r + B\tilde{E}\right] = 0, \tag{4.1.10}$$

the initial condition

$$\tilde{E}(0,t) = 1, \tag{4.1.11}$$

and has a series expansion of the form

$$\tilde{E}(r^2,t) = 1 + \sum_{k=1}^{\infty} t^{2k}e^{(k)}(r^2) \tag{4.1.12}$$

which converges absolutely and uniformly for t and r arbitrarily large (but twice continuously differentiable solutions of (4.1.1) in the form

$$u(x,y) = \mathrm{Re} \int_{-1}^{1} E(z,\bar{z},t) f\left(\frac{z}{2}(1-t^2)\right) \frac{dt}{\sqrt{1-t^2}} \tag{4.1.2}$$

where $z = x+iy$, $\bar{z} = x-iy$, and

$$E(z,z^*,t) = 1 + \sum_{k=1}^{\infty} t^{2k} z^k \int_0^{z^*} P^{(2k)}(z,z^*)dz^* \tag{4.1.3}$$

with the $P^{(2k)}$ defined recursively by

$$\begin{aligned} P^{(2)} &= -2B \\ (2k+1)P^{(2k+2)} &= -2\left[P_z^{(2k)} + B \int_0^{z^*} P^{(2k)} dz^*\right] \quad ; \quad k \geq 1 . \end{aligned} \tag{4.1.4}$$

Recall from section 1.3 that $f(z)$ is an analytic function of z in some neighbourhood of the origin. At this point we make the assumption that $u(x,y)$ is defined in the interior of a bounded domain D containing the origin where D is strictly starlike with respect to the origin, i.e. if P is a point in $\bar{D} = D \cup \partial D$, then the line segment $\overline{OP}$ is contained in D except for possibly the endpoint P. We will further assume that ∂D is twice continuously differentiable. Throughout this chapter whenever we refer to a domain D we will assume it satisfies the conditions described above.

Returning now to (4.1.2)-(4.1.4) we have the following lemma.

<u>Lemma 4.1</u> ([2]): For (4.1.1) the generating function $E(z,z^*,t)$ is a real valued entire function of $r^2 = zz^*$ and t, i.e. $E(z,z^*,t) = \tilde{E}(r^2,t)$.

<u>Proof</u>: Let

$$Q^{(2k)}(z,z^*) = z^k \int_0^{z^*} P^{(2k)}(z,z^*)dz^* \quad ; \quad k = 1,2 \ldots \tag{4.1.5}$$

Then from (4.1.4) we have

$$Q_{z^*}^{(2)} + 2zB(r^2) = 0 \tag{4.1.6a}$$

$$(2k+1)Q_{z^*}^{(2k+2)} + 2z\left[Q_{zz^*}^{(2k)} + B(r^2)Q^{(2k)} - \frac{k}{z} Q_{z^*}^{(2k)}\right] = 0 \tag{4.1.6b}$$

IV The method of ascent for elliptic equations

4.1 Interior Domains.

Although it is possible to extend some of the results of the last three chapters to partial differential equations in three and four independent variables (c.f. [8],[31],[48]) the analysis becomes increasingly more complicated, and hence somewhat less practical for purposes of analytic and numerical approximation. However in certain special cases it is possible to make such an extension in a rather simple and straightforward fashion, and it is this topic which we will consider in this chapter. The special case we have in mind is the elliptic equation

$$\Delta_n u + B(r^2)u = 0 \tag{4.1.1}$$

where $B(r^2)$ is a real valued entire function of $r^2 = x_1^2 + \ldots + x_n^2$, and we will first consider solutions of (4.1.1) which are defined in interior domains. The theory of (4.1.1) in interior domains was developed by R.P. Gilbert in [32], (who described his theory as a "method of ascent") and for exterior domains by Colton and Wendland in [20]. Extensions of this development to the case of parabolic and pseudoparabolic equations are also possible (c.f. [46]), although we will not discuss this topic in the present work.

Equations in the form (4.1.1) arise naturally in the theory of steady state heat conduction and the scattering of acoustic waves (to name but two areas of many possible applications) when the medium is no longer homogeneous but varies smoothly as a function of the variable r (c.f. [6], [20]).

We begin our study of (4.1.1) in interior domains by first considering the case n=2 and using the Bergman operator (section 1.3) to represent real valued, where $\left|\frac{\partial\phi}{\partial\zeta}(\zeta,\tau)\right|^2 = \frac{\partial\phi}{\partial\zeta}(\zeta,\tau)\,\frac{\partial\bar{\phi}}{\partial\zeta}(\zeta,\tau)$ and $|\phi(\zeta,\tau) - \xi|^2 = (\phi(\zeta,\tau)-\xi)(\bar{\phi}(\zeta,\tau)-\bar{\xi})$. The equation (3.3.14) is the solution of the inverse Stefan problem, i.e. for every one parameter family of conformal mappings $\phi(z,t)$, (3.3.14) defines a solution of (3.3.1)-(3.3.3) with the "free" boundary $\Gamma(t)=\{(x,y):\Phi(x,y,t)=0\}$ given by (3.3.4). Note that from the definition of the conformal mappings $\phi(z,t)$, it is seen that (3.3.14) is valid in a region containing $R \cup \partial R x [0,t_o]$. In order to obtain a physically meaningful solution of the inverse Stefan problem we assume $\gamma(x,y,t)=0$ for $(x,y,t)\varepsilon\partial R x[0,t_o] \cap \{(x,y,t):\Phi(x,y,t) \geqslant 0\}$ and choose the conformal mappings $\phi(z,t)$ such that $u(x,y,t) \geqslant 0$ for $\{(x,y,t):\Phi(x,y,t) < 0\}$. We note that from the boundary condition (3.3.3) this last condition is always satisfied (at least for t_o sufficiently small) provided we choose $\phi(z,t)$ such that $\left.\frac{\partial\phi}{\partial t}\right|_{\Gamma(t)} \geqslant 0$. Due to the appearance of the factor $(n!)^2$ in the denominator of the n^{th} term in the series (3.3.14), accurate approximation of the solution to the inverse Stefan problem can be obtained by truncating this series after only a few terms.

We will now obtain the solution of the inverse Stefan problem (3.3.7)-(3.3.9) by first using Stokes theorem to integrate $VL[U] - UM[V]$ over a torus lying in the space of three complex variables and then computing the residue of the resulting integral representation.

Let τ be real and for t on the circle $|t-\tau|=\delta$, $\delta > 0$, let $G(t)$ be a cell whose boundary consists of a curve $C(t)$ lying on the surface $\phi^{-1}(z,t)=\bar{\phi}^{-1}(z^*,t)$ and line segments lying on the characteristic planes $z=\xi$ and $z^*=\bar{\xi}$ respectively which join the point $(\xi,\bar{\xi})$ to $C(t)$. Now use Stokes theorem to integrate the identity

$$\begin{aligned} VL[U] - UM[V] &= (\tfrac{1}{2}VU_{z^*} - \tfrac{1}{2}V_{z^*}U)_z \\ &+ (\tfrac{1}{2}VU_z - \tfrac{1}{2}V_zU)_{z^*} \\ &- (\tfrac{1}{4a}VU)_t \end{aligned} \tag{3.3.13}$$

over the torus $\{(z,z^*,t) : (z,z^*)\varepsilon G(t), |t-\tau|=\delta\}$, making use of the initial conditions (3.3.12a), (3.3.12b) satisfied by V, the fact that $U=0$ on $C(t)$, and the fact that $dzdz^*=0$ on $\partial G(t)\times\Omega$ (complex differentials are interpreted in the sense of exterior differential forms c.f. [7]). After computing the residue at the point $z=\xi$, $z^*=\bar{\xi}$, this calculation gives ([18])

$$\begin{aligned} U(\xi,\bar{\xi},\tau) &= \frac{1}{4\pi i}\int_{|t-\tau|=\delta}\int_{C(t)}[VU_z dz - VU_{z^*}dz^*]dt \\ &= \frac{\lambda\rho}{4\pi k}\int_{|t-\tau|=\delta}\int_{\bar{\phi}^{-1}(\bar{\xi},t)}^{\phi^{-1}(\xi,t)} \frac{1}{t-\tau}\exp\left\{\frac{(\phi(\zeta,t)-\xi)(\bar{\phi}(\zeta,t)-\xi)}{4a(t-\tau)}\right\} \\ &\quad\cdot\left|\frac{\partial\phi(\zeta,t)}{\partial\zeta}\right|^2 g(\phi(\zeta,t),t)d\zeta dt \\ &= \frac{i\lambda\rho}{2k}\sum_{n=0}^{\infty}\frac{1}{(4a)^n(n!)^2}\frac{\partial^n}{\partial\tau^n}\left\{\int_{\bar{\phi}^{-1}(\bar{\xi},\tau)}^{\phi^{-1}(\xi,\tau)}|\phi(\zeta,\tau)-\xi|^{2n}\right. \\ &\quad\left.\cdot\left|\frac{\partial\phi(\zeta,\tau)}{\partial\zeta}\right|^2 g(\phi(\zeta,\tau),\tau)d\zeta\right\} \end{aligned} \tag{3.3.14}$$

second variables respectively. (3.3.9) was arrived at in the following manner. Let $\zeta = \xi_1+i\xi_2$. Then C(t) is the image of $\xi_2=0$ under the mapping $z=\phi(\zeta,t)$. We have $\frac{\partial u}{\partial\nu} = \frac{\partial u}{\partial\xi_1}\frac{\partial\xi_1}{\partial\nu} + \frac{\partial u}{\partial\xi_2}\frac{\partial\xi_2}{\partial\nu}$. But on C(t), ν is in the direction of the level curve ξ_1 = constant, since $\zeta=\xi_1(x,y,t)+i\xi_2(x,y,t)=\phi^{-1}(z,t)$ is a conformal mapping. Hence on C(t) $\frac{\partial\xi_1}{\partial\nu} = 0$, and by the Cauchy Riemann equations and the fact that $\nu = -\frac{\nabla\xi_2}{|\nabla\xi_2|}$, we have $\frac{\partial\xi_2}{\partial\nu} = -\left|\frac{\partial\phi^{-1}(z,t)}{\partial z}\right|$ on C(t).

Therefore $\frac{\partial u}{\partial\nu} = -\frac{\partial u}{\partial\xi_2}\left|\frac{\partial\phi^{-1}(z,t)}{\partial z}\right|$. But from $\phi^{-1}(\phi(z,t),t)=z$ we have

$\frac{\partial\phi^{-1}}{\partial z}(\phi(z,t),t)\,\frac{\partial\phi(z,t)}{\partial z} = 1$, i.e. $\frac{\partial\phi^{-1}(z,t)}{\partial z} = \frac{1}{\frac{\partial\phi}{\partial z}(\phi^{-1}(z,t),t)}$.

Hence on C(t), $\frac{\partial\phi^{-1}(z,t)}{\partial z} = \frac{1}{\frac{\partial\phi}{\partial s}(s,t)}$, and therefore

$\frac{\partial u}{\partial\nu} = -\frac{\partial u}{\partial\xi_2}\frac{1}{\left|\frac{\partial\phi}{\partial s}(s,t)\right|} = -i\left(U_1\frac{\partial\phi}{\partial s} - U_2\frac{\partial\bar\phi}{\partial s}\right)\left|\frac{\partial\phi}{\partial s}(s,t)\right|^{-1}$. (3.3.9) now follows

from (3.3.3), (3.3.5), and the fact that $|\nabla\Phi|^2 = \left|\frac{\partial\phi^{-1}(z,t)}{\partial z}\right|\left|\frac{\partial\bar\phi^{-1}(\bar z,t)}{\partial z}\right|$,

which implies that on C(t) $|\nabla\Phi| = \frac{1}{\left|\frac{\partial\phi}{\partial s}(s,t)\right|}$.

Now let M be the adjoint operator defined by

$$M[V] \equiv \frac{\partial^2 V}{\partial z\partial z^*} + \frac{1}{4a}\frac{\partial V}{\partial t} = 0 \tag{3.3.10}$$

and let V be the fundamental solution of $M[V] = 0$ defined by

$$V(z,z^*,t\,;\,\xi,\bar\xi,\tau) = \frac{1}{t-\tau}\exp\left\{\frac{(z-\xi)(z^*-\bar\xi)}{4a(t-\tau)}\right\} \tag{3.3.11}$$

where $\xi = \xi_1+i\xi_2$, $\bar\xi = \xi_1-i\xi_2$. Note that V satisfies the Goursat data

$$V(z,\bar\xi,t\,;\,\xi,\bar\xi,\tau) = \frac{1}{t-\tau} \tag{3.3.12a}$$

$$V(\xi,z^*,t\,;\,\xi,\bar\xi,\tau) = \frac{1}{t-\tau}\,. \tag{3.3.12b}$$

$$\Phi(x,y,t) = \frac{1}{2i}\left[\phi^{-1}(z,t) - \bar{\phi}^{-1}(z^*,t)\right] \tag{3.3.4}$$

where $z=x+iy$, $z^*=x-iy$. Noting that $z^*=\bar{z}$ if and only if x and y are real it is seen that $\Phi(x,y,t)=0$ corresponds to $\text{Im}\,\zeta=0$, i.e. the interval $(-1,1)$ in the complex ζ plane. Similarly, the region $\Phi(x,y,t) < 0$ corresponds to $\text{Im}\,\zeta < 0$, i.e. the part of Ω which lies in the lower half plane. We finally note that for $z=x+iy\in\Gamma(t)$ we have $\phi^{-1}(z,t)=\bar{\phi}^{-1}(z^*,t)$ and hence

$$\left.\frac{\partial\Phi}{\partial t}\right|_{\Gamma(t)} = \frac{1}{2i}\left.\left[\frac{\partial\phi^{-1}(z,t)}{\partial t} - \frac{\partial\bar{\phi}^{-1}(z^*,t)}{\partial t}\right]\right|_{z^*=\bar{\phi}(\phi^{-1}(z,t),t)} \tag{3.3.5}$$

$$= g(z,t)$$

i.e. $\frac{\partial\Phi}{\partial t}$ restricted to $\Gamma(t)$ can be analytically continued (for each fixed t) to an analytic function of z for $z\in D_t$.

We will now construct a solution of (3.3.1) which has $\Phi(x,y,t)$ (as given by (3.3.4)) as a free boundary. In (3.3.1) we consider x and y as independent complex variables and define the transformation of $\mathbb{C}^2$ into itself by

$$\begin{aligned} z &= x+iy \\ z^* &= x-iy \quad . \end{aligned} \tag{3.3.6}$$

Under this transformation (3.3.1)-(3.3.3) become

$$L[U] \equiv \frac{\partial^2 U}{\partial z\partial z^*} - \frac{1}{4a}\frac{\partial U}{\partial t} = 0 \tag{3.3.7}$$

$$U(\phi(s,t),\bar{\phi}(s,t),t) = 0 \; ; \quad -1 < s < 1 \tag{3.3.8}$$

$$\begin{aligned} &U_1(\phi(s,t),\bar{\phi}(s,t),t)\,\frac{\partial\phi(s,t)}{\partial s} - U_2(\phi(s,t),\bar{\phi}(s,t),t)\,\frac{\partial\bar{\phi}(s,t)}{\partial s} \\ &= \frac{i\lambda\rho}{k}\left|\frac{\partial\phi(s,t)}{\partial s}\right|^2 g(\phi(s,t),t) \quad ; \quad -1 < s < 1 \end{aligned} \tag{3.3.9}$$

where $U(z,z^*,t) = u\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}, t\right)$, $g(z,t)$ is defined by (3.3.5), and subscripts denote differentiation of $U(z,z^*,t)$ with respect to the first and be described by $\Gamma(t) = \{(x,y) : \Phi(x,y,t) = 0\}$ with the water lying in the region $\Phi(x,y,t) < 0$. The differential equation and boundary conditions governing the conduction of heat in the water are given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{a}\frac{\partial u}{\partial t} \quad ; \quad \Phi(x,y,t) < 0 \tag{3.3.1}$$

$$u\Big|_{\partial R} = \gamma(x,y,t) \tag{3.3.2}$$

$$u\Big|_{\Gamma(t)} = 0 \quad , \quad k\frac{\partial u}{\partial \nu}\Big|_{\Gamma(t)} = \frac{\lambda\rho}{|\nabla\Phi|}\frac{\partial\Phi}{\partial t}\Big|_{\Gamma(t)} \tag{3.3.3}$$

where ν is the unit normal with respect to the space variables that points into the region $\Phi(x,y,t) < 0$, ∇ denotes the gradient with respect to the space variables, $u(x,y,t)$ is the temperature, a the diffusivity coefficient, λ the latent heat of fusion, ρ the density,and k the conductivity of the water. The Stefan problem is to find $\Gamma(t)$ and $u(x,y,t)$ given the function $\gamma(x,y,t)$. The inverse Stefan problem which we are interested in is to find $u(x,y,t)$ (an in particular $\gamma(x,y,t) = \lim\limits_{(x,y)\to R} u(x,y,t)$) given $\Gamma(t)$. In general we cannot hope to solve the inverse problem for arbitrary $\Gamma(t)$; however by suitably restricting $\Gamma(t)$ to lie in a certain class of analytic surfaces we will be able to obtain a relatively simple series representation of the solution, and it is to this problem we now address ourselves.

Let D_t, $0 \leq t < t_o$, be a family of simply connected domains which depend analytically on a parameter t such that $\bigcup\limits_{0\leq t<t_o} D_t$ contains $R \cup \partial R\times[0,t_o]$. Let $z=\phi(\zeta,t)$ conformally map the unit disc Ω onto D_t (D_t being such that the image of $(-1,1)$ intersects R) and for $\zeta^*\epsilon\Omega$, $0 \leq t < t_o$, define $\bar{\phi}(\zeta^*,t)$ by $\bar{\phi}(\zeta^*,t) = \phi\overline{(\bar{\zeta}^*,t)}$ where bars denote conjugation. Now set $z^* = \bar{\phi}(\zeta^*,t)$ and note that $z^*=\bar{z}$ if and only if $\zeta^*=\bar{\zeta}$. We now define the function $\Phi(x,y,t)$ for (possibly) complex values of x,y and t by

In physical terms we are asking the question of how must a given solid (.e.g. ice) be heated in order for it to melt in a prescribed manner, and by constructing a variety of such examples a qualitative idea can be obtained on the shape of the free boundary as a function of the initial-boundary conditions. As in the case of the inverse Stefan problem in one space dimension, such an inverse approach leads to two main problems. The first of these is that the inverse problem has its mathematical formulation as a non-characteristic Cauchy problem for the heat equation and is thus improperly posed in the real domain. However such a problem is well posed in the complex domain, and hence we are led to examine solutions of the heat equation in the space of several complex variables. The inverse Stefan problem can now locally be solved by appealing to the Cauchy-Kowalewski theorem (c.f. [21], [29]). However in addition to being far too tedious for practical computation and error estimation, such an approach does not provide us with the required global solution to the Cauchy problem under investigation. Hence we are led to the problem of the analytic continuation of solutions to non-characteristic Cauchy problems for the heat equation. We will accomplish this by using contour integration and the calculus of residues in the space of several complex variables to arrive at an explicit (global) series representation of the solution to the inverse Stefan problem.

We will motivate the mathematical formulation of the inverse Stefan problem in terms of an ice-water system undergoing a change of phase. Assume that a bounded simply connected region R with boundary ∂R is filled with ice at 0° Centigrade. Beginning at time $t=0$ a non-negative temperature $\gamma=\gamma(x,y,t)$ (where $\gamma(x,y,0)=0$) is applied to ∂R. The ice begins to melt and we will let the interphase boundary $\Gamma(t)$ between ice and water

Theorem 3.2.3 can be used to provide a method for approximating solutions to initial-boundary value problems for (3.2.15) in the same manner as we have already done for elliptic equations in two independent variables and parabolic equations in one space variable. In particular we orthonormalize the set $\{u_{nm}(x,y,t)\}$ in the L^2 norm over the base and lateral boundary of the cylinder $D\times(0,T)$ to obtain the complete set $\{\phi_n(x,y,t)\}$. An approximate solution to the initial-boundary value problem on compact subsets of $D\times(0,T)$ is then given by

$$u^N(x,y,t) = \sum_{n=0}^{N} \sum_{m=0}^{M} c_{nm}\phi_n(x,y,t) \tag{3.2.28}$$

where

$$c_{nm} = \iint_{\partial D\times[0,T]} u(x,y,t)\phi_{nm}(x,y,t)\,ds + \iint_{D} u(x,y,0)\phi_{nm}(x,y,0)\,dxdy \tag{3.2.29}$$

and ds denotes an element of surface area on ∂D.

Error estimates can again be found by applying the maximum principle. This procedure is particularly simple in the case of the heat equation where a complete family is given by (3.2.13) or (3.2.14)

3.3 The Inverse Stefan Problem.

In this section we will present an inverse method for constructing analytic solutions to the single phase Stefan problem for the heat equation in two space dimensions (For the case of one space dimension see the Introduction and section 2.3). Our solution of the inverse Stefan problem will be accomplished by assuming a priori that the free boundary is a relatively simple analytic surface and then constructing a solution to the heat equation which has this prescribed surface as a free boundary ([18]). Provided the solution is analytic in a sufficiently large domain we can then determine the initial-boundary data which is compatible with the given "free" boundary.

$$v(x,y,t) = \sum_{n=0}^{\infty} a_n\phi_n(x,y)\exp(-\lambda_n t) \tag{3.2.24}$$

$$a_n = \iint_D v(x,y,0)\phi_n(x,y)d(x,y)dxdy$$

where the series in (3.2.24) converges absolutely and uniformly in $\bar{D}x[0,T]$. By truncating the series in (3.2.24) and appealing to Theorem 1.3.3 and Theorem 1.3.5 we can conclude that there exists a real valued solution $w_2(x,y,t)$ of (3.2.15) which is an entire function of its independent complex variables such that

$$\max_{\bar{D}x[0,T]} |w_2-v| < \varepsilon/2 \quad . \tag{3.2.25}$$

The inequalities (3.2.22) and (3.2.25) now imply that there exists a real valued solution $u_1(x,y,t)$ of (3.2.15) which is an entire function of its independent complex variables such that

$$\max_{\bar{D}x[0,T]} |u_1-u| < \varepsilon \quad . \tag{3.2.26}$$

Representing $u_1(x,y,t)$ in the form $u = \text{Re}\ \underset{\sim}{P}_2\{f\}$ and truncating the Taylor series for $f(z,t)$ to obtain the polynomial $f_n(z,t)$ such that for $(z,t)\varepsilon$ $\partial Dx[0,T]$, $|f-f_n|$ is sufficiently small, leads to the following theorem:

<u>Theorem 3.2.3</u> ([13]): Let $u(x,y,t)$ be a real valued classical solution of (3.2.15) in $Dx(0,T)$ which ∂D is three times continously differentiable, $d(x,y) > 0$ in $\bar{D}$, and $u(x,y,t)$ is continuous in $\bar{D}x[0,T]$. Let $u_{nm}(x,y,t)$ for $n,m=0.1.2....$ be defined by (3.2.10). Then for any $\varepsilon > 0$ there exists integers $N=N(\varepsilon)$, $M=M(\varepsilon)$, and constants a_{nm}, $n=0,1,...,N$, $m=0,1,...,M$, such that

$$\max_{\bar{D}x[0,T]} |u- \sum_{n=0}^{N} \sum_{m=0}^{M} a_{nm}u_{nm}| < \varepsilon. \tag{3.2.27}$$

$$w(x,y,t) = \sum_{n=0}^{N} w_n(x,y)t^n \tag{3.2.20}$$

such that $w(x,y,t) = u(x,y,t)$ for $(x,y,t)\varepsilon\partial D\text{x}[0,T]$. From (3.2.15) and (3.2.19) it is seen that the function $w_n(x,y)$ must satisfy the recursive scheme

$$\begin{aligned}
&\frac{\partial^2 w_N}{\partial x^2} + \frac{\partial^2 w_N}{\partial y^2} + c(x,y)w_N = 0 \quad ; \quad (x,y)\varepsilon D \\
&w_N(x,y) = f_N(x,y) \quad ; \quad (x,y)\varepsilon\partial D \\
&\frac{\partial^2 w_n}{\partial x^2} + \frac{\partial^2 w_n}{\partial y^2} + c(x,y)w_n = (n+1)d(x,y)w_{n+1} \quad ; \quad (x,y)\varepsilon D \\
&w_n(x,y) = f_n(x,y) \quad ; \quad (x,y)\varepsilon\partial D,
\end{aligned} \tag{3.2.21}$$

for $n=0,1,\ldots,N-1$. The existence of the $w_n(x,y)$ for $n=0,1,\ldots,N$ follows from the smoothness of ∂D and the fact that $c(x,y) \leq 0$ in D (c.f. [27]). From Corollary 1.1.1, Theorem 1.3.3, Theorem 1.3.4 and the fact that $w_n(x,y)$ depends continuously on the nonhomogeneous term $(n+1)d(x,y)w_{n+1}(x,y)$, we can conclude that for $\varepsilon > 0$ there exists a real valued solution $w_1(x,y,t)$ of (3.2.15) which is an entire function of its independent complex variable such that

$$\max_{\bar{D}\text{x}[0,T]} |w_1 - w| < \varepsilon/2 \tag{3.2.22}$$

Now let $v(x,y,t) = u(x,y,t) - w(x,y,t)$ and let λ_n and $\phi_n(x,y)$ be the eigenvalues and eigenfunctions respectively that correspond to the eigenvalue problem

$$\begin{aligned}
&\phi_{xx} + \phi_{yy} + c(x,y)\phi + \lambda d(x,y)\phi = 0 \quad ; \quad (x,y)\varepsilon D \\
&\phi(x,y) = 0 \quad ; \quad (x,y)\varepsilon\partial D.
\end{aligned} \tag{3.2.23}$$

From (3.2.19)-(3.2.21) and the expansion theorem for the eigenvalue problem (3.2.23) (c.f. [35]) we can conclude that conduction in a nonhomogeneous medium is governed by the equation

$$\frac{\partial}{\partial x}\left(k(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(k(x,y)\frac{\partial u}{\partial y}\right) = c(x,y)\frac{\partial u}{\partial t} \qquad (3.2.16)$$

where $k(x,y)$ and $c(x,y)$ are positive, continuous known functions and $u=u(x,y,t)$ denotes the temperature in the medium. Writing (3.2.16) in the form

$$k(x,y)(u_{xx}+u_{yy}) + k_x(x,y)u_x + k_y(x,y)u_y = c(x,y)u_t \qquad (3.2.17)$$

and dividing by $\sqrt{k(x,y)}$ we obtain (after rearrangement)

$$v_{xx}+v_{yy} - \frac{\Delta(\sqrt{k(x,y)})}{\sqrt{k(x,y)}}\, v = \frac{c(x,y)}{\sqrt{k(x,y)}}\, v_t \qquad (3.2.18)$$

where $v(x,y,t) = \sqrt{k(x,y)}\, u(x,y,t)$ and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Now let $u(x,y,t)$ be a classical real valued solution of (3.2.15) in a cylindrical domain $D\times(0,T)$ where D is a bounded, simply connected domain whose boundary ∂D is three times continuously differentiable and let $u(x,y,t)$ be continuous in $\bar{D}\times[0,T]$. We assume that $c(x,y)$ and $d(x,y)$ are entire functions of their independent complex variables and that for $(x,y)\varepsilon\bar{D}$ we have $d(x,y) > 0$. By means of the change of variables $u = e^{\alpha t}v$ where $\alpha > 0$ is large, it is seen that without loss of generality we can assume $c(x,y) \leqslant 0$ for $(x,y)\varepsilon D$. From the maximum principle for parabolic equations and the Weierstrass approximation theorem we can assume without loss of generality (for purposes of approximation) that the boundary data assumed by $u(x,y,t)$ on $\partial D\times[0,T]$ is a polynomial in t, i.e.

$$u(x,y,t) = \sum_{n=0}^{N} f_n(x,y)t^n \quad ; \quad (x,y,t)\varepsilon\partial D\times[0,T] \qquad (3.2.19)$$

where (by a further approximation) the $f_n(x,y)$ are Hölder continuous functions defined on ∂D. We now look for a real valued solution of (3.2.15) in the form for $n,m = 0,1,2,\ldots$. Then for any $\varepsilon > 0$ there exists integers $N = N(\varepsilon)$, $M = M(\varepsilon)$, and constants a_{nm}, $n = 0, 1,\ldots,N$, $m = 0,1,\ldots,M$, such that

$$\max_{\bar{D}_o \times [\delta_o, T-\delta_o]} \left| u - \sum_{n=0}^{N} \sum_{m=0}^{M} a_{nm} u_{nm} \right| < \varepsilon . \qquad (3.2.11)$$

We note that for the case of the heat equation (3.1.4) we have from (3.1.15) and the result

$$\int_{-1}^{1} (1-s^2)^{n-\frac{1}{2}} s^{2k} \, ds = \frac{\Gamma(n+\frac{1}{2})\Gamma(k+\frac{1}{2})}{\Gamma(n+k+1)} \qquad (3.2.12)$$

that

$$u_{2n,m}(x,y,t) = \cos n\theta \sum_{k=0}^{m} \frac{\pi\, \Gamma(m+1)\Gamma(n+\frac{1}{2})}{\Gamma(k+1)\Gamma(m-k+1)\Gamma(n+k+1)} r^{2k+n} t^{m-k} \qquad (3.2.12)$$

$$u_{2n+1,m}(x,y,t) = \sin n\theta \sum_{k=0}^{m} \frac{\pi\, \Gamma(m+1)\Gamma(n+\frac{1}{2})}{\Gamma(k+1)\Gamma(m-k+1)\Gamma(n+k+1)} r^{2k+n} t^{m-k}$$

where $x = r\cos\theta$, $y = r\sin\theta$. Noting that in this special case $u_{n,m}(x,y,t)$ is a polynomial in x,y and t, it follows from the uniqueness theorem for Cauchy's problem for the heat equation (c.f. [35]) that another complete family of solutions (on compact subsets) for the heat equation defined in $D\times(0,T)$ is given by

$$v_{n,m}(x,y,t) = h_n(x,t)\, h_m(y,t) \qquad (3.2.14)$$

for $n,m = 0,1,2,\ldots$ where $h_n(x,t)$ is the polynomial defined in (2.3.17).

We now consider the case when (3.1.1) is of the form

$$u_{xx} + u_{yy} + c(x,y)u = d(x,y)u_t \qquad (3.2.15)$$

and show that if ∂D is three times continuously differentiable, $u(x,y,t)$ is continuous in $\bar{D}\times[0,T]$, and $d(x,y) > 0$ in $\bar{D}$, then the family (3.2.10) is in fact complete "up to the boundary", i.e. in (3.2.11) $\bar{D}_o\times[\delta_o, T-\delta_o]$ can be replaced by $\bar{D}\times[0,T]$. Equations of the form (3.2.15) are of particular interest since a wide variety of equations appearing in mathematical physics can be written in the form (3.2.15). For example the equation of heat solution $u_o(x,y,t)$ of (1.1) such that

$$\max_{\bar{D}_o \times [\delta_o, T-\delta_o]} |u-u_o| < \varepsilon \tag{3.2.8}$$

Proof: Let $u_1(x,y,t)$ be an analytic solution of (3.1.1) in $\bar{D}_1 \times [\frac{\delta_o}{2}, T-\frac{\delta_o}{2}]$ such that (3.2.1) is valid. From Theorem 3.2.1 we have that $U_1(z,z^*,t) = u_1(x,y,t)$ is analytic in $D_1 x D_1^* x E$, and from Theorem 3.1.1 we can represent $U_1(z,\bar{z},t)$ in this domain in the form $U_1(z,\bar{z},t) = \mathrm{Re}\ \underset{\sim}{P}_2\{f\}$ where $f(z,t)$ is given by (3.1.18) with U replaced by U_1. (We emphasize again the importance of Theorem 3.2.1 which tells us that $U(z,0,t)$ is analytic in $D_1 x E$. From (3.1.18) this implies $f(\frac{z}{2},t)$ is analytic in $D_1 x E$ and hence $\mathrm{Re}\ \underset{\sim}{P}_2\{f\}$ is analytic in $D_1 x D_1^* x E$ and therefore must equal $U_1(z,\bar{z},t)$ not only locally but in the entire product domain $D_1 x D_1^* x E$.) Since product domains are Runge domains of the first kind (c.f. [28], p.49). we can approximate $U_1(z,0,t)$ (and hence $f(\frac{z}{2},t)$) on compact subsets of $D_1 x E$ by a polynomial. In particular since $\mathrm{Re}\ \underset{\sim}{P}_2\{f\}$ tends to zero as $f(z,t)$ tends to zero in the maximum norm, we can conclude that there exists a polynomial $f_n(z,t)$ and entire solution $u_2(x,y,t) = \mathrm{Re}\ \underset{\sim}{P}_2\{f_n\}$ of (3.1.1) such that

$$\max_{\bar{D}_o \times [\delta_o, T-\delta_o]} |u_2-u_1| < \varepsilon/2 \quad . \tag{3.2.9}$$

The theorem now follows from (3.2.1) and (3.2.9) by the use of the triangle inequality and the fact that $\bar{D}_1 \supset \bar{D}_o$.

As an immediate consequence of Theorem 3.2.2 we have the following corollary, where "Im" denotes "take the imaginary part":

Corollary 3.2.1: Let $u(x,y,t)$ be a real valued classical solution of (3.1.1) in $D x(0,T)$ where $d(x,y,t) > 0$ in $D x(0,T)$, $\bar{D}_o \subset D$, $\delta_o > 0$, and let

$$\begin{aligned} u_{2n,m} &= \mathrm{Re}\ \underset{\sim}{P}_2\{z^n t^m\} \\ u_{2n+1,m} &= \mathrm{Im}\ \underset{\sim}{P}_2\{z^n t^m\} \end{aligned} \tag{3.2.10}$$

$$= \iint_{\partial D_1 x\Omega} H[u_1, R\log r] + 2\pi \oint_{\Omega} \frac{u_1(\xi,\eta,t)}{t-\tau}\, dt$$

$$+ \iiint_{D_1 x\Omega} u_1 \mathcal{M}[R\log r]\, dxdydt \tag{3.2.5}$$

$$= \iint_{\partial D_1 x\Omega} H[u_1, R\log r] + 4\pi^2 i\, u_1(\xi,\eta,\tau)$$

$$+ \iiint_{D_1 x\Omega} u_1 \mathcal{M}[R\log r]\, dxdydt$$

i.e.

$$u_1(\xi,\eta,\tau) = \frac{i}{\pi^2}\left(\iint_{\partial D_1 x\Omega} H[u_1, R\log r] + \iiint_{D_1 x\Omega} u_1 \mathcal{M}[R\log r]\, dxdydt\right). \tag{3.2.6}$$

Returning now to the complex coordinates z, z^*, we see from the fact that $M[R] = 0$ that

$$\mathcal{M}[R\log r] = M[R\log r] = 2\,\frac{\partial R/\partial z - BR}{\zeta^* - z^*} + 2\,\frac{\partial R/\partial z^* - AR}{\zeta - z}, \tag{3.2.7}$$

and hence from (3.1.27) we have that $\mathcal{M}[R\log r]$ is an entire function of its independent complex variables except for an essential singularity at $t=\tau$. Hence, replacing $\bar{\zeta}$ by ζ^*, we see that the second integral in (3.2.6) can be continued to an entire function of ζ and ζ^* for $\tau \in E$. The first integral in (3.2.6) can be continued to an analytic function of ζ, ζ^* and τ for $(\zeta,\zeta^*,\tau) \in D_1 x D_1^* x E$. Hence (3.2.6) shows that $U_1(\zeta,\zeta^*,\tau) = u_1(\xi,\eta,\tau)$ is analytic in $D_1 x D_1^* x E$ and the theorem is established.

With the help of the above theorem on the analytic continuation of analytic solutions to (3.1.1) we can now establish the following version of Runge's Theorem for parabolic equations in two space variables:

<u>Theorem 3.2.2</u> ([17]): Let $u(x,y,t)$ be a real valued classical solution of (3.1.1) in $Dx(0,T)$ where $d(x,y,t) > 0$ in $Dx(0,T)$ and let $\overline{D_o} x [\delta_o, T-\delta_o]$ be a compact subset of $Dx(0,T)$. Then for every $\varepsilon > 0$ there exists an entire

$$D_1 = \{z: z \in D_1\}$$
$$D_1^* = \{z^* : \bar{z}^* \in D_1\} \tag{3.2.2}$$

and E is an ellipse in $\mathbb{C}^1$ containing the interval $[\delta_o, T-\delta_o]$ such that for $(x,y)\epsilon\bar{D}_1$, $u(x,y,t)$ is an analytic function of t in E. This result is the analogue for parabolic equations of the Bergman-Vekua theorem for elliptic equations (c.f. section 1.1).

<u>Theorem 3.2.1</u> ([17]): $U_1(z,z^*,t)$ is analytic in $D_1 x D_1^* x E$.

<u>Proof</u>: From Stokes theorem we have that for u and v analytic in a neighbourhood of $\bar{D}_1 x[\delta_o, T-\delta_o]$

$$\iiint\limits_{D_1 x\Omega} (v\mathcal{L}[u] - u\mathcal{M}[v])dxdydt = \iint\limits_{\partial D_1 x\Omega} H[u,v] \tag{3.2.3}$$

where $\mathcal{L}$ is the differential operator defined by (3.1.1), $\mathcal{M}$ is its adjoint, $\Omega = \{t: |t-\tau|=\delta\}$ such that $\Omega \subset E$, and

$$H[u,v] = \{(vu_x-uv_x+auv)dydt - (vu_y-uv_y+buv)dxdt - (duv)dxdy\}. \tag{3.2.4}$$

The region of integration $D_1 x\Omega$ in (3.2.3) can be geometrically visualised as a three dimensional torus lying in the six dimensional space $\mathbb{C}^3$. Note that on ∂D_1 we have dxdy=0. Now let D_ε be a small disc of radius ε about the point (ξ,η), $u=u_1(x,y,t)$, $v=R(z,\bar{z},t;\zeta,\bar{\zeta},\tau)\log r$ (where $r^2=(z-\zeta)(\bar{z}-\bar{\zeta})$, $\zeta=\xi+i\eta$, $\bar{\zeta}=\xi-i\eta$) and apply (3.2.3) to u and v with the torus $D_1 x\Omega$ replaced by the hollow torus $D_1 \setminus D_\varepsilon x\Omega$. Letting ε tend to zero now gives

$$0 = \lim_{\varepsilon\to 0} \{ \iint\limits_{\partial(D_1\setminus D_\varepsilon)x\Omega} H[u_1, R\log r] + \iiint\limits_{D_1\setminus D_\varepsilon x\Omega} u_1\mathcal{M}[R\log r]dxdydt\}$$

3.2 Complete Families of Solutions.

We will now use the integral operators and Riemann function constructed in the last section to construct a complete family of solutions to (3.1.1) in the space of real valued classical solutions to (3.1.1) defined in a cylinder $D\times(0,T)$ where T is a positive constant. We will assume that D is a bounded, simply connected domain in $\mathbb{R}^2$ containing the origin and that the cofficient $d(x,y,t)$ is greater than zero in $D\times(0,T)$.

Let $u(x,y,t)$ be a real valued classical solution of (3.1.1) in $D\times(0,T)$, $\bar{D}_0$ and $\bar{D}_1$ compact subsets of D such that $D \supset \bar{D}_1 \supset \bar{D}_0$ and let ∂D_1 be analytic. From the existence theorems for solutions of initial-boundary value problems for parabolic equations (c.f. [26],[27]) and the maximum principle for parabolic equations we can conclude that for $\varepsilon > 0$, $\delta_o > 0$, there exists a solution $u_1(x,y,t)$ of (3.1.1) in $D_1\times(\frac{\delta_o}{2}, T-\frac{\delta_o}{2})$ such that $u_1(x,y,t)$ is continuous in $\bar{D}_1\times[\frac{\delta_o}{2}, T-\frac{\delta_o}{2}]$, assumes analytic Dirichlet data on $\partial D_1\times[\frac{\delta_o}{2}, T-\frac{\delta_o}{2}]$, and satisfies

$$\max_{\bar{D}_1\times[\frac{\delta_o}{2}, T-\frac{\delta_o}{2}]} |u_1-u| < \varepsilon/2. \tag{3.2.1}$$

From a result of Friedman ([27]p.212) we can conclude that $u_1(x,y,t)$ is analytic in $\bar{D}_1\times(\frac{\delta_o}{2}, T-\frac{\delta_o}{2})$, i.e. for every point $(x_o,y_o,t_o) \in \bar{D}_1\times(\frac{\delta_o}{2}, T-\frac{\delta_o}{2})$ there exists a ball in $\mathbb{C}^3$ with centre at (x_o,y_o,t_o) such that as a function of the complex variables x,y,t, $u_1(x,y,t)$ is analytic in this ball. By standard compactness arguments we can conclude that $u_1(x,y,t)$ is analytic in some "thin" neighbourhood in $\mathbb{C}^3$ of the product domain $\bar{D}_1\times[\delta_o, T-\delta_o]$. We now want to show that $U_1(z,z^*,t) = u_1(x,y,t)$ can be analytically continued as a function z,z^* and t (where $z^*=\bar{z}$ for x and y real) into the product domain $D_1\times D_1^*\times E$ where

To establish the existence of the Riemann function we let $f(\frac{z}{2},t) = \frac{1}{t-\tau}F(\frac{z}{2},t)$ where

$$F(\frac{z}{2},t) = -\frac{1}{2\pi}\int_\gamma \exp\{\int_0^{z(1-\rho^2)} B(\sigma+\zeta,\zeta^*,t)d\sigma\}\frac{d\rho}{\rho^2} \tag{3.1.28}$$

(with γ defined as in (3.1.18)) and define the solution $V(z,z^*,t;\zeta,\zeta^*,\tau)$ of (3.1.26) by

$$V(z,z^*,t;\zeta,\zeta^*,\tau) = \underset{\sim}{P}_2^* \{\frac{F(z,t)}{t-\tau}\} \tag{3.1.29}$$

where $\underset{\sim}{P}_2^*$ is the integral operator associated with (3.1.26).

Then from the reciprocal relations (c.f. section 1.3)

$$\int_{-1}^{1} f(\frac{z}{2}(1-s^2))\frac{ds}{\sqrt{1-s^2}} = g(z)$$
$$-\frac{1}{2\pi}\int_\gamma g(z(1-\rho^2)\frac{d\rho}{\rho^2} = f(\frac{z}{2}) \tag{3.1.30}$$

we have that

$$V(z,\zeta^*,t;\zeta,\zeta^*,\tau) = \frac{1}{t-\tau}\exp\{\int_0^{(z-\zeta)} B(\sigma+\zeta,\zeta^*,t)d\sigma\}$$
$$= \frac{1}{t-\tau}\exp\{\int_\zeta^z B(\sigma,\zeta^*,t)d\sigma\} \tag{3.1.31}$$
$$V(\zeta, z^*,t;\zeta,\zeta^*,\tau) = \frac{1}{t-\tau}\exp\{\int_{\zeta^*}^{z^*} A(z,\sigma,t)d\sigma\},$$

i.e. $V(z,z^*,t;\zeta,\zeta^*,\tau)$ is in fact the Riemann function $R(z,z^*,t;\zeta,\zeta^*,\tau)$. Note that except for an essential singularity at $t=\tau$ the Riemann function is an entire function of its six independent complex variables.

$$Q^{(2)} = -2(t-t_1)\tilde{C} - 2\tilde{D}$$

$$(2n+1)Q^{(2n+2)} = -2\Big[(t-t_1)Q_z^{(2n)} + (t-t_1)\tilde{B}\, Q^{(2n)} + (t-t_1)\tilde{C}\int_{\zeta^*}^{z^*} Q^{(2n)}d\sigma$$
$$+ (n+1)\tilde{D}\int_{\zeta^*}^{z^*} Q^{(2n)}d\sigma - (t-t_1)\tilde{D}\int_{\zeta^*}^{z^*} Q_t^{(2n)}d\sigma\Big] \,. \tag{3.1.25}$$

By slightly modifying our previous analysis for the case of the operator $\underset{\sim}{P}_2$ (c.f. section 1.1) it can be seen that the operator $\underset{\sim}{P}_2^*$ exists and maps analytic functions of two complex variables defined in some neighbourhood of the point $(0,t_o)$ into the class of analytic solutions of (3.1.1) defined in some neighbourhood of the point (ζ,ζ^*,t_o). It is also easy to see that $E^*(z,z^*,t,t_1,s) = E^*(z,z^*,t;\ \zeta,\zeta^*,t_1,s)$ is an entire function of its seven independent complex variables except for an essential singularity at $t=t_1$.

We make the observation that if, as a function of t, $f(z,t)$ has an isolated singularity at $t=\tau$ for a given $\tau \in \mathbb{C}^1$ then $U(z,z^*,t) = \underset{\sim}{P}_2^*\{f\}$ also has an isolated singularity at $t=\tau$.

We will now use the integral operator $\underset{\sim}{P}_2^*$ associated with the adjoint equation to (3.1.13) to construct the Riemann function for (3.1.1). The Riemann function $R(z,z^*,t;\zeta,\zeta^*,\tau)$ for (3.1.1) is defined to be the (unique) solution of the adjoint equation

$$M[V] = V_{zz^*} - \frac{\partial(AV)}{\partial z} - \frac{\partial(BV)}{\partial z^*} + CV + \frac{\partial}{\partial t}(DV) = 0 \tag{3.1.26}$$

satisfying the initial data

$$R(z,\zeta^*,t;\zeta,\zeta^*,\tau) = \frac{1}{t-\tau}\exp\left\{\int_{\zeta}^{z} B(\sigma,\zeta^*,t)d\sigma\right\}$$
$$R(\zeta,z^*,t;\zeta,\zeta^*,\tau) = \frac{1}{t-\tau}\exp\left\{\int_{\zeta^*}^{z^*} A(\zeta,\sigma,t)d\sigma\right\} \tag{3.1.27}$$

(c.f. [17], [37]).

independent of t, and taking the real part of (3.1.16) and integrating termwise yields the representation

$$U(z,\bar{z}) = \mathrm{Re}\Big[\exp\{-\int_0^{\bar{z}} A(z,\sigma)d\sigma\}. \tag{3.1.20}$$

$$\cdot\int_{-1}^{1} E(z,\bar{z},s)f(\tfrac{z}{2}(1-s^2))\,\frac{ds}{\sqrt{1-s^2}}\Big]$$

where

$$E(z,z^*,s) = 1+\sum_{n=1}^{\infty} s^{2n}z^n \int_0^{z^*} P^{(2n)}(z,\zeta^*)d\zeta^* \tag{3.1.21}$$

with the $P^{(2n)}$ being defined recursively by

$$P^{(2)} = -2\tilde{C} \tag{3.1.22}$$

$$(2n+1)P^{(2n+2)} = -2\Big[P_z^{(2n)} + \tilde{B}P^{(2n)} + \tilde{C}\int_0^{z^*} P^{(2n)}d\zeta^*\Big] ,$$

A comparison of (3.1.20)-(3.1.22) with (1.3.21), (1.2.13) and (1.3.14) shows that the operator defined by (3.1.20) is identical with the Bergman operator $\mathrm{Re}\,\underset{\sim}{B}_2\{f\}$.

In addition to the operator $\underset{\sim}{P}_2$ we will also need to make use of a generalized form of this operator which we will denote by $\underset{\sim}{P}_2^*$ and is defined by

$$U(z,z^*,t) = \underset{\sim}{P}_2^*\{f\} = -\frac{1}{2\pi i}\exp\{-\int_{\zeta^*}^{z^*} A(z,\sigma,t)d\sigma\}. \tag{3.1.23}$$

$$\cdot\oint_{|t-t_1|=\delta}\int_{-1}^{1} E^*(z,z^*,t,t_1,s)f(\frac{(z-\zeta)}{2}(1-s^2),t_1)\,\frac{dsdt_1}{\sqrt{1-s^2}}$$

where $\delta > 0$, $(\zeta,\zeta^*) \in \mathbb{C}^2$, $f(z,t_1)$ is an analytic function of two complex variables in some neighbourhood of the point $(0,t)$, and

$$E^*(z,z^*,t,t_1,s) = \frac{1}{t-t_1} + \sum_{n=1}^{\infty}\frac{s^{2n}(z-\zeta)^n}{(t-t_1)^{n+1}}\int_{\zeta^*}^{z^*} Q^{(2n)}(z,\sigma,t,t_1)d\sigma \tag{3.1.24}$$

with

$$= \frac{1}{2}\int_{-1}^{1} f(\tfrac{z}{2}(1-s^2),t)\frac{ds}{\sqrt{1-s^2}} + \frac{\pi}{2}\bar{f}(0,t)\exp(-\int_0^z \bar{A}(0,\sigma,t)d\sigma)$$

where $\bar{f}(z,t) = \overline{f(\bar{z},\bar{t})}$ and $\bar{A}(z,z^*t) = \overline{A(\bar{z},\bar{z}^*,\bar{t})}$. A solution of the integral equation (3.1.17) is given by (c.f. section 1.2)

$$f(\tfrac{z}{2},t) = -\frac{1}{2\pi i}\int_\gamma \Big[2U(z(1-s^2),0,t) - U(0,0,t)\exp(-\int_0^z \bar{A}(0,\sigma,t)d\sigma)\Big]\frac{ds}{s^2} \tag{3.1.18}$$

where γ is a rectifiable arc joining the points $s=-1$ and $s=+1$ and not passing through the origin. (3.1.17) and (3.1.18) show that if $U(z,\bar{z},t) = u(x,y,t)$ is real valued for x,y and t real, then $f(z,t)$ can be chosen such that $U(z,0,t)$ assumes prescribed (analytic) values. We summarize our results in the following theorem:

<u>Theorem 3.1.1</u> ([16], [17]): Let $u(x,y,t)$ be a real valued analytic solution of (3.1.1) defined in some neighbourhood of the point $(0,0,t_o)$. Then $u(x,y,t) = U(z,\bar{z},t)$ can be represented in the form $u(x,y,t) = \mathrm{Re}\,\underset{\sim}{P}_2\{f\}$ where $f(z,t)$ is an analytic function of z and t in some neighbourhood of the point $(0,t_o)$. Conversely, for every analytic function $f(z,t)$ defined in some neighbourhood of the point $(0,t_o)$, $u(x,y,t) = \mathrm{Re}\,\underset{\sim}{P}_2\{f\}$ defines a real valued analytic solution of (3.1.1) in some neighbourhood of the point $(0,0,t_o)$.

The operator $\underset{\sim}{P}_2$ is in fact closely related to the Bergman operator $\underset{\sim}{B}_2$ for elliptic equations in two independent variables constructed in section 1.2. To see this we consider the case in which the coefficients and solution of (3.1.1) are independent of t and hence $u(x,y,t) = u(x,y)$ satisfies the elliptic equation

$$u_{xx}+u_{yy}+a(x,y)u_x+b(x,y)u_y+c(x,y)u = 0\,. \tag{3.1.19}$$

In this situation the associated analytic function $f(z,t) = f(z)$ is of z,z^*, s and $t-\tau$, i.e. $E(z,z^*,t,\tau,s) = E(z,z^*,t-\tau,s)$. In particular for the special case of the heat equation

$$u_{xx} + u_{yy} = u_t \tag{3.1.14}$$

we have

$$E(z,z^*,t,\tau,s) = \frac{\pi}{t-\tau} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+\frac{1}{2})} \left(\frac{r^2 s^2}{t-\tau} \right)^n \tag{3.1.15}$$

where $r^2 = zz^* = x^2+y^2$.

We have now shown that the operator $\underset{\sim}{P}_2$ defined by

$$U(z,z^*,t) = \underset{\sim}{P}_2\{f\} = -\frac{1}{2\pi i} \exp\left\{ -\int_0^{z^*} A(z,\sigma,t)d\sigma \right\}.$$

$$\cdot \oint_{|t-\tau|=\delta} \int_{-1}^{1} E(z,z^*,t,\tau,s) f\left(\frac{z}{2}(1-s^2),\tau \right) \frac{dsd\tau}{\sqrt{1-s^2}} \tag{3.1.16}$$

exists and maps analytic functions of two complex variables into the class of (complex valued) solutions of (3.1.3). An elementary power series analysis shows that solutions of (3.1.3) which are real valued for t real and $z^*=\bar{z}$ (i.e. x and y real) are uniquely determined by their values on the characteristic plane $z^*=0$. Furthermore, since the coefficients of (3.1.1) are real valued for x,y and t real, the operator $\mathrm{Re}\underset{\sim}{P}_2\{f\}$ (where "Re" denotes "take the real part") defines a real valued solution of (3.1.1) provided we set $z^*=\bar{z}$ and keep t real. Evaulating $\mathrm{Re}\ \underset{\sim}{P}_2\{f\}$ at $z^*=0$ and keeping t real gives

$$U(z,0,t) = \mathrm{Re}\ \underset{\sim}{P}_2\{f\}\Big|_{z^*=0} \tag{3.1.17}$$

$$= -\frac{1}{4\pi i} \oint_{|t-\tau|=\delta} \int_{-1}^{1} \left[f\left(\frac{z}{2}(1-s^2),\tau\right) + \bar{f}(0,\tau)\exp\left(-\int_0^z \bar{A}(0,\sigma,t)d\sigma\right) \right] \cdot$$

$$\cdot \frac{dsd\tau}{(t-\tau)\sqrt{1-s^2}}$$

$$Q^{(2n)} << \frac{M_n 2^n t_o^n (1+\varepsilon)^n}{2n-1} (1-\frac{z}{r})^{-(2n-1)} (1-\frac{z^*}{r})^{-(2n-1)} (1-\frac{t}{t_o})^{-(3n-1)} r^{-n} \tag{3.1.10}$$

where

$$M_1 = \frac{rB_o(1+t_o)}{t_o(1+\varepsilon)} \tag{3.1.11}$$

$$M_{n+1} = M_n(1+\varepsilon)^{-1} \{1+ \frac{B_o r}{(2n-1)^2} (2n-1+r+\frac{4nr}{t_o})\} .$$

Note that for n sufficiently large we have $M_{n+1} \leqslant M_n$, i.e. there exists a positive constant M which is independent of n such that $M_n \leqslant M$ for all n. Now let $\delta_o \geqslant 1$ and $\alpha > 1$ be positive constants such that

$$\begin{array}{ll} |s| \leqslant \delta_o & |z| < \frac{r}{\alpha} \\ |\tau| \leqslant t_o & |z^*| < \frac{r}{\alpha} \\ |t| < \frac{t_o}{2} & \delta_o \leqslant |t-\tau| \end{array} \tag{3.1.12}$$

where r and t_o arbitrarily large (but fixed) positive numbers and δ_o is arbitrarily small (but again fixed). Then from (3.1.10) it is seen that the series (3.1.6) is majorized by the series

$$\frac{1}{\delta_o} + \sum_{n=1}^{\infty} \frac{r M_n 2^{4n-1} s_o^{2n} t_o^n (1+\varepsilon)^n \alpha^{3n-3}}{\delta_o^{n+1}(2n-1)(\alpha-1)^{4n-2}} . \tag{3.1.13}$$

If α is chosen such that $16 s_o^2 t_o(1+\varepsilon)\alpha^3\delta_o^{-1}(\alpha-1)^{-4} < 1$ then the series (3.1.13) is convergent. Since r, t_o and s_o can be arbitrarily large, δ_o arbitrarily small, and ε is independent of r, t_o, s_o and δ_o, we can now conclude that the series (3.1.6) converges absolutely and uniformly on compact subsets of $\{(z,z^*,t,\tau,s):(z,z^*,t,\tau,s)\in \mathbb{C}^5, t\neq\tau\}$, i.e. $E(z,z^*,t,\tau,s)$ exists and is an entire function of its independent complex variables except for an (essential) singularity at $t=\tau$. Note that if the coefficients $a(x,y,t)$, $b(x,y,t), c(x,y,t)$ and $d(x,y,t)$ are independent of t, then $E(z,z^*,t,\tau,s)$ is a function only

Substituting (3.1.6) into (3.1.5) yields the following recursion formula for the $Q^{(2n)}$:

$$Q^{(2)} = -2(t-\tau)\tilde{C} - 2\tilde{D} \tag{3.1.7}$$

$$(2n+1)Q^{(2n+2)} = -2\Big[(t-\tau)Q_z^{(2n)} + (t-\tau)\tilde{B}\,Q^{(2n)} + (t-\tau)\tilde{C}\int_o^{z^*} Q^{(2n)}d\sigma$$

$$+ (n+1)\tilde{D}\int_o^{z^*} Q^{(2n)}d\sigma - (t-\tau)\tilde{D}\int_o^{z^*} Q_t^{(2n)}d\sigma\Big] ,$$

$n=1,2,\ldots$.

It is clear from (3.1.7)that each of the $Q^{(2n)}$, $n=1,2,\ldots$, is uniquely determined. In order to show the existence of $E(z,z^*,t,\tau,s)$ it is now necessary to show the convergence of the series (3.1.6) and it is to this end that we first majorize the functions $Q^{(2n)}$. Let r and t_o be arbitrarily large positive numbers and let B_o be a positive constant such that for $|z| < r$, $|z^*| < r$, $|t| < t_o$, we have

$$\begin{aligned}
\tilde{B}(z,z^*,t) &<< \frac{B_o}{(1-\frac{z}{r})(1-\frac{z^*}{r})(1-\frac{t}{t_o})}\\
\tilde{C}(z,z^*,t) &<< \frac{B_o}{(1-\frac{z}{r})(1-\frac{z^*}{r})(1-\frac{t}{t_o})}\\
\tilde{D}(z,z^*,t) &<< \frac{B_o}{(1-\frac{z}{r})(1-\frac{z^*}{r})(1-\frac{t}{t_o})}
\end{aligned} \tag{3.1.8}$$

where "<<" denotes domination. We also have thefact that for $|\tau| \leq t_o$, $|t| < t_o$,

$$t-\tau << t_o(1-\frac{t}{t_o})^{-1} \quad . \tag{3.1.9}$$

In a straightforward manner which is by now familiar, it can be shown by induction that for any $\varepsilon > 0$ and $|z| < r$, $|z^*| < r$, $|t| < t_o$, $|\tau| \leq t_o$ we have (with respect to the variables z,z^*,t)

$$L[U] \equiv U_{zz^*} + A(z,z^*,t)U_z + B(z,z^*,t)U_{z^*} + C(z,z^*,t)U - D(z,z^*,t)U_t = 0 \tag{3.1.3}$$

where $A = \frac{1}{4}(a+ib)$, $B = \frac{1}{4}(a-ib)$, $C = \frac{1}{4}c$ and $D = \frac{1}{4}d$. Note that the change of variables (3.1.2) is permissible since we are considering analytic solutions of (3.1.1). Note also, however, that in general classical solutions of (3.1) are not analytic, and hence a problem we will eventually have to face is how to apply the results we are about to obtain on analytic solutions of (3.1.1) to the problem of approximating classical solutions of (3.1.1). We now look for solutions of (3.1.3) in the form

$$U(z,z^*,t) = -\frac{1}{2\pi i}\exp\left\{-\int_0^{z^*} A(z,\sigma,t)\,d\sigma\right\}. \tag{3.1.4}$$

$$\cdot \oint_{|t-\tau|=\delta} \int_{-1}^{1} E(z,z^*,t,\tau,s) f\left(\frac{z}{2}(1-s^2),\tau\right) \frac{ds\,d\tau}{\sqrt{1-s^2}}$$

where $\delta > 0$, $f(z,\tau)$ is an analytic function of two complex variables in a neighbourhood of the point $(0,t)$ and $E(z,z^*,t,\tau,s)$ is an (analytic) function to be determined. The first integral in (3.1.4) is an integration in the complex τ plane in a counterclockwise direction about a circle of radius δ with centre at t, and the second integral is an integration over a curvilinear path in the unit disc in the complex s plane joining the points $s=+1$ and $s=-1$. Substituting (3.1.4) into (3.1.3) and integrating by parts shows that $E(z,z^*,t,\tau,s)$ must satisfy the differential equation

$$(1-s^2)E_{z^*s} - \frac{1}{s}E_{z^*} + 2sz(E_{zz^*} + \tilde{B}E_{z^*} + \tilde{C}E - \tilde{D}E_t) = 0 \tag{3.1.5}$$

where $\tilde{B} = B - \int_0^{z^*} A_z\,d\sigma$, $\tilde{C} = -(A_z+AB-C)$, $\tilde{D} = D$. We now look for a solution of (3.1.5) in the form

$$E(z,z^*,t,\tau,s) = \frac{1}{t-\tau} + \sum_{n=1}^{\infty} \frac{s^{2n} z^n}{(t-\tau)^{n+1}} \int_0^{z^*} Q^{(2n)}(z,\sigma,t,\tau)\,d\sigma . \tag{3.1.6}$$

III Parabolic equations in two space variables

3.1 Integral Operators and the Riemann Function.

We now want to obtain results for parabolic equations in two space variables which are analogous to the theory previously developed for elliptic equations in two independent variables and parabolic equations in one space variable. In the present case new problems are presented since the domains of the integral operators we are about to construct lie in the space of analytic functions of several complex variables as opposed to analytic functions of one complex variable as in the previous chapters. Nevertheless considerable progress can be made in using analytic function theory to develop constructive methods for solving initial-boundary value problems for parabolic equations in two space variables. In this section we begin the development of this theory by constructing integral operators which map analytic functions of two complex variables onto analytic solutions of the general linear second order parabolic equation written in normal form as

$$u_{xx}+u_{yy}+a(x,y,t)u_x+b(x,y,t)u_y+c(x,y,t)u = d(x,y,t)u_t \qquad (3.1.1)$$

where the coefficients of (3.1.1) are entire functions of their independent complex variables which are real valued for x,y and t real. At this point no assumptions are made on the positivity of d(x,y,t). Since the analysis of this section is similar to that of sections 1.1 and 1.2, we will make our presentation somewhat briefer than in previous sections.

The change of variables in $\mathbb{C}^2$

$$\begin{aligned} z &= x+iy \\ z^* &= x-iy \end{aligned} \qquad (3.1.2)$$

transforms (3.1.1) into the form

$$E^{(2)}(x,t,\tau) = \frac{x}{t-\tau} + \sum_{j=1}^{\infty} \frac{x^{2j+1}(-1)^j j!}{(2j+1)!(t-\tau)^{j+1}} \quad . \tag{2.3.37}$$

Note that if $s(t)$ is analytic for $0 \leqslant t \leqslant t_o$ then $h(x,t)$ is analytic for $-\infty < x < \infty$, $0 \leqslant t \leqslant t_o$, in particular the temperature $\phi(x)=h(0,t)$ can be obtained by simply evaluating (2.3.36) at x=0. Computing the residue in (2.3.36) leads to the following solution of (2.3.34),(2.3.35):

$$h(x,t) = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \frac{\partial^n}{\partial t^n} \left[x-s(t)\right]^{2n} \quad . \tag{2.3.38}$$

For further discussion of this problem see [36].

then

$$\max_{(x,t)\varepsilon D_o} \left|u(x,t) - \sum_{n=0}^{N} c_n\phi_n(x,t)\right| < M\,\varepsilon \tag{2.3.32}$$

where $M=M(D_o)$ is a constant. Hence an approximate solution to (2.3.1), (2.3.2) on compact subsets of D is given by

$$u^N(x,t) = \sum_{n=0}^{N} c_n\phi_n(x,t). \tag{2.3.33}$$

Since each $\phi_n(x,t)$ is a solution of (2.3.1), error estimates can be found by finding the maximum of $\left|u(x,t) - \sum_{n=0}^{N} c_n\phi_n(x,t)\right|$ on the base and sides of D and applying the maximum principle. A numerical example using this approach can be found in the Appendix to these lectures.

To conclude this section we show how the operator $\underset{\sim}{P}_1$ of section 2.1 can be used to solve the inverse Stefan problem discussed in the Introduction. The (normalized) inverse Stefan problem is to find a solution of

$$h_{xx} = h_t \ ; \quad 0 \leq x < s(t), \ t > 0 \tag{2.3.34}$$

such that on the given analytic arc $x=s(t)$ we have

$$\begin{aligned} &h(s(t),t) = 0, \quad t > 0 \\ &h_x(s(t),t) = -\dot{s}(t), \quad t > 0 \ . \end{aligned} \tag{2.3.35}$$

In the representation (2.1.23) (with $u(x,t) = h(x,t)$)we place the cycle $|t-\tau| = \delta$ on the two dimensional manifold $x=s(t)$ in the space of two complex variables, and note that since $h(s(t),t)=0$ the integral in (2.1.23) which contains $E^{(1)}(x,t,\tau)$ vanishes. We are thus led to the following representation of the solution to the inverse Stefan problem ([14],[36]):

$$h(x,t) = \frac{1}{2\pi i}\oint_{|t-\tau|=\delta} E^{(2)}(x-s(\tau),t,\tau)\dot{s}(\tau)d\tau \ , \tag{2.3.36}$$

where

bounded by the characteristics $t=t_o$, $t=0$, and the analytic curves $x = 2x_1(t) - x_2(t)$, $x=x_2(t)$. Applying Theorem 2.2.3 a second time, but this time continuing $w(x,t)$ across the arc $x=x_2(t)$, shows that $w(x,t)$ can be continued into the region bounded by $t=t_o$, $t=0$, $x=2_1(t) - x_2(t)$, and $x=3x_2(t) - 2x_1(t)$. Due to the fact that $x_1(t) < x_2(t)$ for $0 < t < t_o$, it is seen that by repeating the above procedure we can continue $w(x,t)$ into the entire infinite strip $-\infty < x < \infty$, $0 \leq t \leq t_o$, as a solution of (2.2.1). In particular there exists a rectangle $\bar{R} \supset \bar{D}$ into which $w(x,t)$ can be continued, and we have this established the existence of the desired function $w(x,t)$.

A special case of Theorem 2.3.3 is the following Corollary:

Corollary 2.3.1 ([13]): Let $h(x,t)$ be a classical solution of (2.3.5) in D which is continuous in $\bar{D}$, where $x_1(t)$ and $x_2(t)$ are analytic for $|t| \leq t_o$. Then given $\varepsilon > 0$ there exist constants $a_o,\ldots,a_N$ such that

$$\max_{(x,t)\varepsilon\bar{D}} \left|h(x,t) - \sum_{n=0}^{N} a_n h_n(x,t)\right| < \varepsilon \,.$$

The complete family $\{u_n(x,t)\}$ can be used to approximate the solution of (2.3.1), (2.3.2) on compact subsets of D in the same manner as for elliptic equations. In particular we orthonormalize the set $\{u_n(x,t)\}$ in the L^2 norm over the base and sides of D to obtain the complete set $\{u_n\}$. Let

$$\begin{aligned} c_n = \int_0^{t_o} \psi_1(t)\phi_n(x_1(t),t)dt + \int_{x_1(0)}^{x_2(0)} \phi(x)\phi_n(x,0)dx \\ + \int_0^{t_o}\psi_2(t)\phi_n(x_2(t),t)dt \end{aligned} \tag{2.3.30}$$

and let D_o be a compact subset of D. From the representation of the solution of (2.3.1), (2.3.2) in terms of the Green's function it is seen that if

$$\int_{\partial D\backslash t=t_o} |u- \sum_{n=0}^{N} c_n\phi_n|^2 < \varepsilon \tag{2.3.31}$$

$$u_{2n}(x,t) = \underset{\sim}{T}_2\{h_{2n}\}$$
$$u_{2n+1}(x,t) = \underset{\sim}{T}_1\{h_{2n+1}\} \quad . \tag{2.3.28}$$

Theorem 2.3.2 shows that the set $\{u_n(x,t)\}$ defined by $u_n(x,t) = \underset{\sim}{T}_3\{h_n\}$ is a complete family of solutions for (2.3.1) in a rectangle. We now want to use the reflection principles of section 2.2 to show that the set $\{u_n(x,t)\}$ is complete in $\bar{D} = \{(x,t) : x_1(t) \leqslant x \leqslant x_2(t),\ 0 \leqslant t \leqslant t_o\}$ under the assumption that $x_1(t)$ and $x_2(t)$ are analytic.

<u>Theorem 2.3.3</u> ([13]): Let $u(x,t)$ be a classical solution of (2.2.1) in D which is continuous in $\bar{D}$, where $x_1(t)$ and $x_2(t)$ are analytic for $|t| \leqslant t_o$. Then given $\varepsilon > 0$ there exist constants $a_o,\ldots,a_N$ such that

$$\max_{(x,t)\varepsilon\bar{D}} \left|u(x,t) - \sum_{n=0}^{N} a_n u_n(x,t)\right| < \varepsilon \ .$$

<u>Proof</u>: From Theorem 2.3.2 the theorem will be proved if for a given $\varepsilon > 0$ we can construct a solution $w(x,t)$ of (2.2.1) defined in a rectangle $R = \{(x,t): -x_o < x < x_o,\ 0 < t < t_o\}$ such that $w(x,t)$ is continuous in $\bar{R}$, $\bar{D} \subset \bar{R}$, and

$$\max_{(x,t)\varepsilon\bar{D}} |u(x,t) - w(x,t)| < \varepsilon \ . \tag{2.3.29}$$

From the existence of a solution to the first initial-boundary value problem for (2.2.1) (c.f. the first part of this section), the maximum principle for parabolic equations, and the Weierstrass approximation theorem, it is seen that there exists a solution $w(x,t)$ of (2.2.1) in D satisfying analytic boundary data on $x = x_1(t)$, $x = x_2(t)$, and $t = 0$ such that (2.3.29) is valid. From Theorem 2.2.3 (after making the change of variables (2.2.3)) and the regularity theorems for solutions to initial-boundary value problems for parabolic equations (c.f. [26]) we can conclude that $w(x,t)$ can be uniquely continued as a solution of (2.2.1) across the arc $x=x_1(t)$ into the region

$$E^{(1)}(x,t,\tau) = \frac{1}{t-\tau} + \sum_{j=1}^{\infty} \frac{x^{2j}(-1)^j j!}{(2j)!(t-\tau)^{j+1}} \tag{2.3.25}$$

$$E^{(2)}(x,t,\tau) = \frac{x}{t-\tau} + \sum_{j=1}^{\infty} \frac{x^{2j+1}(-1)^j j!}{(2j+1)!(t-\tau)^{j+1}} \quad .$$

By truncating the Taylor series for $w_3(0,\tau)$ and $w_{3x}(0,t)$, (2.3.4), (2.3.25) show that there exists an entire solution $w_4(x,t)$ of (2.3.5) satisfying polynomial Cauchy data on x=0 such that

$$\max_{(x,t)\in\bar{R}} |w_3(x,t) - w_4(x,t)| < \frac{\varepsilon}{3} \, . \tag{2.3.26}$$

But from (2.3.17) and Holmgren's uniqueness theorem it is seen that there exist positive constants $a_o,\ldots,a_N$ such that

$$w_4(x,t) = \sum_{n=0}^{N} a_n h_n(x,t) \quad , \tag{2.3.27}$$

and the proof of the theorem follows from (2.3.27), (2.3.26), (2.3.23) and the triangle inequality.

From Theorem 2.3.1, Theorem 2.1.2, and the continuity of the kernel of the operator $\underset{\sim}{T}_3$, we can now immediately arrive at the following theorem:

<u>Theorem 2.3.2</u> ([12]): Let u(x,t) be a classical solution of (2.2.1) in R which is continuous in $\bar{R}$. Then given $\varepsilon > 0$ there exists constants $a_o,\ldots a_N$ such that

$$\max_{(x,t)\in\bar{R}} \left|u(x,t) - \sum_{n=0}^{N} a_n u_n(x,t)\right| < \varepsilon$$

where $u_n(x,t) = \underset{\sim}{T}_3\{h_n\}$.

<u>Remark</u>: Observing that $h_{2n}(x,t)$ is an even function of x and $h_{2n+1}(x,t)$ is an odd function of x, it is seen that we can represent the special solutions $u_n(x,t)$ in terms of the operators $\underset{\sim}{T}_1$ and $\underset{\sim}{T}_2$ by

where $v(-x_o,t) = w_1(-x_o,t)$, $v(x_o,t) = w_1(x_o,t)$. Substituting (2.3.20) into (2.3.5) leads to the following recursion scheme for the $v_m(x)$:

$$\frac{d^2 v_M}{dx^2} = 0$$

$$v_M(-x_o) = b_M \quad , \qquad v_M(x_o) = c_M \, , \tag{2.3.21}$$

$$\frac{d^2 v_{m-1}}{dx^2} = m v_m$$

$$v_{m-1}(-x_o) = b_{m-1}, \qquad v_{m-1}(x_o) = c_{m-1} \, ,$$

$m=1,2,\ldots M$. Equation (2.3.21) shows that each $v_m(x)$ is a polynomial in x and is uniquely determined. Now consider $w_2(x,t)=w_1(x,t)-v(x,t)$. By the method of separation of variables it is seen that there exist constants $d_0,\ldots d_L$ such that

$$\max_{(x,t)\in\bar{R}} \left| w_2(x,t) - \sum_{\ell=0}^{L} d_\ell \sin \frac{\ell\pi}{2x_o} (x+x_o) \exp\left(-\frac{\ell^2\pi^2 t}{4x_o^{\,2}}\right) \right| < \frac{\varepsilon}{3} \, .$$

Hence there exists a solution $w_3(x,t)$ of (2.3.5) which is an entire function of the complex variables x and t such that

$$\max_{(x,t)\in\bar{R}} |h(x,t) - w_3(x,t)| < \frac{2\varepsilon}{3} \, . \tag{2.3.23}$$

From Theorem 2.1.1 (see also Theorem 2.2.2) we can represent $w_3(x,t)$ in the form

$$w_3(x,t) = -\frac{1}{2\pi i} \oint_{|t-\tau|=\delta} E^{(1)}(x,t,\tau) w_3(0,\tau)\,d\tau - \frac{1}{2\pi i} \oint_{|t-\tau|=\delta} E^{(2)}(x,t,\tau) w_{3x}(0,\tau)\,d\tau \tag{2.3.24}$$

where

weakened.

We first consider the set $\{h_n(x,t)\}$ of polynomial solutions to the heat equation (2.3.5) defined by

$$h_n(x,t) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2k}\, t^k}{(n-2k)!\,k!} \qquad (2.3.17)$$

$$= (-t)^{n/2}\, H_n\left(\frac{x}{(-4t)^{\frac{1}{2}}} \right)$$

where H_n denotes the Hermite polynomial of degree n(c.f.[45]).

Let x_o be a positive constant, $R = \{(x,t) : x_o < x < x_o,\ \ 0 < t < t_o\}$ and $\bar{R}$ denote the closure of R.

<u>Theorem 2.3.1</u> ([12]): Let h(x,t) be a classical solution of (2.3.5) in R which is continuous in $\bar{R}$. Then given $\varepsilon > 0$ there exist constants $a_o, \ldots, a_N$ such that

$$\max_{(x,t)\varepsilon\bar{R}} \left| h(x,t) - \sum_{n=0}^{N} a_n h_n(x,t) \right| < \varepsilon .$$

<u>Proof</u>: By the Weierstrass approximation theorem and the maximum principle for the heat equation there exists a solution $w_1(x,t)$ of (2.3.5) in R which assumes polynomial initial and boundary data such that

$$\max_{(x,t)\varepsilon\bar{R}} \left| h(x,t) - w_1(x,t) \right| < \frac{\varepsilon}{3} . \qquad (2.3.18)$$

Let

$$w_1(-x_o,t) = \sum_{m=0}^{M} b_m t^m \qquad (2.3.19)$$

$$w_1(x_o,t) = \sum_{m=0}^{M} c_m t^m$$

and look for a solution of (2.3.5) in the form

$$v(x,t) = \sum_{m=0}^{M} v_m(x)\, t^m \qquad (2.3.20)$$

$$\left| \int^{\infty} K^{(1)}(s,x,t)\, \frac{\partial G_o}{\partial \xi}(s,t,x_1(\tau),\tau)ds \right|$$

$$\leq \text{constant} \int_x^{\infty} \frac{\partial G_o}{\partial \xi}(s,t,x_1(\tau),\tau)ds \tag{2.3.16}$$

$$= \text{constant } G_o(x,t,x_1(\tau),\tau)$$

$$\frac{\text{constant}}{\sqrt{t-\tau}} \quad ,$$

which implies for example that the first integral in (2.3.15) is uniformly convergent with respect to x.

The system (2.3.13) is of Volterra type of the second kind with continuous right hand side and hence always has a unique set of continuous solutions $\mu_1(t)$ and $\mu_2(t)$. The solution of the reduced initial-boundary value problem for (2.3.1) is now given by equations (2.3.10) and (2.3.12).

We now turn our attention to the problem of constructing an approximate solution to the initial-boundary value problem (2.3.1), (2.3.2). One method, which is immediate from the above analysis, is to construct an approximate set of solutions to the system of Volterra equations (2.3.13) and then substitute this set of approximate densities $\mu_1(t)$, $\mu_2(t)$ into (2.3.10) and (2.3.12). We will now present an alternate method for obtaining an approximate solution to (2.3.1), (2.3.2) based on the use of a complete family of solutions in a manner analogous to the approach used for elliptic equations in Chapter One. The construction of a complete family of solutions for (2.3.1) is accomplished through the use of the operator $\underset{\sim}{T}_3$ obtained in section 2.1 and the application of the reflection principles obtained in section 2.2. In the rest of this section we will assume that the arcs $x_1(t)$ and $x_2(t)$ are analytic for $0 \leq t \leq t_o$, although through the use of suitable approximation arguements this assumption can be considerably

$$\mu_1(t) + \int_0^t G^{(1)}(x_1(t),t,x_1(\tau),\tau)\mu_1(\tau)d\tau$$

$$+ \int_0^t G^{(2)}(x_1(t),t,x_2(\tau),\tau)\mu_2(\tau)d\tau = \psi_1(t) \tag{2.3.13}$$

$$-\mu_2(t) + \int_0^t G^{(1)}(x_2(t),t,x_1(\tau),\tau)\mu_1(\tau)d\tau$$

$$+ \int_0^t G^{(2)}(x_2(t),t,x_2(\tau),\tau)\mu_2(\tau)d\tau = \psi_2(t)$$

where

$$G^{(1)}(x,t,\xi,\tau) = \frac{\partial G_o}{\partial \xi}(x,t,\xi,\tau)$$

$$+ \int_x^\infty K^{(1)}(s,x,t)\frac{\partial G_o}{\partial \xi}(s,t,\xi,\tau)ds \tag{2.3.14}$$

$$G^{(2)}(x,t,\xi,\tau) = \frac{\partial G_o}{\partial \xi}(x,t,\xi,\tau)$$

$$+ \int_{-\infty}^x K^{(2)}(s,x,t)\frac{\partial G_o}{\partial \xi}(s,t,\xi,\tau)ds.$$

In the derivation of (2.3.13) we have made use of the discontinuity properties of heat potentials of the second kind (c.f. [39]) and the fact that the integrals

$$\int_0^t \int_x^\infty K^{(1)}(s,x,t)\frac{\partial G_o}{\partial \xi}(s,t, x_1(\tau),\tau)\mu_1(\tau)dsd\tau \tag{2.3.15}$$

$$\int_0^t \int_{-\infty}^x K^{(2)}(s,x,t)\frac{\partial G_o}{\partial \xi}(s,t, x_2(\tau),\tau)\mu_2(\tau)dsd\tau$$

are continuous as x tends to $x_1(t)$ or $x_2(t)$. This last statement follows from estimates of the form it is seen that (2.3.4) satisfies the required conditions for $v(x,t)$. Hence without loss of generality we now consider the initial-boundary value problem (2.3.1),(2.3.2) with $\phi(x)=0$. We will call this problem the reduced initial-boundary value problem for (2.3.1).

We look for a solution of the reduced problem in the form

$$
\begin{aligned}
u(x,t) &= \underset{\sim}{A}_1\{h^{(1)}\} + \underset{\sim}{A}_2\{h^{(2)}\} \\
&= h^{(1)}(x,t) + h^{(2)}(x,t) + \int_x^\infty K^{(1)}(s,x,t)h^{(1)}(s,t)ds \qquad (2.3.10) \\
&\quad + \int_{-\infty}^x K^{(2)}(s,x,t)h^{(2)}(s,t)ds
\end{aligned}
$$

where $h^{(1)}(x,0)=h^{(2)}(x,0)=0$, $h^{(1)}(x,t)$ is a solution of (2.3.5) for $x > x_1(t)$, $0 < t < t_o$, and $h^{(2)}(x,t)$ is a solution of (2.3.5) for $x < x_2(t)$, $0 < t < t_o$. Let

$$
G_o(x,t,\xi,\tau) = \frac{1}{\sqrt{4\pi(t-\tau)}} \exp\left[-\frac{(x-\xi)^2}{4(t-\tau)}\right], \qquad (2.3.11)
$$

and represent $h^{(1)}(x,t)$ and $h^{(2)}(x,t)$ as heat potentials of the second kind (c.f. [39])

$$
\begin{aligned}
h^{(1)}(x,t) &= \int_o^t \frac{\partial G_o}{\partial \xi}(x,t,x_1(\tau),\tau)\mu_1(\tau)d\tau \\
h^{(2)}(x,t) &= \int_o^t \frac{\partial G_o}{\partial \xi}(x,t,x_2(\tau),\tau)\mu_2(\tau)d\tau ,
\end{aligned} \qquad (2.3.12)
$$

where $\mu_1(\tau)$ and $\mu_2(\tau)$ are continuous densities to be determined. Substituting (2.3.12) into (2.3.10), interchanging orders of integration, and letting x tend to $x_1(t)$ and x tend to $x_2(t)$ respectively, leads to the following system of Volterra integral equations for $\mu_1(t)$ and $\mu_2(t)$:

We also assume without loss of generality that $\bar{D} \subset \bar{R}$ where

$\bar{R} = \{(x,t): -a \leqslant x \leqslant a, \quad 0 \leqslant t \leqslant t_o\}$.

We first reduce the initial-boundary value problem (2.3.1), (2.3.2) to a problem of the same form but with $\phi(x)=0$. To do this it suffices to construct a particular solution $v(x,t)$ of (2.3.1) such that $v(x,t)$ is a classical solution of (2.3.1) for $-\infty < x < \infty$, $0 < t < t_o$, continuous for $-\infty < x < \infty$, $0 \leqslant t \leqslant t_o$, and satisfies

$$v(x,0) = \phi(x); \quad x_1(0) \leqslant x \leqslant x_2(0) \ , \tag{2.3.3}$$

since in this case the reduced problem can be obtained by considering $u(x,t)-v(x,t)$. We look for $v(x,t)$ in the form

$$v(x,t) = \underset{\sim}{A}_1\{h\} = h(x,t) + \int_x^\infty K^{(1)}(s,x,t)h(s,t)ds \tag{2.3.4}$$

where $h(x,t)$ is a solution of

$$h_{xx} = h_t \quad . \tag{2.3.5}$$

To this end we continue $\phi(x)$ in an arbitrary but continuous manner such that $\phi(a)=0$ and define

$$h(x,0) = 0 \ ; \quad x \geqslant a \quad . \tag{2.3.6}$$

Then for $x_1(0) \leqslant x \leqslant a$ let $h(x,0)$ be the unique solution of the Volterra integral equation

$$\phi(x) = h(x,0) + \int_x^a K^{(1)}(s,x,0)h(s,0)ds. \tag{2.3.7}$$

Note that from (2.3.7) we have $h(a,0)=0$ which agrees with (2.3.6). Now for $x \leqslant x_1(0)$ continue $h(x,0)$ in an arbitrary but continuous manner such that

$$h(x,0) = 0 \ ; \quad x \leqslant -a \ . \tag{2.3.8}$$

If we now define $h(x,t)$ by means of the Poisson formula for the heat equation

$$h(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-a}^{a} \frac{1}{\sqrt{t}} \exp\left[-\frac{(\xi-x)^2}{4t}\right]h(\xi,0)d\xi \ , \tag{2.3.9}$$

defined by (2.2.16). Equations (2.2.15), (2.2.18) and the regulatiry of $K^{(1)}(s,x,t)$ and $K^{(2)}(s,x,t)$ for $-x_o < s < x_o$, $-x_o < x < x_o$, $|t| < t_o$, now imply that $u(x,t)$ can be continued into $R^+ \cup R^- \cup \sigma$ as a solution of $L(u)=0$. The uniqueness of the continuation follows from the uniqueness of the continuation of $h^{(1)}(x,t)$. The proof of Theorem 2.2.3 is now complete.

2.3 Initial-Boundary Value Problems.

We will now use the integral operators and reflection principles obtained in the last two sections to derive constructive methods for solving initial-boundary value problems for parabolic equations in one space variable defined in domains with time dependent boundaries. Without loss of generality we again consider equations written in the canonical form

$$L(u) \equiv u_{xx} - q(x,t)u - u_t = 0 \qquad (2.3.1)$$

and make the assumption that $q(x,t)$ is continuously differentiable for $-\infty < x < \infty$, $|t| \leqslant t_o$ (where t_o is a positive constant), is analytic with respect to t for $|t| \leqslant t_o$, and $q(x,t) \equiv 0$ for $|x| > a$.

Our first aim is to use the operators $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ of section 2.1 to construct a classical solution $u(x,t)$ of (2.3.1) in the domain $D = \{(x,t): x_1(t) < x < x_2(t),\ 0 < t < t_o\}$ such that $u(x,t)$ is continuous in $\bar{D} = \{(x,t): x_1(t) \leqslant x \leqslant x_2(t),\ 0 \leqslant t \leqslant t_o\}$ and satisfies the initial-boundary data

$$\begin{aligned} u(x_1(t),t) &= \psi_1(t) \quad ; \quad 0 \leqslant t \leqslant t_o \\ u(x_2(t),t) &= \psi_2(t) \quad ; \quad 0 \leqslant t \leqslant t_o \\ u(x,0) &= \psi(x) \quad ; \quad x_1(0) \leqslant x \leqslant x_2(0) \end{aligned} \qquad (2.3.2)$$

where $\psi_1(0)=\phi(x_1(0))$ $\psi_2(0) = \phi(x_2(0))$ ([15]). We will assume that $x_1(t)$ and $x_2(t)$ are continuously differentiable for $0 \leqslant t \leqslant t_o$ and that there exist constants α and β such that $x_1(t) \leqslant \alpha < \beta \leqslant x_2(t)$ for $0 \leqslant t \leqslant t_o$.

in R^+. Differentiating (2.2.9) with respect to x and integrating by parts gives

$$u(x,t) = h^{(1)}(x,t) + h^{(2)}(x,t) + \int_o^x K^{(1)}(s,x,t)h^{(1)}(s,t)ds \quad (2.2.15)$$

$$+ \int_o^x K^{(2)}(s,x,t)h^{(2)}(s,t)ds$$

where $h^{(2)}(x,t)$ is defined by

$$h^{(2)}(x,t) = -a(t)h_x^{(1)}(x,t) \quad . \quad (2.2.16)$$

The fact that u(x,t) satisfies (2.2.8) implies that

$h_x^{(2)}(0,t) = -a(t)h_{xx}^{(1)}(0,t) = 0$, i.e. $h_{xx}^{(1)}(0,t) = 0$ for $0 < t < t_o$ since $a(t) \neq 0$ in this interval. Applying the differential equation (2.2.1) to both sides of (2.2.15) and using equations (2.2.11)-(2.2.14) to integrate by parts gives

$$\begin{aligned} 0 = (h_{xx}^{(1)}(x,t)-h_t^{(1)}(x,t)) & -a(t)(h_{xxx}^{(1)}(x,t) -h_{xt}^{(1)}(x,t)) \\ & -a(t)K^{(2)}(x,x,t).(h_{xx}^{(1)}(x,t)-h_t^{(1)}(x,t)) \\ & + \int_o^x K^{(1)}(s,x,t)(h_{xx}^{(1)}(s,t)-h_t^{(1)}(s,t))ds \\ & + \int_o^x a(t)K_s^{(2)}(s,x,t)(h_{ss}^{(1)}(s,t)-h_t^{(1)}(s,t))ds. \end{aligned} \quad (2.2.17)$$

Integrating both sides of (2.2.17) with respect to x gives

$$0 = -a(t)(h_{xx}^{(1)}(x,t)-h_t^{(1)}(x,t))$$

$$+ \int_o^x \Gamma(s,x,t)(h_{ss}^{(1)}(s,t) - h_t^{(1)}(s,t))ds \quad (2.2.8)$$

where $\Gamma(s,x,t)$ is defined by (2.2.10). Since $a(t) \neq 0$ and solutions of nonsingular Volterra integral equations of the second kind are unique, we can conclude that $h^{(1)}(x,t)$ must be a classical solution of (2.2.5) in R^+. From theorem 2.2.1 we can conclude that $h^{(1)}(x,t)$ can be uniquely continued into $R^+ \cup R^- \cup \sigma$ as a solution of (2.2.5) and hence so can $h^{(2)}(x,t)$ as

The proof of the theorem under the hypothesis of condition 3) is a bit more involved. Let $a(t) = \frac{\beta(t)}{\alpha(t)}$ and again assume without loss of generality that $f(t) \equiv 0$. Then the boundary condition (2.2.4) becomes

$$u(0,t) + a(t)u_x(0,t) = 0 \quad . \tag{2.2.8}$$

Let $h^{(1)}(x,t)$ (for $x \geqslant 0$) be the unique solution of the Volterra integral equation

$$\int_0^x u(s,t)ds = - a(t)h^{(1)}(x,t) + \int_0^x \Gamma(s,x,t)h^{(1)}(s,t)ds \tag{2.2.9}$$

where

$$\begin{aligned}\Gamma(s,x,t) &= 1-a(t)K^{(2)}(s,s,t) \\ &+ \int_s^x K^{(1)}(s,\xi,t)d\xi + a(t) \int_s^x K_s^{(2)}(s,\xi,t)d\xi\end{aligned} \tag{2.2.10}$$

and $K^{(1)}(s,x,t), K^{(2)}(s,x,t)$ are the solutions of the initial value problems

$$K_{xx}^{(1)} - K_{ss}^{(1)} + q(x,t)K^{(1)} = K_t^{(1)} \tag{2.2.11}$$

$$K^{(1)}(x,x,t) = - \frac{1}{2} \int_0^x q(s,t)ds \tag{2.2.12a}$$

$$K^{(1)}(0,x,t) = 0 \tag{2.2.12b}$$

and

$$K_{xx}^{(2)} - K_{ss}^{(2)} + (q(x,t) - \frac{\dot{a}(t)}{a(t)})K^{(2)} = K_t^{(2)} \tag{2.2.13}$$

$$K^{(2)}(x,x,t) = - \frac{1}{2} \int_0^x (q(s,t) - \frac{\dot{a}(t)}{a(t)})ds \tag{2.2.14a}$$

$$K_s^{(2)}(0,x,t) = 0 \tag{2.2.14b}$$

respectively. The existence of the kernels $K^{(1)}(s,x,t)$ and $K^{(2)}(s,x,t)$ and the fact that they are twice constinuously differentiable for $-x_o < x < x_o$, $|t| < t_o$, follows from the analysis of section 2.1. The existence and uniqueness of $h^{(1)}(x,t)$ is assured from the fact that $a(t) \neq 0$ for $0 < t < t_o$. (2.2.9) also implies that $h^{(1)}(0,t)=0$ and $h^{(1)}(x,t)$ is twice continuously differentiable in $R^+ \cup \sigma$, and three times differentiable

1) $\beta(t) \equiv 0$, $\alpha(t) \not\equiv 0$ for $|t| < t_o$

2) $\alpha(t) \equiv 0$, $\beta(t) \not\equiv 0$ for $|t| < t_o$.

3) $\alpha(t) \not\equiv 0$ and $\beta(t) \not\equiv 0$ for $|t| < t_o$.

Then $u(x,t)$ can be uniquely continued into $R^+ \cup R^- \cup \sigma$ as a solution of $L(u)=0$.

Remark: Note that since $q(x,t)$ is in general not an analytic function of x, the continuation stated above is in general not an analytic continuation with respect to x.

Proof: We first assume condition 1) holds, i.e. $\beta(t)$ is identically zero and $\alpha(t) \not\equiv 0$ in the disc $|t| < t_o$ in the complex t plane. Let $h(x,t)$ be the analytic solution of (2.2.5) given by (2.2.6) where $h(0,t)={}^{f(t)}/_{\alpha(t)}$ and $h_x(0,t)=0$, and define the solution $v(x,t)$ of $L(u)=0$ by $v(x,t)=\underset{\sim}{T}_2\{h\}$ where $\underset{\sim}{T}_2$ is the operator constructed in section 2.1 (c.f. Theorem 2.1.2). By construction the operator $\underset{\sim}{T}_2$ preserves Cauchy data on σ: $x=0$ and hence from Theorem 2.2.2 and the regularity of the kernel $K(s,x,t)$ of $\underset{\sim}{T}_2$ we can conclude that $v(x,t)$ is a solution of $L(u)=0$ in $R^+ \cup R^- \cup \sigma$ such that $I(v)=f(t)$ on σ. Hence (by considering $u-v$ instead of u) we can assume without loss of generality that $f(t)\equiv 0$, i.e. $u(0,t)=0$. But now from Theorem 2.1.2 we can represent $u(x,t)$ in the form $u(x,t) = \underset{\sim}{T}_1\{h\}$ where $h(x,t)$ is a solution of (2.2.5) and $h(0,t)=0$, and from Theorem 2.2.1 $h(x,t)$ can be uniquely continued into $R^+ \cup R^- \cup \sigma$ as a solution of (2.2.5). This implies that $u(x,t)$ can be continued as a solution of $L(u)=0$ into $R^+ \cup R^- \cup \sigma$. The uniqueness of the continuation follows from the invertibility of the operator $\underset{\sim}{T}_1$ and the uniqueness of the continuation of $h(x,t)$.

The proof of the theorem under the hypothesis that condition 2) holds proceeds in the same manner by appropriate use of the operators $\underset{\sim}{T}_1$ and $\underset{\sim}{T}_2$.

a solution of (2.2.5) by the rule $h(x,t) = h(-x,t)$.

Our next result is concerned with the analytic continuation of solutions to (2.2.5) satisfying analytic Cauchy data on $x=0$:

Theorem 2.2.2 ([9], [36]): Let $h(x,t) \in C^2(R^+) \cap C^1(R^+ \cup \sigma)$ be a solution of (2.2.5) in R^+ such that $h(0,t)$ and $h_x(0,t)$ are analytic for $|t| < t_o$. Then $h(x,t)$ can be uniquely continued into $-\infty < x < \infty$, $-t_o < t < t_o$, as a solution of (2.2.5) that is an analytic function of x and t for $|x| < \infty$, $|t| < t_o$.

Proof: From the results of section 2.1 and Holmgren's uniqueness theorem (c.f. [29]) we can represent $h(x,t)$ in the form

$$h(x,t) = -\frac{1}{2\pi i}\oint_{|t-\tau|=\delta} E^{(1)}(x,t,\tau)h(0,\tau)d\tau - \frac{1}{2\pi i}\oint_{|t-\tau|=\delta} E^{(2)}(x,t,\tau)h_x(0,\tau)d\tau \quad (2.2.6)$$

where

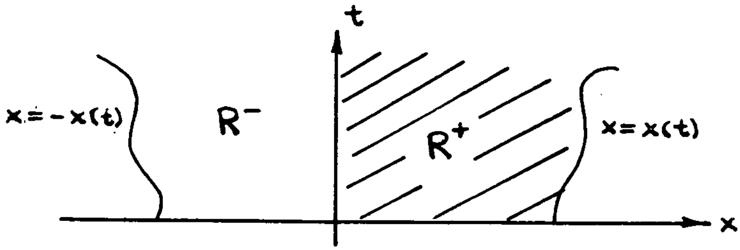
$$E^{(1)}(x,t,\tau) = \frac{1}{t-\tau} + \sum_{j=1}^{\infty} \frac{x^{2j}(-1)^j j!}{(2j)!(t-\tau)^{j+1}}$$
$$(2.2.7)$$
$$E^{(2)}(x,t,\tau) = \frac{x}{t-\tau} + \sum_{j=1}^{\infty} \frac{x^{2j+1}(-1)^j j!}{(2j+1)!(t-\tau)^{j+1}}$$

and $\delta > 0$. The statement of the theorem now follows from (2.2.6) and the analyticity of $E^{(1)}(x,t,\tau)$ and $E^{(2)}(x,t,\tau)$.

We can now prove the following reflection principle for solutions of $L(u)=0$ in R^+ satisfying $I(u)=f(t)$ on σ:

Theorem 2.2.3 ([10], [11]): Let $q(x,t)$, $\alpha(t)$, $\beta(t)$ and $f(t)$ satisfy the assumptions stated previously and let $u(x,t) \in C^2(R^+) \cap C^1(R^+ \cup \sigma)$ be a solution of $L(u) = 0$ in R^+ such that $I(u)=f(t)$ on σ: $x=0$. Suppose one of the following three conditions is satisfied:

Let R^- be the mirror image of R^+ reflected across $\sigma: x=0$



and let $x_o > 0$ be such that $R^+ \cup R^- \cup \sigma \subset B = \{(x,t): -x_o \leq x \leq x_o, |t| < t_o\}$.
We make the following assumptions:

1. $q(x,t) \in C^1(B)$ and is an analytic function of t for $|t| < t_o$ (In particular this implies that $x_1(t)$ should have been analytic for $|t| < t_o$).
2. $\alpha(t)$, $\beta(t)$ and $f(t)$ are analytic for $|t| < t_o$ (This implies that $a_1(t)$, $a_2(t)$ and $g(t)$ should have been analytic for $|t| < t_o$).

We first need to obtain two results on the continuation of solutions to the heat equation

$$h_{xx} = h_t \quad . \tag{2.2.5}$$

The first theorem is the well known reflection principle for solutions of (2.2.5) satisfying homogeneous Dirichlet or Neumann data on $x=0$, and can be proved in the same manner as the reflection principle for solutions to Laplace equation (c.f. [21]) if one uses the Green (or Neumann) function for the heat equation in a rectangle instead of the Green (or Neumann) function for Laplace's equation in a disc:

Theorem 2.2.1: 1) Let $h(x,t) \in C^2(R^+) \cap C^0(R^+ \cup \sigma)$ be a solution of (2.2.5) in R^+ such that $h(0,t)=0$. Then $h(x,t)$ can be uniquely continued into $R^+ \cup R^- \cup \sigma$ as a solution of (2.2.5) by the rule $h(x,t) = -h(-x,t)$.
2) Let $h(x,t) \in C^2(R^+) \cap C^1(R^+ \cup \sigma)$ be a solution of (2.2.5) in R^+ such that $h_x(0,t)=0$. Then $h(x,t)$ can be uniquely continued into $R^+ \cup R^- \cup \sigma$ as

$$L(u) \equiv u_{xx} + q(x,t)u - u_t = 0 \tag{2.2.1}$$

defined in a domain $D^+ = \{(x,t) : x_1(t) < x < x_2(t), \; 0 < t < t_o\}$ where t_o is a positive constant and $x_1(t)$ and $x_2(t)$ are given functions of t. We are primarily interested in the following problem: Let u(x,t) be a solution of L(u) = 0 in D^+ such that $u(x,t) \in C^2(D^+) \cap C^1(\bar{D}^+)$ and satisfies the boundary condition

$$\underset{\sim}{I}(u) \equiv a_1(t)u(x_1(t),t) + a_2(t)u_x(x_1(t),t) = g(t) \tag{2.2.2}$$

on $\sigma: x = x_1(t)$ where $a_1(t), a_2(t)$ and g(t) are given functions of t. Let D^- be the mirror image of D^+ reflected across the arc σ. Under what conditions can u(x,t) be uniquely continued as a solution of (2.2.1) into $D^+ \cup D^- \cup \sigma$? By making the change of variables

$$\begin{aligned} \xi &= x - x_1(t) \\ \tau &= t \end{aligned} \tag{2.2.3}$$

and following this by a change of variables of the form (2.1.2) we can reduce this problem to the case when $x_1(t)=0$, i.e. D^+ is replaced by the domain $R^+ = \{(x,t) : 0 < x < x(t), \; 0 < t < t_o\}$ where x(t) is a given function of t and (2.2.2) becomes

$$I(u) \equiv \alpha(t)u(0,t) + \beta(t)u_x(0,t) = f(t) \tag{2.2.4}$$

on $\sigma: x=0$ with $\alpha(t), \beta(t)$ and f(t) given functions of t.

<u>Remark</u>: Even when (2.2.1) is originally the heat equation, the change of variables (2.2.3), (2.1.2) changes this equation into one of the form L(u)=0 but where q(x,t) now depends on x and t. Furthermore, $a_1(t) \equiv 0$ does not imply that $\alpha(t) \equiv 0$.

for $\frac{1}{2}(s+x) \leqslant -a$, $s \leqslant x$. If $u(x,t)$ is a classical solution of (2.1.27) defined for $x < a$, $0 < t < t_o$, then $u(x,t)$ can be represented in the form $u(x,t) = \underset{\sim}{A}_2\{h\}$ where $h(x,t)$ is a classical solution of (2.1.29) defined in the same domain as $u(x,t)$.

We summarize our results in the following theorem:

Theorem 2.1.3 ([15]): Let the coefficient $q(x,t)$ of (2.1.27) be continuously differentiable for $-\infty < x < \infty$, $|t| < t_o$, an analytic function of t for $|t| < t_o$, and such that $q(x,t) \equiv 0$ for $|x| > a$ where a is a positive constant.

1) If $u(x,t)$ is a classical solution of (2.1.27) for $x \geqslant -a$, $0 < t < t_o$, then $u(x,t)$ can be represented in the form $u(x,t) = \underset{\sim}{A}_1\{h\}$ where $h(x,t)$ is a classical solution of (2.1.39) for $x \geqslant -a$, $0 < t < t_o$. Conversely, for any such $h(x,t)$, $u(x,t) = \underset{\sim}{A}_1\{h\}$ satisfies the above hypothesis on $u(x,t)$.
2) If $u(x,t)$ is a classical solution of (2.1.27) for $x \leqslant a$, $0 < t < t_o$, then $u(x,t)$ can be represented in the form $u(x,t) = \underset{\sim}{A}_2\{h\}$ where $h(x,t)$ is a classical solution of (2.1.39) for $x \leqslant a$, $0 < t < t_o$. Conversely, for any such $h(x,t)$, $u(x,t) = \underset{\sim}{A}_2\{h\}$ satisfies the above hypothesis on $u(x,t)$.

2.2 Reflection Principles.

In this section we will use the integral operators constructed in the previous section to obtain reflection principles for solutions of parabolic equations in one space variable. Such results will be needed later on in this chapter to help construct a complete family of solutions for parabolic equations in a manner somewhat similar to the use of the Bergman-Vekua theorem in constructing a complete family of solutions for elliptic equations. Without loss of generality we consider equations written in the canonical form

$$K^{(2)}_{xx} - K^{(2)}_{ss} + q(x,t)K^{(2)} = K^{(2)}_t \quad ; \quad s < x \tag{2.1.82}$$

$$K^{(2)}(x,x,t) = -\frac{1}{2}\int_{-\infty}^{x} q(s,t)ds \tag{2.1.83}$$

$$K^{(2)}(s,x,t) \equiv 0 \text{ for } \tfrac{1}{2}(s+x) \leqslant -a \tag{2.1.84}$$

$$K^{(2)}(s,x,t) \equiv 0 \text{ for } s > x\,, \tag{2.1.85}$$

and $K^{(2)}(s,x,t)$ is twice continuously differentiable with respect to s,x and t for $s \leqslant x$, $|t| < t_o$. The existence of the kernel $K^{(2)}(s,x,t)$ (and hence the operator $\underset{\sim}{A}_2$) follows in the same manner as that of $K^{(1)}(s,x,t)$, where, for $\frac{1}{2}(s+x) \leqslant -a$, $s \leqslant x$, $K^{(2)}(s,x,t)$ satisfies the integro-differential equation

$$\begin{aligned} K^{(2)}(s,x,t) &= -\frac{1}{2}\int_{-a}^{\frac{1}{2}(s+x)} q(\sigma,t)d\sigma \\ &\quad - \frac{1}{2}\int_{\frac{1}{2}(s+x)}^{x}\int_{s+\sigma-x}^{s+x-\sigma}\left[q(\sigma,t)K^{(2)}(\mu,\sigma,t)-K^{(2)}_t(\mu,\sigma,t)\right]d\mu d\sigma \\ &\quad - \frac{1}{2}\int_{-a}^{\frac{1}{2}(s+x)}\int_{s+\sigma-x}^{\sigma}\left[q(\sigma,t)K^{(2)}(\mu,\sigma,t)-K^{(2)}_t(\mu,\sigma,t)\right]d\mu d\sigma \end{aligned} \tag{2.1.86}$$

and

$$K^{(2)}(s,x,t) = \sum_{n=1}^{\infty} K^{(2)}_n(s,x,t) \tag{2.1.87}$$

where

$$\begin{aligned} K^{(2)}_1(s,x,t) &= -\frac{1}{2}\int_{-a}^{\frac{1}{2}(s+x)} q(\sigma,t)d\sigma \\ K^{(2)}_n(s,x,t) &= -\frac{1}{2}\int_{\frac{1}{2}(s+x)}^{x}\int_{s+\sigma-x}^{s+x-\sigma}\left[q(\sigma,t)K^{(2)}_{n-1}(\mu,\sigma,t) - \frac{\partial}{\partial t}K^{(2)}_{n}(\mu,\sigma,t)\right]d\mu d\sigma \\ &\quad - \frac{1}{2}\int_{-a}^{\frac{1}{2}(s+x)}\int_{s+\sigma-x}^{\sigma}\left[q(\sigma,t)K^{(2)}_{n-1}(\mu,\sigma,t) - \frac{\partial}{\partial t}K^{(2)}_{n-1}(\mu,\sigma,t)\right]d\mu d\sigma \end{aligned} \tag{2.1.88}$$

for $n \geqslant 2$, with

$$|K^{(2)}_n(s,x,t)| \leqslant C^n\left(1-\frac{|t|}{t_o}\right)^{-n}\left(a+\tfrac{1}{2}(s+x)\right)(n-1)!\,\frac{(a+x)^{2n}}{(2n)!} \tag{2.1.89}$$

$$K_n^{(1)}(s,x,t) \ll C^n(1-\frac{t}{t_o})^{-n}(a-\frac{1}{2}(s+x))(n-1)!\,\frac{(a-x)^{2n}}{(2n)!} \qquad (2.1.79)$$

and hence for $\frac{1}{2}(s+x) \leqslant a$, $s \geqslant x$,

$$|K_n^{(1)}(s,x,t)| \leqslant C^n(1-\frac{|t|}{t_o})^{-n}(a-\frac{1}{2}(s+x))\,\frac{(n-1)!}{(2n)!}(a-x)^{2n} \qquad (2.1.80)$$

which implies that the series (2.1.72) is absolutely and uniformly convergent for $\frac{1}{2}(s+x) \leqslant a$, $s \geqslant x \geqslant -a$, $|t| \leqslant t_o - \varepsilon$ where $\varepsilon > 0$ is arbitrarily small, thus establishing the existence of the kernel $K^{(1)}(s,x,t)$. It can easily be verified that $K^{(1)}(s,x,t)$ is twice continuously differentiable with respect to s,x and t for $s \geqslant x$, $|t| < t_o$. We have thus established the existence of the operator $\underset{\sim}{A}_1$ defined by (2.1.59). It is an easy matter to show that every classical solution u(x,t) of (2.1.27) defined for $x > -a$, $0 < t < t_o$ (where $q(x,t)\equiv 0$ for $|x| > a$) can be represented in the form $u(x,t) = \underset{\sim}{A}_1\{h\}$ where h(x,t) is a classical solution of (2.1.29) defined in the same domain as u(x,t). For from (2.1.60) we have that the range of integration in the integral in (2.1.59) is in fact only over the finite interval $x \leqslant s \leqslant 3a$ and the invertibility of the operator $\underset{\sim}{A}_1$ follows from the properties of Volterra integral equations of the second kind in the same manner which we previously showed the invertibility of the operators $\underset{\sim}{T}_1$, $\underset{\sim}{T}_2$ and $\underset{\sim}{T}_3$.

In addition to the operator $\underset{\sim}{A}_1$ we will also need the operator $\underset{\sim}{A}_2$ defined by

$$u(x,t) = \underset{\sim}{A}_2\{h\} = h(x,t) + \int_{-\infty}^{x} K^{(2)}(s,x,t)h(s,t)ds \qquad (2.1.81)$$

which maps classical solutions of (2.1.29) defined for $x < a$, $0 < t < t_o$, onto classical solutions of (2.1.27) defined in the same domain, where $K^{(2)}(s,x,t)$ is the unique solution of

where

$$K_1^{(1)}(s,x,t) = -\frac{1}{2}\int_{\frac{1}{2}(s+x)}^{a} q(\sigma,t)\,d\sigma \tag{2.1.73}$$

$$K_n^{(1)}(s,x,t) = -\frac{1}{2}\int_{x}^{\frac{1}{2}(s+x)}\int_{s+x-\sigma}^{s+\sigma-x}\left[q(\sigma,t)K_{n-1}^{(1)}(\mu,\sigma,t) - \frac{\partial}{\partial t}K_{n-1}^{(1)}(\mu,\sigma,t)\right]d\mu\, d\sigma$$

$$-\frac{1}{2}\int_{\frac{1}{2}(s+x)}^{a}\int_{\sigma}^{s+\sigma-x}\left[q(\sigma,t)K_{n-1}^{(1)}(\mu,\sigma,t) - \frac{\partial}{\partial t}K_{n-1}^{(1)}(\mu,\sigma,t)\right]d\mu\, d\sigma$$

for $n \geqslant 2$. Let C be a positive constant such that for $|t| < t_o$

$$q(x,t) \ll C\left(1 - \frac{t}{t_o}\right)^{-1} \tag{2.1.74}$$

with respect to t, uniformly for $-a \leqslant x \leqslant a$. Without loss of generality we can assume $C \geqslant 1$, $t_o \geqslant 1$. Then

$$K_1^{(1)}(s,x,t) \ll \frac{1}{2}C\left(1 - \frac{t}{t_o}\right)^{-1}\left(a - \frac{1}{2}(s+x)\right) \tag{2.1.75}$$

and, since in both the double integrals defining $K_2^{(1)}(s,x,t)$ we have $a - \frac{1}{2}(\mu+\sigma) \leqslant a - \frac{1}{2}(s+x)$,

$$K_2^{(1)}(s,x,t) \ll \frac{C^2}{2}\left(1 - \frac{t}{t_o}\right)^{-2}\left(a-\frac{1}{2}(s+x)\right). \tag{2.1.76}$$

$$\left(\int_{x}^{\frac{1}{2}(s+x)} 2(\sigma-x)\,d\sigma + \int_{\frac{1}{2}(s+x)}^{a}(s-x)\,d\sigma\right).$$

But in the second integral on the right hand side of (2.1.76) we have $\frac{1}{2}(s+x) \leqslant \sigma$ and hence $s-x \leqslant 2(\sigma-x)$, which implies

$$K_2^{(1)}(s,x,t) \ll C^2\left(1 - \frac{t}{t_o}\right)^{-2}\left(a-\frac{1}{2}(s+x)\right)\frac{(a-x)^2}{2!}. \tag{2.1.77}$$

Using the identity

$$\int_{x}^{a}\frac{(a-\sigma)^{2n}}{(2n)!}(\sigma-x)\,d\sigma = \frac{(a-x)^{2n+2}}{(2n+2)!} \tag{2.1.78}$$

we have by induction that

For $\eta \leqslant 0$ we have

$$\tilde{K}^{(1)}(\xi,\eta,t) = -\frac{1}{2}\int_{\xi}^{\infty} q(s,t)ds \tag{2.1.68}$$

$$-\int_{\eta}^{\infty}\int_{\xi}^{\infty}\left[q(\alpha+\beta,t)\tilde{K}^{(1)}(\alpha,\beta,t)-\tilde{K}_t^{(1)}(\alpha,\beta,t)\right]d\alpha d\beta.$$

Note that (2.1.68) satisfies (2.1.65) since for $\eta=0$ the double integral in (2.1.68) is over the region $\beta \geqslant 0$ where $\tilde{K}^{(1)}(\alpha,\beta,t) \equiv 0$. Now in (2.1.68) make the change of variables

$$\alpha = \frac{\sigma+\mu}{2}$$
$$\beta = \frac{\sigma-\mu}{2} \quad . \tag{2.1.69}$$

Then (2.1.68) becomes

$$K^{(1)}(s,x,t) = -\frac{1}{2}\int_{\frac{1}{2}(s+x)}^{\infty} q(\sigma,t)d \tag{2.1.70}$$

$$-\frac{1}{2}\int_{x}^{\infty}\int_{s+x-\sigma}^{s+\sigma-x}\left[q(\sigma,t)K^{(1)}(\mu,\sigma,t)-K_t^{(1)}(\mu,\sigma,t)\right]d\mu d\sigma$$

and from the assumptions on $q(x,t)$ and (2.1.60), (2.1.63) we have that for $\frac{1}{2}(s+x) \leqslant a$, $s \geqslant x$,

$$K^{(1)}(s,x,t) = -\frac{1}{2}\int_{\frac{1}{2}(s+x)}^{a} q(\sigma,t)d\sigma$$

$$-\frac{1}{2}\int_{x}^{\frac{1}{2}(s+x)}\int_{s+x-\sigma}^{s+\sigma-x}\left[q(\sigma,t)K^{(1)}(\mu,\sigma,t)-K_t^{(1)}(\mu,\sigma,t)\right]d\mu d\sigma \tag{2.1.71}$$

$$-\frac{1}{2}\int_{\frac{1}{2}(s+x)}^{a}\int_{\sigma}^{s+\sigma-x}\left[q(\sigma,t)K^{(1)}(\mu,\sigma,t)-K_t^{(1)}(\mu,\sigma,t)\right]d\mu d\sigma.$$

For $\frac{1}{2}(s+x) < a$, $s \geqslant x$, we now look for a solution of (2.1.71) in the form

$$K^{(1)}(s,x,t) = \sum_{n=1}^{\infty} K_n^{(1)}(s,x,t) \tag{2.1.72}$$

among other conditions, satisfies

$$K^{(1)}(s,x,t) \equiv 0 \text{ for } \frac{1}{2}(s+x) \geqslant a. \tag{2.1.60}$$

(2.1.60) guarantees the existence of the integral (2.1.59) for any classical solution $h(x,t)$ of (2.1.29) defined in $x > -a$, $0 < t < t_o$. Substituting (2.1.59) into (2.1.27) and integrating by parts using (2.1.60) shows that (2.1.59) is a solution of (2.1.27) provided $K^{(1)}(s,x,t)$ satisfies (2.1.60) and

$$K^{(1)}_{xx} - K^{(1)}_{ss} + q(x,t)K^{(1)} = K^{(1)}_t \quad ; \quad s > x \tag{2.1.61}$$

$$K^{(1)}(x,x,t) = -\frac{1}{2}\int_x^\infty q(s,t)ds \quad . \tag{2.1.62}$$

The equations (2.1.60) - (2.1.62) are not enough to uniquely determine $K^{(1)}(s,x,t)$ and so we impose the additional condition

$$K^{(1)}(s,x,t) \equiv 0 \quad \text{for} \quad s < x \; . \tag{2.1.63}$$

We will now construct a function $K^{(1)}(s,x,t)$ satisfying (2.1.60)-(2.1.63) such that $K^{(1)}(s,x,t)$ is twice continuously differentiable with respect to s, x and t for $s \geqslant x$, $|t| < t_o$. In particular this implies that if $h(x,t)$ is a classical solution of (2.1.29) for $x > -a$, $0 < t < t_o$, then $u(x,t)$ as defined by (2.1.59) is a classical solution of (2.1.27) and the domain of regularity of $h(x,t)$ and $u(x,t)$ coincide. Let ξ and η be defined by (2.1.43) and $\tilde{K}^{(1)}(\xi,\eta,t) = K^{(1)}(\xi-\eta, \xi+\eta, t)$. Then (2.1.60)-(2.1.63) become

$$\tilde{K}^{(1)}_{\xi\eta} + q(\xi+\eta,t)\tilde{K}^{(1)} = \tilde{K}^{(1)}_t \quad ; \quad \eta < 0 \tag{2.1.64}$$

$$\tilde{K}^{(1)}(\xi,0,t) = -\frac{1}{2}\int_\xi^\infty q(s,t)ds \tag{2.1.65}$$

$$\tilde{K}^{(1)}(\xi,\eta,t) \equiv 0 \quad \text{for} \quad \xi \geqslant a \tag{2.1.66}$$

$$\tilde{K}^{(1)}(\xi,\eta,t) \equiv 0 \quad \text{for} \quad \eta > 0 \quad . \tag{2.1.67}$$

for $0 \leq x < x_o$, $0 < t < t_o$, and satisfying $h_x(0,t)=0$. Conversely for any such $h(x,t)$, $u(x,t) = \underset{\sim}{T}_2\{h\}$ satisfies the above hypothesis on $u(x,t)$.

3) If $u(x,t)$ is a classical solution of (2.1.27) in R then $u(x,t)$ can be represented in the form $u(x,t) = \underset{\sim}{T}_3\{h\}$ where $h(x,t)$ is a classical solution of (2.1.39) in R. Conversely, for any such $h(x,t)$, $u(x,t)=\underset{\sim}{T}_3\{h\}$ satisfies the above hypothesis on $u(x,t)$.

In the following sections of this chapter we will use the operators $\underset{\sim}{T}_1$ and $\underset{\sim}{T}_2$ to obtain reflection principles for solutions of (2.1.27) and the operator $\underset{\sim}{T}_3$ to construct a complete family of solutions. We also want to show how integral operators can be used to reformulate the first-initial boundary value problem for (2.1.27) as an integral equation in a manner similar to their use in the case of the Dirichlet problem for elliptic equations (c.f. section 1.3). The operators $\underset{\sim}{T}_1$, $\underset{\sim}{T}_2$ and $\underset{\sim}{T}_3$ are not suitable for this purpose since solutions in the range of the operators $\underset{\sim}{T}_1$ and $\underset{\sim}{T}_2$ must satisfy homogeneous boundary data at $x=0$, and solutions in the range of the operator $\underset{\sim}{T}_3$ must be defined in a domain which is symmetric with respect to $x=0$. Hence we will now construct integral operators which are suitable for reformulating the first-initial boundary value problem for (2.1.27) as a Volterra integral equation.

We assume that $q(x,t)$ has been continued in a continuously differentiable manner such that $q(x,t)$ is defined for $-\infty < x < \infty$, $|t| < t_o$, is analytic with respect to t for $|t| < t_o$, and $q(x,t)\equiv 0$ for $|x| \geq a$ where a is a positive constant. We first look for a solution of (2.1.27) for $x \geq -a$, $0 < t < t_o$, in the form

$$u(x,t) = \underset{\sim}{A}_1\{h\} = h(x,t) + \int_x^\infty K^{(1)}(s,x,t)h(s,t)ds \tag{2.1.59}$$

where $h(x,t)$ is a classical solution of (2.1.29) defined for $x > -a$, $0 < t < t_o$, and $K^{(1)}(s,x,t)$ is a function to be determined which, to be shown that h(x,t) is a solution of (2.1.39). From (2.1.39) and (2.1.42) we have that $K(s,x,t) = -K(-s,x,t)$ and $M(s,x,t) = M(-s,x,t)$ and hence we can rewrite (2.1.56) in the form

$$u(x,t) = \frac{1}{2}(h(x,t)-h(-x,t)) + \frac{1}{2}\int_0^x K(s,x,t)[h(s,t)-h(-s,t)]ds \tag{2.1.57}$$
$$+ \frac{1}{2}(h(x,t)+h(-x,t)) + \frac{1}{2}\int_0^x M(s,x,t)[h(s,t)+h(-s,t)]ds.$$

Substituting (2.1.57) into (2.1.27), using (2.1.30), (2.1.31a), (2.1.31b), (2.1.33), (2.1.34a), (2.1.34b), and rewriting the resulting expression in the form of (2.1.56) gives

$$0 = (h_{xx}-h_t) + \frac{1}{2}\int_{-x}^{x}[K(s,x,t)+M(s,x,t)](h_{ss}(s,t)-h_t(s,t))ds \tag{2.1.58}$$

Since solutions of Volterra integral equations of the second kind are unique, we can conclude that h(x,t) is a solution of (2.1.39) in R.

We summarize our results in the following theorem :

<u>Theorem 2.1.2</u> ([10], [12]): Let the coefficient q(x,t) of (2.1.27) be continuously differentiable for $-x_o < x < x_o$, $|t| < t_o$, and an analytic function of t for $|t| < t_o$. Let $R^+ = \{(x,t) : 0 < x < x_o, \; 0 < t < t_o\}$ and $R = \{(x,t) : -x_o < x < x_o, \; 0 < t < t_o\}$.

1) If u(x,t) is a classical solution of (2.1.27) in R^+, continuously differentiable for $0 \leq x < x_o, 0 < t < t_o$, and satisfying u(0,t)=0, then u(x,t) can be represented in the form $u(x,t) = \underset{\sim}{T}_1\{h\}$ where h(x,t) is a classical solution of (2.1.39) in R^+, continuously differentiable for $0 \leq x < x_o$, $0 < t < t_o$, and satisfying h(0,t)=0. Conversely for any such h(x,t), $u(x,t) = \underset{\sim}{T}_1\{h\}$ satisfies the above hypothesis on u(x,t).

2) If u(x,t) is a classical solution of (2.1.27) in R^+, continuously differentiable for $0 \leq x < x_o$, $0 < t < t_o$, and satisfying $u_x(0,t)=0$, then u(x,t) can be represented in the form $u(x,t) = \underset{\sim}{T}_2\{h\}$ where h(x,t) is a classical solution of (2.1.39) in R^+, continuously differentiable

regularity properties as $u(x,t)$ and satisfies $h(0,t)=u(0,t)=0$. This can be seen by using the resolvent operator to express $h(x,t)$ in terms of $u(x,t)$. To show that this solution of the integral equation (2.1.53) is in fact a solution of the heat equation, we substitute (2.1.53) into (2.1.27) and use (2.1.30), (2.1.31a), (2.1.31b) to obtain

$$\begin{aligned} 0 &= u_{xx} + q(x,t)u - u_t \\ &= (h_{xx}-h_t) + \int_0^x K(s,x,t)(h_{ss}(s,t) - h_t(s,t))ds. \end{aligned} \tag{2.1.54}$$

Since solutions of Volterra integral equations of the second kind are unique, we must have

$$h_{xx} - h_t = 0, \tag{2.1.55}$$

i.e. $h(x,t)$ is a solution of (2.1.39) in R^+.

In a similar manner we can show that if $u(x,t)$ is a classical solution of (2.1.27) in R^+, is continuously differentiable for $0 \leqslant x < x_o$, $0 < t < t_o$, and satisfies $u_x(0,t)=0$, then $u(x,t)$ can be represented in the form $u(x,t)=\underset{\sim}{T}_2\{h\}$ where $h(x,t)$ is a solution of (2.1.39) in R^+ such that $h(x,t)$ is continuously differentiable for $0 \leqslant x < x_o$, $0 < t < t_o$, and satisfies $h_x(0,t)=0$.

We now want to combine the results obtained above to construct an integral operator whose domain and range are independent of the boundary data at $x=0$. Let $u(x,t)$ be a classical solution of (2.1.27) in $R = \{(x,t): -x_o < x < x_o,\ 0 < t < t_o\}$. We will show that there exists a classical solution $h(x,t)$ of (2.1.39) in R such that $u(x,t)$ can be represented in the form

$$u(x,t) = \underset{\sim}{T}_3\{h\} = h(x,t) + \frac{1}{2}\int_{-x}^{x} [K(s,x,t) + M(s,x,t)]h(s,t)ds \tag{2.1.56}$$

(2.1.56) is a Volterra equation of the second kind for $h(x,t)$ and hence can be uniquely solved for $h(x,t)$ where $h(x,t)$ is defined in R. It remains

We will now show that the series (2.1.48) converges absolutely and uniformly for (ξ,η,t) on an arbitrary compact subset Ω of $\{(\xi,\eta,t): -x_o < \xi < x_o, -x_o < \eta < x_o, |t| < t_o\}$. To this end let C be a positive constant such that for $(\xi,\eta,t) \in \Omega$ we have with respect to t

$$q(\xi,\eta,t) << C(1-\frac{t}{t_o})^{-1} \tag{2.1.50}$$

Without loss of generality assume $C \geq 1$, $t_o \geq 1$, $x_o \leq 1$. Then from (2.1.49) and the properties of dominants it follows by induction that

$$\tilde{E}_n << \frac{2^n C^n|\xi|^{n-1}|\eta|^{n-1}}{(n-1)!} (1-\frac{t}{t_o})^{-n} \tag{2.1.51}$$

and hence

$$|\tilde{E}_n| \leq \frac{2^n C^n|\xi|^{n-1}|\eta|^{n-1}}{(n-1)!} (1-\frac{t}{t_o})^{-n} \tag{2.1.52}$$

for $(\xi,\eta,t) \in \Omega$. Hence the series (2.1.48) converges absolutely and uniformly for $(\xi,\eta,t) \in \Omega$. In a similar manner it is easily seen that $\tilde{E}(\xi,\eta,t)$ is twice continuously differentiable in Ω, and we can now conclude the existence of the function $\tilde{E}(\xi,\eta,t) = E(s,x,t)$ having the desired properties. Similarly the function $G(s,x,t)$ exists and is twice continuously differentiable in Ω, and we have therefore now established the existence of the operators $\underset{\sim}{T}_1$ and $\underset{\sim}{T}_2$.

We now want to show that if $u(x,t)$ is a classical solution of (2.1.27) in R^+, is continuously differentiable for $0 \leq x < x_o$, $0 < t < t_o$, and satisfies $u(0,t)=0$, then $u(x,t)$ can be represented in the form

$$u(x,t) = \underset{\sim}{T}_1 \{h\} \tag{2.1.53}$$

for some solution $h(x,t)$ of (2.1.39) in R^+ where $h(x,t)$ is continuously differentiable for $0 \leq x < x_o$, $0 < t < t_o$, and satisfies $h(0,t)=0$. (2.1.53) is a Volterra integral equation of the second kind for $h(x,t)$ and hence there exists a solution $h(x,t)$ of (2.1.53) which has the same satisfies (2.1.33) and (2.1.34a), (2.1.34b). Hence it suffices to show the existence of the functions $E(s,x,t)$ and $G(s,x,t)$. We will now do this for $E(s,x,t)$; the existence of $G(s,x,t)$ follows in an identical fashion.

Let

$$x = \xi+\eta \tag{2.1.43}$$

$$s = \xi-\eta$$

and define $\tilde{E}(\xi,\eta,t)$ and $\tilde{q}(\xi,\eta,t)$ by

$$\begin{aligned}\tilde{E}(\xi,\eta,t) &= E(\xi-\eta,\ \xi+\eta,t)\\ \tilde{q}(\xi,\eta,t) &= q(\xi+\eta,t).\end{aligned} \tag{2.1.44}$$

Then (2.1.37), (2.1.38a), (2.1.38b) become

$$\tilde{E}_{\xi\eta} + \tilde{q}(\xi,\eta,t)\tilde{E} = \tilde{E}_t \tag{2.1.45}$$

$$\tilde{E}(\xi,0,t) = -\frac{1}{2}\int_0^{\xi} q(s,t)ds \tag{2.1.46a}$$

$$\tilde{E}(0,\eta,t) = \frac{1}{2}\int_0^{\eta} q(s,t)ds \quad , \tag{2.1.46b}$$

and hence $\tilde{E}(\xi,\eta,t)$ satisfies the Volterra integro-differential equation

$$\begin{aligned}\tilde{E}(\xi,\eta,t) = &-\frac{1}{2}\int_0^{\xi} q(s,t)ds + \frac{1}{2}\int_0^{\eta} q(s,t)ds\\ &+\int_0^{\eta}\int_0^{\xi} (\tilde{E}_t(\xi,\eta,t) - \tilde{q}(\xi,\eta,t)\tilde{E}(\xi,\eta,t))d\xi d\eta \ .\end{aligned} \tag{2.1.47}$$

The solution of (2.1.47) can formally be obtained by iteration in the form

$$\tilde{E}(\xi,\eta,t) = \sum_{n=1}^{\infty} \tilde{E}_n(\xi,\eta,t) \tag{2.1.48}$$

where

$$\begin{aligned}\tilde{E}_1(\xi,\eta,t) &= -\frac{1}{2}\int_0^{\xi} q(s,t)ds + \frac{1}{2}\int_0^{\eta} q(s,t)ds\\ \tilde{E}_{n+1}(\xi,\eta,t) &= \int_0^{\eta}\int_0^{\xi} (\tilde{E}_{nt}(\xi,\eta,t) - \tilde{q}(\xi,\eta,t)\tilde{E}_n(\xi,\eta,t))d\xi d\eta \ .\end{aligned} \tag{2.1.49}$$

(2.1.27) by

$$\underset{\sim}{T}_1\{h\} = h(x,t) + \int_0^x K(s,x,t)h(s,t)ds \tag{2.1.35}$$

$$\underset{\sim}{T}_2\{h\} = h(x,t) + \int_0^x M(s,x,t)h(s,t)ds \tag{2.1.36}$$

where the domain of $\underset{\sim}{T}_1$ is the class of solutions to the heat equation in R^+ satisfying $h(0,t)=0$ and the domain of $\underset{\sim}{T}_2$ is the class of solutions to the heat equation in R^+ satisfying $h_x(0,t)=0$.

We will now show the existence of the functions $K(s,x,t)$ and $M(s,x,t)$. Due to the regularity assumptions on $q(x,t)$ we will in fact show that $K(s,x,t)$ and $M(s,x,t)$ are twice continuously differentiable solutions of (2.1.30) and (2.1.33) for $-x_o < x < x_o$, $-x_o < s < x_o$, $|t| < t_o$. Suppose $E(s,x,t)$ satisfies

$$E_{xx} - E_{ss} + q(x,t)E = E_t \tag{2.1.37}$$

for $-x_o < x < x_o$, $-x_o < s < x_o$, $|t| < t_o$, and assumes the initial data

$$E(x,x,t) = -\frac{1}{2}\int_0^x q(s,t)ds \tag{2.1.38a}$$

$$E(-x,x,t) = \frac{1}{2}\int_0^x q(s,t)ds \tag{2.1.38b}$$

Then

$$K(s,x,t) = \frac{1}{2}\left[E(s,x,t) - E(-s,x,t)\right] \tag{2.1.39}$$

satisfies (2.1.30) and (2.1.31a), (2.1.31b). Similarly, if $G(s,x,t)$ satisfi

$$G_{xx} - G_{ss} + q(x,t)G = G_t \tag{2.1.40}$$

for $-x_o < x < x_o$, $-x_o < s < x_o$, $|t| < t_o$, and assumes the initial data

$$G(x,x,t) = -\frac{1}{2}\int_0^x q(s,t)ds \tag{2.1.41a}$$

$$G(-x,x,t) = -\frac{1}{2}\int_0^x q(s,t)ds \tag{2.1.41b}$$

then

$$M(s,x,t) = \frac{1}{2}\left[G(s,x,t) + G(-s,x,t)\right] \tag{2.1.42}$$

We now look for solutions of (2.1.27) in the form

$$u(x,t) = h(x,t) + \int_0^x K(s,x,t)h(s,t)ds \tag{2.1.28}$$

where $h(x,t)$ is a classical solution of the heat equation

$$h_{xx} = h_t \tag{2.1.29}$$

in R^+, is continuously differential for $0 \leqslant x < x_o$, $0 < t < t_o$, and satisfies the Dirichlet data $h(0,t)=0$ (Note that from (2.1.28) this implies that $u(0,t)=0$ also). Substituting (2.1.28) into (2.1.27) and integrating by parts shows that (2.1.28) is a solution of (2.1.27) provided $K(s,x,t)$ is a solution of

$$K_{xx} - K_{ss} + q(x,t)K = K_t \tag{2.1.30}$$

for $0 < s \leqslant x < x_o$ which satisfies the initial data

$$K(x,x,t) = -\frac{1}{2}\int_0^x q(s,t)ds \tag{2.1.31a}$$

$$K(0,x,t) = 0 \quad . \tag{2.1.31b}$$

Now suppose that instead of satisfying $h(0,t)=0$, $h(x,t)$ satisfies $h_x(0,t)=0$. We again look for a solution of (2.1. 27) in the form

$$u(x,t) = h(x,t) + \int_0^x M(s,x,t)h(s,t)ds \quad . \tag{2.1.32}$$

Then it is seen that (2.1.32) will be a solution of (2.1.27) provided $M(s,x,t)$ is a solution of

$$M_{xx} - M_{ss} + q(x,t)M = M_t \tag{2.1.33}$$

for $0 < s \leqslant x < x_o$ which satisfies the initial data

$$M(x,x,t) = -\frac{1}{2}\int_0^x q(s,t)ds \tag{2.1.34a}$$

$$M_s(0,x,t) = 0 \quad . \tag{2.1.34b}$$

If the functions $K(s,x,t)$ and $M(s,x,t)$ exist, we can now define two operators $\underset{\sim}{T}_1$ and $\underset{\sim}{T}_2$ mapping solutions of the heat equation onto solutions of

The integral operator $\underset{\sim}{P}_1$ suffers from the disadvantage that its range is the class of analytic solutions of $L[u]=0$. However it is known from the general theory of parabolic equations that solutions of $L[u]=0$ are in general not analytic in the t variable, even though the coefficients of $L[u]=0$ are analytic functions of x and t (c.f. [26]).

Hence we will now construct a class of operators whose range is the class of solutions to (2.1.1) which are twice continuously differentiable with respect to x and continuously differentiable with respect to t. Such solutions will be called classical. We assume that in (2.1.1) we have $c(x,t) > 0$, and make the change of variables

$$\xi = \int_0^x \sqrt{c(s,t)}\,ds \qquad (2.1.26)$$

$$\tau = t$$

This transformation reduces (2.1.1) to an equation of the same form but with $c(x,t)=1$. If we now make a change of variables of the form (2.1.2) we arrive at an equation of the form

$$u_{xx} + q(x,t)u = u_t \qquad (2.1.27)$$

and we will henceforth restrict our attention to parabolic equations which are written in this canonical form. We will first consider classical solutions $u(x,t)$ of (2.1.27) defined in the rectangle $R^+ = \{(x,t) : 0 < x < x_o,\ 0 < t < t_o\}$ such that $u(x,t)$ is continuously differentiable for $0 \leq x < x_o$, $0 < t < t_o$, and make the assumptions that $q(x,t)$ is continuously differentiable for $-x_o < x < x_o$, $|t| < t_o$, and is an analytic function of t for $|t| < t_o$. Here x_o and t_o are again positive constants.

Remark 1: The assumptions on $q(x,t)$ can be weakened.

Remark 2: The operators we are about to consider are related to the translation operators of Levitan for ordinary differential equations (c.f. [40 every solution of $L[u]=0$ which is analytic for $|t| < t_o$, $|x| < x_o$, where t_o and x_o are positive constants, can be represented in the form (2.1.23). For let $u(x,t)$ be an analytic solution of $L[u]=0$ for $|x| < x_o$, $|t| < t_o$, and set $u(0,\tau)=f(\tau)$, $u_x(0,\tau)=g(\tau)$. Then $f(\tau)$ and $g(\tau)$ are analytic for $|\tau| < t_o$. Define $w(x,t)$ by

$$w(x,t) = \underset{\sim}{P}_1 \{f,g\} . \tag{2.1.24}$$

Then $w(x,t)$ is an analytic solution of $L[u]=0$ and from (2.1.5a), (2.1.5b), (2.1.6a), (2.1.6b) we have

$$w(0,t) = -\frac{1}{2\pi i} \oint_{|t-\tau|=\delta} \frac{f(\tau)}{t-\tau}\, d\tau = f(t) \tag{2.1.25}$$

$$w_x(0,t) = -\frac{1}{2\pi i} \oint_{|t-\tau|=\delta} \frac{g(\tau)}{t-\tau}\, d\tau = g(t) \quad ;$$

i.e. the Cauchy data for $w(x,t)$ and $u(x,t)$ argree on the noncharacteristic curve $x=0$. From the Cauchy-Kowalewski theorem (c.f.[21]) we can now conclude that $u(x,t) = w(x,t)$, i.e. $u(x,t)$ can be represented in the form (2.1.23). We summarize our results in the following theorem:

<u>Theorem 2.1.1.</u>([9]): Let the coefficients $b(x,t)$ and $c(x,t)$ of (2.1.3) be analytic functions of the complex variables x and t for $|x| < \infty$, $|t| < t_o$. Then if $u(x,t)$ is a solution of (2.1.3) which is analytic for $|x| < x_o$, $|t| < t_o$, $u(x,t)$ can be prepresented in the form $u(x,t)=\underset{\sim}{P}_1 \{f,g\}$ where $f(t)=u(0,t)$ and $g(t)=u_x(0,t)$ are analytic functions of t for $|t| < t_o$. Conversely if $f(t)$ and $g(t)$ are analytic for $|t| < t_o$ then $u(x,t) = \underset{\sim}{P}_1 \{f,g\}$ is a solution of (2.1.3) which is an analytic function of x and t for $|x| < \infty$, $|t| < t_o$.

$$E^{(2)}(x,t,\tau) = \frac{x}{t-\tau} + \sum_{n=3}^{\infty} x^n q^{(n)}(x,t,\tau) \tag{2.1.21}$$

where the $q^{(n)}(x,t,\tau)$ are analytic functions (except for $t=\tau$) to be determined. We again note that if termwise differentiation is permitted the series (2.1.21) satisfies the initial conditions (2.1.6a), (2.1.6b). Substituting (2.1.21) into (2.1.3) leads to the following recursion formulas for the $q^{(n)}(x,t,\tau)$:

$$\begin{aligned} q^{(2)} &= 0 \\ q^{(3)} &= -\frac{c}{6(t-\tau)^2} - \frac{b}{6(t-\tau)} \\ q^{(k+2)} &= -\frac{2}{k+2} q_x^{(k+1)} - \frac{1}{(k+2)(k+1)} \left[q_{xx}^{(k)} + bq^{(k)} - cq_t^{(k)}\right] \; ; \; k \geq 2. \end{aligned} \tag{2.1.22}$$

The recursion scheme (2.1.22) is almost identical to the scheme given in (2.1. 8), and following our previous analysis we can again verify that the series (2.1.21) defines an analytic function of x,t and τ for $|x| < \infty$, $|t| < t_o$, $|\tau| < t_o$, $t \neq \tau$, which satisfies $L[u]=0$ for $t \neq \tau$ and the initial data (2.1.6a), (2.1.6b). At the point $t=\tau$, $E^{(2)}(x,t,\tau)$ has an essential singularity. It is of interest to contrast this singular nature of the functions $E^{(1)}(x,t,\tau)$ and $E^{(2)}(x,t,\tau)$ with the analytic nature of the generating function of the Bergman operator $\underset{\sim}{B}_2$ for elliptic equations.

We have now shown that the integral operator defined by

$$\begin{aligned} u(x,t) = \underset{\sim}{P}_1\{f,g\} = &-\frac{1}{2\pi i} \oint_{|t-\tau|=\delta} E^{(1)}(x,t,\tau) f(\tau) d\tau \\ &-\frac{1}{2\pi i} \oint_{|t-\tau|=\delta} E^{(2)}(x,t,\tau) g(\tau) d\tau \end{aligned} \tag{2.1.23}$$

exists and maps ordered pairs of analytic functions into the class of analytic solutions of $L[u]=0$. It is a simple matter to show that in fact

Then

$$(1-\frac{x}{r}) \geq \frac{\alpha-1}{\alpha} \qquad (1-\frac{\tau}{2t_o}) \geq \frac{1}{2}$$
$$(1-\frac{t}{t_o}) \geq \frac{\delta_1}{(1+\delta_1)} \qquad |t-\tau| < 2t_o \tag{2.1.19}$$

Hence for x,t,τ restricted as in (2.1.18) we have from (2.1.13) that the series (2.1.7) is majorized by

$$\frac{1}{\delta_o} + \sum_{n=2}^{\infty} \frac{M_n 16^n t_o^n (\frac{3}{2}+\varepsilon)^n (\alpha-1)^n (1+\delta_1)^n}{\alpha^{2n}\delta_o^n \delta_1^n} \quad . \tag{2.1.20}$$

Due to the fact that M_n is a bounded function of n, it is seen that if α is chosen sufficiently large then the series (2.1.20) converges. Since δ_o, δ_1 and ε are arbitrarily small (and independent of r) and r can be chosen arbitrarily large, we can now conclude that the series (2.1.20) converges uniformly and absolutely for $|x| \leq r$, $|t| \leq t_o/(1+\delta_1)$, $|\tau| \leq t_o$ and $|t-\tau| \geq \delta_o$ for δ_o and δ_1 arbitrarily small and r arbitrarily large. Since each term of the series (2.1.20) is an analytic function of x, t and τ for $|x| < \infty$, $|t| < t_o$, $|\tau| < t_o$, $\tau \neq t$, we can conclude that $E^{(1)}(x,t,\tau)$ exists and is an analytic function of its independent variables for $|x| < \infty$, $|t| < t_o$, $|\tau| < t_o$ and $t \neq \tau$. At the point $t=\tau$, $E^{(1)}(x,t,\tau)$ has an essential singularity. It is clear from the above discussion that termwise differentiation of the series (2.1.7) is permissible and hence $E^{(1)}(x,t,\tau)$ satisfies the differential equation (2.1.3) and the initial conditions (2.1.5a) and (2.1.5b).

We now turn our attention to the construction of the function $E^{(2)}(x,t,\tau)$. Setting $f(\tau)=0$ in (2.1.4) and substituting this equation into (2.1.3) shows that, as a function of x and t, $E^{(2)}(x,t,\tau)$ must be a solution of $L[u]=0$ for $t \neq \tau$. We now assume that $E^{(2)}(x,t,\tau)$ has the expansion

$$M_{n+2} = (\tfrac{3}{2} + \varepsilon)^{-1} \Big[M_{n+1} + \frac{M_n}{(\frac{3}{2} + \varepsilon)} \Big(\frac{n}{2(n+2)} + \frac{M_o r^2}{2(n+2)(n+1)} + \frac{nM_o r^2}{(n+2)(n+1)t_o}\Big)\Big] \tag{2.1.14}$$

The proof of (2.1.13) now follows by induction once we have shown that M_n is a bounded function of n. For $n \geq n_o = n_o(\varepsilon)$ we have from (2.1.14) that

$$M_{n+2} \leq (\tfrac{3}{2} + \varepsilon)^{-1} \Big[M_{n+1} + \frac{M_n}{(\frac{3}{2} + \varepsilon)} (\tfrac{1}{2} + \tfrac{\varepsilon}{2})\Big] \; ; \; n \geq n_o . \tag{2.1.15}$$

If $M_{n+1} \leq M_n$ for $n \geq n_o$ we are done, for then we have $M_n \leq \max \{ M_1, M_2, \ldots, M_{n_o} \}$. Suppose then that there exists $n_1 \geq n_o$ such that $M_{n_1+1} > M_{n_1}$. Then from (2.1.15) we have

$$\begin{aligned} M_{n_1+2} &< (\tfrac{3}{2} + \varepsilon)^{-1} \Big[M_{n_1+1} + M_{n_1+1} \frac{(\frac{1}{2} + \frac{\varepsilon}{2})}{(\frac{3}{2} + \varepsilon)}\Big] \\ &= \frac{(2 + \frac{3\varepsilon}{2})}{(\frac{3}{2} + \varepsilon)(\frac{3}{2} + \varepsilon)} M_{n_1+1} \qquad (2.1.16) \\ &< M_{n_1+1} \end{aligned}$$

and by induction

$$M_{n_1+m} \leq M_{n_1+1} \tag{2.1.17}$$

for m=1,2,3,... . Hence $M_n \leq \max \{ M_1, M_2, \ldots M_{n_1+1}\}$ and we can conclude that M_n is a bounded function of n.

We now return to the convergence of the series (2.1.7). Let δ_o, δ_1, and $\alpha > 1$ be positive numbers and let

$$\begin{aligned} &|x| \leq r/\alpha && |\tau| \leq t_o \\ &|t| \leq t_o/(1+\delta_1) && |t-\tau| \geq \delta_o . \end{aligned} \tag{2.1.18}$$

If we now define $Q^{(k)}(x,t,\tau)$ by the equation

$$Q^{(k)}(x,t,\tau) = \tau^k p^{(k)}(x,t,t-\tau) \tag{2.1.9}$$

then (2.1.8) yields the following recursion formula for the $Q^{(k)}(x,t,\tau)$:

$$Q^{(1)} = 0$$

$$Q^{(2)} = -\frac{1}{2}\left[c+\tau b\right]$$

$$\begin{aligned} Q^{(k+2)} = -\frac{2\tau}{k+2} Q_x^{(k+1)} - \frac{2\tau}{(k+2)(k+1)} \Big[\tau\, Q_{xx}^{(k)} + \tau b\, Q^{(k)} \\ - \tau c\, Q_t^{(k)} + ck\, Q^{(k)} - \tau c\, Q_\tau^{(k)}\Big] \; ; \; k \geq 1 \; . \end{aligned} \tag{2.1.10}$$

Now let M_o be a positive constant such that

$$c(x,t) << M_o\left(1 - \frac{x}{r}\right)^{-1}\left(1- \frac{t}{t_o}\right)^{-1}$$

$$b(x,t) << M_o\left(1 - \frac{x}{r}\right)^{-1}\left(1- \frac{t}{t_o}\right)^{-1} \tag{2.1.11}$$

for $|x| < r$ and $|t| < t_o$. Using the fact that

$$\tau << 2\, t_o\left(1- \frac{t}{2t_o}\right)^{-1} \tag{2.1.12}$$

we shall now show by induction that there exist positive constants M_n, n=1,2,..., and ε (where ε can be chosen arbitrarily small and is independent of n and M_n is a bounded function of n) such that for $|x| < r$, $|t| < t_o$, $|\tau| < 2t_o$, we have

$$\begin{aligned} Q^{(n+1)} << M_{n+1}\, 4^{n+1}\, t_o^{n+1} \left(\frac{3}{2} + \varepsilon\right)^{n+1} \\ \cdot\left(1- \frac{x}{r}\right)^{-(n+1)}\left(1- \frac{t}{t_o}\right)^{-(n+1)} \left(1- \frac{\tau}{2t_o}\right)^{-(2n+2)} r^{-(n+1)} \; , \end{aligned} \tag{2.1.13}$$

n=0,1,2,... . (2.1.13) is clearly true for n=0 and n=1, and from (2.1.10) it can be shown that (2.1.13) is true for n=k+1 if the M_n are defined by the recursion formula

We now look for a solution of (2.1.3) in the form

$$u(x,t) = -\frac{1}{2\pi i}\oint_{|t-\tau|=\delta} E^{(1)}(x,t,\tau)f(\tau)d\tau - \frac{1}{2\pi i}\oint_{|t-\tau|=\delta} E^{(2)}(x,t,\tau)g(\tau)d\tau \tag{2.1.4}$$

where $t_o - |t| > \delta > 0$ and $f(\tau)$ and $g(\tau)$ are arbitrary analytic functions of τ for $|\tau| < t_o$. We will furthermore ask that $E^{(1)}(x,t,\tau)$ and $E^{(2)}(x,t,\tau)$ satisfy the initial conditions

$$E^{(1)}(0,t,\tau) = \frac{1}{t-\tau} \tag{2.1.5a}$$

$$E_x^{(1)}(0,t,\tau) = 0 \tag{2.1.5b}$$

$$E^{(2)}(0,t,\tau) = 0 \tag{2.1.6a}$$

$$E_x^{(2)}(0,t,\tau) = \frac{1}{t-\tau} \tag{2.1.6b}$$

and be analytic functions of their independent variables for $|x| < \infty$, $|t| < t_o$, $|\tau| < t_o$, $t \neq \tau$. We shall first construct the function $E^{(1)}(x,t,\tau)$. Setting $g(\tau)=0$ and substituting (2.1.4) into $L[u]=0$ shows that, as a function of x and t, $E^{(1)}(x,t,\tau)$ must be a solution of $L[u]=0$ for $t \neq \tau$. We now assume that $E^{(1)}(x,t,\tau)$ has the expansion

$$E^{(1)}(x,t,\tau) = \frac{1}{t-\tau} + \sum_{n=2}^{\infty} x^n p^{(n)}(x,t,\tau) \tag{2.1.7}$$

where the $p^{(n)}(x,t,\tau)$ are analytic functions to be determined. Note that if termwise differentiation is permitted the series (2.1.7) satisfies the initial conditions (2.1.5a) and (2.1.5b). Substituting (2.1.7) into (2.1.3) we are led to the following recursion formula for the $p^{(n)}(x,t,\tau)$:

$$p^{(1)} = 0$$

$$p^{(2)} = -\frac{c}{2(t-\tau)^2} - \frac{b}{2(t-\tau)} \tag{2.1.8}$$

$$p^{(k+2)} = -\frac{2}{k+2}\, p_x^{(k+1)} - \frac{1}{(k+2)(k+1)}\left[p_{xx}^{(k)} + bp^{(k)} - cp_t^{(k)}\right] \; ; \; k \geq 1 \; .$$

II Parabolic equations in one space variable

2.1 Integral Operators.

We now want to develop a theory for parabolic equations in one space variables that is analegous to the theory just developed for elliptic equations in two independent variables. To this end we consider the general linear homogeneous parabolic equation of the second order in one space variable written in normal form as

$$u_{xx} + a(x,t)u_x + b(x,t)u - c(x,t)u_t = 0 \quad . \tag{2.1.1}$$

In the theory we are about to develop we will need to construct a variety of integral operators for (2.1.1), and in each such construction we will impose somewhat different assumptions on the coefficients of (2.1.1). The first operator we will consider will map ordered pairs of analytic functions of a single complex variable onto analytic solutions of (2.1.1). In order to construct this operator we will make the assumption that the coefficients $a(x,t)$, $b(x,t)$ and $c(x,t)$ in (2.1.1) are analytic functions of the complex variables x and t for $|x| < \infty$ and $|t| < t_o$ where t_o is a positive constant. By making the change of dependent variable

$$u(x,t) = v(x,t) \exp \{ -\frac{1}{2} \int_o^x a(\xi,t)d\xi \} \tag{2.1.2}$$

we arrive at an equation for $v(x,t)$ of the same form as (2.1.1) but with $a(x,t)=0$. Hence without loss of generality we can restrict our attention to equations of the form

$$L[u] \equiv u_{xx} + b(x,t)u - c(x,t)u_t = 0 \tag{2.1.3}$$

where $b(x,t)$ and $c(x,t)$ are analytic functions of x and t for $|x| < \infty$, $|t| < t_o$.

P.R. Garabedian and D.G. Korn, Numerical design of transonic airfoils, in Numerical Solution of Partial Differential Equations-II, Bert Hubbard, editor, Academic Press, 1971, 253-271.

D.G. Korn, Transonic design in two dimensions, in Constructive and Computational Methods for Differential and Integral Equations, D.L. Colton and R.P. Gilbert, editors, Springer-Verlag Lecture Note Series Vol.430, Springer-Verlag, 1974, 271-288.

singularity at the point $\log(-\bar{A}) = \zeta$ which corresponds to a dipole of the compressible fluid flow; however $\Psi(\lambda,\theta)$ will not in general be zero on the boundary ∂D of the non-schlicht domain D.

Having obtained a solution $\Psi(\lambda,\theta)$ of (1.5.9) with prescribed singularity at $\zeta = \log(-\bar{A})$, we now have to see what type of flow this solution represents in the physical (x,y) plane. It can be easily seen that the point $\zeta = \log(-\bar{A})$ in the (λ,θ) plane corresponds to the point at infinity in the (x,y) plane and that the flow behaves there as if a dipole were situated at infinity. The curve C* in the (x,y) plane on which ψ vanishes will of course be different in general from the original curve C for which we wanted to solve the boundary value problem; however C* will not be too different from C if the velocities involved are not too near the sonic velocity $q = c$. By choosing complex velocity potentials associated with different curves C and constructing $\Psi(\lambda,\theta)$ as above we now have an inverse method for obtaining subsonic compressible flows past an obstacle originating from a dipole at infinity.

For an example of numerical experiments using the methods of this section see [3] and [4] .

The basic ideas of the approach described above for subsonic fluid flow can also be used to study transonic flow problems. Particular problems of course arise due to the need to continue the solution past the sonic line, and the analysis is by no means trivial. However these difficulties have been overcome and the use of integral operators and inverse methods has recently led to the numerical design of shock-free transonic flow at specified cruising speeds. The interested reader is referred to

S. Bergman, Two-dimensional transonic flow patterns, Amer.J.Math.70(1948),856-891.

From (1.5.31) and (1.5.32) we have

$$\zeta = \log(-\bar{A}) - \left(\frac{\bar{a}_1}{\bar{A}} \right) \frac{1}{\bar{z}^2} + \ldots \tag{1.5.33}$$

and hence for ζ near $\bar{\beta} = \log(-\bar{A})$

$$z = \frac{a}{(\bar{\zeta}-\beta)^{\frac{1}{2}}} + P((\bar{\zeta}-\beta)^{\frac{1}{2}}) \tag{1.5.34}$$

where $P(\tau)$ is a power series in τ about the origin and $a = -\left(\frac{a_1}{A} \right)^{\frac{1}{2}}$.

In the pseudo-logarithmic plane we thus obtain a complex potential $F^*(\zeta)$ defined by means of

$$\overline{F^*(\zeta)} = F(z(\bar{\zeta})) = \frac{A^*}{(\bar{\zeta}-\beta)^{\frac{1}{2}}} + P^*((\bar{\zeta}-\beta)^{\frac{1}{2}}) \tag{1.5.35}$$

where $P^*(\tau)$ is a power series in τ about the origin and A^* is a constant. From (1.5.33) we see that the image of the flow domain in the (λ,θ) plane covers this plane in a non-schlicht manner and has the point $\zeta = \log(-\bar{A})$ as a second order branch point. The stream function $\psi(\lambda,\theta)$ is defined by

$$\psi(\lambda,\theta) = \mathrm{Re}(iF^*(\zeta)) \tag{1.5.36}$$

(c.f. (1.5.30) and (1.5.35))and vanishes on the boundary ∂D of the image of the flow domain D in the ζ plane.

Now associate with $iF^*(\zeta) = \Phi_o(\zeta)$ a solution $\Psi(\lambda,\theta)$ of (1.5.9) defined by (1.5.19) (we will assume that the image of the flow domain in the pseudo-logarithmic plane lies entirely in the region where the operator (1.5.19) is applicable). Note that for small velocities $\ell(\lambda) \approx$ constant and hence $L(\lambda)$ can be assumed to be small for large negative values of λ. This in turn implies the constant C in (1.5.10) is small and hence from the recursion relations (1.5.23), (1.5.24) $U(\lambda,\theta;t)$ is small for large negative values of λ, i.e. for such values of λ $\Psi(\lambda,\theta)$ does not differ too much from $\mathrm{Re}(\Phi_o(\zeta))$, the corresponding solution in the incompressible case. $\Psi(\lambda,\theta)$ has a i.e. ζ must satisfy

$$|\zeta| \leq 2a|\lambda| < 2|\lambda| \tag{1.5.28}$$

and since $\zeta = \lambda+i\theta$ we have

$$|\theta| < \sqrt{3}|\lambda| \tag{1.5.29}$$

i.e. ζ must lie in an angle of 120° symmetric to the λ axis in the left half plane. Conversely, if ζ lies in this angular region it can always be connected with the origin by a path along which t fulfills (1.5.27) for an appropriate value of $a < 1$. Hence we can construct solutions of (1.5.9) in the region (1.5.29) by means of (1.5.19).

We will now show how the operator defined by (1.5.19) can serve as the basis for the development of an inverse approach to solving boundary value problems in the theory of subsonic, compressible fluid flow. We restrict our attention to the case in which the flow domain in the physical (x,y) plane contains the point at infinity and in which the flow originates from a dipole there (i.e. the velocity at infinite is uniform). Let the flow domain be bounded by a closed curve C, and let

$$F(z) = \phi+i\psi = Az+a_o + \frac{a_1}{z} + \ldots \tag{1.5.30}$$

be the complex velocity potential in the case of <u>incompressible</u> flow (expanded about the dipole at infinity). The function F(z) can be obtained by classical methods in analytic function theory (c.f. [6]). Note that on C we have $\psi = 0$. The velocity function of the incompressible flow near infinity is now given by

$$w = - F'(z) = u-iv = -A + \frac{a_1}{z^2} + \ldots, \tag{1.5.31}$$

and since in the case of an incompressible fluid flow $\lambda = \log q$ we have

$$\zeta = \lambda+i\theta = \log \bar{w} . \tag{1.5.32}$$

The system (1.5.21) can be solved explicitly by setting

$$Q_n(\lambda) = n!\ (\varepsilon-\lambda)^{-n}\mu_n \tag{1.5.22}$$

where the μ_n satisfy the recursion formula

$$\mu_o = 1$$

$$\mu_{n+1} = \mu_n \frac{(n+\alpha)(n+\beta)}{(n+1)^2} \tag{1.5.23}$$

with

$$\alpha = \frac{1}{2} -(\frac{1}{4} - C)^{\frac{1}{2}}$$
$$\beta = \frac{1}{2} +(\frac{1}{4} - C)^{\frac{1}{2}} \quad . \tag{1.5.24}$$

From (1.5.10) and (1.5.13), (1.5.15) it is easily seen that

$$G_n(\lambda) << Q_n(\lambda) \ , \tag{1.5.25}$$

and hence the series (1.5.20) is majorized by the series

$$\begin{aligned}\Omega(\lambda,\theta;t) &= \sum_{n=1}^{\infty} \frac{Q_n(\lambda)}{(n-1)!2^n} |\zeta-t|^{n-1} \\ &= \sum_{n=1}^{\infty} n\ \mu_n \frac{|\zeta-t|^{n-1}}{2^n(\varepsilon-\lambda)^n} \\ &= \frac{1}{2}(\varepsilon-\lambda)^{-1} \sum_{n=1}^{\infty} n\ \mu_n (\frac{|\zeta-t|}{2(\varepsilon-\lambda)})^{n-1} \\ &= \frac{1}{2(\varepsilon-\lambda)} \frac{d}{dx} F(\alpha,\beta,1; \frac{|\zeta-t|}{2(\varepsilon-\lambda)})\end{aligned} \tag{1.5.26}$$

where $F(\alpha,\beta,1;x)$ is the hypergeometric function of Gauss. But it is well known that the hypergeometric series for $F(\alpha,\beta,1;x)$ coverges uniformly for $|x| \leqslant a < 1$ provided α and β are not zero or a positive integer, which from (1.5.24) is certainly not the case here. Hence the series (1.5.26) (and hence the series (1.5.20)) converges uniformly for

$$\frac{|\zeta-t|}{2(\varepsilon-\lambda)} \leqslant a < 1 \quad , \lambda < \varepsilon < 0 \tag{1.5.27}$$

rest of the fluid.

We first solve the recursion relation (1.5.14). Let $\phi_o(\zeta)$ be an analytic function of $\zeta = \lambda+i\theta$ and let $g_o(\lambda,\theta) = \text{Re }\phi_o$. Then (1.5.14) will hold if

$$g_n(\lambda,\theta) = \text{Re }\Phi_n(\zeta) \tag{1.5.15}$$

where

$$\frac{d\Phi_n(\zeta)}{d\zeta} = -\frac{1}{2}\Phi_{n-1}(\zeta), \quad n = 1,2,\ldots \quad . \tag{1.5.16}$$

In particular a solution of (1.5.16) is given by

$$\Phi_n(\zeta) = \frac{(-1)^n}{(n-1)!2^n}\int_o^\zeta \Phi_o(t)(\zeta-t)^{n-1}dt \tag{1.5.17}$$

and hence

$$g_n(\lambda,\theta) = \frac{(-1)^n}{(n-1)!2^n}\text{ Re }\left(\int_o^\zeta \Phi_o(t)(\zeta-t)^{n-1}dt\right) \tag{1.5.18}$$

for $n = 1,2,\ldots$. A formal solution of (1.5.9) is thus given by

$$\psi(\lambda,\theta) = \text{Re}(\Phi_o(\zeta) + \int_o^\zeta \Phi_o(t)U(\lambda,\theta;t)dt) \tag{1.5.19}$$

where

$$U(\lambda,\theta;t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!2^n} G_n(\lambda)(\zeta-t)^{n-1}. \tag{1.5.20}$$

Our formal analysis will now be valid provided we can show that the series (1.5.20) converges uniformly to an analytic function for λ and θ in the region of definition of $\psi(\lambda,\theta)$ and for t in the region of integration in (1.5.19). We will do this through the method of dominants. Define the functions $Q_n(\lambda)$ by the recursive scheme

$$\begin{aligned} &Q_o = 1 \\ &Q'_{n+1} = Q''_n + C(\varepsilon-\lambda)^{-2}Q_n \quad ; \quad n = 1,2,\ \ldots \\ &Q_n(-\infty) = 0 \ ; \ n = 1,2,\ \ldots \quad . \end{aligned} \tag{1.5.21}$$

We will now construct an integral operator which maps analytic functions into the class of solutions of (1.5.9). We will make the assumption (valid in particular for the case of an adiabatic gas where the pressure p is given by p = constant ρ^{γ} for some constant γ) that $L(\lambda)$ is an analytic function of λ for $\lambda < 0$ and has a dominant of the form

$$L(\lambda) << C(\varepsilon-\lambda)^{-2} \tag{1.5.10}$$

where $C > 0$, $\varepsilon < 0$ ((1.5.10) is interpreted in the sense that $\left|\frac{d^n L(\lambda)}{d\lambda^n}\right| \leq C \frac{d^n}{d\lambda^n}(\varepsilon-\lambda)^{-2}$ for $n = 0, 1, 2, \ldots$.) Note that the integral operators previously constructed are not applicable in this special case, since $L(\lambda)$ is not in general an entire function of λ. We first look for a formal solution of (1.5.9) in the form

$$\psi(\lambda,\theta) = \sum_{n=0}^{\infty} G_n(\lambda) g_n(\lambda,\theta) \tag{1.5.11}$$

where $g_n(\lambda,\theta)$ is a harmonic function of λ and θ.

Proceeding formally we have

$$0 = \Delta\psi - L\psi = \sum_{n=0}^{\infty} (g_n(G''_n - LG_n) + 2 \frac{\partial g_n}{\partial\lambda} G'_n \tag{1.5.12}$$

and hence we require

$$G_o = 1$$

$$G'_{n+1} = G''_n - LG_n \quad ; \quad n = 1,2, \ldots \tag{1.5.13}$$

$$2 \frac{\partial g_n}{\partial\lambda} = - g_{n-1} \tag{1.5.14}$$

with $g_o(\lambda,\theta)$ an arbitrary harmonic function. We normalize the $G_n(\lambda)$ by imposing the condition

$$G_n(-\infty) = 0, \quad n = 1,2, \ldots$$

which is motivated by the fact that $\lambda = -\infty$ corresponds to $q = 0$, the state of which are polar coordinates in the hodograph plane (u,v). Under this change of variables the nonlinear system (1.5.2) becomes the linear system (c.f. [6]).

$$\phi_\theta = \frac{q}{\rho}\psi_q \qquad (1.5.4)$$
$$\phi_q = -\left(1-\frac{q^2}{c^2}\right)\frac{1}{\rho q}\psi_\theta$$

where $c^2 = c^2(q^2)$ is the square of the local velocity of sound in the medium. If we now make the assumption that the flow is subsonic i.e. $q^2 < c^2$, and transform (1.5.4) into the pseudo-logarithmic plane (λ,θ) by means of the change of variables

$$\lambda = \int^q \frac{1}{q}\left(1-\frac{q^2}{c^2}\right)^{\frac{1}{2}}dq \qquad (1.5.5)$$
$$\theta = \theta$$

we arrive at the system

$$\phi_\theta = \ell(\lambda)\psi_\lambda \qquad (1.5.6)$$
$$\phi_\lambda = -\ell(\lambda)\psi_\theta$$

where
$$\ell(\lambda) = \frac{1}{\rho}\left(1-\frac{q^2}{c^2}\right)^{\frac{1}{2}} \qquad (1.5.7)$$

is a known function depending on the physical nature of the fluid under consideration. Note that in the case of an incompresible fluid flow $(c=\infty)$ the system (1.5.6) reduces (after the introduction of an appropriate scaling factor) to the Cauchy-Riemann equations. Eliminating $\phi(\lambda,\theta)$ from (1.5.6) gives

$$\ell(\lambda)\left[\psi_{\lambda\lambda} + \psi_{\theta\theta}\right] + \ell'(\lambda)\psi_\lambda = 0 \qquad (1.5.8)$$

where $\ell'(\lambda) = \frac{d\ell}{d\lambda}$. Setting $\psi = \ell^{\frac{1}{2}}\psi$ now gives

$$\psi_{\lambda\lambda} + \psi_{\theta\theta} - L(\lambda)\psi = 0 \qquad (1.5.9)$$

where
$$L(\lambda) = \frac{\Delta(\ell^{\frac{1}{2}})}{\ell^{\frac{1}{2}}}, \quad \Delta = \frac{\partial^2}{\partial\lambda^2} + \frac{\partial^2}{\partial\theta^2} .$$

1.5 Inverse Methods in Compressible Fluid Flow.

In this section we will be considering stationary, irrotational flow of a two dimensional compressible fluid, and will derive an inverse method for obtaining flows past an obstacle due to a dipole at infinity. The approach we are about to derive is due to S.Bergman and our presentation is based on the material in [6]. Another excellent survey of the present topic can be found in [41].

Let $\vec{q}$ be the velocity vector of the motion and $\phi(x,y)$ be the velocity potential, i.e.

$$\vec{q} = -\,\mathrm{grad}\phi = (u,v) \tag{1.5.1}$$

where $u = -\frac{\partial\phi}{\partial x}$, $v = -\frac{\partial\phi}{\partial y}$. Let $\rho(x,y)$ denote the density of the fluid where $\rho = \rho(q^2)$, $q^2 = (\frac{\partial\phi}{\partial x})^2 + (\frac{\partial\phi}{\partial y})^2$. Then from the equation of continuity $\mathrm{div}\,(\rho\vec{q}) = 0$ we can assert the existence of a stream function $\psi(x,y)$ such that

$$\begin{aligned} \rho\,\frac{\partial\phi}{\partial x} &= \frac{\partial\psi}{\partial y} \\ \rho\,\frac{\partial\phi}{\partial y} &= -\frac{\partial\psi}{\partial x}, \end{aligned} \tag{1.5.2}$$

and where $\psi(x,y)$ remains constant along each stream line.

We now introduce the new variables

$$\begin{aligned} q &= (u^2+v^2)^{\frac{1}{2}} \\ \theta &= \arctan\frac{v}{u} \end{aligned} \tag{1.5.3}$$

(The inverse mapping is given by (c.f. [6]).

$$x = -\int \left(\frac{1}{q}\left(\cos\theta\phi_q - \frac{\sin\theta}{\rho}\psi_q\right)dq + \frac{1}{q}\left(\cos\theta\phi_\theta - \frac{\sin\theta}{\rho}\psi_\theta\right)d\theta \right)$$

$$y = -\int \left(\frac{1}{q}\left(\sin\theta\phi_q + \frac{\cos\theta}{\rho}\psi_q\right)dq + \frac{1}{q}\left(\sin\theta\phi_\theta + \frac{\cos\theta}{\rho}\psi_\theta\right)d\theta \right))$$

$$||u(x,y) - \sum_{n=0}^{N} a_n\phi_n(x,y)||^2_D < \varepsilon. \tag{1.4.10}$$

In particular since the set $\{\phi_n\}$ is an orthonormal set, the optimum choice of the constants $a_0, \ldots, a_N$ is given by

$$\begin{aligned} a_n &= (u,\phi_n)_D \\ &= -\int_{\partial D} u \frac{\partial\phi_n}{\partial\nu}\, ds \quad . \end{aligned} \tag{1.4.11}$$

From (1.4.9) and (1.4.10) we have

$$|u(\xi,\eta) - \sum_{n=0}^{N} a_n\phi_n(\xi,\eta)|^2 < \varepsilon\, K(\xi,\eta\ ;\ \xi,\eta)\ , \tag{1.4.12}$$

and hence the series

$$u(x,y) = \sum_{n=0}^{\infty} a_n\phi_n(x,y) \tag{1.4.13}$$

$$a_n = (u,\phi_n)_D$$

converges uniformly to $u(x,y)$ in every closed subdomain D_o of D. In particular setting $u(x,y) = K(x,y\ ;\ \xi,\eta)$ we have from (1.4.8)

$$(\phi_n(x,y),\ K(x,y\ ;\ \xi,\eta))_D = \phi_n(\xi,\eta) \tag{1.4.14}$$

and hence for (x,y) and (ξ,η) on compact subsets of D we have the remarkable representation

$$K(x,y\ ;\ \xi,\eta) = \sum_{n=0}^{\infty} \phi_n(x,y)\phi_n(\xi,\eta\ ). \tag{1.4.15}$$

Note that the representation (1.4.15) is in fact independent of the particular orthonormal system $\{\phi_n\}$ we started out with.

Numerical methods based on the kernel function can be found in [5]. Other numerical methods for solving the Dirichlet problem for elliptic equations using the method of integral operators can be found in [1], [24], [33], [34] and [47]

If we define the inner product $(\cdot,\cdot)_D$ by

$$(u,v)_D = \iint_D \left[u_x v_x + u_y v_y - quv\right] dxdy \tag{1.4.6}$$

where $u,c \varepsilon C^1(\bar{D})$ we see that $(\cdot,\cdot)_D$ satisfies all the conditions of an inner product. In particular since $q(x,y) < 0$ for $(x,y)\varepsilon\bar{D}$ we have $||u||_D^2 = (u,u)_D = 0$ if and only if $u \equiv 0$ in D. From Green's formula we have the fact that if $v(x,y)$ is a solution of (1.4.1) then

$$(u,v)_D = -\int_{\partial D} u \frac{\partial v}{\partial \nu} ds \quad , \tag{1.4.7}$$

in particular if $u(x,y)$ is a solution of (1.4.1) then (1.4.5) can be written as the single relation

$$u(\xi,\eta) = (u(x,y), K(x,y\ ;\ \xi,\eta))_D \tag{1.4.8}$$

since both $u(x,y)$ and $K(x,y\ ;\ \xi,\eta)$ are solutions of (1.4.1). Equation (1.4.8) is known as the reproducing property of the kernel function. In particular from Schwarz's inequality we have

$$\begin{aligned} |u(\xi,\eta)|^2 &= |(u,K)_D|^2 \\ &< (u,u)_D (K,K)_D \\ &= K(\xi,\eta\ ;\ \xi,\eta)||u||^2_D. \end{aligned} \tag{1.4.9}$$

Now let $\{u_n\}$ be the family of solutions to (1.4.1) defined by $u_{2n} = \operatorname{Re} \underset{\sim}{B}_2 \{z^n\}$, $u_{2n+1} = \operatorname{Im} \underset{\sim}{B}_2 \{z^n\}$, and orthonormalize this set with respect to $(\cdot,\cdot)_D$ to obtain the set $\{\phi_n\}$. From Theorem 1.3.6 we have that the set $\{\phi_n\}$ is complete in the Dirichlet norm $||.||_D$ over $\bar{D}$ with respect to the class of solutions to (1.1.1) that are Hölder continuously differentiable. In particular if $u(x,y)$ is a solution of (1.4.1) which is Hölder continuously differentiable on ∂D then for any $\varepsilon > 0$ there exists an integer N and constants $a_0,\ldots,a_N$ such that

1.4 The Bergman Kernel Function.

We will restrict ourselves to the self-adjoint elliptic equation

$$u_{xx} + u_{yy} + q(x,y)u = 0 \tag{1.4.1}$$

where $q(x,y)$ is an entire function of its independent complex variables. We again consider solutions $u(x,y)$ of (1.4.1) defined in a domain which is bounded, simply connected, and in class Ah, and make the assumption that $q(x,y) < 0$ in $\bar{D}$. Let $N(x,y\,;\,\xi,\eta)$ and $G(x,y\,;\,\xi,\eta)$ be the Neumann's and Green's function respectively of (1.4.1) in D. Then the <u>kernel function</u> $K(x,y\,;\,\xi,\eta)$ of (1.4.1) in D is defined by

$$K(x,y\,;\,\xi,\eta) = N(x,y\,;\,\xi,\eta) - G(x,y\,;\,\xi,\eta). \tag{1.4.2}$$

Note that since the singularities of the singular parts of $N(x,y\,;\,\xi,\eta)$ and $G(x,y\,;\,\xi,\eta)$ cancel we have that $K(x,y\,;\,\xi,\eta)$ is regular in D both as a function of (x,y) and (ξ,η). Furthermore, due to the symmetry of the Neumann's and Green's function, we have

$$K(x,y\,;\,\xi,\eta) = K(\xi,\eta\,;\,x,y). \tag{1.4.3}$$

From the boundary conditions satisfied by the Neumann's and Green's function we have

$$\begin{aligned} K(x,y\,;\,\xi,\eta) &= N(x,y\,;\,\xi,\eta) \quad ; \quad (x,y)\in\partial D \\ \frac{\partial K}{\partial \nu}(x,y\,;\,\xi,\eta) &= -\frac{\partial G}{\partial \nu}(x,y\,;\,\xi,\eta) \quad ; \quad (x,y)\in\partial D \end{aligned} \tag{1.4.4}$$

where ν is the unit inward normal to ∂D. Hence if $u(x,y)\in C^1(D)$ is a solution of (1.4.1) we have from Green's formulas

$$\begin{aligned} u(\xi,\eta) &= -\int_{\partial D} u(t)\,\frac{\partial K(t;\xi,\eta)}{\partial \nu}\,ds \\ u(\xi,\eta) &= -\int_{\partial D} K(t;\xi,\eta)\,\frac{\partial u(t)}{\partial \nu}\,ds \end{aligned} \tag{1.4.5}$$

where $u(t) = u(x,y)$, $K(t;\xi,\eta) = K(x,y\,;\,\xi,\eta)$ for $(x,y) = (x(t),y(t))\in\partial D$ and ds denotes arclength.

first derivative in $\bar{D}$ (and hence $u(x,y)$ has Hölder continuous first derivatives in $\bar{D}$).

Further generalizations can be found in [47].

From Theorems 1.3.3 and 1.3.4 we can now approximate solutions to the Dirichlet problem for $L[u] = 0$ in D, where $D \varepsilon Ah$, $c(x,y) \leq 0$, in the following manner : Orthonormalize the set $\{u_n\}$ in the L^2 norm over ∂D to obtain the complete set $\{\phi_n\}$ and set

$$c_n = \int_{\partial D} f\phi_n \tag{1.3.37}$$

$$u^N(x,t) = \sum_{n=0}^{N} c_n\phi_n \quad . \tag{1.3.38}$$

(Since Hölder continuous functions can be approximated by continuous functions we can, by the maximum principle, assume that $f(t)$ is merely continuous on ∂D and still conclude from Theorem 1.3.4 and Theorem 1.3.3 that the set $\{\phi_n\}$ is complete in the maximum norm over $\bar{D}$, and hence complete in the L^2 norm over $\bar{D}$). As we have already discussed in the introduction, we can now conclude that the given $\varepsilon > 0$, N sufficiently large and D_o a compact subset of D,

$$\max_{D_o} |u-u^N| < \varepsilon \; . \tag{1.3.39}$$

Since each ϕ_n is a solution of $L[u] = 0$ in D, error estimates can be found in the case when $c(x,y) \leq 0$ in D by applying the maximum principle.

In the next section we will discuss an alternate method for approximating solutions to $L[u] = 0$ by means of a complete family of solutions. The method to be discussed is based on the <u>Bergman kernel function</u> (c.f. [6]).

From Walsh's generalization of Runge's theorem (c.f. [50]) we have that $U(z,0)$ can be uniformly approximated in $\bar{D}$ by polynomials and we therefore have the following extension of Theorem 1.3.1:

<u>Theorem 1.3.3</u> ([49]): Let $D \varepsilon Ah$ and let u_n be defined by (1.3.1). Then the set $\{u_n\}$ is complete in the maximum norm over D for the class of real valued solutions of $L[u] = 0$ which are Hölder continuous in $\bar{D}$. In order to apply Theorem 1.3.3 we need criteria for which a solution $u(x,y)$ of $L[u] = 0$ is Hölder continuous in $\bar{D}$. The following give criteria suitable for the purposes of these lectures.

<u>Theorem 1.3.4</u> ([49]): Suppose $c(x,y) \leq 0$ in D, $D \varepsilon Ah$, and $u(x,y) \varepsilon C^{0}(\bar{D}) \cap C^{2}(D)$ is a real valued solution of $L[u] = 0$ in D such that

$$u(t) = f(t) \qquad \text{on } \partial D$$

where $f(t)$ is Hölder continuous on ∂D. Then $u(x,y)$ is Hölder continuous in $\bar{D}$.

<u>Proof</u>: This follows from the maximum principle, lemma 1.3.3, and (1.3.13), (1.3.18), since if $\mu(t)$ is Hölder continuous on ∂D then $\phi(z)$ is Hölder continuous in $\bar{D}$.

<u>Theorem 1.3.5</u>: Suppose ∂D has Hölder continuous curvature. Then if $u(x,y) \varepsilon C^{0}(\bar{D}) \cap C^{2}(D)$ is a real valued solution of $L[u] = 0$ in D such that

$$u(t) = 0 \qquad \text{on } \partial D$$

then $u(x,y)$ and its first and second derivative are Hölder continuous in $\bar{D}$.

<u>Proof</u>: This follows immediately from the Shauder estimates (c.f. [21]).

We state now the following generalization of Theorems 1.3.2 and 1.3.4, the proof of which can be found in [49].

<u>Theorem 1.3.6</u> ([49]): Let $D \varepsilon Ah$ and $u(x,y) \varepsilon C^{0}(\bar{D}) \cap C^{2}(D)$ be a real valued solution of $L[u] = 0$ in D where $c(x,y) \leq 0$ in D and

$$u(t) = f(t) \qquad \text{on } \partial D,$$

where $\frac{df}{ds}$ is Hölder continuous on ∂D. Then $U(z,0)$ has a Hölder continuous is a solution of

$$w_{xx} + w_{yy} + (a+2\frac{\partial \log v}{\partial x})\frac{\partial w}{\partial x} + (b+2\frac{\partial \log v}{\partial y})\frac{\partial w}{\partial y} = 0 \qquad (1.3.34)$$

in D and $w(x,y)\epsilon C^o(\bar{D})\cap C^2(D)$, $w = 0$ on ∂D. Hence from the maximum principle $w(x,y) \equiv u(x,y) \equiv 0$ in D and the lemma is proved.

Definition 1.3.1: Let $v(x,y) = R(z_o,\bar{z}_o ; z,\bar{z})$ where $z_o = x_o+iy_o\epsilon D$. Then $v(x_o,y_o) = 1$ and hence there exists a neighbourhood of (x_o,y_o) such that $v(x,y) > 0$ in this neighbourhood; such a neighbourhood will be called a Riemann neighbourhood of the point z_o.

Theorem 1.3.2 ([49]): Let u(x,y) be a real valued solution of L[u] = 0 that is Hölder continuous in $\bar{D}$. Then U(z,0) is Hölder continuous in $\bar{D}$.

Proof: Let $\gamma c\partial D$ be a closed arc such that γ lies entirely inside some Riemann neighbourhood. Complete this arc to form a closed curve $\partial D'$ such that D' is of class Ah, D' lies inside D, and $\bar{D}'$ lies in a Riemann neighbourhood. From lemma 1.3.4 and our previous analysis we can represent u(x,y) inside D' as

$$u(x,y) = \int_{\partial D} \mu'(t)K(z,t)ds \qquad (1.3.35)$$

where $\mu'(t)$ is Hölder continuous on $\partial D'$. But from the representation (1.3.13) we see that

$$\phi(z) = \int_{\partial D'} \frac{t\mu'(t)ds}{t-z}, \quad z\epsilon D' \quad . \qquad (1.3.36)$$

From the properties of Cauchy integrals we have that $\phi(z)$ is Hölder continuous on γ and hence so is U(z,0). Covering ∂D by a finite number of overlapping closed arcs γ_i(each γ_i being contained in a Riemann neighbourhood) shows that U(z,0) is Hölder continuous on ∂D and hence, from the properties of Cauchy integrals, Hölder continuous in D.

$$A(t_o)\mu(t_o) + \int_{\partial D} K(t_o,t)\mu(t)ds = 0 \qquad (1.3.30)$$

has only the trivial solution $\mu(t) = 0$ on ∂D.

Lemma 1.3.3 ([49]): If the homogeneous boundary value problem

$$L[u] = 0 \text{ in } D$$

$$u = 0 \text{ on } \partial D \qquad (1.3.31)$$

$$u \in C^o(\bar{D}) \cap C^2(D)$$

has only the trivial solution $u \equiv 0$ in D then there exists a unique Hölder continuous solution $\mu(t)$ of (1.3.23).

Proof: We must show that the only solution of (1.3.30) is the trivial solution $\mu(t) = 0$ on ∂D. Let $\mu_o(t)$ be a solution of (1.3.30). Then

$$u_o(x,y) = \mathrm{Re}\left[H_o(z)\phi_o(z) + \int_o^z H(z,t)\phi_o(t)dt\right] \qquad (1.3.32)$$

with

$$\phi_o(z) = \int_{\partial} \frac{t\mu_o(t)ds}{t-z}, \quad t\varepsilon\partial D \qquad (1.3.33)$$

is a solution of (1.3.31) which is continuous in $\bar{D}$ and satisfies $u_o = 0$ on ∂D. Hence from the hypothesis of the theorem $u_o(x,y) \equiv 0$ in D and hence $\phi_o(z) \equiv 0$ in D. But this is the case if and only if $\mu_o(t) = 0$ for t on ∂D (c.f. [42]) and the lemma is proved.

From the maximum principle the hypothesis of lemma 1.3.2 are satisfied if $c(x,y) \leq 0$ in D. The following lemma gives an alternative sufficient criteria for these hypothesis to be valid.

Lemma 1.3.4 ([49]): If there exists a real valued solution $v(x,y)\varepsilon C^o(\bar{D})\cap C^2(D)$ of $L[u] = 0$ in D such that $v(x,y) \neq 0$ for $(x,y)\varepsilon\bar{D}$ then the homogeneous boundary value problem (1.3.31) has only the trivial solution.

Proof: Let $u(x,y)$ be a solution of (1.3.31). Then $w(x,y) = \dfrac{u(x,y)}{v(x,y)}$

singular integral equation for the unknown function $\mu(t)$. We now rewrite (1.3.23) in the form

$$A(t_o)\mu(t_o) + \frac{B(t_o)}{i\pi}\int_{\partial D}\frac{\mu(t)dt}{t-t_o} + \int_{\partial D}K_o(t_o,t)\mu(t)ds = f(t_o) \qquad (1.3.25)$$

where

$$B(t_o) = i\pi\ \mathrm{Re}\left[t_o\bar{t}_o{}'H_o(t_o)\right] \qquad (1.3.26)$$

$$K_o(t_o,t) = K(t_o,t) - \frac{t'B(t_o)}{i\pi(t-t_o)}$$

and note that $K_o(t_o,t)$ has the form

$$K_o(t_o,t) = \frac{K^*(t_o,t)}{|t-t_o|^{\alpha}} \qquad (1.3.27)$$

where $0 \leqslant \alpha < 1$ and $K^*(t_o,t)$ is a Hölder continuous function on $\partial D\times\partial D$.

From (1.3.24) and (1.3.26) we have

$$\begin{aligned} A(t_o) + B(t_o) &= i\pi t_o\overline{t_o{}'}H_o(t_o) \\ A(t_o) - B(t_o) &= -i\pi\bar{t}_o t_o{}'\overline{H_o(t_o)} \end{aligned} \qquad (1.3.28)$$

and since $H_o(t_o) \neq 0$ for $t_o \in \partial D$ we have that A+B and A−B are nonzero on ∂D. Hence (1.3.23) is of <u>normal type</u> and the general theory of singular integral equations can be applied (c.f. [42]).

From (1.3.28) we have that the <u>index</u> κ of (1.3.23) is

$$\kappa = \frac{1}{2\pi i}\left[\log\frac{A(t_o)-B(t_o)}{A(t_o)+A(t_o)}\right]_{\partial D} = 0 \qquad (1.3.29)$$

(this follows from the facts that $[\log t_o\bar{t}_o{}']_{\partial D} = [\log \bar{t}_o t_o{}']_{\partial D} = 0$ and $H_o(t_o) = \exp(-\int_o^{t} A(t_o,\eta)d\eta)$, which implies $[\log H_o(t_o)]_{\partial D} = [\log \overline{H_o(t_o)}]_{\partial D}$ and hence all three Fredholm theorems hold for equation (1.3.23), in particul (1.3.23) has a unique Hölder continuous solution for any (Hölder continuous) f(t) if and only if the homogeneous equation where ds denotes arclength along ∂D. If such a $\mu(t)$ exists then we can conclude that $\phi(z)$ as given by (1.3.18) is Hölder continuous in $\bar{D}$; c.f. [42]. Substituting (1.3.18) into (1.3.13) gives

$$u(x,y) = \int_{\partial D} K(z,t)\mu(t)ds \tag{1.3.19}$$

where

$$K(z,t) = \mathrm{Re}\left[\frac{tH_o(z)}{t-z} + \int_o^z \frac{tH(z,t_1)}{t-t_1}\,dt_1\right] \tag{1.3.20}$$

and $t\varepsilon\partial D$, $z\varepsilon D$. Note that $K(z,t)$ has the form

$$K(z,t) = \mathrm{Re}\left[\frac{tH_o(z)}{t-z} - tH(z,t)\log(1-\frac{z}{t}) + H^*(z,t)\right] \tag{1.3.21}$$

where

$$H^*(z,t) = \int_o^z \frac{t\left[H(z,t_1)-H(z,t)\right]}{t-t_1}\,dt_1 \tag{1.3.22}$$

is an analytic function of z,t in DxD and $\log(1-\frac{z}{t})$ is understood to be its principal value. Note also that for fixed $t\varepsilon\partial D$, $K(z,t)$ is a solution of $L[u] = 0$ in D.

Now let $z = xiy\varepsilon D$ tend to a point $t_o\varepsilon\partial D$. From the limit properties of Cauchy integrals (c.f. [42])we have

$$A(t_o)\mu(t_o) + \int_{\partial D} K(t_o,t)\mu(t)ds = f(t_o) \tag{1.3.23}$$

where

$$\begin{aligned} A(t_o) &= \mathrm{Re}\left[i\pi t_o\bar{t}_o{}'H_o(t_o)\right] \\ K(t_o,t) &= \mathrm{Re}\left[\frac{tH_o(t_o)}{t-t_o} - tH(t_o,t)\log(1-\frac{t_o}{t}) + H^*(t_o,t)\right] \end{aligned} \tag{1.3.24}$$

and $t' = {}^{dt}/ds = e^{i\theta(t)}$ where $\theta(t)$ is the angle between the positive direction of the tangent to ∂D at the point t to the x axis. Since D is in class Ah we have that $t'(s)$ is Hölder continuous on ∂D. (1.3.23) is a

$$H_o(z) = R(z,0,z,\bar{z})$$

$$H(z,\zeta) = -\frac{\partial}{\partial\zeta} R(\zeta,0\ ;\ z,\bar{z}) \tag{1.3.14}$$

$$\phi(z) = 2U(z,0) - U(0,0)$$

and, as usual, $U(z,z^*) = u\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right)$.

Proof: Integrating by parts in (1.2.1) gives, using (1.3.9),

$$\begin{aligned} U(z,z^*) &= -U(0,0)R(0,0\ ;\ z,z^*) \\ &+U(z,0)R(z,0;z,z^*)-\int_o^z U(\zeta,0)\frac{\partial R}{\partial\zeta}(\zeta,0;z,z^*)d\zeta \\ &+U(0,z^*)R(0,z^*;z,z^*)-\int_o^z U(0,\zeta^*)\frac{\partial R}{\partial\zeta}(0,\zeta^*;z,z^*)d\zeta^* \end{aligned} \tag{1.3.15}$$

and hence from (1.1.7) and (1.3.9)

$$\begin{aligned} U(z,z^*) &= R(z,0;z,z^*)\Phi(z) - \int_o^z \Phi(\zeta)\frac{\partial R}{\partial\zeta}(\zeta,0;z,z^*)d\zeta \\ &+R(0,z^*;z,z^*)\Phi(z^*) - \int_o^{z^*} \Phi(\zeta^*)\frac{\partial R}{\partial\zeta^*}(0,\zeta^*;z,z^*)d\zeta^* \end{aligned} \tag{1.3.16}$$

where

$$\begin{aligned} \Phi(z) &= U(z,0) - \tfrac{1}{2}U(0,0) \\ \Phi(z^*) &= U(0,z^*) - \tfrac{1}{2}U(0,0). \end{aligned} \tag{1.3.17}$$

Since $u(x,y)$ is real valued we have $U(z,\bar{z}) = \overline{U(z,\bar{z})}$ and $\Phi(z) = \overline{\Phi(\bar{z})}$. Hence from lemma 1.3.1 we can write (1.3.16) as (1.3.13) with $\phi(z)$ as given in (1.3.14).

Now assume that $u(x,y)\varepsilon C^o(\bar{D})$ is a solution of $L[u] = 0$ in D such that $u(t) = f(t)$ for $t\varepsilon\partial D$ where $f(t)$ is Hölder continuous on ∂D. Let $\mu(t)$ be an real valued Hölder continuous function for $t\varepsilon\partial D$. We will try and determine $\mu(t)$ such that $u(x,y)$ can be represented in the form (1.3.13) with

$$\phi(z) = \int_{\partial D} \frac{t\mu(t)ds}{t-z} \tag{1.3.18}$$

i.e.

$$A'(0,z^*) = B'(z,0) = 0 \quad . \tag{1.3.9}$$

Hence we can assume that the coefficients $A(z,z^*)$ and $B(z,z^*)$ satisfy (1.3.9) to begin with.

<u>Lemma 1.3.1</u> ([49]): The Riemann function $R(\zeta,\zeta^* : z,z^*)$ of $L[u] = 0$ with real-valued coefficients takes real values when $z^* = \bar{z}$, $\zeta^* = \bar{\zeta}$.

<u>Proof</u>: Since the coefficients of $L[u] = 0$ are real we have

$$\operatorname{Im} R(\zeta,\bar{\zeta} \; ; \; z,\bar{z}) = \frac{1}{2i}\left[R(\zeta,\bar{\zeta} \; ; \; z,\bar{z}) - \bar{R}(\bar{\zeta},\zeta : \bar{z},z)\right] \tag{1.3.10}$$

where $\bar{R}(\zeta,\bar{\zeta} \; ; \; z,\bar{z}) = \overline{R(\bar{\zeta},\zeta \; ; \; \bar{z},z)}$ is a solution of $L[u] = 0$. Extending (1.3.10) into the complex domain and evaluating along the characteristic $z=0$ gives

$$\begin{aligned}\operatorname{Im} R(\zeta,\bar{\zeta} \; ; \; 0,z^*) &= \frac{1}{2i}\left[\exp\left(-\int_{\bar{\zeta}}^{z^*} A(\zeta,\tau)d\tau\right)\right. \\ &\left. - \exp\left(-\int_{\bar{\zeta}}^{z^*} \bar{B}(\sigma,\tau)d\sigma\right)\right] \\ &= 0\end{aligned} \tag{1.3.11}$$

from (1.3.7) and the fact that $A(z,\bar{z}) = \overline{B(z,\bar{z})}$ (since the coefficients of $L[u] = 0$ are real). Similarly

$$\operatorname{Im} R(\zeta,\bar{\zeta} \; ; \; z,0) = 0 \; , \tag{1.3.12}$$

and hence from Theorem 1.2.1 $\operatorname{Im} R(\zeta,\bar{\zeta} \; ; \; z,\bar{z}) \equiv 0$ and the theorem is proved.

For the remainder of this section we assume that the coefficients of (1.3.3) satisfy (1.3.9) and that $L[u] = 0$ has real-valued coefficients.

<u>Lemma 1.3.2.</u> ([49]): Let $u(x,y)$ be a real valued solution of $L[u] = 0$ in D. Then

$$u(x,y) = \operatorname{Re}\left[H_o(z)\phi(z) + \int_o^z H(z,\zeta)\phi(\zeta)d\zeta\right] \tag{1.3.13}$$

where

where "Re" denotes "take the real part" and "Im" denotes "take the imaginary part". Let D_o be a compact subset of D. Then for any $\varepsilon > 0$ there exists an integer $N = N(\varepsilon)$ and constants $a_0, \ldots, a_N$ such that

$$\max_{\bar{D}_o} \left|u - \sum_{n=0}^{N} a_n u_n\right| < \varepsilon \quad . \tag{1.3.2}$$

The problem therefore is to replace D_o in (1.3.2) by $\bar{D}$, i.e. to show the set $\{u_n\}$ is "complete up to the boundary". We will show this through the use of singular integral equations and the method of I.N.Vekua ([49]).

We will first need a few preliminary results concerning the elliptic equation in the complex domain

$$L^*[U] \equiv \frac{\partial^2 U}{\partial z \partial z^*} + A(z,z^*)\frac{\partial U}{\partial z} + B(z,z^*)\frac{\partial U}{\partial z^*} + C(z,z^*)U = 0 \ . \tag{1.3.3}$$

Let

$$\begin{aligned}\Lambda(z,z^*) = \exp\{ -\int_0^z B(\zeta,0)d\zeta - \int_0^{z^*} A(0,\zeta^*)d\zeta^* \\ + \int_0^z \int_0^{z^*} [A(\zeta,\zeta^*)B(\zeta,\zeta^*) - C(\zeta,\zeta^*)]d\zeta^* d\zeta\}\end{aligned} \tag{1.3.4}$$

and set

$$U = \Lambda U' \ . \tag{1.3.5}$$

Then $U'(z,z^*)$ satisfies

$$L'[U'] \equiv \frac{\partial^2 U'}{\partial z \partial z^*} + A'(z,z^*)\frac{\partial U'}{\partial z} + B'(z,z^*)\frac{\partial U'}{\partial z^*} + A'(z,z^*)B'(z,z^*)U' = 0 \tag{1.3.6}$$

where

$$\begin{aligned}A'(z,z^*) &= \int_0^z h(\zeta,z^*)d\zeta \\ B'(z,z^*) &= \int_0^z k(z,\zeta^*)d\zeta^*\end{aligned} \tag{1.3.7}$$

with

$$\begin{aligned}h(z,z^*) &= \frac{\partial A(z,z^*)}{\partial z^*} + A(z,z^*)B(z,z^*) - C(z,z^*) \\ k(z,z^*) &= \frac{\partial B(z,z^*)}{\partial z^*} + A(z,z^*)B(z,z^*) - C(z,z^*) \quad ,\end{aligned} \tag{1.3.8}$$

Corollary 1.2.2 ([2]): Let $u(x,y) = \text{Re } \underset{\sim}{B}_2\{f\}$ and suppose $u(x,y)$ is regular in D (i.e. $u(x,y) \varepsilon C^2(D)$). Then $f(\frac{z}{2})$ is analytic for $z = x+iy \varepsilon D$.

Proof: From Theorem 1.1.2 we have that

$$g(z) = 2U(z,0) - U(0,0)\exp\left(-\int_0^z \bar{A}(0,\zeta)d\zeta\right) \tag{1.2.27}$$

is analytic for $z\varepsilon D$. From (1.2.22), (1.2.25) we have

$$f\left(\frac{z}{2}\right) = -\frac{1}{2\pi}\int_{-1}^{1} g(z(1-t^2))\frac{dt}{t^2} \tag{1.2.28}$$

and, by deforming the path of integration in (1.2.28) if necessary, it is seen that $f(\frac{z}{2})$ is also analytic for $z\varepsilon D$.

1.3. Complete Families of Solutions.

In this section we will make the further assumption on D, that in addition to being bounded and simply connected, D is in class Ah, i.e. the angle $\theta(t)$ between the tangent to ∂D at the point t and the x axis is Hölder continuous along ∂D. Without loss of generality we assume that D contains the origin. We want to construct a set of solutions $\{u_n\}$ to $L[u] = 0$ such that if $u(x,y)\varepsilon C^0(\bar{D}) \cap C^2(D)$ is a real valued solution of $L[u] = 0$ in D then for any $\varepsilon > 0$ there exists an integer $N = N(\varepsilon)$ and constants $a_0, \ldots, a_N$ such that

$$\max_{\bar{D}} \left| u - \sum_{n=0}^{N} a_n u_n \right| < \varepsilon .$$

Then set $\{u_n\}$ is then said to be complete in the maximum norm over $\bar{D}$. From Runge's theorem for analytic functions, Corollary 1.2.2, and the regularity of $E(z,z^*,t)$ we immediately have the following theorem:

Theorem 1.3.1 ([2] [49]): Let $u(x,y)\varepsilon C^2(D)$ be a real valued solution of $L[u] = 0$ in D and let

$$\begin{aligned} u_{2n} &= \text{Re } \underset{\sim}{B}_2\{z^n\} \;; \quad n = 0,1, \ldots \\ u_{2n+1} &= \text{Im } \underset{\sim}{B}_2\{z^n\}; \quad n = 0,1, \ldots \end{aligned} \tag{1.3.1}$$

is real and $f(0) = \frac{1}{\pi} U(0,0)$. Hence to show the invertibility of Re $\underset{\sim}{B}_2$ it follows from (1.2.22) and Corollary 1.2.1 that we must be able to invert the integral equation

$$g(z) = \int_{-1}^{1} f\left(\frac{z}{2}(1-t^2)\right) \frac{dt}{\sqrt{1-t^2}} . \tag{1.2.23}$$

Setting

$$g(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad |z| < \rho \tag{1.2.24}$$

it follows from the definition of the Gamma function that

$$-\frac{1}{2\pi} \int_{-1}^{1} g(z(1-t^2)) \frac{dt}{t^2} = \sum_{n=0}^{\infty} \frac{\Gamma(n+1) a_n z^n}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})} \tag{1.2.25}$$

(where in (1.2.25) the path of integration does not pass through the origin) and, setting $f(^z/2)$ equal to the right hand side of (1.2.25), that

$$\begin{aligned} &\int_{-1}^{1} f\left(\frac{z}{2}(1-t^2)\right) \frac{dt}{\sqrt{1-t^2}} \qquad (12.26) \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(n+1) a_n z^n}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{n-\frac{1}{2}} dt \\ &= \sum_{n=0}^{\infty} a_n z^n \\ &= g(z). \end{aligned}$$

Summarizing the above results gives the following theorem:

<u>Theorem 1.2.2</u> ([2]): Let $u(x,y)$ be a real valued (classical) solution of $L[u] = 0$ in some neighbourhood of the origin. Then $u(x,y) = U(z,\bar{z})$ can be represented in the form $U(z,\bar{z}) = \text{Re } \underset{\sim}{B}_2\{f\}$ where $f(z)$ is analytic in some neighbourhood of the origin in $\mathbb{C}^1$. Conversely, for every analytic function $f(z)$ defined in some neighbourhood of the origin in $\mathbb{C}^1$, Re$\underset{\sim}{B}_2\{f\}$ defines a real valued solution of $L[u] = 0$ in some neighbourhood of the origin.

$$2\alpha^3 t_o(1+\varepsilon)(\alpha-1)^{-4} < 1 , \tag{1.2.20}$$

then the series (1.2.19) is convergent. Since r is an arbitrarily large positive number and ε is arbitrarily small and independent of r, we can now conclude that the series (1.2.13) converges absolutely and uniformly on compact subsets of $\mathbb{C}^3$, i.e. $E(z.z^*,t)$ is an entire function of its independent complex variables.

We have now shown that the operator $\underset{\sim}{B}_2$ defined by

$$\begin{aligned} U(z,z^*) &= \underset{\sim}{B}_2\{f\} \\ &= \exp\left\{ - \int_o^{z^*} A(z,\zeta^*)d\zeta^* \right\} . \\ &\cdot \int_{-1}^{1} E(z,z^*,t) f\left(\frac{z}{2} (1-t^2)\right) \frac{dt}{\sqrt{1-t^2}} \end{aligned} \tag{1.2.21}$$

exists and maps analytic functions which are regular in some neighbourhood of the origin in $\mathbb{C}^1$ <u>into</u> the class of (complex valued) solutions of $L^*[U] = 0$. We now make use of Corollary 1.2.1 to show that the operator Re $\underset{\sim}{B}_2$, where "Re" denotes "take the real part", maps analytic functions <u>onto</u> the class of real valued solutions of $L[u] = 0$. We first note that since the coefficients of $L[u] = 0$ are real valued for x and y real, Re $\underset{\sim}{B}_2\{f\}$ defines a real valued solution of $L[u] = 0$ provided we set $z^* = \bar{z}$. Evaluating Re $\underset{\sim}{B}_2\{f\}$ at $z^* = 0$ gives

$$\begin{aligned} U(z,0) &= (\mathrm{Re}\, \underset{\sim}{B}_2\{f\})_{z^*=0} \\ &= \frac{1}{2}\int_{-1}^{1} \left[f\left(\frac{z}{2}(1-t^2)\right)+\bar{f}(0)\exp\left(-\int_o^z \bar{A}(\sigma,\zeta)d\zeta\right)\right] \frac{dt}{\sqrt{1-t^2}} \\ &= \frac{1}{2}\int_{-1}^{1} f\left(\frac{z}{2}(1-t^2)\right)\frac{dt}{\sqrt{1-t^2}} + \frac{\pi}{2}\,\bar{f}(0)\exp\left(-\int_o^z \bar{A}(0,\zeta)d\zeta\right) \end{aligned} \tag{1.2.22}$$

where $\bar{f}(z) = \overline{f(\bar{z})}$ and $\bar{A}(z,z^*) = \overline{A(\bar{z},\bar{z}^*)}$. From (1.2.22) we have $U(0,0) = \frac{\pi}{2}(f(0) + \bar{f}(0))$ and so without loss of generality we can assume $f(0)$

We will now show by induction that there exist positive constants M_n and ε (where ε is independent of n and M_n is a bounded function of n) such that for $|z| < r$, $|z^*| < r$, we have

$$p^{(2n)} \ll \frac{M_n 2^n (1+\varepsilon)^n}{2n-1} (1-\frac{z}{r})^{-(2n-1)} (1-\frac{z^*}{r})^{-(2n-1)} r^{-n} . \qquad (1.2.16)$$

This is clearly true for n = 1. Now suppose for n = k (1.2.16)is valid. Then from (1.2.14) and (1.2.15) and the straightforward use of the method of dominants we have

$$p^{(2k+2)} \ll \frac{M_k 2^{k+1}(1+\varepsilon)^k}{(2k+1)} \left[1+ \frac{Mr}{2k-1} + \frac{Mr^2}{(2k-1)(2k-1)} \right] \cdot$$

$$\cdot (1-\frac{z}{r})^{-(2k+1)} (1-\frac{z^*}{r})^{-(2k+1)} r^{-k-1} . \qquad (1.2.17)$$

By setting

$$M_{n+1} = M_n (1+\varepsilon)^{-1} \{1+\varepsilon \frac{Mr}{2n-1} + \frac{Mr^2}{(2n-1)(2n-1)}\} \qquad (1.2.18)$$

$$M_1 = M$$

we have shown that (1.2.16) is true for n = k + 1, thus completing the induction step. Note that for n sufficiently large we have $M_{n+1} \leqslant M_n$, i.e. there exists a positive constant M_o which is independent of n such that $M_n \leqslant M_o$ for all n.

We now return to the convergence of (1.2.13). Let $t_o \geqslant 1$ and $\alpha > 1$ be positive constants and let $|t| \leqslant t_o$, $|z| < \frac{r}{\alpha}$, $|z^*| < \frac{r}{\alpha}$. Then $(1-\frac{|z|}{r}) \geqslant \frac{\alpha-1}{\alpha}$, $(1-\frac{|z^*|}{r}) \geqslant \frac{\alpha-1}{\alpha}$, and from (1.2.16) it is seen that the series (1.2.13) is majorised by

$$1 + \sum_{n=1}^{\infty} \frac{rM_n 2^n t_o^n (1+\varepsilon)^n \alpha^{3n-3}}{(2n-1)(\alpha-1)^{4n-2}} . \qquad (1.2.19)$$

If α is chosen such that

lifferential equation (1.2.8) and integrate by parts using

$$f_z = - f_t \frac{(1-t^2)}{2zt} \tag{1.2.11}$$

:o show that if $E(z,z^*,t)$ satisfies

$$(1-t\ )E_{z^*t} - \frac{1}{t} E_{z^*} + 2tz \left[E_{zz^*} + DE_{z^*} + FE \right] = 0 \tag{1.2.12}$$

:hen (1.2.10) yields a solution of (1.2.8).

/e will now show the existence and regularity of $E(z,z^*,t)$. We look for a olution of (1.2.12) in the form

$$E(z,z^*,t) = 1 + \sum_{n=1}^{\infty} t^{2n} z^n \int_0^{z^*} P^{(2n)}(z,z^*)dz^* \quad . \tag{1.2.13}$$

ubstituting (1.2.13) into (1.2.12) gives the following recursion formula 'or the $P^{(2n)}$:

$$P^{(2)} = - 2F \tag{1.2.14}$$

$$(2n+1)P^{(2n+2)} = -2\left[P_z^{(2n)} + DP^{(2n)} + F \int_0^{z^*} P^{(2n)} dz^*\right] \; ; \; n \geq 1 \; .$$

'ote that since D and F are entire functions of z and z^*, so are the $P^{(2n)}$. 'e will now show that the series (1.2.13) converges absolutely and uniformly 'or (t,z,z^*) on compact subsets of $\mathbb{C}^3$. To do this we will again use the ethod of dominants. Let r be an arbitrarily large positive number and M positive constant such that for $|z| < r$, $|z^*| < r$, we have

$$D(z,z^*) << \frac{M}{(1-\frac{z}{r})(1-\frac{z^*}{r})} \tag{1.2.15}$$

$$F(z,z^*) << \frac{M}{(1-\frac{z}{r})(1-\frac{z^*}{r})} \quad .$$

operators to certain classes of elliptic equations in more than two independent variables and to parabolic equations in one and two space variables. In order to construct this operator we will need the assumption that a,b and c are real valued, and we will assume this from now on.

We consider $L[u] = 0$ in its complex form

$$L^*[U] \equiv U_{zz^*} + A(z,z^*)U_z + B(z,z^*)U_z + C(z,z^*)U = 0 \qquad (1.2.6)$$

and make the change of dependent variables

$$V(z,z^*) = U(z,z^*)\exp\left\{ \int_0^{z^*} A(z,\zeta^*)d\zeta^* \right\}. \qquad (1.2.7)$$

Under the change of variables (1.2.7), (1.2.6) becomes

$$V_{zz^*} + D(z,z^*)V_{z^*} + F(z,z^*)V = 0 \qquad (1.2.8)$$

where

$$\begin{aligned} D &= B - \int_0^{z^*} A_z(z,\zeta^*)d\zeta^* \\ F &= -(A_z + AB - C). \end{aligned} \qquad (1.2.9)$$

We look for solutions of (1.2.7) in the form

$$V(z,z^*) = \int_{-1}^{1} E(z,z^*,t)f\left(\frac{z}{2}(1-t^2)\right)\frac{dt}{\sqrt{1-t^2}} \qquad (1.2.10)$$

where $f(z)$ is an analytic function in some neighbourhood of the origin and $E(z,z^*,t)$ is to be determined.

<u>Definition 1.2.1</u>: $E(z,z^*,t)$ is known as the <u>generating function</u> for equation (1.2.8).

<u>Remark</u>: The path of integration in (1.2.10) is assumed to be a curvilinear path in the unit disc in the complex t plane joining the points $t = +1$ and $t = -1$.

Assuming that $E(z,z^*,t)$ is an analatyic function of t for $|t| \leq 1$ and (z,z^*) in some neighbourhood of the origin in $\mathbb{C}^2$ we substitute (1.2.10) into the

Theorem 1.2.1 ([2], [49]): Let $u(x,y)$ be a solution of $L[u] = 0$ in D. Then $u(x,y)$ is uniquely determined from the complex Goursat data $U(z,0)$ and $U(0,z^*)$.

Proof: If $U(z,0) = U(0,z^*) = 0$ then $\alpha_o = f(z) = g(z) = 0$ and hence from (1.2.1) $U(z,z^*) \equiv 0$.

Corollary 1.2.1 ([2], [49]): Let $u(x,y)$ be a real valued solution of $L[u] = 0$ in D. Then $u(x,y)$ is uniquely determined by $U(z,0)$.

Proof: From Theorem 1.1.2 we have that in some ball in $\mathbb{C}^2$ about the origin

$$U(z,z^*) = \sum_{m,n=0}^{\infty} a_{mn} z^m z^{*n}. \tag{1.2.3}$$

Since $u(x,y)$ is real valued $U(z,\bar{z}) = \overline{U(z,\bar{z})}$, i.e.

$$\sum_{m,n=0}^{\infty} a_{mn} z^m \bar{z}^n = \sum_{m,n=0}^{\infty} \overline{a_{mn}}\, \bar{z}^m z^n \tag{1.2.4}$$

and hence

$$a_{mn} = \overline{a_{nm}}\,.$$

Since

$$\begin{aligned} U(z,0) &= \sum_{m=0}^{\infty} a_{mo} z^m \\ U(0.z^*) &= \sum_{n=0}^{\infty} a_{on} z^{*n} \end{aligned} \tag{1.2.5}$$

we have that $U(z,0)$ determines $U(0,z^*)$ and the corollary now follows from Theorem 1.2.1.

We will return later to further discussion of Vekua's integral operator.

We now want to construct another operator which maps analytic functions on to solutions of $L[u] = 0$, the so called Begman integral operator of the first kind (c.f. [2]). We want to do this since it is the Bergman operator which provides the proper motivation for generalizing the method of integral

1.2 Integral Operators.

Let $u(x,y)$ be a real valued solution of $L[u] = 0$ in a bounded simply connected domain D. We make the further assumption that, in addition to being entire functions, the coefficients a, b and c are real valued for real values of their arguments. Our aim is to construct an integral operator which maps analytic functions of a single complex variable onto solutions of $L[u] = 0$ (c.f. [2], [49]). Without loss of generality we assume that the origin is an interior point of D.

One such operator is already given to us from the results of section 1.1. From Theorem 1.1.2 we have that $U(z,z^*) = u\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right)$ is analytic for $(z,z^*)\varepsilon D\times D^*$ and hence from (1.1.17) we have

$$\begin{aligned} U(z,z^*) &= \alpha_o R(0,0;\, z,z^*) \\ &+ \int_o^z f(\zeta)R(\zeta,0,z,z^*)d\zeta \qquad (1.2.1) \\ &+ \int_o^{z^*} g(\zeta^*)R(0,\zeta^*,z,z^*)d\zeta^* \end{aligned}$$

where

$$\alpha_o = U(0,0)$$

$$f(z) = \frac{\partial U(z,0)}{\partial z} + B(z,0)U(z,0) \qquad (1.2.2)$$

$$g(z^*) = \frac{\partial U(0,z^*)}{\partial z^*} + A(0,z^*)U(0,z^*).$$

From Theorem 1.1.2 we have that $f(z)$ and $g(z^*)$ are analytic in D and D* respectively. Conversely, it is easily seen that if $f(z)$ and $g(z^*)$ are any analytic functions in D and D* respectively, then (1.2.1) defines a solution of $L[u] = 0$. Note that there we have not made use of the fact that u, a, b and c are real valued. The operator defined by (1.2.1) is known as <u>Vekua's integral operator</u>. For more details see [49].

where $H[u,v] = \{v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} + (a \frac{\partial x}{\partial \nu} + b \frac{\partial y}{\partial \nu})uv\}\, ds$.

Now let u be a solution of $L[u] = 0$ in D (without loss of generality assume $u \varepsilon C^1(\bar{D})$) and set $v = R(z,\bar{z};\ \zeta,\bar{\zeta}) \log r$ (where $\zeta = \xi + i\eta$, $\bar{\zeta} = \xi - i\eta$, $r^2 = (z-\zeta)(\bar{z}-\bar{\zeta})$). Let Ω_ε be a small circle about the point (ξ,η) and apply (1.1.20) to the region D/Ω_ε instead of D. Letting $\varepsilon \to 0$ and interchanging the roles of (x,y) and (ξ,η) now gives in a straightforward fashion

$$u(x,y) = \frac{1}{2\pi} \int_{\partial D} H[u, R\log r] - \frac{1}{2\pi} \int\int_D uM[R\log r]\, d\xi\, d\eta \qquad (1.1.21)$$

where integration over D and ∂D is now with respect to the point (ξ,η), $R = R(\zeta,\bar{\zeta};\ z,\bar{z})$, and M is a differential operator with respect to the (ξ,η) variables. Since $M[R] = 0$ we have (with respect to the complex variables $(\zeta,\zeta^*,\ z,z^*)$

$$M[R\log r] = 2\, \frac{\partial R/\partial\zeta - BR}{\zeta^* - z^*} + 2\, \frac{\partial R/\partial\zeta^* - AR}{\zeta - z} \qquad (1.1.22)$$

and hence from (1.1.7) we have that $M[R\log r]$ is in fact an entire function of its independent complex variables. Hence the second integral in (1.1.21) can be continued to an entire function of z and z* (replace $\bar{z}$ by z*). The first integral in (1.1.21) can be continued to an analytic function of z and z* for $z \varepsilon D$, $z^* \varepsilon D^*$ (i.e. for z and z* such that $r \neq 0$). Hence (1.1.21) shows that $U(z,z^*)$ is analytic for $(z,z^*) \varepsilon D x D^*$.

<u>Remark</u>: Note that u(x,y) analytic for $(x,y) \varepsilon D$ means that for each point $(x_o,y_o) \varepsilon D$ there exists a neighbourhood N of (x_o,y_o) in $\mathbb{C}^2$ such that u(x,y) is analytic in N. Theorem 1.1.2 provides a <u>global</u> analytic continuation as opposed to this local result.

$$+\int_{z_o^*}^{z^*} R(z_o,\zeta^*,z,z^*)\{ \frac{\partial U(z_o,\zeta^*)}{\partial \zeta^*} + A(z_o,\zeta^*)U(z_o,\zeta^*)\}d\zeta^*$$

$$+\int_{z_o}^{z}\int_{z_o^*}^{z^*} R(\zeta,\zeta^*,z,z^*)L^*[U(\zeta,\zeta^*)]d\zeta^* d\zeta \tag{1.1.17}$$

Setting $U(z,z^*) = R(z_o,z_o^*,z,z^*)$ and using (1.1.7) shows that

$$\int_{z_o}^{z}\int_{z_o^*}^{z} R(\zeta,\zeta^*;z,z^*)L^*[R(z\ ,z_o^*,\zeta,\zeta^*)]d\zeta^* d\zeta = 0 \tag{1.1.18}$$

i.e. with respect to its last two arguments $R(z_o,z_o^*,z,z^*)$ is a solution of $L^*[U] = 0$. (1.1.7) now shows that a function of z,z^*, $R(\zeta,\zeta^*, z,z^*)$ is the Riemann function for $M[v] = 0$.

Corollary 1.1.1 : Let $F(z,z^*)$ be analytic for $(z,z^*)\varepsilon D\times D^*$. Then

$$U_o(z,z^*) = \int_{z_o}^{z}\int_{z_o^*}^{z^*} R(\zeta,\zeta^*\ z,z^*)F(\zeta,\zeta^*)d\zeta^* d\zeta \tag{1.1.19}$$

is a particular solution of $L^*[U] = F(z,z^*)$ analytic for (z,z^*) $D\times D^*$.

Proof: This follows from (1.1.17).

We now want to prove the main result of this section, the Bergman-Vekua Theorem (c.f. [2], [49]).

Theorem 1.1.2 ([2], [49]): Let $u(x,y)$ be a classical solution of $L[u] = 0$ in D. Then $U(z,\bar{z}) = u(x,y)$ is analytic for $(x,y)\varepsilon D$ and $U(z,z^*) = u(\frac{z+z^*}{2}, \frac{z-z^*}{2i})$ can be analytically continued into the domain $D\times D^*$.

Proof: Without loss of generality assume D has a smooth boundary ∂D and let ν denote the inner normal and s the arclength along ∂D. Then from Green's theorem we have for $u,v\varepsilon C^2(D)\cap C^1(\bar{D})$

$$\int\int_D (vL[u] - uM[v]dxdy + \int_{\partial D} H[u,v] = 0 \tag{1.1.20}$$

$$+ \frac{1}{k}(1-\frac{(z-\zeta)}{r})^{-k-1}\ (1-\frac{(z^*-\zeta^*)}{r})^{-k-1}] \tag{1.1.14}$$

$$<< \frac{3^{k+1}M^{k+1}r^{2k+2}}{k!}\ (1-\frac{(z-\zeta)}{r})^{-k-1}\ (1-\frac{(z^*-\zeta^*)}{r})^{-k-1}$$

thus showing (1.1.13) is true for $n = k + 1$ and completing the induction proof. Now let $\alpha > 1$ and $|z-\zeta| \leqslant \frac{r}{\alpha}$, $|z^*-\zeta^*| \leqslant \frac{r}{\alpha}$.

Then $(1-\frac{|z-\zeta|}{r}) \geqslant \frac{\alpha-1}{\alpha}$, $(1-\frac{|z^*-\zeta^*|}{r}) \geqslant \frac{\alpha-1}{\alpha}$, and the series $\sum_{n=0}^{\infty} |u_n| = \sum_{n=0}^{\infty} |R_{n+1}-R_n|$ is majorised by

$$1 + \sum_{n=1}^{\infty} \frac{3^n M^n r^{2n}}{(n-1)!}\ (\frac{\alpha}{\alpha-1})^{2n} < \infty \tag{1.1.15}$$

Hence we have shown that $R(z,z^*;\zeta,\zeta^*)$ exists and is an entire function of its independent variables.

We now want to prove the following theorem:

<u>Theorem 1.1.1</u> ([49]) : As a function of its last two arguements $R(\zeta,\zeta^*,z,z^*)$ is the Riemann function for $M[v] = 0$.

<u>Proof</u>: Let D be abounded simply connected domain $\mathbb{C}^1$ and let $U(z,z^*)$ be an analytic function of z and z* for $(z,z^*)\varepsilon D\times D^*$ where $D^* = \{z^* : \bar{z}^*\varepsilon D\}$. From $M^*[R] = 0$ and (1.1.7) we have

$$\frac{\partial^2 (UR)}{\partial z \partial z} - RL^*[U] = \frac{\partial}{\partial z}\{U(\frac{\partial R}{\partial z^*} - AR)\} + \frac{\partial}{\partial z^*}\{U(\frac{\partial R}{\partial z} - BR)\} \tag{1.1.16}$$

where $R = R(z,z^*;\zeta,\zeta^*)$. Interchange z,z^* and ζ,ζ^* in (1.1.16) and integrate with respect to ζ and ζ^* from z_o to z and z_o^* to z^* where $(z_o,z_o^*)\varepsilon D\times D^*$. Making use of (1.1.7) again we have

$$U(z,z^*) = U(z_o,z_o^*)R(z_o,z_o^*;z,z^*)$$

$$+\int_{z_o}^{z} R(\zeta,z_o^*,z,z^*)\{\frac{\partial U(\zeta,z_o^*)}{\partial \zeta} + B(\zeta,z_o^*)U(\zeta,z_o^*)\}\, d\zeta$$

The generalization of the above definition to series not expanded about the origin and to series of several complex variables is immediate.

From (1.1.9) we have, setting $u_n = R_{n+1}-R_n$,

$$u_{n+1}(z,z^*)=\int_{\zeta}^{z} B(\sigma,z^*)u_n(\sigma,z^*)d\sigma + \int_{\zeta^*}^{z^*} A(z,\tau)u_n(z,\tau)d\tau \tag{1.1.11}$$
$$- \int_{\zeta}^{z}\int_{\zeta^*}^{z^*} C(\sigma,\tau)u_n(\sigma,\tau)d\tau d\sigma$$

with $u_o = 1$. Let M be a positive constant such that for $|z-\zeta| < r$ and $|z^*-\zeta^*| < r$

$$A(z,z^*) << \frac{M}{(1-\frac{z-\zeta}{r})(1-\frac{z^*-\zeta^*}{r})}$$

$$B(z,z^*) << \frac{M}{(1-\frac{z-\zeta}{r})(1-\frac{z^*-\zeta^*}{r})} \tag{1.1.12}$$

$$C(z,z^*) << \frac{M}{(1-\frac{z-\zeta}{r})(1-\frac{z^*-\zeta^*}{r})}$$

where M can be chosen independent of ζ and ζ^* for ζ and ζ^* bounded.

We claim that for $|z-\zeta| < r$, $|z^*-\zeta^*| < r$, $n \geqslant 1$

$$u_n(z,z^*) << \frac{3^n M^n r^{2n}}{(n-1)!}\left(1-\frac{(z-\zeta)}{r}\right)^{-n}\left(1-\frac{(z^*-\zeta^*)}{r}\right)^{-n}. \tag{1.1.13}$$

From the properties of dominants and (1.1.11) it is seen that this is clearly true for n = 1. We will now establish (1.1.13) by induction. Assume (1.1.13) is true for n = k.

Then

$$u_{k+1} << \frac{3^k M^{k+1} r^{2k+2}}{k!}\left[\frac{1}{r}\left(1-\frac{(z-\zeta)}{r}\right)^{-k}\left(1-\frac{(z^*-\zeta^*)}{r}\right)^{-k-1}\right.$$
$$+ \frac{1}{r}\left(1-\frac{(z-\zeta)}{r}\right)^{-k-1}\left(1-\frac{(z^*-\zeta^*)}{r}\right)^{-k} +$$

We will show that there exists a solution of (1.1.8) which is an entire function of z, z^*, ζ and ζ^*. It suffices to show that $R(z,z^*;\zeta,\zeta^*)$ is an analytic function of its independent variables for $|z-\zeta| < r$ and $|z^*-\zeta^*| < r$ for $r > 1$ an arbitrary large positive number.

We define the recursive scheme

$$
\begin{aligned}
R_o(z,z^*) &= 0 \\
R_{n+1}(z,z^*) &= 1+\int_\zeta^z B(\sigma,z^*)R_n(\sigma,z^*)\,d\sigma \\
&\quad +\int_{\zeta^*}^{z^*} A(z,\tau)R_n(z,\tau)\,d\tau \\
&\quad -\int_\zeta^z\int_{\zeta^*}^{z^*} C(\sigma,\tau)R_n(\sigma,\tau)\,d\tau\, d\sigma, \quad n \geq 1
\end{aligned}
\tag{1.1.9}
$$

where $R_n(z,z^*) = R_n(z,z^*;\zeta,\zeta^*)$ and will show that

$$R = \lim_{n\to\infty} R_n = \sum_{n=0}^{\infty} (R_{n+1}-R_n)$$

converges absolutely and uniformly for $|z-\zeta| \leq \frac{r}{\alpha}$, $|z^*-\zeta^*| \leq \frac{r}{\alpha}$, $\alpha > 1$ arbitrary, and, since each R_n is analytic for $|z-\zeta| < r$, $|z^*-\zeta^*| < r$, so is the limit R. To this end we make use of the <u>method of dominants</u>. If we are given two series

$$S = \sum_{n=0}^{\infty} a_n z^n \quad , \quad \tilde{S} = \sum_{n=0}^{\infty} \tilde{a}_n z^n \quad ; \quad |z| < r$$

where $\tilde{a}_n \geq 0$ then we say $\tilde{S}$ dominates S if $|a_n| \leq \tilde{a}_n$, $n = 0,1,2, \ldots$, and write $S << \tilde{S}$. It is easily verified that dominates can be multiplied and if $S << \tilde{S}$ then

$$
\begin{aligned}
&1) \quad \frac{\partial S}{\partial z} << \frac{\partial \tilde{S}}{\partial z} \\
&2) \quad \int_o^z S(z)\,dz << \int_o^z \tilde{S}(z)\,dz \\
&3) \quad S << \frac{\tilde{S}}{(1-az)}, \quad 0 \leq a < \frac{1}{r}.
\end{aligned}
\tag{1.1.10}
$$

$$U(z,z^*) = U\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i} \right)$$

$$V(z,z^*) = v\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i} \right)$$

$$A = \frac{1}{4}(a+ib) \tag{1.1.6}$$

$$B = \frac{1}{4}(a-ib)$$

$$C = \frac{1}{4}c$$

and we are assuming that $u(x,y)$ and $v(x,y)$ are analytic functions of the complex variables x and y. We will show later that this is true for every classical solution of (1.1.1) or (1.1.2).

The Riemann function for (1.1.1) is defined to be the (unique) solution $R(z,z^*;\zeta,\zeta^*)$ of (1.1.5) depending on the complex parameters $\zeta = \xi + i\eta$, $\xi^* = \xi - i\eta$ (where ξ,η are complex variables) which satisfies the initial conditions

$$R(z,\zeta^*;\zeta,\zeta^*) = \exp\left[\int_\zeta^z B(\sigma,\zeta^*)d\sigma \right] \tag{1.1.7}$$

$$R(\zeta,z^*;\zeta,\zeta^*) = \exp\left[\int_{\zeta^*}^{z^*} A(\zeta,\tau)d\tau \right]$$

on the complex hyperplanes $z^* = \zeta^*$ and $z = \zeta$. Note that by Cauchy's theorem the integrals in (1.1.7) are independent of the path of integration. We will now construct $R(z,z^*;\zeta,\zeta^*)$. (1.1.5) and (1.1.7) are equivalent to the integral equation

$$\begin{aligned} R(z,z^*;\zeta,\zeta^*) &- \int_\zeta^z B(\sigma,z^*)R(\sigma,z^*;\zeta,\zeta^*)d\sigma \\ &- \int_{\zeta^*}^{z^*} A(z,\tau)R(z,\tau;\zeta,\zeta^*)d\tau \\ &+ \int_\zeta^z\int_{\zeta^*}^{z^*} C(\sigma,\tau)R(\sigma,\tau;\zeta,\zeta^*)d\tau d\sigma = 1 \end{aligned} \tag{1.1.8}$$

I Elliptic equations in two independent variables

1.1 Analytic Continuation

We are interested here in classical solutions of the second order elliptic equation in two independent variables written in canonical form as

$$L[u] \equiv u_{xx}+u_{yy}+a(x,y)u_x+b(x,y)u_y+c(x,y)u=0 \qquad (1.1.1)$$

where we assume that a,b and c are entire functions of their independent complex variables x and y.

Remark: The assumption that a,b and c are entire functions can be easily relaxed to being analytic in a sufficiently large polydisc in $\mathbb{C}^2$, the space of two complex variables.

We will also need to look at special solutions of the adjoint equation to $L[u]=0$ defined by

$$M[v] \equiv v_{xx} + v_{yy} - \frac{\partial(av)}{\partial x} - \frac{\partial(bv)}{\partial y} + cv = 0. \qquad (1.1.2)$$

In particular we first want to construct a special entire solution of (1.1.2) known as the (complex) Riemann function for $L[u]=0$ ([49]).

To this end we define a mapping of $\mathbb{C}^2 \to \mathbb{C}^2$ by

$$\begin{aligned} z &= x + iy \\ z^* &= x - iy . \end{aligned} \qquad (1.1.3)$$

Note that $z^* = \bar{z}$ if and only if x and y are real.

Under the transformation (1.13) equations (1.1.1) and (1.1.2) become

$$L^*[U] \equiv \frac{\partial^2 V}{\partial z \partial z^*} + A(z,z^*)\frac{\partial U}{\partial z} + B(z,z^*)\frac{\partial U}{\partial z^*} + C(z,z^*)U = 0 \qquad (1.1.4)$$

$$M^*[V] \equiv \frac{\partial^2 V}{\partial z \partial z^*} - \frac{\partial(AV)}{\partial z} - \frac{\partial(BV)}{\partial z^*} + CV = 0 \qquad (1.1.5)$$

where

3) The inverse scattering problem for acoustic waves in a spherically stratified medium.

In order to accomplish the program outlined above we will use the theory of integral operators as developed by Bergman and Vekua for elliptic equations in two independent variables, by Bergman, Colton and Gilbert for ellitpic equations in three independent variables, and by Colton for parabolic equations in one and two space variables. Suitable references in the case of elliptic equations are [2], [6], [8], [30], [31], [49].

Remark: [8] also contains material on parabolic, hyperbolic and pseudoparabolic equations.

It is easily seen that this problem is non linear in $s(t)$. The inverse Stefan problem is, given $s(t)$, to determine $u(x,t)$ for $0<x<s(t)$ and $\phi(t)=u(0,t)$, i.e. now must we heat the water in order to melt the ice along a prescribed curve? The idea is to construct a "catalog" of solutions $u(x,t)$ corresponding to a large class of "free" boundaries $s(t)$ and to then be in a position to solve the Stefan problem (13)-(16) by looking in the "catalog" for a solution whose boundary data at $x=0$ is close to $\phi(t)$. The inverse Stefan problem is linear; however it is <u>improperly posed</u> in the real domain in the sense that $u(x,t)$ does not depend continuously on the initial data on the curve $s(t)$. To see this let $s(t) = 0$ and assume $\frac{k}{\rho c} = 1$. Then

$$u_n(x,t) = \frac{1}{n}\left[e^{nx}\sin(2n^2t+nx)+e^{-nx}\sin(2n^2t-nx)\right] \tag{17}$$

is a solution of (13) such that

$$u_n(0,t) = \frac{2}{n}\sin 2n^2 t \tag{18}$$

$$u_{nx}(0,t) = 0 \tag{19}$$

But although $u_n(0,t)\to 0$ as $n\to\infty$, for any $x>0$, $u_n(x,t)\to\infty$ as $n\to\infty$. However, as a consequence of the Cauchy-Kowalewski theorem, the inverse Stefan problem is well posed in the <u>complex</u> domain and this is where we will later study it.

<u>Remark</u>: The Cauchy-Kowalewski theorem does not provide a practical method for solving the inverse Stefan problem, particularly in higher dimensional space, since the calculations are far too tedious and, more important, the solution may not converge in a large enough domain.

The inverse problems we will study in these lectures are

1) Inverse methods for solving boundary value problems arising in the theory of compressible fluid flow.
2) The inverse Stefan problem for the heat equation in one and two space variables.

The above extension will often be based on the development of methods for the analytic continuation of solutions to elliptic and parabolic equations. (We will for the sake of simplicity often restrict our attention to the case when the coefficients of the partial differential equation under investigation are entire functions of their independent complex variables. In practice this is not a serious restriction since the coefficients are in general obtained from physical measurements and can be approximated on compact sets by polynomials).

In addition to approximating solutions of boundary value problems for partial differential equations by means of a complete family of solutions, or the method of integral equations, we will also be interested in solving various (in general non linear) problems through the use of <u>inverse methods</u> and analytic function theory. The simplest example of such a problem is the inverse Stefan problem for the heat equation in one space variable, which can be formulated as follows. Consider a thin block of ice at $0^{\circ}C$ occupying the interval $0\leqslant x<\infty$ and suppose at $x=0$ the temperature is given by a prescribed function $\phi(t)>0$ where $t\geqslant 0$ denotes time. Then the ice will begin to melt and for $t>0$ the water will occupy an interval $0\leqslant x<s(t)$. If $u(x,t)$ is the temperature of the water we have

$$\frac{k}{\rho c} u_{xx} = u_t \qquad \text{for } 0<x<s(t) \tag{13}$$

$$u(0,t) = \phi(t) \qquad \text{for } t>0 \tag{14}$$

$$u(s(t),t) = 0 \qquad \text{for } t>0 \tag{15}$$

and, from the law of conservation of energy,

$$u_x(s(t),t) = -\frac{\lambda\rho}{k}\frac{ds(t)}{dt}, \tag{16}$$

where λ, k, ρ, and c are thermal constants. $s(t)$ is an unknown free boundary and the Stefan problem is to determine $u(x,t)$ and $s(t)$ from (13)-(16).

(2) to a simple one of quadrature, a method which in fact is well suited to use on a digital computer.

The representation (3) can also be used to obtain a method for approximating the solution to the Dirichlet problem (1), (2) in a different manner than the one just described. This is accomplished by representing $\phi(z)$ in the form

$$\phi(z) = \frac{1}{\pi i} \int_{\partial D} \frac{\mu(t)dt}{t-z} \quad , \quad z \varepsilon D \tag{11}$$

where $\mu(t)$ is a real valued function to be determined, and then using (3) and the limit properties of Cauchy integrals to derive the following integral equation for the unkown potential $\mu(t)$:

$$\mu(t_o) + \int_{\partial D} \left[\operatorname{Re} \frac{t'}{\pi i(t-t_o)}\right] \mu(t)ds = f(t_o) \tag{12}$$

where s denotes arclength, $t' = {}^{dt}/ds$ and $t_o \varepsilon \partial D$. Equation (12) is a Fredholm integral equation of the second kind for the unknown density $\mu(t)$ and this equation always has a unique solution for $f(t) \varepsilon C^o(\partial D)$ (Note that for any $\varepsilon>0, |t_o - t|^{\varepsilon} \operatorname{Re}(\frac{t'}{\pi i(t-t_o)})$ satisfies a Hölder condition for $t \varepsilon \partial D$). If one now appeals to various methods for approximating solutions to Fredholm integral equations of the second kind, one is lead to a constructive method for approximating the solution to the Dirichlet problem (1), (2).

A major part of these lectures will be to extend the methods just described to equations with variable coefficients,in particular to

1) Second order elliptic equations in two independent variables.
2) Second order parabolic equations in one space variable.
3) Second order parabolic equations in two space variables.
4) Certain classes of second order elliptic equations in $n \geqslant 2$ independent variables with spherically symmetric coefficients.

Now orthonormalize the set $\{u_n\}$ in the L^2 norm over ∂D to obtain the complete set $\{\phi_n\}$, i.e.

$$\int_{\partial D} \phi_n \phi_m = 0 \qquad \text{for } n \neq m \tag{6}$$

$$\int_{\partial D} |\phi|^2 = 1.$$

Let

$$c_n = \int_{\partial D} f\phi_n \ . \tag{7}$$

Let D_o be a compact subset of D. From the representation of the solution of (1), (2) in terms of the Green's function it is seen that if

$$\int_{\partial D} |f - \sum_{n=0}^{N} c_n\phi_n|^2 < \varepsilon \tag{8}$$

then

$$\max_{D_o} |u - \sum_{n=0}^{N} c_n\phi_n| < M\varepsilon \tag{9}$$

where $M = M(D_o)$ is a constant. Hence an approximate solution to (1), (2) on compact subsets of D is given by

$$u^N = \sum_{n=0}^{N} c_n\phi_n \ . \tag{10}$$

Since each ϕ_n is a solution of (1), error estimates can be found by finding the maximum of $|f-u^N|$ on ∂D and applying the maximum principle.

Remark: The assumption that ∂D is analytic is made to avoid technical approximation arguments. However the above method remains valid under much weaker assumptions, e.g. ∂D is Hölder continuously differentiable. We will discuss this in more detail during the course of these lectures.

We have now reduced the problem of constructing an approximate solution to (1

Introduction

The simplest example of the type of problem we will want to consider in these lectures is the following approach for approximating solutions of the Dirichlet problem for Laplace's equation. Let D be a bounded simply connected domain in $\mathbb{R}^2$ with analytic boundary ∂D. We wish to approximate (in the maximum norm) the solution of $(u \varepsilon C^2(D) \cap C^o(\bar{D}))$

$$u_{xx} + u_{yy} = 0 \qquad \text{for } (x,y) \varepsilon D \tag{1}$$

$$u(t) = f(t) \qquad \text{for } t \varepsilon \partial D \tag{2}$$

where $f(t) \varepsilon C^o(\partial D)$. From the maximum principle it suffices to approximate the solution of (1), (2) for $f(t)$ analytic. In this case $u(x,y)$ is in fact a solution of (1) in a domain $\tilde{D} \supset \bar{D} (\bar{D} = D \cup \partial D)$.

We have

$$u(x,y) = \mathrm{Re}\{\phi(z)\} \tag{3}$$

where $\phi(z)$ is an analytic function of $z = x+iy$ in $\tilde{D}$ and hence by Runge's theorem the set

$$\begin{aligned} u_{2n}(x,y) &= \mathrm{Re}\{z^n\} \\ u_{2n+1}(x,y) &= \mathrm{Im}\{z^n\} \end{aligned} \tag{4}$$

is a complete family of solutions to (1) in $\tilde{D}$, i.e. for every compact subset $B \subset \tilde{D}$ (in particular for $B=\bar{D}$) and $\varepsilon>0$ there exist constants $a_1, \ldots, a_N$ such that for N sufficiently large

$$\max_B \left| u - \sum_{n=0}^{N} a_n u_n \right| < \varepsilon . \tag{5}$$

Contents

and financial support during the year, and also the Air Force Office of Scientific Research for their continued support of my research efforts through AFOSR Grant 74-2592. Appreciation is also given to Mrs. Norma Sim of the University of Strathclyde for her patient and careful typing of the manuscript.

Preface

These lecture notes are intended to be a companion volume to [8]. In [8] a general survey was given of the analytic theory of partial differential equations, with particular emphasis on improperly posed initial value problems and the analytic continuation of solutions to partial differential equations. The use of integral operators to solve boundary value and initial-boundary value problems arising in mathematical physics was discussed only briefly. In the present set of notes this topic now becomes the main theme, and the interplay between analytic continuation and the approximation of solutions to partial differential equations is developed in some detail. With the idea that these two sets of lectures should be read together, we have minimized overlapping topics, while at the same time keeping each set of lectures self-contained. Indeed the only topics common to [8] and the present volume are integral operators for elliptic equations in two independent variables (which is treated in considerably more detail in the present set of notes) and the inverse Stefan problem for the heat equation in one space variable (which occupy only a few pages in both [8] and the present volume).

The present set of lectures was given during the academic year 1974-75 while the author was a Guest Professor at the University of Konstanz. The prerequisites for the course were a one semester course in partial differential equations and a one semester course in analytic function theory. I would like to particularly thank Professor Wolfgang Watzlawek and the Fachbereich Mathematik of the University of Konstanz for their hospitality

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Research Notes in Mathematics 6

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PART III: SYNOPSIS

This part of my submission is supplementary to my main submission in Part II and consists of 41 papers by myself and with collaborators. Reference numbers refer to the references in Part I.

References [3] - [12] are a study of the analytic behaviour of a class of singular partial differential equations related to the Bessel operator

$$\frac{\partial^2}{\partial y^2} + \frac{2\nu}{y}\frac{\partial}{\partial y} \quad . \tag{12}$$

Particular attention has been paid to values of ν such that $\nu < -\frac{1}{2}$ since it is here that the particular singular nature of the operator becomes evident. This can be seen, for example, by noting that in domains containing the singular line many of the classical boundary value and initial value problems become improperly posed, viz. Dirichlet's problem for

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} = 0, \tag{13}$$

Cauchy's problem for

$$\frac{\partial^2 u}{\partial x^2} + \frac{2\nu}{x}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} \quad , \tag{14}$$

and the scattering problem for

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} + k^2 u = 0 \quad . \tag{15}$$

These problems can often be made well posed by prescribing the behaviour of solutions in the complex domain and this is roughly the subject matter of most of the references [3] - [12]. The main tools in this investigation are analytic function theory and the use of Jacobi and Appell series. In [13] it is shown that the methods used for studying the singular equation (15) also have applications in the non-singular case, i.e. when $\nu = o$.

References [14] - [21] are concerned with my study of function theoretic methods in the investigation of improperly posed Cauchy problems for elliptic and parabolic equations. Such problems arise when inverse methods are used to study free boundary problems arising in fluid dynamics and heat conduction (c.f. [1], [2], [48]). The main emphasis here is to provide a constructive method for obtaining the solution, since the Cauchy-Kowalewski theorem is far too tedious for practical application, and more important, the series solution obtained by means of this theorem may not converge in a large enough domain.

References [22] - [27] are concerned with integral operators for elliptic equations in three or more independent variables. This work generalizes the work of Bergman and Vekua on elliptic equations in two variables to the higher dimensional case. My main contribution was to construct an operator that was invertible and thus applicable to the solution of boundary value problems by means of a complete family of solutions. For a complete discussion of this area the reader is referred to Gilbert's monograph [50].

References [28] - [37] are concerned with my work on parabolic and pseudoparabolic equations which has already been discussed in Part I.

References [38] - [43] represent my contributions to the application of the method of integral operators to problems in scattering theory in a homogeneous or spherically stratified medium. As far as the problem of scattering in a spherically stratified medium is concerned, the advantage of my approach over previous methods is that it reduces the problem to that of solving a Fredholm integral equation defined over the boundary of a three dimensional region instead of over the entire region. Applications have also been made to various inverse problems arising in scattering theory through the use of reflection principles and the analysis of generalized moment problems. Reference [43] represents a survey of some of this work.

3

JACOBI POLYNOMIALS OF NEGATIVE INDEX AND A NONEXISTENCE THEOREM FOR THE GENERALIZED AXIALLY SYMMETRIC POTENTIAL EQUATION

DAVID COLTON

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JACOBI POLYNOMIALS OF NEGATIVE INDEX AND A NONEXISTENCE THEOREM FOR THE GENERALIZED AXIALLY SYMMETRIC POTENTIAL EQUATION*

DAVID COLTON†

Introduction. Expansions in series of the classical orthogonal polynomials, e.g., Laguerre, Jacobi, and Hermite polynomials, are one of the most important tools of the applied mathematician. The theory of such expansions is well known for the range in which the weight function is integrable and orthogonality holds [3]. However, for values of the index of these polynomials such that this is no longer true little has been done even though such expansions arise often in several areas of research, in particular the study of certain classes of singular partial differential equations [1]. It is the purpose of this paper to determine when an analytic function can be expanded in a series of Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$, where $\alpha < -1$, $\beta < -1$. These results will then be used to derive a nonexistence theorem for the equation of generalized axially symmetric potentials [4]. As pointed out in [2] one of the main difficulties in studying "improperly posed" boundary value problems has been the lack of any suitable technique for determining when existence fails. This paper suggests an avenue of approach to this problem for a particular class of equations.

Analysis.

Definition 1. The space of functions analytic on the closed segment $[-1, +1]$ will be denoted by $\mathcal{A}$. If $\alpha > -1$, $\beta > -1$, it is well known [3] that the Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ form a complete set in the space $\mathcal{A}$, i.e., if $f(x) \in \mathcal{A}$, then $f(x)$ can be expanded in a Jacobi series which is convergent in the interior of the greatest ellipse with foci at ± 1 in which $f(x)$ is regular. The following theorem extends this result to the case $\alpha \leqq -1, \beta \leqq -1\ \alpha + \beta, \neq -2, -3, \cdots$.

Theorem 1. *Assume* $\alpha + \beta \neq -2, -3, \cdots, \alpha \leqq -1$, $\beta \leqq -1$. *Then, if* $f(x) \in \mathcal{A}$, $f(x)$ *can be expanded in a Jacobi series which is convergent in the interior of the greatest ellipse with foci at* ± 1 *in which* $f(x)$ *is regular.*

Proof. Let m be an integer greater than $-\alpha - 1$ and $-\beta - 1$. Since $f(x)$ is analytic on $[-1, +1]$, so is $d^m f(x)/dx^m$. Now expand $d^m f(x)/dx^m$ in a Jacobi series of indices $(m + \alpha, m + \beta)$:

$$\frac{d^m f(x)}{dx^m} = \sum_{n=m}^{\infty} a_{n-m} P_{n-m}^{(m+\alpha, m+\beta)}(x), \tag{1}$$

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which is possible by [3, p. 238]. The above series converges uniformly in some ellipse containing $[-1, +1]$; and since $|P_n^{(\alpha,\beta)}(x)|^{1/n} > 1$ for n large enough, $x \notin [-1, +1]$, and all real α, β (see [3, p. 195]), we can use the root test to conclude that $\limsup_{n\to\infty} |a_{n-m}|^{1/n} < 1$. We now use the following relationship between Jacobi polynomials [3, p. 62]:

$$\frac{d^m}{dx^m} P_n^{(\alpha,\beta)}(x) = 2^{-m}(1+\alpha+\beta+n)_m P_{n-m}^{(\alpha+m,\beta+m)}(x), \tag{2}$$

where

$$\alpha_m \triangleq \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+m-1).$$

Define

$$\gamma_k = 2^{k-m}(1+\alpha+\beta+n)_{m-k} P_{n-m+k}^{(\alpha+m-k,\beta+m-k)}(1), \tag{3}$$

$$I_k(x) = \int_1^x \cdots \int_1^x dx, \qquad I_0(x) = 1. \tag{4}$$

By (2) we have

$$\begin{aligned} 2^{-m}(1+\alpha+\beta+n)_m \int_1^x P_{n-m}^{(\alpha+m,\beta+m)}(x)\,dx \\ &= \int_1^x \frac{d^m}{dx^m} P_n^{(\alpha,\beta)}(x)\,dx \\ &= \frac{d^{m-1}}{dx^{m-1}} P_n^{(\alpha,\beta)}(x)\,dx - \gamma_1. \end{aligned} \tag{5}$$

Repeating this m times we have

$$\begin{aligned} \int_1^x \cdots \int_1^x P_{n-m}^{(\alpha+m,\beta+m)}(x)\,dx \\ = 2^{-m}(1+\alpha+\beta+n)_m[P_n^{(\alpha,\beta)}(x) + f_n(x)] \end{aligned} \tag{6}$$

where

$$f_n(x) = -\sum_{l=0}^{m-1} \gamma_{m-l} I_l(x). \tag{7}$$

Since from [3, p. 57] we have

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!} = O(n^\alpha) \tag{8}$$

and $(1+\alpha+\beta+n)_{m-k} = O(n^{m-k})$, we can conclude from (7) that if we express $f_n(x)$ as

$$f_n(x) = \sum_{k=0}^{m-1} \beta_k x^k, \tag{9}$$

then $\beta_k = O(n^{m+\alpha})$ for $0 \leqq k \leqq m - 1$. Therefore, since

$$\limsup_{n\to\infty} | a_{n-m} |^{1/n} < 1,$$

it is possible to integrate (1) termwise m times and then rearrange the series to obtain

$$f(x) = \sum_{n=m}^{\infty} b_n P_n^{(\alpha,\beta)}(x) + h(x), \tag{10}$$

where $h(x)$ is a polynomial of degree at most $m - 1$. From [3, p. 61] we have, for $n \geqq 1$,

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right), \tag{11}$$

where ${}_2F_1$ denotes the hypergeometric function. Hence, since $\alpha + \beta \neq -2, -3, \cdots$ and $P_0^{(\alpha,\beta)} = 1$, $P_n^{(\alpha,\beta)}(x)$ is a polynomial of degree exactly n, and therefore we can expand the polynomial $h(x)$ in a finite Jacobi series, viz.,

$$h(x) = \sum_{n=0}^{m-1} b_n P_n^{(\alpha,\beta)}(x). \tag{12}$$

Putting (10) and (12) together we have

$$f(x) = \sum_{n=0}^{\infty} b_n P_n^{(\alpha,\beta)}(x). \tag{13}$$

Note that by construction the series (13) is convergent in the interior of the greatest ellipse with foci at ± 1 in which $f(x)$ (and hence $d^m f(x)/dx^m$) is regular.

We now show that if $\alpha,\beta \neq -1, -2, -3, \cdots, \alpha + \beta = -2, -3, \cdots$, then the $P_n^{(\alpha,\beta)}(x)$ do not form a complete set in $\mathfrak{A}$ in the sense described above, i.e., there does not exist a region enclosing $[-1, +1]$ in the complex x-plane such that any $f(x) \in \mathfrak{A}$ can be represented by a Jacobi series in some such region.

THEOREM 2. *Let $\alpha,\beta \neq -1, -2, -3, \cdots, \alpha + \beta = -2, -3, \cdots$, and let $f(x)$ be a polynomial of degree $-\alpha - \beta - 1$. Then it is not possible to expand $f(x)$ in a Jacobi series*

$$f(x) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x), \tag{14}$$

where the series converges in some region containing $[-1, +1]$.

Proof. From (11) it is seen that if $\alpha + \beta$ is a negative integer, $P_n^{(\alpha,\beta)}(x)$ is of degree n for $0 \leqq 2n + 1 \leqq -\alpha - \beta$ and $-\alpha - \beta$

$\leqq n$, whereas $P_n^{(\alpha,\beta)}(x)$ is of degree strictly less than n for $2n \geqq -\alpha - \beta$ and $n \leqq -\alpha - \beta - 1$. Since $f(x)$ is a polynomial of degree $-\alpha - \beta - 1$, it is therefore impossible to expand $f(x)$ in a Jacobi series with a finite number of terms. Now suppose it were possible to expand $f(x)$ in an infinite series of Jacobi polynomials, i.e.,

$$f(x) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x), \tag{15}$$

where there does not exist an N such that, for $n > N$, $a_n = 0$ and the series (15) converges in some region containing $[-1, +1]$. Let C be a simple closed curve lying in this region and enclosing $[-1, +1]$. From [3, p. 245] we have, for $\alpha' > -1$, $\beta' > -1$,

$$\frac{1}{\pi i} \int_C (y-1)^{\alpha'}(y+1)^{\beta'} Q_n^{(\alpha',\beta')}(y) P_m^{(\alpha',\beta')}(y)\, dy = \delta_{mn} h_n^{(\alpha',\beta')}, \tag{16}$$

$$h_n^{(\alpha',\beta')} = \frac{2^{\alpha'+\beta'+1}}{2n+\alpha'+\beta'+1} \frac{\Gamma(n+\alpha'+1)\Gamma(n+\beta'+1)}{\Gamma(n+\alpha'+\beta'+1)}, \tag{17}$$

where $Q_n^{(\alpha',\beta')}(y)$ denotes a Jacobi function of the second kind. If we expand $f(x)$ in a Jacobi series for some (fixed) $\alpha' > -1$, $\beta' > -1$ and apply (16), we conclude (since $f(x)$ is a polynomial of degree $-\alpha - \beta - 1$) that, for $n > -\alpha - \beta - 1$,

$$\int_C f(y)(y-1)^{\alpha'}(y+1)^{\beta'} Q_n^{(\alpha',\beta')}(y)\, dy = 0. \tag{18}$$

By analytic continuation with respect to α' and β', (16) and (18) hold for $\alpha', \beta' \neq -1, -2, -3, \cdots$, in particular for $\alpha' = \alpha$, $\beta' = \beta$. Now note that due to the asymptotic expansion of the $P_n^{(\alpha,\beta)}(x)$ (see [3, p. 195]),

$$\frac{|P_n^{(\alpha,\beta)}(x)|^{1/n}}{|x + (x^2-1)^{1/2}|} = 1 + o(1) \quad \text{as} \quad n \to \infty, \qquad x \notin [-1, +1]; \tag{19}$$

the series (15) converges uniformly in every compact subset of its ellipse of convergence, and hence termwise integration is permissible. Equations (16) and (18) now imply that, in the series (15), $a_n = 0$ for $n > -\alpha - \beta - 1$ which is a contradiction.

By using Theorem 2 we can establish a nonexistence theorem for the generalized axially symmetric potential or GASP equation [4]:

$$L_\nu(u) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} = 0. \tag{20}$$

A regular solution $u(x,y)$ of (20) in a domain D symmetric with respect to the axis $y = 0$ is a solution $u(x,y)$ of $L_\nu(u) = 0$ which is an analytic function of x and y^2 in D. Such solutions always exist, and if $2\nu \neq -1$,

$-3, -5, \cdots$, $u(x,y)$ is uniquely determined by its values on the axis $y = 0$ (see [1]). For $\nu < -\frac{1}{2}$ it is a simple matter to show that uniqueness fails to hold for regular solutions $u(x,y)$ of $L_\nu(u) = 0$ if we prescribe only the value of $u(x,y)$ on the boundary of a domain D. For example $u(x,y) = y^2 - (2\nu + 1)x^2 - 1$ vanishes on the ellipse $y^2 - (2\nu + 1)x^2 = 1$ and is a regular solution of $L_\nu(u) = 0$ on the ellipse and its interior, but $u(x,y)$ is not identically zero. The following theorem shows that for $\nu = -1, -2, -3, \cdots$ there exist domains D such that, in general, no solution $u(x,y)$ of $L_\nu(u) = 0$, regular in $\bar{D}$ and satisfying prescribed analytic boundary conditions, exists, i.e., existence as well as uniqueness fails.

THEOREM 3. *Let $f(\xi)$ be an analytic function of $\xi = \cos\theta$ for $\xi \in [-1, +1]$ and let ν be a negative integer. Then, in general, no real-valued solution $u(x,y)$ of $L_\nu(u) = 0$, regular in the closed unit disc $\bar{\Omega} = \{x,y \mid r = +(x^2 + y^2)^{1/2} \leqq 1\}$, exists which assumes the values $f(\xi)$ on $r = 1$.*

Proof. Suppose $u(x,0) = \sum_{n=0}^{\infty} a_n x^n$, $x \in [-1, +1]$. Let $x = r\cos\theta$, $y = r\sin\theta$, and consider

$$u^\dagger(x, y) = \sum_{n=0}^{\infty} a_n r^n \frac{P_n^{(\nu-1/2,\nu-1/2)}(\cos\theta)}{P_n^{(\nu-1/2,\nu-1/2)}(1)} \tag{21}$$

for (x,y) in the open disc Ω. Note that, although the Jacobi polynomials for the values of ν considered here represent degenerate cases of the hypergeometric function, these are still well defined [3, p. 61]. Now note that

$$\begin{aligned} u^\dagger(x, 0) &= \sum_{n=0}^{\infty} a_n x^n \frac{P_n^{(\nu-1/2,\nu-1/2)}(1)}{P_n^{(\nu-1/2,\nu-1/2)}(1)} \\ &= \sum_{n=0}^{\infty} a_n x^n, \qquad 1 \geqq x \geqq 0, \\ u^\dagger(x, 0) &= \sum_{n=0}^{\infty} a_n(-x)^n \frac{P_n^{(\nu-1/2,\nu-1/2)}(-1)}{P_n^{(\nu-1/2,\nu-1/2)}(1)} \\ &= \sum_{n=0}^{\infty} a_n(-x)^n(-1)^n \frac{P_n^{(\nu-1/2,\nu-1/2)}(1)}{P_n^{(\nu-1/2,\nu-1/2)}(1)} \\ &= \sum_{n=0}^{\infty} a_n x^n, \qquad -1 \leqq x \leqq 0. \end{aligned} \tag{22}$$

It is easily verified that $L_\nu(u^\dagger) = 0$ in Ω, and since $u^\dagger(x,0) = u(x,0)$ for $x \in [-1, +1]$, we can conclude that $u^\dagger(x,y) = u(x,y)$ in Ω. From [1] we note that since $u(x,y)$ is regular in $\bar{\Omega}$ and real-valued, $u(x + iy, 0)$ is regular for $|x + iy| \leqq 1$ and hence $\limsup_{n\to\infty} |a_n| < 1$. For $x \in [-1, +1]$, $\alpha,\beta \neq -1, -2, -3, \cdots$, we have (see [3, p. 164])

$$|P_n^{(\alpha,\beta)}(x)| = O(n^{-1/2}). \tag{23}$$

Therefore from (19) and (23) we can conclude that the right-hand side of (21) agrees with $u(x,y)$ on $\bar{\Omega}$, and for fixed $r \leqq 1$ the series converges uniformly in some region in the complex $\xi = \cos \theta$ plane enclosing $[-1, +1]$. But for $r = 1$, $u(x,y) = f(\xi)$, i.e., we have

$$f(\xi) = \sum_{n=0}^{\infty} a_n \frac{P_n^{(\nu-1/2,\nu-1/2)}(\xi)}{P_n^{(\nu-1/2,\nu-1/2)}(1)}, \qquad \xi \in [-1, +1]. \tag{24}$$

Since both sides of (24) are analytic functions of ξ in some region containing $[-1, +1]$ and agree for $\xi \in [-1, +1]$, (24) holds in some region in the complex ξ-plane enclosing $[-1, +1]$. By Theorem 2 there exist analytic functions $f(\xi)$ such that this is impossible, and hence in this case $u(x,y)$ cannot equal $f(\cos \theta)$ on $\partial\Omega$ and the theorem is proved.

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4

Jacobi series which converge to zero, with applications to a class of singular partial differential equations

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1. *Introduction.* Expansions in series of functions are one of the most important tools of the applied mathematician, particularly expansions in series of the classical orthogonal polynomials, e.g. Laguerre, Jacobi and Hermite polynomials. In applied problems, the uniqueness of the particular expansion is usually intrinsic to the analysis, and often implicitly assumed. Indeed, in those cases where the functions in the series are orthogonal, uniqueness can often be proved by an argument that runs as follows. Let $\{\phi_n(x)\}$ $(n = 0, 1, 2, \ldots)$ be a sequence of functions orthogonal with respect to the weight function $\rho(x)$ over the interval $[0, 1]$, and suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x), \tag{1}$$

$$= \sum_{n=0}^{\infty} d_n \phi_n(x), \tag{2}$$

the series being boundedly convergent for $0 \leqslant x \leqslant 1$.

Then

$$0 = \sum_{n=0}^{\infty} (c_n - d_n)\, \phi_n(x), \tag{3}$$

and multiplying this series by $\phi_m(x)\,\rho(x)$ and integrating between 0 and 1, which is permissible, see (1), we find

$$c_m = d_m, \quad (m = 0, 1, 2, \ldots). \tag{4}$$

Even when the $\{\phi_n(x)\}$ are not orthogonal, one can show, as above, that the problem of uniqueness involves the question of whether 0 has a non-trivial representation as a series of the functions in question.

It is perhaps too little understood that care must be exercised in assuming that such expansions are unique, even in the case of the classical orthogonal polynomials. For example, let

$$R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1) \tag{5}$$

be the shifted Jacobi polynomial, the notation on the right above, as all other notation here, being that of (2). We show in this paper that one can determine subsets of $[0, 1]$ of measure 1 where

$$0 = \sum_{n=0}^{\infty} c_n R_n^{(\alpha,\beta)}(x), \tag{6}$$

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yet $c_n \neq 0$ for every n. This result, which holds provided only that $\alpha < -\frac{1}{2}$ and $\beta \neq -1, -2, \ldots$, has important applications in other areas of mathematics. As an example we use it to prove that conditions which are shown to guarantee uniqueness of the solutions of a class of singular partial differential equations cannot be relaxed.

The phenomenon (6) is not confined to Jacobi series, for the above statement is a corollary of a result which holds for sequences $\{g_n(x)\}$ defined by a wide class of generating functions

$$G(x,t) = \sum_{n=0}^{\infty} g_n(x)\,t^n. \tag{7}$$

2. *Results.*

THEOREM 1. *Let*

$$\lim_{n\to\infty} n g_n(x) = 0 \quad (x \in X), \tag{8}$$

where $\{g_n(x)\}$ *is defined by* (7) *for* $|t| < 1$, *so that* $G(x,t)$ *in* (7) *is analytic for* $|t| < 1$. *Assume furthermore that for each* $x \in X$, G *is also analytic at* $t = 1$ *and satisfies*

$$\left.\frac{\partial G(x,t)}{\partial t}\right|_{t=1} = KG(x,1) \quad (K \neq 0). \tag{9}$$

Then

$$0 = \sum_{n=0}^{\infty} (n-K)\,g_n(x) \quad (x \in X). \tag{10}$$

Proof. By (3), the series (10) converges. We have

$$G(x,t) = G(x,1)\,[1+K(t-1)] + O[(t-1)^2] \quad (t \to 1); \tag{11}$$

so

$$\left.\begin{aligned} t\frac{d}{dt}G(x,t) - KG(x,t) &= K(t-1)\,G(x,1) + O[(t-1)^2], \\ &= \sum_{n=0}^{\infty} (n-K)\,g_n(x)t^n, \end{aligned}\right\} \tag{12}$$

and (10) follows by Abel's theorem (4).

We now consider the case where $\{g_n(x)\}$ are the shifted Jacobi polynomials.

In what follows, let

$$X_1 = (0,1), \quad X_2 = [0,1), \quad X_3 = [0,1], \quad \gamma = \alpha+\beta+1. \tag{13}$$

THEOREM 2. *Let* $\alpha < -\frac{1}{2}$, $\beta \neq -1, -2, \ldots$, *and*

(i) *if* $\gamma < 0$, *then* $r = 2$;
(ii) *if both* $\gamma < 0$, $\alpha < -1$, *then* $r = 3$;
(iii) $r = 1$ *if neither of the above prevails.*

Then

$$0 = \sum_{n=0}^{\infty} \frac{(\gamma)_n\,(2n+\gamma)}{(\beta+1)_n} R_n^{(\alpha,\beta)}(x) \quad (x \in X_r). \tag{14}$$

Proof. Our starting point is the generating function given in (5).

$$(1+t)^{-\gamma} H[4xt/(1+t)^2] = \sum_{n=0}^{\infty} \frac{(\gamma)_n\,t^n}{(\beta+1)_n} R_n^{(\alpha,\beta)}(x) \quad (|t| < 1), \tag{15}$$

$$H(z) = {}_2F_1\left(\left.\begin{matrix} \frac{\gamma}{2}, \frac{\gamma+1}{2} \\ \beta+1 \end{matrix}\right| z\right). \tag{16}$$

Since $H(z)$ is analytic for $|z| < 1$, the results for $r = 1, 2$ follow immediately from Theorem 1 and the asymptotic estimates for $R_n^{(\alpha,\beta)}(x)$ given in (6),

$$\left.\begin{aligned} R_n^{(\alpha,\beta)}(x) &= A(\theta)\, n^{-\frac{1}{2}} \cos\{(n+\gamma/2)\,\theta - \tfrac{1}{2}\pi\,(\tfrac{1}{2}+\beta)\}\,[1+O(n^{-1})], \\ n &\to \infty, \quad x = (1-\cos\theta)/2, \quad 0 < x < 1. \end{aligned}\right\} \tag{17}$$

A is a bounded function of θ independent of n. (We leave it to the reader to verify that not only (15) but any generating function of the form $\Psi(t)\, G\,[4xt/(1+t)^2]$, where Ψ, G are analytic in appropriate regions, satisfies the conditions of Theorem 1.)

When $r = 3$, more than Theorem 1 is needed, since $G(1, t)$ is not analytic for $t = 1$. Note, however, that if $\gamma < 0$ and $\alpha < -1$ the convergence of (14) for all $x \in X_3$ may be inferred from (17) and results in (7).

Let, then, $x = 1$ and put

$$L(t) = (1+t)^{-\gamma}\, H[4t/(1+t)^2], \tag{18}$$

$$\left(t\frac{d}{dt}+\frac{\gamma}{2}\right) L\,(t) = \frac{\gamma(\gamma+1)\,t(1-t)}{(1+t)\,(\beta+1)}\, L^*(t) + \frac{\gamma(1-t)}{2(1+t)} L(t) \quad (|t| < 1), \tag{19}$$

where L^* is L with α replaced by $\alpha+1$ and β by $\beta+1$.

Now, the behaviour of $L(t)$ near $t = 1$ is known; see, for example ((8), eq. 2·10). We have

$$L(t) = O[(1-t)^{2\alpha} \ln(1-t)] + O(1) \tag{20}$$

$$L^*(t) = O[(1-t)^{-2\alpha-2} \ln(1-t)] + O(1) \tag{21}$$

$$|t| \to 1, \quad |\arg(1-t)| < \pi.$$

Thus the hypotheses of the theorem guarantee that

$$\lim_{t\to 1-} \left(t\frac{d}{dt}+\frac{\gamma}{2}\right) L(t) = 0. \tag{22}$$

Consequently, Abel's theorem applies, and gives the result for X_3.

The third case is rather interesting, since the polynomials $R_n^{(\alpha,\beta)}(x)$ are orthogonal over the interval $[0, 1]$ (the weight function being $(1-x)^\alpha x^\beta$), the series sums to zero for $0 < x < 1$, and yet its coefficients are not all zero. Of course, the argument of section 1 does not apply here, since the series does not converge boundedly for all $0 \leqslant x \leqslant 1$.

For those values of x, α, β for which the convergence is absolute, (14) follows by substitution of the identity given in (9)

$$(2n+\gamma)\, R_n^{(\alpha,\beta)}(x) = (n+\gamma)\, R_n^{(\alpha+1,\beta)}(x) - (n+\beta)\, R_{n-1}^{(\alpha+1,\beta)}(x) \quad (n \geqslant 1), \tag{23}$$

and rearranging the terms.

A result more general than (14) which applies to series of the hypergeometric polynomials

$${}_{P+2}F_{P+1}\left(\left.\begin{matrix} -n, n+\gamma, a_1, a_2, \ldots, a_P \\ b_1, b_2, \ldots, b_{P+1} \end{matrix}\right| x\right) \tag{24}$$

can be demonstrated by using Theorem 1 on a generating function given in (10).

Kogbetliantz (11) has proved that, if the ultraspherical series

$$\sum_{n=0}^{\infty} c_n R_n^{(\alpha,\alpha)}(x) \tag{25}$$

converges with the sum zero everywhere in $[0, 1]$ (except, perhaps, at 0 and 1 and on a set of interior points of measure zero, where it may diverge, or converge with a sum different from 0) then $c_n = 0$ for all n. Used in his proof, however (but nowhere explicitly stated), is the hypothesis that $\alpha \geqslant -\frac{1}{2}$, see (11), p. 167. The same author has discussed at length the Cesàro summability of the series (14) when $\alpha = \beta$, see (12).

We now turn to an application of the above theorem.

Although uniqueness theorems for linear elliptic partial differential equations defined in a bounded domain D with coefficients continuous in $\bar{D}$ have been known for some time (13) it is only recently that uniqueness theorems have been derived for equations whose coefficients have singularities in the domain in question (14). Here we consider the singular partial differential equation

$$L_\nu(u) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} - \lambda^2 u = 0, \tag{26}$$

in a bounded domain D whose intersection with the x-axis is an open interval, and where ν is a real number, $\lambda > 0$. We shall now establish a uniqueness theorem for this equation and use Theorem 2 to explore its limitations.

In what follows, let ∂D denote the boundary of D.

THEOREM 3. *Let $\nu \geqslant -\frac{1}{2}$ and $g(x,y) \in C^0(\partial D)$. Then there is at most one solution $u(x,y)$ of $L_\nu(u) = 0$ such that $u(x,y) \in C^2(D) \cap C^0(\bar{D})$, $u(x,y) = u(x,-y)$ and $u(x,y) = g(x,y)$ on ∂D.*

This result is the best possible in the following sense: if $\nu < -\frac{1}{2}$, $2\nu \neq -1, -2, \ldots$, there are domains where, if any solution at all of $L_\nu(u) = 0$ exists satisfying the stated conditions, then that solution is not unique.

Proof. Assume $u(x,y)$ satisfies the conditions of the theorem. If u achieves its positive maximum in D and not on ∂D, this point must be on the x-axis, by the Hopf maximum principle (13). The fact that u is even in y implies that

$$\left.\frac{1}{y}\frac{\partial u}{\partial y}\right|_{y=0} = \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0}, \tag{27}$$

and so
$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_0,0)} + (1+2\nu)\left.\frac{\partial^2 u}{\partial y^2}\right|_{(x_0,0)} - \lambda^2 u(x_0,0) = 0, \tag{28}$$

and if $(x_0, 0)$ is this maximum point,

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_0,0)} \leqslant 0, \quad \left.\frac{\partial^2 u}{\partial y^2}\right|_{(x_0,0)} \leqslant 0, \tag{29}$$

By hypothesis, $(1+2\nu) \geqslant 0$, $\lambda^2 u(x_0,0) > 0$. But this makes (28) absurd so $u(x,y)$ cannot achieve its positive maximum in D. By replacing $u(x,y)$ by $-u(x,y)$, one finds similarly that $u(x,y)$ cannot achieve its negative minimum in D. If two solutions of (26) are equal to $g(x,y)$ on ∂D, then their difference satisfies (26) and vanishes on ∂D. But such a solution, not to be identically zero, must possess a positive maximum or a negative minimum in D. This is impossible, and the first part of the theorem is established.

We now use Theorem 2 to prove the last part of the theorem. The domain whose existence is asserted we will take to be the unit disk, Ω.

Consider the Bessel–Gegenbauer series

$$w(x,y) = r^{-\nu} \sum_{n=0}^{\infty} \frac{(n+\nu)\, I_{\nu+n}(\lambda r)}{I_{\nu+n}(\lambda)} C_n^\nu(\cos\theta), \tag{30}$$

where $I_{\nu+n}$ is the modified Bessel function of the first kind. Assume $\nu < -\frac{1}{2}$, $2\nu \neq -1, -2, \ldots$. From the series representation of $I_{\nu+n}$ we conclude that

$$I_{\nu+n}(z) = \frac{(z/2)^{n+\nu}}{\Gamma(n+\nu+1)}[1+O[(n+\nu)^{-1}]] \quad (n \to \infty), \tag{31}$$

and hence the differential operator L_ν can be applied termwise to the series (30) for $r < 1$. Since each term of the series satisfies (26), we infer that $L_\nu(w) = 0$ in Ω. Using (31) we can write (30) as

$$w(x,y) = \sum_{n=0}^{\infty} (n+\nu)\, r^n \left\{1 + \frac{M_n(\nu,r)}{(n+\nu)}\right\} C_n^\nu(\cos\theta), \tag{32}$$

where $M_n(\nu, r)$ is a bounded function of n for $(x,y) \in \bar{\Omega}$.

Also, $|C_n(\cos\theta)| = O(n^{\nu-1})$ uniformly for $\theta \epsilon [0,\pi]$, see (15), and so for $\nu < 0$, the series

$$\sum_{n=0}^{\infty} r^n M_n(\nu,r)\, C_n^\nu(\cos\theta) \tag{33}$$

converges uniformly in $\bar{\Omega}$ and hence defines a continuous function there. By using a known result ((12), eq. (7)) and Theorem 2, we find that

$$\nu(1-r^2)(1-2r\cos\theta+r^2)^{-\nu-1} = \sum_{n=0}^{\infty} (n+\nu)\, r^n C_n^\nu(\cos\theta) \tag{34}$$

in $\bar{\Omega}$. Hence if $\nu < -\frac{1}{2}$ the series on the right-hand side of (32) defines a continuous function in $\bar{\Omega}$ so $w(x,y) \in C^0(\bar{\Omega})$. From (31) and the previously mentioned bound on the Gegenbauer polynomials we infer that $w(x,y) \in C^2(\Omega)$. Obviously, $w(x,y) = w(x,-y)$. From Theorem 2 with $\alpha = \beta = \nu - \frac{1}{2}$ and x replaced by $\frac{1}{2}(1+\cos\theta)$ we can conclude that $w(x,y) = 0$ on $\partial\Omega$. Hence $w(x,y)$ satisfies the conditions of the theorem but is not identically zero. (To show this, let r tend to zero in (30).) The final statement of the theorem now follows.

For $\lambda = 0$, Theorem 3 was proved by Parter (14). In his work the function corresponding to our $w(x,y)$ was constructed from a generating function for Gegenbauer polynomials. This method fails in the case of the equation (26), since no generating function is known which satisfies the equation.

[*Added in proof*]: Prof. Richard Askey has kindly pointed out to us that the Kogbetliantz theorem referred to above is false, and that the problem of characterizing uniqueness sets is still unsolved even for Fourier series (the case $\alpha = \beta = -\frac{1}{2}$).

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5

APPLICATIONS OF A CLASS OF SINGULAR PARTIAL DIFFERENTIAL EQUATIONS TO GEGENBAUER SERIES WHICH CONVERGE TO ZERO

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APPLICATIONS OF A CLASS OF SINGULAR PARTIAL DIFFERENTIAL EQUATIONS TO GEGENBAUER SERIES WHICH CONVERGE TO ZERO*

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1. Introduction. Expansions in series of hypergeometric polynomials arise frequently when the method of separation of variables is applied to a partial differential equation and the resulting solutions are superimposed in an attempt to solve certain boundary value problems. As was pointed out in [8] care must be used in this approach since the solutions obtained by such a procedure will not necessarily be unique due to the existence of nontrivial representations of zero. In particular this occurs in the study of the singular partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} = 0, \tag{1}$$

where $\nu < -1/2$. If $\nu \neq -1, -2, \cdots$ and interest is focused on solutions of (1) which are regular on the singular line $y = 0$, then separation of variables in polar coordinates (r, θ) leads to solutions of the form

$$r^n C_n^\nu(\cos\theta), \qquad n = 0, 1, 2, \cdots, \tag{2}$$

where C_n^ν denotes Gegenbauer's polynomial defined by the generating function

$$(1 - 2r\xi + r^2)^{-\nu} = \sum_{n=0}^{\infty} r^n C_n^\nu(\xi). \tag{3}$$

In view of the representation [8]

$$\lim_{N\to\infty} \sum_{n=0}^{N} (n + \nu) C_n^\nu(\cos\theta) = 0, \qquad \text{uniformly for } \theta \in [0, 2\pi], \tag{4}$$

it is not possible to solve uniquely the Dirichlet problem for the unit disc by a superposition of the solutions given in (2). (We are concerned here with the interior Dirichlet problem. This can be transformed to the exterior problem by means of a generalized Kelvin transformation [3].) The existence of expansions such as (4) leads to the conclusion that Dirichlet's problem for the singular equation (1) defined in domains containing a portion of the singular line $y = 0$ in its interior is in fact an improperly posed problem. Equation (1) (known as the generalized axially symmetric potential equation [7]) is far from being simply a pathological example. The case when 2ν is a negative integer describes axially symmetric Stokes flow in $n = -2\nu + 2$ dimensions, whereas from a mathematical viewpoint, this equation is the simplest example of an elliptic equation with meromorphic coefficients. These remarks serve as motivation for a closer examination of representations of zero by series of Gegenbauer polynomials. The purpose of this paper is to initiate such an investigation through the utilization of some recent

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In this paper, we have made strong use of the fact that L is elliptic. It is clear from results of Payne and Sather [7], Knops and Payne [4] and Levine [5], that for certain special classes of differential equations and geometries, the ellipticity requirement can be relaxed. In all of the above cases, however, the problems were such that the surfaces $f =$ const. could be chosen as hyperplanes. We propose now to see to what extent this requirement may be relaxed or eliminated.

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developments in the analytic theory of partial differential equations. In particular if $2\nu \neq -1, -3, \cdots$, conditions will be given to assure that no nontrivial representation of zero exists, whereas if $2\nu = -1, -3, \cdots$, an upper bound to the number of representations of zero will be given. These results enable one to determine when a solution of the above mentioned Dirichlet problem is unique.

2. A basic lemma and its application. In the analysis that follows it is assumed that $\nu < -\frac{1}{2}$ since for $\nu \geqq -\frac{1}{2}$ the Dirichlet problem for (1) is well-posed [6] and no representation of zero of the form of equation (4) can exist; it is further assumed that the coefficients a_n of the representation $\sum_{n=0}^{\infty} a_n C_n^{\nu}(\cos\theta) = 0$ are all real. We first require a few preliminary definitions.

DEFINITION 1. The m nontrivial representations of zero on the interval $[0, 2\pi]$, $\sum_{n=0}^{\infty} a_{nj} C_n^{\nu}(\cos\theta)$, $j = 1, 2, \cdots, m$, are said to be *~~in~~dependent* if there exist constants $C_1, \cdots, C_m$ independent of n such that $C_1 a_{n_1} + \cdots + C_m a_{nm} = 0$ for all n. Representations which are not ~~in~~dependent are in dependent.

DEFINITION 2. If $\sum_{n=0}^{\infty} a_n C_n^{\nu}(\cos\theta)$ is a nontrivial representation of zero on the interval $[0, 2\pi]$ then the series $\sum_{n=0}^{\infty} a_n C_n^{\nu}(1) z^n$ is called the associated power series of the representation.

Since $\sum_{n=0}^{\infty} a_n C_n^{\nu}(1)$ is convergent the associated power series will converge absolutely and uniformly on compact subsets of the disc $|z| < 1$ in the complex z-plane. In view of the fact that $C_n^{\nu}(1)$ does not equal zero for $2\nu \neq -1, -2, -3, \cdots$ (this follows from (3),) it is clear that if $2\nu \neq -1, -2, -3, \cdots$, then m nontrivial representations of zero are dependent if and only if their associated power series converge to functions which are linearly dependent on the real interval $(-1, +1)$.

In the use of Gegenbauer series to investigate improperly posed problems for singular partial differential equations interest is focused primarily on those representations of zero which converge uniformly for $\theta \in [0, 2\pi]$. This is due to the fact that the solutions of the differential equation being considered are usually required to be continuous in the closure of their domain of definition (cf. [8]). The fact that the Gegenbauer polynomials satisfy

$$|C_n^{\nu}(\cos\theta)| = O(n^{\nu-1}), \qquad \text{uniformly for } \theta \in [0, 2\pi], \tag{5}$$

$$\frac{\partial}{\partial\theta} C_n^{\nu}(\cos\theta) = \sin\theta\, C_{n-1}^{\nu+1}(\cos\theta), \qquad n \geqq 1, \tag{6}$$

leads in a natural manner to the following definition.

DEFINITION 3. A nontrivial representation of zero on the interval $[0, 2\pi]$, $\sum_{n=0}^{\infty} a_n C_n^{\nu}(\cos\theta)$, is said to be of *class* C^m if $\sum_{n=0}^{\infty} a_n n^{\nu+m-1}$ is absolutely convergent.

We observe that a nontrivial representation of zero of class C^m where $m \geqq [-\nu + \frac{1}{2}]$ does not exist since in this case it would be possible to differentiate the series termwise and make use of (6) to conclude the existence of a nontrivial representation of zero of class C^0 for a value of ν greater than $-\frac{1}{2}$. As was previously mentioned, this is not possible. We are now in a position to prove our basic lemma.

BASIC LEMMA. *Assume $2\nu \neq -1, -2, -3, \cdots$ and let $\sum_{n=0}^{\infty} a_n C_n^{\nu}(\cos\theta)$ be a nontrivial representation of zero on the interval $[0, 2\pi]$ which is of class C^1. Then the* *associated power series* $\sum_{n=0}^{\infty} a_n C_n^{\nu}(1) z^n$ *is singular at either* $z = +1, z = -1$, *or both, and nowhere else on the circle* $|z| = 1$.

Proof. Consider the function

$$u(r, \theta) = \sum_{n=0}^{\infty} a_n r^n C_n^{\nu}(\cos \theta). \tag{7}$$

Since $\sum_{n=0}^{\infty} a_n n^{\nu}$ is absolutely convergent it is seen [6] that the series (7) converges uniformly on compact subsets of the unit disc to a solution of (1). Equations (5) and (6) furthermore show that the first partial derivatives of $u(r, \theta)$ are uniformly continuous in the closed disc $r \leqq 1$, $0 \leqq \theta \leqq 2\pi$. Since $u(1, \theta) = 0$ it is possible [3] to analytically continue $u(r, \theta)$ across the unit circle $r = 1$ provided $\theta \neq 0, \pi$, i.e., for all points on the unit circle not lying on the singular line $y = 0$. It is known from [1] and [2] that for $2\nu \neq -1, -2, -3, \cdots$, the associated power series $\sum_{n=0}^{\infty} a_n C_n^{\nu}(1) z^n$ is singular at $z = e^{i\theta}$ if and only if the solution of (1) defined by (7) is singular at $(1, \theta)$. Since (7) is analytic at all points $(1, \theta) \neq (1, 0)$ or $(1, \pi)$, it is possible to conclude that the only possible singular points of the associated power series are at $z = \pm 1$. If neither of these points is a singular point then the associated power series has no singularities on the unit circle in the complex z-plane and hence converges for $|z| < 1 + \delta$ where $\delta > 0$. This implies $\lim_{n\to\infty} |a_n|^{1/n} < 1$, i.e., $\sum_{n=0}^{\infty} a_n C_n^{\nu}(\cos \theta)$ is a nontrivial representation of zero of class C^m where $m > [-\nu + \frac{1}{2}]$. As was observed previously this is impossible and hence the associated power series must be singular at either $z = +1, z = -1$, or both.

From the classical results on the relationship between the coefficients of a power series and the location of singular points on its circle of convergence, many theorems can now be given. Two typical examples of such results are given below.

THEOREM 1. *Assume that* $2\nu \neq -1, -2, -3, \cdots$. *Then there exists no nontrivial representation of zero which is of class* C^1 *and of the form*

$$\sum_{n=0}^{\infty} a_n C_n^{\nu}(\cos \theta) = 0, \qquad \theta \in [0, 2\pi],$$

where $a_n = 0$ *except when* n *belongs to a sequence* n_k *such that* $n_{k+1} > (1 + \delta) n_k$, $\delta > 0$.

Proof. Hadamard's gap theorem shows that the circle $|z| = 1$ is a natural boundary for the associated power series and the result follows by the basic lemma.

THEOREM 2. *Assume that* $2\nu \neq -1, -2, -3, \cdots$; *then there exists no nontrivial representation of zero which is of class* C^1 *and of the form*

$$\sum_{n=0}^{\infty} a_n C_{mn}^{\nu}(\cos \theta) = 0, \qquad \theta \in [0, 2\pi],$$

where m *is an integer greater than or equal to three.*

Proof. The basic lemma and the fact that if the power series $\sum_{n=0}^{\infty} a_n C_{mn}^{\nu}(1) z^{mn}$ has a singularity at $z = +1$ or $z = -1$, then a singularity will also exist at $z = e^{2\pi i/m}$ or $z = e^{\pi i/m}$.

3. The case when $2\nu = -1, -2, -3, \cdots$. As was pointed out in the introduction, the case when 2ν is a negative integer is of particular interest since (1) then describes axially symmetric Stokes flow in $n = (-2\nu + 2)$-dimensional space.

The existence of nontrivial representations of zero by Gegenbauer polynomials leads to the conclusion that the Dirichlet problem for the unit disc is an improperly posed problem. If $\nu = -1, -2, \cdots$ and interest is focused on solutions of (1) which are analytic functions of x and y^2 in a region containing the singular line $y = 0$, then separation of variables in polar coordinates leads to solutions of the form

$$r^n P_n^{(\nu-1/2,\nu-1/2)}(\cos\theta), \qquad n = 0, 1, \cdots, \tag{8}$$

where $P_n^{(\alpha,\beta)}$ denotes Jacobi's polynomial. For $\alpha = \beta = \nu - \frac{1}{2}$ these are essentially renormalized Gegenbauer polynomials (note that from (3), for $\nu = -1, -2, -3, \cdots, C_n^\nu(\xi) \equiv 0$ for $n > -2\nu$) and by using similar techniques theorems analogous to those obtained in § 2 can be derived for these polynomials. If, however, instead of requiring solutions to be *even* analytic functions with respect to y, it is asked that they be *odd*, then it can be shown [7] that any solution $u(x, y)$ of (1) which is analytic in a neighborhood of the singular line $y = 0$ must be of the form

$$u(x, y) = y^{1-2\nu} u^+(x, y), \tag{9}$$

where $u^+(x, y)$ is a solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2 - 2\nu}{y}\frac{\partial u}{\partial y} = 0. \tag{10}$$

Hence if $u(x, y)$ vanishes on the boundary of a domain D containing a portion of the singular line in its interior, then $u(x, y)$ vanishes on the boundary of $D \cap \{(x, y)|y > 0\}$ and hence from the maximum principle for elliptic partial differential equations [3], $u(x, y)$ is identically zero if $u(x, y) \in C^2(D) \cap C^0(\bar{D})$. Using the results of Parter [6] it can be shown that there exists a solution $u(x, y)$ to (1) such that $u(x, y) = y^{1-2\nu} f(x, y)$ on the boundary of a domain D symmetric with respect to the axis $y = 0$, where $f(x, y) = f(x, -y)$ is a prescribed function continuous in the closure $\bar{D}$ of D. Thus Dirichlet's problem for (1) can be made well posed in the case $\nu = -1, -2, -3, \cdots$, and for domains D containing a portion of the singular line in its interior. We therefore turn our attention to the case when 2ν is a negative odd integer.

THEOREM 3. *Assume 2ν is a negative odd integer. Then there exist at most $-2\nu - 1$ independent nontrivial representations of zero which are of class C^0.*

Proof. Suppose there exist -2ν independent nontrivial representations of zero

$$\sum_{n=0}^{\infty} a_{nj} C_n^\nu(\cos\theta), \qquad j = 1, 2, \cdots, -2\nu, \tag{11}$$

and consider the following corresponding solutions of (1) in the unit disc, $\Omega = \{(x, y)|x^2 + y^2 < 1\}$:

$$\sum_{n=0}^{\infty} a_{nj} r^n C_n^\nu(\cos\theta), \qquad j = 1, 2, \cdots, -2\nu. \tag{12}$$

A linear combination of these solutions gives a solution $u(r, \theta)$ to (1) of the form

$$u(r, \theta) = \sum_{n=-2\nu-1}^{\infty} b_n r^n C_n^\nu(\cos \theta) \tag{13}$$

such that $u(1, \theta) = 0$ for $\theta \in [0, 2\pi]$ and $u(r, \theta) \in C^0(\bar{\Omega}) \cap C^2(\Omega)$. Since for $2\nu = -1, -3, \cdots, C_n^\nu(1) = 0$ for $n \geqq -2\nu + 1$, whereas $C_n^\nu(1) \neq 0$ for $n < -2\nu + 1$ (this follows from (3)) we have

$$u(1, 0) = 0 = b_{-2\nu-1}C_{-2\nu-1}^\nu(1) + b_{-2\nu}C_{-2\nu}^\nu(1), \tag{14}$$

$$\begin{aligned} u(1, \pi) = 0 &= b_{-2\nu-1}C_{-2\nu-1}^\nu(-1) + b_{-2\nu}C_{-2\nu}^\nu(-1), \\ &= b_{-2\nu-1}C_{-2\nu-1}^\nu(1) - b_{-2\nu}C_{-2\nu}^\nu(1). \end{aligned} \tag{15}$$

Equations (14) and (15) now imply that $b_{-2\nu-1} = b_{-2\nu} = 0$, i.e., along the singular line $y = 0$, $u(r, \theta) = 0$. Hence $u(r, \theta)$ is a solution of (1) in $\Omega^+ = \Omega \cap \{(x, y) | y > 0\}$, vanishes on the boundary of Ω^+, and $u(r, \theta) \in C^0(\bar{\Omega}^+) \cap C^2(\Omega^+)$. By the maximum principle for elliptic partial differential equations it is seen that $u(r, \theta) \equiv 0$ in Ω^+ and hence in Ω. By noting that, for fixed θ, (13) is a power series in r and that $C_n^\nu(\cos \theta)$ is a polynomial of degree n in $\cos \theta$ it is possible to conclude that $b_n = 0$ for $n = 0, 1, 2, \cdots$. Hence the representations given in (11) are dependent and there cannot exist more than $-2\nu - 1$ nontrivial representations of zero of class C^0.

The methods used in Theorem 3 can be immediately adapted to show that if 2ν is a negative odd integer, then for a given domain D containing a portion of the singular line in its interior there exist at most $-2\nu - 1$ solutions of (1) which are linearly independent in D and vanish on the boundary of D.

By using the methods developed in [1] to examine the analytic theory of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} + \frac{2\mu}{x}\frac{\partial u}{\partial x} = 0 \tag{16}$$

it is possible to derive results analogous to those obtained in §§ 2 and 3 for series of Jacobi polynomials which converge to zero.

For $\nu > 0$ the relationship between the singularities of (7) and the associated power series was given in [4] and [5]. For such values of ν however there do not exist any nontrivial representations of zero of class C^0 (see [6]) and hence such results are not applicable to our investigation.

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THE CONSTRUCTION OF SOLUTIONS FOR BOUNDARY VALUE PROBLEMS BY FUNCTION THEORETIC METHODS*

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1. Introduction. In this paper we develop a *method of ascent* by which one may obtain a general representation formula for solutions of the differential equation of n variables

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + a(r^2) \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} + c(r^2)u = 0, \tag{1.1}$$

with $r^2 = x_1^2 + \cdots + x_n^2$, in terms of a representation formula for solutions of the differential equation of 2 variables

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + a(r^2)\left(x_1 \frac{\partial u}{\partial x_1} + x_2 \frac{\partial u}{\partial x_2}\right) + c(r^2)u = 0. \tag{1.2}$$

Indeed, we find that all regular solutions of (1.1) (about the origin) may be represented in the form

$$u(\mathbf{r}) = h(\mathbf{r}) + \int_0^1 \sigma^{n-1} G(r; 1 - \sigma^2) h(\mathbf{r}\sigma^2)\, d\sigma; \tag{1.3}$$

here $h(\mathbf{r})$ is an arbitrary harmonic function, and

$$G(r, 1 - \sigma^2) \equiv -rR_1(r\sigma^2, 0; r, r), \tag{1.4}$$

where $R(z, z^*; \zeta, \zeta^*)$ is the Riemann function for (1.2), with $z = x_1 + ix_2$, $z^* = x_1 - ix_2$.

The formula (1.3) is a natural extension of the integral formulas of S. Bergman and I. N. Vekua for $n = 2$ variables. Indeed, for $n = 2$ the G-function is an integral transform of Bergman's E-function. Also, by certain manipulations with Vekua's representations one may obtain our formula (1.3) when $n = 2$. However, our (1.3) is actually new even for the case of two variables. We present numerous examples to illustrate its use. In addition, a reduction of the Dirichlet problem to a corresponding Fredholm integral equation is given via (1.3) by equations (4.41), (4.42). It is assumed here that $c(r^2) \leqq 0$ for $\mathbf{r}$ in the closure of the particular domain at hand.

2. Elliptic equations with analytic coefficients of two variables. As a first step in obtaining an approximate method for solving boundary value problems associated with the real, analytic, partial differential equation

$$\mathbf{e}[u] \equiv \Delta u + \alpha(x, y)u_x + \beta(x, y)u_y + \gamma(x, y)u = 0, \tag{2.1}$$

we first seek suitable integral representations of a fairly wide class of solutions.

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PAPER II-4

*On the analytic theory of a class of singular partial differential equations**

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IT IS KNOWN ([16], [18]) that for $2\nu \neq 0, -1, -2, \ldots$ every solution $u(x, y)$ of the generalized axially symmetric potential or GASP equation

$$L_\nu(u) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y} \frac{\partial u}{\partial y} = 0 \tag{1}$$

which depends analytically on the two real variables x and y in a domain containing a segment of the singular line $y = 0$ is uniquely determined by its values on $y = 0$. Furthermore if $\nu > 0$ it has been shown ([7], [14]) that if the function $g(z) = u(z, 0)$ is continued to complex values of z then $u(x, y)$ is singular at $(x, y) = (r \cos \theta, r \sin \theta)$ if and only if $g(z)$ is singular at $z = re^{i\theta}$ or $z = re^{-i\theta}$. It is the purpose of this paper to extend this result into the range $\nu < 0$ and to indicate how our work can be generalized to include other singular partial differential equations, such as those considered by Gilbert and Howard ([6]–[12]). This completely answers the problem first posed by Henrici in [16], p. 201. A partial answer to this problem for the case of Eq. (1) has been given by Erdélyi ([5]) who proved that if $g(z)$ is regular in some y-convex region R (i.e., a region which contains with the point $x + iy$ also the points $x + iyt$, $-1 \leqq t \leqq 1$) and $2\nu \neq 0, -1, -2, \ldots$, then the singular points of $g(z)$ and $u(x, y)$ on the boundary of R coincide with one another.

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Our first result is to extend this theorem of Erdélyi to the more general case in which the region R is only required to be simply connected. This is first proved using two theorems due to Henrici and Colton, and suffers from the disadvantage that the methods used do not readily generalize to the other singular equations discussed in [9]–[12]. We therefore develop an alternate proof which overcomes this difficulty. We finally give a new proof of a theorem of Hyman and Mackie concerning the equation $L_\nu(u) = 0$ when $2\nu = -1, -3, -5, \ldots$, which has the advantage that the methods used lend themselves readily to a similiar investigation of the equations considered in the above mentioned work of Gilbert and Howard.

For the first part of our work we require the following two theorems:

THEOREM 1 (Henrici) *Let $g(z)$ be holomorphic in a simply connected domain η containing the origin and let $2\nu \neq 0, -1, -2, \ldots$ Then there exists a unique solution $u(x, y)$ of $L_\nu(u) = 0$ satisfying $U(z, 0) = g(z)$ for $z \in \eta$; $u(x, y)$ is an analytic function of its arguments in the domain $D(\eta)$ of all complex points (x, y) for which $x + iy \in \eta$, $x - iy \in \eta$.*

Proof [14]

THEOREM 2 (Colton) *Let $x = r\cos\theta$, $y = r\sin\theta$, $\xi = \cos\theta$ and assume $\nu \neq 0, -1, -2, \ldots$ If $u(x, y)$ is a real analytic solution of $L_\nu(u) = 0$ in a domain η containing the origin then $u(x, y) = \tilde{u}(r, \xi)$ is a real analytic function of r and ξ, and for every positive integer k, $\tilde{u}^\dagger(r, \xi) = r^{-k}\dfrac{\partial^k \tilde{u}}{\partial \xi^k}$ is a real analytic solution of $L_{\nu+k}(u) = 0$ in η.*

Proof [2]

We are now in a position to prove the following theorem.

THEOREM 3 *Let $2\nu \neq 0, -1, -2, \ldots$ and let $u(x, y)$ be a real analytic solution of $L_\nu(u) = 0$ in a simply connected domain η containing a portion of the x axis. Then the function $g(x) = u(x, 0)$ can be continued analytically to a function $g(z)$ which is holomorphic for $z \in \eta$.*

Proof Since $u(x, y)$ must be an even function of y ([2], [5]) there is no loss of generality in assuming η is symmetric with respect to the x axis, and since $L_\nu(u) = 0$ is invariant under transformations along the x axis we can assume that the origin is an interior point of η. If $\nu > 0$ the result can be easily proved using Green's Theorem ([5], [13]). Therefore assume $\nu < 0$. From theorem two we can conclude that $\tilde{u}^\dagger(r, \xi) = r^{-k}\dfrac{\partial^k \tilde{u}}{\partial \xi^k}$ is a real analytic solution of $L_{\nu+k}(u) = 0$ in η, and by choosing k large enough we can make $\nu + k > 0$. In some neighborhood of the origin $u(x, y) = \tilde{u}(r, \xi)$ has the representation ([5])

$$\tilde{u}(r, \xi) = \sum_{n=0}^{\infty} a_n r^n C_n^\nu(\xi) \tag{2}$$

where $C_n^\nu(\xi)$ denotes Gegenbauer's polynomial. Since ([3])

$$\frac{d^k}{d\xi^k} C_n^\nu(\xi) = 2^k(\nu)_k\, C_{n-k}^{\nu+k}(\xi); \quad n \geqq k \tag{3}$$

and $C_n^\nu(\xi)$ is a polynomial of degree n for $2\nu \neq 0, -1, -2, \ldots$ we can conclude that

$$\tilde{u}^\dagger(r, \xi) = 2^k(\nu)_k \sum_{n=0}^{\infty} a_{n+k} r^n C_n^{\nu+k}(\xi). \tag{4}$$

Now from [7], [14] we know that

$$u^\dagger(x, 0) = 2^k(\nu)_k \sum_{n=0}^{\infty} a_{n+k} x^n C_n^\nu(1) = 2^k(\nu)_k \sum_{n=0}^{\infty} \frac{a_{n+k}\Gamma(n + 2\nu + 2k)}{n!\,\Gamma(2\nu + 2k)} x^n \tag{5}$$

can be analytically continued into η, since $\nu + k > 0$. From formula (2) we have

$$u(x, 0) = \sum_{n=0}^{\infty} a_n x^n C_n^\nu(1) = \sum_{n=0}^{\infty} \frac{a_n\Gamma(n + 2\nu)}{n!\,\Gamma(2\nu)} x^n. \tag{6}$$

Let

$$g(x) = \frac{d^k}{dx^k}[\Gamma(2\nu)\, u(x, 0)] = \sum_{n=0}^{\infty} \frac{a_{n+k}\Gamma(n + k + 2\nu)}{n!} x^n \tag{7}$$

and note that

$$x^{-k-2\nu}\frac{d^k}{dx^k}[x^{2k+2\nu}g(x)] = \sum_{n=0}^{\infty} \frac{a_{n+k}\Gamma(2k + 2\nu + n)}{n!} x^n = \frac{\Gamma(2\nu + 2_k)}{2^k(\nu)_k} u^\dagger(x, 0). \tag{8}$$

Applying Leibniz's formula to the left hand side of formula (8) gives

$$\frac{\Gamma(2\nu + 2k)}{2^k(\nu)_k} u^\dagger(x, 0) = \sum_{j=0}^{k} \binom{k}{j} \frac{\Gamma(2k + 2\nu)}{\Gamma(2k + 2\nu - j)} x^{k-j} g^{(k-j)}(x). \tag{9}$$

Formula (9) is a kth order non homogeneous ordinary differential equation for $g(x)$ with $x = 0$ as its only finite singular point, at which point we know that $g(x)$ is analytic. From the analytic theory of ordinary differential equations ([18]), $g(x)$ (and hence $u(x, 0)$) can be continued analytically into whatever domain the nonhomogeneous term $u^\dagger(x, 0)$ is analytic, i.e., the domain η. This concludes the proof.

Theorems one and three now imply the following "fundamental theorem" concerning singularities.

THEOREM 4 *Let* $2\nu \neq 0, -1, -2, \ldots$ *and let* $u(x, y)$ *be a real analytic solution of* $L_\nu(u) = 0$ *in a simply connected domain* η *containing a portion of the x axis. Then necessary and sufficient conditions for* $u(x, y)$ *to be singular at* $(x, y) = (r\cos\theta, r\sin\theta)$ *is* $g(z)$, *the analytic continuation of* $u(x, 0)$, *be singular at* $z = re^{i\theta}$ *or* $z = re^{-i\theta}$.

The methods used above do not readily lend themselves to a similar investigation of more general singular equations such as those considered by Gilbert and Howard ([6]–[12]). We therefore give a different proof of theorem four which does not have this defect. The approach used is that of integral operators and is a direct extension of Gilbert and Howard's work into the range $\nu < 0$. The first step is no problem; if $g(z)$ is an analytic function regular about the origin, $g(z) = \sum_{n=0}^{\infty} a_n z^n$, then for $2\nu \neq -1, -2, -3, \ldots$, the operator defined by

$$u(x, y) = A_\nu[g] \equiv \frac{\Gamma(\nu + \frac{1}{2})\Gamma(1 - 2\nu)}{2\pi i\Gamma(\frac{1}{2})} \int_C g(x + iyt)(t^2 - 1)^{2\nu - 1}\, dt \tag{10}$$

maps $g(z)$ onto the GASP function $u(x, y)$ defined by

$$u(x, y) = \tilde{u}(r, \xi) = \sum_{n=0}^{\infty} \frac{n!}{\Gamma(2\nu + n)} r^n C_n^\nu(\xi) \tag{11}$$

where C is a figure eight loop inclosing the points -1 and $+1$ in the complex t plane ([18]). In order to use Gilbert's "envelope method" it is now necessary to construct an inverse operator $A_\nu^{-1}[u]$ which maps $u(x, y)$ back onto $g(z)$ and here is where the difficulty arises, since in [6]–[12] the construction of such an operator depended on the orthogonality properties of the Gegenbauer polynomials over the interval $[-1, +1]$. For $\nu < -\frac{1}{2}$ such a relationship is no longer true. However, we can prove the following result:

THEOREM 5 *Assume* $\nu < 0$, $2\nu \neq -1, -2, -3, \ldots$ *Let* $C_n^\nu(\xi)$ *denote Gegenbauer's polynomial and let* C *be a figure eight loop inclosing the points* ± 1

in the complex ξ plane (which is cut by two lines running from these points to infinity). Then

$$\frac{1}{2\pi i}\int_C (\xi^2 - 1)^{\nu-\frac{1}{2}} C_m^\nu(\xi)\, C_n^\nu(\xi)\, d\xi = h_n \delta_{mn}$$

where δ_{mn} is the Kronecker delta and

$$h_n = \frac{2^{1-2\nu}\cos\pi\nu\Gamma(2\nu + n)}{[\Gamma(\nu)]^2 (n + \nu)\, n!}.$$

Proof Without loss of generality we can assume m is greater than or equal to n. Using the Rodrigues formula for Gegenbauer's polynomial [4] we have

$$\frac{1}{2\pi i}\int_C (\xi^2 - 1)^{\nu-\frac{1}{2}} C_m^\nu(\xi)\, C_n^\nu(\xi)\, d\xi$$
$$= \frac{(2\nu)_m}{2^m\, m!(\nu + \frac{1}{2})_m}\int_C \frac{d^m}{d\xi^m}[(\xi^2 - 1)^{m+\nu-\frac{1}{2}}]C_n^\nu(\xi)\, d\xi. \tag{12}$$

Let P be a point on the curve C. Then the value of $(1 - \xi^2)^{m+\nu-\frac{1}{2}}$ will be the same when the point describing the curve C returns to P as it was originally at P. Hence if m is strictly greater than n we can integrate (12) by parts $n + 1$ times and use the fact that $C_n^\nu(\xi)$ is a polynomial of degree exactly n to obtain

$$\frac{1}{2\pi i}\int_C (\xi^2 - 1)^{\nu-\frac{1}{2}} C_m^\nu(\xi)\, C_n^\nu(\xi)\, d\xi = 0; \quad m \neq n. \tag{13}$$

Now if $m = n$, integrating (12) by parts n times gives

$$\frac{1}{2\pi i}\int_C (\xi^2 - 1)^{\nu-\frac{1}{2}} C_n^\nu(\xi)\, C_n^\nu(\xi)\, d\xi = \frac{(-1)^n (2\nu)_n (\nu)_n}{n!(\nu + \frac{1}{2})_n\, 2\pi i}\int_C (\xi^2 - 1)^{n+\nu-\frac{1}{2}} d\xi \tag{14}$$

since the leading coefficient of $C_n^\nu(\xi)$ is $\frac{(\nu)_n 2^n}{n!}$. As long as $2\nu \neq -1, -2, -3, \ldots$ the integral in (14) does not vanish and can be evaluated by refering to known representations of the beta function in terms of loop integrals ([3])

or alternatively by assuming $n + \nu > -\frac{1}{2}$, shrinking the contour onto the real axis, evaluating in terms of gamma functions, and extending the result to $n + \nu < -\frac{1}{2}$ by analytic continuation. The result is that

$$\frac{1}{2\pi i}\int_C (\xi^2 - 1)^{n+\nu-\frac{1}{2}}\, d\xi = \frac{\sqrt{\pi}}{\Gamma(n + \nu + 1)\,\Gamma(\frac{1}{2} - n - \nu)}. \tag{15}$$

Using the fact ([3]) that

$$\Gamma(\tfrac{1}{2} - n - \nu)\,\Gamma(\tfrac{1}{2} + n + \nu) = \frac{(-1)^n\pi}{\cos\pi\nu} \tag{16}$$

and the Legendre duplication formula

$$\frac{\Gamma(2\nu)}{\Gamma(\nu)} = \frac{2^{2\nu-1}\,\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}} \tag{17}$$

the theorem now follows.

With the help of theorem five we can now find an inverse operator $A_\nu^{-1}[u]$ in the manner done by Gilbert and Howard ([6]–[12]) by considering $u(x, y) = \tilde{u}(r, \xi)$ as a function of the two complex variables r and ξ. We define the kernel

$$K\left(\frac{\sigma}{r}, \xi\right) = \frac{[\Gamma(\nu)]^2}{2^{1-2\nu}\cos\pi\nu}(\xi^2 - 1)^{\nu-\frac{1}{2}}\sum_{n=0}^{\infty}(n + \nu)\left(\frac{\sigma}{r}\right)^n C_n^\nu(\xi) \tag{18}$$

where $\sigma = x + iyt$. From theorem five it may be seen that

$$\sigma^n = \int_C \left(\frac{n!}{\Gamma(2\nu + n)}\, r^n C_n^\nu(\varepsilon)\right) K\left(\frac{\sigma}{r}, \xi\right) d\xi. \tag{19}$$

Consequently if $g(\sigma) = \sum_{n=0}^{\infty} a_n\sigma^n$ and

$$\tilde{u}(r, \xi) = A_\nu[g] = \sum_{n=0}^{\infty}\frac{a_n n!}{\Gamma(2\nu + n)}\, r^n C_n^\nu(\xi) \tag{20}$$

then we have

$$g(\sigma) = A_\nu^{-1}[u] \equiv \int_C \tilde{u}(r, \xi)\, K\left(\frac{\sigma}{r}, \xi\right) d\xi. \tag{21}$$

This procedure can be justified by observing that for $\left|\frac{\sigma}{r}\right| \leqq \varrho < 1$ we have

$$K\left(\frac{\sigma}{r}, \xi\right) = \frac{[\Gamma(\nu)]^2}{2^{1-2\nu}\cos\pi\nu}(\xi^2 - 1)^{\nu-\frac{1}{2}}\left\{t^{1-\nu}\frac{\partial}{\partial t}\left[t^\nu \sum_{n=0}^{\infty} t^n C_n^\nu(\xi)\right]\right\}_{t=\sqrt{\frac{\sigma}{r}}}$$

$$= \frac{\nu[\Gamma(\nu)]^2(\xi^2 - 1)^{\nu-\frac{1}{2}}\left(1 - \frac{\sigma^2}{r^2}\right)}{2^{1-2\nu}\cos\pi\nu\left[1 - 2\xi\left(\frac{\sigma}{r}\right) + \frac{\sigma^2}{r^2}\right]^{\nu+\frac{1}{2}}} \tag{22}$$

This follows from the generating function for Gegenbauer polynomials ([4]):

$$\sum_{n=0}^{\infty} t^n C_n^\nu(\xi) = (1 - 2\xi t + t^2)^{-\nu}; \quad |t| < 1. \tag{23}$$

Using the operators A_ν defined by Eq. (10) and A_ν^{-1} defined by Eqs. (21) and (22), theorem four now follows by using the "envelope method" developed by Gilbert. It should be observed that the methods used here can be easily applied to the other singular partial differential equations considered by Gilbert and Howard, thus extending their results into the range $\nu < 0$. (For equations which are bi-axially symmetric such as those discussed in [9], [10] and [12], one considers Jacobi polynomials instead of Gegenbauer polynomials and the contour C of theorem five must be replaced by a Pochhammer contour ([13]).

If we insist that $u(x, y)$ be a real analytic function of x and y^2 then the above results can be extended to include the case when $\nu = 0, -1, -2, \ldots$; here the role of the Gegenbauer polynomial is replaced by the Jacobi polynomial $P_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(\xi)$. However, for $2\nu = -1, -3, -5, \ldots$ it has been shown by Hyman ([17]) and Mackie ([19]) that $u(x, 0)$ is a polynomial of degree at most -2ν and hence can always be extended to an entire function, regardless of the domain of regularity of $u(x, y)$. We give a new proof of this below, which has the advantage that the methods used here can be easily applied to the equations considered in [6]–[12].

THEOREM 6 *Let $2\nu = -1, -3, -5, \ldots$ and let $u(x, y)$ be a real analytic solution of $L_\nu(u) = 0$ in a domain η containing the origin. Then $u(x, 0)$ is a polynomial of degree at most -2ν.*

Proof In theorem 4.2 of [1] it was indirectly proved that if $f(\xi)$ is an analytic function of ξ on the closed interval $[-1, +1]$ and $2\nu = -1, -3, -5, \ldots,$

then $f(\xi)$ can be expanded in a Gegenbauer series convergent in some ellipse in the complex ξ plane enclosing $[-1, +1]$

$$f(\xi) = \sum_{n=0}^{\infty} a_n C_n^\nu(\xi) \tag{24}$$

where for $n \geqq -2\nu + 1$

$$a_n = \frac{h_n^\nu}{\pi i} \int_{C_0} f(\xi)\, (\xi^2 - 1)^{\nu-\frac{1}{2}}\, Q_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(\xi)\, d\xi. \tag{25}$$

Here $h_n^\nu \neq 0$ is a normalization constant, $Q_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(\xi)$ denotes a Jacobi function of the second kind and C_0 is an ellipse surrounding the interval $[-1, +1]$. Let $N(0, \varepsilon)$ be a neighborhood about the origin of radius ε and contained in the domain η. From the analytic theory of the GASP equation ([2], [5]) it is known that if $2\nu = -1, -3, -5, \ldots$ and $u(x, y)$ is analytic in x and y, then it is analytic in x and y^2 and hence regular in the sense of [1]. From Theorem 3.1 of [1] we know that $u(x, y) = \tilde{u}(r, \xi)$ is an analytic function of ξ for $r \in [0, \varepsilon]$, $\xi \in T$, where T is a region in the complex ξ plane enclosing $[-1, +1]$. Hence for each fixed $r \in [0, \varepsilon]$ we can expand $\tilde{u}(r, \xi)$ in a Gegenbauer series

$$\tilde{u}(r, \xi) = \sum_{n=0}^{\infty} a_n(r)\, C_n^\nu(\xi). \tag{26}$$

In r, ξ coordinates $L_\nu(u) = 0$ becomes

$$\frac{\partial^2 \tilde{u}}{\partial r^2} + \frac{2\nu + 1}{r}\frac{\partial \tilde{u}}{\partial r} + \frac{(1 - \xi^2)}{r^2}\frac{\partial^2 \tilde{u}}{\partial \xi^2} - \frac{(2\nu + 1)\,\xi}{r^2}\frac{\partial \tilde{u}}{\partial \xi} = 0. \tag{27}$$

For $n \geqq -2\nu + 1$ we have

$$a_n(r) = \frac{h_n^\nu}{\pi i} \int_{C_0} \tilde{u}(r, \xi)\, (\xi^2 - 1)^{\nu-\frac{1}{2}}\, Q_n^{(\nu-\frac{1}{2},\, \nu-\frac{1}{2})}(\xi)\, d\xi \tag{28}$$

and since for $r \in [0, \varepsilon]$ we can choose C_0 independent of r, we can use the differential Eq. (27), formula (28) and the analyticity of $\tilde{u}(r, \xi)$ in $N(0, \varepsilon)$ to conclude that for $n \geqq -2\nu + 1$, $a_n(r) = a_n r^n$ where a_n is a constant. For $0 \leqq n \leqq -2\nu$ we can go to the differential equation directly as in Theorem 4.2 of [1] $\Big($consider $\tilde{u}(r, \xi) - \sum_{n=-2\nu+1}^{\infty} a_n r^n C_n^\nu(\xi)\Big)$ to conclude $a_n(r) = a_n r^n$ for these values of n also. Hence

$$\tilde{u}(r, \xi) = \sum_{n=0}^{\infty} a_n r^n C_n^\nu(\xi) \tag{29}$$

and

$$u(x, 0) = \sum_{n=0}^{\infty} a_n x^n C_n^\nu(1) \tag{30}$$

From Eq. (23) it is seen that $C_n^\nu(1) = 0$ for $n \geqq -2\nu + 1$. Hence

$$u(x, 0) = \sum_{n=0}^{-2\nu} a_n x^n C_n^\nu(1) \tag{31}$$

and the theorem is proved.

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Uniqueness Theorems for Axially Symmetric Partial Differential Equations

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Uniqueness Theorems for Axially Symmetric Partial Differential Equations

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I. Introduction. Although existence and uniqueness theorems for linear elliptic partial differential equations in a domain D with coefficients continuous in D have been known for some time [2], similar results for equations whose coefficients have singularities in the domain under consideration are practically unknown. Recently attention has been given to a class of singular equations which appear frequently in both pure and applied mathematics and are known as generalized axially symmetric partial differential equations. Typical examples of equations from this class are the following:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} = 0, \tag{1}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} + k^2 u = 0, \tag{2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} - k^2 u = 0. \tag{3}$$

Here ν and k are real numbers, $k > 0$. Just as a thorough knowledge of Laplace's equation guided the attack on linear elliptic equations with continuous coefficients, it is hoped that a better understanding of equations such as 1), 2) and 3) will give insight towards developing a theory of elliptic equations with singular coefficients.

The results so far have been sparse and incomplete. In 1965 Parter [18] derived existence and uniqueness theorems for equation 1) (known as the generalized axially symmetric potential or GASP equation) for the case in which $\nu \geqq -\frac{1}{2}$ and showed that his results were no longer valid for $\nu < -\frac{1}{2}$. In 1968 Colton [1] proved a uniqueness theorem for the exterior Dirichlet problem for equation 2) (the generalized axially symmetric Helmholtz or GASH equation) for $\nu > -\frac{1}{2}$.

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For $\nu \geqq -\frac{1}{2}$ uniqueness conditions for equation 3) were found by Wimp and Colton [24] and shown to be no longer strong enough for the situation in which $\nu < -\frac{1}{2}$. A rather unusual fact appears: there is a sharp difference in the criteria needed to insure uniqueness for the cases $\nu \geqq -\frac{1}{2}$ and $\nu < -\frac{1}{2}$.

It should also be noted that Huber [14] and Schecter [19] have investigated the situation in which the coefficients of the differential equation are singular for points on the boundary of the domain D but are continuous in the interior of D. Since we are interested in the case in which the coefficients are singular in the interior of D, this work has no direct bearing on our investigations.

In this paper we first consider equation 2) and derive a result similar to that in [1] except we now consider the case in which $u(x, y)$ is regular (see Def. 2.1) in the whole space. Although the analysis in [1] was rather long and relied heavily on the use of special functions, the proof of uniqueness for the situation considered here is relatively short and straightforward. This is due in a large part to the discovery of a new relationship between solutions of axially symmetric equations for different values of the parameter ν. This result is derived in Lemma 2.1 and forms the basis of the work presented here. Finally we turn our attention to equations 1) and 3) and extend the results of Parter, Wimp and Colton into the range $\nu < -\frac{1}{2}$. A seemingly pathological situation occurs: in order to insure uniqueness it is necessary to impose boundary conditions similar to those employed in formulating a well posed boundary value problem for an m^{th} order equation [17], where here m depends on the value of ν.

II. The analytic theory of generalized axially symmetric equations. The analytic theory of equations such as 1), 2) and 3) has been extensively investigated by Weinstein ([22], [23]), Erdélyi [5], Henrici ([12], [13]), Hyman [15], Mackie [16], Gilbert ([6]–[10]), Gilbert and Howard [11], and Colton [1]. That part of the theory which is needed in this paper will now be briefly summarized. For notational convenience let $L_{\nu,k^2}(u) = 0$ denote equations 1), 2) and 3) *i.e.*, $L_{\nu,0}(\mu) = 0$ is equation 1), $L_{\nu,+k^2}(u) = 0$ is equation 2), and $L_{\nu,-k^2}(u) = 0$ is equation 3).

In either of the half planes $y > 0$ and $y < 0$, $L_{\nu,k^2}(u) = 0$ is an elliptic partial differential equation with analytic coefficients, and hence every twice continuously differentiable solution is an analytic function of x and y in each such half plane and can be extended into the complex x and y planes [2]. The line $y = 0$, which will be called the axis, is a singular curve of the regular type with exponents 0 and $1 - 2\nu$ [4]. Consequently, there always exist solutions which are regular on (some portion of) the axis. It is seen from the differential equation that if $\nu \neq 0$ then $\partial u/\partial y = 0$ on the axis for such regular solutions. For $2\nu \neq 0, -1, -2, -3, \cdots$ each regular solution can be continued across the axis as an even function of y *i.e.*, for $2\nu \neq 0, -1, -2, -3, \cdots$ every regular solution is an analytic function of x and y^2 in some domain D that is symmetric with respect to the axis $y = 0$. If $2\nu = 0, -1, -2, \cdots$ the assumption that $u(x, y)$ is an even function of y will be part of the definition of regularity, *viz.*

Definition 2.1. A solution $u(x, y)$ of $L_{\nu,k^2}(u) = 0$ will be called *regular* if it is an analytic function of x and y^2 in some region which is symmetric with respect to the axis $y = 0$.

Since $L_{\nu,k^2}(u)$ is invariant under translations of the x axis, we can assume without loss of generality that the domain of regularity of $u(x, y)$ contains the origen. If $u(x, y)$ is a regular solution of $L_{\nu,k^2}(u)=0$ then $u(x, y)$ is an even function of y and hence can be expressed as $u(x, y) = \tilde{u}(r, \xi)$ where $x = r\xi, y = r(1 - \xi^2)^{1/2}, \xi = \cos\theta$. Since an analytic function of an analytic function is analytic, $\tilde{u}(r, \xi)$ is an analytic function of r and ξ. We are now in a position to prove the following basic lemma, which will play a central role in the forthcoming analysis.

Lemma 2.1. Let $u(x, y) = \tilde{u}(r, \xi)$ be a regular solution of $L_{\nu,k^2}(u) = 0$ in a domain D. Then for every positive integer j, $\tilde{u}^{\dagger}(r, \xi) = r^{-j}\, \partial^j\tilde{u}(r, \xi)/\partial\xi^j$ is a regular solution of $L_{\nu+j,k^2}(u) = 0$ in D.

Proof. If $u(x, y)$ is a regular solution of $L_{\nu,k^2}(u) = 0$ then $u(x, y) = \tilde{u}(r, \xi)$ and $\tilde{u}(r, \xi)$ satisfies the partial differential equation

$$(4)\qquad \frac{\partial^2\tilde{u}}{\partial r^2} + \frac{2\nu + 1}{r}\frac{\partial\tilde{u}}{\partial r} + \frac{(1 - \xi^2)}{r^2}\frac{\partial^2\tilde{u}}{\partial\xi^2} - \frac{(2\nu + 1)\xi}{r^2}\frac{\partial\tilde{u}}{\partial\xi} + k^2\tilde{u} = 0.$$

Let $v = \partial u/\partial\xi$. From 4) we have that v satisfies

$$(5)\qquad \frac{\partial^2 v}{\partial r^2} + \frac{2\nu + 1}{r}\frac{\partial v}{\partial r} - \frac{(2\nu + 3)}{r^2}\frac{\partial v}{\partial\xi} + \frac{(1 - \xi^2)}{r^2}\frac{\partial^2 v}{\partial\xi^2} + \left(k^2 - \frac{2\nu + 1}{r^2}\right)v = 0.$$

Now let $\tilde{u}^{\dagger} = v/r$ (multiply equation 5) through by $1/r$) and obtain

$$(6)\qquad \left(\frac{1}{r}\frac{\partial^2 v}{\partial r^2} + \frac{2\nu + 1}{r^2}\frac{\partial v}{\partial r} - \frac{2\nu + 1}{r^3}v\right) - \frac{(2\nu + 3)\xi}{r^2}\frac{\partial\tilde{u}^{\dagger}}{\partial r} + \frac{(1 - \xi^2)}{r^2}\frac{\partial^2\tilde{u}^{\dagger}}{\partial\xi^2} + k^2\tilde{u}^{\dagger} = 0,$$

$$(7)\qquad \frac{\partial^2\tilde{u}^{\dagger}}{\partial r^2} + \frac{2\nu + 3}{r}\frac{\partial\tilde{u}^{\dagger}}{\partial r} + \frac{(1 - \xi^2)}{r^2}\frac{\partial^2\tilde{u}^{\dagger}}{\partial\xi^2} - \frac{(2\nu + 3)\xi}{r^2}\frac{\partial\tilde{u}^{\dagger}}{\partial r} + k^2\tilde{u}^{\dagger} = 0.$$

Since $u(x, y)$ is an analytic function of x and y^2 in D we have

$$(8)\qquad \frac{\partial\tilde{u}}{\partial\xi} = \frac{\partial u}{\partial(y^2)}\frac{\partial(y^2)}{\partial\xi} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial\xi} = -2\xi r^2\frac{\partial u}{\partial(y^2)} + r\frac{\partial u}{\partial x}$$

and hence $\tilde{u}^{\dagger} = 1/r(\partial\tilde{u}/\partial\xi)$ is regular at $r = 0$. Since $u(r, \xi)$ is an analytic function of r and ξ in D we can conclude that the lemma is true for $j = 1$. Since

$$\frac{1}{r}\frac{\partial\tilde{u}^{\dagger}}{\partial\xi} = \frac{1}{r}\frac{\partial}{\partial\xi}\left(\frac{1}{r}\frac{\partial\tilde{u}}{\partial\xi}\right) = \frac{1}{r^2}\frac{\partial^2\tilde{u}}{\partial\xi^2}$$

the lemma is true for $j = 2$ and by induction for all positive integer j.

III. The equation $L_{\nu,+k^2}(u) = 0$. In this section $u(x, y)$ is assumed to be a regular solution of $L_{\nu,+k^2}(u) = 0$ (*i.e.* equation 2) which is regular in the whole real x, y plane. We first prove a uniqueness theorem for the case in which $\nu \geqq -\frac{1}{2}$.

Theorem 3.1. Assume $\nu \geqq -\frac{1}{2}$ and let $u(x, y) = \tilde{u}(r, \xi)$ be a solution of $L_{\nu,+k^2}(u) = 0$ which is regular in the whole real x, y plane. If

1) *there exists a positive constant $M < \infty$ such that $|y^{\nu+1/2}u(x, y)| \leqq M$ for all real x and y,*
2) *$\lim_{r\to\infty} y^{\nu+1/2}u(x, y) = 0$ pointwise for $\theta \in [0, \pi]$,*

then $u(x, y) \equiv 0$.

Proof. First consider the case $\nu > -\frac{1}{2}$. Since $\tilde{u}(r, \xi)$ is an analytic function of ξ for fixed r, $\xi \in [-1, +1]$, we can expand $\tilde{u}(r, \xi)$ in a Gegenbauer series [20]:

$$\tilde{u}(r, \xi) = \sum_{n=0}^{\infty} hnC_n^{\nu}(\xi) \int_{-1}^{+1} \tilde{u}(r, t)(1 - t^2)^{\nu-1/2}C_n^{\nu}(t)\, dt \tag{9}$$

where $C_n^{\nu}(\xi)$ denotes Gegenbauer's polynomial [3] and h_n is a normalization factor. Now let

$$v_n(r) = h_n \int_{-1}^{+1} \tilde{u}(r, t)(1 - t^2)^{\nu-1/2}C_n^{\nu}(t)\, dt. \tag{10}$$

From equation 4) and the fact that $\tilde{u}(r, \xi)$ is regular in the whole plane it can be shown that $v_n(r)$ is a solution of Bessel's equation and in fact

$$v_n(r) = a_n r^{-\nu} J_{\nu+n}(kr) \tag{11}$$

where a_n is a constant. Making use of hypothesis one and two of the theorem and Lebesgue's dominated convergence theorem we have

$$\begin{aligned}\lim_{r\to\infty} |a_n r^{1/2} J_{\nu+n}(kr)| &\leqq \lim_{r\to\infty} \int_{-1}^{+1} |r^{\nu+1/2}\tilde{u}(r, t)(1 - t^2)^{\nu-1/2}C_n^{\nu}(t)|\, dt \\ &\leqq \lim_{r\to\infty} \int_0^{\pi} |y^{\nu+1/2}\tilde{u}(r, \cos\theta)|\, |\sin^{\nu-1/2}\theta C_n^{\nu}(\cos\theta)|\, d\theta \\ &\leqq 0.\end{aligned} \tag{12}$$

In view of the asymptotic behavior of Bessel's function [3]

$$(2\pi r)^{1/2} J_{\nu+n}(kr) = 2\cos\left[r + \tfrac{1}{2}(\nu + n)\pi - \frac{\pi}{4}\right] + O(1); \qquad r \to \infty, \tag{13}$$

we can conclude that $a_n = 0$ for $n = 0, 1, 2, \cdots$ and hence $v_n(r) = 0$ for $n = 0, 1, 2, \cdots$. By equation 9) and 10) this implies that $\tilde{u}(r, \xi) = u(x, y) \equiv 0$ and the theorem is proved for $\nu > -\frac{1}{2}$.

Now consider the case in which $\nu = -\frac{1}{2}$. Since $u(x, y)$ is an analytic function of x and y^2 we have

$$u(x, y) = u_1(x) + y^2 u_2(x, y) \tag{14}$$

where $u_1(x)$ is an analytic function of x and $u_2(x, y)$ is an analytic function of

x and y^2. The fact that $u(x, y)$ is even in y implies that

$$(15) \qquad -\frac{1}{y}\frac{\partial u}{\partial y}\bigg|_{y=0} = -\frac{\partial^2 u}{\partial y^2}\bigg|_{y=0}$$

and from the differential equation we have that $u(x, 0) = u_1(x)$ must satisfy

$$(16) \qquad \frac{d^2 u_1}{dx^2} + k^2 u_1 = 0$$

i.e., $u_1(x) = ae^{ikx} + be^{-ikx}$ where a and b are constants. By the second hypothesis of the theorem, $\lim_{x\to+\infty} u_1(x) = 0$ and hence $a = b = 0$. Therefore $u(x, y) = y^2 u_2(x, y)$. By Weinstein's recurrence relation [23] we can conclude that $u_2(x, y)$ is a solution of $L_{3/2,+k^2}(u) = 0$. By the hypothesis of the theorem, $|u(x, y)| = |y^2 u_2(x, y)| \leqq M$ and $\lim_{r\to\infty} u(x, y) = \lim_{r\to\infty} y^2 u_2(x, y) = 0$ pointwise for $\theta \in [0, \pi]$. Hence from previously proved results we can conclude that $u_2(x, y) \equiv 0$ and hence $u(x, y) \equiv 0$. The theorem is now completely proved.

We would now like to extend the results of Theorem 3.1 to the case in which $\nu < -\frac{1}{2}$. However, it is not immediately obvious even what the form of such an extension should be since if $\nu < -\frac{1}{2}$ the term $y^{\nu+1/2}$ which appears in the theorem becomes infinitely large as y approaches zero. The following theorem gives the desired result and will be shown by an example to be essentially the sharpest conclusion possible.

Theorem 3.2. Assume $\nu < -\frac{1}{2}$ *and let* $u(x, y) = \tilde{u}(r, \xi)$ *be a solution of* $L_{\nu,+k^2}(u) = 0$ *which is regular in the whole real* x, y *plane. Suppose there exists an ellipse* E *in the complex* ξ *plane inclosing* $[-1, +1]$ *such that* $\tilde{u}(r, \xi)$ *is analytic in the interior* T *of* E *for all fixed* r, $0 \leqq r < \infty$. *If*

1) *there exist positive constants* $M < \infty$ *and* $\delta < \infty$ *such that* $\sup\{r^{\nu+1/2}|\tilde{u}(r, \xi)| : \delta \leqq r < \infty, \xi \in T\} \leqq M$,
2) $\lim_{r\to\infty} r^{\nu+1/2}\tilde{u}(r, \xi) = 0$ *pointwise for* $\xi \in S$ *where* S *is some interval contained in* $[-1, +1]$;

then $u(x, y) \equiv 0$.

Proof. Let m be a positive integer greater than $-\nu - \frac{1}{2}$. Since $\tilde{u}(r, \xi)$ is an analytic function of ξ in T we can write

$$(17) \qquad \frac{\partial^m \tilde{u}(r, \xi_0)}{\partial \xi^m} = \frac{m!}{2\pi i}\oint_C \frac{\tilde{u}(r, t)}{(t - \xi_0)^{m+1}}\, dt$$

where $\xi_0 \in T$ and C is a circle surrounding the point ξ_0 and contained in T. Now consider the ellipse E_ϵ which is a distance ϵ away from E and $[-1, +1]$ and let T_ϵ denote its interior. For each point ξ_0 let C have radius ϵ. Then for every $\xi_0 \in T_\epsilon$ we have from equation 17) and hypothesis 1) of the theorem

$$(18) \qquad \begin{aligned} \left| r^{\nu+1/2}\frac{\partial^m \tilde{u}(r, \xi_0)}{\partial \xi^m}\right| &\leqq \frac{m!}{\epsilon^m} r^{\nu+1/2} \max_{\xi\in C} |\tilde{u}(r, \xi)| \\ &\leqq \frac{m!}{\epsilon^m} M. \end{aligned}$$

i.e., for every $\xi \in T_\epsilon$ we have

$$r^{m+\nu+1/2}\left(r^{-m}\frac{\partial^m \tilde{u}(r,\xi)}{\partial \xi^m}\right) \leq \frac{m!}{\epsilon^m}M. \tag{19}$$

Now let $\{r_j\}$ be a sequence of r values such that $\lim_{j\to\infty} r_j = \infty$. The precise values of r_j will be chosen later. By Vitali's convergence theorem [21] and hypothesis one and two of the theorem we can conclude that $\lim_{j\to\infty} r_j^{\nu+1/2}\tilde{u}(r_j, \xi)$ is an analytic function of ξ in T_ϵ. Since for $\xi \in S$ we have $\lim_{j\to\infty} r_j^{\nu+1/2}\tilde{u}(r_j, \xi) = 0$, by the identity theorem $\lim_{j\to\infty} r_j^{\nu+1/2}\tilde{u}(r_j, \xi) = 0$ for $\xi \in T_\epsilon$. From equation (17) and Vitali's convergence theorem again we can conclude that

$$\lim_{j\to\infty} r_j^{m+\nu+1/2}\left(r_j^{-m}\frac{\partial^m \tilde{u}(r_j,\xi)}{\partial \xi^m}\right) = 0 \tag{20}$$

for $\xi \in T_\epsilon$. Recall from Lemma 2.1 that $r^{-m}\,\partial^m\tilde{u}(r,\xi)/\partial\xi^m = \tilde{u}^\dagger(r,\xi)$ is a regular solution of $L_{\nu+m}(u) = 0$. Now expand $\tilde{u}^\dagger(r,\xi)$ in a Gegenbauer series as in Theorem 3.1:

$$\tilde{u}^\dagger(r,\xi) = r^{-\nu-m}\sum_{n=0}^{\infty} a_n J_{\nu+m+n}(kr)C_n^{\nu+m}(\xi). \tag{21}$$

If $\tilde{u}^\dagger(r,\xi) \not\equiv 0$ there exists an n_0 such that $a_{n_0} \neq 0$. By using equations (19) and (20) we can conclude as in Theorem 3.1 that

$$\lim_{j\to\infty} |a_{n_0} r_j^{1/2} J_{\nu+m+n_0}(kr_j)| = 0. \tag{22}$$

Now if the sequence $\{r_j\}$ is chosen such that

$$\cos\left[r_j - \tfrac{1}{2}(\nu + m + n_0)\pi - \pi/4\right] = 1 \tag{23}$$

we can conclude from equations (13) and (22) that $a_{n_0} = 0$ which contradicts the assumption that $\tilde{u}^\dagger(r,\xi) \not\equiv 0$. Therefore $\tilde{u}^\dagger(r,\xi) \equiv 0$ *i.e.*, $\tilde{u}(r,\xi)$ is a polynomial in ξ of degree at most $m-1$ with coefficients depending on r:

$$\tilde{u}(r,\xi) = \sum_{n=0}^{m-1} g_n(r)\xi^n. \tag{24}$$

From the differential equation we can now conclude that

$$\tilde{u}(r,\xi) = r^{-\nu}\sum_{n=0}^{m-1} b_n J_{\nu+n}(kr)C_n^\nu(\xi) \tag{25}$$

where the b_n are constants. If $\nu = -1, -2, \cdots$ then $C_n^\nu(\xi)$ is to be replaced by Jacobi's polynomial $P_n^{(\nu-1/2,\nu-1/2)}(\xi)$. For details see Theorem 4.2 of [1]. Using equation (13) and the second hypothesis of the theorem we can conclude from equation (25) that for $\xi \in S$

$$\begin{aligned} 0 &= \lim_{r\to\infty}\sum_{n=0}^{m-1} b_n \cos\left(kr - \frac{\pi}{2}(\nu+n) - \pi/4\right)C_n^\nu(\xi) \\ &= \lim_{r\to\infty}\left\{\cos\left(kr - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)\sum_{n=0}^{[(m-1)/2]}(-1)^n b_{2n}C_{2n}^\nu(\xi)\right. \end{aligned} \tag{26}$$

$$+ \sin\left(kr - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \sum_{n=0}^{[(m-2)/2]} (-1)^n b_{2n+1} C_{2n+1}^{\nu}(\xi)\Big\}.$$

Equation (26) implies that for $\xi \varepsilon S$ we have

$$\sum_{n=0}^{[(m-1)/2]} (-1)^n b_{2n} C_{2n}^{\nu}(\xi) = \sum_{n=0}^{[(m-2)/2]} (-1)^n b_{2n+1} C_{2n+1}^{\nu}(\xi) = 0. \tag{27}$$

If $\nu \neq -1, -2, -3, \cdots$ we are done since $C_n^{\nu}(\xi)$ is a polynomial of degree n in ξ and hence $b_n = 0, n = 0, 1, 2, \cdots m - 1$. From equation (25) we can conclude that $\tilde{u}(r, \xi) \equiv 0$. Therefore assume $\nu = -1, -2, -3, \cdots$ and recall that in this case the $C_n^{\nu}(\xi)$ are replaced by $P_n^{(\nu-1/2,\nu-1/2)}(\xi)$. From [20] p. 61 we have

$$P_n^{(\nu-1/2,\nu-1/2)}(\xi) = h_n P_{-n-2\nu}^{(\nu-1/2,\nu-1/2)}(\xi); \qquad 0 \leqq n \leqq -2\nu \tag{28}$$

(where h_n is a normalization factor) and for $0 \leqq n \leqq -\nu$ and $n \geqq -2\nu + 1$, $P_n^{(\nu-1/2,\nu-1/2)}(\xi)$ is a polynomial of degree n in ξ. Hence if (27) vanishes we must have

$$\begin{gathered} b_n = 0 \quad \text{for} \quad n \geqq -2\nu + 1, \\ b_n h_n + (-1)^{n+\nu} b_{-n-2\nu} = 0 \quad \text{for} \quad 0 \leqq n \leqq -2\nu. \end{gathered} \tag{29}$$

Since $J_{\nu+n}(kr) = (-1)^{n+\nu} J_{-\nu-n}(kr)$ we have from equation (25) that $\tilde{u}(r, \xi) \equiv 0$ and the theorem is now completely proved.

Example 3.1. The region T in the hypothesis of Theorem 3.2 cannot be replaced by the closed interval $[-1, +1]$ since in this case $u(x, y) = e^{ikx}$ satisfies all the conditions of the theorem, but $u(x, y) \not\equiv 0$.

IV. The Equations $L_{\nu,0}(u) = 0$ and $L_{\nu,-k^2}(u) = 0$. In this section we will derive uniqueness theorems for $L_{\nu,0}(u) = 0$ and $L_{\nu,-k^2}(u) = 0$ for all values of the parameter ν (except $\nu = -1, -2, \cdots$ in the case of $L_{\nu,0}(u) = 0$), thus extending the results of Parter, Wimp and Colton. Our first result is the following:

Theorem 4.1. Assume $\nu \neq -1, -2, -3, \cdots$ and let m be a non negative integer such that $m \geqq -\nu - \frac{1}{2}$. Let D be a bounded domain whose intersection with the x axis is an open interval and let ∂D denote the boundary of D. Let $\tilde{u}(r, \xi) \varepsilon C^{(m)}(\bar{D})$ be a regular solution of $L_{\nu,0}(u) = 0$ in D. If $\partial^j \tilde{u}(r, \xi)/\partial \xi^j = 0$ on ∂D, $j = 0, 1, \cdots, m$, then $u(x, y) \equiv 0$.

Proof. We can assume that $\nu < -\frac{1}{2}$ (and hence $m \geqq 1$) since if $\nu \geqq -\frac{1}{2}$ the theorem reduces to that of Parter [18]. By Lemma 2.1, $\tilde{u}^{\dagger}(r, \xi) = r^{-m} \partial^m \tilde{u}(r, \xi)/\partial \xi^m$ is a regular solution of $L_{\nu+m,0}(u) = 0$ in D and by hypothesis $\tilde{u}(r, \xi) \varepsilon C^0(\bar{D})$. Since $\partial^m \tilde{u}(r, \xi)/\partial \xi^m = 0$ on ∂D we can conclude from the results of [18] that $\tilde{u}^{\dagger}(r, \xi) \equiv 0$ in D which implies that $\partial^m \tilde{u}(r, \xi)/\partial \xi^m \equiv 0$ in D *i.e.*, $\tilde{u}(r, \xi)$ is a polynomial in ξ of degree at most $m - 1$ with coefficients depending on r:

$$\tilde{u}(r, \xi) = \sum_{n=0}^{m-1} g_n(r) \xi^n. \tag{30}$$

From the differential equation we can conclude (see [1] for details) that

$$\tilde{u}(r, \xi) = \sum_{n=0}^{m-1} a_n r^n C_n^\nu(\xi). \tag{31}$$

The polynomial $C_n^\nu(\xi)$ is of degree exactly n in ξ; denote the coefficient of ξ^n by b_n. Then $\partial^{m-1}\tilde{u}(r, \xi)/\partial\xi^{m-1} = 0$ on ∂D implies that

$$(m-1)!b_{m-1}a_{m-1}r^{m-1} = 0$$

on ∂D and hence $a_{m-1} = 0$. Proceeding in this manner it is seen that $a_n = 0$, $n = 0, 1, \cdots, m-1$ and hence from equation (31) $\tilde{u}(r, \xi) \equiv 0$.

Example 4.1. Theorem 5.1 is not true for the case in which $\nu = -1, -2, -3, \cdots$ as can be seen by considering the function

$$\tilde{u}(r, \xi) = P_0^{(\nu-1/2,\nu-1/2)}(\xi) - r^{-2\nu}h_0 P_{-2\nu}^{(\nu-1/2,\nu-1/2)}(\xi) \tag{33}$$

where h_0 is as in equation (28) and $P_0^{(\nu-1/2,\nu-1/2)}(\xi)$, $P_{-2\nu}^{(\nu-1/2,\nu-1/2)}$ are Jacobi polynomials. Equation (33) defines a solution of $L_{\nu,0}(u) = 0$ which is regular in the entire x, y plane and such that for any integer $j \geqq 0$, $\partial^j\tilde{u}(r, \xi)/\partial\xi^j = 0$ on the unit circle.

Example 4.2. The following example shows that the integer m in the theorem is the best possible choice. From [24] we have

$$\nu(1-r^2)(1-2\xi r+r^2)^{-\nu-1} = \sum_{n=0}^{\infty}(n+\nu)r^n C_n^\nu(\xi) \tag{34}$$

where equation (34) is valid for $\nu \neq -1, -2, -3, \cdots$ and (r, ξ) contained in the closed unit disc $\bar{\Omega}$. It is easily verified that $\tilde{u}(r, \xi) = \nu(1-r^2)(1-2\xi r+r^2)^{-\nu-1}$ is a regular solution of $L_{\nu,0}(u) = 0$ in the open unit disc Ω. Let m be a positive integer less than $-\nu-\frac{1}{2}$. Then $\tilde{u}(r, \xi) \in C^{(m)}(\bar{\Omega})$ and $\partial^j\tilde{u}(r, \xi)/\partial\xi^j = 0$ on $\partial\Omega$ for $j = 0, 1, \cdots m$. But $\tilde{u}(r, \xi)$ is clearly not identically zero on Ω and hence the choice of m in the theorem is the best possible.

Example 4.3. The example

$$u(x, y) = y^2 - (2\nu+1)x^2 - 1; \qquad \nu < -\tfrac{1}{2} \tag{35}$$

shows that even if $u(x, y)$ is regular in $\bar{D}$, $u(x, y) = 0$ on ∂D does not imply that $u(x, y) \equiv 0$ in D when $\nu < -\frac{1}{2}$. The function $u(x, y)$ defined by equation (35) vanishes on the ellipse $y^2 - (2\nu+1)x^2 = 1$, satisfies $L_{\nu,0}(u) = 0$ in the interior T of this ellipse and is regular in $\bar{T}$.

We now prove a result analogous to Theorem 4.1 for the equation $L_{\nu,-k^2}(u) = 0$. In the present situation however, we will be able to show that the theorem is true for all values of the parameter ν, *i.e.*, we do not exclude the cases $\nu = -1, -2, -3, \cdots$ as was done in Theorem 4.1.

Theorem 4.2. *Let $\tilde{u}(r, \xi) \in C^{(m)}(\bar{D})$ be a regular solution of $L_{\nu,-k^2}(u) = 0$ in a bounded domain D ($D \neq \{(x, y) \mid x^2 + y^2 < r_0\}$ where r_0 lies in the finite set $A = \{r_0 \mid I_{\nu+n}(kr_0) = 0$ for some integer n, $\nu \leqq \nu + n < -1\}$, $I_{\nu+n}$ being a modified Bessel function of the first kind) whose intersection with the x axis is an* *open interval and where m is a non negative integer such that $m \geqq -\nu-\frac{1}{2}$. If $\partial^j u(r, \xi)/\partial\xi^j = 0$ on ∂D, $j = 0, 1, \cdots, m$, then $\tilde{u}(r, \xi) \equiv 0$.*

Proof. We can assume that $\nu < -\frac{1}{2}$ (and hence $m \geqq 1$) since if $\nu \geqq -\frac{1}{2}$ the theorem reduces to that of Wimp and Colton [24]. As in Theorem 4.1 we can conclude that

$$\tilde{u}(r, \xi) = \sum_{n=0}^{m-1} g_n(r)\xi^n. \tag{36}$$

From the differential equation (see [1] for details) we can conclude that

$$\tilde{u}(r, \xi) = r^{-\nu} \sum_{n=0}^{m-1} a_n I_{\nu+n}(kr) C_n^\nu(\xi) \tag{37}$$

where the a_n are constants, $I_{\nu+n}(kr)$ is the modified Bessel function of the first kind, and $C_n^\nu(\xi)$ is Gegenbauer's polynomial. If $\nu = -1, -2, \cdots$ then $C_n^\nu(\xi)$ is to be replaced by Jacobi's polynomial $P_n^{(\nu-1/2,\nu-1/2)}(\xi)$. Since $I_{\nu+n}(kr)$ has no real zeros for $r \notin A$ ([3], p. 59) and $C_n^\nu(\xi)$ is a polynomial of degree n in ξ, for $\nu \neq -1, -2, \cdots$ we can conclude from the hypothesis of the theorem that $\tilde{u}(r, \xi) \equiv 0$ if $\nu \neq -1, -2, \cdots$ just as was done in Theorem 4.1. If $\nu = -1, -2, -3, \cdots$ we make use of equation (28) and the fact that $I_{\nu+n}(kr) = (-1)^{n+\nu} I_{-\nu-n}(kr)$ to conclude that

$$\begin{aligned} \tilde{u}(r, \xi) = &\sum_{n=0}^{-\nu} [a_n h_n + (-1)^{n+\nu} a_{-n-2\nu}] I_{\nu+n}(kr) P_n^{(\nu-1/2,\nu-1/2)}(\xi) \\ &+ \sum_{h=-2\nu+1}^{m-1} a_n I_{\nu+n}(kr) P_n^{(\nu-1/2,\nu-1/2)}(\xi). \end{aligned} \tag{38}$$

Since $P_n^{(\nu-1/2,\nu-1/2)}(\xi)$ is a polynomial of degree n in ξ for $0 \leqq n \leqq -\nu$ and $n \geqq -2\nu + 1$, the hypothesis of the theorem imply that $\tilde{u}(r, \xi) = 0$. The theorem is now completely proved.

An example similar to Example 4.2 can be constructed showing that the choice of m is best possible [24].

It should be noted that Theorems 5.1 and 5.2 are natural extensions of the results of [18] and [24] in the sense that when $\nu \geqq -\frac{1}{2}$ we can choose $m = 0$ and the theorems reduce to those of the above mentioned authors.

Theorems 4.1, 4.2, and Examples 4.1, 4.2, 4.3 show that in order to insure uniqueness for the Dirichlet problem for the equations $L_{\nu,0}(u) = 0$ and $L_{\nu,-k^2}(u) = 0$ it is necessary to impose an m^{th} order boundary condition. This resembles the situation of an elliptic equation of order $2m$ with continuous coefficients, where it is necessary to impose the conditions that $u(x, y)$ and its first $m - 1$ normal derivatives vanish on the boundary in order to achieve uniqueness [17]. Finally let us point out that since

$$\frac{\partial\tilde{u}}{\partial\xi} = \frac{-1}{\sin\theta}\frac{\partial\tilde{u}}{\partial\theta},$$

the conditions $\partial^j\tilde{u}/\partial\xi^j = 0$ on ∂D could have been replaced by requiring that $\partial^j\tilde{u}/\partial\theta^j = 0$ on ∂D. Phrased in this manner, the uniqueness problems considered here can be thought of as a particular case of the oblique-derivative problem [17] associated with these equations.

In future work we will use the basic Lemma 2.1 to investigate the domain of regularity of solutions of $L_{\nu,k^2}(u) = 0$.

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8

Cauchy's Problem for a Singular Parabolic Partial Differential Equation

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I. INTRODUCTION

In recent years analytic function theory has been shown to play a basic role in the investigation of existence and uniqueness theorems for solutions to elliptic partial differential equations ([6], [8], [17]). An approach which has proved particularly fruitful is that of integral operators ([1], [12], [17]), from whose use complete families of solutions can be obtained, thus enabling one to construct the Bergman kernel function and solve the Dirichlet and Neumann problems ([2]). For singular elliptic equations serious difficulties arise due to the nonregularities in the kernels of such operators, as well as the failure of Green's representation to hold in a neighborhood of the singular curve. In such cases recourse is often made to the use of operators whose path of integration is a contour in the complex plane ([7], [15]). In this paper we apply integral operator techniques in conjunction with function theoretic methods to establish an existence, uniqueness, and representation theorem to Cauchy's problem for the singular parabolic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{2\nu}{x}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} \tag{1}$$

where ν is a real parameter. Equation (1) has previously been studied by Bragg and Haimo ([4], [13]) and is sometimes known as the generalized heat equation.

The purpose of our work is twofold:

1. To the author's knowledge this is the first time the integral operator techniques described above have been applied to a parabolic equation.

2. It shows not only the sufficiency but also the necessity of relying on function theoretic methods in the investigation of singular initial value problems such as the one considered here.

Cauchy's problem for equation (1) can be formulated as follows: to find a unique function $u(x, t) \in C^2$ satisfying equation (1) for $0 < t < t_0$, $-\infty < x < \infty$, such that $u(x, t)$ continuously assumes given boundary data on the initial line $t = 0$. For $\nu > 0$ this problem has been solved by Cholewinski and Haimo ([5]) provided that for each arbitrarily small positive ϵ, $u(x, t)$ satisfies a bound of the form

$$| u(x, t)| \leqslant Me^{Ax^2}, \quad 0 \leqslant t \leqslant t_0 - \epsilon \tag{2}$$

for positive constants $M = M(\epsilon)$, $A = A(\epsilon)$. The example $u(x, t) = t^{-\nu-\frac{1}{2}}e^{-x^2/4t}$ shows that such a condition is no longer sufficient to insure uniqueness (to say nothing of existence) if $\nu < -\frac{1}{2}$. By use of the fundamental solution to equation (1) Haimo has furthermore obtained expansion theorems in terms of a complete polynomial set for solutions of (1) which are analytic in a neighborhood of the origin, provided again that $\nu > 0$ ([14]). This work is a generalization of Widder's result for the case $\nu = 0$ ([18]) and is based on rather lengthy calculations involving Laguerre polynomials and Bessel functions ([13]). Here by the use of integral operator methods we obtain a somewhat weakened form of Haimo's result as a direct consequence of Widder's work, with the added advantage that the representation theorem is now valid for all real values of ν except $2\nu = -1, -2, -3, \ldots$. From now on the case $\nu = 0$ will always be excluded since in this case (1) becomes the heat equation for which the existence, uniqueness, and representation of solutions to Cauchy's problem are well-known results ([11], [16]).

At this point we make the important observation that if $u(x, t)$ is a solution of equation (1) which is analytic in a neighborhood of the origin, then, provided $2\nu \neq -1, -2, -3, \ldots$, $u(x, t)$ is an even function of x which is uniquely determined by its axial values $u(0, t)$. This follows from the fact that the line $x = 0$ is a singular curve of the regular type with indices 0 and $1 - 2\nu$ such that $\partial u/\partial x = 0$ along the axis $x = 0$ (cf. [10]).

II. Fractional Integration and the Generalized Heat Equation

From the relationship holding between the Riemann-Liouville operator of fractional integration

$$I_{x^2}^{\nu,\alpha} f(x) \stackrel{\text{def}}{=} \frac{2}{\Gamma(\alpha)} \int_0^1 (1 - \xi^2)^{\alpha-1} \xi^{2\nu+1} f(x\xi)\, d\xi; \quad \alpha > 0, \quad \nu > -\tfrac{1}{2} \tag{3}$$

and Bessel's differential operator

$$L_\nu f \overset{\text{def}}{=} \frac{d^2 f}{dx} + \frac{2\nu}{x}\frac{df}{dx} \tag{4}$$

viz. ([10])

$$I_{x^2}^{\nu,\alpha} L_\nu f = L_{\nu+\alpha} I_{x^2}^{\nu,\alpha} f \tag{5}$$

we can relate twice continuously differentiable solutions $v(x, t)$ of the classical one dimensional heat equation which are even functions of x to solutions $u(x, t)$ of equation (1) by the relation

$$u(x, t) = \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu)\,\Gamma(\frac{1}{2})} \int_0^1 v(x\xi, t)(1 - \xi^2)^{\nu-1}\, d\xi \tag{6}$$

provided $\nu > 0$. If we assume further that $v(x, t)$ is analytic at the origin (and hence also $u(x, t)$) then, following Erdélyi ([9]), for x and t sufficiently small, we can rewrite (6) as the contour integral

$$\begin{aligned} u(x, t) &= \frac{\int_C v(x\xi, t)(1 - \xi^2)^{\nu-1} d\xi}{\int_C (1 - \xi^2)^{\nu-1} d\xi} \\ &\overset{\text{def}}{=} \Lambda v(x, t) \end{aligned} \tag{7}$$

where for $\nu > 0$ C is the interval $[0, 1]$ and for $\nu < 0$, $2\nu \neq -1, -3, -5, \ldots$ C is a loop beginning and ending at $\xi = 0$ and encircling $\xi = 1$ counterclockwise. The operator Λ defined above can furthermore be inverted ([9]) to obtain

$$\begin{aligned} v(x, t) &= \frac{\int_C u(x\xi, t)\, \xi^{2\nu}(1 - \xi^2)^{-\nu-1} d\xi}{\int_C \xi^{2\nu}(1 - \xi^2)^{-\nu-1} d\xi} \\ &\overset{\text{def}}{=} \Lambda^{-1} u(x, t) \end{aligned} \tag{8}$$

where C is a loop starting and ending at $\xi = 0$ and encircling $\xi = 1$ if $\nu > 0$, and a loop starting and ending at $\xi = 1$ and encircling $\xi = 0$ if $\nu < 0$, $2\nu \neq -1, -3, -5, \ldots$. In view of the unique dependence of $u(x, t)$ on its axial values for $2\nu \neq -1, -2, -3, \ldots$, and the fact that from (7) we have $u(0, t) = v(0, t)$, it can be deduced from corollary 4.1 of [18] (which solves the Cauchy-Kowalewski boundary value problem for the heat equation "in the large") that if $2\nu \neq -1, -2, -3, \ldots$, every solution of equation (1) which is analytic in a neighborhood of the origin can be uniquely expressed locally in the form of equation (7). If so desired, equations (7) and (8) can now be used for the purpose of analytic continuation.

In [16] Rosenbloom and Widder obtained a set of polynomial solutions to the one dimensional heat equation which are complete in the space of solutions analytic in some neighborhood of the origin ([18]). These are known as heat polynomials and can be expressed as

$$v_n(x, t) = n! \sum_{k=0}^{[n/2]} \frac{x^{n-2k} t^k}{(n - 2k)! k!} \tag{9}$$

This result was subsequently generalized by Bragg and Haimo to the case of equation (1) for $\nu > 0$ ([4], [13]), with the corresponding "generalized" heat polynomials given by

$$P_{n,\nu}(x, t) = \sum_{k=0}^{n} 2^{2k} \binom{n}{k} \frac{\Gamma(\nu + n + \frac{1}{2})}{\Gamma(\nu + n - k + \frac{1}{2})} x^{2n-2k} t^k \tag{10}$$

Observing that

1. $P_{n,\nu}(x, t)$ satisfies equation (1) not only for $\nu > 0$ but for all real values of ν,

2) $P_{n,\nu}(0, t) = 2^{2n} \dfrac{\Gamma(\nu + n + \frac{1}{2})}{\Gamma(\nu + \frac{1}{2})} t^n$,

3) $v_{2n}(0, t) = \dfrac{(2n)!}{n!} t^n$,

we have the immediate result (due to the unique dependence of solution to equation (1) on its axial values):

Lemma 1. *For* $2\nu \neq -1, -2, -3, \ldots$,

$$\Lambda v_{2n}(x, t) = h_n^{\nu} P_{n,\nu}(x, t)$$

where

$$h_n^{\nu} = \frac{\Gamma(n + \frac{1}{2})\, \Gamma(\nu + \frac{1}{2})}{\Gamma(n + \nu + \frac{1}{2})\, \Gamma(\frac{1}{2})}$$

In Lemma 1 use was made of Legendre's duplication formula to evaluate the constant h_n^{ν}. We note in passing that from the relationships ([13], [16])

$$v_{2n}(x, -t) = t^n H_{2n} \left(\frac{x}{(4t)^{\frac{1}{2}}} \right) \tag{11}$$

$$P_{n,\nu}(x, -t) = (-1)^n 2^{2n} n! t^n L_n^{\nu - \frac{1}{2}} \left(\frac{x^2}{4t} \right) \tag{12}$$

where H_{2n} denotes Hermite's polynomial and $L_n^{\nu-\frac{1}{2}}$ Laguerre's polynomial, the following formula, due originally to Uspensky, is an immediate consequence of Lemma 1:

$$\Gamma(n+\nu+1)\int_{-1}^{1}(1-\xi^2)^{\nu-\frac{1}{2}}H_{2n}(x^{\frac{1}{2}}\xi)\,d\xi = (-1)^n\pi^{\frac{1}{2}}(2n)!\,\Gamma(\nu+\tfrac{1}{2})\,L_n^{\nu}(x);$$
$$\nu > -\tfrac{1}{2} \qquad (13)$$

III. Existence, Uniqueness, and Representation of the Solution to Cauchy's Problem

Our first result in this section is to derive an expansion theorem for solutions of equation (1) which are analytic at the origin in terms of generalized heat polynomials. Using different methods, the following theorem has been obtained in a somewhat stronger form by Haimo ([13], [14]) for the case $\nu > 0$.

Theorem 1. *Let $u(x, t)$ be a solution of equation* (1) *which is analytic for $|t| < \sigma$, $|x| < \sigma$, and assume $2\nu \neq -1, -2, -3, \ldots$. Then $u(x, t)$ can be analytically continued into the strip $|t| < \sigma$, $-\infty < x < \infty$, and expanded in a series of the form*

$$u(x, t) = \sum_{n=0}^{\infty} a_n P_{n,\nu}(x, t),$$

the series converging pointwise for $0 < t < \sigma$, $-\infty < x < \infty$.

Proof. Let $v(x, t) = \Lambda^{-1}u(x, t)$. Then $v(x, t)$ is a solution of the heat equation, even with respect to x, and analytic for $|t| < \sigma$, $|x| < \sigma$. Hence from [16] and [18] $v(x, t)$ is analytic in the strip $|t| < \sigma$, $-\infty < x < \infty$, and we can write

$$v(x, t) = \sum_{n=0}^{\infty} b_n v_{2n}(x, t) \qquad (14)$$

where the series converges uniformly for each fixed t, $0 < t < \sigma$, x contained in any compact region of the complex x plane. Hence $u(x, t) = \Lambda v(x, t)$ is analytic in the strip $|t| < \sigma$, $-\infty < x < \infty$, and for $0 < t < \sigma$ we can apply the operator termwise in (14) and use Lemma 1 to obtain

$$u(x, t) = \sum_{n=0}^{\infty} b_n h_n^{\nu} P_{n,\nu}(x, t) \qquad (15)$$

for $0 < t < \sigma$, $-\infty < x < \infty$. Setting $b_n h_n^\nu$ equal to the new constant a_n establishes the theorem. (If $\nu > 0$ we could now use the results of [13] to show (15) in fact converges absolutely in the strip $|t| < \sigma$, $-\infty < x < \infty$, and represents $u(x, t)$ there. However for our purposes this is not required.)

We now proceed to the main result of this paper, i.e. the solution of Cauchy's problem for the generalized heat equation. We first require a preliminary definition.

DEFINITION 1. An entire function of a complex variable is of growth (ρ, τ) if and only if it is of order less than or equal to ρ, and is of type τ if of order ρ.

From [3] we have the result that the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is of growth (ρ, τ) if and only if

$$\limsup_{n\to\infty} \frac{n}{e\rho} \mid a_n \mid^{\rho/n} \leqslant \tau. \tag{16}$$

Note that if $f(z)$ is an entire function of growth (ρ, τ), then $f(z^2)$ is an even entire function of growth $(2\rho, \tau)$.

THEOREM 2. *If $2\nu \neq -1, -2, -3, \ldots$, then there exists a unique solution to Cauchy's problem for equation* (1) *which is of class C^2 for $0 < t < \sigma$, $-\infty < x < \infty$, provided that*

1. *$u(x, t)$ is analytic for $|x| < \sigma$, $|t| < \sigma$, and*

2. *$u(x, 0) = g(x)$ where $g(x)$ is the restriction to the real axis of an even entire function of growth $(2, \frac{1}{4}\sigma)$.*

If $g(x)$ is represented by its Taylor series

$$g(x) = \sum_{n=0}^{\infty} a_n x^{2n}$$

then for $0 < t < \sigma$, $-\infty < x < \infty$, $u(x, t)$ has the representation

$$u(x, t) = \sum_{n=0}^{\infty} a_n P_{n,\nu}(x, t)$$

Proof. Let

$$f(x) = \Lambda^{-1} g(x) = \sum_{n=0}^{\infty} a_n \Lambda^{-1} x^{2n}$$

$$= \sum_{n=0}^{\infty} a_n \frac{\Gamma(\frac{1}{2})\,\Gamma(\nu + n + \frac{1}{2})}{\Gamma(n + \frac{1}{2})\,\Gamma(\nu + \frac{1}{2})} x^{2n},$$

termwise integration being permissible since the Taylor series for $g(x)$ converges uniformly in the complex x plane. As is easily seen from its series development, $f(x)$ is an entire function of growth $(2, \frac{1}{4}\sigma)$ and hence from [16] and [18] we can construct a unique solution of the heat equation, $v(x, t)$, such that $v(x, t)$ is analytic for $|t| < \sigma$, $-\infty < x < \infty$ and $v(x, 0) = f(x)$. This function is given explicitly by

$$v(x, t) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(\nu + n + \frac{1}{2})\, \Gamma(\frac{1}{2})}{\Gamma(n + \frac{1}{2})\, \Gamma(\nu + \frac{1}{2})} v_{2n}(x, t) \tag{17}$$

Let $u(x, t) = \Lambda v(x, t)$. Then $u(x, t)$ is a solution of the generalized heat equation which is analytic for $|t| < \sigma$, $-\infty < x < \infty$. Furthermore

$$u(x, 0) = \Lambda x(v, 0) = \Lambda f(x) = \sum_{n=0}^{\infty} a_n \Lambda \Lambda^{-1} x^{2n} = \sum_{n=0}^{\infty} a_n x^{2n} = g(x), \tag{18}$$

termwise integration being permissible due to the uniform convergence of the Taylor series for $f(x)$. Applying the operator Λ termwise in equation (17) for $0 < t < \sigma$ as was done in Theorem 1 yields the representation

$$u(x, t) = \sum_{n=0}^{\infty} a_n P_{n,\nu}(x, t). \tag{19}$$

The solution $u(x, t)$ is uniquely determined since if a second solution $u_1(x, t)$ existed satisfying conditions (1) and (2) of the theorem, then $w(x, t) = u(x, t) - u_1(x, t)$ would also be a solution of equation (1) satisfying the hypothesis of Theorem 2 with $g(x) = 0$, and hence $v(x, t) = \Lambda^{-1} w(x, t)$ would be a solution of the heat equation analytic at the origin which vanishes along the characteristic $t = 0$. From [18] $v(x, t)$ can be analytically continued into the strip $|t| < \sigma$, $-\infty < x < \infty$, and must be identically zero there, which implies $\Lambda v(x, t) = w(x, t) \equiv 0$, i.e., $u(x, t) = u_1(x, t)$ in a neighborhood of the origin and both can be analytically continued into the strip $|t| < \sigma$, $-\infty < x < \infty$. This shows that the solution $u(x, t)$ we have constructed is unique in the class of solutions analytic in the strip $|t| < \sigma$, $-\infty < x < \infty$. But from the analytic theory of parabolic partial differential equations ([11]) we have that any solution which is of class C^2 in a domain not containing the singular line $x = 0$ must be an analytic function of x for each fixed t and hence if it is analytic for $|t| < \sigma$, $|x| < \sigma$, agrees with the above constructed solution $u(x, t)$. The theorem is now completely proved.

Recall from part I that the example $u(x, t) = t^{-\nu-\frac{1}{2}} e^{-x^2/4t}$ shows that in order to assure uniqueness, condition (1) of Theorem 2 is a necessary as well as a sufficient hypothesis for the case $\nu < -\frac{1}{2}$, even if an *a priori* bound of the form of equation (2) is assumed. Condition 2 now appears as a kind of "compatibility" condition, since the proof of Theorem 2 shows that any solution satisfying the first condition automatically has an analytic continuation into the strip $|t| < \sigma$, $-\infty < x < \infty$, under which $u(x, 0)$ is continued to an even entire function of growth $(2, \frac{1}{4}\sigma)$.

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JOHN'S DECOMPOSITION THEOREM FOR GENERALIZED METAHARMONIC FUNCTIONS

DAVID COLTON

1. *Introduction*

A result of basic importance in the theory of metaharmonic functions [**14**] is John's decomposition theorem [**12, 11**], which states that any metaharmonic function regular in an exterior domain can be uniquely decomposed into the sum of a solution regular in the entire space and one which satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{(n-1)/2}\left(\frac{\partial u}{\partial r}-iu\right)=0 \tag{1}$$

uniformly in all directions. Here r is the modulus of the position vector and n is the dimension of the space. This theorem for example plays a central role in the derivation of asymptotic expansions and uniqueness theorems [**4, 11**]. In this paper we consider regular solutions of the singular partial differential equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{2\nu}{y}\frac{\partial u}{\partial y}+u=0, \tag{2}$$

where ν is a fixed real parameter and which for ν equal to a positive integer can be interpreted as being a metaharmonic function in $n = 2\nu+2$ variables depending only on the two variables $x = x_1$, $y = (x_2{}^2+\ldots+x_n{}^2)^{\frac{1}{2}}$. Solutions regular on some portion of the axis $y = 0$ are known as generalized metaharmonic functions [**10**]. The axis $y = 0$ is a singular curve of the regular type [**8, 9**] with exponents 0 and $1-2\nu$. Consequently there always exist solutions of (2) which are regular on some portion of the axis, and for $2\nu \neq 0, -1, -2, \ldots$ each such regular solution can be continued across the axis as an even function of y. Therefore for $2\nu \neq 0, -1, -2, \ldots$ every solution regular on some portion of the axis is a real analytic function of x and y^2 in some domain D that is symmetric with respect to the singular line $y = 0$. Hence a generalized metaharmonic function can be expressed as $u(x, y) = \tilde{u}(r, \xi)$ where $x = r\cos\theta$, $y = r\sin\theta$, $\xi = \cos\theta$. For $\nu > 0$, we have at our disposal Weinacht's (renormalized) fundamental solution [**15**]

$$\Omega(x, y;\ x_0, y_0) = \frac{1}{[\Gamma(\nu)]^2\, 2^{1-2\nu}}\int_0^{\pi} R^{-\nu} H_\nu^{(1)}(R)\sin^{2\nu-1} t\, dt, \tag{3}$$

where $R = [(x-x_0)^2+y^2+y_0{}^2-2yy_0\cos t]^{\frac{1}{2}}$ and $H_\nu^{(1)}$ denotes a Hankel function of the first kind of order ν (Weinacht's original formula was expressed in terms of Neumann's function rather than Hankel's function). For future reference we note that $\Omega(x, y;\ x_0, y_0)$ is an analytic function of x, y and ν for $\nu > 0$, $(x, y) \neq (x_0, y_0)$. By using Green's second formula for equation (2) [**9**]

$$\int_{\partial D\cap R^+} y^{2\nu}\left(u\frac{\partial w}{\partial n}-w\frac{\partial u}{\partial n}\right) ds = \iint_{D\cap R^+} y^{2\nu}[uL_\nu(w)-wL_\nu(u)]\, dx\, dy, \tag{4}$$

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where $R^+ = \{(x, y) \mid y > 0\}$, $L_\nu \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2\nu}{y}\frac{\partial}{\partial y}$ and n is the outward normal to D, the following theorem for equation (2) can be proved in the same manner as in John's work, viz an application of formula (4) with u a regular solution of equation (2) and w set equal to the fundamental solution Ω (*cf.* [5; p. 315] or [11]):

THEOREM 1. *Assume $\nu > 0$. Let u be a regular solution of equation (2) in the exterior of a bounded domain D. Then u can be uniquely decomposed as*

$$u = U + V, \tag{5}$$

where U and V are regular solutions of (2), U is regular in the entire plane, and V satisfies the radiation condition

$$\lim_{r\to\infty} r^{\nu+\frac{1}{2}}\left(\frac{\partial u}{\partial r} - iu\right) = 0 \tag{6}$$

uniformly for $-1 \leqslant \xi \leqslant 1$.

For $\nu < 0$ this standard method of analysis is no longer applicable due to the fact that the integrals defined in formulae (3) and (4) do not exist for $\nu < 0$ and $\nu < -\frac{1}{2}$ respectively, and it is to this problem we now address ourselves. The approach adopted is to analytically continue the function defined by (3) into the range $\nu < 0$ and then to apply a relationship motivated by formula (4) where the path of integration is chosen to lie on the Riemann surface of the integrand

$$y^{2\nu}\left(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\right).$$

The resulting decomposition theorem is of particular interest in that it now turns out that the radiation condition (6) must hold uniformly for ξ contained in a region lying in the complex ξ plane, and not simply for $-1 \leqslant \xi \leqslant 1$ as in the case of Theorem 1. An example will be given showing that this reult is best possible. This seems to indicate that analytic function theory not only provides a powerful method for studying generalized metaharmonic functions, but, as in the case with singular ordinary differential equations, is in fact the correct and natural avenue of approach.

2. *Analytic continuation of the fundamental solution*

Setting $x = r\cos\theta$, $y = r\sin\theta$, $x_0 = \rho\cos\varphi$, $y_0 = \rho\sin\varphi$, we can express R as

$$R = [r^2+\rho^2-2r\rho(\cos\theta\cos\varphi+\sin\theta\sin\varphi\cos t)]. \tag{7}$$

By using Gegenbauer's addition formulae [7; p. 101, 178]

$$\omega^{-\nu}H_\nu^{(1)}(\omega) = (\tfrac{1}{2}r\rho)^{-\nu}\,\Gamma(\nu)\sum_{n=0}^{\infty}(\nu+n)\,C_n^{\nu}(\cos\phi)\,J_{\nu+n}(r)\,H_{\nu+n}^{(1)}(\rho)$$

$$\nu \neq 0, -1, -2, \ldots;\quad r < \rho;\quad \omega = [\rho^2+r^2-2r\rho\cos\phi]^{\frac{1}{2}}, \tag{8}$$

$$C_n^{\nu}(\cos\theta\cos\varphi+\sin\theta\sin\varphi\cos t)$$
$$= \sum_{m=0}^{n} 2^m(2\nu+2m-1)(n-m)!\,\frac{[(\nu)_m]^2}{(2\nu-1)_{n+m-1}}(\sin\theta)^m C_{n-m}^{\nu+m}(\cos\theta)$$
$$\times(\sin\varphi)^m C_{n-m}^{\nu+m}(\cos\varphi)\,C_m^{\nu-\frac{1}{2}}(\cos t) \tag{9}$$

(where $J_{\nu+n}$ denotes Bessel's function of order $\nu+n$ and C_n^ν is Gegenbauer's polynomial) the orthogonality property of Gegenbauer polynomials [7; p.174]

$$\int_0^\pi \sin^{2\nu} t\, C_n^\nu(\cos t)\, C_m^\nu(\cos t)\, dt = \frac{\pi^{\frac{1}{2}}(2\nu)_n\, \Gamma(\nu+\frac{1}{2})}{\Gamma(\nu)\, n!\, (n+\nu)}\delta_{mn}; \quad \nu > -\tfrac{1}{2}, \quad \nu \neq 0 \tag{10}$$

and Legendre's duplication formula [6; p. 5]

$$\frac{\Gamma(2\nu)}{\Gamma(\nu)} = \frac{2^{2\nu-1}\,\Gamma(\nu+\frac{1}{2})}{\pi^{\frac{1}{2}}}, \tag{11}$$

for $\nu > 0$, and $r < \rho$ we can express formula (3) as

$$\begin{aligned}\Omega\,(x, y; x_0, y_0) &= \tilde{\Omega}(r, \xi;\ \rho, \eta)\\ &= (\tfrac{1}{2}r\rho)^{-\nu} \sum_{n=0}^{\infty} \frac{(n+\nu)\,n!}{\Gamma(2\nu+n)}\, J_{\nu+n}(r)\, H^{(1)}_{\nu+n}(\rho) C_n^\nu(\xi)\, C_n^\nu(\eta) \end{aligned}\tag{12}$$

(where $\xi = \cos\theta$, $\eta = \cos\varphi$) since the series (8) converges uniformly for $r < \rho$, $0 \leqslant \phi \leqslant \pi$. By using the asymptotic formulae [7; p. 4, 8] [13; p. 199]

$$\Gamma(\nu+n+1)\left(\frac{r}{2}\right)^{-\nu-n} J_{\nu+n}(r) = 1+o(1); \quad n \to \infty, \tag{13}$$

$$-\frac{\pi}{i}\,\frac{(\rho/2)^{\nu+n}}{\Gamma(\nu+n)}\, H^{(1)}_{\nu+n}(\rho) = 1+o(1); \quad n \to \infty, \tag{14}$$

$$\frac{|C_n^\nu(\xi)|^{1/n}}{|\xi+(\xi^2-1)^{\frac{1}{2}}|} = 1+o(1); \quad \xi \notin [-1, +1], \quad n \to \infty, \tag{15}$$

it can be seen that the series (8) in fact converges uniformly for $r < \rho$, $\cos\phi$ contained in some ellipse in the complex $\cos\phi$ plane inclosing $[-1, +1]$. If r is restricted such that $r \leqslant \rho' < \rho$ for some constant ρ' then this ellipse can be chosen to be independent of r.

If, instead of using the orthogonality property (10), we use the result [3]

$$\frac{1}{2\pi i}\int_C (\zeta^2-1)^{\nu-\frac{1}{2}}\, C_m^\nu(\zeta)\, C_n^\nu(\zeta)\, d\zeta = \frac{2^{1-2\nu}\cos\pi\nu\,\Gamma(2\nu+n)}{[\Gamma(\nu)]^2(n+\nu)\,n!}\,\delta_{mn}, \tag{16}$$

where C is a figure eight loop inclosing the points ± 1 in the complex ζ plane (which is cut by two lines running from these points to infinity), the preceding analysis yields the result

$$\begin{aligned}&\frac{i}{\sin\pi\nu[\Gamma(\nu)]^2\, 2^{2-2\nu}}\int_C R^{-\nu} H_\nu^{(1)}(R)(\zeta^2-1)^{\nu-1}\, d\zeta\\ &= (\tfrac{1}{2}r\rho)^{-\nu}\sum_{n=0}^{\infty}\frac{(n+\nu)\,n!}{\Gamma(2\nu+n)}\, J_{\nu+n}(r)\, H^{(1)}_{\nu+n}(\rho)\, C_n^\nu(\cos\theta)\, C_n^\nu(\cos\varphi),\end{aligned}\tag{17}$$

where R is given by formula (7) with $\zeta = \cos t$, $\nu > 0$, $r < \rho$. (In [3] formula (16) was proved for $\nu < 0$, $2\nu \neq -1, -2, \ldots$; the general result holds by analytic continuation with respect to ν). The integral in (17) can be continued to complex values of x, y, x_0, y_0 and defines an analytic function of x, y and ν for $(x, y) \neq (x_0, y_0)$ ($\nu = 0, \pm 1, \pm 2, \pm 3, \ldots$ are removable singularities) which agrees with $\Omega(x, y;\ x_0, y_0)$

for $\nu > 0$, $r < \rho$. Hence (17) is the desired analytic continuation of (3) into the range $\nu < 0$, $(x, y) \neq (x_0, y_0)$.

3. *A decomposition theorem for generalised metaharmonic functions*

From the analytic theory of partial differential equations it can be shown [1] that for each fixed $r \geqslant R$ (where R is a sufficiently large positive constant) $\tilde{u}(r, \xi)$ is an analytic function of ξ for $-1 \leqslant \xi \leqslant 1$. Hence $\tilde{u}(r, \xi)$ can be expanded in a Gegenbauer series, provided $2\nu \neq 0, -1, -2, \ldots$ [2], and the coefficients can be determined by use of equation (16). This yields the result that

$$\tilde{u}(r, \xi) = r^{-\nu} \sum_{n=0}^{\infty} [a_n J_{\nu+n}(r) + b_n H^{(1)}_{\nu+n}(r)] C_n^{\nu}(\xi); \quad r \geqslant R, \tag{18}$$

where a_n, b_n are constants and the series converges uniformly for $R \leqslant r \leqslant R_1 < \infty$, where R_1 is an arbitrarily large number, ξ contained in some ellipse in the complex ξ plane inclosing $[-1, +1]$. Motivated by the results of parts one and two, we consider the function $V(r, \xi)$ defined by

$$V(r, \xi) = \frac{a^{2\nu+1} \pi [\Gamma(\nu)]^2}{i 2^{2-\nu} \cos \pi\nu} \int_C \left(\tilde{u}(a, \zeta) \frac{\partial \tilde{\Omega}(a, \zeta; r, \xi)}{da} - \tilde{\Omega}(a, \zeta; r, \zeta) \frac{\partial \tilde{u}(a, \zeta)}{da} \right) (\zeta^2 - 1)^{\nu - 1/2} \, d\zeta, \tag{19}$$

where $R \leqslant a < r$, $\nu < 0$, $2\nu \neq -1, -2, -3, \ldots$, C is a figure eight loop in the complex ζ plane surrounding the points ± 1, and Ω is the continued fundamental solution of part two. Using the formulae

$$\tilde{u}(a, \zeta) = \sum_{n=0}^{\infty} \{a_n [a^{-\nu} J_{\nu+n}(a)] + b_n [a^{-\nu} H^{(1)}_{\nu+n}(a)]\} C_n^{\nu}(\zeta), \tag{20}$$

$$\frac{\partial \tilde{\Omega}(a, \zeta; r, \xi)}{\partial a} = 2^{\nu} \sum_{n=0}^{\infty} \frac{(n+\nu) n!}{\Gamma(2\nu+n)} \frac{d[a^{-\nu} J_{\nu+n}(a)]}{da} [r^{-\nu} H^{(1)}_{\nu+n}(r)] C_n^{\nu}(\xi) C_n^{\nu}(\zeta), \tag{21}$$

$$\tilde{\Omega}(a, \zeta; r, \xi) = 2^{\nu} \sum_{n=0}^{\infty} \frac{(n+\nu) n!}{\Gamma(\nu+n)} [a^{-\nu} J_{\nu+n}(a)][r^{-\nu} H^{(1)}_{\nu+n}(r)] C_n^{\nu}(\xi) C_n^{\nu}(\zeta), \tag{22}$$

$$\frac{\partial \tilde{u}(a, \zeta)}{\partial a} = \sum_{n=0}^{\infty} \left\{ a_n \frac{d[a^{-\nu} J_{\nu+n}(a)]}{da} + b_n \frac{d[a^{-\nu} H_{\nu+n}(a)]}{da} \right\} C_n^{\nu}(\zeta), \tag{23}$$

$$\frac{2i}{\pi a} = J_{\nu+n}(a) \frac{dH^{(1)}_{\nu+n}(a)}{da} - H^{(1)}_{\nu+n}(a) \frac{dJ_{\nu+n}(a)}{da}. \tag{24}$$

the uniform convergence of the series (17) and (20), and the orthogonality property (16), we have

$$\begin{aligned} V(r, \xi) &= \frac{\pi a^{2\nu+1} r^{-\nu}}{2i} \sum_{n=0}^{\infty} b_n \left\{ [a^{-\nu} H^{(1)}_{\nu+n}(a)] \frac{d[a^{-\nu} J_{\nu+n}(a)]}{da} - [a^{-\nu} J_{\nu+n}(a)] \frac{d[a^{-\nu} H^{(1)}_{\nu+n}(a)]}{da} \right\} H^{(1)}_{\nu+n}(r) C_n^{\nu}(\xi) \\ &= r^{-\nu} \sum_{n=0}^{\infty} b_n H^{(1)}_{\nu+n}(r) C_n^{\nu}(\xi), \end{aligned} \tag{25}$$

where the series (25) is uniformly convergent for $a < R_0 \leqslant r \leqslant R_1 < \infty$, ξ contained in some ellipse T in the complex ξ plane inclosing $[-1, +1]$. From Theorem 4.3 of [1] it is seen that (25) is a regular solution of equation (2) for $r > a$ satisfying

$$\lim_{r \to \infty} r^{\nu+\frac{1}{2}} \left(\frac{\partial V}{\partial r} - iV \right) = 0 \tag{26}$$

uniformly for $\xi \in T$. From equation (18) we therefore can write

$$\tilde{u}(r, \xi) = U(r, \xi) + V(r, \xi); \quad r > a, \tag{27}$$

where

$$U(r, \xi) = r^{-\nu} \sum_{h=0}^{\infty} a_n J_{\nu+n}(r) C_n^{\nu}(\xi) \tag{28}$$

is uniformly convergent for $R_1 \geqslant r \geqslant R_0 > a$. In view of equation (13), this implies that $U(r, \xi)$ is a real analytic function of r and ξ in the entire (x, y) plane and this fact along with formula (27) shows that $U(r, \xi)$ is an everywhere regular solution of equation (2). The decomposition (27) is unique since if $U(r, \xi)$ is a generalised metaharmonic function which is regular in the entire plane and also satisfies the radiation condition (26), then the orthogonality property (16) and the series representation (18) shows that $\tilde{u}(r, \xi)$ must be identically zero. We have thus proved the following theorem:

THEOREM 2. *Assume $\nu < 0$, $2\nu \neq -1, -2, -3, \ldots$. Let u be a regular solution of equation* (2) *in the exterior of a bounded domain D. Then u can be uniquely decomposed as*

$$u = U + V, \tag{29}$$

where U and V are solutions of (2), *U is regular in the entire plane, and V satisfies the radiation condition*

$$\lim_{r \to \infty} r^{\nu+\frac{1}{2}} \left(\frac{\partial V}{\partial r} - iV \right) = 0 \tag{30}$$

uniformly for ξ contained in some ellipse T in the complex ξ plane inclosing $[-1, +1]$ *in its interior.*

Example. For $\nu < -\frac{1}{2}$ the ellipse T cannot be replaced by the line segment $[-1, +1]$. For in this case $U(r, \xi) = e^{ir\xi}$ is a solution of equation (2) regular in the entire plane which also satisfies the radiation condition, i.e. the decomposition is no longer unique.

Remark. For $\nu > -\frac{1}{2}$ the condition that (30) holds uniformly in T is implied by the weaker requirement that (30) holds only for $\xi \in [-1, +1]$ [**1**].

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A CONTRIBUTION TO THE VEKUA-RELLICH THEORY OF METAHARMONIC FUNCTIONS.*

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1. **Introduction.** It is the purpose of this paper to use integral operator techniques to investigate the expansion problem for Appell series and to use these results to derive a uniqueness theorem for the generalized axially symmetric reduced wave equation in $n+1$ variables [15]. The importance of our work is two-fold:

(a) It is a significant contribution to the method of generating kernels in the study of polynomial expansions of analytic functions ([2]).

(b) It presents for the first time a uniqueness theorem for an elliptic partial differential equation in more than two variables whose coefficients are singular in its domain of definition.

2. **The expansion problem for Appell series.** An Appell series is a series of the form

$$(1)\qquad \sum a_m V_M{}^{(s)} \equiv \sum_{\mu=0}^{\infty} \sum_{m=\mu} a_M V_M{}^{(s)}(\xi)\,;\; s > -1,\; s \neq 0,$$

where $M = (m_1, m_2, \cdots, m_n)$ is multi-index, $m = |M| \equiv m_1 + \cdots + m_n$, and $V_M{}^{(s)}(\xi) \equiv V_M{}^{(s)}(\xi_1, \cdots, \xi_n)$. The $V_M{}^{(s)}(\xi)$ are uniquely defined by the generating function

$$(2)\qquad (1 - 2(\alpha, \xi) + \| \alpha \|^2)^{-(n+s-1)/2} = \sum \alpha V_M{}^{(s)}(\xi).$$

Here

$$(a, \xi) \equiv \sum_{i=1}^{n} \alpha_i \xi_i, \quad \| \alpha \|^2 = (\alpha, \alpha),$$

and

$$(3)\qquad \alpha^M \equiv \alpha_1{}^{m_1} \cdots \alpha_n{}^{m_n},$$

and the summation in equation (1) and (2) is meant to be an n-fold sum

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over all indices from zero to infinity. In our study of Appell series we also will need to make use of the related polynomials,

$$U_M^{(s)}(\xi) \equiv U_M^{(s)}(\xi_1, \cdots, \xi_n), \tag{4}$$

which are uniquely defined by the generating function

$$\{[(\alpha,\xi)-1]^2+\|\alpha\|^2(1-\|\xi\|^2)\}^{-s/2}=\sum \alpha^M U_M^{(s)}(\xi) \tag{5}$$

and possess the biorthogonality property

$$\begin{aligned}&\int_{S(0;1)}(1-\|\xi\|^2)^{(s-1)/2}V_M^{(s)}U_L^{(s)}(\xi)\,d\xi\\ &\quad=\delta_{LM}\frac{2\pi^{n/2}\Gamma(\frac{1}{2}s+1)(s)_M}{(2m+n+s-1)\Gamma(n/2+s/2-\frac{1}{2})M!},\end{aligned} \tag{6}$$

where $M! \equiv m_1! \cdots m_n!$, and $\delta_{LM} = \delta_{l_1m_1} \cdots \delta_{l_nm_n}$. Here $S(0;1)$ is the real solid n dimensional ball $\{\xi \mid \|\xi\| \leqq 1\}$. For more information concerning the polynomials $U_M^{(s)}$ and $V_M^{(s)}$ the reader is referred to [1].

In this section we will consider the extension of the classical expansion problem for analytic functions of one complex variable to functions of n complex variables in terms of the polynomials $V_M^{(s)}(\xi)$. In particular, is such an expansion possible if the analytic function is known only on a real environment, and in this case what can be said about the region of convergence? For the case of one complex variable an established method of attacking such problems is the method of generating kernels ([2]) which can be briefly described as follows. Suppose we want to represent a given function $f(\xi)$ in the form of a series $\sum a_i\phi_i(\xi)$, with a prescribed sequence of polynomials $\{\phi_i(\xi)\}$. We choose a "suitable" sequence of functions $p_i(s)$ and define the formed kernel

$$K(\xi,\zeta) \equiv \sum \phi_i(\xi)p_i(\zeta). \tag{7}$$

If $f(\xi)$ can be represented as

$$f(\xi) \equiv \mathfrak{D}F = \int_\Gamma K(\xi,\zeta)F(\zeta)\,d\zeta \tag{8}$$

for some path Γ and analytic function $F(\zeta)$, then we have (assuming certain convergence conditions),

$$\begin{aligned}f(\xi) &= \int_\Gamma K(\xi,\zeta)F(\zeta)\,d\zeta\\ &= \int_\Gamma \sum \phi_i(\xi)p_i(\zeta)F(\zeta)\,d\zeta\end{aligned} \tag{9}$$

$$= \sum \phi_i(\xi) \int_\Gamma p_i(\zeta) F(\zeta)\, d\zeta.$$

Thus $f(\xi) = \sum \alpha_i \phi_i(\xi)$ where the "Fourier coefficients" a_i are defined as $a_i \equiv \int_\Gamma p_i(\zeta) F(\zeta)\, d\zeta$. The major difficulties in this approach is constructing a suitable kernel $K(\xi,\zeta)$, analytic function $F(\zeta)$, and determining the range of the operator $\mathfrak{O}$. For hany cases, however, $K(\xi,\zeta)$ may be obtained by simple analytic computations from a known generating function of the polynomials $\{\phi_i(\xi)\}$, and $F(\zeta)$ is the (generalized) Borel transform of $f(\xi)$, defined as

$$F(\zeta) \equiv \mathfrak{O}^{-1} f = \sum \frac{f_k}{\Psi_k \zeta^{k+1}}, \tag{10}$$

with $f(\xi) = \sum f_k \xi^k$ and the Ψ_k determined in a unique manner from $K(\xi,\zeta)$ ([2]). As our notation indicates in equation (10) we may consider the Borel transform as in inverse operator for $\mathfrak{O}$. Consequently the problem of performing a series expansion in terms of the sequence $\{\phi_i\}$ may be seen to reduce to the determination of an inverse operator $\mathfrak{O}^{-1}$. Furthermore, knowledge of such an inverse operator can be instrumental in examining the analytic properties of such series expansions, as has been emphasized in the monograph [14]. Following the approach of [14] we are able to solve the above mentioned expansion problem associated with the polynomials $V_M^{(s)}(\xi)$ for the first time. See for instance [11] pp. 280-282, and [1] pp. 296-297, for a list of the known results concerning these polynomial expansions prior to the present investigation.

The operator $\mathfrak{O}$ is constructed by considering the kernel

$$K_1(\xi,\zeta) \equiv \sum \zeta^{-M} V_M^{(s)}(\xi) = (1 - 2(\frac{1}{\zeta}, \xi) + \| \frac{1}{\zeta} \|^2)^{-(n+s-1)/2} \tag{11}$$

and observing that by Cauchy's theorem for several complex variables

$$(\tfrac{1}{2\pi i})^n \int_\Gamma K_1(\xi,\zeta) \zeta^M \frac{d\zeta}{\zeta} = V_M^{(s)}(\xi) \tag{12}$$

where $\Gamma = \prod_{i=1}^{n} \Gamma_i$, and the Γ_i are chosen such that Γ lies outside a sufficiently large hypersphere $\Delta(0;R) \equiv \{\zeta \mid \|\zeta\|_{\text{a.v.}} \leqq R\}$ where $\|\zeta\|_{\text{a.v.}} \equiv (\sum_{i=1}^{n} |\zeta_i|^2)^{\frac{1}{2}}$. If the holomorphic function $F(\zeta)$ is defined as

$$F(\zeta) \equiv \sum a_M \zeta^M,\ \zeta \in \Delta(0;R'),\ (R' > R) \tag{13}$$

then we may define the function element $\{f(\xi); \Delta(0;\epsilon)\}$, $\epsilon > 0$, by

$$
(14) \quad \begin{aligned} f(\xi) = \mathfrak{D}F &\equiv \left(\tfrac{1}{2\pi i}\right)^n \int_{\Gamma} K_1(\xi,\zeta) F(\zeta) \frac{d\zeta}{\zeta} \\ &= \sum a_M V_M^{(s)}(\xi), \ \xi \in \Delta(0;\epsilon). \end{aligned}
$$

From (12) the kernel $K_1(\xi,\zeta)$ is seen to be singular only on the analytic set,

$$
(15) \qquad \mathcal{K}_1 \equiv \{(\xi,\zeta) \mid 1 - 2\left(\frac{1}{\zeta},\xi\right) + \left\| \frac{1}{\zeta} \right\|^2 = 0\};
$$

hence one may conclude using the Hadamard-Gilbert Theorem[1] (or Landau-Bjorken rules) [13], [22] (See in particular Theorem (1.3.3) of [14].), that if $F(\zeta)$ is holomorphic in $\Delta(0;1)$, then the function element $\{f;\Delta(0,\epsilon)\}$ has a holomorphic extension to a full neighborhood $N\{S\}$ of the real unit ball, $S(0,1) \equiv \{\xi \mid \|\xi\| \leqq 1\}$. We proceed by taking $F(\zeta)$ holomorphic in $\Delta(0;1+\epsilon)$, $\epsilon > 0$ sufficiently small and note that the singular points of $K_1(\zeta,\xi)$ must lie on the analytic set $\mathcal{K}_1$, defined by $1 - \|\xi\|^2 + \left\|\frac{1}{\zeta} - \xi\right\|^2 = 0$. For $\xi \in S(0;1)$ this means $\left\|\frac{1}{\zeta} - \xi\right\|^2 = -\delta$, where $0 \leqq \delta \leqq 1$, which is easily violated by taking either a $\|\zeta\|_{\text{a.v.}}$ sufficiently large or small. Consequently, ignoring the "kernel pinch" which may occur for $\|\xi\| = 1$ (and which does *not* correspond to a singularity of $f(\xi)$ on its principal sheet), it is seen that $f(\xi)$ is holomorphic in a full complex neighborhood of $S(0;1)$. Actually one may compute the singularities of $f(\xi)$, using the Hadamard-Gilbert Theorem, if the singularity manifolds of $F(\zeta)$ are known. For instance, if $F(\zeta)$ should have a singularity on the analytic plane $\zeta_l = \alpha\,(l = 1,2,\cdots,n)$ $|\alpha| > 1$ then a singularity of $f(\xi)$ must correspond to a coincidence,

$$
\begin{aligned} &\mathcal{K}_1 \cap \{\zeta_l = \alpha\} \\ &\quad = \{(\zeta,\xi) \mid S(\zeta,\xi) \equiv 1 - 2\xi_l\alpha^{-1} + \alpha^{-2} + \sum_{i \neq l} (\zeta_i^2 - 2\xi_i\zeta_i^{-1}) = 0\}. \end{aligned}
$$

Eliminating ζ_i $(i \neq l)$ between $S(\zeta,\xi) = 0, \dfrac{\partial S}{\partial \zeta_i} = 0$ $(i \neq l)$ yields the following set of *candidates* for singularities of $f(\xi)$:

$$
(16) \qquad E_l \equiv \{\xi \mid \xi_l = \tfrac{1}{2}\left(\alpha + \frac{1}{\alpha}\right) - \frac{\alpha}{2} \sum_{i \neq l} \xi_i^2\}.
$$

When $\xi_i = 0$ $(i \neq l)$, E_l is a point in the ξ_l-plane which lies on the ellipse

[1] The well known (to field-theoretical physicists) Landau-Bjorken rules (1959) [22], which occur in the study of the singularities of Feynman amplitudes are actually a direct corollary of the Hadamard-Gilbert Theorem (1959) [13]. Actually, Landau merely conjectured these rules, and was unable to provide a valid proof.

with focii at ± 1 and major axis $|\alpha| + |\alpha|^{-1}$. This restriction of E_l is exactly the singular point Nehari [25] found, when he investigated the singularities of Legendre series. This is not at all remarkable, however, since for the case $n = s = 1$, the polynomials $V_M^{(s)}(\xi)$ reduce to the Legendre polynomials $P_m(\xi_1)$.

Remark. $E_l \cap N\{S(0;1)\} = \emptyset$ $(l = 1, 2, \cdots, n)$ for a sufficiently small neighborhood of $S(0;1)$.

We now turn our attention to the construction of an inverse integral operator, $\mathfrak{O}^{-1}$, to the operator defined by (14). We introduce as a kernel for this operator the formal sum,

$$\begin{aligned} K_2(\zeta,\xi) &= \frac{\Gamma(\frac{1}{2}[n+s-1])(1-\|\xi\|^2)^{(s-1)/2}}{2\pi^{n/2}\Gamma(s/2+1)} \cdot \\ &\quad \sum \frac{(2m+n+s-1)}{(s)_m} M!\, U_M^{(s)}(\xi)\zeta^M. \end{aligned} \tag{17}$$

In order to show that the formal series (17) converges uniformly for $\xi \in S(0;1)$ and $\zeta \in \Delta(0;\epsilon)$ we need only to consider the several variables analogue of the Weierstrass comparison theorem, and the generating function expansion (5). We conclude $K_2(\zeta,\xi)$ converges uniformly in any region $\Delta(0;\epsilon) \times N\{S(0;1)\}$ which does not meet the set,

$$\mathfrak{a} \equiv \{[\zeta,\xi] \mid [(\zeta,\xi)-1]^2 + \|\zeta\|^2(1-\|\xi\|^2) = 0\}.$$

From the orthogonality properties (16) between the polynomials $V_M^{(s)}(\xi)$ and $U_M^{(s)}(\xi)$, one has that

$$\zeta^M = \int_{S(0;1)} K_2(\zeta,\xi) V_M^{(s)}(\xi)\, d\xi, \tag{18}$$

and indeed when the series (17) converges uniformly, that

$$\begin{aligned} F(\zeta) = \mathfrak{O}^{-1}f(\xi) &\equiv \left(\frac{1}{2\pi i}\right)^n \int_{S(0;1)} K_2(\zeta,\xi) f(\xi)\, d^n\xi \\ &= \sum a_M \zeta^M. \end{aligned} \tag{19}$$

Actually, the comparison theorem indicates $K_2(\zeta,\xi)$ will converge on a somewhat larger region than mentioned above. The singularities of $K_2(\zeta,\xi)$ may be computed by using a method, introduced in [15], which incorporates a several complex variable analogue of a theorem due to Fabry. Another approach, is to replace the sum (17) by an integral suggested by the generating function (5), and the Beta function representation, namely

$$K_2(\zeta,\xi) = \frac{\Gamma(\frac{1}{2}[n+s-1])}{2\pi^{n/2}\Gamma(\frac{1}{2}s+1)}\{(2\lambda\frac{\partial}{\partial\lambda}+n+s-1)\tilde{K}_2(\lambda\zeta,\xi)\}_{\lambda=1},$$

where

$$\begin{aligned}(20)\qquad \tilde{K}_2(\zeta,\xi) \equiv \zeta_1\cdots\zeta_n \frac{\partial^n}{\partial\zeta_1\cdots\partial\zeta_n}\int_0^1\cdots\int_0^1 & ((\xi_1\zeta_1 t_1 t_2\cdots t_n \\ & + \xi_2\zeta_2[1-t_1]t_2\cdots t_n + \cdots + \xi_n\zeta_n[1-t_{n-1}]t_n - 1)^2 \\ & + (1-\|\xi\|^2)([\zeta_1 t_1\cdots t_n]^2 + [\zeta_2(1-t_1)t_2\cdots t_n]^2 + \cdots \\ & + [\zeta_n(1-t_{n-1})t_n]^2))^{-s/2}\cdot\frac{dt_1}{t_1(1-t_1)}\,\frac{dt_2}{t_2(1-t_2)}\cdots\frac{dt_n}{t_n(1-t_n)^{1-s}}.\end{aligned}$$

Since the terms in the series (17) vanish for all $\mu < n$, it is quickly seen that the integrand for $\tilde{K}_2(\zeta,\xi)$ is regular at $t_1 = t_2 = \cdots = t_n = 0$, and at $t_1 = t_2 = \cdots = t_{n-1} = 1$; hence, the integral for $\tilde{K}_2(\zeta,\xi)$ is valid providing $s > 0$, $\xi \in S(0;1)$ and $\zeta \in \Delta(0;\epsilon)$ with $\epsilon > 0$ sufficiently small. We remark that $K_2(\zeta,\xi)$ and its analytic continuations are clearly regular where $\tilde{K}_2(\zeta,\xi)$ is regular, and hence the singularity structure of $K_2(\zeta,\xi)$ may be determined by investigating the latter function. By the Hadamard-Gilbert Theorem (or Landau-Bjorken rules) the singularities of $\tilde{K}_2(\zeta,\xi)$ must correspond to either envelope-pinches or end-point pinches; see for instance, Theorem (1.3.3) of [14] and also the remark before Theorem (1.3.2). The singularity manifold of the integrand occurring in (20) may be represented as

$$\begin{aligned}(21)\qquad \chi(\zeta,\xi,t) \equiv & (\xi_1\zeta_1 t_1\cdots t_n + \xi_2\zeta_2[1-t_1]t_2\cdots t_n + \cdots \\ & + \xi_n\zeta_n[1-t_{n-1}]t_n - 1)^2 + (1-\|\xi\|^2)([\zeta_1 t_1\cdots t_n]^2 \\ & + [\zeta_2(1-t_1)t_2\cdots t_n]^2 + \cdots + [\zeta_n(1-t_{n-1})t_n]^2) = 0.\end{aligned}$$

The possible singularities of $\tilde{K}_2(\zeta,\xi)$ may be then computed by eliminating each of the $t_1, t_2, \cdots, t_n$ variables from $\chi(\zeta,\xi,t) = 0$ by setting $t_i = 0$, 1, or adding the condition $\frac{\partial\chi}{\partial t_i}(\zeta,\xi,t) = 0$, which specifies a coincidence of singular points in the complex t_i-plane. For instance, if we consider the following end-point pinch $\{t_k = 0;\ t_i = 1,\ \text{for } \forall i \neq k,\ k < n\}$ $\chi(\zeta,\xi,t) = 0$ becomes $(\zeta_{k+1}\xi_{k+1} - 1)^2 + \zeta_{k+1}^2(1-\|\xi\|^2) = 0$.

Remark. Recalling the singularity sets, E_l, given in (16) for a function $f(\xi)$, let us compute a possible singularity of an $F(\zeta) = \mathfrak{D}^{-1}f(\xi)$, corresponding to

$$\xi_l = \tfrac{1}{2}(\alpha_l + \frac{1}{\alpha_l}) - \frac{1}{2\alpha_l}\sum_{\substack{i=1\\ i\neq l}}^{n}\xi_i^2,$$

considering a coincidence with $(\zeta_l\xi_l-1)^2+\zeta_l^2(1-\|\xi\|^2)=0$. One obtains

$$\chi(\zeta,\xi)\equiv 1-2\zeta_l\xi_l+\zeta_l^2\left(-\frac{1}{\alpha_l^2}+\frac{2\xi_l}{\alpha_l}\right)=0.$$

Forming the coincidence pinch with respect to ξ_l, i.e. eliminating ξ_l between the above and $\dfrac{\partial\chi}{\partial\xi_l}\equiv-2\zeta_l+2\zeta_l^2\alpha_l^{-1}=0$ yields $\zeta_l=\alpha_l$ as the only *consistent* singular point. We notice that $\zeta_l=\alpha_l$ was exactly the singularity of $F(\zeta)$ which produced E_l as a candidate for a singular set of $f(\xi)$.

We summarize our previous discussion as a theorem.

THEOREM 1. *Let $F(\zeta)$ as defined by* (13) *be holomorphic in $\Delta(0,1)\subset \boldsymbol{C}^n$, and let $f(\xi)$ be defined as in* (14). *Then $f(\xi)$ is a holomorphic function in some complex neighborhood of $S(0;1)$, on its principal sheet of definition. Furthermore, if $F(\zeta)$ has a singularity on the analytic plane, $\zeta_l=\alpha$, $|\alpha|>1$, then $f(\xi)$ is singular on E_l. On other than the principal sheet, $f(\xi)$ may also be singular on $\|\xi\|=1$.*

THEOREM 2. *The only singular points of $K_2(\zeta,\xi)$ are to be found on the union of analytic sets*

$$\mathcal{K}_2\equiv\bigcup_{l=1}^{n}\{[\zeta,\xi]\mid[\zeta_l\xi_l-1]^2+\zeta_l^2(1-\|\xi\|^2)=0;\quad (\zeta_1,\cdots,\hat{\zeta}_l,\cdots,\zeta_n)\in\boldsymbol{C}^{n-1}\}. \tag{22}$$

The notation $(\zeta_1,\cdots,\hat{\zeta}_l,\cdots,\zeta_n)$ means that ζ_l is missing from the $(n-1)$-tuple.)

Proof. It is of interest to recall that these are just the singular sets predicted by the end-point pinches discussed above. In order to prove this result we represent $K_2(\zeta,\xi)$ by the integral

$$K_2(\zeta,\xi)=\left(\tfrac{1}{2\pi i}\right)^n\int_{\mathcal{D}}\cdots\int[\Sigma U_M^{(s)}(\xi)\eta^M]\cdot\left[\Sigma\frac{2m+n+s-1}{(s)_m}M!\left(\frac{\zeta}{\eta}\right)^M\right]\frac{d\zeta}{\zeta}, \tag{23}$$

$$\text{where }\left(\frac{\zeta}{\eta}\right)^M\equiv\left(\frac{\zeta_1}{\eta_1}\right)^{m_1}\cdots\left(\frac{\zeta_n}{\eta_n}\right)^{m_n},$$

$\mathcal{D}=\prod_{l=1}^{m}\partial\Delta_l(0;1-\epsilon)$ and $\left(\dfrac{\zeta}{\eta}\right)\in\Delta(0;1)$. From the Weierstrass comparison theorem it is clear that the second series converges for all $\binom{\zeta}{\eta} \in \boldsymbol{C}^n$ except for those points on the coordinate planes. In this case we have

$$[\frac{2m+n+s-1}{(s)_m} m!][\frac{2m+n+s+1}{(s)_{m+1}}(m+1)!]^{-1} \to 1 \text{ as } m \to \infty.$$

Hence, by Fabry's theorem [12] pg. 377, on each coordinate, analytic-plane, $\zeta_i = 1$, $(i = 1, 2, \cdots, n)$, $\zeta \equiv (\zeta_1, \cdots, \hat{\zeta}_l, \cdots, \zeta_n) \in \boldsymbol{C}^{n-1}$, is a singular point. Using the extension of the Hadamard multiplication of singularities theorem as given in [16] pg. 35 we conclude that $K_2(\zeta, \xi)$ may be singular only for

$$(24) \qquad (\zeta_l \xi_l - 1)^2 + \zeta_l^2 (1 - \| \xi \|^2) = 0, \ \zeta_k \in \boldsymbol{C}^1, \ k \neq l,$$

as is stated.

THEOREM 3. *Let $f(\xi)$ be an analytic function of n complex variables in some neighborhood $\tilde{N}$ of the unit ball $S(0;1)$. Then $f(\xi)$ can be expanded in an Appell series,*

$$f(\xi) = \sum a_M V_M^{(s)}(\xi), \ s > -1, \ s \neq 0,$$

which converges uniformly for

$$\xi \in \mathfrak{J} \equiv N\{S(0;1)\} \cap N^*\{S(0;1)\} \text{ with } N\{S(0;1)\} \subset \tilde{N},$$

and where the coefficients are given by the formula

$$(25) \qquad a_M = h_M{}^s \int_{S(0;1)} (1 - \| \xi \|)^{(s-1)/2} f(\xi) U_M{}^{(s)}(\xi) \, d^n\xi$$

with

$$h_M{}^s = \frac{(2m+n+s-1)\Gamma(\frac{1}{2}[n+s-1])M!}{2\pi^{n/2}\Gamma(s/2+1)(s)_m},$$

Proof. If $f(\xi)$ is holomorphic in $N\{S(0;1)\}$ then we may choose a $\mathfrak{J} = \mathfrak{J}^* \subset N\{S(0;1)\}$ which is a complex neighborhood of $S(0;1)$. Furthermore from (24) the only singularities of the kernel may be written as

$$(26) \qquad \zeta_l = (1 - \sum_{i \neq l} \xi_i^2)^{-1} [\xi_l \pm i\sqrt{1 - \| \xi \|^2}], \qquad (l = 1, 2, \cdots, n).$$

for either choice of sign in the above $|\zeta_l| \geqq 1$, for $\xi \in S(0;1)$; indeed, $K_2(\zeta, \xi)$ is seen by majorization to be regular *in* $\Delta(0;1) \times S(0;1)$. Hence, by a direct computation of possible singularity coincidences between the kernel and $f(\xi)$, and by computing also the "endpoint pinches," we conclude $F(\zeta)$

is regular in $\Delta(0;1+\epsilon)$, on the principal sheet of $F(\zeta)$. Using this $F(\zeta)$ we may now compute the series representation for $f(\xi)$ as

$$f(\xi)=\mathfrak{D}(\mathfrak{D}^{-1}f(\xi))=\mathfrak{D}\mathfrak{F}(\zeta)=\sum a_M V_M^{(s)}(\xi)\quad(s>-1,s\neq)$$

where the coefficients are defined by (25). The termwise integration is valid since $F(\zeta)$ is holomorphic in $\Delta(0;1)$, and the kernel $K_1(\zeta,\xi)$ is uniformly convergent for $\zeta\in\Delta(0;1+\epsilon)$, $\xi\in\mathcal{J}$, which may be seen by simple estimates.

3. A uniqueness theorem for the generalized axially symmetric reduced wave equation $n+1$ variables. Although existence and uniqueness theorems for linear elliptic partial differential equations in a domain D with coefficients continuous in D have been known for some time ([3]), similar results for equations whose coefficients have singularities in the domain under consideration are practically unknown. Recently attention has been given to a class of singular equations which appear frequently in both pure and applied mathematics and are known as generalized axially symmetric partial differential equations. This class of equations was first studied by Weinstein ([30], [31]) and developed further by many other researchers, in particular Gilbert ([14]) and Gilbert and Howard [17], [18]. Problems of existence and uniqueness of solutions to boundary value problems for the elliptic equations in this class have been studied by Huber ([21]), Parter ([26]), Colton ([4]-[7]), and Colton and Wimp ([9]). Just as a thorough knowledge of the Laplace and Helmholtz equation guided the attack on linear elliptic equations with continuous coefficients, it is hoped that a better understanding of generalized axially symmetric equations will give insight towards developing a theory of elliptic equations with singular coefficients.

In this section we will derive a uniqueness theorem for the exterior Dirichlet problem for the generalized axially symmetric reduced wave equation in $n+1$ variables

$$\mathfrak{L}_{\lambda,s}[u]\equiv\frac{\partial^2u}{\partial x_1^2}+\cdots+\frac{\partial^2u}{\partial x_n^2}+\frac{\partial^2u}{\partial\rho^2}+\frac{s}{\rho}\,\frac{\partial u}{\partial\rho}+\lambda^2u=0\tag{27}$$

where $s>-1$, $s\neq0$, and $\lambda>0$ is real. This is the first time a uniqueness theorem has been obtained for a singular elliptic partial differential equation in more than two variables, such work having been delayed until the necessary results on several complex variables were available to derive the theorem obtained in section two. It should be noted that Huber ([21] has investigated the equation $\mathfrak{L}_{0,s}[u]=0$ in domains D for the case in which D does not contain a portion of the plane $\rho=0$. Since we are concerned with the case in which the coefficients of the differential equation are singular in the interior of D, his work has no direct bearing on our investigations. The analytic theory of equation (27) has been studied by Gilbert ([15], [16]), Gilbert and Howard ([17]), and Henrici ([20]). Uniqueness theorems for the case $s = 0$ by Colton and Gilbert ([8]), Vekua ([33], [34]), Sommerfield ([28]), Rellich ([27]), Magnus ([24]), Levine ([23]), and Wilcox ([32]).[2]

In either of the half spaces $\rho > 0$ and $\rho < 0$ equation (19) is an elliptic partial differential equation with analytic coefficients and hence every twice continuously differentiable solution is a real-analytic function of $(x, \rho) \equiv (x_1, \cdots, x_n, \rho)$ in each such half space ([3]). The plane $\rho = 0$, which will be called the axis, is a singular plane of the regular type with exponents 0 and $1 - s$ ([10]). Consequently, there always exist solutions of equation (19) which are regular (i.e. analytic functions of x and ρ) on (some portion of) the axis. It is seen from the differential equation that if $s \neq 0$ then $\dfrac{\partial u}{\partial \rho} = 0$ on the axis for such regular solutions and if $s > -1$, $s \neq 0$, each regular solution can be continued across the axis as an even function of ρ, i.e. for such values of s every regular solution is an analytic function of x and ρ^2 in some domain D that is symmetric with respect to the axis $\rho = 0$. The case $s = 0$ is the classical n dimensional reduced wave equation and has been treated in [8], [23], [24], [27], [28], and [32]. If $u(x, \rho)$ is a regular solution of $\mathfrak{L}_{\lambda,s}[u] = 0$ and $s > -1$, $s \neq 0$, then $u(x, \rho)$ is an even function of ρ and hence can be expressed as $u(x, \rho) = \tilde{u}(r, \xi)$, $\xi \equiv (\xi_1, \cdots, \xi_n)$, where r, ξ are the hyper-zonal coordinates ([1]) defined as

$$
\begin{aligned}
x_1 &= r\xi_1 \\
x_2 &= r\xi_2 \\
&\;\;\vdots \\
x_n &= r\xi_n \\
\rho &= r\Big(1 - \sum_{i=1}^{n} \xi_i^2\Big)^{\frac{1}{2}} \\
r^2 &= x_1^2 + \cdots + x_n^2 + \rho^2.
\end{aligned}
\tag{28}
$$

[2] It should be noted here that the fundamental results contained in [27] (which are normally attributed in western countries solely to Rellich) were also developed simultaneously by the Russian mathematician, I. N. Vekua ([33], [34]).

In what follows $D \subset \boldsymbol{R}^{n+1}$ will always denote a normal domain symmetric with respect to the axis $\rho = 0$, ∂D will be the boundary of D, F the region exterior to the closure $\bar{D}$ of D.

THEOREM 4. *Assume $s > -1$, $s \neq 0$, and let $u(x,\rho) = \tilde{u}(r,\xi)$ be a regular solution of $\mathfrak{L}_{\lambda,s}[u] = 0$ in F such that $u(x,\rho) \in C^2(\bar{F})$ and for each fixed r, $a \leqq r < \infty$, (where a is such that $r = a$ contains D) $\tilde{u}(r,\xi)$ is an analytic function of ξ in a domain $\mathfrak{J}$ in the complex ξ space containing the (real) closed unit ball $S(0;1)$ where $\mathfrak{J}$ is symmetric with respect to conjugation. If*

(i) $r^{(n+s)/2} \mid \tilde{u}(r,\xi)$ *is uniformly bounded for* $a \leqq r < \infty, \xi \in \mathfrak{J}$

(ii) $\displaystyle\lim_{r\to\infty} \int_{S(0;1)} r^{n+s}(1 - \| \xi \|^2)^{(s-1)/2} \left| \frac{\partial \tilde{u}}{\partial r} - ik\tilde{u} \right|^2 d\xi = 0$

where $S(0;1) \equiv \{\xi \mid \| \xi \|^2 \leqq 1\}$,

(iii) $\tilde{u}(r,\xi) \equiv 0$ *on* ∂D

then $\tilde{u}(r,\xi) \equiv 0$.

Remark. If conditions (i) and (ii) of the theorem are not imposed there will exist "eigenfunctions" of equation (16) and we cannot expect a unique solution to exist ([4]).

Proof. In order to establish this result it is necessary to express (27) first in the hyper-zonal coordinates (28). This was done in [14]: however, the computations involved are not presented there or elsewhere for s *not* a positive integer, and hence we indicate them below. The formal identities,

$$
\begin{aligned}
r^2 \sum_{j=1}^{n} \frac{\partial^2 \Psi}{\partial x_j^2} \equiv (r^2 - \rho^2)\frac{\partial^2 \Psi}{\partial r^2} &+ \left(r[n-1] + \frac{\rho^2}{r}\right) \frac{\partial \Psi}{\partial r} \\
&+ \frac{2\rho^2}{r} \sum_{j=1}^{n} \xi_j \frac{\partial^2 \Psi}{\partial \xi_j \partial r} + \left(1 - n - \frac{3\rho^2}{r^2}\right) \sum_{j=1}^{n} \xi_j \frac{\partial \Psi}{\partial \xi_j} \\
&+ \sum_{j=1}^{n} \frac{\partial^2 \Psi}{\partial \xi_j^2} - \left(1 + \frac{\rho^2}{r^2}\right) \sum_{j,k=1}^{n} \xi_j \xi_k \frac{\partial^2 \Psi}{\partial \xi_j \partial \xi_k},
\end{aligned}
\tag{29}
$$

and

$$
\begin{aligned}
r^2 \frac{\partial^2 \Psi}{\partial \rho^2} \equiv \left(r - \frac{\rho^2}{r}\right) \frac{\partial \Psi}{\partial r} &+ \left(\frac{3\rho^2}{r} - 1\right) \sum_{l=1}^{n} \xi_l \frac{\partial \Psi}{\partial \xi_l} + \rho^2 \frac{\partial^2 \Psi}{\partial \rho^2} \\
&- \frac{2\rho^2}{r} \sum_{l=1}^{n} \xi_l \frac{\partial^2 \Psi}{\partial r \partial \xi_l} + \frac{\rho^2}{r^2} \sum_{l,k=1}^{n} \xi_l \xi_k \frac{\partial^2 \Psi}{\partial \xi_l \partial \xi_k},
\end{aligned}
\tag{30}
$$

may be combined, for $x_{n+1} \equiv \rho$, to form

$$r^2 \Delta_{n+1} \Psi \equiv \frac{1}{r^{n-2}} \frac{\partial}{\partial r} (r^n \frac{\partial \Psi}{\partial r}) - n\Psi$$

$$+ \sum_{l=1}^{n} \frac{\partial}{\partial \xi_l} \{ \frac{\partial \Psi}{\partial \xi_l} - \xi_l (\sum_{l=1}^{n} \xi_j \frac{\partial \Psi}{\partial \xi_j} - \Psi) \}. \tag{31}$$

A separation of variables is possible by choosing the separation constant to be $(\mu + n)(\mu + s - 1)$, with μ arbitrary, obtaining the following two equations for a solution to (27) of the form $P(r)E(\xi)$:

$$r^2 P'' + r(\mu + s)P' + [\lambda^2 r^2 - \mu(n + s + \mu - 1)]P = 0, \tag{32}$$

$$(\mu + n)(\mu + s - 1)E + \sum_{l=1}^{n} \frac{\partial}{\partial \xi_l} \{ \frac{\partial E}{\partial \xi_l} - \xi_l [\sum_{j=1}^{n} \xi_j \frac{\partial E}{\partial \xi_j} + (s-1)E] \} = 0. \tag{33}$$

When $\mu = m$, an integer, (33) is the partial differential equation that $V_M^{(s)}(\xi)$ and $U_M^{(s)}(\xi)$ satisfy [11] Vol. II, pg. 275, 278; hence, one may obtain separated solutions of the form

$$\tilde{u}(r, \xi) = r^{-\frac{1}{2}(n+s-1)} Z_{\mu + \frac{1}{2}(n+s-1)}(\lambda r) V_M^{(s)}(\xi), \tag{34}$$

where $Z_\nu(\lambda r)$ designates a cylinder function of order ν.

We return now to the proof of our theorem (making use of Theorem 3) and expand $\tilde{u}(r, \xi)$ for each fixed $r \geqq a$ in an Appell series,

$$\tilde{u}(r, \xi) = \sum a_M(r) V_M^{(s)}(\xi), \tag{35}$$

which converges in a suitable domain $\mathcal{J}$ satisfying the hypothesis of the theorem. Next, we wish to verify that the Fourier coefficients of such an expansion are of the form $a_M(r) \equiv r^{-\frac{1}{2}(n+s-1)} Z_{\mu+\frac{1}{2}(n+s-1)}(\lambda r)$. To this end we obtain, in view of Theorem 3 and equation (33),

$$h_M^{(s)} \int_{S(0;1)} r^2 U_M^{(s)}(\xi) (1 - \| \xi \|^2)^{(s-1)/2} \mathfrak{L}_{\lambda,s}[\tilde{u}] d^n\xi$$

$$= \frac{1}{r^{n+s-1}} \frac{d}{dr} (r^{n+s} \frac{da_M(r)}{dr}) + \lambda^2 r^2 a_M(r) + n(s-1) a_M(r)$$

$$+ h_M^{(s)} \sum_N a_N(r) \{ \sum_{l=1}^{n} \int_{S(0;1)} (1 - \| \xi \|^2)^{(s-1)/2} U_M^{(s)}(\xi) \frac{\partial}{\partial \xi_l} [\frac{\partial V_N^{(s)}}{\partial \xi_l} \tag{36}$$

$$- \xi_l (\sum_{j=2}^{n} \xi_j \frac{\partial V_N^{(s)}}{\partial \xi_l} + (s-1) V_N^{(s)})] d^n\xi \},$$

$$= \frac{1}{r^{n+s-1}} \frac{d}{dr} (r^{n+s} \frac{d}{dr} a_M(r)) + \lambda^2 r^2 a_M(r) - \mu(n + \mu + s - 1) a_M(r) = 0.$$

The procedure we shall follow at this point is to prove that the $a_M(r) \equiv 0$ for $\forall M$. To this end, we need an identity which we obtain by an application of Green's formula, namely

$$(37) \qquad \int_{S(0;1)} r^{n+s}(1-\|\xi\|^2)^{\frac{1}{2}(s-1)}\left(\tilde{u}\frac{\partial \tilde{u}^*}{\partial r} - \tilde{u}^*\frac{\partial \tilde{u}}{\partial r}\right) d^n\xi = 0,$$

for sufficiently large r, and $\tilde{u}^* \equiv \overline{(\tilde{u})}$. Consequently, it is possible to show, as was done in [6] for GASHE, that

$$(38) \qquad \lim_{r\to\infty} \int_{S(0;1)} r^{n+s}(1-\|\xi\|^2)^{(s-1)/2} |u|^2 d\xi = 0.$$

We now define

$$(39) \quad g(r,\xi) = \tilde{u}(r;\xi)\tilde{u}^*(r;\xi^*) \equiv [\sum a_M(r) V_M^{(s)}(\xi)][\sum \overline{a_M(r)} V_M^{(s)}(\xi)]$$

and observe that $g(r,\xi)$ is a analytic function of ξ and which agrees with $|\tilde{u}(r,s)|^2$ for ξ real valued. Since $V_M^{(s)}(\xi)$ is a polynomial with real coefficients and $\mathcal{J}$ is symmetric with respect to conjugation, it can easily be shown ([6]) using equation (35) and condition (i) of the theorem that $r^{n+s}|g(r,\xi)|$ is uniformly bounded for $a \leqq r < \infty$, $\xi \in \mathcal{J}$. Now define for fixed r and variable ξ

$$(40) \qquad F_r(\xi) = \int_{\xi^{(0)}} r^{n+s}(1-\|\xi\|^2)^{(s-1)/2} g(r,\xi)\, d\xi.$$

Observe that $F_r(\xi)$ is analytic in $\Delta(\xi^{(0)};\zeta)$ (if ζ is chosen sufficiently small) and for $r \geqq a$, $\xi \in \Delta(\xi_0^{(0)};\zeta)$, $|F_r(\xi)|$ is uniformly bounded. By equation (38) we have

$$(41) \qquad \lim_{r\to\infty} F_r(\xi) = 0;\ \xi \in \Delta(\xi^{(0)};\zeta) \cap S(0;1).$$

Hence by a several complex variable version of Vitali's theorem as given in [19] we have

$$(42) \qquad \lim_{r\to\infty} F_r(\xi) = 0;\ \xi \in \Delta(\xi^{(0)};\zeta).$$

Differentiating in equation (40) and applying Theorem (2) a second time we have

$$(43) \qquad \lim_{r\to\infty} r^{n+s} g(r,\xi) = 0;\ \xi \in \Delta(\xi^{(0)};\zeta).$$

Hence

$$(44) \qquad \lim_{r\to\infty} r^{(n+s)/2} \sum a_M(r) V_M^{(s)}(\xi) = 0;\ \xi \in \Delta(\xi^{(0)};\zeta) \cap S(0;1).$$

From equation (35), condition (i) of the theorem and Vitali's Theorem again we can conclude that

$$\lim_{r\to\infty} r^{(n+s)/2}\tilde{u}(r,\xi) = 0;\ \xi \in S(0;1). \tag{45}$$

By the formula for the coefficients $a_M(r)$ given in Theorem (3) and Lebesgue's dominated convergence theorem we have for each M,

$$\lim_{r\to\infty} r^{(n+s)/2} a_M(r) = 0. \tag{46}$$

But since $r^{(n+s-1)/2}a_M(r)$ is a cylinder function, this implies $a_M(r) = 0$ for each M, and hence by equation (35) and the analyticity of $\tilde{u}(r,\xi)$ in F we can conclude that $\tilde{u}(r,\xi) \equiv 0$. The theorem is now proved.

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11

FUNCTION THEORETIC METHODS IN THE THEORY OF BOUNDARY VALUE PROBLEMS FOR GENERALIZED META-HARMONIC FUNCTIONS

BY

DAVID COLTON AND ROBERT P. GILBERT

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FUNCTION THEORETIC METHODS IN THE THEORY OF BOUNDARY VALUE PROBLEMS FOR GENERALIZED METAHARMONIC FUNCTIONS[1]

BY DAVID COLTON AND ROBERT P. GILBERT

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1. **Introduction.** Although existence and uniqueness theorems for linear elliptic partial differential equations in a domain D with coefficients continuous in D have been known for some time, similar results for equations whose coefficients have singularities in the domain under consideration are practically unknown. Recently attention has been given to a class of singular equations which appear frequently in both pure and applied mathematics and are known as generalized axially symmetric partial differential equations [3], [5], [8]. Just as a thorough knowledge of the Laplace and Helmholtz equation guided the attack on linear elliptic equations with continuous coefficients, it is hoped that a better understanding of generalized axially symmetric equations will give insight towards developing a theory of elliptic equations with singular coefficients.

We wish to announce in this note a uniqueness theorem for the exterior Dirichlet problem for the generalized axially symmetric metaharmonic equation

$$L_{\lambda s}[u] \equiv \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} + \frac{\partial^2 u}{\partial \rho^2} + \frac{s}{\rho}\frac{\partial u}{\partial \rho} + \lambda^2 u = 0 \tag{1}$$

where $s > -1$, $s \neq 0$, and $\lambda > 0$ [5]. This is the first time a uniqueness theorem has been obtained for a singular elliptic partial differential equation in more than two variables whose coefficients are singular in its domain of definition, such work in the present case having been delayed due to an insufficient knowledge of certain areas of the theory of several complex variables. Our result depends on first using the Hadamard-Gilbert Theorem [5] to solve the classical expansion problem for Appell series and to then apply this along with Vitali's theorem for several complex variables [6] to obtain the desired uniqueness theorem.

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2. The expansion problem for Appell series. An Appell series is a series of the form

$$\sum_{M} a_M V_M^{(s)}(\xi) \equiv \sum_{\mu=0}^{\infty} \sum_{m=\mu} a_M V_M^{(s)}(\xi); \qquad s > -1,\ s \neq 0 \tag{2}$$

where the polynomials

$$V_M^{(s)}(\xi) \equiv V_M^{(s)}(\xi_1, \cdots, \xi_n); \quad M = (m_1, m_2, \cdots, m_n) \tag{3}$$

are uniquely defined by the generating function

$$(1 - 2(\alpha, \xi) + \|\alpha\|^2)^{-(n+s-1)/2} = \sum \alpha^M V_M^{(s)}(\xi). \tag{4}$$

Here $(\alpha, \xi) \equiv \sum_{i=1}^{n} a_i \xi_i$, $\|\alpha\|^2 = (\alpha, \alpha)$, $\alpha^M \equiv \alpha_1^{m_1} \cdots \alpha_n^{m_n}$, $m = |M| = m_1 + m_2 + \cdots + m_n$ and the summation in equation (2), (4), and in what follows, is meant to be an n-fold sum over all indices from zero to infinity. We also need the related polynomials

$$U_M^{(s)}(\xi) \equiv U_M^{(s)}(\xi_1, \xi_2, \cdots, \xi_n) \tag{5}$$

which are defined by the generating function

$$\{[(\alpha, \xi) - 1]^2 + \|\alpha\|^2(1 - \|\xi\|^2)\}^{-s/2} = \sum \alpha^M U_M^{(s)}(\xi). \tag{6}$$

These polynomials satisfy the biorthogonality relation

$$\begin{aligned} &\int_{S(0,1)} (1 - \|\xi\|^2)^{(s-1)/2} V_M^{(s)}(\xi) U_L^{(s)}(\xi) d^n\xi \\ &\qquad = \delta_{LM} \frac{2\pi^{n/2}\Gamma(s/2 + 1)(s)m}{(2m + n + s - 1)\Gamma(n/2 + s/2 - 1/2)M!}, \end{aligned} \tag{7}$$

$\delta_{LM} \equiv \delta_{l_1 m_1} \cdots \delta_{l_n m_n}$, $M! = m_1! \cdots m_n!$. Here $S(0, 1)$ is the real solid n dimensional ball $\{\xi \mid \|\xi\| \leq 1\}$. For more information concerning the polynomials $V_M^{(s)}(\xi)$ and $U_M^{(s)}(\xi)$ see [1]. We are interested in the classical expansion problem for analytic functions of n complex variables in terms of the polynomials $V_M^{(s)}(\xi)$ viz. if $f(\xi)$ is analytic on $S(0, 1)$ can it be expanded in a series of the form (2) and if so what can be said about the region of convergence?

THEOREM 1. *Let $f(\xi)$ be an analytic function of n complex variables in some neighborhood $\bar{\eta}$ of the unit ball $S(0, 1)$. Then $f(\xi)$ can be expanded in an Appell series,*

$$f(\xi) = \sum a_M V_M^{(s)}(\xi); \qquad s > -1,\ s \neq 0$$

which converges uniformly for $\xi \in \mathfrak{J} \equiv \eta\{S(0, 1)\} \cap \eta^*\{S(0, 1)\}$ *with* $\eta\{S(0, 1)\} \subset \tilde{\eta}$, $\eta^* \equiv \{\xi \mid \xi^* \in \eta\}$ (* *denotes complex conjugation*), *and where the coefficients are given by the formula*

$$a_M = h_M^s \int_{S(0,1)} (1 - \|\xi\|)^{(s-1)/2} f(\xi) U_M^{(s)}(\xi) d^n\xi$$

with

$$h_M^s = \frac{(2m + n + s - 1)\Gamma(\frac{1}{2}[n + s - 1])M!}{2\pi^{n/2}\Gamma(s/2 + 1)(s)_m}.$$

OUTLINE OF PROOF. The technique used is to develop an integral operator approach to the method of generating kernels [2]. We first define a new function $F(\zeta)$ defined by

$$F(\zeta) = Of(\xi) = \left(\frac{1}{2\pi i}\right)^n \int_{S(0,1)} K_1(\zeta, \xi) f(\xi) d^n\xi = \sum a_M \zeta^M \tag{8}$$

where

$$\begin{aligned} K_1(\zeta, \xi) &\equiv \frac{\Gamma(\frac{1}{2}[n + s - 1])(1 - \|\xi\|^2)^{(s-1)/2}}{2\pi^{n/2}\Gamma(s/2 + 1)} \\ &\quad \cdot \sum \frac{2m + n + s - 1}{(s)_m} M! U_M^{(s)}(\xi) \zeta^M. \end{aligned} \tag{9}$$

From the Weierstrass comparison theorem [5] and the generating function expansion (6) it can be shown that equation (8) defines an analytic function of ζ for $\zeta \in \Delta(0, 1)$. Here $\Delta(0, 1)$ denotes the open unit ball in the complex ζ space. We next define the *inverse operator* O^{-1} by

$$\begin{aligned} f(\xi) &= O^{-1}F(\zeta) = \left(\frac{1}{2\pi i}\right)^n \int_\Gamma K_2(\xi, \zeta) F(\zeta) \frac{d^n\zeta}{\zeta} \\ &= \sum a_M V_M^{(s)}(\xi) \end{aligned} \tag{10}$$

where

$$K_2(\xi, \zeta) \equiv \sum \zeta^{-M} V_M^{(s)}(\xi) = \left(1 - 2\left(\frac{1}{\zeta}, \xi\right) + \left\|\frac{1}{\zeta}\right\|^2\right)^{-(n+s-1)/2}$$

and $\Gamma = \prod_{i=1}^n \Gamma_i$ where the Γ_i are chosen such that Γ lies outside a sufficiently large hypersphere $\Delta(0, R)$. By using the Hadamard-Gilbert theorem [5] and the fact that $f(\xi)$ is analytic in $\tilde{\eta}$, one can conclude from (8) that $F(\zeta)$ must in fact be an analytic function in some complex neighborhood of $\overline{\Delta(0, 1)}$ on its principal sheet of definition. Using, this fact in conjunction with the series representation (8) for $F(\zeta)$ shows that the series (10) converges uniformly in some complex neighborhood of $S(0, 1)$ and agrees with $f(\xi)$ there. The formula for the coefficients is arrived at through use of the biorthogonality relation (7).

3. **The uniqueness theorem.** Since the plane $\rho=0$ is a singular curve of the regular type with indices 0 and $1-s$ [4] there always exist solutions of equation (1) which are real analytic on some portion of the axis $\rho=0$, and if $s>-1$, $s\neq 0$, such solutions can be continued across the axis as an even function of ρ. Hence each such analytic solution is analytic in a domain D that is symmetric with respect to the axis $\rho=0$ and can therefore be expressed as $u(x, \rho)=\tilde{u}(r, \xi)$ where r, $\xi=(\xi_1, \cdots, \xi_n)$ are hyper-zonal coordinates.

THEOREM 2. *Assume $s>-1$, $s\neq 0$, and let $u(x, \rho)=\tilde{u}(r, \xi)$ be a real analytic solution of $L_{\lambda,s}[u]=0$ in $\mathfrak{F}$ where $\mathfrak{F}$ is the exterior of a normal domain $\mathfrak{D}$ which is symmetric with respect to the axis $\rho=0$. Let $\tilde{u}(r, \xi)\in C^2(\mathfrak{F})$ and for each fixed r, $a\leqq r<\infty$ (where a is such that $r=a$ contains $\mathfrak{D}$) assume that $\tilde{u}(r, \xi)$ is an analytic function of ξ in a domain $\mathfrak{J}$ in the complex ξ space containing the (real) closed unit ball $S(0, 1)$, where $\mathfrak{J}$ is symmetric with respect to conjugation and independent of r. If*

(i) *there exists a positive constant M such that*

$$r^{(n+s)/2}\left|\tilde{u}(r, \xi)\right| \leqq M \qquad \text{for } a\leqq r<\infty, \quad \xi\in\mathfrak{J},$$

(ii) $\lim_{r\to\infty}\int_{S(0,1)}r^{n+s}(1-\|\xi\|^2)^{(s-1)/2}\left|\partial\tilde{u}/\partial r-i\lambda\tilde{u}\right|^2dS=0,$

(iii) $\tilde{u}(r, \xi)=0$ *on the boundary of* $\mathfrak{D}$

then $\tilde{u}(r, \xi)\equiv 0$.

REMARK. If conditions (i) and (ii) of the theorem are not imposed there will exist eigenfunctions of equation (1) and we cannot expect a unique solution to exist.

OUTLINE OF PROOF. Using Theorem 1 we expand $\tilde{u}(r, \xi)$ for each fixed $r\geqq a$ in an Appell series

$$\tilde{u}(r, \xi) = \sum a_M(r)V_M^{(s)}(\xi); \qquad \xi\in\mathfrak{J}. \tag{11}$$

From the biorthogonality property (7) it can be shown that

$$a_M(r) = r^{-1/2(n+s-1)}Z_{\mu+1/2(n+s-1)}(\lambda r) \tag{12}$$

where $Z_\nu(\lambda r)$ denotes a cyclinder function of order ν. Applying Green's formula and condition (ii) gives

$$\lim_{r\to\infty} \int_{S(0,1)} r^{n+s}(1 - \|\xi\|^2)^{(s-1)/2} |u|^2 d^n\xi = 0. \tag{13}$$

If the polynomials $V_M^{(s)}(\xi)$ satisfied Parseval's relation we could now proceed with the proof by following the approach used by Vekua in [7]. Since this is not the case we resort to techniques first used by Colton in [3]. Define

$$g(r, \xi) \equiv \tilde{u}(r, \xi)\tilde{u}^*(r, \xi^*), \tag{14}$$

and observe that $g(r, \xi)$ is an analytic function of ξ which agrees with $|u(r, \xi)|^2$ for ξ real valued. From condition (i) of the theorem it can be shown that $r^{n+s}|g(r, \xi)|$ is uniformly bounded for $a \leqq r < \infty$, $\xi \in \mathfrak{Z}$. Now define for fixed r and variable ξ

$$F_r(\xi) \equiv \int_0^\xi r^{n+s}(1 - \|\xi\|^2)^{((s-1)/2)} g(r, \xi) d^n\xi, \tag{15}$$

and note that $F_r(\xi)$ is analytic in $\Delta(0, \epsilon)$ (if ϵ is chosen sufficiently small) and for $r \geqq a$, $\xi \in \Delta(0, \epsilon)$, $|F_r(\xi)|$ is uniformly bounded. By applying a version of Vitali's theorem for several complex variables [6], it can be shown using equations (13) and (15) that

$$\lim_{r\to\infty} r^{(n+s)/2}\tilde{u}(r, \xi) = 0; \qquad \xi \in S(0, 1). \tag{16}$$

Theorem 1, equation (12), and the asymptotic behavior of cylinder functions (c.f. [7]) now shows that $a_M(r) = 0$ for every M and hence $\tilde{u}(r, \xi) \equiv 0$.

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DECOMPOSITION THEOREMS FOR THE GENERALIZED METAHARMONIC EQUATION IN SEVERAL INDEPENDENT VARIABLES

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DECOMPOSITION THEOREMS FOR THE GENERALIZED METAHARMONIC EQUATION IN SEVERAL INDEPENDENT VARIABLES

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In this paper solutions of the generalized metaharmonic equation in several independent variables

(1) $$L_{\lambda,s}^{(n)}[u] = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} + \frac{\partial^2 u}{\partial \rho^2} + \frac{s}{\rho}\frac{\partial u}{\partial \rho} + \lambda^2 u = 0$$

where $\lambda > 0$ are uniquely decomposed into the sum of a solution regular in the entire space and one satisfying a generalized Sommerfeld radiation condition. Due to the singular nature of the partial differential equation under investigation it is shown that the radiation condition in general must hold uniformly in a domain lying in the space of several complex variables. This result indicates that function theoretic methods are not only the correct and natural avenue of approach in the study of singular ordinary differential equations, but are basic in the investigation of singular partial differential equations as well.

The techniques employed in the analytic theory of partial differential equations in $n > 2$ variables are in general quite different than in the case of two independent variables since one now needs to study analytic functions of several complex variables instead of a single complex variable (c.f. [9]). This point is aptly illustrated in the present work since although for $n = 1$ the above mentioned decomposition theorem has been previously obtained in [2] ,the methods used there do not immediately generalize to the several variable case considered here. This is due to the fact that in [2] rather explicit evaluations of certain contour integrals over the Riemann surface of multivalued analytic functions were required, and for functions of several complex variables this becomes prohibitively difficult. Hence an entirely different approach is employed ,namely the use of differential recursion relations similar to those first used in [3] and [4] to investigate the analytic theory and uniqueness problems for a class of singular equations closely related to 1). Although the use of contour integration is avoided the approach remains function theoretic in nature.

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For the special case $n = 2$, $s = 0$, (i.e. the nonsingular case) a particularly good discussion of the decomposition problem under consideration (including its application to scattering theory) can be found in [7] pp. 312–320. The results presented in [7] were first obtained by John in [13].

1. Appell series and generalized metaharmonic functions in several variables

An Appell series (c.f. [10]) is a series of the form

$$\sum a_M V_M^{(s)}(\xi) = \sum_{\mu=0}^{\infty} \sum_{|M|=\mu} a_M V_M^{(s)}(\xi) \tag{2}$$

where the polynomials

$$V_M^{(s)}(\xi) = V_M(\xi_1, \cdots, \xi_n); \quad M = (m_1, \cdots, m_n) \tag{3}$$

are uniquely defined by the generating function

$$(1-2(\alpha, \xi)+\|\alpha\|^2)^{-(n+s-1)/2} = \sum \alpha^M V_M^{(s)}(\xi). \tag{4}$$

Here

$$(\alpha, \xi) = \sum_{i=1}^{n} \alpha_i \xi_i, \ \|\alpha\|\|^2 = (\alpha, \alpha), \alpha^M = \alpha_1^{m_1} \cdots \alpha_n^{m_n},$$

$|M| = m_1+m_2+\cdots+m_n$, and the summation in equations (2), (4), and in what follows is meant to be an n-fold sum over all indices from zero to infinity. A related set of polynomials

$$U_M^{(s)}(\xi) = U_M^{(s)}(\xi_1, \cdots, \xi_n) \tag{5}$$

is defined by the generating function

$$\{[(\alpha, \xi)-1]^2+\|\alpha\|^2(1-\|\xi\|^2)\}^{-s/2} = \sum \alpha^M U_M^{(s)}(\xi). \tag{6}$$

For $s > -1$, $s \neq 0$, these polynomials satisfy the biorthogonality relation

$$\begin{aligned} &\int_{S(0;1)} (1-\|\xi\|^2)^{(s-1)/2} V_M^{(s)}(\xi)\, U_L^{(s)}(\xi)\, d^n\xi \\ &\qquad = \delta_{LM} \frac{2\pi^{n/2}\Gamma(s/2+1)(s)_{|M|}}{(2|M|+n+s-1)\Gamma\left(\frac{n}{2}+\frac{s}{2}-\frac{1}{2}\right) M!} \end{aligned} \tag{7}$$

where $\delta_{LM} = \delta_{l_1 m_1} \cdots \delta_{l_n m_n}, M! = m_1! \cdots m_n!$ and $S(0; 1)$ is the real solid n dimensional ball $\{\xi \mid \|\xi\| \leqq 1\}$. A basic result concerning Appell series is the following theorem obtained by the author and R. P. Gilbert in [5] and [6]:

THEOREM 1.1. *Let $f(\xi)$ be an analytic function of n complex variables in some neighbourhood $\tilde{\eta}$ of the unit ball $S(0; 1)$. Then $f(\xi)$ can be expanded in an Appell series*

(8) $$f(\xi) = \sum a_M^{(s)} V_M^{(s)}(\xi);\ s > -1, s \neq 0$$

which converges uniformly for $\xi \in \eta\,\{[S(0;1)\} \cap \eta^*\{S(0;1)\}$ *with* $S(0;1) \subset \eta\{S(0;1)\} \subset \tilde{\eta}$, $\eta^* = \{\xi | \xi^* \in \eta\}$ (* *denotes complex conjugation*), *and where the coefficients are given by the formula*

(9) $$a_M^{(s)} = h_M^s \int_{S(0;1)} (1-||\xi||^2)^{(s-1)/2} f(\xi) U_M^{(s)}(\xi)\, d^n\xi$$

with

$$h_M^s = \frac{(2|M|+n+s-1)\Gamma\left(\frac{n}{2}+\frac{s}{2}-\frac{1}{2}\right) M!}{2\pi^{n/2}\Gamma\left(\frac{s}{2}+1\right)(s)_{|M|}}.$$

There exists a neighbourhood η *of the unit ball* $S(0;1)$ *such that the series* (8) *converges uniformly to a holomorphic function for* $\xi \in \eta$ *if and only if the function*

(10) $$F(\xi) = \sum a_M^{(s)} \xi^M$$

can be continued to a holomorphic function on

$$\overline{\Delta(0;1)} = \{\xi | \sum_{i=1}^{n} |\xi_i|^2 \leqq 1\}.$$

Theorem 1.1 leads to an expansion theorem for solutions to equation (1) which are analytic functions of $(x, \rho) = (x_1, \cdots, x_n, \rho)$. This can be seen as follows. The plane $\rho = 0$ is a singular surface of the regular type with indices 0 and $1-s$ ([11]). Hence there always exist solutions of equation 1) which are analytic on some portion of the plane $\rho = 0$, and if $s \neq 0, -2, -4, \cdots$ such solutions can be continued across this plane as even functions of ρ. Each such regular solution is analytic in a domain D that is symmetric with respect to the plane $\rho = 0$ and can therefore be expressed as $u(x, \rho) = \tilde{u}(r, \xi)$ where r, $\xi = (\xi_1, \cdots, \xi_n)$ are the hyper-zonal coordinates defined as

(11) $$\begin{aligned} x_1 &= r\xi_1 \\ x_2 &= r\xi_2 \\ &\vdots \\ x_n &= r\xi_n \\ \rho &= r(1-\sum_{i=1}^{n} \xi_i^2)^{\frac{1}{2}} \\ r^2 &= x_1^2 + \cdots + x_n^2 + \rho^2. \end{aligned}$$

In these coordinates the differential equation (1) becomes ([9], p. 229)

$$r^2(L^{(n)}_{\lambda,s}[\tilde u]) = \frac{1}{r^{n+s-2}}\frac{\partial}{\partial r}\left(r^{n+s}\frac{\partial \tilde u}{\partial r}\right) + \lambda^2 r^2\tilde u + n(s-1)\tilde u$$
$$(12)\qquad + \sum_{j=1}^{n}\frac{\partial}{\partial \xi_j}\left\{\frac{\partial \tilde u}{\partial \xi_j} - \xi_j\left(\sum_{k=1}^{n}\xi_k\frac{\partial \tilde u}{\partial \xi_k} + (s-1)\tilde u\right)\right\} = 0.$$

From the above discussion and theorem 1.1 it is seen that for each r (where the sphere of radius r is contained in D) it is possible to expand $\tilde u(r, \xi)$ in an Appell series

$$(13)\qquad \tilde u(r, \xi) = \sum a^{(s)}_M(r)V^{(s)}_M(\xi)$$

and then use equations (9), (12) and the differential equation satisfied by the $U^{(s)}_M(\xi)$ ([10] p. 278) to conclude that

$$(14)\qquad \begin{aligned} a^{(s)}_M(r) = r^{-\frac{1}{2}(n+s-1)}[&c^{(s)}_M J_{|M|+(n+s-1)/2}(\lambda r) \\ &+ d^{(s)}_M H^{(1)}_{|M|+(n+s-1)/2}(\lambda r)] \end{aligned}$$

where J_μ is a Bessel function of order μ, $H^{(1)}_\mu$ is a Hankel function of the first kind of order μ, and $c^{(s)}_M$, $d^{(s)}_M$ are constants. Hence we have the following theorem ([5]. [6]):

THEOREM 1.2. *Let $\tilde u(r, \xi)$ be a regular solution of $L^{(s)}_{\lambda,s}[\tilde u] = 0$ in the exterior of a bounded domain D let and $s > -1$, $s \neq 0$. Then for $r \geqq a$ (where a is such that $r = a$ contains D) $\tilde u(r, \xi)$ can be expanded as*

$$(15)\qquad \tilde u(r, \xi) = \sum a^{(s)}_M(r)V^{(s)}_M(\xi)$$

where the coefficients $a^{(s)}_M(r)$ are given by equation (14). *For each fixed r the series* (15) *converges uniformly for ξ contained in some complex neighbourhood of $S(0; 1)$.*

The following theorem can be proved directly from the differential equation (12) satisfied by $\tilde u(r, \xi)$. The reader is referred to [3] for details in the case of two independent variables.

THEOREM 1.3. *Let $\tilde u(r, \xi)$ be a regular solution of $L^{(n)}_{\lambda,s}[\tilde u] = 0$ in a domain D. Then for $1 \leqq i \leqq n$,*

$$\tilde u^+(r, \xi) = \frac{1}{r}\frac{\partial \tilde u(r, \xi)}{\partial \xi_i}$$

is a regular solution of $L^{(n)}_{\lambda,s+2}[\tilde u] = 0$ in D.

By using the relationships ([10], p. 176, 275)

$$(16)\qquad \frac{dC^{\nu}_m(x)}{dx} = 2\nu C^{\nu+1}_{m-1}(x), \qquad m \geqq 1$$

$$(17)\qquad \|b\|^m C^{\frac{1}{2}(n+s-1)}_m\left[\frac{(b, \xi)}{\|b\|}\right] = \sum_{m_1+\cdots+m_n=m} b_1^{m_1}\cdots b_n^{m_n}V^{(s)}_{m_1,\cdots,m_n}(\xi_1, \cdots \xi_n)$$

where $C_m^\nu(x)$ denotes Gegenbauer's polynomial of index ν, one can verify that for $m_i \geqq 1$, $i \leqq l \leqslant n$,

$$\frac{\partial V_{m_1,\cdots,m_n}^{(s)}}{\partial \xi_i}(\xi_1, \cdots, \xi_n) = (n+s-1)V_{m_1,\cdots,m_i-1,\cdots,m_n}^{(s+2)}(\xi_1, \cdots, \xi_n). \tag{18}$$

Equation (18) shows that for $s \neq 0, -1, -2, \cdots$ there exist no nontrivial representations of zero of the form

$$\sum a_M^{(s)} V_M^{(s)}(\xi) = 0 \tag{19}$$

with the series 19) converging uniformly in a complex neighbourhood of $S(0; 1)$. This follows by observing that if such a representation existed the series could be differentiated termwise with respect ot ξ_i, $1 \leqq i \leqq n$, as often as desired, resulting in a series of the form (19) with $s > 0$. Use of the biorthogonality property shows that all the coefficients of this latter series are zero and hence the original series (19) consists of only a finite number of terms. Since for $\nu \neq 0, -1, -2, \cdots$ the Appell polynomials are of degree exactly m, (c.f.[10], p. 274) each of the coefficients in this finite series must be identically zero, i.e. $a_M^{(s)} = 0$ for every M.

Theorem 1.3 now enables us to extend the result of theorem 1.2 to include the cases $s < -1, s \neq -2, -3, \cdots$.

THEOREM 1.4. *If $\tilde{u}(r, \xi)$ is a regular solution of $L_{\lambda,s}^{(n)}[\tilde{u}] = 0$ in the exterior of a bounded domain D and $s \neq 0, -1, -2, \cdots$ then for r sufficiently large $\tilde{u}(r, \xi)$ can be expanded as*

$$\tilde{u}(r, \xi) = \Sigma\, a_M^{(s)}(r)\, V_M^{(s)}(\xi) \tag{20}$$

where the coefficients $a_M^{(s)}(r)$ are given by equation (14). *For each fixed r the series converges uniformly for ξ contained in some complex neighbourhood of $S(0; 1)$.*

PROOF. For $s > -1$ this result is given in theorem 1.2. Let $-2 < s < -1$. Then by theorem 1.3 r^{-1} $\text{grad}_\xi\, \tilde{u}(r, \xi)$ is a vector whose components $\tilde{u}_i(r, \xi)$ are regular solutions of $L_{\lambda,s+2}^{(n)}[\tilde{u}] = 0$ in the exterior of D. Hence by theorem 1.2

$$\tilde{u}_i(r, \xi) = \sum_i a_M^{(s+2)}(r)\, V_M^{(s+2)}(\xi) \tag{21}$$

where the subscript i denotes the dependence of the coefficient on $\tilde{u}_i(r, \xi)$, We furthermore have

$$\frac{\partial u_j}{\partial \xi_i} = \frac{\partial u_i}{\partial \xi_j}. \tag{22}$$

Since the series (21) converges uniformly for ξ contained in a complex neighbourhood of $S(0; 1)$ it is possible to differentiate (21) termwise. Doing this and using equations (18), (22) and the observation following equation (18) we have for $m_i, m_j \geqq 1$

(23) $$ {}_{j}a^{(s+2)}_{m_1,\cdots,m_j-1,\cdots m_n}(r) = {}_{i}a^{(s+2)}_{m_1,\cdots,m_i-1,\cdots m_n}(r). $$

For $|M| \geqq 1$ let

(24) $$ a^{(s)}_M(r) = a^{(s)}_{m_1,\cdots m_n}(r) = (n+s-1)^{-1} r\, {}_{i}a^{(s+2)}_{m_1,\cdots,m_i-1,\cdots m_n}(r) $$

where m_i is a non zero index. By equation (23) the $a^{(s)}_M(r)$ are well defined. Now let $\tilde{u}^+(r, \xi)$ be defined by the formal series

(25) $$ \tilde{u}^+(r, \xi) = \sum_{\mu=1}^{\infty} \sum_{|M|=\mu} a^{(s)}_M(r) V^{(s)}_M(\xi). $$

Since each $\tilde{u}_i(r, \xi)$, $1 \leqq i \leqq n$, is regular for $r > a$ (where the sphere $r = a$ contains D) the results of [5] and [6] show that the associated functions $F_i(\xi) \equiv F_i(r, \xi)$ defined in theorem 1.1 are analytic for $r > a$, $\xi \in \overline{\Delta(0; 1)}$. Equations (21) and (24) now show that

(26) $$ F(\xi) = \sum_{\mu=1}^{\infty} \sum_{|M|=\mu} a^{(s)}_M(r) \xi^M $$

defines a holomorphic function of r, ξ for $r > a$, $\xi \in \overline{\Delta(0; 1)}$. Hence from Cauchy's theorem for several complex variables (c.f. [9], p. 5) there exists a vector $\beta = (\beta_1, \cdots, \beta_n)$ where $\beta_i > 0$ for $1 \leqq i \leqq n$ and $||\beta|| < 1$ such that for r on compact subsets of (a, ∞)

(27) $$ |a^{(s)}_M(r)| \leqq C\beta^M $$

where C, B_i, $1 \leqq i \leqq n$ are positive constants which depend on the size of the compact subset of (a, ∞). By considering the generating function (4) as a power series in $\alpha_1, \cdots, \alpha_n$ it is seen that for fixed $\gamma < 1$ the series in equation (4) converges absolutely and uniformly for $\alpha \in \overline{\Delta(0, \gamma)}$ and ξ lying in some complex neighbourhood of $S(0; 1)$ whose size depends on γ. From the several variable analogue of the Weierstrass comparison theorem and equation (4) it is now possible to conclude that the series (25) converges uniformly for r on compact subsets of (a, ∞) and ξ contained in some complex neighbourhood of $S(0; 1)$ provided γ is chosen close enough to one, i.e. γ and hence the size of the complex neighbourhood of $S(0; 1)$ will depend on the particular compact subset of (a, ∞) that is chosen. A similar analysis shows that termwise differentiation with respect to $r, \xi_1, \cdots, \xi_n$ is permissible for $\xi \in S(0; 1)$ and r on compact subsets of (a, ∞) with the resulting series being uniformly convergent. Hence $\tilde{u}^+(t, \xi)$ defines a regular solution of $L^{(n)}_{\lambda,s}[\tilde{u}] = 0$ for $r > a$, $\xi \in S(0; 1)$. Termwise differentiation now gives

(28) $$ \operatorname{grad}_\xi[\tilde{u}(r, \xi) - \tilde{u}^+(r, \xi)] = 0. $$

Hence $\tilde{u}(r, \xi) - \tilde{u}^+(r, \xi)$ is a solution of equation 12) which depends only on r which implies that

(29) $$\tilde{u}(r, \xi)-\tilde{u}^{+}(r, \xi) = a_0^{(s)}(r)$$

with $a_0^{(s)}(r)$ being of the form defined in equation (14). Hence for $-2 < s < -1$

(30) $$\tilde{u}(r, \xi) = \sum a_M^{(s)}(r)V_M^{(s)}(\xi).$$

This proves the theorem for $-2 < s < -1$ and the complete theorem follows by induction on s.

2. Decomposition theorems

We now proceed to derive decomposition theorems for generalized metaharmonic functions in several independent variables which are analogous to those obtained in [2] for two independent variables. We begin with the case when $s > -1, s \neq 0$.

THEOREM 2.1. *Assume $s > -1$, $s \neq 0$. Let $\tilde{u}(r, \xi)$ be a regular solution of equation* (12) *in the exterior of a bounded domain D. Then $\tilde{u}(r, \xi)$ can be uniquely decomposed as*

(31) $$u(r, \xi) = U(r, \xi)+V(r, \xi)$$

where $U(r, \xi)$ is a regular solution of equation (12) *in the entire $n+1$ dimensional Euclidean space R^{n+1} and $V(r, \xi)$ is a regular solution of equation* 12) *in the exterior of D which satisfies the radiation condition*

(32) $$\lim_{r\to\infty} r^{\frac{1}{2}(n+s)}\left(\frac{\partial V}{\partial r}-i\lambda V\right) = 0$$

uniformly for $\xi \in S(0; 1)$.

PROOF. Let $0 < a < r$ and consider the function $\Omega(a, \zeta; r, \xi)$ defined by the formal series

(33) $$\Omega(a, \zeta; r, \xi) = (ar)^{-\frac{1}{2}(n+s-1)}\,\Sigma\,(h_M^s)^{-1}J_{|M|+(n+s-1)/2}(\lambda a)\cdot \\ \cdot H^{(1)}_{|M|+(n+s-1)/2}(\lambda r)U_M^{(s)}(\zeta)V_M^{(s)}(\xi).$$

By using theorem 1.1 and the asymptotic formulae ([10]. p. 4,8)

(34) $$\Gamma(\mu)\left(\frac{\lambda a}{2}\right)^{-\mu}J_\mu(\lambda a) = 1+o(1);\quad \mu\to\infty$$

(35) $$-\frac{\pi}{i}\,\frac{\left(\frac{\lambda r}{2}\right)^{\mu}}{\Gamma(\mu)}\,H_\mu^{(1)}(\lambda r) = 1+o(1);\quad \mu\to\infty$$

(which hold uniformly for a, r on compact subsets of the positive real axis) it is seen from the several variable analogue of the Weierstrass comparison theorem and Hartog's theorem that for $0 < a < r$ the series (33) converges uniformly to a holomorphic function of the $2n$ variables ζ, ξ for (ζ, ξ) contained in some complex neighbourhood of $S(0; 1) \times S(0; 1)$. (Cauchy's formula for several complex variables is applied to equation 6) in order to obtain bounds for $U_M^{(s)}(\zeta)$.) A similar analysis shows that for a, r, ζ, ξ as indicated above, termwise differentiation is permissible and $\Omega(a, \zeta; r, \xi)$ converges uniformly to *a* solution of equation 12) both as a function of (a, ζ) and of (r, ξ). Now let a be chosen such that D is contained in the sphere of radius a and consider the solution to equation 12) defined for $r > a$ by

$$V(r, \xi) = \frac{a^{n+s}\pi}{2i} \int_{S(0;1)} r^{n+s}(1 - ||\zeta||^2)^{(s-1)2} \left[\tilde{u}(a, \zeta) \frac{\partial \Omega(a, \zeta; r, \xi)}{\partial a} - \Omega(a, \zeta; r, \xi) \frac{\partial \tilde{u}(a, \zeta)}{\partial a} \right] d\zeta^n. \tag{36}$$

Using the relation ([10], p. 80)

$$\frac{2i}{\pi a} = J_\mu(\lambda a) \frac{dH_\mu^{(1)}(\lambda a)}{da} - H_\mu^{(1)}(\lambda a) \frac{dJ_\mu(\lambda a)}{da} \tag{37}$$

equations (7), (15), (33), and the uniform convergence of the series under consideration, we have

$$V(r, \xi) = r^{-\frac{1}{2}(n+s-1)} \sum d_M^{(s)} H_{|M|+(n+s-1)/2}^{(1)} (\lambda r) V_M^{(s)}(\xi) \tag{38}$$

where the series (38) is uniformly convergent for each fixed $r > a$, ξ contained in some complex neighbourhood of $S(0; 1)$. (Details of this last calculation for the case $n = 1$ are provided in [2]). By using the Lommel polynomials to express $H_{|M|+(n+s-1)/2}^{(1)}(\lambda r)$ in terms of $H_{(n+s-1)/2}^{(1)}(\lambda r)$ and $H_{(n+s-3)/2}^{(1)}(\lambda r)$, substituting this relationship into the series (38), and then rearranging terms, it can be seen that the solution $V(r, \xi)$ satisfies the radiation condition (32) uniformly for ξ contained in some complex neighbourhood of $S(0; 1)$. The details of this last operation are identical to the case when $n = 1$ and the reader is referred to [1] and [14] for more information. From equation (15) we therefore can write

$$\tilde{u}(r, \xi) = U(r, \xi) + V(r, \xi); \; r > a \tag{39}$$

where

$$U(r, \xi) = r^{-\frac{1}{2}(n+s-1)} \sum c_M^{(s)} J_{|M|+(n+s-1)/2}(\lambda r) V_M^{(s)}(\xi) \tag{40}$$

is uniformly convergent for each fixed $r > a$, ξ contained in some complex neighbourhood of $S(0; 1)$. From the results of [8] and [9] it is seen that $U(r, \xi)$ can be continued analytically into all of R^{n+1}, and this fact along with equation (39) shows that $U(r, \xi)$ is an everywhere regular solution of equation 12). (It is now clear that $V(r, \xi)$ is regular in the exterior of D and not: only for $r > a$.) The decomposition (31) is unique since if $U(r, \xi)$ is a generalized metaharmonic function which is regular in the entire plane and also satisfies the radiation condition (32), then the biorthogonality property (7) and the series representation (15) shows that $U(r, \xi)$ must be identically zero.

THEOREM 2.2. *Assume* $s < -1$, $s \neq -2, -3, -4, \cdots$. *Let* $\tilde{u}(r, \xi)$ *be a regular solution of equation* (12) *in the exterior of a bounded domain D. Then* $\tilde{u}(r, \xi)$ *can be uniquely decomposed as*

$$\tilde{u}(r, \xi) = U(r, \xi) + V(r, \xi) \tag{41}$$

where $U(r, \xi)$ *is a regular solution of equation* (12) *in the entire* $n+1$ *dimensional Euclidean space* R^{n+1} *and* $V(r, \xi)$ *is a regular solution of equation* (12) *in the exterior of D which satisfies the radiation condition*

$$\lim_{r \to \infty} r^{(n+s)/2} \left(\frac{\partial V}{\partial r} - i\lambda V \right) = 0 \tag{42}$$

uniformly for ξ *contained in some complex domain inclosing* $S(0; 1)$ *in its interior.*

PROOF. First let $-2 < s < -1$. Let the sphere of radius a contain D in its interior. Using theorem 1.4 $\tilde{u}(r, \xi)$ can be expressed as

$$\tilde{u}(r, \xi) = \sum a_M^{(s)}(r) V_M^{(s)}(\xi); \; r > a \tag{43}$$

where the coefficients are given by equation (14) and for each fixed $r > a$ the series (43) can be differentiated with respect to ξ_i, $1 \leqq i \leqq n$. Using this fact, theorem 1.3, and equation 18) we have that $r^{-1} \operatorname{grad}_\xi \tilde{u}(r, \xi)$ is a vector whose components $\tilde{u}_i(r, \xi)$ are regular solutions of $L_{\lambda, s+2}^{(n)}[\tilde{u}] = 0$ for $r > a$ and have the expansion

$$\tilde{u}_i(r, \xi) = \textstyle\sum_i a_M^{(s+2)}(r) V_M^{(s+2)}(\xi); \; r > a \tag{44}$$

where

$${}_i a_M^{(s+2)}(r) = {}_i a_{m_1, \cdots, m_n}^{(s+2)}(r) = r^{-1}(n+s-1)\, {}_i a_{m_1, \cdots, m_i+1, \cdots, m_n}^{(s)}(r). \tag{45}$$

By theorem 2.1 it is possible to conclude that for each i, $1 \leqq i \leqq n$, the series

$$r^{-\frac{1}{2}(n+s-1)} \textstyle\sum_i d_M^{(s+2)} H^{(1)}_{|M|+(n+s+1)/2}(\lambda r) V_M^{(s+2)}(\xi) \tag{46}$$

where

$${}_i d_M^{(s+2)} = {}_i d_{m_1, \cdots, m_n}^{(s+2)} = (n+s-1) d_{m_1, \cdots, m_i+1, \cdots m_n}^{(s)} \tag{47}$$

is uniformly convergent for each fixed $r > a$ and ξ contained in some complex neighbourhood of $S(0; 1)$. By using this fact and arguments similar to those used in theorem 1.4 it can be concluded that

$$V(r, \xi) = r^{-\frac{1}{2}(n+s-1)} \sum d_M^{(s)} H^{(1)}_{|M|+(n+s-1)/2}(\lambda r) V_M^{(s)}(\xi) \tag{48}$$

converges to a solution of $L^{(n)}_{\lambda,s}[\tilde{u}] = 0$ for $r > a$ and ξ contained in some complex neighbourhood of $S(0; 1)$. By arguements analogous to those used in theorem 2.1 it is seen that if $U(r, \xi)$ is defined by

$$(49) \qquad U(r, \xi) = r^{-\frac{1}{2}(n+s-1)} \sum c_M^{(s)} J_{|M|+(n+s-1)/2}(\lambda r)\, V_M^{(s)}(\xi)$$

then $V(r, \xi)$ and $U(r, \xi)$ have the properties ascribed to them in the theorem. We now come to the question of the uniqueness of the decomposition, and this is where it is necessary to require that the radiation condition (42) be valid in a complex domain instead of simply for the closed unit ball $S(0; 1)$. For suppose the decomposition (41) is not unique. Then there exists a nontrivial solution $U(r, \xi)$ of $L^{(n)}_{\lambda,s}[\tilde{u}] = 0$ which is regular in R^{n+1} and also satisfies the complex radiation condition (42). From Vitali's theorem for several complex variables ([12]), and the radiation condition (42), we have that for $1 \leqq i \leqq n$

$$(50) \qquad \lim_{r\to\infty} r^{\frac{1}{2}(n+s+2)} \left(\frac{1}{r} \frac{\partial^2 U}{\partial r \partial \xi_i} - i \frac{\lambda}{r} \frac{\partial U}{\partial \xi_i} \right) = 0$$

uniformly for ξ contained in some complex domain inclosing $S(0; 1)$ in its interior. But

$$(51) \qquad \frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial U}{\partial \xi_i} \right) = \frac{1}{r} \frac{\partial^2 U}{\partial r \partial \xi_i} - \frac{1}{r^2} \frac{\partial U}{\partial \xi_i}$$

or

$$(52) \qquad \lim_{r\to\infty} r^{\frac{1}{2}(n+s+2)} \left(\frac{\partial}{\partial r}\left[\frac{1}{r} \frac{\partial U}{\partial \xi_i} \right] + \frac{1}{r}\left[\frac{1}{r} \frac{\partial U}{\partial \xi_i} \right] - i \left[\frac{\lambda}{r} \frac{\partial U}{\partial \xi} \right] \right) = 0$$

uniformly for ξ contained in the above mentioned complex domain. By theorem 1.3 we have that $U_i = 1/r\, \partial U/\partial \xi_i$ is a regular solution of $L^{(n)}_{\lambda,s+2}[\tilde{u}] = 0$ in R^{n+1}. Hence $U_i(r, \xi)$ can be represented as ([8])

$$(53) \qquad U_i(r, \xi) = r^{-\frac{1}{2}(n+s+1)} \Sigma_i\, c_M^{(s+2)} J_{|M|+(n+s+1)/2}(\lambda r)\, V_M^{(s+2)}(\xi).$$

By using the biorthogonality property 7) and equation (52) it is seen that ${}_i c_M^{(s+2)} = 0$ for all M and hence $U_i(r, \xi)$ is identically zero for each i, $1 \leqq i \leqq n$. Hence $U(r, \xi)$ is a function of r alone i.e.

$$(54) \qquad U(r, \xi) = \text{const.}\; r^{-\frac{1}{2}(n+s-1)} J_{(n+s-1)|2}(\lambda r).$$

The radiation condition (42) and the asymptotic expansion (which can be differentiated with respect to r)

$$(55) \qquad J_\mu(\lambda r) = \sqrt{\frac{2}{\pi \lambda r}} \cos\left(\lambda r - \frac{\mu\pi}{2} - \frac{\pi}{4} \right) + 0\left(\frac{1}{r^{\frac{3}{2}}} \right); \qquad r \to \infty$$

now shows that $U(r, \xi)$ must be identically zero, which is the desired contradiction.

For $s < -2, s \neq -3, -4, \cdots$ the decomposition follows by induction on s. The uniqueness of the decomposition follows by repeated application of theorem 1.3 in the manner just completed and by observing that if a finite series of the form

$$r^{-\frac{1}{2}(n+s-1)} \sum_{\mu=0}^{n} \sum_{|M|=\mu} c_M^{(s)} J_{|M|+(n+s-1)/2}(\lambda r)\, V_M^{(s)}(\xi) \tag{56}$$

satisfies the radiation condition (42) and $s \neq 0, -1, -2, \cdots$ then $c_M^{(s)} = 0$ for each M. This can be seen from the asymptotic expression (55) and the discussion following equation (18).

EXAMPLE 2.1. For $s < -1$ the radiation condition (42) must hold for ξ lying in a complex domain containing $S(0; 1)$ in its interior and cannot be weakened to hold only for $\xi \in S(0; 1)$. For in this latter case

$$U(r, \xi) = R^{-\frac{1}{2}(n-2)} J_{(n-2)/2}(\lambda r) \tag{57}$$

where $R = r(\xi_1^2 + \cdots + \xi_n^2)^{\frac{1}{2}}$ is a solution of $L_{\lambda,s}^{(n)}[\tilde{u}] = 0$ which is regular in the entire plane. But equation (55) shows that $U(r, \xi)$ also satisfies the radiation condition if $s < -1$, i.e. the decomposition is no longer unique.

If $V(r, \xi)$ is a solution of equation 12) for $s < -1$, $s \neq -2, -3, -4 \cdots$ and satisfies the complex radiation condition (42), then by theorem 2.2 $V(r, \xi)$ has the representation

$$V(r, \xi) = r^{-\frac{1}{2}(n+s-1)} \sum d_M^{(s)} H_{|M|+(n+s-1)/2}^{(1)}(\lambda r)\, V_M^{(s)}(\xi). \tag{58}$$

By using the Lommel polynomials to express $H_{|M|+(n+s-1)2}^{(1)}(\lambda r)$ in terms of $H_{(n+s-1)/2}^{(1)}(\lambda r)$ and $H_{(n+s-3)/2}^{(1)}(\lambda r)$, substituting this relationship into the series (58), and then rearranging terms (c.f. [1], [14]) it is seen that $V(r, \xi)$ can be represented asymptotically as

$$V(r, \xi) = \frac{f(\xi)}{r^{\frac{1}{2}(n+s)}} e^{i\lambda r} + 0\left(\frac{1}{r^{\frac{1}{2}(n+s)+1}}\right); \qquad r \to \infty \tag{59}$$

where $V(r, \xi)$ is uniquely determined by its 'scattering amplitude' $f(\xi)$. Example 2.1. shows that if a complex radiation condition is not insisted upon, then $f(\xi)$ no longer uniquely determines $V(r, \xi)$, i.e. the inverse scattering problem (c.f. [8], [9]) is improperly posed.

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A Priori Estimates for Solutions of the Helmholtz Equation in Exterior Domains and Their Application

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1. INTRODUCTION

A priori estimates constitute a powerful tool in the study of boundary value problems for elliptic partial differential equations in bounded domains, and for nonlinear equations such estimates are the essential and usually the most difficult step in the analysis. For equations defined in unbounded domains, such estimates in general are unknown, and their unavailability is one of the main reasons that the theory of exterior boundary value problems is relatively undeveloped (see however [12]). It is the purpose of this paper to derive certain *a priori* estimates (in the form of uniform bounds for derivatives) for solutions of the two-dimensional Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u = 0, \tag{1}$$

defined in an exterior domain, and to demonstrate their use by obtaining uniform asymptotic expansions and a uniqueness theorem for such solutions.

In connection with the first of these applications, Herglotz has shown (c.f. [13]) that if $u(x, y)$ is an entire solution of (1) (i.e., $u(x, y) \in C^2(R^2)$), subject to a boundness condition as $r = +\sqrt{x^2 + y^2} \to \infty$, then certain "local means" of $u(x, y)$ satisfy an asymptotic relation as $r \to \infty$. Hartman ([9]) has shown that $u(x, y)$ itself satisfies the corresponding asymptotic relation in an averaged L^2-sense and obtained an analogue of this result for $u(x, y) \in C^2(F)$, where F is the exterior of a bounded domain D. It is further known ([11]) that if $u(x, y) \in C^2(F)$ and satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{1/2}\left(\frac{\partial u}{\partial r} - iu\right) = 0 \tag{2}$$

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uniformly for $\theta = \tan^{-1} y/x \in [0, 2\pi]$, then it is possible to construct a uniform asymptotic expansion for $u(x, y)$ as $r \to \infty$. Conditions on general $u(x, y) \in C^2(F)$ such that this construction is possible are unknown and are obtained in this paper for the first time. It should be noted that such expansion theorems are easily derived if conditions on the derivatives of $u(x, y)$ are resorted to; see for example the proof of Corollary 2 below.

Dating from the classical results of Helmholtz and Sommerfeld, many criteria have been given to assure that the exterior Dirichlet problem for the Helmholtz equation is well posed (c.f. [10]). By the use of the *a priori* estimates derived in this paper, new uniqueness conditions will be derived and shown to be equivalent to Sommerfeld's radiation condition.

Although for the sake of computational ease all results in this paper are done in two dimensional space, the methods used are easily adaptable to the general n dimensional case.

2. The Space W_T

Let $u(x, y)$ be a regular solution (i.e., $u(x, y)$ is of class C^2) of the Helmholtz equation in the exterior F of a bounded domain D, in particular for $r \geqslant R$ where R is some positive constant, and let (r, θ) be cylindrical coordinates defined as

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta, \end{aligned} \tag{3}$$

where $0 \leqslant \theta \leqslant 2\pi$. Then it is well known ([9]) that for fixed r, $u(x, y)$ has a Fourier expansion of the form

$$u(x, y) = \sum_{n=-\infty}^{\infty} a_n Z_n(r)\, e^{in\theta}, \tag{4}$$

where Z_n is a cylinder function of order n. From the analytic theory of partial differential equations it is known ([1], [5]) that for each fixed $r \geqslant R$, $u(x, y)$ is a holomorphic function of $\cos\theta$ and $\sin\theta$ (considered as independent variables) in some domain T in the space $\mathscr{C}^2$ of two complex variables containing $[-1, +1] \otimes [-1, +1]$ in its interior. In general, of course, the size of T will depend on the value of r, i.e., $T = T(r)$.

Definition 1. Let $u(x, y)$ be a regular solution of the Helmholtz equation for $r \geqslant R$ and assume that for fixed $r \geqslant R$, $u(x, y)$ is a holomorphic function of $(\cos\theta, \sin\theta)$ in some domain $T \subset \mathscr{C}^2$ containing $[-1, +1] \otimes [-1, +1]$ in its interior, *where T is independent of r*. If there exists a positive constant M

such that $|r^{1/2}u(x, y)| \leqslant M$ for $r \geqslant R$, $(\cos\theta, \sin\theta) \in T$, then $u(x, y)$ is said to belong to the space W_T.

Considering solution spaces similar to W_T is a common practice in the theory of potential scattering (c.f. [7]), where the singularities of $u(x, y)$ on the boundary of T correspond to the "bound states" of the nuclear model. Recently spaces of this type have been shown to play an important role in the study of certain classes of partial differential equations with singular coefficients ([1]–[4]).

3. *A Priori* Estimates and Their Application

THEOREM 1. *Every solution $u(x, y)$ of Helmholtz equation regular for $r \geqslant R$ can be uniquely decomposed as*

$$u = U + V,$$

where U is an everywhere regular solution (i.e., entire) and V is a regular solution for $r \geqslant R$ satisfying the Sommerfeld radiation condition.

PROOF. [10].

THEOREM 2. *Let $V(x, y)$ be a regular solution of the Helmholtz equation for $r \geqslant R$ satisfying the Sommerfeld radiation condition. Then $V(x, y) \in W_T$ for some domain T and for every integer $m = m_1 + m_2$ and domain T^* bounded by a contour interior to T, there exists a positive constant $M(m, T^*)$ such that*

$$\left| r^{1/2} \frac{\partial^m V}{\partial r^{m_1} \partial \theta^{m_2}} \right| \leqslant M(m, T^*)$$

for $R \leqslant r < \infty$, $(\cos\theta, \sin\theta) \in T^*$.

PROOF. $V(x, y)$ can be expressed as

$$V(x, y) = \sum_{n=-\infty}^{\infty} a_n \frac{H_n^{(1)}(r)}{H_n^{(1)}(R)} e^{in\theta}, \tag{5}$$

where

$$a_n \frac{H_n^{(1)}(r)}{H_n^{(1)}(R)} = \frac{1}{2\pi} \int_0^{2\pi} V(r\cos\theta, r\sin\theta)\, e^{-in\theta}\, d\theta \tag{6}$$

and $H_n^{(1)}$ denotes a Hankel function of the first kind ([1]). Integrating by parts in (6) gives

$$\frac{\partial^{m_2} V}{\partial \theta^{m_2}} = \sum_{n=-\infty}^{\infty} a_n (in\theta)^{m_2} \frac{H_n^{(1)}(r)}{H_n^{(1)}(R)} e^{in\theta}. \tag{7}$$

By using the formula ([14] p. 297)

$$H_n^{(1)}(r) = H_0^{(1)}(r)\, R_{n,0}(r) + H_1^{(1)}(r)\, R_{n-1,1}(r), \tag{8}$$

where R denotes Lommel's polynomial, and the fact that $V(R\cos\theta, R\sin\theta)$ is a holomorphic function of $(\cos\theta, \sin\theta)$ in some domain T containing $[-1,+1]\otimes[-1,+1]$ (which implies $\overline{\lim}_{|n|\to\infty} |a_n|^{1/|n|} < 1$; see Theorem 3), we can conclude from (7) that

$$\frac{\partial^{m_2} V}{\partial\theta^{m_2}} = H_0^{(1)}(r) \sum_{n=0}^{\infty} \frac{F_n(\theta)}{r^n} + H_1^{(1)}(r) \sum_{n=0}^{\infty} \frac{G_n(\theta)}{r^n}, \tag{9}$$

where the series in (9) converge absolutely uniformly for $r \geqslant R$, $(\cos\theta, \sin\theta) \in T^*$ (where T^* is as defined in the statement of the theorem). Here $F_n(\theta)$, $G_n(\theta)$ are holomorphic functions of $(\cos\theta, \sin\theta)$ in T and the series may be differentiated with respect to r as often as desired. For details the reader is referred to [1] and [11]. From this result the theorem follows readily by using the asymptotic relation ([14] p. 196)

$$H_n^{(1)}(r) = \sqrt{\frac{2}{\pi r}}\, e^{i(r-\lambda\pi/2-\pi/4)} + 0\left(\frac{1}{r^{3/2}}\right); \qquad r\to\infty, \tag{10}$$

which holds uniformly for λ contained in a compact subset of $[0, \infty]$.

Attention is now turned towards entire solutions $U(x, y)$ and as a first step in the analysis a bound on $|r^{1/2} J_\lambda(r)|$ which is independent of r will be obtained, where J_λ denotes a Bessel function of order λ. Although from the asymptotic relation ([14] p. 199, 225)

$$J_\lambda(r) = \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\lambda\pi}{2} - \frac{\pi}{4}\right) + 0\left(\frac{1}{r^{3/2}}\right); \qquad r\to\infty \tag{11}$$

(which holds uniformly for λ contained in a compact subset of $[0, \infty]$) and

$$J_\lambda(r) = \left(\frac{r}{2}\right)^{\lambda} \frac{1}{\Gamma(\lambda+1)} \left[1 + 0\left(\frac{1}{\lambda}\right)\right]; \qquad \lambda\to\infty \tag{12}$$

(which holds uniformly for r contained in a compact subset of $[0, \infty]$), it is clear that $|r^{1/2} J_\lambda(r)|$ is uniformly bounded if one of the variables r, λ is held fixed, this is not the case if both variables are allowed to become large, as can be seen by noting ([14], p. 260) that $\lambda^{1/3} J_\lambda(\lambda)$ is a (bounded) increasing function of λ. (It is of interest to contrast this observation with the result ([9])

$$\frac{1}{r}\int_0^r t J_\lambda^2(t)\, dt \leqslant \text{constant} \tag{13}$$

for all $r, \lambda \geqslant 0$.)

LEMMA 1. $|r^{1/2}J_\lambda(r)| \leqslant M(1 + \lambda^{1/6})$ *for* $r, \lambda \geqslant 0$ *where* M *is a positive constant independent of* r *and* λ.

PROOF. From the uniform asymptotic representations of Bessel functions as developed by Cherry and Langer (c.f. [6]), we have

$$(\lambda x)^{1/2} J_\lambda(\lambda x) = \lambda^{1/6}\sqrt{\frac{2}{\phi}}\, Ai(-\lambda^{2/3}\phi)[1 + 0(\lambda^{-1})] \tag{14}$$

uniformly in x, $0 < x < \infty$, as $\lambda \to \infty$, $\operatorname{Re} \lambda \geqslant 0$ (except that the error term needs some slight modification near the zeros of $Ai(-\lambda^{2/3}\phi)$). Here the real-valued function $\phi(x)$ is defined by

$$\begin{aligned} \frac{2}{3}[-\phi(x)]^{3/2} &= -(1 - x^2)^{1/2} + \log\frac{1 + (1 - x^2)^{1/2}}{x}; \qquad 0 < x \leqslant 1 \\ &= (x^2 - 1)^{1/2} - \cos^{-1} x^{-1}; \qquad 1 \leqslant x < \infty, \end{aligned} \tag{15}$$

and $Ai(z)$ is the Airy function defined by

$$Ai(z) = \tfrac{1}{3}z^{1/2}[J_{-1/3}(\tfrac{2}{3}z^{3/2}) + J_{1/3}(\tfrac{2}{3}z^{3/2})]. \tag{16}$$

From Eq. (15) and (16) it is seen that $|\sqrt{2/\sigma}\, Ai(-\lambda^{2/3}\phi)|$ is uniformly bounded for $x \geqslant x_0 > 0$, $\lambda \geqslant \lambda_0 > 0$, and hence Eq. (14) implies there exists a constant M, independent of r and λ such that

$$|(\lambda x)^{1/2} J_\lambda(\lambda x)| \leqslant M_1\lambda^{1/6} \tag{17}$$

for $x \geqslant x_0 > 0$, $\lambda \geqslant \lambda_0 > 0$, e.g.

$$|r^{1/2}J_\lambda(r)| \leqslant M_1\lambda^{1/6} \tag{18}$$

for $r \geqslant \lambda_0 x_0$, $\lambda \geqslant \lambda_0$. From the discussion preceding lemma one and the fact that $r^{1/2}J_\lambda(r)$ is a continuous function of r and λ in the quarter plane $r \geqslant 0$, $\lambda \geqslant 0$, it can be concluded that there exists a constant M_2 such that

$$|r^{1/2}J_\lambda(r)| \leqslant M_2 \tag{19}$$

for $0 \leqslant \lambda \leqslant \lambda_0$, $r \geqslant 0$, and a similar inequality exists for $0 \leqslant r \leqslant x_0\lambda_0$, $\lambda \geqslant 0$. The conclusion of the lemma now follows.

THEOREM 3. *Let* $U(x, y)$ *be an entire solution of the Helmholtz equation such that* $U(x, y) \in W_T$ *for some domain* T. *Then for every integer* $m = m_1 + m_2$ *and domain* T^* *bounded by a contour interior to* T, *there exists a positive constant* $M(m, T^*)$ *such that*

$$\left| r^{1/2}\frac{\partial^m U}{\partial r^{m_1}\partial\theta^{m_2}} \right| \leqslant M(m, T^*)$$

for $0 \leqslant r < \infty$, $(\cos\theta, \sin\theta) \in T^*$.

PROOF. $U(x, y)$ can be expressed ([9]) as

$$r^{1/2}U(x, y) = \sum_{n=-\infty}^{\infty} a_n r^{1/2} J_n(r)\, e^{in\theta}, \tag{20}$$

where

$$a_n r^{1/2} J_n(r) = \frac{1}{2\pi} \int_0^{2\pi} r^{1/2} U(r\cos\theta, r\sin\theta)\, e^{-in\theta}\, d\theta. \tag{21}$$

Now let $\{r_i\}$ be a sequence of r values such that $\cos(r_i - \pi/4) = 1$, $i = 1, 2, 3,\ldots$. Since $U(x, y) \in W_T$, by Vitali's theorem for several complex variables ([8]) there exists a subsequence $\{r_j\}$, $j = 1, 2, 3,\ldots$ such that

$$\lim_{j\to\infty} r_j^{1/2} U(r_j \cos\theta, r_j \sin\theta) = f(\cos\theta, \sin\theta) \tag{22}$$

where $f(\cos\theta, \sin\theta)$ is a holomorphic function of $(\cos\theta, \sin\theta)$ in T. Expanding in a Fourier series gives

$$f(\cos\theta, \sin\theta) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}, \tag{23}$$

where $\overline{\lim}_{|n|\to\infty} | b_n |^{1/|n|} < 1$. (This last inequality follows from the fact that $h(r, \theta) = \sum_{n=-\infty}^{\infty} b_n r^{|n|} e^{in\theta}$ is a harmonic function for $r < 1$ which has no singularities on the circle $r = 1$. Hence the series representation will actually converge uniformly and absolutely for $r \leqslant r_0$ where r_0 in some constant such that $r_0 > 1$.) By letting r run through the sequence $\{r_j\}$ in (21) and using Lebesgue's dominated convergence theorem, it is seen by use of formula (11) (and the relation $J_{-n}(r) = (-1)^n J_n(r)$ for n an integer) that $| a_{2m} | = | b_{2m} |$, $b_{2m+1} = 0$, $m = 0, 1, 2,\ldots$ and hence

$$\overline{\lim_{|m|\to\infty}}\, | a_{2m} |^{1/2|m|} < 1. \tag{24}$$

In a similar fashion by choosing $\{r_i\}$ such that $\cos(r_i - \pi/4) = 0$, $i = 1, 2,\ldots$, it is seen that

$$\overline{\lim_{|m|\to\infty}}\, | a_{2m+1} |^{1/2|m|+1} < 1. \tag{25}$$

Equations (24) and (25) together imply that

$$\overline{\lim_{|n|\to\infty}}\, | a_n |^{1/|n|} < 1. \tag{26}$$

Now by use of formula (21) it is seen that

$$r^{1/2} \frac{\partial^{m_1} U(r\cos\theta, r\sin\theta)}{\partial r^{m_1}} = \sum_{n=-\infty}^{\infty} a_n r^{1/2} J_n^{(m_1)}(r)\, e^{in\theta}, \tag{27}$$

where

$$J_n^{(m_1)}(r) = \frac{\partial^{m_1} J_n(r)}{\partial r^{m_1}}.$$

Repeated use of the relations ([14] pp. 45, 15).

$$\begin{aligned} J_{n-1}(r) - J_{n+1}(r) &= 2J_n^{(1)}(r) \\ J_{-n}(r) &= (-1)^n J_n(r) \end{aligned} \tag{28}$$

in conjunction with lemma one yields

$$|r^{1/2} J_n^{(m_1)}(r)| \leqslant M(1 + (|n| + m_1)^{1/6}) \tag{29}$$

for $r \geqslant 0$, n a positive or negative integer, and hence (26) and (29) imply that the series (27) is absolutely uniformly convergent and

$$\left| r^{1/2} \frac{\partial^{m_1} U(r\cos\theta, r\sin\theta)}{\partial r^{m_1}} \right| \leqslant M(m_1, T^*) \tag{30}$$

for $0 \leqslant r < \infty$, $(\cos\theta, \sin\theta) \in T_*$, where M is a positive constant independent of r and θ, and T_* is a domain bounded by a contour interior to T. From Cauchy's theorem for several complex variables ([8]) it is seen that

$$r^{1/2} \frac{\partial^{m_1+m_2+m_3} U(r\cos\theta, r\sin\theta)}{\partial r^{m_1} \partial(\cos\theta)^{m_2} \, \partial(\sin\theta)^{m_3}} = \frac{m_2! \, m_3!}{(2\pi i)^{m_2+m_3}} \int_L \frac{r^{1/2} \, \partial^{m_1} U/\partial r^{m_1} \, d(\cos\phi) \, d(\sin\phi)}{(\cos\phi - \cos\theta)^{m_2+1} (\sin\phi - \sin\theta)^{m_3+1}}, \tag{31}$$

where $L = L_1 \otimes L_2$ is a product of regular contours L_i $(i = 1, 2)$ in the $\cos\theta$, $\sin\theta$ plane respectively such that L is contained in T_*. Hence for a domain T^* bounded by a contour interior to T_*, (31) implies that

$$\left| r^{1/2} \frac{\partial^{m_1+m_2+m_3} U}{\partial r^{m_1} \partial(\cos\theta)^{m_2} \, \partial(\sin\theta)^{m_3}} \right| \leqslant M(m_1, m_2, m_3, T^*) \tag{32}$$

for $0 \leqslant r < \infty$, $(\cos\theta, \sin\theta) \in T^*$, where M is a positive constant independent of r and θ (L is chosen to lie in $T_* - T^*$). By repeatedly using the relation

$$\frac{\partial U}{\partial\theta} = \frac{\partial U}{\partial(\sin\theta)} \cos\theta - \frac{\partial U}{\partial(\cos\theta)} \sin\theta, \tag{33}$$

the statement of the theorem now follows.

COROLLARY 1. *Let $u(x, y) \in W_T$ be a regular solution of the Helmholtz equation for $r \geqslant R$. Then there exist functions $f_i(\theta)$ $(i = 1, 2, 3)$ analytic on $[0, 2\pi]$ such that as $r \to \infty$*

$$\begin{aligned} r^{1/2}u(r\cos\theta, r\sin\theta) &= e^{ir}f_1(\theta) + \cos(r - \pi/4) f_2(\theta) \\ &\quad + \sin(r - \pi/4) f_3(\theta) + o(1) \end{aligned} \tag{34}$$

$$\begin{aligned} r^{1/2}\frac{\partial u(r\cos\theta, r\sin\theta)}{\partial r} &= ie^{ir}f_1(\theta) - \sin(r - \pi/4) f_2(\theta) \\ &\quad + \cos(r - \pi/4) f_3(\theta) + o(1) \end{aligned} \tag{35}$$

uniformly for $\theta \in [0, 2\pi]$. If $u(x, y)$ is entire $f_1(\theta) \equiv 0$ and if $u(x, y)$ satisfies the Sommerfeld radiation condition $f_2(\theta) \equiv f_3(\theta) \equiv 0$.

PROOF. In view of theorems one and two and the results of [11], it suffices to consider only entire solutions $U(x, y)$. By theorem three $| r^{1/2}(\partial^2 U/\partial\theta^2)|$ is uniformly bounded for $\theta \in [0, 2\pi]$ and integration by parts in formula (21) shows that $| a_n r^{1/2} J_n(r)| \leqslant (M/n^2)$ where M is a positive constant independent of r and n. Letting $r \to \infty$ and using formula (11) shows that $| a_n | \leqslant (M/n^2)$ and hence

$$\begin{aligned} \lim_{r\to\infty} \Bigg| r^{1/2}U(x, y) &- \cos(r - \pi/4) \sum_{n=-\infty}^{\infty} (-1)^n a_{2n} e^{i2n\theta} \\ &- \sin(r - \pi/4) \sum_{n=-\infty}^{\infty} (-1)^{n+1} a_{2n+1} e^{i(2n+1)\theta} \Bigg| \\ &\leqslant \lim_{r\to\infty} \sum_{n=-\infty}^{\infty} \left| a_n r^{1/2} J_n(r) - a_n \cos\left(r - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right| = 0, \end{aligned} \tag{36}$$

since each term in the series is uniformly bounded by $(2M/n^2)$ and tends to zero as $r \to \infty$. Formula (34) now follows with

$$\begin{aligned} f_2(\theta) &= \sum_{n=-\infty}^{\infty} (-1)^n a_{2n} e^{i2n\theta} \\ f_3(\theta) &= \sum_{n=-\infty}^{\infty} (-1)^{n+1} a_{2n+1} e^{i(2n+1)\theta}. \end{aligned} \tag{37}$$

By formula (26) it is seen that $f_2(\theta), f_3(\theta)$ are analytic functions of θ for $\theta \in [0, 2\pi]$. Formula (35) is derived in the same manner by considering $(\partial U/\partial r)$ instead of U.

Corollary 2. *Let $u(x, y)$ be a regular solution of the Helmholtz equation for $r \geqslant R$. Then the following conditions are equivalent*:

(1) *$u(x, y) \in W_T$ for some domain T and* $\lim_{r\to\infty} r^{1/2}(\partial u/\partial r - iu) = 0$ *pointwise for $\theta_0 \leqslant \theta \leqslant \theta_1$, where θ_0, θ_1 are constants such that* $0 \leqslant \theta_0 < \theta_1 \leqslant 2\pi$.

(2) $\lim_{r\to\infty} r^{1/2}(\partial u/\partial r - iu) = 0$ *uniformly for* $0 \leqslant \theta \leqslant 2\pi$.

Proof. The fact that (2) implies (1) follows immediately from Theorem 2. Conversely Corollary 1 and the identity theorem for analytic functions of a single complex variable shows that (1) implies (2).

From Corollary 2 and the Sommerfeld-Rellich-Vekua uniqueness theorem ([5]) it can be concluded that if $u(x, y)$ is a solution of the Helmholtz equation in the exterior F of a bounded domain D such that $u(x, y) \in C^2(\bar{F})$, satisfies condition (1) of Corollary 2, and vanishes on the boundary of D, then $u(x, y) \equiv 0$ in F.

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Cauchy's Problem for Almost Linear Elliptic Equations in Two Independent Variables

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I. INTRODUCTION

It is well-known (cf. [4] p. 108) that due to its unstable nature the Cauchy problem for elliptic partial differential equations is an improperly posed problem in the sense of Hadamard. Nevertheless, situations arise in mathematical physics for which it becomes necessary to solve such a problem, in particular when it is desired to construct an inverse solution to what is essentially a free boundary problem ([3]). In such cases the differential equation and prescribed data are often analytic and, hence, permit an application of the Cauchy-Kowalewski theorem. This approach is not very satisfactory, however, since what is actually required is a method that can be adapted for numerical integration. For the case of quasilinear equations in two independent variables (x, y), Garabedian has introduced a method which overcomes this difficulty by using characteristic coordinates to reduce the differential equation to a canonical system and then solving a one parameter family of related (stable) hyperbolic Cauchy problems ([3], [4] p. 623–633). In this paper we present a new method for solving the Cauchy problem for the case of almost-linear elliptic equations in a manner that is suitable for numerical computation. This method is based on the use of conjugate coordinates and reduces the Cauchy problem to finding a fixed point of a contraction mapping.

II. CONJUGATE COORDINATES AND THE CAUCHY PROBLEM

We seek a solution of the almost linear elliptic partial differential equation (written in normal form)

$$u_{xx} + u_{yy} = g(x, y, u, u_x, u_y) \tag{1}$$

which satisfies the Cauchy data

$$
\begin{aligned}
u(x, y) &= \Phi(x + iy), \qquad x + iy \in L, \\
\frac{\partial u(x, y)}{\partial n} &= \Omega(x + iy), \qquad x + iy \in L,
\end{aligned} \tag{2}
$$

where L is a given analytic arc, n is the unit outward normal to L and g, Φ, and Ω are assumed to have certain regularity properties to be described shortly. By the use of a conformal transformation, we can assume without loss of generality that the arc L is in fact a segment of the x axis containing the origin (i.e., $y = 0$ in Eq. (2)). By introducing *conjugate coordinates* ([5], [6])

$$
\begin{aligned}
z &= x + iy, \\
z^* &= x - iy,
\end{aligned} \tag{3}
$$

Eq. (1) becomes an equation of hyperbolic form:

$$
U_{zz^*} = f(z, z^*, U, U_z, U_{z^*}), \tag{4}
$$

where

$$
u\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right) = U(z, z^*),
$$

and the Cauchy data is transformed into

$$
\begin{aligned}
U(z, z^*) &= \Phi(z) \qquad \text{on } z = z^*, \\
\frac{\partial U(z, z^*)}{\partial z} - \frac{\partial U(z, z^*)}{\partial z^*} &= -i\Omega(z) \qquad \text{on } z = z^*.
\end{aligned} \tag{5}
$$

We assume at this point that as a function of its first two arguments, $f(z, z^*, \xi_1, \xi_2, \xi_3)$ is holomorphic in a bicylinder $G \times G^*$, where $G^* = \{z \mid z^* \in G\}$, and G is simply connected, and as a function of its last three variables it is holomorphic in a sufficiently large ball about the origin. We further assume that G contains the origin and is symmetric with respect to conjugation, i.e., $G = G^*$, and that $\Phi(z)$ and $\Omega(z)$ are holomorphic for all $z \in G$. The domain G described above is known as a *fundamental domain* ([5], [6]).

Now suppose $U(z, z^*)$ is a solution of Eq. (4) which is bounded and holomorphic in $G \times G^*$ and define a new function $s(z, z^*) = U_{zz^*}(z, z^*)$. It then follows that

$$U(z, z^*) = \int_0^z \int_0^{z^*} s(\xi, \xi^*)\, d\xi^*\, d\xi + \int_0^z \varphi(\xi)\, d\xi + \int_0^{z^*} \psi(\xi^*)\, d\xi^* + U(0, 0), \tag{6}$$

$$U_z(z, z^*) = \int_0^{z^*} s(z, \xi^*)\, d\xi^* + \varphi(z), \tag{7}$$

$$U_{z^*}(z, z^*) = \int_0^z s(\xi, z^*)\, d\xi + \psi(z^*), \tag{8}$$

where $\varphi(z) = U_z(z, 0)$ and $\psi(z^*) = U_{z^*}(0, z^*)$. Note that $s(z, z^*)$ must satisfy the equation

$$s(z, z^*) = f\Big[z, z^*, \int_0^z \int_0^{z^*} s(\xi, \xi^*)\, d\xi^*\, d\xi + \int_0^z \varphi(\xi)\, d\xi + \int_0^{z^*} \psi(\xi^*)\, d\xi^*$$
$$+ U(0, 0), \int_0^{z^*} s(z, \xi^*)\, d\xi^* + \varphi(z), \int_0^z s(z, \xi^*)\, d\xi + \psi(z^*)\Big] \tag{9}$$

and, conversely, if $s(z, z^*)$ satisfies (9) then a solution of (4) is given by (6). The initial conditions (5) become

$$\int_0^z \int_0^z s(\xi, \xi^*)\, d\xi^*\, d\xi + \int_0^z \varphi(\xi)\, d\xi + \int_0^z \psi(\xi^*)\, d\xi^* + U(0, 0) = \Phi(z) \tag{10}$$

or, differentiating in the z plane,

$$\int_0^z s(z, \xi^*)\, d\xi^* + \int_0^z s(\xi, z)\, d\xi + \varphi(z) + \psi(z) = \Phi'(z) \tag{11}$$

and

$$\int_0^z s(z, \xi^*)\, d\xi^* + \varphi(z) - \int_0^z s(\xi, z)\, d\xi - \psi(z) = -i\Omega(z). \tag{12}$$

Equations (11) and (12) now yield the following expressions for $\varphi(z)$ and $\psi(z)$ in terms of the function $s(z, z^*)$:

$$\varphi(z) = \tfrac{1}{2}[\Phi'(z) - i\Omega(z)] - \int_0^z s(z, \xi^*)\, d\xi^*, \tag{13}$$

$$\psi(z) = \tfrac{1}{2}[\Phi'(z) + i\Omega(z)] - \int_0^z s(\xi, z)\, d\xi. \tag{14}$$

Hence, we can express the functions $\varphi(z)$ and $\psi(z)$ as operators on the function $s(z, z^*)$. In particular, if we define the operators B_i, $i = 1, 2, 3,$

by the right sides of (6), (7), and (8), respectively, where $\varphi(z)$ and $\psi(z)$ are determined from Eq. (13) and (14) (note that $U(0, 0) = \Phi(0)$), then $s(z, z^*)$ satisfies the equation

$$s(z, z^*) = f(z, z^*, B_1[s(z, z^*)], B_2[s(z, z^*)], B_3[s(z, z^*)]). \tag{15}$$

III. The Solution of Cauchy's Problem

The approach to be used in this section is patterned after the ideas of [1], and [2] (see also [5] p. 154–164). Consider the class $HB(\Delta\rho, \Delta\rho^*)$ of functions of two complex variables which are holomorphic and bounded in $\Delta\rho \times \Delta\rho^*$, where $\Delta\rho = \{z \mid |z| < \rho\}$, $\Delta\rho^* = \{z \mid z^* \in \Delta\rho\}$. If a norm is defined on $HB(\Delta\rho, \Delta\rho^*)$ by

$$\| s \|_\lambda = \sup\{e^{-\lambda(|z|+|z^*|)} \mid s(z, z^*)|\}, \tag{16}$$

where $(z, z^*) \in \Delta\rho \times \Delta\rho^*$ and $\lambda > 0$ is fixed, $HB(\Delta\rho, \Delta\rho^*)$ becomes a Banach space which we denote $A\rho$. We shall now show that the operator T defined by

$$Ts(z, z^*) = f(z, z^*, B_1[s(z, z^*)], B_2[s(z, z^*)], B_3[s(z, z^*)]) \tag{17}$$

maps a closed ball of the Banach space $A\rho$ into itself, and is a contraction mapping, thus providing a constructive method for obtaining the unique solution to our Cauchy problem.

By hypothesis, f is holomorphic in a compact subset of the space of five complex variables and, hence, from Schwarz's lemma for functions of several complex variables ([5] p. 38, 159), a Lipschitz condition holds there with respect to the last three arguments, i.e.,

$$\begin{aligned} &| f(z, z^*, \xi_1, \xi_2, \xi_3) - f(z, z^*, \xi_1^0, \xi_2^0, \xi_3^0)| \\ &\qquad \leqslant C_0\{| \xi_1 - \xi_1^0 | + | \xi_2 - \xi_2^0 | + | \xi_3 - \xi_3^0 |\}, \end{aligned} \tag{18}$$

where C_0 is a positive constant. Hence, for $s_1, s_2 \in A\rho$ and ρ sufficiently small,

$$\| Ts_1 - Ts_2 \|_\lambda \leqslant C_0\{\| B_1s_1 - B_1s_2 \|_\lambda + \| B_2s_1 - B_2s_2 \|_\lambda + \| B_3s_1 - B_3s_3 \|_\lambda\}. \tag{19}$$

From estimates of the form

$$\left| \int_0^z s(\xi, z^*)\, d\xi \right| \leqslant \int_0^{|z|} \| s \|_\lambda\, e^{\lambda|\xi|+\lambda|z^*|}\, | d\xi | \leqslant \frac{1}{\lambda} e^{\lambda|z|+\lambda|z^*|} \| s \|_\lambda , \tag{20}$$

i.e.,

$$\left\| \int_0^z s(\xi, z^*)\, d\xi \right\|_\lambda \leqslant \frac{\| s \|_\lambda}{\lambda} \tag{21}$$

(where we have assumed $s(z, z^*)$ is regular in the polydisc $\Delta\rho \times \Delta\rho^*$, so that the curvilinear path of integration may be replaced by a straight line-segment) it can be seen that

$$\| B_i s_1 - B_i s_2 \|_\lambda \leqslant \frac{N_i}{\lambda} \| s_1 - s_2 \|_\lambda , \qquad i = 1, 2, 3, \tag{22}$$

where the N_i are positive constants independent of λ and $\lambda > 0$. Hence,

$$\| Ts_1 - Ts_2 \|_\lambda \leqslant \frac{M}{\lambda} \| s_1 - s_2 \|_\lambda , \tag{23}$$

where M is a positive constant independent of λ. Inequality (23) implies that

$$\| Ts \|_\lambda \leqslant \frac{M}{\lambda} \| s \|_\lambda + \| T_0 \|_\lambda < \frac{M}{\lambda} \| s \|_\lambda + M_0 , \tag{24}$$

where M_0 is a positive constant. Therefore, for $\| s \|_\lambda < M_0$ and λ sufficiently large, $\| Ts \|_\lambda < M_0$, i.e., T takes a closed ball in $A\rho$ into itself. Equation (23) also implies that, for λ sufficiently large,

$$\| Ts_1 - Ts_2 \|_\lambda < \| s_1 - s_2 \|_\lambda , \tag{25}$$

i.e., T is a contraction mapping. The existence and uniqueness of a solution to the equation $Ts = s$ in $A\rho$ is now immediate. We have proved the following:

THEOREM 1. *Let G be a fundamental domain for the elliptic equation* (1) *and let $f(z, z^*, \xi_1, \xi_2, \xi_3)$ be holomorphic in $G \times G^* \times B^{(3)}$, where $B^{(3)}$ is a sufficiently large ball about the origin. Assume, further, that $G = G^*$ and the functions $\Phi(z)$, $\Omega(z)$ are holomorphic in G. Then, for ρ sufficiently small, Eq.* (17), (13), (14), *and* (6) *provide a constructive method for obtaining a unique solution of Eq.* (1) *in $|z| \leqslant \rho$, satisfying the Cauchy data* (2).

It is important to note here that the unstable dependence of the solution of the elliptic equation (1) on the (real) Cauchy data (2) appears exclusively in the step where this data is extended to complex values of the independent variable x. When this can be done in an elementary way, for example, by direct substitution via the transformation (3), no instabilities will occur when one uses the contraction mapping operator T to obtain approximations to the desired solution.

For the case where Eq. (1) is linear, Henrici ([5], [6]) has used conjugate coordinates and the Riemann function to obtain a solution of Cauchy's problem. Hence, Theorem 1 can be considered as an extension of Henrici's results to the case of almost linear elliptic equations.

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Cauchy's Problem for Almost Linear Elliptic Equations in Two Independent Variables, II

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1. INTRODUCTION

Due to the unstable dependence of the solution on the initial data, Cauchy's problem for elliptic equations is well known to be improperly posed in the sense of Hadamard (cf. [2, p. 108]). Such problems arise, however, in the study of free boundary problems (cf. [2, p. 622]), in mathematical physics and hence attention has been focused on methods of solution that are suitable for analytic approximation and numerical computation. We note that the Cauchy–Kowalewski theorem is no more suitable here than it is for hyperbolic equations. For second-order equations in two independent variables, the approximation problem is in satisfactory condition [1; 2, p. 623–631; 4]. However, the situation for higher order equations or equations in more than two independent variables is not so well off from a computational viewpoint, since the only available method is to convert an elliptic problem in only $n + 1$ variables into a hyperbolic problem in no less than $2n + 1$ variables [2, p. 614-621]. In this note, we show how previous results obtained by the author in [1] for second-order almost linear (or semilinear) equations can be adapted to give approximation techniques for a quite general class of higher order equations.

In [1], the equation

$$\Delta u = f(x, y, u, u_x, u_y) \tag{1.1}$$

was considered, with Cauchy data prescribed on a given analytic arc L. Without loss of generality, we assumed L was the x axis. In *conjugate coordinates* [3, 5]

$$z = x + iy,$$

and (1.2)

$$z^* = x - iy,$$

Eq. (1.1) became

$$U_{zz^*} = F(z, z^*, U, U_z, U_{z^*}), \tag{1.3}$$

with initial data prescribed on the plane $z = z^*$. Under the assumption that $F(z, z^*, \xi_1, \xi_2, \xi_3)$ was an analytic function of its five variables, it was shown in [1] that $S(z, z^*) = U_{zz^*}$ is the (unique) fixed point of a contraction mapping in an appropriate Banach space of analytic functions, and that U could be easily obtained from S by integration and a knowledge of the Cauchy data. We now show how the Cauchy problem

$$\begin{gathered}\Delta^n u = f\left(x, y, u, u_x, u_y, \ldots, \frac{\partial^{l+m}\Delta^j u}{\partial x^l\, \partial y^m}, \ldots, \frac{\partial \Delta^{n-1}u}{\partial x}, \frac{\partial \Delta^{n-1}u}{\partial y}\right), \\ l = 0, 1, \ldots, n; \qquad m = 0, 1, \ldots, n; \qquad l + m + 2j \leqslant 2n - 1,\end{gathered} \tag{1.4}$$

$$u(x, 0) = \varphi_0(x), \quad \frac{\partial^k u(x, 0)}{\partial y^k} = \varphi_k(x); \qquad k = 1, 2, \ldots, 2n - 1, \tag{1.5}$$

can be reduced to a Cauchy problem for

$$\Delta u = f(x, y, A_1(u), \ldots, A_N(u)), \tag{1.6}$$

where A_i, $i = 1, 2, \ldots, N$, are operators satisfying a certain type of Lipschitz condition in an appropriate Banach space. This latter problem will then be solved using techniques similar to those used in solving Cauchy's problem for Eq. (1.1). Note that again there is no loss of generality in assuming that the Cauchy data is prescribed along the x axis.

II. Reduction and Solution of Higher Order Cauchy Problems

In complex form, the Cauchy problem (1.4), (1.5) becomes

$$\frac{\partial^{2n} U}{\partial z^n\, \partial z^{*n}} = F\left(z, z^*, U, \ldots, \frac{\partial^{p+q} U}{\partial z^p\, \partial z^{*q}}, \ldots, \frac{\partial^{2n-1} U}{\partial z^{n-1}\, \partial z^{*n}}, \frac{\partial^{2n-1} U}{\partial z^n\, \partial z^{*n-1}}\right), \tag{2.1}$$

$$\begin{gathered}U(z, z^*) = \varphi_0(z); \qquad z = z^*, \\ i^k \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right)^k U(z, z^*) = \varphi_k(z); \qquad z = z^*, \quad k = 1, \ldots, 2n - 1,\end{gathered} \tag{2.2}$$

where $U(z, z^*) = u((z + z^*/2, (z - z^*)/2i)$, $p = 0, 1, \ldots, n$; $q = 0, 1, \ldots, n$; $p + q \leqslant 2n - 1$. We assume that, as a function of its first two arguments, $F(z, z^*, \xi_1, \ldots, \xi_N)$ is holomorphic in a bicylinder $\mathfrak{S} \times \mathfrak{S}^*$, where $\mathfrak{S}^* = \{z \mid z^* \in \mathfrak{S}\}$, and as a function of its last N variables, it is holomorphic in a sufficiently large ball about the origin. We further assume that $\mathfrak{S}$ is simply connected, contains the origin, is symmetric with respect to conjugation, i.e., $\mathfrak{S} = \mathfrak{S}^*$, and that $\varphi_k(z)$, $k = 0, 1, \ldots, 2n - 1$, are holomorphic in $\mathfrak{S}$.

We would like to emphasize that it is necessary for us to restrict ourselves to equations of the form (1.4) in order that there do not appear any terms of the form $\partial^{n+1}U/\partial z^{n+1}$, $\partial^{n+1}U/\partial z^{*n+1}$, $\partial^{2n-1}U/\partial z^{n-2}\,\partial z^{*n+1}$ etc., when Eq. (1.4) is written in terms of conjugate coordinates. For example, our analysis is not applicable to equations such as

$$\Delta^n u = f\left(x, y, u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^{2n-1} u}{\partial x^{2n-1}}\right). \tag{2.3}$$

We note that the same type of restriction was also encountered by I. N. Vekua [5, p. 174-228] in his study of the analytic theory of higher order linear elliptic equations in two independent variables.

We now proceed with the reduction of the Cauchy problem (1.4), (1.5) to the second-order operator Eq. (1.6). Let

$$U^{(1)} = \frac{\partial^2 U}{\partial z\,\partial z^*} = \frac{1}{4}\Delta u. \tag{2.4}$$

Then

$$\begin{aligned} U_z &= \int_z^{z^*} U^{(1)}(z, \xi^*)\,d\xi^* + U_z(z, z) \\ &= \int_z^{z^*} U^{(1)}(z, \xi^*)\,d\xi^* + \tfrac{1}{2}[\varphi_0'(z) - i\varphi_1(z)], \end{aligned} \tag{2.5}$$

$$\begin{aligned} U_{z^*} &= \int_{z^*}^{z} U^{(1)}(\xi, z^*)\,d\xi + U_{z^*}(z, z) \\ &= \int_{z^*}^{z} U^{(1)}(\xi, z^*)\,d\xi + \tfrac{1}{2}[\varphi_0'(z) + i\varphi_1(z)], \end{aligned} \tag{2.6}$$

$$\begin{aligned} U &= \int_{z^*}^{z} U_\xi(\xi, z^*)\,d\xi + U(z, z) \\ &= \int_{z^*}^{z} \left\{\int_\xi^{z^*} U^{(1)}(\xi, \xi^*)\,d\xi^* + \tfrac{1}{2}[\varphi_0'(\xi) - i\varphi_1(\xi)]\right\} d\xi + \varphi_0(z). \end{aligned} \tag{2.7}$$

By using Eqs. (2.4)–(2.7), $\partial^{p+q}U/\partial z^p\,\partial z^{*q}$, for $p = 0, 1,\ldots, n$; $q = 0, 1,\ldots, n$; $p + q \leqslant 2n - 1$, can all be computed in terms of a linear combination of $\partial^{p+q}U^{(1)}/\partial z^p\,\partial z^{*q}$ and its integrals, $p = 0, 1,\ldots, n-1$; $q = 0, 1,\ldots, n-1$; $p + q \leqslant 2n - 3$. Furthermore, Eqs. (2.4) and (1.5) allow $\partial^k u^{(1)}(x, 0)/\partial y^k$, $k = 1, 2,\ldots, 2n-3$, to be computed in terms of the Cauchy data for u. Hence we are led to the following Cauchy problem for $U^{(1)}$:

$$\frac{\partial^{2n-2}U^{(1)}}{\partial z^{n-1}\,\partial z^{*n-1}} = F(z, z^*, A_1^{(1)}(U^{(1)}),\ldots, A_N^{(1)}(U^{(1)})), \tag{2.8}$$

$$\begin{aligned} U^{(1)}(z, z^*) &= \varphi_0^{(1)}(z); \qquad z = z^*, \\ i^k\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right)^k U^{(1)}(z, z^*) &= \varphi_k^{(1)}(z); \qquad z = z^*, \quad k = 1, 2,\ldots, 2n-3, \end{aligned} \tag{2.9}$$

where $A_i^{(1)}$, $i = 1, 2, \ldots, N$, are integral operators on HB into HB, $HB \equiv HB(\varDelta\rho, \varDelta\rho^*)$ being the Banach space of functions of two complex variables which are holomorphic and bounded in

$$\varDelta\rho \times \varDelta\rho^*, \qquad \varDelta\rho = \{z \mid |z| < \rho\}, \qquad \varDelta\rho^* = \{z \mid z^* \in \varDelta\rho\},$$

with norm

$$\| S \| = \sup_{\varDelta\rho \times \varDelta\rho^*} | S(z, z^*)|. \tag{2.10}$$

More precisely, $A_1^{(1)}$ is defined by Eq. (2.7), $A_2^{(1)}$ by Eq. (2.6), $A_3^{(1)}$ by Eq. (2.5), and $A_i^{(1)}$, $i > 3$, is obtained by repeated differentiation of Eq. (2.5) or (2.6). It is easily seen that each $A_i^{(1)}$, $i = 1, \ldots, N$, satisfies the condition

$$\begin{aligned}
&\| A_i^{(1)}(U_1^{(1)}) - A_i^{(1)}(U_2^{(1)})\| \\
&\quad \leqslant M_i^{(1)} \Big\{\| U_1^{(1)} \quad U_2^{(1)} \| + \cdots + \left\| \frac{\partial^{p+q} U_1^{(1)}}{\partial z^p\, \partial z^{*q}} - \frac{\partial^{p+q} U_2^{(1)}}{\partial z^p\, \partial z^{*q}} \right\| + \cdots \\
&\quad + \left\| \frac{\partial^{2n-3} U_1^{(1)}}{\partial z^{n-2}\, \partial z^{*n-1}} - \frac{\partial^{2n-3} U_2^{(1)}}{\partial z^{n-2}\, \partial z^{*n-1}} \right\| + \left\| \frac{\partial^{2n-3} U_1^{(1)}}{\partial z^{n-1}\, \partial z^{*n-2}} - \frac{\partial^{2n-3} U_2^{(1)}}{\partial z^{n-1}\, \partial z^{*n-2}} \right\| \Big\}
\end{aligned} \tag{2.11}$$

for some positive constant $M_i^{(1)}$. Repeating this process $n - 1$ times, we are led to a Cauchy problem of the form

$$U_{zz^*}^{(n-1)} = F(z, z^*, A_1^{(n-1)}(U^{(n-1)}), \ldots, A_N^{(n-1)}(U^{(n-1)})), \tag{2.12}$$

$$\begin{aligned}
U^{(n-1)}(z, z^*) &= \varphi_0^{(n-1)}(z); \qquad z = z^*, \\
i\left(\frac{\partial U^{(n-1)}}{\partial z} - \frac{\partial U^{(n-1)}}{\partial z^*}\right) &= \varphi_1^{(n-1)}(z); \qquad z = z^*,
\end{aligned} \tag{2.13}$$

where

$$U^{(n-1)}(z, z^*) = \frac{\partial^2 U^{(n-2)}}{\partial z\, \partial z^*} \tag{2.14}$$

and $A_i^{(n-1)}$, $i = 1, \ldots, N$, are integral operators on HB into HB which satisfy the condition

$$\begin{aligned}
&\| A_i^{(n-1)}(U_1^{(n-1)}) - A_i^{(n-1)}(U_2^{(n-1)})\| \\
&\quad \leqslant M_i^{(n-1)} \Big\{\| U_1^{(n-1)} - U_2^{(n-1)} \| + \left\| \frac{\partial U_1^{(n-1)}}{\partial z} - \frac{\partial U_2^{(n-1)}}{\partial z} \right\| \\
&\quad + \left\| \frac{\partial U_1^{(n-1)}}{\partial z^*} - \frac{\partial U_2^{(n-1)}}{\partial z^*} \right\| \Big\}
\end{aligned} \tag{2.15}$$

for some positive constant $M_i^{(n-1)}$.

We note that the operators $A_i^{(k)}$, $i = 1, \ldots, N$, $k = 1, \ldots, n - 1$, all turn out to be integral operators satisfying a condition such as (2.11), due to the fact that we restricted ourselves to a rather special class of semilinear equations. For equations not of the form (1.4) (e.g., Eq. (2.3)), the operator $A_i^{(k)}$ would fail to satisfy such conditions for $k > k_0$, where k_0 is some integer less than $n-1$.

We now proceed to use the contraction mapping principle to find a solution of Eqs. (2.12), (2.13). By hypothesis, F is holomorphic in a compact subset of the space of $N+2$ complex variables and, hence, from Schwarz's lemma for functions of several complex variables [3, p. 38], a Lipschitz condition holds there with respect to the last N arguments, i.e.,

$$\begin{aligned} &| F(z, z^*, \xi_1, \ldots, \xi_N) - F(z, z^*, \xi_1^0, \ldots, \xi_N^0)| \\ &\quad \leqslant C_0\{| \xi_1 - \xi_1^0 | + \cdots + | \xi_N - \xi_N^0 |\}, \end{aligned} \tag{2.16}$$

where C_0 is a positive constant. Hence, by (2.15) and (2.16), there exists a positive constant C_1 such that

$$\begin{aligned} &\| F(z, z^*, A_1^{(n-1)}(U_1^{(n-1)}), \ldots, A_N^{(n-1)}(U_1^{(n-1)})) \\ &\quad - F(z, z^*, A_1^{(n-1)}(U_2^{(n-1)}), \ldots, A_N^{(n-1)}(U_2^{(n-1)}))\| \\ &\quad \leqslant C_1 \left\{\| U_1^{(n-1)} - U_2^{(n-1)} \| + \left\| \frac{\partial U_1^{(n-1)}}{\partial z} - \frac{\partial U_2^{(n-1)}}{\partial z} \right\| \right. \\ &\quad \left. + \left\| \frac{\partial U_1^{(n-1)}}{\partial z^*} - \frac{\partial U^{(n-1)}}{\partial z^*} \right\| \right\}. \end{aligned} \tag{2.17}$$

It should be noted that $A_i^{(n-1)}$ are in fact integral operators on $U_1^{(n-1)}$ and its derivatives with respect to z and z^*, i.e. $A_i^{(n-1)}(U_1^{(n-1)}) \equiv A_i^{(n-1)}(U_1^{(n-1)}, \partial U_1^{(n-1)}/\partial z, \partial U_1^{(n-1)}/\partial z^*)$.

Now define the operators B_i, $i = 1, 2, 3$, by

$$s(z, z^*) = U_{zz^*}^{(n-1)}(z, z^*), \tag{2.18}$$

$$\begin{aligned} B_1(s) \equiv U^{(n-1)}(z, z^*) = &\int_0^z \int_0^{z^*} s(\xi, \xi^*)\, d\xi^*\, d\xi + \int_0^z \gamma(\xi)\, d\xi \\ &+ \int_0^{z^*} \psi(\xi^*)\, d\xi^* + \varphi_0^{(n-1)}(0), \end{aligned} \tag{2.19}$$

$$B_2(s) \equiv U_z^{(n-1)}(z, z^*) = \int_0^{z^*} s(z, \xi^*)\, d\xi^* + \gamma(z), \tag{2.20}$$

$$B_3(s) \equiv U_{z^*}^{(n-1)}(z, z^*) = \int_0^z s(\xi, z^*)\, d\xi + \psi(z^*), \tag{2.21}$$

where [1]

$$\gamma(z) = \frac{1}{2}\left[\frac{d\varphi_0^{(n-1)}(z)}{dz} - i\varphi_1^{(n-1)}(z)\right] - \int_0^z s(z, \xi^*)\, d\xi^*, \tag{2.22}$$

$$\psi(z) = \frac{1}{2}\left[\frac{d\varphi_0^{(n-1)}(z)}{dz} + i\varphi_1^{(n-1)}(z)\right] - \int_0^z s(\xi, z)\, d\xi. \tag{2.23}$$

Finding a solution to the Cauchy problem (2.12), (2.13) is now equivalent to finding a fixed point in the Banach space HB of the operator $T: HB \to HB$, defined by

$$Ts = F(z, z^*, A_1^{(n-1)}(B_1(s)), \ldots, A_N^{(n-1)}(B_1(s))). \tag{2.24}$$

For a given a, $0 \leqslant a < 1$, and ρ sufficiently small, it is easily seen from Eqs. (2.18)–(2.23) that

$$\| B_i s_1 - B_i s_2 \| \leqslant \frac{a}{3C_1} \| s_1 - s_2 \| ; \qquad i = 1, 2, 3. \tag{2.25}$$

Hence, from Eqs. (2.17) and (2.24), we have

$$\| Ts_1 - Ts_2 \| \leqslant a \| s_1 - s_2 \| \tag{2.26}$$

and

$$\| Ts \| \leqslant a \| s \| + \| To \| \leqslant \| s \| + (1 - a)\, M_0\,, \tag{2.27}$$

for some positive constant M_0. Hence, if $\| s \| < M_0$, then $\| Ts \| < M_0$, i.e., T is a contraction mapping of a closed ball of HB into itself. Hence T has a (unique) fixed point $s(z, z^*)$ and, therefore, Eqs. (2.19), (2.22), (2.23) give the solution $U^{(n-1)}(z, z^*)$ to (2.12), (2.13). Now, refering back to Eqs. (2.7) and (2.14), we see that

$$U^{(k-1)}(z, z^*) = \int_{z^*}^{z} \left\{ \int_{\xi}^{z^*} U^{(k)}(\xi, \xi^*)\, d\xi^* + \frac{1}{2}\left[\frac{d\varphi_0^{(k)}(\xi)}{d\xi} - i\varphi_1^{(k)}(\xi)\right]\right\} d\xi$$
$$+ \varphi_0^{(k)}(z). \tag{2.28}$$

Hence, from a knowledge of $U^{(n-1)}(z, z^*)$, we immediately obtain the solution $U(z, z^*)$ to our original Cauchy problem (2.1), (2.2), by a series of quadratures.

THEOREM. *There exists a constructive procedure, suitable for analytic approximations, for solving the Cauchy problem* (1.4), (1.5). *Such a procedure is given explicitly by* (2.1)–(2.28).

It is important to note that the unstable dependence of the solution of the elliptic Eq. (1.4) on the (real) Cauchy data (1.5) appears exclusively in the step where this data is extended to complex values of the independent variable x. When this can be done in an elementary way, for example, by direct substitution via the transformation (1.2), no instabilities will occur when one uses the contraction mapping operator T to obtain approximations to the desired solution.

We finally note that if equation (1.4) is linear and one uses exponential majorization (c.f. [1], [3]), then the above techniques yield *global* solutions to Cauchy's problem. In particular if the norm (2.10) is taken over $\mathfrak{S} \times \mathfrak{S}^*$ instead of $\Delta\rho \times \Delta\rho^*$, we obtain an extension of Henrici's theorem ([4], p. 196) to higher order elliptic equations.

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Cauchy's Problem for a Class of Fourth Order Elliptic Equations in Two Independent Variables

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Integral operator techniques are used to construct the solution to Cauchy's problem for a class of fourth order elliptic equations in two independent variables. If the Cauchy data is prescribed along an arbitrary analytic arc C, then approximate solutions can be obtained on compact subsets of domains which are conformally symmetric with respect to C. This improves upon results previously obtained by Henrici, Pucci, and Colton.

1. INTRODUCTION

Cauchy's problem for elliptic equations is one of the classic examples of an improperly posed problem in partial differential equations due to the lack of continuous dependence on the initial data (c.f. [4], p. 108). Until recent years such a situation was considered as little more than an interesting pathological example, under the assumption that there do not exist any physical situations corresponding to improperly posed mathematical problems. However, the increasing use of inverse methods to solve free boundary problems in mathematical physics (c.f. [5], [6], [8]) has led mathematicians to finally consider the long ignored problem of deriving stable, constructive methods for solving the elliptic Cauchy problem ([2], [4], [9], [10], [13], [14]). Despite its apparant generality, the Cauchy–Kowalewski theorem is unsuitable in this regard, since the actual computation of the

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solution is far too tedious for practical application, and even if it is constructed the series solution may not converge in the full region where the solution is needed in a particular example.

A fruitful approach to the study of the elliptic Cauchy problem has been through the use of function theoretic methods in partial differential equations, particularly in the case of second order equations in two independent variables ([2], [5], [6], [9]). Probably the most satisfying result is that due to Henrici ([7], p. 195–199, [9]) in which an integral representation of the solution $u(x, y)$ is obtained. This representation enables one to determine the domain of regularity D of $u(x, y)$, and to construct a sequence of solutions to the differential equation which tend uniformly to $u(x, y)$ on closed subdomains of D. The area of higher order equations in two independent variables is not so well developed, although such results are needed in order to study various free boundary problems arising in elasticity (c.f. [12]). Both Pucci ([14], [15]) and Colton ([3]) have given iterative procedures for the construction of solutions to higher order equations, but these methods suffer from the facts that the approximations no longer satisfy the differential equation, and are valid only in a rectangle and disc respectively.

In this paper we give new methods for constructing solutions to Cauchy's problem for the equation

$$e_2(u) \equiv \Delta^2 u + a_1(x, y)\Delta u + a_2(x, y)\frac{\partial u}{\partial x} + a_3(x, y)\frac{\partial u}{\partial y} + a_4(x, y)u = 0$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{1.1}$$

where $a_i(x, y)$, $i = 1, 2, 3, 4$, are analytic functions of x and y in some domain of the x, y plane, and where $u(x, y)$ is required to satisfy Cauchy data along an arbitrary analytic arc C. Equations of the form (1.1) arise in many areas of elasticity, for example the bending of thin plates, the theory of shallow elastic shells, etc. (c.f. [16], [18]). By using some results of Vekua to reduce the Cauchy problem to a system of Volterra integral equations in the space of several complex variables, we are able to overcome the objections to the iterative procedures of Pucci and Colton, and extend the results of Henrici in [9] to the fourth order equation $e_2(u) = 0$.

2. REFORMULATION OF CAUCHY'S PROBLEM IN THE SPACE OF SEVERAL COMPLEX VARIABLES

Following Vekua ([18]) we can use conjugate coordinates

$$z = x + iy \tag{2.1}$$
$$z^* = x - iy$$

to rewrite $e_2(u) = 0$ as the formally hyperbolic equation

$$\frac{\partial^4 U}{\partial z^2\,\partial z^{*2}} + A_1(z, z^*)\frac{\partial^2 U}{\partial z\,\partial z^*} + A_2(z, z^*)\frac{\partial U}{\partial z} + A_3(z, z^*)\frac{\partial U}{\partial z^*} + A_4(z, z^*)U = 0 \tag{2.2}$$

where

$$U(z, z^*) = u\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right).$$

Note that z and z^* are conjugate if and only if x and y are real. If x and y are allowed to take on complex values, then z and z^* will be independent complex variables. A simply connected domain D in the $z = x+iy$ plane is known as a *fundamental domain* ([18]) for $e_2(u) = 0$ if all the coefficients $A_i(z, z^*)$, $i = 1, 2, 3, 4$, of equation (2.2) are analytic functions of z, z^* in $(D, \bar{D})$. Finally we can rewrite equation (2.2) as

$$\frac{\partial^4 U}{\partial z^2\,\partial z^{*2}} + \frac{\partial^2(B_1 U)}{\partial z\,\partial z^*} + \frac{\partial(B_2 U)}{\partial z} + \frac{\partial(B_3 U)}{\partial z^*} + B_4 U = 0 \tag{2.3}$$

where $B_i \equiv B_i(z, z^*)$, $i = 1, 2, 3, 4$, can be expressed in terms of the $A_i(z, z^*)$.

It is shown in Vekua ([18], pp. 184–196) that every regular solution $u(x, y)$ of $e_2(u) = 0$ (i.e. $u \in C^4(D)$) can be analytically continued into the domain of complex values of x, y and the resulting function $U(z, z^*)$ is an analytic solution of $e_2(u) = 0$ in the domain $(D, \bar{D})$. More specifically, any regular solution of $e_2(u) = 0$ in a fundamental domain D can be expressed in terms of a set of constants a_0, a_1, and functions $x_k(z)$, $x_k^*(z^*)$, $k = 1, 2$, holomorphic in D and $\bar{D}$ respectively, by the formula

$$\begin{aligned} U(z, z^*) &= \sum_{k=0}^{1} a_k G_k(0, 0, z, z^*) \\ &\quad + \sum_{k=0}^{1}\left\{\int_0^z G_k(t, 0, z, z^*)x_k(t)\,dt + \int_0^{z^*} G_k(0, t^*, z, z^*)x_k^*(t^*)\,dt^*\right\} \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} G_k(t, t^*, z, z^*) &= \frac{(z-t)^k(z^*-t^*)^k}{k!k!} \\ &\quad + \int_t^z d\xi \int_{t^*}^{z^*} \frac{(\xi-t)^k(\xi^*-t^*)^k}{k!k!}\,\Gamma(z, z^*, \xi, \xi^*)\,d\xi^* \end{aligned}$$

$$\Gamma(z, z^*, \xi, \xi^*) = \sum_{j=1}^{\infty} K_j(z, z^*, \xi, \xi^*) \tag{2.5}$$

$$K_1(z, z^*, \xi, \xi^*) = -(z-\xi)(z^*-\xi^*)B_4(\xi, \xi^*)-(z-\xi)B_3(\xi, \xi^*)$$
$$-(z^*-\xi^*)B_2(\xi, \xi^*)-B_1(\xi, \xi^*)$$

$$K_j(z, z^*, \xi, \xi^*) = \int_\xi^z dt \int_{\xi^*}^{z^*} K_1(z, z^*, t, t^*)K_{j-1}(t, t^*, \xi, \xi^*)\, dt^*.$$

It is shown in [18] that $G_k(t, t^*, z, z^*)$ is an analytic function in the domain $t, z \in D$, $t^*, z^* \in \bar{D}$. We will make use of this basic formula of Vekua to construct solutions to the Cauchy problem for $e_2(u) = 0$.

Let the Cauchy data be given by

$$\left.\begin{aligned} u &= f_o(z) \\ \frac{\partial^i u}{\partial v_i} &= f_i(z) \end{aligned}\right\} ; i = 1, 2, 3,\ z = x+iy \in C \tag{2.6}$$

where v is the positive normal to the analytic arc C, and $f_i(z)$, $i = 0, 1, 2, 3$, are analytic functions defined in a domain D which contains C. We require D to be *conformally symmetric* ([19]) with respect to C, i.e. there exists a conformal mapping which transforms C into an interval of the real axis and D into a domain which is symmetric with respect to the real axis. Without loss of generality we can assume that C passes through the origin. The data (2.6) allows us to evaluate the derivatives of $U(z, z^*)$ for $z = x+iy \in C$, $z^* = \bar{z}$, as follows: By assumption there exists a conformal mapping $z = \varphi(\zeta)$ which maps an interval I of the real axis of the ζ plane onto C, and a domain D' which is symmetric with respect to the real axis onto D. Since C passes through the origin we can assume without loss of generality that $\varphi(0) = 0$. Define the analytic function $\bar{\varphi}(\zeta^*)$ by $\bar{\varphi}(\zeta^*) = \overline{\varphi(\overline{\zeta^*})}$. Then for $s \in I$ we have after a rather long but straightforward calculation:

$$\begin{aligned}
U_1(\varphi(s), \bar{\varphi}(s)) &= \frac{F_o'(s)-iF_1(s)}{2\varphi'(s)} \\
U_2(\varphi(s), \bar{\varphi}(s)) &= \frac{F_o'(s)+iF_1(s)}{2\bar{\varphi}'(s)} \\
U_{12}(\varphi(s), \bar{\varphi}(s)) &= \frac{1}{4|\varphi'(s)|^2}(F_o''(s)+F_2(s)) \\
U_{112}(\varphi(s), \bar{\varphi}(s)) &= \frac{1}{8|\varphi'(s)|^2\varphi'(s)}(F_o'''(s)-iF_o''(s) \\
&\quad +F_2'(s)-iF_3(s))-\frac{\varphi''(s)}{4|\varphi'(s)|^2(\varphi'(s))^2}(F_o''(s)+F_2(s))
\end{aligned} \tag{2.7}$$

$$U_{122}(\varphi(s), \bar{\varphi}(s)) = \frac{1}{8|\varphi'(s)|^2\bar{\varphi}'(s)}(F_o'''(s)+iF_1''(s)$$
$$+F_2'(s)+iF_3(s)) - \frac{\bar{\varphi}''(s)}{4|\varphi'(s)|^2(\bar{\varphi}'(s))^2}(F_o''(s)+F_2(s))$$

where

$$U_1(z, z^*) \equiv \frac{\partial U(z, z^*)}{\partial z} \qquad U_{12}(z, z^*) \equiv \frac{\partial^2 U(z, z^*)}{\partial z\, \partial z^*}$$
$$U_2(z, z^*) \equiv \frac{\partial U(z, z^*)}{\partial z^*} \qquad U_{112}(z, z^*) \equiv \frac{\partial^3 U(z, z^*)}{\partial z^2\, \partial z^*}$$
$$U_{122}(z, z^*) \equiv \frac{\partial^3 U(z, z^*)}{\partial z\, \partial z^{*2}}$$

$$\begin{aligned} F_o(s) &= f_0(\varphi(s)) \\ F_1(s) &= |\varphi'(s)|\, f_1(\varphi(s)) \\ F_2(s) &= |\varphi'(s)|^2[f_2(\varphi(s)) - i\Phi_1(s) f_1(\varphi(s))] \\ F_3(s) &= |\varphi'(s)|^3[f_3(\varphi(s)) - \Phi_3(s)(\Phi_1(s) f_1(\varphi(s)) + if_2(\varphi(s))) \\ &\qquad - i\Phi_2(s)|\varphi'(s)| f_1(\varphi(s))] \end{aligned} \tag{2.8}$$

$$\Phi_1(s) = \left(\frac{\partial}{\partial\zeta} - \frac{\partial}{\partial\zeta^*}\right)\left(\frac{1}{\sqrt{[\varphi'(\zeta)\bar{\varphi}'(\zeta^*)]}}\right)\Bigg|_{\zeta=\zeta^*=s}$$
$$\Phi_2(s) = \left[\frac{1}{\sqrt{[\varphi'(\zeta)\bar{\varphi}'(\zeta^*)]}}\left(\frac{\partial}{\partial\zeta} - \frac{\partial}{\partial\zeta^*}\right)\right]^2\left(\frac{1}{\sqrt{[\varphi'(\zeta)\bar{\varphi}'(\zeta^*)]}}\right)\Bigg|_{\zeta=\zeta^*=s}$$
$$\Phi_3(s) = |\varphi'(s)|\left(\frac{\partial}{\partial\zeta} - \frac{\partial}{\partial\zeta^*}\right)\left(\frac{1}{\sqrt{[\varphi'(\zeta)\bar{\varphi}'(\zeta^*)]}}\right)^3\Bigg|_{\zeta=\zeta^*=s}$$
$$|\varphi'(s)| = \sqrt{[\varphi'(\zeta)\bar{\varphi}'(\zeta^*)]}\Big|_{\zeta=\zeta^*=s}$$

In Eqs. (2.7) and (2.8) set $s = \varphi^{-1}(z)$, $z \in C$. Since $\varphi(\zeta)$ is conformal in D' it is clear that if the square roots are appropriately chosen, then $U(z, z^*)$ and its derivatives with respect to z and z^* can be analytically continued off the curve C (as a function of z) into the same domain D that the Cauchy data $f_i(z)$ are analytic. It is from this data that we will obtain a system of Volterra integral equations to determine the constants a_0, a_1, and the functions $x_k(z)$, $x_k^*(z)$, $k = 1, 2$, such that Eq. (2.4) gives the solution of Cauchy's problem. We note in passing that an alternative approach to the one we are following here is to use conformal mapping techniques to reduce the Cauchy problem (1.1), (2.6) to one in which the data is prescribed along the x axis. However, under this operation the form of Eq. (1.1) is changed, the expression for $G_k(t, t^*, z, z^*)$, $k = 1, 2$, become more complicated, and attempts to express the functions $x_k(z)$, $x_k^*(z^*)$, $k = 1, 2$, as the solution of a system of Volterra integral equations run into serious difficulties.

3. REDUCTION OF CAUCHY'S PROBLEM TO A SYSTEM OF VOLTERRA INTEGRAL EQUATIONS IN THE COMPLEX DOMAIN

From Eqs. (2.4), (2.5), and the fact that the curve C passes through the origin, we have

$$\begin{aligned} U(0,0) &= f(0) = a_o \\ U_{12}(0,0) &= \frac{1}{8|\varphi'(0)|^2\varphi'(0)}(F_o''(0)+F_2(0)) = a_1 \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &U_1(z,z^*) - \sum_{k=0}^{1} a_k \frac{\partial G_k}{\partial z}(0,0,z,z^*) \\ &= x_o(z) + \sum_{k=0}^{1} \left\{ \int_o^z \frac{\partial G_k}{\partial z}(t,0,z,z^*) x_k(t)\,dt + \int_o^{z^*} \frac{\partial G_k}{\partial z}(0,t^*,z,z^*) x_k^*(t^*)\,dt^* \right\} \end{aligned} \tag{3.2a}$$

$$\begin{aligned} &U_2(z,z^*) - \sum_{k=0}^{1} a_k \frac{\partial G_k}{\partial z}(0,0,z,z^*) \\ &= x_o^*(z^*) + \sum_{k=0}^{1} \left\{ \int_o^z \frac{\partial G_k}{\partial z^*}(t,0,z,z^*) x_k(t)\,dt + \int_o^{z^*} \frac{\partial G_k}{\partial z^*}(0,t^*,z,z^*) x_k^*(t^*)\,dt^* \right\} \end{aligned} \tag{3.2b}$$

$$\begin{aligned} &U_{121}(z,z^*) - \left[U_1(z,z^*) - \sum_{k=0}^{1} a_k \frac{\partial G_k}{\partial z}(0,0,z,z^*) \right] \left[\frac{\partial^2 G_o}{\partial z\,\partial z^*}(t,0,z,z^*) \right]_{t=z} \\ &= x_1(z) + \sum_{k=0}^{1} \left\{ \int_o^z \left(\frac{\partial^3 G_k}{\partial z^2\,\partial z^*}(t,0,z,z^*) - \frac{\partial G_k}{\partial z}(t,0,z,z^*) \right.\right. \\ &\quad \left. \times \left[\frac{\partial^2 G_o}{\partial z\,\partial z_*}(t,0,z,z^*) \right]_{t=z} \right) x_k(t)\,dt + \int_o^{z^*} \left(\frac{\partial^3 G_k}{\partial z^2\,\partial z^*}(0,t^*,z,z^*) \right. \\ &\quad \left.\left. - \frac{\partial G_k}{\partial z}(0,t^*,z,z^*) \left[\frac{\partial^2 G_o}{\partial z\,\partial z^*}(t,0,z,z^*) \right]_{t=z} \right) x_k^*(t^*)\,dt^* \right\} \end{aligned} \tag{3.2c}$$

$$U_{122}(z,z^*) - \left[U_2(z,z^*) - \sum_{k=0}^{1} a_k \frac{\partial G_k}{\partial z^*}(0,0,z,z^*) \right] \left[\frac{\partial^2 G_o}{\partial z\,\partial z^*}(0,t^*,z,z^*) \right]_{t^*=z^*} \tag{3.2d}$$

$$= x_1^*(z^*) + \sum_{k=0}^{1} \Bigg\{ \int_o^z \left(\frac{\partial^3 G_k}{\partial z^{*2}\,\partial z}(t, 0, z, z^*) - \frac{\partial G_k}{\partial z^*}(t, 0, z, z^*) \right.$$

$$\times \left. \left[\frac{\partial^2 G_o}{\partial z\,\partial z^*}(0, t^*, z, z^*) \right] \right)_{t=z} x_k(t)\,dt + \int_o^{z^*} \left(\frac{\partial^3 G_k}{\partial z^{*2}\,\partial z}(0, t^*, z, z^*) \right.$$

$$\left. - \frac{\partial G_k}{\partial z^*}(0, t^*, z, z^*) \left[\frac{\partial^2 G_o}{\partial z\,\partial z^*}(0, t^*, z, z^*) \right] \right)_{t^*=z^*} x_k^*(t^*)\,dt^* \Bigg\}.$$

Assume now that the domain of regularity D of the Cauchy data $f_i(z,)$ $i = 0, 1, 2, 3$, is a conformally symmetric fundamental domain of the equation $e_2(u) = 0$. Setting $z = \varphi(\zeta)$, $z^* = \bar{\varphi}(\zeta^*)$ in Eqs. (3.2a)–(3.2d), and restricting ζ to the real interval I, yields a system of integral equations of the form

$$g_i(s) = y_i(s) + \sum_{k=0}^{1} \left\{ \int_o^{\varphi(s)} R_{i,k}(t, \varphi(s), \bar{\varphi}(s)) x_k(t)\,dt \right. \tag{3.3}$$

$$\left. + \int_o^{\bar{\varphi}(s)} R_{i,k}^*(t^*, \varphi(s), \bar{\varphi}(s)) x_k^*(t^*)\,dt^* \right\}; \quad i = 1, 2, 3, 4, \quad s \in I$$

where $g_i(s)$ are known functions expressible in terms of the Cauchy data and $G_k(t, t^*, z, z^*)$ (Eqs. (2.7), (2.8), (3.1)), $R_{i,k}(t, \varphi(s), \bar{\varphi}(s))$ and $R_{i,k}^*(t^*, \varphi(s), \bar{\varphi}(s))$ are expressible in terms of $G_k(t, t^*, z, z^*)$ and its derivatives with respect to z and z^* (Eqs. (3.2a)–(3.2d)), and $y_i(s)$ are defined by

$$\begin{aligned} y_1(s) &= x_o(\varphi(s)) \\ y_2(s) &= x_1(\varphi(s)) \\ y_3(s) &= x_o^*(\bar{\varphi}(s)) \\ y_4(s) &= x_1^*(\bar{\varphi}(s)). \end{aligned} \tag{3.4}$$

Setting $s = \varphi^{-1}(z)$ and making the change of variables

$$t^* = \bar{\varphi}(\varphi^{-1}(\tau)) \tag{3.5}$$

(note that

$$\frac{d}{d\tau}[\bar{\varphi}(\varphi^{-1}(\tau))] = \frac{\bar{\varphi}'(\varphi^{-1}(\tau))}{\varphi'(\varphi^{-1}(\tau))} \neq 0$$

for $\tau \in D$) transforms the system (3.3) into

$$g_i(\varphi^{-1}(z)) = w_i(z) + \sum_{k=0}^{1}\left\{\int_o^z R_{i,k}(t, z, \bar{\varphi}(\varphi^{-1}(z)))w_k(t)\,dt \right. \tag{3.6}$$

$$\left. + \int_o^z R^*_{i,k}(\bar{\varphi}(\varphi^{-1}(\tau)), z\, \bar{\varphi}(\varphi^{-1}(z)))\,\frac{\bar{\varphi}'(\varphi^{-1}(\tau))}{\varphi'(\varphi^{-1}(\tau))}\,w_{k+2}(\tau)\,d\tau\right\}$$

$$i = 1, 2, 3, 4, \qquad z = x+iy \in C$$

where

$$\begin{aligned} w_1(z) &= x_o(z) \\ w_2(z) &= x_1(z) \\ w_3(z) &= x^*_o(\bar{\varphi}(\varphi^{-1}(z))) \\ w_4(z) &= x^*_1(\bar{\varphi}(\varphi^{-1}(z))). \end{aligned} \tag{3.7}$$

Since the kernels and the terms not involving $w_i(z)$ are regular in D, Eq. (3.6) defines a system of Volterra integral equations for $w_i(z)$, $z \in D$, and hence can be explicitly solved for these unknown functions (c.f. [7], [17], [18]). By the properties of Volterra integral equations in the complex plane ([11], [18]), it is clear that $w_i(z)$ must be analytic in D, and in particular $w_3(\varphi(\bar{\varphi}^{-1}(z^*))) = x^*_o(z^*)$, $w_4(\varphi(\bar{\varphi}^{-1}(z^*))) = x^*_1(z^*)$ are regular in $\bar{D}$. From Eq. (2.4) it is seen that $U(z, z^*)$ is regular in $(D, \bar{D})$ and hence $u(x, y)$ is regular in D. Conversely, using Eqs. (2.7) and (2.8) to solve for $f_i(z)$, $i = 1, 2, 3$, in terms of $U(z, \bar{\varphi}(\varphi^{-1}(z)))$ and its derivatives with respect to z and z^*, $z \in C$, it is seen that if $u(x, y)$ is a regular solution of the Cauchy problem in D, then the Cauchy data are holomorphic on C, and can be continued analytically into the whole of D (see also [9], pp. 199–200, in this regard).

By using the resolvent to express $w_i(z)$ in terms of $g_i(\varphi^{-1}(z))$ and using a complete family of analytic functions to approximate the resolvent kernel and $g_i(\varphi^{-1}(z))$ in D, one can approximate $w_i(z)$ on compact subsets of D, and then use Eq. (2.4) to approximate $u(x, y)$ on compact subsets of D.

We summarize our results in the following two theorems, which parallel those of Henrici for the case of second order equations ([9]):

THEOREM 1 *Let the fundamental domain D of $e_2(u) = 0$ be conformally symmetric with respect to the analytic arc C, and let the Cauchy data $f_i(z)$, $i = 0, 1, 2, 3$, be holomorphic throughout D. Then the solution of the Cauchy problem* (1.1), (2.6) *exists, is regular in D, and there is a constructive procedure for analytically approximating this solution on compact subsets of D.*

THEOREM 2 *Let the fundamental domain D of $e_2(u) = 0$ be conformally symmetric with respect to the analytic arc C, let $u(x, y)$ be a solution of*

$e_2(u) = 0$ which is regular in D, and let ν be the positive normal to C. Then the functions

$$f_0(z) = u$$

$$f_1(z) = \frac{\partial u}{\partial \nu}$$

$$f_2(z) = \frac{\partial^2 u}{\partial \nu^2} \qquad (z = x+iy \in C)$$

$$f_3(z) = \frac{\partial^3 u}{\partial \nu^3}$$

are holomorphic on C and can be continued analytically into the whole of D.

We note in conclusion that representations of the solutions of $e_2(u) = 0$, other than that given by Eq. (2.4), have been obtained by Bergman in [1]. The methods of this paper could also be used in conjunction with Bergman's operators to construct solutions to Cauchy's problem for $e_2(u) = 0$.

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IMPROPERLY POSED INITIAL VALUE PROBLEMS FOR SELF-ADJOINT HYPERBOLIC AND ELLIPTIC EQUATIONS*

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Abstract. Integral representations are obtained for the solution to Cauchy's problem for hyperbolic equations along a convex time-like surface, the exterior characteristic initial value problem for hyperbolic equations, and Cauchy's problem for elliptic equations along an analytic surface. Each of these problems is improperly posed in the real domain and hence our representations are constructed by integrating over appropriate regions in the space of one and several complex variables.

1. Introduction. Until about twenty years ago the problem of constructing approximate solutions to improperly posed initial value problems in partial differential equations was ignored by most mathematicians on the basis that such problems did not correspond to meaningful physical phenomena and hence such efforts were at best misguided and at worst fruitless. However, during the past two decades it came to be realized that such problems do in fact arise in mathematical physics. One such appearance is in the form of inverse free boundary problems in fluid mechanics (cf. [14]), and another is in boundary value problems where part of the boundary is inaccessible to measurement and hence the boundary data is incomplete (cf. [24], [26]). The physical origin of these problems has led to two different mathematical approaches.

In the case of inverse free boundary value problems the interest lies in constructing a "catalogue" of explicit solutions, and hence analytic data is prescribed on some analytic surface and it is desired to construct an approximate solution to a well-defined initial value problem. On the other hand, in the situation where the boundary data is incomplete, the initial data is not known exactly and approximations are constructed by assuming an a priori bound on the solution and then applying a Rayleigh–Ritz procedure [29].

Alternatively one can assume that the initial data itself satisfies an a priori bound, approximate it by a polynomial in some appropriate region (cf. [23]), and then treat the resulting initial value problem in the manner developed for inverse free boundary problems.

In all approaches the basic problem remains the same: the initial value problem is improperly posed in the sense that the solution does not depend continuously on the (real) initial data and hence one cannot approximate the solution by simply constructing the solution corresponding to approximate initial data.

In this paper we consider three classic examples of improperly posed initial value problems in partial differential equations: Cauchy's problem for hyperbolic equations along a time-like manifold [3], [21], [22], [11, pp. 754–760]; the exterior characteristic initial value problem for hyperbolic equations [12], [18], [25]; and Cauchy's problem for elliptic equations [4], [5], [6], [14], [17]. Each of these

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problems is improperly posed in the sense that the solution (if it exists) does not depend continuously on the initial data and possesses coherence properties [12], [14], [21], [11, pp. 754–760]. (It should be noted, however, that in the case of analytic coefficients uniqueness is no problem since it is assured by Holmgren's theorem [12], [21], [13, pp. 185–188].)

We shall first treat in detail the exterior characteristic initial value problem and Cauchy's problem along a time-like manifold for the self-adjoint hyperbolic equation

$$u_{x_1x_1} = u_{x_2x_2} + u_{x_3x_3} + q(x_1, x_2, x_3)u - f(x_1, x_2, x_3), \tag{1.1}$$

where $q(x_1, x_2, x_3)$ and $f(x_1, x_2, x_3)$ are analytic functions of their independent variables. We shall then briefly show how to modify these results to treat Cauchy's problem for the elliptic equation

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} + q(x_1, x_2, x_3)u = f(x_1, x_2, x_3). \tag{1.2}$$

For the special case of equation (1.1) when $q \equiv 0$ (i.e., the wave equation) the problems we are considering have been studied by Pucci [25] and Cannon [3] who showed existence, uniqueness, and continuous dependence on the data (in the complex domain) under the assumption that the initial data was analytic in one of its variables and differentiable to a sufficiently high order in the remaining variable (our results show that in the case of Cauchy's problem the smoothness conditions imposed by Cannon on the initial data can be weakened somewhat). It should also be noted that in the case of Cauchy's problem similar results had previously been given for general hyperbolic equations in two space variables by Titt [27] through the use of contraction mapping and majorization arguments.

However, our aim (and that of Cannon and Pucci) is more ambitious in that we want to obtain the solution as a linear functional of the data when the data is analytic in one of its variables and is prescribed either along a smooth time-like surface or on intersecting characteristic planes. Such an approach is advantageous in that it leads in a natural manner to results on existence, continuous dependence on the initial data, and approximation procedures. In the special case when the manifold on which the initial data is prescribed is noncharacteristic and analytic, and when the initial data is analytic in all of its independent variables, our work can be compared in some respects to that of Hill [19] and Garabedian [13, pp. 211–224].

Our results for hyperbolic equations and their analogue for elliptic equations in three independent variables are of additional interest in that they provide integral operators analogous to those of Riemann and Vekua in two independent variables [13], [30]. In the elliptic case these operators have several advantages (and some disadvantages) over the author's previous construction of integral operators in [7] (which can be viewed as an extension of Bergman's operators in two independent variables [1]) and a brief comparison of these two approaches will be discussed in § 3. It should be noted that in the elliptic case it is assumed that the initial data and the initial surface are analytic, and hence in this case the initial value problem under consideration could be solved locally via the Cauchy–Kowalewski theorem (cf. [20, pp. 116–119]). However, in addition to no longer being able to represent the solution by quadrature, this approach is far too tedious for practical application, and even if a series solution is constructed it may not converge in the full region where the solution is needed in a particular example (cf. [29]).

2. The hyperbolic equation (1.1). We shall now construct integral representations of the solutions to the Cauchy problem along a time-like manifold and the exterior characteristic initial value problem for (1.1). For convenience's sake we make the assumption that $q(x_1, x_2, x_3)$ and $f(x_1, x_2, x_3)$ are entire functions of the (complex) variables x_1, x_2 and x_3. It will be clear from our analysis that this assumption can be relaxed to assuming only $q(x_1, x_2, x_3)$ and $f(x_1, x_2, x_3)$ to be analytic in some polydisc in $\mathbb{C}^3$, the space of three complex variables. We also need the following definition [27].

DEFINITION 2.1. A function $g(x_1, x_2)$ of two real variables x_1 and x_2 is said to be *partially analytic with respect to* x_1 for $x_1 = a$ in the interval $\alpha \leqq x_2 \leqq \beta$ provided it can be represented by a series of the form

$$g(x_1, x_2) = b_0(x_2) + b_1(x_2)(x_1 - a) + b_2(x_2)(x_1 - a)^2 + \cdots \tag{2.1}$$

whose coefficients are continuous functions of x_2 in the interval $\alpha \leqq x_2 \leqq \beta$ and provided that the series (2.1) converges absolutely and uniformly for $\alpha \leqq x_2 \leqq \beta$, $|x_1 - a| \leqq \gamma$. The region $\alpha \leqq x_2 \leqq \beta$, $|x_1 - a| \leqq \gamma$ is known as the *region of partial analyticity*. The extension to more variables is evident.

We now introduce the coordinates

$$x = x_3 - x_1, \qquad y = x_1 + x_3, \qquad z = x_2 \tag{2.2}$$

and rewrite (1.1) in the form

$$L[u] \equiv u_{zz} + 4u_{xy} + Q(x, y, z)u = F(x, y, z), \tag{2.3}$$

where $F(x, y, z) = f(x_1, x_2, x_3)$ and $Q(x, y, z) = q(x_1, x_2, x_3)$. Let u and v be "well-behaved" functions to be prescribed shortly. Integrate the identity

$$vL[u] - uL[v] = (2u_y v - 2uv_y)_x + (2u_x v - 2uv_x)_y + (vu_z - uv_z)_z \tag{2.4}$$

over the torus $D \times \Omega$, where Ω is the circle $|z - \xi| = \delta > 0$ in the complex plane and D is the region in the Euclidean plane $\mathbb{R}^2$ bounded by a contour C consisting of a vertical segment C_1 joining a point B on the smooth, monotonically decreasing curve $y = y(x)$ to a point P above this curve, plus a horizontal segment C_2 joining P to a point A on $y = y(x)$, plus the arc C_3 defined by $y = y(x)$ joining A and B (see Fig. 1).

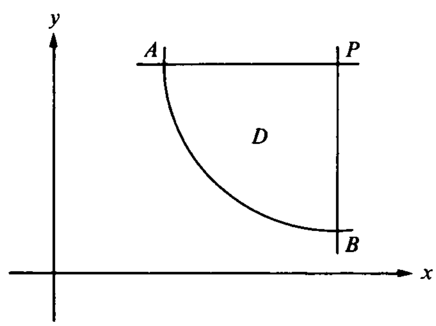

FIG. 1

Note that the integrals are to be interpreted in the sense of the calculus of exterior differential forms (cf. [2], [13, pp. 167, 213]), which attaches a meaning to them even when the differential dz is complex. Note also that the cylinder $y = y(x)$ in Euclidean three-space $\mathbb{R}^3$ is time-like with respect to the hyperbolic equation (2.3). For our purpose it is important that the curve $y = y(x)$ be monotonically decreasing and hence that the region D be as in Fig. 1 rather than as in Fig. 2. This is because of the fact that the curve AB in Fig. 2 is not time-like but space-like. Furthermore, we shall later on allow the curve C_3 to degenerate to a segment of the vertical characteristic plane through A and a segment of the horizontal characteristic plane through B. In the case of Fig. 1 this will correspond to an exterior characteristic initial value problem, whereas for Fig. 2 this becomes a (well-posed) interior characteristic value problem.

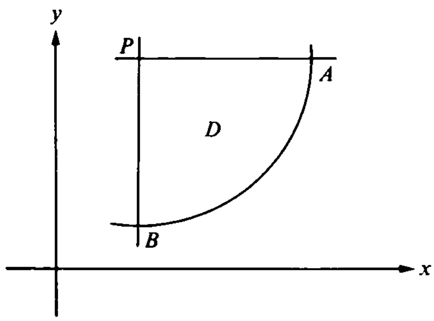

FIG. 2

The result of integrating (2.4) over the torus $D \times \Omega$, and then preforming an integration by parts on the right-hand side of the resulting identity, is, in the notation of the calculus of exterior differential forms,

$$\begin{aligned}
&\iiint_{D\times\Omega} (vL[u] - uL[v])\, dx\, dy\, dz \\
(2.5)\qquad &\quad + \int_\Omega [2v(A, z)u(A, z) + 2v(B, z)u(B, z) - 4v(P, z)u(P, z)]\, dz \\
&\quad + 4 \iint_{C_1\times\Omega} uv_y\, dy\, dz - 4 \iint_{C_2\times\Omega} uv_x\, dx\, dz \\
&\quad + 2 \iint_{C_3\times\Omega} [(uv_y - vu_y)\, dy\, dz - (uv_x - vu_x)\, dx\, dz] = 0,
\end{aligned}$$

where we have made use of the fact that $dx\, dy = 0$ on $\partial D \times \Omega$. Note that an expression of the form $v(A, z)$ is a function of three independent variables, i.e., $v(A, z) = v(x, y, z)$, where (x, y) are the Cartesian coordinates of the point A in $\mathbb{R}^2$.

We now choose u and v such that equation (2.5) reduces to an expression for the solution u of $L[u] = f$ satisfying prescribed Cauchy data on a smooth convex surface, where C_3 is the intersection of this surface with the plane $z = \zeta$, i.e., C_3 is a function of ζ. It is further assumed that the normal to the initial surface is never parallel to the z-axis and that C_3 is an analytic function of ζ. First let u be a twice continuously differentiable solution of $Lu = f$, where u and its partial derivatives of order less than or equal to two are partially analytic with respect to z in some neighborhood of the curve $y = y(x)$ and such that u satisfies prescribed Cauchy data on this curve. For v we construct a fundamental solution of $L[v] = 0$ which satisfies the boundary conditions

$$v_y = 0 \quad \text{on } C_1 \times \Omega, \tag{2.6}$$

$$v_x = 0 \quad \text{on } C_2 \times \Omega, \tag{2.7}$$

and such that at the point $(P, z) = (\xi, \eta, z)$,

$$v(P, z) = \frac{1}{8\pi i(z - \zeta)} + \text{analytic function of } (z - \zeta). \tag{2.8}$$

Note that conditions (2.6) and (2.7) are analogues to the boundary conditions satisfied by the Riemann function in two independent variables, and imply that in (2.5) the integrals over $C_1 \times \Omega$ and $C_2 \times \Omega$ vanish. We shall now show that the function v exists and possesses the necessary regularity properties for it to be substituted into (2.5).

Recall [13, pp. 152–168] that a fundamental solution $S = S(x, y, z; \xi, \eta, \zeta)$ of $L[u] = 0$ is of the form

$$S = U/R + W, \tag{2.9}$$

where $R = \sqrt{(z - \zeta)^2 + (x - \xi)(y - \eta)}$, $U = \sum_{l=0}^{\infty} U_l R^{2l}$, and W is a regular solution of $L[u] = 0$. The terms U_l, $l = 0, 1, 2, \cdots$, can be computed recursively. When the coefficients of the differential equation are entire, so is U, both as a function of (x, y, z) and the parameter point (ξ, η, ζ) (cf. [13, pp. 161, 167]). The term U_0 is given by the formula

$$U_0 = P_{00} \exp\left(- \int_0^s (C - 3/2)\frac{ds}{s} \right), \tag{2.10}$$

where (in the case of (2.3)) s is a parameter measured along the geodesics of the metric whose arc length element ds is given by the quadratic form

$$ds^2 = dx^2 + 4\, dx\, dy, \tag{2.11}$$

C is defined by

$$C = \frac{1}{4}\left[\frac{\partial^2 R^2}{\partial z^2} + 4 \frac{\partial^2 R^2}{\partial x \partial y} \right], \tag{2.12}$$

and P_{00} is a constant. Equations (2.10) and (2.12) imply $U_0 = P_{00}$, a constant. We choose $P_{00} = 1/(8\pi i)$. Hence we have

$$S = \frac{1}{8\pi i R} + \sum_{l=1}^{\infty} U_l R^{2l-1} + W. \tag{2.13}$$

Now let us look at the singularities of $1/R$ in the complex z-plane for x and y in the region D of Fig. 1. In this case $(x - \xi)(y - \eta) \geqq 0$. If we cut the complex z-plane along a line parallel to the imaginary axis between $\zeta \pm i\sqrt{(x - \xi)(y - \eta)}$, $1/R$ is an analytic function of z outside this cut. (Note that if the region of integration were the region D in Fig. 2, we would have $(x - \xi)(y - \eta) \leqq 0$ and the complex z-plane would have had to be cut along the real axis.) In particular, $1/R$ is analytic for $|z - \zeta|^2 > |(x - \xi)(y - \eta)|$, i.e., for

$$\text{(2.14)} \qquad \frac{|(x - \xi)(y - \eta)|}{|z - \zeta|^2} < 1.$$

Hence if W is, for example, an entire solution of $L[u] = 0$, S is regular for all points (x, y, z) and (ξ, η, ζ) satisfying the inequality (2.14). Thus if the point (ξ, η, ζ) is sufficiently near to the curve $y = y(x)$, S can be substituted for v in (2.5). The range of validity of (2.5) with S substituted for v can now be extended by analytic continuation, provided S satisfies (2.6) and (2.7) and the domain of regularity (as a function of z) of the Cauchy data is known.

We now turn our attention to choosing W such that (2.6) and (2.7) are satisfied by S. From (2.13) and the definition of R it is seen that one way this can be accomplished is to construct a solution W of $L[u] = 0$ satisfying the boundary conditions

$$\text{(2.15)} \qquad W = -\sum_{l=1}^{\infty} U_l(z - \zeta)^{2l-1}$$

on the characteristic plane $x = \xi$, and

$$\text{(2.16)} \qquad W = -\sum_{l=1}^{\infty} U_l(z - \zeta)^{2l-1}$$

on the characteristic plane $y = \eta$. (Note that $1/(8\pi iR)$ satisfies the boundary conditions (2.6)–(2.8). Furthermore, due to the form of equations (2.6)–(2.8), there exist boundary conditions different from (2.15) and (2.16) that could be chosen to define the function W.) This defines a characteristic initial value problem for $L[u] = 0$ with analytic (in fact entire) initial data. Hence from Hormander's generalized Cauchy–Kowalewski theorem [20, pp. 116–119] we can construct an entire solution W of $L[u] = 0$ which satisfies the initial data (2.15) and (2.16). Equation (2.13) now gives a suitable function $v = S$ to be substituted into (2.5). Note that from (2.13) we have that S satisfies condition (2.8). In the special case when $q = \text{const.} = \lambda^2$ a possible choice for the function $S = S_\lambda$ is

$$\text{(2.17)} \qquad S_\lambda = \frac{\cos \lambda R}{8\pi iR}.$$

Now in (2.5) let $v = S$ and let u be a twice continuously differentiable solution of $L[u] = f$ whose partial derivatives of order less than or equal to two are partially analytic with respect to z. From (2.8) we have

$$\text{(2.18)} \qquad 4\int_\Omega v(P, z)u(P, z)\,dz = u(\xi, \eta, \zeta).$$

Hence (2.5) becomes

$$
\begin{aligned}
u(\xi,\eta,\zeta) = &+ 2\int_\Omega [S(A,z;\xi,\eta,\zeta)u(A,z) + S(B,z;\xi,\eta,\zeta)u(B,z)]\,dz \\
&- 2\iint_{C_3\times\Omega} [u(x,y,z)S_x(x,y,z;\xi,\eta,\zeta) \\
&\qquad - S(x,y,z;\xi,\eta,\zeta)u_x(x,y,z)]\,dx\,dz \\
&+ 2\iint_{C_3\times\Omega} [u(x,y,z)S_y(x,y,z;\xi,\eta,\zeta) \\
&\qquad - S(x,y,z;\xi,\eta,\zeta)u_y(x,y,z)]\,dy\,dz \\
&+ \iiint_{D\times\Omega} S(x,y,z;\xi,\eta,\zeta)F(x,y,z)\,dx\,dy\,dz.
\end{aligned} \tag{2.19}
$$

Equation (2.19) is the desired integral representation of u in terms of its Cauchy data along a smooth time-like convex surface, where C_3 denotes the intersection of this surface with the plane $z = \zeta$. Equation (2.19) also shows that at the point (ξ, η, ζ), $u(\xi, \eta, \zeta)$ depends continuously on its Cauchy data in $C_3 \times G$, where G is an arbitrarily small neighborhood containing the branch line $\zeta \pm i\sqrt{(x - \xi)(y - \eta)}$ for all points $(x, y) \in C_3$.

The solution of the exterior characteristic initial value problem for $L[u] = f$ can now be obtained in a manner analogous to the method used to solve the characteristic initial value problem for hyperbolic equations in two variables [13, p. 131] by setting $v = S(x, y, z; \xi, \eta, \zeta)$ in (2.4) and integrating this identity over the rectangle $ATBP$ in Fig. 3. In other words, we allow the curve C_3 to degenerate onto the characteristics $C_4 = AT$ and $C_5 = TB$ (where C_4 and C_5 are independent of ζ).

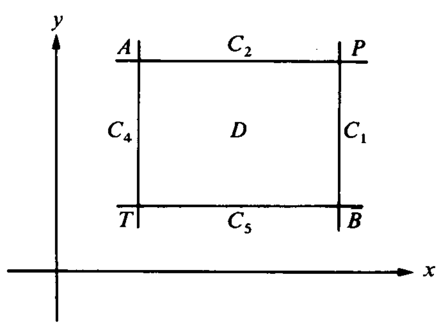

FIG. 3

Performing this deformation, and integrating by parts along the characteristics to eliminate the partial derivatives of u there, leads to

$$
\begin{aligned}
u(\xi,\eta,\zeta) = &+ 4\int_\Omega [S(A,z;\xi,\eta,\zeta)u(A,z) + S(B,z;\xi,\eta,\zeta)u(B,z) \\
&- S(T,z;\xi,\eta,\zeta)u(T,z)]\,dz
\end{aligned}
$$

(cont.)

$$+ 4 \iint_{C_4 \times \Omega} S_y(x, y, z; \xi, \eta, \zeta) u(x, y, z) \, dy \, dz \tag{2.20}$$

$$- 4 \iint_{C_5 \times \Omega} S_x(x, y, z; \xi, \eta, \zeta) u(x, y, z) \, dx \, dz$$

$$+ \iiint_{D \times \Omega} S(x, y, z; \xi, \eta, \zeta) F(x, y, z) \, dx \, dy \, dz.$$

Equation (2.20) gives the integral representation of the solution u of $L[u] = f$ as a linear functional of its initial data on two intersecting characteristic planes which is valid in the wedge bisected by the plane $y = x$ and bounded by the two characteristic planes, i.e., equation (2.20) gives the solution of the exterior characteristic initial value problem.

3. The elliptic equation (1.2). Similar integral representations to those developed in § 2 for hyperbolic equations can also be found for the elliptic equation (1.2), provided we make the further assumptions that the initial data is analytic in each of its independent variables and that, in the case of Cauchy's problem, the surface on which the data is prescribed is also analytic. To see this we make use of the fact that twice continuously differentiable solutions of (1.2) are analytic functions of their independent variables (cf. [13, p. 164]) and introduce the change of variables

$$x = x_1, \qquad z = x_2 + ix_3, \qquad z^* = x_2 - ix_3 \tag{3.1}$$

defining a nonsingular map of $\mathbb{C}^3$ into itself. The elliptic equation (1.2) can then be written as

$$u_{xx} + 4u_{zz^*} + Q(x, z, z^*)u = F(x, z, z^*), \tag{3.2}$$

which is formally of the same hyperbolic form as equation (2.3). Repeating the analysis of § 2 now leads to the integral representations (2.19) and (2.20) (with z replaced by x, x replaced by z, and y replaced by z^*) for the solution of the Cauchy and complex Goursat problems, respectively. (In the case of Cauchy's problem, $z = z(z^*)$ is the expression in conjugate coordinates of the intersection of the plane $x = \zeta$ with the initial surface.) In this case our analysis is reminiscent of Vekua's [15], [30] and Henrici's [15], [17] development of the analytic theory of elliptic equations in two independent variables. It is also similar to the integral operators constructed by Colton in [7] (see also [8], [9], [10], [16] and [28]).

The operators constructed in this paper have several advantages over the approach used in [7]:

(i) The form of the integral representations arises in a natural manner.

(ii) The integral representation of the solution to Cauchy's problem can be readily obtained. In particular, this considerably improves upon the results in [10] where the Cauchy data was required to be prescribed on the plane $x_1 = 0$ instead of on an analytic surface as in the present work, and where furthermore the coefficient $q(x_1, x_2, x_3)$ was required to be independent of x_1.

(iii) The nonhomogeneous equation can be treated.

On the other hand, several disadvantages must be mentioned. One of these is that difficulties arise in treating non-self-adjoint equations since the leading (singular) term of the fundamental solution S in general no longer satisfies the Goursat data as it does in the self-adjoint case. Extensions to higher dimensions also run into difficulties due to logarithmic terms appearing in the construction of S in an even number of independent variables and also due to the fact that the geodesic distance R between two points no longer has a pole-like singularity along the characteristics. The author is at present looking into these problems, and the results will hopefully be reported in a future paper.

We finally note in passing that different representations than those obtained in this paper can be derived for the solutions to improperly posed Cauchy problems for elliptic and hyperbolic equations by means of an appropriate change of variables in the complex domain and use of a fundamental solution (cf. [3], [13, pp. 614–621]). In this case the fundamental solution is not required to satisfy prescribed boundary data along the characteristics. On the other hand, new problems are created since the representation now includes terms involving the derivative of an improper integral and/or the finite parts of divergent integrals.

Note added in proof. The fact that the solution W of $L[u] = 0$ satisfying the Goursat data (2.15), (2.16) is entire follows from the results of Jan Persson in his paper *Linear Goursat problems for entire functions when the coefficients are variable*, Ann. Scoula Norm. Sup. Pisa, 23 (1969), pp. 87–98.

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THE NONCHARACTERISTIC CAUCHY PROBLEM FOR PARABOLIC EQUATIONS IN ONE SPACE VARIABLE*

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Abstract. An integral operator is constructed which maps ordered pairs of analytic functions onto analytic solutions of linear second order parabolic equations in one space variable with analytic coefficients. This operator is then used to construct a solution to the noncharacteristic Cauchy problem for parabolic equations in one space variable. Applications are made to the inverse Stefan problem and the analytic continuation of solutions to parabolic equations.

1. Introduction. Consider a thin block of ice at 0°C occupying the interval $0 \leqq x < \infty$ and suppose at $x = 0$ the temperature is given by a prescribed function $\varphi(t) > 0$ where $t \geqq 0$ denotes time. Then the ice will begin to melt and for $t > 0$ the water will occupy an interval $0 \leqq x < s(t)$. If $u(x, t)$ is the temperature of the water we have

$$\begin{aligned} \frac{k}{\rho c} u_{xx} - u_t &= 0 \qquad \text{for } 0 < x < s(t), \\ u(0, t) &= \varphi(t) \quad \text{for } t > 0, \\ u(s(t), t) &= 0 \qquad \text{for } t > 0, \end{aligned} \tag{1.1}$$

where c denotes heat capacity, ρ the density, and k the conductivity of the water. In (1.1) it is assumed that c, ρ and k are constants. The curve $x = s(t)$ is a free boundary and is not given a priori. However, from the law of conservation of energy we have

$$u_x(s(t), t) = \frac{-\lambda\rho}{k} \frac{ds(t)}{dt}, \tag{1.2}$$

where λ is the latent heat of fusion. Equations (1.1) and (1.2) constitute a free boundary problem (the Stefan problem) for the heat equation. In the more general case when c, ρ and k are not constants, but are functions of x and t, we arrive at a free boundary problem for a parabolic equation in one space variable with variable coefficients.

Free boundary problems for parabolic equations are in general quite difficult to solve, and in recent years attention has been given to a study of the inverse problem, i.e., given $s(t)$ to find $\varphi(t)$ (c.f., [2], [3], [4], [6, pp. 71–80]). In physical terms this means we are asking how to heat the water in order to melt the ice along a prescribed curve, and in certain situations (e.g., the growing of crystals) it is in this inverse problem that we are primarily interested. Such an inverse approach leads mathematically to the problem of solving a noncharacteristic Cauchy problem for a parabolic equation and difficulties arise due to the fact that this problem is improperly posed in the sense of Hadamard (c.f., [4], [5]). However, as a

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consequence of the Cauchy–Kowalewski theorem, the noncharacteristic Cauchy problem is well-posed in the complex domain, and hence we are led to impose the requirement that $s(t)$ be an analytic function of t.

However, even after making the assumption that $s(t)$ is analytic, we are still left with serious problems in providing a constructive approach for solving the inverse Stefan problem for parabolic equations with (possibly) variable coefficients. For example, even though a local solution can always be constructed via the Cauchy–Kowalewski theorem, such an approach is far too tedious for practical application, and (more seriously) may not converge in the full region in which the solution is needed, i.e., in a region containing (a portion of) the positive t-axis. On the other hand, in the special case when the coefficients of the parabolic equation are independent of time (e.g., the heat equation), a constructive method for solving the inverse Stefan problem has been given by C. D. Hill [4]. In theory Hill's approach also applies when the coefficients are time-dependent. However, in practice, this is not the case, since Hill's work is based on the construction of a fundamental solution $S(x, t; \xi, \tau)$ given by the series expansion

$$S(x, t; \xi, \tau) = \sum_{j=0}^{\infty} S_j(x, t; \xi)\frac{j!}{(t-\tau)^{j+1}}, \tag{1.3}$$

where in the case of time-dependent coefficients each $S_j(x, t; \xi)$, $j = 0, 1, 2, \cdots$, is in turn a solution of a nonhomogeneous, noncharacteristic Cauchy problem for a parabolic equation with time-dependent coefficients. To construct $S(x, t; \xi, \tau)$ via this method (and to determine its domain of regularity) is as tedious and impractical as using the Cauchy–Kowalewski theorem, and hence in this general case it is desirable to derive new methods for solving the noncharacteristic Cauchy problem.

Our approach to this problem is based on the construction of an integral operator which maps noncharacteristic Cauchy data onto solutions of (a canonical form of) the parabolic equation being investigated. The kernel of this operator can be expanded in an infinite series, each term of which is determined by a simple three term recursion relation. To guarantee the global existence of our operator we will make the assumption that the coefficients of the differential equation are entire functions of x and analytic in t for $|t| < t_0$ where t_0 is some positive constant. We will show that as a consequence of this assumption every solution of a linear parabolic equation in one space variable (with analytic coefficients) which is analytic in some (complex) neighborhood of the origin has an automatic analytic continuation into an infinite strip parallel to the x axis containing this neighborhood. This theorem generalizes analogous results obtained by Widder for the heat equation [7] and Hill for parabolic equations with time-independent coefficients [4].

2. Integral operators for parabolic equations. Consider the general linear homogeneous parabolic equation of the second order in one space variable written in normal form

$$u_{xx} + a(x, t)u_x + b(x, t)u - c(x, t)u_t = 0. \tag{2.1}$$

We shall make the assumption that the coefficients $a(x,t)$, $b(x, t)$ and $c(x, t)$ are analytic functions of the (complex) variables x and t for $|x| < \infty$ and $|t| < t_0$. By making the change of dependent variable

$$u(x,t) = v(x,t)\exp\left\{-\frac{1}{2}\int_0^x a(\xi,t)\,d\xi\right\} \tag{2.2}$$

we arrive at an equation for $v(x,t)$ of the same form as (2.1) but with $a(x,t) = 0$. Hence without loss of generality we can restrict our attention to equations of the form

$$L[u] \equiv u_{xx} + b(x,t)u - c(x,t)u_t = 0, \tag{2.3}$$

where $b(x,t)$ and $c(x,t)$ are analytic functions of x and t for $|x| < \infty$, $|t| < t_0$.

We now look for a solution of (2.3) in the form

$$u(x,t) = -\frac{1}{2\pi i}\oint_{|t-\tau|=\delta} E^{(1)}(x,t,\tau)f(\tau)\,d\tau - \frac{1}{2\pi i}\oint_{|t-\tau|=\delta} E^{(2)}(x,t,\tau)g(\tau)\,d\tau, \tag{2.4}$$

where $t_0 - |t| > \delta > 0$ and $f(\tau)$ and $g(\tau)$ are arbitrary analytic functions of τ for $|\tau| < t_0$. We shall furthermore ask that $E^{(1)}(x,t,\tau)$ and $E^{(2)}(x,t,\tau)$ satisfy the initial conditions

$$E^{(1)}(0,t,\tau) = \frac{1}{t-\tau}, \tag{2.5a}$$

$$E_x^{(1)}(0,t,\tau) = 0, \tag{2.5b}$$

$$E^{(2)}(0,t,\tau) = 0, \tag{2.6a}$$

$$E_x^{(2)}(0,t,\tau) = \frac{1}{t-\tau} \tag{2.6b}$$

and be analytic functions of their independent variables for $|x| < \infty$, $|t| < t_0$, $|\tau| < t_0$, $t \neq \tau$. We shall first construct the function $E^{(1)}(x,t,\tau)$. Setting $g(\tau) = 0$ and substituting (2.4) into the differential equation shows that, as a function of x and t, $E^{(1)}(x,t,\tau)$ must be a solution of $L[u] = 0$ for $t \neq \tau$. We now assume that $E^{(1)}(x,t,\tau)$ has the expansion

$$E^{(1)}(x,t,\tau) = \frac{1}{t-\tau} + \sum_{n=2}^{\infty} x^n P^{(n)}(x,t,\tau), \tag{2.7}$$

where the $P^{(n)}(x,t,\tau)$ are (analytic) functions to be determined. Note that if termwise differentiation is permitted the series (2.7) satisfies the initial conditions (2.5a) and (2.5b). Observe that, in contrast to the kernel (1.3) of Hill's integral operator, we are expanding the kernel $E^{(1)}(x,t,\tau)$ (and later the kernel $E^{(2)}(x,t,\tau)$) in powers of x (instead of powers of $1/(t-\tau)$). This will allow us to determine the coefficients $P^{(n)}(x,t,\tau)$ via a simple three term recursion relation instead of being forced to determine each coefficient as a solution of a noncharacteristic Cauchy problem for a parabolic equation as in Hill's work [4]. Indeed, if we substitute (2.7) into

$L[u] = 0$, we are immediately led to the following recursion formula for the $P^{(n)}(x, t, \tau)$:

$$\begin{aligned} &P^{(1)} = 0, \\ &P^{(2)} = -\frac{c}{2(t-\tau)^2} - \frac{b}{2(t-\tau)}, \\ &P^{(k+2)} = -\frac{2}{k+2}P_x^{(k+1)} - \frac{1}{(k+2)(k+1)}[P_{xx}^{(k)} + bP^{(k)} - cP_t^{(k)}], \quad k \geqq 1. \end{aligned} \tag{2.8}$$

We now let

$$\tilde{P}^{(k)}(x, t, \tau) = P^{(k)}(x, t, t - \tau). \tag{2.9}$$

Then (2.8) becomes

$$\begin{aligned} &\tilde{P}^{(1)} = 0, \\ &\tilde{P}^{(2)} = -\frac{c}{2\tau^2} - \frac{b}{2\tau}, \end{aligned} \tag{2.10}$$

$$\tilde{P}^{(k+2)} = -\frac{2}{k+2}\tilde{P}_x^{(k+1)} - \frac{1}{(k+2)(k+1)}[\tilde{P}_{xx}^{(k)} + b\tilde{P}^{(k)} - c\tilde{P}_t^{(k)} - c\tilde{P}_\tau^{(k)}], \quad k \geqq 1.$$

If we now define $Q^{(k)}(x, t, \tau)$ by the equation

$$\tilde{P}^{(k)}(x, t, \tau) = \tau^{-k}Q^{(k)}(x, t, \tau), \tag{2.11}$$

then (2.10) yields the following recursion formula for the $Q^{(k)}(x, t, \tau)$:

$$\begin{aligned} &Q^{(1)} = 0, \\ &Q^{(2)} = -\tfrac{1}{2}[c + \tau b], \\ &Q^{(k+2)} = -\frac{2\tau}{k+2}Q_x^{(k+1)} - \frac{2\tau}{(k+2)(k+1)}[\tau Q_{xx}^{(k)} + \tau b Q^{(k)} - \tau c Q_t^{(k)} \\ &\qquad\qquad + ckQ^{(k)} - \tau c Q_\tau^{(k)}], \qquad k \geqq 1. \end{aligned} \tag{2.12}$$

Now let M_0 be a positive constant such that

$$\begin{aligned} c(x, t) &\ll M_0(1 - x/r)^{-1}(1 - t/t_0)^{-1}, \\ b(x, t) &\ll M_0(1 - x/r)^{-1}(1 - t/t_0)^{-1} \end{aligned} \tag{2.13}$$

for $|x| < r$ and $|t| < t_0$. In (2.13) the symbol "$\ll$" means "is dominated by" (c.f., [1]). The main properties of dominants we will use are the following: If $f(x) \ll g(x)$ for $|x| < r$, then

$$\frac{df(x)}{dx} \ll \frac{dg(x)}{dx} \quad \text{for } |x| < r, \tag{2.14a}$$

$$f(x) \ll g(x)(1 - x/r)^{-1} \quad \text{for } |x| < r. \tag{2.14b}$$

Similar properties apply to functions of several complex variables. Using the property

$$\tau \ll 2t_0\left(1 - \frac{\tau}{2t_0}\right)^{-1} \tag{2.15}$$

for $|\tau| < 2t_0$ we shall now show by induction that there exist positive constants M_n, $n = 1, 2, \cdots$, and ε (where ε can be chosen arbitrarily small and is independent of n, and M_n is a bounded function of n) such that for $|x| < r$, $|t| < t_0$, $|\tau| < 2t_0$, we have

$$\begin{aligned} Q^{(n+1)} \ll {} & M_{n+1}4^{n+1}t_0^{n+1}(3/2 + \varepsilon)^{n+1} \\ & \cdot\left(1 - \frac{x}{r}\right)^{-(n+1)}\left(1 - \frac{t}{t_0}\right)^{-(n+1)}\left(1 - \frac{\tau}{2t_0}\right)^{-(2n+2)} r^{-(n+1)}, \\ & \qquad n = 0, 1, 2, \cdots. \end{aligned} \tag{2.16}$$

Equation (2.16) is clearly true for $n = 0$ and $n = 1$. Assume now that it is true for $n = k - 1$ and $n = k$. Then from (2.12)–(2.15) we have

$$\begin{aligned} Q^{(k+2)} \ll {} & \left\{\frac{(k+1)}{(k+2)}M_{k+1}4^{k+2}t_0^{k+2}(3/2+\varepsilon)^{k+1}\right. \\ & \left. + \frac{M_k4^{k+1}t_0^{k+1}(3/2+\varepsilon)^k}{(k+2)(k+1)}(2t_0k(k+1) + 2M_0t_0r^2 + 4M_0kr^2)\right\} \\ & \cdot\left(1 - \frac{x}{r}\right)^{-(k+2)}\left(1 - \frac{t}{t_0}\right)^{-(k+2)}\left(1 - \frac{\tau}{2t_0}\right)^{-(2k+4)} r^{-(k+2)} \\ \ll {} & 4^{k+2}t_0^{k+2}(3/2+\varepsilon)^{k+1}\left[M_{k+1} + \frac{kM_k}{2(k+2)(3/2+\varepsilon)}\right. \\ & \left. + \frac{M_0r^2M_k}{2(k+2)(k+1)(3/2+\varepsilon)} + \frac{kM_0r^2M_k}{(k+2)(k+1)t_0(3/2+\varepsilon)}\right] \\ & \cdot\left(1 - \frac{x}{r}\right)^{-(k+2)}\left(1 - \frac{t}{2t_0}\right)^{-(k+2)}\left(1 - \frac{\tau}{2t_0}\right)^{-(2k+4)} r^{-(k+2)}. \end{aligned} \tag{2.17}$$

If we now set

$$\begin{aligned} M_{k+2} = (3/2+\varepsilon)^{-1}\left[M_{k+1} + \frac{M_k}{(3/2+\varepsilon)}\left(\frac{k}{2(k+2)} + \frac{M_0r^2}{2(k+2)(k+1)}\right.\right. \\ \left.\left. + \frac{kM_0r^2}{(k+2)(k+1)t_0}\right)\right], \end{aligned} \tag{2.18}$$

we have shown that (2.16) is true for $n = k + 1$, thus completing the induction step. It remains to be shown from (2.18) that M_k is a bounded function of k. For $k \geqq k_0 = k_0(\varepsilon)$ we have from (2.18) that

$$M_{k+2} \leqq (3/2+\varepsilon)^{-1}\left[M_{k+1} + \frac{M_k}{(3/2+\varepsilon)}(1/2 + \varepsilon/2)\right], \qquad k \geqq k_0. \tag{2.19}$$

If $M_{k+1} \leqq M_k$ for $k \geqq k_0$ we are done, for then we have $M_k \leqq \max\{M, M_2, \cdots, M_{k_0}\}$. Suppose then that there exists $k_1 \geqq k_0$ such that $M_{k_1+1} > M_{k_1}$. Then from (2.19) we have

$$\begin{aligned} M_{k_1+2} &< (3/2+\varepsilon)^{-1}\left[M_{k_1+1} + M_{k_1+1}\frac{(1/2+\varepsilon/2)}{(3/2+\varepsilon)}\right] \\ &= \frac{(2+3/2\varepsilon)}{(3/2+\varepsilon)(3/2+\varepsilon)}M_{k_1+1} < M_{k_1+1}, \end{aligned} \tag{2.20}$$

and by induction

$$M_{k_1+m} \leqq M_{k_1+1} \tag{2.21}$$

for $m = 1, 2, 3, \cdots$. Hence $M_k \leqq \max\{M_1, M_2, \cdots, M_{k_1+1}\}$ and we can conclude that M_k is a bounded function of k.

We now return to the convergence of the series (2.7). Let δ_0, δ_1 and $\alpha > 1$ be positive numbers and let

$$\begin{aligned} &|x| \leqq r/\alpha, && |\tau| \leqq t_0, \\ &|t| \leqq t_0/(1+\delta_1), && |t-\tau| \geqq \delta_0. \end{aligned} \tag{2.22}$$

Then

$$\begin{aligned} &\left(1-\frac{x}{r}\right) \geqq \frac{\alpha-1}{\alpha}, && \left(1-\frac{\tau}{2t_0}\right) \geqq \frac{1}{2}, \\ &\left(1-\frac{t}{t_0}\right) \geqq \frac{\delta_1}{1+\delta_1}, && |t-\tau| \leqq t_0\left(\frac{2+\delta_1}{1+\delta_1}\right) < 2t_0. \end{aligned} \tag{2.23}$$

From (2.9) and (2.11) we have

$$P^{(k)}(x,t,\tau) = (t-\tau)^{-k}Q^{(k)}(x,t,t-\tau). \tag{2.24}$$

Hence for x, t and τ restricted as in (2.22) we have from (2.16) and (2.24) that the series (2.7) is majorized by

$$\frac{1}{\delta_0} + \sum_{n=2}^{\infty} \frac{M_n 16^n t_0^n (3/2+\varepsilon)^n(\alpha-1)^n(1+\delta_1)^n}{\alpha^{2n}\delta_0^n\delta_1^n}. \tag{2.25}$$

Owing to the fact that M_n is a bounded function of n it is seen that if α is chosen sufficiently large then the series (2.25) converges. Since δ_0, δ_1 and ε are arbitrarily small (and independent of r) and r can be chosen arbitrarily large, we can now conclude that the series (2.7) converges uniformly and absolutely for $|x| \leqq r$, $|t| \leqq t_0/(1+\delta_1)$, $|\tau| \leqq t_0$ and $|t-\tau| \geqq \delta_0$ for δ_0 and δ_1 arbitrarily small and r arbitrarily large. Since each term of the series (2.7) is an analytic function of the variables x, t and τ for $|x| < \infty$, $|t| < t_0$, $|\tau| < t_0$, $\tau \neq t$, we can conclude that $E^{(1)}(x,t,\tau)$ exists and is an analytic function of its independent variables for $|x| < \infty$, $|t| < t_0$, $|\tau| < t_0$ and $t \neq \tau$. At the point $t = \tau$, $E^{(1)}(x,t,\tau)$ has an essential singularity. It is clear from our majorization argument that termwise differentiation of the series (2.7) is permissible and hence $E^{(1)}(x,t,\tau)$ satisfies the differential equation (2.3) and the initial conditions (2.5a), (2.5b).

We now turn our attention to the construction of the function $E^{(2)}(x, t, \tau)$. Setting $f(\tau) = 0$ in (2.4) and substituting this equation into (2.3) shows that, as a function of x and t, $E^{(2)}(x, t, \tau)$ must be a solution of $L[u] = 0$ for $t \neq \tau$. We now assume that $E^{(2)}(x, t, \tau)$ has the expansion

$$E^{(2)}(x, t, \tau) = \frac{x}{t - \tau} + \sum_{n=3}^{\infty} x^n p^{(n)}(x, t, \tau), \tag{2.26}$$

where the $p^{(n)}(x, t, \tau)$ are (analytic) functions to be determined. We again note that if termwise differentiation is permitted the series (2.26) satisfies the initial conditions (2.6a), (2.6b). Substituting (2.26) into (2.3) leads to the following recursion formulas for the $p^{(n)}(x, t, \tau)$:

$$\begin{aligned} p^{(2)} &= 0, \\ p^{(3)} &= -\frac{c}{6(t - \tau)^2} - \frac{b}{6(t - \tau)}, \\ p^{(k+2)} &= -\frac{2}{k + 2} p_x^{(k+1)} - \frac{1}{(k + 2)(k + 1)}[p_{xx}^{(k)} + bp^{(k)} - cp_t^{(k)}], \quad k \geqq 2. \end{aligned} \tag{2.27}$$

The recursion scheme (2.27) is essentially identical to the scheme given in (2.8) and following our previous analysis showing the convergence of the series (2.7) we can again verify that the series (2.26) defines an analytic function of x, t and τ for $|x| < \infty, |t| < t_0, |\tau| < t_0, t \neq \tau$, which satisfies $L[u] = 0$ and the initial data (2.6a), (2.6b). At the point $t = \tau$, $E^{(2)}(x, t, \tau)$ has an essential singularity.

We have now shown that the integral operator defined by (2.4) exists and maps ordered pairs of analytic functions onto analytic solutions of $L[u] = 0$. It is a simple matter to show that in fact every solution of $L[u] = 0$ which is analytic for $|t| < t_0$, $|x| < x_0$, can be represented in the form of (2.4). For let $u(x, t)$ be an analytic solution of $L[u] = 0$ and set $u(0, \tau) = f(\tau)$, $u_x(0, \tau) = g(\tau)$. Then $f(\tau)$ and $g(\tau)$ are analytic for $|\tau| < t_0$. Define

$$\begin{aligned} w(x, t) = &-\frac{1}{2\pi i}\oint_{|t-\tau|=\delta} E^{(1)}(x, t, \tau) f(\tau)\, d\tau \\ &-\frac{1}{2\pi i}\oint_{|t-\tau|=\delta} E^{(2)}(x, t, \tau) g(\tau)\, d\tau. \end{aligned} \tag{2.28}$$

Then $w(x, t)$ is an analytic solution of $L[u] = 0$ and from (2.5a), (2.5b), (2.6a), (2.6b) we have

$$\begin{aligned} w(0, t) &= -\frac{1}{2\pi i}\oint_{|t-\tau|=\delta} \frac{f(t)}{t - \tau}\, d\tau = f(t), \\ w_x(0, t) &= -\frac{1}{2\pi i}\oint_{|t-\tau|=\delta} \frac{g(\tau)}{t - \tau}\, d\tau = g(t); \end{aligned} \tag{2.29}$$

i.e., the Cauchy data for $w(x, t)$ and $u(x, t)$ agree on the noncharacteristic curve $x = 0$. From the Cauchy–Kowalewski theorem we can now conclude that $u(x, t) = w(x, t)$, i.e., $u(x, t)$ can be represented in the form of (2.4).

3. The noncharacteristic Cauchy problem. Consider the parabolic equation (2.1) where the coefficients $a(x,t)$, $b(x,t)$, $c(x,t)$ are analytic functions of the (complex) variables x and t for $|x| < \infty$ and $|t - t_0| < t_0$. Suppose we wish to construct a solution of this equation which satisfies the Cauchy data

$$\begin{aligned} u(s(t),t) &= f(t), \\ u_x(s(t),t) &= g(t), \end{aligned} \tag{3.1}$$

where $x = s(t)$ is a noncharacteristic curve and $f(t)$, $g(t)$ and $s(t)$ are analytic for $|t - t_0| < t_0$. We note that the inverse Stefan problem is of this form where $f(t) = 0$ and $g(t) = -(\lambda\rho/k)\, ds(t)/dt$. By making the nonsingular change of variables

$$\xi_1 = x - s(t), \qquad \xi_2 = t - t_0, \tag{3.2}$$

we arrive at an equation of the same form as (2.1) with the coefficients analytic for $|\xi_1| < \infty$ and $|\xi_2| < t_0$. Under the transformation (3.2) the curve $x = s(t)$ is transformed into the straight line $\xi_1 = 0$. If we now apply the change of variables (2.2) we arrive at an equation of the form (2.3) in the variables ξ_1 and ξ_2 with Cauchy data prescribed along $\xi_1 = 0$. As shown at the end of the last section, this problem can be solved by using the operator defined by (2.4). Hence, if the coefficients and interphase boundary are analytic in appropriate regions, we have a constructive method for solving the noncharacteristic Cauchy problem (2.1), (3.1). Note that due to the factor of $(k + 2)^{-1}$ which appears in each term of the recursion formulas (2.8) and (2.27), the convergence of the series expansions of $E^{(1)}(x,t,\tau)$ and $E^{(2)}(x,t,\tau)$ is in general quite rapid. Hence close approximations can usually be made by truncating these series after a few terms and using the resulting approximate E-functions in (2.4).

As a simple example of the above method we shall now construct a solution to the (normalized) inverse Stefan problem for the heat equation

$$u_{xx} = u_t, \tag{3.3}$$

$$u(s(t),t) = 0, \tag{3.4a}$$

$$u_x(s(t),t) = -ds(t)/dt \tag{3.4b}$$

in the special case when $s(t) = t$. The transformation (3.2) (with $t_0 = 0$) reduces this problem to

$$w_{\xi_1\xi_1} + w_{\xi_1} = w_{\xi_2}, \tag{3.5}$$

$$w(0,\xi_2) = 0, \tag{3.6a}$$

$$w_{\xi_1}(0,\xi_2) = -1, \tag{3.6b}$$

where $w(\xi_1,\xi_2) = u(\xi_1 + \xi_2, \xi_2) = u(x,t)$. If we now set

$$w(\xi_1,\xi_2) = v(\xi_1,\xi_2)\exp(-\tfrac{1}{2}\xi_1), \tag{3.7}$$

equations (3.5), (3.6a), (3.6b) become

$$v_{\xi_1\xi_1} - \tfrac{1}{4}v_{\xi_1} = v_{\xi_2}, \tag{3.8}$$

(3.9a) $$v(0, \xi_2) = 0,$$

(3.9b) $$v_{\xi_1}(0, \xi_2) = -1.$$

From the form of the differential equation (3.8) and the initial conditions (3.9a), (3.9b) it is seen that we only need to compute the coefficient of $(\xi_2 - \tau)^{-1}$ in the series expansion for $E^{(2)}(\xi_1, \xi_2, \tau)$. From (2.26) we have

(3.10)
$$\begin{aligned} E^{(2)}(\xi_1, \xi_2, \tau) &= \frac{2}{\xi_2 - \tau}\left[\frac{\xi_1}{2} + \sum_{k=1}^{\infty} \frac{(\frac{1}{2}\xi_1)^{2k+1}}{(2k+1)!}\right] \\ &\quad + \text{terms involving higher powers of } (\xi_2 - \tau)^{-1} \\ &= \frac{2\sinh(\frac{1}{2}\xi_1)}{\xi_2 - \tau} \\ &\quad + \text{terms involving higher powers of } (\xi_2 - \tau)^{-1}, \end{aligned}$$

and hence, from (2.4) we have

(3.11)
$$\begin{aligned} v(\xi_1, \xi_2) &= \frac{1}{2\pi i}\oint_{|\xi_2 - \tau| = \delta} \frac{2\sinh(\frac{1}{2}\xi_1)}{\xi_2 - \tau}\, d\tau \\ &= -2\sinh(\tfrac{1}{2}\xi_1). \end{aligned}$$

Then $w(\xi_1, \xi_2) = e^{-\xi_1} - 1$ and the solution of (3.3), (3.4a), (3.4b) is given by

(3.12) $$u(x, t) = e^{t-x} - 1.$$

In particular we see from (3.12) that the temperature distribution $\varphi(t)$ needed at $x = 0$ in order to make the ice melt along the curve $x = t$ is given by

(3.13) $$\varphi(t) = u(0, t) = e^t - 1.$$

We now conclude by stating a result on the analytic continuation of solutions to parabolic equations with analytic coefficients which generalizes those obtained by Widder for the heat equation [7] and Hill for parabolic equations with time-independent coefficients [4]. The theorem follows immediately from the transformation (2.2), the representation (2.4), and the fact that $E^{(1)}(x, t, \tau)$ and $E^{(2)}(x, t, \tau)$ are analytic for $|x| < \infty$, $|t| < t_0$, $|\tau| < t_0$, $t \neq \tau$.

THEOREM. *Let $u(x, t)$ be a solution of* (2.1) *which is an analytic function of the complex variables x and t for $|t| < t_0$, $|x| < x_0$. Suppose the coefficients $a(x, t)$, $b(x, t)$ and $c(x, t)$ are analytic functions of the complex variables x and t for $|x| < \infty$, $|t| < t_0$. Then $u(x, t)$ can be analytically continued into the strip $|x| < \infty$, $|t| < t_0$.*

An important application of this theorem is the conclusion that the solution of the inverse Stefan problem can always be analytically continued into a domain containing the line $x = 0$, provided the coefficients of the parabolic equation are analytic for $|x| < \infty$, $|t - t_0| < t_0$, and the interphase boundary is an analytic function of t for $|t - t_0| < t_0$. In particular the above theorem implies that $u(0, t) = \varphi(t)$ is an analytic function of t for $|t - t_0| < t_0$. Thus we can conclude that if $u(0, t) = \varphi(t)$ is not analytic then neither is the interphase boundary $s(t)$. This partially answers the problem posed by Rubinstein in [6, p. 353].

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THE NONCHARACTERISTIC CAUCHY PROBLEM FOR PARABOLIC EQUATIONS IN TWO SPACE VARIABLES[1]

DAVID COLTON

ABSTRACT. An integral representation is obtained for the solution of the noncharacteristic Cauchy problem for second order parabolic equations in two space variables with entire, time independent coefficients. This is accomplished through the use of contour integration techniques and the calculus of residues in the space of several complex variables.

I. **Introduction.** In this note we will give an integral representation of the solution to the noncharacteristic Cauchy problem for second order parabolic equations in two space variables with entire, time independent coefficients. Although improperly posed in the real domain, such problems nevertheless arise when inverse methods are used to study free boundary problems for parabolic equations, for example in the case of the inverse Stefan problem for the heat equation (cf. [2]). The noncharacteristic Cauchy problem for parabolic equations becomes properly posed if the behavior of the solution in the complex domain is taken into consideration, and hence we make use of contour integration techniques and the calculus of residues in order to obtain our desired integral representation. The representation of the solution obtained in this paper is valid in the large (in the space of several complex variables) as opposed to the local solution obtained via the Cauchy-Kowalewski theorem. This is of crucial importance as far as the applications are concerned. Such a representation will also allow us to obtain results on the analytic continuation of solutions to parabolic equations.

The problem considered here has previously been studied by C. D. Hill in [3] through the use of a one-parameter family of conformal mappings (thus making the coefficients of the differential equation dependent on time) and the construction of a new fundamental solution for parabolic equations. In Hill's work this new fundamental solution is constructed by

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recursively solving an infinite family of complex Goursat problems for nonhomogeneous parabolic equations in two space variables. Our approach is much simpler. In particular we avoid the use of conformal mappings and construct our fundamental solution in one step through the use of a generalized version of the Cauchy-Kowalewski theorem.

In the analysis which follows C^2 denotes the space of two complex variables.

II. The noncharacteristic Cauchy problem for parabolic equations. We consider the general linear, second order parabolic equation in two space variables (with time independent coefficients) written in normal form

$$u_{xx} + u_{yy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u - d(x, y)u_t = 0 \tag{2.1}$$

and make the assumption that the coefficients in equation (2.1) are entire functions of their independent (complex) variables. Now let $u(x, y, t)$ be an analytic solution of equation (2.1) and make the nonsingular change of variables in C^2 defined by

$$z = x + iy, \qquad z^* = x - iy. \tag{2.2}$$

Note that $z^*=\bar{z}$ if and only if x and y are real. Under such a transformation equation (2.1) assumes the form

$$\begin{aligned} L[U] = U_{zz^*} &+ A(z, z^*)U_z + B(z, z^*)U_{z^*} \\ &+ C(z, z^*)U - D(z, z^*)U_t = 0 \end{aligned} \tag{2.3}$$

where

$$U(z, z^*, t) = u\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}, t\right),$$

$$\begin{aligned} A(z, z^*) &= \frac{1}{4}\left[a\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right) + ib\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right)\right], \\ B(z, z^*) &= \frac{1}{4}\left[a\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right) - ib\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right)\right], \\ C(z, z^*) &= \frac{1}{4}c\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right), \\ D(z, z^*) &= \frac{1}{4}d\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right). \end{aligned} \tag{2.4}$$

We now introduce the adjoint equation

$$M[V] = V_{zz^*} - \frac{\partial(AV)}{\partial z} - \frac{\partial(BV)}{\partial z^*} + CV + DV_t = 0. \tag{2.5}$$

Let V be a solution of $M[V]=0$ (V will be prescribed more precisely in a few moments) and use Stokes theorem to integrate the identity

$$\begin{aligned} VL[U] - UM[V] = {} & (AVU + \tfrac{1}{2}VU_{z^*} - \tfrac{1}{2}V_{z^*}U)_z \\ & + (BVU + \tfrac{1}{2}VU_z - \tfrac{1}{2}V_zU)_{z^*} - (DVU)_t \end{aligned} \tag{2.6}$$

over the torus $G\times\Omega$, where Ω is the circle $|t-\tau|=\delta>0$ in the complex t plane and G is a two-dimensional cell in (z, z^*) space bounded by a noncharacteristic analytic curve C_3 and line segments C_1 and C_2 which lie in the characteristic planes $z=\zeta$ and $z^*=\bar{\zeta}$ and join the point $R=(\zeta, \bar{\zeta})$ to the curve C_3 at the points Q and P respectively (see Figure 1 below).

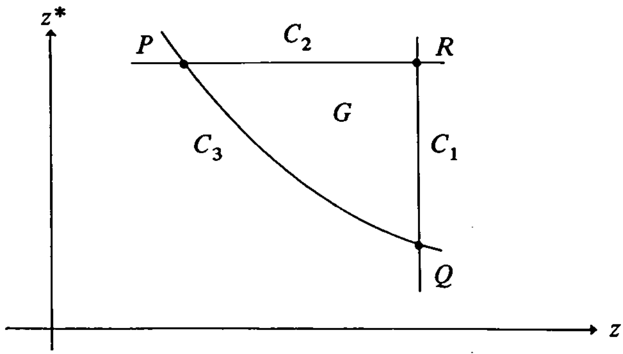

FIGURE 1

The result of this integration (after performing an integration by parts on the right-hand side of the resulting identity) is, in the notation of the calculus of exterior differential forms,

$$\begin{aligned} 0 = {} & \iint_{C_1\times\Omega} (AV - V_{z^*})U\,dz^*\,dt - \iint_{C_2\times\Omega} (BV - V_z)U\,dz\,dt \\ & + \int_{\Omega} [V(R,t)U(R,t) - \tfrac{1}{2}V(P,t)U(P,t) - \tfrac{1}{2}V(Q,t)U(Q,t)]\,dt \\ & + \iint_{C_3\times\Omega} [(AVU + \tfrac{1}{2}VU_{z^*} - \tfrac{1}{2}V_{z^*}U)\,dz^*\,dt \\ & \qquad\qquad - (BVU + \tfrac{1}{2}VU_z - \tfrac{1}{2}V_zU)\,dz\,dt], \end{aligned} \tag{2.7}$$

where we have made use of the fact that $dz\,dz^*=0$ on $\partial C\times\Omega$. Note that an expression of the form $U(R, t)$ is a function of three independent variables, i.e. $U(R,t)=U(\zeta, \bar{\zeta}, t)$ where $(\zeta, \bar{\zeta})$ are the Cartesian coordinates of the point R in C^2.

We now want to choose V and the domain G such that equation (2.7) reduces to an expression for the solution of equation (2.1) satisfying prescribed Cauchy data on a noncharacteristic analytic surface S.

We first assume that the intersection of the plane $t=\tau$ with the surface S is a one-dimensional curve $C_3'=C_3'(\tau)$. Suppose C_3' is described by the equation $F(x, y; \tau)=0$. Since S is analytic we can write

$$F\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}; \tau\right)=0$$

and this is the equation for C_3' in (z, z^*) space. We will choose $C_3=C_3(\tau)$ to be an analytic curve lying on this complex extension of $C_3'(\tau)$ and intersecting the characteristic planes $z=\zeta$ and $z^*=\bar{\zeta}$ at the points Q and P, respectively. We now turn our attention to the construction of V. In particular we ask that $V=V(z, z^*; \zeta, \bar{\zeta}; t-\tau)$ be an entire function of z, z^* and $t-\tau$, except for a simple pole with residue one at the point $(z, z^*, t)=(\zeta, \bar{\zeta}, \tau)$. We also ask that V satisfy the initial conditions

$$V_{z^*} = AV \tag{2.8}$$

on $C_1\times\Omega$, and

$$V_z = BV \tag{2.9}$$

on $C_2\times\Omega$. The function V can be constructed in the following manner. In equation (2.5) let $t-\tau=1/\xi$. Then equation (2.5) becomes

$$W_{zz^*} - \partial(AW)/\partial z - \partial(BW)/\partial z^* + CW - \xi^2 DW_\xi = 0 \tag{2.10}$$

where $W(z, z^*; \zeta, \bar{\zeta}; \xi)=V(z, z^*; \zeta, \bar{\zeta}; 1/\xi)$. Integrating equations (2.8) and (2.9) along the characteristics, and requiring V to have a simple pole with residue one at $(z, z^*, t)=(\zeta, \bar{\zeta}, \tau)$, gives

$$V(\zeta, z^*; \zeta, \bar{\zeta}; t-\tau) = \frac{1}{t-\tau}\exp\left\{\int_{\bar{\zeta}}^{z^*} A(\zeta, \sigma)\, d\sigma\right\}, \tag{2.11}$$

$$V(z, \bar{\zeta}; \zeta, \bar{\zeta}; t-\tau) = \frac{1}{t-\tau}\exp\left\{\int_{\zeta}^{z} B(\sigma, \bar{\zeta})\, d\sigma\right\}, \tag{2.12}$$

i.e.

$$W(\zeta, z^*; \zeta, \bar{\zeta}; \xi) = \xi\exp\left\{\int_{\bar{\zeta}}^{z^*} A(\zeta, \sigma)\, d\sigma\right\}, \tag{2.13}$$

$$W(z, \bar{\zeta}; \zeta, \bar{\zeta}; \xi) = \xi\exp\left\{\int_{\zeta}^{z} B(\sigma, \bar{\zeta})\, d\sigma\right\}. \tag{2.14}$$

From Hörmander's generalized Cauchy-Kowalewski theorem [4] we can locally construct (by iteration) a unique solution W satisfying equations

(2.10), (2.13), and (2.14). From [5] we can conclude that W is in fact an entire function of z, z^*, and ξ. Setting $\xi=1/(t-\tau)$ in the expression obtained in this manner for W gives us the function $V(z, z^*; \zeta, \bar{\zeta}; t-\tau)$ with the desired properties.

If in equation (2.7) we now let $C_3=C_3(\tau)$ lie on the complex extension of the intersection of S with the plane $t=\tau$ and let $V=V(z, z^*; \zeta, \bar{\zeta}; t-\tau)$ be the function just constructed, we have

$$\begin{aligned} U(\zeta, \bar{\zeta}, \tau) = {} & \frac{1}{4\pi i}\int_\Omega [V(P, t)U(P, t) + V(Q, t)U(Q, t)]\, dt \\ & - \frac{1}{2\pi i}\iint_{C_3(\tau)\times\Omega} [(AVU + \tfrac{1}{2}VU_{z^*} - \tfrac{1}{2}V_{z^*}U)\, dz^*\, dt \\ & \qquad\qquad - (BVU + \tfrac{1}{2}VU_z - \tfrac{1}{2}V_zU)\, dz\, dt]. \end{aligned} \tag{2.15}$$

For $(\zeta, \bar{\zeta})$ sufficiently near the initial surface S, and for δ sufficiently small, equation (2.15) gives the desired integral representation of the solution to the noncharacteristic Cauchy problem for equation (2.1), provided we first deform the surface $C_3\times\Omega$ until it lies on the complex extension of the initial surface S. (Recall from equation (2.4) that $U(z, \bar{z}, t)=u(x, y, t)$.) Equation (2.15) can now be used to obtain a global solution via the straightforward use of analytic continuation of the Cauchy data and deformation of the region of integration. In particular such a procedure yields results on the analytic continuation of solutions to parabolic equations along characteristic hyperplanes in terms of the domain of regularity of the Cauchy data of these solutions along noncharacteristic analytic surfaces. For example if for each fixed t in a (complex) neighborhood of $t=\tau$ the Cauchy data for $U(z, z^*, t)$ is regular in a domain D which is conformally symmetric (cf. [1]) with respect to $C_3'(\tau)$, then we can conclude that the restriction of $U(z, z^*, t)$ to the plane $t=\tau$ is an analytic function of z and z^* in $D\times D^*$ where $D^*=\{z^*: \bar{z}^*\in D\}$. It is of interest to contrast this result with the corresponding theorem for elliptic equations obtained in [1].

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THE INVERSE STEFAN PROBLEM *)

by

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I. Introduction

The use of inverse methods to solve boundary value problems for partial differential equations dates back to the time of Euler and has played an important role in various areas of the mechanics of continua (c.f. [13]). In recent years such methods have been used with particular success in the investigation of free boundary problems in fluid mechanics ([8]). Crudly speaking, inverse methods can be described by the statement that, instead of finding a solution to a given problem, a physically reasonable problem is found for a given solution. Such an approach often leads to the formulation of problems which are improperly posed in the real domain, and hence it thus becomes necessary to examine the behavior of solutions to partial differential equations in the complex domain. Indeed, the use of inverse methods in applied mathematics has motivated many of the more important recent developments in the analytic theory of elliptic partial differential equations ([7], chapter 16; [9], chapter 5; [2]; pp. 133-141).

In this talk we will outline some recent progress we have made in the investigation of various topics in the analytic theory of parabolic equations arising out of the study of free boundary problems in heat conduction. As a simple example of such a problem consider the following single phase Stefan problem for one dimensional heat equation: Find functions $s(t)$ und $u(x,t)$ such that

$$(1.1)\qquad \begin{aligned} &u_{xx}-u_t = o && \text{for } o < x < s(t)\\ &u(o,t) = \varphi(t) && \text{for } t \geqslant o\\ &u(s(t),t) = o && \text{for } t \geqslant o\\ &u_x(s(t),t) = -\frac{ds(t)}{dt} && \text{for } t \geqslant o \end{aligned}$$

where $\varphi(t)$ is a prescribed continuous function for $t \geqslant o$. Such a problem arises physically in connection with the melting of solids (c.f. [14]) and constitutes a free boundary problem of the one dimensional heat equation. In more general situations we are concerned with free boundary problems similar to (1.1), but with the one dimensional heat equation replaced by a second order parabolic equation in one, two or three space

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variables with (possibly) variable coefficients and (in the case of two or three space variables) the functions $s(t)$ and $\varphi(t)$ replaced by functions of several independent variables.

Due to the difficulties inherent in solving free boundary problems such as the one described above (particularly in more than one space variable) we propose to consider the inverse problem, i.e. (in the case of one space dimension) given $s(t)$ to find $u(x,t)$ and $\varphi(t)$. The succes of such an approach lies in being able to obtain the (global) solution to a non-characteristic Cauchy problem for a parabolic equation and difficulties arise due to the fact that this problem is improperly posed in the sense of Hadamard (c.f. [12]). However, as a consequence of the Cauchy-Kowalewski theorem, the non-characteristic Cauchy problem is well posed in the complex domain, and hence in order to make our inverse problem properly posed we are led to the requirement that the free boundary be an analytic function of its independent variables. We note that in this situation a local solution can always be constructed via the Cauchy-Kowalewski theorem. However, this approach is unsuitable for our purposes since the calculations involved are far too tedious for practical applications and (more seriously) the power series solution constructed via such a method may not converge in the full region where the solution is needed (i.e. in a region containing a portion of the surface $x = o$). Hence the approach to be presented in this talk will instead be based on the construction of integral operators which map analytic functions of one and several complex variables onto solutions of linear parabolic equations. These operators will then be used in conjunction with contour integration and the calculus of residues in the space of several complex variables to derive explicit integral representations of the solution to the non-characteristic Cauchy problem for parabolic equations. In addition to their application in solving inverse free boundary problems these integral representations will also allow us to obtain a variety of results on the analytic continuation of analytic solutions to parabolic equations.

II. The Non-Characteristic Cauchy Problem for Parabolic Equations in One Space Variable

Consider the general linear homogeneous parabolic equation of the second order in one space variable written in normal form

$$u_{xx} + a(x,t)u_x + b(x,t)u - c(x,t)u_t = o \tag{2.1}$$

where the coefficients $a(x,t)$, $b(x,t)$ and $c(x,t)$ are analytic functions of the (complex) variables x and t for $|x| < \infty$ and $|t-t_o| < t_o$ for some positive positive constant t_o. Suppose we wish to construct a solution of this equation which satisfies the Cauchy data

$$\begin{aligned} u(s(t),t) &= f(t) \\ u_x(s(t),t) &= g(t) \end{aligned} \tag{2.2}$$

where $x = s(t)$ is a non-characteristic curve and $f(t)$, $g(t)$ and $s(t)$ are analytic for $|t-t_o| < t_o$. By making the non-singular change of variables

$$\begin{aligned} \xi_1 &= x - s(t) \\ \xi_2 &= t - t_o \end{aligned} \tag{2.3}$$

we arrive at an equation of the same form as equation (2.1) with the coefficient analytic for $|\xi_1| < \infty$ and $|\xi_2| < t_o$. Under the transformation (2.3) the curve $x = s(t)$ is transformed into the straight line $\xi_1 = o$. Hence without loss of generality we can assume in equations (2.1) and (2.2) that $s(t) = o$, $a(x,t)$, $b(x,t)$ and $c(x,t)$ are analytic for $|x| < \infty$, $|t| < t_o$, and $f(t)$ and $g(t)$ are analytic for $|t| < t_o$. By making the change of dependent variables

$$u(x,t) = v(x,t) \exp\left\{-\frac{1}{2}\int_o^x a(\xi,t)d\xi\right\} \tag{2.4}$$

we arrive at an equation for $v(x,t)$ of the same form as equation (2.1) but with $a(x,t) = o$. Hence we can restrict our attention to Cauchy problems of the form

$$u_{xx} + b(x,t)u - c(x,t)u_t = o \tag{2.5}$$

$$\begin{aligned} u(o,t) &= f(t) \\ u_x(o,t) &= g(t) \end{aligned} \tag{2.6}$$

where $b(x,t)$ and $c(x,t)$ are analytic functions of x and t for $|x| < \infty$, $|t| < t_o$, and $f(t)$ and $g(t)$ are analytic for $|t| < t_o$.

In [3] it is shown that the solution to the Cauchy problem (2.5) and (2.6) can be written in the form

$$\begin{aligned} u(x,t) = &-\frac{1}{2\pi i}\oint_{|t-\tau|=\delta} E^{(1)}(x,t,\tau)f(\tau)d\tau \\ &-\frac{1}{2\pi i}\oint_{|t-\tau|=\delta} E^{(2)}(x,t,\tau)g(\tau)d\tau \end{aligned} \tag{2.7}$$

where $\delta > 0$ and

$$E^{(1)}(x,t,\tau) = \frac{1}{t-\tau} + \Sigma_{n=2}^{\infty} x^n P^{(1,n)}(x,t,\tau) \tag{2.8}$$

$$E^{(2)}(x,t,\tau) = \frac{1}{t-\tau} + \Sigma_{n=3}^{\infty} x^n P^{(2,n)}(x,t,\tau) \tag{2.9}$$

with the functions $P^{(1,n)}(x,t,\tau)$ and $P^{(2,n)}(x,t,\tau)$ defined by the following three term recursion formulas:

$$\begin{aligned} P^{(1,1)} &= 0 \\ P^{(1,2)} &= -\frac{c}{2(t-\tau)^2} - \frac{b}{2(t-\tau)} \end{aligned} \tag{2.10}$$

$$P^{(1,k+2)} = -\frac{2}{k+2}P_x^{(1,k+1)} - \frac{1}{(k+2)(k+1)}[P_{xx}^{(1,k)} + bP^{(1,k)} - cP_t^{(1,k)}]\ ;\ k \geq 1$$

$$p^{(2,2)} = 0$$

(2.11) $$p^{(2,3)} = -\frac{c}{6(t-\tau)^2} - \frac{b}{6(t-\tau)}$$

$$p^{(2,k+2)} = -\frac{2}{k+2}\, P_x^{(2,k+1)} - \frac{1}{(k+2)(k+1)}\, [P_{xx}^{(2,k)} + bP^{(2,k)} - cP_t^{(2,k)}]\ ;\ k \geq 2\ .$$

The series (2.8) and (2.9) converge absolutely and uniformly on compact subsets of the punctured strip $|x| < \infty$, $|t| < t_o$, $|\tau| < t_o$, $t \neq \tau$, and hence define analytic functions in this region. As a consequence of this fact we have the following theorem and corollary:

Theorem: ([3]): Let $u(x,t)$ be a solution of equation (2.1) which is an analytic function of the complex variables x and t for $|t| < t_o$, $|x| < x_o$. Then $u(x,t)$ can be analytically continued into the strip $|x| < \infty$, $|t| < t_o$.

Corollary ([3]): If $u(o,t) = \varphi(t)$ is not analytic then neither is the interphase boundary $s(t)$.

III. Integral Operators for Parabolic Equations in Two Space Variables

In order to construct a solution to the non-characteristic Cauchy problem for parabolic equations in more than one space variable it is necessary for us to first construct an integral operator which maps analytic functions of two complex variables onto analytic solutions of linear parabolic equations of the second order in two space variables. We will restrict our attention to equations with time-independent coefficients, although with minor modifications this restriction could be dropped. In particular we consider the parabolic equation (written in normal form)

(3.1) $$u_{xx} + u_{yy} + a(x,y)\, u_x + b(x,y)\, u_y + c(x,y)u = d(x,y)\, u_t$$

and make the assumption that the coefficients of equation (3.1) are entire functions of their independent (complex) variables. With minor modifications we could have assumed only that these coefficients are analytic in some polydisc in the space of two complex variables. If we define the non-singular transformation of the space $\mathbb{C}^2$ of two complex variables into itself by

(3.2) $$z = x + iy$$
$$z^* = x - iy$$

equation (3.1) assumes the form

(3.3) $$L[U] = U_{zz^*} + A(z,z^*)U_z + B(z,z^*)U_{z^*} + C(z,z^*)\, U - D(z,z^*)\, U_t = 0$$

where $U = u, \quad A = \frac{1}{4}(a+ib), \quad B = \frac{1}{4}(a-ib), \quad c = \frac{c}{4}, \quad D = \frac{d}{4}$.

Note that $z^* = \bar{z}$ if and only if x and y are real.

Now let $f(z,t)$ be an analytic function of two complex variables in a neighborhood of the origin (exept for a possible nonessential singularity of the first kind at $t = \tau$) and define the operator $\underset{\sim}{P}$ by

(3.4) $$U(z,z^*,t) = \underset{\sim}{P}\{f\} =$$

$$= -\frac{1}{2\pi i} \exp\left\{-\int_{\zeta}^{z^*} A(z,\sigma)d\sigma\right\} \oint_{|t-\eta|=\delta} \int_{-1}^{1} E(z,z^*,t-\eta,s)\, f\left(\frac{(z-\zeta)}{2}(1-s^2),\eta\right) \frac{ds\,d\eta}{\sqrt{1-s^2}}$$

where $\delta > 0$,

(3.5) $$E(z,z^*,t,s) = \frac{1}{t} + \Sigma_{n=1}^{\infty}\; s^{2n}(z-\zeta)^n \int_{\bar{\zeta}}^{z^*} P^{(2n)}(z,\sigma,t)\, d\sigma$$

and the functions $P^{(2n)}(z,z^*,t)$ are defined by the recursion formula

(3.6) $$P^{(2)} = -\frac{2}{t}\tilde{C} - \frac{2}{t^2}\tilde{D}$$

$$(2n+1)P^{(2n+2)} = -2[P_z^{(2n)} + \tilde{B}P^{(2n)} + \tilde{C}\int_{\bar{\zeta}}^{z^*} P^{(2n)}\, d\sigma - \tilde{D}\int_{\bar{\zeta}}^{z^*} P_t^{(2n)} d\sigma]\;;\; n \geq 1$$

with $$\tilde{B} = B - \int_{\bar{\zeta}}^{z^*} A_z dz^*, \tilde{C} = -(A_z + AB - C), \tilde{D} = D \;.$$

In [4] it is shown that $E(z,z^*,t,s)$ exists and is an entire function of its independent variables (including the variables ζ and $\bar{\zeta}$) except for an essential singularity at $t = o$ and that the operator $\underset{\sim}{P}$ maps the analytic function $f(z,t)$ onto a solution of equation (3.3). If $f(z,t)$ is analytic in some neighborhood of the origin in $\mathbb{C}^2$ then $U(z,z^*,t) = \underset{\sim}{P}\{f\}$ is analytic in some neighborhood of the point $(z,z^*,t) = (\zeta,\bar{\zeta},0)$ in the space of three complex variables; if $f(z,t)$ has a singularity at $t = \tau$ then so does $U(z,z^*,t)$. It is furthermore shown in [4] that if the coefficients of equation (3.1) are real valued for x and y real, then <u>every</u> real valued analytic solution $u(x,y,t) = U(z,\bar{z},t)$ of equation (3.1) defined in some neighborhood of the origin can be represented in the form $u = \text{Re}\, \underset{\sim}{P}\{f\}$ where $f(z,t)$ is an analytic function in some neighborhood of the origin in $\mathbb{C}^2$ and „Re" denotes „take the real part". In the case in which $u(x,y,t)$ is independent of t, the operator $\text{Re}\, \underset{\sim}{P}$ reduces to Berman's integral operator of the first kind for elliptic equations ([1]).

In the next part of this talk we will be interested in the special singular solution of the adjoint equation

(3.7)
$$M[V] = V_{zz^*} - \frac{\partial}{\partial z}(AV) - \frac{\partial}{\partial z^*}(BV) + CV + DV_t = 0$$

defined by

(3.8)
$$V(z,z^*,t) = \underset{\sim}{P}'\left\{\frac{f(z)}{t-\tau}\right\} =$$
$$= \exp\left\{\int_{\bar{\zeta}}^{z^*} A(z,\sigma)d\sigma\right\}\int_{-1}^{1} E'(z,z^*,t-\tau,s)\,f\!\left(\frac{(z-\zeta)}{2}(1-s^2)\right)\frac{ds}{\sqrt{1-s^2}}$$

where $\underset{\sim}{P}'$ and $E'(z,z^*,t,s)$ are (respectively) the integral operator and E– function associated with equation (3.7) and

(3.9)
$$f(z) = -\frac{1}{2\pi}\int_\gamma \exp\left\{\int_0^{2z(1-\rho^2)} B(\sigma+\zeta,\bar{\zeta})d\sigma\right\}\frac{d\rho}{\rho^2}$$

where γ is a rectifiable arc joining the points $\rho = -1$ and $\rho = +1$ and not passing through the origin. In particular equations (3.8) and (3.9) imply (c.f. [1], p. 12) that $V(z,z^*,t)$ is a solution of $M[V] = 0$ satisfying the Goursat data

(3.10)
$$V(z,\bar{\zeta},t) = \frac{1}{t-\tau}\exp\left\{\int_\zeta^z B(\sigma,\bar{\zeta})\,d\sigma\right\}$$

(3.11)
$$V(\zeta,z^*,t) = \frac{1}{t-\tau}\exp\left\{\int_{\bar{\zeta}}^{z^*} A(\zeta,\sigma)\,d\sigma\right\}$$

Note that in the special case when $a = b = c = o$, $d = 1$, we have

(3.12)
$$V(z,z^*,t) = \frac{1}{t-\tau}\exp\left\{\frac{(z-\zeta)(z^*-\bar{\zeta})}{4(t-\tau)}\right\}$$

i.e. (modulo a constant factor) the classical fundamental solution to the backward heat equation.

IV. The Non-Characteristic Cauchy Problem for Parabolic Equations in Two Space Variables

We again consider equation (3.1) and its complex version (3.3) under the assumptions on the coefficients stated in section three. Let S be an analytic surface in $\mathbb{R}^2 \times [0,T]$ (where $\mathbb{R}^2$ denotes two dimensional Euclidean space and T is a positive constant) and assume that for $\tau \in [0,T]$ the intersection of S with the plane $t = \tau$ is a one dimensional analytic curve $C_3' = C_3'(\tau)$. In this section we will describe a procedure for constructing a global solution of equation (3.1) satisfying prescribed analytic Cauchy data on this surface S. Let $u(x,y,t) = U(z,z^*,t)$ be the local solution to this problem constructed (for example) via the Cauchy-Kowalewski theorem. Then our approach to this problem will be based on integrating the quantity $VL[U] - UM[V]$ over a torus $G \times \Omega$, where $V(z,z^*,t)$ is the function defined by equations (3.8) and (3.9), Ω is the circle $|t-\tau| = \delta > 0$

in the complex t plane, and G is a two dimensional cell in (z,z^*) space. By preforming this integration and using the calculus of residues we will obtain a (local) integral representation of the solution to the non-characteristic Cauchy problem as a linear functional of the data. Such a representation will then provide a method for the (global) analytic continuation of this solution into the complex domain.

We now describe an appropriate choice for the domain G. Suppose $C_3'(\tau)$ is described by the equation $F(x,y;\tau) = 0$. Since $C_3'(\tau)$ is an analytic curve we can write $F(\frac{z+z^*}{2}, \frac{z-z^*}{2i}; \tau) = 0$ and this is the equation for the complex extension of $C_3'(\tau)$ into (z,z^*) space. For points $(\zeta,\bar{\zeta})$ sufficiently near the complex extension of $C_3'(\tau)$, the characteristic planes $z=\zeta=\xi_1+i\xi_2$ and $z^*=\bar{\zeta}=\xi_1-i\xi_2$ intersect this surface at two points Q and P respectively. Let $C_3 = C_3(\tau)$ be a curve lying on the complex extension of $C_3'(\tau)$ and joining the points Q and P. We now let G be a cell whose boundary consists of the curve C_3 and line segments C_1 and C_2 which lie in the characteristic planes $z = \zeta$ and $z^* = \bar{\zeta}$ and join the point $R = (\zeta,\bar{\zeta})$ to the curve C_3 at the points Q and P respectively (see figure 1 below).

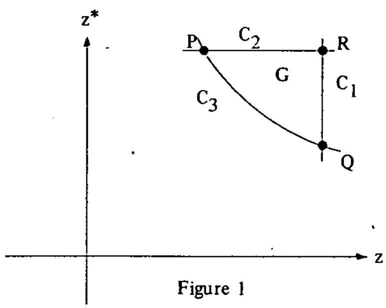

Figure 1

We now use Stokes theorem to integrate the identity

$$\text{(4.1)} \quad \begin{aligned} VL[U] - UM[V] &= (AVU + \tfrac{1}{2}VU_{z^*} - \tfrac{1}{2}V_{z^*}U)_z \\ &+ (BVU + \tfrac{1}{2}VU_z - \tfrac{1}{2}V_zU)_{z^*} - (DVU)_t \end{aligned}$$

over the torus $G \times \Omega$ (where δ is chosen sufficiently small).and make use of equations (3.10) and (3.11) and the fact that $dzdz^* = 0$ on $\partial G \times \Omega$ to arrive at the following (local) integral representation of the solution to the non-characteristic Cauchy problem for equation (3.1):

$$\text{(4.2)} \quad \begin{aligned} u(\xi_1,\xi_2,\tau) = U(\zeta,\bar{\zeta},\tau) &= \frac{1}{4\pi i}\oint_{\Omega} [V(P,t)U(P,t) + V(Q,t)U(Q,t]dt \\ &- \frac{1}{2\pi i}\int\oint_{C_3(\tau)\times\Omega} [(AVU + \tfrac{1}{2}VU_{z^*} - \tfrac{1}{2}V_{z^*}U)dz^*dt \\ &- (BVU + \tfrac{1}{2}VU_z - \tfrac{1}{2}V_zU)\, dzdt] \ . \end{aligned}$$

Equation (4.2) can now be used to obtain a global solution via the straightforward use of analytic continuation of the Cauchy data and deformation of the region of integration. In particular if we invoke the concept of conformal symmetry we can obtain the following theorem:

Definition ([11]): Let $\mathcal{D}$ be a domain in the plane and $\mathcal{C}$ an analytic arc lying in $\mathcal{D}$. Then $\mathcal{D}$ is conformally symmetric with respect to $\mathcal{C}$ if there exists a conformal mapping which transforms $\mathcal{C}$ into an interval of the real axis and $\mathcal{D}$ into a domain which is symmetric with respect to the real axis.

Theorem ([5]): Let $U(z,z^*,t)$ be an analytic solution of equation (3.3) and for each fixed t in a (complex) neighborhood of $t = \tau$ let the Cauchy data for $U(z,z^*,t)$ be regular (as a function of z) in a domain $\mathcal{D}$ which is conformally symmetric with respect to $C_3'(\tau)$. Then the restriction of $U(z,z^*,t)$ to the plane $t = \tau$ is an analytic function of z and z^* in $\mathcal{D} \times \mathcal{D}^*$ where $\mathcal{D}^* = \{z^* : \overline{z^*} \in \mathcal{D}.\}$

V. The Non-Characteristic Cauchy Problem for the Heat Equation in Three Space Variables

We will now obtain an integral representation for the solution to the non-characteristic Cauchy problem for the three (space) dimensional heat equation

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} = u_t \tag{5.1}$$

in a manner analogous to that used in the previous section to solve the non-characteristic Cauchy problem for parabolic equations in two space variables. The extension of the results in the present section to the case of parabolic equations in three space variables with variable coefficients is presently being investigated by Michael Stecher ([15]).

We first introduce the non-singular change of variables in $\mathbb{C}^4$ (the space of four complex variables) defined by

$$\begin{aligned} z &= x_1 + ix_2 \\ z^* &= x_1 - ix_2 \\ x &= x_3 \\ t &= t \end{aligned} \tag{5.2}$$

and rewrite equation (5.1) in the form

$$L[U] = 4U_{zz^*} + U_{xx} - U_t = 0 \tag{5.3}$$

where $U(z,z^*,x,t) = u(x_1,x_2,x_3,t)$ is assumed to be an analytic function of its independent variables. Let $M[V] = 0$ be the adjoint equation defined by

$$M[V] = 4V_{zz^*} + V_{xx} + V_t = 0 \tag{5.4}$$

and let $U(z,z^*,x,t) = u(x_1,x_2,x_3,t)$ be a solution of equation (5.1) assuming prescribed analytic Cauchy data on a noncharacteristic - analytic surface S. Then (following the ideas of the previous section) our approach will be to construct a special singular solution of $M[V] = 0$ and then integrate $VL[U] - VM[V]$ over a hypertorus $G \times \Omega_0 \times \Omega_1$ where $\Omega_0 = |x-\xi_3| = \delta_0 > 0$, $\Omega_1 = |t-\tau| = \delta_1 > 0$, and G is a two dimensional cell in (z,z^*) space. The domain G is the same as described in the previous section. In particular assume that the intersection of S with the hyperplanes $x = \xi_3$ and $t = \tau$ is a one dimensional analytic curve $C_3' = C_3'(\xi_3,\tau)$ and let $C_3 = C_3(\xi_3,\tau)$ be a curve lying on the complex extension of C_3' in (z,z^*) space which intersects the characteristic planes $z = \zeta = \xi_1 + i\xi_2$ and $z^* = \bar{\zeta} = \xi_1 - i\xi_2$ at the points Q and P respectively. Now define G to be the cell whose boundary consists of the curve C_3 and characteristic line segments C_1 and C_2 joining the point $R = (\zeta,\bar{\zeta})$ to the points Q and P respectively (c.f. figure 1).

We now turn our attention to the construction of the function $V(z,z^*,x,t)$ satisfying $M[V] = 0$. Motivated by the analysis of the previous section we will require $V(z,z^*,x,t)$ to satisfy

$$V_z = 0 \quad \text{on } z^* = \bar{\zeta} \tag{5.5}$$

$$V_{z^*} = 0 \quad \text{on } z = \zeta \tag{5.6}$$

and on the intersection of the planes $z^* = \bar{\zeta}$, $z = \zeta$,

$$-\frac{1}{4\pi^2} \oint\oint_{\Omega_0 \times \Omega_1} (UV)dxdt = U(\zeta,\bar{\zeta},\xi_3,\tau) \tag{5.7}$$

for every analytic solution U of $L[U] = 0$. However at this point our analogy with the results of section four breaks down since it is not possible to set $V(z,z^*,x,t)$ equal to the standard fundamental solution of the backward heat equation as we did in the case of the two (space) dimensional heat equation (c.f. equation (3.12)). This is due to the fact that in three space dimensions the fundamental solution to the backward heat equation has a branch point at $t = \tau$ and thus does not satisfy equation (5.7). Hence we must construct a new singular solution to equation (5.4) satisfying equations (5.5) - (5.7). We note that a similar problem was also encountered by C.D. Hill in his investigation of the non-characteristic Cauchy problem for parabolic equations in one space variable ([12]).

We begin our construction of $V(z,z^*,x,t)$ by first considering the equation

$$4W_{zz^*} + W_t = 0 \; . \tag{5.8}$$

From the results of section three of this talk we can represent any real valued (for $z^* = \bar{z}$) analytic solution of equation (5.8) (in a neighborhood of the origin) in the form

$$W(z,\bar{z},t) = \operatorname{Re}\left\{-\frac{1}{2\pi i}\oint_{|t-\eta|=\delta}\int_{-1}^{1} E(r^2,t-\eta,s)\, f\left(\frac{z}{2}(1-s^2),\eta\right)\frac{dsd\eta}{\sqrt{1-s^2}}\right\} \tag{5.9}$$

where $\delta > 0$, $f(z,t)$ is an analytic function of two complex variables in some neighborhood of the origin in $\mathbb{C}^2$ (except for a possible nonessential singularity of the first kind at $t = 0$, in which case $W(z,\bar{z},t)$ is singular at $t = 0$) and $E(r^2,t,s)$ is defined by

$$E(r^2,t,s) = \frac{1}{t} \Sigma_{n=0}^{\infty} \frac{\Gamma(1/2)}{\Gamma(n+1/2)} \left(\frac{r^2 s^2}{rt}\right)^n \tag{5.10}$$

where $r^2 = z\bar{z} = x_1^2 + x_2^2$. If we now set

$$f(z,t) = g(z)/t \quad , \tag{5.11}$$

where $g(z)$ is an analytic function of z in some neighborhood of the origin, and take the operator „Re" inside the integral sign in equation (5.9), we arrive at

$$w(x_1,x_2,t) = \int_{-1}^{1} E(r^2,t,s)H(x_1\sqrt{1-s^2}), x_2\sqrt{1-s^2})\frac{ds}{\sqrt{1-s^2}} \tag{5.12}$$

where $w(x_1,x_2,t) = W(z,\bar{z},t)$ is a singular solution of the backward heat equation in two space variables and $H(x_1,x_2) = \text{Re } g(\frac{z}{2})$ is a harmonic function of the variables x_1 and x_2. If we define a new harmonic function $h(x_1,x_2)$ by the equation

$$h(x_1,x_2) = \int_{-1}^{1} H(x_1\sqrt{1-s^2})., x_2\sqrt{1-s^2})\frac{ds}{\sqrt{1-s^2}} \quad , \tag{5.13}$$

then equation (5.12) becomes

$$w(x_1,x_2,t) = \frac{h(x_1,x_2)}{t} + \int_0^1 \sigma G(r^2,1-\sigma^2,t)\, h(x_1\sigma^2,x_2\sigma^2)d\sigma \tag{5.14}$$

where

$$G(r^2, 1-\sigma^2,t) = \frac{r^2}{2t^2} \quad \exp\left(\frac{(1-\sigma^2)r^2}{4t}\right). \tag{5.15}$$

We now use the „method of ascent" ([10]; see also [1], p. 68 and [16], p. 59) to extend the representation (5.14) from two space variables to three space variables. This operation gives a solution $w(\underset{\sim}{x},t)$ of the backward heat equation in three space variables in the form ([6])

$$w(\underset{\sim}{x},t) = \frac{h(\underset{\sim}{x})}{t} + \frac{r^2}{2t^2} \int_0^1 \sigma^2 \exp\left(\frac{(1-\sigma^2)r^2}{4t}\right) h(\underset{\sim}{x}\sigma^2)d\sigma \tag{5.16}$$

where $\underset{\sim}{x} = (x_1,x_2,x_3)$, $r = |\underset{\sim}{x}|$, and $h(\underset{\sim}{x})$ is a harmonic function of x_1,x_2, and x_3 defined in some neighborhood of the origin.

It is easily verified that the representation (5.16) is still valid if we allow $h(\underset{\sim}{x})$ to have a weak singularity at the origin. Hence in equation (5.16) we can set

(5.17) $$h(\underset{\sim}{x}) = \frac{1}{r} \ .$$

If we now make the change of variables defined by equation (5.2) and translate the origin to the point $(\zeta,\bar{\zeta},\xi_3,\tau)$ we arrive at the desired singular solution $V(z,z^*,x,t)$ of $M[V] = 0$ satisfying equations (5.5) - (5.7), viz.

(5.18) $$V(z,z^*,x,t) = \frac{1}{R(t-\tau)} + \frac{R}{2(t-\tau)^2} \exp\left(\frac{R^2}{4(t-\tau)}\right) \int_0^1 \exp\left(-\frac{R^2\sigma^2}{4(t-\tau)}\right) d\sigma$$

where $R = \sqrt{(x-\xi_3)^2 + (z-\zeta)(z^* - \bar{\zeta})}$.

We now use Stokes theorem to integrate the identity

(5.19) $$\begin{aligned} VL[U] - UM[V] &= (2U_{z^*}V - 2UV_{z^*})_z \\ &+ (2U_zV - 2UV_z)_{z^*} - (UV_x - U_xV)_x - (UV)_t \end{aligned}$$

over the hypertorus $G \times \Omega_0 \times \Omega_1$, making use of equations (5.5) - (5.7) and the fact that $dzdz^* = 0$ on $\partial G \times \Omega_0 \times \Omega_1$. Note that $V(z,z^*,x,t)$ is an analytic function of x outside the branch cut between $\xi_3 \pm i\sqrt{(z-\zeta)(z^* - \bar{\zeta})}$ and hence for the point $R = (\zeta,\bar{\zeta})$ sufficiently near the curve $C_3(\xi_3,\tau)$ (and δ_0 and δ_1 sufficiently small) the integration over $G \times \Omega_0 \times \Omega_1$ is well defined. The result of this integration is the following (local) integral representation of the solution to the non-characteristic Cauchy problem for equation (5.1):

(5.20) $$\begin{aligned} u(\xi_1,\xi_2,\xi_3,\tau) &= U(\zeta,\bar{\zeta},\xi,\tau) = \\ &= \frac{1}{8\pi^2} \oint\oint_{\Omega_0 \times \Omega_1} [V(P,x,t)U(P,x,t) + V(Q,x,t)U(Q,x,t)]\, dxdt \\ &- \frac{1}{8\pi^2} \int\oint\oint_{C_3(\xi_3,\tau) \times \Omega_0 \times \Omega_1} [(U_{z^*}V - UV_{z^*})\,dz^*dxdt - (U_zV - UV_z)dzdxdt]. \end{aligned}$$

The representation (5.20) can now be extended to a global solution through the use of analytic continuation of the Cauchy data and deformation of the region of integration. In particular we have the following theorem:

<u>Theorem</u>: Let $U(z,z^*,x,t)$ be an analytic solution of equation (5.3). Let $\mathfrak{D}$ be a domain in the z plane which is conformally symmetric with respect to $C_3'(\xi_3,\tau)$ and let $\mathcal{B}$ be a domain in the complex x plane which contains the branch cut joining $\xi_3 \pm i\sqrt{(z-\zeta)(z^*-\zeta^*)}$ for all (z,z^*) and (ζ,ζ^*) in $\mathfrak{D} \times \mathfrak{D}^*$ where $\mathfrak{D}^* = \{z^* : \bar{z}^* \in \mathfrak{D}\}$. Suppose that for each fixed t in a (complex) neighborhood of $t = \tau$ the Cauchy

data for $U(z,z^*,x,t)$ is regular (as a function of z and x) in $\mathfrak{D} \times \mathfrak{B}$. Then the restriction of $U(z,z^*,x,t)$ to the intersection of the hyperplanes $x = \xi_3$ and $t = \tau$ is an analytic function of z and z^* in $\mathfrak{D} \times \mathfrak{D}^*$.

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THE INVERSE STEFAN PROBLEM FOR THE HEAT EQUATION IN TWO SPACE VARIABLES

THE INVERSE STEFAN PROBLEM FOR THE HEAT EQUATION IN TWO SPACE VARIABLES

DAVID COLTON

1. *Introduction.* The Stefan problem is a particular free boundary problem for the heat equation which arises in the investigation of the melting of solids. In the case of one space dimension there are numerous results available concerning the existence, uniqueness, and stability of the solution [*c.f.* **6**]. However the case of several space variables is considerably more difficult. This is due in large part to the fact that the geometry of the problem can become quite complicated, and smooth initial and boundary data do not necessarily lead to smooth solutions. In particular, under heating, a connected solid can melt into two (or more) disconnected solids, thus leading to a problem in which the free boundary varies in a discontinuous manner. These difficulties have motivated several researchers to look for "weak" solutions to the Stefan problem [*c.f.* **1**, **3**, **5**]. Although this approach is quite general and leads to numerical schemes for solving the problem under consideration, there are several drawbacks to this method, among them being the fact that no information is obtained concerning the structure of the interphase boundary, nor is there much information on the regularity of these weak solutions.

In this note we will outline an inverse method for constructing analytic solutions to the (single phase) Stefan problem for the heat equation in two space dimensions. This will be accomplished, by assuming *a priori* that the free boundary is a relatively simple analytic surface, and then constructing a solution to the heat equation which has this prescribed surface as a free boundary. Provided the solution is analytic in a sufficiently large domain, we can then determine those initial-boundary data that are compatible with the given "free" boundary. In physical terms we are asking the question "How must a given solid (*e.g.* ice) be heated in order for it to melt in a prescribed manner?" By constructing a variety of such examples a qualitative idea can be obtained on the shape of the free boundary as a function of the initial-boundary conditions. Such an inverse approach leads to two main problems. The first of these is that the inverse problem has its mathematical formulation as a non-characteristic Cauchy problem for the heat equation and is thus improperly posed in the real domain. However such a problem is well posed in the complex domain, and hence we are led to examine solutions of the heat equation in the space of several complex variables. The inverse Stefan problem can now be solved locally by appealing to the Cauchy–Kowalewski theorem. However in addition to being far too tedious for practical computation and error estimation, such an approach does not provide us with the required global solution to the Cauchy problem under investigation. Hence we are led to the problem of the analytic continuation of solutions to non-characteristic Cauchy problems for the heat equation. We will accomplish this, by using contour integration, and the calculus of residues in the space of several complex variables, to arrive at an explicit (global) series representation of the solution to the inverse Stefan problem.

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The inverse approach described in this note was previously described (for the more general case of parabolic equations with variable coefficients) in the research announcement [2]. However this paper did not provide an explicit formula for the solution that was suitable for analytic approximations. This defect will now be remedied through the introduction of a one parameter family of conformal mappings which provides an explicit parametric representation of the " free " boundary in the complex domain, and leads to a representation of the solution in terms of an infinite series of one dimensional integrals.

2. *Mathematical formulation of the inverse Stefan problem.* We will motivate the mathematical formulation of the inverse Stefan problem in terms of an ice–water system undergoing a change of phase. Assume that a bounded simply connected region R with boundary ∂R is filled with ice at 0°C. Beginning at time $t = 0$ a non-negative temperature $\gamma = \gamma(x, y, t)$ (where $\gamma(x, y, 0) = 0$) is applied to ∂R. The ice begins to melt and we will let the interphase boundary $\Gamma(t)$ between ice and water be described by $\Gamma(t) = \{(x, y) : \Phi(x, y, t) = 0\}$, with the water lying in the region $\Phi(x, y, t) < 0$. The differential equation and boundary conditions governing the conduction of heat in the water are given by [*c.f.* **6**]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{a}\frac{\partial u}{\partial t}, \qquad \Phi(x, y, t) < 0, \tag{2.1}$$

$$u|_{\partial R} = \gamma, \tag{2.2}$$

$$u|_{\Gamma(t)} = 0, \qquad k\left.\frac{\partial u}{\partial \nu}\right|_{\Gamma(t)} = \frac{\lambda\rho}{|\nabla\Phi|}\left.\frac{\partial \Phi}{\partial t}\right|_{\Gamma(t)}, \tag{2.3}$$

where ν is the normal, with respect to the space variables, that points into the region $\Phi(x, y, t) < 0$, $u(x, y, t)$ is the temperature, a the diffusivity coefficient, λ the latent heat of fusion, and k the conductivity of the water. The Stefan problem is to find $\Gamma(t)$ and $u(x, y, t)$ given the function $\gamma(x, y, t)$. The inverse Stefan problem (which interests us) is to find $u(x, y, t)$ and

$$\gamma(x, y, t) = \lim_{(x, y)\to\partial R} u(x, y, t)$$

given $\Gamma(t)$. In general we cannot hope to solve the inverse problem for arbitrary $\Gamma(t)$; however, by suitably restricting $\Gamma(t)$ to lie in a certain class of analytic surfaces, we shall be able to obtain a relatively simple series representation of the solution, and it is to this problem we now address ourselves.

Let $\mathscr{D}_t$, $0 \leqslant t < t_0$, be a family of simply connected domains which depend analytically on a parameter t such that

$$\bigcup_{0\leqslant t<t_0} \mathscr{D}_t \quad \text{contains} \quad R \cup \partial R \times [0, t_0).$$

Let $z = \phi(\zeta, t)$ conformally map the unit disc Ω onto $\mathscr{D}_t$ (we assume that $\mathscr{D}_t$ has been chosen such that the image of the interval $(-1, 1)$ intersects the region R) and for $\zeta^* \in \Omega$, $0 \leqslant t < t_0$, define $\bar{\phi}(\zeta^*, t)$ by

$$\bar{\phi}(\zeta^*, t) = \overline{\phi(\bar{\zeta}^*, t)},$$

where bars denote conjugation [*c.f.* 4]. Now set $z^* = \bar{\phi}(\zeta^*, t)$ and note that $z^* = \bar{z}$, if, and only if, $\zeta^* = \bar{\zeta}$. We now define the function $\Phi(x, y, t)$ for (possible) complex values of x, y and t by

$$\Phi(x, y, t) = \frac{1}{2i}[\phi^{-1}(z, t) - \bar{\phi}^{-1}(z^*, t)], \tag{2.4}$$

where $z = x + iy$, $z^* = x - iy$. Noting that $z^* = \bar{z}$, if, and only if, x and y are real, it is seen that $\Phi(x, y, t) = 0$ corresponds to $\operatorname{Im}\zeta = 0$, *i.e.* the interval $(-1, 1)$ in the complex ζ plane. Similarly, the region $\Phi(x, y, t) < 0$ corresponds to $\operatorname{Im}\zeta < 0$, *i.e.* the part of Ω which lies in the lower half plane. We finally note that, for $z = x + iy \in \Gamma(t)$, we have $\phi^{-1}(z, t) = \bar{\phi}^{-1}(z^*, t)$, and hence

$$\begin{aligned}\left.\frac{\partial\Phi}{\partial t}\right|_{\Gamma(t)} &= \frac{1}{2i}\left.\left[\frac{\partial\phi^{-1}(z, t)}{\partial t} - \frac{\partial\bar{\phi}^{-1}(z^*, t)}{\partial t}\right]\right|_{z^* = \bar{\phi}(\phi^{-1}(z, t), t)} \\ &= g(z, t),\end{aligned} \tag{2.5}$$

i.e. $\partial\Phi/\partial t$ restricted to $\Gamma(t)$ can be analytically continued (for each fixed t) to an analytic function of z for $z \in \mathscr{D}_t$.

3. *Solution of the inverse Stefan problem.* In equation (2.1) we consider x and y as independent complex variables and define a transformation of the space of two complex variables into itself by

$$z = x + iy, \qquad z^* = x - iy. \tag{3.1}$$

Under this transformation equations (2.1) and (2.3) become

$$L[U] \equiv \frac{\partial^2 U}{\partial z\,\partial z^*} - \frac{1}{4a}\frac{\partial U}{\partial t} = 0, \tag{3.2}$$

$$U(\phi(s, t), \bar{\phi}(s, t)) = 0, \qquad -1 < s < 1, \tag{3.3}$$

$$\begin{aligned}U_1(\phi(s, t), \bar{\phi}(s, t), t)\frac{\partial\phi(s, t)}{\partial s} &- U_2(\phi(s, t), \bar{\phi}(s, t), t)\frac{\partial\bar{\phi}(s, t)}{\partial s} \\ &= \frac{i\lambda\rho}{k}\left|\frac{\partial\phi(s, t)}{\partial s}\right|^2 g(\phi(s, t), t); \; -1 < s < 1,\end{aligned} \tag{3.4}$$

where

$$U(z, z^*, t) = u\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}, t\right),$$

$g(z, t)$ is defined by equation (2.5), and subscripts denote differentiation of $U(z, z^*, t)$ with respect to the first and second variables respectively.

Now let M be the adjoint operator, defined by

$$M[V] \equiv \frac{\partial^2 V}{\partial z\,\partial z^*} + \frac{1}{4a}\frac{\partial V}{\partial t} = 0, \tag{3.5}$$

and let V be the fundamental solution of $M[V] = 0$, defined by

$$V(z, z^*, t;\ \xi, \bar{\xi}, \tau) = \frac{1}{t-\tau} \exp\left\{\frac{(z-\xi)(z^*-\bar{\xi})}{4a(t-\tau)}\right\}, \tag{3.6}$$

where $\xi = \xi_1 + i\xi_2$, $\bar{\xi} = \xi_1 - i\xi_2$. Note that V satisfies the Goursat data

$$V(z, \bar{\xi}, t;\ \xi, \bar{\xi}, \tau) = \frac{1}{t-\tau}, \tag{3.7a}$$

$$V(\xi, z^*, t;\ \xi, \bar{\xi}, \tau) = \frac{1}{t-\tau}. \tag{3.7b}$$

We will now obtain the solution to the inverse Stefan problem (3.2)–(3.4) by first using Stokes's theorem to integrate $VL[U] - UM[V]$ over a torus lying in the space of three complex variables and then computing the residue of the resulting integral representation.

Let τ be real and for t on the circle $|t-\tau| = \delta$, $\delta > 0$, let $G(t)$ be a cell whose boundary consists of a curve $C(t)$ lying on the surface

$$\phi^{-1}(z, t) = \bar{\phi}^{-1}(z^*, t)$$

and line segments lying on the characteristic planes $z = \xi$ and $z^* = \bar{\xi}$ respectively which join the point $(\xi, \bar{\xi})$ to $C(t)$. Integrating $VL[U] - UM[V]$ over the torus $\{(z, z^*, t) : (z, z^*) \in G(t),\ |t-\tau| = \delta\}$ and making use of the initial conditions (3.7a), (3.7b) satisfied by V gives

$$\begin{aligned}
U(\xi, \bar{\xi}, \tau) &= \frac{1}{4\pi i} \int_{|t-\tau|=\delta} \int_{C(t)} [VU_z\, dz - VU_{z^*}\, dz^*]\, dt \\
&= \frac{\lambda\rho}{4\pi k} \int_{|t-\tau|=\delta} \int_{\bar{\phi}^{-1}(\bar{\xi}, t)}^{\phi^{-1}(\xi, t)} \frac{1}{t-\tau} \exp\left\{\frac{(\phi(\zeta, t)-\xi)(\bar{\phi}(\zeta, t)-\bar{\xi})}{4a(t-\tau)}\right\} \\
&\qquad \cdot \left|\frac{\partial\phi(\zeta, t)}{\partial\zeta}\right|^2 g(\phi(\zeta, t), t)\, d\zeta\, dt \\
&= \frac{i\lambda\rho}{2k} \sum_{n=0}^{\infty} \frac{1}{(4a)^n (n!)^2} \\
&\qquad \times \frac{\partial^n}{\partial\tau^n} \left\{ \int_{\bar{\phi}^{-1}(\bar{\xi}, \tau)}^{\phi^{-1}(\xi, \tau)} |\phi(\zeta, \tau) - \xi|^{2n} \cdot \left|\frac{\partial\phi(\zeta, \tau)}{\partial\zeta}\right|^2 g(\phi(\zeta, \tau), \tau)\, d\zeta \right\}
\end{aligned} \tag{3.8}$$

where

$$\left|\frac{\partial\phi}{\partial\zeta}(\zeta, \tau)\right|^2 = \frac{\partial\phi}{\partial\zeta}(\zeta, \tau)\frac{\partial\bar{\phi}}{\partial\zeta}(\zeta, \tau), \qquad |\phi(\zeta, \tau) - \xi|^2 = (\phi(\zeta, \tau) - \xi)(\bar{\phi}(\zeta, \tau) - \bar{\xi}).$$

Equation (3.8) is the main result of this note. From the definition of the conformal mappings $\phi(\zeta, t)$ it is seen that equation (3.8) is valid in a region containing

$R \cup \partial R \times [0, t_0)$. In order to obtain a physically meaningful solution of the inverse Stefan problem we assume

$$\gamma(x, y, t) = 0 \quad \text{for} \quad (x, y, t) \in \partial R \times [0, t_0) \cap \{(x, y, t) : \Phi(x, y, t) \geqslant 0\},$$

and choose the conformal mappings $\phi(\zeta, t)$, such that $u(x, y, t) \geqslant 0$ for $\{(x, y, t) : \Phi(x, y, t) < 0\}$. We note that, from the boundary condition (2.3), this last condition is always satisfied (at least for t_0 sufficiently small) provided we choose $\phi(\zeta, t)$ such that $(\partial\Phi/\partial t)|_{\Gamma(t)} \geqslant 0$. Due to the appearance of the factor $(n!)^2$ in the denominator of each term in the series (3.8), accurate approximations of the solution to the inverse Stefan problem can be obtained by truncating this series after only a few terms.

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Integral Operators for Elliptic Equations in Three Independent Variables, I

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Integral Operators for Elliptic Equations in Three Independent Variables, 1

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An integral operator is obtained which maps analytic functions of two complex variables onto real valued solutions of an elliptic partial differential equation in three independent variables. An inverse operator is also constructed. These operators are then used for purposes of analytic continuation and to construct a complete family of solutions.

1. INTRODUCTION

The theory of integral operators for second order elliptic partial differential equations in two independent variables with analytic coefficients has been extensively investigated by Bergman [1] and Vekua [21]. These operators map analytic functions of one complex variable onto the class of twice continuously differentiable (i.e. C^2) solutions of the elliptic equation being considered. In recent years their results have been extended by Henrici [14], [15] and Gilbert [10] to include certain classes of equations with singular coefficients. Of the many applications of these integral operators, two are of central importance:

1) Integral operators can be used to construct complete families of solutions. (Once such a family is obtained, it is possible to construct the kernel function [2], [9], and thus solve the classical boundary value problems connected with the partial differential equation being studied.)

2) The investigation of the analytic structure of a solution to an elliptic equation can be reduced to studying the analytic function associated with it via the integral operator. (In particular this has many applications in scattering theory; c.f. [10], chapter five, and [4].)

The above applications depend on the fact that the integral operetor maps the space of analytic functions *onto* the class of C^2 solutions of the differential equation, and that furthermore, given a C^2 solution of the differential equation, the analytic function associated with it can be determined in a reasonably easy manner.

Due to both its physical and mathematical interest, several attempts have been made in recent years to extend the results of Bergman and Vekua to equations in three and four independent variables. (The latter case arises in studying analytic solutions of hyperbolic equations in three space variables and one time variable; c.f. [5].) Up to now, only partial results have been obtained. Considering the elliptic equation

$$\Delta_3 u+q(x, y, z)\, u = 0$$
$$\Delta_3 \equiv \frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^2}+\frac{\partial}{\partial z^2} \tag{1.1}$$

as a prototype, Bergman [1] has obtained integral operators which map certain subspaces of analytic functions of two complex variables onto *subspaces* of the class of C^2 solutions of Eq. (1.1). Bergman further assumed that $q(x, y, z)$ was independent of x. The analytic function associated with $u(x, y, z)$ by Bergman's operator can be easily determined from "Goursat data" of $u(x, y, z)$ in the space of several complex variables. More recently, Colton and Gilbert [6] have constructed an operator which maps certain subspaces of ordered pairs of analytic functions of two complex variables *onto* the space of C^2 solutions of Eq. (1.1), again under the assumption that $q(x, y, z)$ is independent of x. The associated analytic functions are given in terms of the Cauchy data for $u(x, y, z)$. In the general case when $q(x, y, z)$ is no longer independent of x, Tjong [19, 20] gave an integral operator which maps analytic functions onto an unspecified subspace of solutions to Eq. (1.1). Gilbert and Lo [13] showed that if $q(x, y, z) < 0$, then Tjong's operator in fact maps analytic functions onto the whole space of C^2 solutions of Eq. (1.1). However, in the work of Gilbert and Lo the associated analytic function is constructed by solving a Neumann problem for Eq. (1.1), and is thus impractical for use in examining the analytic structure of solutions of Eq. (1.1). Finally, the special case when $q(x, y, z) = B(r^2)$ has been examined by Bergman [1], Gilbert and Howard [12] and Gilbert [11], and work on the four dimensional analogue of Eq. (1.1) has been initiated by Colton and Gilbert [6].

In the present paper we overcome the restrictions and difficulties of the above mentioned work and construct an integral operator which maps a subspace of analytic functions of two complex variables onto the space of real C^2 solutions of Eq. (1.1). The associated analytic function is given in a simple manner in terms of the "Goursat data" for $u(x, y, z)$ in the space of several complex variables. The function $q(x, y, z)$ will in general depend on x, and no assumption will be made on whether $q(x, y, z)$ is positive or negative. This operator is then used to derive a result on the singularity manifold of C^2 solutions of Eq. (1.1), and to construct a complete family of solutions for Eq. (1.1). For the sake of simplicity we assume that $q(x, y, z)$ is an entire function of x, y, and z (considered as complex variables), although with slight modifications our results remain valid when $q(x, y, z)$ is only assumed to be analytic inside some ball in the space of three complex variables. A large part of our work in section two is based on the ideas of Tjong [19, 20], with the important difference being that we replace the distinguished plane $x = 0$ in her work by a complex hyperplane in the space of three complex variables. This is a fundamental step in showing that our operator is onto, since it leads to a well posed (in the space of three complex variables!) Goursat problem for a hyperbolic equation instead of the improperly posed initial value problem implicit in Tjong's work. In this sense our work is a natural generalization of Bergman's and Vekua's results for two independent variables, since they also showed that their operators were onto by considering a complex Goursat problem for a hyperbolic equation. Our methods furthermore suggest the possibility of extending our results to other classes of elliptic equations in three independent variables, and this investigation will be the subject matter of a second paper in this series.

2. CONSTRUCTION OF THE OPERATOR P_3

We now proceed to construct the integral operator which was discussed in the introduction. The various properties of this operator, including the fact that it maps analytic functions onto the class of C^2 solutions of Eq. (1.1), will be discussed in section three. We first give the following theorem in order to motivate the analysis which follows.

Throughout this paper we will assume that $q(x, y, z)$ is a real valued entire function of the variables x, y, z.

THEOREM 2.1. *Let $X = x$, $Z = \frac{1}{2}(y+iz)$, $Z^* = \frac{1}{2}(-y+iz)$, and let $u(x, y, z)$ be a real valued C^2 solution of Eq.* (1.1) *in a neighborhood of the origin. Then $U(X, Z, Z^*) \equiv u(x, y, z)$ is an analytic function of X, Z, Z^* in some neighborhood of the origin in $\mathbb{C}^3$, the space of three complex variables, and is uniquely determined by the function $F(X, Z^*) \equiv U(X, O, Z^*)$.*

Proof The fact that $U(X, Z, Z^*)$ is analytic follows from the well known result that twice continuously differentiable solutions of linear second order elliptic equations with analytic coefficients are analytic functions of their independent variables (c.f. [9], p. 164). Hence locally we can write

$$U(X, Z, Z^*) = \sum_{l,m,n=0}^{\infty} a_{mnl} X^l Z^n Z^{*m} \tag{2.1}$$

$$U(X, O, Z^*) = \sum_{m,l=0}^{\infty} a_{m0l} X^l Z^{*m} \tag{2.2}$$

$$U(X, Z, O) = \sum_{n,l=0}^{\infty} a_{0nl} X^l Z^n. \tag{2.3}$$

Since $u(x, y, z)$ is real valued, we have that for x, y, z real

$$U(X, Z, Z^*) = \overline{U(X, Z, Z^*)}, \tag{2.4}$$

where the bar denotes complex conjugation. This implies that for x, y, z real

$$\sum_{l,m,n=0}^{\infty} a_{mnl} X^l Z^n Z^{*m} = \sum_{l,m,n=0}^{\infty} \overline{a_{mnl}} X^l (-Z^*)^n (-Z)^m \tag{2.5}$$

or

$$a_{mnl} = (-1)^{n+m} \overline{a_{nml}}. \tag{2.6}$$

Equations (2.2) and (2.3) now show that $U(X, Z, O)$ is uniquely determined from $U(X, O, Z^*)$. However in the X, Z, Z^* coordinates, Eq. (1.1) becomes an equation of hyperbolic type, viz.

$$U_{XX} - U_{ZZ^*} + Q(X, Z, Z^*)\, U = 0, \tag{2.7}$$

where for x, y, z real,

$$Q(X, Z, Z^*) \equiv q(x, y, z). \tag{2.8}$$

Hence from Hormander's generalized Cauchy-Kowalewski theorem ([16], pp. 116–119) we have that $U(X, Z, Z^*)$ is uniquely determined from the Goursat data $U(X, O, Z^*)$ and $U(X, Z, O)$, which we have already seen are determined from $U(X, O, Z^*)$ alone. This proves the theorem.

We note in passing that the above Goursat problem is well posed in $\mathbb{C}^3$, but not in real Euclidean space $\mathbb{R}^3$. For an interesting discussion of this latter case, see [7].

Motivated by Theorem 2.1 we now construct an integral operator which maps analytic functions onto solutions of Eq. (1.1) such that the associated analytic function is determined in a simple manner from the function $F(X, Z^*) = U(X, O, Z^*)$. We first introduce the following notation:

$$\begin{aligned} \xi_1 &= 2\zeta Z \\ \xi_2 &= X + 2\zeta Z \\ \xi_3 &= X + 2\zeta^{-1} Z^* \end{aligned} \tag{2.9}$$

$$\mu = \tfrac{1}{2}(\xi_2 + \xi_3) = X + \zeta Z + \zeta^{-1} Z^* \tag{2.10}$$

where $1 - \varepsilon < |\zeta| < 1 + \varepsilon$, $0 < \varepsilon < \frac{1}{2}$. These variables are introduced in order to enable us to construct an integral operator whose kernel can be expanded in an infinite series of recursively determinable analytic functions.

THEOREM 2.2 *Let D be a neighborhood of the origin in the μ plane, $B = \{\zeta : 1-\varepsilon < |\zeta| < 1+\varepsilon\}$, G a neighborhood of the origin in the ξ_1, ξ_2, ξ_3 space and $T = \{t : |t| \leqq 1\}$. Let $f(\mu, \zeta)$ be an analytic function of two complex variables in the product domain $D \times B$, and $E^*(\xi_1, \xi_2, \xi_3, \zeta, t) \equiv E(X, Z, Z^*, \zeta, t)$ be a regular solution of the partial differential equation*

$$\mu t(4E^*_{13} + 2E^*_{23} - E^*_{22} - E^*_{33} - Q^*E^*) + (1-t^2)E^*_{1t} - \frac{1}{t}E^*_1 = 0 \tag{2.11}$$

in $G \times B \times T$, where $Q^*(\xi_1, \xi_2, \xi_3, \zeta) \equiv Q(X, Z, Z^*)$, and

$$E^*_i = \frac{\partial E^*}{\partial \xi_i}, \quad E^*_{ij} = \frac{\partial^2 E^*}{\partial \xi_i \partial \xi_j}, \quad E^*_{it} = \frac{\partial^2 E^*}{\partial \xi_i \partial t}; \quad i, j = 1, 2, 3.$$

Then

$$U(X, Z, Z^*) \equiv \mathbf{P}_3\{f\}$$
$$= \frac{1}{2\pi i}\int_{|\zeta|=1}\int_\gamma E(X, Z, Z^*, \zeta, t) f(\mu(1-t^2), \zeta) \tag{2.12}$$
$$\times \frac{dt}{\sqrt{1-t^2}}\frac{d\zeta}{\zeta},$$

where γ is a path in T joining $t = -1$ and $t = +1$, is a (complex valued) solution of Eq. (1.1) *which is regular in a neighborhood of the origin in X, Z, Z^* space.*

Proof Since the Jacobian of the transformation (2.9) is equal to -4, we can conclude that $U(X, Z, Z^*) = \mathbf{P}_3\{f\}$ is regular in a neighborhood of the origin in the X, Z, Z^* space. Straightforward differentiation and integration by parts in Eq. (2.12) leads to

$$U_{ZZ^*} - U_{XX} - QU$$
$$= \frac{1}{2\pi i}\int_{|\zeta|=1}\int_\gamma \frac{f(\mu(1-t^2), \zeta)}{\mu t}\Big\{\mu t(4E^*_{13} + 2E^*_{23} - E^*_{22} - E^*_{33} - Q^*E^*)$$
$$+ (1-t^2)E^*_{1t} - \frac{1}{t}E^*_1\Big\}\frac{dt}{\sqrt{1-t^2}}\frac{d\zeta}{\zeta}. \tag{2.13}$$

Hence if E^* satisfies Eq. (2.11), then Eq. (2.12) defines a solution of Eq. (1.1).

We now must show the existence of the integral operator $\mathbf{P}_3$, i.e. we must show that a function $E(X, Z, Z^*, \zeta, t)$ satisfying the conditions of Theorem 2.2 exists. Our proof is based on the ideas of Tjong [19. 20].

THEOREM 2.3 *Let* $D_r = \{(\xi_1, \xi_2, \xi_3): |\xi_i| < r,\ i = 1, 2, 3\}$ *where* r *is an arbitrary positive number, and* $B_{2\varepsilon} = \{\zeta: |\zeta - \zeta_0| < 2\varepsilon\}$, $0 < \varepsilon < \frac{1}{2}$, *where* ζ_0 *is arbitrary with* $|\zeta_0| = 1$. *Then for each* n, $n = 0, 1, 2, \ldots$, *there exists a unique function* $p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta)$ *which is regular in* $\bar{D}_r \times \bar{B}_{2\varepsilon}$ *and satisfies*

$$p_1^{(n+1)} = \frac{1}{2n+1}\{p_{22}^{(n)} + p_{33}^{(n)} - 4p_{13}^{(n)} - 2p_{23}^{(n)} + Q^* p^{(n)}\} \tag{2.14}$$

$$p^{(0)}(\xi_1, \xi_2, \xi_3, \zeta) \equiv 1$$

$$p^{(n+1)}(0, \xi_2, \xi_3, \zeta) = 0; \qquad n = 0, 1, 2, \ldots \tag{2.15}$$

where

$$p_i^{(n)} = \frac{\partial p^{(n)}}{\partial \xi_i}, \quad p_{ij}^{(n)} = \frac{\partial p^{2(n)}}{\partial \xi_i \partial \xi_j}; \qquad i, j = 1, 2, 3.$$

Furthermore the function

$$E^*(\xi_1, \xi_2, \xi_3, \zeta, t) = 1 + \sum_{n=1}^{\infty} t^{2n} \mu^n p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta) \tag{2.16}$$

is a solution of Eq. (2.11) *which is regular in the product domain* $G_R \times B \times T$, *where* R *is an arbitrary postive number, and*

$$\begin{aligned} G_R &= \{(\xi_1, \xi_2, \xi_3): |\xi_i| < R, \qquad i = 1, 2, 3\} \\ B &= \{\zeta: 1 - \varepsilon < |\zeta| < 1 + \varepsilon\}, \qquad 0 < \varepsilon < \tfrac{1}{2} \\ T &= \{t: |t| \leqq 1\}. \end{aligned} \tag{2.17}$$

The function defined in (2.16) *satisfies*

$$E^*(0, \xi_2, \xi_3, \zeta, t) = 1. \tag{2.18}$$

Proof For $n = 0$, Eqs. (2.14) and (2.15) become

$$p_1^{(1)} = Q^*(\xi_1, \xi_2, \xi_3, \zeta) \tag{2.19}$$

$$p^{(1)}(0, \xi_2, \xi_3, \zeta) = 0,$$

and hence

$$p^{(1)}(\xi_1, \xi_2, \xi_3, \zeta) = \int_0^{\xi_1} Q^*(\xi_1', \xi_2, \xi_3, \zeta)\, d\xi_1' \tag{2.20}$$

is uniquely determined and regular in $\bar{D}_r \times \bar{B}_{2\varepsilon}$. By induction $p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta)$ exists, is uniquely determined, and is regular in $\bar{D}_r \times \bar{B}_{2\varepsilon}$. Now consider the formal series defined in Eq. (2.16). Straightforward differentiation and collection of terms shows that

$$\mu t(4E_{13}^* + 2E_{23}^* - E_{22}^* - E_{33}^* - Q^*E^*) + (1-t^2)E_{1\,t}^* - \frac{1}{t} E_1^*$$
$$= \sum_{n=0}^{\infty} t^{2n+1}\mu^{n+1}(4p_{13}^{(n)} + 2p_{23}^{(n)} - p_{22}^{(n)} - p_{33}^{(n)} - Q^*p^{(n)} + (2n+1)p_1^{(n+1)}), \tag{2.21}$$

and hence if the $p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta)$ are defined by Eqs. (2.14) and (2.15), then the series in Eq. (2.16) formally satisfies Eq. (2.11). It remains to be shown that $E^*(\xi_1, \xi_2, \xi_3, \zeta, t)$ is regular in $G_R \times B \times T$, i.e. the series (2.16) converges absolutely and uniformly in this region. To this end we note that since $\bar{B}$ is a compact subset of the ζ plane, there are finitely many points ζ_j with $|\zeta_j| = 1$, $j = 1, 2, \ldots, N$, such that B is covered by the union of sets

$$N_j = \{\zeta : |\zeta - \zeta_j| < \tfrac{3}{2}\varepsilon\}; \qquad j = 1, 2, \ldots, N. \tag{2.22}$$

Hence it is sufficient to show that the series converges absolutely and uniformly in $\bar{G}_R \times \bar{N}_j \times T$. To this end we majorize the $p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta)$ in $\bar{D}_r \times \bar{B}_{2\varepsilon}$. Since $Q(X, Z, Z^*)$ is an entire function, it follows that $Q^*(\xi_1, \xi_2, \xi_3, \zeta)$ is regular in $\bar{D}_r \times \bar{B}_{2\varepsilon}$, and hence we have

$$Q^*(\xi_1, \xi_2, \xi_3, \zeta) \ll C\left(1 - \frac{\xi_1}{r}\right)^{-1}\left(1 - \frac{\xi_2}{r}\right)^{-1}\left(1 - \frac{\xi_3}{r}\right)^{-1}\left(1 - \frac{\zeta - \zeta_0}{2\varepsilon}\right)^{-1} \tag{2.23}$$

for some $C > 0$ and $(\xi_1, \xi_2, \xi_3, \zeta)$ in $D_r \times B_{2\varepsilon}$. In Eq. (2.23) the symbol "$\ll$" means "is dominated by". The use of dominants, or majorants as they are sometimes called, is a standard tool in the theory of several complex variables, and the reader unfamiliar with their use is referred to [1] or [10] for further details. We will now show by induction that in $D_r \times B_{2\varepsilon}$ we have

$$p_1^{(n)} \ll M(8+\delta)^n(2n-1)^{-1}\left(1 - \frac{\xi_1}{r}\right)^{-(2n-1)} \cdot \left(1 - \frac{\xi_2}{r}\right)^{-(2n-1)}\left(1 - \frac{\xi_3}{r}\right)^{-(2n-1)}\left(1 - \frac{\zeta - \zeta_0}{2\varepsilon}\right)^{-n} r^{-n} \tag{2.24}$$

where M and δ are positive constants independent of n. Equations (2.20) and (2.23) show that (2.24) is true for $n = 1$. Suppose now that (2.24) is true for $n = k$. Then

$$p_1^{(k)} \ll M(8+\delta)^k(2k-1)^{-1}\left(1 - \frac{\xi_1}{r}\right)^{-(2k-1)} \cdot \times \left(1 - \frac{\xi_2}{r}\right)^{-(2k-1)}\left(1 - \frac{\xi_3}{r}\right)^{-(2k-1)}\left(1 - \frac{\zeta - \zeta_0}{2\varepsilon}\right)^{-k} r^{-k}.$$

$$\ll M(8+\delta)^k(2k-1)^{-1}\left(1-\frac{\xi_1}{r}\right)^{-(2k+1)} \cdot \tag{2.25}$$
$$\times\left(1-\frac{\xi_2}{r}\right)^{-(2k-1)}\left(1-\frac{\xi_3}{r}\right)^{-(2k-1)}\left(1-\frac{\zeta-\zeta_0}{2\varepsilon}\right)^{-k} r^{-k}$$

and using the fact that dominance is preserved under the operation of integration we have

$$p^{(k)} \ll M(8+\delta)^k(2k)^{-1}(2k-1)^{-1}\left(1-\frac{\xi_1}{r}\right)^{-2k} \cdot \tag{2.26}$$
$$\times\left(1-\frac{\xi_2}{r}\right)^{-(2k-1)}\left(1-\frac{\xi_3}{r}\right)^{-(2k-1)}\left(1-\frac{\zeta-\zeta_0}{2\varepsilon}\right)^{-k} r^{-k+1}.$$

From Eq. (2.26), and the fact that dominance is preserved under the operation of differentiation, we have

$$p_{22}^{(k)} \ll M(8+\delta)^k\left(1-\frac{\xi_1}{r}\right)^{-(2k+1)}\left(1-\frac{\xi_2}{r}\right)^{-(2k+1)}$$
$$\times\left(1-\frac{\xi_3}{r}\right)^{-(2k+1)}\left(1-\frac{\zeta-\zeta_0}{2\varepsilon}\right)^{-(k+1)} r^{-(k+1)}$$
$$p_{33}^{(k)} \ll M(8+\delta)^k\left(1-\frac{\xi_1}{r}\right)^{-(2k+1)}\left(1-\frac{\xi_2}{r}\right)^{-(2k+1)} \cdot$$
$$\times\left(1-\frac{\xi_3}{r}\right)^{-(2k+1)}\left(1-\frac{\zeta-\zeta_0}{2\varepsilon}\right)^{-(k+1)} r^{-(k+1)}$$
$$p_{13}^{(k)} \ll M(8+\delta)^k\left(1-\frac{\xi_1}{r}\right)^{-(2k+1)}\left(1-\frac{\xi_2}{r}\right)^{-(2k+1)} \tag{2.27}$$
$$\times\left(1-\frac{\xi_3}{r}\right)^{-(2k+1)}\left(1-\frac{\zeta-\zeta_0}{2\varepsilon}\right)^{-(k+1)} r^{-(k+1)}.$$
$$p_{23}^{(k)} \ll M(8+\delta)^k(2k-1)(2k)^{-1}\left(1-\frac{\xi_1}{r}\right)^{-(2k+1)}$$
$$\times\left(1-\frac{\xi_2}{r}\right)^{-(2k+1)}\left(1-\frac{\xi_3}{r}\right)^{-(2k+1)}\left(1-\frac{\zeta-\zeta_0}{2\varepsilon}\right)^{-(k+1)} r^{-(k+1)}.$$

From Eqs. (2.23) and (2.26), and the fact that dominance is preserved under multiplication, we have

$$Q^*p^{(k)} \ll CM(8+\delta)^k(2k)^{-1}(2k-1)^{-1}\left(1-\frac{\xi_1}{r}\right)^{-(2k+1)} \tag{2.28}$$

$$\times\left(1-\frac{\xi_2}{r}\right)^{-(2k+1)}\left(1-\frac{\xi_3}{r}\right)^{-(2k+1)}\left(1-\frac{\zeta-\zeta_0}{2\varepsilon}\right)^{-(k+1)} r^{-k+1}.$$

Equations (2.28), (2.27), and (2.14) now show that

$$p_1^{(k+1)} \ll \frac{M(8+\delta)^k}{2k+1}\left(6+\frac{2k-1}{k}+\frac{Cr^2}{2k(2k-1)}\right)\left(1-\frac{\xi_1}{r}\right)^{-(2k+1)}$$
$$\times\left(1-\frac{\xi_2}{r}\right)^{-(2k+1)}\left(1-\frac{\xi_3}{r}\right)^{-(2k+1)}\left(1-\frac{\zeta-\zeta_0}{2\varepsilon}\right)^{-(k+1)} r^{-(k+1)}. \tag{2.29}$$

For k sufficiently large we have

$$\left(6+\frac{2k-1}{k}+\frac{Cr^2}{2k(2k-1)}\right) < 8+\delta, \tag{2.30}$$

and hence if M is chosen sufficiently large to begin with, we have shown (2.24) is true for $n = k+1$, thus completing the proof by induction. Equation (2.26) now implies that in $\bar{D}_r \times \bar{N}_j$ we have

$$\begin{aligned}|p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta)| \leqq{}& M(8+\delta)^n(2n)^{-1}(2n-1)^{-1}\\ &\times\left(1-\frac{|\xi_1|}{r}\right)^{-2n}\left(1-\frac{|\xi_2|}{r}\right)^{-(2n-1)}\\ &\times\left(1-\frac{|\xi_3|}{r}\right)^{(-2n-1)}\left(1-\frac{|\zeta-\zeta_j|}{2\varepsilon}\right)^{-n} r^{-n+1}.\end{aligned} \tag{2.31}$$

Now consider $|t^{2n}\mu^n p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta)|$ in $\bar{D}_{\alpha r} \times \bar{N}_j \times T$ where

$$D_{\alpha r} = \{(\xi_1, \xi_2, \xi_3): |\xi_i| < \frac{r}{\alpha};\ \alpha > 1,\ i = 1, 2, 3\}. \tag{2.32}$$

Then in $\bar{D}_{\alpha r} \times \bar{N}_j \times T$ we have

$$\begin{aligned}&1-\frac{|\xi_i|}{r} \geqq \frac{\alpha-1}{\alpha}; \qquad i = 1, 2, 3\\ &1-\frac{|\zeta-\zeta_j|}{2\varepsilon} \geqq \frac{1}{4}\\ &|\mu| = \frac{1}{2}|\xi_2+\xi_3| \leqq \frac{r}{\alpha}\\ &|t| \leqq 1.\end{aligned} \tag{2.33}$$

Thus, from Eqs. (2.31) and (2.33) we have

$$\left|t^{2n}\mu^n p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta)\right| \leqq M\, r(\alpha-1)^2\, \alpha^{-2}(2n)^{-1}(2n-1)^{-1} \times \{4\alpha^5(8+\delta)(\alpha-1)^{-6}\}^n. \tag{2.34}$$

If we choose α such that

$$4\alpha^5(8+\delta)(\alpha-1)^{-6} < 1, \tag{2.35}$$

then the series (2.16) converges absolutely and uniformly in $\bar{D}_{\alpha r} \times \bar{N}_j \times T$. By taking $r = \alpha R$ we can now conclude that $E^*(\xi_1, \xi_2, \xi_3, \zeta, t)$ is regular in $\bar{G}_R \times \bar{N}_j \times T$ and hence in $G_R \times B \times T$. Equation (2.18) follows from Eq. (2.15).

Theorem 2.3 shows that the operator $\mathbf{P}_3$ exists. In the special case when $Q(X, Z, Z^*)$ is a constant, $Q(X, Z, Z^*) \equiv \lambda$, the generating function $E(X, Z, Z^*, \zeta, t)$ can be expressed in closed form. In particular we have from Eq. (2.14) that in this case

$$p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta) = \frac{\lambda^n \xi_1^n 2^n}{(2n)!} \tag{2.36}$$

and hence

$$\begin{aligned} E^*(\xi_1, \xi_2, \xi_3, \zeta, t) &= \sum_{n=0}^{\infty} \frac{t^{2n}\mu^n \lambda^n \xi_1^n 2^n}{(2n)!} \\ &= \cosh \sqrt{2\mu\lambda\xi_1}\, t^2 \\ &= \cosh \sqrt{\lambda\xi_1(\xi_2+\xi_3)}\, t^2 \end{aligned} \tag{2.37}$$

or

$$E(X, Z, Z^*, \zeta, t) = \cosh \sqrt{4\lambda(\zeta XZ + \zeta^2 Z^2 + ZZ^*)}\, t^2. \tag{2.38}$$

It would be interesting to determine other equations and/or different integral operators for which $E(X, Z, Z^*, \zeta, t)$ can be expressed in terms of well known special functions.

3. SPECIAL PROPERTIES OF THE OPERATOR $\mathbf{P}_3$

We now turn our attention towards the analytic structure and special properties of the operator constructed in section two. It is in this regard that the operator $\mathbf{P}_3$ is superior to the operator of Tjong [19, 20] which was discussed in the introduction. We will show in particular that every real valued C^2 solution $u(x, y, z)$ of Eq. (1.1) which is defined in some neighborhood of the origin can be expressed locally in the form

$$u(x, y, z) = \operatorname{Re} \mathbf{P}_3\{f\}, \tag{3.1}$$

where "Re" denotes "take the real part" and $f(\mu, \zeta)$ is given in terms of the Goursat data for $U(X, Z, Z^*) \equiv u(x, y, z)$. This leads to results on the analytic structure of $u(x, y, z)$ in $\mathbb{C}^3$, the space of three complex variables, and to the construction of a complete family of solutions to Eq. (1.1) with respect to bounded simply connected domains. The main result of this section is the following theorem:

THEOREM 3.1 *Let $u(x, y, z)$ be a real valued C^2 solution of Eq.* (1.1) *in some neighborhood of the origin in $\mathbb{R}^3$. Then there exists an analytic function of two complex variables $f(\mu, \zeta)$ which is regular for μ in some neighborhood of the origin and $|\zeta| < 1+\varepsilon$, $\varepsilon > 0$, such that locally $u(x, y, z) = \mathrm{Re}\, \mathbf{P}_3\{f\}$. In particular denote by $U(X, Z, Z^*) \equiv u(x, y, z)$ the extension of $u(x, y, z)$ to the X, Z, Z^* space, and let*

$$F(X, Z^*) = U(X, O, Z^*)$$

$$g(\mu, \zeta) = 2\frac{\partial}{\partial\mu}[\mu\int_0^1 F(t\mu, (1-t)\mu\zeta)dt] - F(\mu, 0)\,. \tag{3.2}$$

Then

$$f(\mu, \zeta) = -\frac{1}{2\pi}\int_{\gamma'} g(\mu(1-t^2), \zeta)\frac{dt}{t^2}, \tag{3.3}$$

where γ' is a rectifiable arc joining the points $t = -1$ and $t = +1$ and not passing through the origin.

Remark It can be shown that $g(\mu, \zeta)$ can be expressed in terms of $f(\mu, \zeta)$ by the formula (c.f. [10], p. 114)

$$g(\mu, \zeta) = \int_\gamma f(\mu(1-t^2), \zeta)\frac{dt}{\sqrt{1-t^2}} \tag{3.4}$$

where the path γ is defined as in Theorem 2.2.

Proof of Theorem Since $u(x, y, z)$ is twice continuously differentiable we can conclude from the analytic theory of partial differential equations that $u(x, y, z)$ is an analytic function of its three independent variables in some neighborhood of the origin. Furthermore, since $q(x, y, z)$ is real valued, $\mathrm{Re}\, \mathbf{P}_3\{f\}$ (where x, y, z are real) is a real valued solution of Eq. (1.1) for any function $f(\mu, \zeta)$ which is analytic in the product domain $D \times B$ (see Theorem 2.2). Now suppose that locally $g(\mu, \zeta)$, $f(\mu, \zeta)$, and $Q(X, Z, Z^*) \equiv q(x, y, z)$ have the expansions (c.f. [1], p. 12)

$$g(\mu, \zeta) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{nm}\mu^n\zeta^m$$

$$f(\mu, \zeta) \equiv -\frac{1}{2\pi}\int_{\gamma'} g(\mu(1-t^2), \zeta)\frac{dt}{t^2}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{nm}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})}\mu^n\zeta^m \tag{3.5}$$

$$Q(X, Z, Z^*) = \sum_{l,m,n=0}^{\infty} b_{mnl}X^lZ^nZ^{*m},$$

and define the analytic functions $\bar{Q}(X, Z, Z^*)$ and $\bar{f}(\mu, \zeta)$ by

$$\bar{Q}(X, Z, Z^*) = \sum_{l,m,n=0}^{\infty} \overline{b_{mnl}}\, X^lZ^nZ^{*m} \tag{3.6}$$

and

$$\bar{f}(\mu, \zeta) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \overline{a_{nm}}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})}\mu^n\zeta^m \tag{3.7}$$

respectively. Let $\bar{E}(X, Z, Z^*, \zeta, t)$ be the generating function corresponding to the differential equation $U_{XX}-U_{ZZ^*}+\bar{Q}(X, Z, Z^*)U = 0$. Then for x, y, z real we can write

$$\operatorname{Re}\mathbf{P}_3\{f\} = \frac{1}{4\pi i}\int_{|\zeta|=1}\int_{\gamma} E(X, Z, Z^*, \zeta, t)f(\mu(1-t^2), \zeta)\frac{dt}{\sqrt{1-t^2}}\frac{d\zeta}{\zeta}$$
$$+\frac{1}{4\pi i}\int_{|\zeta|=1}\int_{\gamma} \bar{E}(X, -Z^*, -Z, \zeta, t)\bar{f}(\bar{\mu}(1-t^2), \zeta)\frac{dt}{\sqrt{1-t^2}}\frac{d\zeta}{\zeta} \tag{3.8}$$

where $\bar{\mu} = X-\zeta Z^*-\zeta^{-1}Z$. Now from Theorem 2.1 we know that $U(X, Z, Z^*)$ is uniquely determined by the function $F(X, Z^*) = U(X, O, Z^*)$, and hence using Eqs. (2.18) and (3.8) we try and determine $f(\mu, \zeta)$ from the equation

$$F(X, Z^*) = \frac{1}{4\pi i}\int_{|\zeta|=1}\int_{\gamma} f(\mu_1(1-t^2), \zeta)\frac{dt}{\sqrt{1-t^2}}\frac{d\zeta}{\zeta}$$
$$+\frac{1}{4\pi i}\int_{|\zeta|=1}\int_{\gamma} \bar{E}(X, -Z^*, O, \zeta, t)\bar{f}(\mu_2(1-t^2), \zeta)\frac{dt}{\sqrt{1-t^2}}\frac{d\zeta}{\zeta} \tag{3.9}$$

where $\mu_1 = X+\zeta^{-1}Z^*$ and $\mu_2 = X-\zeta Z^*$. To this end we first write $\bar{E}^*(\xi_1, \xi_2, \xi_3, \zeta, t) \equiv \bar{E}(X, Z, Z^*, \zeta, t)$ in its series expansion

$$\bar{E}^*(\xi_1, \xi_2, \xi_3, \zeta, t) = 1+\sum_{n=1}^{\infty} t^{2n}\mu^n\bar{p}^{(n)}(\xi_1, \xi_2, \xi_3, \zeta), \tag{3.10}$$

where from Theorem 2.3 we have

$$\bar{p}^{(1)} = \int_0^{\xi_1} \bar{Q}^*(\xi_1, \xi_2, \xi_3, \zeta)d\xi_1' \tag{3.11}$$

$$\bar{p}_1^{(n+1)} = \frac{1}{2n+1}\{\bar{p}_{22}^{(n)} + \bar{p}_{33}^{(n)} - 4\bar{p}_{13}^{(n)} - 2\bar{p}_{23}^{(n)} + \bar{Q}^*\bar{p}^{(n)}\}$$

$$\bar{p}^{(0)}(\xi_1, \xi_2, \xi_3, \zeta) = 1 \tag{3.12}$$

$$\bar{p}^{(n+1)}(0, \xi_2, \xi_3, \zeta) = 0; \qquad n = 0, 1, 2, \ldots$$

with $\bar{Q}^*(\xi_1, \xi_2, \xi_3, \zeta) \equiv \bar{Q}(X, Z, Z^*)$. From Eqs. (2.9) and (3.12) we have

$$\begin{aligned}\bar{p}^{(1)} &= \int_0^{\xi_1} \bar{Q}\left(\xi_2 - \xi_1', \frac{1}{2\zeta}\xi_1', \frac{\zeta}{2}(\xi_3 - \xi_2 + \xi_1')\right)d\xi_1' \\ &= 2\zeta\int_0^Z \bar{Q}\left(\xi_2 - 2\zeta\tau, \tau, \frac{\zeta}{2}(\xi_3 - \xi_2 + 2\zeta\tau)\right)d\tau \\ &= 2\zeta\int_0^Z \bar{Q}\left(X + 2\zeta Z - 2\zeta\tau, \tau, \frac{\zeta}{2}(2\zeta^{-1}Z^* - 2\zeta Z + 2\zeta\tau)\right)d\tau,\end{aligned} \tag{3.13}$$

i.e. $\bar{p}^{(1)}$ is an entire function of X, Z, Z^*, and ζ, and vanishes for $\zeta = 0$. A similar calculation using Eq. (3.12) and induction shows that the same statement can be made about $\bar{p}^{(n)}$ for $n = 1, 2, 3, \ldots$. (This behavior is a special characteristic of our operator $\mathbf{P}_3$ and is not true, for example, of the operator developed by Tjong in [19] and [20]). Due now to the uniform convergence of the series in Eq. (3.10), we can substitute this series into Eq. (3.9) and integrate termwise to conclude that

$$\begin{aligned}F(X, Z^*) &= \frac{1}{4\pi i}\int_{|\zeta|=1}\int_\gamma f(\mu_1(1-t^2), \zeta)\frac{dt}{\sqrt{1-t^2}}\frac{d\zeta}{\zeta} \\ &\quad + \frac{1}{4\pi i}\int_{|\zeta|=1}\int_\gamma \bar{f}(\mu_2(1-t^2), \zeta)\frac{dt}{\sqrt{1-t^2}}\frac{d\zeta}{\zeta} \\ &= \frac{1}{4\pi i}\int_{|\zeta|=1} g(\mu_1, \zeta)\frac{d\zeta}{\zeta} \\ &\quad + \frac{1}{4\pi i}\int_{|\zeta|=1} \bar{g}(\mu_2, \zeta)\frac{d\zeta}{\zeta}\end{aligned} \tag{3.14}$$

where $g(\mu, \zeta)$ is defined by Eqs. (3.4) and (3.5), and

$$\bar{g}(\mu, \zeta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \overline{a_{nm}}\mu^n\zeta^m. \tag{3.15}$$

To complete the proof of the theorem it now suffices to show that Eq. (3.2) gives the solution of Eq. (3.14). In order to show this we let

$$F(X, Z^*) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm} X^n Z^{*m}, \tag{3.16}$$

and note that since $u(x, y, z)$ is real valued, the coefficients c_{n0}, $n = 0, 1, 2, \ldots$, are all real. Using Eqs. (3.5), (3.14), (3.15), (3.16), and equating coefficients of $X^n Z^{*m}$ gives

$$\begin{aligned} 2n!m!\, c_{nm} &= (n+m)!\, a_{n+m\ m}; \qquad n \geqq 0, \quad m > 0 \\ 2c_{n0} &= a_{n0} + \overline{a_{n0}}; \qquad n \geqq 0. \end{aligned} \tag{3.17}$$

Hence, without loss of generality, we assume the coefficients a_{n0}, $n = 0, 1, 2, \ldots$ are real. Equation (3.17) then becomes

$$\begin{aligned} c_{nm} &= \frac{1}{2} \frac{\Gamma(n+m+1)}{\Gamma(n+1)\Gamma(m+1)}\, a_{n+m,\, m}; \qquad n \geqq 0, \quad m > 0 \\ c_{n0} &= a_{n0}; \qquad n \geqq 0, \end{aligned} \tag{3.18}$$

and therefore

$$\begin{aligned} F(X, Z^*) &= \frac{1}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\Gamma(n+m+1)}{\Gamma(n+1)\Gamma(m+1)}\, a_{n+m,\, m} X^n Z^{*m} \\ &\quad + \frac{1}{2} \sum_{n=0}^{\infty} c_{n0} X^n \\ &= \frac{1}{2} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-m+1)\Gamma(m+1)}\, a_{nm} X^{n-m} Z^{*m} \\ &\quad + \frac{1}{2} \sum_{n=0}^{\infty} c_{n0} X^n \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{\Gamma(n+1)}{\Gamma(n-m+1)\Gamma(m+1)}\, a_{nm} X^{n-m} Z^{*m} \\ &\quad + \frac{1}{2} \sum_{n=0}^{\infty} c_{n0} X^n . \end{aligned} \tag{3.19}$$

From the definition of the beta function $B(x, y)$,

$$B(x, y) \equiv \int_0^1 t^{x-1}(1-t)^{y-1}\, dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{3.20}$$

we have

$$\int_0^1 F(tX, (1-t)Z^*)\, dt = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{a_{nm}}{n+1} X^{n-m} Z^{*m} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{c_{n0}}{n+1} X^n \tag{3.21}$$

or

$$\frac{\partial}{\partial\mu}\left[\mu\int_0^1 F(t\mu,(1-t)\mu\zeta)dt\right]$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{nm}\mu^n\zeta^m+\frac{1}{2}\sum_{n=0}^{\infty}c_{n0}\mu^n \tag{3.22}$$
$$=\frac{1}{2}g(\mu,\zeta)+\frac{1}{2}F(\mu,0)\,.$$

Equation (3.2) follows immediately from Eq. (3.22) and this proves the theorem.

Note When $Q(X, Z, Z^*) \equiv 0$ then our operator $\mathbf{P}_3$ reduces to the well known Bergman-Whittaker operator $\mathbf{B}_3$ [1, 10]. Theorem 3.1 gives new results even in this case, since if $H(X, Z, Z^*)$ is a real valued harmonic function (of the variables x, y, z), then for x, y, z real

$$H(X,Z,Z^*)=\operatorname{Re}\mathbf{B}_3\{g\}$$
$$=\operatorname{Re}\left(\frac{1}{2\pi i}\int_{|\zeta|=1}g(\mu,\zeta)\frac{d\zeta}{\zeta}\right) \tag{3.23}$$

and Eq. (3.2) shows we can write $(\operatorname{Re}\mathbf{B}_3)^{-1}H$ as

$$g(\mu,\zeta)=(\operatorname{Re}\mathbf{B}_3)^{-1}H$$
$$=2\frac{\partial}{\partial\mu}\left[\mu\int_0^1 H(t\mu,0,(1-t)\mu\zeta)\,dt\right]-H(\mu,0,0)\,. \tag{3.24}$$

It is of interest to compare Eq. (3.24) with the inversion formula given by Bergman for complex valued harmonic functions (c.f. [1], p. 41, [10], p. 58).

COROLLARY 3.1 *Let $u(x, y, z)$ be a real valued C^2 solution of Eq.* (1.1) *in some neighborhood of the origin in $\mathbb{R}^3$, and let $U(X, Z, Z^*)$ be its extension to the X, Z, Z^* space. Then the location of the possible singular points of $U(X, Z, Z^*)$ in $\mathbb{C}^3$ can be determined from the location of the singularities of $U(X, O, Z^*)$ in $\mathbb{C}^2$.*

Proof The location of the possible singularities of$U(X, Z, Z^*)$ can be determined from a knowledge of the singularity manifold of $f(\mu, \zeta)$ and $E(X, Z, Z^*, \zeta, t)$ by using the operator $\mathbf{P}_3$ and Gilbert's "envelope method" (see in particular Theorem 1.3.4 of [10]). But the location of possible singularities of $f(\mu, \zeta)$ can in turn be determined from the location of the singularities of $U(X, O, Z^*)$ through using Gilbert's "envelope method" again in conjunction with Eqs. (3.2) and (3.3) (c.f. Theorem 1.3.1 of [10]). We note that from the proofs of Theorems 2.3 and 3.1 it is seen that $E(X, Z, Z^*, \xi, t)$ is an entire function of its five independent (complex) variables.

We now turn our attention to the problem of constructing a complete family of solutions to Eq. (1.1) in a simply connected domain G. This will be accomplished by using some results of Malgrange, Lax, Browder, and Garabedian to construct a family of entire solutions of Eq. (1.1) which are dense in the space of solutions of Eq. (1.1) which are twice continuously differentiable in G. We first require a few preliminary definitions. In what follows $\mathbf{L}$ denotes a second order linear elliptic operator with twice continuously differentiable coefficients, and $\mathbf{L}^*$ is the formal adjoint of $\mathbf{L}$.

DEFINITION 3.1 Solutions of an equation $\mathbf{L}u = 0$ are said to have the Runge approximation property if, whenever G_1 and G_2 are two simply connected domains, G_1 a subset of G_2, any solution in G_1 can be approximated uniformly in compact subsets of G_1 by a sequence of solutions in G_2.

DEFINITION 3.2. Solutions of an equation $\mathbf{L}u = 0$ are said to have the unique continuation property if whenever a solution vanishes in an open set it vanishes identically in its domain of definition.

We can now state the following result due to Malgrange [18], Lax [17], and Browder [3]: Solutions of $\mathbf{L}u = 0$ have the Runge approximation property if and only if solutions of $\mathbf{L}^*u = 0$ have the unique continuation property. Since elliptic equations with analytic coefficients have the unique continuation property as a consequence of the analylicity of their solutions, we can conclude that Eq. (1.1) has the Runge approximation property. Hence to construct a complete family of solutions to Eq. (1.1) with respect to a bounded, simply connected domain G, it suffices to construct a complete family of solutions with respect to a sphere containing G in its interior. We are now in a position to prove the following theorem. In what follows"Im" denotes "take the imaginary part".

THEOREM 3.2 *Let G be a bounded, simply connected domain in $\mathbb{R}^3$, and define*

$$
\begin{aligned}
u_{2n,m}(x, y, z) &= \operatorname{Re} \mathbf{P}_3\{\mu^n \zeta^m\};\ 0 \leqq n < \infty,\ m = 0, 1, \ldots, n \\
u_{2n+1,m}(x, y, z) &= \operatorname{Im} \mathbf{P}_3\{\mu^n \zeta^m\};\ 0 \leqq n < \infty,\ m = 0, 1, \ldots, n.
\end{aligned}
\tag{3.25}
$$

Then the set $\{u_{nm}\}$ is a complete family of solutions for Eq. (1.1) *in the space of real valued C^2 solutions of Eq.* (1.1) *defined in G.*

Proof: Let $u(x, y, z)$ be a real valued C^2 solution of Eq. (1.1) in G, and let G_1 be a compact subset of G. Then by the Runge approximation property for any $\varepsilon > 0$ there exists a real valued solution $u_1(x, y, z)$ of Eq. (1.1) which is regular in a sphere S containing G in its interior such that

$$\max_{(x,y,z)\in G_1} |u-u_1| < \frac{\varepsilon}{3}. \tag{3.26}$$

We now try and approximate $u_1(x, y, z)$ in S by an entire solution of Eq. (1.1). From Garabedian's work on Cauchy's problem for analytic systems ([9], pp. 614–619) we can conclude that since $u_1(x, y, z)$ is regular in S, the Cauchy data for $u_1(x, y, z)$ must be regular in some convex region B in $\mathbb{C}^2$, the space of two complex variables, and that $u_1(x, y, z)$ depends continuously on this data in S. Now since convex domains are Runge domains of the first kind ([8], p. 229), on compact subsets of B we can approximate the (real valued) Cauchy data for $u_1(x, y, z)$ by polynomials and construct a (real valued) solution $u_2(x, y, z)$ of Eq. (1.1) with polynomial Cauchy data. Since Eq. (1.1) is linear with the Laplacian as its principle part, and $q(x, y, z)$ is an entire function of the (complex) variables x, y, z, the Cauchy-Kowalewski Theorem implies that $u_2(x, y, z)$ is also entire. Due to the continuous dependence of $u_1(x, y, z)$ on its Cauchy data in B, we therefore have that there exists a domain G_2, $G \subset \bar{G}_2 \subset S$, and a (real valued) entire solution $u_2(x, y, z)$ of Eq. (1.1), such that

$$\max_{(x,y,z)\in G_2} |u_1-u_2| < \frac{\varepsilon}{3}. \tag{3.27}$$

Now since $u_2(x, y, z)$ is an entire function of x, y, z, $U_2(X, Z, Z^*)$, the extension of $u_2(x, y, z)$ to the X, Z, Z^* space, is an entire function of X, Z, Z^*. In particular $U_2(X, Z, Z^*)$ is regular in the product domain $D_R = \{|X| \leqq R\} \times \{|Z| \leqq R\} \times \{|Z^*| \leqq R\}$ for arbitrarily large R. Then $U_2(X, O, Z^*)$ is regular in the product domain $\{|X| \leqq R\} \times \{|Z^*| \leqq R\}$. From Hormander's generalized Cauchy-Kowalewski Theorem ([16], pp. 116–119) and Theorem 2.1, it is seen that $U_2(X, Z, Z^*)$ depends continuously on $U_2(X, O, Z^*)$ (for $\{(X, Z^*): |X| \leqq R, |Z^*| \leqq R\}$) in some smaller product domain $D_{R_1} = \{|X| \leqq R_1\} \times \{|Z| \leqq R_1\} \times \{|Z^*| \leqq R_1\}$. Now choose R large enough such that if (x, y, z) is in $\bar{G}_2$ then (X, Z, Z^*) is in D_{R_1}. Since product domains are Runge domains of the first kind ([8], p. 49), we can approximate $U_2(X, O, Z^*)$ by a polynomial in $\{|X| \leqq R\} \times \{|Z^*| \leqq R\}$ and use Theorem 2.1 and Hormander's version of the Cauchy-Kowalewski Theorem to construct an entire solution $U_3(X, Z, Z^*)$ of Eq. (2.7) which is real for x, y, z real and has polynomial Goursat data. The above discussion implies that there exists a real valued entire solution $u_3(x, y, z)$ of Eq. (1.1) with polynomial Goursat data (in the variables X, Z, Z^*) such that

$$\max_{(x,y,z)\in G_2} |u_2-u_3| < \frac{\varepsilon}{3}. \tag{3.28}$$

Since $U_3(X, O, Z^*)$ is a polynomial, Theorem 3.1 implies that there exists a polynomial (with possibly complex coefficients)

$$h_N(\mu, \zeta) = \sum_{n=0}^{N} \sum_{m=0}^{n} a_{nm}\mu^n\zeta^m \tag{3.29}$$

such that

$$u_3(x, y, z) = \operatorname{Re} \mathbf{P}_3\{h_N\}. \tag{3.30}$$

Equations (3.26)–(3.30) now imply that

$$\max_{(x,y,z) \in G_1} \left|u(x, y, z) - \operatorname{Re} \mathbf{P}_3\{h_N\}\right| < \varepsilon. \tag{3.31}$$

Since ε can be arbitrarily small, and G_1 was an arbitrary compact subset of G, the proof of Theorem 3.2 is now complete.

In passing we would like to point out that if we were considering linear elliptic equations in *two* independent variables we could construct the function analogous to $u_3(x, y, z)$ directly from the results of Vekua ([21], p. 36). Henrici's work on the two variable elliptic Cauchy problem ([14], pp. 195–200) implies that a similar statement can also be made about the two variable analogue of $u_2(x, y, z)$. Finally, we note that when $q(z, y, z) \equiv 0$, and $\mathbf{P}_3$ reduces to the Bergman-Whittaker operator $\mathbf{B}_3$, the functions $u_{nm}(x, y, z)$ become ([10], p. 48)

$$u_{2n,m}(x, y, z) = \frac{n!}{(n+m)!} r^n P_n^m(\cos\theta) \operatorname{Re}(i^m e^{im\varphi}) \tag{3.32}$$

$$u_{2n+1,m}(x, y, z) = \frac{n!}{(n+m)!} r^n P_n^m(\cos\theta) \operatorname{Im}(i^m e^{im\varphi})$$

where r, θ, φ are spherical coordinates and P_n^m denotes the associated Legendre polynomial.

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Integral Operators for Elliptic Equations in Three Independent Variables, II

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An integral operator is obtained which maps analytic functions of two complex variables onto solutions of a homogeneous linear elliptic partial differential equation in three independent variables. An inversion formula is given and used to construct a complete family of solutions for the elliptic equation under investigation.

1. INTRODUCTION

Bergman [1] and Vekua [5] (see also [3]) have both constructed integral operators which map analytic functions of a single complex variable onto solutions of linear elliptic equations in two independent variables. In recent years many attempts have been made to extend these results to the case of three independent variables, attention being given primarily to the elliptic equation

$$\begin{aligned} \Delta_3 u + q(x, y, z)u &= 0 \\ \Delta_3 &\equiv (\partial^2/\partial x^2) + (\partial^2/\partial y^2) + (\partial^2/\partial z^2). \end{aligned} \tag{1.1}$$

For a discussion of both old and new results on equations of this form the reader is referred to [2], which will henceforth be referred to as *I*. In this paper we will give the complete generalization of the work of Bergman and Vekua to the case of three independent variables. In particular we consider the (homogeneous) linear elliptic equation in three independent variables

$$\Delta_3 u + a(x, y, z)u_x + b(x, y, z)u_y + c(x, y, z)u_z + d(x, y, z)u = 0 \tag{1.2}$$

where $a(x, y, z)$, $b(x, y, z)$, $c(x, y, z)$, and $d(x, y, z)$ are real valued analytic functions of their independent variables, and construct an integral operator which maps analytic functions of *two* complex variables onto real valued twice continuously differentiable (i.e. C^2) solutions of Eq. (1.2). As an application of this result we will construct a complete family of solutions of Eq. (1.2) with respect to a bounded, simply connected domain in Euclidean three space $\mathbb{R}^3$. Much of our analysis will be based on the ideas of *I*, and we will assume the reader has access to this paper. Several new ideas do make their appearance however. For example, in addition to the construction of the kernel of our integral operator, it is now also necessary to construct a real valued solution $u_0(x, y, z)$ of Eq. (1.2) which is constant along two complex characteristic hyperplanes passing through the origin. In the special case when $d(x, y, z) \equiv 0$ we can choose $u_0(x, y, z) \equiv 1$. In the general case $u_0(x, y, z)$ is constructed by a standard iterative procedure.

As in *I*, we assume for the sake of simplicity that $a(x, y, z)$, $b(x, y, z)$, $c(x, y, z)$, and $d(x, y, z)$ are (real valued) entire functions of their independent (complex) variables.

2. THE OPERATOR C_3

We first introduce the new independent (complex) variables

$$\begin{aligned} X &= x \\ Z &= \tfrac{1}{2}(y+iz) \\ Z^* &= \tfrac{1}{2}(-y+iz). \end{aligned} \tag{2.1}$$

In X, Z, Z^* coordinates Eq. (1.2) becomes

$$\begin{aligned} U_{XX}-U_{ZZ^*}&+A(X,Z,Z^*)U_X+B(X,Z,Z^*)U_Z \\ &+C(X,Z,Z^*)U_{Z^*}+D(X,Z,Z^*)U=0 \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} U(X,Z,Z^*) &\equiv u(x,y,z) \\ A(X,Z,Z^*) &\equiv a(x,y,z) \\ B(X,Z,Z^*) &\equiv \tfrac{1}{2}[b(x,y,z)+ic(x,y,z)] \\ C(X,Z,Z^*) &\equiv \tfrac{1}{2}[-b(x,y,z)+ic(x,y,z)] \\ D(X,Z,Z^*) &\equiv d(x,y,z). \end{aligned} \tag{2.3}$$

It is convenient for us to rewrite Eq. (2.2) in standard form by making the substitution

$$V(X, Z, Z^*) = U(X, Z, Z^*) \exp\left[-\int_0^Z C(X, Z', Z^*)\, dZ'\right]. \tag{2.4}$$

Substitution of Eq. (2.4) into Eq. (2.2) yields the following equation satisfied by $V(X, Z, Z^*)$:

$$V_{XX} - V_{ZZ^*} + \tilde{A}(X, Z, Z^*) V_X + \tilde{B}(X, Z, Z^*) V_Z + \tilde{D}(X, Z, Z^*) V = 0 \tag{2.5}$$

where

$$\begin{aligned} \tilde{A}(X, Z, Z^*) &= A(X, Z, Z^*) + 2\int_0^Z C_X(X, Z', Z^*)\, dZ' \\ \tilde{B}(X, Z, Z^*) &= B(X, Z, Z^*) - \int_0^Z C_{Z^*}(X, Z', Z^*)\, dZ' \\ \tilde{D}(X, Z, Z^*) &= D(X, Z, Z^*) + A(X, Z, Z^*) \int_0^Z C_X(X, Z', Z^*)\, dZ' \\ &\quad + B(X, Z, Z^*) C(X, Z, Z^*) - C_{Z^*}(X, Z, Z^*) \\ &\quad + \int_0^Z C_{XX}(X, Z', Z^*)\, dZ' + \left[\int_0^Z C_X(X, Z', Z^*)\, dZ'\right]^2. \end{aligned} \tag{2.6}$$

It is clear that Eq. (2.4) defines a one to one mapping of solutions of Eq. (2.2) onto solutions of Eq. (2.5).

The following theorem motivates the analysis which follows and can be proved in the same manner as Theorem 2.1 of *I*.

THEOREM 2.1 *Let $u(x, y, z)$ be a real valued C^2 solution of Eq. (1.2) in a neighborhood of the origin. Then $U(X, Z, Z^*) \equiv u(x, y, z)$ is an analytic function of X, Z, Z^* in some neighborhood of the origin in $\mathbb{C}^3$, the space of three complex variables, and is uniquely determined by the function $U(X, 0, Z^*)$.*

By introducing the change of variables

$$\begin{aligned} \xi_1 &= 2\zeta Z \\ \xi_2 &= X + 2\zeta Z \\ \xi_3 &= X + 2\zeta^{-1} Z^* \end{aligned} \tag{2.7}$$

$$\mu = \tfrac{1}{2}(\xi_2 + \xi_3) = X + \zeta Z + \zeta^{-1} Z^* \tag{2.8}$$

where $1-\varepsilon < |\zeta| < 1+\varepsilon$, $0 < \varepsilon < \frac{1}{2}$, the following theorem can be proved by straightforward substitution and integration by parts.

THEOREM 2.2 *Let D be a neighborhood of the origin in the μ plane, $B = \{\zeta : 1-\varepsilon < |\zeta| < 1+\varepsilon\}$, G a neighborhood of the origin in the ξ_1, ξ_2, ξ_3 space and $T = \{t : |t| \leqq 1\}$. Let $f(\mu, \zeta)$ be an analytic function of two complex* *variables in the product domain $D\times B$, and $E^*(\xi_1, \xi_2, \xi_3, \zeta, t) = E(X, Z, Z^*, \zeta, t)$ be a regular solution of the partial differential equation*

$$\begin{aligned}\mu t(4E^*_{13}+2E^*_{23}-E^*_{22}-E^*_{33}-\tilde{D}^*E^*)+(1-t^2)E^*_{1t}\\ -(1/t)E^*_1-\tilde{A}^*[(E^*_2+E^*_3)\mu t+\tfrac{1}{2}(1-t^*)E^*_t-(1/2t)E^*]\\ -\tilde{B}^*\zeta[(2E^*_1+2E^*_2)\mu t+\tfrac{1}{2}(1-t^2)E^*_t-(1/2t)E^*]=0\end{aligned} \tag{2.9}$$

in $G\times B\times T$, where

$$\begin{aligned}\tilde{A}^*(\xi_1, \xi_2, \xi_3, \zeta) &\equiv \tilde{A}(X, Z, Z^*)\\ \tilde{B}^*(\xi_1, \xi_2, \xi_3, \zeta) &\equiv \tilde{B}(X, Z, Z^*)\\ \tilde{D}^*(\xi_1, \xi_2, \xi_3, \zeta) &\equiv \tilde{D}(X, Z, Z^*)\end{aligned} \tag{2.10}$$

and

$$E^*_i = \frac{\partial E^*}{\partial \xi_i}, \quad E^*_{ij} = \frac{\partial^2 E^*}{\partial \xi_i \partial \xi_j}, \quad E^*_t = \frac{\partial E^*}{\partial t}, \quad E^*_{1t} = \frac{\partial^2 E^*}{\partial \xi_1 \partial t}.$$

Then

$$\begin{aligned}V(X, Z, Z^*) &\equiv \underset{\sim}{C}'_3\{f\}\\ &= (1/2\pi i)\int_{|\zeta|=1}\int_\gamma E(X, Z, Z^*, \zeta, t)f(\mu(1-t^2), \zeta)(dt/\sqrt{1-t^2})(d\zeta/\zeta),\end{aligned} \tag{2.11}$$

where γ is a path in T joining $t=-1$ and $t=+1$, is a (complex valued) solution of Eq. (2.5) *which is regular in a neighborhood of the origin in X, Z, Z^* space.*

From Eq. (2.4) we have the following corollary to Theorem 2.2.

COROLLARY 2.1 *The function*

$$\begin{aligned}U(X, Z, Z^*) &\equiv \underset{\sim}{C}_3\{f\}\\ &= (1/2\pi i)\int_{|\zeta|=1}\int_\gamma \exp\left[\int_0^Z C(X, Z', Z^*)\,dZ'\right]\\ &\quad\times E(X, Z, Z^*, \zeta, t)f(\mu(1-t^2), \zeta)(dt/\sqrt{1-t^2})(d\zeta/\zeta)\end{aligned} \tag{2.12}$$

is a (complex valued) solution of Eq. (2.2) *which is regular in a neighborhood of the origin in X, Z, Z^* space.*

The next theorem shows that a function $E(X, Z, Z^*, \zeta, t)$ satisfying the conditions of Theorem 2.2 exists.

THEOREM 2.3 *Let* $D_r = \{(\xi_1, \xi_2, \xi_3): |\xi_i| < r,\ i = 1, 2, 3\}$ *where* r *is an arbitrary positive number, and* $B_{2\varepsilon} = \{\zeta: |\zeta - \zeta_0| < 2\varepsilon\}$, $0 < \varepsilon < \frac{1}{2}$, *where* ζ_0 *is arbitrary with* $|\zeta_0| = 1$. *Then for each* n, $n = 1, 2, \ldots$ *there exists a unique function* $p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta)$ *which is regular in* $\bar{D}_r \times \bar{B}_{2\varepsilon}$ *and satisfies*

$$p_1^{(n+1)} - \tfrac{1}{2}(\tilde{A}^* + \tilde{B}^*\zeta)p^{(n+1)} = 1/(2n+1)\,\{p_{22}^{(n)} + p_{33}^{(n)} - 4p_{13}^{(n)} - 2p_{23}^{(n)} + (\tilde{A}^* + 2\tilde{B}^*\zeta)p_2^{(n)} + \tilde{A}^* p_3^{(n)} + 2\tilde{B}^*\zeta p_1^{(n)} + \tilde{D}^* p^{(n)}\} \tag{2.13}$$

$$p^{(1)}(\xi_1, \xi_2, \xi_3, \zeta) = \exp\left[\tfrac{1}{2}\textstyle\int_0^{\xi_1} (\tilde{A}^* + \tilde{B}^*\zeta)\, d\xi_1'\right] \tag{2.14}$$

$$p^{(n+1)}(0, \xi_2, \xi_3, \zeta) = 0; \quad n = 1, 2, \ldots \tag{2.15}$$

where

$$p_i^{(n)} = \partial p^{(n)}/\partial \xi_i, \quad p_{ij}^{(n)} = \partial p^{(n)}/\partial \xi_i \partial \xi_j.$$

Furthermore, the function

$$E^*(\xi_1, \xi_2, \xi_3, \zeta, t) = \sum_{n=1}^{\infty} t^{2n} \mu^n p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta) \tag{2.16}$$

is a solution of Eq. (2.9) *which is regular in the product domain* $G_R \times B \times T$, *where* R *is an arbitrary positive number, and*

$$\begin{aligned} G_R &= \{(\xi_1, \xi_2, \xi_3): |\xi_i| < R,\ i = 1, 2, 3\} \\ B &= \{\zeta: 1 - \varepsilon < |\zeta| < 1 + \varepsilon\},\ 0 < \varepsilon < \tfrac{1}{2} \\ T &= \{t: |t| \leq 1\}. \end{aligned} \tag{2.17}$$

The function defined in (2.16) *satisfies*

$$E^*(0, \xi_2, \xi_3, \zeta, t) = \mu t^2. \tag{2.18}$$

Remark It is not possible to have $E^*(0, \xi_2, \xi_3, \zeta, t) = 1$ as in *I*, since in this case Eq. (2.9) cannot be satisfied due to the appearance of the term $(1/2t)E^*(\tilde{A}^* + \tilde{B}^*\zeta)$.

Proof of Theorem It is clear from Eqs. (2.13)–(2.15) that each $p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta)$ exists, is uniquely determined, and is regular in $\bar{D}_r \times \bar{B}_{2\varepsilon}$. Now consider the formal series defined in Eq. (2.16). Straightforward differentiation and collection of terms shows that if the $p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta)$ are defined by Eqs. (2.13)–(2.15) then the series in Eq. (2.16) formally satisfies Eq. (2.9). It remains to be shown that the series converges absolutely and uniformly in $G_R \times B \times T$. To this end we first introduce new functions $q^{(n)}(\xi_1, \xi_2, \xi_3, \zeta)$ defined by

$$q^{(n)} = p^{(n)} \exp\left[-\tfrac{1}{2}\textstyle\int_0^{\xi_1} (\tilde{A}^* + \tilde{B}^*\zeta)\, d\xi_1'\right]; \quad n = 1, 2, \ldots. \tag{2.19}$$

Equations (2.13)–(2.15) now imply that the functions $q^{(n)}(\xi_1, \xi_2, \xi_3, \zeta)$ satisfy the recurrence scheme

$$q_1^{(n+1)} = 1/(2n+1)\{q_{33}^{(n)}+q_{22}^{(n)}-2q_{23}^{(n)}-4q_{13}^{(n)}+R^*q_3^{(n)} + S^*q_2^{(n)}+T^*q_1^{(n)}+W^*q^{(n)}\} \tag{2.20}$$

$$q^{(1)}(\xi_1, \xi_2, \xi_3. \zeta) = 1$$
$$q^{(n+1)}(0, \xi_2, \xi_3, \zeta) = 0; \quad n = 1, 2, 3 \ldots \tag{2.21}$$

where

$$\begin{aligned}
R^* &= \tilde{A}^*-2(\tilde{A}^*+\tilde{B}^*\zeta)-(\partial/\partial\xi_2)\textstyle\int_0^{\xi_1} (\tilde{A}^*+\tilde{B}^*\zeta)\, d\xi_1'+(\partial/\partial\xi_3) \\
&\qquad \times \textstyle\int_0^{\xi_1} (\tilde{A}^*+\tilde{B}^*\zeta)\, d\xi_1' \\
S^* &= \tilde{A}^*+2\tilde{B}^*\zeta-(\partial/\partial\xi_3)\textstyle\int_0^{\xi_1} (\tilde{A}^*+\tilde{B}^*\zeta)\, d\xi_1'+(\partial/\partial\xi_2) \\
&\qquad \times \textstyle\int_0^{\xi_1} (\tilde{A}^*+\tilde{B}^*\zeta)\, d\xi_1' \\
T^* &= 2\tilde{B}^*\zeta-2(\partial/\partial\xi_3)\textstyle\int_0^{\xi_1} (\tilde{A}^*+\tilde{B}^*\zeta)\, d\xi_1' \qquad (2.22)\\
W^* &= \tilde{D}^*+\tilde{B}^*\zeta(\tilde{A}^*+\tilde{B}^*\zeta)+\tfrac{1}{2}(\tilde{A}^*+2\tilde{B}^*\zeta)(\partial/\partial\xi_2)\textstyle\int_0^{\xi_1} (\tilde{A}^*+\tilde{B}^*\zeta)\, d\xi_1' \\
&\quad +\tfrac{1}{2}\tilde{A}^*(\partial/\partial\xi_3)\textstyle\int_0^{\xi_1} (\tilde{A}^*+\tilde{B}^*\zeta)\, d\xi_1'-2(\partial/\partial\xi_3)(\tilde{A}^*+\tilde{B}^*\zeta) \\
&\quad -(\tilde{A}^*+\tilde{B}^*\zeta)(\partial/\partial\xi_3)\textstyle\int_0^{\xi_1} (\tilde{A}^*+\tilde{B}^*\zeta)\, d\xi_1'-(\partial^2/\partial\xi_2\partial\xi_3) \\
&\qquad \times \textstyle\int_0^{\xi_1} (\tilde{A}^*+\tilde{B}^*\zeta)\, d\xi_1' \\
&\quad -\tfrac{1}{2}[(\partial/\partial\xi_2)\textstyle\int_0^{\xi_1} (\tilde{A}^*+\tilde{B}^*\zeta)\, d\xi_1'][(\partial/\partial\xi_3)\textstyle\int_0^{\xi_1} (\tilde{A}^*+\tilde{B}^*\zeta)\, d\xi_1'] \\
&\quad +\tfrac{1}{2}(\partial^2/\partial\xi_2^2)\textstyle\int_0^{\xi_1} (\tilde{A}^*+\tilde{B}^*\zeta)\, d\xi_1' \\
&\quad +\tfrac{1}{4}[(\partial/\partial\xi_2)\textstyle\int_0^{\xi_1} (\tilde{A}^*+\tilde{B}^*\zeta)\, d\xi_1']^2 \\
&\quad +\tfrac{1}{2}(\partial^2/\partial\xi_3^2)\textstyle\int_{0i}^{\xi_1} (\tilde{A}^*+\tilde{B}^*\zeta)\, d\xi_1' \\
&\quad +\tfrac{1}{4}[(\partial/\partial\xi_3)(\tilde{A}^*+\tilde{B}^*\zeta)\, d\xi_1']^2.
\end{aligned}$$

To prove the theorem it now suffices to show that the series

$$\sum_{n=1}^{\infty} t^{2n}\mu^n q^{(n)}(\xi_1, \xi_2, \xi_3, \zeta) \tag{2.23}$$

converges absolutely and uniformly in $G_R \times B \times T$. But the recursion scheme (2.20), (2.21) is of the same basic form as that considered in Theorem 2.3 of *I*, and the convergence of the series in Eq. (2.23) can hence be demonstrated by following the proof of this theorem. The reader is referred to *I* for further details.

3. INVERSION OF THE OPERATOR Re $\underset{\sim}{C}_3$ AND A COMPLETE FAMILY OF SOLUTIONS

An examination of Corollary 2.1 and Eq. (2.18) shows that the operator $\underset{\sim}{C}_3$ maps analytic functions of two complex variables into the space of (complex valued) solutions of Eq. (2.2) which vanish at the origin. However for purposes of application it is of central importance that the integral operator constructed map analytic functions *onto* the space of real valued C^2 solutions of Eq. (1.2), i.e. onto the space of real valued analytic solutions of Eq. (2.2). In this section we will show that the operator defined by

$$U(X, Z, Z^*) = U(0, 0, 0)U_0(X, Z, Z^*) + \mathrm{Re}\, \underset{\sim}{C}_3\{f\} \tag{3.1}$$

(where "Re" denotes "take the real part") satisfies this requirement if $U_0(X, Z, Z^*)$ is the unique ([4], pp. 116–119) solution of Eq. (2.2) which satisfies the Goursat data

$$\begin{aligned} U_0(X, 0, Z^*) &= 1 \\ U_0(X, Z, 0) &= 1. \end{aligned} \tag{3.2}$$

We will also give a specific formula for computing the Taylor coefficients of $f(\mu, \zeta)$.

We first note that for x, y, z real we have $u_0(x, y, z) \equiv U_0(X, Z, Z^*)$ is a solution of Eq. (1.2), and since $a(x, y, z)$, $b(x, y, z)$, $c(x, y, z)$, and $d(x, y, z)$ are real valued we can conclude that

$$\mathrm{Re}\, U_0(X, Z, Z^*) = \tfrac{1}{2}(U_0(X, Z, Z^*) + \overline{U}_0(X, -Z^*, -Z)) \tag{3.3}$$

where

$$U_0(X, Z, Z^*) = \sum_{l,m,n=0}^{\infty} \alpha_{mnl} X^l Z^n Z^{*m} \tag{3.4}$$

$$\overline{U}_0(X, Z, Z^*) = \sum_{l,m,n=0}^{\infty} \overline{\alpha_{mnl}} X^l Z^n Z^{*m}$$

is also a solution of Eq. (1.2). Equation (3.3) allows us to extend Re $U_0(X, Z, Z^*)$ to complex values of x, y, z and shows that Re $U_0(X, Z, Z^*)$ is a solution of Eq. (2.2) which satisfies the Goursat data

$$\begin{aligned} \mathrm{Re}\, U_0(X, 0, Z^*) &= 1 \\ \mathrm{Re}\, U_0(X, Z, 0) &= 1. \end{aligned} \tag{3.5}$$

Hence by the uniqueness part of Hormander's generalized Cauchy-Kowalewski theorem ([4], pp. 116–119) we can conclude that

$$\mathrm{Re}\, U_0(X, Z, Z^*) = U_0(X, Z, Z^*), \tag{3.6}$$

i.e. $U_0(X, Z, Z^*)$ is real valued for x, y, z real.

In the special case when $D(X, Z, Z^*) \equiv 0$, it is clear that

$$U_0(X, Z, Z^*) = 1. \tag{3.7}$$

In the general case when $D(X, Z, Z^*) \neq 0$, $U_0(X, Z, Z^*)$ can be constructed via the recursive scheme ([4], p. 116)

$$\begin{aligned} U_0(X, Z, Z^*) &= 1 + \lim_{n \to 0} W_n(X, Z, Z^*) \\ W_0 &\equiv 0 \\ W_{n+1} &= \int_0^Z \int_0^{Z^*} [(\partial^2 W_n/\partial X^2) + A(\partial W_n/\partial X) + B(\partial W_n/\partial Z) \\ &\qquad + C(\partial W_n/\partial Z^*) + DW_n - D]\, dZ' dZ^{*\prime}; \\ &\qquad n = 0, 1, \ldots. \end{aligned} \tag{3.8}$$

Since $a(x, y, z)$, $b(x, y, z)$, $c(x, y, z)$, and $d(x, y, z)$ are entire functions of their independent (complex) variables, the sequence $W_n(X, Z, Z^*)$ converges uniformly to $U_0(X, Z, Z^*)$ in any compact polydisc of the X, Z, Z^* space ([6], [4], pp. 116–118). This says in particular that $U_0(X, Z, Z^*)$ is an entire function of X, Z, and Z^*.

The main result of this section is the following theorem:

THEOREM 3.1 *Let $u(x, y, z)$ be a real valued C^2 solution of Eq.* (1.2) *in some neighborhood of the origin in $\mathbb{R}^3$ and denote by $U(X, Z, Z^*) \equiv u(x, y, z)$ the extension of $u(x, y, z)$ to the X, Z, Z^* space. Then there exists an analytic function of two complex variables $f(\mu, \zeta)$ which is regular for μ in some neighborhood of the origin and $|\zeta| < 1+\varepsilon$, $\varepsilon > 0$, such that locally*

$$U(X, Z, Z^*) = U(0, 0, 0)U_0(X, Z, Z^*) + \operatorname{Re} \underset{\sim}{C}_3\{f\}. \tag{3.9}$$

In particular, if

$$\begin{aligned} U(X, 0, Z^*) - U(0, 0, 0) &= \sum_{\substack{n=0\, m=0 \\ n+m \neq 0}}^{\infty} \sum^{\infty} \gamma_{nm} X^n Z^{*m} \\ C(X, Z, Z^*) &= \sum_{l, m, n=0}^{\infty} c_{lmn} X^l Z^m Z^{*n} \\ \bar{C}(X, Z, Z^*) &= \sum_{l, m, n=0}^{\infty} \overline{c_{lmn}} X^l Z^m Z^{*n}, \end{aligned} \tag{3.10}$$

then

$$f(\mu, \zeta) = 3/2\pi \int_{\gamma'} g(\mu(1-t^2), \zeta) \frac{(1-t^2)}{t^4}\, dt, \tag{3.11}$$

where

$$g(\mu, \zeta) = \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} a_{nm}\mu^n\zeta^m \tag{3.12}$$

with

$$a_{n-1,0} = \gamma_{n0};\ n \geqq 1$$

$$a_{n+m-1,m} = \frac{2\Gamma(n+1)\Gamma(m+1)}{\Gamma(n+m+1)}\gamma_{nm} - \sum_{k=0}^{n-1} \frac{\Gamma(n+1)}{\Gamma(n+m-1)\Gamma(k+1)}\delta_{km}\gamma_{n-k,0};$$

$$n \geqq 0,\ m > 0 \tag{3.13}$$

$$\delta_{km} = \left(\frac{\partial^{k+m}}{\delta X^k \partial Z^{*m}} \exp\left[\int_0^{-Z^*} \bar{C}(X, Z', 0)\, dZ'\right]\right) X = Z^* = 0.$$

In Eq. (3.11) γ' *is a rectifiable arc joining the points* $t = -1$ *and* $t = +1$ *and not passing through the origin.*

Remark In Eq. (3.13) the finite series is omitted when $n = 0$.

Proof of Theorem Note that since $a(x, y, z)$, $b(x, y, z)$, $c(x, y, z)$, $d(x, y, z)$ are real, the function $U(X, Z, Z^*)$ defined by Eq. (3.9) is a real valued solution of Eq. (2.2). Now suppose that locally $\tilde{A}(X, Z, Z^*)$, $\tilde{B}(X, Z, Z^*)$, and $\tilde{D}(X, Z, Z^*)$ have the expansions

$$\begin{aligned} \tilde{A}(X, Z, Z^*) &= \sum_{l,m,n=0}^{\infty} a_{lmn}X^lZ^mZ^n \\ \tilde{B}(X, Z, Z^*) &= \sum_{l,m,n=0}^{\infty} b_{lmn}X^lZ^mZ^{*n} \\ \tilde{D}(X, Z, Z^*) &= \sum_{l,m,n=0}^{\infty} d_{lmn}X^lZ^mZ^{*n}, \end{aligned} \tag{3.14}$$

and define the analytic functions $\bar{g}(\mu, \zeta)$, $\bar{f}(\mu, \zeta)$, $\bar{A}(X, Z, Z^*)$, $\bar{B}(X, Z, Z^*)$, and $\bar{D}(X, Z, Z^*)$ by

$$\begin{aligned} \bar{g}(\mu, \zeta) &= \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} \overline{a_{nm}}\, \mu^n\zeta^m \\ \bar{f}(\mu, \zeta) &= 3/2\pi \int_{\gamma'} \bar{g}(\mu(1-t^2), \zeta) \frac{(1-t^2)}{t^4}\, dt \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} \overline{a_{nm}} \frac{\Gamma(n+2)}{\Gamma(n+\frac{1}{2})\Gamma(3/2)} \mu^n\zeta^m \\ \bar{A}(X, Z, Z^*) &= \sum_{l,m,n=0}^{\infty} \overline{a_{lmn}} X^lZ^mZ^{*n} \\ \bar{B}(X, Z, Z^*) &= \sum_{l,m,n=0}^{\infty} \overline{b_{lmn}} X^lZ^mZ^{*n} \\ \bar{D}(X, Z, Z^*) &= \sum_{l,m,n=0}^{\infty} \overline{d_{lmn}} X^lZ^mZ^{*n}. \end{aligned} \tag{3.15}$$

Let $\bar{E}(X, Z, Z^*, \zeta\ t)$ be the generating function corresponding to the partial differential equation

$$V_{XX} - V_{ZZ^*} + \bar{A}(X, Z, Z^*)V_X + \bar{B}(X, Z, Z^*)V_Z + \bar{D}(X, Z, Z^*)V = 0. \tag{3.16}$$

Then for x, y, z real we can write

$$\begin{aligned} \operatorname{Re} \underset{\sim}{C}_3\{f\} = {} & 1/(4\pi i) \int_{|\zeta|=1} \int_\gamma \exp\left[\int_0^Z C(X, Z', Z^*)\, dZ'\right] \\ & \times E(X, Z, Z^*, \zeta, t) f(\mu(1-t^2), \zeta)(dt/\sqrt{1-t^2})(d\zeta/\zeta) \\ & + 1/(4\pi i) \int_{|\zeta|=1} \int_\gamma \exp\left[\int_0^{-Z^*} \bar{C}(X, Z', -Z)\, dZ'\right. \\ & \times \bar{E}(X, -Z^*, -Z, \zeta, t) \bar{f}(\bar{\mu}(1-t^2), \zeta)(dt/\sqrt{1-t^2})(d\zeta/\zeta); \end{aligned} \tag{3.17}$$

where $\bar{\mu} = X - \zeta Z^* - \zeta^{-1}Z$. Now from Theorem 2.1 we know that $U(X, Z, Z^*)$ is uniquely determined by $U(X, 0, Z^*)$, and hence using Eqs. (2.17) and (3.2), and extending Eq. (3.17) to complex values of x, y, z, we try and determine $f(\mu, \zeta)$ from the integral equation

$$\begin{aligned} U(X, 0, Z^*) - U(0, 0, 0) = {} & 1/(4\pi i) \int_{|\zeta|=1} \int_\gamma \mu_1 f(\mu_1(1-t^2), \zeta) \\ & \times [(t^2 dt)/\sqrt{(1-t^2)}](d\zeta/\zeta) \\ & + 1/(4\pi i) \int_{|\zeta|=1} \int_\gamma \exp\left[\int_0^{-Z^*} \bar{C}(X, Z', 0)\, dZ'\right. \\ & \times \bar{E}(X, -Z^*, 0, \zeta, t) \bar{f}(\mu_2(1-t^2), \zeta)(dt/\sqrt{1-t^2})(d\zeta/\zeta), \end{aligned} \tag{3.18}$$

where $\mu_1 = X + \zeta^{-1}Z^*$ and $\mu_2 = X - \zeta Z^*$. We first note that if we write $\bar{E}^*(\xi_1, \xi_2, \xi_3, \zeta, t) \equiv \bar{E}(X, Z, Z^*, \zeta, t)$ in its series expansion

$$\bar{E}^*(\xi_1, \xi_2, \xi_3, \zeta, t) = \sum_{n=1}^{\infty} t^{2n} \mu^n \bar{p}^{(n)}(\xi_1, \xi_2, \xi_3, \zeta), \tag{3.19}$$

then each $\bar{p}^{(n)}$, $n = 2, 3, \ldots$, is an entire function of X, Z, Z^*, and ζ, and vanishes for $\zeta = 0$. This can be easily seen by using Eqs. (2.19)–(2.21) and then following the analysis in Theorem 3.1 of *I*. Indeed, it was to insure this behavior of the $\bar{p}^{(n)}$ that we transformed Eq. (2.2) into the standard form of Eq. (2.5), eliminating thereby any term involving U_{Z^*}. Due now to the uniform convergence of the series in Eq. (3.19), we can substitute this series into Eq. (3.18) and integrate termwise to conclude that

$$\begin{aligned} U(X, 0, Z^*) - U(0, 0, 0) = {} & 1/(4\pi i) \int_{|\zeta|=1} \int_\gamma \mu_1 f(\mu_1(1-t^2), \zeta) \\ & [(t^2 dt)/(\sqrt{1-t^2})](d\zeta/\zeta) \\ & + 1/(4\pi i) \int_{|\zeta|=1} \int_\gamma \exp\left[\int_0^{-Z^*} \bar{C}(X, Z', 0)\, dZ'\right] \\ & \times \bar{p}^{(1)}(X, -Z^*, 0, \zeta) \mu_2 \bar{f}(\mu_2(1-t^2), \zeta) \\ & (t^2/\sqrt{1-t^2})(d\zeta/\zeta) \\ = {} & 1/(4\pi i) \int_{|\zeta|=1} \mu_1 g(\mu_1 \zeta)\, d\zeta/\zeta \end{aligned}$$

$$+1/(4\pi i)\int_{|\zeta|=1}\mu_2 \exp\left[\int_0^{-Z^*}\bar{C}(X,Z',0)\,dZ'\right]$$
$$\times\tilde{p}^{(1)}(X,-Z^*,0,\zeta)\bar{g}(\mu_2,\zeta)\,d\zeta/\zeta$$

where

$$\tilde{p}^{(1)}(X,Z,Z^*,\zeta)\equiv\tilde{p}^{(1)}(\xi_1,\xi_2,\xi_3,\zeta)=\exp\left[\tfrac{1}{2}\int_0^{\xi_1}(\bar{A}^*+\bar{B}^*\zeta)\,d\xi_1'\right]$$
$$=\exp\left[\zeta\int_0^Z\bar{A}(X+2\zeta Z\tau,\tau,(\zeta/2)(2\zeta^{-1}Z^*-2\zeta Z+2\zeta\tau))\,d\tau\right]$$
$$\times\exp\left[\zeta^2\int_0^Z\bar{B}(X+2\zeta Z\tau,\tau,(\zeta/2)(2\zeta^{-1}Z^*-2\zeta Z+2\zeta\tau))\,d\tau\right] \tag{3.21}$$

and we have used the easily verifiable results

$$g(\mu,\zeta)=\int_\gamma t^2 f(\mu(1-t^2),\zeta)\,dt/\sqrt{1-t^2}. \tag{3.22}$$

To complete the proof of the theorem it now suffices to show that Eqs. (3.12), (3.13) gives the solution of the integral Eq. (3.20). To this end we use Leibnitz's formula to repeatedly differentiate both sides of Eq. (3.20) and evaluate the result at $X = Z^* = 0$. (This calculation is simplified by the observation that terms involving derivatives with respect to X and Z^* of $\tilde{p}^{(1)}(X, -Z^*, 0)$ and of terms involving derivatives with respect to Z^* of $\mu_2\bar{g}(\mu_2,\zeta)$ vanish due to the fact that in such cases the integrand becomes an analytic functions of ζ). The result of this calculation is that

$$2n!m!\gamma_{nm}=(n+m)!a_{n+m-1,m}+\sum_{k=0}^{n-1}(\Gamma(n+1)/\Gamma(k+1))\delta_{km}\bar{a}_{n-k-1,0}, \tag{3.23}$$

where

$$\delta_{km}=\left(\frac{\partial^{k+m}}{\partial X^k\partial Z^{*m}}\exp\left[\int_0^{-Z^*}\bar{C}(X,Z',0)\,dZ'\right]\right)_{X=Z^*=0} \tag{3.24}$$

and the finite series in Eq. (3.23) is omitted when $n = 0$. Noting that $\delta_{k0} = 0$ for $k \geqq 1$, we have

$$2\gamma_{n0}=a_{n-1,0}+\bar{a}_{n-1,0}, \tag{3.25}$$

and since $U(X, 0, 0)$ is real we can assume without loss of generality that the coefficients a_{n0}, $n = 0, 1, 2, \ldots$ are real, i.e.

$$\gamma_{n0}=a_{n-1,0}=\bar{a}_{n-1,0}. \tag{3.26}$$

Equation (3.13) now follows from Eqs. (3.23)–(3.26). Since $U(X, 0, Z^*)$ is regular in some neighborhood of the origin, Eq. (3.13) implies that $g(\mu,\zeta)$ (and hence $f(\mu,\zeta)$) is regular for μ in some neighborhood of the origin and $|\zeta| < 1+\varepsilon$, $\varepsilon > 0$. The theorem is now proved.

Equation (3.13) shows that if $U(X, Z, Z^*)$ has polynomial Goursat data, then the associated analytic function $f(\mu,\zeta)$ is an entire function of μ and ζ.

Using this fact in conjunction with the results proved in Theorem 3.2 of I now enables us to prove the following theorem. In the statement of the theorem "*Im*" denotes "take the imaginary part".

THEOREM 3.2 *Let G be a bounded, simply connected domain in $\mathbb{R}^3$, and define*

$$u_0(x, y, z) \equiv U_0(X, Z, Z^*) \tag{3.27}$$

$$u_{2n,m}(x, y, z) = \text{Re}\ \underset{\sim}{C}_3\{\mu^n\zeta^m\}; \quad 0 \leqq n < \infty, m = 0, 1, \ldots, n+1,$$

$$u_{2n+1,m}(x, y, z) = \text{Im}\ C_3\{\mu^n\zeta^m\}; \quad 0 \leqq n < \infty,\ m = 0, 1, \ldots, n+1.$$

Then the set $\{u_0\} \cup \{u_{nm}\}$ is a complete family of solutions for Eq. (1.2) *in the space of real valued C^2 solutions of Eq.* (1.2) *defined in G.*

Proof From the analysis of Theorem 3.2 of I it suffices to show that if $U(X, Z, Z^*)$ is a real valued solution of Eq. (2.2) with polynomial Goursat data, then given $R > 0$ there exists a polynomial $f_1(\mu, \zeta)$ such that

$$\max_{\substack{|X| \leqq R \\ |Z| \leqq R \\ |Z^*| \leqq R}} |U(X, Z, Z^*) - U_1(X, Z, Z^*)| < \varepsilon \tag{3.28}$$

where

$$\begin{aligned} U_1(X, Z, Z^*) &= U_1(0, 0, 0)U_0(X, Z, Z^*) + \text{Re}\ \underset{\sim}{C}_3\{f_1\} \\ U(0, 0, 0) &= U_1(0, 0, 0). \end{aligned} \tag{3.29}$$

From Eqs. (3.11)–(3.13) we can construct an entire function $f(\mu, \zeta)$ such that

$$U(X, Z, Z^*) = U(0, 0, 0)U_0(X, Z, Z^*) + \text{Re}\ C_3\{f\}. \tag{3.30}$$

From the proof of Theorem 2.3 we can conclude that there exists a positive constant $M = M(R)$ such that

$$\begin{aligned} \max_{\substack{|X| \leqq R \\ |Z| \leqq R \\ |Z^*| \leqq R}} |\text{Re}\ C_3\{f\}| &\leqq \max_{\substack{|X| \leqq R \\ |Z| \leqq R \\ |Z^*| \leqq R}} |\underset{\sim}{C}_3\{f\}| \\ &\leqq M \max_{\substack{|\mu| \leqq 3R \\ |\zeta| = 1}} |f(\mu, \zeta)|. \end{aligned} \tag{3.31}$$

Now let $f_1(\mu, \zeta)$ be a polynomial such that

$$\max_{\substack{|\mu| \leqq 3R \\ |\zeta| = 1}} |f(\mu, \zeta) - f_1(\mu, \zeta)| < \varepsilon/M. \tag{3.32}$$

This polynomial can be constructed, for example, by truncating the Taylor series for $f(\mu, \zeta)$. We now have

$$\begin{aligned} &\max_{\substack{|X| \leqq R\\ |Z| \leqq R\\ |Z^*| \leqq R}} \left|U(X, Z, Z^*) - U_1(X, Z, Z^*)\right| \\ &\leqq \max_{\substack{|X| \leqq R\\ |Z| \leqq R\\ |Z^*| \leqq R}} \left|\operatorname{Re} C_3\{f - f_1\}\right| \\ &\leqq M \max_{\substack{|\mu| \leqq 3R\\ |\zeta| = 1}} \left|f(\mu, \zeta) - f_1(\mu, \zeta)\right| < \varepsilon \end{aligned} \tag{3.33}$$

and the theorem is proved.

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BERGMAN OPERATORS FOR ELLIPTIC EQUATIONS IN THREE INDEPENDENT VARIABLES

BY

DAVID COLTON

[illegible]

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BERGMAN OPERATORS FOR ELLIPTIC EQUATIONS IN THREE INDEPENDENT VARIABLES

BY DAVID COLTON

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Introduction. S. Bergman [1] and I. N. Vekua [7] have both constructed integral operators which map analytic functions of one complex variable onto solutions of the elliptic equation

$$u_{xx} + u_{yy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0. \tag{1}$$

We wish to announce in this note the extension of these results to the three-variable case, i.e. the equation

$$\begin{aligned} u_{xx} + u_{yy} + u_{zz} + a(x, y, z)u_x + b(x, y, z)u_y \\ + c(x, y, z)u_z + d(x, y, z)u = 0 \end{aligned} \tag{2}$$

where a, b, c, d are real valued entire functions of the (complex) variables x, y, z. (With minor modifications we could have assumed only that a, b, c, d are analytic in some ball containing the origin.) Partial results on integral operators for equation (2) (in the special case when $a=b=c=0$) have been obtained by Bergman [1], Tjong [6], Colton and Gilbert [4], and Gilbert and Lo [5].

Main results. Let $X=x$, $Z=\frac{1}{2}(y+iz)$, $Z^*=\frac{1}{2}(-y+iz)$. Then equation (2) becomes

$$\begin{aligned} U_{XX} - U_{ZZ^*} + A(X, Z, Z^*)U_X + B(X, Z,Z^*)U_Z \\ + C(X, Z, Z^*)U_{Z^*} + D(X, Z, Z^*)U = 0 \end{aligned} \tag{3}$$

where

$$\begin{aligned} U(X, Z, Z^*) &= u(x, y, z), \\ A(X, Z, Z^*) &= a(x, y, z), \\ B(X, Z, Z^*) &= \tfrac{1}{2}(b(x, y, z) + ic(x, y, z)) \\ C(X, Z, Z^*) &= \tfrac{1}{2}(-b(x, y, z) + ic(x, y, z)) \\ D(X, Z, Z^*) &= d(x, y, z). \end{aligned} \tag{4}$$

The substitution

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$$V(X, Z, Z^*) = U(X, Z, Z^*) \exp\left[-\int_0^Z C(X, Z', Z^*)\, dZ'\right] \tag{5}$$

yields the following equation for $V(X, Z, Z^*)$,

$$\begin{aligned} V_{XX} - V_{ZZ^*} + \tilde{A}(X, Z, Z^*)V_X + \tilde{B}(X, Z, Z^*)V_Z \\ + \tilde{D}(X, Z, Z^*)V = 0, \end{aligned} \tag{6}$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{D}$ are expressible in terms of the coefficients A, B, C, D. Let $U_0(X, Z, Z^*)$ be the real valued, entire solution of equation (3) which satisfies the Goursat data $U_0(X, 0, Z^*) = U_0(X, Z, 0) = 1$. Note that in the special case when $D=0$ we can choose $U_0 \equiv 1$. In the general case when $D \neq 0$, U_0 can be constructed via the recursive scheme

$$U_0 = 1 + \lim_{n\to\infty} W_n,$$

$$\begin{aligned} W_{n+1} = \int_0^Z \int_0^{Z^*} \Big(\frac{\partial^2 W_n}{\partial X^2} + A \frac{\partial W_n}{\partial X} + B \frac{\partial W_n}{\partial Z} \\ + C \frac{\partial W_n}{\partial Z^*} + D W_n - D \Big)\, dZ' dZ^{*\prime}, \end{aligned} \tag{7}$$

$$W_0 = 0.$$

By introducing the variables

$$\begin{aligned} \xi_1 &= 2\zeta Z, \\ \xi_2 &= X + 2\zeta Z, \\ \xi_3 &= X + 2\zeta^{-1} Z^*, \\ \mu &= \tfrac{1}{2}(\xi_2 + \xi_3) = X + \zeta Z + \zeta^{-1} Z^*, \end{aligned} \tag{8}$$

where ζ is a complex variable such that $1-\epsilon < |\zeta| < 1+\epsilon$, $0 < \epsilon < \frac{1}{2}$, we can now state the following theorem. In the theorems which follow "Re" denotes "take the real part" and "Im" denotes "take the imaginary part."

THEOREM 1. *Let*

$$E^*(\xi_1, \xi_2, \xi_3, \zeta, t) = \sum_{n=1}^{\infty} t^{2n} \mu^n p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta) \tag{9}$$

where

$$p_1^{(n+1)} - \tfrac{1}{2}(\tilde{A}^* + \tilde{B}^*\zeta)p^{(n+1)}$$
$$= \frac{1}{2n+1}\{p_{22}^{(n)} + p_{33}^{(n)} - 4p_{13}^{(n)} - 2p_{23}^{(n)} + (\tilde{A}^* + 2\tilde{B}^*\zeta)p_2^{(n)}$$
$$+ \tilde{A}^* p_3^{(n)} + 2\tilde{B}^*\zeta p_1^{(n)} + \tilde{D}^* p^{(n)}\},$$

(10)

$$p^{(1)}(\xi_1, \xi_2, \xi_3, \zeta) = \exp\left[\frac{1}{2}\int_0^{\xi_1} (\tilde{A}^* + \tilde{B}^*\zeta)\, d\xi_1'\right],$$
$$p^{(n+1)}(0, \xi_2, \xi_3, \zeta) = 0, \quad n = 1, 2, \cdots,$$
$$p_i^{(n)} = \partial p^{(n)}/\partial\xi_i, \qquad p_{ij}^{(n)} = \partial^2 p^{(n)}/\partial\xi_i\partial\xi_j,$$

with $\tilde{A}^*(\xi_1, \xi_2, \xi_3, \zeta) = \tilde{A}(X, Z, Z^*)$, $\tilde{B}^*(\xi_1, \xi_2, \xi_3, \zeta) = \tilde{B}(X, Z, Z^*)$, $\tilde{D}^*(\xi_1, \xi_2, \xi_3, \zeta) = \tilde{D}(X, Z, Z^*)$. *Then the following is true:*

(1) $E^*(\xi_1, \xi_2, \xi_3, \zeta, t) = E(X, Z, Z^*, \zeta, t)$ *is regular in* $G_R \times B \times T$ *where* $G_R = \{(\xi_1, \xi_2, \xi_3) : |\xi_i| < R, i = 1, 2, 3\}$, $B = \{\zeta : 1-\epsilon < |\zeta| < 1+\epsilon\}$, $T = \{t : |t| \leqq 1\}$, *and R is an arbitrarily large positive number.*

(2) *If* $U(X, Z, Z^*)$ *is a real valued (for* (x, y, z) *real) solution of equation* (3) *which is regular in some neighborhood of the origin, then there exists an analytic function* $f(\mu, \zeta)$ *which is regular for* μ *in some neighborhood of the origin and* $|\zeta| < 1+\epsilon$, *such that locally*

$$U(X, Z, Z^*) = U(0, 0, 0)U_0(X, Z, Z^*) + \mathrm{Re}\, \mathbf{C}_3\{f\}, \tag{11}$$

where

$$\mathbf{C}_3\{f\} = \frac{1}{2\pi i}\int_{|\zeta|=1}\int_{-1}^{+1} \exp\left[\int_0^Z C(X, Z', Z^*)\, dZ'\right]$$
(12)
$$\cdot E(X, Z, Z^*, \zeta, t) f(\mu(1 - t^2), \zeta)\, \frac{dt}{(1 - t^2)^{1/2}}\, \frac{d\zeta}{\zeta}.$$

(3) *If*

$$U(X, 0, Z^*) - U(0, 0, 0) = \sum_{n=0}^{\infty} \sum_{m=0;\, n+m\neq 0}^{\infty} \gamma_{nm} X^n Z^{*m}, \tag{13}$$

$$\overline{C(X, Z, Z^*)} = \overline{C(X, -Z^*, -Z)}, \qquad x, y, z \text{ real}, \tag{14}$$

then

$$f(\mu, \zeta) = \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} a_{nm} \frac{\Gamma(n + 2)}{\Gamma(n + \frac{1}{2})\Gamma(\frac{3}{2})} \mu^n \zeta^m, \tag{15}$$

where

$$a_{n-1,0} = \gamma_{n0}, \qquad n \geqq 1,$$

$$(16)\qquad \begin{aligned} a_{n+m-1,m} &= \frac{2n!m!}{(n+m)!}\gamma_{nm} - \sum_{k=0}^{n-1}\frac{n!}{(n+m)!k!}\delta_{km}\gamma_{n-k,0}, \quad n \geqq 0, m>0,\\ \delta_{km} &= \left(\frac{\partial^{k+m}}{\partial X^k \partial Z^{*m}}\exp\left[\int_0^{-Z^*}\bar{C}(X, Z', 0)\, dZ'\right]\right)_{X=Z^*=0}. \end{aligned}$$

(*The finite series in equation* (16) *is omitted when* $n=0$.)

The fact that *every* real valued twice continuously differentiable solution of equation (2) (i.e., a regular solution of equation (3)) can be represented in the form of equation (11) now leads to the following theorem:

THEOREM 2. *Let G be a bounded, simply connected domain in Euclidean three space, and, for x, y, z real, define*

$$u_0(x, y, z) = U_0(X, Z, Z^*)$$

$$(17)\qquad u_{2n,m}(x, y, z) = \operatorname{Re} \mathbf{C}_3\{\mu^n\zeta^m\}, \quad 0 \leqq n < \infty, m = 0, 1, \cdots, n+1,$$

$$u_{2n+1,m}(x, y, z) = \operatorname{Im} \mathbf{C}_3\{\mu^n\zeta^m\}, \quad 0 \leqq n < \infty, m = 0, 1, \cdots, n+1.$$

Then the set $\{u_0\}\cup\{u_{nm}\}$ is a complete family of solutions for equation (2) *in the space of real valued solutions of equation* (2) *defined in G.*

Special cases. (a) $A=B=C=0$.

THEOREM 3. *Assume $A=B=C=0$, and let*

$$(18)\qquad \tilde{E}^*(\xi_1, \xi_2, \xi_3, \zeta, t) = 1 + \sum_{n=1}^{\infty} t^{2n}\mu^n p^{(n+1)}(\xi_1, \xi_2, \xi_3, \zeta)$$

where the $p^{(n)}$ are defined by equation (10) *with $\tilde{A}=\tilde{B}=0$. Then*

(1) *$\tilde{E}^*(\xi_1, \xi_2, \xi_3, \zeta, t) = \tilde{E}(X, Z, Z^*, \zeta, t)$ is regular in $G_R \times B \times T$.*

(2) *Every real valued solution $U(X, Z, Z^*)$ of equation* (3) *which is regular in some neighborhood of the origin can be represented locally in the form*

$$(19)\qquad U(X, Z, Z^*) = \operatorname{Re} \mathbf{P}_3\{f\}$$

where

$$(20)\qquad \mathbf{P}_3\{f\} = \frac{1}{2\pi i}\int_{|\zeta|=1}\int_{-1}^{+1}\tilde{E}(X, Z, Z^*, \zeta, t) f(\mu(1-t^2), \zeta)\frac{dt}{(1-t^2)^{1/2}}\frac{d\zeta}{\zeta},$$

and

$$f(\mu, \zeta) = -\frac{1}{2\pi}\int_{\gamma'} g(\mu(1-t^2), \zeta)\frac{dt}{t^2} \tag{21}$$

$$g(\mu, \zeta) = 2\frac{\partial}{\partial\mu}\left[\mu\int_0^1 U(t\mu, 0, (1-t)\mu\zeta)\,dt\right] - U(\mu, 0, 0). \tag{22}$$

In equation (21) γ' *is a rectifiable arc joining the points* $t=-1$ *and* $t=+1$ *and not passing through the origin.*

(b). $A=B=C=D=0$.

In the special case when $A=B=C=D=0$, the operator $\boldsymbol{P}_3$ reduces to the well-known Bergman-Whittaker operator $\boldsymbol{B}_3$ [1] and equation (22) gives a new inversion formula for the operator Re $\boldsymbol{B}_3$.

Complete proofs of the results stated in this announcement will appear in [2] and [3].

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BERGMAN OPERATORS FOR ELLIPTIC EQUATIONS IN FOUR INDEPENDENT VARIABLES*

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Abstract. Integral operators are obtained which map analytic functions of three complex variables onto solutions of linear elliptic partial differential equations in four independent variables. An inversion formula is given and used to construct a complete family of solutions for the elliptic equation under investigation.

1. Introduction. The theory of integral operators for elliptic partial differential equations was initiated by S. Bergman [1] and I. N. Vekua [20], both of whom constructed operators which map analytic functions of a single complex variable onto twice continuously differentiable (class C^2) solutions of the elliptic equation

$$\Delta_2 u + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0. \tag{1.1}$$

These operators were then used to construct complete families of solutions and to investigate the analytic properties of solutions to (1.1). Recently, Colton [3], [4], [5] was able to extend the results of Bergman and Vekua to the case of three independent variables, that is, the equation

$$\Delta_3 u + a(x, y, z)u_x + b(x, y, z)u_y + c(x, y, z)u_z + d(x, y, z)u = 0. \tag{1.2}$$

More specifically, integral operators were obtained in [3], [4] and [5] which map analytic functions of *two* complex variables onto C^2-solutions of (1.2), and were then used for purposes of analytic continuation and to construct a complete family of solutions to (1.2). This work was the culmination of the efforts of several mathematicians, among them Bergman [1], Tjong [18], [19], Colton and Gilbert [6] and Gilbert and Lo [14]. In this paper we indicate how the approach used to treat equation (1.2) can be extended to treat elliptic equations in four independent variables, that is, the equation

$$\begin{aligned}\Delta_4 u + a(x_1, x_2, x_3, x_4)u_{x_1} + b(x_1, x_2, x_3, x_4)u_{x_2} + c(x_1, x_2, x_3, x_4)u_{x_3}\\ + d(x_1, x_2, x_3, x_4)u_{x_4} + f(x_1, x_2, x_3, x_4)u = 0.\end{aligned} \tag{1.3}$$

Our methods unfortunately do not appear applicable to elliptic equations in more than four variables, and so at present it seems that the use of integral operators in investigating the analytic theory of elliptic equations is restricted to equations in two, three and four variables.

Until a few years ago integral operators for elliptic equations in four independent variables were available only for the harmonic equation and certain classes of equations with spherically symmetric coefficients [11], [12], [13], [16]. Recently, however, Colton and Gilbert obtained an integral operator which mapped analytic functions of *three* complex variables onto an unspecified *subspace* of solutions to (1.3) in the special case when $a = b = c = d = 0$ (see [6]).

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If in addition the coefficient $f(x_1, x_2, x_3, x_4)$ was independent of x_1, then Colton and Gilbert were able to construct an operator which mapped ordered pairs of analytic functions of three complex variables onto the space of C^2-solutions of (1.3). This last result was then used to investigate Cauchy's problem for certain classes of elliptic equations in four independent variables and hyperbolic equations in three space variables and one time variable [6], [7]. In the present paper we overcome the problem of showing that our operator maps analytic functions onto the whole space of real-valued C^2-solutions of (1.3) by carefully choosing new independent variables, reducing the question of invertibility to the problem of showing that a Goursat problem for an ultrahyperbolic equation in the space of four complex variables is well-posed, and then solving an integral equation associated with this Goursat problem. We shall furthermore give an explicit formula for constructing the analytic function associated with a given real-valued C^2-solution of (1.3) by our integral operator. A special case of this last result is a new inversion formula for the operator Re $\mathbf{G}_4$, where $\mathbf{G}_4$ is Gilbert's generalization of the Bergman–Whittaker operator [11, pp. 75–82] and "Re" denotes "take the real part." As an application of our main theorem we shall construct a complete family of solutions for (1.3) in a bounded, simply connected domain in Euclidean four-space R^4.

For the sake of brevity we only consider the special case of (1.3) when $a = b = c = d = 0$. The extension to the more general case can easily be made by combining the results of this paper with the approach used in [4] for the case of three independent variables. We furthermore assume that the coefficient $f(x_1, x_2, x_3, x_4)$ is an entire function of x_1, x_2, x_3 and x_4 (considered as complex variables), although with slight modification our results remain valid when $f(x_1, x_2, x_3, x_4)$ is only assumed to be analytic inside some polydisc in the space of four complex variables. It will also always be assumed that $f(x_1, x_2, x_3, x_4)$ is real-valued for x_1, x_2, x_3 and x_4 real. Since much of our analysis is based on the ideas of [3], it might be helpful if the reader had access to this paper.

2. The operator $\mathbf{P}_4$. In this section we consider the partial differential equation

$$\Delta_4 u + f(x_1, x_2, x_3, x_4)u = 0, \tag{2.1}$$

where $f(x_1, x_2, x_3, x_4)$ is a real-valued (for x_1, x_2, x_3, x_4 real) entire function of its independent (complex) variables. Our first result is the following theorem which is central to the analysis which follows.

THEOREM 2.1. *Let* $Y = \frac{1}{2}(x_1 + ix_2)$, $Y^* = \frac{1}{2}(x_1 - ix_2)$, $Z = \frac{1}{2}(x_3 + ix_4)$, $Z^* = -\frac{1}{2}(x_3 - ix_4)$, *and let* $u(x_1, x_2, x_3, x_4)$ *be a real-valued* C^2*-solution of* (2.1) *in a neighborhood of the origin. Then* $U(Y, Y^*, Z, Z^*) = u(x_1, x_2, x_3, x_4)$ *is an analytic function of* Y, Y^*, Z, Z^* *in some neighborhood of the origin in* $\mathbb{C}^4$, *in the space of four complex variables, and is uniquely determined by the function* $U(Y, 0, Z, Z^*)$.

Remark. Note that $Y = \overline{Y^*}$, $Z = \overline{-Z^*}$ if and only if x_1, x_2, x_3, x_4 are real.

Proof of Theorem 2.1. The fact that $U(Y, Y^*, Z, Z^*)$ is analytic follows from the fact that C^2-solutions of second order linear elliptic equations with analytic coefficients are analytic functions of their independent variables (cf. [10, p. 164]).

Hence, locally we can write

$$U(Y, Y^*, Z, Z^*) = \sum_{l,m,n,p=0}^{\infty} c_{lmnp} Y^l Y^{*m} Z^n Z^{*p}, \tag{2.2}$$

$$U(Y, 0, Z, Z^*) = \sum_{l,n,p=0}^{\infty} c_{l0np} Y^l Z^n Z^{*p}, \tag{2.3}$$

$$U(0, Y^*, Z, Z^*) = \sum_{m,n,p=0}^{\infty} c_{0mnp} Y^{*m} Z^n Z^{*p}. \tag{2.4}$$

Since $u(x_1, x_2, x_3, x_4)$ is real-valued, we have that for x_1, x_2, x_3, x_4 real,

$$U(Y, Y^*, Z, Z^*) = \overline{U(Y, Y^*, Z, Z^*)}, \tag{2.5}$$

where the bar denotes complex conjugation. This implies that for x_1, x_2, x_3, x_4 real,

$$\sum_{l,m,n,p=0}^{\infty} c_{lmnp} Y^l Y^{*m} Z^n Z^{*p} = \sum_{l,m,n,p=0}^{\infty} \overline{c_{lmnp}} Y^{*l} Y^m (-Z^*)^n (-Z)^p \tag{2.6}$$

or

$$c_{lmnp} = (-1)^{n+p} \overline{c_{mlpn}}. \tag{2.7}$$

Equations (2.3), (2.4) and (2.7) now show that $U(0, Y^*, Z, Z^*)$ is uniquely determined from $U(Y, 0, Z, Z^*)$. However in the Y, Y^*, Z, Z^* variables, (2.1) becomes an equation of ultrahyperbolic type, viz.

$$U_{YY^*} - U_{ZZ^*} + F(Y, Y^*, Z, Z^*)U = 0, \tag{2.8}$$

where

$$F(Y, Y^*, Z, Z^*) \equiv f(x_1, x_2, x_3, x_4). \tag{2.9}$$

From Hormander's generalized Cauchy–Kowalewski theorem [15, pp. 116–119], [2], we have that $U(Y, Y^*, Z, Z^*)$ is uniquely determined from the Goursat data $U(0, Y^*, Z, Z^*)$ and $U(Y, 0, Z, Z^*)$, which we have already seen are determined from $U(Y, 0, Z, Z^*)$ alone. The theorem is now proved.

We now begin to construct an integral operator which maps $U(Y, 0, Z, Z^*)$ onto $U(Y, Y^*, Z, Z^*)$. We first introduce the following notation:

$$\begin{aligned} \xi_1 &= \eta^{-1}\zeta^{-1}Y^*, \\ \xi_2 &= \eta^{-1}\zeta^{-1}Y^* + \eta^{-1}Z^*, \\ \xi_3 &= \eta^{-1}Z^* + Y, \\ \xi_4 &= \zeta^{-1}Z + Y, \end{aligned} \tag{2.10}$$

$$\mu = \xi_2 + \xi_4 = Y + \zeta^{-1}Z + \eta^{-1}Z^* + \eta^{-1}\zeta^{-1}Y^*, \tag{2.11}$$

where ζ, η are complex variables such that $1 - \varepsilon < |\zeta| < 1 + \varepsilon$, $1 - \varepsilon < |\eta| < 1 + \varepsilon$, $0 < \varepsilon < \frac{1}{2}$. Noting that the Jacobian of the transformation (2.10) is equal to $-(\eta\zeta)^{-2} \neq 0$, one can prove the following theorem by straightforward differentiation and integration by parts (cf. [6, Theorem 4.1]).

THEOREM 2.2. *Let D be a neighborhood of the origin in the μ-plane, $B = \{(\zeta, \eta): 1 - \varepsilon < |\zeta| < 1 + \varepsilon,\ 1 - \varepsilon < |\eta| < 1 + \varepsilon\}$, G a neighborhood of the origin in $(\xi_1, \xi_2, \xi_3, \xi_4)$-space and $T = \{t : |t| \leqq 1\}$. Let $f(\mu, \zeta, \eta)$ be an analytic function of three complex variables in the product domain $D \times B$ and $E^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) \equiv E(Y, Y^*, Z, Z^*, \zeta, \eta, t)$ be a regular solution of the partial differential equation*

$$(2.12)\quad 2\mu t(E^*_{13} + E^*_{14} + E^*_{23} - E^*_{34} + \eta\zeta F^* E^*) + (1 - t^2)E^*_{1t} - \frac{1}{t}E^*_1 = 0$$

in $G \times B \times T$, where $F^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta) \equiv F(Y, Y^*, Z, Z^*)$, and

$$E^*_i = \frac{\partial E^*}{\partial \xi_i}, \quad E^*_{ij} = \frac{\partial^2 E^*}{\partial \xi_i \partial \xi_j}, \quad E^*_{1t} = \frac{\partial^2 E^*}{\partial \xi_i \partial t}, \qquad i, j = 1, 2, 3, 4.$$

Then,

$$U(Y, Y^*, Z, Z^*) = \mathbf{P}_4\{f\}$$

$$(2.13)\qquad = -\frac{1}{4\pi^2}\int_{|\zeta|=1}\int_{|\eta|=1}\int_{\gamma} E(Y, Y^*, Z, Z^*, \zeta, \eta, t) f(\mu(1 - t^2), \zeta, \eta) \cdot \frac{dt}{\sqrt{1 - t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta},$$

where γ is a path in T joining $t = -1$ and $t = +1$, is a (complex-valued) solution of (2.1) *which is regular in a neighborhood of the origin in (Y, Y^*, Z, Z^*)-space.*

We must now show that the integral operator $\mathbf{P}_4$ exists; that is, we must show the existence of a function $E(Y, Y^*, Z, Z^*, \zeta, \eta, t)$ satisfying the conditions of Theorem 2.2.

THEOREM 2.3. *Let $D_r = \{(\xi_1, \xi_2, \xi_3, \xi_4): |\xi_i| < r, i = 1, 2, 3, 4\}$, where r is an arbitrary positive number, and let $B_{2\varepsilon} = \{(\zeta, \eta): |\zeta - \zeta_0| < 2\varepsilon, |\eta - \eta_0| < 2\varepsilon\}, 0 < \varepsilon < \frac{1}{2}$, where ζ_0, η_0 are arbitrary with $|\zeta_0| = |\eta_0| = 1$. Then, for each n, $n = 0, 1, 2, \cdots$, there exists a unique function $p^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$ which is regular in $\bar{D}_r \times \bar{B}_{2\varepsilon}$ (the bar denoting closure) and which satisfies*

$$(2.14)\quad p^{(n+1)}_1 = -\frac{1}{2n + 1}\{2p^{(n)}_{13} + 2p^{(n)}_{14} + 2p^{(n)}_{13} - 2p^{(n)}_{34} + \eta\zeta F^* p^{(n)}\},$$

$$(2.15)\quad \begin{aligned} &p^{(0)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta) = 1, \\ &p^{(n+1)}(0, \xi_2, \xi_3, \xi_4, \zeta, \eta) = 0, \qquad n = 0, 1, 2, \cdots, \end{aligned}$$

where

$$p^{(n)}_i = \frac{\partial p^{(n)}}{\partial \xi_i}, \qquad p^{(n)}_{ij} = \frac{\partial^2 p^{(n)}}{\partial \xi_i \partial \xi_j}, \qquad i, j = 1, 2, 3, 4.$$

The function

$$(2.16)\quad E^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) = 1 + \sum_{n=1}^{\infty} t^{2n}\mu^n p^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$$

is a solution of (2.12) *which is regular in the product domain $G_R \times B \times T$, where R*

is an arbitrary positive number, and

$$G_R = \{(\xi_1, \xi_2, \xi_3, \xi_4): |\xi| < R, i = 1,2,3,4\},$$

$$(2.17) \qquad B = \{(\zeta, \eta): 1 - \varepsilon < |\zeta| < 1 + \varepsilon, 1 - \varepsilon < |\eta| < 1 + \varepsilon\}, \ 0 < \varepsilon < \tfrac{1}{2},$$

$$T = \{t : |t| \leqq 1\}.$$

The function defined in (2.16) *satisfies*

$$(2.18) \qquad E^*(0, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) = 1.$$

Proof. It is easily verified from (2.14) and (2.15) that $p^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$ exists, is uniquely determined and is regular in $\bar{D}_r \times \bar{B}_{2\varepsilon}$ for $n = 0, 1, 2, \cdots$. Straightforward differentiation and collection of terms shows that the series (2.16) formally satisfies (2.12). It remains to be shown that this series converges absolutely and uniformly in $G_R \times B \times T$. To this end, note that since $\bar{B}$ is a compact subset of the (ζ, η)-space, there are finitely many points (ζ_j, η_j) with $|\zeta_j| = |\eta_j| = 1$, $j = 1, 2, \cdots, N$, such that B is covered by the union of sets

$$(2.19) \qquad N_j = \{(\zeta, \eta): |\zeta - \zeta_j| < \tfrac{3}{2}\varepsilon, |n - \eta_j| < \tfrac{3}{2}\varepsilon\}, \quad j = 1, 2, \cdots, N.$$

Hence it is sufficient to show that the series converges absolutely and uniformly in $\bar{G}_R \times \bar{N}_j \times T$. To this end we majorize the $p^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$ in $D_R \times B_{2\varepsilon}$. Since $F(Y, Y^*, Z, Z^*)$ is an entire function, it follows that $F^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$ is regular in $\bar{D}_r \times \bar{B}_{2\varepsilon}$, and hence we have

$$(2.20) \qquad \begin{aligned} F^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta) \ll C&\left(1 - \frac{\xi_1}{r}\right)^{-1}\left(1 - \frac{\xi_2}{r}\right)^{-1}\left(1 - \frac{\xi_3}{r}\right)^{-1} \\ &\cdot\left(1 - \frac{\xi_4}{r}\right)^{-1}\left(1 - \frac{\zeta - \zeta_0}{2\varepsilon}\right)^{-1}\left(1 - \frac{\eta - \eta_0}{2\varepsilon}\right)^{-1} \end{aligned}$$

for some $C > 0$ and $(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$ in $D_r \times B_{2\varepsilon}$. In (2.20) the symbol "$\ll$" means "is dominated by." The use of dominants is a standard tool in the analytic theory of partial differential equations, and the reader unfamiliar with their use is referred to [1] or [11] for further details. From (2.14), (2.15) and (2.10) it is a somewhat lengthy but straightforward procedure to show by induction that in $D_r \times B_{2\varepsilon}$ we have

$$(2.21) \qquad \begin{aligned} p_1^{(n)} \ll M(8 + \delta)^n(2n - 1)&\left(1 - \frac{\xi_1}{r}\right)^{-(2n-1)}\left(1 - \frac{\xi_2}{r}\right)^{-(2n-1)}\left(1 - \frac{\xi_3}{r}\right)^{-(2n-1)} \\ &\cdot\left(1 - \frac{\xi_4}{r}\right)^{-(2n-1)}\left(1 - \frac{\zeta - \zeta_0}{2\varepsilon}\right)^{-n}\left(1 - \frac{\eta - \eta_0}{2\varepsilon}\right)^{-n} r^{-n}, \end{aligned}$$

where M and δ are positive constants independent of n. (For details of the proof of closely related results the reader is referred to [3], [6] and [18].) Equation (2.21) now implies (after some slight manipulation) that in $D_r \times B_{2\varepsilon}$ we have

$$(2.22) \qquad \begin{aligned} p^{(n)} \ll M(8 + \delta)^n(2n)^{-1}(2n - 1)^{-1}&\left(1 - \frac{\xi_1}{r}\right)^{-2n}\left(1 - \frac{\xi_2}{r}\right)^{-(2n-1)} \\ \cdot\left(1 - \frac{\xi_3}{r}\right)^{-(2n-1)}&\left(1 - \frac{\xi_4}{r}\right)^{-(2n-1)}\left(1 - \frac{\zeta - \zeta_0}{2\varepsilon}\right)^{-n}\left(1 - \frac{\eta - \eta_0}{2\varepsilon}\right)^{-n} r^{-n} \end{aligned}$$

which implies that in $\bar{D}_r \times \bar{N}_j$ we have

$$
\begin{aligned}
|p^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)| \leqq M(8 + \delta)^n (2n)^{-1}(2n-1)^{-1}\left(1 - \frac{|\xi_1|}{r}\right)^{-2n} \\
\cdot \left(1 - \frac{|\xi_2|}{r}\right)^{-(2n-1)}\left(1 - \frac{|\xi_3|}{r}\right)^{-(2n-1)} \\
\cdot \left(1 - \frac{|\xi_4|}{r}\right)^{-(2n-1)}\left(1 - \frac{|\zeta - \zeta_j|}{2\varepsilon}\right)^{-n} \\
\cdot \left(1 - \frac{|\eta - \eta_j|}{2\varepsilon}\right)^{-n} r^{-n}.
\end{aligned}
\tag{2.23}
$$

Now consider $|t^{2n}\mu^n p^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)|$ in $\bar{D}_{\alpha r} \times \bar{N}_j \times T$, where

$$D_{\alpha r} = \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_i| < r/\alpha ; \alpha > 1, i = 1, 2, 3, 4\}.$$

In $\bar{D}_{\alpha r} \times \bar{N}_j \times T$ we have

$$
\begin{aligned}
&1 - \frac{|\xi_i|}{r} \geqq \frac{\alpha - 1}{\alpha}, \qquad i = 1, 2, 3, 4, \\
&1 - \frac{|\zeta - \zeta_j|}{2\varepsilon} \geqq \frac{1}{4}, \qquad 1 - \frac{|\eta - \eta_j|}{2\varepsilon} \geqq \frac{1}{4}, \\
&|\mu| = |\xi_2 + \xi_4| \leqq \frac{2r}{\alpha}, \qquad |t| \leqq 1.
\end{aligned}
\tag{2.24}
$$

Thus, from (2.23) and (2.24) we have that in $\bar{D}_{\alpha r} \times \bar{N}_j \times T$,

$$
\begin{aligned}
|t^{2n}\mu^n p^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)| \leqq Mr(\alpha - 1)^3 (2n-1)^{-1}(2n)^{-1}\alpha^{-3} \\
\cdot (32\alpha^7(8 + \delta)(\alpha - 1)^{-8})^n.
\end{aligned}
\tag{2.25}
$$

If we choose α such that

$$32\alpha^7(8 + \delta)(\alpha - 1)^{-8} < 1, \tag{2.26}$$

then the series (2.16) converges absolutely and uniformly in $\bar{D}_{\alpha r} \times \bar{N}_j \times T$. By taking $r = \alpha R$ we can now conclude that $E^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta, t)$ is regular in $\bar{G}_R \times \bar{N}_j \times T$ for each $j = 1, 2, \cdots, N$, and hence in $G_R \times B \times T$. Equation (2.18) follows from (2.15).

We now want to show that *every* real-valued C^2-solution $u(x_1, x_2, x_3, x_4)$ of (2.1) which is defined in some neighborhood of the origin in R^4 can be represented locally in the form

$$u(x_1, x_2, x_3, x_4) = \operatorname{Re} \mathbf{P}_4\{f\}. \tag{2.27}$$

We shall furthermore show that the associated analytic function $f(\mu, \zeta, \eta)$ has a simple representation in terms of the Goursat data $U(Y, 0, Z, Z^*)$ for $U(Y, Y^*, Z, Z^*) \equiv u(x_1, x_2, x_3, x_4)$. These results will then enable us to construct a complete family of solutions (in the L^∞-norm) for (2.1).

THEOREM 2.4. *Let $u(x_1, x_2, x_3, x_4)$ be a real-valued C^2-solution of* (2.1) *in some neighborhood of the origin in R^4. Then there exists an analytic function of three complex variables $f(\mu, \zeta, \eta)$ which is regular for μ in some neighborhood of the origin and $|\zeta| < 1 + \varepsilon, |\eta| < 1 + \varepsilon, \varepsilon > 0$, such that locally $u(x_1, x_2, x_3, x_4) = \operatorname{Re} \mathbf{P}_4\{f\}$. In particular, denote by $U(Y, Y^*, Z, Z^*) \equiv u(x_1, x_2, x_3, x_4)$ the extension of $u(x_1, x_2, x_3, x_4)$ to the (Y, Y^*, Z, Z^*)-space, and let*

$$g(\mu, \zeta, \eta) = \frac{\partial^2}{\partial \mu^2}\left\{\int_0^1 \int_0^1 \mu^2(1 - t)[2U(\mu t, 0, \xi(1 - t)\mu\zeta, (1 - t)(1 - \xi)\mu\eta) - U(0, 0, \xi(1 - t)\mu\zeta, (1 - t)(1 - \xi)\mu\eta)]\, dt\, d\xi\right\}. \tag{2.28}$$

Then,

$$f(\mu, \zeta, \eta) = -\frac{1}{2\pi}\int_{\gamma'} g(\mu(1 - t^2)\zeta, \eta)\frac{dt}{t^2}, \tag{2.29}$$

where γ' is a rectifiable arc joining the points $t = -1$ and $t = +1$ and not passing through the origin.

Remark. Equation (2.29) can be inverted by the formula (cf. [11, p. 114])

$$g(\mu, \zeta, \eta) = \int_{\gamma} f(\mu(1 - t^2), \zeta, \eta)\frac{dt}{\sqrt{1 - t^2}}, \tag{2.30}$$

where the path γ is defined in Theorem 2.2.

Proof of Theorem 2.4. The fact that $u(x_1, x_2, x_3, x_4)$ is a strong solution of (2.1) implies that $u(x_1, x_2, x_3, x_4)$ is an analytic function of its independent variables in some neighborhood of the origin. Furthermore, since $F(Y, Y^*, Z, Z^*)$ is real-valued (for $Y = \overline{Y^*}, Z = -\overline{Z^*}$), $\operatorname{Re} \mathbf{P}_4\{f\}$ is a real-valued solution of (2.1) for any function $f(\mu, \zeta, \eta)$ which is analytic in the product domain $D \times B$ (see Theorem 2.2). Now suppose that locally $g(\mu, \zeta, \eta)$, $f(\mu, \zeta, \eta)$ and $F(Y, Y^*, Z, Z^*)$ have the expansions

$$\begin{aligned} g(\mu, \zeta, \eta) &= \sum_{n=0}^{\infty} \sum_{\substack{k=0 \\ k+l \leq n}}^{n} \sum_{l=0}^{n} a_{nkl}\mu^n\eta^k\zeta^l, \\ f(\mu, \zeta, \eta) &= -\frac{1}{2\pi}\int_{\gamma'} g(\mu(1 - t^2), \zeta, \eta)\frac{dt}{t^2} \\ &= \sum_{n=0}^{\infty} \sum_{\substack{k=0 \\ k+l \leq n}}^{n} \sum_{l=0}^{n} a_{nkl}\frac{\Gamma(n + 1)}{\Gamma(n + \frac{1}{2})\Gamma(\frac{1}{2})}\mu^n\eta^k\zeta^l \\ F(Y, Y^*, Z, Z^*) &= \sum_{l,m,n,p=0}^{\infty} b_{lmnp}Y^l Y^{*m}Z^n Z^{*p}, \end{aligned} \tag{2.31}$$

and let the analytic functions $\bar{g}(\mu, \zeta, \eta)$, $\bar{f}(\mu, \zeta, \eta)$, $\bar{F}(Y, Y^*, Z, Z^*)$ be defined by replacing a_{nkl} and b_{lmnp} by $\overline{a_{nkl}}$ and $\overline{b_{lmnp}}$, respectively, in (2.31). Let $\bar{E}(Y, Y^*, Z, Z^*, \zeta, \eta, t)$ be the generating function corresponding to the differential equation

$U_{YY^*} - U_{ZZ^*} + \bar{F}(Y, Y^*, Z, Z^*)U = 0$. Then for x_1, x_2, x_3, x_4 real we can write

$$\begin{aligned}\operatorname{Re} \mathbf{P}_4\{f\} = &-\frac{1}{8\pi^2} \int_{|\zeta|=1} \int_{|\eta|=1} \int_\gamma E(Y, Y^*, Z, Z^*, \zeta, \eta, t) \\ &\qquad \cdot f(\mu(1-t^2), \zeta, \eta) \frac{dt}{\sqrt{1-t^2}} \frac{d\eta}{\eta} \frac{d\zeta}{\zeta} \\ &-\frac{1}{8\pi^2} \int_{|\zeta|=1} \int_{|\eta|=1} \int_\gamma \bar{E}(Y^*, Y, -Z^*, -Z, \zeta, \eta, t) \\ &\qquad \cdot \bar{f}(\bar{\mu}(1-t^2), \zeta, \eta) \frac{dt}{\sqrt{1-t^2}} \frac{d\eta}{\eta} \frac{d\zeta}{\zeta},\end{aligned} \tag{2.32}$$

where $\bar{\mu} = Y^* - \zeta^{-1}Z^* - \eta^{-1}Z + \eta^{-1}\zeta^{-1}Y$. Now from Theorem 2.1 we know that $U(Y, Y^*, Z, Z^*)$ is uniquely determined by the function $U(Y, 0, Z, Z^*)$, and hence, using (2.18) and (2.32) we try to determine $f(\mu, \zeta, \eta)$ from the equation

$$\begin{aligned}U(Y, 0, Z, Z^*) = &-\frac{1}{8\pi^2} \int_{|\zeta|=1} \int_{|\eta|=1} \int_\gamma f(\mu_1(1-t^2)\zeta, \eta) \frac{dt}{\sqrt{1-t^2}} \frac{d\eta}{\eta} \frac{d\zeta}{\zeta} \\ &-\frac{1}{8\pi^2} \int_{|\zeta|=1} \int_{|\eta|=1} \int_\gamma \bar{E}(0, Y, -Z^*, -Z, \zeta, \eta) \\ &\qquad \cdot \bar{f}(\mu_2(1-t^2), \zeta, \eta) \frac{dt}{\sqrt{1-t^2}} \frac{d\eta}{\eta} \frac{d\zeta}{\zeta},\end{aligned} \tag{2.33}$$

where $\mu_1 = Y + \zeta^{-1}Z + \eta^{-1}Z^*$ and $\mu_2 = \eta^{-1}\zeta^{-1}Y - \zeta^{-1}Z^* - \eta^{-1}Z$. To this end we first write $\bar{E}^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) \equiv \bar{E}(Y, Y^*, Z, Z^*, \zeta, \eta, t)$ in its series expansion

$$\bar{E}^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) = 1 + \sum_{n=1}^{\infty} t^{2n}\mu^n \bar{p}^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta), \tag{2.34}$$

where, from Theorem 2.3, we have

$$\bar{p}^{(1)} = -\eta\zeta \int_0^{\xi_1} \bar{F}^*(\xi_1', \xi_2, \xi_3, \xi_4, \zeta, \eta)\, d\xi_1', \tag{2.35}$$

$$\begin{aligned}&\bar{p}_1^{(n+1)} = -\frac{1}{2n+1}\left\{2\bar{p}_{13}^{(n)} + 2\bar{p}_{14}^{(n)} + 2\bar{p}_{23}^{(n)} - 2\bar{p}_{34}^{(n)} + \eta\zeta\bar{F}^*\bar{p}^{(n)}\right\}, \\ &\bar{p}^{(n+1)}(0, \xi_2, \xi_3, \xi_4, \zeta, \eta) = 0,\end{aligned} \tag{2.36}$$

with $\bar{F}^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta) \equiv \bar{F}(Y, Y^*, Z, Z^*)$. From (2.10) and (2.35) we have

$$\begin{aligned}\bar{p}^{(1)} &= -\eta\zeta \int_0^{\xi_1} \bar{F}(\xi_3 + \xi_1' - \xi_2, \eta\zeta\xi_1', \zeta(\xi_4 - \xi_3 + \xi_2 - \xi_1'), \eta(\xi_2 - \xi_1'))\, d\xi_1' \\ &= -\int_0^{Y^*} \bar{F}(Y + \eta^{-1}\zeta^{-1}\tau - \eta^{-1}\zeta^{-1}Y^*, \tau, \\ &\qquad \zeta(\zeta^{-1}Z - \eta^{-1}\zeta^{-1}Y^* - \eta^{-1}\zeta^{-1}\tau), \\ &\qquad \eta(\eta^{-1}\zeta^{-1}Y^* + \eta^{-1}Z^* - \eta^{-1}\zeta^{-1}\tau))\, d\tau.\end{aligned} \tag{2.37}$$

The fact that $\bar{F}(Y, Y^*, Z, Z^*)$ is an analytic function of Y, Y^*, Z and Z^* now implies that

$$-\frac{1}{8\pi^2}\int_{|\zeta|=1}\int_{|\eta|=1}\int_{\gamma} t^2\mu_2\tilde{p}^{(1)}(0, Y, -Z^*, -Z, \zeta, \eta) \cdot \bar{f}(\mu_2(1-t^2), \zeta, \eta)\frac{dt}{\sqrt{1-t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta} = 0, \tag{2.38}$$

$$\tilde{p}^{(1)}(Y, Y^*, Z, Z^*, \zeta, \eta) \equiv \bar{p}^{(1)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$$

since the Laurent series of the integrand has no terms involving $\zeta^l\eta^m$ for both $l > -2$ and $m > -2$. A similar calculation using (2.36) and induction shows that

$$-\frac{1}{8\pi^2}\int_{|\zeta|=1}\int_{|\eta|=1}\int_{\gamma} t^{2n}\mu_2^n\tilde{p}^{(n)}(0, Y, -Z, Z^*, \zeta, \eta) . \bar{f}(\mu_2(1-t^2), \zeta, \eta)\frac{dt}{\sqrt{1-t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta} = 0, \tag{2.39}$$

$$\tilde{p}^{(n)}(Y, Y^*, Z, Z^*, \zeta, \eta) \equiv \bar{p}^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$$

for $n = 1, 2, 3, \cdots$. Because of the uniform convergence of the series in (2.34), we can substitute this series into (2.33) and integrate termwise to conclude that

$$U(Y, 0, Z, Z^*) = -\frac{1}{8\pi^2}\int_{|\zeta|=1}\int_{|\eta|=1} g(\mu_1, \zeta, \eta)\frac{d\eta}{\eta}\frac{d\zeta}{\zeta} - \frac{1}{8\pi^2}\int_{|\zeta|=1}\int_{|\eta|=1} \bar{g}(\mu_2, \zeta, \eta)\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}, \tag{2.40}$$

where we have made use of (2.30). To complete the proof of the theorem it now suffices to show that (2.28) gives the solution of the integral equation (2.40). In order to show this we let

$$U(Y, 0, Z, Z^*) = \sum_{n,k,l=0}^{\infty} \gamma_{nkl} Y^n Z^{*k} Z^l \tag{2.41}$$

and equate coefficients of $Y^n Z^{*k} Z^l$ on both sides of (2.40). This gives

$$\begin{aligned} 2n!k!l!\gamma_{nkl} &= (n+k+l)!a_{n+k+l,k,l}, \qquad n > 0, \\ 2k!l!\gamma_{0kl} &= (k+l)!a_{k+l,k,l} + (k+l)!(-1)^{k+l}\overline{a_{k+l,l,k}}. \end{aligned} \tag{2.42}$$

Since $U(0, 0, Z, Z^*)$ is real-valued for x_3 and x_4 real, we have from (2.41) that $\overline{\gamma_{0kl}} = (-1)^{k+l}\gamma_{0lk}$, and hence, we can assume without loss of generality that

$(-1)^{k+l}\overline{a_{k+l,l,k}} = a_{k+l,k,l}$. Equations (2.41) and (2.42) now give

$$U(Y,0,Z,Z^*) = \tfrac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(n+k+l+1)}{\Gamma(n+1)\Gamma(k+1)\Gamma(l+1)}a_{n+k+l,k,l}Y^nZ^{*k}Z^l + \tfrac{1}{2}U(0,0,Z,Z^*)$$

$$(2.43)\qquad = \tfrac{1}{2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=k+l}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n-k-l+1)\Gamma(k+1)\Gamma(l+1)} \cdot a_{nkl}Y^{n-k-l}Z^{*k}Z^l + \tfrac{1}{2}U(0,0,Z,Z^*)$$

$$= \tfrac{1}{2}\sum_{n=0}^{\infty}\sum_{\substack{k=0\\k+l\leq n}}^{n}\sum_{l=0}^{n}\frac{\Gamma(n+1)}{\Gamma(n-k-l+1)\Gamma(k+1)\Gamma(l+1)} \cdot a_{nkl}Y^{n-k-l}Z^{*k}Z^l + \tfrac{1}{2}U(0,0,Z,Z^*).$$

From the definition of the beta function (cf. [8, p. 9]) we can now write

$$\int_0^1 (1-t)[U(tY,0,(1-t)Z,(1-t)Z^*) - \tfrac{1}{2}U(0,0,(1-t)Z,(1-t)Z^*)]\,dt$$

$$(2.44)\qquad = \tfrac{1}{2}\sum_{n=0}^{\infty}\sum_{\substack{k=0\\k+l\leq n}}^{n}\sum_{l=0}^{n}\frac{\Gamma(k+l+2)}{(n+2)(n+1)\Gamma(l+1)\Gamma(k+1)}a_{nkl}Y^{n-k-l}Z^{*k}Z^l,$$

and hence,

$$\int_0^1\int_0^1 (1-t)[U(tY,0,\xi(1-t)Z,(1-t)(1-\xi)Z^*)$$

$$(2.45)\qquad - \tfrac{1}{2}\,U(0,0,\xi(1-t)Z,(1-t)(1-\xi)Z^*)]\,dt\,d\xi$$

$$= \tfrac{1}{2}\sum_{n=0}^{\infty}\sum_{\substack{k=0\\k+l\leq n}}^{n}\sum_{l=0}^{n}\frac{a_{nkl}}{(n+2)(n+1)}Y^{n-k-l}Z^{*k}Z^l,$$

which implies that

$$\frac{\partial^2}{\partial\mu^2}\left\{\int_0^1\int_0^1 \mu^2(1-t)[U(\mu t,0,\xi(1-t)\mu\zeta,(1-t)(1-\xi)\mu\eta) - \tfrac{1}{2}U(0,0,\xi(1-t)\mu\zeta,(1-t)(1-\xi)\mu\eta)]\,dt\,d\xi\right\}$$

$$(2.46)\qquad = \tfrac{1}{2}\sum_{n=0}^{\infty}\sum_{\substack{k=0\\k+l\leq n}}^{n}\sum_{l=0}^{n}a_{nkl}\mu^n\eta^k\zeta^l = \tfrac{1}{2}g(\mu,\zeta,\eta).$$

Equation (2.28) follows immediately from (2.46), and this proves the theorem.

Note. When $F(Y,Y^*,Z,Z^*) \equiv 0$, our operator $\mathbf{P}_4$ reduces to Gilbert's operator $\mathbf{G}_4$ (see [11, pp. 75–82]), and (2.28) gives a new inversion formula for the operator $\operatorname{Re}\mathbf{G}_4$. It is of interest to compare (2.28) with the inversion formula given by Kreyszig for complex-valued harmonic functions in four independent variables [16], [11, p. 78].

Theorems 2.2, 2.3 and 2.4 can now be used to construct a complete family of solutions in the L^∞-norm for (2.1). The proof of the following theorem exactly parallels that for the case of three independent variables, and the reader is referred to Theorem 3.2 of [3] for further details. Briefly, the proof proceeds as follows: Since (2.1) is elliptic and has analytic coefficients, it possesses the unique continuation property and hence the Runge approximation property (cf. [17]). Hence it suffices to find a complete family of (real-valued) solutions defined in an arbitrarily large sphere S in R^4. From Garabedian's work on Cauchy's problem for analytic systems [10, pp. 614–619] it is possible to conclude that the Cauchy data for solutions of (2.1) defined in S must be regular in some convex region B in $\mathbb{C}^3$, the space of three complex variables. Since convex domains are Runge domains of the first kind [9, p. 229] and solutions of (2.1) defined in S depend continuously on their Cauchy data in B, we can approximate solutions in S by solutions having polynomial Cauchy data, that is, entire solutions of (2.1). Such (real-valued) entire solutions can then be approximated by (real-valued) solutions having polynomial Goursat data in the (Y, Y^*, Z, Z^*)-space. But by Theorem 2.4, real-valued solutions $u(x_1, x_2, x_3, x_4)$ of (2.1) with polynomial Goursat data can be represented in the form

$$u(x_1, x_2, x_3, x_4) = \operatorname{Re} \mathbf{P}_4\{h_N\}, \tag{2.47}$$

where

$$h_N(\mu, \zeta, \eta) = \sum_{n=0}^{N} \sum_{\substack{k=0 \\ k+l \leqq n}}^{n} \sum_{l=0}^{n} a_{nkl}\mu^n\eta^k\zeta^l, \tag{2.48}$$

from which follows the theorem below. In the statement of the theorem "Im" denotes "take the imaginary part."

THEOREM 2.5. *Let G be a bounded, simply connected domain in R^4, and define*

$$\begin{aligned} u_{2n,k,l}(x_1, x_2, x_3, x_4) &= \operatorname{Re} \mathbf{P}_4\{\mu^n\eta^k\zeta^l\}, \\ u_{2n+1,k,l}(x_1, x_2, x_3, x_4) &= \operatorname{Im} \mathbf{P}_4\{\mu^n\eta^k\zeta^l\}, \end{aligned} \tag{2.49}$$

where $0 \leqq n < \infty$, $l = 0, 1, \cdots, n$, $k = 0, 1, \cdots, n$, $k + l \leqq n$. Then the set $\{u_{nkl}\}$ is a complete family of solutions in the L^∞-norm for (2.1) *in the space of real-valued C^2-solutions of* (2.1) *defined in G.*

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26

INTEGRAL REPRESENTATIONS OF SOLUTIONS TO A CLASS OF FOURTH ORDER ELLIPTIC EQUATIONS IN THREE INDEPENDENT VARIABLES

DAVID COLTON

1. *Introduction.* Both S. Bergman [1] and I. N. Vekua [13] have constructed integral operators which map ordered pairs of analytic functions of one complex variable onto solutions of fourth order elliptic equations in two independent variables. Such operators play an important role in the investigation of the analytic properties of solutions to higher order elliptic equations and in the approximation of solutions to boundary value problems associated with these equations. Unfortunately, little progress has been made in developing an analogous theory for elliptic equations in more than two independent variables. Recently, however, Colton and Gilbert [7] constructed integral operators for a class of fourth order elliptic equations with spherically symmetric coefficients in $p+2$ $(p \geqslant 0)$ independent variables, and at present Dean Kukral [11], a student of R. P. Gilbert, is in the process of trying to extend some recent results of Colton [3, 4, 5] for second order equations in three independent variables to the fourth order case.

In the present paper we extend the results of [3] in a different direction from that of Kukral, and construct an integral operator which maps ordered pairs of analytic functions of *two* complex variables onto real valued solutions of the equation

$$(\Delta_3 + c^{(2)}(x, y, z))(\Delta_3 + c^{(1)}(x, y, z))\, u = 0, \tag{1.1}$$

where $c^{(1)}(x, y, z)$ and $c^{(2)}(x, y, z)$ are real valued entire functions of their independent (complex) variables. (With minor modifications we could have assumed these functions to be analytic only in some neighbourhood of the origin in $\mathbb{C}^3$, the space of three complex variables.) The advantages of our approach is that our integral operators are easy to construct, the inverse operator is readily obtainable, and the class of equations of the form (1.1) include many of the better known fourth order equations in mathematical physics (for example in the special case when $c^{(1)} = c^{(2)} = 0$ equation (1.1) becomes the biharmonic equation, whereas the case $c^{(1)} \neq c^{(2)} \neq 0$ appears in the propagation of time harmonic elastic waves [*cf.* 12]. As an application of our integral representations we will construct a complete family of solutions in the maximum norm to equation (1.1).

2. *Preliminaries.* From [3] and [5] we have that every real valued solution $u^{(j)}(x, y, z)$, $j = 1, 2$, of

$$\Delta_3 u^{(j)} + c^{(j)}(x, y, z)\, u^{(j)} = 0, \tag{2.1}$$

which is regular in some neighbourhood of the origin can be represented locally in the form

$$u^{(j)}(x, y, z) = \operatorname{Re} \mathbf{P}_3^{(j)}\{f^{(j)}\}, \tag{2.2}$$

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where

$$\mathbf{P}_3^{(j)}\{f^{(j)}\} = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \int\limits_{-1}^{+1} E^{(j)}(X, Z, Z^*, \zeta, t) f^{(j)}(\mu(1-t^2), \zeta) \frac{dt}{\sqrt{(1-t^2)}} \frac{d\zeta}{\zeta}, \tag{2.3}$$

$X = x$, $Z = \frac{1}{2}(y + iz)$, $Z^* = \frac{1}{2}(-y + iz)$, $\mu = X + \zeta Z + \zeta^{-1} Z^*$ and $f^{(j)}(\mu, \zeta)$ is an analytic function of two complex variables in some neighbourhood of the origin in $\mathbb{C}^2$, the space of two complex variables. The *generating function* $E^{(j)}(X, Z, Z^*, \zeta, t) = \tilde{E}^{(j)}(\xi_1, \xi_2, \xi_3, \zeta, t)$ is given by

$$\tilde{E}^{(j)}(\xi_1, \xi_2, \xi_3, \zeta, t) = 1 + \sum_{n=1}^{\infty} t^{2n} \mu^n p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta), \tag{2.4}$$

where

$$\left.\begin{aligned} \xi_1 &= 2\zeta Z, \\ \xi_2 &= X + 2\zeta Z, \\ \xi_3 &= X + 2\zeta^{-1} Z^*, \end{aligned}\right\} \tag{2.5}$$

and the series converges uniformly and absolutely for $|\xi_i| < R$, $i = 1, 2, 3$, $1 - \varepsilon < |\zeta| < 1 + \varepsilon$, $|t| \leqslant 1$, where $0 < \varepsilon < \frac{1}{2}$ and R is an arbitrarily large positive number. In equation (2.2) " Re " denotes " take the real part ". In equation (2.4) the functions $p^{(n)}(\xi_1, \xi_2, \xi_3, \zeta)$ are defined recursively by

$$\left.\begin{aligned} p_1^{(n+1)} &= \frac{1}{2n+1}\{p_{22}^{(n)} + p_{33}^{(n)} - 4p_{13}^{(n)} - 2p_{23}^{(n)} + \tilde{C}^{(j)} p^{(n)}\}, \\ p^{(0)}&(\xi_1, \xi_2, \xi_3, \zeta) = 1, \\ p^{(n+1)}&(0, \xi_2, \xi_3, \zeta) = 0, \quad n = 0, 1, 2, \ldots, \end{aligned}\right\} \tag{2.6}$$

where $\tilde{C}^{(j)}(\xi_1, \xi_2, \xi_3, \zeta) = C^{(j)}(X, Z, Z^*) = c^{(j)}(x, y, z)$, and the subscripts denote differentiation with respect to the ξ_i, $i = 1, 2, 3$, variables. The analytic function $f^{(j)}(\mu, \zeta)$ is given by the formula

$$f^{(j)}(\mu, \zeta) = -\frac{1}{2\pi} \int\limits_{\gamma} g^{(j)}(\mu(1-t^2), \zeta) \frac{dt}{t^2}, \tag{2.7}$$

where γ is a rectifiable arc joining the points $t = -1$ and $t = +1$ and not passing through the origin, and

$$g^{(j)}(\mu, \zeta) = 2\frac{\partial}{\partial \mu}\left[\mu \int\limits_0^1 U^{(j)}(t\mu, 0, (1-t)\,\mu\zeta)\,dt\right] - U^{(j)}(\mu, 0, 0) \tag{2.8}$$

with $U^{(j)}(X, Z, Z^*) = u^{(j)}(x, y, z)$.

The results outlined above will now be used to obtain integral operators for equation (1.1). In the analysis which follows, D is a neighbourhood of the origin in the μ plane, $B_\varepsilon = \{\zeta : 1 - \varepsilon < |\zeta| < 1 + \varepsilon\}$ where $0 < \varepsilon < \frac{1}{2}$,

$$G_R = \{(\xi_1, \xi_2, \xi_3) : |\xi_i| < R, \quad i = 1, 2, 3\},$$

and $T = \{t : |t| \leqslant 1\}$.

3. *General representations of solutions to equation* (1.1). We first note the obvious fact that any solution of

$$\Delta_3 u^{(1)} + c^{(1)}(x, y, z)\, u^{(1)} = 0, \tag{3.1}$$

is also a solution of equation (1.1), in particular for any function $f^{(1)}(\mu, \zeta)$ analytic in some neighbourhood of the origin in $\mathbb{C}^2$,

$$U^{(1)}(X, Z, Z^*) = \operatorname{Re} \mathbf{P}_3^{(1)}\{f^{(1)}\}, \tag{3.2}$$

is a real valued solution of equation (1.1). This solution cannot, however, represent *all* real valued solutions, since from equations (2.7) and (2.8) we have that if $U^{(1)}(X, 0, Z^*) = 0$ then $f^{(1)}(\mu, \zeta) = 0$ which implies $U^{(1)}(X, Z, Z^*) \equiv 0$. But there clearly exist non-trivial real valued solutions $u(x, y, z) = U(X, Z, Z^*)$ of equation (1.1) such that $U(X, 0, Z^*) = 0$ (*c.f.* [**10**; p. 116–119]). Hence we now turn our attention to constructing a class of solutions to equation (1.1) which vanish along the hyperplane $Z = 0$, but are not identically zero.

THEOREM 3.1. *Let $f(\mu, \zeta)$ be an analytic function of two complex variables in the product domain $D \times B_\varepsilon$ and let $\tilde{E}^{(2)}(\xi_1, \xi_2, \xi_3, \zeta, t)$ be the generating function corresponding to the equation*

$$\Delta_3 u^{(2)} + c^{(2)}(x, y, z)\, u^{(2)} = 0. \tag{3.3}$$

Suppose $\tilde{E}(\xi_1, \xi_2, \xi_3, \zeta, t) = E(X, Z, Z^*, \zeta, t)$ is a regular solution of the partial differential equation

$$\mu t(4\tilde{E}_{13} + 2\tilde{E}_{23} - \tilde{E}_{22} - \tilde{E}_{33} - \tilde{C}^{(1)}\tilde{E}) + (1 - t^2)\tilde{E}_{1t} - \frac{1}{t}\tilde{E}_1 + \mu t \tilde{E}^{(2)} = 0, \tag{3.4}$$

in $G_R \times B_\varepsilon \times T$ for some $R > 0$, $0 < \varepsilon < \frac{1}{2}$, where the subscripts denote differentiation in the ξ_i, $i = 1, 2, 3$, variables. Then

$$\begin{aligned} U(X, Z, Z^*) &= \mathbf{T}_3\{f\} \\ &= \frac{1}{2\pi i} \int_{|\zeta|=1} \int_{-1}^{+1} E(X, Z, Z^*, \zeta, t) f(\mu(1 - t^2), \zeta) \frac{dt}{\sqrt{(1 - t^2)}} \frac{d\zeta}{\zeta} \end{aligned} \tag{3.5}$$

is a (complex valued) solution of equation (1.1) *which is regular in a neighbourhood of the origin in X, Z, Z^* space.*

Proof. Since the Jacobian of the transformation (2.5) is equal to -4, we can conclude that $U(X, Z, Z^*) = \mathbf{T}_3\{f\}$ is regular in a neighbourhood of the origin in the X, Z, Z^* space. Straightforward differentiation and integration by parts in equation (3.5) leads to

$$\begin{aligned} &\Delta_3 u + c^{(1)}(x, y, z)\, u \\ &\quad = U_{XX} - U_{ZZ^*} + C^{(1)} U \\ &\quad = -\frac{1}{2\pi i} \int_{|\zeta|=1} \int_{-1}^{+1} \frac{f(\mu(1 - t^2), \zeta)}{\mu t} \Big\{ \mu t(4\tilde{E}_{13} + 2\tilde{E}_{23} - \tilde{E}_{22} - \tilde{E}_{33} - \tilde{C}^{(1)}\tilde{E}) \\ &\qquad\qquad + (1 - t^2)\tilde{E}_{1t} - \frac{1}{t}\tilde{E}_1 \Big\} \frac{dt}{\sqrt{(1 - t^2)}} \frac{d\zeta}{\zeta} \\ &\quad = \frac{1}{2\pi i} \int_{|\zeta|=1} \int_{-1}^{+1} E^{(2)}(X, Z, Z^*, \zeta, t) f(\mu(1 - t^2), \zeta) \frac{dt}{\sqrt{(1 - t^2)}} \frac{d\zeta}{\zeta}, \end{aligned} \tag{3.6}$$

which is a solution of equation (3.3), i.e. $\mathbf{T}_3\{f\}$ is a solution of equation (1.1).

We now want to show that the integral operator $\mathbf{T}_3$ exists, *i.e.* we must construct a function $E(X, Z, Z^*, \zeta, t)$ satisfying the conditions of Theorem 3.1. From previous considerations we will furthermore require that

$$E(X, 0, Z^*, \zeta, t) = 0. \tag{3.7}$$

To this end we write $\tilde{E}(\xi_1, \xi_2, \xi_3, \zeta, t) = E(X, Z, Z^*, \zeta, t)$ in the form

$$\tilde{E}(\xi_1, \xi_2, \xi_3, \zeta, t) = \sum_{n=1}^{\infty} t^{2n} \mu^n q^{(n)}(\xi_1, \xi_2, \xi_3, \zeta), \tag{3.8}$$

and impose the initial condition on the $q^{(n)}$

$$q^{(n)}(0, \xi_2, \xi_3, \zeta) = 0, \qquad n = 1, 2, 3, \ldots . \tag{3.9}$$

Recalling from §2 that $\tilde{E}^{(2)}(\xi_1, \xi_2, \xi_3, \zeta, t)$ can be expanded in the series defined by equations (2.4)–(2.6), we substitute the series (3.8) into equation (3.4) and arrive at the following recursion formula for the $q^{(n)}$:

$$q_1^{(n+1)} = \frac{1}{2n+1}\{q_{22}^{(n)} + q_{33}^{(n)} - 4q_{13}^{(n)} - 2q_{23}^{(n)} + \tilde{C}^{(1)} q^{(n)} - p^{(n)}\}$$
$$n = 0, 1, 2, \ldots, \tag{3.10}$$

$$q^{(0)}(\xi_1, \xi_2, \xi_3, \zeta) = 0, \tag{3.11}$$

where the subscripts again denote differentiation with respect to the ξ_i, $i = 1, 2, 3$, variables, and the $p^{(n)}$ are given recursively by equation (2.6) with $j = 2$. Hence the functions $q^{(n)}(\xi_1, \xi_2, \xi_3, \zeta)$ can be determined recursively. We now must show the series (3.8) converges in $G_R \times B_\varepsilon \times T$ for R arbitrarily large. To show this first suppose B_ε is covered by the union of sets

$$N_k = \{\zeta : |\zeta - \zeta_k| < \tfrac{3}{2}\varepsilon\} \qquad k = 1, 2, \ldots, N,$$

where $|\zeta_k| = 1$. Then from [3] we have that in $\bar{G}_R \times \bar{N}_k$ (where the bar denotes closure)

$$\begin{aligned} p^{(n)} &\ll M^{(k)}(8 + \delta^{(k)})^n (2n)^{-1}(2n-1)^{-1}\left(1 - \frac{\xi_1}{r}\right)^{-2n}\left(1 - \frac{\xi_2}{r}\right)^{-(2n-1)} \\ &\qquad \times \left(1 - \frac{\xi_3}{r}\right)^{-(2n-1)}\left(1 - \frac{\zeta - \zeta_k}{2\varepsilon}\right)^{-n} r^{-n+1} \\ &\ll M^{(k)}(8 + \delta^{(k)})^n (2n)^{-1}(2n-1)^{-1}\left(1 - \frac{\xi_1}{r}\right)^{-(2n+1)}\left(1 - \frac{\xi_2}{r}\right)^{-(2n+1)} \\ &\qquad \times \left(1 - \frac{\xi_3}{r}\right)^{-(2n+1)}\left(1 - \frac{\zeta - \zeta_k}{2\varepsilon}\right)^{-(n+1)} r^{-n+1}, \end{aligned} \tag{3.12}$$

where $\delta^{(k)}$ and $M^{(k)}$ are positive constants and "$\ll$" means "is dominated by" (*c.f.* [1]). It is now a matter of straightforward induction to show from equation (3.10) that

$$\left.\begin{aligned} q_1^{(n)} &\ll M(8 + \delta)^n (2n)^{-1}(2n-1)^{-1}\left(1 - \frac{\xi_1}{r}\right)^{-(2n-1)} \\ &\left(1 - \frac{\xi_2}{r}\right)^{-(2n-1)}\left(1 - \frac{\xi_3}{r}\right)^{-(2n-1)}\left(1 - \frac{\zeta - \zeta_k}{2\varepsilon}\right)^{-n} r^{-n}, \end{aligned}\right\} \tag{3.13}$$

and from this to conclude that the series (3.8) converges absolutely and uniformly in $\bar{G}_R \times \bar{N}_k \times T$, and hence in $G_R \times B_\varepsilon \times T$. For details of the proof of this last step the reader is referred to an almost identical majorization argument in Theorem 2.3 of [**3**]. We now have the following theorem:

THEOREM 3.2. *The series defined in equations* (3.8)–(3.10) *converges absolutely and uniformly in $G_R \times B_\varepsilon \times T$ (where R is an arbitrary positive number) and is a regular solution of equation* (3.4) *satisfying the initial condition* (3.7).

We will now show that every real valued solution $u(x, y, z)$ of equation (1.1) which is four times continuously differentiable (i.e. in class C^4) in some neighbourhood of the origin can be represented locally in the form

$$u(x, y, z) = \operatorname{Re}\{\mathbf{P}_3^{(1)}\{f^{(1)}\} + \mathbf{T}_3\{f^{(2)}\}\}, \tag{3.14}$$

where $f^{(j)}(\mu, \zeta)$, $j = 1, 2$, are analytic functions of two complex variables in some neighbourhood of the origin in $\mathbb{C}^2$. We will furthermore give an explicit formula for calculating $f^{(j)}(\mu, \zeta)$, $j = 1, 2$, in terms of the values of $u(x, y, z) = U(X, Z, Z^*)$ and its derivatives along the characteristic hyperplane $Z = 0$. We first need the following lemma:

LEMMA 3.1. *Let $u(x, y, z)$ be a real valued C^4 solution of equation* (1.1) *in a neighbourhood of the origin and let $U(X, Z, Z^*) = u(x, y, z)$ be the extension of $u(x, y, z)$ to the X, Z, Z^* space. Then $U(X, Z, Z^*)$ is an analytic function of X, Z, Z^* in some neighbourhood of the origin in $\mathbb{C}^3$ and is uniquely determined by the functions $F^{(1)}(X, Z^*) = U(X, 0, Z^*)$ and $F^{(2)}(X, Z^*) = (U_{XX} - U_{ZZ^*} + C^{(1)}U)_{Z=0}$.*

Proof. Since $u(x, y, z)$ is a strong solution of an elliptic equation with analytic coefficients, $U(X, Z, Z^*)$ is an analytic function of X, Z and Z^*. Now let

$$V = U_{XX} - U_{ZZ^*} + C^{(1)}U. \tag{3.15}$$

Then $V(X, Z, Z^*)$ is a real valued (for x, y, z real) solution of

$$V_{XX} - V_{ZZ^*} + C^{(2)}V = 0, \tag{3.16}$$

and hence is uniquely determined by the function $V(X, 0, Z^*) = F^{(2)}(X, Z^*)$ [**3**; Theorem 2.1]. But $U(X, Z, Z^*)$ satisfies equation (3.15) and since $u(x, y, z)$ is real valued it follows from a simple power series argument that $U(X, Z, 0)$ is uniquely determined from $U(X, 0, Z^*) = F^{(1)}(X, Z^*)$. Since $V(X, Z, Z^*)$ is known, we now have from equation (3.15) and Hormander's generalized Cauchy–Kowalewski theorem [**10**; p. 116–119] that $U(X, Z, Z^*)$ is completely determined.

THEOREM 3.3. *Let $u(x, y, z)$ be a real valued C^4 solution of equation* (1.1) *in some neighbourhood of the origin. Then there exists an ordered pair of analytic functions of two complex variables $(f^{(1)}, f^{(2)})$, where $f^{(j)}(\mu, \zeta)$, $j = 1, 2$, are regular for μ in some neighbourhood of the origin and $|\zeta| < 1 + \varepsilon$, $\varepsilon > 0$, such that locally the representation* (3.14) *is valid. In particular, we have*

$$f^{(j)}(\mu, \zeta) = -\frac{1}{2\pi}\int_\gamma g^{(j)}(\mu(1 - t^2), \zeta)\frac{dt}{t^2}, \tag{3.17}$$

where

$$g^{(j)}(\mu, \zeta) = 2\frac{\partial}{\partial\mu}\left[\mu\int_0^1 F^{(j)}(t\mu, (1 - t)\mu\zeta)\,dt\right] - F^{(j)}(\mu, 0), \tag{3.18}$$

and $F^{(j)}(\mu, \zeta)$, $j = 1, 2$, are defined in Lemma 3.1. In equation (3.17) γ is a rectifiable arc joining the points $t = -1$ and $t = +1$ and not passing through the origin.

Remark. It can be shown that $g^{(j)}(\mu, \zeta)$, $j = 1, 2$, can be expressed in terms of $f^{(j)}(\mu, \zeta)$ by the formula [**1**; p. 15]:

$$g^{(j)}(\mu, \zeta) = \int_{-1}^{+1} f^{(j)}(\mu(1 - t^2), \zeta) \frac{dt}{\sqrt{(1 - t^2)}}. \tag{3.19}$$

Proof of theorem. From our previous analysis we know that equation (3.14) is a solution of equation (1.1) for arbitrary analytic functions $f^{(1)}$ and $f^{(2)}$. Evaluating (3.14) at $Z = 0$ gives

$$F^{(1)}(X, Z^*) = (\operatorname{Re} \mathbf{P}_3^{(1)}\{f^{(1)}\})_{Z=0}, \tag{3.20}$$

and from §2 we have $f^{(1)}$ given by equations (3.17) and (3.18) with $j = 1$. Applying the operator

$$\frac{\partial^2}{\partial X^2} - \frac{\partial}{\partial Z\, \partial Z^*} + C^{(1)} = \Delta_3 + c^{(1)}$$

to the right-hand side of equation (3.14) gives (from Theorem 3.1)

$$F^{(2)}(X, Z^*) = (\operatorname{Re} P_3^{(2)}\{f^{(2)}\})_{Z=0}, \tag{3.21}$$

and again equations (3.17) and (3.18), $j = 2$, follow from §2. This completes the proof of the theorem.

Example 3.1. In the special case when $c^{(1)} = c^{(2)} = 0$, Theorems 3.1–3.3 imply that every real valued C^4 solution of the biharmonic equation $\Delta_3^2 u = 0$ can be represented as

$$\begin{aligned} u(x, y, z) = \operatorname{Re}\Bigg\{ & \int_{|\zeta|=1} \int_{-1}^{+1} f^{(1)}(\mu(1 - t^2), \zeta) \frac{dt}{\sqrt{(1 - t^2)}} \frac{d\zeta}{\zeta} \\ & + \int_{|\zeta|=1} \int_{-1}^{+1} 2Z\zeta f^{(2)}(\mu(1 - t^2), \zeta) \frac{dt}{\sqrt{(1 - t^2)}} \frac{d\zeta}{\zeta} \Bigg\} \\ = \operatorname{Re}\Bigg\{ & \int_{|\zeta|=1} g^{(1)}(\mu, \zeta) \frac{d\zeta}{\zeta} + 2Z \int_{|\zeta|=1} g^{(2)}(\mu, \zeta)\, d\zeta \Bigg\}, \end{aligned} \tag{3.22}$$

where $f^{(j)}(\mu, \zeta)$ and $g^{(j)}(\mu, \zeta)$; $j = 1, 2$, are analytic functions of two complex variables (see §2 and equation (3.19)). But from [**1**; p. 43], it is seen that the integrals in the second line of equation (3.22) are harmonic functions of x, y, and z. Hence the representation (3.22) becomes

$$u(x, y, z) = \operatorname{Re}\{h^{(1)}(x, y, z) + (y + iz)\, h^{(2)}(x, y, z)\}, \tag{3.23}$$

where $h^{(j)}(x, y, z)$, $j = 1, 2$, are (possibly complex valued) harmonic functions. It is of interest to compare the representations (3.22) and (3.23) with that for biharmonic functions in two independent variables (*c.f.* [**8**; p. 269], [**12**; p. 175–179]).

We now use Theorem 3.3 to construct a complete family of solutions in the maximum norm for equation (1.1). The proof of this result is almost identical to that for the case of second order equations, and the reader is referred to Theorem 3.2 of [3] for further details. Briefly, the proof is based on the fact that since equation (1.1) is elliptic and has analytic coefficients, it possesses the unique continuation property and hence the Runge approximation property [2]. Hence it suffices to find a complete family of (real valued) solutions defined in an arbitrarily large sphere S in Euclidean three space R^3. By using the results of Garabedian on Cauchy's problem for analytic systems [8; p. 614–621] we can construct entire solutions of equation (1.1) with polynomial Cauchy data which approximate solutions in S. Such solutions can in turn be approximated in S by entire solutions where the functions $F^{(j)}(X, Z^*)$, $j = 1, 2$, are polynomials, and hence the following theorem (In the theorem below "Im" denotes "take the imaginary"):

THEOREM 3.4. *Let G be a bounded, simply connected domain in R^3, and define*

$$\left.\begin{aligned} u_{2n,m}^{(1)}(x,y,z) &= \operatorname{Re} \mathbf{P}_3^{(1)}\{\mu^n \zeta^m\} \\ u_{2n+1,m}^{(1)}(x,y,z) &= \operatorname{Im} \mathbf{P}_3^{(1)}\{\mu^n \zeta^m\} \\ u_{2n,m}^{(2)}(x,y,z) &= \operatorname{Re} \mathbf{T}_3\{\mu^n \zeta^m\} \\ u_{2n+1,m}^{(2)}(x,y,z) &= \operatorname{Im} \mathbf{T}_3\{\mu^n \zeta^m\} \end{aligned}\right\} \tag{3.24}$$

where $0 \leqslant n < \infty$, $m = 0, 1, \ldots, n$. Then the set $\{u_{nm}^{(1)}\} \cup \{u_{nm}^{(2)}\}$ is a complete family of solutions in the maximum norm for equation (1.1) *in the space of real valued C^4 solutions of equation* (1.1) *defined on G.*

4. *Generalizations.* The techniques used in this paper can also be applied to other classes of decomposable equations, for example to fourth order equations in four independent variables using the results of [6], to fourth order equations with spherically symmetric coefficients in $p + 2$ variables using the "method of ascent" [9], to more general fourth order equations in three variables using the results of [4], and finally, by repeated applications of the methods described in this paper, to equations of order greater than four.

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ON THE NUMERICAL TREATMENT OF PARTIAL DIFFERENTIAL EQUATIONS BY FUNCTION THEORETIC METHODS*

Robert P. Gilbert and David L. Colton**

1. Introduction

In the second section of this paper we shall show how constructive analytic methods may be used to solve boundary value problems for linear elliptic equations. We discuss in particular partial differential equations with analytic coefficients in two and three dimensions. The case in which the coefficients are only required to be smooth can also be handled if we first approximate these coefficients by polynomials.

In the third section we indicate how a direct numerical procedure is generated by an analytic procedure. This is done in detail for an n-dimensional equation with smooth, radially symmetric coefficients. Bounds are given on the discretization, and iterative errors. This approach

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is quite competative with the finite difference method, as was shown by the numerical example contained in the appendix of a previous paper [24].

In the fourth and fifth sections we investigate iterative methods for solving boundary value problems associated with semilinear equations in two dimensions. Our procedure depends on known methods for approximating the kernel function [8] and on the theory of generalized analytic functions [35]. Also in this regard see [24] page 347. It appears that these results can be extended to n-dimensions; however, we shall report on this at a later date.

2. Boundary Value Problems: Analytic Methods

In order to present our numerical schemes for solving boundary value problems we must first develop some analytic procedures for formulating these problems. In this section we do this for the following elliptic equations:

(2.1)

$$\underset{\sim}{\mathcal{E}}_2[u] \equiv \Delta_2 u + a(x,y)u_x + b(x,y)u_y - c(x,y)u = 0 ,$$

$$\underset{\sim}{E}_3[u] \equiv \Delta_3 u - F(x,y,z)u = 0 , \tag{2.2}$$

$$\underset{\sim}{e}_n[u] \equiv \Delta_n u - B(r^2)u = 0 , \tag{2.3}$$

$$\text{with} \quad r = |\underset{\sim}{x}| , \quad \underset{\sim}{x} \equiv (x_1, \ldots, x_n) .$$

Here $\Delta_n \equiv \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$, and the functions $c(x,y)$, $F(x,y,z)$, $B(r^2)$ are to be non-negative in their respective (real) domains of definition, D. For the purposes of simplicity of exposition we shall further assume, initially that these functions have entire extensions to $\mathbb{C}^2$, $\mathbb{C}^3$, and $\mathbb{C}$ respectively. Later we can relax these conditions to merely requiring that the coefficients be smooth.

The domains for which we shall pose the boundary value problems are to be bounded, star-like with respect to the origin, and have a Lyapunov boundary. We shall refer to such domains as being appropriate.

The approach we use is to derive a general integral representation for the solutions of each equation, and to then use this representation to formulate the respective boundary value problem as either a Fredholm integral equation, or to develop a complete family of functions for the purposes of approximating the solution. For Equation (2.1) the Fredholm integral equation method was developed by Vekua [34] via the theory of singular integral equations [29]. Our contribution is to develop an efficient iterative procedure for solving this equation by using Bergman's method for constructing [6] the integral representation. The approach to equations (2.2) and (2.3) is due to one of us [26], [22], [23] and is new.

Once the boundary value problems of Equations (2.1), (2.2), (2.3) have been reduced formally to a Fredholm equation, we are in the domain of numerical analysis. For instance, we are now faced with the various problems of numerically solving the integral equations which arise from the general integral representations. We shall discuss this in detail later for the case of equation (2.3).

A. The Equation $\underset{\sim}{\mathcal{E}}_2[u] = 0$:

As mentioned above we assume that the coefficinets of $\underset{\sim}{\mathcal{E}}_2[u] = 0$ have analytic continuation to $\mathbb{C}^2$ (or at least to $[D + \partial D] \times [D^* + \partial D^*]$), which permits us to transform (2.1) to a complex-valued hyperbolic equation,

$$(2.4) \qquad U_{ZZ^*} + AU_Z + BU_{Z^*} + CU = 0 \ , \quad B = \bar{A}$$

$$U(Z,Z^*) \equiv u\left(\frac{Z+Z^*}{2}, \frac{Z-Z^*}{2i}\right),$$

$$Z = x + iy \ , \quad Z^* = x - iy \ ,$$

$(Z,Z^*) \in [D + \partial D] \times [D^* + \partial D^*]$; see [6], [20], and [21]. Bergman [6] has given the following integral representation for the solutions of (2.4),

(2.5)

$$U(Z,Z^*) = \exp\left[-\int_0^{Z^*} A(Z,s)ds\right] \cdot \left[g(Z) + \sum_{n\geq 1} \frac{Q^{(n)}(Z,Z^*)}{2^{2n}B(n,n+1)} \int_0^Z \int_0^{Z_1} \cdots \int_0^{Z_n} g(Z_n)dZ_n \ \ldots \ dZ_1\right],$$

where $g(Z)$ is taken to be analytic in $D + \partial D$. The functions $Q^{(n)}(Z,Z^*)$ are defined by

$$Q^{(n)}(Z,Z^*) = \int_0^{Z^*} P^{(2n)}(Z,s)ds \ , \tag{2.6}$$

and the $P^{(2n)}(Z,Z^*)$ are defined recursively by the system,

$$P^{(2)} \equiv -2F \equiv -2(A_Z-AB+C) \ , \tag{2.7}$$

and

(2.8)

$$(2n+1)P^{(2n+2)} = -2 \left[P_Z^{(2n)} + \left\{ \Phi' - \int_0^{Z^*} A_Z dZ^* + B \right\} P^{(2n)} + F \int_0^{Z^*} P^{(2n)} dZ^* \right] , \quad n \geq 1 \ .$$

Here $\Phi = \Phi(Z)$ is an arbitrary analytic function of one complex variable in $D + \partial D$. Another representation of solutions to (2.4) is given by

$$U(Z,Z^*) = \exp\left[- \int_0^{Z^*} A(Z,\zeta)d\zeta \right] \left[g(Z) + \sum_{n\geq 1} \frac{Q^{(n)}(Z,Z^*)}{2^{2n}B(n,n+1)} \int_0^{Z} (Z-\zeta)^{n-1} g(\zeta)d\zeta \right] . \tag{2.9}$$

When $g(0) = 0$, this solution is identical to (2.5). It is straightforward to show that the representation (2.9) is identical with Vekua's representation, in terms of the complex Riemann function $R(t, t^*; Z, Z^*)$, namely

(2.10)

$$U(Z, Z^*) = R(Z, 0; Z, Z^*)g(Z) + \int_0^Z \Big\{ -R_1(t, 0; Z, Z^*) + B(t, 0)R(t, 0; Z, Z^*) \Big\} \, g(t)dt \, .$$

It is well known [29], [34] that if $g(Z)$ is holomorphic in D and also in the Hölder class, $\mathcal{H}^{\alpha}(D + \partial D)$, $0 < \alpha \leqslant 1$, then it has a representation as

$$(2.11) \qquad G(Z) = \int_{\partial D} \frac{t\mu(t)ds}{t - Z} \, , \quad Z \in D \, ,$$

where $\mu(t)$ is real valued and Hölder continuous. It is this device which Vekua exploits to rearrange (2.10) into a Fredholm equation for a density $\mu(t)$ such that $U(Z, \bar{Z}) = u(x, y)$ satisfies the Dirichlet data $u(x, y) = f(Z)$, $Z \in \partial D$. Since the Bergman solution (2.9) is identical to (2.10), we may exploit the already proved result of Vekua [34] that the resulting integral equation of the second kind is invertible. Furthermore, the Bergman formulation suggests an iterative procedure for solving this equation. Substituting (2.11) into (2.9) and inverting orders of integration, which is permissible, leads to [26]

(2.12)

$$u(x,y) = \mathrm{Re} \left\{ \hat{H}_0(Z) \int_{\partial D} t\mu(t) \left[\frac{1}{t-Z} + \sum_{n=1}^{\infty} \frac{Q^{(n)}(Z,\bar{Z})}{2^{2n}B(n,n+1)} \int_0^Z \frac{(Z-\zeta)^{n-1}}{t - \zeta} \, d\zeta \right] ds \right\}$$

where $\hat{H}_o(Z) \equiv \exp\left[- \int_0^{\bar{Z}} A(Z, \sigma)\,d\sigma \right]$. (Note, that in what follows, $\hat{H}_o(Z)$ etc. does not mean $\hat{H}_o$ is an analytic function of Z, rather it is a function of the point Z.) By computing the residue as $Z \to t_o \in \partial D$, one obtains the singular integral equation for the density $\mu(t)$, namely [26]

(2.13)

$$f(t_o) = \mathrm{Re}\left\{ \hat{H}_o(t_o) \left[\pi i t_o \mu(t_o) \bar{t}_o' + \int_{\partial D} t\mu(t) \left(\frac{1}{t-t_o} + \sum_{n=1}^{\infty} \frac{Q^{(n)}(t,t_o)}{2^{2n}B(n,n+1)} \int_0^{t_o} \frac{(t_o-\zeta)^{n-1}}{t-\zeta}\,d\zeta \right) \right] ds \right\}$$

where $t'(x) \equiv \frac{dt}{ds}$. An alternate form of this equation is

(2.14)

$$f(t_o) = A(t_o)\mu(t_o) + \frac{B(t_o)}{i\pi} \int_{\partial D} \frac{\mu(t)\,dt}{t-t_o} + \int_{\partial D} F_1(t_o,t)\mu(t)\,ds + \int_{\partial D} F_2(t_o,t)\mu(t)\,ds ,$$

(2.15) where $A(t_o) \equiv \mathrm{Re}\left[\pi i\ t_o\ \overline{t_o'}\ \hat{H}_o(t_o) \right]$,

$$i\pi \mathrm{Re}\left[t_o\ \overline{t_o'}\ \hat{H}_o(t) \right] ,$$

(2.16) $$F_1(t_o, t) = \mathrm{Re}\left\{ \frac{t\hat{H}_o(t_o)}{t - t_o} \right\} - \frac{t'B(t_o)}{i\pi(t-t_o)}$$

and

(2.17)

$$F_2(t_o,t) = \text{Re}\left\{ t \sum_{n=1}^{\infty} \frac{\hat{H}_o(t_o)Q^{(n)}(t_o,\bar{t}_o)}{2^{2n}B(n,n+1)} \int_0^{t_o} \frac{(t_o-\zeta)^{n-1}}{t-\zeta}\, d\zeta \right\}.$$

By using the Poincare-Bertrand formula [29] one may reduce (2.14) to the form of a Fredholm equation [26].

$$\mu(t_o) + \int_{\partial D} K(t_o,t)\mu(t)ds = F(t_o) \,, \tag{2.18}$$

with

$$F(t_o) \equiv \frac{1}{\pi^2|t_o|^2|\hat{H}_o(t_o)|^2}\left[A(t_o)f(t_o) - \frac{B(t_o)}{\pi i}\int_{\partial D} \frac{f(t)}{t-t_o}\, dt \right] , \tag{2.19}$$

and

(2.20)

$$K(t_o,t) \equiv \frac{1}{\pi^2|t_o|^2|\hat{H}_o(t_o)|^2}\left[A(t_o)\left(\text{Re}\left\{\frac{t\hat{H}_o(t_o)}{t-t_o}\right\} - \frac{t'B(t_o)}{i\pi(t-t_o)} \right) + A(t_o)F_2(t_o,t) - \frac{B(t_o)}{\pi i}\int_{\partial D} \text{Re}\left\{\frac{t\hat{H}_o(t)}{t-\tau}\right\}\frac{d\tau}{\tau-t_o} - \frac{B(t_o)}{\pi i}\int_{\partial D} \frac{F_2(\tau,t)d\tau}{\tau-t_o} \right].$$

An iterative procedure may be obtained for solving (2.18) by replacing $F_2(\tau, t)$ in (2.20) by truncating its series representation (2.17). The series (2.17) converges rapidly in general, and this leads to a useful method. For equation (2.3) when $n = 2$, we have a special case of (2.1). We postpone to that case the numerical treatment of the corresponding integral equations. Numerical results have been published for this problem in [24].

B. <u>The Equation</u> $\underset{\sim}{E}_3[u] = 0$:

In her thesis Bwee Lan Tjong [33] obtained a generalization of the Bergman-Whittaker operator to the case of the equation $\underset{\sim}{E}_3[u] = 0$.[†] It is the following integral representation:

(2.21)

$$u(x,y,z) \equiv \psi(X,Z,Z^*) = \underset{\sim}{T}f$$
$$\equiv \int_{|\zeta|=1} \int_{\gamma} E(X,Z,Z^*,\zeta,t) f(w,\zeta) \frac{dt}{\sqrt{1-t^2}} \frac{d\zeta}{\zeta} ,$$

where γ is a rectifiable curve from $t = -1$ to $+1$. Here the variables X, Z, Z^*, and w are defined as $X = x$, $Z = \frac{1}{2}[y + iz]$, $Z^* = \frac{1}{2}[-y + iz]$

[†]It was subsequently shown by Colton and Gilbert [13] that it was possible to find an analogous operator for the four-dimensional equation,

$$\Delta_4\psi - F(\underset{\sim}{x})\psi = 0 , \qquad \underset{\sim}{x} = (x_1,x_2,x_3x_4) .$$

In this case this operator generalizes the integral operator $\underset{\sim}{G}_4$; see [22] for further details concerning $\underset{\sim}{G}_4$.

(2.22) $$w = (1-t^2)u \;, \quad \text{with} \quad u = X + \zeta Z + \zeta^{-1}Z^* \;.$$

Furthermore, the kernel $E(X,Z,Z^*,\zeta,t)$ satisfies the partial differential equation,

(2.23) $$ut\left(\frac{\partial^2\hat{E}}{\partial\xi_1^2} + \frac{\partial^2\hat{E}}{\partial\xi_2^2} + \frac{\partial^2\hat{E}}{\partial\xi_3^2} + 2\frac{\partial^2\hat{E}}{\partial\xi_1\partial\xi_2} + 2\frac{\partial^2\hat{E}}{\partial\xi_1\partial\xi_3} - 2\frac{\partial^2\hat{E}}{\partial\xi_2\partial\xi_3} + \hat{F}\hat{E}\right) + (1-t^2)\frac{\partial^2\hat{E}}{\partial\xi_1\partial t} - \frac{1}{t}\frac{\partial\hat{E}}{\partial\xi_1} = 0 \;,$$

where $\hat{E}$ is obtained from E under the change of X, Z, Z^* - variables to $\xi_1 = X$, $\xi_2 = X + 2\zeta Z$, $\xi_3 = X + 2\zeta^{-1}Z^*$. She also gives a series representation for $\hat{E}$ of the form

(2.24) $$\hat{E}(\xi_1, \xi_2, \xi_3, \zeta, t) = 1 + \sum_{n\geq 1} t^{2n}u^n p^{(n)}(\underset{\sim}{\xi}; \zeta) \;,$$

where the $p^{(n)}(\underset{\sim}{\xi}; \zeta)$ satisfy the differential equations

(2.25) $$p_1^{(n+1)} = -\frac{1}{2n+1}\left\{p_{11}^{(n)} + p_{22}^{(n)} + p_{33}^{(n)} + 2p_{12}^{(n)} + 2p_{13}^{(n)} - 2p_{23}^{(n)} + \hat{F}p^{(n)}\right\}$$

with $p^{(n+1)}(0, \xi_2, \xi_3, \zeta) = 0$, which gives rise to an alternate representation for solutions,

(2.26)

$$\psi = \frac{1}{2\pi i} \int_{|\zeta|=1} g(u,\zeta) \frac{d\zeta}{\zeta} +$$

$$+ \sum_{n\geq 1} \frac{1}{2\pi i B(n,\frac{1}{2})} \int_{|\zeta|=1} \left\{ p^{(n)}(\xi,\zeta) \int_0^u (u-s)^{n-1} g(s,\zeta)ds \right\} \frac{d\zeta}{\zeta} ,$$

where $g(u,\zeta) = \int_\gamma f(w,\zeta) \frac{dt}{\sqrt{1-t^2}}$, and γ is an arc from -1 to +1.

It was shown by Gilbert and Lo [26] that the solution having the representations (2.21, 2.26) was a general solution, i.e. any solution of (2.2) which is in $C^1(D+\partial D)$, where D is appropriate, has such a representation. This was done by making use of an inversion of the Bergman-Whittaker operator,

$$H(X,Z,Z^*) = (\underset{\sim}{B}_3 g)(X,Z,Z^*) \tag{2.27}$$

$$\equiv \frac{1}{2\pi i} \int_{|\zeta|=1} g(u,\zeta) \frac{d\zeta}{\zeta} ,$$

namely [26], $\quad g(u,\zeta) = (B_3^{-1}H)(u,\zeta)$

$$(B_3^{-1}H)(u,\zeta) \equiv \frac{1}{2\pi} \int_{\partial D_0} \rho(\underset{\sim}{Y}) \frac{1}{\underset{\sim}{N}\cdot(\underset{\sim}{X}-\underset{\sim}{Y})} d\omega_y ; \tag{2.28}$$

here $\underset{\sim}{N}(\zeta)$ is the isotropic vector introduced earlier by the definition $u = \underset{\sim}{N} \cdot \underset{\sim}{X}$, and $\rho(\underset{\sim}{Y})$ is the single-layer desnity which generates the potential $H(\underset{\sim}{X})$ for $\underset{\sim}{X} \in D_0 \subset\subset D$. Using (2.28) in (2.26) yields

$$\psi(\underset{\sim}{X}) = H(\underset{\sim}{X}) + \int_{\partial D_0} \rho(\underset{\sim}{Y})K(\underset{\sim}{X},\underset{\sim}{Y})d\omega_y \ , \tag{2.29}$$

(2.30) where $$K(\underset{\sim}{X},\underset{\sim}{Y}) \equiv \frac{1}{2\pi i}\int_{|\zeta|=1} P(\underset{\sim}{X},\underset{\sim}{Y};\zeta)\frac{d\zeta}{\zeta} \ ,$$

$$P(\underset{\sim}{X},\underset{\sim}{Y};\zeta) \equiv \sum_{n\geq 1}\frac{1}{B(n,\frac{1}{2})}\, p^{(n)}(\xi;\zeta)\Phi_n(u;\ \underset{\sim}{N}\cdot\underset{\sim}{Y}) \ , \tag{2.31}$$

and

(2.32)

$$\Phi_n(u;\ \underset{\sim}{N}\cdot\underset{\sim}{Y}) = [(\underset{\sim}{N}\cdot\underset{\sim}{Y}-u)^{n-1} - (\underset{\sim}{N}\cdot\underset{\sim}{Y})^{n-1}]\log(u-\underset{\sim}{N}\cdot\underset{\sim}{Y})$$
$$- \sum_{\nu=1}^{n-1}\binom{n-1}{\nu}\frac{(\underset{\sim}{N}\cdot\underset{\sim}{Y})^{n-\nu-1}}{\nu} \ .$$

The functions $\Phi_n(u;\ \underset{\sim}{N}\cdot\underset{\sim}{Y})$ are <u>universal functions</u>, and the kernel $K(\underset{\sim}{X},\underset{\sim}{Y})$ is <u>fixed for each particular differential equation</u> (2.2).

It is easy to show that the integral in (2.29) is a compact operator on the class of functions $\mathcal{C}[\partial D_0]$. Furthermore, if we formulate the Neumann problem for (2.2) with $\frac{\partial\psi}{\partial\nu_x} = f(\underset{\sim}{X})$, $\underset{\sim}{X} \in \partial D_0$, we are led to the Fredholm integral equation

$$f(\underset{\sim}{X}) = -\rho(\underset{\sim}{X}) + \frac{1}{2\pi}\int_{\partial D_0} \rho(Y)\frac{\partial}{\partial\nu_x}\left(\frac{1}{|\underset{\sim}{X}-\underset{\sim}{Y}|}\right)d\omega_y \tag{2.33}$$
$$+ \int_{\partial D_0} \rho(\underset{\sim}{Y})\frac{\partial}{\partial\nu_x}K(\underset{\sim}{X},\underset{\sim}{Y})d\omega_y \ ,$$

which we can show, following the discussion in Garabedian [20] for Laplace's equation, is uniquely soluble.

We now turn to the Dirichlet problem associated with (2.2), and we assume that the data is sufficiently smooth [20] page 347. Let us consider the class of all harmonic functions that are in $\mathfrak{C}[D + \partial D]$, and let us designate this class by $\mathcal{H}[D]$. It has recently been shown by du Plessis [30] that the harmonic polynomials are complete for simply connected domains in $\mathbb{R}^n$ with respect to the uniform norm. With this in mind we wish to obtain a complete system of solutions for $\underset{\sim}{E}[u] = 0$, [14], [26]. One method of doing this is by means of the representation† [26] $\psi(\underset{\sim}{X}) = (\hat{\underset{\sim}{T}}H)(\underset{\sim}{X})$,

(2.34)
$$(\hat{\underset{\sim}{T}}H)(\underset{\sim}{X}) \equiv H(X) - \frac{a}{4\pi^3 i} \sum_{n\geq 1} \frac{1}{B(n,\frac{1}{2})} \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta'$$
$$\cdot \left\{ H(\underset{\sim}{X}' a^2/R^2) \int_{|\zeta|=1} p^{(n)}(\xi;\zeta) D(u,\hat{u};n) \frac{d\zeta}{\zeta} \right\},$$

where $\hat{u} \equiv X' + \zeta(1 - \frac{1}{\alpha})Z' + \zeta^{-1}(1 - \frac{1}{\alpha})^{-1} Z^{*\prime}$, and the coefficients $D(u, \hat{u}; n)$ are defined as

$$(2.35) \qquad D(u, \hat{u}; n) \equiv \int_0^u (u-s)^{n-1} A(s,\hat{u}) \partial s \ ,$$

---

†This representation comes about by making use of an inversion of $\underset{\sim}{B}_3^{-1}$ which is developed in [26]; see also [22] for another representation of $\underset{\sim}{B}_3^{-1}$ when the harmonic functions are regular at infinity.

with

$$(2.36) \qquad A(s,\hat{u}) \equiv \int_0^1 d\alpha \int_0^1 \frac{d\beta\sqrt{\beta}}{\sqrt{1-\beta}} \left[\frac{12s\alpha\beta(1-\alpha)\hat{u}+a}{(4s\alpha\beta(1-\alpha)\hat{u}-a)^3} \right] .$$

(Here a is chosen so that D is contained in a sphere of radius a.) Indeed, we have the

<u>Thoerem 2.1</u>: Let $h_{n,m}(\underset{\sim}{X})$, $(m = 0, \pm 1, \pm 2, \ldots, +n;$ $n = 0, 1, 2, \ldots)$ represent the spherical harmonics. Then the functions given via (2.35) by

(2.37)

$$\psi_{n,m}(\underset{\sim}{X}) \equiv (\hat{\underset{\sim}{T}}h_{n,m})(\underset{\sim}{X}) \ , \quad (m = 0, \pm 1, \ldots, \pm n;\ n = 0, 1, \ldots)$$

are a complete system of solutions for $\underset{\sim}{E}[u] = 0$, with respect to uniform convergence in D.

<u>Proof</u>: To see that this is true, let $\hat{\psi}(\underset{\sim}{X}) \in \mathbf{E}(D)^{\dagger}$ be a solution in D which does not lie in the space spanned by the $\psi_{n,m}(\underset{\sim}{X})$. <u>In</u> D, $\psi(\underset{\sim}{X}) \in C^{\infty}$, and hence there exists a harmonic function $H(\underset{\sim}{X}) \in \mathcal{H}[D_0]$, where $D_0 \subset\subset D$ is an arbitrary compact <u>appropriate</u> domain, such that $\hat{\psi}(\underset{\sim}{X}) = (\underset{\sim}{T}H)(\underset{\sim}{X}) \in \mathbf{E}[D_0]$. Furthermore, $\hat{H}(\underset{\sim}{X})$ may be uniformly approximated in D_0 by the harmonic polynomials. Hence, since D_0 is arbitrary, $\hat{\psi}(\underset{\sim}{X})$ may be uniformly approximated in D by the $\psi_{n,m}(X)$, a contradiction.

Given a complete system of solutions $\{\psi_{\nu}(\underset{\sim}{X})\}$ there are various procedures we can use for approximating boundary

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†The class of solutions of $\underset{\sim}{E}[u] = 0$ that are in $C[D+\partial D]$.

value problems. One such method is to approximate the boundary data by means of a linear combination,

$$f(\underset{\sim}{X}) \approx \sum_{j=1}^{N} c_j \psi_j(\underset{\sim}{X}) , \qquad \underset{\sim}{X} \in \partial D \tag{2.38}$$

such that $\max\limits_{X \in \partial D} |f - \sum\limits_{j=1}^{N} c_j \psi_j| < \varepsilon$, for $\varepsilon > 0$ suitably small. The maximum principle for $\underset{\sim}{E}[u] = 0$, then says the solution is within an ε of the approximate solution in $D + \partial D$.

Another approach is to introduce the Dirichlet inner-product [7], [8], [20], for $\underset{\sim}{E}[u] = 0$, namely

$$(\psi,\Phi) \equiv \int_D [\nabla\psi \cdot \nabla\Phi + F\psi\Phi]d\underset{\sim}{x} , \tag{2.39}$$

and to obtain an orthonormal system $\{\Phi_j(\underset{\sim}{X})\}$, by means of the Gram-Schmidt process. One then expands the data as a Fourier series,

$$f(\underset{\sim}{X}) \simeq \sum_{j=1}^{N} a_j \Phi_j(\underset{\sim}{X}) , \qquad \underset{\sim}{X} \in \partial D$$

$$a_j = (f,\Phi_j) = -\int_D f \frac{\partial \Phi_j}{\partial \nu} d\omega , \tag{2.40}$$

and this yields a solution, $\psi_N(\underset{\sim}{X}) = \sum\limits_{j=1}^{N} a_j \Phi_j(\underset{\sim}{X})$, to $\underset{\sim}{E}[u] = 0$, which approximates ψ with respect to the Dirichlet norm. Furthermore, for complete systems, the theory of the kernel function tells us that the Fourier Series (2.40) converge uniformly <u>in</u> D as $N \to \infty$.

Yet another approach is to compute a truncated kernel function

$$(2.41) \qquad K_N(\underset{\sim}{X},\underset{\sim}{Y}) = \sum_{j=1}^{N} \Phi_j(\underset{\sim}{X})\Phi_j(\underset{\sim}{Y}) \ ;$$

then $\psi_N(\underset{\sim}{X})$ may be written as

$$\psi_N(\underset{\sim}{X}) = (f(\underset{\sim}{Y}), K_N(\underset{\sim}{X},\underset{\sim}{Y})) \ ,$$

and one has the estimate, for an arbitrary $\varepsilon > 0$ and a sufficiently large N,

$$(2.42) \ |\psi(\underset{\sim}{X}) - \psi_N(\underset{\sim}{X})|^2 \leqslant \|\psi - \psi_N\|^2 \ K(\underset{\sim}{X},\underset{\sim}{X}) < \varepsilon K(\underset{\sim}{X},\underset{\sim}{X}) \ ,$$

$$\underset{\sim}{X} \in D \ .$$

A similar procedure holds for the Neumann problem; however, here the Fourier coefficients are given as

$$a_j = (\Phi_j,\psi) = -\int_{\partial D} \Phi_j \frac{\partial\psi}{\partial\nu} \, d\omega \ ,$$

where $\frac{\partial\psi}{\partial\nu}(\underset{\sim}{X}) = g(\underset{\sim}{X})$, $\underset{\sim}{X} \in \partial D$, is the Neumann data.

Integral operator techniques related to those above can also be used to construct global approximations to solutions of Cauchy's problem for elliptic and hyperbolic equations in three and four independent variables [13], [14].

C. The Equation $\underset{\sim}{e}_n[u] = 0$:

It may be shown by direct substitution that if $E(r, t; n)$ is a solution of

$$(2.43) \qquad (1-t^2)E_{rt} + (n-3)(t^{-1}-t)E_r + rt(E_{rr} + \frac{n-2}{r} E_r + BE) = 0 ,$$

which satisfies

$$(2.44) \qquad \lim_{t\to 0^+} (t^{n-3}E_r)r^{-1} = 0 , \quad \lim_{t\to 1^-} ((1-t^2)^{\frac{1}{2}}E_r)r^{-1} = 0$$

$$\lim_{r\to 0^+} E = 1 ,$$

then if $H(\underset{\sim}{x})$ is harmonic,

$$(2.45) \qquad u(\underset{\sim}{x}) = \int_0^1 t^{n-2} E(r,t;n) H(\underset{\sim}{x}[1-t^2]) \frac{dt}{(1-t^2)^{\frac{1}{2}}}$$

is a solution of $\underset{\sim}{e}_n(n) = 0$. The representation (2.45) may be reformulated in terms of a new harmonic function $h(\underset{\sim}{x})$ as

(2.46)
$$u(\underset{\sim}{x}) = (\underset{\sim}{I}+\underset{\sim}{G})h(\underset{\sim}{x}) \equiv h(\underset{\sim}{x}) + \int_0^1 \sigma^{n-1}G(r,1-\sigma^2)h(\underset{\sim}{x}\sigma^2)d\sigma$$

where

$$(2.47) \qquad h(\underset{\sim}{x}) \equiv \int_0^1 t^{n-2}H(\underset{\sim}{x}[1-t^2]) \frac{dt}{\sqrt{1-t^2}} ,$$

and $G(r, t)$ is a solution of the Goursat problem

$$2(1-t)G_{rr} - G_r + r(G_{rr} - BG) = 0 \tag{2.48}$$

$$G(0,t) = 0 \ , \quad G(r,0) = \int_0^r rB(r^2)dr \ .$$

It is an interesting fact that $G(r,t)$ is independent of the dimension n of D. It is a <u>more</u> interesting fact that $G(r,t)$ is related to the Riemann function of

$$\frac{\partial^2 U}{\partial Z \partial Z^*} - \frac{1}{4} B(ZZ^*)U = 0 \ ,$$

when $B(\zeta)$ is analytic, namely by [22], [23]

$$G(r,1-\sigma^2) \equiv -2r \ R_1(r\sigma^2,0;r,r) \ . \tag{2.49}$$

If $B(\zeta)$ is entire, then (2.45) is invertible for all <u>appropriate</u> domains in $\mathbb{R}^n$. This follows since (2.45) can be put in the form of a Volterra integral equation by a simple change of integration parameter. Actually, it is only necessary for $B(r^2) \in \mathcal{C}[0,a]$, where a is the radius of D, for (2.45) to be invertible. The following results were shown to be true in [24]:

<u>Theorem 2.2</u>. Let $\mathcal{H}[D]$ be the class of harmonic functions in $\mathcal{C}[D+\partial D]$, where $B(r^2) \geqslant 0$ and $B(r^2) \in \mathcal{C}[0,a]$ (i.e. $B(r^2) \in \mathcal{C}_+[0,a]$). Let $\mathcal{E}[D]$ be the class of solutions of (2.3) in $\mathcal{C}[D+\partial D]$. If D is appropriate, and if $u(\underset{\sim}{x}) \equiv (\underset{\sim}{I}+\underset{\sim}{G})h(\underset{\sim}{x})$, then $u \in \mathcal{E}[D]$ if and only if $h \in \mathcal{H}[D]$.

<u>Theorem 2.3</u>. If D is appropriate, and $B(r^2) \in \mathcal{C}_+^1[0,a]$, then the Dirichlet problem, $u \in \mathcal{E}[D]$, $u\big|_{\partial D} = f(\underset{\sim}{x}) \in \mathcal{H}[\partial D]$,

has a unique solution, which has a representation of the form $u(\underset{\sim}{x}) = (\underset{\sim}{I}+\underset{\sim}{G})h(\underset{\sim}{x})$, where $h(\underset{\sim}{x}) \in \mathcal{H}[D]$ and

$$(2.50)\quad h(\underset{\sim}{x}) = \frac{\Gamma(n/2)}{(n-2)\pi^{n/2}} \int_{\partial D} \mu(\underset{\sim}{y}) \frac{\partial}{\partial \nu_y} \left(\frac{1}{|\underset{\sim}{x}-\underset{\sim}{y}|^{n-2}} \right) d\omega_y \ .$$

The density $\mu(\underset{\sim}{x})$ is a solution of the Fredholm integral equation

$$(2.51)\quad f(\underset{\sim}{x}) = \mu(\underset{\sim}{x}) + \int_{\partial D} K(\underset{\sim}{x},\underset{\sim}{y})\mu(\underset{\sim}{y})d\omega_y \ , \qquad \underset{\sim}{x} \in \partial D \ .$$

with

$$(2.52)\qquad K(\underset{\sim}{x},\underset{\sim}{y}) \equiv \frac{\Gamma(n/2)}{(n-2)\pi^{n/2}} \left\{ \frac{\partial}{\partial \nu_y} \frac{1}{|\underset{\sim}{x}-\underset{\sim}{y}|^{n-2}} \right.$$

$$\left. + \int_0^1 \sigma^{n-1} G(r,1-\sigma^2) \cdot \frac{\partial}{\partial \nu_y} \left(\frac{1}{|\underset{\sim}{x}\sigma^2-\underset{\sim}{y}|^{n-2}} \right) d\sigma \right\}.$$

<u>Theorem 2.4</u>. Let D be appropriate, and $B(r^2) \in \mathcal{C}_+[0,a]$. Then the operator G is monotone in the sense of Collatz on $\mathcal{C}[D]$.

<u>Theorem 2.5</u>. Let $B(r^2) \in \mathcal{C}^1_+[0,a]$, and let $\psi(n;m_k;\underline{+};\underset{\sim}{x})$ be defined by the integrals,

(2.53)

$$\psi(n;m_k;\underline{+};\underset{\sim}{x}) = \int_0^1 t^{n-2} E(r,t;n) H(n;m_k;\underline{+},\underset{\sim}{x}[1-t^2]) \frac{dt}{\sqrt{1-t^2}} \ ,$$

where the $H(n;m_k;\underline{+};\underset{\sim}{x})$ are the homogeneous harmonic polynomials of degree n, $(n = 0, 1, 2, \ldots;\ 0 \leq m_{n-2} \leq \leq m_{n-3} \leq \ldots \leq m_1 \leq n)$, and

$$(2.54) \qquad E(r,t;n) \equiv 1 + t^2 \int_0^1 \sigma^{n-2}\, G(r,[1-\sigma^2]t^2)\,d\sigma \ .$$

Then the $\psi(n;m_k;\pm,\underset{\sim}{x})$ form a complete family of solutions of $\underset{\sim}{e}_n[u] = 0$, with respect to appropriate domains.

Proof: As mentioned before, the du Plessis theorem tells us the harmonic polynomials are complete in the uniform norm, for simply-connected domains. The remainder of the theorem comes about by realizing that the above representation is invertible. This follows from a formal identity involving $E(r,t;n)$, and $G(r,t)$. First, however, let us note that if $H(\underset{\sim}{x})$ is harmonic in an appropriate domain D, then so is $h(\underset{\sim}{x}) = (\underset{\sim}{J}H)(\underset{\sim}{x}) \equiv \int_0^1 t^{n-2} H(\underset{\sim}{x}[1-t^2]) \dfrac{dt}{\sqrt{1-t^2}}$.
From before, the function $u(\underset{\sim}{x}) = (\underset{\sim}{I}+\underset{\sim}{G})h(\underset{\sim}{x})$ is then seen to be harmonic in D also. Furthermore, if $u(\underset{\sim}{x}) \in \mathcal{E}[D]$, then $h(\underset{\sim}{x}) \in \mathcal{H}[D]$, which follows from Theorem 2.2. Actually, in [23] the inverse operator $(\underset{\sim}{I}+\underset{\sim}{G})^{-1}$ is given as a Volterra integral operator. That $H(\underset{\sim}{x})$ is also in $\mathcal{H}[D]$ follows from the claim: a representation for J^{-1} is given by

(2.55)

$$(J^{-1}h)(x) \equiv i^n \frac{(n-1)}{\pi} \int_{\mathcal{L}^{-1}}^{+1} t^{-n}\, h(\underset{\sim}{x}[1-t^2])(1-t^2)^{n/2-1}\, dt \ ,$$

where $\mathcal{L}$ is a rectifiable arc from -1 to +1 which does not pass through the origin.

It follows if $u(\underset{\sim}{x}) \in \mathcal{E}[D]$, then the harmonic function $H(\underset{\sim}{x})$, which is the preimage of $u(\underset{\sim}{x})$ under the mapping $(\underset{\sim}{I}+\underset{\sim}{G})\underset{\sim}{J}$, is uniquely determined by the integral representation $H(\underset{\sim}{x}) = [\underset{\sim}{J}^{-1}(\underset{\sim}{I}+\underset{\sim}{G})^{-1}u](\underset{\sim}{x})$. Furthermore, $H(\underset{\sim}{x}) \in \mathcal{C}[D+\partial D]$. From this it follows directly that the functions

$$(2.56) \qquad \Phi(n;m_k;\underline{+};\underset{\sim}{x}) \equiv (\underset{\sim}{I}+\underset{\sim}{G})\underset{\sim}{J}\; H(n;m_k;\underline{+};\underset{\sim}{x})$$

form a complete family of solutions for $\underset{\sim}{e}_n[u] = 0$ for appropriate domains. It remains for us to identify these functions with the $\psi(n;m_k;\underline{+};\underset{\sim}{x})$. To this end we first show that $E(r,t)$ is given by the above integral identity (2.54). Substitution into equation (2.43) for the E-functions reveals, after a few manipulations plus integration by parts, that it is indeed a solution of this partial differential equation. We leave this for the reader to verify. We must next show that the function defined by (2.54) satisfies the boundary data (2.44). Since $E_r t^{n-3} r^{-1} = t^{n-1} r^{-1} \int_0^1 \sigma^{n-2} G_r(r,[1-\sigma^2]t^2)d\sigma$, we have that $\lim_{t\to 0^+} (E_r t^{n-3} r^{-1}) = \lim_{t\to 0^+} t^{n-1} \frac{B(r^2)}{n-1} = 0$ for $n \geqslant 2$. Likewise, one has

(2.57)

$$\lim_{t\to 1^-} [(1-t^2)^{\frac{1}{2}} E_r r^{-1}] = \lim_{t\to 1^-} (1-t^2)^{\frac{1}{2}} r^{-1} \int_0^1 \sigma^{n-2} G_r(r,1-\sigma^2)\, d\sigma =$$

$$= 2 \lim_{t\to 1^-} (1-t^2)^{\frac{1}{2}} \int_0^1 \sigma^{n-2} g_1(r^2, 1-\sigma^2)d\sigma = 0 ,$$

where $g(r^2,t) \equiv G(r,t)$ is clearly a $\mathcal{C}^1$ function of r^2. (See [24] for a proof that $G(r,t)$ is an even, $\mathcal{C}^1$-function of r.) Finally, using the initial condition $G(0,t) = 0$, it is immediate that $\lim_{r\to 0^+} E = 1$ from the representation (2.54). Consequently, the conditions (2.43-2.44) for the E-function are satisfied, and since (2.43-2.44) represents an over determined system, the E-function is then seen to be uniquely defined either by (2.54) and (2.48), or by (2.43-2.44).

3. Boundary Value Problems: Numerical Treatment

We next turn to the numerical evaluation of the G-function. There are several approaches to doing this. One such approach was mentioned earlier in a paper by one of us [24] for the special case when $B(r^2)$ was an analytic function of r^2. In this case it is known [23] that $G(r,t)$ may be found in the form

$$(3.1) \qquad G(r,t) = \sum_{\ell=1}^{\infty} c_\ell(r^2)t^{\ell-1} ,$$

where the expansion coefficients $c\ (r^2)$ satisfy the following recursion formulae

(3.2)

$$c_1'(r^2) = rB(r^2) , \qquad \left(\frac{d}{dr} \equiv {}' \right)$$

$$2\ell c_{\ell+1}'(r^2) = -rc_\ell''(r^2) + (2\ell-1)c_\ell'(r^2) + rP(r^2)c_\ell(r^2) \ .$$

$$\ell \geqslant 1 \ .$$

An approximate G-function may then be found by the truncated series

(3.3) $$G_N(r,t) \equiv \sum_{\ell=1}^{N} c_\ell(r^2)t^{\ell-1} \ .$$

It is easy to show that $G_N(r,t) \to G(r,t)$ uniformly on $[0,a] \times [0,1]$.

In order to compute $G(r,t)$ it is sometimes advantageous to introduce a change of variables, $w(\rho,t) = (1-t)G(r,t)$, $\rho = r\sqrt{1-t}$, $\tau = t$, and to extend the definition of $B(r^2)$ to $[0,\infty)$ by the scheme [24]

(3.4)

$$\tilde{B}(r^2) \equiv \begin{cases} B(r^2), & 0 \leqslant r \leqslant a \\ B(a^2)\exp\left[-\dfrac{\varepsilon^2}{\varepsilon^2-(r-a)^2}\right], & a \leqslant r \leqslant a+\varepsilon \\ 0 \ , & r > a+\varepsilon \ , \end{cases}$$

where $\varepsilon > 0$ is taken arbitrarily small. The differential equation for $w(\rho,\tau)$ is then

(3.5) $$\frac{\partial^2 w}{\partial\rho\partial\tau} = \frac{1}{2}\,\frac{\rho\,\tilde{B}(\frac{\rho^2}{1-\tau})}{(1-\tau)^2}\,w \ ,$$

and it satisfies the data $w(0,\tau) = 0$, $w(\rho,0) = \int_0^\rho \rho B(\rho^2)d\rho$.

In the case where $B(r^2)$ is not analytic, but merely $\mathcal{C}^1[0,a]$ various procedures are useful. One possibility is to approximate $B(r^2)$ by a polynomial and use the series procedure. Another such method is to use the finite difference Riemann function approach as developed in an interesting paper by Aziz and Hubbard [5]. We outline this latter method below. Let $R \equiv [0,a] \times [0,1]$ be a closed rectangle in the x, y plane, and let R_k be the set of grid points (mk, nk), m, n being positive integers. The real number k is the mesh constant. The "Goursat data" is to be given on the characteristic, mesh-point-surfaces, $(mk,0)$ and $(0,nk)$. Following Aziz and Hubbard we introduce a mesh Riemann function for the finite difference equation,

$$L_k U \equiv U_{XY} + AU_X + BU_Y + CU = f(x,y) , \tag{3.6}$$

where $x = mk - \frac{1}{2}k$, $y = nk - \frac{1}{2}k$,

$$U_X \equiv k^{-1}\left[U(x + \frac{1}{2}k,y) - U(x - \frac{1}{2}k,y) \right] , \tag{3.7}$$

$$U_Y \equiv k^{-1}\left[U(x,y + \frac{1}{2}) - U(x,y - \frac{1}{2}k) \right] , \tag{3.8}$$

(3.9)

$$U_{XY} \equiv k^{-2}\left[U(x + \frac{1}{2}, y + \frac{1}{2}k) - U(x + \frac{1}{2}k, y - \frac{1}{2}k) - U(x - \frac{1}{2}k, y + \frac{1}{2}k) + U(x - \frac{1}{2}k, y - \frac{1}{2}k) \right] .$$

Here, the functions have been extended to the "half-mesh-points" by an averaging procedure, [5]. The mesh Riemann function is the solution of the adjoint finite difference problem

(3.10)

$$\underset{\sim}{L}_k^* V(x,y) \equiv V_{XY} - (AV)_X - (BV)_Y + CV = 0 \ ,$$

$$0 < x = mk - \frac{1}{2}k < \xi < a \ , \quad \text{and} \quad 0 < y = nk - \frac{1}{2}k < \eta < 1 \ ,$$

$$V(x,\eta;\xi,\eta)_X - B(x,\eta)V(x,\eta;\xi,\eta) = 0 \ , \quad 0 < x = mk - \frac{1}{2}k < \xi < a \ ,$$

(3.11)

$$V(\xi,y;\xi,\eta)_Y - A(\xi,y)V(\xi,y;\xi,\eta) = 0 \ , \quad 0 < y = nk - \frac{1}{2}k < \eta < 1 \ .$$

One may thereby solve the finite difference problem,

$$W(\rho,\tau)_{PT} = \frac{1}{2}\frac{\rho}{(1-\tau)^2}\tilde{B}\left(\frac{\rho^2}{1-\tau}\right)W(\rho,\tau) \ ,$$

(3.12)

$$W(0,\tau) = 0 \ , \quad W(\rho,0) = k\sum_{m=1}^{M}(mk - \frac{1}{k}k)\tilde{B}([mk - \frac{1}{2}k]^2) \ ,$$

$\rho = Mk$, M a positive integer, by means of the finite difference Riemann formula [5]. One obtains the following representation

(3.13)

$$W(\rho,\tau) = V(0,0;\rho,\tau)W(0,0) + \sum_{m=1}^{M}\left\{(mk - \frac{1}{2}k)\tilde{B}([mk - \frac{1}{2}k]^2)\ \cdot\right.$$

$$\cdot\ V(mk - \tfrac{1}{2}k, 0; \rho, \tau) \Big\} \ , \quad \rho = Mk \ .$$

The dependence of this solution on the mesh constant k shall be indicated functionally by $W(\rho,\tau;k)$.

Recalling the definition $G(r,t) \equiv (1-t)^{-1}w(r\sqrt{1-t},t)$, we introduce the discrete function $G(r,t;k)$ by replacing $w(\rho,\tau)$ by $W(\rho,\tau;k)$. The function $G(r,t;k)$ is no longer defined on a rectangular grid; however, ignoring this point as being merely of technical interest, we represent the approximate operator $\underset{\sim}{G}_k$ by (for $n \geqslant 3$)

$$(3.14) \qquad (\underset{\sim}{G}_k h)(\underset{\sim}{x}) \equiv \int_0^1 \sigma^{n-3}\, W(r\sigma, 1-\sigma^2;k)h(\underset{\sim}{x}\sigma^2)d\sigma \ ,$$

where it is understood that the discrete function $W(r\sigma,1-\sigma^2;k)$ has been extended to the continuum by a smooth interpolation. This leads to a sequence of integral transformations

$$(3.15) \qquad u(\underset{\sim}{x};k) = (\underset{\sim}{I}+\underset{\sim}{G}_k)h(\underset{\sim}{x}) \ , \quad h(\underset{\sim}{x}) \in \mathcal{H}[D] \ .$$

The functions $u(\underset{\sim}{x};k)$ may be considered as belonging to a family of approximate solutions of (2.3). In order to obtain an approximate solution of the boundary value problem, we replace $h(x)$ in (3.15) by a double layer potential, as above, and compute the residue. We obtain a sequence of integral equations

$$(3.16) \qquad f(\underset{\sim}{x}) = (\underset{\sim}{I}+\underset{\sim}{K}_k)\mu(\underset{\sim}{x};k) \ ,$$

which may easily be shown to be of Fredholm type. The kernel $K(\underset{\sim}{x},\underset{\sim}{y};k)$ of $\underset{\sim}{K}_k$ is given by (2.52) with $G(r,t)$ replaced by the interpolated discrete function $G(r,t;k)$.

If we introduce the usual operator norm,

(3.17) $$\|K\| \equiv \max_{\underset{\sim}{x} \in \partial D} \int_{\partial D} |K(\underset{\sim}{x},\underset{\sim}{y})| d\omega_y \quad ,$$

then it is clear from the following

(3.18)

$$\frac{\Gamma(n/2)}{(n-2)\pi^{n/2}} \int_{\partial D} \frac{\partial}{\partial \nu_y} \left(\frac{1}{\|x-y\|^{n-2}} \right) d\omega_y = \begin{cases} 1 \quad , & \underset{\sim}{x} \in D^0 \\ 1/2 \quad , & \underset{\sim}{x} \in \Gamma \\ 0 \quad , & \underset{\sim}{x} \in D' \quad , \end{cases}$$

that $\|\underset{\sim}{K}\| < \infty$, and $\|\underset{\sim}{K}_k\| < \infty$ for $k > 0$ sufficiently small. Further more, we may show that $\|\underset{\sim}{K}-\underset{\sim}{K}_k\| \to 0$ as $k \to 0$. This will follow from the discretization error estimate on $w(\rho,\tau) - W(\rho,\tau;k)$ as given in [5]. This estimate applied to our case is,

(3.19)

$$|w(\rho,\tau) - W(\rho,\tau;k)| \leqslant \ell(\rho,\tau) \int_0^\rho d\tilde{\rho} \int_0^\tau d\tilde{\tau} \left[e_k^{K\{(\rho-\tilde{\rho})+(\tau-\tilde{\tau})\}} \right] \quad ,$$

where

(3.20) $$\ell(\rho,\tau) = \max_{\substack{0 \leqslant \tilde{\rho} \leqslant \rho \\ 0 \leqslant \tilde{\tau} \leqslant \tau}} |\underset{\sim}{L}_k \, w(\tilde{\rho},\tilde{\tau})| \quad ,$$

(3.21) $$2K^2 = \max_{\substack{0\le\rho\le a\\ 0\le\tau\le 1}} \left| \frac{\rho}{(1-\tau)^2} \tilde{B}\left(\frac{\rho^2}{(1-\tau)}\right)\right| ,$$

and $e_k^{f(x)}$ is a finite difference analogue of the exponential function $e^{f(x)}$. In [5] $e_k^{f(x)}$ is given as

(3.22)
$$e_k^{f(x)} = e_k^{f(0)} \prod_{m=1}^{M} \left[\frac{1 + \frac{1}{2}k\ f(mk - \frac{1}{2}k)_x}{1 - \frac{1}{2}k\ f(mk - \frac{1}{2}k)_x}\right] , \quad x = mk .$$

We remark that the constant K given above is bounded because of the construction of $\tilde{B}\left(\frac{\rho^2}{1-\tau}\right)$ as a function which vanishes smoothly between $\frac{\rho^2}{1-\tau} = a$ and $a + \varepsilon$, $\varepsilon > 0$ arbitrarily small.

In [5] a bound on (ρ,τ) is given as Mk^2, with estimates on the constant M. In order to show that $\|\underset{\sim}{K}-\underset{\sim}{K}_k\| \to 0$ as $k \to 0$, it suffices to know that $|w(\rho,\tau) - W(\rho,\tau;k)| \le \tilde{M}k^2$, which follows by an elementary computation. That $\|\underset{\sim}{K}-\underset{\sim}{K}_k\| \to 0$ can be seen directly from

(3.23)
$$K(\underset{\sim}{x},\underset{\sim}{y})-K(\underset{\sim}{x},\underset{\sim}{y};k) \equiv \frac{\Gamma(n/2)}{(n-2)\pi^{n/2}} \int_0^1 \sigma^{n-3} \left[w(r\sigma,1-\sigma^2)-W(r\sigma,1-\sigma^2;k)\right]$$
$$\cdot \frac{\partial}{\partial\nu_y}\left(\frac{1}{\|x\ ^2-y\|^{n-2}}\right) d\sigma ,$$

which implies

(3.24)
$$\|\underset{\sim}{K}-\underset{\sim}{K}_k\| = \max_{x\in\partial D} \int_{\partial D} |K(\underset{\sim}{x},\underset{\sim}{y}) - K(\underset{\sim}{x},\underset{\sim}{y};k)|d\omega_y \le$$

$$\leq \max_{x\in\partial D} \frac{\Gamma(n/2)}{(n-2)\pi^{n/2}} \int_{\partial D} \int_0^1 \sigma^{n-3} |w(r\sigma,1-\sigma^2) - W(r\sigma,1-\sigma^2;k)|$$

$$\cdot \left| \frac{\partial}{\partial \nu_y} \left(\frac{1}{\|x\sigma^2 - y\|^{n-2}} \right) \right| d\sigma\, d\omega_y$$

$$\leq k^2 \tilde{M} \max_{x\in\partial D} \frac{\Gamma(n/2)}{(n-2)\pi^{n/2}} \int_0^1 \sigma^{n-3} \int_{\partial D} \left| \frac{\partial}{\partial \nu_y} \left(\frac{1}{\|\underset{\sim}{x}\sigma^2 - \underset{\sim}{y}\|^{n-2}} \right) \right| d\sigma\, d\omega_y$$

$$\leq k^2 \tilde{M} , \quad \text{for} \quad n \geq 3 .$$

From this we conclude $\|\underset{\sim}{K} - \underset{\sim}{K}_k\| \approx O(k^2)$ as $k \to 0$. Using similar estimates we can show that

(3.25)

$$\lim_{\|\underset{\sim}{x}_1 - \underset{\sim}{x}_2\| \to 0} \left\{ \int_{\partial D} |K(\underset{\sim}{x}_1, \underset{\sim}{y}; k) - K(\underset{\sim}{x}_2, \underset{\sim}{y}; k)| d\omega_y \right\} = 0$$

uniformly in $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$. This plus the fact that $\|K\| < \infty$, implies via the Arzela-Ascoli Theorem that the $\underset{\sim}{K}_k$ are compact. We already know that $\underset{\sim}{K}$ is compact, and furthermore, that $(\underset{\sim}{I}+\underset{\sim}{K})^{-1}$ exists. Two norm inequalities from Taylor [32] pg. 164, namely that when $\|\underset{\sim}{K} - \underset{\sim}{K}_k\| < \frac{1}{\|(I+K)^{-1}\|}$ then

$$\|(I+K_k)^{-1}\| \leq \frac{\|(\underset{\sim}{I}+\underset{\sim}{K})^{-1}\|}{1 - \|(\underset{\sim}{I}+\underset{\sim}{K})^{-1}\| \cdot \|\underset{\sim}{K} - \underset{\sim}{K}_k\|}$$

and

(3.27)

$$\|(\underset{\sim}{I}+\underset{\sim}{K}_k)^{-1} - (\underset{\sim}{I}+\underset{\sim}{K})^{-1}\| \leqslant \frac{\|(\underset{\sim}{I}+\underset{\sim}{K})^{-1}\|^2 \cdot \|\underset{\sim}{K}-\underset{\sim}{K}_k\|}{1-\|(\underset{\sim}{I}+\underset{\sim}{K})^{-1}\| \cdot \|\underset{\sim}{K}-\underset{\sim}{K}_k\|}$$

tell us that $(\underset{\sim}{I}+\underset{\sim}{K}_k)^{-1}$ exists and that the unique solutions (for $k > 0$ sufficiently small) of

$$f(\underset{\sim}{x}) = (\underset{\sim}{I}+\underset{\sim}{K}_k)\mu(\underset{\sim}{x};k) \tag{3.28}$$

tend to the unique solution

$$f(\underset{\sim}{x}) = (\underset{\sim}{I}+\underset{\sim}{K})\mu(\underset{\sim}{x}) \tag{3.29}$$

We next turn our attention to the numerical solution of the sequence of equations (3.28) above. The approach we use is an extension of the Nyström method to kernels with weak singularities and is due primarily to Anselone [1], [2]; see also in this regard Atkinson [3], [4].

We replace the integration in (3.28) by numerical quadrature. To this end, we first reparametrize (3.28) by introducing the following representation for ∂D, $\underset{\sim}{y} = \underset{\sim}{y}(\underset{\sim}{t})$, $\underset{\sim}{t} \in \mathcal{A}$, $\mathcal{A} \equiv \{\underset{\sim}{t} \mid 0 \leqslant \max t_i \leqslant 1;\ \ i = 1, 2, \ldots, (n-1)\}$, $\tilde{K}(\underset{\sim}{s},\underset{\sim}{t};k) \equiv K(\underset{\sim}{x}(\underset{\sim}{s}), \underset{\sim}{y}(\underset{\sim}{t});k)$. Then (3.28) becomes with $f(\underset{\sim}{x}(\underset{\sim}{s})) = F(\underset{\sim}{s})$, $\mu(\underset{\sim}{x}(\underset{\sim}{s});k) = \rho^{(k)}(\underset{\sim}{s})$

(3.30)

$$F(\underset{\sim}{s}) = \rho(\underset{\sim}{s}) + \int_0^1 dt_1 \int_0^1 dt_2 \cdots \int_0^1 dt_{n-1} \tilde{K}(\underset{\sim}{s},\underset{\sim}{t};k)\rho(\underset{\sim}{t}) ,$$

which may be then replaced by the numerical quadrature

(3.31)

$$\rho_{(m)}^{(k)}(\underset{\sim}{s}) + \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \cdots \sum_{j_{n-1}=1}^{m} \Phi_{j_1 \cdots j_{n-1}} \tilde{K}(\underset{\sim}{s}, \underset{\sim}{t}_j; k)\, \rho_{(m)}^{(k)}(\underset{\sim}{t}_j) = F(\underset{\sim}{s}) .$$

The Nyström method is to now set the points $\underset{\sim}{s} = \underset{\sim}{t}_i$, $(i_\ell = 1, 2, \ldots, m)$ $(\ell = 1, 2, \ldots, n-1)$ and solve the linear system

(3.32)

$$z_i^{(k)} + \sum_{j_1=1}^{m} \cdots \sum_{j_{n-1}=1}^{m} \Phi_j\, K(\underset{\sim}{t}_i, \underset{\sim}{t}_j; k)\, z_j^{(k)} = F(\underset{\sim}{t}_i) ,$$

$(i_\ell = 1,2,\cdots,m)$, $(\ell = 1,2,\cdots,n-1)$, $i = (i_1,\cdots,i_{n-1})$.

The solution of (3.31) above is then given in terms of the solutions $z_j^{(k)}$ of (3.32) by

(3.33)

$$\rho_{(m)}^{(k)}(\underset{\sim}{s}) = \left[F(\underset{\sim}{s}) - \sum_{j_1=1}^{m} \cdots \sum_{j_{n-1}=1}^{m} \Phi_j\, \tilde{K}(\underset{\sim}{s}, \underset{\sim}{t}_j; k) z_j^{(k)} \right] .$$

Let us put the above equations into a formal setting. Let $\mathcal{C}[\mathcal{A}]$ denote the Banach space of continuous functions on $\mathcal{A}$, with the uniform norm. Let $\underset{\sim}{\mathcal{K}}^{(k)}$ be the integral operator on $\mathcal{C}[\mathcal{A}]$ with the kernel $\tilde{K}(\underset{\sim}{s}, \underset{\sim}{t}; k)$:

$$(3.34) \qquad (\underset{\sim}{\mathcal{K}}^{(k)}\rho)(\underset{\sim}{s}) \equiv \int \tilde{K}(\underset{\sim}{s}, \underset{\sim}{t}; k)\rho(t)dt , \quad \underset{\sim}{s} \in \mathcal{A} .$$

Let $\underset{\sim}{\mathcal{K}}^{(k)}_{(m)}$ be the numerical quadrature operator on $C[\mathcal{A}]$ given by

$$(3.35) \qquad (\underset{\sim}{\mathcal{K}}^{(k)}_{(m)}\rho)(\underset{\sim}{s}) \equiv \sum_j \Phi_j \tilde{K}(\underset{\sim}{s},\underset{\sim}{t}_j;k)\rho(t_j) \ , \quad \underset{\sim}{s} \in \mathcal{A} .$$

We have shown earlier that $\underset{\sim}{\mathcal{K}}^{(k)}$ is compact on $C[\mathcal{A}]$. Furthermore, since $\underset{\sim}{\mathcal{K}}^{(k)}_{(m)}$ is a finite rank operator it is also compact.

The equations (3.30) and (3.31) may now be written in the operator notation as

$$(3.36) \qquad (\underset{\sim}{I}+\underset{\sim}{\mathcal{K}}^{(k)})\rho^{(k)}(\underset{\sim}{s}) = F(\underset{\sim}{s}) \ , \quad \underset{\sim}{s} \in \mathcal{A},$$

and

$$(3.37) \qquad (\underset{\sim}{I}+\underset{\sim}{\mathcal{K}}^{(k)}_{(m)})\rho^{(k)}_{(m)}(\underset{\sim}{s}) = F(\underset{\sim}{s}) \ , \quad \underset{\sim}{s} \in \mathcal{A}.$$

As it was mentioned above, the work of Anselone [1], [2] concerning the solutions of (3.36) and (3.37) is influential to our approach. In particular, he has given estimates on the difference between the solutions of these equations. For instance, if $(\underset{\sim}{I}+\underset{\sim}{\mathcal{K}}^{(k)})^{-1}$ exists (which we showed was true), and if

$$(3.38) \qquad \|(\underset{\sim}{\mathcal{K}}^{(k)} - \underset{\sim}{\mathcal{K}}^{(k)}_{(m)})\underset{\sim}{\mathcal{K}}^{(k)}_{(m)}\| < \frac{1}{\|(\underset{\sim}{I}+\underset{\sim}{\mathcal{K}}^{(k)})^{-1}\|} ,$$

then $(\underset{\sim}{I}+\underset{\sim}{\mathcal{K}}^{(k)}_{(m)})^{-1}$ exists, and is bounded in norm by

$$(3.39) \qquad \|(\underset{\sim}{I}+\underset{\sim}{\mathcal{K}}^{(k)}_{(m)})^{-1}\| \leq$$

$$\leqslant \frac{1 + \|\underset{\sim}{\mathcal{K}}^{(k)}_{(m)}\| \cdot \|(\underset{\sim}{I}+\underset{\sim}{\mathcal{K}}^{(k)})^{-1}\|}{1 - \|(\underset{\sim}{\mathcal{K}}^{(k)}-\underset{\sim}{\mathcal{K}}^{(k)}_{(m)})\underset{\sim}{\mathcal{K}}^{(k)}_{(m)}\| \cdot \|(\underset{\sim}{I}+\underset{\sim}{\mathcal{K}}^{(k)})^{-1}\|} .$$

In addition, if $(\underset{\sim}{I}+\underset{\sim}{\mathcal{K}}^{(k)}_{(m)})^{-1}$ exists, say for m sufficiently large, then one has [1], [3],

(3.40)

$$\|\rho^{(k)}_{(m)}(\underset{\sim}{s}) - \rho^{(k)}(\underset{\sim}{s})\| \leqslant \|(\underset{\sim}{I}+\underset{\sim}{\mathcal{K}}^{(k)}_{(m)})^{-1}\| \cdot$$

$$\cdot \frac{\|(\underset{\sim}{\mathcal{K}}^{(k)}-\underset{\sim}{\mathcal{K}}^{(k)}_{(m)})F\| + \|\rho^{(k)}_{(m)}\| \cdot \|(\underset{\sim}{\mathcal{K}}^{(k)}-\underset{\sim}{\mathcal{K}}^{(k)}_{(m)})\underset{\sim}{\mathcal{K}}^{(k)}\|}{1 - \|(\underset{\sim}{\mathcal{K}}^{(k)}-\underset{\sim}{\mathcal{K}}^{(k)}_{(m)})\underset{\sim}{\mathcal{K}}^{(k)}\| \cdot \|(\underset{\sim}{I}+\underset{\sim}{\mathcal{K}}^{(k)}_{(m)})^{-1}\|} .$$

These estimates imply the following theorem:

<u>Theorem 3.1</u>. Let $D \subset \mathbb{R}^n$, $n \geqslant 3$, be an appropriate domain, $B(r^2) \in \mathcal{C}^1_+[0,a]$, and $f(x) \in \mathcal{C}^0[\partial D]$. Then the solution of the Dirichlet problem, $\Delta_n u - B(r^2)u = 0$, $u(\underset{\sim}{x}) = f(\underset{\sim}{x})$ for $\underset{\sim}{x} \in \partial D$, has the following approximate solution:

$$u^k_m(\underset{\sim}{x}) = (\underset{\sim}{I}+\underset{\sim}{G}_k)h^k_m(\underset{\sim}{x}) \ , \quad \underset{\sim}{x} \in D+\partial D \ , \tag{3.41}$$

where

(3.42)

$$h^k_m(\underset{\sim}{x}) = \frac{\Gamma(n/2)}{(n-2)\pi^{n/2}} \int_{\partial D} \mu^k_m(y) \frac{\partial}{\partial \nu_y} \left(\frac{1}{\|x-y\|^{n-2}}\right) d\omega_y \ ,$$

and $\mu^k_m(\underset{\sim}{y}) \equiv \rho^{(k)}_{(m)}(\underset{\sim}{t}(\underset{\sim}{y}))$, $\underset{\sim}{y} \in \partial D$. Furthermore, the error $|u(\underset{\sim}{x}) - u^k_m(\underset{\sim}{x})|$ is bounded by

$$(3.43) \qquad |u(\underset{\sim}{x}) - u_m^k(\underset{\sim}{x})| \leq C_1 \|\rho - \rho_{(m)}^{(k)}\| + C_2 k^2 ,$$

$$\underset{\sim}{x} \in D+\partial D ,$$

where C_1 and C_2 are constants and

(3.44)

$$\|\rho - \rho_{(m)}^{(k)}\| \leq \frac{\|(\underset{\sim}{I}+\underset{\sim}{\mathcal{K}}^{(k)})^{-1}\| \cdot \|\underset{\sim}{\mathcal{K}}^{(k)}-\underset{\sim}{\mathcal{K}}\| \cdot \|\underset{\sim}{\rho}^k\|}{1 - \|(\underset{\sim}{I}+\underset{\sim}{\mathcal{K}}^{(k)})^{-1}\| \cdot \|\underset{\sim}{\mathcal{K}}^{(k)}-\underset{\sim}{\mathcal{K}}\|}$$

$$+ \frac{\|(\underset{\sim}{\mathcal{K}}^{(k)}-\underset{\sim}{\mathcal{K}}_{(m)}^{(k)})f\| + \|\rho_{(m)}^{(k)}\| \cdot \|(\mathcal{K}^{(k)}-\mathcal{K}_{(m)}^{(k)})\mathcal{K}^{(k)}\|}{1 - \|(\underset{\sim}{\mathcal{K}}^{(k)}-\underset{\sim}{\mathcal{K}}_{(m)}^{(k)})\underset{\sim}{\mathcal{K}}^{(k)}\| \cdot \|(\underset{\sim}{I}+\mathcal{K}_{(m)}^{(k)})^{-1}\|} .$$

<u>Proof</u>: The bound on $\|\rho - \rho_{(m)}^{(k)}\|$ is found by the triangle inequality, equation (3.40), and an estimate on $\|\rho - \rho^{(k)}\|$. One has the identity (for k sufficiently small so that $(\underset{\sim}{I}+\underset{\sim}{\mathcal{K}}^{(k)})^{-1}$ exists)

(3.45)

$$(\underset{\sim}{I}+\underset{\sim}{\mathcal{K}})^{-1} - (\underset{\sim}{I}+\underset{\sim}{\mathcal{K}}^{(k)})^{-1} = (\underset{\sim}{I}+\underset{\sim}{\mathcal{K}}^{(k)})^{-1} (\underset{\sim}{\mathcal{K}}^{(k)}-\underset{\sim}{\mathcal{K}})$$

$$\cdot [\underset{\sim}{I}-(\underset{\sim}{I}+\underset{\sim}{\mathcal{K}}^{(k)})^{-1} (\underset{\sim}{\mathcal{K}}^{(k)}-\underset{\sim}{\mathcal{K}}]^{-1} (\underset{\sim}{I}+\underset{\sim}{\mathcal{K}}^{(k)})^{-1} .$$

The estimate for $\|\rho - \rho^{(k)}\|$ follows immediately from this. The estimate for $|u(x) - u_m^k(x)|$ follows from the maximum principal for solutions of $\Delta_n u - B(r^2)u = 0$, with $B(r^2) \geq 0$ for $\underset{\sim}{x} \in D$, the obvious inequality

$$|u-u_m^k| \leq |h-h_m^k| + |\underset{\sim}{G}(h-h_m^k)| + |(\underset{\sim}{G}-\underset{\sim}{G}_k) h_m^k| ,$$

and the fact that the operator $\underset{\sim}{G}$ is monotone in the sense of Collatz [10], [24]. The constants C_1 and C_2 may be estimated without difficulty.

An alternate procedure for computing an approximate G-function is to use the Cauchy-Euler Polygon Method. In this regard see the work of Diaz [D.1,2].

One chooses a sequence of subdivisions of the rectangle $\mathcal{R} \equiv [0,a] \times [0,1]$, i.e. for each (m,n), we form a subdivision.

(3.46)
$$0 = \rho_{0,m} < \rho_{1,m} < \cdots < \rho_{m,m} = a \ ,$$

$$0 = \tau_{0,n} < \tau_{1,n} < \cdots < \tau_{n,n} = a \ ,$$

and on each of the sub-rectangles $\mathcal{R}_{k,\ell} \equiv [\rho_{km}, \rho_{k+1,m}] \times [\tau_{\ell n}, \tau_{\ell+1,n}]$ we consider the "miniature problem"

$$\frac{\partial^2 w}{\partial\rho\partial\tau} = A_{k,\ell}\, w \ ; \quad A_{k,\ell} \equiv A(\rho_{k,m}, \tau_{\ell,n}) \ ,$$

where

(.47)
$$A(\rho,\tau) \equiv \frac{1}{2}\,\frac{\rho}{(1-\tau)^2}\,\tilde{B}\left(\frac{\rho^2}{1-\tau}\right) .$$

$$w(\rho,\tau_\ell) = D_{k\ell} + B_{k\ell}(\rho-\rho_k) \ , \quad \rho_k \leqslant \rho \leqslant \rho_{k+1} \ ,$$

$$w(\rho_k,\tau) = D_{k\ell} + C_{k\ell}(\tau-\tau_\ell) \ , \quad \tau_\ell \leqslant \tau \leqslant \tau_{\ell+1} \ .$$

Hence, in $\mathcal{R}_{k,\ell}$ we have,

$$(3.48)\qquad w(\rho,\tau) = A_{k,\ell}(\rho-\rho_k)(\tau-\tau_k) + B_{k\ell}(\rho-\rho_k) + C_{kk}(\tau-\tau_\ell) + D_{k\ell} .$$

On the rectangles having sides on $\tau = 0$ the initial data $w(\rho,0) = \int_0^\rho \rho\ B(\rho^2)d\rho$ is linearly approximated; on rectangles having sides on $\rho = 0$, the data is chosen to be zero. The general form of the solution in $\mathcal{R}_{k,\ell}$ has been given by Diaz in [18,19] and is for our case,

(3.49)

$$\begin{aligned} w_{mn}(\rho,\tau) &= w_{k0} + w_{0\ell} - w_{00} + \frac{w_{k+1,0}-w_{k,0}}{\rho_{k+1}-\rho_k}(\rho-\rho_k) \\ &+ \sum_{i=1}^{k}\sum_{j=1}^{\ell} A_{i-1,j-1}(\rho_i-\rho_{i-1})(\tau_j-\tau_{j-1}) \\ &+ \sum_{j=1}^{\ell} A_{k,j-1}(\rho-\rho_k)(\tau_j-\tau_{j-1}) \\ &+ \sum_{i=1}^{k} A_{i-1,\ell}(\rho_i-\rho_{i-1})(\tau-\tau_\ell) \\ &+ A_{k\ell}(\rho-\rho_k)(\tau-\tau_\ell) . \end{aligned}$$

Here we are using the notation $w_{ij} \equiv w(\rho_{im},\tau_{jn})$ and $w_{mn}(\rho,\tau)$ is the approximating solution computed by subdividing $\mathcal{R}$ into $m \times n$ smaller rectangles.

Such a construction is useful to use in order to obtain an ε-approximating solution as is done in the case of ordinary differential equations; i.e., given an $\varepsilon > 0$ we choose a subdivision such that in each $\mathcal{R}_{k\ell}$ the differential equation is almost satisfied by $w_{mn}(\rho,\tau)$. More precisely, we choose m and n such that $\left|\frac{\partial^2 w_{mn}}{\partial\rho\partial\tau} - Aw\right| < \varepsilon$. If this is the case we can obtain bounds on the difference between an actual solution and an ε-approximating solution.

Lemma 3.1. Let $K^2 = \max\limits_{(\rho,\tau)\in\mathcal{R}} A(\rho,\tau)$, and $w_i(\rho,\tau)$, $(i=1,2)$ be ε_i-approximating solutions of (3.5), which satisfy the required data. Then for all $(\rho,\tau)\in\mathcal{R}$ one has the estimate

$$|w_1(\rho,\tau)-w_2(\rho,\tau)| \leqslant \varepsilon\rho\tau[1+\rho\tau K^2 e^{K(\rho+\tau)}] \,, \tag{3.50}$$

where $\varepsilon = \varepsilon_1 + \varepsilon_2$.

Proof: Our proof is modeled after the ordinary differential equation case in Coddington and Levinson [17], Chapter I. First we notice that the $w_i(\rho,\tau)$ satisfy

(3.51)
$$\left|w_i(\rho,\tau) - \int_0^\rho \rho\ \tilde{B}(\rho^2)d\rho - \frac{1}{2}\int_0^\rho\int_0^\tau \frac{\rho}{(1-\tau)^2}\ \tilde{B}\left(\frac{\rho^2}{1-\tau}\right) w_i(\rho,\tau)d\tau d\rho\right| \leqslant \varepsilon_i\ \rho\tau.$$

We notice that because of the extension of $\tilde{B}(r^2)$ to $[0,\infty)$, $\left|\frac{\rho}{2(1-\tau)^2}\ \tilde{B}\left(\frac{\rho^2}{1-\tau}\right)\right|$ is continuous on $\mathcal{R}$, and hence its maximum does exist on $\mathcal{R}$; we set it equal to K^2. If $c(\rho,\tau) \equiv |w_1(\rho,\tau)-w_2(\rho,\tau)|$, then by adding (3.51) for $i = 1,2$, we obtain

(3.52)
$$c(\rho,\tau) \leqslant \frac{1}{2}\int_0^\rho\int_0^\tau c(\xi,\eta)\,\frac{\xi}{(1-\eta)^2}\,\tilde{B}\left(\frac{\xi^2}{1-\eta}\right)d\eta d\xi + \varepsilon\rho\tau \ .$$

Defining $C(\rho,\tau) \equiv \int_0^\rho\int_0^\tau c(\xi,\eta)d\eta d\xi$, (3.52) becomes

$$\frac{\partial^2 C}{\partial\rho\partial\tau} \leqslant K^2 C(\rho,\tau) + \varepsilon\rho\tau \ . \tag{3.53}$$

Since the Riemann function for $C_{\rho\tau} - K^2C = 0$ is $I_0(2K\sqrt{(\rho-\xi)(\tau-\eta)})$ we obtain the following estimate,

$$C(\rho,\tau) \leqslant \varepsilon\int_0^\rho\int_0^\tau \xi\eta\, I_0(2K\sqrt{(\rho-\xi)(\tau-\eta)})d\xi d\eta \ .$$

Since $I_0(x)$ is given by the series expansion

$$I_0(x) = \sum_{m\geqslant 0}\left(\frac{x}{2}\right)^{2m}\frac{1}{(m!)^2} \ ,$$

we have

$$C(\rho,\tau) \leqslant \varepsilon\rho^2\tau^2\, I_0(2K\sqrt{\rho\tau}) \ ,$$

which upon substitution into (3.52) yields

$$\begin{aligned} c(\rho,\tau) &\leqslant \varepsilon\rho\tau[1 + \rho\tau K^2\, I_0(2K\sqrt{\rho\tau})] \\ &\leqslant \varepsilon\rho\tau[1 + \rho\tau K^2\, e^{K(\rho+\tau)}] \ . \end{aligned} \tag{3.54}$$

This is the stated result.

Remark: Since $A(\rho,\tau)$ is uniformly continuous on $\mathcal{R}$, it is possible to put a "uniform mesh" on $\mathcal{R}$ in order to obtain an ε-approximating solution.

We now introduce an approximate G-function $G_\varepsilon(r,t)$ by $G_\varepsilon(r,t) \equiv w_\varepsilon(r\sqrt{1-t},\ t)(1-t)^{-1}$. Then we have

(3.55)

$$|(\underset{\sim}{G}-\underset{\sim}{G}_\varepsilon)h(\underset{\sim}{x})| = \left|\int_0^1 \sigma^{n-3}[w(r\sigma,1-\sigma^2) - w_\varepsilon(r\sigma,1-\sigma^2)] \cdot h(x\sigma^2)\,d\sigma\right|$$

$$\leq \varepsilon r \int_0^1 \sigma^{n-2}(1-\sigma^2)[1+K^2 r\sigma(1-\sigma^2)e^{K(r\sigma+1-\sigma^2)}] \cdot |h(\underset{\sim}{x}\sigma^2)|\,d\sigma$$

$$\leq \varepsilon r\,\|h\| \left(\int_0^1 \sigma^{n-2}(1-\sigma^2)\,d\sigma\right) \cdot \left(1+k^2 re^{K(r+1)}\right)$$

$$\leq \varepsilon\,\frac{2r}{n}\left(\frac{n-2}{n}\right)^{n-2}\left(1+K^2 re^{K(r+1)}\right) \cdot \|h\|_{\partial D}\ ;$$

$$\|h\|_{\partial D} \equiv \max_{\partial D}|h|\ .$$

We next introduce the approximate kernel $K_\varepsilon(\underset{\sim}{x},\underset{\sim}{y})$ by replacing $G(r,t)$ in (2.52) by $G_\varepsilon(r,t)$, and then estimate $\|\underset{\sim}{K}-\underset{\sim}{K}_\varepsilon\|$ using the $\|\cdot\|$ defined by (3.17). We obtain

(3.56)

$$\left|\int_{\partial D} K(\underset{\sim}{x},\underset{\sim}{y})-K_\varepsilon(\underset{\sim}{x},\underset{\sim}{y})d\omega_y\right| = \frac{\Gamma(n/2)}{(n-2)\pi^{n/2}}\int_0^1 \sigma^{n-1}\left|\int_{\partial D}(G-G_\varepsilon)\cdot\frac{\partial}{\partial\nu_y}\frac{1}{\|\underset{\sim}{x}\sigma^2-\underset{\sim}{y}\|^{n-2}}\right| d\omega_y\, d\sigma$$

$$\leq \frac{\Gamma(n/2)}{\pi^{n/2}}\int_0^1 \sigma^{n-1}\max_D |G-G_\varepsilon| \int_{\partial D}\frac{|\cos(\nu,\underset{\sim}{x}\sigma^2-\underset{\sim}{y})|}{\|\underset{\sim}{x}\sigma^2-\underset{\sim}{y}\|^{n-1}} d\omega_y\, d\sigma$$

$$\leq \int_0^1 \max_D |G-G_\varepsilon|\sigma^{n-1}\, d\sigma$$

$$\leq \varepsilon\,\frac{2r}{n}\left(\frac{n-2}{n}\right)^{n-2}\left(1+k^2re^{K(r+1)}\right) .$$

Hence, $\|K-K_\varepsilon\| \simeq O(\varepsilon)$ as $\varepsilon \to 0$.

Arguing as we did previously we may show that $(\underset{\sim}{I}+\underset{\sim}{K}_\varepsilon)^{-1}$ exists, and furthermore, that $\|(\underset{\sim}{I}+\underset{\sim}{K}_\varepsilon)^{-1} - (\underset{\sim}{I}+\underset{\sim}{K})^{-1}\| \to 0$ as $\varepsilon \to 0$; hence, it is sufficient for us to consider the sequence of integral equations

$$(\underset{\sim}{I}+\underset{\sim}{K}_\varepsilon)\,\mu_\varepsilon(\underset{\sim}{x}) = f(\underset{\sim}{x})$$

to obtain the approximate solutions $\mu_\varepsilon(\underset{\sim}{x}) \to \mu(\underset{\sim}{x})$ as $\varepsilon \to 0$.

4. Nonlinear Equations in Two Independent Variables: The Dirichlet Problem

In this section we will obtain rapidly convergent analytic approximations to solutions of Dirichlet's problem for the equation

$$\Delta u = f(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}) \tag{4.1}$$

defined in a simply connected domain D with Liapunov boundary C ([16]). Without loss of generality we assume $u = 0$ on C. We require $f(x,y,\xi_1,\xi_2,\xi_3)$ to satisfy the conditions

$$H_1: \quad f(x,y,0,0,0) \in L_p(D+C) \ , \quad p > 2 \ ,$$

$$H_2: \quad |f(x,y,\xi_1^0,\xi_2^0,\xi_3^0) - f(x,y,\xi_1^1,\xi_2^1,\xi_3^1)|$$
$$\leq f_0(x,y)\{|\xi_1^1-\xi_1^1| + |\xi_2^0-\xi_2^1| + |\xi_3^0-\xi_3^1|\}$$

where $f_0(x,y) \in L_p(D+C)$ and H_2 holds for $|\xi_1| + |\xi_2| + |\xi_3| < R$, R being a sufficiently large, but fixed, positive constant. In order to obtain the geometric convergence of our approximation sequence we rewrite equation (4.1) as

$$\begin{aligned} \Delta u - \alpha u &= f(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}) - \alpha u \\ &= g(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}) \ . \end{aligned} \tag{4.2}$$

where α is an arbitrary but fixed positive constant. It is easily seen that g satisfies the conditions H_1 and H_2 with f replaced by g. Now let $G(x,y;\xi,\eta)$ be the Green's function for $\Delta u - \alpha u$ in D and define the operators $\underset{\sim}{\Pi}_0$, $\underset{\sim}{\Pi}_1$, $\underset{\sim}{\Pi}_2$, and $\underset{\sim}{T}$ as follows:

$$(4.3)\quad u(x,y) = (\underset{\sim}{\Pi}_0\rho)(x,y) \equiv \iint_D G(x,y;\xi,\eta)\rho(\xi,\eta)d\xi d\eta$$

$$(4.4)\quad \frac{\partial u(x,y)}{\partial x} = (\underset{\sim}{\Pi}_1\rho)(x,y) \equiv \iint_D \frac{\partial}{\partial x} G(x,y;\xi,\eta)\rho(\xi,\eta)d\xi d\eta$$

$$(4.5)\quad \frac{\partial u(x,y)}{\partial y} = (\underset{\sim}{\Pi}_2\rho)(x,y) \equiv \iint_D \frac{\partial}{\partial y} G(x,y;\xi,\eta)\rho(\xi,\eta)d\xi d\eta$$

$$(4.6)\quad \Delta u - \alpha u = \rho(x,y) = (\underset{\sim}{T}_\rho)(x,y) = g(x,y,\Pi_0\rho,\Pi_1\rho,\Pi_2\rho)$$

where $\rho \in L_p(D+C)$ and the derivatives in equation (4.3)-(4.6) are to be interpreted in a generalized or Sobolev sense. In [16] it was shown that for D sufficiently small $\underset{\sim}{T}$ is a contraction mapping of a closed ball in $L_p(D+C)$ into itself and hence has a unique fixed point ρ in $L_p(D+C)$. The (generalized) solution of the Dirichlet problem is now given by

$$(4.7)\qquad u(x,y) = (\underset{\sim}{\Pi}_0\rho)(x,y)\ .$$

From the theory of integral operators whose kernel has a weak singularity and Sobolev's lemma we can conclude that $u(x,y)$ is continuously differentiable in $D + C$.

We will now construct a sequence of functions which converge geometrically to $u(x,y)$. Let $K(x,y;\xi,\eta) \equiv K(P,Q)$ be the kernel function ([8]) for $\Delta u - \alpha u$ in D. Then the Green's function $G(x,y;\xi,\eta) \equiv G(P,Q)$ can be represented as

$$(4.8)\quad G(P,Q) = \frac{1}{2\pi}\left[\log \overline{PQ} - \int_C \log \overline{PT}\, \frac{\partial K}{\partial n_T}(T,Q)\, ds_T\right]$$

where $\overline{PQ}$ denotes the distance from the point P to the point Q. Now let D_1 be a square containing D in its interior and let $S(P,Q)$ be Neumann's function for $\Delta u - \alpha u$ in D_1. Note that $S(P,Q)$ can be constructed in a variety of ways, including the method of images [16]. Following Bergman and Schiffer [8] we define $i^{(\nu)}(P,Q)$, $\nu = 1, 2, 3, \ldots,$ recursively by

$$(4.9) \qquad i^{(1)}(P,Q) = 4 \int_C S(Q,T) \frac{\partial S(T,P)}{\partial n_T} \, ds_T$$

$$(4.10) \quad i^{(\nu)}(P,Q) = -\int_C i^{(\nu-1)}(P,T) \frac{\partial i^{(1)}}{\partial n_T} (T,Q) \, ds_T \; ; \qquad \nu \geqslant 2 .$$

We can then express the kernel function $K(P,Q)$ in terms of $i^{(\nu)}(P,Q)$ by ([8], p. 315)

$$(4.11) \qquad K(P,Q) = \sum_{m=0}^{\infty} \left[\sum_{\nu=0}^{m} (-1)^{\nu} \binom{m}{\nu} i^{(\nu+1)}(P,Q) \right]$$

and approximate Green's function by $G_N(X,y;\xi,\eta) \equiv G_N(P,Q)$ where

$$(4.12) \quad G_N(P,Q) = \frac{1}{2\pi} \left[\log \overline{PQ} - \int_C \log \overline{PT} \, \frac{\partial K_N(T,Q)}{\partial n_T} \, ds_T \right]$$

$$(4.13) \qquad K_N(P,Q) = \sum_{m=0}^{N} \sum_{\nu=0}^{m} (-1)^{\nu} \binom{m}{\nu} i^{(\nu+1)}(P,Q) .$$

Now define the operators $\underset{\sim}{\Pi}_0^{(N)}$, $\underset{\sim}{\Pi}_1^{(N)}$, $\underset{\sim}{\Pi}_2^{(N)}$, $\underset{\sim}{T}^{(N)}$ by

(4.14)

$$u^{(N)}(x,y) = (\underset{\sim}{\Pi}_0^{(N)}\rho)(x,y) \equiv \iint_D G_N(x,y;\xi,\eta)\rho(\xi,\eta)d\xi d\eta$$

(4.15)

$$\frac{\partial u^{(N)}(x,y)}{\partial x} = (\underset{\sim}{\Pi}_1^{(N)}\rho)(x,y) \equiv \iint_D \frac{\partial G_N(x,y;\xi,\eta)}{\partial x}\rho(\xi,\eta)d\xi d\eta$$

(4.16)

$$\frac{\partial u^{(N)}(x,y)}{\partial y} = (\underset{\sim}{\Pi}_2^{(N)}\rho)(x,y) \equiv \iint_D \frac{\partial G_N(x,y;\xi,\eta)}{\partial y}\rho(\xi,\eta)d\xi d\eta$$

(4.17)

$$\Delta u^{(N)} - \alpha u^{(N)} = \rho(x,y) = (\underset{\sim}{T}^{(N)}\rho)(x,y)$$

$$\equiv g(x,y,\underset{\sim}{\Pi}_0^{(N)}\rho,\underset{\sim}{\Pi}_1^{(N)}\rho,\underset{\sim}{\Pi}_2^{(N)}\rho)$$

where $\rho \in L_p(D+C)$ and the derivatives in equation (4.14)-(4.17) are to be interpreted in a generalized or Solobev sense. Again it can be shown ([23]) that if D is sufficiently small then $\underset{\sim}{T}^{(N)}$ is a contraction mapping of a closed ball in $L_p(D+C)$ into itself, and hence has a unique fixed point $\rho^{(N)}(x,y)$ in $L_p(D+C)$. Our candidate for an approximation to our original Dirichlet problem is now given by

$$u^{(N)}(x,y) = (\underset{\sim}{\Pi}_0^{(N)}\rho^{(N)})(x,y) . \tag{4.18}$$

We can again show that $u^{(N)}(x,y)$ is continuously differentiable in $D + C$ ([16]). Due to the particular choice of $S(P,Q)$ and the fact that $\alpha > 0$ it can in fact be shown ([16]) that the sequence $u^{(N)}(x,y)$ converges geometrically in $D + C$ to the solution of the Dirichlet problem for equation (4.1). More precisely we have the following theorem ([16]):

Theorem: Let $u(x,y)$ be the unique solution of the Dirichlet problem for equation (4.1), which exists for D sufficiently small. If the sequence $u^{(N)}(x,y)$, $N = 1, 2, 3, \ldots$ is defined by equations (4.9)-(4.18) then for D sufficiently small

$$\max_{(x,y)\in D+C} |u^{(N)}(x,y) - u(x,y)| = O\left(\frac{1}{\lambda_1^{2N}}\right)$$

where $\lambda_1 > 1$ is the first eigenvalue of the Fredholm integral equation

$$\phi_\nu(P) = 2\lambda_\nu \int_C \frac{\partial S(P,Q)}{\partial n_Q} \phi_\nu(Q)\,ds_Q \ , \qquad P \in C \ .$$

An analogous theorem can be proved showing how to approximate solutions of Riquier's problem for higher order elliptic equations ([15]).

5. Nonlinear Equations in Two Independent Variables: The Cauchy Problem

In trying to solve free boundary problems by inverse methods it frequently becomes necessary to construct solutions to elliptic Cauchy problems (c.f. [20], Chapter 16). Garabedian and Lieberstein have used such an approach in a particularly elegant manner to study detached shock wave problems in fluid mechanics ([26]). Many problems in elasticity [31] also lend themselves to the use of inverse methods. Such problems in elasticity involve fourth order equations as opposed to the second order equations arising in fluid mechanics. With applications to elasticity in mind we outline below a constructive approach for solving the Cauchy problem

$$(5.1) \qquad \Delta^2 u = \tilde{g}(x,y,u,u_x,u_y,\Delta u, \frac{\partial \Delta u}{\partial x}, \frac{\partial \Delta u}{\partial y})$$

$$(5.2) \qquad \frac{\partial^\ell u(x,y)}{\partial n^\ell} = \tilde{\phi}_\ell(x+iy) \; ; \quad x+iy \in L \, ,$$

$$\ell = 0, 1, 2, 3$$

where L is a given analytic arc, n is the outward normal to L, and $\tilde{g}$, $\tilde{\phi}_\ell$, $\ell = 0, 1, 2, 3$, are assumed to have certain regularity properties to be described shortly. We first use a conformal mapping $z = f(\zeta)$ to map L onto a segment of the x-axis containing the origin. Since in the use of inverse methods the arc L is often assumed to be a portion of an algebraic curve, for example an ellipse, the mapping $z = f(\zeta)$ can be either calculated explicitly or approximated accurately by numerical methods [25]. Under this conformal transformation the Cauchy problem (5.1), (5.2) assumes the form

$$(5.3) \quad \Delta_1 \left(\frac{\Delta_1 u}{|f'(\zeta)|^2} \right) = g(\xi,\eta,u,u_\xi,u_\eta,\Delta_1 u, \frac{\partial \Delta_1 u}{\partial \xi}, \frac{\partial \Delta_1 u}{\partial \eta})$$

$$\frac{\partial^\ell u(\xi,0)}{\partial \eta^\ell} = \phi_\ell(\xi) \; ; \quad \ell = 0, 1, 2, 3$$

where $\zeta = \xi + i\eta$, $\Delta_1 = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}$. In conjugate coordinates

$$\zeta = \xi + i\eta$$

$$\zeta^* = \xi - i\eta$$

equation (5.3), (5.4) become

(5.4)
$$\frac{\partial^2}{\partial\zeta\partial\zeta^*}\left(\frac{1}{|f'(\zeta)|^2}\frac{\partial^2 U}{\partial\zeta\partial\zeta^*}\right)$$

$$= G\left(\zeta,\zeta^*,U,\frac{\partial U}{\partial\zeta},\frac{\partial U}{\partial\zeta^*},\frac{\partial^2 U}{\partial\zeta\partial\zeta^*},\frac{\partial^3 U}{\partial\zeta^2\partial\zeta^*},\frac{\partial^3 U}{\partial\zeta\partial\zeta^{*2}}\right)$$

$$U(\zeta,\zeta^*) = \phi_0(\zeta) \ ; \quad \zeta = \zeta^*$$

(5.5)
$$i^\ell\left(\frac{\partial}{\partial\zeta} - \frac{\partial}{\partial\zeta^*}\right)^\ell U(\zeta,\zeta^*) = \phi_\ell(\zeta) \ ; \quad \zeta = \zeta^* \ ; \quad \ell = 1,\ 2,\ 3$$

where $U(\zeta,\zeta^*) = u\left(\frac{\zeta+\zeta^*}{2}, \frac{\zeta-\zeta^*}{2i}\right)$, $G\left(\zeta,\zeta^*,U,\ldots,\frac{\partial^3 U}{\partial\zeta^2\partial\zeta^*}\right)$ $= \frac{1}{16} g\left(\frac{\zeta+\zeta^*}{2}, \frac{\zeta-\zeta^*}{2i}, \ldots, 4i\left(\frac{\partial}{\partial\zeta} - \frac{\partial}{\partial\zeta^*}\right)\frac{\partial^2 U}{\partial\zeta\partial\zeta^*}\right)$. We now assume that as a function of its first two arguments $G(\zeta,\zeta^*,Z_1, \ldots,Z_6)$ is holomorphic in a bicylinder $\theta \times \theta^*$ where $\theta^* = \{\zeta|\zeta^* \in \theta\}$, and as a function of its last six variables it is holomorphic in a sufficiently large ball about the origin. We further assume that θ is simply connected, contains the origin, is symmetric with respect to conjugation, i.e. $\theta = \theta^*$, and that $\phi_l(\zeta)$, $l = 0, 1, 2, 3,$ are holomorphic for all $\zeta \in \theta$.

Now let

$$U^{(1)} = \frac{1}{|f'(\zeta)|^2} \frac{\partial^2 U}{\partial\zeta\partial\zeta^*} . \tag{5.6}$$

Then ([12]) we can define the operators $\underset{\sim}{A}_i$, $i = 1,2,3,4,5,6,$ by

(5.7)

$$\begin{aligned}\underset{\sim}{A}_2(U^{(1)}) &\equiv \frac{\partial U}{\partial\zeta}\\ &= \int_{\zeta}^{\zeta^*} |f'(\zeta)|^2\, U^{(1)}(\zeta,\xi^*)d\xi^* + \frac{1}{2}\left[\frac{d\phi_0}{d\zeta}(\zeta) + i\phi_1(\zeta)\right]\end{aligned}$$

(5.8)

$$\begin{aligned}\underset{\sim}{A}_3(U^{(1)}) &\equiv \frac{\partial U}{\partial\zeta^*}\\ &= \int_{\zeta^*}^{\zeta} |f'(\zeta)|^2 U^{(1)}(\xi,\zeta^*)d\xi + \frac{1}{2}\left[\frac{d\phi_0}{d\zeta}(\zeta) + i\phi_1(\zeta)\right]\end{aligned}$$

(5.9)

$$\begin{aligned}\underset{\sim}{A}_1(U)^{(1)}) \equiv U \equiv \int_{\zeta^*}^{\zeta} &\left\{\int_{\zeta}^{\zeta^*} |f'(\zeta)|^2\, U^{(1)}(\xi,\xi^*)d\xi^*\right.\\ &\left. + \frac{1}{2}\left[\frac{d\phi_0^*}{d\xi}(\xi) - i\phi_1(\xi)\right]\right\} d\xi + \phi_0(\zeta) .\end{aligned}$$

$$\underset{\sim}{A}_4(U^{(1)}) \equiv \frac{\partial^2 U}{\partial\zeta\partial\zeta^*} = |f'(\zeta)|^2\, U^{(1)} \tag{5.10}$$

$$(5.11)\qquad \underset{\sim}{A}_5(U^{(1)}) \equiv \frac{\partial^3 U}{\partial\zeta^2\partial\zeta^*} = \frac{\partial}{\partial\zeta}\,[|f'(\zeta)|^2\,U^{(1)}] \quad ,$$

$$(5.12)\qquad \underset{\sim}{A}_6(U^{(1)}) \equiv \frac{\partial^3 U}{\partial\zeta\partial\zeta^{*2}} = \frac{\partial}{\partial\zeta^*}\,[|f'(\zeta)|^2\,U^{(1)}] \; .$$

The Cauchy problem (5.4), (5.5) now becomes

$$(5.13)\qquad \frac{\partial 2_U(1)}{\partial\zeta\partial\zeta^*} = G(\zeta,\zeta^*,\underset{\sim}{A}_1(U^{(1)}),\cdots,\underset{\sim}{A}_6(U^{(1)}))$$

$$U^{(1)}(\zeta,\zeta^*) = \phi_0^{(1)}(\zeta) \;\; ; \qquad \zeta = \zeta^*$$

$$(5.14)\qquad i\left(\frac{\partial U^{(1)}}{\partial\zeta} - \frac{\partial U^{(1)}}{\partial\zeta^*}\right) = \phi_1^{(1)}(\zeta) \;\; ; \qquad \zeta = \zeta^*$$

where $\phi_0^{(1)}$, $\phi_1^{(1)}$ can be computed from equation (5.6) and the original Cauchy data ϕ_ℓ, $\ell = 0, 1, 2, 3$ ([12]). We next define the operator $\underset{\sim}{B}$ by (also see [21], Chapter III)

$$(5.15)\qquad s(\zeta,\zeta^*) = \frac{\partial^2 U^{(1)}}{\partial\zeta\partial\zeta^*}$$

$$\underset{\sim}{B}(s) \equiv U^{(1)}(\zeta,\zeta^*) = \int_0^{\zeta}\int_0^{\zeta^*} s(\xi,\xi^*)d\xi^* d\xi + \int_0^{\zeta}\gamma(\xi)d\xi \; +$$

$$+ \int_0^{\zeta^*}\psi(\xi^*)d\xi^* + \phi_0^{(1)}(0)$$

where [11]

(5.17)

$$\gamma(\zeta) = \frac{1}{2}\left[\frac{d\phi_0^{(1)}(\zeta)}{d\zeta} - i\phi_1^{(1)}(\zeta)\right] - \int_0^{\zeta} s(\zeta,\xi^*)d\xi^*$$

(5.18)
$$\psi(\zeta) = \frac{1}{2}\left|\frac{d\phi_0^{(1)}(\zeta)}{d\zeta} + i\phi_1^{(1)}(\zeta)\right| - \int_0^\zeta s(\xi,\zeta)d\xi \ .$$

Let $HB \equiv HB(\Delta\rho,\Delta\rho^*)$ be the Banach space of functions of two complex variables which are holomorphic and bounded in $\Delta\rho \times \Delta\rho^*$, $\Delta\rho = \{\zeta \mid |\zeta| < \rho\}$, $\Delta\rho^* = \{\zeta^* | \zeta^* \in \Delta\rho\}$, with norm

(5.19) $$\|s\| = \sup_{\Delta\rho\times\Delta\rho} |s(\zeta,\zeta^*)| \ .$$

Finding a solution of the Cauchy problem (4.13), (4.14) is now equivalent to finding a fixed point in the Banach space HB of the operator $\underset{\sim}{T}:HB \to HB$ defined by

(5.20) $$\underset{\sim}{T}s = G(\zeta,\zeta^*,\underset{\sim}{A}_1(B(s)),\cdots,\underset{\sim}{A}_6(B(s))) \ .$$

It was shown in [12] that due to the hypothesis imposed upon G that $\underset{\sim}{T}$ is a contraction mapping of a closed ball of HB into itself and hence has a unique fixed point $s(\zeta,\zeta^*)$. Equations (5.16) and (5.9) now allow us to construct $U(\zeta,\zeta^*)$, and use of the inverse mapping to $z = f(\zeta)$ yields the solution of our original Cauchy problem (5.1), (5.2). If equation (5.1) is linear, and one uses exponential majorization ([11], [21]), the above procedure yields global solutions.

Theorem: There exists a constructive procedure, suitable for analytic and numerical approximations, for solving the Cauchy problem (2.1), (2.2). Such a procedure is given explicitly by equations (5.3) - (5.20).

The Cauchy-Kowalewski Theorem also provides a method for constructing solutions to elliptic Cauchy problems. However, this method is not satisfactory for purposes of approximation theory of numerical computation. The difficulties which arise in devising efficient procedures for solving initial value problems for elliptic equations is due to the fact that such problems are improperly posed due to the lack of continuous dependence on the initial data ([20]). In our analysis this unstable dependence appears exclusively in the step where this data is extended to complex values of the independent variables ξ, η. When this can be done in an elementary way, for example by direct substitution via the conjugate-coordinate transformation, no instabilities will occur when one uses the contraction mapping operator $\underset{\sim}{T}$ to obtain approximations to the desired solution.

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Pseudoparabolic Equations in One Space Variable*

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I. INTRODUCTION

In this paper we will study the first initial-boundary value problem for the pseudoparabolic equation

$$\mathscr{L}[u] = u_{xtx} + d(x, t)\, u_t + \eta u_{xx} + a(x)\, u_x + b(x)u = q(x, t) \qquad (1.1)$$

defined in the rectangle $D(H, T) = \{x, t) \mid 0 < x < H, 0 < t < T\}$. We make the assumption that the coefficient $d(x, t)$ is continuously differentiable in $\bar{D}(H, T) = \{(x, t) \mid 0 \leqslant x \leqslant H, 0 \leqslant t \leqslant T\}$, $q(x, t)$ is continuous in $\bar{D}(H, T)$, $a(x)$ is continuously differentiable in $0 \leqslant x \leqslant H$, and $b(x)$ is continuous in $0 \leqslant x \leqslant H$. In equation (1.1) η is a constant. Equations of the form (1.1) have been called pseudoparabolic by Showalter and Ting ([6]) not only because well posed initial-boundary value problems for parabolic equations are well posed for equation (1.1), but also because of the fact that in certain cases the solution of a parabolic initial-boundary value problem can be obtained as the limit of solutions to the corresponding problem for a related class of pseudoparabolic equations. Equations of the form of equation (1.1) arise in the study of nonsteady simple shearing flow of second order fluids (c.f. [2], [3], [8]) and also in the theory of the consolidation of clay ([7]) and the theory of seepage of homogeneous fluids through fissured rocks ([1]).

Our main contribution in this paper is the introduction of a special solution of equation (1.1) analogous to the Riemann function for hyperbolic equations. This function can be constructed by iteration and we will refer to it as the Riemann function for equation (1.1). We will then use this Riemann function to reduce the solution of the first initial-boundary value problem for equation (1.1) to that of solving a one dimensional Volterra integral equation. This is made possible by the fact that both the lines t = constant and x = constant

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are characteristics of equation (1.1). Our approach leads in a direct and natural manner to sufficient conditions for the existence, uniqueness and continuous dependence on the boundary data of the solution to the first initial-boundary value problem for equation (1.1). In this regard our work is connected with that of Showalter and Ting ([6]) who used a Hilbert space approach to study initial-boundary value problems for pseudoparabolic equations in n space variables, $n \geqslant 1$. However in contrast to the work of Showalter and Ting our work allows some of the coefficients to be time dependent and also gives a constructive method for obtaining the solution to the initial-boundary value problem under investigation. In another direction our work is related to that of [4] in which a Riemann function was constructed for a class of pseudoparabolic equations in two space dimensions and used to investigate the analytic behaviour of solutions to such equations.

II. The Riemann Function

We define the adjoint equation to $\mathscr{L}[u] = 0$ to be

$$\mathscr{M}[v] = v_{xtx} + d(x, t)\, v_t - \eta v_{xx} + (a(x)v)_x - b(x)v = 0. \tag{2.1}$$

Now let $(\xi, \tau) \in D(H, T)$ and integrate the identity

$$v_t \mathscr{L}[u] - u_t \mathscr{M}[v] = \frac{\partial}{\partial x}[u_{xt}v_t - u_t v_{xt} - au_t v + \eta u_x v_t + \eta u_t v_x] + \frac{\partial}{\partial t}[au_x v + buv - \eta u_x v_x] \tag{2.2}$$

over the rectangle R which is bounded by the lines $x = 0$, $x = \xi$, $t = 0$ and $t = \tau$. An application of Green's formula gives

$$\begin{aligned}\int_0^\tau \int_0^\xi (v_t \mathscr{L}[u] - u_t \mathscr{M}[v])\, dx\, dt &= \int_{\partial R} (u_{xt}v_t - u_t v_{xt} - au_t v + \eta u_x v_t + \eta u_t v_x)\, dt \\ &\quad - (au_x v + buv - \eta u_x v_x)\, dx.\end{aligned} \tag{2.3}$$

Suppose there exists a function $v(x, t; \xi, \tau)$ satisfying $\mathscr{M}[v] = 0$ in R and the boundary conditions

$$v_x(\xi, t; \xi, \tau) = \frac{1}{\eta}[1 - e^{\eta(t-\tau)}] \tag{2.4a}$$

$$v(\xi, t; \xi, \tau) = 0 \tag{2.4b}$$

$$v(x, \tau; \xi, \tau) = 0, \tag{2.4c}$$

where if $\eta = 0$ the boundary condition (2.4a) is to be interpreted in its limiting form as $\eta \to 0$. Then if there exists a solution u of $\mathscr{L}[u] = q$ in $D(H, T)$ satisfying

$$u(0, t) = f(t) \tag{2.5a}$$

$$u_x(0, t) = g(t) \tag{2.5b}$$

$$u(x, 0) = h(x) \tag{2.5c}$$

where $f(t), g(t) \in C^1[0, T]$, $h(x) \in C^2[0, H]$, we have from equation (2.3) that

$$\begin{aligned} u(\xi, \tau) = h(\xi) &+ \int_0^{\xi} [a(x) h'(x) v(x, 0; \xi, \tau) - \eta h'(x) v_x(x, 0; \xi, \tau) \\ &+ b(x) h(x) v(x, 0; \xi, \tau)] \, dx \\ &+ \int_0^{\tau} [g'(t) v_t(0, t; \xi, \tau) - f'(t) v_{xt}(0, t; \xi, \tau) \\ &- a(0) f'(t) v(0, t; \xi, \tau) + \eta g(t) v_t(0, t; \xi, \tau) \\ &+ \eta f'(t) v_x(0, t; \xi, \tau)] \, dt + \int_0^{\tau} \int_0^{\xi} q(x, t) v_t(x, t; \xi, \tau) \, dx \, dt. \end{aligned} \tag{2.6}$$

Equation (2.6) gives the solution of the Goursat problem (1.1), (2.5) in terms of the *Riemann function* $v(x, t; \xi, \tau)$. In particular if we can show that $v(x, t; \xi, \tau)$ exists and is sufficiently smooth, then we can use equation (2.6) to verify directly the existence of a function $u(x, t)$ satisfying $\mathscr{L}[u] = q$ and the initial data (2.5). We now turn our attention to this construction. Rewrite equation (2.1) in the form

$$v_{xtx} = F(x, t, v, v_t, v_x, v_{xx}) \tag{2.7}$$

where $F(x, t, v_t, v_x, v_{xx}) = \eta v_{xx} - d(x, t) v_t - (a(x)v)_x + b(x)v$. Let

$$s(x, t) = v_{xxt}(x, t) \tag{2.8}$$

and define the operators $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \mathbf{B}_4$ by

$$\begin{aligned} \mathbf{B}_1(s) &= v(x, t) = \int_{\xi}^{x} \int_{\tau}^{t} (x - x_1) s(x_1, t_1) \, dx_1 \, dt_1 + \frac{1}{\eta} (x - \xi)(1 - e^{\eta(t-\tau)}) \\ \mathbf{B}_2(s) &= v_t(x, t) = \int_{\xi}^{x} (x - x_1) s(x_1, t) \, dx_1 - (x - \xi) e^{\eta(t-\tau)} \\ \mathbf{B}_3(s) &= v_x(x, t) = \int_{\xi}^{x} \int_{\tau}^{t} s(x_1, t_1) \, dx_1 \, dt_1 + \frac{1}{\eta} (1 - e^{\eta(t-\tau)}) \\ \mathbf{B}_4(s) &= v_{xx}(x, t) = \int_{\tau}^{t} s(x, t_1) \, dt_1 . \end{aligned} \tag{2.9}$$

Let $C(R)$ be the Banach space of continuous functions defined and continuous in the closed rectangle R with norm

$$\| s \|_\lambda = \max_{(x,t)\in R} \{e^{-\lambda[(\xi-x)+(\tau-t)]} \mid s(x, t)|\} \tag{2.10}$$

where $\lambda > 0$ is fixed, $s(x, t) \in C(R)$. The existence of a Riemann function $v(x, t; \xi, \tau)$ is now reduced to finding a fixed point of the operator $\mathbf{T}$ in $C(R)$ where

$$\mathbf{T}s = F(x, t, \mathbf{B}_1(s), \mathbf{B}_2(s), \mathbf{B}_3(s), \mathbf{B}_4(s)). \tag{2.11}$$

Due to the linearity and continuity of the coefficients of $\mathscr{M}[v] = 0$ we have for $(x, t) \in R$ there exists a constant C such that

$$\begin{aligned} \| \mathbf{T}s_1 - \mathbf{T}s_2 \|_\lambda \leqslant C\{\| \mathbf{B}_1 s_1 - \mathbf{B}_1 s_2 \|_\lambda + \| \mathbf{B}_2 s_1 - \mathbf{B}_2 s_2 \|_\lambda \\ + \| \mathbf{B}_3 s_1 - \mathbf{B}_3 s_2 \|_\lambda + \| \mathbf{B}_4 s_1 - \mathbf{B}_4 s_2 \|_\lambda\}. \end{aligned} \tag{2.12}$$

From estimates of the form

$$\begin{aligned} \left| \int_\xi^x s(x_1, t)\, dx_1 \right| &\leqslant \int_x^\xi \| s \|_\lambda\, e^{\lambda[(\xi-x_1)+(\tau-t)]}\, dx_1 \\ &\leqslant \frac{1}{\lambda} \| s \|_\lambda [e^{\lambda[(\xi-x)+(\tau-t)]} - e^{\lambda(\tau-t)}], \end{aligned} \tag{2.13}$$

i.e.

$$\begin{aligned} \left\| \int_\xi^x s(x_1, t)\, dx_1 \right\|_\lambda &\leqslant \frac{1}{\lambda} \| s \|_\lambda [1 - e^{-\lambda(\xi-x)}] \\ &\leqslant \frac{1}{\lambda} \| s \|_\lambda, \end{aligned} \tag{2.14}$$

we have

$$\| \mathbf{B}_i s_1 - \mathbf{B}_i s_2 \|_\lambda \leqslant \frac{C_i}{\lambda} \| s_1 - s_2 \|_\lambda, \qquad i = 1, 2, 3, 4 \tag{2.15}$$

where the C_i are positive constants independent of λ.

From equation (2.12) this implies that

$$\| \mathbf{T}s_1 - \mathbf{T}s_2 \|_\lambda \leqslant \frac{M}{\lambda} \| s_1 - s_2 \|_\lambda \tag{2.16}$$

where M is a positive constant independent of λ. We also have from equation (2.16) that

$$\| \mathbf{T}s \|_\lambda \leqslant \frac{M}{\lambda} \| s \|_\lambda + M_0 \tag{2.17}$$

where M_0 is a positive constant.

Equations (2.16) and (2.17) imply that for λ sufficiently large $\mathbf{T}$ takes a closed ball of $C(R)$ into itself and is a contraction mapping. Thus by the

Banach contraction mapping principle there exists an $s \in C(R)$ such that $s = \mathbf{T}s$ and we have established the existence of a Riemann function $v(x, t; \xi, \tau)$. Note that by construction we have shown that with respect to its first two variables $v(x, t; \xi, \tau)$ is a strong solution of $\mathscr{M}[v] = 0$ and is furthermore continuous with respect to its four independent variables for $(x, t) \in R$, $x \leqslant \xi \leqslant H$, $t \leqslant \tau \leqslant T$.

We now wish to establish some further regularity properties of $v(x, t; \xi, \tau)$. Let $(\alpha, \beta) \in R$ and let $R_{\alpha\beta}$ be the rectangle bounded by the lines $x = \alpha$, $x = \xi$, $t = \beta$ and $t = \tau$. By the same method we used to construct $v(x, t; \xi, \tau)$ we can construct a solution $w(x, t; \alpha, \beta)$ of $\mathscr{L}[u] = 0$ in $R_{\alpha\beta}$ which satisfies the boundary conditions

$$w_x(\alpha, t; \alpha, \beta) = \frac{1}{\eta}\left[e^{-\eta(t-\beta)} - 1\right] \tag{2.18a}$$

$$w(\alpha, t; \alpha, \beta) = 0 \tag{2.18b}$$

$$w(x, \beta; \alpha, \beta) = 0. \tag{2.18c}$$

Integrating the identity (2.2) over $R_{\alpha\beta}$ (setting $u = w$) and applying Green's, formula gives

$$w(\xi, \tau; \alpha, \beta) = v(\alpha, \beta; \xi, \tau), \tag{2.19}$$

i.e. as a function of its last two variables $v(x, t; \xi, \tau)$ is a solution of $\mathscr{L}[u] = 0$. We can furthermore easily show that if $s = w_{\xi\xi\tau}$ then s_α, s_β, $s_{\alpha\beta}$ are continuous for $\alpha \leqslant \xi \leqslant H$, $\beta \leqslant \tau \leqslant T$ (c.f. [5], pp. 116-117). It is now possible to show directly that equation (2.6) is the unique (strong) solution of $\mathscr{L}[u] = q$ satisfying the Goursat data (2.5) and that $u(x, t)$ depends continuously on the Goursat data $f(t)$, $g(t)$ and $h(x)$ and their derivatives. (In the case when the coefficients of equation (1.1) are entire functions of their independent (complex) variables it is not difficult to show that $v(x, t; \xi, \tau)$ is also an entire function of its independent variables. In this case equation (2.6) shows that if $u(x, t)$ is a solution of equation (1.1) which is analytic in some neighborhood of the origin $|x| < x_0$, $|t| < t_0$, and $h(x) = q(x, t) = 0$, then $u(x, t)$ can be analytically continued into the strip $-\infty < x < \infty$, $|t| < t_0$, a result analogous to the analytic behavior of solutions to parabolic equations.)

III. The First Initial-Boundary Value Problem for $\mathscr{L}[u] = q$

The first initial-boundary value problem for $\mathscr{L}[u] = q$ is to find a solution of $\mathscr{L}[u] = q$ in $D(H, T)$, continuously differentiable in $\bar{D}(H, T)$, which satisfies

$$u(0, t) = f(t) \tag{3.1a}$$

$$u(x, 0) = h(x) \tag{3.1b}$$

$$u(H, t) = \varphi(t). \tag{3.1c}$$

We will require $f(t)$, $\varphi(t) \in C^1[0, T]$ and $h(x) \in C^2[0, H]$. To find a solution of this problem we return to equation (2.6) and set $\xi = H$. After an integration by parts in the second integral on the right hand side we arrive at

$$\gamma(\tau) = g(\tau)\, v_t(0, \tau; H, \tau) + \int_0^\tau [v_t(0, t; H, \tau) - v_{tt}(0, t; H, \tau)]\, g(t)\, dt \tag{3.2}$$

where

$$\begin{aligned}\gamma(\tau) = \varphi(\tau) - h(H) - \int_0^H & [h'(x)(a(x)\, v(x, 0; H, \tau) - \eta v_x(x, 0; H, \tau)) \\ & + h(x)\, b(x)\, v(x, 0; H, \tau)]\, dx \\ & + h'(0)[v_t(0, 0; H, \tau) + \eta v_x(0, 0; H, \tau)] \\ & + \int_0^\tau f'(t)[v_{xt}(0, t; H, \tau) - a(0)\, v(0, t; H, \tau) + \eta v_x(0, t; H, \tau)]\, dt \\ & - \int_0^\tau \int_0^H q(x, t)\, v_t(x, t; H, \tau)\, dx\, dt. \end{aligned} \tag{3.3}$$

Note that due to the assumption that $f(t)$, $\varphi(t) \in C^1[0, T]$ and the coefficient $d(x, t)$ is continuously differentiable in $\bar{D}(H, T)$, we can conclude that $\gamma(\tau)$ and the kernel of the integral equation (3.2) is continuously differentiable with respect to τ for $0 \leqslant \tau \leqslant T$ (this follows from the construction of $v(x, t; \xi, \tau)$) and hence if a solution $g(\tau)$ of the integral equation exists, then $g(\tau) \in C^1[0, T]$. To show the existence of a unique solution to the integral equation (3.2) on the interval $0 \leqslant \tau \leqslant T$ it is sufficient to show that $v_t(0, \tau; H, \tau)$ is never equal to zero on this interval (c.f. [9]). To this end consider the function

$$\mu(x) = v_t(x, \tau; H, \tau) \tag{3.4}$$

for an arbitrary (but fixed) τ in the interval $0 \leqslant \tau \leqslant T$. Then from the differential equation (2.1) and the boundary condition (2.4c) we have

$$\mu_{xx} + d(x, \tau)\mu = 0. \tag{3.5}$$

Hence if we require $d(x, t) \leqslant 0$ for $(x, t) \in \bar{D}(H, T)$ we can conclude from the theory of ordinary differential equations that if $\mu(0) = 0$ then $\mu(x) \equiv 0$, since by equation (2.4b) $\mu(H) = 0$. This then implies that $\mu_x(H) = v_{xt}(H, \tau; H, \tau) = 0$. But by equation (2.4a) we have $v_{xt}(H, \tau; H, \tau) = -1$. Hence the assumption $\mu(0) = 0$ leads to a contradiction if we also assume $d(x, t) \leqslant 0$ in $\bar{D}(H, T)$. Making this assumption, solving equation (3.2) for $g(\tau)$, and then substituting into equation (2.6) gives the unique solution of the first initial-boundary value problem for $\mathscr{L}[u] = q$. We summarize our result in the following theorem:

THEOREM. *Let $d(x, t)$ be continuously differentiable and nonpositive in $\bar{D}(H, T)$, $q(x, t)$ continuous in $D(H, T)$, and assume $a(x) \in C^1[0, H]$, $b(x) \in C[0, H]$. Let $f(t)$, $\varphi(t) \in C^1[0, T]$ and $h(x) \in C^2[0, H]$. Then there exists a unique solution to $\mathscr{L}[u] = q(x, t)$ satisfying the initial-boundary data* (3.1).

The following example shows that in general the assumption that $d(x, t) \leqslant 0$ in $\bar{D}(H, T)$ is necessary (In reference [2] Coleman, Duffin, and Mizel give several theorems and examples which show that the assumption $d(x, t) \leqslant 0$ is essential to the theorem stated above. They show that for the equation $v_t = v_{xx} - v_{xtx}$ uniqueness can fail, and, if H is sufficiently small, there may exist no solutions in $\bar{D}(H, T)$).

EXAMPLE. $u(x, t) = t \sin kx$ is a solution of

$$u_{xtx} + k^2 u_t = 0 \tag{3.6}$$

for $(x, t) \in D(\pi/k, T)$, T arbitrary, and is continuously differentiable in $\bar{D}(\pi/k, T)$. But $u(x, t)$ satisfies the initial-boundary data $u(0, t) = u(\pi/k, t) = u(x, 0) = 0$, i.e. the solution of the first initial-boundary value problem for equation (3.6) in $D(\pi/k, T)$ is not unique.

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ON THE ANALYTIC THEORY OF PSEUDOPARABOLIC EQUATIONS†

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1. Introduction

THE analytic theory of linear elliptic partial differential equations in two independent variables has been extensively investigated during the past thirty years by such mathematicians as Bergman, Vekua, and Lewy [see (**1**), (**10**), (**7**)]. This area of research has by now developed into an important and elegant field in its own right which bridges the gap between analytic function theory and partial differential equations. In this paper we show that under the assumption of analytic initial conditions, a corresponding theory can be developed for pseudoparabolic equations. Such equations have the form of (1.1) below and acquire their name not only from the fact that well-posed initial-boundary value problems for parabolic equations are also well posed for this class of equations, but also from the fact that the solution of the parabolic initial-boundary value problem can in certain cases be obtained as the limit of a sequence of solutions to the corresponding pseudoparabolic problem [cf. (**8**)]. Equations of pseudoparabolic type make their appearance in physics in connection with certain problems associated with the flow of a viscous fluid. For example the velocity of a nonsteady simple shearing flow of a second-order fluid satisfies a pseudoparabolic equation with constant coefficients in one space dimension. Similarly, the hydrostatic excess pressure within a portion of clay during consolidation satisfies an equation of the form (1.1) below. Yet another example occurs in the theory of seepage of homogeneous liquids in fissured rocks. In this case the average pressure of the liquid in the fissures satisfies an equation of pseudoparabolic type. Considerable research has been done on these problems in recent years, and the reader is referred to the bibliography at the end of (**2**) and (**8**) for specific references. In this paper we will obtain integral operators, reflection laws, and regularity theorems for a general class of pseudoparabolic equations with time independent coefficients. This is accomplished through the discovery

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and use of a fundamental solution and an analogue of Vekua's complex Riemann function [see (**5**), (**10**)]. More specifically we will consider the pseudoparabolic equation

$$\mathbf{M}\left[\frac{\partial u}{\partial t}\right]+\gamma\mathbf{L}[u] = 0, \tag{1.1}$$

where **M** and **L** are linear second order elliptic operators in two independent variables with analytic coefficients and Laplacian as their principal part, **M** is self adjoint, and γ is a constant. (A similar class of equations was considered in (**8**) by Showalter and Ting who used a Hilbert space approach to study initial-boundary value problems associated with (1.1).) By making the preliminary substitution

$$u(x,y,t) = e^{-\gamma t}w(x,y,t),$$

it is seen that without loss of generality we can consider the equation

$$\mathscr{L}[u] \equiv M\left[\frac{\partial u}{\partial t}\right]+L[u] = 0, \tag{1.2}$$

where
$$M = \Delta+d(x,y),$$
$$L = a(x,y)\frac{\partial}{\partial x}+b(x,y)\frac{\partial}{\partial y}+c(x,y). \tag{1.3}$$

2. The fundamental solution and Riemann function

We will first construct a fundamental solution and Riemann function for (1.2) and use these functions to obtain results on the regularity of solutions to (1.2). We define the adjoint of $\mathscr{L}[u] = 0$ to be the equation

$$\mathscr{M}[v] \equiv M[v_t]-L^*[v] = 0, \tag{2.1}$$

where $L^*[v] \equiv -(av)_x-(bv)_y+cv$ and the subscripts denote differentiation. A function S of the form

$$S(x,y,t;\xi,\eta,\tau) = A(x,y,t;\xi,\eta,\tau)\log\frac{1}{r}+B(x,y,t;\xi,\eta,\tau), \tag{2.2}$$

where $r = [(x-\xi)^2+(y-\eta)^2]^{\frac{1}{2}}$, will be called a *fundamental solution* of equation (1.2) if it satisfies the following conditions:

(1) as a function of (x,y,t), S satisfies $\mathscr{M}[S] = 0$ and is an analytic function of its arguments except at $r = 0$, where S has a logarithmic singularity,

(2) at the parameter point $x = \xi$, $y = \eta$, $t = \tau$ we have $A_t = 1$,

(3) the functions A and B are analytic functions of (x,y,t) at $r = 0$ and vanish at $t = \tau$.

We now proceed to the construction of S. Considering x, y, ξ, η as independent complex variables, we first make the nonsingular transformation

$$\begin{aligned} z &= x+iy, & \zeta &= \xi+i\eta, \\ z^* &= x-iy, & \zeta^* &= \xi-i\eta. \end{aligned} \tag{2.3}$$

Then (2.1) can be written in the form

$$\begin{aligned} \mathscr{M}[V] &\equiv M[V_t]-L^*[V] \\ &= V_{zz^*t}+\delta V_t+(\alpha V)_z+(\beta V)_{z^*}-\gamma V \\ &= 0, \end{aligned} \tag{2.4}$$

where $V(z,z^*,t) = v(x,y,t), \alpha = \frac{1}{4}(a+ib), \beta = \frac{1}{4}(a-ib), \gamma = \frac{1}{4}c, \delta = \frac{1}{4}d$. Noting that, in the new variables, $r^2 = (z-\zeta)(z^*-\zeta^*)$, and substituting (2.2) into (2.4), gives

$$\mathscr{M}[S] = \mathscr{M}[A]\log\frac{1}{r}-\frac{A_{zt}+\beta A}{2(z^*-\zeta^*)}-\frac{A_{z^*t}+\alpha A}{2(z-\zeta)}+\mathscr{M}[B] = 0. \tag{2.5}$$

(Where no confusion can arise we will often use the same symbols S, A, B, etc. to denote the new functions of z, z^*, t, ζ, ζ^*, τ obtained from the original ones by means of the transformation (2.3).) Since A and B are regular at $z = \zeta$ and $z^* = \zeta^*$, and a multi-valued logarithmic singularity cannot be cancelled solely by poles, we have

$$\mathscr{M}[A] = 0. \tag{2.6}$$

In order to cancel the poles at $z^* = \zeta^*$ and $z = \zeta$ we have

$$\begin{aligned} \left[\frac{\partial^2}{\partial z\partial t}+\beta(z,\zeta^*)\right]A(z,\zeta^*,\tau;\zeta,\zeta^*,\tau) &= 0, \\ \left[\frac{\partial^2}{\partial z^*\partial t}+\alpha(\zeta,z^*)\right]A(\zeta,z^*,t;\zeta,\zeta^*,\tau) &= 0. \end{aligned} \tag{2.7}$$

Once we have determined A satisfying (2.6) and (2.7), then B can be any solution of

$$\mathscr{M}[B] = \frac{A_{zt}+\beta A}{2(z^*-\zeta^*)}+\frac{A_{z^*t}+\alpha A}{2(z-\zeta)}. \tag{2.8}$$

Now expand A in powers of $t-\tau$, taking note of condition (3) satisfied by S:

$$A(z,z^*,t;\zeta,\zeta^*,\tau) = \sum_{j=1}^{\infty} A_j(z,z^*;\zeta,\zeta^*)\frac{(t-\tau)^j}{j!}. \tag{2.9}$$

The boundary conditions (2.7) imply that on the characteristics $z = \zeta$, $z^* = \zeta^*$, the A_j satisfy the first order partial differential equations

$$\begin{aligned} \frac{\partial A_1}{\partial z}(z, \zeta^*; \zeta, \zeta^*) &= 0, \\ \frac{\partial A_1}{\partial z^*}(\zeta, z^*; \zeta, \zeta^*) &= 0, \end{aligned} \tag{2.10 a}$$

$$\begin{aligned} \frac{\partial A_{j+1}}{\partial z}(z, \zeta^*; \zeta, \zeta^*) + \beta(z, \zeta^*) A_j(z, \zeta^*; \zeta, \zeta^*) &= 0; \quad j = 1, 2, \ldots, \\ \frac{\partial A_{j+1}}{\partial z^*}(\zeta, z^*; \zeta, \zeta^*) + \alpha(\zeta, z^*) A_j(\zeta, z^*; \zeta, \zeta^*) &= 0; \quad j = 1, 2, \ldots. \end{aligned} \tag{2.10 b}$$

Condition (2) satisfied by S implies that

$$A_1(\zeta, \zeta^*; \zeta, \zeta^*) = 1, \tag{2.11 a}$$

$$A_j(\zeta, \zeta^*; \zeta, \zeta^*) = 0; \quad j = 2, 3, \ldots. \tag{2.11 b}$$

Equations (2.10) and (2.11) now yield

$$\begin{aligned} A_1(z, \zeta^*; \zeta, \zeta^*) &= 1, \\ A_1(\zeta, z^*; \zeta, \zeta^*) &= 1, \end{aligned} \tag{2.12 a}$$

and

$$\begin{aligned} A_{j+1}(z, \zeta^*; \zeta, \zeta^*) &= -\int_{\zeta}^{z} \beta(\sigma, \zeta^*) A_j(\sigma, \zeta^*; \zeta, \zeta^*)\, d\sigma; \quad j = 1, 2, \ldots, \\ A_{j+1}(\zeta, z^*; \zeta, \zeta^*) &= -\int_{\zeta^*}^{z^*} \alpha(\zeta, \rho) A_j(\zeta, \rho; \zeta, \zeta^*)\, d\rho; \quad j = 1, 2, \ldots. \end{aligned} \tag{2.12 b}$$

Substituting the series (2.9) into equation (2.6) gives

$$\begin{aligned} \mathscr{M}[A] &= M[A_t] - L^*[A] \\ &= M[A_1] + \sum_{j=1}^{\infty} (M[A_{j+1}] - L^*[A_j]) \frac{(t-\tau)^j}{j!} \\ &= 0. \end{aligned} \tag{2.13}$$

Equating powers of $(t-\tau)$ to zero gives

$$M[A_1] = 0, \tag{2.14 a}$$

$$M[A_{j+1}] = L^*[A_j]. \tag{2.14 b}$$

Equations (2.12) and (2.14) now determine a recursive sequence of characteristic initial value problems for the coefficients A_j. Note that in particular $A_1(z, z^*; \zeta, \zeta^*)$ is the complex Riemann function [see (**5**), (**10**)] for $M[u] = 0$. This motivates our calling the function

$$A(z, z^*, t; \zeta, \zeta^*, \tau)$$

the Riemann function for equation (1.2). We must now show the existence of $A(z, z^*, t; \zeta, \zeta^*, \tau)$. To this end we have the following lemma:

LEMMA 2.1. *Assume that α, β, γ, and δ are analytic functions of the two complex variables z, z^* in the cylindrical domain $D \times D^*$, where D is simply connected and $D^* = \{z^* \mid \bar{z}^* \in D\}$. Then the series (2.9) converges absolutely and uniformly for $|t-\tau| \leqslant R$, z, $\zeta \in \Omega$, z^*, $\zeta^* \in \Omega^*$, where R is an arbitrarily large positive number, Ω is a compact subset of D, and Ω^* is a compact subset of D^*. In particular $A(z, z^*, t; \zeta, \zeta^*, \tau)$ is an analytic function of its six independent variables for all (complex) t, τ and z, $\zeta \in D$, z^*, $\zeta^* \in D^*$.*

Proof. From (**10**) 17, it is clear that $A_1(z, z^*; \zeta, \zeta^*)$ is an analytic function of z, z^*, ζ, ζ^* for z, $\zeta \in D$, z^*, $\zeta^* \in D^*$. From (2.12 b) and (**10**) 19, 24, we can write, for $j = 1, 2, 3, \ldots$,

$$
\begin{aligned}
A_{j+1}(z, z^*; \zeta, \zeta^*) = &-\int_{\zeta}^{z} A_1(\sigma, \zeta^*; z, z^*)\beta(\sigma, \zeta^*)A_j(\sigma, \zeta^*; \zeta, \zeta^*)\, d\sigma - \\
&-\int_{\zeta^*}^{z^*} A_1(\zeta, \rho; z, z^*)\alpha(\zeta, \rho)A_j(\zeta, \rho; \zeta, \zeta^*)\, d\rho + \\
&+\int_{\zeta}^{z}\int_{\zeta^*}^{z^*} A_1(\sigma, \rho; z, z^*)L^*[A_j(\sigma, \rho; \zeta, \zeta^*)]\, d\rho d\sigma
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
= &-\int_{\zeta}^{z} A_1(\sigma, z^*; z, z^*)\beta(\sigma, z^*)A_j(\sigma, z^*; \zeta, \zeta^*)\, d\sigma - \\
&-\int_{\zeta^*}^{z^*} A_1(z, \rho; z, z^*)\alpha(z, \rho)A_j(z, \rho; \zeta, \zeta^*)\, d\rho + \\
&+\int_{\zeta}^{z}\int_{\zeta^*}^{z^*} \Big[A_1(\sigma, \rho; z, z^*)\gamma(\sigma, \rho) + \\
&+\frac{\partial}{\partial\sigma}A_1(\sigma, \rho; z, z^*)\alpha(\sigma, \rho) + \frac{\partial}{\partial\rho}A_1(\sigma, \rho; z, z^*)\beta(\sigma, \rho)\Big] \times \\
&\times A_j(\sigma, \rho; \zeta, \zeta^*)\, d\rho d\sigma,
\end{aligned}
\tag{2.16}
$$

where we have integrated by parts in (2.16). By induction it is clear that $A_j(z, z^*; \zeta, \zeta^*)$ is analytic for z, $\zeta \in D$, z^*, $\zeta^* \in D^*$. Now let K be an upper bound on $|A_1(z,z^*;\zeta,\zeta^*)\beta(z,z^*)|$, $|A_1(z,z^*;\zeta,\zeta^*)\alpha(z,z^*)|$, and

$$\left|A_1(z,z^*;\zeta,\zeta^*)\gamma(z,z^*)+\frac{\partial}{\partial z}A_1(z,z^*;\zeta,\zeta^*)\alpha(z,z^*)+\right.$$
$$\left.+\frac{\partial}{\partial z^*}A_1(z,z^*;\zeta,\zeta^*)\beta(z,z^*)\right|$$

for z, $\zeta\in\Omega$, z^*, $\zeta^*\in\Omega^*$, let l be an upper bound on the length of the paths of integration in (2.16), and let $|A_1(z,z^*,\zeta,\zeta^*)|\leqslant C$ for z, $\zeta\in\Omega$, $z^*,\zeta^*\in\Omega^*$. Then from (2.16) we have by induction that

$$|A_j(z,z^*;\zeta,\zeta^*)|\leqslant CK^jl^j(2+l)^j \quad (z,\zeta\in\Omega;z^*,\zeta^*\in\Omega^*). \tag{2.17}$$

Equation (2.17) implies that the series (2.9) is absolutely and uniformly convergent for $|t-\tau|\leqslant R$, z, $\zeta\in\Omega$, z^*, $\zeta^*\in\Omega^*$. Since Ω and Ω^* are arbitrary compact subsets, and the uniform limit of analytic functions is analytic, the lemma is now completely proved.

We now turn our attention to the construction of $B(z,z^*,t;\zeta,\zeta^*,\tau)$. Setting

$$B(z,z^*,t;\zeta,\zeta^*,\tau)=\sum_{j=1}^{\infty}B_j(z,z^*;\zeta,\zeta^*)\frac{(t-\tau)^j}{j!}, \tag{2.18}$$

substituting the series (2.18) into (2.8), and equating the powers of $(t-\tau)^j$ to zero gives

$$M[B_{j+1}]=L^*[B_j]+\frac{(\partial/\partial z)A_{j+1}+\beta A_j}{2(z^*-\zeta^*)}+\frac{(\partial/\partial z^*)A_{j+1}+\alpha A_j}{2(z-\zeta)};$$
$$j=1,2,\ldots, \tag{2.19}$$
$$M[B_1]=\frac{\partial A_1/\partial z}{2(z^*-\zeta^*)}+\frac{\partial A_1/\partial z^*}{2(z-\zeta)}.$$

Since B is an arbitrary solution of (2.8) we can impose, in particular, the initial conditions

$$B_j(z,\zeta^*;\zeta,\zeta^*)=B_j(\zeta,z^*;\zeta,\zeta^*)=0;\quad j=1,2,3,\ldots. \tag{2.20}$$

Noting that by construction the right-hand side of (2.19) is regular for z, $\zeta\in D$, z^*, $\zeta^*\in D^*$, we can follow the proof of Lemma 2.1 and show the uniform convergence of the series (2.18) for $|t-\tau|\leqslant R$, z, $\zeta\in\Omega$, z^*, $\zeta^*\in\Omega^*$. In particular we have the following lemma:

LEMMA 2.2. *Assume that α, β, γ, and δ are analytic functions of the two complex variables z, z^* in the cylindrical domain $D\times D^*$ where D is simply connected and $D^*=\{z^*\mid\overline{z^*}\in D\}$. Then $B(z,z^*,t;\zeta,\zeta^*,\tau)$ is an analytic* *function of its six independent variables for all (complex)* t, τ, and z, $\zeta \in D$, z^*, $\zeta^* \in D^*$.

Now let D be a simply connected domain in two-dimensional Euclidean space, let $T = \{t \mid 0 \leqslant t < T_0\}$ where T_0 is a fixed constant, and let $\mathfrak{C}(D\times T) = \{u(x,y,t) \mid u, u_x, u_y \in C^1(D\times T); u_{xyt}, u_{xxt}, u_{yyt} \in C^0(D\times T)\}$. Let $u(x,y,t) \in \mathfrak{C}(D\times T)$ be a solution of (1.2) in $D\times T$. We wish to show that if α, β, γ, δ and $U(z,z^*,0) = u(x,y,0)$ are analytic in $D\times D^*$, then so is $U(z,z^*,t) = u(x,y,t)$ for each fixed $t \in T$. We first show that without loss of generality we can assume $U(z,z^*,0) = 0$. Let

$$f(z,z^*) = L[U(z,z^*,0)]$$

and define
$$C(z,z^*,\zeta,\zeta^*,t) = \sum_{j=1}^{\infty} C_j(z,z^*,\zeta,\zeta^*)\frac{t^j}{j!}, \tag{2.21}$$

where
$$M[C_1] = f(z,z^*), \tag{2.22 a}$$

$$M[C_{j+1}] = -L[C_j]; \quad j = 1, 2,\dots, \tag{2.22 b}$$

and
$$C_j(z,\zeta^*,\zeta,\zeta^*) = C_j(\zeta,z^*,\zeta,\zeta^*) = 0; \quad j = 1, 2,\dots, \tag{2.23}$$
$$(\zeta,\zeta^*) \in D\times D^*.$$

Then by using the same techniques as in our previous analysis it is easily verified that $C(z,z^*,\zeta,\zeta^*,t)$ is analytic for all complex t, z, $\zeta \in D$, z^*, $\zeta^* \in D^*$, and satisfies

$$\mathscr{L}[C] = f(z,z^*), \tag{2.24}$$

$$C(z,z^*,\zeta,\zeta^*,0) = 0. \tag{2.25}$$

Hence
$$V(z,z^*,t) = U(z,z^*,t) - U(z,z^*,0) + C(z,z^*,\zeta,\zeta^*,t)$$

satisfies $\mathscr{L}[V] = 0$, $V(z,z^*,0) = 0$, and for each fixed t the domain of regularity of $V(z,z^*,t)$ and $U(z,z^*,t)$ coincide. We are now in a position to prove the following theorem:

THEOREM 2.1. *Let* $u(x,y,t) \in \mathfrak{C}(D\times T)$ *be a solution of* (1.2) *in* $D\times T$ *and assume that* $U(z,z^*,0)$ $(= u(x,y,0))$ *and the coefficients* α, β, γ, *and* δ *are analytic in the cylindrical domain* $D\times D^*$. *Then for each fixed* $t \in T$, $U(z,z^*,t) = u(x,y,t)$ *is an analytic function of the two complex variables* z, z^* *in* $D\times D^*$.

Proof. Without loss of generality we can assume that $U(z,z^*,0) = 0$. By slightly shrinking D we can furthermore assume without loss of generality that $u(x,y,t) \in \mathfrak{C}(\bar{D}\times T)$ and that D has a smooth boundary.

Now note that for $u, v \in \mathfrak{C}(D\times T)$ the expression

$$v_t\mathscr{L}[u]-u_t\mathscr{M}[v] = \frac{\partial}{\partial x}[u_{xt}v_t-u_tv_{xt}-au_tv]+$$

$$+\frac{\partial}{\partial y}[u_{yt}v_t-u_tv_{yt}-bu_tv]+\frac{\partial}{\partial t}[cuv+au_xv+bu_yv] \quad (2.26)$$

is a divergence, and hence by the divergence theorem

$$\iiint\limits_{D\times T}(v_t\mathscr{L}[u]-u_t\mathscr{M}[v])\,dxdydt = \iint\limits_{\partial(D\times T)}(u_{xt}v_t-u_tv_{xt}-au_tv)\,dydt-$$

$$-(u_{yt}v_t-u_tv_{yt}-bu_tv)\,dxdt+(cuv+au_xv+bu_yv)\,dxdy. \quad (2.27)$$

Now let $T_0 = \tau$, let v be the fundamental solution S, let u satisfy $\mathscr{L}[u] = 0$, and integrate (2.26) over the region $D\times T-\Omega\times T$ (instead of $D\times T$) where Ω is a thin cylinder surrounding the singular line $r = 0$. Note that u vanishes on the plane $t = 0$, S vanishes on the plane $t = \tau$, and the left-hand side of (2.26) is identically zero. Computing the residue as Ω shrinks down onto the singular line $r = 0$ gives

$$0 = 2\pi\int_0^\tau u_t(\xi,\eta,t)\,dt+$$

$$+\int_0^\tau\int_{\partial D}[(u_{xt}S_t-u_tS_{xt}-au_tS)\,dy-(u_{yt}S_t-u_tS_{yt}-bu_tS)\,dx]\,dt, \quad (2.28)$$

or

$$u(\xi,\eta,\tau) = -\frac{1}{2\pi}\int_0^\tau\int_{\partial D}[(u_{xt}S_t-u_tS_{xt}-au_tS)\,dy-$$

$$-(u_{yt}S_t-u_tS_{yt}-bu_tS)\,dx]\,dt. \quad (2.29)$$

But from Lemmas 2.1, 2.2, and the definition of S it is seen from (2.29) that for each fixed τ, $U(\zeta,\zeta^*,\tau) = u(\xi,\eta,\tau)$ is an analytic function of ζ, ζ^* in $D\times D^*$. This completes the proof of the theorem.

Example. In Theorem 2.1 it is not possible to relax the assumption of analyticity of the initial data $U(z,z^*,0) = u(x,y,0)$. For example, consider the special case of equation (1.1) when $\mathbf{M} = \mathbf{L}$ and $\gamma = 1$. Then $u(x,y,t) = e^{-t}u(x,y,0)$ is a solution of (1.1) and is not analytic for each fixed t unless the initial data $u(x,y,0)$ is analytic.

We also note in passing that if in addition to satisfying the hypothesis of Theorem 2.1, $u(x,y,t) = 0$ on $\partial D\times T$, then $U(z,z^*,t) = u(x,y,t)$ is also an analytic function of t for each fixed $(z,z^*)\in D\times D^*$ [see (**8**)].

Hence by Hartog's theorem [see (5)] we can conclude that $U(z, z^*, t)$ is an analytic function of the three complex variables z, z^*, t in the product domain $D \times D^* \times T$.

3. Integral operators and the reflection principle

We now turn our attention to the construction of integral operators which map analytic functions onto solutions of (1.2), and to the derivation of an analogue of Lewy's reflection principle [(7); see also (4), 641–50]. We will consider only the case of reflection across plane boundaries; the more general case can be reduced to this special case through the use of a conformal mapping.

We first note the formal identity

$$V_t \mathscr{L}[U] - U_t \mathscr{M}[V] = \frac{\partial}{\partial z}(U_{tz^*} V_t - \alpha U_t V) - $$

$$- \frac{\partial}{\partial z^*}(U_t V_{tz} + \beta U_t V) + \frac{\partial}{\partial t}(\alpha U_z V + \beta U_{z^*} V + \gamma U V). \quad (3.1)$$

Now let V be the Riemann function A constructed in § 2 and let $\mathscr{L}[U] = 0$. Note that by Lemma 2.1 and Theorem 2.1 both U and A are analytic functions of (z, z^*) in $D \times D^*$ and so the derivatives in (3.1) are well defined. Integrating (3.1) over a three-dimensional cell $G \subseteq D \times D^* \times T$ in the complex domain with piecewise smooth boundary ∂G and applying Stokes's theorem [cf. (4) 167, 213], we arrive at

$$0 = \iint_{\partial G} (U_{tz^*} A_t - \alpha U_t A)\, dz^* dt +$$

$$+ \iint_{\partial G} (U_t A_{tz} + \beta U_t A)\, dz dt + \iint_{\partial G} (\alpha U_z A + \beta U_{z^*} A + \gamma U A)\, dz dz^* \quad (3.2)$$

$$= \iint_{\partial G} (U_t A_t)_{z^*}\, dz^* dt - \iint_{\partial G} U_t(A_{tz^*} + \alpha A)\, dz^* dt +$$

$$+ \iint_{\partial G} U_t(A_{tz} + \beta A)\, dz dt + \iint_{\partial G} (\alpha U_z A + \beta U_{z^*} A + \gamma U A)\, dz dz^*. \quad (3.3)$$

Now let ∂G be homologous to the boundary of a cube with corners $(\zeta_0, \zeta_0^*, 0)$, $(\zeta, \zeta_0^*, 0)$, $(\zeta, \zeta^*, 0)$, $(\zeta_0, \zeta^*, 0)$ and $(\zeta_0, \zeta_0^*, \tau)$, (ζ, ζ_0^*, τ), (ζ, ζ^*, τ), (ζ_0, ζ^*, τ). Then integrating (3.3), paying attention to the boundary conditions satisfied by A, gives

$$U(\zeta,\zeta^*,\tau) = U(\zeta,\zeta^*,0)+\int_0^\tau U_t(\zeta,\zeta_0^*,t)A_t(\zeta,\zeta_0^*,t;\zeta,\zeta^*,\tau)\,dt-$$

$$-\int_0^\tau \big(U_t(\zeta_0,\zeta_0^*,t)A_t(\zeta_0,\zeta_0^*,t;\zeta,\zeta^*,\tau)-U_t(\zeta_0,\zeta^*,t)A_t(\zeta_0,\zeta^*,t;\zeta,\zeta^*,\tau)\big)\,dt-$$

$$-\int_0^\tau\int_{\zeta_0^*}^{\zeta^*} U_t(\zeta_0,z^*,t)\big(A_{tz^*}(\zeta_0,z^*,t;\zeta,\zeta^*,\tau)+$$

$$+\alpha(\zeta_0,z^*)A(\zeta_0,z^*,t;\zeta,\zeta^*,\tau)\big)\,dz^*dt$$

$$-\int_0^\tau\int_{\zeta_0}^{\zeta} U_t(z,\zeta_0^*,t)\times$$

$$\times\big(A_{tz}(z,\zeta_0^*,t;\zeta,\zeta^*,\tau)+\beta(z,\zeta_0^*)A(z,\zeta_0^*,t;\zeta,\zeta^*,\tau)\big)\,dzdt+$$

$$+\int_{\zeta_0}^{\zeta}\int_{\zeta_0^*}^{\zeta^*} A(z,z^*,0;\zeta,\zeta^*,\tau)\times$$

$$\times\big(\alpha(z,z^*)U_z(z,z^*,0)+\beta(z,z^*)U_{z^*}(z,z^*,0)+\gamma(z,z^*)U(z,z^*,0)\big)\,dzdz^*. \tag{3.4}$$

It is easily seen that (3.4), where $U(\zeta,\zeta^*,0)$, $U_t(\zeta,\zeta_0^*,t)$, $U_t(\zeta_0,\zeta^*,t)$ are arbitrary analytic functions of ζ and ζ^* for $\zeta\in D$, $\zeta^*\in D^*$, continuous in $D\times D^*\times T$, give all solutions of $\mathscr{L}[U]=0$ in $\mathfrak{C}(D\times T)$ which are analytic in $D\times D^*$ for each fixed t. Using Theorem 2.1 and interchanging the roles of ζ, ζ^*, τ and z, z^*, t respectively in (3.4) now gives us the following theorem:

THEOREM 3.1. *Let $U(z,z^*,0)$ $(=u(x,y,0))$ and the coefficients α,β,γ, and δ be analytic in the cylindrical domain $D\times D^*$, and let $u(x,y,t)\in\mathfrak{C}(D\times T)$ be a solution of $\mathscr{L}[u]=0$ in $D\times T$. Then there exist continuous functions $\phi^{(1)}(z,t)$, $\phi^{(2)}(z^*,t)$, $\phi^{(3)}(z,z^*)$, analytic for $z\in D$, $z^*\in D^*$, such that $u(x,y,t)$ has the representation*

$$u(x,y,t) = \phi^{(3)}(z,z^*)+\int_0^t \phi^{(1)}(z,\tau)A_\tau(z,z_0^*,\tau;z,z^*,t)\,d\tau-$$

$$-\int_0^t \big(\phi^{(2)}(z_0^*,\tau)A_\tau(z_0,z_0^*,\tau;z,z^*,t)-\phi^{(2)}(z^*,\tau)A_\tau(z_0,z^*,\tau;z,z^*,t)\big)\,d\tau-$$

$$-\int_0^t\int_{z_0^*}^{z^*}\phi^{(2)}(\zeta^*,\tau)\times$$

$$\times\big(A_{\tau\zeta^*}(z_0,\zeta^*,\tau;z,z^*,t)+\alpha(z_0,\zeta^*)A(z_0,\zeta^*,\tau;z,z^*,t)\big)\,d\zeta^*d\tau-$$

$$-\int_0^t \int_{z_0}^z \phi^{(1)}(\zeta, \tau) \times$$
$$\times \left(A_{\tau\zeta}(\zeta, z_0^*, \tau; z, z^*, t) + \beta(\zeta, z_0^*) A(\zeta, z_0^*, \tau; z, z^*, t) \right) d\zeta d\tau +$$
$$+ \int_{z_0}^z \int_{z_0^*}^{z^*} A(\zeta, \zeta^*, 0; z, z^*, t) \times$$
$$\times \left(\alpha(\zeta, \zeta^*) \phi_\zeta^{(3)}(\zeta, \zeta^*) + \beta(\zeta, \zeta^*) \phi_{\zeta^*}^{(3)}(\zeta, \zeta^*) + \gamma(\zeta, \zeta^*) \phi^{(3)}(\zeta, \zeta^*) \right) d\zeta d\zeta^*$$
$$(z = x+iy, z^* = x-iy, (x, y) \in D), \tag{3.5}$$

where $\phi^{(1)}(z, t)$, $\phi^{(2)}(z^*, t)$, $\phi^{(3)}(z, z^*)$ *are defined by*

$$\begin{aligned} \phi^{(1)}(z, t) &= U_t(z, z_0^*, t), \\ \phi^{(2)}(z^*, t) &= U_t(z_0, z^*, t), \\ \phi^{(3)}(z, z^*) &= U(z, z^*, 0), \end{aligned} \tag{3.6}$$

and $z_0 = x_0+iy_0$, $z_0^* = x_0-iy_0$ *where* (x_0, y_0) *is a fixed point in* D.

The integral operator defined in Theorem 3.1 has many applications analogous to integral operators for elliptic equations in two independent variables [cf. (**1**), (**5**), (**10**)]. For example, by using the theorems of Faber and Chebyshev to approximate $\phi^{(1)}$, $\phi^{(2)}$, and $\phi^{(3)}$ by polynomials, we can construct a complete family of solutions in the maximum norm to $\mathscr{L}[u] = 0$, and thus be in a position to approximate the solutions to the standard initial-boundary value problems associated with (1.2). We will not pursue this matter any further at this time, but instead now turn our attention to deriving a principle of reflection for pseudoparabolic equations analogous to that of Lewy [see (**7**)] for elliptic equations.

To be more precise, let $D \times T$ be a simply connected cylindrical domain in the half space $y < 0$ whose boundary contains a portion σ of the plane $y = 0$. Without loss of generality we assume that the origin is contained in σ. Let $u(x, y, t) \in \mathfrak{C}(D \times T) \cap C^2(\bar{D} \times T)$ be a solution of $\mathscr{L}[u] = 0$ in $D \times T$, and suppose that on σ we have

$$u(x, 0, t) = U(x, x, t) = \rho(x, t), \tag{3.7}$$

where $\rho(z, t)$ is a continuously differentiable function of z and t in $D \cup \sigma \cup D^* \times T$ and for each fixed $t \in T$ is an analytic function of z in $D \cup \sigma \cup D^*$. Assume further that $U(z, z^*, 0) = u(x, y, 0)$ and that the coefficients α, β, γ, and δ are analytic functions of z and z^* in

$$D \cup \sigma \cup D^* \times D \cup \sigma \cup D^*.$$

Returning now to (3.3), let ∂G be homologous to the boundary of the triangular wedge with corners $(\zeta, \bar{\zeta}, 0)$, $(\bar{\zeta}^*, \zeta^*, 0)$, $(\zeta, \zeta^*, 0)$ and $(\zeta, \bar{\zeta}, \tau)$,

$(\bar{\zeta}^*, \zeta^*, \tau)$, (ζ, ζ^*, τ). Then integrating (3.3), paying attention again to the boundary conditions satisfied by A, gives

$$
\begin{aligned}
U(\zeta, \zeta^*, \tau) = {} & U(\zeta, \zeta^*, 0) - \int_0^\tau U_t(\zeta, \bar{\zeta}, t) A_t(\zeta, \bar{\zeta}, t; \zeta, \zeta^*, \tau)\, dt - \\
& - \frac{1}{\sqrt{2}} \int_0^\tau \int_{\bar{\zeta}}^{\zeta^*} [(U_t(s, \bar{s}, t) A_t(s, \bar{s}, t; \zeta, \zeta^*, \tau))_{\bar{s}} - U_t(s, \bar{s}, t) \times \\
& \qquad \times (A_{t\bar{s}}(s, \bar{s}, t; \zeta, \zeta^*, \tau) + \alpha(s, \bar{s}) A(s, \bar{s}, t; \zeta, \zeta^*, \tau))]\, d\bar{s}dt - \\
& - \frac{1}{\sqrt{2}} \int_0^\tau \int_{\zeta}^{\bar{\zeta}^*} U_t(s, \bar{s}, t) \times \\
& \qquad \times (A_{ts}(s, \bar{s}, t; \zeta, \zeta^*, \tau) + \beta(s, \bar{s}) A(s, \bar{s}, t; \zeta, \zeta^*, \tau))\, ds dt + \\
& + \int_{\bar{\zeta}}^{\zeta^*} \int_{\zeta}^{s} A(z, s, 0; \zeta, \zeta^*, \tau) \times \\
& \qquad \times (\alpha(z, s) U_z(z, s, 0) + \beta(z, s) U_s(z, s, 0) + \gamma(z, s) U(z, s, 0))\, dz ds.
\end{aligned} \tag{3.8}
$$

Now for ζ in $D \cup \sigma$, $U(\zeta, 0, \tau)$ can be determined from (3.8). In (3.4) set $\zeta_0 = \zeta_0^* = 0$, differentiate with respect to τ, and set $\zeta^* = \zeta$. Since $U(\zeta, \zeta, \tau)$ and $U(\zeta, \zeta^*, 0)$ are known for $\zeta \in D \cup \sigma \cup D^*$ and

$$(\zeta, \zeta^*) \in D \cup \sigma \cup D^* \times D \cup \sigma \cup D^*$$

respectively, this differentiated version of (3.4) now becomes a Volterra integral equation for $U_\tau(0, \zeta^*, \tau)$, $\zeta^* \in D \cup \sigma$, $\tau \in T$. Since

$$A_\tau(\zeta, 0, \tau; \zeta, \zeta^*, \tau) = A_\tau(0, \zeta^*, \tau; \zeta, \zeta^*, \tau) = 1,$$

this equation can be solved for $U_\tau(0, \zeta^*, \tau)$, $\zeta^* \in D \cup \sigma$, $\tau \in T$. Furthermore, since for each fixed τ the kernel and terms not involving $U_\tau(0, \zeta^*, \tau)$ are analytic for $\zeta^* = \zeta \in D$ and continuous in $D \cup \sigma$, so must the solution $U_\tau(0, \zeta^*, \tau)$. But for $\zeta^* \in D^* \cup \sigma$, $U(0, \zeta^*, \tau)$ (and hence $U_\tau(0, \zeta^*, \tau)$) is known from (3.8). Due to the continuity properties imposed upon $U(\zeta, \zeta^*, \tau)$ in $\bar{D} \times T$, it is seen that for each fixed $\tau \in T$, $U_\tau(0, \zeta^*, \tau)$ is analytic for $\zeta^* \in D^*$ and continuous in $D^* \cup \sigma$. Hence the above construction of $U_\tau(0, \zeta^*, \tau)$, $\zeta^* \in D \cup \sigma$, furnishes for each fixed τ the analytic continuation of $U_\tau(0, \zeta^*, \tau)$ into $D \cup \sigma \cup D^*$ [(9) 157]. In a similar fashion $U_\tau(\zeta, 0, \tau)$ can be analytically continued into $D \cup \sigma \cup D^*$ for each fixed τ. From the representation of these functions in terms of their resolvents, and the continuity properties of $U(\zeta, \zeta^*, \tau)$ in $\bar{D} \times T$, it is also seen that $U_\tau(0, \zeta^*, \tau)$ and $U_\tau(\zeta, 0, \tau)$ are continuous in $D \cup \sigma \cup D^* \times T$. Equation

(3.4) now defines a solution of $\mathscr{L}[u] = 0$ in class $\mathfrak{C}(D \cup \sigma \cup D^* \times T)$. The continuation is unique since for each fixed τ the continuation is analytic. We summarize the above analysis in the following theorem:

THEOREM 3.2. *Let $D \times T$ be a simply connected cylindrical domain in the half space $y < 0$ whose boundary contains a portion σ of the plane $y = 0$. Let $u(x, y, t) \in \mathfrak{C}(D \times T) \cap C^2(\bar{D} \times T)$ be a solution of $\mathscr{L}[u] = 0$ in $D \times T$ and on σ suppose that we have $u(x, 0, t) = \rho(x, t)$ where*

$$\rho(z, t) \in C^1(D \cup \sigma \cup D^* \times T)$$

and for each fixed $t \in T$ is an analytic function of z in $D \cup \sigma \cup D^*$. Let $U(z, z^*, 0)$ $(= u(x, y, 0))$ and the coefficients α, β, γ, and δ be analytic functions of z and z^* in $D \cup \sigma \cup D^* \times D \cup \sigma \cup D^*$. Then $u(x, y, t)$ can be uniquely continued as a solution of $\mathscr{L}[u] = 0$ in class $\mathfrak{C}(D \cup \sigma \cup D^* \times T)$ into all of $D \cup \sigma \cup D^* \times T$.

It is of interest to note that Theorem 3.2 is a reflection law for a partial differential equation in three independent variables whose domain of dependence is of the same dimension (one) as in Lewy's theory for elliptic equations in two independent variables. This is in sharp contrast to the analytic continuation properties of elliptic equations in three independent variables [see (**3**)], although it does possess analogies with the reflection laws discovered by Hill [see (**6**)] for parabolic equations in three independent variables. In general, it appears that the analytic behaviour of pseudoparabolic equations is not completely analogous to that of either elliptic or parabolic equations, but occupies a position somewhere in between these two cases.

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Integral Operators and the First Initial Boundary Value Problem for Pseudoparabolic Equations with Analytic Coefficients

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I. INTRODUCTION

This paper was motivated by the desire to derive constructive methods for solving the first initial boundary value problem for the equation

$$p_t - \eta \Delta_n p_t = \kappa \Delta_n p; \qquad n = 2, 3, \tag{1.1}$$

where $p_t = \partial p/\partial t$ and κ and η are positive constants. (Equation (1.1) has previously been studied for the case $n = 1$ in [4] and will not be treated here.) This equation arises, for example, in the theory of seepage of liquids in fissured rocks [1], in whlch case p denotes the pressure in the fissures and η and κ are constants determined by the physical properties of the rock. The specific problem which arises is to construct a solution of Eq. (1.1) which assumes given initial conditions at $t = 0$ and prescribed boundary values on the cylinder $\partial D \times T$ (where D is a simply connected domain in $\mathbb{R}^n$ with Lyapunov boundary ∂D and $T = \{t\colon 0 \leqslant t \leqslant t_0\}$ where t_0 is a fixed, but arbitrarily large, positive constant). As in the theory of seepage in a porous medium, the steady-state initial conditions are of greatest interest (i.e., the harmonic initial distributions $p^{(1)}$ which satisfy Eq. (1.1)). Setting

$$p = e^{-(\kappa/\eta)t} u + p^{(1)}, \tag{1.2}$$

it is, therefore, seen that without loss of generality we can consider the equation,

$$\Delta_n u_t - (1/\eta)\, u_t + (\kappa/\eta^2)\, u = 0, \tag{1.3}$$

and assume that $u = 0$ at $t = 0$.

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In this paper we will consider the more general problem of solving the first initial boundary value problem (with homogeneous initial conditions) for the equations

$$\Delta_2 u_t + c(x, y)\, u_t + d(x, y)u = 0, \tag{1.4}$$

$$\Delta_n u_t + A(r^2)\, u_t + B(r^2)u = 0; \qquad n \geqslant 2, \tag{1.5}$$

where $c(x, y)$ and $d(x, y)$ are real valued (for x and y real) entire functions of their independent (complex) variables and $A(r^2)$ and $B(r^2)$ are real valued entire functions of $r^2 = x_1^2 + \cdots + x_n^2$. Our basic goal is to reduce the problem of finding a solution of the first initial boundary value problem for Eqs. (1.4) and (1.5) to that of solving an integral equation. For Eq. (1.4) this is accomplished by using the fundamental solution which was previously constructed by the author in [5]. However, for Eq. (1.5) a fundamental solution has not yet been constructed and we adopt an approach based on the use of integral operators. This involves first constructing integral operators for Eq. (1.4) which are analogous to Bergman's operators for elliptic equations [2, 9] and then using this as a basis for a "method of ascent" [6, 10, 11] to construct integral operators for Eq. (1.5). The result is an integral operator which maps solutions of the equation,

$$\Delta_n u_t = 0, \tag{1.6}$$

onto solutions of Eq. (1.5), and through the use of such an operator it is possible to reduce the first initial boundary value problem for Eq. (1.5) to the problem of solving an integral equation. An interesting aspect of our analysis is that the integral equations which arise are of neither Fredholm nor Volterra type, but of the form $f = (\mathbf{I} + \mathbf{T} + \mathbf{L})\mu$ where $\mathbf{T}$ is a Fredholm operator and $\mathbf{L}$ is a Volterra operator. We will show that under the assumption that $c(x, y) \leqslant 0$ and $A(r^2) \leqslant 0$, respectively, such equations are always solvable.

Equations of the form of Eqs. (1.4) and (1.5) were first systematically studied by Sobolev [18] and Galpern [7], and more recently by Showalter [13–16], Ting [20] and Showalter and Ting [17], who refer to such equations as being of pseudoparabolic or Sobolev–Galpern type. The approach of these authors is based on Hilbert space methods and yield quite general existence and uniqueness theorems in $\mathbb{R}^n \times T$ for n an arbitrary integer. In the case of one space dimension ($n = 1$) these equations have also been studied through the use of integral operators [4], Laplace transforms [19], and separation of variables [3]. In addition, the analytic theory of pseudoparabolic equations in two space dimensions has been studied by Colton [5].

II. Bergman Operators

We will now construct an integral operator which maps analytic functions of a single complex variable depending on a parameter t onto the class of real valued strong solutions of Eq. (1.4) which vanish at $t = 0$. Let D be a simply connected domain of $\mathbb{R}^2$ and $T = \{t: 0 \leqslant t \leqslant t_0\}$ where t_0 is a positive constant. Then from [5] we have that strong solutions of Eq. (1.4) defined in $D \times T$ and vanishing at $t = 0$ are, for each fixed t, analytic functions of $z = x + iy$ and $z^* = x - iy$ in $D \times D^*$ where $D^* = \{z^*: \bar{z}^* \in D\}$. Hence, we can rewrite Eq. (1.4) in the form

$$U_{zz^*t} + C(z, z^*)\, U_t + D(z, z^*)U = 0, \tag{2.1}$$

where

$$\begin{aligned} U(z, z^*, t) &= u\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}, t\right), \\ C(z, z^*) &= \frac{1}{4}\, c\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right), \\ D(z, z^*) &= \frac{1}{4}\, d\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right). \end{aligned} \tag{2.2}$$

We now look for a solution of Eq. (2.1) in the form

$$U(z, z^*, t) = \int_0^t \int_{-1}^{+1} E(z, z^*, t - \tau, s)\, f_\tau\left(\frac{z}{2}(1 - s^2), \tau\right) \frac{ds\, d\tau}{(1 - s^2)^{1/2}}, \tag{2.3}$$

where $f(z, t)$ is an analytic function of z and continuously differentiable with respect to t. Without loss of generality we can assume that $f(z, 0) = 0$. Substituting (2.3) into (2.1) and integrating by parts (c.f. [2, 9]) shows that $E(z, z^*, t, s)$ must satisfy the singular partial differential equation

$$(1 - s^2)\, E_{z^*st} - (1/s)\, E_{z^*t} + 2sz(E_{zz^*t} + CE_t + DE) = 0, \tag{2.4}$$

provided we impose the boundary conditions

$$\begin{aligned} E(z, z^*, 0, s) &= 0, \\ E_{z^*t}(0, z^*, t, s) &= 0, \\ E_{z^*t}(z, z^*, t, 0) &= 0, \end{aligned} \tag{2.5}$$

and require $E(z, z^*, t, s)$ to be analytic for $t \in T$, $s \in I = \{s: |s| \leqslant 1\}$, and $(z, z^*) \in D \times D^*$. Following Bergman [2] we now look for a solution of Eq. (2.4) in the form

$$E(z, z^*, t, s) = t + \sum_{k=1}^{\infty} s^{2k} z^k \int_0^{z^*} P^{(2k)}(z, \zeta^*, t)\, d\zeta^*, \tag{2.6}$$

where we require $P^{(2k)}(z, z^*, 0) = 0$ for $k = 1, 2, \ldots$. Substituting (2.6) into (2.4) and integrating with respect to t yields the following recursion formulas for the $P^{(2k)}(z, z^*, t)$:

$$P^{(2)} = -2tC - t^2 D,$$
$$(2k+1)\, P^{(2k+2)} = -2 \Big[P_z^{(2k)} + C \int_0^{z^*} P^{(2k)}(z, \zeta^*, t)\, d\zeta^* + D \int_0^t \int_0^{z^*} P^{(2k)}(z, \zeta^*, \tau)\, d\zeta^*\, d\tau \Big]; \quad k \geqslant 0. \tag{2.7}$$

Hence, each of the $P^{(2k)}$, $k = 1, 2, \ldots$, is uniquely determined. We now must show that the series (2.6) converges uniformly in $D \times D^* \times T \times I$. We will do a bit more than this and show that due to the fact that $C(z, z^*)$ and $D(z, z^*)$ are entire functions of z and z^* the series (2.6) converges for arbitrary values of z and z^* (uniformly on compact subsets in the space of two complex variables). Let r be an arbitrarily large positive number and let C_0 be a positive constant chosen such that (as functions of z)

$$C(z, z^*) \ll \frac{C_0}{1 - (z/r)}, \qquad D(z, z^*) \ll \frac{C_0}{1 - (z/r)}, \tag{2.8}$$

for $|z| < r$, $|z^*| < r$, where "$\ll$" denotes domination (c.f. [2]). We will now show by induction that there exist positive constants M and δ which are independent of k such that for $|z| < r$, $|z^*| < r$, $|t| \leqslant t_0$,

$$P^{(2k)} \ll M 2^k (1 + \delta)^k (2k - 1)^{-1} [1 - (z/r)]^{-(2k-1)} r^{-k}. \tag{2.9}$$

From Eqs. (2.7) and (2.8) this is obviously true for $k = 1$. Now suppose for $k = j$ we have

$$P^{(2j)} \ll M_j 2^j (1 + \delta)^j (2j - 1)^{-1} [1 - (z/r)]^{-(2j-1)} r^{-j}, \tag{2.10}$$

where for the time being we allow M_j to depend on j. Then from Eqs. (2.7) and (2.8) and the standard use of the theory of dominants we have

$$P^{(2j+2)} \ll \frac{M_j 2^{j+1} (1 + \delta)^j}{2j + 1} \left\{ 1 + \frac{C_0 r^2 + C_0 r^2 t_0}{2j - 1} \right\} [1 - (z/r)]^{-(2j+1)}\, r^{-j-1} \tag{2.11}$$

(the main property of dominants we have used in deriving Eq. (2.11) is that if $f \ll g$ then $f \ll g[1 - (z/r)]^{-1}$). By setting

$$M_{j+1} = M_j (1 + \delta)^{-1} \left\{ 1 + \frac{C_0 r^2 + C_0 r^2 t_0}{2j - 1} \right\}, \tag{2.12}$$

we have shown that Eq. (2.10) is true for j replaced by $j + 1$. But for j sufficiently large we have $M_{j+1} \leqslant M_j$, i.e., there exists a positive constant M which is independent of j such that $M_j \leqslant M$ for all j. Equation (2.9) is now established for all k.

We now return to the convergence of the series (2.6). First, let $D_{\alpha r} = \{z: |z| < r/\alpha\}$ and $D^*_{\alpha r} = \{z^*: |z^*| < r/\alpha\}$ where $\alpha > 1$ is fixed. We will show that for α sufficiently large the series (2.6) converges in $D_{\alpha r} \times D^*_{\alpha r} \times T \times I$. Using the estimate $[1 - (|z|/r)] \geqslant (\alpha - 1/\alpha)$ and the fact that if $f \ll M[1-(z/r)]^{-1}$ then $|f| \leqslant M[1 - (|z|/r)]^{-1}$ we have from Eq. (2.9) that the series (2.6) is majorized in $D_{\alpha r} \times D^*_{ar} \times T \times I$ by

$$t_0 + \sum_{k=1}^{\infty} \frac{rM2^k(1+\delta)^k \alpha^{k-1}}{(2k-1)(\alpha-1)^{2k-1}}. \tag{2.13}$$

If α is chosen such that $2(1 + \delta)\alpha(\alpha - 1)^{-2} < 1$ then the series (2.13) converges. Since r is an arbitrarily large positive number and δ is arbitrarily small and independent of r, we now have that the series (2.6) converges absolutely and uniformly for $|z| < r$, $|z^*| < r$, $|t| \leqslant t_0$ and $|s| \leqslant 1$.

We have now proved that the operator defined by Eq. (2.3) exists and maps analytic functions $f(z, t)$ into the class of (complex valued) strong solutions of Eq. (1.4) with homogeneous initial data. It is important for our purposes that this mapping, in fact, be onto the class of real valued solutions of Eq. (1.4). However, this is obviously not the case (even if we take the real part of the right side of Eq. (2.3)) since if $U(z, z^*t)$ can be represented in the form of Eq. (2.3), it must be true that $U_t(z, z^*, 0) = 0$. There are obviously solutions of Eq. (1.4) which do not satisfy this property (c.f. [5]). However, if we note that if $u(x, y, t)$ is a solution of Eq. (1.4) then so is $u_t(x, y, t)$, we can differentiate Eq. (2.3) with respect to t, take the real part of both sides, and arrive at a new operator which maps analytic functions onto solutions of Eq. (1.4) with homogeneous initial data. The step of taking the real part of both sides of Eq. (2.3) is justified since $c(x, y)$ and $d(x, y)$ are real valued for x and y real. We are, thus, led to consider the operator defined (for x, y real) by

$$u(x, y, t) = \mathrm{Re} \int_0^t \int_{-1}^{+1} E_t(z, \bar{z}, t - \tau, s) f_\tau\left(\frac{z}{2}(1 - s^2), \tau\right) \frac{ds\, d\tau}{(1 - s^2)^{1/2}}, \tag{2.14}$$

where "Re" denotes "take the real part." We will now show that every real valued strong solution $u(x, y, t)$ of Eq. (1.4) with homogeneous initial data can be represented in the form of Eq. (2.14). An elementary power series analysis (c.f. [21, pp. 55–56]) coupled with the results of [5] shows that such solutions are uniquely determined by their values on the characteristic $z^* = 0$.

Extending Eq. (2.14) to complex values of x and y and evaluating at $z^* = 0$ leads to the equation

$$U(z, 0, t) = \frac{1}{2} \int_{-1}^{+1} f\left(\frac{z}{2}(1 - s^2), t\right) \frac{ds}{(1 - s^2)^{1/2}} + \frac{\pi}{2} \bar{f}(0, t), \qquad (2.15)$$

where $\bar{f}(z, t) = \overline{f(\bar{z}, t)}$ and we have used the fact that $f(z, 0) = 0$. Equation (2.15) shows that $f(z, t)$ can be chosen such that $U(z, 0, t)$ assumes prescribed values (c.f. [2, pp. 12–13]), and, thus, $u(x, y, t) = U(z, \bar{z}, t)$ can be represented in the form of Eq. (2.14). Finally, we note that by comparing the recursion formulas (2.7) with those of Bergman [2, p. 13] we have

$$E_t(z, z^*, 0, s) = \tilde{E}(z, z^*, s), \qquad (2.16)$$

where $\tilde{E}(z, z^*, s)$ is Bergman's generating function for the elliptic equation $\Delta_2 u + c(x, y)u = 0$ (c.f. [2, 9]). We summarize the results of this section in the following theorem.

THEOREM 2.1. *Let $u(x, y, t)$ be a real valued strong solution of Eq.* (1.4) *defined in a cylindrical domain $D \times T$ where D is simply connected and suppose $u(x, y, 0) = 0$. Then $u(x, y, t)$ can be represented in the form of Eq.* (2.14) *where $f(z, t)$ is an analytic function of z and continuously differentiable with respect to t such that $f(z, 0) = 0$ and $E(z, z^*, t, s)$ is defined by Eqs.* (2.6) *and* (2.7). *$E(z, z^*, t, s)$ is an entire function of z and z^* and is analytic in t and s for $|t| \leqslant t_0$ and $|s| \leqslant 1$. $E_t(z, z^*, 0, s) = \tilde{E}(z, z^*, s)$ where $\tilde{E}(z, z^*, s)$ is Bergman's generating function for the elliptic equation $\Delta_2 u + c(x, y)u = 0$.*

The operator defined by Eq. (2.14) can be used to construct a complete family of solutions for Eq. (1.4). This is accomplished by setting $f(z, t) = z^l t^k$ and letting l and k be arbitrary nonnegative integers. Such a complete family can be used to approximate solutions of the first initial boundary value problem for Eq. (1.4) satisfying homogeneous initial conditions. We note that the operator presented in [5] can also be used to construct a complete family of solutions. However, the operator derived here is considerably easier to construct and is, therefore, more suitable for computational purposes.

III. THE METHOD OF ASCENT

We will now use the ideas of [6, 10] to extend the results of Section 2 to include Eq. (1.5). We first consider the case when $n = 2$. In this situation it is easily verified (c.f. [2, pp. 27–28]) that $E_t(z, \bar{z}, t, s)$ depends only on $r^2 = z\bar{z}$, t and s, and, hence, we can rewrite Eq. (2.14) in the form

$$u(x, y, t) = \int_0^t \int_{-1}^{+1} E_t(r^2, t - \tau, s)\, H_\tau[x(1 - s^2)^{1/2}, y(1 - s^2)^{1/2}, \tau] \frac{ds\, d\tau}{(1 - s^2)^{1/2}}, \tag{3.1}$$

where

$$H_t(x, y, t) = \operatorname{Re} f_t[(z/2), t] \tag{3.2}$$

is a harmonic function of x and y for each fixed t, i.e. $H(x, y, t)$ is a solution of the pseudoparabolic equation (1.6) for $n = 2$. From the previously imposed condition that $f(z, 0) = 0$ we have that $H(x, y, 0) = 0$. From Eqs. (2.4)–(2.6) it can be shown that $E(r^2, t, s)$ satisfies the partial differential equation

$$(1 - s^2)\, E_{rst} - (1/s)\, E_{rt} + rs[E_{rrt} + (1/r)\, E_{rt} + AE_t + BE] = 0; \tag{3.3}$$

the initial conditions

$$\begin{aligned} E(r^2, 0, s) &= 0, \\ E_t(0, t, s) &= 1; \end{aligned} \tag{3.4}$$

and has a series expansion of the form

$$E(r^2, t, s) = t + \sum_{k=1}^{\infty} e^{(k)}(r^2, t)\, s^{2k}, \tag{3.5}$$

which converges absolutely and uniformly for $|s| \leqslant 1$ and r and t arbitrarily large (but bounded).

Now define $h(x, y, t)$ by

$$h(x, y, t) = \int_{-1}^{+1} H[x(1 - s^2)^{1/2}, y(1 - s^2)^{1/2}, t] \frac{ds}{(1 - s^2)^{1/2}}. \tag{3.6}$$

Then Eq. (3.1) can be rewritten (c.f. [6, 10]) as

$$u(x, y, t) = h(x, y, t) + \int_0^t \int_0^1 \sigma G_t(r^2, 1 - \sigma^2, t - \tau)\, h_\tau(x\sigma^2, y\sigma^2, \tau)\, d\sigma\, d\tau, \tag{3.7}$$

where $h(x, y, t)$ is again a solution of $\Delta_2 u_t = 0$ satisfying the initial condition $h(x, y, 0) = 0$ and $G(r^2, \rho, t)$ is defined by

$$G(r^2, \rho, t) = \sum_{k=1}^{\infty} \frac{2e^{(k)}(r^2, t)\, \Gamma(k + \frac{1}{2})}{\Gamma(\frac{1}{2})\, \Gamma(k)} \rho^{k-1}. \tag{3.8}$$

From the analysis of section two it is clear that Eq. (3.7) defines a mapping of the class of real valued solutions of the equation $\Delta_2 u_t = 0$ which vanish at $t = 0$ and are defined in a domain $D \times T$ (where D is starlike with respect to the origin) onto the class of real valued solutions of Eq. (1.5) (for $n = 2$) which vanish at $t = 0$ and are defined in $D \times T$.

We now want to generalize the representation (3.7) from $n = 2$ to general n. To this end we first look for solutions of Eq. (1.5) in the form

$$u(\mathbf{x}, t) = \int_0^t \int_0^1 s^{n-2} E(r^2, t - \tau, s; n)\, H_\tau(\mathbf{x}(1 - s^2), \tau) \frac{ds\, d\tau}{(1 - s^2)^{1/2}}, \tag{3.9}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and $H(\mathbf{x}, t)$ is a real valued solution of Eq. (1.6) which vanishes at $t = 0$. We require that $E(r^2, t, s; n)$ be an entire function of r^2 and t, analytic in s for $|s| \leqslant 1$, and satisfy the initial conditions $E(r^2, 0, s; n) = 0$, $E_t(0, t, s; n) = 1$. We now temporarily replace the path of integration from zero to one by a loop starting from $s = +1$, passing counterclockwise around the origin and onto the second sheet of the Riemann surface of the integrand, and then back up to $s = +1$, and substitute the resulting expression into the differential equation (1.5). If $u(\mathbf{x}, t)$ is to be a solution of Eq. (1.5) it is then easily verified by integrating by parts that $E(r^2, t, s; n)$ must satisfy the singular partial differential equation

$$(1 - s^2) E_{rst} + (n - 3/s) E_{rt} + rs[E_{rrt} + (1/r) E_{rt} + AE_t + BE] = 0. \tag{3.10}$$

We now look for a solution of Eq. (3.10) in the form

$$E(r^2, t, s; n) = t + \sum_{k=1}^{\infty} e^{(k)}(r^2, t; n)\, s^{2k}. \tag{3.11}$$

Substituting (3.11) into Eq. (3.9) and making use of the initial condition $E(r^2, 0, s; n) = 0$ yields the following recursion formulas for the determination of the $e^{(k)}(r^2, t; n)$:

$$\begin{aligned} (n-1)\, e_r^{(1)} &= -trA - (t^2/2)\, rB \\ (2k + n - 3)\, e_r^{(k)} &= (2k - 3)\, e_r^{(k-1)} - r e_{rr}^{(k-1)} - rAe^{(k-1)} \\ &\quad - rB \int_0^t e^{(k-1)}\, d\tau; \quad k \geqslant 2. \end{aligned} \tag{3.12}$$

From the initial condition $E_t(0, t, s; n) = 1$ we have the initial conditions

$$e^{(k)}(0, t; n) = 0; \qquad k = 1, 2, \ldots, \tag{3.13}$$

for each of the $e^{(k)}(r^2, t; n)$. Hence, each of the $e^{(k)}(r^2, t; n)$ in Eq. (3.11) is uniquely determined. We must now show the series (3.11) converges uniformly for r and t arbitrarily large (but bounded) and $|s| \leqslant 1$. We first note that for $n = 2$ the $e^{(k)}(r^2, t; 2)$ are identical with the function $e^{(k)}(r^2, t)$ defined by Eq. (3.5). This follows from the facts that the form of the series expansion for $E(r^2, t, s)$ and $E(r^2, t, s; 2)$ are the same and these functions satisfy the same differential equation and initial conditions. Hence, the series (3.11) converges when $n = 2$. Now define new functions $c^{(k)}(r^2, t; n)$ by the formula

$$c^{(k)}(r^2, t; n) = \frac{2e^{(k)}(r^2, t; n)\, \Gamma(k + n/2 - \frac{1}{2})}{\Gamma(n/2 - \frac{1}{2})\, \Gamma(k)}; \quad k \geqslant 1. \tag{3.14}$$

Then from Eq. (3.12) and (3.13) it is seen that the $c^{(k)}(r^2, t; n)$ satisfy the recursion formula

$$\begin{aligned} c_r^{(1)} &= -trA - (t^2/2)\, rB \\ 2(k-1)\, c_r^{(k)} &= (2k-3)\, c_r^{(k-1)} - rc_{rr}^{(k-1)} - rAc^{(k-1)} - rB \int_0^t c^{(k-1)}\, d\tau; \, k \geqslant 2. \end{aligned} \tag{3.15}$$

and the initial conditions

$$c^{(k)}(0, t; n) = 0; \qquad k \geqslant 1. \tag{3.16}$$

Equations (3.15) and (3.16) imply that the $c^{(k)}(r^2, t; n)$ are in fact independent of n. Since we know the series (3.11) is convergent when $n = 2$, we can now conclude from Eq. (3.14) and the fact that the $c^{(k)}(r^2, t; n)$ are independent of n that the series (3.11) converges absolutely and uniformly for r and t arbitrarily large (but bounded) and $|s| \leqslant 1$. This establishes the existence of the operator defined by Eqs. (3.9) and (3.11).

Motivated again by the results of section 2, we differentiate the representation (3.9) with respect to t and define a new operator mapping solutions of equation (1.6) onto solutions of Eq. (1.5). If in this operator we now set

$$h(\mathbf{x}, t) = \int_0^1 s^{n-2} H(\mathbf{x}(1 - s^2), t) \frac{ds}{(1 - s^2)^{1/2}}, \tag{3.17}$$

we arrive (c.f. [6, 10]) at the following integral operator which maps real valued solutions of Eq. (1.6) which vanish at $t = 0$ into the class of real valued solutions of equation (1.5) which vanish at $t = 0$ (we again assume $h(\mathbf{x}, t)$ and $u(\mathbf{x}, t)$ are defined in a domain of the form $D \times T$ where D is starlike with respect to the origin):

$$u(\mathbf{x}, t) = h(\mathbf{x}, t) + \int_0^t \int_0^1 \sigma^{n-1} G_t(r^2, 1 - \sigma^2, t - \tau)\, h_\tau(\mathbf{x}\sigma^2, \tau)\, d\sigma\, d\tau. \tag{3.18}$$

In Eq. (3.18) $G(r^2, \rho, t)$ is defined by Eq. (3.8) and is independent of n. This is the basis for refering to the approach used in this section as a "method of ascent." Such techniques were first used by R. P. Gilbert [10, 11] (see also [2, p. 68]) in his investigation of the elliptic equation

$$\Delta_n u + A(r^2)u = 0. \tag{3.19}$$

Subsequently this approach was extended by Colton and Gilbert [6] to treat the fourth order elliptic equation

$$\Delta_n^2 u + A(r^2)\,\Delta_n u + B(r^2)u = 0. \tag{3.20}$$

We now want to show that the operator defined by Eq. (3.18) is invertible, i.e. Eq. (3.18) defines an operator which maps solutions of Eq. (1.6) which vanish at $t = 0$ onto the class of solutions of Eq. (1.5) which vanish at $t = 0$. To this end we differentiate Eq. (3.18) with respect to t and rewrite the resulting expression as the Volterra integral equation

$$\begin{aligned}\Phi_t(r;\theta;\phi,t) = \psi_t(r;\theta;\phi,t) &+ \int_0^r K^{(1)}(r,\rho,0)\,\psi_t(\rho;\theta;\phi,t)\,d\rho \\ &+ \int_0^t \int_0^r K^{(2)}(r,\rho,t-\tau)\,\psi_t(\rho;\theta;\phi,\tau)\,d\rho\,d\tau,\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}\Phi(r;\theta;\phi,t) &= r^{(n-2)/2}u(r;\theta;\phi,t),\\ \psi(r;\theta;\phi,t) &= r^{(n-2)/2}h(r;\theta;\phi,t),\\ K^{(1)}(r,\rho,0) &= (1/2r)\,G_t(r^2, 1-(\rho/r), 0),\\ K^{(2)}(r,\rho,t) &= (1/2r)\,G_{tt}(r^2, 1-(\rho/r), t),\end{aligned} \tag{3.22}$$

and $(r;\theta;\phi)$ are spherical coordinates. From the recursion formula (3.15) it is seen that each $c^{(k)}(r^2, t; n)$ is of the form

$$c^{(k)}(r^2, t; n) = r^{2k}\tilde{c}^{(k)}(r^2, t; n), \tag{3.23}$$

where $\tilde{c}^{(k)}(r^2, t; n)$ is an entire function of r^2 and t. This follows from the fact that the differential operator $(2k-3)(d/dr) - r(d^2/dr^2)$ annihilates r^{2k-2}. Hence, the functions $K^{(1)}(r, \rho, 0)$ and $K^{(2)}(r, \rho, t)$ defined in Eq. (3.22) are entire functions of r and t and analytic in ρ for $|\rho| \leqslant |r|$. Equations of the form of Eq. (3.21) have previously been studied by Vekua (c.f. [21, pp. 11–16]), and using his techniques we can easily show that for each continuous function $\Phi_t(r;\theta;\phi,t)$ there exists a unique continuous solution $\psi_t(r;\theta;\phi,t)$ of the integral equation (3.21). Furthermore, it can be verified without excessive hardship that if $r^{(2-n)/2}\,\Phi(r;\theta;\phi,t)$ is a (strong) solution of Eq. (1.5) which vanishes at $t = 0$ then the function $r^{(2-n)/2}\psi(r;\theta;\phi,t)$, where $\psi(r; \theta; \phi, t)$ is a solution of the integral equation (3.21), is a (strong) solution of Eq. (1.6) which vanishes at $t = 0$. We can now conclude that the operator defined by Eq. (3.18) is invertible, since if $\Phi(r; \theta; \phi, 0) = 0$ then $\Phi(r; \theta; \phi, t)$ is uniquely determined by $\Phi_t(r; \theta; \phi, t)$. (In the case when $A(r^2) \leqslant 0$ an alternate proof of the invertibility of the operator defined in Eq. (3.18) can be deduced from the results of Section 4 of this paper.)

In passing we note that by comparing Eqs. (3.8), (3.12), and (3.14) with the corresponding equations for Gilbert's G-function for Eq. (3.19) [10] we have that

$$G_t(r^2, \rho, 0) = \tilde{G}(r, \rho), \tag{3.24}$$

where $\tilde{G}(r, \rho)$ denotes Gilbert's G-function for Eq. (3.19). In particular, Gilbert's method of ascent for elliptic equations appears as the limiting case of Eq. (3.18) obtained by differentiating both sides of this equation with respect to t and then letting t tend to zero.

We summarize our results in the following theorem.

THEOREM 3.1. *Let $u(\mathbf{x}, t)$ be a real valued strong solution of Eq.* (1.5) *defined in a domain $D \times T$ where D is starlike with respect to the origin and suppose $u(\mathbf{x}, 0) = 0$. Then $u(\mathbf{x}, t)$ can be represented in the form of Eq.* (3.18) *where $h(\mathbf{x}, t)$ is a solution of Eq.* (1.6) *such that $h(\mathbf{x}, 0) = 0$ and $G(r^2, \rho, t)$ is defined by Eq.* (3.8). *$G(r^2, \rho, t)$ is an entire function of r^2 and t and is analytic for $|\rho| \leqslant 1$. $G_t(r^2, \rho, 0) = \tilde{G}(r, \rho)$ where $\tilde{G}(r, \rho)$ is Gilbert's G-function for the elliptic equation* (3.19).

COROLLARY. *Let $u(\mathbf{x}, t)$ be a real valued strong solution of Eq.* (1.5) *defined in a domain $D \times T$ where D is starlike with respect to the origin and suppose $u(\mathbf{x}, 0) = 0$. Then, for each fixed t, $u(\mathbf{x}, t)$ is an analytic function of the variables $x_1, \ldots, x_n$ for $\mathbf{x} \in D$.*

Proof of Corollary. This follows from the representation (3.18), the analyticity of $G(r^2, \rho, t)$, and the fact that if $h(\mathbf{x}, t)$ is a solution of Eq. (1.6) in $D \times T$ and vanishes at $t = 0$, then $h(\mathbf{x}, t)$ is a harmonic function of $x_1, \ldots, x_n$ in D (for each fixed t), and, hence, is analytic in these variables.

IV. THE FIRST INITIAL-BOUNDARY VALUE PROBLEM

The (first) initial-boundary value problem for Eqs. (1.4) and (1.5) is to find a strong solution of the differential equation in $D \times T$ (where D is bounded, simply connected and has Lyapunov boundary ∂D) which is continuously differentiable with respect to t in the closure of $D \times T$, vanishes at $t = 0$, and assumes prescribed boundary values on $\partial D \times T$. From the results of [17] such a solution exists and is unique provided $c(x, y) \leqslant 0$ and $A(r^2) \leqslant 0$ respectively in the closure of D. Our purpose is to reformulate this initial-boundary value problem into the problem of solving an integral equation whose form is suitable for solution by iteration or numerical methods, thus giving a constructive method of exhibiting the solution.

We first consider Eq. (1.4). In [5] we have defined a fundamental solution of Eq. (1.4) to be a function of the form

$$S(x, y, t; \xi, \eta, \tau) = A(x, y, t; \xi, \eta, \tau) \log(1/r) + B(x, y, t; \xi, \eta, \tau), \tag{4.1}$$

where $A(x, y, t; \xi, \eta, \tau) = \tilde{A}(z, z^*, t; \zeta, \zeta^*, \tau)$ and $B(x, y, t; \xi, \eta, \tau) = \tilde{B}(z, z^*, t; \zeta, \zeta^*, \tau)$ have the series expansions

$$\begin{aligned} \tilde{A}(z, z^*, t; \zeta, \zeta^*, \tau) &= \sum_{j=1}^{\infty} \tilde{A}^{(j)}(z, z^*; \zeta, \zeta^*) \frac{(t-\tau)^j}{j!}, \\ \tilde{B}(z, z^*, t; \zeta, \zeta^*, \tau) &= \sum_{j=1}^{\infty} \tilde{B}^{(j)}(z, z^*; \zeta, \zeta^*) \frac{(t-\tau)^j}{j!}; \end{aligned} \tag{4.2}$$

which converge absolutely and uniformly for arbitrary values of t and τ and $z, \zeta \in \Omega$, z^*, $\zeta^* \in \Omega^*$ where $\zeta = \xi + i\eta$, $\zeta^* = \xi - i\eta$, $z = x + iy$, $z^* = x - iy$, Ω is an arbitrary compact subset of the complex plane, and $\Omega^* = \{z^*: \bar{z}^* \in \Omega\}$. The coefficients $\tilde{A}^{(j)}(z, z^*; \zeta, \zeta^*)$ and $\tilde{B}^{(j)}(z, z^*; \zeta, \zeta^*)$ can be determined recursively and satisfy (among other initial conditions)

$$\begin{aligned} &\tilde{A}^{(1)}(\zeta, \zeta^*; \zeta, \zeta^*) = 1, \\ &\tilde{A}^{(j)}(\zeta, \zeta^*; \zeta, \zeta^*) = 0 \qquad \text{for } j \geqslant 2, \\ &\tilde{B}^{(j)}(\zeta, \zeta^*; \zeta, \zeta^*) = 0 \qquad \text{for } j \geqslant 1. \end{aligned} \tag{4.3}$$

Motivated by the use of double layer potentials to solve the Dirichlet problem for elliptic equations, we look for a solution of the first initial-boundary value problem in the form

$$u(x, y, t) = \frac{1}{\pi} \int_0^t \int_{\partial D} \mu(\xi, \eta, \tau) \frac{\partial^2}{\partial \nu \, \partial \tau} S(\xi, \eta, \tau; x, y, t) \, ds \, d\tau, \tag{4.4}$$

where $\mu(\xi, \eta, \tau)$ is a potential to be determined, ν is the inner normal to ∂D, and $ds\, d\tau$ is an element of surface area of $\partial D \times T$. Since as a function of its last three variables $S(\xi, \eta, \tau; x, y, t)$ is a solution of Eq. (1.4) and $S_\tau(\xi, \eta, t; x, y, t)$ is a fundamental solution of $\Delta_2 u + c(x, y)u = 0$, which is independent of t, it is easily verified that if $\mu(\xi, \eta, \tau)$ is continuous in the closure of $D \times T$ then Eq. (4.4) defines a strong solution of Eq. (1.4) which is continuously differentiable with respect to t in the closure of $D \times T$. Now suppose we want to determine $\mu(\xi, \eta, \tau)$ such that for (x, y, t) on $\partial D \times T$ we have

$u(x, y, t) = f(x, y, t)$, where $f(x, y, t)$ is a prescribed function on $\partial D \times T$ which is continuously differentiable with respect to t. Differentiating Eq. (4.4) with respect to t, letting (x, y, t) approach $\partial D \times T$, and using the well known properties of logarithmic potentials (c.f. [8, pp. 334–339]), leads to the following integral equation for $\mu(\xi, \eta, \tau)$:

$$\begin{aligned} f_t(x, y, t) &= \mu(x, y, t) + \frac{1}{\pi} \int_{\partial D} \mu(\xi, \eta, t) \frac{\partial^2}{\partial \nu\, \partial \tau} S(\xi, \eta, t; x, y, t)\, ds \\ &\quad + \frac{1}{\pi} \int_0^t \int_{\partial D} \mu(\xi, \eta, \tau) \frac{\partial^3}{\partial \nu\, \partial \tau\, \partial t} S(\xi, \eta, \tau; x, y, t)\, ds\, d\tau. \end{aligned} \tag{4.5}$$

Note that no residue arises from the second integral in Eq. (4.5) as (x, y, t) approaches $\partial D \times T$ due to the conditions imposed by Eq. (4.3).

We will now show that the integral Eq. (4.5) can always be solved for $\mu(x, y, t)$ provided that $f_t(x, y, t)$ is continuous and $c(x, y) \leqslant 0$ for (x, y) in the closure of D. Equation (4.5) can be written in the form

$$f_t = (\mathbf{I} + \mathbf{T})\mu + \mathbf{L}\mu, \tag{4.6}$$

where

$$\begin{aligned} \mathbf{T}\mu &= \frac{1}{\pi} \int_{\partial D} \mu(\xi, \eta, t) \frac{\partial^2}{\partial \nu\, \partial \tau} S(\xi, \eta, t; x, y, t)\, ds \\ \mathbf{L}\mu &= \frac{1}{\pi} \int_0^t \int_{\partial D} \mu(\xi, \eta, \tau) \frac{\partial^3}{\partial \nu\, \partial \tau\, \partial t} S(\xi, \eta, \tau; x, y, t)\, ds\, d\tau. \end{aligned} \tag{4.7}$$

Note that $\mathbf{T}$ is a Fredholm operator and $\mathbf{L}$ is a Volterra operator with a continuous kernel (due to Eqs. (4.1)–(4.3)). Since $S_\tau(\xi, \eta, t; x, y, t)$ is a (normalized) fundamental solution for the equation $\Delta_2 u + c(x, y)u = 0$ and $c(x, y) \leqslant 0$, the operator $(\mathbf{I} + \mathbf{T})^{-1}$ exists (c.f. [8, pp. 364–365]). Furthermore, by Fubini's theorem, $\mathbf{L}$ and $\mathbf{T}$ commute (and, hence, so do $\mathbf{L}$ and $(\mathbf{I} + \mathbf{T})^{-1}$), and due to $\mathbf{L}$ being a Volterra operator, $\|(\mathbf{I} + \mathbf{T})^{-m}\mathbf{L}^m\| < 1$ for m sufficiently large ($\|\cdot\|$ denotes the L_2 operator norm). Thus, the operator $(\mathbf{I} + (\mathbf{I} + \mathbf{T})^{-1}\mathbf{L})^{-1}$ exists. Hence, from Eq. (4.6) we have

$$(\mathbf{I} + \mathbf{T})^{-1} f_t = \mu + (\mathbf{I} + \mathbf{T})^{-1}\mathbf{L}\mu \tag{4.8}$$

and

$$\begin{aligned} \mu &= (\mathbf{I} + (\mathbf{I} + \mathbf{T})^{-1}\mathbf{L})^{-1}(\mathbf{I} + \mathbf{T})^{-1} f_t \\ &= (\mathbf{I} + \mathbf{T} + \mathbf{L})^{-1} f_t\,. \end{aligned} \tag{4.9}$$

The continuity of f_t implies that μ is continuous in the closure of $D \times T$, and, hence, Eqs. (4.4) and (4.9) give the desired solution of the first initial-boundary value problem.

Equation (4.5) lends itself to various procedures for approximating solutions of the first initial-boundary value problem for Eq. (1.4). For example possible approaches would be to replace the integral equation (4.5) by a system of algebraic equations (c.f. [12, pp. 98–103]) or to use the method of moments (c.f. [12, pp. 150–154]). However, the systematic study of integral equations of mixed Fredholm and Volterra type such as the ones arising here has (to the author's knowledge) not yet been undertaken by mathematicians working in the area of integral equations or numerical analysis. It would be desirable to complete such an investigation and hopefully this paper will give some motivation for mathematicians to begin working on integral equations of this type.

We summarize the results obtained up to this point in the following theorem.

THEOREM 4.1. *Let D be a bounded simply connected domain in $\mathbb{R}^2$ with Lyapunov boundary ∂D and $T = \{t: 0 \leqslant t \leqslant t_0\}$ where t_0 is a positive constant. Assume that $c(x, y) \leqslant 0$ in the closure of D. Then Eqs.* (4.4) *and* (4.9) *define the* (*unique*) *strong solution to Eq.* (1.4) *in $D \times T$ which is continuously differentiable with respect to t in the closure of $D \times T$, vanishes at $t = 0$, and assumes prescribed boundary values $f(x, y, t)$ on $\partial D \times T$.*

We now turn our attention to developing a constructive method for solving the first initial-boundary value problem for Eq. (1.5) in the case when $n > 2$. A straightforward generalization of the analysis just completed for the case of Eq. (1.4) is no longer possible since a fundamental solution for pseudoparabolic equations in more than two space dimensions has not yet been constructed. Motivated by the work of Gilbert [10, 11] we will overcome this problem through the use of Theorem 3.1 and the well known fundamental solution for Laplace's equation. In the following discussion we will assume that the domain D, in addition to the hypothesis given at the beginning of this section, is also starlike with respect to the origin.

We begin by differentiating both sides of Eq. (3.18) with respect to t to arrive at the general representation

$$\begin{aligned} u_t(\mathbf{x}, t) &= h_t(\mathbf{x}, t) + \int_0^1 \sigma^{n-1} G_t(r^2, 1-\sigma^2, 0)\, h_t(\mathbf{x}\sigma^2, t)\, d\sigma \\ &\quad + \int_0^t \int_0^1 \sigma^{n-1} G_{tt}(r^2, 1-\sigma^2, t-\tau)\, h_\tau(\mathbf{x}\sigma^2, \tau)\, d\sigma\, d\tau. \end{aligned} \tag{4.10}$$

Since $h(\mathbf{x}, t)$ is a solution of Eq. (1.6), it is clear that, for each fixed t, $h_t(\mathbf{x}, t)$ is harmonic. Hence, for $n > 2$ we can represent $h_t(\mathbf{x}, t)$ as a double layer potential

$$h_t(\mathbf{x}, t) = \frac{\Gamma(n/2)}{\pi^{n/2}} \int_{\partial D} \mu(\mathbf{y}, t) \frac{\partial}{\partial \nu}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|^{n-2}}\right) ds, \tag{4.11}$$

where ν is the inner normal on ∂D, $\mu(\mathbf{y}, t)$ is a potential to be determined, and $\mathbf{x} \in D$ (for $n = 2$ we would represent $h_t(\mathbf{x}, t)$ as a double layer logarithmic potential). Substituting (4.11) into Eq. (4.10), interchanging orders of integration, and letting $\mathbf{x}$ approach the boundary of D, leads to the following integral equation for the determination of $\mu(\mathbf{x}, t)$ (where $u(\mathbf{x}, t) = f(\mathbf{x}, t)$ on ∂D):

$$\begin{aligned} f_t(\mathbf{x}, t) &= \mu(\mathbf{x}, t) + \frac{\Gamma(n/2)}{\pi^{n/2}} \int_{\partial D} \mu(\mathbf{y}, t)\, K^{(1)}(\mathbf{x}, \mathbf{y}, t)\, ds \\ &\quad + \frac{\Gamma(n/2)}{\pi^{n/2}} \int_0^t \int_{\partial D} \mu(\mathbf{y}, \tau)\, K^{(2)}(\mathbf{x}, \mathbf{y}, t-\tau)\, ds\, d\tau, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} K^{(1)}(\mathbf{x}, \mathbf{y}, t) &= \frac{\partial}{\partial \nu}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|^{n-2}}\right) \\ &\quad + \int_0^1 \sigma^{n-1} G_t(r^2, 1-\sigma^2, 0) \frac{\partial}{\partial \nu}\left(\frac{1}{|\mathbf{x}\sigma^2 - \mathbf{y}|^{n-2}}\right) d\sigma \\ K^{(2)}(\mathbf{x}, \mathbf{y}, t) &= \int_0^1 \sigma^{n-1} G_{tt}(r^2, 1-\sigma^2, t) \frac{\partial}{\partial \nu}\left(\frac{1}{|\mathbf{x}\sigma^2 - \mathbf{y}|^{n-2}}\right) d\sigma. \end{aligned} \tag{4.13}$$

We note that the kernels $K^{(1)}(\mathbf{x}, \mathbf{y}, t)$ and $K^{(2)}(\mathbf{x}, \mathbf{y}, t)$ have weak singularities at $x = y$. Hence, Eq. (4.12) is again of the form

$$f_t = (\mathbf{I} + \mathbf{T})\mu + \mathbf{L}\mu, \tag{4.14}$$

where $\mathbf{T}$ is a Fredholm operator and $\mathbf{L}$ is a Volterra operator. From Theorem 3.1 it is seen that the operator $\mathbf{I} + \mathbf{T}$ is identical with the operator defined in Eq. (4.42) of [10], and, hence, if $A(r^2) \leqslant 0$ in the closure of D, $(\mathbf{I} + \mathbf{T})^{-1}$ exists. Repeating the analysis which led to Eq. (4.9) we have that $(\mathbf{I} + \mathbf{T} + \mathbf{L}^{-})^{1}$ exists and

$$\mu = (\mathbf{I} + \mathbf{T} + \mathbf{L})^{-1} f_t\,. \tag{4.15}$$

Equations (4.15), (4.10), and (4.11) now give the solution of the first initial-boundary value problem for Eq. (1.5). We summarize this result in the following theorem.

THEOREM 4.2. *Let D be a bounded domain in $\mathbb{R}^n$, $n > 2$, which is starlike with respect to the origin and has Lyapunov boundary ∂D and let $T = \{t: 0 \leqslant t \leqslant t_0\}$ where t_0 is a positive constant. Assume that $A(r^2) \leqslant 0$ in the closure of D. Then Eqs.* (4.10), (4.11), *and* (4.15) *define the (unique) strong solution to Eq.* (1.5) *in $D \times T$ which is continuously differentiable with respect to t in the closure of $D \times T$, vanishes at $t = 0$, and assumes prescribed boundary values $f(\mathbf{x}, t)$ on $\partial D \times T$.*

In closing we would like to point out that the assumption made throughout this paper that the coefficients of Eqs. (1.4) and (1.5) are entire functions can obviously be weakened to the requirement that these coefficients only be analytic in a sufficiently large ball about the origin.

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Integral Operators and Reflection Principles for Parabolic Equations in One Space Variable

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1. INTRODUCTION

In this paper we will construct integral operators which map solutions of the heat equation in one space variable onto solutions of linear parabolic equations in one space variable with analytic coefficients. These operators will then be used to obtain reflection principles for solutions to parabolic equations which are partially analytic with respect to the space variable. The distinguishing feature of our approach is that we are able to construct operators whose domain is the space of solutions to the heat equation and not the space of analytic functions as in previous work in this area [1, 4]. This allows us to consider solutions of parabolic equations with analytic coefficients which are partially analytic with respect to the space variable instead of the smaller class of solutions which are analytic in both the space and time variables. (In this connection we note that if the coefficients of a parabolic equation are analytic, then any strong solution is in fact partially analytic with respect to the space variable [2].) In the context of the general analytic theory of partial differential equations the approach given in this paper seems to be the natural one in the sense that integral operators for elliptic equations reduce in the case of the harmonic equation to taking the real part of an analytic function (cf. [3]), whereas in the case of the heat equation our operators reduce to the identity operator.

An important application of our integral operators is the derivation of a reflection principle for parabolic equations in one space variable. This is of particular interest in the sense that it is the first time a reflection principle has been given for partially analytic solutions of parabolic equations (except in the trivial case of the heat equation). For analytic solutions of parabolic equations in one space variable such a continuation is immediate since any analytic solution of a parabolic equation in one space variable can be analytically continued into a strip bounded by the characteristics, regardless

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of what the (analytic) data is on a noncharacteristic curve [1, 4]. Our approach to the reflection problem is of further interest in that it also suggests the possibility of obtaining continuation theorems for nonanalytic solutions of parabolic equations. This is due to the fact that the reflection principle obtained in this paper ultimately rests on constructing a solution $E(s, x, t)$ of a Goursat problem for the equation

$$E_{xx} - E_{ss} + q(x, t)E = E_t \tag{1.1}$$

where $q(x, t)$ is a function depending only on the coefficients of the parabolic equation under investigation. In the case when the coefficients of the parabolic equation are independent of t, so are $q(x, t)$ and $E(s, x, t)$, and hence $E(s, x, t)$ is the solution of a Goursat problem for a hyperbolic equation. In this case the assumption of analyticity of the coefficients can be relaxed and results on the continuation of nonanalytic solutions to parabolic equations can be obtained.

2. Integral Operators

Consider the general linear parabolic equation of second order in two independent variables written in normal form as

$$u_{xx} + a(x, t)u_x + b(x, t)u = c(x, t)u_t \tag{2.1}$$

where the coefficients $a(x, t)$, $b(x, t)$, and $c(x, t)$ are analytic in the rectangle $D = \{(x, t)\colon -x_0 < x < x_0\,, 0 < t < t_0\}$, x_0 and t_0 are positive constants, and $c(x, t) > 0$ for $(x, t) \in D$. The one-to-one analytic transformation

$$\begin{aligned} \xi &= \int_0^x (c(s, t))^{1/2}\, ds \\ \tau &= t \end{aligned} \tag{2.2}$$

reduces (2.1) into an equation of the same form but with $c(x, t) = 1$. Hence we may assume $c(x, t) = 1$ in (2.1) to begin with. If we now set

$$u(x, t) = v(x, t) \exp \left\{-\tfrac{1}{2} \int_0^x a(s, t)\, ds\right\} \tag{2.3}$$

we arrive at an equation for $v(x, t)$ of the same form as (2.1) but with $a(x, t) = 0$. Hence, without loss of generality, we can restrict ourselves to equations of the canonical form

$$u_{xx} + q(x, t)u = u_t \tag{2.4}$$

where $q(x, t)$ is analytic for $(x, t) \in D$. We now look for solutions of (2.4) in the form

$$u(x, t) = h(x, t) + \int_0^x K(s, x, t)\, h(s, t)\, ds \tag{2.5}$$

where $h(x, t)$ is a solution of the heat equation

$$h_{xx} = h_t \tag{2.6}$$

and satisfies the Dirichlet data $h(0, t) = 0$. Substituting (2.5) into (2.4) gives

$$\begin{aligned} 0 = u_{xx} + q(x, t)u - u_t &= q(x, t)\, h(x, t) + K(x, x, t)\, h_x(x, t) \\ &+ (K\,(x, x, t) + 2K_x(x, x, t))\, h(x, t) \\ &+ \int_0^x (K_{xx} + q(x, t)\, K - K_t)\, h(s, t)\, ds \\ &- \int_0^x K(s, x, t)\, h_t(s, t)\, ds. \end{aligned} \tag{2.7}$$

But

$$\begin{aligned} &\int_0^x K(s, x, t)\, h_t(s, t)\, ds \\ &\quad = \int_0^x K(s, x, t)\, h_{ss}(s, t)\, ds \\ &\quad = h_s(s, t)\, K(s, x, t) \Big|_{s=0}^{s=x} - \int_0^x K_s(s, x, t)\, h_s(s, t)\, ds \\ &\quad = h_x(x, t)\, K(x, x, t) - h_x(0, t)\, K(0, x, t) - K_s(s, x, t)\, h(s, t) \Big|_{s=0}^{s=x} \\ &\qquad + \int_0^x K_{ss}(s, x, t)\, h(s, t)\, ds \\ &\quad = h_x(x, t)\, K(x, x, t) - h_x(0, t)\, K(0, x, t) - h(x, t)\, K_s(x, x, t) \\ &\qquad + \int_0^x K_{ss}(s, x, t)\, h(s, t)\, ds. \end{aligned} \tag{2.8}$$

Substituting (2.8) into (2.7) gives

$$\begin{aligned} 0 = K(0, x, t)\, h_x(0, t) &+ 2(K_s(x, x, t) + K_x(x, x, t) + \tfrac{1}{2} q(x, t))\, h(x, t) \\ &+ \int_0^x (K_{xx} - K_{ss} + q(x, t)\, K - K_t)\, h(s, t)\, ds. \end{aligned} \tag{2.9}$$

Now suppose $E(s, x, t)$ satisfies

$$E_{xx} - E_{ss} + q(x, t)E = E_t \tag{2.10}$$

for $(x, t) \in D$, $-x_0 < s < x_0$, and assumes the Goursat data

$$E(x, x, t) = -\tfrac{1}{2}\int_0^x q(s, t)\, ds \tag{2.11}$$

$$E(-x, x, t) = \tfrac{1}{2}\int_0^x q(s, t)\, ds \tag{2.12}$$

on the characteristic planes $s = x$ and $s = -x$ respectively. Define

$$K(s, x, t) = \tfrac{1}{2}[E(s, x, t) - E(-s, x, t)]. \tag{2.13}$$

Then $K(s, x, t)$ satisfies (2.10) and the initial data

$$K(x, x, t) = -\tfrac{1}{2}\int_0^x q(s, t)\, ds \tag{2.14}$$

$$K(0, x, t) = 0, \tag{2.15}$$

i.e., (2.9) is satisfied identically and hence (2.5) is a solution of (2.4).

Now suppose that instead of satisfying $h(0, t) = 0$, $h(x, t)$ satisfies $h_x(0, t) = 0$. We again look for a solution of (2.4) in the form

$$u(x, t) = h(x, t) + \int_0^x M(s, x, t)\, h(s, t)\, ds. \tag{2.16}$$

Then the equation for $M(s, x, t)$ corresponding to (2.9) is

$$\begin{aligned} 0 = {} & -M_s(0, x, t)\, h(0, t) + 2(M_s(x, x, t) + M_x(x, x, t) + \tfrac{1}{2}q(x, t))\, h(x, t) \\ & + \int_0^x (M_{xx} - M_{ss} + q(x, t)M - M_t)\, h(s, t)\, ds. \end{aligned} \tag{2.17}$$

If $G(s, x, t)$ satisfies (2.10) for $(x, t) \in D$, $-x_0 < s < x_0$, and assumes the Goursat data

$$G(x, x, t) = -\tfrac{1}{2}\int_0^x q(s, t)\, ds \tag{2.18}$$

$$G(-x, x, t) = -\tfrac{1}{2}\int_0^x q(s, t)\, ds \tag{2.19}$$

on the characteristic planes $s = x$ and $s = -x$, respectively, then

$$M(s, x, t) = \tfrac{1}{2}[G(s, x, t) + G(-s, x, t)] \tag{2.20}$$

satisfies (2.10) and the initial data

$$M(x, x, t) = -\tfrac{1}{2}\int_0^x q(s, t)\, ds \tag{2.21}$$

$$M_s(0, x, t) = 0, \tag{2.22}$$

i.e., (2.17) is satisfied identically and hence (2.16) is a solution of (2.4).

If the functions $E(s, x, t)$ and $G(s, x, t)$ exist, we can now define two operators $\mathbf{T}_1$ and $\mathbf{T}_2$ mapping solutions of the heat equation onto solutions of (2.4) by

$$\mathbf{T}_1 h = h(x, t) + \int_0^x K(s, x, t)\, h(s, t)\, ds \tag{2.23}$$

$$\mathbf{T}_2 h = h(x, t) + \int_0^x M(s, x, t)\, h(s, t)\, ds, \tag{2.24}$$

where the domain of T_1 is the class of solutions to the heat equation satisfying $h(0, t) = 0$, and the domain of $\mathbf{T}_2$ is the class of solutions to the heat equation satisfying $h_x(0, t) = 0$.

To show the existence of the operators $\mathbf{T}_1$ and $\mathbf{T}_2$ we must show the existence of the functions $E(s, x, t)$ and $G(s, x, t)$. We will now do this for $E(s, x, t)$; the existence of $G(s, x, t)$ follows in an identical fashion. Let

$$\begin{aligned} x &= \xi + \eta \\ s &= \xi - \eta \end{aligned} \tag{2.25}$$

and define $\tilde{E}(\xi, \eta, t)$ and $\tilde{q}(\xi, \eta, t)$ by

$$\begin{aligned} \tilde{E}(\xi, \eta, t) &= E(\xi - \eta, \xi + \eta, t) \\ \tilde{q}(\xi, \eta, t) &= q(\xi + \eta, t). \end{aligned} \tag{2.26}$$

Then (2.10)–(2.12) become

$$\tilde{E}_{\xi\eta} + \tilde{q}(\xi, \eta, t)\tilde{E} = \tilde{E}_t\,, \tag{2.27}$$

$$\tilde{E}(\xi, 0, t) = -\tfrac{1}{2}\int_0^\xi q(s, t)\, ds, \tag{2.28}$$

$$\tilde{E}(0, \eta, t) = \tfrac{1}{2}\int_0^\eta q(s, t)\, ds, \tag{2.29}$$

and hence $\tilde{E}(\xi, \eta, t)$ satisfies the Volterra integral–differential equation

$$\tilde{E}(\xi, \eta, t) = -\tfrac{1}{2}\int_0^{\xi} q(s, t)\, ds + \tfrac{1}{2}\int_0^{\eta} q(s, t)\, ds + \int_0^{\eta}\int_0^{\xi} (\tilde{E}_t(\xi, \eta, t) - \tilde{q}(\xi, \eta, t)\, \tilde{E}(\xi, \eta, t))\, d\xi\, d\eta. \quad (2.30)$$

The function $\tilde{q}(\xi, \eta, t)$ is analytic for $-x_0 < \xi + \eta < x_0$, $0 < t < t_0$. Let x_1, t_1, be such that $0 < x_1 < x_0$, $0 < t_1 < t_0$. Then by standard compactness arguments there exists a positive number $\delta = \delta(x_1)$ and a domain B in the complex (ξ, η) space containing the square $|\xi| + |\eta| \leqslant x_1$ in the real domain such that $\tilde{q}(\xi, \eta, t)$ is analytic in the (six-dimensional) product domain $\Omega = B \times \{t\colon |t - t_1| < \delta\}$ and is continuous in its closure. We will now show that the solution $\tilde{E}(\xi, \eta, t)$ of the integral–differential equation (2.30) exists and is analytic in Ω. Since x_1 and t_1 are arbitrary points in the interval $(0, x_0)$ and $(0, t_0)$, respectively, this will show that $E(s, x, t)$ exists and is analytic for $(x, t) \in D$ and $-x_0 < s < x_0$. By making a preliminary linear change of variables we can assume without loss of generality that $t_1 = 0$ and $\tilde{q}(\xi, \eta, t)$ is an analytic function of t in some neighborhood of the origin. We can also assume that $|\xi| + |\eta| \leqslant 2x$, for $(\xi, \eta) \in B$.

The solution of the integral–differential equation (2.30) can formally be obtained by iteration in the form

$$\tilde{E}(\xi, \eta, t) = \tilde{E}_1(\xi, \eta, t) + \tilde{E}_2(\xi, \eta, t) + \cdots + \tilde{E}_n(\xi, \eta, t) + \cdots \quad (2.31)$$

where

$$\tilde{E}_1(\xi, \eta, t) = -\tfrac{1}{2}\int_0^{\xi} q(s, t)\, ds + \tfrac{1}{2}\int_0^{\eta} q(s, t)\, ds \quad (2.32)$$

and

$$\tilde{E}_{n+1}(\xi, \eta, t) = \int_0^{\eta}\int_0^{\xi} (\tilde{E}_{nt}(\xi, \eta, t) - \tilde{q}(\xi, \eta, t)\, \tilde{E}_n(\xi, \eta, t))\, d\xi\, d\eta. \quad (2.33)$$

We will be done if we show that the series converges uniformly in Ω. To this end let C be a positive constant such that for $(\xi, \eta, t) \in \Omega$ we have

$$q(\xi, \eta, t) \ll (C/x_1)(1 - t/\delta)^{-1} \quad (2.34)$$

where "$\ll$" denotes domination with respect to t (cf. [3]). Without loss of generality assume $C \geqslant 1$ and $\delta \leqslant 1$. We will show by induction that for $(\xi, \eta, t) \in \Omega$

$$\tilde{E}_n \ll (2^n C^n |\xi|^{n-1} |\eta|^{n-1} \delta^{-n+1})/(n-1)!\,(1 - t/\delta)^{-n}. \quad (2.35)$$

This is clearly true for $n = 1$. Assume now that (2.35) is valid for $n = k$. Then using the standard properties of dominants we have

$$\begin{aligned}\tilde{E}_{k+1} &\ll \int_0^{|\eta|}\int_0^{|\xi|} (2^k C^k \mid \xi \mid^{k-1} \mid \eta \mid^{k-1} \delta^{-k+1})/(k-1)!(k/\delta + C) \\ &\quad \times (1 - t/\delta)^{-k-1} | \, d\xi \mid \mid d\eta \mid \\ &\ll (2^k C^k \mid \xi \mid^k \mid \eta \mid^k \delta^{-k+1})/(k)(k)!(k/\delta + C)(1 - t/\delta)^{-k-1} \\ &\ll 2^{k+1} C^{k+1} \mid \xi \mid^k \mid \eta \mid^k \delta^{-k}/(k)!(1 - t/\delta)^{-k-1},\end{aligned} \tag{2.36}$$

and hence, (2.35) is true for $n = k + 1$ and the induction proof is completed. Equation (2.35) implies that for $(\xi, \eta, t) \in \Omega$ we have

$$| \tilde{E}_n | \leqslant 2^n C^n \mid \xi \mid^{n-1} \mid \eta \mid^{n-1} \delta^{-n+1}/(n-1)! \, (1 - \mid t \mid/\delta)^{-n}; \tag{2.37}$$

hence, the series (2.31) converges uniformly on compact subsets of Ω and defines an analytic function of its independent variables in this region. We have now established the existence of the operators $\mathbf{T}_1$ and $\mathbf{T}_2$.

3. Reflection Principles

We will now show how the operators constructed in the previous section can be used to obtain reflection principles for solutions to (2.1). Let $u(x, t)$ be a strong solution of (2.1) in the region $0 < x < s(t)$, $0 < t < t_0$, which vanishes along the noncharacteristic analytic arc $x = s(t)$ and is continuously differentiable for $0 < x \leqslant s(t)$, $0 < t < t_0$. By making the change of variables

$$\begin{aligned}\xi &= s(t) - x \\ \tau &= t\end{aligned} \tag{3.1}$$

we arrive at an equation of the same form as (2.1) but with the arc $x = s(t)$ replaced by $\xi = 0$. The transformations (2.2) and (2.3) leave the boundary condition $u(0, t) = 0$ invariant. Hence, without loss of generality, we can assume that $u(x, t)$ is a strong solution of (2.4) defined in the region $0 < x < x_0$, $0 < t < t_0$, is continuously differentiable for $0 \leqslant x < x_0$, $0 < t < t_0$, and satisfies the boundary condition $u(0, t) = 0$. We now want to represent $u(x, t)$ in the form

$$u(x, t) = \mathbf{T}_1 h = h(x, t) + \int_0^x K(s, x, t) \, h(s, t) \, ds \tag{3.2}$$

where $h(x, t)$ is a solution of the heat equation satisfying $h(0, t) = 0$. Equation (3.2) is a Volterra integral equation of the second kind for $h(x, t)$;

hence, there exists a unique solution $h(x, t)$ of Eq. (3.2) which has the same regularity properties as $u(x, t)$ and satisfies $h(0, t) = u(0, t) = 0$. This can be seen by using the resolvent operator to express $h(x, t)$ in terms of $u(x, t)$. To show that this solution of the integral equation (3.2) is in fact a solution of the heat equation we substitute (3.2) into (2.4) and use the properties of the kernel $K(s, x, t)$ (cf. Eqs. (2.7)–(2.15)) to obtain

$$\begin{aligned} 0 &= u_{xx} + q(x, t)u - u_t \\ &= (h_{xx} - h_t) + \int_0^x K(s, x, t)(h_{ss}(s, t) - h_t(s, t))\, ds. \end{aligned} \tag{3.3}$$

Since solutions of Volterra integral equations of the second kind are unique, we must have

$$h_{xx} - h_t = 0, \tag{3.4}$$

i.e., $h(x, t)$ is a solution of the heat equation. Since $h(x, t)$ is a strong solution of the heat equation in $0 < x < x_0$, $0 < t < t_0$, is continuously differentiable in $0 \leqslant x < x_0$, $0 < t < t_0$, and vanishes at $x = 0$, $h(x, t)$ can be reflected across the $x = 0$ axis by the formula

$$h(x, t) = -h(-x, t). \tag{3.5}$$

Hence, $h(x, t)$ is in fact a strong solution of the heat equation in $D = \{(x, t): -x_0 < x < x_0, 0 < t < t_0\}$ and is partially analytic in D with respect to x. Since the kernel $K(s, x, t)$ is also analytic with respect to s and x for $-x_0 < s < x_0$, $-x_0 < x < x_0$, Eq. (3.2) now provides for each fixed t the (unique) analytic continuation of $u(x, t)$ into the domain $D \cap \{(x, t): x \leqslant 0\}$.

Reflection principles associated with (2.4) and the boundary condition $u_x(0, t) = 0$ can be obtained in a similar manner by using the operator $\mathbf{T}_2$.

For reflection principles for analytic solutions of elliptic and parabolic equations in two space variables the reader is referred to [6 and 5], respectively.

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GENERALIZED REFLECTION PRINCIPLES FOR PARABOLIC EQUATIONS IN ONE SPACE VARIABLE

DAVID COLTON

I. Introduction. In [1] the author has obtained reflection principles for solutions to the parabolic equation

$$u_{xx} + a(x, t)u_x + b(x, t)u = u_t \tag{1.1}$$

satisfying the boundary condition $u(0, t) = 0$, and for solutions of

$$u_{xx} + q(x, t)u = u_t \tag{1.2}$$

satisfying $u_x(0, t) = 0$, under the assumption that the coefficients in (1.1) and (1.2) are analytic. It is the purpose of this paper to extend these results to include the case when $u(x, t)$ is a solution of (1.1) satisfying the boundary data

$$\alpha(t)u(0, t) + \beta(t)u_x(0, t) = f(t) \tag{1.3}$$

where $\alpha(t)$, $\beta(t)$, and $f(t)$ are analytic functions and $\beta(t) \neq 0$ for $0 < t < t_0$, t_0 being a positive constant. This result provides an analogue for parabolic equations of Lewy's reflection principle for elliptic equations ([5]). Our approach to the reflection problem (1.1), (1.3), is based on the construction of an integral operator which maps solutions of the heat equation onto solutions of parabolic equations with variable coefficients. However it is of interest to note that our results are new even for the heat equation. We also want to emphasize that the reflection principle obtained here is valid for strong (in particular not necessarily analytic) solutions of (1.1). In this context we observe that for analytic solutions of parabolic equations in one space variable with analytic coefficients reflection principles are trivial in the sense that any analytic solution of a parabolic equation in one space variable can be analytically continued into a strip bounded by the characteristics, regardless of what the (analytic) data is on the t axis ([2], [4]).

II. Reduction to Canonical Form. We consider (1.1) under the assumption that the coefficients $a(x, t)$ and $b(x, t)$ are analytic in the rectangle $D = \{(x, t): -x_0 < x < x_0, 0 < t < t_0\}$ where x_0 and t_0 are positive constants. Setting

$$u(x, t) = v(x, t) \exp\left\{-\frac{1}{2}\int_0^x a(s, t)\, ds\right\} \tag{2.1}$$

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we arrive at an equation for $v(x, t)$ of the form (1.2) with $q(x, t)$ analytic for $(x, t) \in D$. The boundary condition (1.3) assumes the form

$$(\alpha(t) - \tfrac{1}{2}\beta(t)a(0, t))v(0, t) + \beta(t)v_x(0, t) = f(t). \tag{2.2}$$

We will now show that without loss of generality we can assume $f(t) \equiv 0$. Let $g(t)$ be defined by

$$g(t) = \frac{xf(t)}{\beta(t)}. \tag{2.3}$$

Then $w(x, t) = v(x, t) - g(t)$ is a solution of an equation of the form

$$w_{xx} + q(x, t)w - w_t = p(x, t) \tag{2.4}$$

where $q(x, t)$ and $p(x, t)$ are analytic for $(x, t) \in D$ and on the axis $x = 0$ $w(x, t)$ satisfies

$$(\alpha(t) - \tfrac{1}{2}\beta(t)a(0, t))w(0, t) + \beta(t)w_x(0, t) = 0. \tag{2.5}$$

Now let $K(s, x, t)$ be a solution of

$$K_{xx} - K_{ss} + q(x, t)K = K_t \tag{2.6}$$

for $-x_0 < x < x_0$, $-x_0 < s < x_0$, $0 < t < t_0$, which satisfies the initial data

$$K(x, x, t) = -\frac{1}{2}\int_0^x q(s, t)\, ds \tag{2.7a}$$

$$K(0, x, t) = 0. \tag{2.7b}$$

The existence of such a function was established in [1] where it was further shown that if $q(x, t)$ is analytic in D then $K(s, x, t)$ is analytic for $-x_0 < x < x_0$, $-x_0 < s < x_0$, $0 < t < t_0$. Let $\rho(x, t)$ be the (unique) solution of the Volterra integral equation

$$p(x, t) = \rho(x, t) + \int_0^x K(s, x, t)\rho(s, t)\, ds. \tag{2.8}$$

Due to the analyticity of $p(x, t)$ and $K(s, x, t)$ it can be easily verified that $\rho(x, t)$ is also analytic in D. Let $h(x, t)$ be the unique analytic solution of

$$h_{xx} - h_t = \rho(x, t) \tag{2.9a}$$

$$h(0, t) = h_x(0, t) = 0 \tag{2.9b}$$

for $(x, t) \in D$. The existence of $h(x, t)$ follows from the results of [4]. Now define $z(x, t)$ by

$$z(x, t) = h(x, t) + \int_0^x K(s, x, t)h(s, t)\, ds. \tag{2.10}$$

From the results of [1] it can be seen that $z(x, t)$ is an analytic solution of (2.4) for $(x, t) \in D$ which satisfies the Cauchy data $z(0, t) = z_x(0, t) = 0$, i.e., $z(x, t)$

satisfies the boundary condition (2.5). Hence $w(x, t) - z(x, t)$ is a solution of (1.2), (2.2) for $0 < x < x_0$, $0 < t < t_0$, with $f(t) \equiv 0$, i.e., without loss of generality we can assume that $f(t) \equiv 0$ in (2.2) to begin with.

Now suppose that $u(x, t)$ is a solution of (1.1) for $0 < x < x_0$, $0 < t < t_0$, continuously differentiable for $0 \leq x < x_0$, $0 < t < t_0$, and satisfies the boundary condition (1.3). Then from the above discussion it is seen that the problem of continuing $u(x, t)$ as a solution of (1.1) into all of D (i.e., "reflecting" $u(x, t)$ across the axis $x = 0$) can be reduced to the problem of reflecting solutions of

$$u_{xx} + q(x, t)u = u_t \tag{2.11}$$

which are defined in $0 < x < x_0$, $0 < t < t_0$, continuously differentiable in $0 \leq x < x_0$, $0 < t < t_0$, and satisfy the boundary condition

$$(\alpha(t) - \tfrac{1}{2}\beta(t)a(0, t))u(0, t) + \beta(t)u_x(0, t) = 0. \tag{2.12}$$

(2.11) and (2.12) will be referred to as the canonical form of the reflection problem (1.1), (1.3).

III. A Reflection Principle. In this section we will prove the following theorem (where by a strong, or classical, solution of (1.1) we mean a solution of (1.1) which is twice continuously differentiable with respect to x, continuously differentiable with respect to t, and satisfies (1.1) pointwise):

THEOREM: *Let $u(x, t)$ be a strong solution of* (1.1) *in $0 < x < x_0$, $0 < t < t_0$, continuously differentiable in $0 \leq x < x_0$, $0 < t < t_0$, and satisfying* (1.3) *on the axis $x = 0$, where $a(x, t)$ and $b(x, t)$ are analytic in D, $\alpha(t)$, $\beta(t)$ and $f(t)$ are analytic for $0 < t < t_0$, and $\beta(t) \not\equiv 0$ for $0 < t < t_0$. If either*
1) $2\alpha(t) - \beta(t)a(0, t) \not\equiv 0$ *for $0 < t < t_0$, or*
2) $2\alpha(t) - \beta(t)a(0, t) \equiv 0$ *for $0 < t < t_0$,*
then $u(x, t)$ can be uniquely continued as a strong solution of (1.1) *into all of D.*

Proof. Without loss of generality we can assume the reflection problem (1.1), (1.3) has been reduced to the canonical form (2.11), (2.12), where $q(x, t)$ is analytic for $(x, t) \in D$. If condition 2) of the theorem is true, the theorem follows from the results of [1]. Hence we now assume condition 1) is true. We first divide both sides of (2.12) by $\alpha(t) - \frac{1}{2}\beta(t)a(0, t)$ and rewrite this equation in the form

$$u(0, t) + a(t)u_x(0, t) = 0 \tag{3.1}$$

where $a(t) = \beta(t)[\alpha(t) - \frac{1}{2}\beta(t)a(0, t)]^{-1} \not\equiv 0$ is analytic for $0 < t < t_0$. Let $h^{(1)}(x, t)$ be a solution of the heat equation

$$h_{xx}^{(1)} = h_t^{(1)} \tag{3.2}$$

which is twice continuously differentiable for $0 \leq x < x_0$, $0 < t < t_0$, and satisfies the boundary condition

$$h^{(1)}(0, t) = 0. \tag{3.3}$$

Define $h^{(2)}(x, t)$ by

$$h^{(2)}(x, t) = -a(t)h_x^{(1)}(x, t). \tag{3.4}$$

Then it can easily be verified that $h^{(2)}(x, t)$ is a strong solution of

$$h_{xx}^{(2)} + \frac{a'(t)}{a(t)} h^{(2)} = h_t^{(2)} \tag{3.5}$$

in $0 < x < x_0$, $0 < t < t_0$, is continuously differentiable in $0 \leq x < x_0$, $0 < t < t_0$ and satisfies the boundary condition

$$h_x^{(2)}(0, t) = 0. \tag{3.6}$$

We now look for a solution of (2.11), (2.12) in the form

$$\begin{aligned} u(x, t) = h^{(1)}(x, t) + h^{(2)}(x, t) &+ \int_0^x K^{(1)}(s, x, t)h^{(1)}(s, t)\, ds \\ &+ \int_0^x K^{(2)}(s, x, t)h^{(2)}(s, t)\, ds \end{aligned} \tag{3.7}$$

where $K^{(1)}(s, x, t)$ and $K^{(2)}(s, x, t)$ are functions to be determined. Substituting (3.7) into (2.11) and using (3.2)–(3.6) to integrate by parts gives

$$\begin{aligned} 0 &= u_{xx} + q(x, t)u - u_t \\ &= K^{(1)}(0, x, t)h_x^{(1)}(0, t) - K_s^{(2)}(0, x, t)h^{(2)}(0, t) \\ &+ 2(K_s^{(1)}(x, x, t) + K_x^{(1)}(x, x, t) + \tfrac{1}{2}q(x, t))h^{(1)}(x, t) \\ &+ 2\left(K_s^{(2)}(x, x, t) + K_x^{(2)}(x, x, t) + \tfrac{1}{2}q(x, t) - \frac{a'(t)}{a(t)}\right)h^{(2)}(x, t) \\ &+ \int_0^x (K_{xx}^{(1)} - K_{ss}^{(1)} + q(x, t)K^{(1)} - K_t^{(1)})h^{(1)}(s, t)\, ds \\ &+ \int_0^x \left(K_{xx}^{(2)} - K_{ss}^{(2)} + \left(q(x, t) - \frac{a'(t)}{a(t)}\right)K^{(2)} - K_t^{(2)}\right)h^{(2)}(s, t)\, ds. \end{aligned} \tag{3.8}$$

(3.8) will be satisfied if $K^{(1)}(s, x, t)$ satisfies

$$K_{xx}^{(1)} - K_{ss}^{(1)} + q(x, t)K^{(1)} = K_t^{(1)} \tag{3.9}$$

for $(x, t) \in D$, $-x_0 < s < x_0$, and the initial data

$$K^{(1)}(x, x, t) = -\frac{1}{2}\int_0^x q(s, t)\, ds \tag{3.10a}$$

$$K^{(1)}(0, x, t) = 0, \tag{3.10b}$$

and $K^{(2)}(s, x, t)$ satisfies

$$(3.11) \qquad K_{xx}^{(2)} - K_{ss}^{(2)} + \left(q(x, t) - \frac{a'(t)}{a(t)}\right)K^{(2)} = K_t^{(2)}$$

for $(x, t) \in D$, $-x_0 < s < x_0$, and the initial data

$$(3.12a) \qquad K^{(2)}(x, x, t) = -\frac{1}{2}\int_0^x \left(q(s, t) - \frac{a'(t)}{a(t)}\right) ds$$

$$(3.12b) \qquad K_s^{(2)}(0, x, t) = 0.$$

The existence of the functions $K^{(1)}(s, x, t)$ and $K^{(2)}(s, x, t)$ and their analyticity for $(x, t) \in D$, $-x_0 < s < x_0$, follows from the results of [1]. We note that from the initial conditions (3.10) and (3.12) it is seen that if $u(x, t)$ is a solution of (2.11) defined by (3.7) then

$$(3.13) \qquad u(0, t) + a(t)u_x(0, t) = h^{(2)}(0, t) + a(t)h_x^{(1)}(0, t) = 0,$$

i.e., $u(x, t)$ satisfies the boundary condition (3.1).

We will now show that if $u(x, t)$ is any (strong) solution of (2.11) defined in $0 < x < x_0$, $0 < t < t_0$, continuously differentiable in $0 \le x < x_0$, $0 < t < t_0$, and satisfying the boundary condition (3.1), then it can be represented in the form (3.7) where $h^{(1)}(x, t)$ and $h^{(2)}(x, t)$ are defined by (3.2)–(3.4). Let $h^{(1)}(x, t)$ be the unique solution of the Volterra integral equation

$$(3.14) \qquad \int_0^x u(s, t)\, ds = -a(t)h^{(1)}(x, t) + \int_0^x \Gamma(s, x, t)h^{(1)}(s, t)\, ds,$$

where

$$(3.15) \qquad \Gamma(s, x, t) = 1 - a(t)K^{(2)}(s, s, t) + \int_s^x K^{(1)}(s, \xi, t)\, d\xi + a(t)\int_s^x K_s^{(2)}(s, \xi, t)\, d\xi.$$

The existence and uniqueness of $h^{(1)}(x, t)$ is assured from the fact that $a(t) \neq 0$ for $0 < t < t_0$. Equation (3.14) also implies that $h^{(1)}(0, t) = 0$ and $h^{(1)}(x, t)$ is twice continuously differentiable for $0 \le x < x_0$, $0 < t < t_0$, and three times differentiable for $0 < x < x_0$, $0 < t < t_0$. Differentiating (3.14) with respect to x and integrating by parts gives

$$(3.16) \qquad u(x, t) = h^{(1)}(x, t) + h^{(2)}(x, t) + \int_0^x K^{(1)}(s, x, t)h^{(1)}(s, t)\, ds + \int_0^x K^{(2)}(s, x, t)h^{(2)}(s, t)\, ds$$

where $h^{(2)}(x, t)$ is defined by (3.4). The fact that $u(x, t)$ satisfies (3.1) implies that $h_x^{(2)}(0, t) = -a(t)h_{xx}^{(1)}(0, t) = 0$, i.e. $h_{xx}^{(1)}(0, t) = 0$ for $0 < t < t_0$ since $a(t) \neq 0$ in this interval. Applying the differential equation (2.11) to both sides of (3.16), using equations (3.5) and (3.9)–(3.12), and integrating by parts gives

$$(3.17)\qquad \begin{aligned} 0 = {} & (h_{xx}^{(1)}(x, t) - h_t^{(1)}(x, t)) - a(t)(h_{xxx}^{(1)}(x, t) - h_{xt}^{(1)}(x, t)) \\ & - a(t)K^{(2)}(x, x, t)(h_{xx}^{(1)}(x, t) - h_t^{(1)}(x, t)) \\ & + \int_0^x K^{(1)}(s, x, t)(h_{ss}^{(1)}(s, t) - h_t^{(1)}(s, t))\, ds \\ & + \int_0^x a(t)K_s^{(2)}(s, x, t)(h_{ss}^{(1)}(s, t) - h_t^{(1)}(s, t))\, ds. \end{aligned}$$

Integrating both sides of (3.17) with respect to x gives

$$(3.18)\qquad \begin{aligned} 0 = {} & -a(t)(h_{xx}^{(1)}(x, t) - h_t^{(1)}(x, t)) \\ & + \int_0^x \Gamma(s, x, t)(h_{ss}^{(1)}(s, t) - h_t^{(1)}(s, t))\, ds \end{aligned}$$

where $\Gamma(s, x, t)$ is defined by (3.15). Since $a(t) \neq 0$ and solutions of nonsingular Volterra integral equations of the second kind are unique, we can conclude that $h^{(1)}(x, t)$ must be a (strong) solution of (3.2) and $h^{(2)}(x, t)$ a (strong) solution of (3.5).

The conclusion of the theorem now follows immediately from the well known reflection principle for solutions $h(x, t)$ of the heat equation satisfying the boundary condition $h(0, t) = 0$. This is due to the fact that $u(x, t)$ can be represented in the form (3.7) where $h^{(1)}(x, t)$ and $h^{(2)}(x, t)$ satisfy (3.2)–(3.4). By the reflection principle for solutions of the heat equation which satisfy homogeneous Dirichlet data on $x = 0$ we can conclude that $h^{(1)}(x, t)$ is in fact a strong solution of (3.2) in all of D, and hence $h^{(2)}(x, t)$ is also a strong solution of (3.5) throughout D. Since $K^{(1)}(s, x, t)$ and $K^{(2)}(s, x, t)$ are analytic for $-x_0 < x < x_0$, $-x_0 < s < x_0$, $0 < t < t_0$, (3.7) implies that $u(x, t)$ can be continued as a strong solution of (2.11) into all of D. The uniqueness of the continuation follows from the fact that strong solutions of parabolic equations with analytic coefficients are analytic in the space variable ([3]).

IV. Concluding Remarks. It would be desirable to remove the conditions 1), 2) of the above theorem. In the special case when the coefficients of the differential equation (1.1) and the boundary condition (1.3) are independent of t one of the two conditions in the theorem is always satisfied and hence these conditions no longer need be incorporated in the theorem.

The more general parabolic equation

$$(4.1)\qquad u_{xx} + a(x, t)u_x + b(x, t)u = c(x, t)u_t$$

where $c(x, t) > 0$ for $(x, t) \in D$ can be reduced to (1.1) by a simple change of independent variables (c.f. [1]). Under this change of variables the form of the boundary condition (1.3) remains the same, i.e., $\beta(t)$ remains analytic and non-vanishing for $0 < t < t_0$.

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THE APPROXIMATION OF SOLUTIONS TO INITIAL BOUNDARY VALUE PROBLEMS FOR PARABOLIC EQUATIONS IN ONE SPACE VARIABLE

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THE APPROXIMATION OF SOLUTIONS TO INITIAL BOUNDARY VALUE PROBLEMS FOR PARABOLIC EQUATIONS IN ONE SPACE VARIABLE*

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1. Introduction. In a recent paper [3] the author has constructed integral operators which map solutions of the heat equation

$$h_{xx} = h_t \tag{1.1}$$

onto solutions of the parabolic equation

$$u_{xx} + q(x, t)u = u_t \tag{1.2}$$

and used these operators to obtain reflection principles for Eq. (1.2) which are analogous to the Schwarz reflection principle for analytic functions of a complex variable. (We note that the more general equation

$$v_{xx} + a(x, t)v_x + b(x, t)v = v_t \tag{1.3}$$

can be reduced to an equation of the form (1.2) by the change of variables

$$v(x, t) = u(x, t) \exp\left\{-\frac{1}{2}\int_0^x a(s, t)\, ds\right\}).$$

In this paper we will show how these operators can be used to obtain approximate solutions to the first initial boundary value problem for Eq. (1.2) (or (1.3)) in a rectangle and quarter plane. More specifically, our approach provides an analogue for Eqs. (1.2) and (1.3) of the method of separation of variables and the "method of images" for the heat equation, and is an extension of the use of integral operator methods for approximating solutions of boundary value problems for elliptic equations (cf. [1, 2, 6, 10]) to the case of initial boundary value problems for parabolic equations. Numerical examples using the methods described in this paper will be published elsewhere.

2. The first initial boundary value problem in a rectangle. Let $u(x, t)$ be a (strong) solution of Eq. (1.2) in the rectangle $R = \{(x, t): -1 < x < 1, 0 < t < T\}$ such that $u(x, t)$ continuously assumes the initial-boundary data

$$u(-1, t) = f(t), \quad u(1, t) = g(t); \quad 0 \le t < T, \quad u(x, 0) = h(x); \quad -1 \le x \le 1. \tag{2.1}$$

(A strong, or classical, solution of Eq. (1.2) is a solution of Eq. (1.2) which is twice

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continuously differentiable with respect to x, continuously differentiable with respect to t, and satisfies Eq. (1.2) pointwise.) Let $\bar{R}$ denote the closure of R and assume that $q(x, t) \in C^1(\bar{R})$, and that for each fixed x, $-1 \leq x \leq 1$, $q(x, t)$ is an analytic function of t for $|t - \frac{1}{2}T| \leq \frac{1}{2}T$. (This domain of regularity is chosen in order to guarantee the global existence of the integral operators used in this paper—cf. [3].) Our aim is to construct a function $w(x, t)$ which is a solution of Eq. (1.2) in R and approximates $u(x, t)$ arbitrarily closely in the maximum norm on compact subsets of R. This will be accomplished by constructing a complete family of solutions to Eq. (1.2) in the maximum norm and then minimizing the L_2 norm of a finite linear combination of these solutions over the base and vertical sides of R.

We first consider Eq. (1.1). In [9] Rosenbloom and Widder have constructed a set of polynomial solutions to Eq. (1.1) which are defined by

$$\begin{aligned} h_n(x, t) &= n! \sum_{k=0}^{[n/2]} \frac{x^{n-2k} t^k}{(n-2k)!\, k!} \\ &= (-t)^{n/2} H_n\left(\frac{x}{(-4t)^{1/2}}\right), \end{aligned} \tag{2.2}$$

where $H_n(z)$ denotes the Hermite polynomials. In [12] Widder showed that the set $\{h_n(x, t)\}$ was complete in the space of solutions to Eq. (1.1) which are analytic in a neighborhood of the origin, i.e. if $h(x, t)$ is a solution of Eq. (1.1) which is analytic for $|x| \leq x_0$, $|t| \leq t_0$ (where x and t are complex variables) then on the rectangle $-x_0 \leq x \leq x_0$, $-t_0 \leq t \leq t_0$, $h(x, t)$ can be approximated in the maximum norm by a finite linear combination of members of the set $\{h_n(x, t)\}$. The lemma below shows that the set $\{h_n(x, t)\}$ is in fact complete for the space of strong solutions of Eq. (1.1) which are defined in R and continuous in $\bar{R}$.

LEMMA 2.1. Let $h(x, t)$ be a (strong) solution of Eq. (1.1) in R which is continouus in $\bar{R}$. Then, given $\epsilon > 0$, there exist constants $a_1, \cdots, a_N$ such that

$$\max_{(x,t)\in\bar{R}} \left| h(x, t) - \sum_{n=0}^{N} a_n h_n(x, t) \right| < \epsilon.$$

Proof: By the Weierstrass approximation theorem and the maximum principle for the heat equation [7], there exists a solution $w_1(x, t)$ of Eq. (1.1) in R which assumes polynomial initial and boundary data such that

$$\max_{(x,t)\in\bar{R}} |h(x, t) - w_1(x, t)| < \epsilon/3. \tag{2.3}$$

Let

$$w_1(-1, t) = \sum_{m=0}^{M} b_m t^m, \qquad w_1(1, t) = \sum_{m=0}^{M} c_m t^m,$$

and look for a solution of Eq. (1.1) in the form

$$v(x, t) = \sum_{m=0}^{M} v_m(x) t^m \tag{2.4}$$

where $v(-1, t) = w_1(-1, t)$, $v(1, t) = w_1(1, t)$. Substituting Eq. (2.4) into Eq. (1.1) leads to the following recursion scheme for the $v_m(x)$:

$$\begin{aligned}
&v_M'' = 0; && v_M(-1) = b_M, && v_M(1) = c_M,\\
&v_{M-1}'' = Mv_M; && v_{M-1}(-1) = b_{M-1}, && v_{M-1}(1) = c_{M-1},\\
&\quad\vdots && && \vdots\\
&v_0'' = v_1; && v_0(-1) = b_0, && v_0(1) = c_0.
\end{aligned} \tag{2.5}$$

Eq. (2.5) shows that each $v_m(x)$ is a polynomial in x and is uniquely determined. Now consider $w_2(x, t) = w_1(x, t) - v(x, t)$. By the method of separation of variables it is seen that there exist constants $d_1, \cdots, d_L$ such that

$$\max_{(x,t)\in\bar{R}} \left| w_2(x, t) - \sum_{l=0}^{L} d_l \sin \frac{l\pi}{2}(x + 1) \exp\left(-\frac{l^2\pi^2 t}{4}\right)\right| < \frac{\epsilon}{3}. \tag{2.6}$$

Hence there exists a solution $w_3(x, t)$ of Eq. (1.1) which is an entire function of the complex variables x and t such that

$$\max_{(x,t)\in\bar{R}} |h(x, t) - w_3(x, t)| < \frac{2\epsilon}{3}. \tag{2.7}$$

From the previously mentioned results of [12] there exist positive constants $a_1, \cdots, a_n$ such that

$$\max_{(x,t)\in\bar{R}} \left| w_3(x, t) - \sum_{n=0}^{N} a_n h_n(x, t)\right| < \frac{\epsilon}{3}, \tag{2.8}$$

and the proof of the lemma now follows immediately from the triangle inequality.

We now want to construct a complete family of solutions to Eq. (1.2) which is analogous to the family $\{h_n(x, t)\}$ for the heat equation. To accomplish this we make use of the integral operators constructed in [3]. Let $u(x, t) \in C^0(\bar{R})$ be a (strong) solution of Eq. (1.2) in R such that $u(O, t) = 0$. Then from [3] we have that $u(x, t)$ can be represented in the form

$$u(x, t) = h(x, t) + \int_0^x K(s, x, t)h(s, t)\, ds \tag{2.9}$$

where $h(x, t)$ is a solution of Eq. (1.1) in R satisfying $h(0, t) = 0$ and $K(s, x, t)$ is defined by

$$K(s, x, t) = \tfrac{1}{2}[E(s, x, t) - E(-s, x, t)] \tag{2.10}$$

where $\tilde{E}(\xi, \eta, t) = E(\xi - \eta, \xi + \eta, t)$ can be constructed by the recursion scheme

$$\tilde{E}(\xi, \eta, t) = \lim_{n\to\infty} \tilde{E}_n(\xi, \eta, t)$$

$$\tilde{E}_1(\xi, \eta, t) = -\frac{1}{2}\int_0^\xi q(s, t)\, ds + \frac{1}{2}\int_0^\eta q(s, t)\, ds,$$

$$\begin{aligned}\tilde{E}_{n+1}(\xi, \eta, t) = &-\frac{1}{2}\int_0^\xi q(s, t)\, ds + \frac{1}{2}\int_0^\eta q(s, t)\, ds,\\
&+ \int_0^\eta \int_0^\xi \left(\frac{\partial}{\partial t}\tilde{E}_n(\xi, \eta, t) - q(\xi + \eta, t)\tilde{E}_n(\xi, \eta, t)\right) d\xi\, d\eta; \qquad n \geq 1.\end{aligned} \tag{2.11}$$

The sequence $\{\tilde{E}_n\}$ converges uniformly for $(x, t) \in \bar{R}$, $-1 \leq s \leq 1$. The convergence of the sequence $\{\tilde{E}_n\}$ is quite rapid and good approximations can be found by terminating the recursion process after several iterations. Error estimates for such an approximating procedure can be found in [3]. If instead of the condition $u(0, t) = 0$ we have that $u(x, t)$ satisfies $u_x(0, t) = 0$, then we can represent $u(x, t)$ in the form

$$u(x, t) = h(x, t) + \int_0^x M(s, x, t)h(s, t)\, ds \tag{2.12}$$

where $h(x, t)$ is a solution of Eq. (1.1) in R satisfying $h_x(0, t) = 0$ and $M(s, x, t)$ is defined by

$$M(s, x, t) = \tfrac{1}{2}[G(s, x, t) + G(-s, x, t)] \tag{2.13}$$

where $\tilde{G}(\xi, \eta, t) = G(\xi - \eta, \xi + \eta, t)$ can be constructed via the recursion scheme

$$\tilde{G}(\xi, \eta, t) = \lim_{n\to\infty} \tilde{G}_n(\xi, \eta, t),$$

$$\begin{aligned} \tilde{G}_{n+1}(\xi, \eta, t) = &-\frac{1}{2}\int_0^\xi q(s, t)\, ds - \frac{1}{2}\int_0^\eta q(s, t)\, ds \\ &+ \int_0^\eta \int_0^\xi \left(\frac{\partial}{\partial t}\tilde{G}_n(\xi, \eta, t) - q(\xi, \eta, t)\tilde{G}_n(\xi, \eta, t)\right) d\xi\, d\eta; \qquad n \geq 1. \end{aligned} \tag{2.14}$$

The sequence $\{\tilde{G}_n\}$ again converges rapidly and uniformly for $(x, t) \in \bar{R}$, $-1 \leq s \leq 1$. Observing that for $n \geq 0$, $h_{2n}(x, t)$ is an even function of x and that for $n \geq 0$, $h_{2n+1}(x, t)$ is an odd function of x, we now define the particular solutions $u_n(x, t)$ of Eq. (1.2) by

$$\begin{aligned} u_{2n}(x, t) &= h_{2n}(x, t) + \int_0^x M(s, x, t)h_{2n}(s, t)\, ds; \qquad n \geq 0, \\ u_{2n+1}(x, t) &= h_{2n+1}(x, t) + \int_0^x K(s, x, t)h_{2n+1}(s, t)\, ds; \qquad n \geq 0. \end{aligned} \tag{2.15}$$

Lemma 2.2: Let $u(x, t)$ be a (strong) solution of Eq. (1.2) in R which is continuous in $\bar{R}$. Then, given $\epsilon > 0$, there exist constants $a_1, \cdots, a_N$ such that

$$\max_{(x,t)\in\bar{R}} \left| u(x, t) - \sum_{n=0}^N a_n u_n(x, t) \right| < \epsilon.$$

Proof: We first show that $u(x, t)$ can be represented in the form

$$u(x, t) = h(x, t) + \frac{1}{2}\int_{-x}^x [K(s, x, t) + M(s, x, t)]h(s, t)\, ds \tag{2.16}$$

where $h(x, t)$ is a solution of Eq. (1.1) in R. Eq. (2.16) is a Volterra integral equation of the second kind for $h(x, t)$ and can be uniquely solved for $h(x, t)$ where $h(x, t)$ is defined in R and continuous in $\bar{R}$ [11]. It remains to be shown that $h(x, t)$ is a solution of Eq. (1.1). From Eqs. (2.10) and (2.13) we have that $K(s, x, t) = -K(-s, x, t)$ and $M(s, x, t) = M(-s, x, t)$ and hence we can rewrite Eq. (2.16) in the form

$$\begin{aligned} u(x, t) = &\tfrac{1}{2}(h(x, t) - h(-x, t)) + \frac{1}{2}\int_0^x K(s, x, t)[h(s, t) - h(-s, t)]\, ds \\ &+ \tfrac{1}{2}(h(x, t) + h(-x, t)) + \frac{1}{2}\int_0^x M(s, x, t)[h(s, t) + h(-s, t)]\, ds. \end{aligned} \tag{2.17}$$

Applying the differential operator (1.2) to both sides of Eq. (2.17), using the fact that $K(s, x, t)$ and $M(s, x, t)$ are solutions of the following initial boundary value problems [3]

$$K_{xx} - K_{ss} + q(x, t)K = K_t , \tag{2.18a}$$

$$K(x, x, t) = -\frac{1}{2}\int_0^x q(s, t)\, ds, \qquad K(0, x, t) = 0, \tag{2.18b}$$

$$M_{xx} - M_{ss} + q(x, t)M = M_t , \tag{2.19a}$$

$$M(x, x, t) = -\frac{1}{2}\int_0^x q(s, t)\, ds, \qquad M_x(0, x, t) = 0, \tag{2.19b}$$

and rewriting the resulting expression in the form of Eq. (2.16), gives

$$0 = (h_{xx} - h_t) + \frac{1}{2}\int_{-x}^{x} [K(s, x, t) + M(s, x, t)](h_{ss}(s, t) - h_t(s, t))\, ds. \tag{2.20}$$

Since solutions of Volterra integral equations of the second kind are unique we can conclude that $h(x, t)$ is a solution of Eq. (1.1) in R.

Using Lemma 2.1, we now approximate $h(x, t)$ by a linear combination of the polynomials defined in Eq. (2.2) such that

$$\max_{(x,t)\in\bar{R}} \left| h(x, t) - \sum_{n=0}^{N} a_n h_n(x, t) \right| < \frac{\epsilon}{1 + C} \tag{2.21}$$

where

$$C = \max_{\substack{(x,t)\in\bar{R} \\ -1\leq s\leq 1}} |K(s, x, t) + M(s, x, t)|.$$

Eqs. (2.16), (2.17) and the fact that $h_{2n}(x, t)$ is an even function of x and $h_{2n+1}(x, t)$ is an odd function of x for $n \geq 0$ now show that

$$\max_{(x,t)\in\bar{R}} \left| u(x, t) - \sum_{n=0}^{N} a_n u_n(x, t) \right| < \epsilon. \tag{2.22}$$

THEOREM 2.1: Let $u(x, t)$ be a (strong) solution of Eq. (1.2) in R which is continuous in $\bar{R}$ and satisfies the initial-boundary data (2.1). Let R_0 be a compact subset of R. Let N be a positive integer and define a_{kn} and b_k, $k = 0, 1, \cdots, N$, $n = 0, 1, \cdots, N$, by the formulas

$$a_{kn} = \int_0^T u_n(-1, t)u_k(-1, t)\, dt + \int_{-1}^{1} u_n(x, 0)u_k(x, 0)\, dx + \int_0^T u_n(1, t)u_k(1, t)\, dt,$$

$$b_k = \int_0^T f(t)u_k(-1, t)\, dt + \int_{-1}^{1} h(x)u_k(x, 0)\, dx + \int_0^T g(t)u_k(1, t)\, dt. \tag{2.23}$$

Then there exists a unique solution $c_1, \cdots, c_N$ of the linear algebraic system

$$\sum_{n=0}^{N} a_{kn}c_n = b_k ; \qquad k = 0, 1, \cdots, N, \tag{2.24}$$

and given $\epsilon > 0$ we have

$$\max_{(x,t)\in R_0} \left| u(x, t) - \sum_{n=0}^{N} c_n u_n(x, t) \right| < \epsilon \tag{2.25}$$

for N sufficiently large.

Proof: Let $G(x, t, \xi, \tau)$ be the Green's function for Eq. (1.2) in R. Then $u(x, t)$ can be represented in the form

$$u(x, t) = \int_0^t \frac{\partial}{\partial \xi} G(x, t, -1, \tau) f(\tau)\, d\tau - \int_0^t \frac{\partial}{\partial \xi} G(x, t, 1, \tau) g(\tau)\, d\tau + \int_{-1}^1 G(x, t, \xi, 0) h(\xi)\, d\xi \tag{2.26}$$

where $G(x, t, \xi, \tau)$ is continuous for $(x, t, \xi, \tau) \in R_0 \times \partial R$. Hence for $(x, t) \in R_0$ we have by Schwarz's inequality

$$\max_{(x,t)\in R_0} |u(x, t)|^2 \leq C\left[\int_0^T |f(\tau)|^2\, d\tau + \int_0^T |g(\tau)|^2\, d\tau + \int_{-1}^1 |h(\xi)|^2\, d\xi\right] \tag{2.27}$$

where

$$C = \max_{(x,t)\in R_0} \left\{\int_0^T \left|\frac{\partial}{\partial \xi} G(x, t, -1, \tau)\right|^2 d\tau + \int_0^T \left|\frac{\partial}{\partial \xi} G(x, t, 1, \tau)\right|^2 d\tau + \int_{-1}^1 |G(x, t, \xi, 0)|^2\, d\xi. \right. \tag{2.28}$$

From Lemma 2.2 we can conclude that for N sufficiently large there exist constants $C_1, \cdots, C_N$ such that

$$\max_{(x,t)\in R} \left|u(x, t) - \sum_{n=0}^N c_n u_n(x, t)\right|^2 < \frac{\epsilon^2}{2C(T+1)} \tag{2.29}$$

and Eq. (2.27) (applied to $u(x, t) - \sum_{n=0}^N c_n u_n(x, t)$ instead of $u(x, t)$) shows that a suitable choice of the constants $c_1, \cdots, c_N$ can be determined by minimizing the quadratic functional

$$Q(c_1, \cdots, c_N) = \int_0^T \left|f(\tau) - \sum_{n=0}^N c_n u_n(-1, \tau)\right|^2 d\tau + \int_0^T \left|g(\tau) - \sum_{n=0}^N c_n u_n(1, \tau)\right|^2 d\tau + \int_{-1}^1 \left|h(\xi) - \sum_{n=0}^N c_n u_n(\xi, 0)\right|^2 d\xi. \tag{2.30}$$

We note that $Q(c_1, \cdots, c_N)$ is always positive or zero and hence its only stationary point represents a minimum. This minimum can be found by solving the set of equations $\partial Q/\partial c_k = 0$ and this leads to the system (2.23), (2.24). Since the set $\{u_n(x, t)\}_{n=0}^N$ is linearly independent (this follows from the fact that the set $\{v_n(x, t)\}_{n=0}^N$ is linearly independent) the coefficient matrix (a_{kn}) is nonsingular and hence the system (2.23), (2.24) has a unique solution. (Here use has been made of the fact that if a solution of Eq. (1.2) vanishes on the base and vertical sides of R it must be identically zero throughout R [7]). If $c_1, \cdots, c_N$ is the solution of the system (2.24) then Eqs. (2.27) and (2.29) imply the validity of Eq. (2.25).

We note in passing that error estimates for the above approximation procedure can be found if one can estimate the maximum of $|u(x, t) - \sum_{n=0}^N c_n u_n(x, t)|$ on the base and vertical sides of R. The maximum principle for parabolic equations [7] then immediately gives estimates for $|u(x, t) - \sum_{n=0}^\infty c_n u_n(x, t)|$ in the interior of R.

3. The first initial boundary value problem in a quarter plane. In this section we will derive constructive methods for approximating solutions of Eq. (1.2) which satisfy the initial-boundary data

$$\begin{aligned} u(0, t) &= 0; \qquad 0 \le t < T, \\ u(x, 0) &= f(x); \qquad 0 \le x < \infty, \end{aligned} \tag{3.1}$$

where we assume $f(x)$ is continuous, $f(0) = 0$, and there exist positive constants M and A such that

$$|f(x)| \le M \exp Ax^2; \qquad 0 \le x < \infty. \tag{3.2}$$

We will look for a solution $u(x, t)$ of Eq. (1.2) in $0 < x < \infty, 0 < t < T < 1/4A$ which is continuous for $0 \le x < \infty$, $0 \le t < T$, satisfies the initial-boundary data (3.1), and satisfies a bound of the form

$$|u(x, t)| \le M_1 \exp A_1 x^2; \qquad 0 \le x < \infty, \qquad 0 \le t < T \tag{3.3}$$

for some positive constants M_1 and A_1 (cf. [7], Ch. 4). For the sake of simplicity we will only consider the case when $q(x, t) = q(x)$ is independent of t, and make the assumption that $q(x)$ is continuously differentiable for $0 \le x < \infty$ and is bounded in absolute value by a positive constant C for $0 \le x < \infty$. In order to exploit the construction of the kernel $K(s, x, t)$ already given in Eqs. (2.10), (2.11) we will assume without loss of generality that $q(x)$ has been extended to a continuously differentiable function defined for $-\infty < x < \infty$. The method we will use to solve the initial-boundary value problem (1.2), (3.1), is basically an application of the reflection principle (or "method of images") for parabolic equations derived in [3].

We look for a solution of Eqs. (1.2) and (3.1) in the form

$$u(x, t) = h(x, t) + \int_0^x K(s, x) h(s, t)\, ds \tag{3.9}$$

where $K(s, x)$ is defined by Eqs. (2.10) and (2.11) (noting that $q(x, t) = q(x)$ is independent of t and hence so is $K(s, x, t) = K(s, x)$) and $h(x, t)$ is a (strong) solution of Eq. (1.1) for $0 \le x < \infty, 0 < t < T$, satisfying $h(0, t) = 0$. Note that by the reflection principle for the heat equation we can conclude that $h(x, t)$ is in fact a solution of the heat equation for $-\infty < x < \infty, 0 < t < T$ and hence $u(x, t)$ is a strong solution of Eq. (1.2) in this region. Evaluating Eq. (3.9) at $t = 0$ leads to a Volterra integral equation of the second kind for the unknown function $h(x, 0)$ and from this data along with $h(0, t) = 0$ it is possible to construct $h(x, t)$ in the region $0 < x < \infty, 0 < t < T$, provided we know that $h(x, 0)$ satisfies a bound of the form (3.2). However, the construction of $h(x, 0)$ and the estimation of its rate of growth is based on the construction and rate of growth of the resolvent kernel for Eq. (3.9). But the resolvent kernel is obtained by an iteration procedure involving the kernel $K(s, x)$ which in turn is constructed by the iteration procedure (2.11). Hence, in order to solve the initial boundary value problem (1.2), (3.1) by the use of the integral operator (3.9), it is important to provide a better method of constructing the resolvent kernel for Eq. (3.9). We will now show how this can be done by reducing the construction of the resolvent kernel to the problem of solving a Goursat problem for a hyperbolic equation.

We look for a solution $h(x, t)$ of Eq. (1.1) in the form

$$h(x, t) = u(x, t) + \int_0^x \Gamma(s, x)u(s, t)\, ds \tag{3.10}$$

where $u(x, t)$ is a solution of Eq. (1.2) in $0 < x < \infty$, $0 < t < T$, is continuously differentiable for $0 \le x < \infty$, $0 < t < T$, continuous for $0 \le x < \infty$, $0 \le t < T$, and satisfies the boundary condition $u(0, t) = 0$ for $0 \le t < T$. Substituting Eq. (3.10) into Eq. (1.1) and integrating by parts shows that $h(x, t)$ will be a solution of Eq. (1.1) provided $\Gamma(s, x)$ satisfies the Goursat problem

$$\Gamma_{xx} - \Gamma_{ss} - q(s)\Gamma = 0 \tag{3.11a}$$

$$\Gamma(x, x) = \frac{1}{2}\int_0^x q(s)\, ds, \qquad \Gamma(0, x) = 0. \tag{3.11b}$$

From [5, p. 119], it is seen that the unique solution $\tilde{\Gamma}(\xi, \eta) = \Gamma(\xi - \eta, \xi + \eta)$ of Eqs. (3.11a), (3.11b) is given by the iterative scheme

$$\begin{aligned} \tilde{\Gamma}(\xi, \eta) &= \lim_{n\to\infty} \tilde{\Gamma}_n(\xi, \eta), \\ \tilde{\Gamma}_1(\xi, \eta) &= \frac{1}{2}\int_\eta^\xi q(s)\, ds, \\ \tilde{\Gamma}_{n+1}(\xi, \eta) &= \frac{1}{2}\int_\eta^\xi q(s)\, ds - \int_0^\eta \int_\eta^\xi q(\xi + \eta)\tilde{\Gamma}_n(\xi, \eta)\, d\xi\, d\eta; \qquad n \ge 1. \end{aligned} \tag{3.12}$$

Hence the existence of the operator (3.10) is established. From the initial conditions (3.11b) and (2.18b) satisfied by the kernels $\Gamma(s, x)$ and $K(s, x)$ respectively, it is seen that the operators (3.9) and (3.10) leave the Cauchy data assumed by $h(x, t)$ and $u(x, t)$ invariant. Hence from the uniqueness of the solution to Cauchy's problem for parabolic equations [8] we can conclude that the operators defined by Eqs. (3.9) and (3.10) are inverses of one another, i.e. $\Gamma(s, x)$ is the resolvent kernel of the operator (3.9).

We now want to obtain an estimate on the rate of growth of $\Gamma(s, x)$ for $0 \le s \le x$, $0 \le x < \infty$. Since $x = \xi + \eta$, $s = \xi - \eta$, it is seen that under these restrictions on s and x we have $\xi \ge \eta$, $\eta \ge 0$. Since $|q(x)| < C$ for $0 \le x < \infty$, it is seen from Eq. (3.12) that for $\xi \ge \eta$, $\eta \ge 0$, $|\tilde{\Gamma}(\xi, \eta)| \le P(\xi, \eta)$ where $P(\xi, \eta)$ is defined by the recursion scheme

$$\begin{aligned} P(\xi, \eta) &= \lim_{n\to\infty} P_n(\xi, \eta), \\ P_1(\xi, \eta) &= C\xi, \\ P_{n+1}(\xi, \eta) &= C\xi + C\int_0^\eta \int_0^\xi P_n(\xi, \eta)\, d\xi\, d\eta. \end{aligned} \tag{3.13}$$

Hence

$$\begin{aligned} P(\xi, \eta) &= \sum_{k=0}^\infty \frac{C^{k+1}\xi^{k+1}\eta^k}{(k+1)!\, k!} \\ &\le C\xi \sum_{k=0}^\infty \frac{C^k\xi^k\eta^k}{k!\, k!} \\ &= C\xi I_0(2(C\xi\eta)^{1/2}) \end{aligned} \tag{3.14}$$

where $I_0(z)$ denotes the modified Bessel function of the first kind. From the asymptotic expansion of $I_0(z)$ (cf. [4]) we can now conclude that there exists a positive constant C_1 such that

$$0 < P(\xi, \eta) < C_1 \xi \exp(2(C\xi\eta)^{1/2}); \tag{3.15}$$

i.e. for $0 \le s \le x$, $0 \le x < \infty$,

$$|\Gamma(s, x)| \le C_1 x \exp(\sqrt{C}\, x). \tag{3.16}$$

From the above analysis and the fact that $K(s, x)$ satisfies a Goursat problem of the same form as $\Gamma(s, x)$ (cf. Eqs. (2.18a), (2.18b)), it is seen that for $0 \le s \le x$, $0 \le x < \infty$, $K(s, x)$ also satisfies the inequality

$$|K(s, x)| \le C_1 x \exp(\sqrt{C}\, x). \tag{3.17}$$

We now return to the initial boundary value problem (1.2), (3.1). From Eqs. (3.2), (3.10) and (3.16) we have

$$g(x) = h(x, 0) = f(x) + \int_0^x \Gamma(s, x) f(s)\, ds \tag{3.18}$$

and for $0 < x < \infty$

$$\begin{aligned} |g(x)| &\le M \exp(Ax^2)[1 + C_1 x^2 \exp \sqrt{C}\, x] \\ &\le C_2 \exp[(A + \epsilon)x^2] \end{aligned} \tag{3.19}$$

for $\epsilon > 0$ fixed but arbitrarily small and C_2 a positive constant. Using the "method of images", we now define the solution $h(x, t)$ of Eq. (1.1) by

$$h(x, t) = \int_0^\infty \{s(x - y, t) - s(x + y, t)\} g(y)\, dy \tag{3.20}$$

where

$$s(x, t) = \frac{1}{(4\pi t)^{1/2}} \exp\left(-\frac{x^2}{4t}\right). \tag{3.21}$$

From [7] it is seen that $h(x, t)$ is a strong solution of Eq. (1.1) for $-\infty < x \infty$, $0 < t < T$, is continuous for $-\infty < x < \infty$, $0 \le t \le T$, assumes the initial-boundary data $h(0, t) = h(0, t) = 0$, $0 \le t < T$, $h(x, 0) = g(x)$, $0 \le x \le \infty$, and satisfies $|h(x, t)| \le M_2 \exp A_2 x^2$ for suitable constants M_2 and A_2 and $0 \le x < \infty$, $0 \le t < T$. Since from our previous discussion we have

$$f(x) = g(x) + \int_0^x K(s, x) g(s)\, ds, \tag{3.22}$$

it is seen that Eqs. (3.18), (3.20) and (3.9) now define the solution of the initial boundary value problem (1.2), (3.1) for $0 \le x < \infty$, $0 \le t \le T$. From Eq. (3.17) and the bound on $h(x, t)$ we can conclude that the inequality (3.3) is valid.

For (x, t) restricted to compact subsets of $0 < x < \infty$, $0 < t < T$, approximations of the solution to the initial boundary value problem (1.2), (3.1) can be obtained by using the recursion schemes (2.10)–(2.11) and (3.12) to approximate the kernels $K(s, x)$ and $\Gamma(s, x)$ respectively. Error estimates for such an approximation procedure can be found from estimates of the form (3.13), (3.14). For (x, t) again restricted to compact subsets of $0 < x < \infty, 0 < t < T$, the improper integral (3.20) can be accurately approximated by a proper integral by setting $s(x, t) = 0$ for $|x|$ sufficiently large. This is particularly useful if $f(x)$ satisfies a bound of the form $|f(x)| \leq M \exp Ax$ instead of the bound in Eq. (3.2), since in this case an estimate of the form (3.19) leads to a similar bound for $g(x)$, thus speeding up the convergence of the integral (3.20). The problem of dealing with the improper integral (3.20) is avoided completely if we make the assumption that $q(x)$ and $f(x)$ both vanish for $x \geq x_0$. In this case we have from Eq. (3.18) that

$$g(x) = \int_0^{x_0} \Gamma(s, x) f(s)\, ds \tag{3.23}$$

for $x \geq x_0$. But for $x \geq 3x_0$ we have $\xi \geq \eta = \frac{1}{2}(x - s) \geq x_0$, and hence from Eq. (3.12) it is seen that $\Gamma(s, x) \equiv 0$ for $0 \leq s \leq x_0$, $x > 3x_0$. Therefore from Eq. (3.23) it is seen that $g(x) \equiv 0$ for $x > 3x_0$ and the integral (3.20) reduces to a proper integral.

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COMPLETE FAMILIES OF SOLUTIONS FOR PARABOLIC EQUATIONS WITH ANALYTIC COEFFICIENTS*

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Abstract. A complete family of solutions is constructed for the general linear second order parabolic equation in one space variable with entire coefficients defined in a domain with moving boundary and for a class of second order parabolic equations in two space variables with entire coefficients defined in a cylindrical domain. The construction is based on the use of integral operators and results on the analytic continuation of solutions to partial differential equations with analytic coefficients. A numerical example is given which uses a complete family of solutions to approximate the solution to the first initial-boundary value problem for a parabolic equation in one space variable defined in a cylindrical domain.

1. Introduction. One of the more important applications of integral operators for elliptic equations is their use in constructing a complete family of solutions for the equation under investigation and thus providing a method for approximating the solutions to a wide variety of boundary value problems associated with ~~equations~~ of elliptic type (cf. [1], [7], [9], [11], [13], [15]). In recent papers ([3], [5]) the author has constructed an integral operator for the parabolic equation

$$u_{xx} + a(x, t)u_x + b(x, t)u = u_t \tag{1.1}$$

and showed how this operator could be used to construct a complete family of solutions to (1.1) in a rectangle. These last two papers lay the foundation for using integral operator methods to solve initial-boundary value problems for parabolic equations in a manner analogous to their use in the solution of elliptic boundary value problems. It is the purpose of this paper to extend the results of [5] in three different directions:

(a) Instead of (1.1) we will consider the general linear second order parabolic equation

$$u_{xx} + a(x, t)u_x + b(x, t)u = c(x, t)u_t. \tag{1.2}$$

(b) We will construct a set of solutions to (1.2) which are complete with respect to the maximum norm over the closure of domains with moving boundaries instead of only in a rectangle.

(c) We will show how these results can be extended to the case of parabolic equations in two space variables defined in cylindrical domains.

Numerical experiments on using the methods described in this paper to solve initial-boundary value problems for parabolic equations are presently being carried out by Y. F. Chang of the Data Systems and Services Department at Indiana University, and we hope to report on this in detail in the near future. A preliminary numerical example taken from this work is given in § 4 of this paper.

2. Complete families of solutions for parabolic equations in one space variable. We consider (1.2) and for the sake of simplicity assume that the coefficients $a(x, t)$, $b(x, t)$ and $c(x, t)$ are entire functions of their independent (complex) variables.

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We will further assume that $c(x, t) > 0$ for $-\infty < x < \infty$, $0 \leqq t \leqq t_0$ and, again for the sake of simplicity, that

$$\int_{-\infty}^{0} \sqrt{c(s, t)}\, ds = \int_{0}^{\infty} \sqrt{c(s, t)}\, ds = \infty. \tag{2.1}$$

We note at this point that due to the analyticity of the coefficients, every classical solution of (1.2) (i.e., a solution of (1.2) that is twice continuously differentiable with respect to x and continuously differentiable with respect to t) in a domain D is in fact analytic with respect to x and infinitely differentiable (but not necessarily analytic) with respect to t. Our aim is to construct a complete family of solutions with respect to the maximum norm for (1.2) defined in a region D bounded by the characteristics $t = 0$ and $t = t_0$ as well as the analytic curves $x = s_1(t)$ and $x = s_2(t)$, where $s_1(t) < s_2(t)$ for $0 \leqq t \leqq t_0$. The one-to-one analytic transformation

$$\xi = \int_0^x \sqrt{c(s, t)}\, ds, \qquad \tau = t, \tag{2.2}$$

reduces (1.2) to an equation of the same form but with $c(x, t) = 1$. The domain D is transformed into a domain in the (ξ, τ)-plane of the same form as that described above. Hence we can assume $c(x, t) = 1$ in (1.2) to begin with. If we now set

$$u(x, t) = v(x, t) \exp \left\{ -\frac{1}{2} \int_0^x a(s, t)\, ds \right\}, \tag{2.3}$$

we arrive at an equation for $v(x, t)$ of the same form as (1.2) but with $a(x, t) = 0$. Hence, without loss of generality, we can restrict ourselves to equations of the canonical form

$$u_{xx} + q(x, t)u = u_t, \tag{2.4}$$

where (due to the assumption (2.1)) $q(x, t)$ is analytic for $-\infty < x < \infty, 0 \leqq t \leqq t_0$, and consider classical solutions of (2.4) which continuously assume the initial-boundary data

$$\begin{aligned} &u(s_1(t), t) = f(t), \quad u(s_2(t), t) = g(t), \qquad 0 \leqq t \leqq t_0, \\ &u(x, 0) = h(x), \qquad s_1(0) \leqq x \leqq s_2(0), \end{aligned} \tag{2.5}$$

where $x = s_1(t)$ and $x = s_2(t)$ are analytic arcs satisfying

$$s_1(t) < s_2(t) \quad \text{for } 0 \leqq t \leqq t_0, \quad f(0) = h(s_1(0)), \quad g(0) = h(s_2(0)),$$

and $f(t)$, $g(t)$ and $h(x)$ are continuous functions of their independent variables.

Now suppose that for a given $\varepsilon > 0$ we are able to construct a solution $w(x, t)$ of (2.4) defined in a rectangle $R = \{(x, t)\colon -x_0 \leqq x \leqq x_0, 0 \leqq t \leqq t_0\}$ such that $D \subset R$ and

$$\max_{(x,t)\in \bar{D}} |u(x, t) - w(x, t)| < \varepsilon/2, \tag{2.6}$$

where $\bar{D}$ denotes the closure of D. Let $h_n(x, t)$ be defined by

$$h_n(x, t) = \sum_{k=0}^{[n/2]} \frac{x^{n-2k} t^k}{(n - 2k)!\, k!}, \tag{2.7}$$

and let $u_n(x, t)$ be the solution of (2.4) defined by

$$u_n(x, t) = h_n(x, t) + \int_{-x}^{x} P(s, x, t)h_n(s, t)\, ds, \tag{2.8}$$

where $P(s, x, t)$ is the (unique) solution of the initial value problem

$$P_{xx} - P_{ss} + q(x, t)P = P_t, \tag{2.9a}$$

$$P(x, x, t) = -\frac{1}{2}\int_0^x q(s, t)\, ds, \tag{2.9b}$$

$$P(-x, x, t) = 0. \tag{2.9c}$$

The existence of the function $P(s, x, t)$ and the fact that $u_n(x, t)$ is a solution of (2.4) follows from the results of [3] and [5]. In particular, $\tilde{P}(\xi, \eta, t) = P(\xi - \eta, \xi + \eta, t)$ can be constructed by the iterative scheme

$$\begin{aligned} \tilde{P}(\xi, \eta, t) &= \lim_{n\to\infty} \tilde{P}_n(\xi, \eta, t), \\ \tilde{P}_1(\xi, \eta, t) &= -\frac{1}{2}\int_0^\xi q(s, t)\, ds, \\ \tilde{P}_{n+1}(\xi, \eta, t) &= -\frac{1}{2}\int_0^\xi q(s, t)\, ds \\ &\quad + \int_0^\eta \int_0^\xi \left(\frac{\partial}{\partial t}\tilde{P}_n(\xi, \eta, t) - q(\xi + \eta, t)\tilde{P}_n(\xi, \eta, t)\right) d\xi\, d\eta, \\ &\qquad\qquad n \geqq 1. \end{aligned} \tag{2.10}$$

The convergence of the sequence $\{\tilde{P}_n\}$ is quite rapid and good approximations can be found by terminating the recursion process after several iterations. From the results of [5] we can now conclude that there exists an integer N and constants $a_1, \cdots, a_N$ such that

$$\max_{(x,t)\in R} \left| w(x, t) - \sum_{n=0}^{N} a_n u_n(x, t) \right| < \varepsilon/2 \tag{2.11}$$

and hence from (2.6),

$$\max_{(x,t)\in \bar{D}} \left| u(x, t) - \sum_{n=0}^{N} a_n u_n(x, t) \right| < \varepsilon; \tag{2.12}$$

i.e., the set $\{u_n(x, t)\}$ is a complete family of solutions for (2.4) defined in D. If we first orthonormalize the set $\{u_n(x, t)\}$ over the base and sides of D, it is seen that on compact subsets of D we can approximate the solution to the first initial-boundary value problem for (2.4) in D by the sum $\sum_{n=0}^{N} a_n \varphi_n(x, t)$, where

$$\begin{aligned} a_n &= \int_0^{t_0} f(t)\varphi_n(s_1(t), t)\, dt + \int_{s_1(0)}^{s_2(0)} h(x)\varphi_n(x, 0)\, dx \\ &\quad + \int_0^{t_0} g(t)\varphi_n(s_2(t), t)\, dt \end{aligned} \tag{2.13}$$

and the set $\{\varphi_n(x, t)\}$ is obtained by applying the Gram–Schmidt process to the set $\{u_n(x, t)\}$. Since each $\varphi_n(x, t)$ is a solution of (2.4), error estimates can be obtained by either applying the maximum principle for parabolic equations or the pointwise bounds for solutions established by Sigillito in [14]. Thus the problem we are considering will be solved if we can construct a function $w(x, t)$ defined in R and satisfying (2.6), and we now turn our attention to this problem.

From the existence theorem for the first initial-boundary value problem for parabolic equations, the maximum principle for parabolic equations and the Weierstrass approximation theorem, it is seen that there exists a solution $w(x, t)$ of (2.4) in D satisfying analytic boundary data on $x = s_1(t)$, $x = s_2(t)$ and $t = 0$ such that (2.6) is valid. From the reflection principle for parabolic equations ([3], [4]) (and the regularity theorems for solutions to initial-boundary value problems for parabolic equations—c.f. [8]) we can conclude that $w(x, t)$ can be uniquely continued as a solution of (2.4) across the arc $s_1(t)$ into the region bounded by the characteristics $t = t_0$, $t = 0$, and the analytic curves $x = 2s_1(t) - s_2(t)$, $x = s_2(t)$. Applying the reflection principle a second time, but this time continuing $w(x, t)$ across the arc $s_2(t)$, shows that $w(x, t)$ can be continued into the region bounded by $t = t_0$, $t = 0$, $x = 2s_1(t) - s_2(t)$ and $x = 3s_2(t) - 2s_1(t)$. Due to the fact that $s_1(t) < s_2(t)$ for $0 \leqq t \leqq t_0$, it is seen that by repeating the above procedure we can continue $w(x, t)$ into the entire infinite strip $-\infty < x < \infty$, $0 \leqq t \leqq t_0$. In particular, there exists a rectangle $R \supset D$ into which $w(x, t)$ can be continued and we have thus established the existence of the desired function $w(x, t)$.

We now make use of the above results to construct a complete family of solutions to (1.2) without first reducing it to the canonical form (2.4). This is desirable from a computational point of view in order to eliminate the problem of inverting the transformation (2.2). From the above analysis and the fact that $P(s, x, t)$ is analytic for $-\infty < s < \infty$, $-\infty < x < \infty$, $0 \leqq t \leqq t_0$ (cf. [3]) it is seen from equations (2.2)–(2.3) and (2.7)–(2.8) that every classical solution of (1.2) in D can be approximated arbitrarily closely in the maximum norm over $\bar{D}$ by a solution of (1.2) which is an analytic function of x and t in the strip $-\infty < x < \infty$, $0 \leqq t \leqq t_0$. Hence from the results of [2] we have that a complete family of solutions to (1.2) with respect to the maximum norm over $\bar{D}$ is given by

$$(2.14)\qquad \begin{aligned} u_{2n}(x, t) &= \frac{1}{2\pi i}\exp\left\{-\frac{1}{2}\int_0^x a(s, t)\,ds\right\}\oint_{|t-\tau|=\delta} E^{(1)}(x, t, \tau)\tau^n\,d\tau, \\ u_{2n+1}(x, t) &= \frac{1}{2\pi i}\exp\left\{-\frac{1}{2}\int_0^x a(s, t)\,ds\right\}\oint_{|t-\tau|=\delta} E^{(2)}(x, t, \tau)\tau^n\,d\tau, \\ &\qquad n = 0, 1, 2, \cdots, \end{aligned}$$

where

$$(2.15)\qquad \begin{aligned} E^{(1)}(x, t, \tau) &= \frac{1}{t-\tau} + \sum_{n=2}^{\infty} x^n p^{(1,n)}(x, t, \tau), \\ E^{(2)}(x, t, \tau) &= \frac{x}{t-\tau} + \sum_{n=3}^{\infty} x^n p^{(2,n)}(x, t, \tau) \end{aligned}$$

with

$$p^{(1,1)} = 0,$$

$$p^{(1,2)} = -\frac{c(x,t)}{2(t-\tau)^2} - \frac{q(x,t)}{2(t-\tau)},$$

$$\text{(2.16a)} \quad p^{(1,k+2)} = -\frac{2}{k+2}\, p_x^{(1,k+1)} - \frac{1}{(k+2)(k+1)}[p_{xx}^{(1,k)} + q(x,t)p^{(1,k)} - c(x,t)p_t^{(1,k)}], \qquad k \geqq 1,$$

$$p^{(2,2)} = 0,$$

$$p^{(2,3)} = -\frac{c(x,t)}{6(t-\tau)^2} - \frac{q(x,t)}{6(t-\tau)},$$

$$\text{(2.16b)} \quad p^{(2,k+2)} = -\frac{2}{k+2} p_x^{(2,k+1)} - \frac{1}{(k+2)(k+1)}[p_{xx}^{(2,k)} + q(x,t)p^{(2,k)} - c(x,t)p_t^{(2,k)}], \qquad k \geqq 2,$$

and

$$\text{(2.17)} \qquad q(x,t) = b(x,t) - \tfrac{1}{2}\left[a_x(x,t) + a^2(x,t) - c(x,t)\int_0^x a_t(s,t)\,ds\right].$$

The convergence of the series (2.15) for $t \neq \tau$ and estimates on the rate of this convergence can be found in [2]. An approximation of the solution $u_n(x, t)$ can be obtained by truncating the series (2.15) and computing the residue in (2.14).

3. Complete families of solutions for parabolic equations in two space variables. In this section we will show how the methods developed in [5] and the previous section of this paper can be extended to include the case of the parabolic equation in two space variables

$$\text{(3.1)} \qquad u_{xx} + u_{yy} + c(x,y)u = d(x,y)u_t$$

defined in a cylindrical domain $\Omega \times T$, where $T = [0, t_0]$ and Ω is a bounded simply connected domain whose boundary $\partial\Omega$ is three times continuously differentiable. We will assume for the sake of simplicity that $c(x, y)$ and $d(x, y)$ are entire functions of their independent (complex) variables and that furthermore $c(x, y) \leqq 0$, $d(x, y) > 0$, for $(x, y) \in \bar{\Omega} = \Omega \cup \partial\Omega$.

Let $u(x, y, t)$ be a (classical) solution of (3.1) which continuously assumes prescribed initial-boundary data on $\partial\Omega \times T$ and $\Omega_0 = \{(x, y, t):(x, y) \in \bar{\Omega}, t = 0\}$. From the maximum principle for parabolic equations and the Weierstrass approximation theorem, we can assume, without loss of generality, that the boundary data assumed by $u(x, y, t)$ on $\partial\Omega \times T$ is a polynomial in t, i.e.,

$$u(x, y, t) = \sum_{n=0}^{N} f_n(x, y)t^n, \qquad (x, y, t) \in \partial\Omega \times T, \tag{3.2}$$

where the $f_n(x, y)$ are Hölder continuous functions defined on $\partial\Omega$. We now look for a solution of (3.1) in the form

$$w(x, y, t) = \sum_{n=0}^{N} w_n(x, y)t^n \tag{3.3}$$

such that $w(x, y, t) = u(x, y, t)$ for $(x, y, t) \in \partial\Omega \times T$. From (3.1) and (3.2) it is seen that the functions $w_n(x, y)$ must satisfy the recursive scheme

$$\begin{aligned}
&\frac{\partial^2 w_N}{\partial x^2} + \frac{\partial^2 w_N}{\partial y^2} + c(x, y)w_N = 0, && (x, y) \in \Omega,\\
&w_N(x, y) = f_N(x, y), && (x, y) \in \partial\Omega,\\
&\frac{\partial^2 w_n}{\partial x^2} + \frac{\partial^2 w_n}{\partial y^2} + c(x, y)w_n = (n + 1)\, d(x, y)w_{n+1}, && (x, y) \in \Omega,\\
&w_n(x, y) = f_n(x, y), && (x, y) \in \partial\Omega,
\end{aligned} \tag{3.4}$$

for $n = 0, 1, \cdots, N - 1$. The existence of the $w_n(x, y)$ for $n = 0, 1, \cdots, N$ follows from the smoothness of $\partial\Omega$ and the fact that $c(x, y) \leqq 0$ in $\bar{\Omega}$. From the results of Vekua ([15, p. 156, p. 19]) and the fact that $w_n(x, y)$ depends continuously on the nonhomogeneous term $(n + 1)\, d(x, y)w_{n+1}(x, y)$, we can conclude that for $\varepsilon > 0$ there exists a solution $w_1(x, y, t)$ of (3.1) which is an entire function of its independent (complex) variables such that

$$\max_{\bar{\Omega} \times T} |w_1(x, y, t) - w(x, y, t)| < \varepsilon/2. \tag{3.5}$$

Now let $v(x, y, t) = u(x, y, t) - w(x, y, t)$ and let λ_n and $\varphi_n(x, y)$ be the eigenvalues and eigenfunctions, respectively, that correspond to the eigenvalue problem

$$\begin{aligned}
&u_{xx} + u_{yy} + c(x, y)u + \lambda\, d(x, y)u = 0, && (x, y) \in \Omega,\\
&u(x, y) = 0, && (x, y) \in \partial\Omega.
\end{aligned} \tag{3.6}$$

From (3.2)–(3.4) and the expansion theorem for the eigenvalue problem (3.6) (c.f. [10, p. 229]) we can conclude that

$$\begin{aligned}
&v(x, y, t) = \sum_{n=0}^{\infty} a_n\varphi_n(x, y) \exp(-\lambda_n t),\\
&a_n = \iint_\Omega v(x, y, 0)\varphi_n(x, y)\, d(x, y)\, dx\, dy,
\end{aligned} \tag{3.7}$$

where the series in (3.7) converges absolutely and uniformly in $\bar{\Omega} \times T$. By truncating the series in (3.7) and again appealing to the results of Vekua, we can conclude that there exists a solution $w_2(x, y, t)$ of (3.1) which is an entire function of its independent (complex) variables such that

$$\max_{\bar{\Omega} \times T} |w_2(x, y, t) - v(x, y, t)| < \varepsilon/2. \tag{3.8}$$

The inequalities (3.5) and (3.8) now imply that there exists a solution $\tilde{u}(x, y, t)$ of (3.1) which is an entire function of its independent complex variables such that

$$\max_{\bar{\Omega}\times T} |\tilde{u}(x, y, t) - u(x, y, t)| < \varepsilon. \tag{3.9}$$

The above analysis shows that in order to approximate classical solutions of (3.1) with respect to the maximum norm over $\bar{\Omega} \times T$, it suffices to construct a family of solutions which are complete in the maximum norm over $\bar{\Omega} \times T$ with respect to the class of solutions to (3.1) which are entire functions of their independent complex variables. From the results of [6] it is seen that such a complete family of solutions is given by

$$\begin{aligned} u_{2n,m}(x, y, t) &= \operatorname{Re}\left[-\frac{z^n}{2\pi i}\oint_{|t-\tau|=\delta}\int_{-1}^{1} E(z, \bar{z}, t-\tau, s)\tau^m(1-s^2)^{n-(1/2)}\,ds\,d\tau\right], \\ u_{2n+1,m}(x, y, t) &= \operatorname{Im}\left[-\frac{z^n}{2\pi i}\oint_{|t-\tau|=\delta}\int_{-1}^{1} E(z, \bar{z}, t-\tau, s)\tau^m(1-s^2)^{n-(1/2)}\,ds\,d\tau\right], \\ & \qquad n, m = 0, 1, 2, \cdots, \end{aligned} \tag{3.10}$$

where "Re" denotes "take the real part", "Im" denotes "take the imaginary part", $z = x + iy$, $\bar{z} = x - iy$, and

$$E(z, z^*, t, s) = \frac{1}{t} + \sum_{n=1}^{\infty} s^{2n}z^n \int_0^{z^*} P^{(2n)}(z, \zeta^*, t)\,d\zeta^* \tag{3.11}$$

with

$$\begin{aligned} P^{(2)} &= -\frac{2C(z, z^*)}{t} - \frac{2D(z, z^*)}{t^2}, \\ (2n+1)P^{(2n+2)} &= -2\left[P_z^{(2n)} + C(z, z^*)\int_0^{z^*} P^{(2n)}\,d\zeta^* - D(z, z^*)\int_0^{z^*} P_t^{(2n)}\,d\zeta^*\right], \\ & \qquad n \geqq 1, \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} C(z, z^*) &= \frac{1}{4}c\left(\frac{z+z^*}{2}; \frac{z-z^*}{2i}\right), \\ D(z, z^*) &= \frac{1}{4}d\left(\frac{z+z^*}{2}; \frac{z-z^*}{2i}\right). \end{aligned} \tag{3.13}$$

Estimates on the rate of convergence of the series (3.11) can be found in [6], and approximations of the solution $u_{n,m}(x, y, t)$ can be obtained by truncating the series (3.11) and computing the residue in (3.10). In particular, for the special case of the heat equation ($c = 0, d = 1$), we have

$$E(z, \bar{z}, t-\tau, s) = \frac{1}{t-\tau}\sum_{k=0}^{\infty} \frac{(-1)^k\Gamma(\frac{1}{2})}{\Gamma(k+\frac{1}{2})}\left(\frac{r^2s^2}{t-\tau}\right)^k, \tag{3.14}$$

where $r^2 = z\bar{z} = x^2 + y^2$, and using the result

$$\int_{-1}^{1} (1 - s^2)^{n-(1/2)} s^{2k}\, ds = \frac{\Gamma(n + \frac{1}{2})\Gamma(k + \frac{1}{2})}{\Gamma(n + k + 1)}, \tag{3.15}$$

we have

$$\begin{aligned} u_{2n,m}(x, y, t) &= \cos n\theta \sum_{k=0}^{m} \frac{\Gamma(\frac{1}{2})\Gamma(m + 1)\Gamma(n + \frac{1}{2})}{\Gamma(k + 1)\Gamma(m - k + 1)\Gamma(n + k + 1)} r^{2k+n} t^{m-k}, \\ u_{2n+1,m}(x, y, t) &= \sin n\theta \sum_{k=0}^{m} \frac{\Gamma(\frac{1}{2})\Gamma(m + 1)\Gamma(n + \frac{1}{2})}{\Gamma(k + 1)\Gamma(m - k + 1)\Gamma(n + k + 1)} r^{2k+n} t^{m-k}, \end{aligned} \tag{3.16}$$

where $x = r\cos\theta$, $y = r\sin\theta$. Noting that since in this special case $u_n(x, y, t)$ is a polynomial in x, y and t, it follows from the results of § 2 and the uniqueness theorem for Cauchy's problem for the heat equation that another complete family of solutions for the heat equation defined in $\Omega \times T$ is given by

$$v_{n,m}(x, y, t) = h_n(x, t)h_m(y, t) \tag{3.17}$$

for $n, m = 0, 1, 2, \cdots$, where $h_n(x, t)$ is defined in (2.7).

4. A numerical example. In this section we given an example of the use of the methods discussed in [5] and this paper to approximate the solution of the initial-boundary value problem

$$u_{xx} - x^2 u = u_t, \qquad -1 < x < 1, \quad 0 < t < 1, \tag{4.1}$$

$$\begin{aligned} &u(-1, t) = \exp(-\tfrac{1}{2} - t), \qquad u(1, t) = \exp(-\tfrac{1}{2} - t), \qquad 0 \leqq t \leqq 1, \\ &u(x, 0) = \exp(-\tfrac{1}{2}x^2), \qquad -1 \leqq x \leqq 1. \end{aligned} \tag{4.2}$$

Initial-boundary value problems for (4.1) defined in a domain with moving boundary can of course be treated in an identical manner. A complete family of solutions for (4.1) was constructed by using the operator (2.8). Since the coefficients of (4.1) are independent of t, so is $P(s, x, t)$, i.e., $P(s, x, t) = P(s, x)$. As an approximation to the kernel $P(s, x)$ we used $P_{10}(s, x)$ as defined by (2.10). A short calculation using (2.10) shows that

$$\max_{\substack{-1 \leqq x \leqq 1 \\ 1 \leqq s \leqq 1}} |P(s, x) - P_{10}(s, x)| \leqq 1.6 \times 10^{-20}. \tag{4.3}$$

The set $\{u_n(x, t)\}$ obtained from (2.8) was then orthonormalized over the base and vertical sides of the rectangle $-1 \leqq x \leqq 1$, $0 \leqq t \leqq 1$, to obtain the set $\{\varphi_n(x, t)\}$ and the solution to the initial-boundary value problem (4.1), (4.2) was approximated by the sum

$$u^*(x, t) = \sum_{n=0}^{14} a_n \varphi_n(x, t) \tag{4.4}$$

with the coefficients a_n, $n = 0, 1, \cdots, 14$, given by (2.13). Note that since the solution of the initial-boundary value problem (4.1), (4.2) is an even function of x, the odd coefficients $a_1, a_3, \cdots, a_{13}$ in (4.4) all turn out to be identically zero.

The exact solution of the initial-boundary value problem (4.1), (4.2) is

$$u(x, t) = \exp\left(-\tfrac{1}{2}x^2 - t\right). \tag{4.5}$$

In Table 1 we give the values of $u^*(x, t)$ at selected grid points and also the relative error defined by

$$\text{relative error} = \frac{u^*(x, t) - u(x, t)}{u(x, t)}. \tag{4.6}$$

TABLE 1

x	t	Approximate solution	Relative error
0	0	1.00000	-6.9580×10^{-9}
0.2	0	0.98020	-3.3762×10^{-9}
0.4	0	0.92312	4.2696×10^{-9}
0.6	0	0.83527	8.0803×10^{-9}
0.8	0	0.72615	-4.3613×10^{-10}
1.0	0	0.60653	-2.2466×10^{-8}
0	0.2	0.81873	4.0571×10^{-10}
0.2	0.2	0.80252	9.3730×10^{-10}
0.4	0.2	0.75578	2.7202×10^{-9}
0.6	0.2	0.68386	6.1830×10^{-9}
0.8	0.2	0.59452	1.1356×10^{-8}
1.0	0.2	0.49659	1.5536×10^{-8}
0	0.4	0.67032	2.2209×10^{-9}
0.2	0.4	0.65705	1.7415×10^{-9}
0.4	0.4	0.61878	3.2910×10^{-11}
0.6	0.4	0.55990	-3.6697×10^{-9}
0.8	0.4	0.48675	-1.0332×10^{-8}
1.0	0.4	0.40657	-2.0325×10^{-8}
0	0.6	0.54881	-1.1541×10^{-9}
0.2	0.6	0.53794	-8.6797×10^{-10}
0.4	0.6	0.50662	3.4421×10^{-10}
0.6	0.6	0.45841	3.6095×10^{-9}
0.8	0.6	0.39852	1.0898×10^{-8}
1.0	0.6	0.33287	2.4115×10^{-8}
0	0.8	0.44933	2.7676×10^{-9}
0.2	0.8	0.44043	2.5721×10^{-9}
0.4	0.8	0.41478	1.5339×10^{-9}
0.6	0.8	0.37531	-1.8415×10^{-9}
0.8	0.8	0.32628	-1.0323×10^{-8}
1.0	0.8	0.27253	-2.7649×10^{-8}
0	1.0	0.36788	-7.3333×10^{-11}
0.2	1.0	0.36059	4.9411×10^{-10}
0.4	1.0	0.33960	2.3005×10^{-9}
0.6	1.0	0.30728	4.1011×10^{-9}
0.8	1.0	0.26714	-5.4443×10^{-9}
1.0	1.0	0.22313	-8.4473×10^{-8}

Since $u(x, t)$ and $u^*(x, t)$ are even functions of x, values of the approximate solution and relative error are only given for $0 \leqq x \leqq 1$, $0 \leqq t \leqq 1$. Note that since each $\varphi_n(x, t)$ is a solution of (4.1), the maximum error (in absolute value) occurs on the base or vertical sides of the rectangle $-1 \leqq x \leqq 1$, $0 \leqq t \leqq 1$; in this case at the points $(x, t) = (\pm 1, 1)$, where the relative error is 8.4473×10^{-8} in absolute value.

The computation time to construct $u^*(x, t)$ (i.e., to find the coefficients a_n, the Taylor coefficients of $\varphi_n(x, t)$, and to evaluate $u^*(x, t)$ at selected grid points) using the CDC 6600 computer was approximately six seconds.

5. Concluding remarks. The main problem in constructing a complete family of solutions through the use of integral operators as discussed in this paper, is to show that every classical solution in a given domain can be approximated with respect to the maximum norm over the closure of the domain by a solution of the parabolic equation that is an entire function of its independent complex variables. In the case of both one and two space variables, this was established through the use of results on the (global) analytic continuation of solutions to partial differential equations, in particular, the reflection principle for parabolic equations in one space variable ([3], [4]) and the results of Vekua which are based on knowledge of the domain of regularity in the complex domain of solutions to elliptic equations in two independent variables (cf. [15, p. 32]). What has been established is the analogue of Runge's theorem in analytic function theory for classical solutions to parabolic equations in one and two space variables. In order to extend our results to parabolic equations in two space variables defined in domains with moving boundaries and to parabolic equations in more than two space variables, it is necessary to obtain sharper results on the analytic continuation (with respect to the space variables) of classical solutions to parabolic equations with analytic coefficients in several independent variables. This is a difficult problem and only partial results have been obtained so far. One notable result in this direction is the reflection principle obtained by C. D. Hill for analytic solutions of parabolic equations in two space variables ([12]). It is to be hoped that more refined results in this direction will be forthcoming in the not too distant future.

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BERGMAN OPERATORS FOR PARABOLIC EQUATIONS IN TWO SPACE VARIABLES[1]

DAVID COLTON

ABSTRACT. An integral operator is constructed which maps analytic functions of two complex variables onto the class of real valued analytic solutions of linear second order parabolic equations in two space variables with real valued, analytic, time independent coefficients. When the solution of the parabolic equation is independent of the time variable the operator reduces to Bergman's integral operator for elliptic equations in two independent variables.

I. **Introduction.** Although the analytic theory of elliptic equations has been extensively investigated by many mathematicians (cf. the monographs [**1**], [**4**], [**8**]), little has been done in developing an analogous theory for parabolic equations (however see [**2**], [**5**], [**6**]). An important method in the investigation of the analytic behaviour of solutions to elliptic equations has been the use of a variety of integral operators which map analytic functions onto solutions of the elliptic equation. In order to undertake a similar study of parabolic equations it would be desirable to have similar tools at our disposal. An initial step in this direction was taken by Bergman in [**2**] (see also [**1**, pp. 74–78]) who constructed an integral operator for certain classes of parabolic equations in two space variables. However in addition to having a very complicated structure and being applicable to only a limited class of equations, the operator constructed by Bergman is not an onto mapping. In particular Bergman's operator maps analytic functions into a subclass of solutions of the differential equation which have a Taylor expansion of a certain form. In this note we will overcome the difficulties inherent in Bergman's approach and construct an integral operator which maps analytic functions of two complex variables onto real valued analytic solutions of the general linear second order parabolic equation in two independent variables with real valued, analytic, time independent coefficients. (Our analysis can easily be modified to include the case in which the coefficients also depend on time.) In particular we

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will consider the parabolic equation (written in normal form)

$$u_{xx} + u_{yy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y)u_t \tag{1.1}$$

and make the assumption that the coefficients of equation (1.1) are entire functions of their independent (complex) variables (with minor modifications we could have assumed only that these coefficients are analytic in some polydisc in the space of two complex variables) and are real valued for x and y real. When the solution of equation (1.1) is independent of t we will show that our operator reduces to that of Bergman for elliptic equations in two independent variables.

An alternate method to that of Bergman for constructing integral operators for elliptic equations in two independent variables has been given by Vekua [**8**]. In [**6**] Hill has constructed an integral operator for parabolic equations which is analogous to that of Vekua for elliptic equations. The advantages (and disadvantages) of our operator in comparison with that of Hill are comparable to a similar comparison between the operators of Bergman and Vekua for elliptic equations (cf. [**1**, p. 2]). We will not enter into such a discussion at this time, except to point out that the kernel of our operator is considerably easier to construct than that of Hill since the kernel of Hill's operator is expressed as an infinite series, each of whose terms is computed by solving a complex Goursat problem for an elliptic equation in two independent variables.

II. **An integral operator for equation (1.1).** We first define the nonsingular transformation of the space C^2 of two complex variables into itself by

$$z = x + iy, \qquad z^* = x - iy. \tag{2.1}$$

Under such a transformation equation (1.1) assumes the form

$$U_{zz^*} + A(z, z^*)U_z + B(z, z^*)U_{z^*} + C(z, z^*)U = D(z, z^*)U_t \tag{2.2}$$

where

$$U(z, z^*, t) = u\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}, t\right),$$

$$\begin{aligned} A(z, z^*) &= \frac{1}{4}\left[a\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right) + ib\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right)\right], \\ B(z, z^*) &= \frac{1}{4}\left[a\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right) - ib\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right)\right], \\ C(z, z^*) &= \frac{1}{4}c\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right), \\ D(z, z^*) &= \frac{1}{4}d\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right). \end{aligned} \tag{2.3}$$

Setting

$$V(z, z^*, t) = U(z, z^*, t)\exp\left\{\int_0^{z^*} A(z, \zeta^*)\, d\zeta^*\right\} \tag{2.4}$$

reduces equation (2.2) to the canonical form

$$V_{zz^*} + \tilde{B}(z, z^*)V_{z^*} + \tilde{C}(z, z^*)V = \tilde{D}(z, z^*)V_t \tag{2.5}$$

where

$$\begin{aligned} \tilde{B}(z, z^*) &= B(z, z^*) - \int_0^{z^*} A_z(z, \zeta^*)\, d\zeta^*, \\ \tilde{C}(z, z^*) &= -(A_z + AB - C), \\ \tilde{D}(z, z^*) &= D(z, z^*). \end{aligned} \tag{2.6}$$

We now proceed to construct an integral operator which maps analytic functions of two complex variables onto analytic solutions of equation (2.5). In particular we look for solutions of equation (2.5) in the form

$$\begin{aligned} &V(z, z^*, t) \\ &= \frac{-1}{2\pi i} \oint_{|t-\tau|=\delta} \int_{-1}^{+1} E(z, z^*, t-\tau, s) f\left(\frac{z}{2}(1-s^2), \tau\right) \frac{ds\, d\tau}{(1-s^2)^{1/2}} \end{aligned} \tag{2.7}$$

where $\delta > 0$, $f(z, t)$ is an analytic function of two complex variables in a neighborhood of the origin in C^2, and $E(z, z^*, t, s)$ is a function to be determined. The first integral in equation (2.7) is an integration in the complex τ plane in a counterclockwise direction about a circle of radius δ with centre at t, and the second integral is an integration over a curvilinear path in the unit disc in the complex s plane joining the points $s=+1$ and $s=-1$. Substituting equation (2.7) into equation (2.5) and integrating by parts (cf. [**1**, p. 11]) show that $E(z, z^*, t, s)$ must satisfy the differential equation

$$(1-s^2)E_{z^*s} - (1/s)E_{z^*} + 2sz(E_{zz^*} + \tilde{B}E_{z^*} + \tilde{C}E - \tilde{D}E_t) = 0, \tag{2.8}$$

provided we also assume that $E(z, z^*, t, s)$ is an analytic function of s for $|s| \leqq 1$, t for $\delta_0 \leqq |t| \leqq \delta_1$ (where $\delta_0 < \delta < \delta_1$), and (z, z^*) in some neighbourhood of the origin in C^2. Motivated by Bergman's analysis for elliptic equations in two independent variables we now look for a solution of equation (2.8) in the form

$$E(z, z^*, t, s) = \frac{1}{t} + \sum_{n=1}^{\infty} s^{2n} z^n \int_0^{z^*} P^{(2n)}(z, \zeta^*, t)\, d\zeta^*. \tag{2.9}$$

Substituting equation (2.9) into equation (2.8) yields the following recursion formula for the coefficients $P^{(2n)}$:

$$P^{(2)} = -\frac{2}{t}\tilde{C} - \frac{2}{t^2}\tilde{D},$$

$$(2.10)\qquad \begin{aligned}(2n+1)&P^{(2n+2)}\\ &= -2\left[P_z^{(2n)} + \tilde{B}P^{(2n)} + \tilde{C}\int_0^{z^*} P^{(2n)}\,d\zeta^* - \tilde{D}\int_0^{z^*} P_t^{(2n)}\,d\zeta^*\right],\\ &\qquad n = 1, 2, \cdots.\end{aligned}$$

Setting $P^{(2n)}(z, z^*, t) = t^{-n-1}Q^{(2n)}(z, z^*, t)$ in equation (2.10) yields the following recursion formula for the $Q^{(2n)}$:

$$Q^{(2)} = -2t\tilde{C} - 2\tilde{D},$$

$$(2.11)\qquad \begin{aligned}(2n+1)&Q^{(2n+2)}\\ &= -2\Big[tQ_z^{(2n)} + t\tilde{B}Q^{(2n)} + t\tilde{C}\int_0^{z^*} Q^{(2n)}\,d\zeta^*\\ &\qquad + (n+1)\tilde{D}\int_0^{z^*} Q^{(2n)}\,d\zeta^* - t\tilde{D}\int_0^{z^*} Q_t^{(2n)}\,d\zeta^*\Big],\\ &\qquad n = 1, 2, \cdots.\end{aligned}$$

It is clear from equation (2.10) that each of the $P^{(2n)}$, $n=1, 2, \cdots$, is uniquely determined. In order to show the existence of the function $E(z, z^*, t, s)$ it is now necessary to show the convergence of the series (2.9). To this end we first majorize the functions $Q^{(2n)}(z, z^*, t)$. Let r be an arbitrarily large positive number and let B_0 be a positive constant chosen such that for $|z|<r$, $|z^*|<r$, we have

$$(2.12)\qquad \begin{aligned}\tilde{B}(z, z^*) &\ll \frac{B_0}{(1-z/r)(1-z^*/r)},\\ \tilde{C}(z, z^*) &\ll \frac{B_0}{(1-z/r)(1-z^*/r)},\\ \tilde{D}(z, z^*) &\ll \frac{B_0}{(1-z/r)(1-z^*/r)},\end{aligned}$$

where "$\ll$" denotes domination (cf. [**1**], [**4**]). We will now show by induction that there exist positive constants M_n and ε (where ε is independent of n and M_n is a bounded function of n) such that for $|z|<r$,

$|z^*|<r$, $|t|<\delta_1$, we have

$$
\begin{aligned}
(2.13)\qquad Q^{(2n)} \ll & \frac{M_n 2^{2n}\delta_1^n(1+\varepsilon)^n}{2n-1} \\
& \times \left(1-\frac{z}{r}\right)^{-(2n-1)}\left(1-\frac{z^*}{r}\right)^{-(2n-1)}\left(1-\frac{t}{2\delta_1}\right)^{-(2n-1)} r^{-n}.
\end{aligned}
$$

This is clearly true for $n=1$. Now suppose for $n=k$ equation (2.13) is valid. Then from equations (2.11) and (2.12) and the straightforward use of the theory of dominants we have

$$
\begin{aligned}
(2.14)\qquad Q^{(2k+2)} \ll & \frac{M_k 2^{2k+2}\delta_1^{k+1}(1+\varepsilon)^k}{2k+1} \\
& \times \left\{1+\frac{B_0}{2k-1}\left(r+\frac{r^2}{(2k-1)}+\frac{r^2(k+1)}{(2k-1)2\delta_1}+\frac{r^2}{2\delta_1}\right)\right\} \\
& \times \left(1-\frac{z^*}{r}\right)^{-(2k+1)}\left(1-\frac{z}{r}\right)^{-(2k+1)}\left(1-\frac{t}{2\delta_1}\right)^{-(2k+1)} r^{-k-1}.
\end{aligned}
$$

In the derivation of equation (2.14) we have made use of the fact that $t \ll 2\delta_1(1-t/2\delta_1)^{-1}$ and that if $f \ll g$ then

$$f \ll g(1-z/r)^{-j}(1-z^*/r)^{-k}(1-t/2\delta_1)^{-l}$$

for arbitrary positive integers j, k, and l.

By setting

$$(2.15)\qquad M_{n+1} = M_n(1+\varepsilon)^{-1}\left\{1+\frac{B_0 r}{(2n-1)^2}\left(2n-1+r+\frac{3nr}{2\delta_1}\right)\right\}$$

we have shown that equation (2.13) is true for $n=k+1$, thus completing the induction step. Note that for n sufficiently large we have $M_{n+1} \leqq M_n$, i.e. there exists a positive constant M which is independent of n such that $M_n \leqq M$ for all n.

We now turn to the convergence of the series (2.9). Let $s_0 \geqq 1$ and $\alpha>1$ be positive constants and let $|s|\leqq s_0$, $|z|<r/\alpha$, $|z^*|<r/\alpha$, and $\delta_0 \leqq |t| \leqq \delta_1$. Then $(1-|z|/r) \geqq (\alpha-1)/\alpha$, $(1-|z^*|/r)\geqq(\alpha-1)/\alpha$, $(1-|t|/2\delta_1)\geqq \frac{1}{2}$, and from equation (2.14) it is seen that the series (2.9) is majorized by the series

$$(2.16)\qquad \frac{1}{\delta_0}+\sum_{n=1}^{\infty}\frac{rM_n 2^{4n-1}s_0^{2n}\delta_1^n(1+\varepsilon)^n\alpha^{3n-2}}{\delta_0^{n+1}(2n-1)(\alpha-1)^{4n-2}}.$$

If α is chosen such that $16 s_0^2\delta_1(1+\varepsilon)\alpha^3\delta_0^{-1}(\alpha-1)^{-4}<1$, then the series (2.16) is convergent. Since r is an arbitrarily large positive number and ε is arbitrarily small and independent of r, we can now conclude that the series (2.9) converges absolutely and uniformly for $|z|<r$, $|z^*|<r$, $|s|\leqq s_0$,

$\delta_0 \leqq |t| \leqq \delta_1$ for r, δ_1, and s_0 arbitrarily large and $\delta_0 > 0$ arbitrarily small, i.e. $E(z, z^*, t, s)$ is an entire function of its independent variables except for an (essential) singularity at $t=0$.

We have now shown that the operator $\boldsymbol{P}_2$ defined by

$$\begin{aligned} U(z, z^*, t) &= \boldsymbol{P}_2\{f\} \\ &= \frac{-1}{2\pi i} \exp\left\{-\int_0^{z^*} A(z, \zeta^*)\, d\zeta^*\right\} \\ &\quad \times \oint_{|t-\tau|=\delta} \int_{-1}^{+1} E(z, z^*, t-\tau, s) f\left(\frac{z}{2}(1-s^2)\tau\right) \frac{ds\, d\tau}{(1-s^2)^{1/2}} \end{aligned} \tag{2.17}$$

exists and maps analytic functions which are regular in some neighbourhood of the origin in C^2 into the class of (complex valued) solutions of equation (2.2). An elementary power series analysis (cf. [**8**, pp. 55–56]) coupled with Hormander's generalized Cauchy-Kowalewski theorem [**7**] shows that solutions of equation (2.2) which are real valued for t real and $z^* = \bar{z}$ (i.e. x and y real) are uniquely determined by their values on the characteristic plane $z^*=0$. Furthermore, since the coefficients of equation (1.1) are real valued for x and y real, the operator Re $\boldsymbol{P}_2\{f\}$ (where "Re" denotes "take the real part") defines a real valued solution of equation (1.1) provided we set $z^* = \bar{z}$ and keep t real. Evaluating Re $\boldsymbol{P}_2\{f\}$ at $z^*=0$ and keeping t real gives

$$\begin{aligned} U(z, 0, t) &= \frac{-1}{4\pi i} \oint_{|t-\tau|=\delta} \int_{-1}^{+1} \left[f\left(\frac{z}{2}(1-s^2), \tau\right) \right. \\ &\qquad \left. + \bar{f}(0, \tau) \exp\left(-\int_0^z \bar{A}(0, \zeta^*)\, d\zeta^*\right)\right] \\ &\qquad \times \frac{ds\, d\tau}{(1-s^2)^{1/2}(t-\tau)} \\ &= \frac{1}{2} \int_{-1}^{+1} f\left(\frac{z}{2}(1-s^2), t\right) \frac{ds}{(1-s^2)^{1/2}} + \frac{\pi}{2} \bar{f}(0, t) \\ &\qquad \times \exp\left(-\int_0^z \bar{A}(0, \zeta^*)\, d\zeta^*\right) \end{aligned} \tag{2.18}$$

where

$$\bar{f}(z, t) = \overline{f(\bar{z}, \bar{t})} \quad \text{and} \quad \bar{A}(z, z^*) = \overline{A(\bar{z}, \bar{z}^*)}.$$

A solution of the integral equation (2.18) is given by [**1**, p. 12]

$$\begin{aligned} f\left(\frac{z}{2}, t\right) = -\frac{1}{2\pi} \int_\gamma \Bigg[& 2U(z(1-s^2), 0, t) \\ & U(0, 0, t) \exp\left(-\int_0^z \bar{A}(0, \zeta^*)\, d\zeta^*\right)\Bigg] \frac{ds}{s^2} \end{aligned} \tag{2.19}$$

where γ is a rectifiable arc joining the points $s=-1$ and $s=+1$ and not passing through the origin. Equations (2.18) and (2.19) show that if $U(z, \bar{z}, t)$ is real valued for t real, then $f(z, t)$ can be chosen such that $U(z, 0, t)$ assumes prescribed values. We thus have the following theorem:

THEOREM. *Let $u(x, y, t)$ be a real valued analytic solution of equation* (1.1) *defined in some neighborhood of the origin. Then $u(x, y, t)=U(z, \bar{z}, t)$ can be represented in the form*

$$
\begin{aligned}
U(z, \bar{z}, t) &= \operatorname{Re} P_2\{f\} \\
&= \operatorname{Re}\left[\frac{-1}{2\pi i} \exp\left\{-\int_0^{\bar{z}} A(z, \zeta^*)\, d\zeta^*\right\}\right. \\
&\qquad \left. \cdot \oint_{|t-\tau|=\delta} \int_{-1}^{+1} E(z, \bar{z}, t-\tau, s) f\left(\frac{z}{2}(1-s^2), \tau\right) \frac{ds\, d\tau}{(1-s^2)^{1/2}}\right]
\end{aligned} \tag{2.20}
$$

where $E(z, z^*, t, s)$ is defined by equations (2.9) *and* (2.10) *and is an entire function of its independent variables except for an essential singularity at $t=0$, and $f(z, t)$ is defined by equation* (2.19) *and is analytic in some neighborhood of the origin in C^2. Conversely, for every analytic function $f(z, t)$ defined in some neighbourhood of the origin in C^2, equation* (2.20) *defines a real valued analytic solution of equation* (1.1) *in some neighbourhood of the origin.*

The representation (2.20) can now be used to analytically continue solutions of parabolic equations. For the type of theorems which can be obtained the reader is referred to the results for elliptic equations in two independent variables obtained in [**1**]. The operator defined by equation (2.20) is in fact closely related to Bergman's operator for elliptic equations in two independent variables. To see this we consider the case in which $u(x, y, t)=u(x, y)$ is independent of t and hence satisfies the elliptic equation

$$u_{xx} + u_{yy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0. \tag{2.21}$$

In this situation the associated analytic function $f(z, t)=f(z)$ is independent of t, and termwise integration in equation (2.20) yields the representation

$$
\begin{aligned}
U(z, \bar{z}) = \operatorname{Re}\Bigg[&\exp\left\{-\int_0^{\bar{z}} A(z, \zeta^*)\, d\zeta^*\right\} \\
&\times \int_{-1}^{+1} E(z, \bar{z}, s) f\left(\frac{z}{2}(1-s^2)\right) \frac{ds}{(1-s^2)^{1/2}}
\end{aligned} \tag{2.22}
$$

where

$$(2.23)\qquad E(z, z^*, s) = 1 + \sum_{n=1}^{\infty} s^{2n} z^n \int_0^{z^*} P^{(2n)}(z, \zeta^*)\, d\zeta^*$$

with the $P^{(2n)}$ being defined recursively by

$$P^{(2)} = -2\tilde{C},$$

$$(2.24)\quad (2n + 1)P^{(2n+2)} = -2\left[P_z^{(2n)} + \tilde{B}P^{(2n)} + \tilde{C}\int_0^{z^*} P^{(2n)}\, d\zeta^*\right], \qquad n = 1, 2, \cdots$$

A comparison of equations (2.22)–(2.24) with the corresponding formula in [1] shows that the operator defined by equation (2.22) is identical with Bergman's operator for elliptic equations in two independent variables.

In closing we note that it is also of interest to compare our integral representation (2.20) for parabolic equations in two space variables with the corresponding representation for elliptic equations in three independent variables obtained in [3].

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21.—Runge's Theorem for Parabolic Equations in Two Space Variables.*†

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Synopsis

Let u be a real valued strong solution defined in a cylindrical domain of a linear second-order parabolic equation in two space variables with entire coefficients. Then it is shown that on compact subsets of its domain of definition u can be approximated arbitrarily closely in the maximum norm by an entire solution of the parabolic equation.

1. Introduction

In this paper we are concerned with the problem of approximating solutions of the general linear second-order parabolic equation

$$u_{xx}+u_{yy}+a(x, y, t)u_x+b(x, y, t)u_y+c(x, y, t)u = d(x, y, t)u_t \tag{1.1}$$

defined in a cylindrical domain $\mathfrak{D}\times(0, T)$ where $\mathfrak{D}$ is a bounded simply connected domain in two-dimensional Euclidean space $\mathbb{R}^2$ (without loss of generality we will assume that $\mathfrak{D}$ contains the origin). It is assumed that the coefficients of (1.1) are entire functions of their independent (complex) variables, are real valued for x, y and t real, and that $d(x, y, t)>0$ in $\mathfrak{D}\times(0, T)$. We will focus our attention on the problem of approximating real valued strong solutions of (1.1) in $\mathfrak{D}\times(0, T)$, i.e. real valued solutions of (1.1) in $\mathfrak{D}\times(0, T)$ that are twice continuously differentiable with respect to x and y and continuously differentiable with respect to t (we note that from the regularity properties of solutions to parabolic equations [cf. 7] a strong solution of (1.1) is analytic in x and y and infinitely differentiable, but not necessarily analytic, with respect to t). In particular we will obtain the following generalisation of Runge's theorem in analytic function theory: If $u(x, y, t)$ is a real valued strong solution of (1.1) in $\mathfrak{D}\times(0, T)$ and $\mathfrak{D}_0\times[\delta_0, T-\delta_0]$ is a compact subset of $\mathfrak{D}\times(0, T)$, then for every $\varepsilon>0$ there exists an entire solution $u_0(x, y, t)$ of (1.1) (i.e. $u_0(x, y, t)$ is an entire function of its three independent complex variables and satisfies (1.1) for $(x, y, t)\in\mathbb{C}^3$, the space of three complex variables) such that

$$\max_{\mathfrak{D}_0\times[\delta_0,\, T-\delta_0]} |u-u_0|<\varepsilon. \tag{1.2}$$

In the case when the coefficients of (1.1) are independent of t and $a = b = 0$ a stronger version of this result has been proved in [6] and used there, in conjunction with the method of integral operators, to obtain constructive methods for solving initial-boundary value problems for parabolic equations. However, the methods

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used in [6] were based on separation of variables and the use of known results for elliptic equations in two independent variables, and are no longer applicable to the general case now under consideration. Our approach in this paper will instead be based on the construction of integral operators for (1.1) and the derivation of a result on the analytic continuation of analytic solutions to parabolic equations in two space variables. Fundamental to our investigation is the construction and use of a special solution to (1.1) known as the Riemann function for parabolic equations. In the special case of time-independent coefficients this function was constructed in [4] and [10]. However, the methods used in these papers are not immediately applicable to the case of time dependent coefficients, and we will therefore construct the Riemann function for (1.1) through the use of integral operators. These integral operators are of interest in their own right since it is not only the first time that integral operators have been obtained for the general linear second-order parabolic equation in two space variables with time-dependent coefficients, but these operators are also of a form that are suitable for the development of constructive methods for solving the standard initial-boundary value problems associated with (1.1) [cf. **6**].

The results obtained in this paper can be considered as the analogue for parabolic equations in two space variables of the fundamental results obtained by Bergman [**1**] and Vekua [**11**] on the analytic continuation and approximation of solutions to elliptic equations in two independent variables. In particular it is hoped that the present results on parabolic equations will lead to constructive methods for solving initial-boundary value problems for parabolic equations in a manner similar to that used by Bergman and Vekua to solve boundary value problems for elliptic equations. The missing step is to derive an analogue of Walsh's generalisation of Runge's theorem in analytic function theory [**12**] to the case of parabolic partial differential equations, i.e. to show that (1.2) is valid over the closure of $\mathfrak{D}\times(0, T)$ and not merely on compact subsets of this cylinder (in the case of elliptic equations the appropriate results can be found in [**2** and **11**]). The author is presently investigating this problem and will hopefully be able to report some progress in this direction in the not too distant future.

2. Integral Operators and the Riemann Function

In this part of the paper we will construct integral operators which map analytic functions of two complex variables into analytic solutions of (1.1) and use one of these operators to construct the Riemann function for (1.1). Since these constructions closely follow our previous analysis for the case of time-independent coefficients [cf. **3**, **5**], we will try to make the presentation as brief as possible, referring the reader to earlier work for more details.

The change of variables in $\mathbb{C}^2$

$$\begin{aligned} z &= x+iy \\ z^* &= x-iy \end{aligned} \tag{2.1}$$

transforms (1.1) into the form

$$L[U] = U_{zz^*}+A(z, z^*, t)U_z+B(z, z^*, t)U_{z^*}+C(z, z^*, t)U-D(z, z^*, t)U_t = 0, \tag{2.2}$$

where $A = \frac{1}{4}(a+ib)$, $B = \frac{1}{4}(a-ib)$, $C = \frac{1}{4}c$ and $D = \frac{1}{4}d$. We now look for solutions of (2.2) in the form

$$U(z, z^*, t) = -\frac{1}{2\pi i}\exp\left\{-\int_0^{z^*} A(z, \sigma, t)d\sigma\right\}$$
$$\times \oint_{|t-\tau|=\delta}\int_{-1}^{1} E(z, z^*, t, \tau, s) f\left(\frac{z}{2}(1-s^2), \tau\right)\frac{dsd\tau}{(1-s^2)^{1/2}}, \quad (2.3)$$

where $\delta > 0, f(z, t)$ is an analytic function of two complex variables in a neighbourhood of the origin in C^2, and $E(z, z^*, t, \tau, s)$ is an (analytic) function to be determined. The first integral in (2.3) is an integration in the complex τ plane in a counterclockwise direction about a circle of radius δ with centre at t, and the second integral is an integration over a curvilinear path in the unit disc in the complex s plane joining the points $s = +1$ and $s = -1$. Substituting (2.3) into (2.2) and integrating by parts shows that $E(z, z^*, t, \tau, s)$ must satisfy the differential equation

$$(1-s^2)E_{z^*s} - \frac{1}{s}E_{z^*} + 2sz(E_{zz^*} + \tilde{B}E_{z^*} + \tilde{C}E - \tilde{D}E_t) = 0, \quad (2.4)$$

where

$$\tilde{B} = B - \int_0^{z^*} A_z d\sigma,\ \tilde{C} = -(A_z + AB - C),\ \tilde{D} = D.$$

We now look for a solution of (2.4) in the form

$$E(z, z^*, t, \tau, s) = \frac{1}{t-\tau} + \sum_{n=1}^{\infty}\frac{s^{2n}z^n}{(t-\tau)^{n+1}}\int_0^{z^*} Q^{(2n)}(z, \sigma, t, \tau)d\sigma. \quad (2.5)$$

Substituting (2.5) into (2.4) yields the following recursion formula for the $Q^{(2n)}$:

$$Q^{(2)} = -2(t-\tau)\tilde{C} - 2\tilde{D} \quad (2.6)$$

$$(2n+1)Q^{(2n+2)} = -2\Big[(t-\tau)Q_z^{(2n)} + (t-\tau)\tilde{B}Q^{(2n)} + (t-\tau)\tilde{C}\int_0^{z^*} Q^{(2n)}d\sigma$$
$$+ (n+1)\tilde{D}\int_0^{z^*} Q^{(2n)}d\sigma - (t-\tau)\tilde{D}\int_0^{z^*} Q_t^{(2n)}d\sigma\Big].$$

It is clear from (2.6) that each of the $Q^{(2n)}$, $n = 1, 2, \ldots$, is uniquely determined. In order to show the existence of $E(z, z^*, t, \tau, s)$ it is now necessary to show the convergence of the series (2.5) and it is to this end that we first majorise the functions $Q^{(2n)}$. Let r and t_0 be arbitrarily large positive numbers and let B_0 be a positive constant such that for $|z| < r$, $|z^*| < r$, $|t| < t_0$ we have

$$\left.\begin{aligned}
\tilde{B}(z, z^*, t) &\ll \frac{B_0}{\left(1-\frac{z}{r}\right)\left(1-\frac{z^*}{r}\right)\left(1-\frac{t}{t_0}\right)} \\
\tilde{C}(z, z^*, t) &\ll \frac{B_0}{\left(1-\frac{z}{r}\right)\left(1-\frac{z^*}{r}\right)\left(1-\frac{t}{t_0}\right)}, \\
\tilde{D}(z, z^*, t) &\ll \frac{B_0}{\left(1-\frac{z}{r}\right)\left(1-\frac{z^*}{r}\right)\left(1-\frac{t}{t_0}\right)}.
\end{aligned}\right\} \quad (2.7)$$

where '$\ll$' denotes 'domination' [cf. **1**, **9**]. We also have the fact that for $|\tau| \leqq t_0$, $|t| < t_0$,

$$t-\tau \ll t_0 \left(1-\frac{t}{t_0}\right)^{-1}. \tag{2.8}$$

It is now a straightforward matter [cf. 3] to show by induction that for any $\varepsilon > 0$ and $|z| < r$, $|z^*| < r$, $|t| < t_0$, $|\tau| \leqq t_0$ we have (with respect to the variables z, z^*, t)

$$Q^{(2n)} \ll \frac{M_n 2^n t_0^n (1+\varepsilon)^n}{2n-1} \left(1-\frac{z}{r}\right)^{-(2n-1)} \left(1-\frac{z^*}{r}\right)^{-(2n-1)} \left(1-\frac{t}{t_0}\right)^{-(3n-1)} r^{-n}, \tag{2.9}$$

where

$$\begin{aligned} M_1 &= \frac{rB_0(1+t_0)}{t_0(1+\varepsilon)} \\ M_{n+1} &= M_n(1+\varepsilon)^{-1}\left\{1+\frac{B_0 r}{(2n-1)^2}\left(2n-1+r+\frac{4nr}{t_0}\right)\right\}. \end{aligned} \tag{2.10}$$

Note that for n sufficiently large we have $M_{n+1} \leqq M_n$, i.e. there exists a positive constant M which is independent of n such that $M_n \leqq M$ for all n. Now let $s_0 \geqq 1$ and $\alpha > 1$ be positive constants such that

$$\begin{array}{ll} |s| \leqq s_0 & |z| < \dfrac{r}{\alpha} \\ |r| \leqq t_0 & |z^*| < \dfrac{r}{\alpha} \\ |t| < \frac{1}{2}t_0 & \delta_0 \leqq |t-\tau|, \end{array} \tag{2.11}$$

where r and t_0 are arbitrarily large (but fixed) positive numbers and δ_0 is arbitrarily small (but again fixed). Then from (2.9) it is seen that the series (2.5) is majorised by the series

$$\frac{1}{\delta_0} + \sum_{n=1}^{\infty} \frac{rM_n 2^{4n-1} s_0^{2n} t_0^n (1+\varepsilon)^n \alpha^{3n-3}}{\delta_0^{n+1}(2n-1)(\alpha-1)^{4n-2}}. \tag{2.12}$$

If α is chosen such that $16 s_0^2 t_0 (1+\varepsilon)\alpha^3 \delta_0^{-1}(\alpha-1)^{-4} < 1$ then the series (2.12) is convergent. Since r, t_0 and s_0 can be arbitrarily large, δ_0 arbitrarily small, and ε is independent of r, t_0, s_0 and δ_0, we can now conclude that the series (2.5) converges absolutely and uniformly on compact subsets of

$$\{(z, z^*, t, \tau, s) : (z, z^*, t, \tau, s) \in C^5,\ t \neq \tau\},$$

i.e. $E(z, z^*, t, \tau, s)$ exists and is an entire function of its independent complex variables except for an (essential) singularity at $t = \tau$.

We have now shown that the operator $\boldsymbol{P}_l$ (the subscript is to denote the fact that the integral operator $\boldsymbol{P}_l$ is associated with the differential operator L defined in (2.2)) defined by

$$U(z, z^*, t) = \boldsymbol{P}_l\{f(z, t)\} = -\frac{1}{2\pi i}\exp\left\{-\int_0^{z^*} A(z, \sigma, t)d\sigma\right\}$$
$$\times \oint_{|t-\tau|=\delta}\int_{-1}^{1} E(z, z^*, t, \tau, s) f\left(\frac{z}{2}(1-s^2), \tau\right)\frac{ds d\tau}{(1-s^2)^{1/2}} \tag{2.13}$$

exists and maps analytic functions of two complex variables into the class of (complex valued) solutions of (2.2). Since the coefficients of (1.1) are real valued for x, y and t real, the operator Re $\boldsymbol{P}_t\{f(z, t)\}$ (where 'Re' denotes 'take the real part') defines a real valued solution of (1.1) provided we set $z^* = \bar{z}$ and keep t real. Since real valued analytic solutions of (1.1) in a neighbourhood of the origin are uniquely determined by their values on the characteristic plane $z^* = 0$ [cf. 3] we can now follow the analysis contained in [3] to conclude that if $(u(x, y, t) = U(z, \bar{z}, t)$ is a real valued analytic solution of (1.1) in a neighbourhood of the origin then we can represent $U(z, \bar{z}, t)$ in the form

$$U(z, \bar{z}, t) = \operatorname{Re} \boldsymbol{P}_t\{f(z, t)\}, \tag{2.14}$$

where

$$f\left(\frac{z}{2}, t\right) = -\frac{1}{2\pi}\int_\gamma\left[2U(z(1-s^2), 0, t) - U(0, 0, t)\exp\left(-\int_\delta^z \bar{A}(0, \sigma, t)d\sigma\right)\right]\frac{ds}{s^2},$$

$$\bar{A}(z, z^*, t) = \overline{A(\bar{z}, \bar{z}^*, \bar{t})} \tag{2.15}$$

and γ is a rectifiable arc joining the points $s = -1$ and $s = +1$ and not passing through the origin.

In addition to the operator Re $\boldsymbol{P}_t$ defined alone we will also need to make use of a generalised form of the operator $\boldsymbol{P}_t$ which will be denoted by $\boldsymbol{P}_t^*$ and is defined by

$$U(z, z^*, t) = \boldsymbol{P}_t^*\{f(z, t)\} = -\frac{1}{2\pi i}\exp\left\{-\int_{\zeta^*}^{z^*} A(z, \sigma, t)d\sigma\right\}$$

$$\times \int_{|t-t_1|=\delta}\int_{-1}^{1} E(z, z^*, t, t_1, s) f\left(\frac{(z-\zeta)}{2}(1-s^2), t_1\right)\frac{ds\,dt_1}{(1-s^2)^{1/2}}, \tag{2.16}$$

where $\delta > 0$, $(\zeta, \zeta^*) \in C^2$, $f(z, t)$ is an analytic function of two complex variables in some neighbourhood of the origin, and

$$E(z, z^*, t, t_1, s) = \frac{1}{t-t_1} + \sum_{n=1}^{\infty}\frac{s^{2n}(z-\zeta)^n}{(t-t_1)^{n+1}}\int_{\zeta^*}^{z^*} Q^{(2n)}(z, \sigma, t, t_1)d\sigma \tag{2.17}$$

with

$$Q^{(2)} = -2(t-t_1)\tilde{C} - 2\tilde{D} \tag{2.18}$$

$$(2n+1)Q^{(2n+2)} = -2\Bigg[(t-t_1)Q_z^{(2n)} + (t-t_1)\tilde{B}Q^{(2n)} + (t-t_1)\tilde{C}\int_{\zeta^*}^{z^*} Q^{(2n)}d\sigma$$

$$+ (n+1)\tilde{D}\int_{\zeta^*}^{z^*} Q^{(2n)}d\sigma - (t-t_1)\tilde{D}\int_{\zeta^*}^{z^*} Q_t^{(2n)}d\sigma\Bigg].$$

By slightly modifying our previous analysis for the case of the operator $\boldsymbol{P}_t$ it can be seen that the operator $\boldsymbol{P}_t^*$ exists and maps analytic functions of two complex variables defined in some neighbourhood of the origin into the class of analytic solutions of (2.2) defined in some neighbourhood of the point $(z, z^*, t) = (\zeta, \zeta^*, 0)$. It is also easy to see that $E(z, z^*, t, t_1; s) = E(z, z^*, t; \zeta, \zeta^*; t_1, s)$ is an entire function of its seven independent complex variables except for an (essential) singularity at $t = t_1$.

We make the observation that if, as a function of t, $f(z, t)$ has an isolated singularity at $t = \tau$ for a given $\tau \in C^1$, then $U(z, z^*, t) = \boldsymbol{P}_t^*\{f(z, t)\}$ also has an isolated singularity at $t = \tau$.

We will now use the integral operator P_m^* associated with the adjoint equation to (2.2) to construct the Riemann function for (1.1). The Riemann function

$$R(z, z^*, t; \zeta, \zeta^*, \tau)$$

for (1.1) is defined to be the (unique) solution of the adjoint equation

$$M[V] \equiv V_{zz^*} - \frac{\partial(AV)}{\partial z} - \frac{\partial(BV)}{\partial z^*} + CV + \frac{\partial}{\partial t}(DV) = 0 \tag{2.19}$$

satisfying the initial data

$$\begin{aligned} R(z, \zeta^*, t; \zeta, \zeta^*, \tau) &= \frac{1}{t-\tau} \exp\left\{\int_\zeta^z B(\sigma, \zeta^*, t)d\sigma\right\} \\ R(\zeta, z^*, t; \zeta, \zeta^*, \tau) &= \frac{1}{t-\tau} \exp\left\{\int_{\zeta^*}^{z^*} A(\zeta, \sigma, t)d\sigma\right\}. \end{aligned} \tag{2.20}$$

One approach towards constructing the Riemann function has been suggested by C. D. Hill in [10]. This method is based on expanding R in a series

$$R(z, z^*, t; \zeta, \zeta^*, \tau) = \sum_{n=0}^{\infty} R_n(z, z^*, t; \zeta, \zeta^*)n!(t-\tau)^{-n-1} \tag{2.21}$$

and then determining the coefficients R_n in a recursive fashion. This procedure leads to an infinite recursive sequence of analytic characteristic initial value problems for (2.19) which one must then solve and establish the convergence of the series (2.21) We will present an alternate approach based on using the integral operator P_m^* associated with the differential equation (2.19). Indeed, if we have the operator P_m^* at our disposal, the existence of the Riemann function for (1.1) is immediate. To see this let $f\left(\frac{z}{2}, t\right) = \frac{1}{t-\tau} F\left(\frac{z}{2}, t\right)$, where

$$F\left(\frac{z}{2}, t\right) = -\frac{1}{2\pi}\int_\gamma \exp\left\{\int_0^{z(1-\rho^2)} B(\sigma+\zeta, \zeta^*, t)d\sigma\right\}\frac{d\rho}{\rho^2} \tag{2.22}$$

with γ defined as in (2.15)) and define the solution $V(z, z^*, t; \zeta, \zeta^*, \tau)$ of (2.19) by

$$V(z, z^*, t; \zeta, \zeta^*, \tau) = P_m^*\left\{\frac{F(z, t)}{t-\tau}\right\} \tag{2.23}$$

Then from the reciprocal relations [cf. 1, 9]

$$\int_{-1}^{1} f\left(\frac{z}{2}(1-s^2)\right)\frac{ds}{(1-s^2)^{1/2}} = g(z) \tag{2.24}$$

$$-\frac{1}{2\pi}\int_\gamma g(z(1-\rho^2))\frac{d\rho}{\rho^2} = f\left(\frac{z}{2}\right) \tag{2.25}$$

we have that

$$\begin{aligned} V(z, \zeta^*, t; \zeta, \zeta^*, \tau) &= \frac{1}{t-\tau} \exp\left\{\int_0^{(z-\zeta)} B(\sigma+\zeta, \zeta^*, t)d\sigma\right\} \\ &= \frac{1}{t-\tau} \exp\left\{\int_\zeta^z B(\sigma, \zeta^*, t)d\sigma\right\} \\ V(\zeta, z^*, t; \zeta, \zeta^*, \tau) &= \frac{1}{t-\tau} \exp\left\{\int_{\zeta^*}^{z^*} A(z, \sigma, t)d\sigma\right\}, \end{aligned} \tag{2.26}$$

i.e. $V(z, z^*, t; \zeta, \zeta^*, \tau)$ is in fact the Riemann function $R(z, z^*, t; \zeta, \zeta^*, \tau)$. Note that except for an essential singularity at $t = \tau$ the Riemann function is an entire function of its six independent complex variables.

3. Analytic Continuation and Runge's Theorem

We will now use the results obtained in section 2 to derive the generalisation of Runge's theorem stated in the Introduction. Let $\mathfrak{D}$ be a bounded simply connected domain in $\mathbb{R}^2$ containing the origin and let $u(x, y, t)$ be a real valued strong solution of (1.1) in $\mathfrak{D}\times(0, T)$. Let $\mathfrak{D}_0$ and $\mathfrak{D}_1$ be compact subsets of $\mathfrak{D}$ such that $\mathfrak{D}\supset\mathfrak{D}_1\supset\mathfrak{D}_0$ and let $\partial\mathfrak{D}_1$ be analytic. From the well-known existence theorems for solutions of initial-boundary value problems for parabolic equations [7] and the maximum principle for parabolic equations we can conclude that for $\varepsilon, \delta_0>0$ there exists a solution $u_1(x, y, t)$ of (1.1) in $\mathfrak{D}_1\times[\frac{1}{2}\delta_0, T-\frac{1}{2}\delta_0]$ assuming analytic Dirichlet data on $\partial\mathfrak{D}_1\times[\frac{1}{2}\delta_0, T-\frac{1}{2}\delta_0]$ such that

$$\max_{\mathfrak{D}_1\times[\frac{1}{2}\delta_0,\, T-\frac{1}{2}\delta_0]} |u_1-u|<\varepsilon/2. \tag{3.1}$$

From a result of Friedman [7, 212] we can conclude that $u_1(x, y, t)$ is analytic in $\mathfrak{D}_1\times(\frac{1}{2}\delta_0, T-\frac{1}{2}\delta_0)$, i.e. for every point $(x_0, y_0, t_0)\in\mathfrak{D}_1\times(\frac{1}{2}\delta_0, T-\frac{1}{2}\delta_0)$ there exists a ball in $\mathbb{C}^3$ with centre at (x_0, y_0, t_0) such that as a function of the complex variables x, y, t, $u_1(x, y, t)$ is analytic in this ball. By standard compactness arguments we can conclude that $u_1(x, y, t)$ is analytic in some thin neighbourhood in $\mathbb{C}^3$ of the product domain $\mathfrak{D}_1\times[\delta_0, T-\delta_0]$. We now want to show that $U_1(z, z^*, t) = u_1(x, y, t)$ can be analytically continued as a function of z, z^* and t (where $z^* = \bar{z}$ for x and y real) into the interior of the product domain $\mathfrak{D}_1\times\mathfrak{D}_1^*\times\mathscr{E}$ where

$$\begin{aligned}\mathfrak{D}_1 &= \{z\colon z\in\mathfrak{D}_1\}\\ \mathfrak{D}_1^* &= \{z^*\colon \overline{z^*}\in\mathfrak{D}_1\}\end{aligned} \tag{3.2}$$

and $\mathscr{E}$ is an ellipse in $\mathbb{C}^1$ containing the interval $[\delta_0, T-\delta_0]$ such that for $(x, y)\in\mathfrak{D}_1$, $u_1(x, y, t)$ is an analytic function of t in $\mathscr{E}$.

Theorem. *$U_1(z, z^*, t)$ is analytic in the interior of $\mathfrak{D}_1\times\mathfrak{D}_1^*\times\mathscr{E}$.*

Proof. From Stokes theorem we have that for u and v analytic in a neighbourhood of $\mathfrak{D}_1\times[\delta_0, T-\delta_0]$

$$\iiint_{\mathfrak{D}_1\times\Omega}(v\mathscr{L}[u]-u\mathscr{M}[v])dxdydt = \iint_{\partial\mathfrak{D}\cdot\times\Omega} H[u, v], \tag{3.3}$$

where $\mathscr{L}$ is the differential operator defined by (1.1), $\mathscr{M}$ is its adjoint,

$$\Omega = \{t\colon|t-\tau| = \delta\}$$

such that $\Omega\subset\mathscr{E}$, and

$$H[u, v] = \{(vu_x-uv_x+auv)dydt-(vu_y-uv_y+buv)dxdt-(duv)dxdy\}. \tag{3.4}$$

The region of integration $\mathfrak{D}_1\times\Omega$ in (3.3) can be geometrically visualised as a three-dimensional torus lying in the six-dimensional space $\mathbb{C}^3$. Note that on $\partial\mathfrak{D}_1$ we have $dxdy = 0$. Now let $\mathfrak{D}_\varepsilon$ be a small disc of radius ε about the point (ξ, η), $u = u_1(x, y, t)$, $v = R(z, \bar{z}, t; \zeta, \bar{\zeta}, \tau)\log r$ (where $r^2 = (z-\zeta)(\bar{z}-\bar{\zeta})$, $\zeta = \xi+i\eta$, $\bar{\zeta} = \xi+i\eta$) and apply (3.3) to u and v with the torus $\mathfrak{D}_1\times\Omega$ replaced by the hollow torus $\mathfrak{D}_1/\mathfrak{D}_\varepsilon\times\Omega$.

Letting $\varepsilon \to 0$ now gives

$$0 = \lim_{\varepsilon \to 0} \left\{ \iint_{\partial(\mathfrak{D}_1/\mathfrak{D}_\varepsilon)\times\Omega} H[u_1, R \log r] + \iiint_{(\mathfrak{D}_1/\mathfrak{D}_\varepsilon)\times\Omega} u_1 \mathscr{M}[R \log r] dxdydt \right\}$$

$$= \iint_{\partial\mathfrak{D}_1\times\Omega} H[u_1, R \log r] + 2\pi \oint_\Omega \frac{u_1(\xi, \eta, t)}{t-\tau} dt + \iiint_{\mathfrak{D}_1\times\Omega} u_1 \mathscr{M}[R \log r] dxdydt \quad (3.5)$$

$$= \iint_{\partial\mathfrak{D}_1\times\Omega} H[u_1, R \log r] + 4\pi^2 i u_1(\xi, \eta, \tau) + \iiint_{\mathfrak{D}_1\times\Omega} u_1 \mathscr{M}[R \log r] dxdydt,$$

i.e.

$$u_1(\xi, \eta, \tau) = \frac{i}{4\pi^2} \left(\iint_{\partial\mathfrak{D}_1\times\Omega} H[u_1, R \log r] + \iiint_{\mathfrak{D}_1\times\Omega} u_1 \mathscr{M}[R \log r] dxdydt \right). \quad (3.6)$$

Returning now to the complex coordinates z, z^* we see from the fact that $M[R] = 0$ that

$$\mathscr{M}[R \log r] = M[R \log r] = 2\frac{\partial R/\partial z - BR}{\zeta^* - z^*} + 2\frac{\partial R/\partial z^* - AR}{\zeta - z}, \quad (3.7)$$

and hence from (2.20) we have that $\mathscr{M}[R \log r]$ is an entire function of its independent complex variables except for an essential singularity at $t = \tau$. Hence, replacing $\bar{\zeta}$ by ζ^*, we see that the second integral in (3.6) can be continued to an entire function of ζ and ζ^* for $\tau \in \mathscr{E}$. The first integral in (3.6) can be continued to an analytic function of ζ, ζ^* and τ for (ζ, ζ^*, τ) in the interior of $\mathfrak{D}_1 \times \mathfrak{D}_1^* \times \mathscr{E}$. Hence (3.6) shows that $U_1(\zeta, \zeta^*, \tau) = u_1(\xi, \eta, \tau)$ is analytic in the interior of $\mathfrak{D}_1 \times \mathfrak{D}_1^* \times \mathscr{E}$ and the theorem is established.

By using the operator Re $\boldsymbol{P}_l$ (the generalised) Runge's theorem now follows as a corollary of the above theorem:

COROLLARY (Runge's Theorem). *Let $u(x, y, t)$ be a real valued strong solution of (1.1) in $\mathfrak{D} \times (0, T)$ and let $\mathfrak{D}_0 \times [\delta_0, T-\delta_0]$ be a compact subset of $\mathfrak{D} \times (0, T)$. Then for every $\varepsilon > 0$ there exists an entire solution $u_0(x, y, t)$ of (1.1) such that*

$$\max_{\mathfrak{D}_0 \times [\delta_0, T-\delta_0]} | u - u_0 | < \varepsilon. \quad (3.8)$$

Proof. Let $u_1(x, y, t)$ be an analytic solution of (1.1) in $\mathfrak{D}_1 \times [\frac{1}{2}\delta_0, T-\frac{1}{2}\delta_0]$ such that (3.1) is valid. From the theorem $U_1(z, z^*, t) = u_1(x, y, t)$ is analytic in the interior of $\mathfrak{D}_1 \times \mathfrak{D}_1^* \times \mathscr{E}$, and from the results of section two we can represent $U_1(z, \bar{z}, t)$ in this domain in the form $U_1(z, \bar{z}, t) = \text{Re}\, \boldsymbol{P}_l\{f(z, t)\}$(where $f(z, t)$ is given by (2.15) with U replaced by U_1). Since product domains are Runge domains of the first kind [8, 49] we can approximate $U_1(z, 0, t)$ (and hence $f\left(\frac{z}{2}, t\right)$) on compact subsets of $\mathfrak{D}_1 \times \mathscr{E}$ by a polynomial. In particular since Re $\boldsymbol{P}_l\{f(z, t)\} \to 0$ as $f(z, t) \to 0$ in the maximum norm we can conclude that there exists a polynomial $f_n(z, t)$ and entire solution $u_2(x, y, t) = \text{Re}\, \boldsymbol{P}_l\{f_n(z, t)\}$ of (1.1) such that

$$\max_{\mathfrak{D}_0 \times [\delta_0, T-\delta_0]} | u_2 - u_1 | < \varepsilon/2. \quad (3.9)$$

The corollary now follows from (3.1) and (3.9) by use of the triangle inequality.

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Complete Families of Solutions to the Heat Equation and Generalized Heat Equation in $\mathbb{R}^n$

by

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II. The Heat Equation.

Let D be a bounded, simply connected domain in Euclidean n-space $\mathbb{R}^n$ with ∂D in class C^{2i+2} where $i=1+[\frac{n}{4}+\frac{1}{2}]$, T a positive constant, $\underset{\sim}{x}=(x_1,\ldots,x_n)\varepsilon\mathbb{R}^n$, and $u(\underset{\sim}{x},t)\varepsilon C^2(Dx(0,T))\cap C^o(\bar{D}x[0,T])$ a solution of the heat equation

$$\begin{aligned} &\Delta_n u = u_t \\ &\Delta_n \doteq \frac{\partial^2}{\partial x_1} + \ldots + \frac{\partial^2}{\partial x_n} \end{aligned} \tag{2.1}$$

in $Dx(0,T)$. We are interested in the problem of approximating $u(\underset{\sim}{x},t)$ in the maximum norm over $\bar{D}x[0,T]$ by a linear combination of the particular solutions of (2.1) defined by

$$\begin{aligned} &h_m(\underset{\sim}{x},t) = h_{m_1}(x_1,t)h_{m_2}(x_2,t)\ldots h_{m_n}(x_n,t) \\ &\qquad m = (m_1,\ldots,m_n)\varepsilon\,\mathbb{N}^n \end{aligned} \tag{2.2}$$

where

$$h_p(x,t) = \sum_{k=o}^{[\frac{p}{2}]} \frac{x^{p-2k}t^k}{(p-2k)!k!} \tag{2.3}$$

are the so called heat polynomials introduced by Rosenbloom and Widder in [15]. This problem has already been solved in the case when n=1 and n=2 in [6] and [7]. However in these papers the proofs depended strongly on the dimension n, and do not generalize immediately to the general case now under consideration. In this section we will solve the above mentioned approximation problem for general n by the use of known regularity results for solutions to parabolic equations together with a new class of integral operators for parabolic equations constructed by W. Rundell and M. Stecher in [16]. We note that without loss of generality we can assume that $u(\underset{\sim}{x},0)=0$. This follows from the fact that by the maximum principle for (2.1) and the Weierstrass approximation theorem we can approximate $u(\underset{\sim}{x},t)$ in the maximum norm over $\bar{D}x[0,T]$ by a

∂D_1. For $\underset{\sim}{x}'\varepsilon\partial D_1$ define $f(\underset{\sim}{x}',t)$ by $f(\underset{\sim}{x}',t)=u_1(\underset{\sim}{x},t)$ where $\underset{\sim}{x}$ is the point on ∂D associated with $\underset{\sim}{x}'\varepsilon\partial D_1$ under the above deformation, and let $u_2(\underset{\sim}{x},t)\varepsilon C^2(D_1 x(-1,T))\cap C^o(\bar{D}_1 x[-1,T])$ be the solution of equation (2.1) in $D_1 x(-1,T)$ satisfying the initial-boundary data

$$u_2(\underset{\sim}{x}',t) = f(\underset{\sim}{x}',t) \quad ; \quad \underset{\sim}{x}'\varepsilon\partial D_1 \tag{2.8}$$

$$u_2(x,-1)=0 \quad . \tag{2.9}$$

From the Weierstrass approximation theorem, the maximum principle for the heat equation, and the existence theorem for the heat equation (c.f. [14]) we can construct a solution $u_o(\underset{\sim}{x},t)\varepsilon C^2(D_1 x(-1,T))\cap C^o(\bar{D}_1 x[-1,T])$ of equation (2.1) such that $u_o(\underset{\sim}{x},t)$ has analytic boundary data on $D_1 x[-1,T]$ and

$$\max_{\bar{D}_1 x[-1,T]} | u_o(\underset{\sim}{x},t) - u_2(\underset{\sim}{x},t) | < \varepsilon_1 \tag{2.10}$$

for $\varepsilon_1 > 0$ arbitrarily small. From [8], p.140-141, we can conclude that there exists a positive constant C which is independent of d for $d \leqslant d_o$ such that

$$|\nabla_{\underset{\sim}{x}} u_o(\underset{\sim}{x},t)| \leqslant C \tag{2.11}$$

for $(\underset{\sim}{x},t)\varepsilon\bar{D}_1 x[-1+\delta,T]$, $\delta>0$ arbitrarily small. In particular the constant C depends only on δ, d_o, the boundary data of $u_o(\underset{\sim}{x},t)$, and D. Hence from the mean value theorem, for $t\varepsilon[-1+\delta,T]$

$$|u_o(\underset{\sim}{x}',t) - u_o(\underset{\sim}{x},t)| \leqslant Cd, \tag{2.12}$$

and from equation (2.10) and the triangle inequality

$$|u_2(\underset{\sim}{x}',t) - u_o(\underset{\sim}{x},t)| \leqslant \varepsilon_1+Cd \tag{2.13}$$

for $t\varepsilon[-1+\delta,T]$. But $u_2(\underset{\sim}{x}',t)=u_1(x,t)$ and hence

$$|u_1(\underset{\sim}{x},t) - u_o(\underset{\sim}{x},t)| \leqslant \varepsilon_1+Cd \tag{2.14}$$

for $\underset{\sim}{x}\varepsilon\partial D$, $t\varepsilon[-1+\delta,T]$. We now note that from the maximum principle $u_2(\underset{\sim}{x},t)=0$ for $(\underset{\sim}{x},t)\varepsilon\bar{D}_1 x[-1,0]$, and hence equations (2.10) and (2.14) imply

$$\Delta_n w_k = (k+1)w_{k+1} \quad ; \quad \underset{\sim}{x}\varepsilon D_1$$
$$w_k(\underset{\sim}{x}) = \psi_k(\underset{\sim}{x}) \quad ; \quad \underset{\sim}{x}\varepsilon \partial D_1 \tag{2.18b}$$

for $k=0,1,\ldots k_o-1$. The existence of functions $w_k \varepsilon C^2(D_1) \cap C^o(\bar{D}_1)$ satisfying equation (2.18) follows from the analyticity of $\psi_k(\underset{\sim}{x})$, the regularity of ∂D_1, and the existence of solutions to the Dirichlet problem for the Poisson equation (c.f. [14]). Hence $w(\underset{\sim}{x},t)$ as defined by equation (2.17) exists. Now $w_k(\underset{\sim}{x})$ is a solution of

$$\Delta_n^{k_o-k+1} w_k = 0 \tag{2.19}$$

for $k=0,1,\ldots k_o$, $\underset{\sim}{x}\varepsilon D_1$, and hence from an easy generalization of a result in [2], p.229-230, we have

$$w_k = \sum_{j=1}^{k_o-k+1} r^{2(j-1)} h_j(\underset{\sim}{x}) \tag{2.20}$$

where $r=|\underset{\sim}{x}|$ and the $h_j(\underset{\sim}{x})$ are harmonic functions in D_1. From the Runge approximation property for elliptic equations ([13]) and the analyticity of harmonic functions, we have that each function $h_j(\underset{\sim}{x})$, $j=1,\ldots,k_o-k+1$, can be approximated in the maximum norm on compact subsets of D_1 by a finite linear combination of harmonic polynomials. From equations (2.17) and (2.20) we can now conclude that there exists a solution $v_o(x,t)$ of equation (2.1) that is an entire function of its independent complex variables such that

$$\max_{\bar{D}\times[-1,T]} |w(\underset{\sim}{x},t) - v_o(\underset{\sim}{x},t)| < \varepsilon \tag{2.21}$$

for $\varepsilon > 0$ arbitrarily small.

Now let $v_1(\underset{\sim}{x},t) = u_o(\underset{\sim}{x},t) - w(\underset{\sim}{x},t)$ and let λ_j and $\phi_j(\underset{\sim}{x})$ be the eigenvalues and eigenfunctions respectively of the eigenvalue problem

$$\Delta_n u + \lambda u = 0 \quad ; \quad \underset{\sim}{x}\varepsilon D_1 \tag{2.22}$$
$$u(\underset{\sim}{x}) = 0 \quad ; \quad \underset{\sim}{x}\varepsilon \partial D_1 \quad .$$

inequality we have completed the proof of lemma 2.2.

We are now in a position to prove the following theorem:

Theorem 2.1: Let $u(\underset{\sim}{x},t)\varepsilon C^2(Dx(0,T))\cap C^o(\bar{D}x[0,T])$ be a solution of equation (2.1) in $Dx(0,T)$. Then for every $\varepsilon > 0$ there exists an integer M and constants a_m, $|m| \leq M$, such that

$$\max_{\bar{D}x[0,T]} \left|u(\underset{\sim}{x},t) - \sum_{|m|\leq M} a_m h_m(\underset{\sim}{x},t)\right| < \varepsilon ,$$

where the $h_m(\underset{\sim}{x},t)$ are defined in equations (2.2) and (2.3).

Proof: Without loss of generality we can assume $u(\underset{\sim}{x},0)=0$ (See the discussion before lemma 2.1). From lemma 2.2. it suffices to approximate $u_1(\underset{\sim}{x},t)$ in the maximum norm over $\bar{D}x[0,T]$ where $u_1(\underset{\sim}{x},t)$ is a solution of equation (2.1) that is an entire function of its independent complex variables. But from the results of [16] we can write $u_1(\underset{\sim}{x},t)$ in the form

$$u_1(\underset{\sim}{x},t) = h(\underset{\sim}{x},t) + \frac{1}{2\pi i}\oint_{|t-\tau|=\delta}\int_0^1 \sigma^{n-1}G(r,1-\sigma^2,t-\tau)h(\underset{\sim}{x}\sigma^2,\tau)\,d\sigma d\tau \tag{2.27}$$

where $\delta > 0$ is arbitrary, $h(\underset{\sim}{x},t)$ is an entire function of its independent complex variables such that $\Delta_n h=0$ for each fixed t, and

$$G(r,\xi,t) = \frac{r^2}{2t^2}\exp\left(\frac{\xi r^2}{4t}\right) . \tag{2.28}$$

Let Ω be a sphere in $\mathbb{R}^n$ such that $\Omega \supset \bar{D}$ and let $\{h_j(\underset{\sim}{x})\}$ denote the set of harmonic polynomials. Then from the Runge approximation property for elliptic equations we can approximate $h(\underset{\sim}{x},t)$ on $\Omega \times \{t : |t|\leq T+\delta\}$ by a finite sum of the form

$$\sum_{j=0}^{j_o}\sum_{k=0}^{k_o} a_{jk}h_j(\underset{\sim}{x})t^k \tag{2.29}$$

where the a_{jk}, $j=0,1,\ldots j_o$, $k=0,1,\ldots,k_o$, are constants, and hence from equation (2.27) we have that for every $\varepsilon > 0$ there exist integers j_o

and k_o and constants a_{jk} such that

$$\max_{\bar{D}x[0,T]} \left| u_1(\underset{\sim}{x},t) - \sum_{j=0}^{j_o} \sum_{k=0}^{k_o} a_{jk} u_{jk}(\underset{\sim}{x},t) \right| < \varepsilon \tag{2.30}$$

where the $u_{jk}(\underset{\sim}{x},t)$ are polynomial solutions of equation (2.1) defined by

$$u_{jk}(\underset{\sim}{x},t) = h_j(\underset{\sim}{x})t^k + \frac{1}{2\pi i} \oint_{|t-\tau|=\delta} \int_0^1 \sigma^{n-1} G(r,1-\sigma^2,t-\tau) h_j(\underset{\sim}{x}\sigma^2)\tau^k d\sigma d\tau. \tag{2.31}$$

Since the $u_{jk}(\underset{\sim}{x},t)$ are polynomials in $x_1,\ldots x_n$ and t, there exist an integer M and constants $b_m = b_m(j,k)$, $|m| \leq M$, such that

$$u_{jk}(\underset{\sim}{x},0) = \sum_{|m|\leq M} b_m h_m(\underset{\sim}{x},0) \ . \tag{2.32}$$

From the uniqueness theorem for Cauchy's problem for the heat equation and the uniqueness of analytic continuation we have that

$$u_{jk}(\underset{\sim}{x},t) = \sum_{|m|\leq M} b_m h_m(\underset{\sim}{x},t) \tag{2.33}$$

for all $\underset{\sim}{x}$ and t, and the conclusion of the theorem now follows from equations (2.30) and (2.33).

obviously fulfills (3.2) if we take $U=\{x_o:|x_o|<\varepsilon\}\times\{\underset{\sim}{x}:|\underset{\sim}{x}-\underset{\sim}{x}^1|<\delta\}$.
But it also is a solution of $L_{k-1}w=0$ since

$$L_{k-1}w = \Delta\phi-\phi_t + (2k-1)u + x_o u_{x_o} + \int_0^{x_o} \xi(\Delta u-u_t)(\xi,\underset{\sim}{x},t)d\xi$$

and

$$(2k-1)u(x_o,\underset{\sim}{x},t)+x_o u_{x_o}(x_o,\underset{\sim}{x},t)+\int_0^{x_o}\xi(\Delta u-u_t)(\xi,\underset{\sim}{x},t)d\xi = (2k-1)u(0,\underset{\sim}{x},t)$$

which implies $L_{k-1}w=0$ if ϕ is a solution of (3.2).

<u>Lemma 3.2</u>: For k=1,2,3,... all functions $u\varepsilon S_k$ are infinitely differentiable in $Dx(0,T)$.

<u>Proof</u>: It is clear that $u\varepsilon S_k$ is infinitely differentiable at all points $(x_o,\underset{\sim}{x},t)\varepsilon Dx(0,T)$ with $x_o\neq 0$. Now first let k=1 and $(0,\underset{\sim}{x},t)\varepsilon Dx(0,T)$. Then according to lemma 3.1 we have a neighbourhood $U\subset D$ of $(0,\underset{\sim}{x})$ and a solution w of $L_o w=0$ in $Ux(0,T)$ such that (3.2) holds. On the other hand it can be verified directly that

$$u(x_o,\underset{\sim}{x},t) = \int_0^1 v(x_o\xi,\underset{\sim}{x},t)d\xi \qquad (3.5)$$

with $v(x_o,\underset{\sim}{x},t)=u(x_o,\underset{\sim}{x},t) + x_o u_{x_o}(x_o,\underset{\sim}{x},t)$.

Therefore $w_{x_o x_o} = v$ in $Ux(0,T)$ and since $L_o w=0$ in $Ux(0,T)$, w, and hence v, is infinitely differentiable in $Ux(0,T)$. From (3.5) it follows that u is infinitely differentiable in $Ux(0,T)$.

For k=2,3,... the assertion of lemma 3.2 now follows by induction: If every $w\varepsilon S_k$ is infinitely differentiable then also all $u\varepsilon S_{k+1}$ are infinitely differentiable because the representation (3.2) implies that

$$u(x_o,\underset{\sim}{x},t) = x_o^{-1}w_{x_o}(x_o\underset{\sim}{x},t) = \frac{1}{2k}\left(w_t(x_o,\underset{\sim}{x},t) - \sum_{i=0}^{n} w_{x_i x_i}(x_o,\underset{\sim}{x},t)\right)$$

in a neighbourhood of a point of the singular plane.

<u>Lemma 3.3</u>: For every $u\varepsilon S_k$ even with respect to x_o (k=1,2,3,...) there is a $v\varepsilon S_o$ such that

$$u(x_o,\underset{\sim}{x},t) = \int_0^1 v(x_o\xi,\underset{\sim}{x},t)(1-\xi^2)^{k-1}d\xi \qquad (3.6)$$

for $(x_o,\underset{\sim}{x},t)\varepsilon Dx(0,T)$.

If $m = (m_o, m_1, \ldots, m_n)$ is a multi-index and $\lambda>0$, we use the notation

$$h_{m,\lambda}(x_o, \underset{\sim}{x}, t) = h_{m_o,\lambda}(x_o, t) h_{m_1}(x_1, t) \ldots h_{m_n}(x_n, t)$$

where the heat polynomials $h_{m_j}(x_j, t)$ $(j=1,\ldots,n)$ are defined as in (2.3) and $h_{m_o,\lambda}$ is a generalized heat polynomial (see [3], [10]) defined by

$$h_{r,\lambda}(x_o, t) = \sum_{j=0}^{r} 2^{2j} \binom{r}{j} \frac{1}{\Gamma(r-j+\lambda+\frac{1}{2})} x_o^{2r-2j} t^j .$$

It is easy to see (c.f. [5], [17], [18]) that

$$\int_0^1 h_{2r}(x_o\xi, t)(1-\xi^2)^{k-1} d\xi = \gamma_{r,k} h_{r,k}(x_o, t) \qquad (3.8)$$

for certain constants $\gamma_{r,k}$.

Theorem 3.1: Let $D_o \subset D$ have the following properties: (i) D_o is simply connected, (ii) $\bar{D}_o \subset D$, (iii) ∂D_o is of class C^{2i+2} where $i=1 + \left[\frac{n+1}{4} + \frac{1}{2}\right]$. Let $\delta>0$. Then for every $\varepsilon>0$ and for every $u\varepsilon S_k$ even with respect to x_o $(k=1,2,3,\ldots)$ there exist $M\varepsilon \mathbb{N}$ and $a_m \varepsilon \mathbb{R}$, $(|m| \leq M)$, such that

$$\max_{\bar{D}_o \times [\delta, T-\delta]} |u(x_o, \underset{\sim}{x}, t) - \sum_{|m|\leq M} a_m h_{m,k}(x_o, \underset{\sim}{x}, t)| < \varepsilon \qquad (3.9)$$

Proof: According to lemma 3.3 there is a $v \varepsilon S_o$ such that (3.6) holds. By Theorem 2.1 for every $\varepsilon>0$ there exist $M \varepsilon \mathbb{N}$ and $b_m \varepsilon \mathbb{R}$ such that

$$\max_{\bar{D}_o \times [\delta, T-\delta]} |v(x_o, \underset{\sim}{x}, t) - \sum_{|m|\leq M} b_m h_m(x_o, \underset{\sim}{x}, t)| < \varepsilon. \qquad (3.10)$$

Since v is even with respect to x_o all heat polynomials h_{m_o} appearing in (3.10) have even index and applying (3.6) and (3.8) it easily follows that (3.9) holds with $a_{m_o} = b_{m_o} \gamma_{m_o,k}$.

Remark: It can be expected that a result analogous to theorem 2.1 holds for all $k>0$ since the methods used in the proofs above require k to be an integer only at the point where the integral operator connecting solutions of the heat equation and of the generalized heat equation has to be inverted. In the case of arbitrary $k>0$ one would have to use regularity results for fractional integration operators.

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ON THE INVERSE SCATTERING PROBLEM FOR AXIALLY SYMMETRIC SOLUTIONS OF THE HELMHOLTZ EQUATION

By DAVID COLTON

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1. Introduction

IN this paper we consider axially symmetric solutions of the three-dimensional Helmholtz equation which are of class C^2 (i.e. regular) in the exterior of a bounded domain D. In cylindrical coordinates (r, z, Φ) such solutions satisfy the equation

$$u_{zz}+u_{rr}+\frac{1}{r}u_r+u = 0, \tag{1}$$

where we have assumed the axis of symmetry to be $r = 0$. If we further assume that u satisfies the Sommerfeld radiation condition

$$\lim_{R\to\infty} R\left(\frac{\partial u}{\partial R}-iu\right) = 0; \qquad R = +\sqrt{(r^2+z^2)}, \tag{2}$$

then u may be regarded as being generated by volume sources, surface sources, or point singularities, all of which are inside D.

A solution u of equation (1) satisfying (2) behaves asymptotically like

$$u \sim \frac{e^{iR}}{R} f(\cos\theta); \quad R \to \infty, \tag{3}$$

where (R,θ) are polar coordinates defined by $z = R\cos\theta$, $r = R\sin\theta$. The function $f(\cos\theta)$ is known as the *radiation pattern* [see (**9**)]. We are concerned here with the *inverse scattering problem* associated with equation (1), i.e. given a radiation pattern $f(\cos\theta)$ to determine u and the location of the sources which generate u. The class of radiation patterns has been previously determined by Müller in (**9**) and is neither the class of continuous functions nor the class of analytic functions; it can best be characterized by constructing an associated set of harmonic functions (we incorporate here the slight correction to Müller's result as given by Hartman and Wilcox in (**5**)):

THEOREM 1 (Müller). *A necessary and sufficient condition for a function $f(\cos\theta)$, defined on $[0, 2\pi]$, to be a radiation pattern is that there exist an (axially symmetric) harmonic function $h(z,r) = \tilde{h}(R,\theta)$ which is regular* *in the entire space and is such that* $\tilde{h}(1,\theta) = f(\cos\theta)$, *and further has the property that*

$$\int_0^{2\pi} |\tilde{h}(R,\theta)|^2\, d\theta$$

is an entire function of R of order one and finite type C. When this condition holds, there exists a unique function $u(z,r) = \tilde{u}(R,\theta)$ *which satisfies the Sommerfeld radiation condition and is a regular solution of the (axially symmetric) Helmholtz equation for* $R > C$ *such that*

$$\tilde{u}(R,\theta) \sim \frac{e^{iR}}{R} f(\cos\theta) + O\left(\frac{1}{R^2}\right); \quad R \to \infty.$$

Müller's theorem shows that for a given radiation pattern $f(\cos\theta)$ the sources which generate u must lie within a closed disc of radius C, where C is uniquely determined by $f(\cos\theta)$. It is the purpose of this paper to give more precise information on the location of these sources. More specifically let Ω be a circle orthogonal to the circle $R = C$ and let $\mathfrak{S}$ be the conjugate indicator diagram [(**1**) 73–77] of $h(2iz, 0)$ considering $z = Re^{i\theta}$ as a complex variable. Then we will show that $\tilde{u}(R,\theta)$ is regular in Ω provided Ω does not intersect $\mathfrak{S} \cup \bar{\mathfrak{S}} \cup \{(R,\theta) \mid \theta = 0, \pi\}$ where the bar denotes complex conjugation. A statement of the result may be found in Theorem 2 at the end of this paper.

2. Analytic continuation of solutions to the inverse scattering problem

From Müller's theorem it is seen that every radiation pattern $f(\cos\theta)$ can be expressed as a uniformly convergent Legendre series

$$f(\cos\theta) = \sum_{n=0}^{\infty} a_n P_n(\cos\theta), \tag{4}$$

where $P_n(\cos\theta)$ denotes Legendre's polynomial. Corresponding to this radiation pattern is the solution of equation (1) having (4) as its pattern [see (**9**)]

$$u(z,r) = \tilde{u}(R,\theta) = \sqrt{(\tfrac{1}{2}\pi)} \sum_{n=0}^{\infty} a_n i^{n+1} H^{(1)}_{n+\frac{1}{2}}(R) P_n(\cos\theta), \tag{5}$$

where $H^{(1)}_{n+\frac{1}{2}}(R)$ denotes Hankel's function of the first kind and the series (5) converges absolutely and uniformly for $R \geqslant C' > C$. The axially symmetric harmonic function associated with the far field pattern (4) is given by

$$h(z,r) = \tilde{h}(R,\theta) = \sum_{n=0}^{\infty} a_n R^n P_n(\cos\theta), \tag{6}$$

and satisfies the partial differential equation

$$u_{zz}+u_{rr}+\frac{1}{r}u_r = 0. \tag{7}$$

From Müller's theorem it is seen that $\sum_{n=0}^{\infty} |a_n|^2 R^{2n}$ is an entire function of order one and exponential type C, i.e. [see (**1**) 11]

$$\limsup_{n\to\infty} n|a_n|^{1/n} = \tfrac{1}{2}eC. \tag{8}$$

We now wish to use some recent function-theoretic developments in the theory of partial differential equations to show how $u(z,r)$ can be analytically continued across the circle $R = C$.

By the identity theorem for axially symmetric harmonic functions in (**13**), $h(z,r)$ is uniquely determined by the values it takes on the axis $r = 0$, i.e. the function

$$h(z,0) = \sum_{n=0}^{\infty} a_n z^n. \tag{9}$$

Equation (8) shows that by allowing z to assume complex values $h(z,0)$ defines an entire function of order one and type $C/2$. Now let $f(z)$ be the Borel transform of $h(2iz,0)$ defined by

$$f(z) = \sum_{n=0}^{\infty} a_n 2^n i^n n!\, z^{-n-1}. \tag{10}$$

A classical result of Pólya [cf. (**1**) 75] states that $f(z)$ is regular in the exterior of the conjugate indicator diagram of $h(2iz,0)$. We denote the conjugate indicator diagram of $h(2iz,0)$ by $\mathfrak{S}$ and recall that since $h(2iz,0)$ is of order one and type C then $\mathfrak{S}$ is a closed convex set contained in the disc $|z| \leqslant C$. For further properties of $\mathfrak{S}$ the reader is referred to (**1**) 73.

We now define an analytic function $g(z)$ by

$$g(z) = \sqrt{(\tfrac{1}{2}\pi)} \sum_{n=0}^{\infty} a_n i^{n+1} H^{(1)}_{n+\frac{1}{2}}(C')\left(\frac{z}{C'}\right)^{-n-1}, \tag{11}$$

from which we will construct a solution of equation (7) which agrees with $u(z,r)$ on $R = C'$. From (8) and the asymptotic relation [see (**2**) 4, 8]

$$\lim_{n\to\infty} \frac{(C'/2)^{n+\frac{1}{2}}}{\Gamma(n+\frac{1}{2})} H^{(1)}_{n+\frac{1}{2}}(C') = -\frac{i}{\pi}, \tag{12}$$

it is clear that $g(z)$ is regular for $|z| > C'$. The following lemma gives a more refined result which is needed for our purposes.

LEMMA 1. *$g(z)$ is regular in the exterior of $\mathfrak{S}$.*

Proof. Since $f(z)$ is regular in the exterior of $\mathfrak{S}$, by Hadamard's multiplication of singularities theorem [(**11**) 157] it suffices to show that the singularities of

$$G(z) = \sum_{n=0}^{\infty} \frac{H^{(1)}_{n+\frac{1}{2}}(C')}{n!\,2^n}\left(\frac{z}{C'}\right)^{-n} \tag{13}$$

lie on the closed interval $[0, 1]$. To this end we note the following formulae [(**2**) 78, 100]:

$$(s^2-t^2)^{-\frac{1}{4}}H^{(1)}_{\frac{1}{2}}[w(s^2-t^2)^{\frac{1}{2}}] = \sum_{n=0}^{\infty}\left(\frac{wt^2}{2}\right)^n s^{-\frac{1}{2}-n}H^{(1)}_{n+\frac{1}{2}}(ws)/n!, \tag{14}$$

$$H^{(1)}_{\frac{1}{2}}(\xi) = -i(\tfrac{1}{2}\pi\xi)^{-\frac{1}{2}}e^{i\xi}. \tag{15}$$

Setting $s = 1$, $w = C'$, $t^2 = 1/z$ in (14) and then using (15) gives

$$G(z) = -i\sqrt{\left(\frac{2}{\pi C'}\right)}\left(1-\frac{1}{z}\right)^{-\frac{1}{4}} e^{iC'(1-1/z)^{\frac{1}{2}}}. \tag{16}$$

Equation (16) shows that $G(z)$ is regular in the extended z plane except for branch points at $z = 0$ and $z = +1$. This result establishes the lemma.

From the identity theorem for axially symmetric harmonic functions previously cited it is now possible to construct a unique axially symmetric harmonic function $v(z, r)$ such that $v(z, 0) = g(z)$:

$$v(z,r) = \tilde{v}(R,\theta) = \sqrt{\left(\frac{2}{\pi}\right)}\sum_{n=0}^{\infty} a_n\, i^{n+1}H^{(1)}_{n+\frac{1}{2}}(C')\left(\frac{R}{C'}\right)^{-n-1}P_n(\cos\theta), \tag{17}$$

where the series (17) converges uniformly for $R > C'$, $\theta \in [0, 2\pi]$. From (**4**) in conjunction with Kelvin's transformation [(**3**) 84] it is seen that $\tilde{v}(R, \theta)$ is singular at the point (R, θ) if and only if $g(z)$ is singular at either $z = Re^{i\theta}$ or $z = Re^{-i\theta}$. Hence $v(z, r) = \tilde{v}(R, \theta)$ can be analytically continued as a real analytic function of z and r into the exterior of $\mathfrak{S} \cup \bar{\mathfrak{S}}$. By the law of permanence of functional equations [(**7**) 31] $v(z, r)$ satisfies (7) in this region. On the circle $R = C'$ we have $\tilde{v}(C', \theta) = \tilde{u}(C', \theta)$.

Now let Ω be a circle which is orthogonal to the circle $R = C'$ and which intersects neither $\mathfrak{S} \cup \bar{\mathfrak{S}}$ nor the axis $r = 0$. Then Ω is a fundamental domain [see (**12**)] for equation (7) and since a conformal transformation maps orthogonal circles on to orthogonal circles, Ω is conformally symmetric in the sense of Henrici (**6**) with respect to the circle $R = C'$. By Theorem 5.2 of (**6**) and the above discussion it is seen that $v(z, r)$ is an analytic function of $\eta = z+ir$ on the arc

$$A = \{(z,r) \mid \sqrt{(z^2+r^2)} = C'\} \cap \Omega$$

and can be continued analytically as a function of η into the whole of Ω. We denote this function by $V(\eta)$. Note that $V(\eta)$ agrees with $v(z,r)$ only on the arc A since $v(z,r)$ was continued past the circle $R = C'$ as a real analytic function of the two independent variables r and z, whereas $V(\eta)$ has been continued as an analytic function of the single complex variable $\eta = z+ir$.

LEMMA 2. *$u(z,r)$ is regular in Ω.*

Proof. In conjugate coordinates

$$\eta = z+ir,$$
$$\eta^* = z-ir, \tag{18}$$

equation (1) becomes [see (**12**)] the formal hyperbolic equation

$$U_{\eta\eta^*}-\frac{1}{2(\eta-\eta^*)}U_\eta+\frac{1}{2(\eta-\eta^*)}U_{\eta^*}+\tfrac{1}{4}U = 0, \tag{19}$$

where

$$U(\eta,\eta^*) = u\left(\frac{\eta+\eta^*}{2},\frac{\eta-\eta^*}{2i}\right). \tag{20}$$

Now let $\eta = \phi(\zeta)$ be a conformal mapping of the upper half of the ζ plane on to the exterior of the circle $R = C'$ and let $\eta^* = \bar{\phi}(\zeta^*) = \overline{\phi(\overline{\zeta^*})}$ where the bar denotes complex conjugation. Under the transformation

$$\eta = \phi(\zeta),$$
$$\eta^* = \bar{\phi}(\zeta^*) \tag{21}$$

(19) is transformed into a linear, formally hyperbolic equation with analytic coefficients for the function $W(\zeta,\zeta^*) = U(\phi(\zeta),\bar{\phi}(\zeta^*))$. $W(\zeta,\zeta^*)$ is regular for (ζ,ζ^*) contained in $(\phi^{-1}(\Omega\cap R\geqslant C'),\bar{\phi}^{-1}(\Omega\cap R\geqslant C'))$, and on $\zeta=\zeta^*$

$$W(\zeta,\zeta^*) = V(\phi(\zeta)). \tag{22}$$

By construction $V(\phi(\zeta))$ is regular in the circle $\phi^{-1}(\Omega)$. Note also that $\phi^{-1}(\Omega)$ is symmetric with respect to the real axis in the complex ζ plane. Hence by Lewy's reflection principle (**8**) we can analytically continue $W(\zeta,\zeta^*)$ into all of $(\phi^{-1}(\Omega),\bar{\phi}^{-1}(\Omega))$. By transforming back we find that $U(\eta,\eta^*)$ is analytic for (η,η^*) contained in $(\Omega,\overline{\Omega})$, i.e. $u(z,r)$ is regular in Ω.

Note that since C' can be arbitrarily close to C, the circle Ω can be taken orthogonal to C without affecting the validity of Lemma 2. Putting our results together we can now state the following theorem:

THEOREM 2. *Let $f(\cos\theta)$ be the radiation pattern of a solution $\tilde{u}(R,\theta)$ of the three-dimensional axially symmetric Helmholtz equation, where (R,θ) are polar coordinates, and let $\{a_n\}_{n=0}^{\infty}$ be the Legendre coefficients of $f(\cos\theta)$.*

Then $F(z) = \sum_{n=0}^{\infty} a_n(2iz)^n$, $z = Re^{i\theta}$, *is an entire function of order one and finite exponential type* C. *If* $\mathfrak{S}$ *is the conjugate indicator diagram of* $F(z)$ *then* $\tilde{u}(R,\theta)$ *is regular in* $R > C$ *and also in any circle* Ω *orthogonal to the circle* $R = C$ *such that* Ω *does not intersect* $\mathfrak{S} \cup \bar{\mathfrak{S}} \cup \{(R,\theta) \mid \theta = 0, \pi\}$.

In passing we note the existence of an integral operator T by which $F(z)$ can be represented directly as $F(z) = Tf$ [see (**10**)].

Weston, Bowman, and Ar in (**14**) have considered the analytic continuation of solutions to the inverse scattering problem for the vector Helmholtz equation. In their work the scattering body D is given *a priori* and assumed to have an analytic boundary.

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8.—Constructive Methods for Solving the Exterior Neumann Problem for the Reduced Wave Equation in a Spherically Symmetric Medium.*

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(MS received 7 July 1975. Read 1 December 1975)

Synopsis

An integral operator is constructed which maps solutions of the reduced wave equation defined in exterior domains onto solutions of $\Delta_n u+\lambda^2(1+B(r))u = 0$ (*) defined in exterior domains, where $B(r)$ is a continuously differentiable function of compact support. This operator is then used to construct a solution to the exterior Neumann problem for (*) satisfying the Sommerfeld radiation condition at infinity. Such problems arise in connection with the scattering of acoustic waves in a non-homogeneous medium, and this paper gives a method for solving these problems which is suitable for analytic and numerical approximations.

1. Introduction

The mathematical problem we are about to consider has its origin in the following problem connected with the scattering of acoustic waves in a non-homogeneous medium. Let an incoming plane acoustic wave of frequency ω moving in the direction of the z-axis be scattered off a bounded obstacle D which is surrounded by a pocket of rarefied or condensed air in which the local speed of sound is given by $c(r)$, where $r = |q|$ for $q \in R^3$. Assume that this pocket of air is contained in a ball of radius a and that for $r \geqq a$ we have $c(r) = c_0 =$ constant. Let $U(q)$ be the velocity potential (factoring out a term of the form $\exp(i\omega t)$) and set $B(r) = \left(\frac{c_0}{c(r)}\right)^2 - 1$. Then assuming $|\nabla c(r)| << \lambda c(r)$ where $\lambda = \frac{\omega}{c_0}$ we are led to the following boundary value problem, where $u_s(q)$ is the velocity potential of the scattered wave, and ν denotes the outward normal to ∂D:

$$U(q) = \exp(i\lambda z)+u_s(q) \tag{1.1}$$

$$\Delta_3 U+\lambda^2(1+B(r))U = 0 \quad \text{in } R^3 \bar{D} \tag{1.2}$$

$$\frac{\partial U}{\partial \nu} = 0 \quad \text{on } \partial D \tag{1.3}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u_s}{\partial r} - i\lambda u_s\right) = 0, \tag{1.4}$$

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where (1.4) is assumed to hold uniformly in all directions. Now let $u_s(q) = v(q)+u(q)$ where $v(q) \in C^2(R^3\backslash\bar{D}) \cap C^1(R^3\backslash D)$ is such that $\exp(i\lambda z)+v(q)$ is a solution of (1.2) in $R^3\backslash\bar{D}$ and $v(q)$ satisfies (1.4). If such a function $v(q)$ can be found, then the boundary value problem (1.1–1.4) for $U(q)$ can be reduced to the following boundary value problem for $u(q)$:

$$\Delta_3 u + \lambda^2(1+B(r))u = 0 \quad \text{in } R^3\backslash\bar{D} \tag{1.5}$$

$$\frac{\partial u}{\partial \nu} = f(q) \quad \text{on } \partial D \tag{1.6}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - i\lambda u\right) = 0, \tag{1.7}$$

where $f(q) = -\dfrac{\partial}{\partial \nu}(\exp(i\lambda z)+v(q))$.

The existence of the functions $u(q)$ and $v(q)$ described above (and the uniqueness of the function $u_s(q)$) follows from the results of Werner [9], Leis [6] and Jaeger [4]. The purpose of this paper is to give a constructive method for finding $u(q)$ and $v(q)$ in a manner which is suitable for obtaining analytic and numerical approximations. The approach which we are about to develop is based on the use of integral operators for partial differential equations, and is motivated by the work of Vekua [8] and Gilbert [3] on the use of integral operators to solve interior boundary value problems for partial differential equations with variable coefficients. However the integral operators constructed by Vekua and Gilbert are not suitable for our purposes since they are defined only for solutions which are regular in interior domains, and we are now concerned with solutions of (1.5) which are defined in exterior domains and satisfy the Sommerfeld radiation condition (1.7) at infinity. This difficulty will be overcome through the construction of an integral operator which maps solutions of (1.5) with $B(r) = 0$ onto solutions of (1.1) with $B(r) \neq 0$ such that (1.7) is satisfied. The discovery of such an operator initiates the theory of integral operators in exterior domains, and we hope to develop this theory more completely in future work.

2. An Integral Operator

In this section we will construct the integral operator mentioned in the introduction, and in the next section we will use this operator to obtain a constructive method for finding the functions $v(q)$ and $u(q)$. We make the assumption that $B(r)$ is a real valued continuously differentiable function of r with compact support contained in the interval $[0, a]$, where $a>0$, and that $u(q)$ is defined in the exterior of a bounded domain D containing the origin where D is strictly starlike with respect to the origin, i.e. if P is a point in $\bar{D}$ then the line segment $\overline{OP}$ is contained in D except for possibly the endpoint P. It turns out that the analysis of this section is essentially independent of the dimension of the space, and hence we consider the equation

$$\Delta_n u + \lambda^2(1+B(r))u = 0 \tag{2.1}$$

in place of (1.5) and latter on set $n = 3$.

We now look for a solution u of (2.1) defined in the exterior of D in the form

$$u(r, \theta) = h(r, \theta) + \int_r^\infty s^{n-3}K(r, s; \lambda)h(s, \theta)ds, \tag{2.2}$$

where $(r, \theta) = (r, \theta_1, \ldots, \theta_{n-1})$ are spherical coordinates, $h(r, \theta)$ is a solution of

$$\Delta_n h + \lambda^2 h = 0 \tag{2.3}$$

in the exterior of D, and $K(r, s; \lambda)$ is a function to be determined. We assume

$$K(r, s; \lambda) = 0 \quad \text{for } rs \geqq a^2 \tag{2.4}$$

and note that if $h(r, \theta)$ satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{(n-1)/2}\left(\frac{\partial u}{\partial r} - i\lambda u\right) = 0 \tag{2.5}$$

then by (2.4) so will $u(r, \theta)$. We now substitute (2.2) into (2.1) and integrate by parts using (2.4). The result of this calculation is that (2.2) will be a solution of (2.1) provided $K(r, s; \lambda)$ is a twice continuously differentiable solution of

$$r^2\left[K_{rr} + \frac{n-1}{r}K_r + \lambda^2(1+B(r))K\right] = s^2\left[K_{ss} + \frac{n-1}{s}K_s + \lambda^2 K\right] \tag{2.6}$$

for $s > r$ satisfying (2.4) and the initial condition

$$K(r, r; \lambda) = -\tfrac{1}{2}\lambda^2 r^{2-n}\int_r^\infty sB(s)ds. \tag{2.7}$$

Now let $\xi = \log r$, $\eta = \log s$, and define $M(\xi, \eta; \lambda)$ by

$$M(\xi, \eta; \lambda) = \exp\left[\left(\frac{n-2}{2}\right)(\xi+\eta)\right]K(\exp\xi, \exp\eta; \lambda), \tag{2.8}$$

i.e.

$$K(r, s; \lambda) = (rs)^{-\left(\frac{n-2}{2}\right)}M(\log r, \log s; \lambda). \tag{2.9}$$

Then $M(\xi, \eta; \lambda)$ satisfies the differential equation

$$M_{\xi\xi} - M_{\eta\eta} + \lambda^2(\exp 2\xi - \exp 2\eta + \exp 2\xi B(\exp\xi))M = 0 \tag{2.10}$$

for $\eta > \xi$ and the auxilliary conditions

$$M(\xi, \xi; \lambda) = -\tfrac{1}{2}\lambda^2\int_\xi^\infty \exp(2\tau)B(\exp\tau)d\tau \tag{2.11}$$

$$M(\xi, \eta; \lambda) = 0 \quad \text{for } \tfrac{1}{2}(\xi+\eta) \geqq \log a. \tag{2.12}$$

We assume that in addition to (2.10)–(2.12),

$$M(\xi, \eta; \lambda) = 0 \quad \text{for } \xi > \eta. \tag{2.13}$$

Note that $M(\xi, \eta; \lambda)$, if it exists, is independent of the dimension n, and in this sense the operator (2.2) resembles the 'method of ascent' of Gilbert [3] and Eichler [2] for elliptic equations defined in interior domains.

We now proceed to construct a solution of (2.10)–(2.13). Our approach is based on the ideas of Levitan [7]. Let

$$\begin{aligned} x &= \tfrac{1}{2}(\xi+\eta) \\ y &= \tfrac{1}{2}(\xi-\eta) \end{aligned} \tag{2.14}$$

and define $\tilde{M}(x, y; \lambda)$ by

$$\tilde{M}(x, y; \lambda) = M(x+y, x-y; \lambda). \tag{2.15}$$

Then $\tilde{M}(x, y; \lambda)$ satisfies

$$\tilde{M}_{xy} - \lambda^2 F(x+y, x-y)\tilde{M} = 0; \ y<0 \tag{2.16}$$

$$\tilde{M}(x, 0; \lambda) = -\tfrac{1}{2}\lambda^2 \int_x^\infty \exp(2\tau) B(\exp \tau) d\tau \tag{2.17}$$

$$\tilde{M}(x, y; \lambda) = 0 \quad \text{for } x \geqq \log a \tag{2.18}$$

$$\tilde{M}(x, y; \lambda) = 0 \quad \text{for } y>0, \tag{2.19}$$

where in (2.16)

$$F(\xi, \eta) = -[\exp 2\xi - \exp 2\eta + \exp 2\xi B(\exp \xi)]. \tag{2.20}$$

For $y \leqq 0$, (2.16–2.18) imply that $\tilde{M}(x, y; \lambda)$ is the solution of the integral equation

$$\tilde{M}(x, y; \lambda) = -\tfrac{1}{2}\lambda^2 \int_x^\infty \exp(2\tau) B(\exp \tau) d\tau - \lambda^2 \int_y^\infty \int_x^\infty F(\alpha+\beta, \alpha-\beta)\tilde{M}(\alpha, \beta; \lambda) d\alpha d\beta. \tag{2.21}$$

Note that (2.19) implies that the solution of the integral equation (2.21) satisfies the initial condition (2.17, 2.18 and 2.19), and the fact that $B(r)$ has compact support guarantee the existence of the integrals appearing in (2.21). Now in (2.21) make the change of variables

$$\begin{aligned} \alpha &= \tfrac{1}{2}(\tau+\mu) \\ \beta &= \tfrac{1}{2}(\tau-\mu). \end{aligned} \tag{2.22}$$

Then (2.21) becomes

$$\begin{aligned} M(\xi, \eta; \lambda) = &-\tfrac{1}{2}\lambda^2 \int_{\frac{1}{2}(\xi+\eta)}^\infty \exp(2\tau) B(\exp \tau) d\tau \\ &-\tfrac{1}{2}\lambda^2 \int_\xi^\infty \int_{\eta+\xi-\tau}^{\eta+\tau-\xi} F(\tau, \mu) M(\tau, \mu; \lambda) d\mu d\tau. \end{aligned} \tag{2.23}$$

Now note that in (2.23) if $\eta+\xi-\tau>\tau$, then $\mu>\tau$, and hence $M(\tau, \mu; \lambda)$ is not identically zero. On the other hand, if $\eta+\xi-\tau<\tau$, then μ may be less than τ, and in such cases $M(\tau, \mu; \lambda) = 0$. Taking these facts into consideration we have that, for $\eta>\xi$, $M(\xi, \eta; \lambda)$ is the solution of the integral equation

$$\begin{aligned} M(\xi, \eta; \lambda) = &-\tfrac{1}{2}\lambda^2 \int_{\frac{1}{2}(\xi+\eta)}^\infty \exp(2\tau) B(\exp \tau) d\tau \\ &-\tfrac{1}{2}\lambda^2 \int_\xi^{\frac{1}{2}(\xi+\eta)} \int_{\eta+\xi-\tau}^{\eta+\tau-\xi} F(\tau, \mu) M(\tau, \mu; \lambda) d\mu d\tau \\ &-\tfrac{1}{2}\lambda^2 \int_{\frac{1}{2}(\xi+\eta)}^\infty \int_\tau^{\eta+\tau-\xi} F(\tau, \mu) M(\tau, \mu; \lambda) d\mu d\tau. \end{aligned} \tag{2.24}$$

We now want to solve (2.24) through the method of successive approximations. We look for a solution of (2.24) in the form

$$M(\xi, \eta; \lambda) = \sum_{j=0}^\infty M_j(\xi, \eta; \lambda), \tag{2.25}$$

where

$$M_0(\xi, \eta; \lambda) = -\tfrac{1}{2}\lambda^2 \int_{\frac{1}{2}(\xi+\eta)}^{\log a} \exp(2\tau) B(\exp \tau) d\tau$$

$$\begin{aligned} M_j(\xi, \eta; \lambda) = &-\tfrac{1}{2}\lambda^2 \int_{\xi}^{\frac{1}{2}(\xi+\eta)} \int_{\eta+\xi-\tau}^{\eta+\tau-\xi} F(\tau, \mu) M_{j-1}(\tau, \mu; \lambda) d\mu d\tau \\ &-\tfrac{1}{2}\lambda^2 \int_{\frac{1}{2}(\xi+\eta)}^{\log a} \int_{\tau}^{\eta+\tau-\xi} F(\tau, \mu) M_{j-1}(\tau, \mu; \lambda) d\mu d\tau. \end{aligned} \tag{2.26}$$

Note that the region of integration in (2.26) is only in the half-space $\frac{1}{2}(\xi+\eta) \leqq \log a$ since $M_0(\xi, \eta; \lambda) = 0$ for $\frac{1}{2}(\xi+\eta) \geqq \log a$ and this implies that for $\frac{1}{2}(\xi+\eta) \geqq \log a$, $M_j(\xi, \eta; \lambda) = 0$ for each j. Assume $\eta \geqq \xi \geqq -\xi_0$ is a positive constant, and let

$$C = \tfrac{1}{2} |\lambda|^2 \max_{\substack{-\xi_0 \leqq \xi \leqq \log a \\ -\xi_0 \leqq \eta \leqq \xi_0 + \log a}} \{\exp(2\xi) |B(\exp \xi)|, |F(\xi, \eta)|\}. \tag{2.27}$$

Then for $\eta \geqq \xi \geqq -\xi_0$, $\frac{1}{2}(\xi+\eta) \leqq \log a$, we have

$$|M_0(\xi, \eta; \lambda)| \leqq C(\log a - \tfrac{1}{2}(\xi+\eta)) \leqq C(\log a - \xi) \tag{2.28}$$

and

$$\begin{aligned} |M_1(\xi, \eta; \lambda)| \leqq 2C^2 &\int_{\xi}^{\frac{1}{2}(\xi+\eta)} (\log a - \tau)(\tau - \xi) d\tau \\ &+ C^2 \int_{\frac{1}{2}(\xi+\eta)}^{\log a} (\log a - \tau)(\eta - \xi) d\tau. \end{aligned}$$

But in the second integral on the right-hand side of (2.29) we have $\frac{1}{2}(\xi+\eta) \leqq \tau$, which implies $\eta \leqq 2\tau - \xi$, and hence $\eta - \xi \leqq 2(\tau - \xi)$. Therefore from (2.29) we have

$$|M_1(\xi, \eta; \lambda)| \leqq 2C^2 \int_{\xi}^{\log a} (\log a - \tau)(\tau - \xi) d\tau. \tag{2.30}$$

But for $j \geqq 0$ we have

$$\frac{1}{(2j+1)!} \int_{\xi}^{\log a} (\log a - \tau)^{2j+1} (\tau - \xi) d\tau = \frac{(\log a - \xi)^{2j+3}}{(2j+3)!}, \tag{2.31}$$

and hence

$$|M_1(\xi, \eta; \lambda)| \leqq \frac{2C^2}{3!} (\log a - \xi)^3. \tag{2.32}$$

By induction we have

$$\begin{aligned} |M_j(\xi, \eta; \lambda)| &\leqq \frac{2C^{j+1}}{(2j+1)!} (\log a - \xi)^{2j+1} \\ &\leqq \frac{2C^{j+1}}{(2j+1)!} (\log a + \xi_0)^{2j+1} \end{aligned} \tag{2.33}$$

for $j \geqq 0$, and hence the series (2.25) is absolutely and uniformly convergent for $\eta \geqq \xi \geqq -\xi_0$. This establishes the existence of the function $M(\xi, \eta; \lambda)$ and hence the kernel $K(r, s; \lambda)$. It is easily seen that since $B(r)$ is continuously differentiable,

$K(r, s; \lambda)$ is twice continuously differentiable for $s \geqq r>0$. We note that $M(\xi, \eta; \lambda)$ is an entire function of λ and that

$$\lambda^{-2-2j}M_j(\xi, \eta; \lambda) = N_j(\xi, \eta) \tag{2.34}$$

is independent of λ. In particular $s^{n-2}K(r, s; \lambda)$ has the Taylor expansion

$$s^{n-3}K(r, s; \lambda) = s^{(n-4)/2}r^{(2-n)/2} \sum_{j=9}^{\infty} \lambda^{2j+2}N_j(\log r, \log s),$$

which is uniformly convergent for all complex values of λ.

3. The Construction of v(q) and u(q)

We now want to use the integral operator (2.2) (for $n = 3$) to construct the functions $v(q)$ and $u(q)$ described in the introduction. Since $\exp(i\lambda z)$ is a solution of (2.3) we have that in spherical coordinates (r, θ, ϕ)

$$u(r, \theta; \phi) = \exp(i\lambda r \cos\theta) + \int_r^\infty K(r, s; \lambda) \exp(i\lambda s \cos\theta)ds \tag{3.1}$$

is a solution of (1.5), and from (2.4) it is clear that we can chose $v(q) = v(r, \theta, \phi)$ to be

$$v(r, \theta, \phi) = \int_r^\infty K(r, s; \lambda) \exp(i\lambda s \cos\theta)ds. \tag{3.2}$$

We now turn our attention to constructing a solution $u(q)$ of (1.5–1.7). To do this we will use the integral operator (2.2) in conjunction with the work of Jones [5] on the exterior Neumann problem for the Helmholtz equation (2.3). To describe the work of Jones, let $\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots$ be the eigenvalues of the interior Dirichlet problem for (2.3) in D (for $n = 3$). Then in [5] Jones has shown that if $\lambda<\lambda_{M+2}$ and $h(q)$ is a solution of (2.3) (for $n = 3$) satisfying prescribed Neumann data on ∂D and the Sommerfeld radiation condition (1.7) at infinity, there exists a continuous density $\psi(p)$ such that $h(q)$ can be represented in the form

$$h(q) = \int_{\partial D} \psi(p)\Gamma(p, q; \lambda)d\omega_p, \tag{3.3}$$

where

$$\Gamma(p, q; \lambda) = \frac{\exp(i\lambda R)}{R} + \sum_{m=0}^{M} \sum_{n=-m}^{m} b_{mn}\psi_{mn}(p)\psi_{mn}(q), \tag{3.4}$$

$R = |p-q|$, $d\omega_p$ is the element of surface area at the point $p \in \partial D$, the b_{mn} are non-zero real constants (arbitrary, but fixed), and

$$\psi_{mn}(p) = h_m^{(1)}(\lambda |p|)S_{mn}\left(\frac{p}{|p|}\right), \tag{3.5}$$

where $h_m^{(1)}$ denotes a spherical Hankel function and S_{mn} a spherical harmonic. Jones has further shown that for a given λ a suitable value of M can be chosen as follows: Let $\mu_1, \ldots, \mu_j, \ldots$ be the eigenvalues of the interior Dirchlet problem for (2.3) (for $n = 3$) in the unit sphere (which can be computed from a knowledge of the zeros of the spherical Bessel functions) and let r_0 be the radius of the smallest sphere contained in D and r_1 the radius of the largest sphere containing D. Then

$$\frac{\mu_j}{r_0} \geqq \lambda_j \geqq \frac{\mu_j}{r_1}. \tag{3.6}$$

In order to construct a solution $u(q)$ of (1.3–1.7) we will look for a solution in the form

$$u(q) = h(q) + \int_{|q|}^{\infty} K(|q|, s; \lambda) h\left(s \frac{q}{|q|}\right) ds, \tag{3.7}$$

where $h(q)$ is a solution of (2.3) (for $n = 3$) having the representation (3.3) in terms of an unknown continuous density $\psi(p)$ to be determined. Note that $h(q)$, and hence $u(q)$, satisfies the Sommerfeld radiation condition (1.7). Substituting (3.3) into (3.7) and interchanging the orders of integration gives

$$u(q) = \int_{\partial D} \psi(p)\Gamma(p, q; \lambda) d\omega_p + \int_{\partial D} \psi(p) \left\{ \int_{|q|}^{\infty} K(|q|, s; \lambda) \Gamma\left(p, s \frac{q}{|q|}; \lambda\right) ds \right\} d\omega_p. \tag{3.8}$$

We will show shortly that for p, q on ∂D

$$\left| \frac{\partial}{\partial \nu_q} \left\{ \int_{|q|}^{\infty} K(|q|, s; \lambda) \Gamma\left(p, s \frac{q}{|q|}; \lambda\right) ds \right\} \right| \leqq \frac{\text{constant}}{|p-q|}, \tag{3.9}$$

where $\frac{\partial}{\partial \nu_q}$ denotes differentiation with respect to q in the direction of the outward normal at q. Assuming this fact for the time being, we let $q \in \partial D$, evaluate (3.8) at $q' \in R^3 \backslash \bar{D}$, and apply the operator $\nu_q . \nabla$ to both sides of (3.8). Letting q' tend to q and using (3.9) and the discontinuity properties of the derivatures of single layer potentials, we arrive at the following integral equation for $\psi(q)$:

$$\begin{aligned} -\frac{1}{2\pi} f(q) = \psi(q) &- \frac{1}{2\pi} \int_{\partial D} \psi(p) \frac{\partial}{\partial \nu_q} \Gamma(p, q; \lambda) d\omega_p \\ &- \frac{1}{2\pi} \int_{\partial D} \psi(p) \frac{\partial}{\partial \nu_q} \left\{ \int_{|q|}^{\infty} K(|q|, s; \lambda) \Gamma\left(p, s \frac{q}{|q|}; \lambda\right) ds \right\} d\omega_p \\ &\stackrel{\text{def}}{=} (I - T(\lambda))\psi. \end{aligned} \tag{3.10}$$

We now show that, under the assumption that D is strictly starlike, the estimate (3.9) is valid, and hence the derivation of (3.10) is correct. We first observe that from (2.4) and the facts that $K(|q|, s; \lambda)$ and $\psi_{mn}(p)\psi_{mn}(q)$ are twice continuously differentiable for $p, q \in \partial D$, there exists a positive constant C such that for $p, q \in \partial D$

$$\left| \frac{\partial}{\partial \nu_q} \left\{ \int_{|q|}^{\infty} K(|q|, s; \lambda) \Gamma\left(p, s \frac{q}{|q|}; \lambda\right) ds \right\} \right| \leqq C \int_{|q|}^{\infty} \left| p - s \frac{q}{|q|} \right|^{-2} ds. \tag{3.11}$$

Hence we now examine the function $\left| p - s \frac{q}{|q|} \right|$ for p and q on ∂D and $s \geqq |q|$.

In case $p \cdot q < 0$, the expression

$$\left| p - s \frac{q}{|q|} \right|^2 = |p|^2 + s^2 - 2s \frac{q \cdot p}{|q|}, \quad s \geqq |q|, \tag{3.12}$$

has its minimum at $s = |q|$, hence we observe that either

$$\left| p - s \frac{q}{|q|} \right| \geqq |p - q| \tag{3.13}$$

for $s \geqq |q|$, or there exists an $s_0 > |q|$ such that

$$\left| p - s\frac{q}{|q|} \right|^2 \geqq \left| p - s_0 \frac{q}{|q|} \right|^2, \tag{3.14}$$

where

$$\left(p - s_0 \frac{q}{|q|} \right) \cdot q = 0 \quad \text{and} \quad p \cdot q \geqq 0. \tag{3.15}$$

In the second case we have from (3.15) that $s_0 |q| = p \cdot q$ and hence

$$\begin{aligned} p - s_0 \frac{q}{|q|} &= p - q \frac{p \cdot q}{|q|^2} \\ &= p - q - q \frac{(p-q)\cdot q}{|q|^2}. \end{aligned} \tag{3.16}$$

Therefore

$$\left| p - s_0 \frac{q}{|q|} \right|^2 = |p-q|^2 - \frac{((p-q)\cdot q)^2}{|q|^2}. \tag{3.17}$$

Since D is strictly starlike, there exists a positive constant $\alpha < 1$ which is independent of p and q such that

$$|(p-q)\cdot q| \leqq \alpha |p-q| |q| \tag{3.18}$$

uniformly for all $p, q \in \partial D$ with $p \cdot q \geqq 0$.

Hence from (3.17) we have

$$\left| p - s_0 \frac{q}{|q|} \right|^2 \geqq (1-\alpha^2)|p-q|^2 \tag{3.19}$$

uniformly for all $p, q \in \partial D$. We now return to (3.13 and 3.14). If (3.13) is valid, then

$$\begin{aligned} \left| p - s\frac{q}{|q|} \right|^2 &= |p-q|^2 + \frac{2}{|q|}(s-|q|)(q-p)\cdot q + (s-|q|)^2 \\ &\geqq |p-q|^2 + (s-|q|)^2, \end{aligned} \tag{3.20}$$

whereas in the case of (3.14), (3.15) we have

$$\begin{aligned} \left| p - s\frac{q}{|q|} \right|^2 &= \left| p - s_0 \frac{q}{|q|} \right|^2 + (s-s_0)^2 \\ &\geqq (1-\alpha^2)|p-q|^2 + (s-s_0)^2. \end{aligned} \tag{3.21}$$

We can now conclude from the above discussion that for $p, q \in \partial D$

$$\begin{aligned} \int_{|q|}^{\infty} \left| p - s\frac{q}{|q|} \right|^{-2} ds &\leqq \frac{1}{(1-\alpha^2)} \int_{|q|}^{\infty} \frac{ds}{|p-q|^2 + (s-s_1)^2} \\ &= \frac{1}{(1-\alpha^2)|p-q|} \arctan\left(\frac{s-s_1}{|p-q|} \right)\Bigg|_{s=|q|}^{s=\infty} \\ &\leqq \frac{4\pi}{(1-\alpha^2)|p-q|}, \end{aligned} \tag{3.22}$$

where $s_1 = |q|$ in the case of (3.13) and $s_1 = s_0$ in the case of (3.14): The estimate (3.9) now follows from (3.11 and 3.22).

A constructive method for determining the desired function $u(q)$ can now be obtained if we can show that the Fredholm integral equation (3.10) can be uniquely solved for the unknown density $\psi(p)$, i.e. that the operator $I-T(\lambda)$ is invertible. We will accomplish this by proving two theorems. The first theorem below proceeds along classical lines [cf. **8**, **9**] except for the conclusion, where we make use of the integral operator (3.7).

THEOREM 1. *Let* $\lambda>0$ *and let* $u(q)\in C^2(R^3\backslash\bar{D})\cap C^1(R^3\backslash D)$ *be a solution of* (1.5) *in the exterior of* D *satisfying the Sommerfeld radiation condition* (1.7) *at infinity and the boundary condition* $\dfrac{\partial u}{\partial \nu}=0$ *on* ∂D. *Then* $u(q)\equiv 0$ *for* $q\in R^3\backslash D$.

Remark. The same Theorem holds for the Dirichlet problem and can be proved in the same way.

Proof. Let Ω *be a ball of radius* $r>a$, where $B(r)=0$ for $r\geqq a$. Then from Green's formula we have

$$\iint_{\Omega\backslash D}(u\Delta\bar{u}-\bar{u}\Delta u)dV=\int_{\partial D}\left(\bar{u}\frac{\partial u}{\partial\nu}-u\frac{\partial\bar{u}}{\partial\nu}\right)d\omega-\int_{\partial\Omega}\left(\bar{u}\frac{\partial u}{\partial r}-u\frac{\partial\bar{u}}{\partial r}\right)d\omega, \tag{3.23}$$

where dV denotes an element of volume and $d\omega$ an element of surface area. Since λ and $B(r)$ are real and $\dfrac{\partial u}{\partial\nu}=\dfrac{\partial\bar{u}}{\partial\nu}=0$ on ∂D, we have from (3.23) that

$$\int_{\partial\Omega}\left(\bar{u}\frac{\partial u}{\partial r}-u\frac{\partial\bar{u}}{\partial r}\right)d\omega=0. \tag{3.24}$$

But, for $r>a$, $u(q)$ is a solution of $\Delta_3 h+\lambda^2 h=0$ satisfying the Sommerfeld radiation condition (1.7), and hence for $r>a$

$$u(q)=\sum_{m=0}^{\infty}\sum_{n=-m}^{m}a_{mn}h_m^{(1)}(\lambda\mid q\mid)S_{mn}\left(\frac{q}{\mid q\mid}\right), \tag{3.25}$$

where the series converges absolutely and uniformly. By the orthogonality of the functions $S_{mn}\left(\dfrac{q}{\mid q\mid}\right)$ and the formula

$$\overline{h_m^{(1)}(\lambda r)}\frac{d}{dr}h_m^{(1)}(\lambda r)-h_m^{(1)}(\lambda r)\frac{d}{dr}\overline{h_m^{(1)}(\lambda r)}=\frac{4i}{\pi\lambda^2r^2} \tag{3.26}$$

we have from (3.24) and (3.25) that

$$\sum_{m=0}^{\infty}\sum_{n=-m}^{m}\mid a_{mn}\mid^2=0, \tag{3.27}$$

which implies that $u(q)=0$ for $r>a$. Let $h(q)$ be the solution of $\Delta_3 h+\lambda^2 h=0$ associated with $u(q)$ by the integral operator (3.7). Then from (2.4) and the fact that $h(q)$ can be determined from $u(q)$ by inverting an integral equation of Volterra type (which implies that $h(q)$ has the same smoothness properties that $u(q)$ does), we can conclude that $h(q)\in C^2(R^3\backslash\bar{D})\cap C^1(R^3\backslash D)$ and $h(q)=0$ for $r>a$. Hence, since twice continuously differentiable solutions of the Helmholtz equation are analytic functions of their independent variables, we can conclude that $h(q)=0$ for $q\in R^3\backslash D$, and hence from (3.7) $u(q)=0$ for $q\in R^3\backslash D$.

We can now establish the following result on the invertibility of the Fredholm operator $I - T(\lambda)$:

THEOREM 2. *Let $\lambda > 0$ and define the operator $T_0(\lambda)$ by*

$$T_0(\lambda)\psi = \frac{1}{2\pi}\int_{\partial D} \psi(p)\frac{\partial}{\partial \nu_q}\Gamma(p, q; \lambda)d\omega_p; \; q \in \partial D.$$

Then $(I-T(\lambda))^{-1}$ exists if and only if $(I-T_0(\lambda))^{-1}$ exists, where all mappings are understood to be in the space C^0.

Proof. Since $T(\lambda)$ and $T_0(\lambda)$ are integral operators with weakly singular kernels the Fredholm alternative is valid. Now let ψ be a solution of $(I-T(\lambda))\psi = 0$. Then the potential defined by (3.3) generates by (3.7) a solution $u(q)$ of (1.5) in the exterior of D such that $u(q)$ satisfies the Sommerfeld radiation condition, and, since

$$(I-T(\lambda))\psi = 0,$$

we have $\dfrac{\partial u}{\partial \nu} = 0$ for $q \in \partial D$. From Theorem 1 we can now conclude that $u(q) = 0$ in the exterior of D. By inverting the Volterra equation (3.7) we can conclude that $h(q) = 0$ in the exterior of D and hence $(I-T_0(\lambda))\psi = 0$ for $q \in \partial D$. If $(I-T_0(\lambda))^{-1}$ exists, then we can conclude that $\psi(p) = 0$, and hence by the Fredholm alternative $(I-T(\lambda))^{-1}$ exists.

Conversely, if ψ is a solution of $(I-T_0(\lambda))\psi = 0$, then $h(q)$ as defined by (3.3) is zero for $q \in R^3 \backslash D$ and hence from (3.7) $u(q) = 0$ for $q \in R^3 \backslash D$. Then $\dfrac{\partial u}{\partial \nu} = 0$ for $q \in \partial D$ and $(I-T(\lambda))\psi = 0$. Hence if $(I-T(\lambda))^{-1}$ exists we can conclude that $\psi(p) = 0$ and it follows from the Fredholm alternative that $(I-T_0(\lambda))^{-1}$ exists.

From the previously described work of Jones we now have the following Corollary:

COROLLARY. *Let M be such that $\lambda < \lambda_{M+2}$, where λ_j denotes the jth eigenvalue for the interior Dirichlet problem for $\Delta_3 h + \lambda^2 h = 0$ in D. Then $(I-T(\lambda))^{-1}$ exists.*

Given the fact that $(I-T(\lambda))^{-1}$ exists, we can now use any one of a variety of methods for obtaining numerical approximations to the density $\psi(p)$ [cf. **1**]. We also note that all our conclusions remain valid in any function space on ∂D where the weakly singular integral operators define completely continuous mappings such that the spectrum does not change. For example, under the hypothesis that ∂D is twice continuously differentiable, this is true in all the spaces $C^0 \subset L^\infty \subset L^p \subset L^1$, $1 < p < \infty$.

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The Scattering of Acoustic Waves by a Spherically Stratified Inhomogeneous Medium*

by

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Abstract

Integral operators are used to solve the direct and inverse problems of the scattering of acoustic waves by a spherically stratified inhomogeneous medium of compact support. The results are valid for all values of the wave number and an arbitrarily large index of refraction. In the limiting case of small wave number or small inhomogeneities the results are in agreement with those of Rorres and Born.

I. Introduction.

In previous work ([1], [2], [3]) the author has used the theory of integral operators to study the problems of the scattering of acoustic waves by a bounded obstacle situated in a spherically stratified medium. In this paper we shall continue our development of the use of integral operators in scattering theory, and consider the problem of the scattering of acoustic waves by a spherically stratified inhomogeneous medium in the absence of an obstacle. The integral operator used in [2] and [3] is no longer applicable to this new problem and instead we shall make use of R.P. Gilbert's "method of ascent" ([1], [6]) in order to construct the desired solution. Our results are valid for arbitrarily large wave numbers as well as for arbitrarily large inhomogeneities, provided the inhomogeneities are confined to a bounded region of space. For small wave numbers our results are in agreement with Rorres ([8]), whereas for small inhomogeneities they are in agreement with the Born approximation (c.f. [7]). Thus our results show that in the special case of scattering by a spherically stratified inhomogeneity of compact support the methods of Rorres and Born are in fact uniformly valid.

The scattering of a plane wave (moving in the direction of z axis) by a spherically stratified inhomogeneity of compact support is described by the following system of equations for the unknown velocity potential $u(\underset{\sim}{x})$ (where we have factored out a term of the form $e^{i\omega t}$):

$$\Delta_3 u + k^2 B(r) u = 0 \qquad \text{in } \mathbb{R}^3 \tag{1.1}$$

$$u(\underset{\sim}{x}) = e^{ikz} + u_s(\underset{\sim}{x}) \tag{1.2}$$

$$\lim_{r\to\infty} r \left(\frac{\partial u_s}{\partial r} - iku_s \right) = 0 \tag{1.3}$$

where k is the wave number, $r = |\underset{\sim}{x}|$, $B(r)$ is the index of refraction, and $u_s(\underset{\sim}{x})$ is the velocity potential of the scattered wave. We make the assumption that $B(r) \in C^2(\mathbb{R}^3)$ and $B(r) \equiv 1$ for $r \geqslant 1$ (i.e. the problem has been normalized such that the inhomogeneities lie inside the unit ball). Condition (1.3) is the Sommerfeld radiation condition for outward scattering. We shall consider two problems associated with the system (1.1)-(1.3):

Direct Problem: Given $B(r)$, find $u(\underset{\sim}{x})$.

Inverse Problem: Given the far field pattern

$$f(\theta;k) = \lim_{r\to\infty} re^{-ikr}u_s(\underset{\sim}{x}) \tag{1.4}$$

for $0 < k_o < k < k_1$ where k_o and k_1 are fixed constants, find $B(r)$ for $0 \leqslant r \leqslant 1$.

We shall show in the next two sections how both of these problems can be solved through the use of the theory of integral operators for partial differential equations (c.f. [1], [6]).

II. The Direct Problem.

We begin our analysis by observing that every solution of (1.1) regular in a neighbourhood of the origin can be represented in the form ([6])

$$u(\underset{\sim}{x}) = h(\underset{\sim}{x}) - 2r\int_0^1 \sigma^2 R_3(r,r;r\sigma^2,0)h(\underset{\sim}{x}\sigma^2)d\sigma \tag{2.1}$$

where $h(\underset{\sim}{x})$ is a harmonic function regular in a neighbourhood of the origin, $R(x,y;\xi,\eta)$ is the Riemann function for

$$u_{xy} + \frac{k^2}{4} B(\sqrt{xy})u = 0, \tag{2.2}$$

and the subscript denotes differentiation with respect to ξ. Hence for $r \leqslant 1$ we can represent $u(\underset{\sim}{x})$ in the form

$$u(\underset{\sim}{x}) = \sum_{n=0}^{\infty} b_n u_n(r) P_n(\cos\theta); \; r < 1 \tag{2.3}$$

where

$$u_n(r) = r^n \left[1-2r \int_0^1 \sigma^{2n+2} R_3(r,r;r\sigma^2,0)\,d\sigma \right] \tag{2.4}$$

and $P_n(\cos\theta)$ denotes Legendre's polynomial. Since $B(r) \equiv 1$ for $r \geqslant 1$ we have, using Sonine's formula to expand e^{ikz},

$$\begin{aligned} u(\underset{\sim}{x}) &= \sqrt{\frac{\pi}{2kr}} \sum_{n=0}^{\infty} a_n H^{(1)}_{n+\frac{1}{2}}(kr) P_n(\cos\theta) + e^{ikz} \\ &= \sqrt{\frac{\pi}{2kr}} \sum_{n=0}^{\infty} \left[a_n H^{(1)}_{n+\frac{1}{2}}(kr) + (2n+1) i^n J_{n+\frac{1}{2}}(kr) \right] P_n(\cos\theta) \; ; \; r \geqslant 1 \end{aligned} \tag{2.5}$$

for $r \geqslant 1$, where $H^{(1)}_{n+\frac{1}{2}}(kr)$ denotes a Hankel function of the first kind and $J_{n+\frac{1}{2}}((kr)$ a Bessel function. In (2.3) and (2.5) the constants a_n and b_n, $n=0,1,2,\ldots$, are to be determined. We now require that the representations (2.3) and (2.5) agree, along with their first derivatives, at $r = 1$. This implies that $u(x)$ will be a weak solution of (1.1) in $\mathbb{R}^3$ and hence, since $B(r) \in C^2(\mathbb{R}^3)$, $u(\underset{\sim}{x})$ is in fact a classical solution of (1.1) in $\mathbb{R}^3$ (c.f. [5], p.56). These considerations lead to the following system of algebraic equations for the determination of a_n and b_n :

$$\begin{aligned} & b_n \left[1-2 \int_0^1 \sigma^{2n+2} R_3(1,1;\sigma^2,0)\,d\sigma \right] - a_n \sqrt{\frac{\pi}{2k}}\, H^{(1)}_{n+\frac{1}{2}}(k) \\ &= \sqrt{\frac{\pi}{2k}} (2n+1) i^n J_{n+\frac{1}{2}}(k) \end{aligned} \tag{2.6}$$

$$b_n \frac{d}{dr}\Big[r^n - 2r^{n+1}\int_0^1 \sigma^{2n+2} R_3(r,r;r\sigma^2 0)\,d\sigma\Big]_{r=1} - a_n\sqrt{\frac{\pi}{2}}\,\frac{d}{dr}\Big[\sqrt{\frac{1}{kr}}\,H^{(1)}_{n+\frac{1}{2}}(kr)\Big]_{r=1}$$

$$= (2n+1)i^n \sqrt{\frac{\pi}{2}}\,\frac{d}{dr}\Big[\sqrt{\frac{1}{kr}}\,J_{n+\frac{1}{2}}(kr)\Big]_{r=1} \quad .$$

We note that it follows from the uniqueness of the solution to (1.1)-(1.3) that the system (2.6) always has a unique solution, i.e. the determinant of the coefficients of a_n and b_n is non-zero (c.f. [4], chapter 5. Although the results in [4] are derived for the case of potential scattering, they are also valid for the case of acoustic scattering since the asymptotic estimates are for r tending to infinity with k fixed). Using Cramer's rule to solve (2.6) for a_n and b_n and the asymptotic estimates

$$\sqrt{\frac{\pi}{2k}}\,H^{(1)}_{n+\frac{1}{2}}(k) = \frac{i\Gamma(n+\frac{1}{2})2^n}{\sqrt{\pi}\,k^{n+1}}\Big[1+0\,(\tfrac{1}{n})\Big]$$

$$\frac{d}{dr}\Big[\sqrt{\frac{\pi}{2kr}}\,H^{(1)}_{n+\frac{1}{2}}(kr)\Big]_{r=1} = -\frac{i\Gamma(n+3/2)2^n}{\sqrt{\pi}\,k^{n+1}}\Big[1+0\,(\tfrac{1}{n})\Big]$$

$$(2n+1)i^n\sqrt{\frac{\pi}{2k}}\,J_{n+\frac{1}{2}}(k) = \frac{\sqrt{\pi}\,k^n i^n}{2^n\Gamma(n+\frac{1}{2})}\Big[1+0\,(\tfrac{1}{n})\Big]$$

$$(2n+1)i^n\sqrt{\frac{\pi}{2}}\,\frac{d}{dr}\Big[\sqrt{\frac{1}{kr}}\,J_{n+\frac{1}{2}}(kr)\Big]_{r=1} = \frac{\sqrt{\pi}\,k^n i^n}{2^n\Gamma(n-\frac{1}{2})}\Big[1+0\,(\tfrac{1}{n})\Big] \tag{2.7}$$

$$1 - 2\int_0^1 \sigma^{2n+2} R_3(1,1;\sigma^2,0)\,d\sigma = 1 + 0\,(\tfrac{1}{n})$$

$$\frac{d}{dr}\Big[r^n - 2r^{n+1}\int_0^1 \sigma^{2n+2} R_3(r,r;r\sigma^2,0)\,d\sigma\Big]_{r=1} = n\Big[1+0\,(\tfrac{1}{n})\Big]$$

shows that

$$a_n = O\left(\frac{k^{2n+1}}{2^{2n}\Gamma(n+\frac{1}{2})\Gamma(n+3/2)}\right)$$
$$b_n = O\left(\frac{k^n}{2^n\ \Gamma(n+\frac{1}{2})}\right) \quad . \tag{2.8}$$

We can now conclude that the series (2.3) and (2.5) are absolutely and uniformly convergent, and hence (2.3), (2.5) and (2.6) determine the (unique) solution of the direct scattering problem. It is of interest to compare the relatively short analysis above using the theory of integral operators with the more involved derivation of the same result (for small wave numbers!) by Rorres using the theory of integral equations ([8]). For yet another approach see [4], section 8.4.

III. The Inverse Problem.

We now consider the inverse scattering problem, i.e. from a knowledge of the far field pattern $f(\theta;k)$ to determine the index of refraction $B(r)$. From (1.4), (2.5) and the asymptotic estimate

$$H^{(1)}_{n+\frac{1}{2}}(kr) = (-i)^{n+1}\sqrt{\frac{2}{\pi kr}}\, e^{ikr}\left[1+O\left(\frac{1}{kr}\right)\right] \tag{3.1}$$

it can be easily shown that

$$f(\theta;k) = \frac{1}{k}\sum_{n=0}^{\infty}(-i)^{n+1}a_n P_n(\cos\theta). \tag{3.2}$$

Hence it can be assumed that $a_n = a_n(k)$ is known for $0 < k_o < k < k_1$. (2.6) now implies that

$$\left[\sqrt{\frac{\pi}{2k}}\,(2n+1)i^n J_{n+\frac{1}{2}}(k) + \sqrt{\frac{\pi}{2k}}\, a_n(k)H^{(1)}_{n+\frac{1}{2}}(k)\right] \cdot$$
$$\cdot \frac{d}{dr}\left[r^n - 2r^{n+1}\int_0^1 \sigma^{2n+2}R_3(r,r;r\sigma^2 0)\,d\sigma\right]_{r=1} \tag{3.3}$$

$$=\Big[(2n+1)i^n \sqrt{\frac{\pi}{2}}\ \frac{d}{dr}\Big(\sqrt{\frac{1}{kr}}\,J_{n+\frac{1}{2}}(kr)\Big)_{r=1} + a_n(k)\sqrt{\frac{\pi}{2}}\,\frac{d}{dr}\Big(\sqrt{\frac{1}{kr}}\,H^{(1)}_{n+\frac{1}{2}}(kr)\Big)_{r=1}\Big].$$

$$\cdot\Big[1-2\int_0^1 \sigma^{2n+2}R_3(1,1,\sigma^2,0)\,d\sigma\Big].$$

From (3.3), the expansions

$$\sqrt{\frac{1}{kr}}\,J_{n+\frac{1}{2}}(kr) = \sqrt{\frac{1}{2}}\sum_{m=0}^{\infty}(-1)^m\frac{(kr)^{2m+n}}{2^{2m+n}m!\,\Gamma(m+n+3/2)} \tag{3.4}$$

$$\sqrt{\frac{1}{kr}}\,H^{(1)}_{n+\frac{1}{2}}(kr) = \sqrt{\frac{1}{2}}\sum_{m=0}^{\infty}(-1)^m\Big[\frac{(kr)^{2m+n}}{2^{2m+n}m!\,\Gamma(m+n+3/2)} + i\,\frac{(-1)^{n+1}(kr)^{2m-n-1}}{2^{2m-n-1}m!\,\Gamma(m-n+\frac{1}{2})}\Big],$$

and the fact that the Riemann function depends analytically on the parameter k^2 we have that $a_n(k)$ must be analytic in a neighbourhood of the origin and have a Taylor expansion of the form

$$a_n(k) = a_{no}\,k^{2n+3} + a_{n1}\,k^{2n+5} + \ldots \quad . \tag{3.5}$$

We now recall that $R(x,y;\xi,\eta)$ is the (unique) solution of the integral equation

$$R(x,y;\xi,\eta) = 1 - \frac{k^2}{4}\int_\xi^x\int_\eta^y B(\sqrt{\sigma\tau})R(\sigma,\tau;\xi,\eta)\,d\sigma d\tau, \tag{3.6}$$

which implies that

$$\begin{aligned} R_3(1,1;\sigma^2 0) &= \frac{k^2}{4}\int_0^1 B(\sigma\sqrt{\xi})\,d\xi + O(k^4) \\ &= \frac{k^2}{2\sigma^2}\int_0^\sigma s\,B(s)\,ds + O(k^4) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \frac{d}{dr}\Big[R_3(r,r;r\sigma^2 0)\Big]_{r=1} &= \frac{k^2}{4}B(\sigma) + \frac{\sigma k^2}{8}\int_0^1\sqrt{\xi}\,\frac{d}{dr}B(\sigma\sqrt{\xi})\,d\xi + O(k^4) \\ &= \frac{k^2}{2}B(\sigma) - \frac{k^2}{4}\int_0^1 B(\sigma\sqrt{\xi})\,d\xi + O(k^4) \\ &= \frac{k^2}{2}B(\sigma) - \frac{k^2}{2\sigma^2}\int_0^\sigma s\,B(s)\,ds + O(k^4) \quad . \end{aligned} \tag{3.8}$$

Substituting (3.5), (3.7) and (3.8) into (3.3) and equating the coefficients of k^{n+2} gives

$$\frac{1}{2n+3} - a_{no} i^{n+1}(2n+1) \left(\frac{(2n)!}{2^n n!} \right)^2 = \int_0^1 \sigma^{2n+2} B(\sigma)\, d\sigma \qquad (3.9)$$

where we have used the identity

$$\frac{\Gamma(n+3/2)}{\Gamma(-n+\frac{1}{2})} = (n+\tfrac{1}{2})(-1)^n \left(\frac{(2n)!}{2^{2n} n!} \right)^2 . \qquad (3.10)$$

(3.9) is in agreement with the Born approximation ([7]) for $B(r) - 1$ small and with that of Rorres ([8]) for k small. Note however that our derivation makes no assumption on the magnitude of either $B(r)$ or k. We also observe that when $B(r) = 1$ (i.e. a homogeneous medium), $a_{no} = 0$ as to be expected.

Under the assumption that a_{no} is known (from the far field pattern) for $n = 0,1,2,\ldots$, (3.9) defines a moment problem for the determination of $B(r)$. $B(r)$ can now be determined in a variety of ways, for example by using (3.9) to compute the Fourier cosine transform of $r^2 B(r)$ (c.f. [8]) or by expanding $B(r)$ in the (complete) set obtained by orthonormalizing the functions $\{r^{2n+2}\}_{n=0}^{\infty}$ with respect to the inner product in $L^2[0,1]$ (c.f. [2]). We finally note that the determination of $B(r)$ from the far field pattern is an improperly posed problem in the sense that $B(r)$ does not depend continuously on the far field data $f(\theta;k)$. This can be seen from (3.9) where small variations in a_{no} can cause large variations in the integral on the right hand side if n is large.

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The Inverse Scattering Problem for Acoustic Waves in a Spherically Stratified Medium*

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I. Introduction.

The inverse problem we will consider in this paper has its origins in the following problem connected with the scattering of acoustic waves in a nonhomogeneous medium. Let an incoming plane acoustic wave of frequency ω moving in the direction of the z axis be scattered off a "soft" sphere Ω of radius one which is surrounded by a pocket of rarefied or condensed air in which the local speed of sound is given by $c(r)$ where $r = |\underset{\sim}{x}|$ for $\underset{\sim}{x} \in R^3$. Let $u_s(\underset{\sim}{x})e^{i\omega t}$ be the velocity potential of the scattered wave and let $r, \Theta, \emptyset$ be spherical coordinates in R^3. Then from a knowledge of the far field pattern $f(\Theta,\emptyset;\lambda)$ for $\lambda = \frac{\omega}{c_o}$ contained in some finite interval $0 < \lambda_o \leq \lambda \leq \lambda_1$, we would like to determine the unknown function $c(r)$. Under the assumptions that $\nabla c(r) << \lambda c(r)$ and $c(r) = c_o =$ constant for $r \geq a > 1$, we can formulate this problem mathematically as follows (c.f.[I]): Let $B(r) = \left(\frac{c_o}{c(r)}\right)^2 - 1$ and set $u_s(\underset{\sim}{x}) = v(\underset{\sim}{x}) + u(\underset{\sim}{x})$ where $u(\underset{\sim}{x})$ satisfies

$$\Delta_3 u + \lambda^2(1+B(r))u = 0 \quad \text{in} \quad R^3/\Omega \tag{1.1}$$

$$u(\underset{\sim}{x}) = -(e^{i\lambda z} + v(\underset{\sim}{x})) \quad \text{on} \quad \partial\Omega \tag{1.2}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - i\lambda u\right) = 0 \tag{1.3}$$

and $v(\underset{\sim}{x})$ is such that $e^{i\lambda z} + v(\underset{\sim}{x})$ is a solution of (1.1) in R^3/Ω where $v(\underset{\sim}{x}) = 0$ for $r \geq a$. Then given

$$f(\Theta,\emptyset;\lambda) = \lim_{r\to\infty} r\, e^{-i\lambda r} u(\underset{\sim}{x}) \tag{1.4}$$

we want to determine the function $B(r)$. The approach we will use in this paper is to use the theory of integral operators for partial differential equations in unbounded domaine as recently initiated by Colton and Wendland in [1] to reduce the inverse scattering problem described above to a generalized moment problem for the unknown function $B(r)$. In this sense our work has some relation to the work of Rorres, who solved the inverse scattering problem for small frequencies in the absence of an obstacle by reducing the inverse problem to a moment problem over a finite interval ([5]).

II. Integral Operators and the Inverse Scattering Problem.

Assume that $B(r)$ is continuously differentiable and vanishes for $r \geq a$, and let $h(r,\Theta,\emptyset)$ be a solution of

$$\Delta_3 h + \lambda^2 h = 0 \tag{2.1}$$

in R^3/Ω. Then in [1] it was shown that every solution of (1.1) in R^3/Ω can be represented in the form

$$\begin{aligned} u(r,\Theta,\emptyset) &= \underset{\sim}{K}[h] \\ &= h(r,\Theta,\emptyset) + \int_r^\infty K(r,s;\lambda)h(s,\Theta,\emptyset)ds \end{aligned} \tag{2.2}$$

where $h(r,\Theta,\emptyset)$ is a solution of (2.1) in R^3/Ω, and, for $1 \leq r \leq s < \infty$, $K(r,s;\lambda)$ can be represented in the form

$$K(r,s;\lambda) = (rs)^{-1/2} \sum_{j=o}^{\infty} \lambda^{2j+2} N_j \,(\log r,\, \log s) \tag{2.3}$$

with

$$N_o(\log r,\, \log s) = -\frac{1}{2} \int_{1/2 \log rs}^{\log a} e^{2\tau} B(e^\tau)d\tau \tag{2.4}$$

and the functions $N_j(\log r, \log s)$, $j = 0,1,\ldots,$ being determined recursively. Each N_j (log r, log s) is independent of λ , vanishes identically for $rs \geq a^2$, and satisfies a bound of the form

$$\max_{1 \leq r \leq s < \infty} |N_j(\log r, \log s)| \leq \frac{C}{(2j+1)!} \tag{2.5}$$

where C is a constant which is independent of j and depends only on the maximum of $|B(r)|$ in the interval $1 \leq r \leq a$. In particular (2.5) implies that the series (2.3) is uniformly convergent for $1 \leq r \leq s < \infty$ and is an entire function of λ . Since $e^{i\lambda z}$ is a solution of (2.1), the above considerations imply that a suitable choice for $v(\underset{\sim}{x})$ is given by

$$v(\underset{\sim}{x}) = v(r,\theta) = \int_r^\infty K(r,s;\lambda)e^{i\lambda s \cos\theta}ds \quad . \tag{2.6}$$

Now let $J_{n+1/2}(\lambda r)$ and $H^{(1)}_{n+1/2}(\lambda r)$ denote respectively a Bessel function and Hankel function of the first kind, and define $j_{n+1/2}(r)$ and $h_{n+1/2}(r)$ by

$$\begin{aligned} j_{n+1/2}(r) &= \underset{\sim}{K}\left[(\lambda r)^{-1/2} J_{n+1/2}(\lambda r)\right] \\ h_{n+1/2}(r) &= \underset{\sim}{K}\left[(\lambda r)^{-1/2} H^{(1)}_{n+1/2}(\lambda r)\right] \quad . \end{aligned} \tag{2.7}$$

Then from the representation ([2],p.64)

$$e^{i\lambda z} = \sqrt{\frac{\pi}{2\lambda r}} \sum_{h=o}^{\infty} (2n+1)i^n J_{n+1/2}(\lambda r)P_n(\cos\theta) \tag{2.8}$$

where $P_n(\cos\theta)$ denotes Legendre's polynomial, it is easily verified that the solution of (1.1) - (1.3) is given by

$$u(\underset{\sim}{x}) = u(r,\theta) = -\sqrt{\frac{\pi}{2}} \sum_{n=o}^{\infty} \frac{(2n+1)i^n j_{n+1/2}(1)}{h_{n+1/2}(1)} h_{n+1/2}(r)P_n(\cos\theta) . \tag{2.9}$$

Note that from the uniqueness of the solution to (1.1) - (1.3) ([1]) we can conclude that $h_{n+1/2}(1) \neq 0$, and the convergence of the series (2.9) for $1 \leq r < \infty$, $0 \leq \theta \leq \pi$ follows from (2.7) and well known estimates for Bessel functions and Legendre polynomials for large values of n (c.f.[2],p.22-23 and p.205). From the fact that

$$h_{n+1/2}(r) = (\lambda r)^{-1/2} H^{(1)}_{n+1/2}(\lambda r) \quad ; \; r \geq a \tag{2.10}$$

and the asymptotic estimate ([2],p.85)

$$H^{(1)}_{n+1/2}(\lambda r) = (-i)^{n+1} \sqrt{\frac{2}{\pi \lambda r}}\, e^{i\lambda r} \left[1 + O(\frac{1}{\lambda r})\right] \tag{2.11}$$

we can conclude (c.f.[3]) that the far field pattern $f(\theta,\phi;\lambda) = f(\theta;\lambda)$ is given by

$$f(\theta;\lambda) = \sum_{n=0}^{\infty} \frac{i(2n+1) j_{n+1/2}(1)}{\lambda\, h_{n+1/2}(1)} P_n(\cos\theta) \, . \tag{2.12}$$

Recall once again that although the far field pattern $f(\theta;\lambda)$ is assumed to be known, the functions $j_{n+1/2}(r)$ and $h_{n+1/2}(r)$ are unknown since $B(r)$ is as of yet unknown. However if we expand $f(\theta;\lambda)$ in a *a* Legendre series

$$f(\theta;\lambda) = \sum_{n=0}^{\infty} \tilde{a}_n(\lambda) \, P_n(\cos\theta) \; , \tag{2.13}$$

then from (2.12) and (2.13) we have

$$\begin{aligned} \frac{j_{n+1/2}(1)}{h_{n+1/2}(1)} &= a_n(\lambda) \\ &= a_{n_0} \lambda^{2n+1} + a_{n_1} \lambda^{2n+3} + \ldots \end{aligned} \tag{2.14}$$

where

$$a_n(\lambda) = \frac{\lambda \tilde{a}_n(\lambda)}{i(2n+1)} \tag{2.15}$$

are known (analytic) functions of λ. The fact that $a_n(\lambda)$ has a zero of order $2n+1$ at the origin follows from (2.3), (2.7), and the series representations (c.f.[2],p.4)

$$(\lambda r)^{-1/2} J_{n+1/2}(\lambda r) = \sqrt{\frac{1}{2}} \sum_{m=0}^{\infty} (-1)^m \frac{(\lambda r/2)^{2m+n}}{m!\Gamma(m+n+3/2)} \tag{2.16}$$

$$(\lambda r)^{-1/2} H^{(1)}_{n+1/2}(\lambda r) = \sqrt{\frac{1}{2}} \sum_{m=0}^{\infty} (-1)^m \left[\frac{(\lambda r/2)^{2m+n}}{m!\Gamma(m+n+3/2)} + i \frac{(-1)^{n+1} (\lambda r/2)^{2m-n-1}}{m!\Gamma(m-n+1/2} \right]$$

Equating like powers of λ in (2.14) we have

$$a_{n_o} = i \frac{(-1)^n \Gamma(-n+1/2)(1/2)^{2n+1}}{\Gamma(n+3/2)} \tag{2.17}$$

and

$$- \frac{(1/2)^{n+2}}{\Gamma(n+5/2)} + \int_1^{\infty} N_o(\log 1, \log s) \frac{(1/2)^n s^{n-1/2}}{\Gamma(n+3/2)} ds$$

$$= a_{n_o} i \frac{(-1)^n (1/2)^{-n+1}}{\Gamma(-n+3/2)} + a_{n_1} i \frac{(-1)^n (1/2)^{-n-1}}{\Gamma(-n+1/2)} \tag{2.18}$$

$$+ a_{n_o} i \int_1^{\infty} N_o(\log 1, \log s) \frac{(-1)^{n+1} (1/2)^{-n-1} s^{-n-3/2}}{\Gamma(-n+1/2)} ds .$$

Note that the coefficient a_{n_o} is independent of $B(r)$.

From (2.4) we have

$$\int_1^{\infty} N_o(\log 1, \log s) s^m ds = - \frac{1}{2} \int_1^{a^2} \int_{s^{1/2}}^{a} \xi B(\xi) s^m d\xi ds$$

$$= - \frac{1}{2} \int_1^{a} \int_1^{\xi^2} \xi B(\xi) s^m ds d\xi \tag{2.19}$$

$$= - \frac{1}{2(m+1)} \int_1^{a} s^{2m+3} B(s) ds + \frac{1}{2(m+1)} \int_1^{a} sB(s) ds$$

and hence using (2.17) and (2.19) we can rewrite (2.18) in the form

$$\mu_n = \int_1^a B(s)\left[s^{2n+2} + s^{-2n} - 2s\right] ds \qquad (2.20)$$

where
$$\mu_n = -\ (2n+1)\left[\frac{-\ \overset{(2n+1)}{\cancel{10n+11}}}{\cancel{2}(2n+3)(1-2n)} + a_{n_1}\, i\ \frac{(-1)^{n+1}(1/2)^{-2n-1}\Gamma(n+3/2)}{\Gamma(-n+1/2)}\right]$$

The μ_n are known from the far field pattern, and hence the problem of determining the function $B(r)$ has been reduced to solving the generalized moment problem (2.20), (2.21) (note that if we assume that $B(r)$ is real valued, then from (2.17) and (2.18) we have that a_{n_1} i is purely imaginary, and hence μ_n is real for $n = 0,1,\ldots$.)

III. The Generalized Moment Problem.

We will assume the existence of a continuously differentiable function $B(r)$ such that (2.20), (2.21) is valid, and address ourselves to the problem of the uniqueness of $B(r)$ and the approximation of $B(r)$ in the L^2 norm over the interval $[1,a]$. As will be clear from the analysis which follows, necessary and sufficient conditions on the sequence $\mu_n, n = 0,1,\ldots$, for (2.20), (2.21) to determine a function $B(r) \in L^2[1,a]$ can be obtained from known results on the classical Hausdorff moment problem over the interval $\left[\frac{1}{a^2}, a^2\right]$ (c.f.[4]). We however restrict ourselves solely to the problem of uniqueness and approximation since in the context of the present paper it is these problems which are of paramount interest. This is due to the fact that the sequence μ_n (or equivalently the sequence a_{n_1}) is obtained from physical measurement and it is assumed a priori that the sequence μ_n is a (generalized) moment sequence for some function $B(r)$ to be determined.

The basic problems of uniqueness and approximation can be settled by appealing to the following theorem:

<u>Theorem:</u> The functions

$$p_n(r) = r^{2n+2} + r^{-2n} - 2r ,$$

$n = 0,1,2,\ldots,$ are complete in $L^2[1,a]$.

<u>Proof:</u> Let $f(r)$ be a continuous function on the interval $[1,a]$ .Since the space of continuous functions on $[1,a]$ is dense in $L^2[1,a]$, to prove the theorem it suffices to show that if

$$\int_1^a f(s)p_n(s)ds = 0 \tag{3.1}$$

for $n = 0,1,2,\ldots,$ then $f(r) = 0$ for $r \in [1,a]$. For $r \in \left[\frac{1}{a},1\right]$ define $f(r)$ by

$$f(r) = r^{-4}f\left(\frac{1}{r}\right) \quad ; \; r \in \left[\frac{1}{a},1\right] . \tag{3.2}$$

Then

$$\int_1^a f(s)s^{-2n}ds = \int_{1/a}^1 f(s)s^{2n+2}ds \tag{3.3}$$

and hence from (3.1)

$$0 = \int_1^a f(s)\Big[p_n(s) - p_{n+1}(s)\Big]ds$$

$$= \int_1^a f(s)\Big[s^{2n+2} + s^{-2n} - s^{2n+4} - s^{-2n-2}\Big]ds \qquad (3.4)$$

$$= \int_{1/a}^a f(s)\Big[s^{2n+2} - s^{2n+4}\Big]ds$$

$$= \int_{1/a}^a [f(s) - s^2 f(s)]s^{2n+2}ds$$

$$= \frac{1}{2}\int_{1/a^2}^{a^2} f(s^{1/2})\Big[s^{1/2} - s^{3/2}\Big]s^n ds$$

for $n = 0,1,2...$. Since the set $\{r^n\}_{n=0}^{\infty}$ is complete in $L^2\left[\frac{1}{a^2}, a^2\right]$, we have from (3.4) that

$$f\left(r^{1/2}\right)\left(r^{1/2} - r^{3/2}\right) = 0 \qquad (3.5)$$

for $r \in \left[\frac{1}{a^2}, a^2\right]$, and hence $f(r) = 0$ for $r \in [1,a]$ and the theorem is proved.

The uniqueness of the function $B(r)$ follows immediately from the above theorem. Furthermore the function $B(r)$ can be approximated in $L^2[1,a]$ by orthonormalizing the set $\{p_n(r)\}_{n=0}^{\infty}$ over the interval $[1,a]$ to obtain the orthonormal set $\{\varphi_n(r)\}_{n=0}^{\infty}$ and then approximating $B(r)$ in $L^2[1,a]$ by the function

$$B_N(r) = \sum_{n=0}^{N} b_n\varphi_n(r) \qquad (3.6)$$

where

$$b_n = \int_1^a \varphi_n(s)B(s)ds. \qquad (3.7)$$

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A Reflection Principle for Solutions to the Helmholtz Equation and an Application to the Inverse Scattering Problem*

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I. Introduction.

A classical result in potential theory is the Schwarz reflection principle for solutions of Laplace's equation which vanish on a portion of a spherical boundary. The question naturally arises whether or not such a property is also true for solutions of the Helmholtz equation. This has been answered in the affirmative by Diaz and Ludford ([4]; see also [10]) in the limiting case of the plane, and it is the purpose of this paper to show that a reflection principle is also valid for spheres of finite radius. As an application of this result we will study the problem of the analytic continuation of solutions to the Helmholtz equation defined in the exterior of a bounded domain in three dimensional Euclidean space $\mathbb{R}^3$. We shall show that through the use of the reflection principle derived in this paper, this problem can be reduced to the problem of the analytic continuation of an analytic function of two complex variables, which in turn can be performed through a variety of known methods (c.f. [7]).

II. Integral Operators and the Reflection Principle.

We consider solutions of the Helmholtz equation

$$\Delta_n u + \lambda u = 0 \tag{2.1}$$

defined in $D \setminus \bar{S}$ where D is a bounded starlike domain containing the open ball

$$S = \{\underset{\sim}{x} : r = \sqrt{x_1^2 + .. + x_n^2} < a\}, \ \underset{\sim}{x} = (x_1, \ldots, x_n).$$

On the surface r=a we assume that $u(r,\theta) = u(\underset{\sim}{x})$ continuouslsy assumes the boundary data

$$u(a,\theta) = 0 \tag{2.2}$$

where $(r,\theta) = (r, \theta_1, \ldots \theta_{n-1})$ are spherical coordinates.

We shall obtain a reflection principle for solutions of (2.1),(2.2) through the use of an integral operator which maps harmonic functions defined in $D\setminus\bar{S}$ and vanishing on $r = a$ onto solutions of (2.1),(2.2). In this connection our approach resembles in some ways the "method of ascent" as developed by Gilbert ([8]), Eichler ([5]) and Colton and Wendland ([3]), except that we are now concerned with solutions defined in a multiply connected domain instead of a simply connected domain.

We look for a solution of (2.1) in the form

$$u(r,\theta) = h(r,\theta) + \int_a^r s^{n-3}K(r,s;\lambda)h(s,\theta)ds \tag{2.3}$$

where $h(r,\theta) \in C^2(D\setminus\bar{S})\cap C^0(D\setminus S)$ is a solution of

$$\Delta_n h = 0 \tag{2.4}$$

such that

$$h(a,\theta) = 0. \tag{2.5}$$

Substituting (2.3) into (2.1) and integrating by parts using (2.5) we can show that (2.3) will be a solution of (2.1) provided $K(r,s;\lambda)$ satisfies

$$r^2\left[K_{rr} + \frac{(n-1)}{r}K_r + \lambda K\right] = s^2\left[K_{ss} + \frac{n-1}{s}K_s\right] \tag{2.6}$$

and the initial data

$$K(r,r;\lambda) = -\frac{\lambda}{4}r^{2-n}(r^2-a^2) \tag{2.7}$$

$$K(r,a;\lambda) = 0. \tag{2.8}$$

Setting

$$\begin{aligned}\xi &= \log r\\ \eta &= \log s\end{aligned} \tag{2.9}$$

we transform (2.6)-(2.8) into the initial value problem

$$M_{\xi\xi} - M_{\eta\eta} + \lambda e^{2\xi} M = 0 \tag{2.10}$$

$$M(\xi, \log a;\lambda) = 0 \tag{2.11}$$

$$M(\xi,\xi;\lambda) = -\frac{\lambda}{4}(e^{2\xi} - a^2) \tag{2.12}$$

for the function

$$M(\xi,\eta;\lambda) = \exp\left\{ \frac{(n-2)}{2}(\xi+\eta)\right\} K(e^{\xi},e^{\eta};\lambda) \tag{2.13}$$

defined in the cone $\{(\xi,\eta) : \xi \leqslant \eta,\ \eta \leqslant \log a, \text{ or } \xi \geqslant \eta,\ \eta \geqslant \log a\}$. (2.10)-(2.12) is a Goursat problem for a hyperbolic equation and has a unique (analytic) solution in this cone (c.f. [6], pp.118-119). Hence we can conclude that the operator (2.3) exists. It is easy to show (c.f. [2]) that if $u(r,\theta) \in C^2(D\setminus\bar{S})\ \ C^0(D\setminus S)$ is any solution of (2.1), (2.2), then $u(r,\theta)$ can be represented in the form (2.3) for some harmonic function satisfying (2.5).

Before turning to the proof of the reflection principle for solutions $u(r,\theta)$ satisfying (2.1),(2.2), we take this opportunity to construct another integral operator which in a sense is complimentary to (2.3) and which we shall use in the next section of this paper. This operator is of the form

$$u(r,\theta) = h(r,\theta) + \int_a^r s^{n-3}\tilde{K}(r,s;\lambda)h(s,\theta)ds \tag{2.14}$$

where $h(r,\theta) \in C^2(D\setminus\bar{S})\cap C^1(D\setminus S)$ is a solution of (2.4) such that

$$h_r(a,\theta) + \frac{(n-2)}{2a} h(a,\theta) = 0. \tag{2.15}$$

In order for $u(r,\theta)$ as defined by (2.14) to be a solution of (2.1) we must have $\tilde{K}(r,s;\lambda)$ be a solution of (2.6) satisfying (2.7) and the initial data

$$\tilde{K}_s(r,a;\lambda) + \frac{(n-2)}{2a}\tilde{K}(r,a;\lambda) = 0. \tag{2.16}$$

This can be verified directly by substituting (2.14) into (2.1) and integrating by parts using (2.15). Using the change of variables (2.9) and setting

$$\tilde{M}(\xi,\eta;\lambda) = \exp\left\{\frac{(n-2)}{2}(\xi+\eta)\right\}\tilde{K}(e^{\xi},e^{\eta};\lambda) \tag{2.17}$$

we obtain the initial value problem

$$\tilde{M}_{\xi\xi} - \tilde{M}_{\eta\eta} + \lambda e^{2\xi}\tilde{M} = 0 \tag{2.18}$$

$$\tilde{M}_{\eta}(\xi, \log a;\lambda) = 0 \tag{2.19}$$

$$\tilde{M}(\xi,\xi;\lambda) = -\frac{\lambda}{4}(e^{2\xi} - a^2). \tag{2.20}$$

To solve (2.18)-(2.20) we introduce the function $E(\xi,\eta;\lambda)$ defined as the (unique) solution of the characteristic initial value problem

$$E_{\xi\xi} - E_{\eta\eta} + \lambda e^{2\xi}E = 0 \tag{2.21}$$

$$E(\xi,\xi;\lambda) = -\frac{\lambda}{4}(e^{2\xi} - a^2) \tag{2.22}$$

$$E(\xi, -\xi+2\log a;\lambda) = -\frac{\lambda}{4}(e^{2\xi} - a^2). \tag{2.23}$$

The existence of a unique (analytic) solution to (2.21)-(2.23) in the cone $\{(\xi,\eta) : \xi \leqslant \eta,\ \eta+\xi \leqslant 2\log a, \text{ or } \xi \geqslant \eta,\ \eta+\xi \geqslant 2\log a\}$ follows from standard results on hyperbolic equations (c.f. [6], pp.118-119). A solution of (2.18)-(2.20) is now given by

$$\tilde{M}(\xi,\eta;\lambda) = \frac{1}{2}\left[E(\xi,\eta;\lambda) + E(\xi, -\eta + 2\log a;\lambda)\right] \tag{2.24}$$

and we have established the existence of the operator (2.14). It is again easy to show that if $u(r,\theta) \in C^2(D\setminus\bar{S}) \cap C^1(D\setminus S)$ is any solution of (2.1) satisfying

$$u_r(a,\theta) + \frac{(n-2)}{2a}u(a,\theta) = 0, \tag{2.25}$$

then $u(r,\theta)$ can be represented in the form (2.14) for some harmonic function satisfying (2.15).

We are now in a position to prove the following reflection principle for solutions of (2.1), (2.2):

Theorem 1 (Reflection Principle): Let $u(r,\theta) \in C^2(D\setminus\bar{S}) \cap C^o(D\setminus S)$ be a solution of (2.1), (2.2) and let D^* denote the set obtained by inverting $D\setminus S$ across ∂S, i.e. $(r,\theta) \in D^*$ if and only if $(\frac{a^2}{r},\theta) \in D\setminus S$. Then $u(r,\theta)$ is a twice continuously differentiable (and hence analytic) solution of (2.1) in $D\setminus S\cup D^*$.

Remark: The fact that twice continuously differentiable solutions of (2.1) are in fact analytic follows from classical regularity theorems for solutions of (2.1) (c.f. [6]).

Proof of Theorem: From our previous discussion we can represent $u(r,\theta)$ in the form (2.3) where $h(r,\theta)$ satisfies (2.4) and (2.5). Furthermore $u(r,\theta) \in C^2(D\setminus\bar{S}) \cap C^o(D\setminus S)$ implies that $h(r,\theta) \in C^2(D\setminus\bar{S}) \cap C^o(D\setminus S)$. Hence from the reflection principle for harmonic functions $h(r,\theta)$ is harmonic in $D\setminus S\cup D^*$ and hence by (2.3) $u(r,\theta)$ is twice continuously differentiable in $D\setminus S\cup D^*$.

III. An Application to the Inverse Scattering Problem.

Suppose an incoming plane acoustic wave of frequency ω moving in the direction of the z axis is scattered off a "soft" bounded obstacle Ω in $\mathbb{R}^3$ and that $u(r,\theta,\phi)e^{i\omega t}$ is the velocity potential of the scattered wave, where (r,θ,ϕ) denote spherical coordinates. Then $u(r,\theta,\phi)$ will be a solution of (2.1) in the exterior of Ω for n=3, $\lambda = \frac{\omega^2}{c^2} = k^2$, where c is the speed of sound. At infinity $u(r,\theta,\phi)$ has the asymptotic behaviour

$$u(r,\theta,\phi) \sim \frac{e^{ikr}}{r} f(\theta,\phi) \tag{3.1}$$

where $f(\theta,\phi)$ is the far field pattern (c.f. [13]). The inverse scattering problem is to determine Ω, given the fact that $f(\theta,\phi)$ is known exactly. From the results of Müller ([13]) we can determine $u(r,\theta,\phi)$ outside the smallest ball S containing Ω in its interior, where S can be determined from a knowledge of $f(\theta,\phi)$. In particular if the radius of S is a, one can write ([9], [13]).

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} h_n^{(1)}(kr) S_{nm}(\theta,\phi); \qquad r \geqslant a \quad (3.2)$$

where the coefficients a_{nm} are determined from the far field pattern $f(\theta,\phi)$, $h_n^{(1)}$ denotes a spherical Hankel function, S_{nm} a spherical harmonic, and the series (3.2) is uniformly convergent for $r \geqslant a$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$. Hence to find Ω we must analytically continue $u(r,\theta,\phi)$ as given by (3.2) across the boundary of S and look for the locus of points where $u(r,\theta,\phi) + \exp(ikr\cos\theta) = 0$. This problem of analytic continuation has been studied by many research workers, in particular, Weston, Bowman and Ar ([15]), Colton ([1]), Sleeman ([14]), Millar ([12]) and Hartman and Wilcox ([9]). In this section we shall contribute to this study by using the integral operators and reflection principle of section II to relate the domain of regularity of $u(r,\theta,\phi)$ to that of an analytic function of two complex variables. The advantage of such a relationship is that once this has been done there is a number of known methods for determining the domain of regularity of analytic functions of several complex variables; in particular see [8], section 1.3. We first prove the following theorem (Compare this result to that of Millar in the simpler case of two dimensions ([11])):

Theorem 2: Let $h(r,\theta,\phi)$ be the (unique) harmonic function defined in the exterior of the ball S such that $h(a,\theta,\phi) = u(a,\theta,\phi)$ on ∂S. If $h(r,\theta,\phi)$ can be continued to a harmonic function defined in the exterior of a starlike domain $D \subset S$, then $u(r,\theta,\phi)$ can be continued as a solution of (2.1) (with $n=3$, $\lambda = k^2$) into the exterior of D.

Proof: Let $\tilde{h}(r,\theta,\phi)$ be the harmonic function defined by

$$\tilde{h}(r,\theta,\phi) = \frac{1}{2}\left[h(r,\theta,\phi) + \left(\frac{a}{r}\right) h\left(\frac{a^2}{r},\theta,\phi\right)\right] \qquad (3.3)$$

where we have made use of Kelvin's inversion formula. Then $\tilde{h}(r,\theta,\phi)$ is regular in $D^* \cup S \setminus D$ where D^* denotes the set obtained by inverting $\bar{S} \setminus D$ across ∂S. Furthermore we have

$$\tilde{h}_r(a,\theta,\phi) + \frac{1}{2a}\tilde{h}(a,\theta,\phi) = 0. \qquad (3.4)$$

Hence from (2.14) we have that

$$\tilde{u}(r,\theta,\phi) = \tilde{h}(r,\theta,\phi) + \int_a^r \tilde{K}(r,s;k^2)\tilde{h}(s,\theta,\phi)\,ds \qquad (3.5)$$

is a solution of

$$\Delta_3 u + k^2 u = 0 \qquad (3.6)$$

om $D^* \cup S \setminus D$ and

$$u(a,\theta,\phi) = u(a,\theta,\phi). \qquad (3.7)$$

Therefore $w(r,\theta,\phi) = u(r,\theta,\phi) - \tilde{u}(r,\theta,\phi)$ is a solution of (3.6) in D^* such that $w(a,\theta,\phi) = 0$, and hence by Theorem 1 $w(r,\theta,\phi)$ is an analytic solution of (3.6) in $D^* \cup S \setminus D$. We can now conclude that $u(r,\theta,\phi)$ is analytic in $D^* \cup S \setminus D$ and since $u(r,\theta,\phi)$ is already known to be analytic in the exterior of S, the Theorem follows.

In order to apply Theorem 2 it is necessary to have a method for determining the location of the singularities of the harmonic function $h(r,\theta,\phi)$. But this theory has been extensively developed by Gilbert

([7], [8]). In particular since

$$h(r,\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} h_n^{(1)}(ka)\left(\frac{r}{a}\right)^{-n-1} S_{nm}(\theta,\phi) \qquad (3.8)$$

for $r \geqslant a$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$, we have (c.f. [7], chapter 3, or [8], chapter 7) that the singular points of $h(r,\theta,\phi)$ can be determined from aknowledge of the singular points of the analytic function of two complex variables

$$g(z_1,z_2) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} h_n^{(1)}(ka) z_1^n z_2^m . \qquad (3.9)$$

As previously pointed out, methods for determining the singular points of (3.9) can be found in [7]. It should be observed that in the case when $u(r,\theta,\phi) = u(r,\theta)$ is axially symmetric (i.e. independent of ϕ) then $g(z_1,z_2) = g(z_1)$ is an analytic function of a single complex variable, and all calculations are considerably simplified (c.f. [1]).

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The Propagation of Acoustic Waves in a Spherically Stratified Medium†

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I. Introduction.

The use of integral operators to solve problems in applied mathematics dates back to the time of B. RIEMANN and his investigation of certain problems in gas dynamics ([18]). More recently I.N. VEKUA and S. BERGMAN have used the method of integral operators to develop a systematic theory of elliptic equations in two independent variables with the aim of studying various problems in the theory of elasticity and fluid dynamics ([22], [2], [14]). D. COLTON has recently extended this approach to the case of parabolic equations and mathematical problems arising in the theory of heat conduction ([5]). Further applications of integral operators to problems in mathematical physics have been made by A.V. BITSADZE ([13]), R.P. GILBERT ([11]), F. BAUER, P. GARABEDIAN and D. KORN ([1]), and B. LEVITAN ([16]), to mention but a few of many researchers. The basic theme in all of the above-mentioned work in the theory of integral operators has been the interplay between problems in pure and applied mathematics, with particular attention being paid to (well posed) boundary value problems, (improperly posed) inverse problems, and the unique continuation of solutions to partial differential equations. In this paper we shall survey some of the recent results we have obtained on the use of integral operators in the study of wave propagation in a nonhomogeneous medium. In view of the fundamental contributions that I.N. VEKUA has made to both the theory of integral operators and mathematical problems in wave propagation, it gives us particular pleasure to dedicate this paper to him on the occasion of his seventieth birthday.

II. Integral Operators.

The partial differential equation which appears in the course of our investigations is of the form

$$\Delta_3 u + k^2(1+B(r))u = 0, \tag{2.1}$$

where Δ_3 denotes the Laplacian in $\mathbb{R}^3$, k is a parameter (the wave number), and $B(r)$ is a continuously differentiable function of $r = |\underset{\sim}{x}|$ having compact support, i.e. $B(r) = 0$ for $r \geqslant a$ where a is a constant. In our study we shall need two integral operators associated with (2.1) which are in a sense complimentary. The first of these is due to R.P. GILBERT ([12]) and is a generalization of an operator constructed by I.N. VEKUA for the case when $B(r) = 0$ ([22], pp. 57-61). This operator is of the form

$$\begin{aligned} u(\underset{\sim}{x}) &= (\underset{\sim}{I} - \underset{\sim}{G}_1)h \\ &= h(\underset{\sim}{x}) - \int_0^r \left(\frac{s}{r}\right)^{\frac{1}{2}} R_3(r,r;s,0)\, h(s,\theta,\phi)ds \\ &= h(\underset{\sim}{x}) - 2r \int_0^1 \sigma^2 R_3(r,\, r;r\sigma^2,0)\, h(\underset{\sim}{x}\sigma^2)d\sigma \end{aligned} \tag{2.2}$$

where (r,θ,ϕ) denote spherical coordinates, $h(\underset{\sim}{x}) = h(r,\theta,\phi)$ is a harmonic function regular in a neighbourhood of the origin, $R(x,y;\xi,\eta)$ is the Riemann function for

$$R_{xy} + \frac{k^2}{4}(1 + B(\sqrt{xy}))R = 0, \tag{2.3}$$

and the subscript denotes differentiation with respect to ξ. It can be verified that every solution of (2.1) regular in a neighbourhood of the origin can be represented in the form (2.2) for some harmonic function $h(\underset{\sim}{x})$, and conversely if $h(\underset{\sim}{x})$ is harmonic then $u(\underset{\sim}{x})$ as defined by (2.2) is a solution of (2.1) in some neighbourhood of the origin. The second operator we shall need is due to D. COLTON and W. WENDLAND ([9]) and maps solutions of the Helmholtz equation

$$\Delta_3 h + k^2 h = 0 \tag{2.4}$$

defined in the exterior of a bounded, starlike (with respect to the origin) domain D onto solutions of (2.1) defined in the exterior of D by the relation

$$u(\underset{\sim}{x}) = (\underset{\sim}{I} - \underset{\sim}{G}_2)h$$

$$= h(\underset{\sim}{x}) - \int_r^{\infty} G(r,s;k)\ h(s,\theta,\phi)\,ds \qquad (2.5)$$

where $G(r,s;k)$ is the twice continuously differentiable solution of

$$r^2\left[G_{rr} + \frac{2}{r}G_r + k^2(1+B(r))G\right] = s^2\left[G_{ss} + \frac{2}{s}G_s + k^2G\right] \qquad (2.6)$$

for $s > r$ satisfying the rather unusual boundary conditions (see figure 1 below)

$$G(r,s;k) = 0 \quad \text{for } rs \geqslant a^2 \qquad (2.7)$$

$$G(r,r;k) = \frac{k^2}{2r}\int_r^{\infty} sB(s)\,ds \qquad (2.8)$$

$$G(r,s;k) = 0 \quad \text{for } s < r. \qquad (2.9)$$

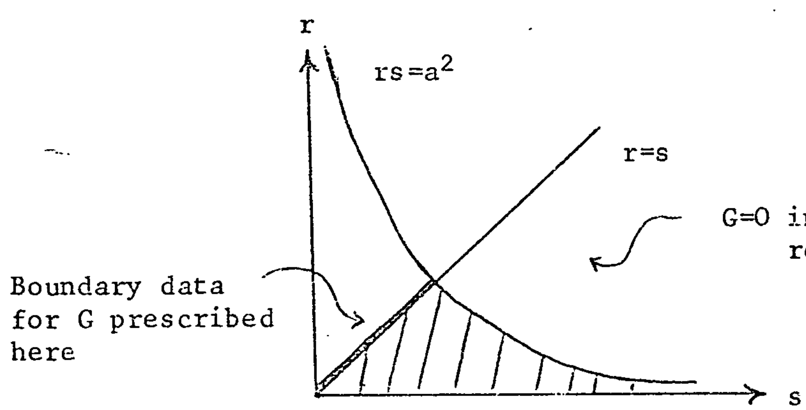

Figure 1

The solution of (2.6) - (2.9) can be obtained in the form

$$G(r,s;k) = (rs)^{-\frac{1}{2}} \sum_{j=0}^{\infty} k^{2j+2} N_j\ (\log r,\ \log s) \qquad (2.10)$$

with

$$N_o(\log r,\ \log s) = \frac{1}{2}\int_{(rs)^{\frac{1}{2}}}^{a} \xi B(\xi)\,d\xi \qquad (2.11)$$

and the functions $N_j(\log r, \log s)$, $j = 1, 2, \ldots$, being determined recursively. Due to the fact that $B(r) = 0$ for $r \geqslant a$ it can be shown ([9]) that the series (2.10) is uniformly convergent for $0 < \delta \leqslant r \leqslant s < \infty$

(where δ is an arbitrarily small constant) and is an entire function of k. It can easily be shown that every solution of (2.1) regular in the exterior of D can be represented in the form (2.5) for some solution $h(\underset{\sim}{x})$ of (2.4) regular in the exterior of D. Note also that if $h(\underset{\sim}{x})$ satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial h}{\partial r} - ikh\right) = 0 \tag{2.12}$$

then by (2.7) so will $u(\underset{\sim}{x}) = (\underset{\sim}{I} - \underset{\sim}{G}_2)h$.

III. The Scattering of Acoustic Waves.

We shall now show how the integral operators constructed in the previous section can be used to solve boundary value problems arising in the theory of the scattering of acoustic waves in a stratified (or layered) medium. (For a general discussion of wave propagation in a stratified medium we refer the reader to the book by L. BREKHOVSKIKH ([4])). We shall consider the case when a plane wave of frequency ω moving in the direction of the z-axis is scattered by a quasi-homogeneous, spherically stratified medium of compact support which may contain a "hard" scattering body D. After factoring out a term of the form $e^{i\omega t}$ we are led to the following two problems for determining the velocity potential $u(\underset{\sim}{x})$. (In the problems below we denote the local speed of sound by $c(r)$, assume that $c(r) = c_o$ = constant for $r > a$, and set $B(r) = \left(\frac{c_o}{c(r)}\right)^2 - 1$):

Problem 1 (No Obstacle Present): Determine $u(\underset{\sim}{x})$ from the equations

$$u(\underset{\sim}{x}) = e^{ikz} + u_s(\underset{\sim}{x}) \tag{3.1}$$

$$\Delta_3 u + k^2(1+B(r))u = 0 \quad \text{in } \mathbb{R}^3 \tag{3.2}$$

$$\lim_{r\to\infty} \left(\frac{\partial u_s}{\partial r} - iku_s\right) = 0 \tag{3.3}$$

where $u_s(\underset{\sim}{x})$ is the velocity potential of the scattered wave.

Problem 2 (Obstacle Present): Let D be a bounded domain, starlike with respect to the origin, with smooth boundary ∂D and outward normal ν. Determine $u(\underset{\sim}{x})$ from the equations

$$u(\underset{\sim}{x}) = e^{ikz} + u_s(\underset{\sim}{x}) \tag{3.4}$$

$$\Delta_3 u + k^2(1+B(r))u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D} \tag{3.5}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D \tag{3.6}$$

$$\lim_{r\to\infty} \left(\frac{\partial u_s}{\partial r} - iku_s\right) = 0 \tag{3.7}$$

where $u_s(\underset{\sim}{x})$ again denotes the velocity potential of the scattered wave and $\bar{D}$ denotes the closure of D.

We first consider Problem 1. This is of course a classical problem and has been considered by a wide variety of research workers; we refer in particular to C. RORRES ([19]) and (for the closely related case of potential scattering) to A. MESSIAH ([17]). We shall show how a constructive method for solving Problem 1 can be obtained in a few lines through the use of the operator $\underset{\sim}{I} - \underset{\sim}{G}_1$. In particular, from (2.2) we can represent $u(\underset{\sim}{x})$ for $r \leqslant a$ in the form

$$u(\underset{\sim}{x}) = \sum_{n=0}^{\infty} b_n \, u_n(r) \, P_n(\cos\theta); \quad r \leqslant a \tag{3.8}$$

where

$$u_n(r) = r^n\left[1 - 2r\int_0^1 \sigma^{2n+2} \, R_3(r,r;\, r\sigma^2,0)\,d\sigma\right], \tag{3.9}$$

$P_n(\cos\theta)$ denotes Legendre's polynomial, and the constants b_n, $n = 0, 1, 2, \ldots$, are to be determined. Since $B(r) = 0$ for $r \geqslant a$ we have, using Sonine's formula to expand e^{ikz},

$$u(\underset{\sim}{x}) = \sum_{n=0}^{\infty} \left[a_n \, h_n^{(1)}(kr) + (2n+1)i^n \, j_n(kr)\right]P_n(\cos\theta); \quad r \geqslant a \tag{3.10}$$

for $r \geqslant a$, where $h_n^{(1)}(kr)$ denotes a spherical Hankel function of the first kind, $j_n(kr)$ a spherical Bessel function, and the constants

a_n, n = 0, 1, 2, ... are to be determined. We now require that the representations (3.8) and (3.10) agree, along with their first derivatives, at r = a. This leads to a two by two system of algebraic equations for the determination of a_n and b_n and it can be easily shown that this system always has a unique solution (for any value of the wave number k). Using Cramer's rule to solve this system for the a_n and b_n, and applying standard asymptotic estimates for the Bessel and Hankel functions in conjunction with

$$a^n\left[1-2a\int_0^1 \sigma^{2n+2}\, R_3(a,a;\ a\sigma^2,0)\,d\sigma\right] = a^n\left[1+O(\tfrac{1}{n})\right] \tag{3.11}$$

$$\frac{d}{dr}\left[r^n - 2r^{n+1}\int_0^1 \sigma^{2n+2}\, R_3(r,r;r\sigma^2,0)\,d\sigma\right]_{r=a} = na^n\left[1+O(\tfrac{1}{n})\right], \tag{3.12}$$

show that

$$\begin{aligned} a_n &= O\left(\frac{a^{2n}\,k^{2n}}{2^{2n}\,\Gamma(n+\frac{1}{2})\,\Gamma(n+3/2)}\right) \\ b_n &= O\left(\frac{a^n\,k^n}{2^n\,\Gamma(n+\frac{1}{2})}\right) . \end{aligned} \tag{3.13}$$

We can now conclude that the series (3.8) and (3.10) are absolutely and uniformly convergent, and hence we have a constructive method for solving Problem 1. More precise estimates for the coefficients a_n (and hence b_n) shall be obtained in the next section when we consider the inverse problem to Problem 1.

We now turn our attention to Problem 2. This problem has also been studied by several mathematicians, in particular R. LEIS ([15]) and P. WERNER ([24]). Our aim, as with the case of Problem 1, is to use the theory of integral operators to provide a constructive method for solving the problem, in particular one that is suitable for numerical computation. We shall accomplish this by using the operator $\underset{\sim}{I} - \underset{\sim}{G}_2$ to reformulate Problem 2 as a Fredholm integral equation over ∂D. As in the case of a homogeneous medium (c.f. [22]), particular problems arise due to the presence of eigenvalues for the interior Dirichlet problem for (3.5). We shall overcome this difficulty by using some recent results of D.S. JONES for the Helmholtz equation ([13]) which were in turn motivated by the work of F. URSELL ([21]). In particular we look for a solution of Problem 2 in the form $u(\underset{\sim}{x}) = (\underset{\sim}{I} - \underset{\sim}{G}_2)\, h(\underset{\sim}{x})$ and represent $h(\underset{\sim}{x})$ in the form

$$h(\underset{\sim}{x}) = e^{ikz} + \int_{\partial D} \mu(\underset{\sim}{\xi})\, \Gamma(\underset{\sim}{\xi},\underset{\sim}{x};k)\, d\omega_{\underset{\sim}{\xi}} \tag{3.14}$$

where

$$\Gamma(\underset{\sim}{\xi},\underset{\sim}{x};k) = \frac{e^{ikR}}{R} + \sum_{n=0}^{N} \sum_{m=-n}^{n} b_{mn}\, \psi_{mn}(\underset{\sim}{\xi})\psi_{mn}(\underset{\sim}{x}), \tag{3.15}$$

$R = |\underset{\sim}{\xi} - \underset{\sim}{x}|$, $d\omega_{\underset{\sim}{\xi}}$ is an element of surface area at the point $\underset{\sim}{\xi} \in \partial D$, the b_{mn} are nonzero real constants (arbitrary, but fixed), and

$$\psi_{mn}(\underset{\sim}{x}) = h_n^{(1)}(kr)\; S_{nm}(\theta,\phi), \tag{3.16}$$

where $h_n^{(1)}$ denotes a spherical Hankel function of the first kind and S_{nm} a spherical harmonic. In (3.14), (3.15), $\mu(\underset{\sim}{\xi})$ is a contiouous density to be determined and N is an integer such that $k < k_{N+2}$ where k_j denotes the jth eigenvalue for the interior Dirichlet problem for the Helmholtz equation (2.4) defined in D. Methods for computing N can be found in [13]. Substituting (3.14) into the representation $u(\underset{\sim}{x}) = (\underset{\sim}{I} - \underset{\sim}{G}_2)h(\underset{\sim}{x})$, interchanging orders of integration, and applying the boundary condition (3.6) now leads to the following Fredholm integral equation with a weakly singular kernel for the determination of $\mu(\underset{\sim}{x})$:

$$\begin{aligned} \frac{1}{2\pi} f(\underset{\sim}{x}) &= \mu(\underset{\sim}{x}) - \frac{1}{2\pi} \int_{\partial D} \mu(\underset{\sim}{\xi}) \frac{\partial}{\partial \nu_{\underset{\sim}{x}}} \Gamma(\underset{\sim}{\xi},\underset{\sim}{x};k)\, d\omega_{\underset{\sim}{\xi}} \\ &\quad + \frac{1}{2\pi} \int_{\partial D} \mu(\underset{\sim}{\xi}) \frac{\partial}{\partial \nu_{\underset{\sim}{x}}} \left\{ \int_r^{\infty} G(r,s;k)\, \Gamma(\underset{\sim}{\xi}, s\frac{\underset{\sim}{x}}{r};k)\, ds \right\} d\omega_{\underset{\sim}{\xi}} \\ &\overset{\text{def.}}{=} (\underset{\sim}{I} - \underset{\sim}{T}(k))\mu \end{aligned} \tag{3.17}$$

where $\underset{\sim}{x} \varepsilon \partial D$ and

$$f(\underset{\sim}{x}) = \frac{\partial}{\partial \nu} \left[e^{ikz} - \int_r^\infty G(r,s;k) e^{iks\cos\theta} ds\right]. \tag{3.18}$$

(Note that the quantity in brackets in (3.18) represents the unique continuation of the function e^{ikz} as a solution of (3.5)). We now have to show that the operator $\underset{\sim}{I} - \underset{\sim}{T}(k)$ is invertible. To this end (motivated by the work of I.N. VEKUA in [22]) the following Theorem was established in [9]:

Theorem: Let $k > 0$ and define the operator $\underset{\sim}{T}_o(k)$ by

$$\underset{\sim}{T}_o(k)\mu = \frac{1}{2\pi} \int_{\partial D} \mu(\xi) \frac{\partial}{\partial \nu_{\underset{\sim}{x}}} \Gamma(\underset{\sim}{\xi},\underset{\sim}{x};k) d\omega_{\underset{\sim}{\xi}}; \quad \underset{\sim}{x} \varepsilon \partial D.$$

Then $(\underset{\sim}{I} - \underset{\sim}{T}(k))^{-1}$ exists if and only if $(\underset{\sim}{I} - \underset{\sim}{T}_o(k))^{-1}$ exists, where all mappings are understood to be in the space C^o.

From the previously mentioned work of D.S. JONES on the Helmholtz equation, we now have the following Corollary:

Corollary: Let N be such that $k < k_{N+2}$ where k_j denotes the jth eigenvalue for the interior Dirichlet problem for $\Delta_3 h + k^2 h = 0$ in D. Then $(\underset{\sim}{I} - \underset{\sim}{T}(k))^{-1}$ exists.

Given the fact that $(\underset{\sim}{I} - \underset{\sim}{T}(k))^{-1}$ exists, we can now use any one of a variety of methods for obtaining numerical approximations to the density $\mu(\underset{\sim}{x})$ (c.f. [10]) and hence the solution to Problem 2.

IV. Inverse Problems.

We now consider the inverse problems to Problems 1 and 2, i.e. given the far field pattern

$$f(\theta,\phi;k) = \lim_{r\to\infty} r\, e^{-ikr} u_s(\underset{\sim}{x}) \tag{4.1}$$

for $0 < k_o < k < k_1$ where k_o and k_1 are fixed constants, to determine $B(r)$ (and hence the speed of sound $c(r)$). We first consider the inverse of Problem 1. From (3.10) it is easily established that

$$f(\theta,\phi;k) = f(\theta;k) = \frac{1}{k}\sum_{n=0}^{\infty} (-i)^{n+1} a_n P_n(\cos\theta) . \tag{4.2}$$

We note, however, that if $f(\theta;k)$ is obtained from experimental data we can only assume that the coefficients of the first N harmonics are known, i.e. $a_n = a_n(k)$ is known for $0 \leqslant n \leqslant N$ and $0 < k_o < k < k_1$. Our problem then is to obtain an optimal evaluation of $B(r)$, $0 \leqslant r \leqslant a$, given a knowledge of the first N coefficients $a_1(k)$, $a_2(k)$, ..., $a_N(k)$. From the representations (3.8) – (3.10) and the continuity of u and $\frac{\partial u}{\partial r}$ across $r = a$ we have

$$\begin{aligned}\left[(2n+1)i^n j_n(ka) + a_n(k)\, h_n^{(1)}(ka)\right] \frac{du_n(a)}{dr} &= \\ &= u_n(a) \frac{d}{dr}\left[(2n+1)i^n j_n(kr) + a_n(k)\, h_n^{(1)}(kr)\right]_{r=a} .\end{aligned} \tag{4.3}$$

From the power series expansions of the Bessel and Hankel functions and the fact that the Riemann function depends analytically on the parameter k^2 we can conclude from (4.3) that $a_n(k)$ must be analytic in a neighbourhood of the origin and have a Taylor expansion of the form

$$a_n(k) = a_{no}k^{2n+3} + a_{n1}k^{2n+5} + \ldots \tag{4.4}$$

We now recall that the Riemann function $R(x,y;\xi,\eta)$ is the (unique) solution of the integral equation

$$R(x,y;\xi,\eta) = 1 - \frac{k^2}{4}\int_\xi^x \int_\eta^y (1 + B(\sqrt{\sigma\tau}))R(\sigma,\tau;\xi,\eta)\,d\sigma d\tau \tag{4.5}$$

which implies that

$$R_3(a,a;a\sigma^2,0) = \frac{k^2}{2a\sigma^2}\int_o^{a\sigma} s(1 + B(s))ds + O(k^4) \tag{4.6}$$

and

$$\frac{d}{dr}\left[R_3(r,r;r\sigma^2,0)\right]_{r=a} = \frac{k^2}{2}(1 + B(a\sigma)) - \frac{k^2}{2a^2\sigma^2}\int_o^{a\sigma} s(1 + B(s))ds + O(k^4) . \tag{4.7}$$

Substituting (4.4), (4.6) and (4.7) into (4.3) and equating the coefficient of k^{n+2} now leads to the moment problem

$$a_{no} i^{n-1} (2n+1) \left(\frac{(2n)!}{2^n \; n!}\right)^2 = \int_0^a \sigma^{2n+2} \; B(\sigma)\, d\sigma \tag{4.8}$$

for the determination of B(r) ([7]). Hence, under the assumption that a_{no} is known for $0 \leqslant n \leqslant N$, an optimal choice for B(r) (in the space $L^2[0,a]$) can be found by orthonormalizing the set $\{\sigma^{2n+2}\}$ over the interval $[0,a]$ and using the relation (4.8) to determine the first N Fourier coefficients of the orthonormal expansion of B(r). We note that (4.8) is in agreement with the moment problem obtained by assuming B(r) is small and using the Born approximation ([17]) and also with the results of C. RORRES for the case when k is small ([19]). However our derivation makes no assumption on the magnitude of either B(r) or k. This verifies a conjecture (viz. that the moment problem associated with the inverse scattering problem is independent of the magnitude of B(r) or k) of B. SLEEMAN ([20]) in the special case of a spherically stratified medium of compact support. We also note that the determination of B(r) from the far field pattern is an improperly posed problem in the sense that B(r) does not depend continuously on the far field data $f(\theta;k)$. This can be seen from (4.8) where small variations in a_{no} can cause large variations in the integral on the right hand side if n is large. Finally we observe that the completeness of the set $\{\sigma^{2n+2}\}$ in $L^2[0,a]$ implies the following Theorem:

<u>Theorem</u>: For problem 1 the far field pattern uniquely determines the speed of sound in the nonhomogeneous medium.

We now turn our attention to the inverse of Problem 2, i.e. given the far field pattern $f(\theta,\phi;k)$, to determine B(r) in $S(0;a)\setminus D$, where $S(0;a)$ denotes the ball of radius a in $\mathbb{R}^3$. We again assume that $f(\theta,\phi;k)$ is only known to within a certain amount of error, i.e. for a given integer N

$$f_N(\theta,\phi;k) = \frac{1}{k} \sum_{n=0}^{N} \sum_{m=-n}^{n} (-i)^{n+1} \, a_{nm} \, S_{nm}(\theta,\phi) \tag{4.9}$$

is known, and from this information we want to determine an optimal evaluation of $B(r)$ in $S(0;a)\setminus D$. From the results of Section III of this paper and a Theorem of I.N. VEKUA ([23]) we can conclude that the solution of Problem 2 can be approximated in an L^2 sense by a finite linear combination of functions of the form

$$u_{nm}(\underset{\sim}{x};k) = (\underset{\sim}{I} - \underset{\sim}{G}_2)\left[h_n^{(1)}(kr)\; S_{nm}(\theta,\phi)\right] \tag{4.10}$$

where, as in Section III, $h_n^{(1)}(kr)$ denotes a spherical Hankel function of the first kind and $S_{nm}(\theta,\phi)$ a spherical harmonic. If we now orthonormalize the set $\{u_{nm}(\underset{\sim}{x};k)\}$ with respect to the inner product (defined over the space of solutions to (3.5) satisfying (3.7)) given by

$$(f,g) = \int_{\partial D} \frac{\partial f}{\partial \nu}\,\frac{\partial g}{\partial \nu}\, d\omega \tag{4.11}$$

to obtain the set $\{\Psi_{nm}(\underset{\sim}{x};k)\}$, we have that the solution of Problem 2 can be approximated by a function of the form

$$u_N(\underset{\sim}{x}) = \sum_{n=0}^{N}\;\sum_{n=-m}^{m} \tilde{a}_{nm}(k)\; \Psi_{nm}(x;k) \tag{4.12}$$

where

$$\tilde{a}_{nm}(k) = \int_{\partial D} f(\underset{\sim}{x})\, \frac{\partial}{\partial \nu}\, \Psi_{nm}(\underset{\sim}{x};k)\, d\omega \tag{4.13}$$

and $f(\underset{\sim}{x})$ is given by (3.18). From (4.12) we have that the far field pattern of $u_N(\underset{\sim}{x})$ is given by

$$f_N(\theta,\phi;k) = \sum_{n=0}^{N}\;\sum_{m=-n}^{n} \tilde{a}_{nm}(k)\; \Phi_{nm}(\theta,\phi;k) \tag{4.14}$$

where each $\Phi_{nm}(\theta,\phi;k)$ is a finite linear combination of the $S_{jk}(\theta,\phi)$, $0 \leq j \leq n$, $-j \leq k \leq j$, with coefficients depending on $B(r)$. In the case of the inverse problem these coefficients (as well as the $\tilde{a}_{nm}(k)$) are of course unknown. However (4.9) is known for the inverse problem, and equating (4.9) to (4.14) and identifying like powers of k leads to a moment problem of the form

$$\gamma_{nm}(N) = \iint_{S(0;a)\setminus D} B(r)\, p_{nm}(r,\theta,\phi)\, dV; \quad 0 \leqslant n \leqslant N, \ -n \leqslant m \leqslant n \tag{4.15}$$

where the $\gamma_{nm}(N)$ are known constants and the $p_{nm}(r,\theta,\phi)$ are known functions. An optimal choice for $B(r)$ can now be obtained by orthonormalizing the set $\{p_{nm}\}$ over the annulus $S(0;a)\setminus D$ and using the relation (4.15) to determine the first N Fourier coefficients of the orthonormal expansion of $B(r)$. For an example of the calculations involved in the above procedure for the case of scattering by a 'soft' sphere of radius one the reader is referred to [8].

Integral operators can also be used to investigate other inverse problems arising in acoustic scattering theory, for example the problem of determining the shape of the scattering obstacle from a knowledge of the far field pattern. For a discussion of this and other inverse problems in acoustic scattering theory the reader is referred to the author's survey article [6].

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